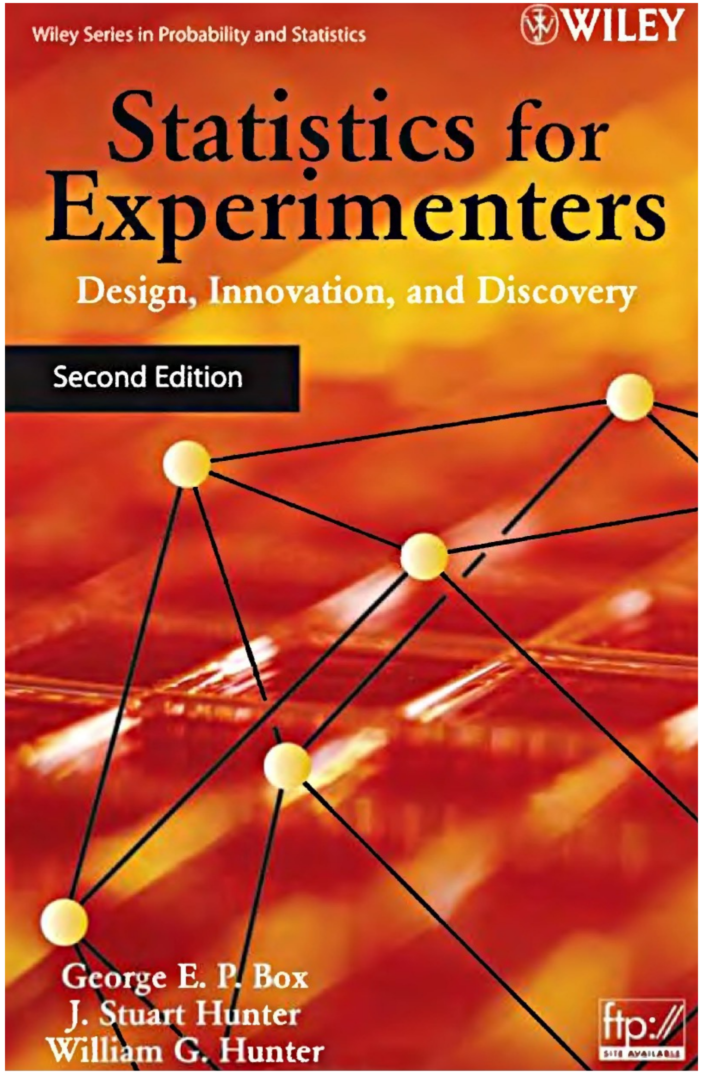
Powertrain Calibration Optimisation

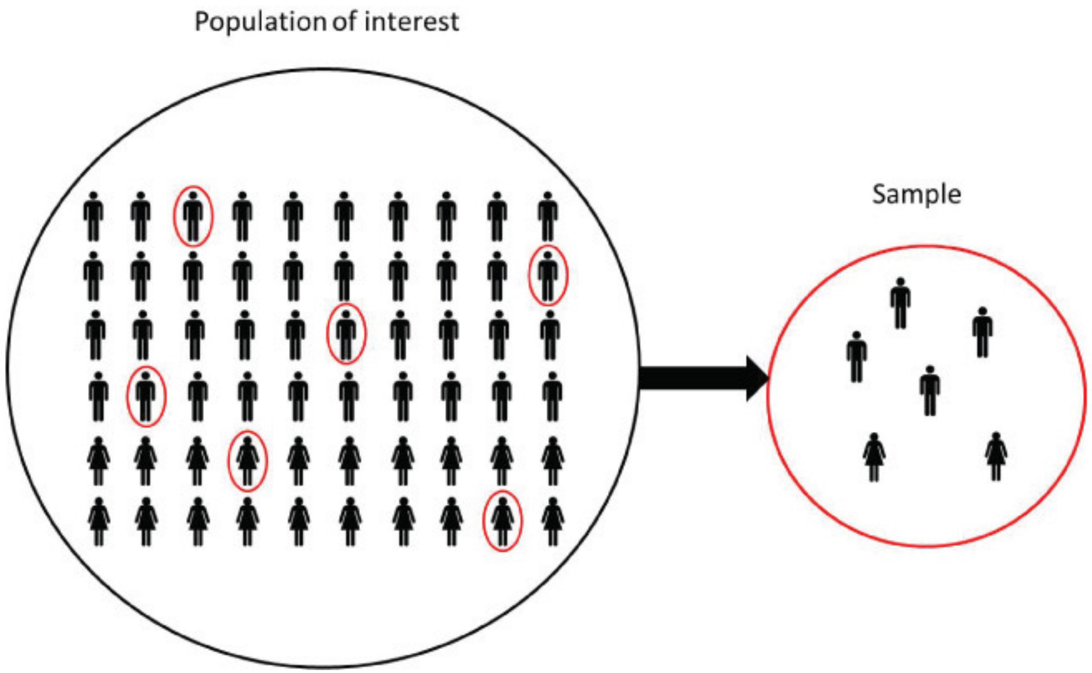
Introduction to Statistics

Overview

- Basic concepts
- Continuous distributions
- Estimation
- Significance tests
- Regression
- ANOVA


Wiley Series in Probability and Statistics
WILEY
Statistics for Experimenters
Design, Innovation, and Discovery
Second Edition
George E. P. Box
J. Stuart Hunter
William G. Hunter
ftp://

Population vs sample



From a set of data, $\bar{y}$ and $s = \sum_N \frac{(x_i - \eta)^2}{N-1}$ form **estimates** of the *population* mean, μ and standard deviation, σ

Degrees of Freedom

- How many choices?
- Degrees of freedom relate to the number of ‘observations’ that are free to vary when estimating statistical parameters

$$mean = \frac{x_1 + x_2 + \cdots x_n}{n}$$

In calculating the mean only $n-1$ observations are ‘free to vary’

Making measurements – location and spread

Mean, $\bar{x} = \frac{x_1+x_2+\cdots+x_N}{N} = \sum_N \frac{x_i}{N}$

Also known as the expectation of x i.e. $E(x)$.

Standard deviation, s, variance, s^2

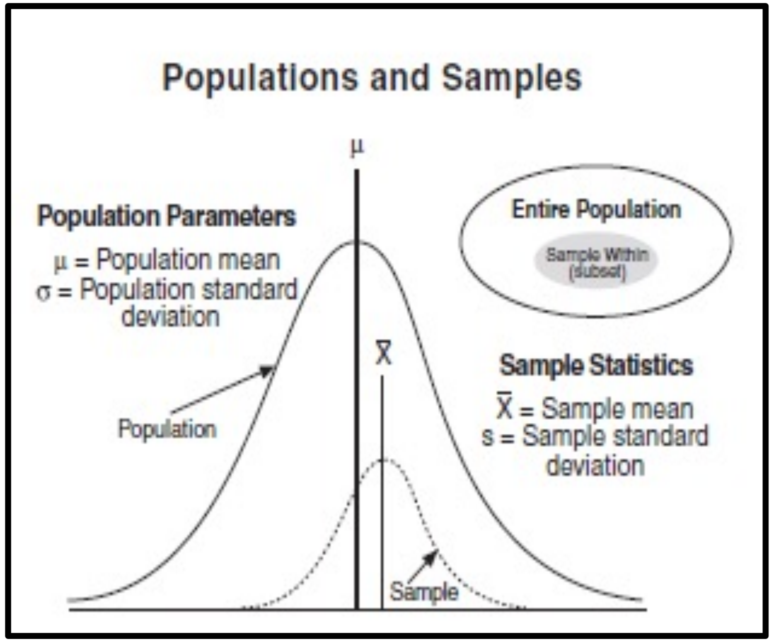
$$s^2 = \frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+\cdots+(x_N-\bar{x})^2}{N-1} = \sum_N \frac{(x_i-\bar{x})^2}{N-1}$$

sample variance

For ease of calculation,

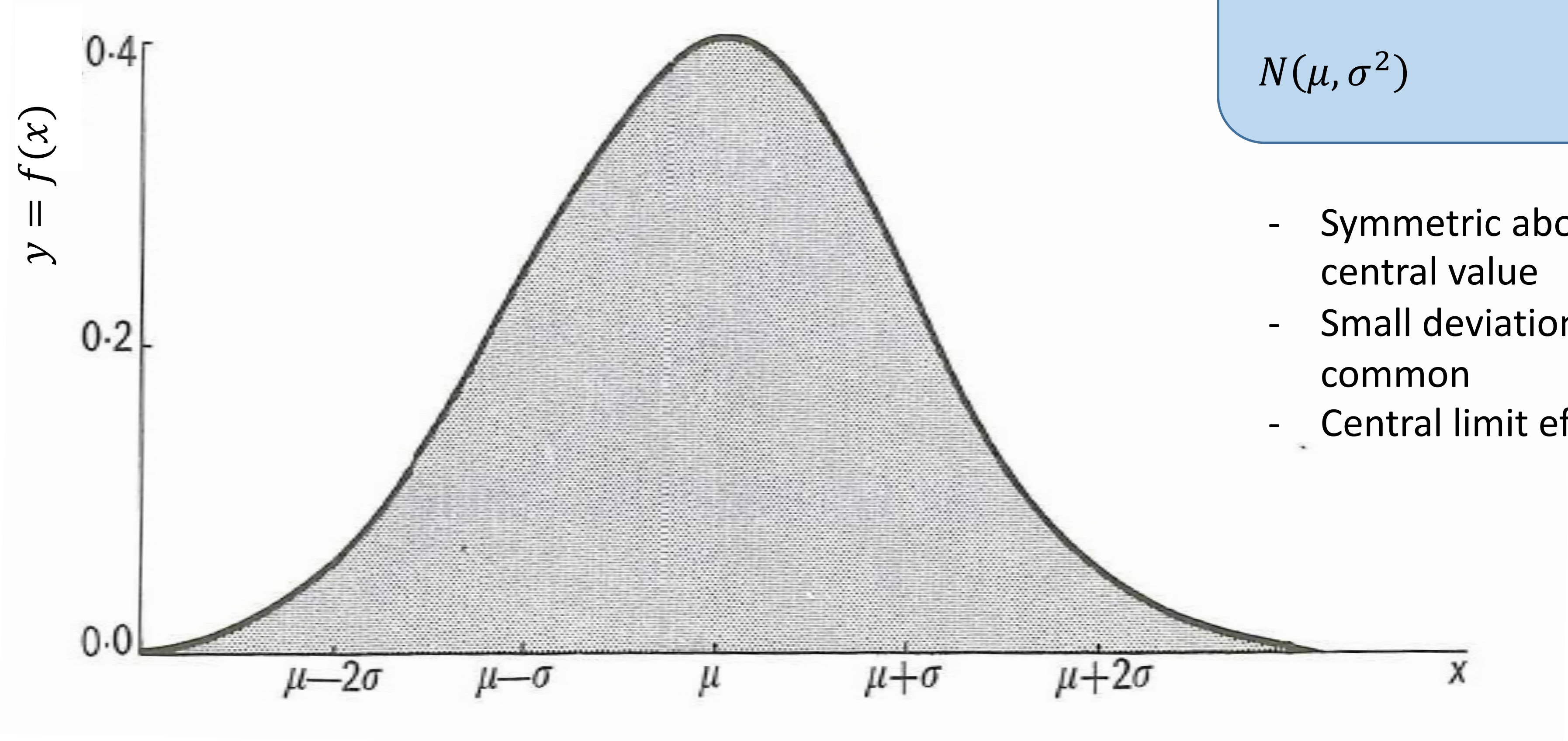
$$s^2 = \frac{\sum_N x_i^2 - N\bar{x}^2}{N-1}$$

Why $N-1$?



Probability distributions

Normal Probability Distribution

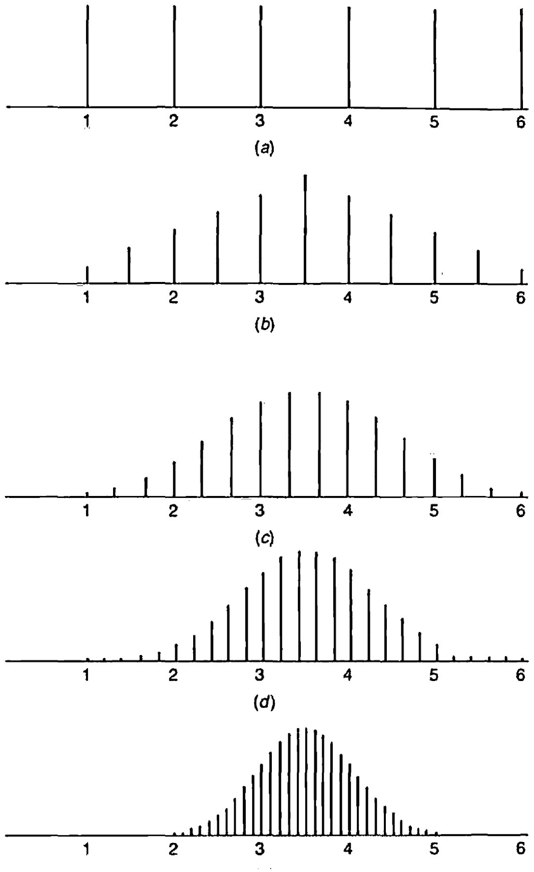


$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(x-\mu)^2/2\sigma^2}$$

$$N(\mu, \sigma^2)$$

- Symmetric about some central value
- Small deviations more common
- Central limit effect

Central limit effect



Average scores of (100 rolls)

One die

Two die

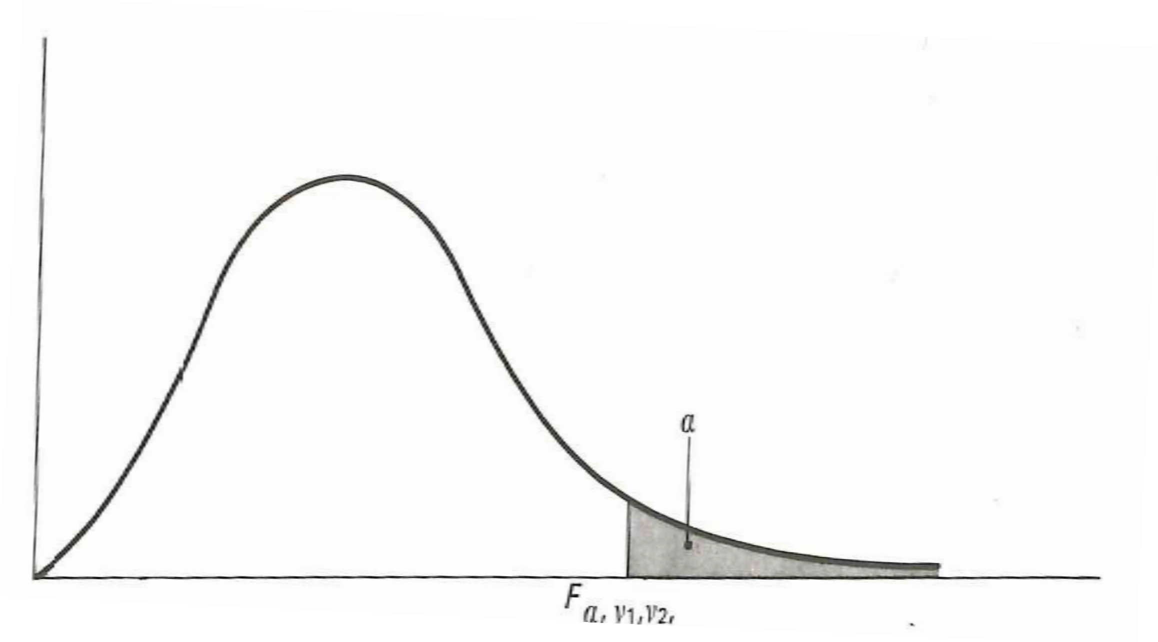
Three die

Five die

Ten die

In many experiments the error is an aggregate of a number of component errors and the distribution will tend to be “normal”

F-distribution



<u>Can obtain ratio of two sample variances;</u>

F statistic is ${s_1}^2/{s_2}^2$

F depends on the estimates and the dof of the variance estimates

Degrees of freedom of population variances;

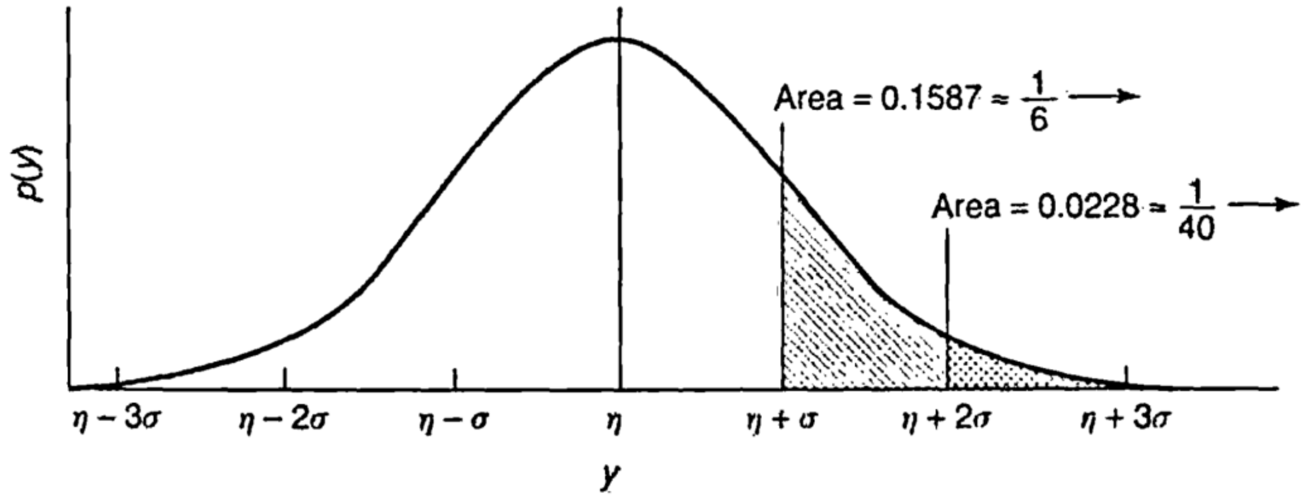
$\nu_1 = n_1 - 1$
$\nu_2 = n_2 - 1$

So F test statistic is designated

F_{ν_1,ν_2}

Probability

η and σ^2 fully characterise a distribution, $N(\eta, \sigma^2)$



Probability density is given by a point on the line p(y)

$p(y > \eta + \sigma) = \frac{1}{6}$ i.e. the area under the curve.

Often it is easier to express probability in terms of the standard deviate;

$$z = \frac{y - \eta}{\sigma}$$

$$= p(y > \eta + \sigma)$$

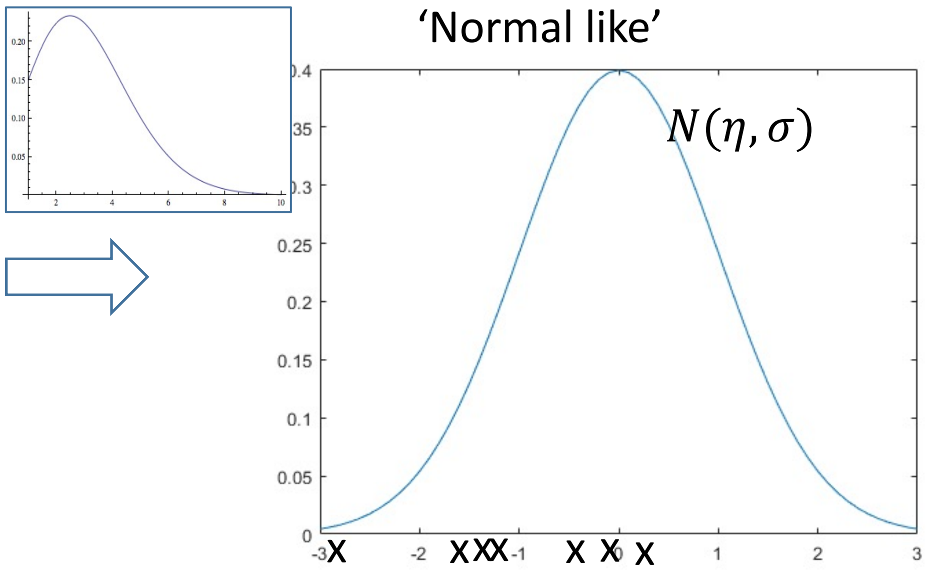
$$= p(y - \eta > \sigma)$$

$$= p\left(\frac{y-\eta}{\sigma} > 1\right)$$

$$= p(z > 1)$$

Standard Error of the Mean

- Take n random samples from a distribution with mean, η and standard deviation σ. Repeat.
- The sample means will form a distribution with the same mean, η but a <u>**smaller standard deviation**</u> $\sigma/\sqrt{n}$ (the **standard error of the sample mean**).



For a sample of size n the sample mean is $\bar{x}$

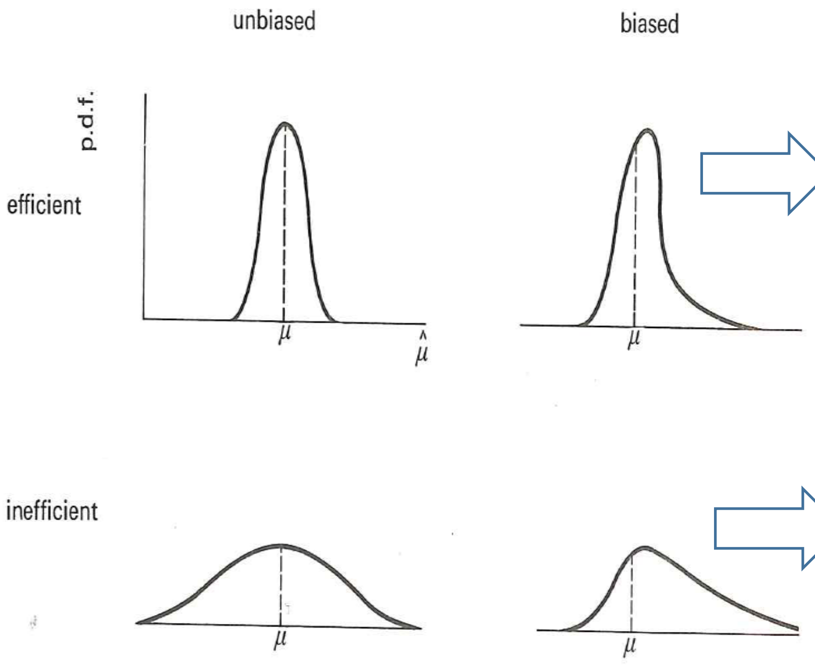
The standard error is an estimate of the standard deviation of the sample means for sample size n

$$SE_m = \frac{\sigma}{\sqrt{n}}$$

Intuitively it is a measure of how sample size affects the likely accuracy of the sample mean relative to the population mean.

Bias and efficiency

- **Bias** – an estimator is said to be biased if the mean of its sampling distribution is not equal to the value it is estimating.
- **Efficiency** – an efficient unbiased estimator is the minimum variance unbiased estimator (MVUE).



$$bias(\hat{\mu}) = \mathrm{E}(\mu) - \mu$$

$$var(\hat{\mu}_1) < var(\hat{\mu}_2)$$

more efficient estimator $\hat{\mu}_1$

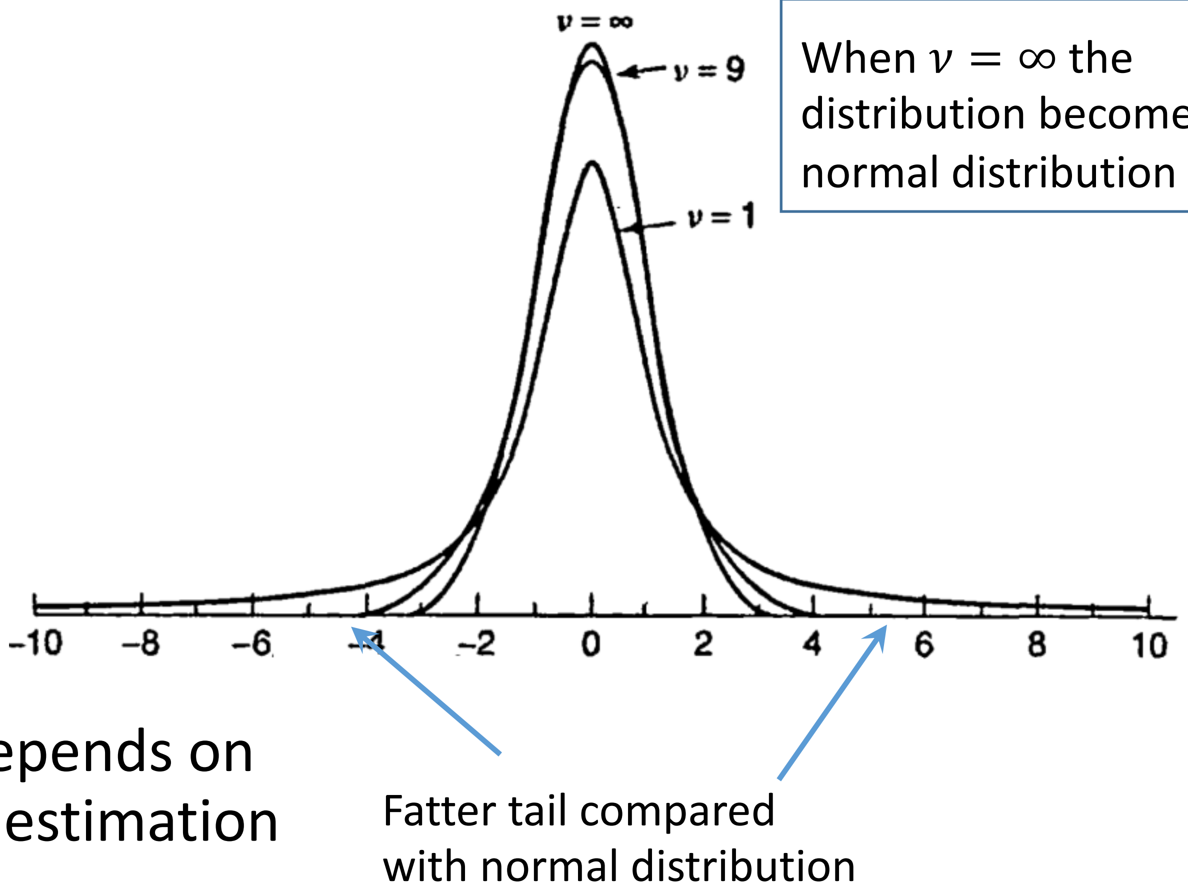
If σ is unknown (which is normally the case)

A substitution can be made for σ *using* s, the sample standard deviation;

$$z = \frac{y - \eta}{\sigma}$$

i.e.

$$t = \frac{y - \eta}{s}$$



When $\nu = \infty$ the distribution becomes the normal distribution

the 'student' or 't' distribution depends on degrees of freedom available for estimation of s.

Significance testing

Testing a theory about the population

Null hypothesis	H_0	What we are testing ….
Alternative hypothesis	H_1	An alternative ….

- Test statistic
- Level of significance
- One tailed and two tailed tests

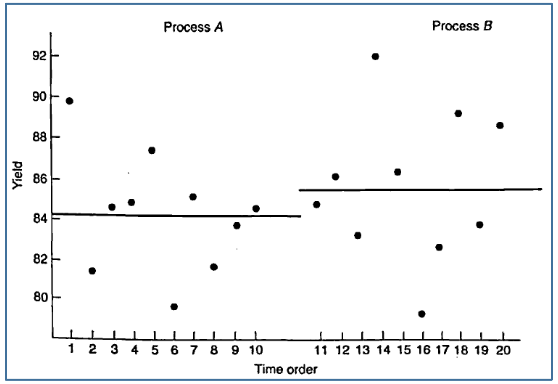
How to know if a treatment is significant?

Batch Process A

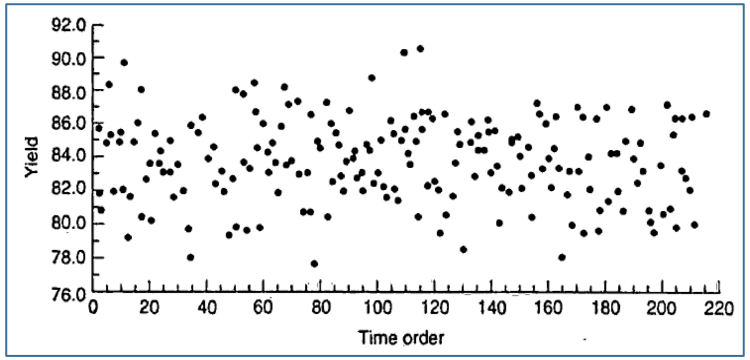
Batch Process B

$\bar{y}_a = 84.24$

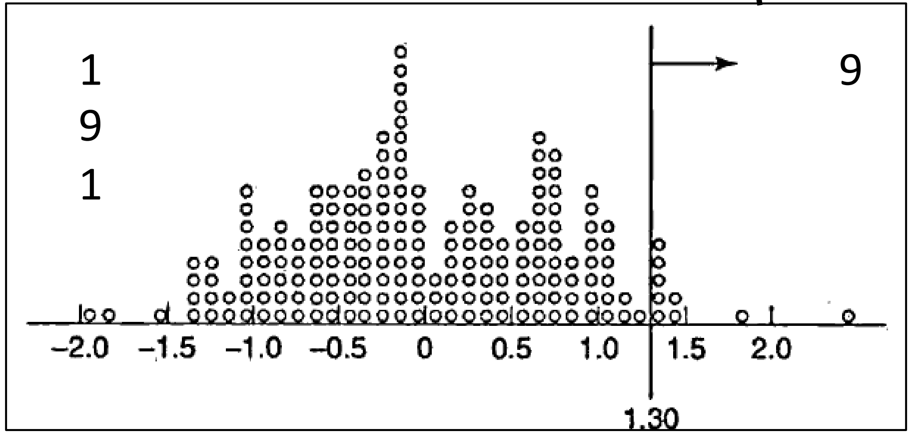
$\bar{y}_b = 85.54$



Previous yield data



$$\frac{9}{191} = 0.047$$



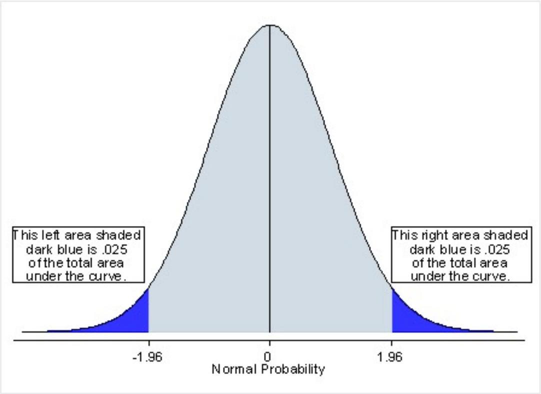
Difference in means of sequential batches

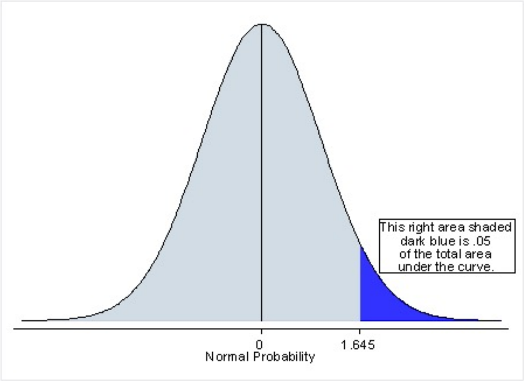
An example - composition of a chemical compound

- The iron content of a compound should be 12.1%. Tests on nine different samples are being used to examine this assumption.

- Null hypothesis i.e. there is no difference in the sample $(n = 9)$ mean
  - $H_0: \mu = 12.1\%$

- Alternative hypothesis
  - $H_1: \mu \neq 12.1\%$





Example (continued)

The analysis of nine samples gave the following values for % content of iron.

11.7	12.2	10.9	11.4	11.3	12.0	11.1	10.7	11.6

$$\bar{y} = 11.43$$
$$s^2 = 0.24$$
$$s = 0.49$$

$$t = \frac{(\bar{y} - \eta)}{s/\sqrt{n}} = \frac{(11.43 - 12.1)}{0.49/\sqrt{9}} = -4.1$$

Standard deviation of the mean (estimate)

Degrees of freedom: eight because nine samples and one DoF used for population mean, $\bar{x}$

Example (continued)

1. Two tailed test
2. 5% level of significance
3. Eight degrees of freedom (from tables)

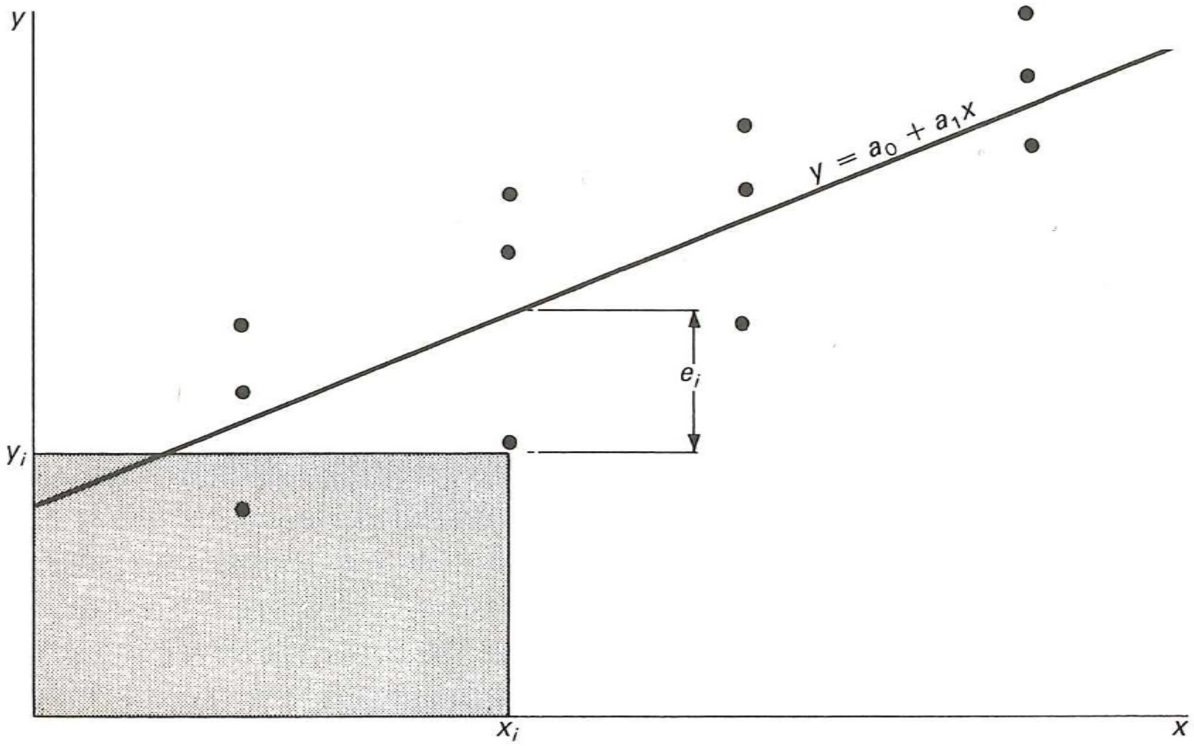
Test statistic is designated: $t_{0.025,8} = 2.31$ (from tables)

In fact, $t_{0.005,8} = 3.36$ (from tables)

So even at the 1% level, the result is significant.

Regression

Fitting a line or curve to the data in order to predict the mean value of the dependent variable for a given value of the controlled variable

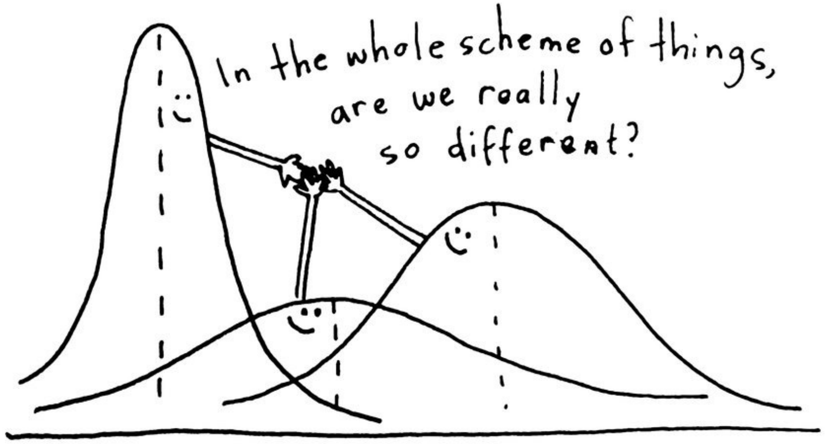


Model = $f(x, \beta)$

$$\min\left(\sum y_i - f(x_i, \beta)\right)$$

Analysing variance (ANOVA)
For comparing more than two entities

	A	B	C	D
	62	63	68	56
	60	67	66	62
	63	71	71	60
	59	64	67	61
	63	65	68	63
	59	66	68	64
Treatment avg	**61**	**66**	**68**	**61**
Overall avg	**64**	**64**	**64**	**64**



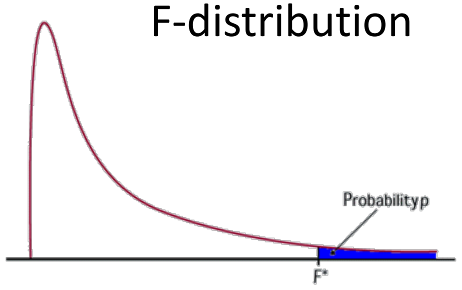
Deviation from overall average | Deviations **between** treatments | deviations **within** treatments

	A	B	C	D	$y_{ti} - \bar{y}$	$\bar{y}_t - \bar{y}$	$y_{ti} - \bar{y}_t$
y_{ti}	62	63	68	56	-2 -1 4 -8	-3 2 4 -3	1 -3 0 -5
	60	67	66	62	-4 3 2 -2	-3 2 4 -3	-1 1 -2 1
	63	71	71	60	-1 7 7 -4	-3 2 4 -3	2 5 3 -1
	59	64	67	61	-5 0 3 -3	-3 2 4 -3	-2 -2 -1 0
	63	65	68	63	-1 1 4 -1	-3 2 4 -3	2 -1 0 2
	59	66	68	64	5 2 4 0	-3 2 4 -3	-2 0 0 3
Sum of squares					**340**	**228**	**112**
Degrees of freedom					**23**	**3**	**20**

y_{ti} individual coagulation results
$\bar{y}_t$ treatment average
$\bar{y}$ overall average

ANOVA Table

Source of variation	Sum of squares	d.f.	$\boldsymbol{\frac{\chi^2}{\nu}}$
Between treatments	$\sum(\bar{y}_t - \bar{\boldsymbol{y}})^2 = 228$	$n-1=3$	$\frac{\sum(\bar{y}_t - \bar{\boldsymbol{y}})^2}{n-1} = 76$
Within treatments	$\sum(\boldsymbol{y}_{ti} - \bar{\boldsymbol{y}}_t)^2 = 112$	$n-1=20$	$\frac{\sum(y_{ti} - \bar{y}_t)^2}{n-1} = 5.6$
Total about the overall average	**340**	**23**	



$$\left. s_1^2/\sigma_1^2 \middle/ s_1^2/\sigma_1^2 \right. = \left. \frac{\sum(\bar{y}_t - \bar{y})^2}{n-1} \middle/ \frac{\sum(y_{ti} - \bar{y}_t)^2}{n-1} \right. \sim F_{\nu_1,\nu_2}$$

$$F_{3,20} = 13.6$$

Significant at 0.001 i.e. we can be confident that treatments do result in different means, we can reject H_0