FPV-Tutorial - SS23

Materialien für Manuel's FPV-Tutorium im Sommersemester 2023

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Zuletzt aktualisiert: 12. Juli 2023

FPV Tutorial - SS23

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Materialien für Manuel's FPV-Tutorium im Sommersemester 2023.

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Week 1: Implications, Assertions and Stronges Postconditions

Implications

Implications are the the key for understanding FPV. They show up in topics such as *Weakest Preconditions*, *Strongest Postconditions*, *Proof by Induction / Structural Induction*…

Definition of Implications

As you remeber from the “Diskrete Strukturen” course, an implication is a statement of the form $A \implies B$. It is read as:

- “A implies B”
- “If A is true, then B is true”

It’s syntactic sugar for the following statement:

$$A \implies B \iff \neg A \lor B$$

This is a very important statement, because it can be used to simplify complex statements, if you can’t remember the specific rules for implications.

Truth Table

A	B	$A \implies B$
F	F	T
F	T	T
T	F	F
T	T	T

Examples

Example 1:

$$\begin{aligned} & x = 1 \implies x \geq 0 \\ \iff & \neg(x = 1) \lor (x \geq 0) \\ \iff & (x \neq 1) \lor (x \geq 0) \\ \iff & \mathit{true} \end{aligned}$$

Example 2:

$$\begin{aligned} & A \implies (B \implies A) \\ \iff & \neg A \lor (B \implies A) \\ \iff & \neg A \lor (\neg B \lor A) \\ \iff & \neg A \lor A \lor \neg B \\ \iff & \mathit{true} \lor \neg B \\ \iff & \mathit{true} \end{aligned}$$

Assertions

Assertions are used to **annotate** specific points in a program and to **check** if a given expression is true at that point. If the expression is false, the program will terminate.

This is usefull if you only want to allow certain values for a variable, because otherwise the program would not work as expected. They can also be used to prove the correctness of a program. Which is the main topic of this course.

Example for MiniJava

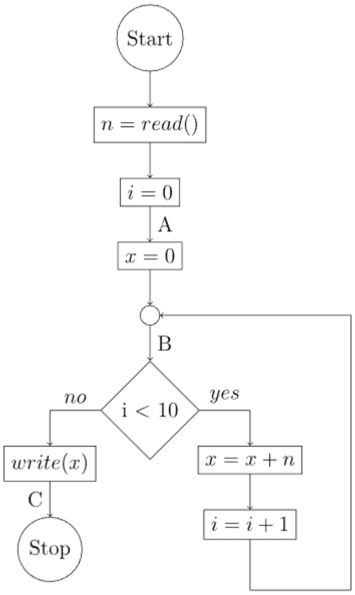


Abbildung 1: Flow Diagram

This corresponds to the following program:

```
void main() {
    var n = read(); //reads an arbitrary integer
    var i = 0;
    assert(A);

    var x = 0;
    assert(B)
    while (i < 10) {
        x = x + n;
        i = i + 1;
        assert(B);
    };

    write(x);
    assert(C);
}
```

The challenge is to find strong and precise assertions for the specific points in the program which allow us to prove the correctness of the program. In this case, we want to prove that the program **always** prints $n \cdot 10$ to the console. This corresponds to Assertion $C \iff x = n \cdot 10$.

Remember that whenever the programm-flow reaches an assertion, the assertion must be true. Otherwise the program will terminate.

Strength of Assertions

Two assertions A and B can have different strengths. This happens for example if assertion A is more precise than assertion B.

For example, the assertion $A = 5$ is stronger than the assertion $B = 5 \vee 6$, because it is more specific. The assertion A only allows the value 5, while the assertion B allows the values 5 and 6.

This makes sense intuitively. But in order to use it in practice, we need to define what it means for an assertion to be stronger than another assertion.

Definition of Assertions-Strength

We say that an assertion A is stronger than an assertion B, if A implies B.

Using this definition, we can compare different assertions and determine if they are:

- **Equivalent**: $A \implies B$ and $B \implies A$
- **Ordered (eg. A is stronger)**: $A \implies B$
- **Uncomparable**: $A \not\Longrightarrow B$ and $B \not\Longrightarrow A$

Special Assertions

Remember that $true$ and $false$ are also valid assertions. They are called **tautologies** and **contradictions** respectively.

How do they fit into the strength definition?

- **Tautologies**: $A \implies true$ for all A
  - This means that every assertion is stronger than $true$ thereby making $true$ the weakest assertion.
- **Contradictions**: $false \implies A$ for all A
  - This means that $false$ is stronger than every assertion thereby making $false$ the strongest assertion.

In practice those assertions show up in the following cases:

- **Tautologies**: If you have no information about the variables at a specific time in the program, you can use $true$ as an assertion to express this.
- **Contradictions**: If you have a point that is **never** reached in the program, you can use $false$ as an assertion to express this. The only way for the program to meet all assertions is to never reach such a point.

Strongest Postconditions

The strongest postcondition of a statement s and a precondition A is the strongest assertion B that holds after the statement s has been executed.

Example

Consider the following program:

```
void main() {
    var i=2;
    var x=6;
    assert(x=3*i && i>=0);

    i=i+1;

    //state at this point:
    //i = 3
    //x = 6
    //since the i in the assertions refers to the old value of i, before the statement i=i+1 was exe

    //can we find a new assertion which explicitly computes the new value of x?
    assert(C);
}
```

What is the strongest postcondition of the statement `i=i+1` and the precondition `x==3*i && i>=0`? In other words what is the strongest assertion which we can insert in the second assertion?

This can be written as:

$$\mathbf{SP}[[\mathbf{i} = \mathbf{i} + \mathbf{1}]](\mathbf{x} = \mathbf{3} * \mathbf{i} \wedge \mathbf{i} \geq \mathbf{0})$$

To compute the assertion after the statement `i=i+1` we basically need to **undo** the statement `i=i+1` because the original assertion refered to the old value of i, before it was updated.

> Note: This only works for updates of variables. Other assignments might be alot more complicated.

We first compute the **undo** of the statement `i=i+1`:

$$\mathbf{Undo}[[\mathbf{i} = \mathbf{i} + \mathbf{1}]] \equiv \mathbf{i} = \mathbf{i} - \mathbf{1}$$

Then we replace the varibale i (which has already gotten updated) inside the assertion with the **undo-ed** statement:

$$\begin{aligned} B &:= x = 3 * i \wedge i \geq 0 \\ &\longrightarrow x = 3 * (i - 1) \wedge (i - 1) \geq 0 \\ &\equiv x = 3(i - 1) \wedge i \geq 1 \\ &=: C \end{aligned}$$

In total we have:

$$\begin{aligned} C &:= \mathbf{SP}[[\mathbf{i} = \mathbf{i} + \mathbf{1}]](\mathbf{x} = \mathbf{3} * \mathbf{i} \wedge \mathbf{i} \geq \mathbf{0}) \\ &\equiv x = 3 * (i - 1) \wedge i \geq 1 \end{aligned}$$

Week 2: Preconditions, Postconditions and Local Consistency

Weakest Preconditions

Weakest Preconditions are used calculate the minimum requirements, which need to hold bevore an assignment, so that a given Assertion after the assignment holds.

Its written as:

$$\mathbf{WP}[[\mathbf{s}]](\mathbf{e})$$

Where s is a statement and e is an assertion.

Example

Consider the following program:

```
void main() {
    var r = 5;
    assert(A);
    var t = 3*r;
    assert(t>=0);
}
```

We want to find the minimal requirements which need to hold at `assert(A)` so that `assert(t>=0)` holds after `var t = 3*r;`.

We can calculate this using the following formula:

$$\begin{aligned} &\mathbf{WP}[[\mathbf{t} = \mathbf{3} * \mathbf{r}]](\mathbf{t} \geq \mathbf{0}) \\ \equiv\ & 3 * r \geq 0 \\ \equiv\ & r \geq 0 \ \ =: A \end{aligned}$$

Now we know, that for the assertion `t>=0` to hold after `var t = 3*r;`, the assertion `r>=0` needs to hold before `var t = 3*r;`.

Local Consistency

Two assertions A and B are locally consistent, if A is **stronger** than the **weakest precondition** of B. This is written as:

$$A \implies \mathbf{WP}[[\mathbf{s}]](\mathbf{B})$$

Note that it is not required that $A = \mathbf{WP}[[\mathbf{s}]](\mathbf{B})$. Because a stronger assertion than required is also fine.

Local consistency is important: It mathematically proves that whenever the assertion A holds, then the assertion B holds after the statement s. This can be used to prove that a program actually computes what it is supposed to compute.

Example Local Consistency

Consider the following program:

```
void main() {
    var x = 30;
    assert(x>25); //A
    x=x+5;
    assert(x!=0); //B
}
```

$$\begin{aligned} &A \equiv x > 25 \\ &B \equiv x \neq 0 \\ &s \equiv x = x + 5 \end{aligned}$$

At the moment all the Assertions are arbitrary, and there is no garantee that they actually hold during the execution of the program.

To prove them, we need to:

1. Show that all the assertions are locally consistent
2. We arrive at *true* at the start of the program

Local Consistency of A and B

We can calculate the weakest precondition of B and s as follows:

$$\begin{aligned} &\mathbf{WP}[[\mathbf{s}]](\mathbf{B}) \\ \equiv\ &\mathbf{WP}[[\mathbf{x} = \mathbf{x} + \mathbf{5}]](\mathbf{x} \neq \mathbf{0}) \\ \equiv\ &x + 5 \neq 0 \\ \equiv\ &x \neq -5 \ \ =: B' \end{aligned}$$

We can check the local consistency of A and B by checking if $A \implies B'$ holds.

This is the case, because:

$$\begin{aligned} &A \implies B' \\ \equiv\ &x > 25 \implies x \neq -5 \\ \equiv\ &\mathit{true} \end{aligned}$$

So we proved that A and B are locally consistent. This means that whenever A holds, then B holds after the statement s.

Weakest Precondition of A

If we compute the weakest precondition of A and `x=30;` we get:

$$\begin{aligned} &\mathbf{WP}[[\mathbf{x} = \mathbf{30}]](\mathbf{x} > \mathbf{25}) \\ \equiv\ &30 > 25 \\ \equiv\ &\mathit{true} =: A' \end{aligned}$$

This is obviously also locally consistent, because $\mathit{true} \implies A' \equiv \mathit{true} \implies \mathit{true} \equiv \mathit{true}$.

Since we arrived at $true$, we know that the whole chain of assertions from the start to the end of the program holds and is locally consistent.

This means that we proved that when the assertion at the start (aka. $true$) holds,then the assertion A and consequently Assertion B holds.

In this case, we proved that in all instances of the program, the variable x cannot be 0 at the end.

Week 3: Loop Invariants

What is a loop invariant?

A loop invariant is a assertion that holds in each iteration of a loop. Finding such a loop invariant is needed to calculate weakest preconditions for programs with loops, because the **normal** way of finding the preconditions does not work for loops.

The Problem with Loops

Lets say you are trying to calculate the weakest precondition for the following program:

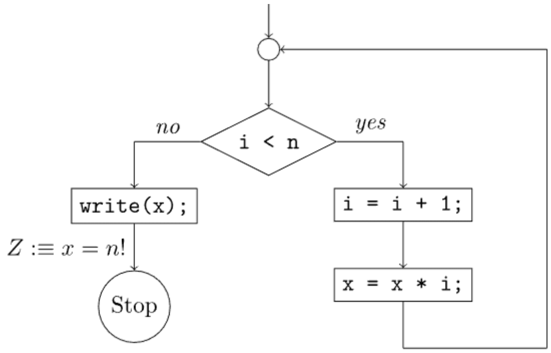


Abbildung 2: Program with loop

The normal way of finding the weakest precondition would be to start at the end of the program and work your way backwards.

$$X = \mathbf{WP}[\mathbf{write(x)}](\mathbf{x = n!}) \qquad = x = n!$$

$$\begin{aligned}
I &= \mathbf{WP}[\mathbf{i < n}](\mathbf{X}, \mathbf{K}) &&= (i < n \wedge K) \vee (i \geq n \wedge X) \\
K &= \mathbf{WP}[\mathbf{i := i + 1}](\mathbf{J}) &&= J[(i+1)/i] \\
J &= \mathbf{WP}[\mathbf{x := x * i}](\mathbf{I}) &&= I[x * i/x]
\end{aligned}$$

So we came to a conclusion that in order for us to compute the weakest precondition I we need to calculate I, K and J. But J itself depends on I so we have a circular dependency and therefore we cannot calculate I directly.

Finding a Loop Invariant

Since we cannot directly compute loop invariants we need to find a way to come up with them indirectly.

We can do this by guessing a loop invariant I that is **strong enough** to prove the correctness of the program. We do this by checking that our assertions (which are constructed using our guessed loop invariant) are locally consistent.

If we have shown local consistency we just need to check if the starting point is annotated with *true*. Then we have successfully proven the correctness of the assertion at the end of the program.

Evaluating different Loop Invariants

For the program above a suitable loop invariant would be: $I := x = i! \land 0 < i \leq n$. But how do we come up with this loop invariant?

For this we look at some other loop invariants and evaluate them:

Example Loop Invariants

1. $I := x \geq 0$:
    - This loop invariant is **not strong enough** to prove the correctness of the program. Since it it fails the local consistency check. ($I \not\Longrightarrow \mathbf{WP}[\mathbf{i} < \mathbf{n}](\mathbf{X}, \mathbf{K})$)
    - It was obvious that this loop invariant fails, because it does not contain any precise information about the value of x, which is needed to prove $x = n!$.
2. $I := i = 0 \land x = 1 \land n = 0$:
    - This loop invariant is way to strong, and is overall a bad choice because it would fail for any $n \neq 0$.
3. $I := x = i! \land 0 < i \leq n$:
    - This loop invariant is **strong enough** to prove the correctness of the program. Since it passes the local consistency check. ($I \implies \mathbf{WP}[\mathbf{i} < \mathbf{n}](\mathbf{X}, \mathbf{K})$)
    - Using this loop invariant we can prove that *true* holds at the start of the program, which means that the program and its assertions is correct.
    - Why does this loop invariant work?
        - It encapsulates all “relevant” information about the variables which change in the loop (x and i).
        - Combined with the false-branch of the if-statement it follows that $i \leq n \land i \geq n \implies i = n$. Which is exactly what is needed to prove $x = n!$ after we exit the loop.
        - It is weak enough to not disturb the proving of *true* at the start of the program.

Tips for Finding a Loop Invariant

There exist some old videos from the lecture “EIDI2” from the year 2017 that explain how to find loop invariants. The video is in german and is not relevant for this years course, but it still contains some useful tips for finding loop invariants.

- https://ttt.in.tum.de/recordings/Info2_2017_11_24-1/Info2_2017_11_24-1.mp4 [Nico Hartmann 2017]

Week 4: Termination Proofs

Why are Termination Proofs Necessary?

Every program containing a loop is potentially dangerous. Under the right circumstances, a loop can run forever, causing the program to hang. This is called an **infinite loop**.

In general, programs which don't eventually halt are of little use and can be considered as faulty.

For example, the following program contains an infinite loop and will never print "Finished":

```
let i = 17;
let j = 5;
while (i > j){
   i += 2;
   j += 1;
}
console.log("Finished");
```

On the other hand, the following program will always halt:

```
let i = 17;
let j = 5;
while (i > j){
   i += 1;
   j += 2;
}
console.log("Finished");
```

But how can we be sure that a program will always halt? In some cases it is not so obvious as in the examples above.

This is where **termination proofs** come in.

What is a Termination Proof?

In an assertion proof, you generally try to prove that a certain variable only takes on positive values inside a loop. Furthermore, you try to prove that the variable is decreased by at least one in each iteration of the loop.

This means that the variable will eventually reach zero (or less) and the cannot be entered again. Because this would violate the Assertion we defined.

But just coming up with arbitrary assertions and then claiming that they prove termination is not enough. We also need to show that those assertions are **localy consistent**.

How to do a Termination Proof?

Bevore we can perform a termination proof, it is necessary to understand what the loop actually does.

In the second example above our intuition tells us that the loop will eventually terminate. Because the variable `j` is increased by two in each iteration, while `i` is only increased by one. This means that `j` will eventually overtake `i` and the loop will terminate.

With this understanding we can define an auxiliary variable `r` which represents this intuition.

Since we only want to prove that the loop terminates, we just need to prove *true* at the end of the program.

In general, we need to insert the following assertions / statements

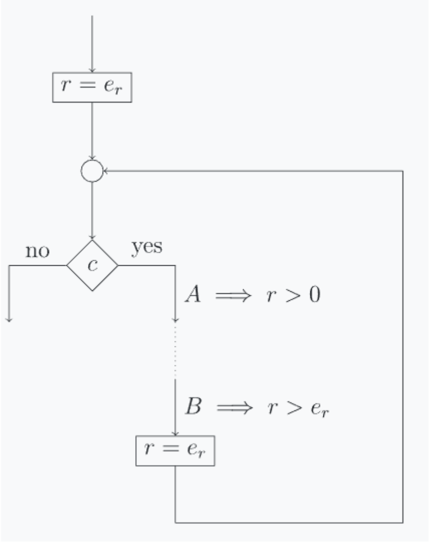


Abbildung 3: Flowchart with auxiliary variable

Notice that we need both these assertions to prove termination:

- $r > 0$ at the beginning of the loop
- $r > r_e$ at the end of the loop, right bevore $r = r_e$

Now the task is to show the following, we have proven that the loop terminates:

- local consistency of all assertions
- arrived at *true* at the start of the program
- The special assertions $r \geq 0$ and $r > r_e$ are also localy consistent

Week 5: OCaml

Basic Syntax

Comments

```
(* This is a comment *)
```

Variables

```
let x = 1;;
```

Functions

```
let f x = x + 1;;
let succ: int -> int = fun x -> x + 1;;
```

Tuples

```
let x = (1, true);;
```

Lists

```
let x = [1; 2; 3];;
let y = 1 :: 2 :: 3 :: [];;
```

Records

```
type person = { name: string; age: int };;
let x = { name = "John"; age = 42 };;
let name = x.name;;
```

If-then-else

```
let x = if 3 < 6 then "1" else "2";;
```

Pattern Matching

```
let x = match 3 with
  | 1 -> "1"
  | 2 -> "2"
  | _ -> "else";;

let y = match "up" with
    | "up" -> (0, 1)
    | "down" -> (0, -1)
    | "left" -> (-1, 0)
    | "right" -> (1, 0)
```

Example Programs

Advanced Hello World

```
let welcome_string = "lmaCO ot emocleW ,!dlroW olleH"

let rec char_iterator sentence =
  if sentence = "" then ()
  else
    let head = String.get sentence 0 in
    let tail = String.sub sentence 1 (String.length sentence - 1) in
    char_iterator tail;
    print_char head

(* Call Function: Result will be visible in the console when loaded via utop *)
let _ = char_iterator welcome_string
```

Debuging OCaml

Using the #use command in utop

This method works best for single files containing the code you want to debug. The #use command works by just copy-pasting the entire content of the file into the utop-environment. It also shadows previous definitions of variables and functions.

1. Enter into the root-directory of the project and run dune build to initially build the project.

   ```
   dune build
   ```

2. Open utop via dune utop src

   ```
   dune utop src
   ```

   In all projects you will encounter, dune is configured in a way that will allow this command to work.

   This will open utop with the project's src-directory as the current working directory.

3. Reload the files

   - Instead of closing and reopening utop every time you change something in the files, you can use the #use command to load the files again.
   - But you need to be careful, because older definitions may still be around after reloading the file.

   ```
   #use "src/main.ml";;
   ```

Week 6: List Module

Lists in OCaml

Lists in ocaml are basically linked lists. They are defined as follows:

```
let my_list = [1; 2; 3; 4; 5]
```

Since they are linked lists, you don't have direct access to the elements in the list. You can only access the head and the tail of the list. The head is the first element in the list and the tail is the rest of the list. The head is an element and the tail is a list. The tail can be an empty list.

```
let my_list = [1; 2; 3; 4; 5]

let head = List.hd my_list (* head = 1 *)
let tail = List.tl my_list (* tail = [2; 3; 4; 5] *)
```

Since Linked lists are recursive in nature, most of the functions that operate on lists are recursive. For example, the length function is defined as follows:

Example Length

```
let rec length l =
  match l with
  | [] -> 0
  | _ :: xs -> 1 + length xs
```

- This basically means that the length of list $l = \underbrace{[a_1, a_2, \ldots, a_n]}_{\text{list}}$ can be recursively defined as
  - $\text{length}(\underbrace{[a_1, a_2, \ldots, a_n]}_{\text{l}}) = 1 + \text{length}(\underbrace{[a_2, \ldots, a_n]}_{\text{tail l}})$.
- Since we need a base case for the recursion, we define the length of an empty list to be 0.
  - $\text{length}([]) = 0$.

Many other functions on lists are defined recursively. For example, the reverse function is defined as follows:

Binary Search Tree

Binary Search Tree in OCaml

A binary search tree is a binary tree where the value of the left child is less than the value of the parent and the value of the right child is greater than the value of the parent. The following is an example of a binary search tree:

```
type tree =
  | Empty
  | Node of int * tree * tree

let my_tree =
    Node(6,
      Node(3,
        Node(1, Empty, Empty),
        Node(4, Empty, Empty)
      ),
      Node(8,
        Node(7, Empty, Empty),
        Node(9, Empty, Empty)
      )
    )
```

The Tree type is defined as follows:

```
type tree =
  | Empty
  | Node of  int * tree * tree
```

This means that a tree is either an empty node or a node with a left subtree, a value and a right subtree. The left and right subtrees are also trees. The value is an integer.

The syntax `Node of int * tree * tree` means that the constructor `Node` takes three arguments. The first argument is an integer and the second and third arguments are trees. The result of the constructor is a value of type tree.

Since Binary Search trees are again recursive in nature, most of the functions that operate on them are recursive. For example, the function `insert` is defined as follows:

Example Insert

```
let rec insert v t =
  match t with
  | Empty -> Node(v, Empty, Empty)
  | Node(y, l, r) ->
    if v < y then
      Node(y, insert v l, r)
    else if v > y then
      Node(y, l, insert v r)
    else
      t
```

It takes a value `v` and a tree `t` and returns a new tree with the value `v` inserted into the tree `t`. If the value `v` is already present in the tree `t`, then the same tree `t` is returned.

It works as follows:

- If the tree is empty, then the value `v` is inserted into the tree as the root node.
- If the tree is not empty:
  - If the value `v` is less than the value of the root node, then the value `v` is inserted into the left subtree.
  - If the value `v` is greater than the value of the root node, then the value `v` is inserted into the right subtree.
  - Else nothing is done and the same tree is returned.

As you can see from this example, it is not possible to change the value of a node in a tree. This is because the tree is *immutable*.

If you want to change the value of a node, you have to create a new tree with the new value.

This means, that if we need to insert a value in the right subtree of a node, we have to create a completely new tree with the new value inserted, and then replace the right subtree of the node with the new tree.

At first this seams unnecessary and inefficient, but in practise it is not that big of a deal. This is because the compilers of functional programs are in general very good at optimizing the code, since they can make more assumptions about the code due to the functional nature of the language. This means that the compiler can do a lot of optimizations that are not possible in imperative languages.

Week 7: Advanced List Operations

List Reduce

The `List` module contains a number of useful functions for working with lists. You can find the documentation for the `List` module here.

Lets look at the `List.reduce` functions:

`List.fold_left`

Fold left is a function that takes a reducer function, an initial value, and a list. It then applies the reducer function to each element of the list, starting with the initial value, and then *eating* itself through the list from left to right.

It works just like the `Stream.reduce` function in Java

```
# List.fold_left (fun acc x -> acc + x) 0 [7; 8; -9; 10];;
- : int = 16
```

If we split up the steps we get something like this:

list		7	8	-9	10
fold_left	acc	l[0]	l[1]	l[2]	l[3]
	0	7	8	-9	10
		7	8	-9	10
			15	-9	10
				6	10
					16

In each step we combine the accumulator with the next element in the list, and then use that as the accumulator for the next step.

It is called `fold_left` because it implicitly groups the elements of the list from left to right.

$$list = [7, 8, -9, 10]$$
$$fold\_left(f, 0, list) = f(f(f(f(0, 7), 8), -9), 10)$$

`List.fold_right`

`List.fold_right` is similar to `List.fold_left` except that it groups the elements of the list from right to left.

It has a different type signature than `List.fold_left`:

```
# List.fold_right (fun x acc -> acc + x) [7; 8; -9; 10] 0;;
- : int = 16
```

list	7	8	-9	10	
fold_right	l[0]	l[1]	l[2]	l[3]	acc

list	7	8	-9	10	
	7	8	-9	10	**0**
	7	8	-9	10	
	7	8	1		
	7	9			
	16				

$$list = [7, 8, -9, 10]$$
$$fold\_right(f, list, 0) = f(7, f(8, f(-9, f(10, 0))))$$

Week 8: Lazy Lists

A lazy list is a list that is not fully evaluated until it is needed. This is useful for representing infinite lists, or lists that are too large to fit in memory.

Examples of infinite lists include the list of all natural numbers, the list of all prime numbers, and the list of all Fibonacci numbers.

Lazy lists in OCaml

One way to implement lazy lists in OCaml is to use the `Lazy` module. But we are going to define our own lazy list type, which we will call `'a llist`.

```
type 'a llist = Cons of 'a * (unit -> 'a llist)
```

This type definition is a bit tricky. It says that a lazy list consists of a value of type `'a` and a function that returns another lazy list. The function is called a *thunk*. It is a function that takes no arguments and returns a lazy list. The thunk is used to delay the evaluation of the tail of the list.

Here is an example of a lazy list of natural numbers:

```
let rec from n = Cons (n, fun () -> from (n + 1))
```

It basically says that the lazy list `from n` consists of the value `n` and a thunk that returns the lazy list `from (n + 1)`.

Here is an example of a lazy list of Fibonacci numbers:

```
let rec fib_step a b = Cons (a, fun () -> fib_step b (a + b))
```

The parameters `a` and `b` are the two previous Fibonacci numbers. The lazy list `fib_step a b` consists of the value `a` and a thunk that returns the lazy list `fib_step b (a + b)`. Of course, the first two Fibonacci numbers are 0 and 1, so we can define the lazy list of all Fibonacci numbers as follows:

```
let fib = fib_step 0 1
```

This list is infinite, but since its generation is lazy, we can still make use of it.

Partial Application

Functions in OCaml are curried by default. This means that a function that takes two arguments is actually a function that takes one argument and returns a function that takes another argument. For example, the function `(+)` is defined as follows:

```
let (+) x y = x + y
```

This is equivalent to the following definition:

```
let (+) = fun x -> fun y -> x + y
```

The function `(+)` takes one argument `x` and returns a function that takes another argument `y` and returns the sum of `x` and `y`.

We can partially apply a function by passing it only some of its arguments. For example, we can define a function `inc` that adds 1 to its argument as follows:

```
let inc = (+) 1

let x = inc 5
# val x : int = 6
```

The List module defines a function `map` that takes a function `f` and a list `xs` and returns a list of the results of applying `f` to the elements of `xs`. We can partially apply `map` to define a function `double` that doubles all the elements of a list:

```
let double = List.map (( * ) 2)

let xs = double [1; 2; 3]
# val xs : int list = [2; 4; 6]
```

Also the `inc` function defined above can be used with `map`:

```
let xs = List.map inc [1; 2; 3]
# val xs : int list = [2; 3; 4]
```

Which is equivalent to:

```
let xs = List.map ((+) 1) [1; 2; 3]
# val xs : int list = [2; 3; 4]
```

Week 9: Side Effects, Units and Exceptions

Side Effects

In functional programming, side effects are a way of interacting with the outside world. For example, printing to the screen, reading from a file, or reading from the keyboard are all side effects. They are called side effects because they happen outside of the control of the program. In other words, the program cannot control when the user types something on the keyboard, or when the user clicks the mouse, or when data arrives from the network.

This also means that repeated calls to a function with side effects may produce different results. For example, if a function reads from the keyboard, then each time it is called, it may read a different value. This is in contrast to a function that does not have side effects, which will always return the same value when called with the same arguments. Those so called `pure` functions are the ones we have been writing so far.

Side effects are not necessarily bad. In fact, they are necessary for a program to be useful. However, they do make it harder to reason about the program. For example, if a function has side effects, then it is not enough to look at the function to understand what it does. You also need to know what side effects it has. This is in contrast to a pure function, which you can understand just by looking at the function itself.

Example File IO

Let's look at an example of a function with side effects. The following function reads the first line of a file and returns it as a string.

```
let read_file (filename : string) : string =
  let ic = open_in filename in
  let row = input_line ic in
  close_in ic;
  row
```

This function has side effects because it reads from a file. It is not a pure function because it does not always return the same value when called with the same arguments.

Writing to a file works similarly. The following function writes a string to a file.

```
let write_file (filename : string) (row : string) : unit =
  let oc = open_out filename in
  output_string oc row;
  close_out oc
```

Exceptions

Exceptions are a way of handling errors. They are called exceptions because they are exceptional. They may occur in the normal course of a program, but they are not expected to occur. For example, if a function reads from a file, then it may fail if the file does not exist. This is an exceptional case because the file is expected to exist.

But sometimes we want exceptions to occur. For example if we read to the end of a file, then we want to know that we have reached the end of the file. Ocaml has a special exception for this case called End_of_file. We can use this exception to detect when we have reached the end of a file.

```
let rec read_file (filename : string) : string list =
  let ic = open_in filename in
  let rec read_loop () =
    try
      let row = input_line ic in
      row :: read_loop ()
    with End_of_file ->
        []
    in
    let result = read_loop () in
    close_in ic;
    result
```

The function read_file reads all the lines of a file and returns them as a list of strings.

Week 10: Modules

In Ocaml, a module is a collection of types, values, and functions. Modules are used to organize code and to avoid name clashes. Modules can be nested, and they can be parameterized by other modules.

10.1 Module Types

A module type is a specification of the types, values, and functions that a module must provide. Module types are used to specify the interface of a module. A module type can be parameterized by other module types.

A module type is defined using the `module type` keyword. For example, the following module type specifies that a module must provide a type `t` and a function `f`:

```
module type ModType = sig
  type t
  val f : t -> t
end
```

In Java, a module type is similar to an interface with a generic type parameter.

```
interface ModType<T> {
  T f(T x);
}
```

10.2 Module

A Module is a similar to a class in Java. It can also implement a module type. A module can be defined using the `module` keyword. For example, the following module implements the module type `ModType`:

```
module Mod : ModType with type t = int = struct
  type t = int
  let f x = x + 1
end
```

In Java, a module is similar to a class that implements an interface.

```
class Mod implements ModType<Integer> {
  public Integer f(Integer x) {
    return x + 1;
  }
}
```

We can call the function `f` in the module `Mod` as follows:

```
Mod.f 1
(* returns 2 *)
```

10.3 Functors

A functor is a function that takes a module as an argument and returns a module. A functor can be defined in the following way:

```
module Functor (M : ModType with type t = int) : ModType with type t = M.t = struct
  type t = M.t
  let f x = M.f (M.f x)
end
```

This particular functor takes a module of type `ModType` and returns a module of type `ModType`. The returned Module differs from the input module in that the function `f` is applied twice.

It can be used in the following way:

```
module Mod2 = Functor(Mod)

Mod2.f 1
(* returns 3 , because inc is applied twice*)
```

Week 11: Big Step Semantics

Big step semantics is a way to define the semantics of a programming language by recursively reducing the program-expression to a value. The reduction is done in a top-down manner, starting from the root of the expression tree. The reduction stops when the expression is reduced to a value.

Big Step for OCaml Expressions

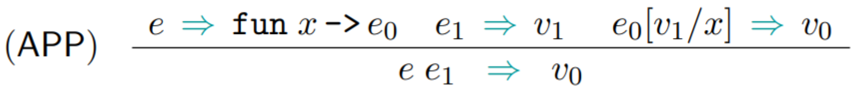
Now we are going to define the big step semantics for OCaml expressions.

Tuple

A tuple is a sequence of expressions separated by commas and enclosed in parentheses. The big step semantics for a tuple is to evaluate each expression in the tuple in order, and return a tuple of the values of the expressions.

`(E1, E2, ..., En) => (V1, V2, ..., Vn)`

where `E1, E2, ..., En` are expressions, `V1, V2, ..., Vn` are values, and `n >= 0`.



$$(\text{APP})\quad \frac{e \Rightarrow \texttt{fun}\ x\ \texttt{->}\ e_0 \quad e_1 \Rightarrow v_1 \quad e_0[v_1/x] \Rightarrow v_0}{e\ e_1 \Rightarrow v_0}$$

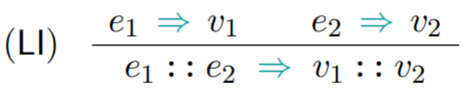
Abbildung 4: Tuple Evaluation

List

Lists work similarly to tuples. The big step semantics for a list is to decompose the list into its head and tail, evaluate the head and tail, and return a list with the evaluated head and tail.

`H::T => V::W`

where `H` is an expression, `T` is a list, `V` is a value, and `W` is a list.



$$(\text{LI})\quad \frac{e_1 \Rightarrow v_1 \qquad e_2 \Rightarrow v_2}{e_1 :: e_2 \Rightarrow v_1 :: v_2}$$

Abbildung 5: List Evalueation

Global Definition

A global definition is a definition of a value at the top level of a program. The big step semantics for a global definition is to look up the value of the expression, and return its value.

$$\text{(GD)}\quad \frac{f = e \qquad e \Rightarrow v}{f \Rightarrow v}$$

Abbildung 6: Global Definition

Local Definition

A local definition is a definition of a value inside a `let` expression. The big step semantics for a local definition is to evaluate the expression, and then substitute the value of the expression for the variable in the body of the `let` expression.

$$\text{(LD)}\quad \frac{e_1 \Rightarrow v_1 \qquad e_0[v_1/x] \Rightarrow v_0}{\texttt{let}\ x = e_1\ \texttt{in}\ e_0 \Rightarrow v_0}$$

Abbildung 7: Local Definition

Function Call

In order to evaluate a function call, we need to evaluate the function expression and the argument expression. Then we substitute the value of the argument expression for the parameter in the body of the function expression.

$$\text{(LD)}\quad \frac{e_1 \Rightarrow v_1 \qquad e_0[v_1/x] \Rightarrow v_0}{\texttt{let}\ x = e_1\ \texttt{in}\ e_0 \Rightarrow v_0}$$

Abbildung 8: Function Call

Pattern Matching

The big step semantics for pattern matching is to evaluate the expression, and then match the value of the expression with the patterns. If the value matches a pattern, then we substitute the value of the expression for the variable in the body of the pattern.

Built-in Operators

Built in operators cannot be evaluated directly, we need to compose them into their mathematical expressions first. For example, `1 + 2` is composed of the integer `1`, the operator `+`, and the integer `2`. The big step semantics for a built-in operator is to evaluate the expressions in the operator, and then apply the operator to the values of the expressions.

Example

In this example, an expression using recursive functions will be evaluated:

```
let rec f = fun x -> x + 1
    and s = fun y -> y * y
```

We are going to calculate the value of the expression:

```
f 16 + s 2
```

Using the rules defined above, we can evaluate the expression and arrive at the value `21`.

$$(\mathsf{PM})\quad \frac{e_0 \Rightarrow v' \equiv p_i[v_1/x_1,\dots,v_k/x_k] \qquad e_i[v_1/x_1,\dots,v_k/x_k] \Rightarrow v}{\texttt{match}\ e_0\ \texttt{with}\ p_1\ \texttt{->}\ e_1\ \texttt{|}\ \dots\ \texttt{|}\ p_m\ \texttt{->}\ e_m \Rightarrow v}$$

Abbildung 9: Pattern Matching

$$(\mathsf{OP})\quad \frac{e_1 \Rightarrow v_1 \qquad e_2 \Rightarrow v_2 \qquad v_1\ \mathsf{op}\ v_2 \Rightarrow v}{e_1\ \mathsf{op}\ e_2 \Rightarrow v}$$

Abbildung 10: Built-in Operators

```
let rec f = fun x -> x+1
    and s = fun y -> y*y
```

$$\frac{\frac{\texttt{(GD)}\frac{\texttt{f = fun x -> x+1}}{\texttt{f}\Rightarrow\texttt{fun x -> x+1}} \quad \texttt{(OP)}\frac{\texttt{16+1}\Rightarrow\texttt{17}}{\texttt{16+1}\Rightarrow\texttt{17}}}{\texttt{f 16}\Rightarrow\texttt{17}}\texttt{(APP)} \quad \frac{\texttt{(GD)}\frac{\texttt{s = fun y -> y*y}}{\texttt{s}\Rightarrow\texttt{fun y -> y*y}} \quad \texttt{(OP)}\frac{\texttt{2*2}\Rightarrow\texttt{4}}{\texttt{2*2}\Rightarrow\texttt{4}}}{\texttt{s 2}\Rightarrow\texttt{4}}\texttt{(APP)} \quad \texttt{17+4}\Rightarrow\texttt{21}}{\texttt{f 16 + s 2}\Rightarrow\texttt{21}}\texttt{(OP)}$$

// uses of $v \Rightarrow v$ have mostly been omitted

Abbildung 11: Example Big Step Program

Week 12: Equational Reasoning

Equational reasoning is a powerful technique for proving properties of programs. We mostly use it to veryfy that a function behaves as expected, the main idea is to repeatedly substitute equal expressions and to perform simplifications until we show that the result is the expected one. We often do this by performing induction on the structure of the input.

An important aspect to note is that if a function has an auxiliary variable. For example:

```
let rec fact_aux n acc =
  if n = 0 then acc
  else fact_aux (n - 1) (n * acc)

let fact_iter n = fact_aux n 1
```

During the induction step it is often necessary to prove a generalized version of the function for the proof to go through. In this case we would need to prove that `fact_aux n acc = acc * fact n`.

```
let fact_aux n acc = acc * fact n
```

Example: Prove that `fact n = n * fact (n - 1)`

Let fact be defined as:

```
let rec fact n = match n with 0 -> 1
  | n -> n * fact (n - 1)
```

```
We want to prove that:
     fact_iter n = n * fact n
Which corresponds to:
     fact_aux n 1 = n * fact n

As mentioned before, we need to prove a generalized version of the function:
     fact_aux n acc = acc * fact n

Proof of the generalized version:

Base case n = 0:
            fact_aux 0 acc
(fact_aux)  = if 0 = 0 then acc else fact_aux (0 - 1) (0 * acc)
(match)     = acc
(arith)     = acc
(match)     = acc * 1
(fact)      = acc * (match 0 with 0 -> 1 | n -> n * fact (n - 1))
            = acc * fact 0


Induction step n+1: (IH: fact_aux n acc = acc * fact n)
            fact_aux (n+1) acc
(fact_aux)  = if (n+1) = 0 then acc else fact_aux ((n+1) - 1) ((n+1) * acc)
(match)     = fact_aux ((n+1) - 1) ((n+1) * acc)
```

```
(arith)     = fact_aux n ((n+1) * acc)
(IH)        = (n+1) * acc * fact n
(arith)     = (n+1) * acc * fact ((n+1) - 1)
(match)     = acc * ((n+1) * fact ((n+1) - 1))
(fact)      = acc * (match n+1 with 0 -> 1 | n -> n * fact (n - 1))
            = acc * fact (n+1)
```

The proof of the generalized version is complete, this means that:

```
fact_aux n acc = acc * fact n
```

To proof the original statement we need instantiate the generalized version with `acc = 1`:

```
            fact_aux n 1
(#)         = 1 * fact n
(arith)     = fact n
```

Which completes the proof. We can be sure that `fact n = n * fact (n - 1)` and that the helper function `fact_aux` is correct.