FPV Tutorübung

Woche 11

Big Step

Manuel Lerchner

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Quiz

Artemis 6.3.2

Courses > Funktionale Programmierung und Verifikation (Sommersemester 2023) > Exercises > Week 11 Quiz

Week 11 Quiz

Points: 20

Open quiz

Exercise details

Release date:	Jul 3, 2023 08:00
Submission due:	Jul 7, 2023 20:00
Complaint possible:	Yes

Passwort:

What is Big Step?

- A way to formally calculate the **value** of expressions (Recursive style)

```
int -> int
let sq = fun x→ x*x

int * int * int -> int
let add = fun(x,y,z) → x+y+z


(* What is the value of x? *)
int
let x = add (2,3+4,sq 3)
```

- *add* (2, 3+4,sq 3) (Function call)
  1. Find the value of the called function
     1. *add*: Global Definition
        1. Extract the value
  2. Find the value of the argument
     1. It‘s a tuple! Simplify all entries
        1. 2 => 2
        2. 3+4 => 7 (Arith)
        3. *sq* 3 = Function call
           1. *sq*: Global Definition
              1. Extract the value
              2. Argument 3 => 3
              3. Substitute: x*x -> 3*3 => 9
  3. Substitute the function variables with actual values
     1. Substitute x+y+z with x=2, y=7, z=9
     2. 2+7+9 =>18

Big Step Rules 1

Tuples

$$(\text{TU}) \quad \frac{e_1 \Rightarrow v_1 \qquad \ldots \qquad e_k \Rightarrow v_k}{(e_1, \ldots, e_k) \Rightarrow (v_1, \ldots, v_k)}$$

Lists

$$(\text{LI}) \quad \frac{e_1 \Rightarrow v_1 \qquad e_2 \Rightarrow v_2}{e_1 :: e_2 \Rightarrow v_1 :: v_2}$$

Global definitions

$$(\text{GD}) \quad \frac{f = e \qquad e \Rightarrow v}{f \Rightarrow v}$$

Big Step Rules 2

Local definitions

$$(\text{LD}) \quad \frac{e_1 \Rightarrow v_1 \qquad e_0[v_1/x] \Rightarrow v_0}{\texttt{let}\ x = e_1\ \texttt{in}\ e_0 \Rightarrow v_0}$$

Function calls

$$(\text{APP}) \quad \frac{e \Rightarrow \texttt{fun}\ x \texttt{ -> } e_0 \qquad e_1 \Rightarrow v_1 \qquad e_0[v_1/x] \Rightarrow v_0}{e\ e_1 \Rightarrow v_0}$$

$$(\text{APP'}) \quad \frac{e_0 \Rightarrow \texttt{fun}\ x_1 \ldots x_k \texttt{ -> } e \qquad e_1 \Rightarrow v_1 \ \ldots\ e_k \Rightarrow v_k \qquad e[v_1/x_1, \ldots, v_k/x_k] \Rightarrow v}{e_0\ e_1\ \ldots\ e_k \Rightarrow v}$$

Big Step Rules 3

Pattern Matching

$$(\text{PM}) \quad \frac{e_0 \Rightarrow v' \equiv p_i[v_1/x_1, \ldots, v_k/x_k] \qquad e_i[v_1/x_1, \ldots, v_k/x_k] \Rightarrow v}{\texttt{match}\ e_0\ \texttt{with}\ p_1\ \texttt{->}\ e_1\ \texttt{|}\ \ldots\ \texttt{|}\ p_m\ \texttt{->}\ e_m \Rightarrow v}$$

Built-in operators

$$(\text{OP}) \quad \frac{e_1 \Rightarrow v_1 \qquad e_2 \Rightarrow v_2 \qquad v_1\ \mathsf{op}\ v_2 \Rightarrow v}{e_1\ \mathsf{op}\ e_2 \Rightarrow v}$$

Example 1

$$(\text{OP}) \quad \frac{(\text{OP}) \quad \dfrac{17 \Rightarrow 17 \qquad 4 \Rightarrow 4 \qquad 17 + 4 \Rightarrow 21}{17 + 4 \Rightarrow 21} \qquad 21 \Rightarrow 21 \qquad 21 = 21 \Rightarrow \texttt{true}}{17 + 4 = 21 \Rightarrow \texttt{true}}$$

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We define these functions:

```
let rec f = fun l ->
  match l with [] -> 1 | x::xs -> x + g xs
and g = fun l ->
  match l with [] -> 0 | x::xs -> x * f xs
```

Consider the following expressions. Find the values they evaluate to and construct a big-step proof for that claim.

1. `let f = fun a -> (a+1,a-1)::[] in f 7`
2. `f [3;6]`
3. `(fun x -> x 3) (fun y z -> z y) (fun w -> w + w)`

Unless specified otherwise, all rules used in a big-step proof tree must be annotated and all axioms ($v \Rightarrow v$) must be written down. You may create aliases $\pi_{subscript}$ for big step trees and $\tau_{subscript}$ for espressions/values.

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```
let f = fun a -> [(a+1,a-1)] in f 7 ⇒
```

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$$(\text{LD})\ \frac{e_1 \Rightarrow v_1 \qquad e_0[v_1/x] \Rightarrow v_0}{\texttt{let}\ x = e_1\ \texttt{in}\ e_0 \Rightarrow v_0}$$

$$\text{LD}\ \frac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \qquad \texttt{(fun a -> [(a+1,a-1)]) 7} \Rightarrow}{\texttt{let f = fun a -> [(a+1,a-1)] in f 7} \Rightarrow}$$

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$$\text{(APP')}\quad \frac{e_0 \Rightarrow \texttt{fun}\ x_1 \ldots x_k\ \texttt{->}\ e \qquad e_1 \Rightarrow v_1 \ \ldots\ e_k \Rightarrow v_k \qquad e[v_1/x_1,\ldots,v_k/x_k] \Rightarrow v}{e_0\ e_1\ \ldots\ e_k \Rightarrow v}$$

$$\pi_0 = \quad \frac{\qquad}{\texttt{[(7+1,7-1)]} \Rightarrow}$$

$$\text{LD}\ \frac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \quad \text{APP'}\ \dfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \quad \texttt{7} \Rightarrow \texttt{7} \quad \pi_0}{\texttt{(fun a -> [(a+1,a-1)]) 7} \Rightarrow}}{\texttt{let f = fun a -> [(a+1,a-1)] in f 7} \Rightarrow}$$

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$$(\mathrm{LI})\ \frac{e_1 \Rightarrow v_1 \qquad e_2 \Rightarrow v_2}{e_1 :: e_2 \Rightarrow v_1 :: v_2}$$

$$\pi_0 = \mathrm{LI}\ \frac{\dfrac{}{\texttt{(7+1,7-1)} \Rightarrow \qquad} \qquad \texttt{[]} \Rightarrow \texttt{[]}}{\texttt{[(7+1,7-1)]} \Rightarrow}$$

$$\mathrm{LD}\ \frac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \quad \mathrm{APP'}\ \dfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \quad \texttt{7} \Rightarrow \texttt{7} \quad \pi_0}{\texttt{(fun a -> [(a+1,a-1)]) 7} \Rightarrow}}{\texttt{let f = fun a -> [(a+1,a-1)] in f 7} \Rightarrow}$$

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$$(\mathsf{TU})\quad \frac{e_1 \Rightarrow v_1 \qquad \ldots \qquad e_k \Rightarrow v_k}{(e_1, \ldots, e_k) \Rightarrow (v_1, \ldots, v_k)}$$

$$\pi_0 = \mathrm{LI}\ \frac{\mathrm{TU}\ \dfrac{\texttt{7+1} \Rightarrow \qquad \texttt{7-1} \Rightarrow}{\texttt{(7+1,7-1)} \Rightarrow} \qquad \texttt{[]} \Rightarrow \texttt{[]}}{\texttt{[(7+1,7-1)]} \Rightarrow}$$

$$\mathrm{LD}\ \frac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \quad \mathrm{APP'}\ \dfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \quad \texttt{7} \Rightarrow \texttt{7} \quad \pi_0}{\texttt{(fun a -> [(a+1,a-1)]) 7} \Rightarrow}}{\texttt{let f = fun a -> [(a+1,a-1)] in f 7} \Rightarrow}$$

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$$(\text{OP})\quad \frac{e_1 \Rightarrow v_1 \qquad e_2 \Rightarrow v_2 \qquad v_1\ \mathsf{op}\ v_2 \Rightarrow v}{e_1\ \mathsf{op}\ e_2 \Rightarrow v}$$

$$\pi_0 = \text{LI}\ \dfrac{\text{TU}\ \dfrac{\text{OP}\ \dfrac{7 \Rightarrow 7 \quad 1 \Rightarrow 1 \quad 7+1 \Rightarrow 8}{\texttt{7+1} \Rightarrow} \quad \text{OP}\ \dfrac{7 \Rightarrow 7 \quad 1 \Rightarrow 1 \quad 7-1 \Rightarrow 6}{\texttt{7-1} \Rightarrow}}{\texttt{(7+1,7-1)} \Rightarrow} \qquad \texttt{[]} \Rightarrow \texttt{[]}}{\texttt{[(7+1,7-1)]} \Rightarrow}$$

$$\text{LD}\ \dfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \quad \text{APP'}\ \dfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \quad \texttt{7} \Rightarrow \texttt{7} \quad \pi_0}{\texttt{(fun a -> [(a+1,a-1)]) 7} \Rightarrow}}{\texttt{let f = fun a -> [(a+1,a-1)] in f 7} \Rightarrow}$$

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$$\pi_0 = \mathrm{LI}\ \cfrac{\mathrm{TU}\ \cfrac{\mathrm{OP}\ \cfrac{7 \Rightarrow 7 \quad 1 \Rightarrow 1 \quad 7 + 1 \Rightarrow 8}{\texttt{7+1} \Rightarrow 8} \quad \mathrm{OP}\ \cfrac{7 \Rightarrow 7 \quad 1 \Rightarrow 1 \quad 7 - 1 \Rightarrow 6}{\texttt{7-1} \Rightarrow 6}}{\texttt{(7+1,7-1)} \Rightarrow} \qquad \texttt{[]} \Rightarrow \texttt{[]}}{\texttt{[(7+1,7-1)]} \Rightarrow}$$

$$\mathrm{LD}\ \cfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \quad \mathrm{APP'}\ \cfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \quad \texttt{7} \Rightarrow \texttt{7} \quad \pi_0}{\texttt{(fun a -> [(a+1,a-1)]) 7} \Rightarrow}}{\texttt{let f = fun a -> [(a+1,a-1)] in f 7} \Rightarrow}$$

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$$\pi_0 = \text{LI}\ \dfrac{\text{TU}\ \dfrac{\text{OP}\ \dfrac{\texttt{7} \Rightarrow \texttt{7} \quad \texttt{1} \Rightarrow \texttt{1} \quad 7 + 1 \Rightarrow \texttt{8}}{\texttt{7+1} \Rightarrow \texttt{8}} \quad \text{OP}\ \dfrac{\texttt{7} \Rightarrow \texttt{7} \quad \texttt{1} \Rightarrow \texttt{1} \quad 7 - 1 \Rightarrow \texttt{6}}{\texttt{7-1} \Rightarrow \texttt{6}}}{\texttt{(7+1,7-1)} \Rightarrow \texttt{(8,6)}} \quad \texttt{[]} \Rightarrow \texttt{[]}}{\texttt{[(7+1,7-1)]} \Rightarrow}$$

$$\text{LD}\ \dfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \quad \text{APP'}\ \dfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \quad \texttt{7} \Rightarrow \texttt{7} \quad \pi_0}{\texttt{(fun a -> [(a+1,a-1)]) 7} \Rightarrow}}{\texttt{let f = fun a -> [(a+1,a-1)] in f 7} \Rightarrow}$$

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$$\pi_0 = \text{LI}\ \dfrac{\text{TU}\ \dfrac{\text{OP}\ \dfrac{7 \Rightarrow 7 \quad 1 \Rightarrow 1 \quad 7+1 \Rightarrow 8}{\texttt{7+1} \Rightarrow \texttt{8}} \quad \text{OP}\ \dfrac{7 \Rightarrow 7 \quad 1 \Rightarrow 1 \quad 7-1 \Rightarrow 6}{\texttt{7-1} \Rightarrow \texttt{6}}}{\texttt{(7+1,7-1)} \Rightarrow \texttt{(8,6)} \quad \texttt{[]} \Rightarrow \texttt{[]}}}{\texttt{[(7+1,7-1)]} \Rightarrow \texttt{[(8,6)]}}$$

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T01: Big Steps Ü1

$$\pi_0 = \mathrm{LI}\,\dfrac{\mathrm{TU}\,\dfrac{\mathrm{OP}\,\dfrac{7 \Rightarrow 7 \quad 1 \Rightarrow 1 \quad 7 + 1 \Rightarrow 8}{\texttt{7+1} \Rightarrow 8} \quad \mathrm{OP}\,\dfrac{7 \Rightarrow 7 \quad 1 \Rightarrow 1 \quad 7 - 1 \Rightarrow 6}{\texttt{7-1} \Rightarrow 6}}{\texttt{(7+1,7-1)} \Rightarrow \texttt{(8,6)} \quad \texttt{[]} \Rightarrow \texttt{[]}}}{\texttt{[(7+1,7-1)]} \Rightarrow \texttt{[(8,6)]}}$$

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$$\text{LD}\ \dfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \quad \text{APP'}\ \dfrac{\texttt{fun a -> [(a+1,a-1)]} \Rightarrow \texttt{fun a -> [(a+1,a-1)]} \quad 7 \Rightarrow 7 \quad \pi_0}{\texttt{(fun a -> [(a+1,a-1)]) 7} \Rightarrow \texttt{[(8,6)]}}}{\texttt{let f = fun a -> [(a+1,a-1)] in f 7} \Rightarrow \texttt{[(8,6)]}}$$

T01: Big Steps Ü2

```
f [3;6] ⇒
```

T01: Big Steps Ü2

$$(\text{APP'}) \quad \frac{e_0 \Rightarrow \texttt{fun}\ x_1 \ldots x_k\ \texttt{->}\ e \quad e_1 \Rightarrow v_1 \ \ldots\ e_k \Rightarrow v_k \quad e[v_1/x_1, \ldots, v_k/x_k] \Rightarrow v}{e_0\ e_1\ \ldots\ e_k \Rightarrow v}$$

$$\text{APP'} \quad \frac{\pi_f \quad \texttt{[3;6]} \Rightarrow \texttt{[3;6]} \qquad \texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow}{\texttt{f [3;6]} \Rightarrow}$$

T01: Big Steps Ü2

$$(\text{PM})\quad \frac{e_0 \Rightarrow v' \equiv p_i[v_1/x_1,\ldots,v_k/x_k] \qquad e_i[v_1/x_1,\ldots,v_k/x_k] \Rightarrow v}{\texttt{match}\ e_0\ \texttt{with}\ p_1\ \texttt{->}\ e_1\ \texttt{|}\ \ldots\ \texttt{|}\ p_m\ \texttt{->}\ e_m \ \Rightarrow\ v}$$

$$\text{APP'}\ \frac{\pi_f \quad \texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{PM}\ \dfrac{\texttt{[3;6]} \Rightarrow \texttt{[3;6]} \qquad \texttt{3+g [6]} \Rightarrow}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow}}{\texttt{f [3;6]} \Rightarrow}$$

T01: Big Steps Ü2

$$(\text{OP})\quad \frac{e_1 \Rightarrow v_1 \qquad e_2 \Rightarrow v_2 \qquad v_1\ \mathsf{op}\ v_2 \Rightarrow v}{e_1\ \mathsf{op}\ e_2 \Rightarrow v}$$

$$\text{APP'}\ \dfrac{\pi_f \quad \texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{PM}\ \dfrac{\texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{OP}\ \dfrac{\texttt{3} \Rightarrow \texttt{3} \qquad \texttt{g [6]} \Rightarrow}{\texttt{3+g [6]} \Rightarrow}}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow}}{\texttt{f [3;6]} \Rightarrow}$$

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$$(\text{APP}')\quad \frac{e_0 \Rightarrow \texttt{fun}\ x_1 \ldots x_k \texttt{->} e \quad e_1 \Rightarrow v_1 \ \ldots\ e_k \Rightarrow v_k \quad e[v_1/x_1, \ldots, v_k/x_k] \Rightarrow v}{e_0\ e_1\ \ldots\ e_k \ \Rightarrow\ v}$$

$$\pi_0 = \frac{}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow}$$

$$\text{APP'}\ \frac{\pi_f \quad \texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{PM}\ \dfrac{\texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{OP}\ \dfrac{\texttt{3} \Rightarrow \texttt{3} \quad \text{APP'}\ \dfrac{\pi_g \quad \texttt{[6]} \Rightarrow \texttt{[6]} \quad \pi_0}{\texttt{g [6]} \Rightarrow}}{\texttt{3+g [6]} \Rightarrow}}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow}}{\texttt{f [3;6]} \Rightarrow}$$

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$$\text{(PM)}\quad \frac{e_0 \Rightarrow v' \equiv p_i[v_1/x_1,\dots,v_k/x_k] \qquad e_i[v_1/x_1,\dots,v_k/x_k] \Rightarrow v}{\texttt{match}\ e_0\ \texttt{with}\ p_1\ \texttt{->}\ e_1\ \texttt{|}\ \dots\ \texttt{|}\ p_m\ \texttt{->}\ e_m \ \Rightarrow\ v}$$

$$\pi_0 = \text{PM}\ \frac{\texttt{[6]} \Rightarrow \texttt{[6]} \qquad \texttt{6*f []} \Rightarrow}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow}$$

$$\text{APP'}\ \dfrac{\pi_f \quad \texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{PM}\ \dfrac{\texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{OP}\ \dfrac{\texttt{3} \Rightarrow \texttt{3} \quad \text{APP'}\ \dfrac{\pi_g \quad \texttt{[6]} \Rightarrow \texttt{[6]} \quad \pi_0}{\texttt{g [6]} \Rightarrow}}{\texttt{3+g [6]} \Rightarrow}}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow}}{\texttt{f [3;6]} \Rightarrow}$$

T01: Big Steps Ü2

$$(\mathsf{OP})\quad \frac{e_1 \Rightarrow v_1 \qquad e_2 \Rightarrow v_2 \qquad v_1\ \mathsf{op}\ v_2 \Rightarrow v}{e_1\ \mathsf{op}\ e_2 \Rightarrow v}$$

$$\pi_0 = \mathrm{PM}\ \frac{\texttt{[6]} \Rightarrow \texttt{[6]} \quad \mathrm{OP}\ \dfrac{6 \Rightarrow 6 \qquad \texttt{f []} \Rightarrow}{\texttt{6*f []} \Rightarrow}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow}$$

$$\mathrm{APP'}\ \frac{\pi_f \quad \texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \mathrm{PM}\ \dfrac{\texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \mathrm{OP}\ \dfrac{3 \Rightarrow 3 \quad \mathrm{APP'}\ \dfrac{\pi_g \quad \texttt{[6]} \Rightarrow \texttt{[6]} \quad \pi_0}{\texttt{g [6]} \Rightarrow}}{\texttt{3+g [6]} \Rightarrow}}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow}}{\texttt{f [3;6]} \Rightarrow}$$

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$$\text{(APP')}\quad \frac{e_0 \Rightarrow \texttt{fun}\ x_1 \ldots x_k \texttt{->} e \quad e_1 \Rightarrow v_1 \ \ldots\ e_k \Rightarrow v_k \quad e[v_1/x_1, \ldots, v_k/x_k] \Rightarrow v}{e_0\ e_1\ \ldots\ e_k \Rightarrow v}$$

$$\pi_0 = \text{PM}\ \frac{\texttt{[6]} \Rightarrow \texttt{[6]}\ \ \text{OP}\ \dfrac{\texttt{6} \Rightarrow \texttt{6}\ \ \text{APP'}\ \dfrac{\pi_f\ \ \texttt{[]} \Rightarrow \texttt{[]} \qquad \texttt{match [] with [] -> 1 | x::xs -> x+g xs} \Rightarrow}{\texttt{f []} \Rightarrow}}{\texttt{6*f []} \Rightarrow}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow}$$

$$\text{APP'}\ \frac{\pi_f\ \ \texttt{[3;6]} \Rightarrow \texttt{[3;6]}\ \ \text{PM}\ \dfrac{\texttt{[3;6]} \Rightarrow \texttt{[3;6]}\ \ \text{OP}\ \dfrac{\texttt{3} \Rightarrow \texttt{3}\ \ \text{APP'}\ \dfrac{\pi_g\ \ \texttt{[6]} \Rightarrow \texttt{[6]}\ \ \pi_0}{\texttt{g [6]} \Rightarrow}}{\texttt{3+g [6]} \Rightarrow}}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow}}{\texttt{f [3;6]} \Rightarrow}$$

T01: Big Steps Ü2

$$(\text{PM})\ \frac{e_0 \Rightarrow v' \equiv p_i[v_1/x_1,\ldots,v_k/x_k] \qquad e_i[v_1/x_1,\ldots,v_k/x_k] \Rightarrow v}{\texttt{match } e_0 \texttt{ with } p_1 \texttt{ -> } e_1 \mid \ \ldots \ \mid p_m \texttt{ -> } e_m \ \Rightarrow \ v}$$

$$\pi_0 = \text{PM}\ \frac{\texttt{[6]} \Rightarrow \texttt{[6]}\ \ \text{OP}\ \dfrac{\texttt{6} \Rightarrow \texttt{6}\ \ \text{APP'}\ \dfrac{\pi_f\ \ \texttt{[]} \Rightarrow \texttt{[]}\ \ \text{PM}\ \dfrac{\texttt{[]} \Rightarrow \texttt{[]}\quad \texttt{1} \Rightarrow \texttt{1}}{\texttt{match [] with [] -> 1 | x::xs -> x+g xs} \Rightarrow}}{\texttt{f []} \Rightarrow}}{\texttt{6*f []} \Rightarrow}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow}$$

$$\text{APP'}\ \frac{\pi_f\ \ \texttt{[3;6]} \Rightarrow \texttt{[3;6]}\ \ \text{PM}\ \dfrac{\texttt{[3;6]} \Rightarrow \texttt{[3;6]}\ \ \text{OP}\ \dfrac{\texttt{3} \Rightarrow \texttt{3}\ \ \text{APP'}\ \dfrac{\pi_g\ \ \texttt{[6]} \Rightarrow \texttt{[6]}\ \ \pi_0}{\texttt{g [6]} \Rightarrow}}{\texttt{3+g [6]} \Rightarrow}}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow}}{\texttt{f [3;6]} \Rightarrow}$$

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$$\pi_0 = \text{PM}\ \dfrac{\texttt{[6]} \Rightarrow \texttt{[6]} \quad \text{OP}\ \dfrac{\texttt{6} \Rightarrow \texttt{6} \quad \text{APP'}\ \dfrac{\pi_f \quad \texttt{[]} \Rightarrow \texttt{[]} \quad \text{PM}\ \dfrac{\texttt{[]} \Rightarrow \texttt{[]} \quad \texttt{1} \Rightarrow \texttt{1}}{\texttt{match [] with [] -> 1 | x::xs -> x+g xs} \Rightarrow \texttt{1}}}{\texttt{f []} \Rightarrow}}{\texttt{6*f []} \Rightarrow}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow}$$

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$$\pi_0 = \text{PM}\ \dfrac{\texttt{[6]} \Rightarrow \texttt{[6]} \quad \text{OP}\ \dfrac{\texttt{6} \Rightarrow \texttt{6} \quad \text{APP'}\ \dfrac{\pi_f \quad \texttt{[]} \Rightarrow \texttt{[]} \quad \text{PM}\ \dfrac{\texttt{[]} \Rightarrow \texttt{[]} \quad \texttt{1} \Rightarrow \texttt{1}}{\texttt{match [] with [] -> 1 | x::xs -> x+g xs} \Rightarrow \texttt{1}}}{\texttt{f []} \Rightarrow \texttt{1}}}{\texttt{6*f []} \Rightarrow}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow}$$

$$\text{APP'}\ \dfrac{\pi_f \quad \texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{PM}\ \dfrac{\texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{OP}\ \dfrac{\texttt{3} \Rightarrow \texttt{3} \quad \text{APP'}\ \dfrac{\pi_g \quad \texttt{[6]} \Rightarrow \texttt{[6]} \quad \pi_0}{\texttt{g [6]} \Rightarrow}}{\texttt{3+g [6]} \Rightarrow}}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow}}{\texttt{f [3;6]} \Rightarrow}$$

T01: Big Steps Ü2

$$\pi_0 = \text{PM}\ \dfrac{\texttt{[6]} \Rightarrow \texttt{[6]} \quad \text{OP}\ \dfrac{\texttt{6} \Rightarrow \texttt{6} \quad \text{APP'}\ \dfrac{\pi_f \quad \texttt{[]} \Rightarrow \texttt{[]} \quad \text{PM}\ \dfrac{\texttt{[]} \Rightarrow \texttt{[]} \quad \texttt{1} \Rightarrow \texttt{1}}{\texttt{match [] with [] -> 1 | x::xs -> x+g xs} \Rightarrow \texttt{1}}}{\texttt{f []} \Rightarrow \texttt{1}} \quad 6 * 1 \Rightarrow \texttt{6}}{\texttt{6*f []} \Rightarrow \texttt{6}}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow}$$

$$\text{APP'}\ \dfrac{\pi_f \quad \texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{PM}\ \dfrac{\texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{OP}\ \dfrac{\texttt{3} \Rightarrow \texttt{3} \quad \text{APP'}\ \dfrac{\pi_g \quad \texttt{[6]} \Rightarrow \texttt{[6]} \quad \pi_0}{\texttt{g [6]} \Rightarrow}}{\texttt{3+g [6]} \Rightarrow}}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow}}{\texttt{f [3;6]} \Rightarrow}$$

T01: Big Steps Ü2

$$\pi_0 = \text{PM}\ \dfrac{\texttt{[6]} \Rightarrow \texttt{[6]} \quad \text{OP}\ \dfrac{\texttt{6} \Rightarrow \texttt{6} \quad \text{APP'}\ \dfrac{\pi_f \quad \texttt{[]} \Rightarrow \texttt{[]} \quad \text{PM}\ \dfrac{\texttt{[]} \Rightarrow \texttt{[]} \quad \texttt{1} \Rightarrow \texttt{1}}{\texttt{match [] with [] -> 1 | x::xs -> x+g xs} \Rightarrow \texttt{1}}}{\texttt{f []} \Rightarrow \texttt{1}} \quad 6 * 1 \Rightarrow \texttt{6}}{\texttt{6*f []} \Rightarrow \texttt{6}}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow \texttt{6}}$$

$$\text{APP'}\ \dfrac{\pi_f \quad \texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{PM}\ \dfrac{\texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{OP}\ \dfrac{\texttt{3} \Rightarrow \texttt{3} \quad \text{APP'}\ \dfrac{\pi_g \quad \texttt{[6]} \Rightarrow \texttt{[6]} \quad \pi_0}{\texttt{g [6]} \Rightarrow}}{\texttt{3+g [6]} \Rightarrow}}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow}}{\texttt{f [3;6]} \Rightarrow}$$

T01: Big Steps Ü2

$$\pi_0 = \text{PM}\ \dfrac{\texttt{[6]} \Rightarrow \texttt{[6]}\quad \text{OP}\ \dfrac{6 \Rightarrow 6\quad \text{APP'}\ \dfrac{\pi_f\quad \texttt{[]} \Rightarrow \texttt{[]}\quad \text{PM}\ \dfrac{\texttt{[]} \Rightarrow \texttt{[]}\quad \texttt{1} \Rightarrow \texttt{1}}{\texttt{match [] with [] -> 1 | x::xs -> x+g xs} \Rightarrow \texttt{1}}}{\texttt{f []} \Rightarrow \texttt{1}}\quad 6 * 1 \Rightarrow \texttt{6}}{\texttt{6*f []} \Rightarrow \texttt{6}}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow \texttt{6}}$$

$$\text{APP'}\ \dfrac{\pi_f\quad \texttt{[3;6]} \Rightarrow \texttt{[3;6]}\quad \text{PM}\ \dfrac{\texttt{[3;6]} \Rightarrow \texttt{[3;6]}\quad \text{OP}\ \dfrac{\texttt{3} \Rightarrow \texttt{3}\quad \text{APP'}\ \dfrac{\pi_g\quad \texttt{[6]} \Rightarrow \texttt{[6]}\quad \pi_0}{\texttt{g [6]} \Rightarrow \texttt{6}}}{\texttt{3+g [6]} \Rightarrow}}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow}}{\texttt{f [3;6]} \Rightarrow}$$

T01: Big Steps Ü2

$$\pi_0 = \text{PM}\ \dfrac{\texttt{[6]} \Rightarrow \texttt{[6]} \quad \text{OP}\ \dfrac{\texttt{6} \Rightarrow \texttt{6} \quad \text{APP'}\ \dfrac{\pi_f \quad \texttt{[]} \Rightarrow \texttt{[]} \quad \text{PM}\ \dfrac{\texttt{[]} \Rightarrow \texttt{[]} \quad \texttt{1} \Rightarrow \texttt{1}}{\texttt{match [] with [] -> 1 | x::xs -> x+g xs} \Rightarrow \texttt{1}}}{\texttt{f []} \Rightarrow \texttt{1}} \quad 6 * 1 \Rightarrow \texttt{6}}{\texttt{6*f []} \Rightarrow \texttt{6}}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow \texttt{6}}$$

$$\text{APP'}\ \dfrac{\pi_f \quad \texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{PM}\ \dfrac{\texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{OP}\ \dfrac{\texttt{3} \Rightarrow \texttt{3} \quad \text{APP'}\ \dfrac{\pi_g \quad \texttt{[6]} \Rightarrow \texttt{[6]} \quad \pi_0}{\texttt{g [6]} \Rightarrow \texttt{6}} \quad 3 + 6 \Rightarrow \texttt{9}}{\texttt{3+g [6]} \Rightarrow \texttt{9}}}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow}}{\texttt{f [3;6]} \Rightarrow}$$

T01: Big Steps Ü2

$$\pi_0 = \text{PM}\ \dfrac{\texttt{[6]} \Rightarrow \texttt{[6]} \quad \text{OP}\ \dfrac{6 \Rightarrow 6 \quad \text{APP'}\ \dfrac{\pi_f \quad \texttt{[]} \Rightarrow \texttt{[]} \quad \text{PM}\ \dfrac{\texttt{[]} \Rightarrow \texttt{[]} \quad \texttt{1} \Rightarrow \texttt{1}}{\texttt{match [] with [] -> 1 | x::xs -> x+g xs} \Rightarrow \texttt{1}}}{\texttt{f []} \Rightarrow \texttt{1}} \quad 6 * 1 \Rightarrow 6}{\texttt{6*f []} \Rightarrow \texttt{6}}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow \texttt{6}}$$

$$\text{APP'}\ \dfrac{\pi_f \quad \texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{PM}\ \dfrac{\texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{OP}\ \dfrac{3 \Rightarrow 3 \quad \text{APP'}\ \dfrac{\pi_g \quad \texttt{[6]} \Rightarrow \texttt{[6]} \quad \pi_0}{\texttt{g [6]} \Rightarrow \texttt{6}} \quad 3 + 6 \Rightarrow 9}{\texttt{3+g [6]} \Rightarrow \texttt{9}}}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow \texttt{9}}}{\texttt{f [3;6]} \Rightarrow}$$

T01: Big Steps Ü2

$$\pi_0 = \text{PM}\ \frac{\texttt{[6]} \Rightarrow \texttt{[6]} \quad \text{OP}\ \dfrac{\texttt{6} \Rightarrow \texttt{6} \quad \text{APP'}\ \dfrac{\pi_f \quad \texttt{[]} \Rightarrow \texttt{[]} \quad \text{PM}\ \dfrac{\texttt{[]} \Rightarrow \texttt{[]} \quad \texttt{1} \Rightarrow \texttt{1}}{\texttt{match [] with [] -> 1 | x::xs -> x+g xs} \Rightarrow \texttt{1}}}{\texttt{f []} \Rightarrow \texttt{1}} \quad 6 * 1 \Rightarrow \texttt{6}}{\texttt{6*f []} \Rightarrow \texttt{6}}}{\texttt{match [6] with [] -> 0 | x::xs -> x*f xs} \Rightarrow \texttt{6}}$$

$$\text{APP'}\ \frac{\pi_f \quad \texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{PM}\ \dfrac{\texttt{[3;6]} \Rightarrow \texttt{[3;6]} \quad \text{OP}\ \dfrac{\texttt{3} \Rightarrow \texttt{3} \quad \text{APP'}\ \dfrac{\pi_g \quad \texttt{[6]} \Rightarrow \texttt{[6]} \quad \pi_0}{\texttt{g [6]} \Rightarrow \texttt{6}} \quad 3 + 6 \Rightarrow \texttt{9}}{\texttt{3+g [6]} \Rightarrow \texttt{9}}}{\texttt{match [3;6] with [] -> 1 | x::xs -> x+g xs} \Rightarrow \texttt{9}}}{\texttt{f [3;6]} \Rightarrow \texttt{9}}$$

T01: Big Steps Ü3

```
(fun x -> x 3) (fun y z -> z y) (fun w -> w+w) ⇒
```

T01: Big Steps Ü3

$$(\text{APP'})\quad \frac{e_0 \Rightarrow \texttt{fun}\ x_1 \ldots x_k \texttt{->} e \quad e_1 \Rightarrow v_1 \ \ldots\ e_k \Rightarrow v_k \quad e[v_1/x_1, \ldots, v_k/x_k] \Rightarrow v}{e_0\ e_1\ \ldots\ e_k \ \Rightarrow\ v}$$

$$\pi_0 = \frac{}{\texttt{(fun x -> x 3) (fun y z -> z y)} \Rightarrow}$$

$$\text{APP'}\ \frac{\pi_0 \quad \texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w}}{\texttt{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow}$$

T01: Big Steps Ü3

$$(\text{APP'})\quad \frac{e_0 \Rightarrow \texttt{fun}\ x_1 \ldots x_k \texttt{->} e \quad e_1 \Rightarrow v_1 \;\ldots\; e_k \Rightarrow v_k \quad e[v_1/x_1, \ldots, v_k/x_k] \Rightarrow v}{e_0\ e_1\ \ldots\ e_k \Rightarrow v}$$

$$\pi_0 = \text{APP'}\ \frac{\texttt{fun x -> x 3} \Rightarrow \texttt{fun x -> x 3} \quad \texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \qquad \texttt{(fun y z -> z y) 3} \Rightarrow}{\texttt{(fun x -> x 3) (fun y z -> z y)} \Rightarrow}$$

$$\text{APP'}\ \frac{\pi_0 \quad \texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \quad .}{\texttt{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow}$$

T01: Big Steps Ü3

$$(\text{APP'})\quad \frac{e_0 \Rightarrow \texttt{fun}\ x_1 \ldots x_k \texttt{->}\ e \quad e_1 \Rightarrow v_1 \ \ldots\ e_k \Rightarrow v_k \quad e[v_1/x_1, \ldots, v_k/x_k] \Rightarrow v}{e_0\ e_1\ \ldots\ e_k \Rightarrow v}$$

$$\pi_0 = \text{APP'}\ \frac{\texttt{fun x -> x 3} \Rightarrow \texttt{fun x -> x 3} \quad \texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \text{APP'}\ \dfrac{\texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \texttt{3} \Rightarrow \texttt{3} \quad \texttt{fun z -> z 3} \Rightarrow \texttt{fun z -> z 3}}{\texttt{(fun y z -> z y) 3} \Rightarrow}}{\texttt{(fun x -> x 3) (fun y z -> z y)} \Rightarrow}$$

$$\text{APP'}\ \frac{\pi_0 \quad \texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \quad .}{\texttt{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow}$$

T01: Big Steps Ü3

$$\pi_0 = \text{APP'}\ \frac{\texttt{fun x -> x 3} \Rightarrow \texttt{fun x -> x 3} \quad \texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \text{APP'}\ \dfrac{\texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \texttt{3} \Rightarrow \texttt{3} \quad \texttt{fun z -> z 3} \Rightarrow \texttt{fun z -> z 3}}{\texttt{(fun y z -> z y) 3} \Rightarrow \texttt{fun z -> z 3}}}{\texttt{(fun x -> x 3) (fun y z -> z y)} \Rightarrow}$$

$$\text{APP'}\ \frac{\pi_0 \quad \texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \quad .}{\texttt{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow}$$

T01: Big Steps Ü3

$$\pi_0 = \text{APP'}\ \frac{\texttt{fun x -> x 3} \Rightarrow \texttt{fun x -> x 3} \quad \texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \text{APP'}\ \dfrac{\texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \texttt{3} \Rightarrow \texttt{3} \quad \texttt{fun z -> z 3} \Rightarrow \texttt{fun z -> z 3}}{\texttt{(fun y z -> z y) 3} \Rightarrow \texttt{fun z -> z 3}}}{\texttt{(fun x -> x 3) (fun y z -> z y)} \Rightarrow \texttt{fun z -> z 3}}$$

$$\text{APP'}\ \frac{\pi_0 \quad \texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \quad .}{\texttt{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow}$$

T01: Big Steps Ü3

$$\pi_0 = \text{APP'}\ \dfrac{\texttt{fun x -> x 3} \Rightarrow \texttt{fun x -> x 3} \quad \texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \text{APP'}\ \dfrac{\texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \texttt{3} \Rightarrow \texttt{3} \quad \texttt{fun z -> z 3} \Rightarrow \texttt{fun z -> z 3}}{\texttt{(fun y z -> z y) 3} \Rightarrow \texttt{fun z -> z 3}}}{\texttt{(fun x -> x 3) (fun y z -> z y)} \Rightarrow \texttt{fun z -> z 3}}$$

$$\text{APP'}\ \dfrac{\pi_0 \quad \texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \quad \texttt{(fun w -> w+w) 3} \Rightarrow}{\texttt{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow}$$

T01: Big Steps Ü3

$$(\text{APP'})\quad \frac{e_0 \Rightarrow \texttt{fun}\ x_1 \dots x_k \texttt{->} e \quad e_1 \Rightarrow v_1 \quad \dots \quad e_k \Rightarrow v_k \quad e[v_1/x_1,\dots,v_k/x_k] \Rightarrow v}{e_0\ e_1\ \dots\ e_k \Rightarrow v}$$

$$\pi_0 = \text{APP'}\ \frac{\texttt{fun x -> x 3} \Rightarrow \texttt{fun x -> x 3} \quad \texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \text{APP'}\ \frac{\texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \texttt{3} \Rightarrow \texttt{3} \quad \texttt{fun z -> z 3} \Rightarrow \texttt{fun z -> z 3}}{\texttt{(fun y z -> z y) 3} \Rightarrow \texttt{fun z -> z 3}}}{\texttt{(fun x -> x 3) (fun y z -> z y)} \Rightarrow \texttt{fun z -> z 3}}$$

$$\text{APP'}\ \frac{\pi_0 \quad \texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \quad \text{APP'}\ \frac{\texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \quad \texttt{3} \Rightarrow \texttt{3} \quad \texttt{3+3} \Rightarrow}{\texttt{(fun w -> w+w) 3} \Rightarrow}}{\texttt{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow}$$

T01: Big Steps Ü3

$$(\mathsf{OP})\quad \frac{e_1 \Rightarrow v_1 \qquad e_2 \Rightarrow v_2 \qquad v_1\ \mathsf{op}\ v_2 \Rightarrow v}{e_1\ \mathsf{op}\ e_2 \Rightarrow v}$$

$$\pi_0 = \text{APP'}\ \frac{\texttt{fun x -> x 3} \Rightarrow \texttt{fun x -> x 3} \quad \texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \text{APP'}\ \dfrac{\texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \texttt{3} \Rightarrow \texttt{3} \quad \texttt{fun z -> z 3} \Rightarrow \texttt{fun z -> z 3}}{\texttt{(fun y z -> z y) 3} \Rightarrow \texttt{fun z -> z 3}}}{\texttt{(fun x -> x 3) (fun y z -> z y)} \Rightarrow \texttt{fun z -> z 3}}$$

$$\text{APP'}\ \frac{\pi_0 \quad \texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \quad \text{APP'}\ \dfrac{\texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \quad \texttt{3} \Rightarrow \texttt{3} \quad \text{OP}\ \dfrac{\texttt{3} \Rightarrow \texttt{3} \quad \texttt{3} \Rightarrow \texttt{3} \quad \texttt{3} + \texttt{3} \Rightarrow \texttt{6}}{\texttt{3+3} \Rightarrow}}{\texttt{(fun w -> w+w) 3} \Rightarrow}}{\texttt{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow}$$

T01: Big Steps Ü3

$$\pi_0 = \text{APP'}\ \frac{\texttt{fun x -> x 3} \Rightarrow \texttt{fun x -> x 3} \quad \texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \text{APP'}\ \dfrac{\texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \texttt{3} \Rightarrow \texttt{3} \quad \texttt{fun z -> z 3} \Rightarrow \texttt{fun z -> z 3}}{\texttt{(fun y z -> z y) 3} \Rightarrow \texttt{fun z -> z 3}}}{\texttt{(fun x -> x 3) (fun y z -> z y)} \Rightarrow \texttt{fun z -> z 3}}$$

$$\text{APP'}\ \frac{\pi_0 \quad \texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \quad \text{APP'}\ \dfrac{\texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \quad \texttt{3} \Rightarrow \texttt{3} \quad \text{OP}\ \dfrac{3 \Rightarrow 3 \quad 3 \Rightarrow 3 \quad 3 + 3 \Rightarrow 6}{3+3 \Rightarrow 6}}{\texttt{(fun w -> w+w) 3} \Rightarrow}}{\texttt{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow}$$

T01: Big Steps Ü3

$$\pi_0 = \text{APP'}\ \dfrac{\texttt{fun x -> x 3} \Rightarrow \texttt{fun x -> x 3} \quad \texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \text{APP'}\ \dfrac{\texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad 3 \Rightarrow 3 \quad \texttt{fun z -> z 3} \Rightarrow \texttt{fun z -> z 3}}{\texttt{(fun y z -> z y) 3} \Rightarrow \texttt{fun z -> z 3}}}{\texttt{(fun x -> x 3) (fun y z -> z y)} \Rightarrow \texttt{fun z -> z 3}}$$

$$\text{APP'}\ \dfrac{\pi_0 \quad \texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \quad \text{APP'}\ \dfrac{\texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \quad 3 \Rightarrow 3 \quad \text{OP}\ \dfrac{3 \Rightarrow 3 \quad 3 \Rightarrow 3 \quad 3 + 3 \Rightarrow 6}{\texttt{3+3} \Rightarrow 6}}{\texttt{(fun w -> w+w) 3} \Rightarrow 6}}{\texttt{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow}$$

T01: Big Steps Ü3

$$\pi_0 = \text{APP'}\,\frac{\texttt{fun x -> x 3} \Rightarrow \texttt{fun x -> x 3} \quad \texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \text{APP'}\,\dfrac{\texttt{fun y z -> z y} \Rightarrow \texttt{fun y z -> z y} \quad \texttt{3} \Rightarrow \texttt{3} \quad \texttt{fun z -> z 3} \Rightarrow \texttt{fun z -> z 3}}{\texttt{(fun y z -> z y) 3} \Rightarrow \texttt{fun z -> z 3}}}{\texttt{(fun x -> x 3) (fun y z -> z y)} \Rightarrow \texttt{fun z -> z 3}}$$

$$\text{APP'}\,\frac{\pi_0 \quad \texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \quad \text{APP'}\,\dfrac{\texttt{fun w -> w+w} \Rightarrow \texttt{fun w -> w+w} \quad \texttt{3} \Rightarrow \texttt{3} \quad \text{OP}\,\dfrac{\texttt{3} \Rightarrow \texttt{3} \quad \texttt{3} \Rightarrow \texttt{3} \quad 3 + 3 \Rightarrow \texttt{6}}{\texttt{3+3} \Rightarrow \texttt{6}}}{\texttt{(fun w -> w+w) 3} \Rightarrow \texttt{6}}}{\texttt{(fun x -> x 3) (fun y z -> z y) (fun w -> w+w)} \Rightarrow \texttt{6}}$$

T02: Multiplication

Given the following function definition:

```
let rec mul a b =
  match a with 0 -> 0 | _ -> b + mul (a - 1) b
```

Prove that **`mul`** a b terminates for all inputs a and b. Here a and b are mini-OCaml expressions that evaluate to non-negative integers.

Unless specified otherwise, all rules used in a big-step proof tree must be annotated and all axioms ($v \Rightarrow v$*) must be written down.*

Tipp: Induction on parameter a

- Base Case: n = 0
- n -> n+1

T02: Multiplication

We know that a and b are expressions which evaluate to integers. We will use n and m to refer respectively to the values of a and b.

To show that the expression $\texttt{mul} \;\; 0 \;\; b$ terminates using a big-step proof, we need to show that it evaluates to some value. Since `mul` multiplies the values of a and b, and we assume n and m are the values of a and b, we will show that the expression evaluates to $n \cdot m$. More precisely, we will prove: if $a \Rightarrow n$ and $b \Rightarrow m$, then $\texttt{mul} \;\; a \;\; b \Rightarrow n \cdot m$. The proof proceeds by an induction on n, or in other words, on the value of the first argument.

- **Base Case**: When n is 0. The statement to show is: if $a \Rightarrow 0$ and $b \Rightarrow m$, then $\texttt{mul} \;\; a \;\; b \Rightarrow 0 \cdot m$, where $0 \cdot m$ is simply 0.

$$\text{APP'} \frac{\pi_{mul} \quad a \Rightarrow 0 \quad b \Rightarrow m \quad \text{PM} \dfrac{0 \Rightarrow \texttt{0} \quad \texttt{0} \Rightarrow 0}{\texttt{match} \;\; 0 \;\; \texttt{with 0 -> 0 | \_ ->} \;\; b \;\; \texttt{+ mul} \;\; \texttt{(}0 \;\texttt{- 1)} \;\; b \Rightarrow 0}}{\texttt{mul} \;\; a \;\; b \Rightarrow 0}$$

The induction hypothesis is: if $a \Rightarrow n$ and $b \Rightarrow m$, then $\texttt{mul} \;\; a \;\; b \Rightarrow n \cdot m$.

T02: Multiplication

- **Inductive Case**: Assume the statement holds for $n \in \mathbb{N}$ and show it holds for $n+1$. The induction hypothesis is: if $a \Rightarrow n$ and $b \Rightarrow m$, then `mul` $a\ \ b \Rightarrow n \cdot m$. The statement to show is: if $a \Rightarrow n+1$ and $b \Rightarrow m$, then `mul` $a\ \ b \Rightarrow (n+1) \cdot m$.

$$\text{APP'}\ \frac{\pi_{mul} \quad a \Rightarrow n+1 \quad b \Rightarrow m \qquad \texttt{match}\ n+1\ \texttt{with 0 -> 0 | \_ ->}\ m\ \texttt{+ mul}\ \texttt{((}n+1\texttt{) - 1)}\ m \Rightarrow}{\texttt{mul}\ a\ \ b \Rightarrow}$$

T02: Multiplication

- **Inductive Case**: Assume the statement holds for $n \in \mathbb{N}$ and show it holds for $n+1$. The induction hypothesis is: if $a \Rightarrow n$ and $b \Rightarrow m$, then $\texttt{mul}\ a\ b \Rightarrow n \cdot m$. The statement to show is: if $a \Rightarrow n+1$ and $b \Rightarrow m$, then $\texttt{mul}\ a\ b \Rightarrow (n+1) \cdot m$.

$$\text{APP'}\ \dfrac{\pi_{mul} \quad a \Rightarrow n+1 \quad b \Rightarrow m \quad \text{PM}\ \dfrac{n+1 \Rightarrow n+1 \qquad \dfrac{m\ \texttt{+ mul}\ ((n+1)\ \texttt{- 1})\ m \Rightarrow}{}}{\texttt{match}\ n+1\ \texttt{with 0 -> 0 | \_ ->}\ m\ \texttt{+ mul}\ ((n+1)\ \texttt{- 1})\ m \Rightarrow}}{\texttt{mul}\ a\ b \Rightarrow}$$

T02: Multiplication

- **Inductive Case**: Assume the statement holds for $n \in \mathbb{N}$ and show it holds for $n+1$. The induction hypothesis is: if $a \Rightarrow n$ and $b \Rightarrow m$, then $\texttt{mul}\ \ a\ \ b \Rightarrow n \cdot m$. The statement to show is: if $a \Rightarrow n+1$ and $b \Rightarrow m$, then $\texttt{mul}\ \ a\ \ b \Rightarrow (n+1) \cdot m$.

$$\text{APP'}\ \dfrac{\pi_{mul} \quad a \Rightarrow n+1 \quad b \Rightarrow m \quad \text{PM}\ \dfrac{n+1 \Rightarrow n+1 \quad \text{OP}\ \dfrac{m \Rightarrow m \qquad \texttt{mul ((}n+1\texttt{) - 1)}\ m \Rightarrow}{m\ \texttt{+ mul ((}n+1\texttt{) - 1)}\ m \Rightarrow}}{\texttt{match}\ n+1\ \texttt{with 0 -> 0 | \_ ->}\ m\ \texttt{+ mul ((}n+1\texttt{) - 1)}\ m \Rightarrow}}{\texttt{mul}\ a\ b \Rightarrow}$$

T02: Multiplication

- **Inductive Case**: Assume the statement holds for $n \in \mathbb{N}$ and show it holds for $n+1$. The induction hypothesis is: if $a \Rightarrow n$ and $b \Rightarrow m$, then `mul` a $b \Rightarrow n \cdot m$. The statement to show is: if $a \Rightarrow n+1$ and $b \Rightarrow m$, then `mul` a $b \Rightarrow (n+1) \cdot m$.

$$
\text{APP'}\ \dfrac{\pi_{mul} \quad a \Rightarrow n+1 \quad b \Rightarrow m \quad \text{PM}\ \dfrac{n+1 \Rightarrow n+1 \quad \text{OP}\ \dfrac{m \Rightarrow m \quad by\ I.H.\ \dfrac{\text{OP}\ \dfrac{n+1 \Rightarrow n+1 \quad \mathtt{1} \Rightarrow 1 \quad (n+1) - 1 \Rightarrow n}{(n+1)\ \mathtt{-}\ \mathtt{1} \Rightarrow n} \qquad m \Rightarrow m}{\mathtt{mul}\ ((n+1)\ \mathtt{-}\ \mathtt{1})\ m \Rightarrow n \cdot m}}{m\ \mathtt{+}\ \mathtt{mul}\ ((n+1)\ \mathtt{-}\ \mathtt{1})\ m \Rightarrow}}{\mathtt{match}\ n+1\ \mathtt{with\ 0\ \text{->}\ 0\ |\ \_\ \text{->}}\ m\ \mathtt{+}\ \mathtt{mul}\ ((n+1)\ \mathtt{-}\ \mathtt{1})\ m \Rightarrow}}{\mathtt{mul}\ a\ b \Rightarrow}
$$

IH: *mul* a b = a*b

T02: Multiplication

- **Inductive Case**: Assume the statement holds for $n \in \mathbb{N}$ and show it holds for $n+1$. The induction hypothesis is: if $a \Rightarrow n$ and $b \Rightarrow m$, then `mul` a $b \Rightarrow n \cdot m$. The statement to show is: if $a \Rightarrow n+1$ and $b \Rightarrow m$, then `mul` a $b \Rightarrow (n+1) \cdot m$.

$$\text{APP'}\,\dfrac{\pi_{mul} \quad a \Rightarrow n+1 \quad b \Rightarrow m \quad \text{PM}\,\dfrac{n+1 \Rightarrow n+1 \quad \text{OP}\,\dfrac{m \Rightarrow m \quad \textit{by I.H.}\,\dfrac{\text{OP}\,\dfrac{n+1 \Rightarrow n+1 \quad 1 \Rightarrow 1 \quad (n+1) - 1 \Rightarrow n}{(n+1)\ \texttt{-}\ 1 \Rightarrow n} \quad m \Rightarrow m}{\texttt{mul}\ ((n+1)\ \texttt{-}\ 1)\ m \Rightarrow n \cdot m} \quad m + (n \cdot m) \Rightarrow}{m\ \texttt{+}\ \texttt{mul}\ ((n+1)\ \texttt{-}\ 1)\ m \Rightarrow}}{\texttt{match}\ n+1\ \texttt{with 0 -> 0 | \_ ->}\ m\ \texttt{+}\ \texttt{mul}\ ((n+1)\ \texttt{-}\ 1)\ m \Rightarrow}}{\texttt{mul}\ a\ b \Rightarrow}$$

T02: Multiplication

- **Inductive Case**: Assume the statement holds for $n \in \mathbb{N}$ and show it holds for $n+1$. The induction hypothesis is: if $a \Rightarrow n$ and $b \Rightarrow m$, then `mul` a $b \Rightarrow n \cdot m$. The statement to show is: if $a \Rightarrow n+1$ and $b \Rightarrow m$, then `mul` a $b \Rightarrow (n+1) \cdot m$.

$$\text{APP'}\ \dfrac{\pi_{mul} \quad a \Rightarrow n+1 \quad b \Rightarrow m \quad \text{PM}\ \dfrac{n+1 \Rightarrow n+1 \quad \text{OP}\ \dfrac{m \Rightarrow m \quad \textit{by I.H.}\ \dfrac{\text{OP}\ \dfrac{n+1 \Rightarrow n+1 \quad 1 \Rightarrow 1 \quad (n+1) - 1 \Rightarrow n}{(n+1)\ \texttt{-}\ 1 \Rightarrow n} \quad m \Rightarrow m}{\texttt{mul}\ ((n+1)\ \texttt{-}\ 1)\ m \Rightarrow n \cdot m} \quad m + (n \cdot m) \Rightarrow (n+1) \cdot m}{m\ \texttt{+}\ \texttt{mul}\ ((n+1)\ \texttt{-}\ 1)\ m \Rightarrow}}{\texttt{match}\ n+1\ \texttt{with 0 -> 0 | \_ ->}\ m\ \texttt{+ mul}\ ((n+1)\ \texttt{-}\ 1)\ m \Rightarrow}}{\texttt{mul}\ a\ b \Rightarrow}$$

T02: Multiplication

- **Inductive Case**: Assume the statement holds for $n \in \mathbb{N}$ and show it holds for $n+1$. The induction hypothesis is: if $a \Rightarrow n$ and $b \Rightarrow m$, then $\texttt{mul}\ \ a\ \ b \Rightarrow n \cdot m$. The statement to show is: if $a \Rightarrow n+1$ and $b \Rightarrow m$, then $\texttt{mul}\ \ a\ \ b \Rightarrow (n+1) \cdot m$.

$$
\text{APP'}\ \dfrac{\pi_{mul} \quad a \Rightarrow n+1 \quad b \Rightarrow m \quad \text{PM}\ \dfrac{n+1 \Rightarrow n+1 \quad \text{OP}\ \dfrac{m \Rightarrow m \quad by\ I.H.\ \dfrac{\text{OP}\ \dfrac{n+1 \Rightarrow n+1 \quad 1 \Rightarrow 1 \quad (n+1) - 1 \Rightarrow n}{(n+1)\ \texttt{-}\ 1 \Rightarrow n} \quad m \Rightarrow m}{\texttt{mul}\ \ ((n+1)\ \texttt{-}\ 1)\ \ m \Rightarrow n \cdot m} \quad m + (n \cdot m) \Rightarrow}{m\ \texttt{+}\ \texttt{mul}\ \ ((n+1)\ \texttt{-}\ 1)\ \ m \Rightarrow (n+1) \cdot m}}{\texttt{match}\ \ n+1\ \ \texttt{with 0 -> 0 | \_ ->}\ \ m\ \texttt{+}\ \texttt{mul}\ \ ((n+1)\ \texttt{-}\ 1)\ \ m \Rightarrow}}{\texttt{mul}\ \ a\ \ b \Rightarrow}
$$

T02: Multiplication

- **Inductive Case**: Assume the statement holds for $n \in \mathbb{N}$ and show it holds for $n+1$. The induction hypothesis is: if $a \Rightarrow n$ and $b \Rightarrow m$, then `mul` $a \ b \Rightarrow n \cdot m$. The statement to show is: if $a \Rightarrow n+1$ and $b \Rightarrow m$, then `mul` $a \ b \Rightarrow (n+1) \cdot m$.

$$
\text{APP'}\ \dfrac{\pi_{mul} \quad a \Rightarrow n+1 \quad b \Rightarrow m \quad \text{PM}\ \dfrac{n+1 \Rightarrow n+1 \quad \text{OP}\ \dfrac{m \Rightarrow m \quad \text{by I.H.}\ \dfrac{\text{OP}\ \dfrac{n+1 \Rightarrow n+1 \quad 1 \Rightarrow 1 \quad (n+1) - 1 \Rightarrow n}{(n+1)\ \texttt{-}\ 1 \Rightarrow n} \quad m \Rightarrow m}{\texttt{mul}\ ((n+1)\ \texttt{-}\ 1)\ m \Rightarrow n \cdot m} \quad m + (n \cdot m) \Rightarrow}{m\ \texttt{+}\ \texttt{mul}\ ((n+1)\ \texttt{-}\ 1)\ m \Rightarrow (n+1) \cdot m}}{\texttt{match}\ n+1\ \texttt{with 0 -> 0 | \_ ->}\ m\ \texttt{+}\ \texttt{mul}\ ((n+1)\ \texttt{-}\ 1)\ m \Rightarrow (n+1) \cdot m}}{\texttt{mul}\ a\ b \Rightarrow}
$$

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$$

T03: Threesum

Use big-step operational semantics to show that the function

```
let rec threesum = fun l ->
  match l with [] -> 0 | x::xs -> 3 * x + threesum xs
```

terminates for all inputs and computes three times the sum of the input list's elements.

Unless specified otherwise, all rules used in a big-step proof tree must be annotated and all axioms ($v \Rightarrow v$) must be written down.

Tipp: Induction on parameter L

- Base Case L= [ ]
- L -> x_{n+1} :: L

T03: Threesum

$$\pi_{ts} = \text{GD}\ \frac{\texttt{threesum = } \tau_{ts} \qquad \tau_{ts} \Rightarrow \tau_{ts}}{\texttt{threesum} \Rightarrow \tau_{ts}}$$

Now, we do an induction on the length n of the list.

- **Base Case**: $n = 0$ (`l = []`)

$$\text{APP}\ \frac{\pi_{ts} \quad \texttt{[]} \Rightarrow \texttt{[]} \quad \text{PM}\ \frac{\texttt{[]} \Rightarrow \texttt{[]} \quad \texttt{0} \Rightarrow \texttt{0}}{\texttt{match [] with [] -> 0 | x::xs -> 3*x + threesum xs} \Rightarrow \texttt{0}}}{\texttt{threesum []} \Rightarrow \texttt{0}}$$

We assume `threesum xs` terminates with $3\sum_{i=1}^{n} x_i$ for an input `xs = ` $[x_n; \ldots; x_1]$ of length $n \geq 0$.

T03: Threesum

- **Inductive Step**: We assume `threesum xs` terminates with $3\sum_{i=1}^{n} x_i$ for an input $\texttt{xs} = [x_n; \ldots; x_1]$ of length $n \geq 0$. Now, show that $\texttt{threesum}\ \ x_{n+1}\texttt{::xs}$ terminates with $3\sum_{i=1}^{n+1} x_i$:

$$\texttt{threesum}\ (x_{n+1}\texttt{::xs}) \Rightarrow$$

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