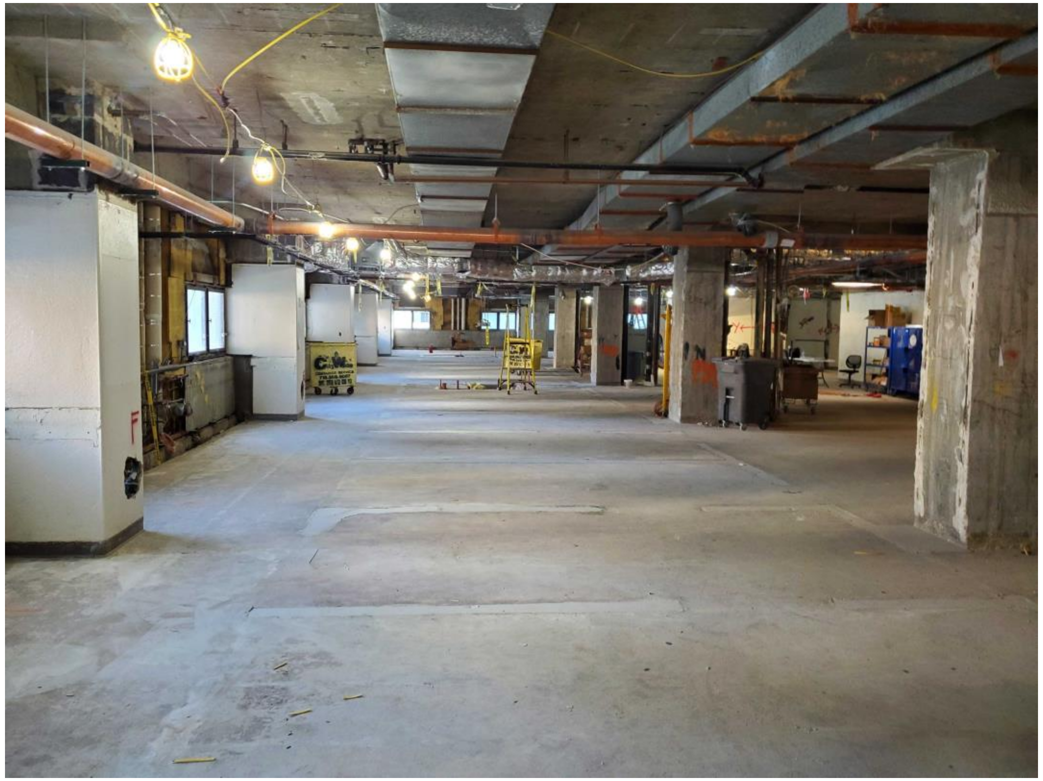
Lecture 5

Inverse Kinematics

Where Logan Is:
Bellevue Hospital Renovation, Downtown Manhattan



Announcements

- Lab 1: Due Today
- HW 2: Due next Friday, 2/25
  - You have everything you need to do this homework BEFORE this lecture.
- Lab 2: Going out this weekend.
  - Grad Students: must work independently
  - Undergrad's: May work in groups of 2 but each must have their own written solution
  - THIS IS A <u>VERY</u> TIME-CONSUMING PROJECT, START EARLY
  - Due 5/11 for a 10% bonus (and enjoyment of spring break)
  - Due **5/25** for full credit.

March

Schedule

Month	Date	Topics	Lecture	Reading	Homework	Laboratories	Other Assignments
Jan	14	Intro / Spatial descriptions		Craig 1 - 2			
	21	Rigid body transformations, Euler angles, Homogeneous Transformations			PS 1 out (due 2/4)	Lab 1 out (due 2/18)	
	28	Forward kinematics, DH parameters, examples		Craig 3			
Feb	4	FK examples Representing rotations, Quaternions,		Craig 4	PS 2 out (due 2/25)		
	11	~~No class! -- Half class to catch up on 2/9~~		Craig 5			
	18	IK w/ PUMP & Inverse kinematics				Lab 2* out (due 3/25)	
	25	Kinematics - Differential kinematics / Jacobians Kinematics - Redundancy, pseudoinverse, wrenches		Craig 6	PS 3 out (due 3/27)		
Mar	4	Dynamics: Acceleration and inertia					
	11	Dynamics: Newton-Euler, Lagrangian Actuation, sensing, and design			PS 4 out (due 5/1)		
	18	***No Class - Spring Break***					
	25	Linear Control of manipulators		Craig 8		Lab 3 out (due 4/22)	Project Proposals due
Apr	1	P, PI, PID control of manipulators Manipulator Control: Joing Space Control		Craig 9	PS 5 out (due 5/15)		
	8	Nonlinear control of manipulators		Craig 10			Exam (take-home) after Spring Break
	15	Force Control: Project Presentations		Craig 11			
	22	Additional Topics Project Presentations					Project submissions due 4/24

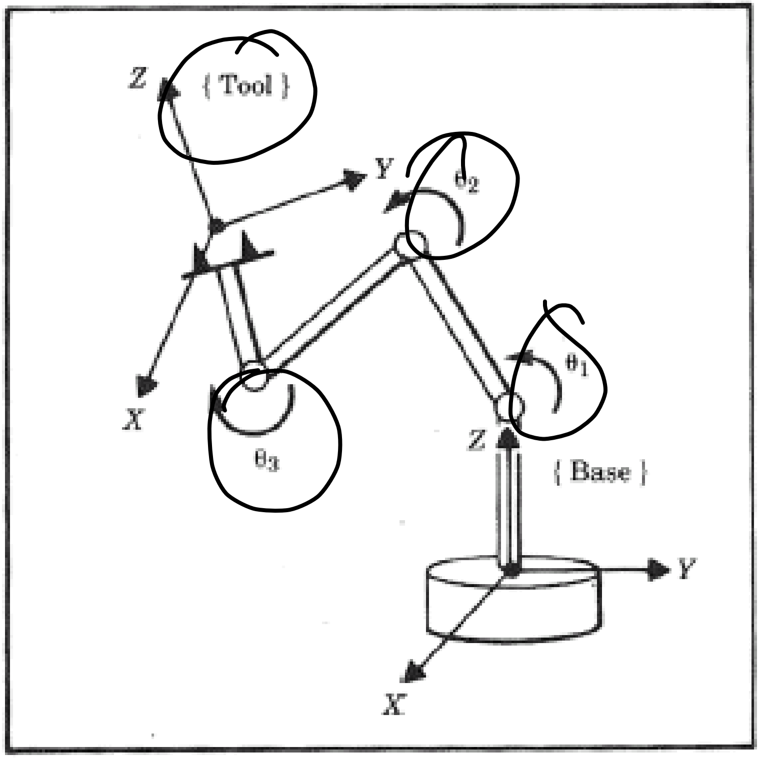
Recap

Direct vs. Inverse Kinematics

Lec 1

- **Direct (Forward) Kinematics**
  - Given: Joint angles and links geometry
  - Compute: Position and orientation of the end effector relative to the base frame

- **Inverse Kinematics**
  - Given: Position and orientation of the end effector relative to the base frame
  - Compute: All possible sets of joint angles and link geometries which could be used to attain the given position and orientation of the end effector



Solvability – PUMA 560

Given : PUMA 560 - 6 DOF, ${}^0_6T$

Solve: $\theta_1 \cdots \theta_6$

$$
{}^0_6T = {}^0_1T\,{}^1_2T\,{}^2_3T\,{}^3_4T\,{}^4_5T\,{}^5_6T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$r_{11} = c_1\left[c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6\right] + s_1(s_4c_5c_6 + c_4s_6),$$
$$r_{21} = s_1\left[c_{23}(c_4c_5c_6 - s_4s_6) - s_{23}s_5c_6\right] - c_1(s_4c_5c_6 + c_4s_6),$$
$$r_{31} = -s_{23}(c_4c_5c_6 - s_4s_6) - c_{23}s_5c_6,$$

$$r_{12} = c_1\left[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6\right] + s_1(c_4c_6 - s_4c_5s_6),$$
$$r_{22} = s_1\left[c_{23}(-c_4c_5s_6 - s_4c_6) + s_{23}s_5s_6\right] - c_1(c_4c_6 - s_4c_5s_6),$$
$$r_{32} = -s_{23}(-c_4c_5s_6 - s_4c_6) + c_{23}s_5s_6,$$

$$r_{13} = -c_1(c_{23}c_4s_5 + s_{23}c_5) - s_1s_4s_5,$$
$$r_{23} = -s_1(c_{23}c_4s_5 + s_{23}c_5) + c_1s_4s_5,$$
$$r_{33} = s_{23}c_4s_5 - c_{23}c_5,$$

$$p_x = c_1\left[a_2c_2 + a_3c_{23} - d_4s_{23}\right] - d_3s_1,$$
$$p_y = s_1\left[a_2c_2 + a_3c_{23} - d_4s_{23}\right] + d_3c_1,$$
$$p_z = -a_3s_{23} - a_2s_2 - d_4c_{23}.$$

Total Number of Equations: 12

Independent Equations: 3 - Rotation Matrix
3 - Position Vector

Type of Equations: Non-linear

Solvability

can we even reach the goal.

- **Existence of Solutions**
- **Multiple Solutions** → 2 or even more → how do we decide
- **Method of solutions**
  - Closed form solution ← best case
    - Algebraic solution
    - Geometric solution
  - Numerical solutions ← not covered in this class

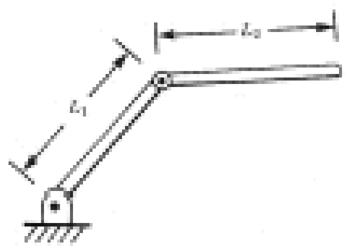
Solvability – Existence of Solution

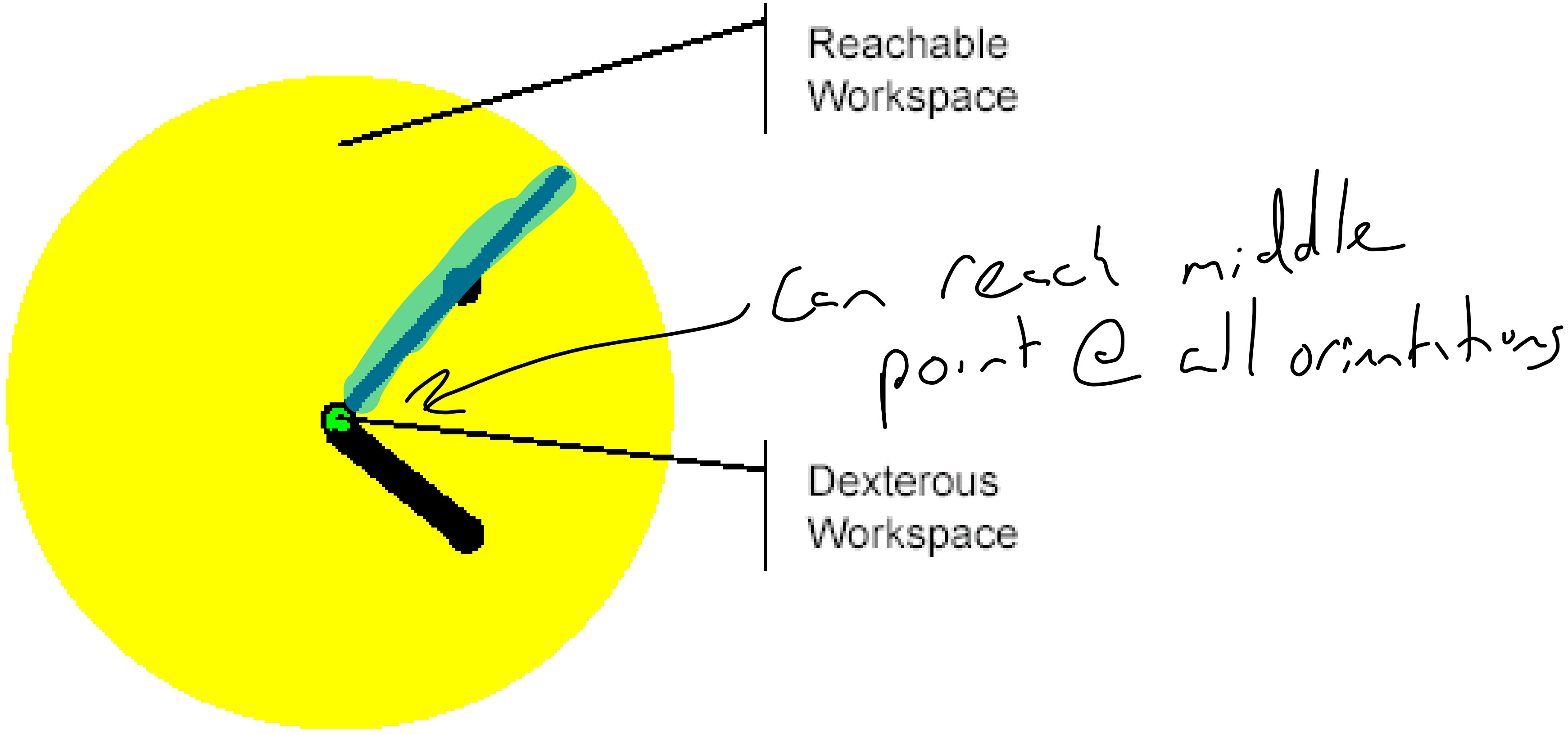
- For a solution to exist, ${}^{0}_{N}T$ must be in the ***workspace*** of the manipulator
- ***Workspace*** - Definitions
  - ***Dexterous Workspace (DW)***: The subset of space in which the robot end effector can reach ***in all orientations*.**
  - ***Reachable Workspace (RW):*** The subset of space in which the robot end effector can reach in ***at least 1 orientation***
- The Dexterous Workspace is a subset of the Reachable Workspace

RW $\supseteq$ DW

Solvability - Existence of Solution - Workspace - 2R

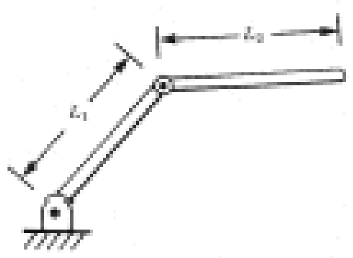
Example 1 ($L_1 = L_2$)

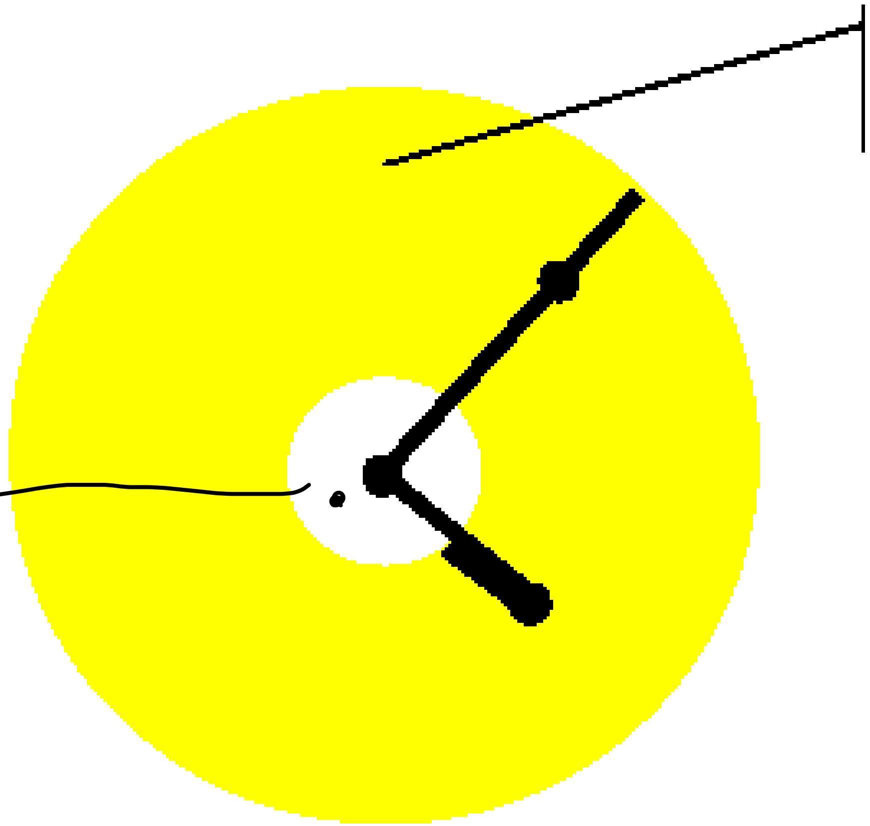




Solvability - Existence of Solution - Workspace - 2R

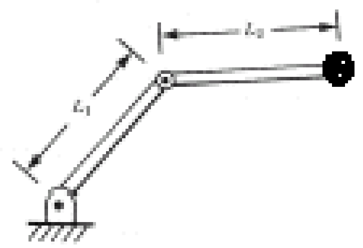
Example 2 ($L_1 \neq L_2$)

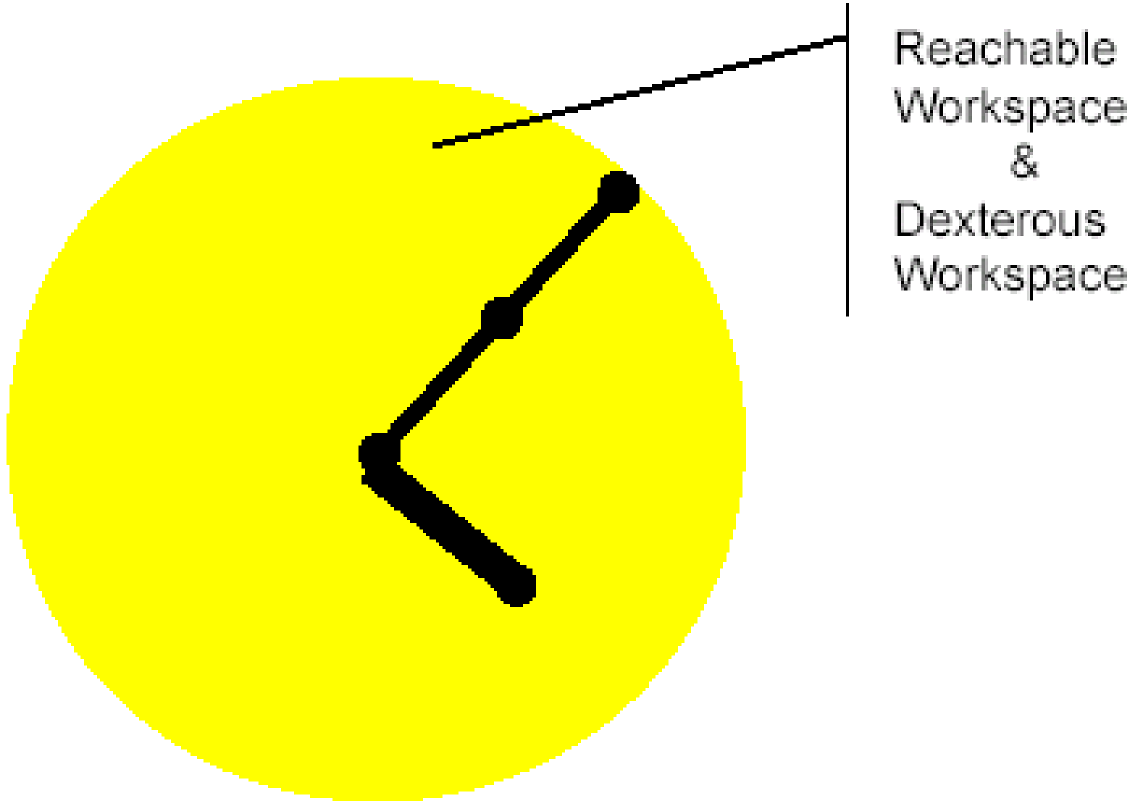




Solvability - Existence of Solution - Workspace - 3R
Example 3 **($L_1 = L_2$)**

$L_3 = 0$

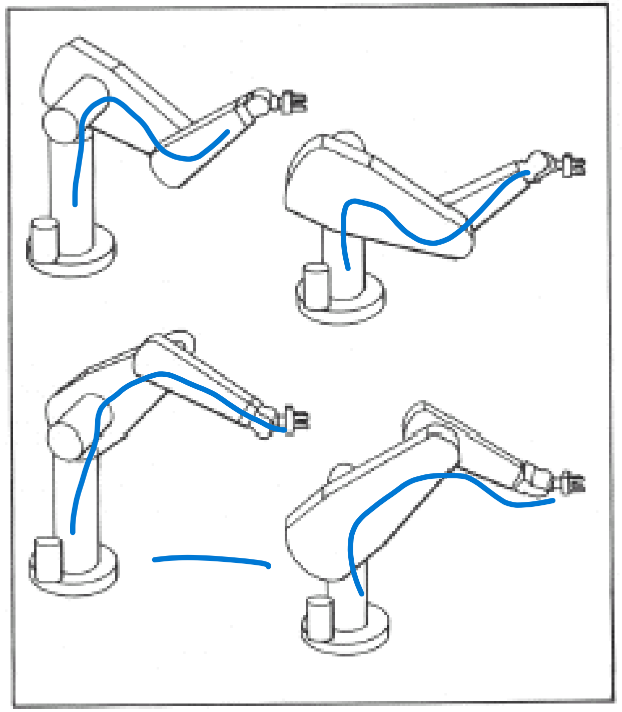




Solvability – Multiple Solutions

x, y, z positions
are all the same

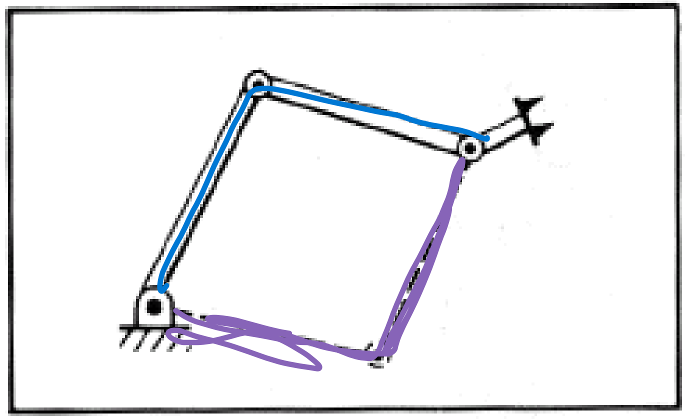
- Multiple solutions are a common problem that can occur when solving inverse kinematics because the system has to be able to chose one
- The number of solutions depends on the number of joints in the manipulator but is also a function of the link parameters *(a_i, α_i, θ_i, d_i)*
- Example: The PUMA 560 can reach certain goals with 8 different arm configurations (solutions)

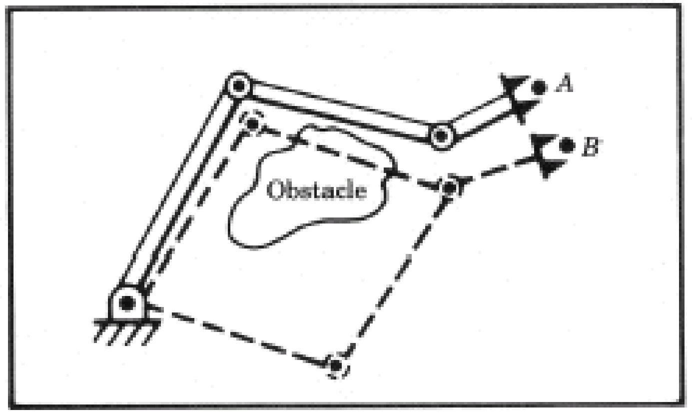


$$\theta_4' = \theta_4 + 180^o$$
$$\theta_5' = -\theta_5$$
$$\theta_6' = \theta_6 + 180^o$$

Solvability – Multiple Solutions

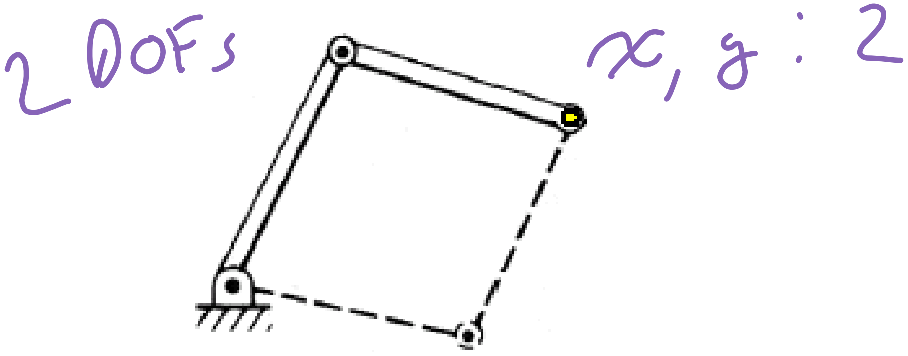
- **Problem:** The fact that a manipulator has multiple solutions may cause problems because the system has to be able to choose one
- **Solution:** Decision criteria
  - The closest (geometrically)
    - Minimizing the amount that each joint is required to move
    - Note 1: input argument - present position of the manipulator
    - Note 2: Joint Weight - Moving small joints (wrist) instead of moving large joints (Shoulder & Elbow)
  - Obstacles exist in the workspace
    - Avoiding collision

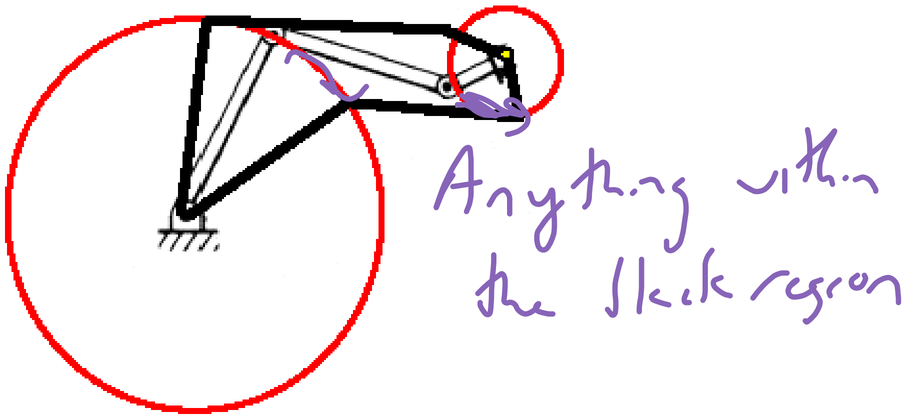




Solvability – Multiple Solutions – Number of Solutions

- Task Definition - Position the end effector in a specific point in the plane (2D)
- No. of DOF = No. of DOF of the task
  - Number of solutions:
    - 2 (elbow up/down)
- No. of DOF > No. of DOF of the task
  - Number of solutions: ∞
  - Self Motion - The robot can be moved without moving the end effector from the goal

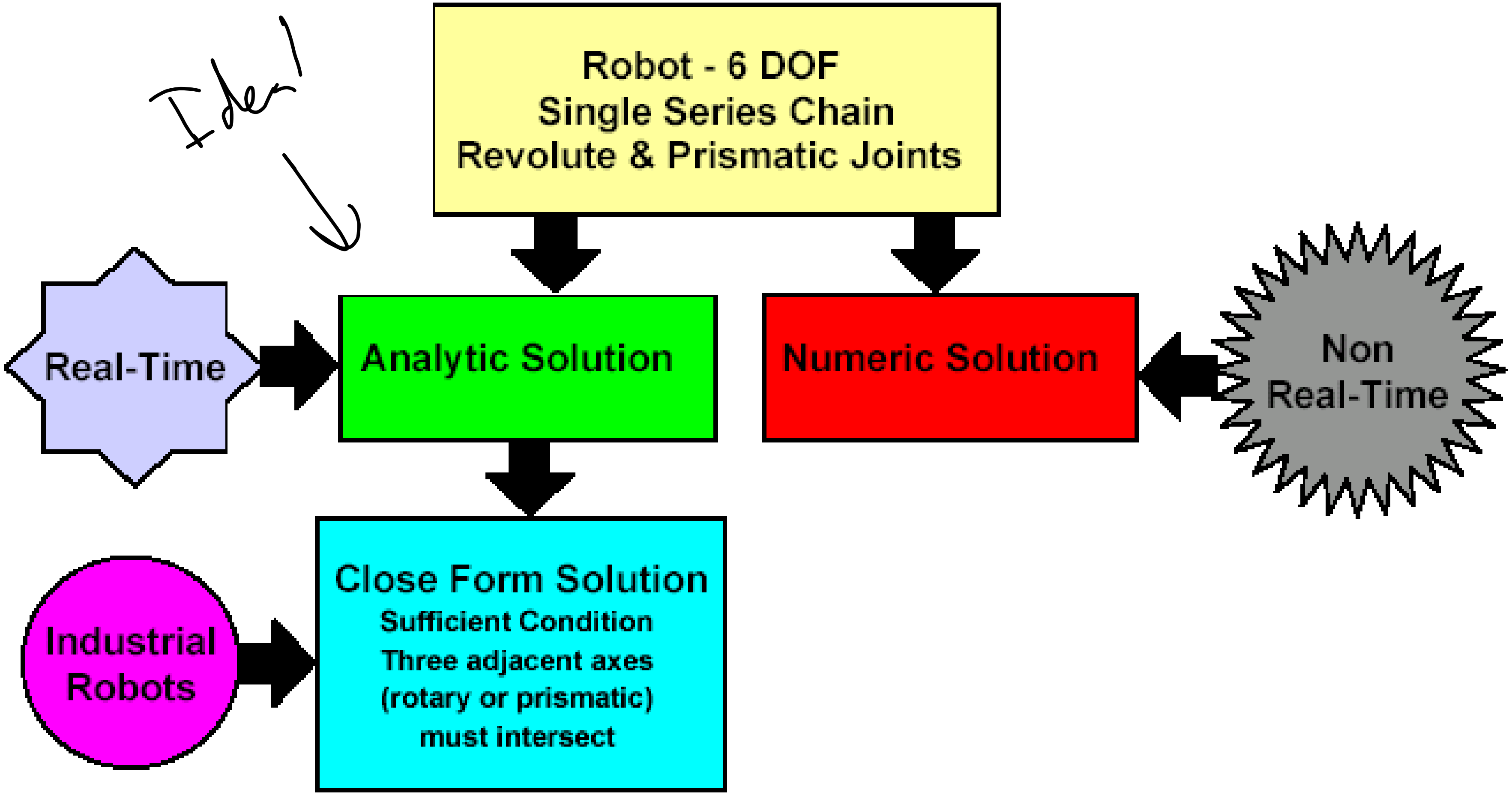




Solvability – Methods of Solutions

- **Solution** (Inverse Kinematics)- A "solution" is the set of joint variables associated with an end effector's desired position and orientation.
- **No general algorithms** that lead to the solution of inverse kinematic equations.
- **Solution Strategies**
  - **Closed form Solutions** - An analytic expression includes all solution sets.
    - **Algebraic Solution** - Trigonometric (Nonlinear) equations
    - **Geometric Solution** - Reduces the larger problem to a series of plane geometry problems.
  - **Numerical Solutions** - Iterative solutions will not be considered in this course.

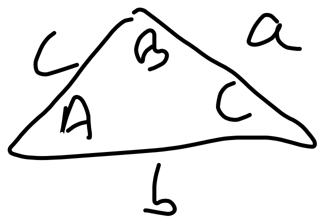
Solvability



Recap Over

Mathematical Equations

- Law of Sines / Cosines - For a general triangle



Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos A$$

- Sum of Angles

↳ Trig Identities

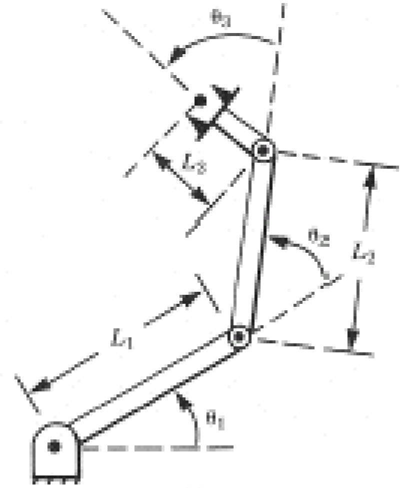
$$\sin(\theta_1 + \theta_2) = s_{12} = c_1 s_2 + s_1 c_2$$

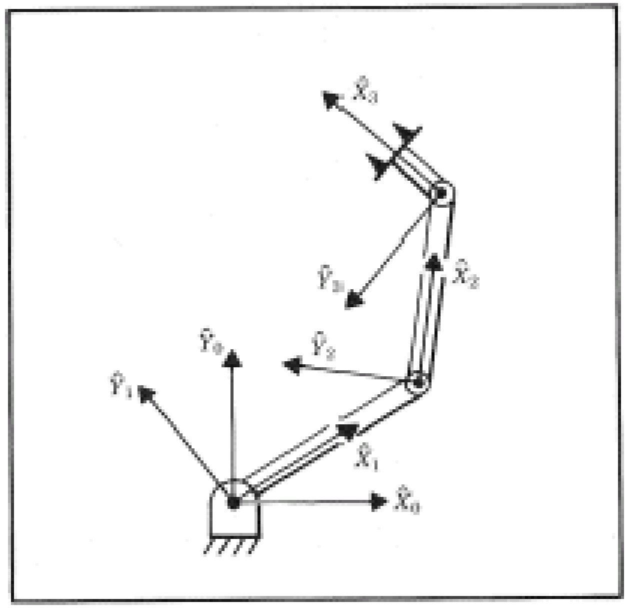
$$\cos(\theta_1 + \theta_2) = c_{12} = c_1 c_2 - s_1 s_2$$

Inverse Kinematics - Planar RRR (3R) - Algebraic Solution

$^{0}_{3}T$

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3





Inverse Kinematics - Planar RRR (3R) - Algebraic Solution

$$
{}^{i-1}_{\;\;i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3

$$
{}^0_1T = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^1_2T = \begin{bmatrix} c2 & -s2 & 0 & L1 \\ s2 & c2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^2_3T = \begin{bmatrix} c3 & -s3 & 0 & L2 \\ s3 & c3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Inverse Kinematics - Planar RRR (3R) - Algebraic Solution

$$ {}^B_WT = {}^0_3T = {}^0_1T\,{}^1_2T\,{}^2_3T = \begin{bmatrix} c1c2c3 - c1s2s3 - s1s2c3 - s1c2s3 & -c1c2s3 - c1s2c3 + s1s2s3 - s1c2c3 & 0 & c1(L2c2 + L1) - s1s2L2 \\ s1cc3 - s1s2s3 + c1s2c3 + c1c2s3 & -s1c2s3 - s1s2c3 - c1s2s3 + c1c2c3 & 0 & s1(L2c2 + L1) + c1s2L2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

angle sum

trig identities

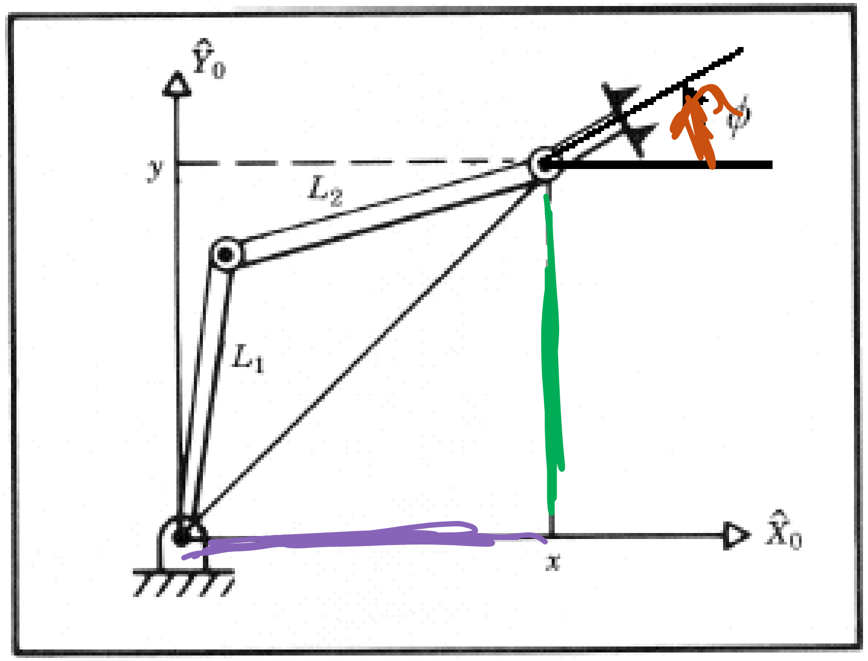
- Using trigonometric identities to simplify ${}^B_WT$, the solution to the forward kinematics is:

$$ {}^B_WT = {}^0_3T = \begin{bmatrix} c_{123} & -s_{123} & 0 & L_1c_1 + L_2c_{12} \\ s_{123} & c_{123} & 0 & L_1s_1 + L_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

- **where** $c_{123} = \cos(\theta_1 + \theta_2 + \theta_3)$ $\quad s_{123} = \sin(\theta_1 + \theta_2 + \theta_3)$

Inverse Kinematics - Planar RRR (3R) - Algebraic Solution

- *Given:*
  - ***Direct Kinematics:*** *The homogenous transformation from the base to the wrist* ${}^{B}_{W}T$
  - ***Goal Point Definition***: For a planar manipulator, specifying the goal can be accomplished by specifying three parameters: The position of the wrist in space (x, y) and the orientation of link 3 in the plane relative to the $\hat{X}$ axis (ϕ)



Inverse Kinematics - Planar RRR (3R) - Algebraic Solution

- ***Problem:***

What are the joint angles $\overbrace{(\theta_1, \theta_2, \theta_3)}^{\text{unknowns}}$ as a function of the wrist position and orientation (x, y, ϕ)

- ***Solution:***
- The goal in terms of position and orientation of the wrist expressed in terms of the homogeneous transformation is defined as follows

$$
{}^{B}_{W}T_{Goal} = \overbrace{\begin{bmatrix} c_\phi & -s_\phi & 0 & x \\ s_\phi & c_\phi & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}^{\text{In terms of knowns}} \qquad {}^{B}_{W}T = {}^{0}_{3}T = \overbrace{\begin{bmatrix} c_{123} & -s_{123} & 0 & L_1c_1 + L_2c_{12} \\ s_{123} & c_{123} & 0 & L_1s_1 + L_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}^{\text{unknowns}}
$$

Inverse Kinematics - Planar RRR (3R) - Algebraic Solution

$${}^{B}_{W}T_{Goal} = {}^{0}_{3}T$$

- A set of four nonlinear equations which must be solved for $\theta_1, \theta_2, \theta_3$

$$c_\phi = c_{123}$$

$$s_\phi = s_{123}$$

$$x = l_1 c_1 + l_2 c_{12}$$

$$y = l_1 s_1 + l_2 s_{12}$$

This is a pretty common technique

- Solving for θ_2
- If we square x and y add them while making use of $c_{12} = c_1 c_2 - s_1 s_2$; $s_{12} = c_1 s_2 + s_1 c_2$ we obtain

$$x^2 + y^2 = l_1^2 + l_2^2 + 2 l_1 l_2 c_2$$

Inverse Kinematics - Planar RRR (3R) - Algebraic Solution

$$x^2 + y^2 = l_1^2 c_1^2 + l_2^2 c_{12}^2 + 2 l_1 l_2 c_1 c_{12}$$
$$+ l_1^2 s_1^2 + l_2^2 s_{12}^2 + 2 l_1 l_2 s_1 s_{12}$$

$$= l_1^2 + l_2^2 + 2 l_1 l_2 \left[c_1 c_{12} + s_1 s_{12} \right]$$

$$= l_1^2 + l_2^2 + 2 l_1 l_2 \left[c_1 (c_1 c_2 - s_1 s_2) + s_1 (c_1 s_2 + s_1 c_2) \right]$$

$$= l_1^2 + l_2^2 + 2 l_1 l_2 \left[\boxed{c_1^2 c_2 + s_1^2 c_2} \; \cancel{- c_1 s_1 s_2 + c_1 s_1 s_2} \right] \to 0$$

$$= l_1^2 + l_2^2 + 2 l_1 l_2 c_2$$

Inverse Kinematics - Planar RRR (3R) - Algebraic Solution

$x^2+y^2=l_1^2+l_2^2+2l_1l_2c_2$

- Solving for c_2 we obtain

$$c_2 = \frac{x^2+y^2-l_1^2-l_2^2}{2l_1l_2}$$

- Note: In order for a solution to exist, the right hand side must have a value between -1 and 1. Physically if this constraints is not satisfied, then the goal point is too far away for the manipulator to reach.

- Assuming the goal is in the workspace, and making use of $c_2^2 + s_2^2 = 1$ we write an expression for s_2 as

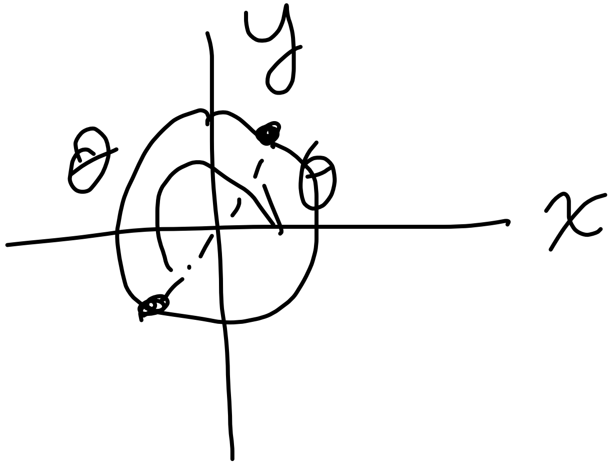
$$s_2 = \pm\sqrt{1-c_2{}^2}$$

- Note: The choice of the sign corresponds to the multiple solutions in which we can choose the "elbow-up" or the "elbow-down" solution

Inverse Kinematics - Planar RRR (3R) - Algebraic Solution

- Finally, we compute θ_2 using the two argument arctangent function

$$\theta_2 = \text{Atan2}(s_2, c_2) = A\tan 2\Big(\overset{y}{\pm\sqrt{1-c_2^2}}, \overset{x}{\frac{x^2+y^2-l_1^2-l_2^2}{2l_1l_2}}\Big)$$



$\operatorname{atan}\left(\frac{y}{x}\right) = \theta$

↳ can give two answers

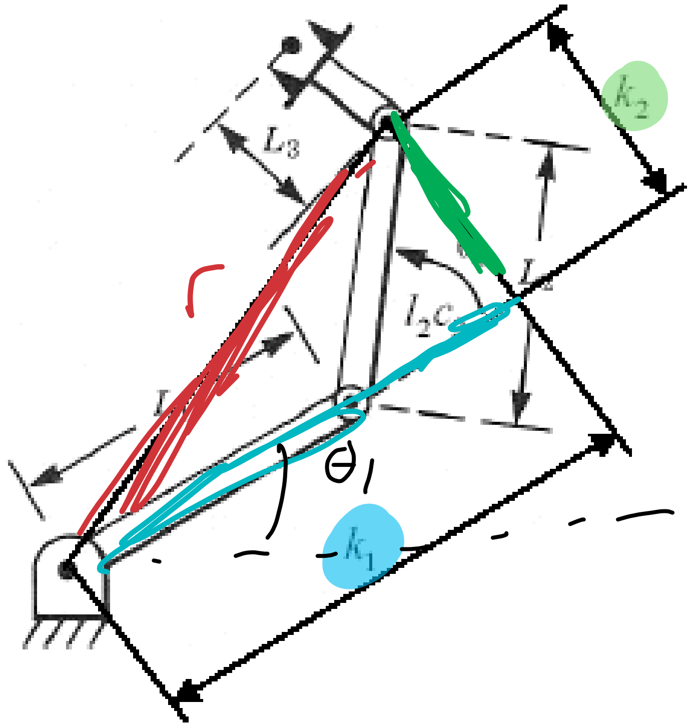
Inverse Kinematics - Planar RRR (3R) - Algebraic Solution

- Solving for θ_1
- For solving θ_1 we rewrite the the original nonlinear equations using a **change of variables** as follows

From earlier

$$x = l_1 c_1 + l_2 c_{12}$$
$$y = l_1 s_1 + l_2 s_{12}$$

$$x = k_1 c_1 + k_2 s_1$$
$$y = k_1 s_1 + k_2 c_1$$



- where

$$k_1 = l_1 + l_2 c_2$$
$$k_2 = l_2 s_2$$

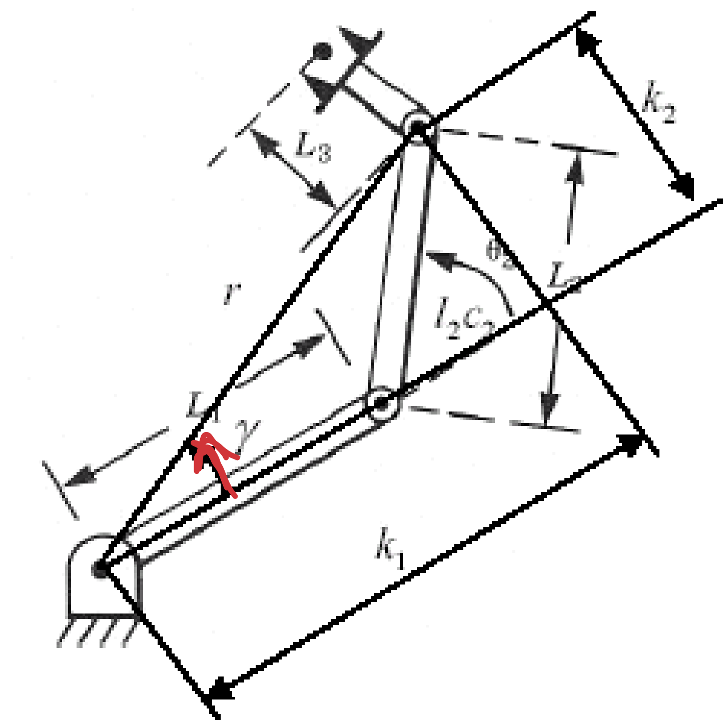
Inverse Kinematics - Planar RRR (3R) - Algebraic Solution

- Changing the way in which we write the constants k_1 and k_2

$$r = +\sqrt{k_1^2 + k_2^2}$$
$$\gamma = A\tan 2(k_2, k_1)$$

- Then

$$k_1 = r\cos\gamma$$
$$k_2 = r\sin\gamma$$



Inverse Kinematics - Planar RRR (3R) - Algebraic Solution

- Based on the previous two transformations, the equations can be rewritten as:

From Earlier $x = k_1 c_1 + k_2 s_1$ $y = k_1 s_1 + k_2 c_1$

$k_1 = r\cos\gamma$

$k_2 = r\sin\gamma$

$$x = r\cos\gamma\cos\theta_1 - r\sin\gamma\sin\theta_1$$

$$y = r\cos\gamma\sin\theta_1 + r\sin\gamma\cos\theta_1$$

divide by r

- or

$$\frac{x}{r} = \cos\gamma\cos\theta_1 - \sin\gamma\sin\theta_1$$

$$\frac{y}{r} = \cos\gamma\sin\theta_1 + \sin\gamma\cos\theta_1$$

angle sums

- or

$$\frac{x}{r} = \cos(\gamma + \theta_1)$$

$$\frac{y}{r} = \sin(\gamma + \theta_1)$$

Inverse Kinematics - Planar RRR (3R) - Algebraic Solution

- Using the two argument arctangent we finally get a solution for θ_1

$$\gamma + \theta_1 = A\tan 2\left(\frac{y}{r}, \frac{x}{r}\right) = A\tan 2(y, x)$$

$$\theta_1 = A\tan 2(y, x) - \overbrace{A\tan 2(k_2, k_1)} \rightarrow \text{definition of } \gamma$$

$$k_1 = l_1 + l_2 c_2$$
$$k_2 = l_2 s_2$$

both of these are known (since we solved for θ_2 first)

- Note:

(1) When a choice of a sign is made in the solution of θ_2 above, it will cause a sign change in k_2 thus affecting θ_1

(2) If $x = y = 0$ then the solution becomes undefined - in this case θ_1 is arbitrary

Inverse Kinematics - Planar RRR (3R) - Algebraic Solution

$\theta_1 + \theta_2 + \theta_3 = \phi$

- Solving for θ_3

- Based on the original equations,

$$c_\phi = c_{123}$$
$$s_\phi = s_{123}$$

- We can solve for the sum of $\theta_1, \theta_2, \theta_3$

$$\theta_1 + \theta_2 + \theta_3 = A\tan 2(s_\phi, c_\phi) = \phi$$

$$\theta_3 = \phi - \theta_1 + \theta_2$$

- Note: It is typical with manipulators that have two or more links moving in a plane that in the course of a solution, expressions for sum of joint angles arise

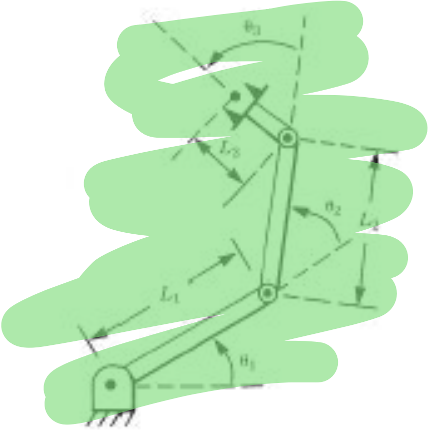
Central Topic – Inverse Manipulator Kinematics - Examples

- ***Geometric Solution – Concept***
  - Decompose spatial geometry into several plane geometry problems
  - ***Examples*** - *Planar RRR (3R) manipulators* - Geometric Solution
- ***Algebraic Solution - Concept***

$$ {}^{0}_{N}T = {}^{0}_{1}T \ldots {}^{N-1}_{N}T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Direct Kinematics Goal (Numeric values)

  - ***Examples*** - *PUMA 560 - Algebraic Solution*

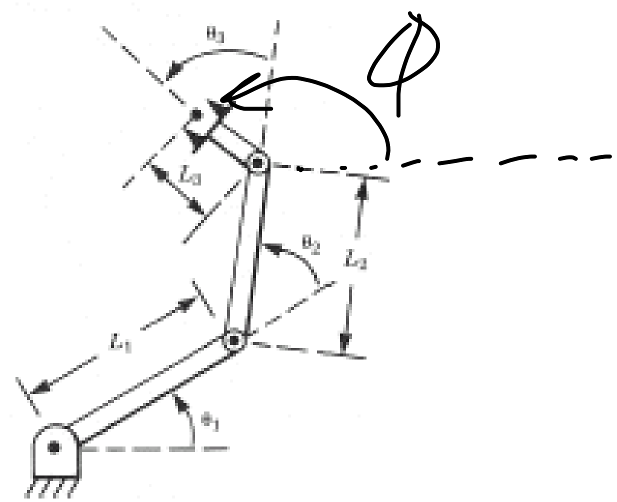


Inverse Kinematics - Planar RRR (3R) - Geometric Solution

- ***Given:***
  - ***Manipulator Geometry***
  - ***Goal Point Definition***: The position x, y and orientation ϕ of the wrist in space

- ***Problem:***

What are the joint angles ($\theta_1, \theta_2, \theta_3$) as a function of the goal (wrist position and orientation)



Inverse Kinematics - Planar RRR (3R) - Geometric Solution

Make a temporary variable: α

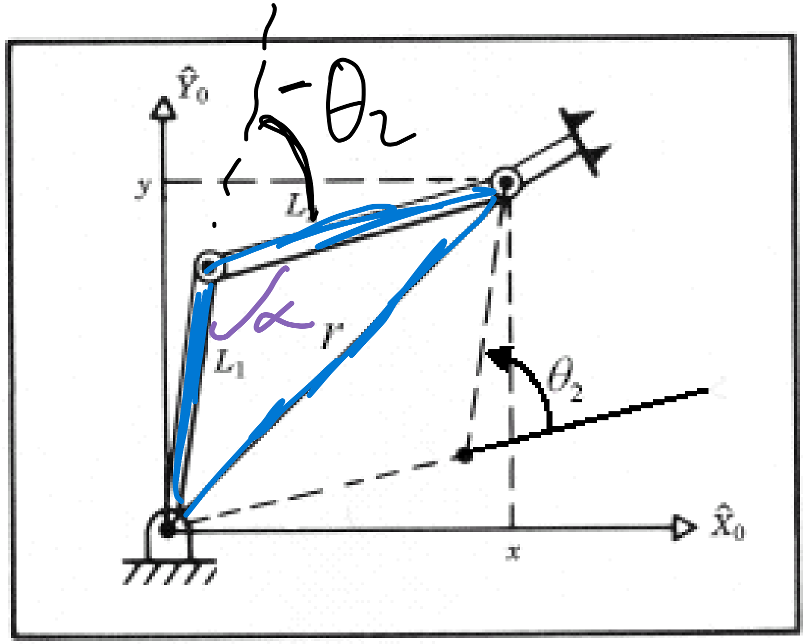
$180 - \theta_2 = \alpha$

- **Solution:**

Make a triangle

- We can apply the law of cosines to solve for θ_2

$$r^2 = x^2 + y^2 = l_1^2 + l_2^2 - 2 l_1 l_2 \cos\overbrace{(180+\theta_2)}^{\alpha}$$



- Since

$$\cos(180+\theta_2) = -\cos\theta_2$$

- We have

$$c_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}$$

$\Rightarrow$ for algebraic square and add gave us this

Inverse Kinematics - Planar RRR (3R) - Geometric Solution

← Same as before

- Note : Condition - Should be checked by the computational algorithm to verify existence of solutions.

$$l_1 + l_2 \geq \sqrt{x^2 + y^2}$$

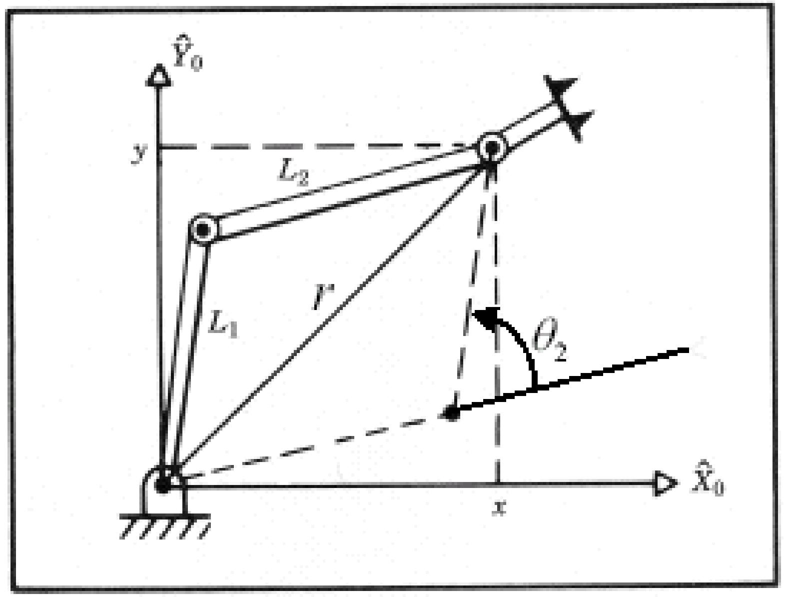
- Assuming that the solution exist it lies in the range of

$$-180^o \leq \theta_2 \leq 0^o$$

$$0 \leq \theta_2 \leq 180^\circ$$

- The other possible solution may found by symmetry to be

$$\theta_2^{'} = -\theta_2$$



To get θ_1 we're going to draw another triangle

Inverse Kinematics - Planar RRR (3R) - Geometric Solution

$l_2^2 = l_1^2 + (x^2+y^2) - 2 l_1 \sqrt{x^2+y^2} \cos\psi$

- By definition

$$\theta_2 < 0$$

$$\theta_1 = \beta \pm \psi$$

$$\theta_2 > 0$$

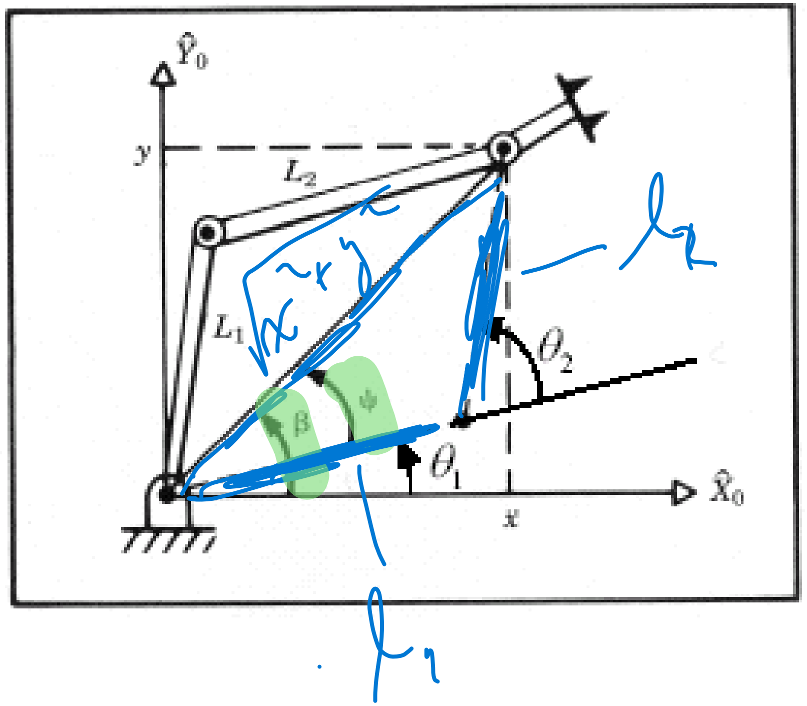
- Defining β as a function of x, y

$$\beta = A\tan 2(y, x)$$

- Applying the law of cosine to find

$$\cos\psi = \frac{x^2 + y^2 + l_1^2 - l_2^2}{2 l_1 \sqrt{x^2 + y^2}}$$

- Note: $0^0 \le \psi \le 180^o$



$\psi = \text{atan}2(\pm\sqrt{1-\cos\psi}, \cos\psi) \Rightarrow \theta_1 = \beta \pm \psi$

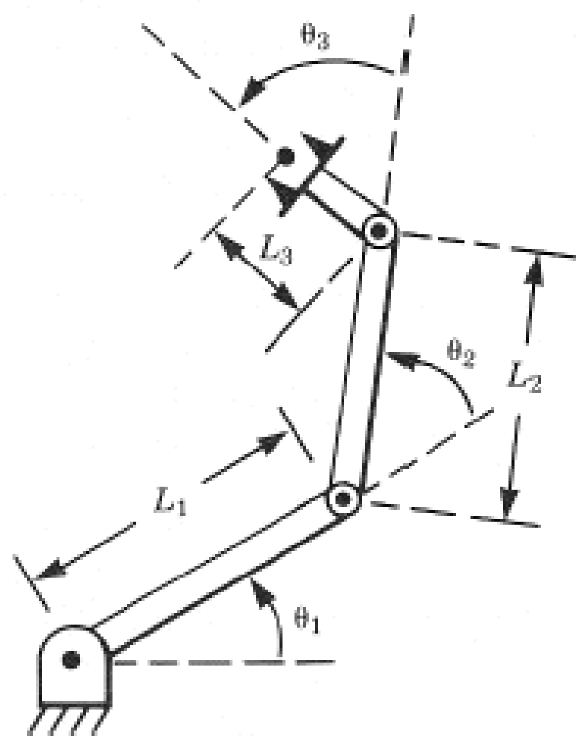
Inverse Kinematics - Planar RRR (3R) - Geometric Solution

Same as Before

- Angle in the plane add up to define the orientation of the last link

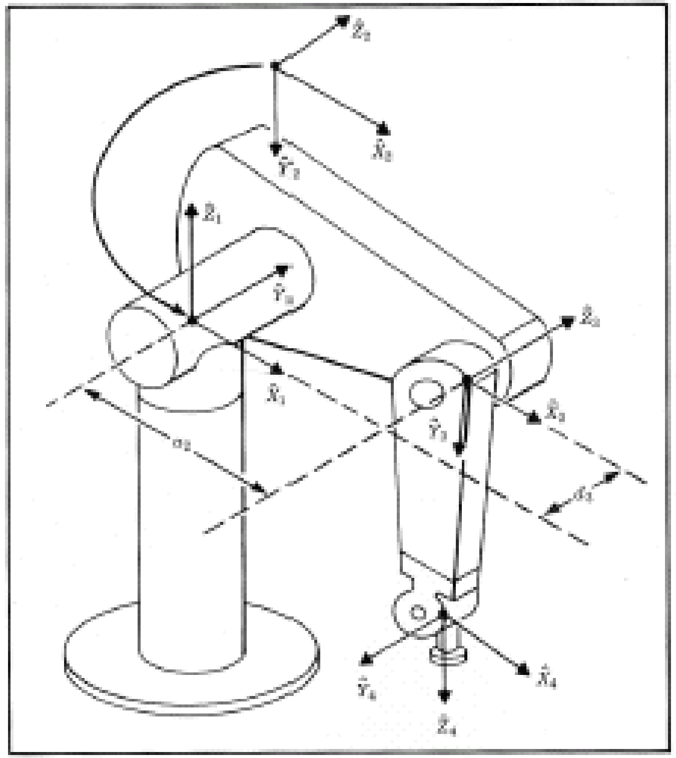
$$\phi = \theta_1 + \theta_2 + \theta_3$$

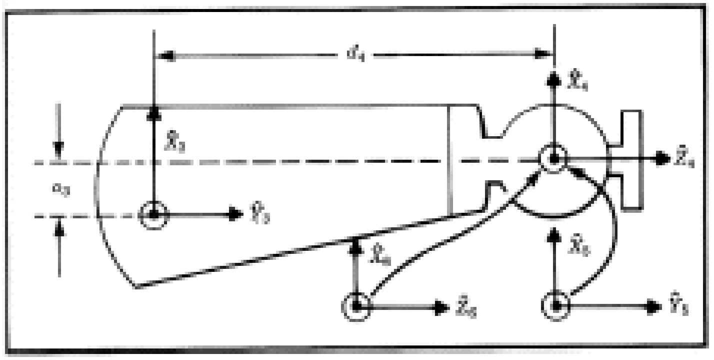
$$\theta_3 = \phi - \theta_1 + \theta_2$$



Inverse Kinematics - PUMA 560 - Algebraic Solution

- **Given:**
  - ***Direct Kinematics:*** *The homogenous transformation from the base to the wrist* ${}^{B}_{W}T$ → Known → Chapter 3 in book
  - ***Goal Point Definition:*** The position and orientation of the wrist in space





Inverse Kinematics - PUMA 560 - Algebraic Solution

- ***Problem:***

  What are the joint angles ($\theta_1 \cdots \theta_6$) as a function of the wrist position and orientation (or when ${}^0_6T$ is given as numeric values)

$$
{}^0_6T = \underbrace{\overbrace{{}^0_1T(\theta_1)\ {}^1_2T(\theta_2)\ {}^2_3T(\theta_3)\ {}^3_4T(\theta_4)\ {}^4_5T(\theta_5)\ {}^5_6T(\theta_6)}^{\text{Functions of Unknowns}}}_{\text{Direct Kinematics}} = \underbrace{\overbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}}^{\text{Known}}}_{\text{Goal}}
$$

Inverse Kinematics - PUMA 560 - Algebraic Solution

- ***Solution (General Technique):*** Multiplying each side of the direct kinematics equation by a an inverse transformation matrix for separating out variables in search of solvable equation

- Put the dependence on θ_1 on the left hand side of the equation by multiplying the direct kinematics eq. with $[{}^0_1T(\theta_1)]^{-1}$ gives

First step

green is what will solve for on each step

I Identity

$$[{}^0_1T(\theta_1)]^{-1}\ {}^0_6T = \boxed{[{}^0_1T(\theta_1)]^{-1}\ {}^0_1T(\theta_1)}\ {}^1_2T(\theta_2)\ {}^2_3T(\theta_3)\ {}^3_4T(\theta_4)\ {}^4_5T(\theta_5)\ {}^5_6T(\theta_6)$$

$$[{}^0_3T(\theta_1,\theta_2,\theta_3)]^{-1}\ {}^0_6T = \boxed{[{}^0_3T(\theta_1)]^{-1}\ {}^0_1T(\theta_1)\ {}^1_2T(\theta_2)\ {}^2_3T(\theta_3)}\ {}^3_4T(\theta_4)\ {}^4_5T(\theta_5)\ {}^5_6T(\theta_6)$$

$$[{}^0_4T(\theta_1,\theta_2,\theta_3,\theta_4)]^{-1}\ {}^0_6T = \boxed{[{}^0_4T(\theta_1,\theta_2,\theta_3,\theta_4)]^{-1}\ {}^0_1T(\theta_1)\ {}^1_2T(\theta_2)\ {}^2_3T(\theta_3)\ {}^3_4T(\theta_4)}\ {}^4_5T(\theta_5)\ {}^5_6T(\theta_6)$$

$$[{}^0_5T(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)]^{-1}\ {}^0_6T = \boxed{[{}^0_5T(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)]^{-1}\ {}^0_1T(\theta_1)\ {}^1_2T(\theta_2)\ {}^2_3T(\theta_3)\ {}^3_4T(\theta_4)\ {}^4_5T(\theta_5)}\ {}^5_6T(\theta_6)$$

Inverse Kinematics - PUMA 560 - Algebraic Solution

Division for scalar

- Put the dependence on θ_1 on the left hand side of the equation by multiplying the direct kinematics eq. with $[{}^0_1T(\theta_1)]^{-1}$ gives

Known

$$[{}^0_1T(\theta_1)]^{-1}\ {}^0_6T = \underbrace{[{}^0_1T(\theta_1)]^{-1}\ {}^0_1T(\theta_1)}_{I}\ {}^1_2T(\theta_2)\ {}^2_3T(\theta_3)\ {}^3_4T(\theta_4)\ {}^4_5T(\theta_5)\ {}^5_6T(\theta_6)$$

$${}^0_1T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[{}^A_BT]^{-1} = {}^B_AT = \begin{bmatrix} {}^A_BR^T & -{}^A_BR^T\,{}^AP_{BORG} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$

$${}^1_0T = [{}^0_1T]^{-1} = \begin{bmatrix} c\theta_1 & s\theta_1 & 0 & 0 \\ -s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse Kinematics - PUMA 560 - Algebraic Solution

${}^1_0T = \left[{}^0_1T\right]^{-1}$

${}^0_6T \rightarrow$ In terms of $x, y, z,$ orientation

$$\underbrace{\begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{{}^1_0T = \left[{}^0_1T\right]^{-1}} \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{{}^0_6T} = {}^1_6T$$

${}^1_6T \rightarrow$ In terms of $\theta_2 \rightarrow \theta_6$

Inverse Kinematics - PUMA 560 - Algebraic Solution

$$
{}^{1}_{2}T = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s\theta_2 & -c\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{2}_{3}T = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_2 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{3}_{4}T = \begin{bmatrix} c\theta_4 & -s\theta_4 & 0 & a_3 \\ 0 & 0 & 1 & d_4 \\ -s\theta_4 & -c\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{4}_{5}T = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s\theta_5 & c\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{5}_{6}T = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s\theta_5 & -c\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

RHS

$$
{}^{1}_{6}T = \begin{bmatrix} {}^{1}r_{11} & {}^{1}r_{12} & {}^{1}r_{13} & {}^{1}p_x \\ {}^{1}r_{21} & {}^{1}r_{22} & {}^{1}r_{23} & {}^{1}p_y \\ {}^{1}r_{31} & {}^{1}r_{32} & {}^{1}r_{33} & {}^{1}p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Contain 4, 5, 6

$$
\begin{aligned}
{}^{1}r_{11} &= c_{23}\left[c_4c_5c_6 - s_4s_6\right] - s_{23}s_5s_6, \\
{}^{1}r_{21} &= -s_4c_5c_6 - c_4s_6, \\
{}^{1}r_{31} &= -s_{23}\left[c_4c_5c_6 - s_4s_6\right] - c_{23}s_5c_6, \\
\\
{}^{1}r_{12} &= -c_{23}\left[c_4c_5s_6 + s_4c_6\right] + s_{23}s_5s_6, \\
{}^{1}r_{22} &= s_4c_5s_6 - c_4c_6, \\
{}^{1}r_{32} &= s_{23}\left[c_4c_5s_6 + s_4c_6\right] + c_{23}s_5s_6, \\
\\
{}^{1}r_{13} &= -c_{23}c_4s_5 - s_{23}c_5, \\
{}^{1}r_{23} &= s_4s_5, \\
{}^{1}r_{33} &= s_{23}c_4s_5 - c_{23}c_5,
\end{aligned}
$$

$$
\begin{aligned}
{}^{1}p_x &= a_2c_2 + a_3c_{23} - d_4s_{23}, \\
{}^{1}p_y &= d_3, \\
{}^{1}p_z &= -a_3s_{23} - a_2s_2 - d_4c_{23}.
\end{aligned}
$$

← no angles

Focus on these terms

Inverse Kinematics - PUMA 560 - Algebraic Solution

$$\begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^1_6T$$

Previous slide

$\Rightarrow$

$^1p_y = d_3$

- Equating the (2,4) elements from both sides of the equation we have

$$-s_1 p_x + c_1 p_y = d_3$$

- To solve the equation of this form we make the trigonometric substitution

$$p_x = \rho \cos\phi$$
$$p_y = \rho \sin\phi$$

ρ is our radius

Inverse Kinematics - PUMA 560 - Algebraic Solution

$-s_1 p_x + c_1 p_y = d_3$

$-s_1 \rho \cos\phi + c_1 \rho \sin\phi = d_3$

$c_1 s\phi - s_1 c\phi = \frac{d_3}{\rho}$

$$\rho = \sqrt{p_x^2 + p_y^2}$$

$$\phi = \cancel{A\tan 2(p_x, p_y)} \; A\tan 2(p_y, p_x)$$

- Substituting p_x, p_y with ρ, ϕ we obtain

$$c_1 s_\phi - s_1 c_\phi = \frac{d_3}{\rho}$$

- Using the difference of angles formula

Trig Identity

$$\sin(\phi - \theta_1) = \frac{d_3}{\rho}$$

Inverse Kinematics - PUMA 560 - Algebraic Solution

- Based on $$\sin^2(\phi-\theta_1)+\cos^2(\phi-\theta_1)=1$$
- and so $$\cos(\phi-\theta_1)=\pm\sqrt{1-\frac{d_3^2}{\rho^2}}$$

$$\phi-\theta_1 = A\tan 2\left(\frac{d_3}{\rho},\pm\sqrt{1-\frac{d_3^2}{\rho^2}}\right)$$

- The solution for θ_1 may be written

$$\theta_1 = \overbrace{A\tan 2(p_y, p_x)}^{\phi} - A\tan 2\left(\frac{d_3}{\rho},\pm\sqrt{1-\frac{d_3^2}{\rho^2}}\right)$$

- Note: we have found two possible solutions for θ_1 corresponding to the +/- sign

Inverse Kinematics - PUMA 560 - Algebraic Solution

- Equating the (1,4) element and (3,4) element

$$\begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^1_6\underline{T}$$

- We obtain

$$c_1 p_x + s_1 p_y = \overbrace{a_3 c_{23} - d_4 s_{23} + a_2 c_2}^{{}^1p_x \text{ from } {}^1_6T}$$

$$-p_z = \underbrace{a_3 c_{23} + d_4 s_{23} + a_2 c_2}_{{}^1p_z \text{ from } {}^1_6T}$$

Inverse Kinematics - PUMA 560 - Algebraic Solution

- If we square the following equations and add the resulting equations

$$-s_1 p_x + c_1 p_y = d_3$$

$$c_1 p_x + s_1 p_y = a_3 c_{23} - d_4 s_{23} + a_2 c_2$$

$$-p_z = a_3 s_{23} + d_4 c_{23} + a_2 s_2$$

LHS

$$s_1^2 p_x^2 + c_1^2 p_y^2 + c_1^2 p_x^2 + s_1^2 p_y^2 + p_z^2 + 2 p_x p_y c_1 s_1 - 2 p_x p_y c_1 s_1$$

$$= p_x^2 + p_y^2 + p_z^2$$

RHS

$$d_3^2 + a_3^2 + d_4^2 + a_2^2 \cancel{- 2 a_3 d_4 c_{23} s_{23}} \cancel{+ 2 a_3 d_4 c_{23} s_{23}} + 2 a_2 a_3 c_2 c_{23} + 2 a_2 a_3 s_2 s_{23} - 2 a_2 d_4 c_2 s_{23} + 2 a_2 d_4 s_2 c_{23}$$

$$p_x^2 + p_y^2 + p_z^2 - a_2^2 - a_3^2 - d_3^2 - d_4^2 = 2a_2 \overbrace{(a_3 c_2 c_{23} + a_3 s_2 s_{23} - d_4 c_2 c_{23} + d_4 s_2 s_{23})}^{\text{Angle sum term}}$$

Inverse Kinematics - PUMA 560 - Algebraic Solution

- we obtain

$$a_3 c_3 - d_4 s_3 = K$$

← Same form as θ_1

- where

$$K = \frac{p_x^2 + p_y^2 + p_z^2 - a_2^2 - a_3^2 - d_3^2 - a_4^2}{2a_2}$$

- Note that the dependence on θ_1 has be removed. Moreover the eq. for θ_3 is of the same form as the eq. for θ_1 and so may be solved by the same kind of trigonometric substitution to yield a solution for θ_3

$$\theta_3 = A\tan 2(a_3, d_4) - A\tan 2\left(K, \pm\sqrt{a_3^2 + d_4^2 - K^2}\right)$$

- Note that the +/- sign leads to two different solution for θ_3

Inverse Kinematics - PUMA 560 - Algebraic Solution

A lot simpler

$$[{}^0_3T(\theta_1,\theta_2,\theta_3)]^{-1}\ {}^0_6T = \boxed{[{}^0_3T(\theta_1)]^{-1}\ {}^0_1T(\theta_1)\ {}^1_2T(\theta_2)\ {}^2_3T(\theta_3)}\ {}^3_4T(\theta_4)\ {}^4_5T(\theta_5)\ {}^5_6T(\theta_6)$$

$${}^3_6T$$

$$\begin{bmatrix} c_1c_{23} & s_1c_{23} & -s_{23} & -a_2c_3 \\ -c_1s_{23} & -s_1s_{23} & -c_{23} & a_2s_3 \\ -s_1 & c_1 & 0 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & -c_4s_5 & a_3 \\ s_5c_6 & -s_5s_6 & -c_5 & d_4 \\ -s_4c_5c_6 - c_4s_6 & s_4c_5s_6 - c_4c_6 & s_4s_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Equating the (1,4) element and (2,4) element we obtain

$$c_1c_{23}p_x + s_1c_{23}p_y - s_{23}p_z - a_2c_3 = a_3$$

$$-c_1s_{23}p_x - s_1s_{23}p_y - c_{23}p_z + a_2s_3 = d_4$$

$\Rightarrow$ Linear with respect to c_{23} and s_{23}

- These equations may be solved simultaneously for s_{23} and c_{23} resulting in

Inverse Kinematics - PUMA 560 - Algebraic Solution

$$s_{23} = \frac{(-a_3 - a_2 c_3)p_z + (c_1 p_x + s_1 p_y)(a_2 s_3 - d_4)}{p_z^2 + (c_1 p_x + s_1 p_y)^2}$$

$$c_{23} = \frac{(a_2 s_3 - d_4)p_z - (-a_3 - a_2 c_3)(c_1 p_x + s_1 p_y)}{p_z^2 + (c_1 p_x + s_1 p_y)^2}$$

- Since the denominator are equal and positive, we solve for the sum of θ_2 and θ_3 as

We already know θ_3

$$\theta_{23} = A\tan 2[(-a_3 - a_2 c_2)p_z + (c_1 p_x + s_1 p_y)(a_2 s_3 - d_4),$$
$$(a_2 s_3 - d_4)p_z - (-a_3 - a_2 c_3)(c_1 p_x + s_1 p_y)]$$

- The equation computes four values of θ_{23} according to the four possible combination of solutions for θ_1 and θ_3

Now we know $\theta_1, \theta_2, \theta_3$

Inverse Kinematics - PUMA 560 - Algebraic Solution

- Then, four possible solutions for θ_2 are computed as

$$\theta_2 = \theta_{23} - \theta_3$$

- Equating the (1,3) and the (3,3) elements

$^3T_6$

$$\begin{bmatrix} c_1c_{23} & s_1c_{23} & -s_{23} & -a_2c_3 \\ -c_1s_{23} & -s_1s_{23} & -c_{23} & a_2s_3 \\ -s_1 & c_1 & 0 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & -c_4s_5 & a_3 \\ s_5c_6 & -s_5s_6 & -c_5 & d_4 \\ -s_4c_5c_6 - c_4s_6 & s_4c_5s_6 - c_4c_6 & s_4s_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- we get

$$r_{13}c_1c_{23} + r_{23}s_1c_{23} - s_{23}r_{33} = -c_4s_5$$

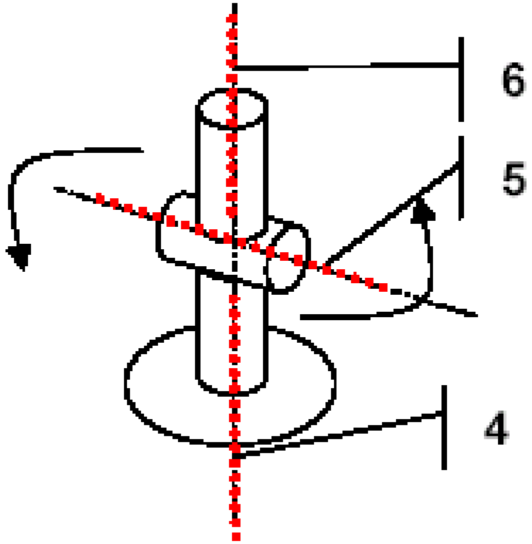
$$-r_{13}s_1 + r_{23}c_1 = s_4s_5$$

$$c_4 = -\frac{r_{13}c_1c_{23} + r_{23}s_1c_{23} - s_{23}r_{33}}{s_5}$$

$$s_4 = \frac{-r_{13}s_1 + r_{23}c_1}{s_5}$$

Inverse Kinematics - PUMA 560 - Algebraic Solution

- As long as $s_5 \neq 0$ we can solve for θ_4
  $$\theta_4 = A\tan 2(-r_{13}s_1 + r_{23}c_1, -r_{13}c_1c_{23} - r_{23}s_1c_{23} + s_{23}r_{33})$$
- When $\theta_5 = 0$ the manipulator is in a ***singular configuration*** in which joint axes 4 and 6 line up and cause the same motion of the last link of the robot. In this case all that can be solved for is the sum or difference of θ_4 and θ_6. This situation is detected by checking whether both arguments of Atan2 are near zero. If so θ_4 is chosen arbitrary (usually chosen to be equal to the present value of joint 4), and θ_6 is computed later, it will be computed accordingly



Inverse Kinematics - PUMA 560 - Algebraic Solution

$$[{}^0_4T(\theta_1,\theta_2,\theta_3,\theta_4)]^{-1}\ {}^0_6T = \boxed{[{}^0_4T(\theta_1,\theta_2,\theta_3,\theta_4)]^{-1}\ {}^0_1T(\theta_1)\ {}^1_2T(\theta_2)\ {}^2_3T(\theta_3)\ {}^3_4T(\theta_4)}\ {}^4_5T(\theta_5)\ {}^5_6T(\theta_6)$$

$$\begin{bmatrix} c_1c_{23}c_4+s_1s_4 & s_1c_{23}c_4-c_1s_4 & -s_{23}c_4 & -a_2c_3c_4+d_3s_4-a_3c_4 \\ -c_1c_{23}s_4+s_1c_4 & -s_1c_{23}s_4-c_1c_4 & s_{23}s_4 & a_2c_3s_4+d_3c_4-a_3s_4 \\ -c_1s_{23} & -s_1s_{23} & c_{23} & a_2s_3-d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} c_5c_6 & -c_5s_6 & -s_5 & 0 \\ s_6 & c_6 & 0 & 0 \\ s_5c_6 & -s_5s_6 & c_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{{}^4_6T}$$

- Equating the (1,3) and the (3,3) elements we get

$$r_{13}(c_1c_{23}c_4+s_1s_4)+r_{23}(s_1c_{23}c_4-c_1s_4)-r_{33}(s_{23}c_4)=s_5$$

$$r_{13}(-c_1s_{23})+r_{23}(-s_1s_{23})+r_{33}(-c_{23})=c_5$$

Inverse Kinematics - PUMA 560 - Algebraic Solution

- We can solve for θ_5

$$\theta_5 = A\tan 2(s_5, c_5)$$

$$[{}^0_5T(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)]^{-1}\ {}^0_6T = \boxed{[{}^0_5T(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)]^{-1}\ {}^0_1T(\theta_1)\ {}^1_2T(\theta_2)\ {}^2_3T(\theta_3)\ {}^3_4T(\theta_4)\ {}^4_5T(\theta_5)}\ {}^5_6T(\theta_6)$$

$$r_{11}(c_1c_{23}s_4 + s_1c_4) - r_{21}(s_1c_{23}s_4 + c_1c_4) + r_{31}(s_{23}s_4) = s_6$$

$$r_{11}[(c_1c_{23}c_4 + s_1s_4)c_5 - c_1s_{23}s_5] + r_{21}[(s_1c_{23}c_4 - c_1s_4)c_5 - s_1s_{23}s_5] - r_{31}(s_{23}c_4c_5 + c_{23}s_5) = c_6$$

$$\theta_6 = A\tan 2(s_6, c_6)$$

Inverse Kinematics - PUMA 560 - Algebraic Solution

- Summary - Number of Solutions

- Four solution

$$\theta_1 = A\tan 2(p_y, p_x) - A\tan 2\left(\frac{d_3}{\rho}, \pm\sqrt{1-\frac{d_3^2}{\rho^2}}\right)$$

$$\theta_3 = A\tan 2(a_3, d_4) - A\tan 2\left(K, \pm\sqrt{a_3^2 + d_4^2 - K^2}\right)$$

- For each of the four solutions the wrist can be flipped

$$\theta_4^{'} = \theta_4 + 180^o$$

$$\theta_5^{'} = -\theta_5$$

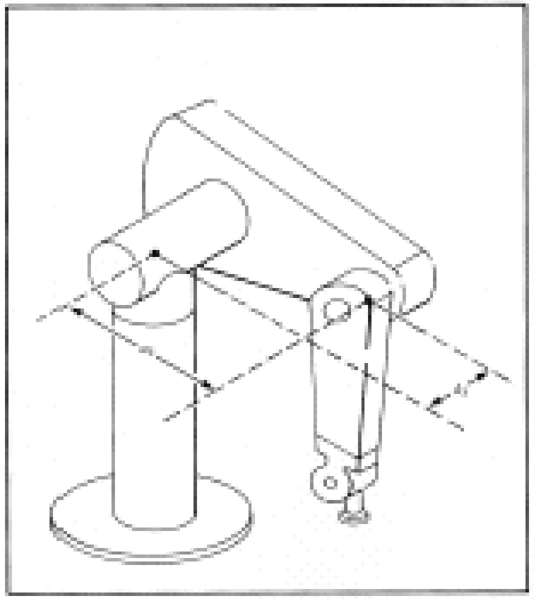
$$\theta_6^{'} = \theta_6 + 180^o$$

Inverse Kinematics - PUMA 560 - Algebraic Solution

- After all eight solutions have been computed, some or all of them may have to be discarded because of joint limit violations.
- Of the remaining valid solutions, usually the one closest to the present manipulator configuration is chosen.

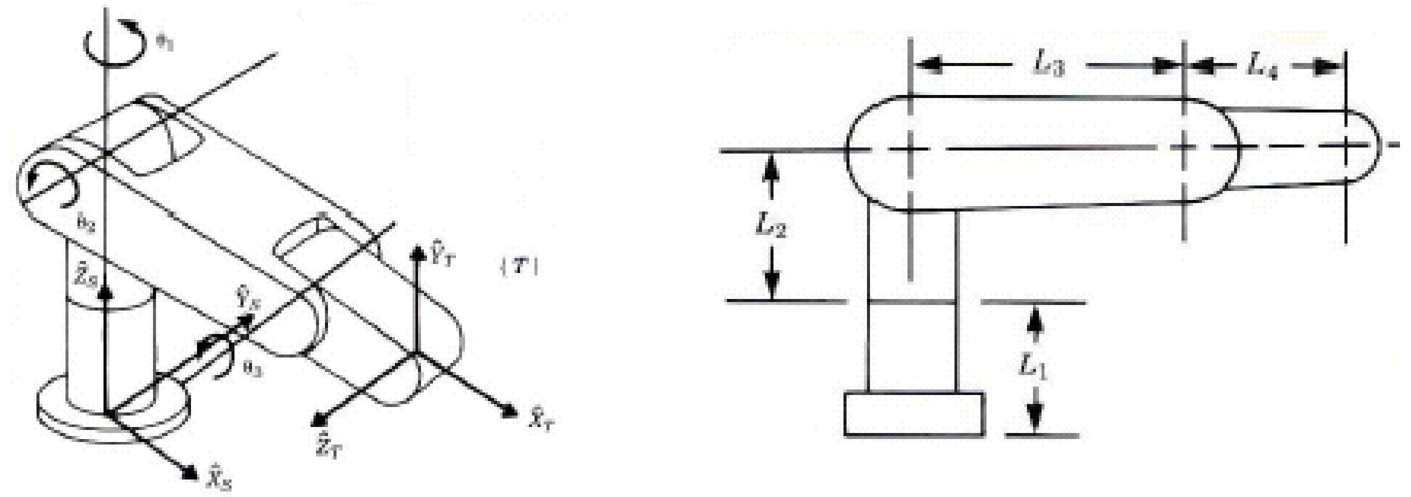
Central Topic - Inverse Manipulator Kinematics -Examples

- **Geometric Solution - Concept**
  - Decompose spatial geometry into several plane geometry
  - **Example** - 3D - RRR (3R) manipulators - Geometric Solution
- **Algebraic Solution** (closed form) –
  - Piepers Method - Last three consecutive axes intersect at one point
  - **Example** - Puma 560



Algebraic Solution by Reduction to Polynomial - Example

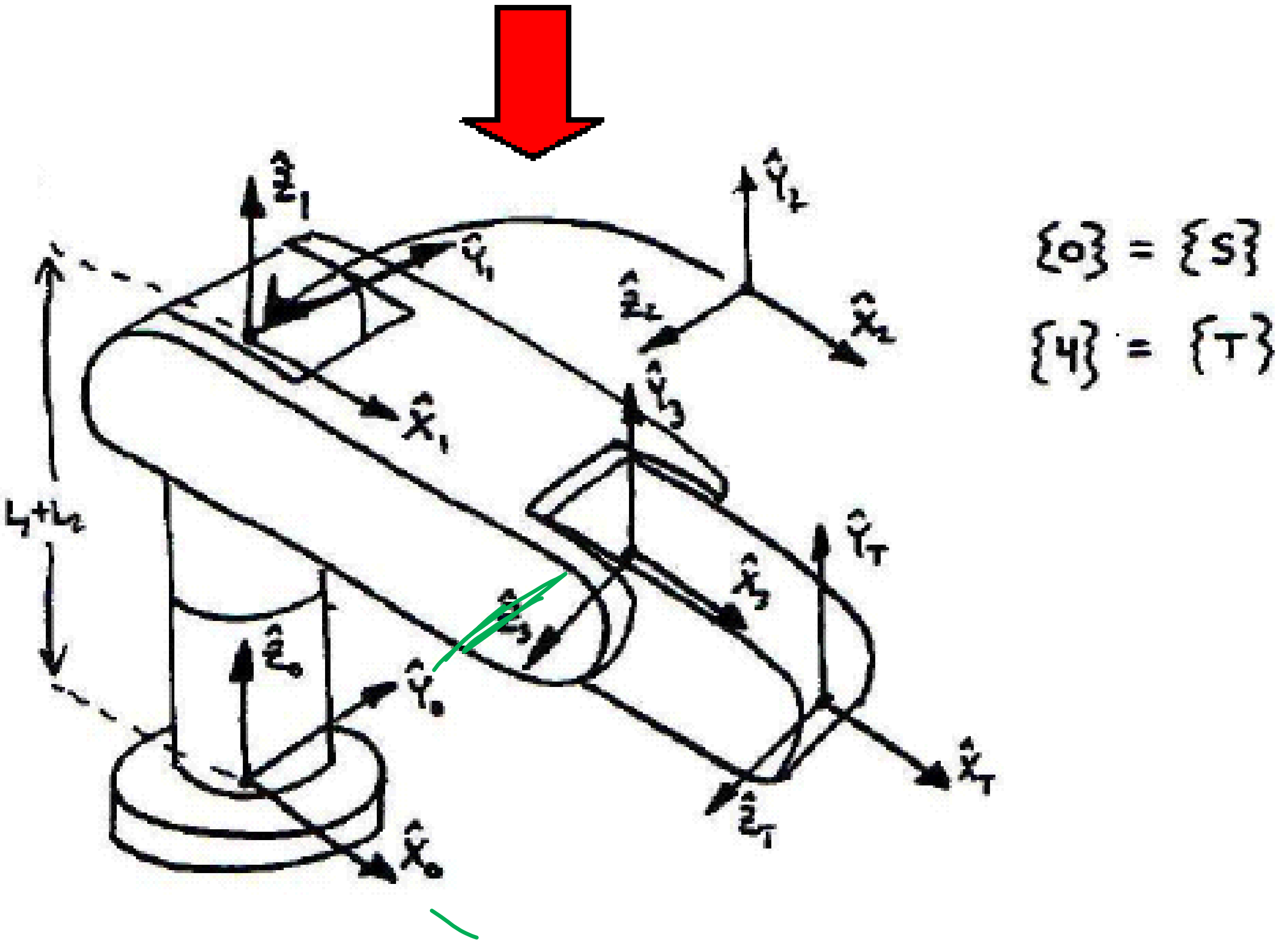
- *Given:*
  - ***Manipulator Geometry***
  - ***Goal Point Definition*: The position x_d, y_d, z_d of the wrist in space**



- *Problem:*

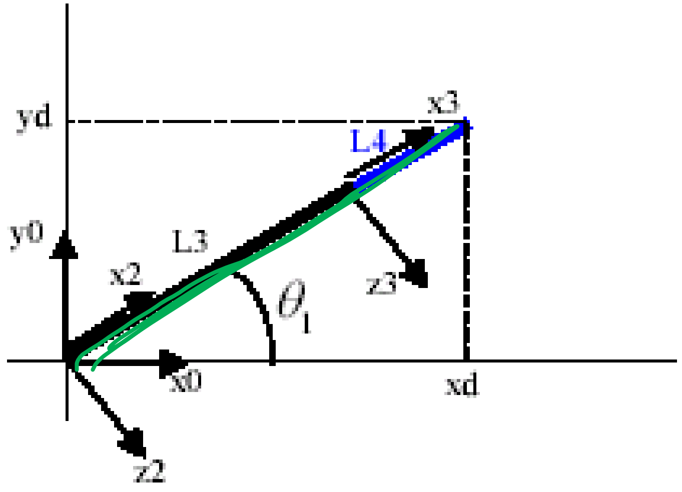
What are the joint angles ($\theta_1, \theta_2, \theta_3$) as a function of the goal (wrist position and orientation)

Algebraic Solution by Reduction to Polynomial - Example



Algebraic Solution by Reduction to Polynomial - Example

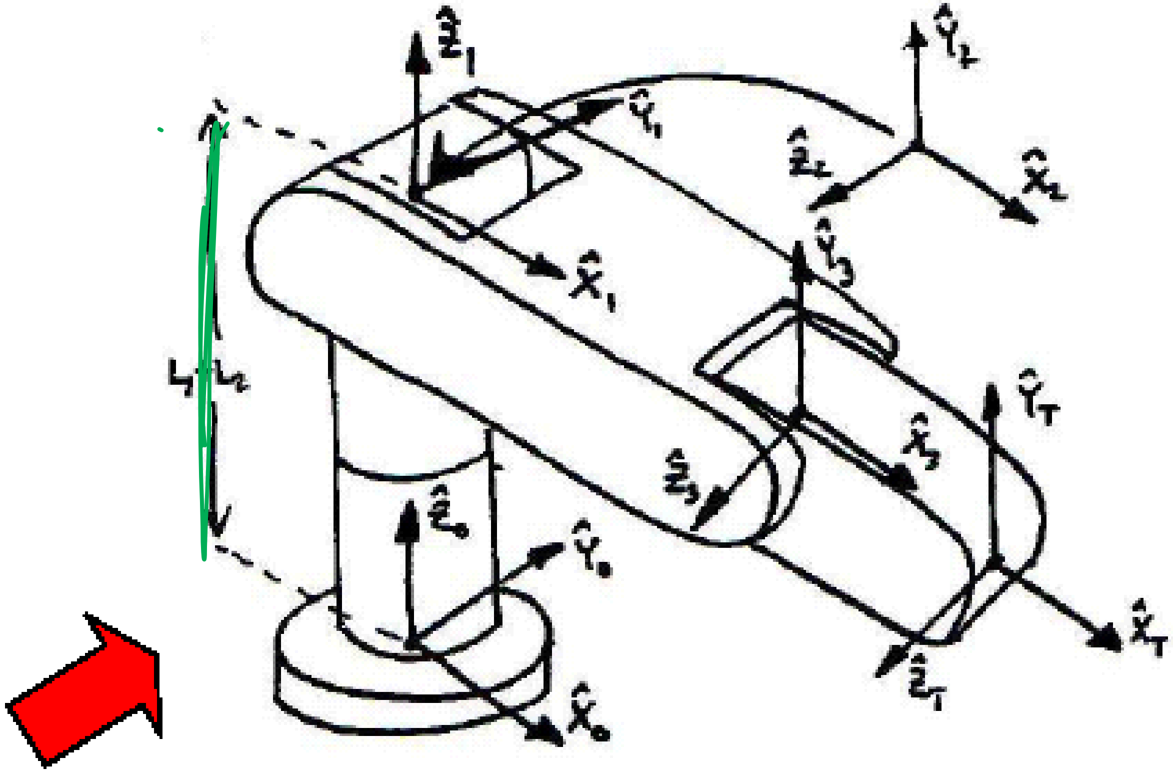
- The planar geometry - top view of the robot



$$\boxed{\theta_1 = A\tan 2(y_d, x_d)}$$

$$r_1 = \sqrt{x_d^2 + y_d^2}$$

Algebraic Solution by Reduction to Polynomial - Example



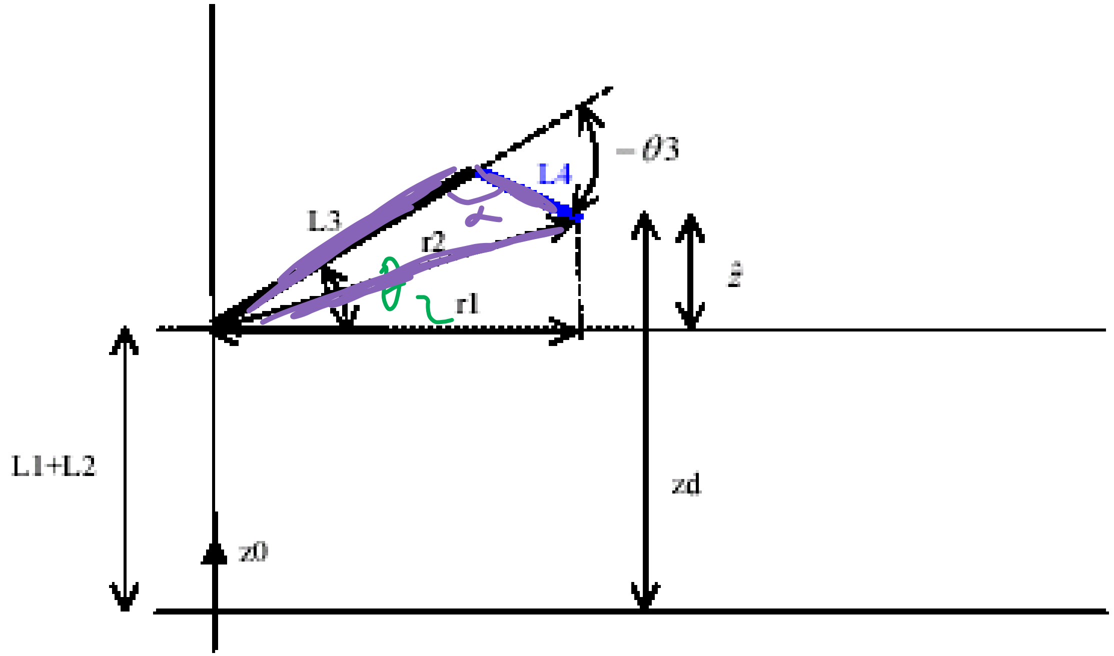
$\{0\} = \{S\}$

$\{4\} = \{T\}$

Algebraic Solution by Reduction to Polynomial - Example

- The planar geometry - side view of the robot:

$r_1^2 = x_d^2 + y_d^2$



$$r_2 = \sqrt{r_1^2 + \hat{z}^2} = \sqrt{x_d^2 + y_d^2 + \hat{z}^2} = \sqrt{x_d^2 + y_d^2 + \left(z_d - (L_1 + L_2)\right)^2}$$

- where

$$\hat{z} = z_d - (L_1 + L_2)$$

Algebraic Solution by Reduction to Polynomial - Example

- By applying the law of cosines, we get

$$r_2^2 = L_3^2 + L_4^2 - 2L_3L_4\cos(\overbrace{180+\theta_3}^{\alpha}) = L_3^2 + L_4^2 + 2L_3L_4\cos(\theta_3)$$

- Rearranging gives

$$c_3 = \frac{r_2^2 - (L_3^2 + L_4^2)}{2L_3L_4}$$

- and

$$s_3 = \sqrt{1-c_3^2}$$

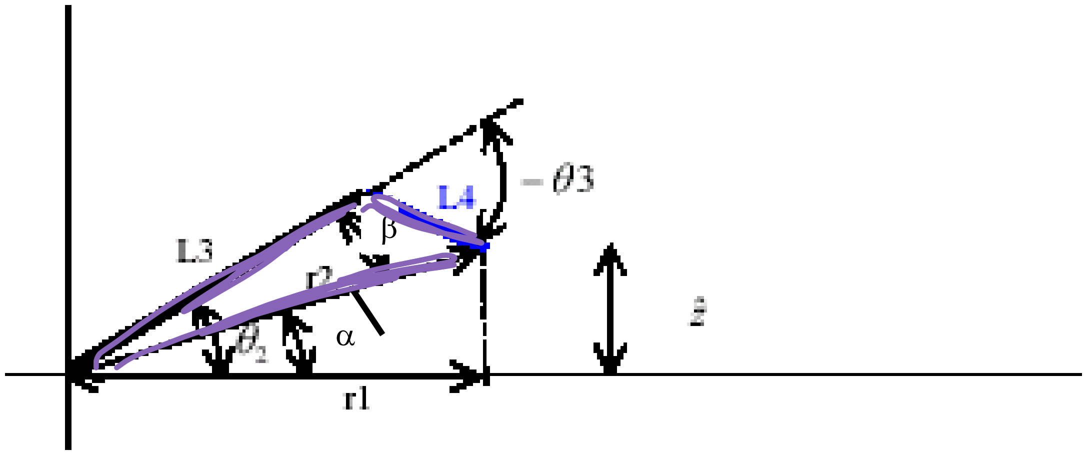
- Solving for θ_3 we get

$$\boxed{\theta_3 = \mathrm{Atan2}(\pm\sqrt{1-c_3^2}, c_3)}$$

we now know θ_1 and θ_3

- Where c_3 is defined above in terms of known parameters L_3, L_4, x_d, y_d, and z_d

Algebraic Solution by Reduction to Polynomial - Example



- Finally we need to solve for θ_2

$$\theta_2 = \alpha + \beta$$

$$\alpha = A\tan 2(\hat{z}, r_1)$$

- where $r_1 = \sqrt{x_d^2 + y_d^2} \qquad \hat{z} = z_d - (L_1 + L_2)$

Algebraic Solution by Reduction to Polynomial - Example

- Based on the law of cosines we can solve for β

$$L_4^2 = r_2^2 + L_3^2 - 2r_2L_3\cos(\beta)$$

$$c_\beta = \frac{r_2^2 + L_3^2 - L_4^2}{2r_2L_3}$$

$$\beta = A\tan 2(\pm\sqrt{1-c_\beta^2}, c_\beta)$$

$$\boxed{\theta_2 = \overbrace{A\tan 2(z_d - (L_1 + L_2), \sqrt{x_d^2 + y_d^2})}^{\alpha} + \overbrace{A\tan 2(\pm\sqrt{1-c_\beta^2}, c_\beta)}^{\beta}}$$

Algebraic Solution by Reduction to Polynomial - Example

- Summary

$$\theta_1 = A\tan 2(y_d, x_d)$$

$$\theta_2 = A\tan 2(z_d - (L_1 + L_2), \sqrt{x_d^2 + y_d^2}) +$$

$$A\tan 2(\pm\sqrt{1 - \left(\frac{x_d^2 + y_d^2 + (z_d - (L_1 + L_2))^2 + L_3^2 - L_4^2}{2\sqrt{x_d^2 + y_d^2 + (z_d - (L_1 + L_2)}L_3)^2}\right)^2}, \frac{x_d^2 + y_d^2 + (z_d - (L_1 + L_2))^2 + L_3^2 - L_4^2}{2\sqrt{x_d^2 + y_d^2 + (z_d - (L_1 + L_2)}L_3)^2})$$

$$\theta_3 = A\tan(\pm\sqrt{1 - \left[\frac{x_d^2 + y_d^2 + (z_d - (L_1 + L_2))^2 - (L_3^2 + L_4^2)}{2L_3L_4}\right]^2}, \frac{x_d^2 + y_d^2 + (z_d - (L_1 + L_2))^2 - (L_3^2 + L_4^2)}{2L_3L_4})$$

Algebraic Solution by Reduction to Polynomial

- Transcendental equations are difficult to solve because they are a function of $c\theta$, $s\theta$

$$f(c\theta, s\theta) = k$$

- Making the following substitutions yields an expression in terms of a single variable *u*
- Using this substitution, transcendental equations are converted into polynomial equations

$$u = \tan\frac{\theta}{2}$$

$$\cos\theta = \frac{1-u^2}{1+u^2}$$

$$\sin\theta = \frac{2u}{1+u^2}$$

Algebraic Solution by Reduction to Polynomial - Example

- Transcendental equation

$$ac\theta + bs\theta = c$$

- Substitute $c\theta, s\theta$ with the following equations

$$\cos\theta = \frac{1-u^2}{1+u^2}$$

$$\sin\theta = \frac{2u}{1+u^2}$$

- yields

$$a(1-u^2) + 2bu = c(1+u^2)$$

$$(a+c)u^2 - 2bu + (c-a) = 0$$

quadratic formula

Algebraic Solution by Reduction to Polynomial - Example

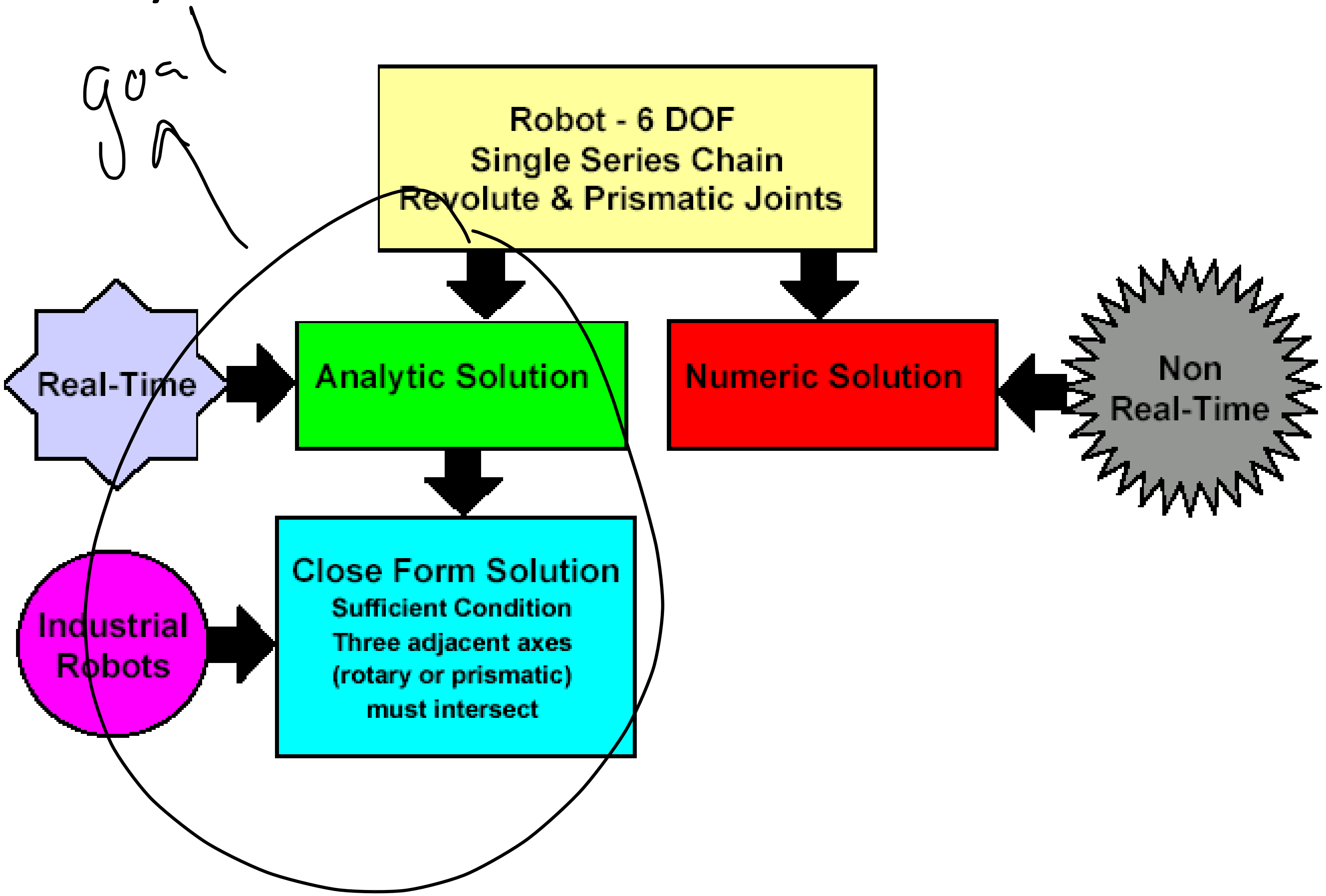
- Which is solved by the quadratic formula to be

$$u = \frac{b \pm \sqrt{b^2 + a^2 - c^2}}{a+c}$$

$$\theta = 2\tan^{-1}\left(\frac{b \pm \sqrt{b^2 + a^2 - c^2}}{a+c}\right)$$

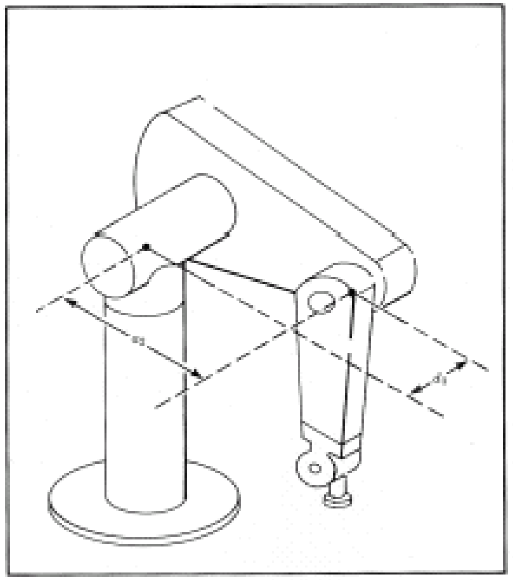
- Note
  - If u is complex there is no real solution to the original transcendental equation
  - If $a + c = 0$ then $\theta = 180^o$

Solvability



Pieper's Solution - Three consecutive Axes Intersect

- **Pieper's Solution**
  - Closed form solution for a serial 6 DOF in which **three consecutive axes intersect at a point** (including robots with three consecutive parallel axes, since they meet at a point at infinity)
- Pieper's method applies to the majority of commercially available industrial robots
  - Example: (Puma 560)
    - All 6 joints are revolute joints
    - The last 3 joints are intersecting



Pieper's Solution - Three consecutive Axes Intersect

- ***Given:***
  - ***Manipulator Geometry:*** 6 DOF & DH parameters
    - All 6 joints are revolute joints
    - The last 3 joints are intersecting
  - ***Goal Point Definition***: The position and orientation of the wrist in space

$$
{}^{0}_{6}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T\,{}^{3}_{4}T\,{}^{4}_{5}T\,{}^{5}_{6}T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

- ***Problem:***
  - What are the joint angles ($\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6$) as a function of the goal (wrist position and orientation)

where 3 axes intersect

Pieper's Solution - Three consecutive Axes Intersect

- When the last three axes of a 6 DOF robot intersect, the origins of link frame {4}, {5}, and {6} are all located at the point of intersection. This point is given in the base coordinate system as

$$ {}^{0}P_{4org} = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T\,{}^{3}P_{4org} $$

- From the general forward kinematics method for determining homogeneous transforms using DH parameters, we know:

$$
\begin{array}{cc}
 & \begin{array}{cc} \quad\quad\quad\quad\quad {}^{i-1}_{i}R \quad\quad\quad\quad\quad & {}^{i-1}P_{iorg} \end{array} \\
{}^{i-1}_{i}T = & \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{array}
$$

Pieper’s Solution - Three consecutive Axes Intersect

- For i=4

$$
\begin{array}{cc}
 & \begin{array}{cc} \qquad\qquad {}^3_4R \qquad\qquad & {}^3P_{4org} \end{array} \\
{}^3_4T = & \begin{bmatrix}
c\theta_4 & -s\theta_4 & 0 & a_3 \\
s\theta_4 c\alpha_3 & c\theta_4 c\alpha_3 & -s\alpha_3 & -s\alpha_3 d_4 \\
s\theta_4 s\alpha_3 & c\theta_4 s\alpha_3 & c\alpha_3 & c\alpha_3 d_4 \\
0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

- Using the fourth column and substituting for ${}^3P_{4org}$ we find

position of wrist

$$
{}^0P_{4org} = {}^0_1T\,{}^1_2T\,{}^2_3T\,{}^3P_{4org} = {}^0_1T\,{}^1_2T\,{}^2_3T \begin{bmatrix} a_3 \\ -s\alpha_3 d_4 \\ c\alpha_3 d_4 \\ 1 \end{bmatrix}
\qquad
{}^0P_{4org} = {}^0_1T\,{}^1_2T\,{}^2_3T\,{}^3P_{4org} = {}^0_1T\,{}^1_2T \begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix}
$$

Pieper's Solution - Three consecutive Axes Intersect

- where

$$\begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix} = {}^2_3T \begin{bmatrix} a_3 \\ -s\alpha_3 d_4 \\ c\alpha_3 d_4 \\ 1 \end{bmatrix}$$

entirely a function of θ_3

$$\begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix} = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_2 \\ s\theta_3 c\alpha_2 & c\theta_3 c\alpha_2 & -s\alpha_2 & -s\alpha_2 d_3 \\ s\theta_3 s\alpha_2 & c\theta_3 s\alpha_2 & c\alpha_2 & c\alpha_2 d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_3 \\ -s\alpha_3 d_4 \\ c\alpha_3 d_4 \\ 1 \end{bmatrix}$$

$$f_1(\theta_3) = a_3 c_3 + d_4 s\alpha_3 s_3 + a_2$$
$$f_2(\theta_3) = a_3 c\alpha_2 s_3 - d_4 s\alpha_3 c\alpha_2 c_3 - d_4 s\alpha_2 c\alpha_3 - d_3 s\alpha_2$$
$$f_3(\theta_3) = a_3 s\alpha_2 s_3 - d_4 s\alpha_3 s\alpha_2 c_3 + d_4 c\alpha_2 c\alpha_3 + d_3 c\alpha_2$$

Pieper's Solution - Three consecutive Axes Intersect

- Repeating the same process again

$$ {}^{0}P_{4org} = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T\,{}^{3}P_{4org} = {}^{0}_{1}T\,{}^{1}_{2}T \begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix} $$

$$ \begin{bmatrix} g_1(\theta_2) \\ g_2(\theta_2) \\ g_3(\theta_2) \\ 1 \end{bmatrix} = {}^{1}_{2}T \begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix} $$

$$ \begin{bmatrix} g_1(\theta_2) \\ g_2(\theta_2) \\ g_3(\theta_2) \\ 1 \end{bmatrix} = \overbrace{\begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & a_1 \\ s\theta_2 c\alpha_1 & c\theta_2 c\alpha_2 & -s\alpha_1 & -s\alpha_1 d_2 \\ s\theta_2 s\alpha_1 & c\theta_2 s\alpha_1 & c\alpha_1 & c\alpha_1 d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}}^{{}^{2}_{3}T} \begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix} $$

Pieper's Solution - Three consecutive Axes Intersect

$$g_1(\theta_2) = c_2 f_1 + s_2 f_2 + a_1$$
$$g_2(\theta_2) = s_3 s\alpha_1 s_3 f_1 + c_2 c\alpha_1 f_2 - s\alpha_1 f_3 - d_2 s\alpha_1$$
$$g_3(\theta_2) = s_2 s\alpha_1 s_2 f + c_2 s\alpha_1 f_2 + c\alpha_1 f_3 + d_2 c\alpha_1$$

- Repeating the same process for the last time

$${}^0P_{4org} = {}^0_1T\,{}^1_2T\,{}^2_3T\,{}^3P_{4org} = {}^0_1T \begin{bmatrix} g_1(\theta_2) \\ g_2(\theta_2) \\ g_2(\theta_2) \\ 1 \end{bmatrix}$$

$${}^0P_{4org} = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & a_0 \\ s\theta_1 c\alpha_0 & c\theta_1 c\alpha_0 & -s\alpha_0 & -s\alpha_0 d_1 \\ s\theta_1 s\alpha_0 & c\theta_1 s\alpha_0 & c\alpha_0 & c\alpha_0 d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} g_1(\theta_2) \\ g_2(\theta_2) \\ g_3(\theta_2) \\ 1 \end{bmatrix}$$

Pieper's Solution - Three consecutive Axes Intersect

- ***Frame {0}*** - The frame attached to the base of the robot or link 0 called frame {0} This frame does not move and for the problem of arm kinematics can be considered as the ***reference frame.***

- Assign {0} to match {1} when the first joint **variable is zero**

$$\theta_1 \neq 0 \qquad \alpha_0 = d_1 = a_0 = 0$$

$$^{0}P_{4org} = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & a_0 \\ s\theta_1 c\alpha_0 & c\theta_1 c\alpha_0 & -s\alpha_0 & -s\alpha_0 d_1 \\ s\theta_1 s\alpha_0 & c\theta_1 s\alpha_0 & c\alpha_0 & c\alpha_0 d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} g_1(\theta_2) \\ g_2(\theta_2) \\ g_3(\theta_2) \\ 1 \end{bmatrix}$$ substitution

$$^{0}P_{4org} = \begin{bmatrix} c_1 g_1 - s_1 g_2 \\ s_1 g_1 + c_1 g_2 \\ g_3 \\ 1 \end{bmatrix}$$ $\Rightarrow$ really simple position of wrist

Pieper's Solution - Three consecutive Axes Intersect

$${}^{0}P_{4ORG} = \begin{bmatrix} c_1 g_1 - s_1 g_2 \\ s_1 g_1 + c_1 g_2 \\ g_3 \end{bmatrix}$$

- Through algebraic manipulation of these equations, we can solve for the desired joint angles ($\theta_1, \theta_2, \theta_3$).

- The first step is to square the magnitude of the distance from the frame {0} origin to frame {4} origin.

$g_1^2 s_1^2 + g_1^2 c_1^2$

same for g_2

$$r^2 = \left({}^0P_{4orgx}\right)^2 + \left({}^0P_{4orgy}\right)^2 + \left({}^0P_{4orgz}\right)^2 = g_1^2 + g_2^2 + g_3^2$$

- Using the previously define function for g_i we have

$$r^2 = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2 f_3 + a_1(c_2 f_1 - s_2 f_2)$$

Pieper’s Solution - Three consecutive Axes Intersect

rewritten from above

$$r^2 = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2f_3 + a_1(c_2f_1 - s_2f_2)$$

$$Z = {}^0P^2_{4orgz} = g_3$$

→ 3 row of ${}^0P_{4ORG}$

- Applying a substitution of temporary variables, we can write the magnitude squared term along with the z-component of the {0} frame origin to the {4} frame origin distance.

$$r^2 = (k_1c_2 + k_2s_2)2a_1 + k_3$$

$$Z = (k_1s_2 - k_2c_2)s\alpha_1 + k_4$$

$$k_1 = f_1$$

$$k_2 = -f_2$$

$$k_3 = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2f_3$$

$$k_4 = f_3c\alpha_1 + d_2c\alpha_1$$

- These equations are useful because dependence on θ_1 has been eliminated, and dependence on θ_2 takes a simple form

Pieper's Solution - Three consecutive Axes Intersect

- Consider 3 cases while solving for θ_3 :
- **Case 1 -** $a_1 = 0$

$$r^2 = k_3$$

$$k_3 = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2 f_3$$

$$f_1(\theta_3) = a_3 c_3 + d_4 s\alpha_3 s_3 + a_2$$
$$f_2(\theta_3) = a_3 c\alpha_2 s_3 - d_4 s\alpha_3 c\alpha_2 c_3 - d_4 s\alpha_2 c\alpha_3 - d_3 s\alpha_2$$
$$f_3(\theta_3) = a_3 s\alpha_2 s_3 - d_4 s\alpha_3 s\alpha_2 c_3 + d_4 c\alpha_2 c\alpha_3 + d_3 c\alpha_2$$

- Solution Methodology - Reduction to Ploynomial => Quadratic Equation

$$u = \tan\frac{\theta}{2} \qquad \cos\theta = \frac{1-u^2}{1+u^2} \qquad \sin\theta = \frac{2u}{1+u^2}$$

Pieper’s Solution - Three consecutive Axes Intersect

- **Case 2 -** $s\alpha_1 = 0$

$$Z = k_4$$

$$k_4 = f_3 c\alpha_1 + d_2 c\alpha_1$$

$$f_3(\theta_3) = a_3 s\alpha_2 s_3 - d_4 s\alpha_3 s\alpha_2 c_3 + d_4 c\alpha_2 c\alpha_3 + d_3 c\alpha_2$$

- Solution Methodology - Reduction to Ploynomial => Quadratic Equation

$$u = \tan\frac{\theta}{2} \qquad \cos\theta = \frac{1-u^2}{1+u^2} \qquad \sin\theta = \frac{2u}{1+u^2}$$

Pieper's Solution - Three consecutive Axes Intersect

- **Case 3 (General case)** : We can find θ_3 through the following algebraic manipulation:

$$\frac{r^2 - k_3}{2a_1} = (k_1 c_2 + k_2 s_2)$$

$$\frac{Z - k_4}{s\alpha_1} = (k_1 s_2 - k_2 c_2)$$

- squaring both sides, we find

$$\left(\frac{r^2 - k_3}{2a_1}\right)^2 = (k_1 c_2 + k_2 s_2)^2 = k_1^2 c_2^2 + k_2^2 s_2^2 + 2k_1 k_2 c_2 s_2$$

$$\left(\frac{Z - k_4}{s\alpha_1}\right)^2 = (k_1 s_2 - k_2 c_2)^2 = k_1^2 s_2^2 + k_2^2 c_2^2 - 2k_1 k_2 c_2 s_2$$

Pieper’s Solution - Three consecutive Axes Intersect

- Adding these two equations together and simplifying using the trigonometry identity (Reduction to Ploynomial), we find a fourth order equation for θ_3

$$\left(\frac{r^2 - k_3}{2a_1}\right)^2 + \left(\frac{Z - k_4}{s\alpha_1}\right)^2 = k_1^2 + k_2^2$$

$$k_1 = f_1$$
$$k_2 = -f_2$$
$$k_3 = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2 f_3$$
$$k_4 = f_3 c\alpha_1 + d_2 c\alpha_1$$

$$f_1(\theta_3) = a_3 c_3 + d_4 s\alpha_3 s_3 + a_2$$
$$f_2(\theta_3) = a_3 c\alpha_2 s_3 - d_4 s\alpha_3 c\alpha_2 c_3 - d_4 s\alpha_2 c\alpha_3 - d_3 s\alpha_2$$
$$f_3(\theta_3) = a_3 s\alpha_2 s_3 - d_4 s\alpha_3 s\alpha_2 c_3 + d_4 c\alpha_2 c\alpha_3 + d_3 c\alpha_2$$

Pieper's Solution - Three consecutive Axes Intersect

- With θ_3 solved, substitute into r^2, Z to find θ_2

$$r^2 = (k_1 c_2 + k_2 s_2)2a_1 + k_3$$
$$Z = (k_1 s_2 - k_2 c_2)s\alpha_1 + k_4$$

$$k_1 = f_1$$
$$k_2 = -f_2$$
$$k_3 = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2 f_3$$
$$k_4 = f_3 c\alpha_1 + d_2 c\alpha_1$$

$$f_1(\theta_3) = a_3 c_3 + d_4 s\alpha_3 s_3 + a_2$$
$$f_2(\theta_3) = a_3 c\alpha_2 s_3 - d_4 s\alpha_3 c\alpha_2 c_3 - d_4 s\alpha_2 c\alpha_3 - d_3 s\alpha_2$$
$$f_3(\theta_3) = a_3 s\alpha_2 s_3 - d_4 s\alpha_3 s\alpha_2 c_3 + d_4 c\alpha_2 c\alpha_3 + d_3 c\alpha_2$$

Pieper's Solution - Three consecutive Axes Intersect

- With θ_2, θ_3 solved, substitute into ${}^0P_{4org}$ to find

$$ {}^0P_{4org} = \begin{bmatrix} c_1 g_1 - s_1 g_2 \\ s_1 g_1 + c_1 g_2 \\ g_3 \\ 1 \end{bmatrix} $$

$$ {}^0P_{4orgx} = c_1 g_1 - s_1 g_2 $$

$$ {}^0P_{4orgy} = s_1 g_1 + c_1 g_2 $$

$$ g_1(\theta_2) = c_2 f_1 + s_2 f_2 + a_1 $$
$$ g_2(\theta_2) = s_3 s\alpha_1 s_3 f_1 + c_2 c\alpha_1 f_2 - s\alpha_1 f_3 - d_2 s\alpha_1 $$
$$ g_3(\theta_2) = s_2 s\alpha_1 s_2 f + c_2 s\alpha_1 f_2 + c\alpha_1 f_3 + d_2 c\alpha_1 $$

$$ f_1(\theta_3) = a_3 c_3 + d_4 s\alpha_3 s_3 + a_2 $$
$$ f_2(\theta_3) = a_3 c\alpha_2 s_3 - d_4 s\alpha_3 c\alpha_2 c_3 - d_4 s\alpha_2 c\alpha_3 - d_3 s\alpha_2 $$
$$ f_3(\theta_3) = a_3 s\alpha_2 s_3 - d_4 s\alpha_3 s\alpha_2 c_3 + d_4 c\alpha_2 c\alpha_3 + d_3 c\alpha_2 $$

- Solve for θ_1 using the reduction to polynomial method

Pieper's Solution - Three consecutive Axes Intersect

- To complete our solution we need to solve for $\theta_4, \theta_5, \theta_6$

- Since the last three axes intersect these joint angle affect the orientation of only the last link. We can compute them based only upon the rotation portion of the specified goal ${}^0_6R$

$$ {}^4_6R\Big|_{\theta_4=0} = {}^0_4R^{-1}\Big|_{\theta_4=0} \quad {}^0_6R $$

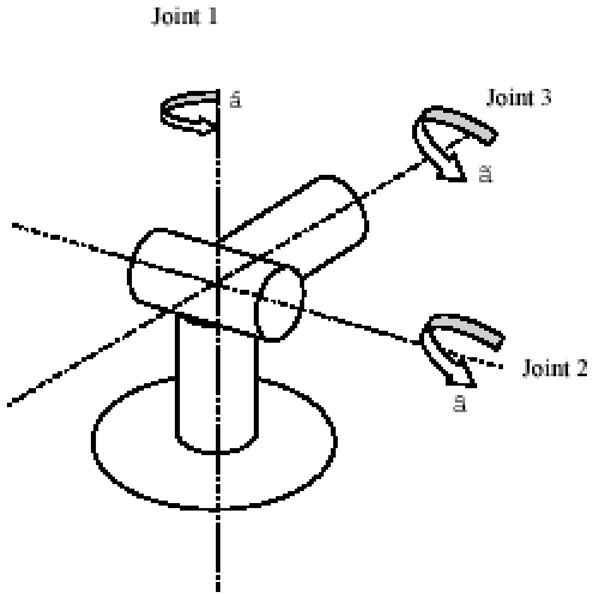
- ${}^0_4R\Big|_{\theta_4=0}$ - The orientation of link frame {4} relative to the base frame {0} when $\theta_4 = 0$

represents a frame @ wrist

- $\theta_4, \theta_5, \theta_6$ are the Euler angles applied to ${}^4_6R\Big|_{\theta_4=0}$

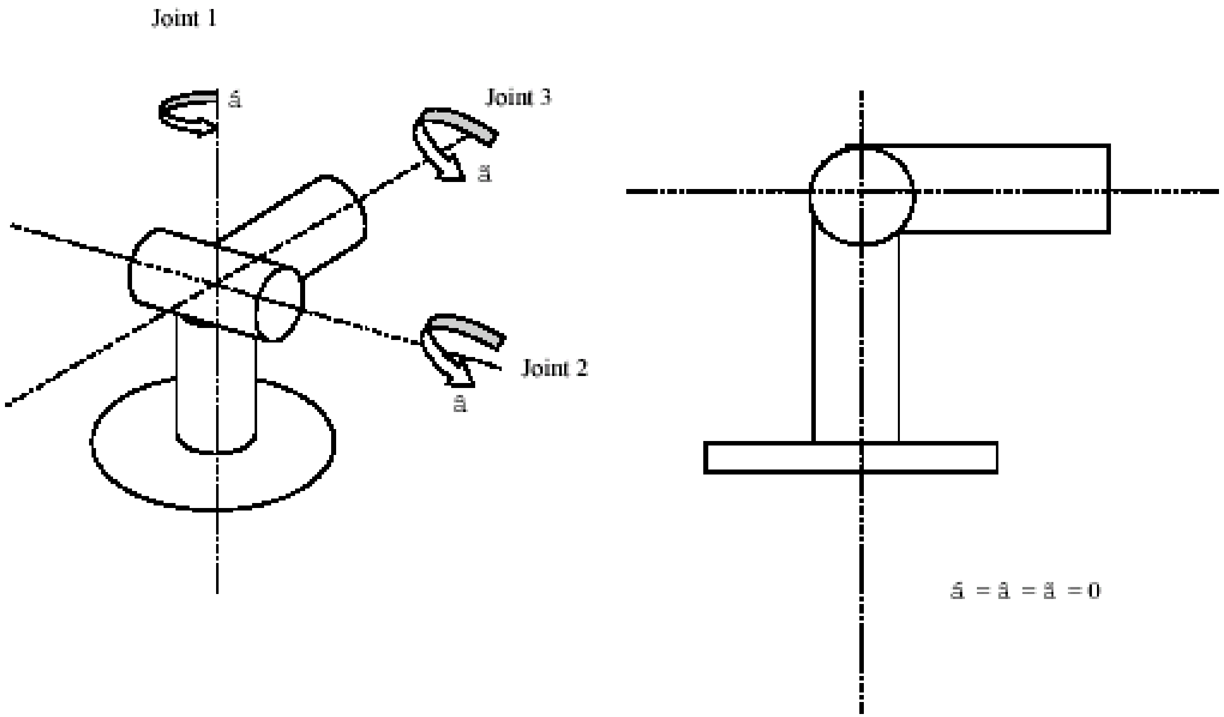
Central Topic - Inverse Manipulator Kinematics - Examples

- **Algebraic Solution** (closed form) –
  - Piepers Method (Continued)
    - Last three consecutive axes intersect at one point



Three consecutive Axes Intersect - wrist

- Consider a 3 DOF non-planar robot whose axes all intersect at a point.



Mapping - Rotated Frames - Z-Y-X Euler Angles

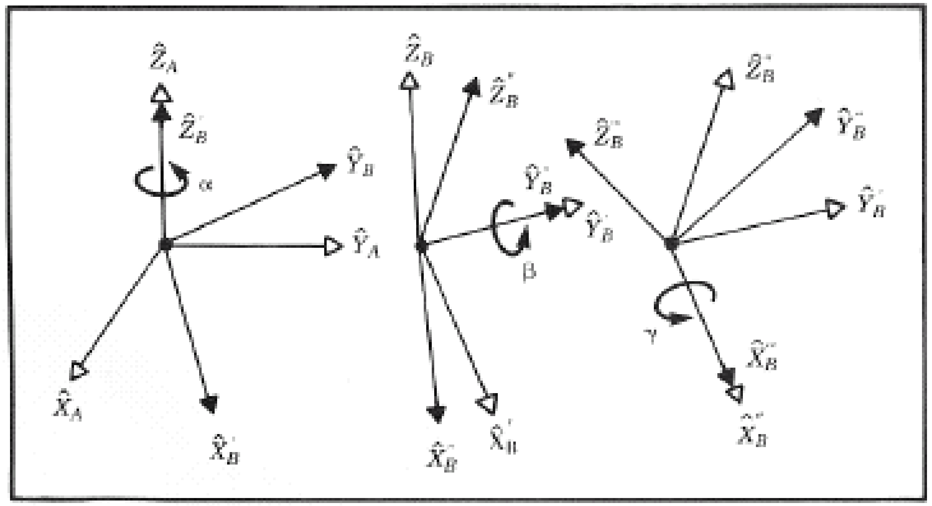
Start with frame {B} coincident with a known reference frame {A}.

- Rotate frame {B} about $\hat{Z}_A$ by an angle α
- Rotate frame {B} about $\hat{Y}_B$ by an angle β
- Rotate frame {B} about $\hat{X}_B$ by an angle γ

} **Euler Angles**

$\alpha \sim \theta_4$
$\beta \sim \theta_5$
$\gamma \sim \theta_6$

Note *- Each rotation is preformed about an axis of the the moving reference frame {B}, rather then a fixed reference frame {A}.*



Mapping - Rotated Frames - ZYX Euler Angles

$$ {}^{A}_{B}R_{X'Y'Z'}(\alpha,\beta,\gamma) = R_Z(\alpha)R_Y(\beta)R_X(\gamma) = \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix} $$

$$ {}^{A}_{B}R_{X'Y'Z'}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} $$

$$ {}^{A}_{B}R_{X'Y'Z'}(\alpha,\beta,\gamma) = {}^{0}_{3}R = {}^{0}_{1}R\,{}^{1}_{2}R\,{}^{2}_{3}R $$

Three consecutive Axes Intersect - wrist

- Because, in this example, our robot can perform no translations, we can write

$$
{}^{0}_{3}T = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma & 0 \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma & 0 \\ -s\beta & c\beta s\gamma & c\beta c\gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

- The above transform provides the solution to the forward kinematics.

Three consecutive Axes Intersect - wrist

- The inverse kinematics problem.
  - Given a particular rotation - Goal (again, this robot can perform no translations)
  - Solve: Find the Z-Y-X Euler angles

$$ {}^{0}_{3}T_{d} = \begin{bmatrix} & [R] & & 0 \\ & & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{0}_{3}T(\alpha, \beta, \gamma) $$

Three consecutive Axes Intersect - wrist

$$\underbrace{\begin{bmatrix} r11 & r12 & r13 & 0 \\ r21 & r22 & r23 & 0 \\ r31 & r32 & r33 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\textbf{Goal}} = \underbrace{\begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma & 0 \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma & 0 \\ -s\beta & c\beta s\gamma & c\beta c\gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\textbf{Direct Kinematics}}$$

Three consecutive Axes Intersect - wrist

- Using elements r_{11} and r_{21}, we can solve for angle α

$$r_{11} = c\alpha c\beta$$
$$r_{21} = s\alpha c\beta$$

- when $\beta \neq \pm\frac{n\pi}{2}$ where n is an odd integer by using the Atan2 function we obtain

$$\alpha = \begin{cases} \text{Atan2}(r_{21}, r_{11}) \text{ when } c\beta \geq \mathbf{0} \\ \text{Atan2}(-r_{21}, -r_{11}) \text{ when } c\beta < \mathbf{0} \end{cases}$$

Three consecutive Axes Intersect - wrist

- Similarly, we find angle γ by

$$r_{32} = c\beta s\gamma$$
$$r_{33} = c\beta c\gamma$$

- when $\beta \neq \pm\dfrac{n\pi}{2}$ where n is an odd integer by using the Atan2 function we obtain

$$\gamma = \begin{cases} \text{Atan2}(r_{32}, r_{33}) \text{ when } c\beta \geq \mathbf{0} \\ \text{Atan2}(-r_{32}, -r_{33}) \text{ when } c\beta < \mathbf{0} \end{cases}$$

Three consecutive Axes Intersect - wrist

- The third angle, β , can be found from

$$r_{11}^2 + r_{21}^2 = c\beta^2\left(c\alpha^2 + s\alpha^2\right)$$

$$r_{31} = -s\beta$$

$$c\beta = \pm\sqrt{r11^2 + r21^2}$$

- Using the Atan2 function, we find

$$\beta = \mathrm{Atan2}\left(-r_{31}, \pm\sqrt{r_{11}{}^2 + r_{21}{}^2}\right)$$

Three consecutive Axes Intersect - wrist

- Note: Two answers exist for angle β which will result in two answers each for angles α and γ .

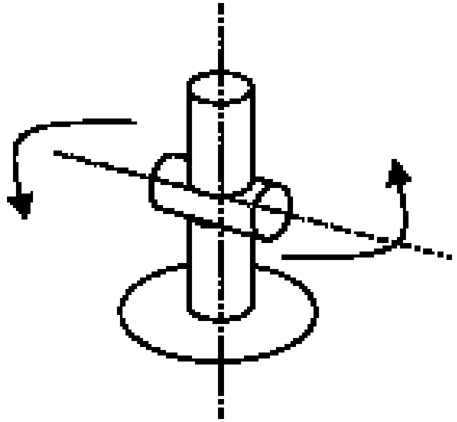
$$\alpha = \begin{cases} \text{Atan2}(r_{21}, r_{11}) \text{ when } c\beta \geq \mathbf{0} \\ \text{Atan2}(-r_{21}, -r_{11}) \text{ when } c\beta < \mathbf{0} \end{cases}$$

$$\beta = \text{atan2}\left(-r31, \pm\sqrt{r11^2 + r21^2}\right)$$

$$\gamma = \begin{cases} \text{Atan2}(r_{32}, r_{33}) \text{ when } c\beta \geq \mathbf{0} \\ \text{Atan2}(-r_{32}, -r_{33}) \text{ when } c\beta < \mathbf{0} \end{cases}$$

Three consecutive Axes Intersect - wrist

- What do we do if $\beta = 90^o$?
- This is troublesome because $\cos(90^o) = 0$.
- Applying the difference of angles formula, we find:



$$\begin{bmatrix} r11 & r12 & r13 & 0 \\ r21 & r22 & r23 & 0 \\ r31 & r32 & r33 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & s(\gamma-\alpha) & c(\gamma-\alpha) & 0 \\ 0 & c(\gamma-\alpha) & s(\gamma-\alpha) & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- We are left with $(\gamma-\alpha)$ for every case. This means we can't solve for either, just their difference.

Three consecutive Axes Intersect - wrist

- One solution is to define two cases such that

$$\gamma - \alpha = \text{Atan2}(r_{12}, r_{22}) \quad \text{if} \quad \beta = 90^0$$

$$\gamma + \alpha = \text{Atan2}(r_{12}, r_{22}) \quad \text{if} \quad \beta = -90^0$$

Three consecutive Axes Intersect - wrist

- Unfortunately, while this seems like a simple solution, it is troublesome in practice because α is never exactly zero. This leads to singularity problems
- For this example, the singular case results in the capability for self-rotation. That is, the middle link can rotate while the end effector's orientation never changes.