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Algorithms HW 2

1.9) Sum:

$$x \equiv x' \bmod N, \quad y \equiv y' \bmod N \Rightarrow x+y \equiv x'+y' \bmod N$$

This means they will have the same remainder when divided by N

$23 \equiv 9 \bmod 7 \rightarrow 23 - 9 = 14 = 2N$
$2 \equiv 2 \bmod 7$
$\frac{14}{2} = 7 = N$

Always goes back to a multiple of N.

$$x+y \equiv x'+y' \bmod N \longrightarrow (x+y) - (x'+y') = cN$$

$y \equiv y' \bmod N \longrightarrow y - y' = aN$ (a is a constant $\in \mathbb{Z}$)
$x \equiv x' \bmod N \longrightarrow x - x' = bN$ (b is a constant $\in \mathbb{Z}$

$$(y-y') + (x-x') = aN + bN$$

$$(y+x) - (y'+x') = (a+b)N \quad (a+b \text{ is constant } \in \mathbb{Z})$$

$$x+y \equiv x'+y' \bmod N$$

Since $(a+b)N$ means that the difference will be a multiple of N, then we can conclude that the remainders of the additions will be the same, and thus holds.

Multiplication: $x \equiv x' \bmod N, \quad y \equiv y' \bmod N \Rightarrow xy \equiv x'y' \bmod N$ mod N

$\left.\begin{array}{l} x - x' = aN \\ y - y' = bN \end{array}\right\} \rightarrow (x-x')(y-y') = abN^2 \longrightarrow$ Need to get $xy - x'y'$

$x = x' + aN$
$y = y' + bN$

$$(x-x')(y-y') = abN^2$$

$$xy - xy' - x'y + x'y' = abN^2$$
$$xy - (xy' + x'y) + x'y' = abN^2$$

$$xy - (y'(x'+aN) + x'(y'+bN)) + x'y' = abN^2$$

Need to get rid of all the remaining x and y because we already have the xy term.

$$xy - (y'x' + y'aN + x'y' + x'bN) + x'y' = abN^2$$

$$xy - x'y' - (aNy' + bNx') = abN^2$$

$$xy - x'y' = abN^2 + (ay' + bx')N$$

$$xy - x'y' = \underbrace{(abN + ay' + bx')N}$$

This shows us that the difference will be a multiple of N, so the two remainders must be the same.

$$\Rightarrow xy \equiv x'y' \bmod N$$

1.11) Is $4^{1536} - 9^{4824}$ divisible by 35?

Try to find a simpler number for N than 35.

$$\begin{array}{c} 35 \\ \wedge \\ 7 \quad 5 \end{array}$$

5, 7 are both prime, so they will be the smallest factors of 35 to help simplify the problem.
If 4^{1536} and 9^{4824} are both multiples of 5 and 7, then their difference will be a multiple of 35 (as shown by previous problem)

<u>Divisible by 7:</u>

$4^{1536} \mod 7$

$4^2 = 16$
$4^3 = 64$ r1

$64 \to 1 \mod 7$

$(4^3)^{512} \to 1^{512}$

$4^{1536} = 1 \mod 7$

$9^{4824} \mod 7$

$7 \times 9 = 63$
$7 \times 10 = 70$
$7 \times 11 = 77$
$7 \times 12 = 84$

If we use this one, then we can reuse the work for checking 4^{1536}

$9^2 = 81 = 4 \mod 7$

$\Rightarrow (4)^{2412} \mod 7$

$= (4^3)^{804} \mod 7$

$4^3 = 64 = 1 \mod 7$

$= 1^{804} = 1 \mod 7$

$$4^{1536} - 9^{4824} = (1 \mod 7) - (1 \mod 7) = 1 - 1 \mod 7$$
$$= 0 \mod 7$$

$\boxed{\text{So difference is divisible by 7}}$

<u>Divisible by 5:</u>

$4 \times 3 = 12$
$4 \times 4 = 16$
$4 \times 5 = 20$

Cannot be 5 b/c exponent not div by 5 evenly

4^{1536}

$\to (4^4)^{384} \mod 5$

$= 1^{384} \mod 5 = 1$

$9^{4824} \mod 5$

$9 \times 9 = 81$

$81 \mod 5 = \underline{\underline{1}}$

$9^{4824} = (9^2)^{2412} = (81)^{2412}$

$1^{2412} \mod 5 = \underline{\underline{1}}$

$$4^{1536} - 9^{4824} \mod 5 = 1 \mod 5 - 1 \mod 5$$
$$= 1 - 1 \mod 5 = 0 \mod 5$$

$\boxed{\text{So the difference is divisible by 5}}$

Since the difference is divisible by both 5 and 7, it must also be divisible by 35.

1.12) What is $2^{2^{2006}} \bmod 3$

$4 \times 5 = 20$, $3 \times 7 = 21$ } wont work: $2006 \% 5 \neq 0$

$2^{2^{2006}} = 4^{2006} \bmod 3$

$4^{2006} = (4^2)^{1003} = 16^{1003}$

$(16 \bmod 3)^{1003} = 1^{1003} = 1$

(Tinkering)

$4^2 = 16$

$16 \bmod 3 = 1$

$$2^{2^{2006}} \bmod 3 = 4^{2006} \bmod 3 = (4^2 \bmod 3)^{1003}$$

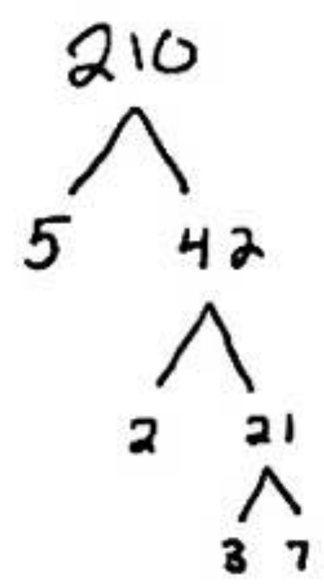
$$= 1^{1003} \bmod 3 = \boxed{1}$$

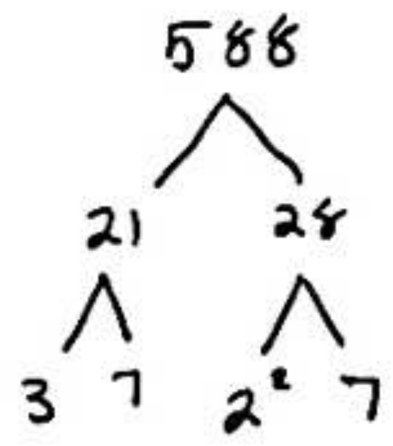
1.18) Compute $\gcd(210, 588)$

Factorization:

Factors of 210: 2, 3, 5, 6, 7, 10, 14, 15, 21, 30, 35, 42, 70, 105

Factors of 588: 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, 49, 84, 98, 147, 196, 294





$588 = 3 \times 7^2 \times 2^2$

$\gcd(210, 588) = 42$

$210 = 5 \times 2 \times 3 \times 7$

$7 \times 3 \times 2 = 42$

Euler's Algorithm:

$\gcd(588, 210)$

$\downarrow$

$\gcd(210, 588 \bmod 210)$

$\gcd(210, 168)$

$\downarrow$

$\gcd(168, 210 \bmod 168)$

$\gcd(168, 42)$

$\downarrow$

$\gcd(42, 168 \bmod 4)$

$\gcd(42, 0)$

$b = 0$, return 42

42

42

42

$$\boxed{\gcd(588, 210) = 42}$$

1.31) Consider $N! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot N$

a) how many bits long is $N!$ in $\Theta(n)$ form

From Stirling's Approximation, we know that the bit length of a factorial is $O(\log_2(\text{Factorial}))$

$$\Longrightarrow \log_2(N!) = \log_2(N \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1) = \underbrace{\log_2(N) + \log_2(n-1) + \ldots \log_2(2) + \log_2(1)}$$

Magnitude of all the terms is relatively close to $\log_2(n)$, so for our approximation we will treat them like they are all the same magnitude of $\log_2(n)$

$$\Longrightarrow \Theta(\log_2 n!) = \boxed{\Theta(n \log_2 n)}$$

b) Algorithm for computing $N!$ with runtime analysis.

$N! = n(n-1)(n-2)\ldots(1)$ $\rightarrow$ Because multiplication is communitive, we can use a for loop to multiply from 1 to N
$N! = (1)(2)\ldots(n-1)(N)$

```
NFactorial (N)
  result = 1
  for i in range(1, N+1):      ← Will go through n-iterations
    result = result * i        ← Multiplication of 2 n-bit numbers will take O(n²)
  return result
```

Bit length $O(n)$ x Bit length $O(n)$

\# of iterations $O(n)$ x time for multiplication $O(n^2)$

$$\boxed{O(n^3) \text{ time}}$$

Lab 2 Results:

- 3 trials of 100, 1000, 10000 digit numbers

Results:		
	Regular	Div and Conq
100	4.29E-06	8.30E-05
	2.62E-06	2.72E-05
	2.86E-06	3.81E-05
Avg	3.26E-06	4.94E-05
1000	3.10E-06	4.60E-05
	2.15E-06	2.88E-05
	2.15E-06	4.10E-05
Avg	2.46E-06	3.86E-05
10000	4.29E-06	4.60E-05
	2.86E-06	4.51E-05
	3.81E-06	4.60E-05
Avg	3.66E-06	4.57E-05