CSC236 Assignment 2

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Due Date: 2020.3.12

1(a).

$$
\begin{aligned}
T(n) &= n + 2T(\tfrac{n}{2}) \\
&= n + 2(\tfrac{n}{2} + 2T(\tfrac{n}{4})) \\
&= 2n + 4T(\tfrac{n}{4}) \\
&= 2n + 4(\tfrac{n}{4} + 2T(\tfrac{n}{8}) \\
&= 3n + 2^3\, T(\tfrac{n}{2^3}) \\
&\ldots\ldots \\
&= m\cdot n + 2^m\cdot T(\tfrac{n}{2^m})
\end{aligned}
$$

where m is the smallest int that $\frac{n}{2^m} \leq k$. So, $m = \lceil \log_2 \frac{n}{k} \rceil$

$= \lceil \log_2 \frac{n}{k} \rceil \cdot n + C\cdot 2^{\lceil \log 2 \frac{n}{k} \rceil}$

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1(b).

If $n \leq k$: $T(n) = C \in \Theta(1)$

If $n > k$: $T(n) = \lceil \log_2 \frac{n}{k} \rceil \cdot n + C\cdot 2^{\lceil \log 2 \frac{n}{k} \rceil}$

$\in \Theta(n\cdot \log_2 n + \frac{n}{k})$

$\in \Theta(n\cdot \log n)$

It depends on the relationship between n and k, so, it depends on k.

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1(c).

From 1(a) we get

$$T(n) = \lceil \log_2 \frac{n}{k} \rceil \cdot n + C \cdot 2^{\wedge \lceil \log 2 \frac{n}{k} \rceil}$$

$$T'(n) = \lceil \log_2 \frac{n}{k} \rceil \cdot n + \frac{n^2}{2^{2 \lceil \log 2 \frac{n}{k} \rceil}} \cdot 2^{\wedge \lceil \log 2 \frac{n}{k} \rceil}$$

replace C by $\frac{n^2}{2^{2 \lceil \log 2 \frac{n}{k} \rceil}}$ which is n^2

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1(d).

If $n \le k$: $T(n) = C \in \Theta(1)$

$T'(n) = n^2 \le k^2 \in \Theta(1)$

So, $T'(n) \in \Theta(T(n))$

If $n > k$: $T(n) = \lceil \log_2 \frac{n}{k} \rceil \cdot n + C \cdot 2^{\wedge \lceil \log 2 \frac{n}{k} \rceil} \in \Theta(n \cdot \log n)$

$T'(n) = \lceil \log_2 \frac{n}{k} \rceil \cdot n + \frac{n^2}{2^{2 \lceil \log 2 \frac{n}{k} \rceil}} \cdot 2^{\wedge \lceil \log 2 \frac{n}{k} \rceil}$

$T'(n) \in \Theta(n \cdot \log_2 \frac{n}{k} + \frac{n}{k} \cdot \frac{n^2}{(\frac{n}{k})^2})$

$T'(n) \in \Theta(n \cdot \log n + n)$

$T'(n) \in \Theta(n \cdot \log n)$

So, $T'(n) \in \Theta(T(n))$

So, $T'(n) \in \Theta(T(n))$.

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2(a).

Pre: Input is an non-empty Array A, All elements in A are positive integers.

Post: umax(A) = x where:

$((\exists i\in\mathbb{N}, \forall j\in \mathbb{N}, i\neq j\Longrightarrow A[i]>A[j]) \Longrightarrow x = A[i]) \wedge$

$((\exists i\in\mathbb{N},\forall j\in\mathbb{N},\exists k\in\mathbb{N}, i\neq j\wedge i\neq k \Longrightarrow(A[i]\geq A[j]\wedge A[i]=A[k])) \Longrightarrow x <0)$

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2(b).

Let A be [1,3,3]

It does not have a unique max value. It should return a negative int, but it returns 1.

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2(c).

The post-condition in 2(a) is too loose, here we need a stricter one.

Post: umax(A) = x where:

$((\exists i\in\mathbb{N}, \forall j\in \mathbb{N}, i\neq j\Longrightarrow A[i]>A[j]) \Longrightarrow x = A[i]) \wedge$

$((\exists i\in\mathbb{N},\forall j\in\mathbb{N},\exists k\in\mathbb{N}, i\neq j\wedge i\neq k \Longrightarrow(A[i]\geq A[j]\wedge A[i]=A[k])) \Longrightarrow x = -A[i])$

the loose one says when there are no unique max in A, return any negative number. The stricter one says in that condition, return the negative value of the max in A. If we can prove the stricter one, then the loose one is also proved.

The pre-condition stays the same.

P(n): an arbitrary array A, A satisfies the pre-condition $\wedge$ len(A)=n $\Longrightarrow$ umax(A) satisfies the post-condition.

WTS: $\forall n \in \mathbb{N}^+$, P(n)

Assume A is an arbitrary array, A satisfies pre-condition $\wedge$ len(A)=n

BaseCase: n = 1

len(A)=1, reaches line 3, return A[0], which is the only element in A. Obvious, it's the unique max of A. So, P(n) holds

Inductive: $n \geq 1$

IH: Assume P(n)

WTS P(n+1)

Let A be an arbitrary array that satisfies pre-condition and len(A) = n+1.

Call umax(A): head = A[0], tail = A[1:], len(tail) = n, tmax = umax(tail)

Case1: $tmax > 0$:

by IH and post-condition, tmax is the unique max of tail,

① $\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, 1 \leq i,j \leq n \wedge A[i]=tamx \wedge i \neq j \Longrightarrow A[j] < A[i] = tmax$

Case1.1: head = tmax:

by head = A[0] = tmax and ① we can get:

$\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, 1 \leq i,j \leq n \wedge A[i]=tamx \wedge i \neq j \Longrightarrow A[j] < A[i]=tmax=A[0]$

A[0] and tmax are both the max of A, they are not unique, by post, we should return the negative of max, which we actually returns -1·head.

So, P(n+1) holds.

Case1.2: head >abs (tmax)

abs(tmax) = tmax, head > tmax

by head = A[0] > tmax and ① we can get:

$\forall i \in \mathbb{N}, 0 \leq i \leq n \wedge i \neq 0 \Longrightarrow A[i] < A[0]$

So, A[0] is the unique max of A, by post, we should return unique max, which we actually returns head. So, P(n+1) holds.

Case1.3: head < tmax

by head = A[0] < tmax and ① we can get:

$\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, 0 \leq i,j \leq n \wedge A[i]=tamx \wedge i \neq j \Longrightarrow A[j] < A[i] = tmax$

tmax is the unique max of A, by post, we should return unique max, which we actually returns tmax. So, P(n+1) holds.

Case2: tmax<0

by IH and post-condition, -tmax is the max but not unique max of tail,

② $\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, \exists k \in \mathbb{N}, 1 \leq i,j,k \leq n \wedge A[i]=-tamx \wedge i \neq j \wedge i \neq k \Longrightarrow A[j] \leq A[i] = A[k]$

Case2.1: head > abs(tmax)

abs(tmax) = - tmax, by head = A[0] > -tmax and ② we can get:

$\forall i \in \mathbb{N}, 0 \leq i \leq n \wedge i \neq 0 \Longrightarrow A[i] < A[0]$

So, A[0] is the unique max of A, by post, we should return unique max, which we actually returns head. So, P(n+1) holds.

Case2.2: head < abs(tmax)

abs(tmax) = - tmax, by head = A[0] < -tmax and ② we can get:

$\exists i \in \mathbb{N}, \forall j \in \mathbb{N}, \exists k \in \mathbb{N}, 0 \leq i,j,k \leq n \wedge A[i]=-tamx \wedge i \neq j \wedge i \neq k \Longrightarrow A[j] \leq A[i] = A[k]$

So, -tmax is the max of A, but it's not unique, by post-condition, we should return the negative value of max, which we actually returns tmax. So, P(n+1) holds.

So, $\forall n \in \mathbb{N}^+$, P(n). Any input A that satisfying pre-condition, umax(A) will satisfies post-condition. umax is correct.

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Theorem 1.1 (R "correctness"). Given any $A \in \mathbb{N}^*$, R(A) terminates and returns a list of natural numbers B such that $\forall x \in \mathbb{N}$, Count(B, x) = Pairs(A, x)

Assume A satisfies the pre-condition, majority of A is X.

$Curr_j$: the value of A after jth R operation on A

$Curr_0 = A$

$Curr_1 = R(A)$ line14

After i iterations in while loop, we get $Curr_{i+1}$

For Count(A,X) and Pairs(A,X), here use the same as in problem set.

Loop Inviriant: in any $Curr_j$ X is still the majority.

P(j): For any A that satisfies the precondition, and it's majority is X. In maj(A), $Curr_j$'s majority is X.

WTS: $\forall j \in \mathbb{N}$, P(j)

Let $j \in \mathbb{N}$

Base Case: $j = 0$. $Curr_0 = A$, by definition, X is majority. P(0)holds

Inductive Steps:

For $j \geq 0$: Assume P(j) holds.

For $j + 1$:

Count($Curr_j$, X)>len($Curr_j$/2) by IH

$Curr_{j+1} = R(Curr_j)$

$\forall n \in \mathbb{N}$, Count($Curr_{j+1}$, n) = Pairs($Curr_j$, n) Theorem 1.1

So, in $Curr_{j+1}$, #X = Pairs($Curr_j$, X),

#other elements = $\sum_{n\in \mathbb{N}, n\neq X} \text{Pairs}(\text{Currj}, n)$.

PAIRS($Curr_j$, X) > $\sum_{n\in \mathbb{N}, n\neq X} \text{Pairs}(\text{Currj}, n)$ by Lamma1.

So, X is still the majority in $Curr_{j+1}$, P(j+1) holds.

Partial Correctness: Assume loop terminates after j itera.

We get len($Curr_{j+1}$) = len(prev) = len($Curr_j$). len($Curr_j$) = len(R($Curr_j$))

By loop inviriante: X is majority of $Curr_{j+1}$ and $Curr_j$.

By Lamma2: there is only 1 identical element in $Curr_{j+1}$ and $Curr_j$.

In conclusion: X is the only thing in $Curr_{j+1}$.

So, $Curr_{j+1}$[0] = X. Program returns $Curr_{j+1}$[0], satisfies postcondition.

Termination: Let mj be # non-majority elements in $Curr_j$.

For example = $Curr_j$ = [1,1,1,1,1,2,2,3,3] mj = 4

mj ∈ ℕ, and the list of mj is dereasing(mj+1 < mj) by Lamma3.

mj will derease to 0, by Well Ordering. When it's 0, there are no non-majority elements in $Curr_j$, the majority is still X, by loop invariant. At this time $Curr_j$ is a array of X's, $Curr_j$ is identical.

In the next iteration:

prev = $Curr_j$, $Curr_{j+1}$ = R($Curr_j$), len (prev) = len($Curr_j$) = len($Curr_{j+1}$)

By Lamma2. This violates loop condition, loop terminates, program terminates.

Lamma1:

A satisfies precondition, A's majority is X

$\Longrightarrow \mathrm{Pairs(A, X)} > \sum_{n\in \mathbb{N},\mathrm{n}\neq \mathrm{X}} \mathrm{Pairs(A, n)}$

Proof:

Assume : A satisfies precondition, A's majority is X.

Set m = len(A), k = #X in A.

Think A as a circular, since first and last element may form a pair.

X can be put in A adjacently or not. So let's say we have p groups of X's in A, groups are not adjacent with each other, inside a group, there are adjacent X's, or just 1 X. $p < k$ by Lamma4. If there is 1 X in a group, then no pairs in this group, and at least 1 X in a group, otherwise, it's not a group. So we get p X's in p groups, with k-p X's left. Put any one of these k-p X's in any group of X will from a new pair of X, which is made of itself and another X already in the group. So, we have k-p pairs of X in A. There are m – k other elements in A, with p gaps between p groups of X(A is circular). To maximize the number of non-X pairs, we assume any two non-X elements can form a pair, if they are adjacent. So, now we have p groups of non-X elements. By the same way we count # X pairs, we have m–k–p non-X pairs.

(k-p) – (m–k–p) = 2k – m > 0 By k > m/2, X is majority of A.

So, $\mathrm{Pairs(A, X)} > \sum_{n\in \mathbb{N},\mathrm{n}\neq \mathrm{X}} \mathrm{Pairs(A, n)}$, Lamma1 proved!

Lamma2:

A is a non-empty array of natural number,

$len(A) = len(R(A)) \Leftrightarrow$ A is identical(only 1 kind of element in A)

Proof:

Assume A is a non-empty array of natural number

WTS: $len(A) = len(R(A)) \Longrightarrow$ A is identical

Assume for contradiction: $\exists a,b,\ a\neq b,\ a,b\in A,\ len(A) = len(R(A))$.

By Lamma6 and Theorem 1.1 , $\forall i\in A$, we need $Count(A,i) = Pairs(A,i)$ to satisfy our assumption for contradiction. By Lamma5, A must be identical. So, $\Longrightarrow \Longleftarrow$, there can't be $a\neq b$.

So, $len(A) = len(R(A)) \Longrightarrow$ A is identical proved!

WTS: A is identical $\Longrightarrow len(A) = len(R(A))$

Lets say the only element in A is X, there are n X in A. Each X form a pair with X right adjacent to it. We get n pairs of X(A is circular).

$len(A) = n$, $len(R(A)) = n$, by Theorem 1.1.

So, A is identical $\Longrightarrow len(A) = len(R(A))$ is proved!

Lamma2 is proved!

Lamma3:

A satisfies the precondition, A's majority is X, elements in A is not identical $\Longrightarrow$ # non-X elements in A $>$ # non-X elements in R(A)

Proof:

Assume A satisfies the precondition, A's majority is X, elements in A is not identical.

Set m $=$ len(A), k $=$ #X in A.

To prove Lamma3, we only need to prove for any arbitaray non-X element Y, #Y in A $>$ #Y in R(A), by Theorem 1.1.

Let Y be such.

By Lamma5 and the fact that A is not identical, Count(A,Y) $\neq$Pairs(A,Y).

By Lamma6 Count(A,Y) $\geq$ Pairs(A,Y).

In conclusion: Count(A,Y) $>$ Pairs(A,Y) $=$ Count(R(A),Y), Theorem 1.1

non-X elements in A $= \sum_{n\in \mathbb{N}, \mathrm{n}\neq \mathrm{X}} \mathrm{Count}(\mathrm{A}, \mathrm{n})$

non-X elements in R(A) $= \sum_{n\in \mathbb{N}, \mathrm{n}\neq \mathrm{X}} \mathrm{Count}(\mathrm{R}(\mathrm{A}), \mathrm{n})$

$\sum_{n\in \mathbb{N}, \mathrm{n}\neq \mathrm{X}} \mathrm{Count}(\mathrm{R}(\mathrm{A}), \mathrm{n}) = \sum_{n\in \mathbb{N}, \mathrm{n}\neq \mathrm{X}} \mathrm{Pairs}(\mathrm{A}, \mathrm{n})$, Theorem 1.1

So, # non-X elements in A $>$ # non-X elements in R(A)

Lamma3 proved!

Lamma4:

A satisfies the precondition, A's majority is X

$\Longrightarrow$ # group of X $<$ # X(we use the same definition of group in Lamma1)

Proof:

Assume A satisfies the precondition, A's majority is X.

Set $m = \mathrm{len}(A)$, $k = \#X$ in A, $p = \#$group of X.

Assume for contradiction: $p \geq k$.

Think A as a circular, since first and last element may form a pair.

So we get p gaps among p groups of X. There are at leat 1 non-X element in a gap, otherwise, to groups of X are adjacent, then they are 1 group.

So, there are at least p non-X elements in A.

By $p \geq k$, # non-X elements $\geq$ # X. This contradict to the fact that X is majority of A. So, $p < k$.

Lamma4 is proved!

Lamma5:

A is a non-empty array of natural number, $X \in A$

$\Longrightarrow$Count(A,X) = Pairs(A,X) $\Leftrightarrow$ A is identical of X.

Proof:

Assume A is a non-empty array of natural number, $X \in A$

Think A as a circular, since first and last element may form a pair.

WTS: Count(A,X) = Pairs(A,X) $\Longrightarrow$ A is identical of X.

Assume for contradiction:

Count(A,X) = Pairs(A,X) $\wedge$ $\exists b \in A \wedge b \neq X$

Set Count(A,X) = n, len(A) = m.

Each X form a pair with X right adjacent to it. There are at least one X, whose right element is b. So, at most we can get n – 1 pairs of X.

Pairs(A,X) = n – 1, this violate the fact that Count(A,X) = Pairs(A,X). So, there can be such a b in A. So, Count(A,X) = Pairs(A,X) $\Longrightarrow$ A is identical of X.

WTS: A is identical of X $\Longrightarrow$ Count(A,X) = Pairs(A,X)

Assume A is identical of X.

Set len(A) = Count(A,X) = n by assumption.

There are n X in A. Each X form a pair with X right adjacent to it. We get n pairs of X(A is circular). Pairs(A,X) = n. So, Count(A,X) = Pairs(A,X)

Lamma5 is proved!

Lamma6:

A is a non-empty array of natural number, $X \in A \Longrightarrow Count(A,X) \geq Pairs(A,X)$

Proof:

Assume A is a non-empty array of natural number, $X \in A$.

Think A as a circular, since first and last element may form a pair.

WTS: $Count(A,X) \geq Pairs(A,X)$

Set $k = Count(A,X)$, $m = len(A)$, $p = \#$ groups of X in A

$k \geq 1$, since $X \in A$

Case1: for all groups of X, there is only 1 X in the group.

We get 0 pairs of X.

$Pairs(A,X)=0\leq1\leq k=Count(A,X)$, Lamma6 holds.

Case2: exist at lest 1 group of X contains more than 1 X, A is not identical.

$p > 1$(A is not identical)

Set there are g groups that contains more than 1 X, $g\geq1$.

For any group that contains more than 1 X:

If it contains h X's, $h>1$. It has h -1 pairs in the group.

So, in total, we get $k - g$ pairs of X in A.

$Pairs(A,X) = k - g$, $Count(A,X) = k$

$Count(A,X) \geq Pairs(A,X)$, Lamma6 holds.

Case3: A is identical of X.

Each X can form a pair with the X right adjacent to it. We get k pairs of X in A(A is circular).

Pairs(A,X) = k, Count(A,X) = k

Count(A,X) $\geq$ Pairs(A,X), Lamma6 holds.

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