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1. Let $G = (V, E)$. Let $\mathcal{F}' = (G, s, t, c')$ be a be a flow network such that for any $(u, v) \in E$,

$$c'(u, v) = |E| \cdot c(u, v) + 1$$

**Claim 1:** Any minimum cut in $\mathcal{F}'$ is a minimum cut in $\mathcal{F}$.

We will prove this by contradiction. Let $(S, T)$ be a min-cut in $\mathcal{F}'$ and let $X$ be the set of edges that are crossings of this cut. Then

$$cap(S, T) = \sum_{(u,v)\in X} c'(u, v) = \sum_{(u,v)\in X} |E| \cdot c(u, v) + 1 = |X| + |E| \sum_{(u,v)\in X} c(u, v)$$

Assume that $(S, T)$ is not a min-cut of $\mathcal{F}$. Then there exist a min-cut $(S', T')$ of $\mathcal{F}$ such that $cap(S', T') < cap(S, T)$. Let $X'$ be the set of edges that are crossings of this cut. Then

$$\begin{aligned}
cap(S', T') &= \sum_{(u,v)\in X'} c'(u, v) \\
&= |X'| + |E| \sum_{(u,v)\in X'} c(u, v) \\
&\leq |E| + |E| \sum_{(u,v)\in X'} c(u, v) \\
&\leq |E| + |E|([\sum_{(u,v)\in X} c(u, v)] - 1) \\
&\quad \text{(since } X \text{ is not a min-cut of } \mathcal{F} \text{ and all cuts have integral capacities)} \\
&= |E| \sum_{(u,v)\in X} c(u, v) \\
&\leq |X| + |E| \sum_{(u,v)\in X} c(u, v) = cap(S, T) && \text{(contradiction!)}
\end{aligned}$$

Therefore $(S, T)$ is a min-cut in $\mathcal{F}$.

**Claim 2:** $(S, T)$ is also a cut with the smallest number of crossings in $\mathcal{F}$.

Assume that $(S, T)$ is not the cut with smallest number of crossings. Then there exist cut $(S', T')$ such that has fewer crossings than $(S, T)$. Let $X'$ be the set of edges that are crossings of this cut. Then

$$\begin{aligned}
cap(S', T') = \sum_{(u,v)\in X'} c'(u, v) &= |X'| + |E| \sum_{(u,v)\in X'} c(u, v) \\
&\leq |X'| + |E| \sum_{(u,v)\in X} c(u, v) \\
&< |X| + |E| \sum_{(u,v)\in X} c(u, v) = cap(S, T) && \text{(contradiction!)}
\end{aligned}$$

Therefore $(S, T)$ is a minimum cut with the smallest number of crossings in $\mathcal{F}$. To find such cut, we simply apply Ford-Fulkerson algorithm on $\mathcal{F}'$.