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In our algorithm, we modify the original graph G to new graph G' . In the original graph, for each node v (except s and t), if v has more than one income paths and more than one outcome paths, then we modify it. For such v , we add a new node v' , a new edge (v, v') , move all outcome path of v to v' . In a other word, v' inherited all outcome paths from v , and v lose all original outcome paths and have only one outcome path (v, v') . Then we do Edge-Disjoint-Paths algorithm in lecture slide on new graph G' and get max set of path. Finally, for each path in the set, return to it's original shape in G , and return the set.

Pseudocode:

```
01 Node-Disjoint-Paths (G, s, t):
02     G' = Modify(G, s, t)
03     P = Edge-Disjoint-Paths(G', s, t) (exactly the same as in lecture slide 14, page 20 ~ 21)
04     P' = Restore(P)
05     return P'
06
07 Modify(G=(V,E), s, t)          O(nm)
08     G' = G
09     for all node v not s or t in V':    O(n)
10         in = 0
11         out = 0
12         outset = {}
13         loop through all edge e in E':    O(m)
14             if e is income edge to v:
15                 in ++
16             if e is outcome edge to v:
17                 out ++
18                 outset add e
19         if in > 1 and out > 1:
20             V' add v'
21             E' add (v, v')
22             for all edges in outset, set the start vertex to v'
23         return G'
24
25 Restore(P)
26     P' = P
27     for all path p in P':
28         loop through p, for all edge looks like a -> v -> v' -> b, reduce it to a -> v -> b
29         (key is to find node who has apostrophe in the name)
30     return P'
```

Time complexity: $O(nm)$

Modify takes $O(nm)$ time, reason illustrated in pseudocode. After Modify the number of nodes and edges are less than the double of origin. Edge-Disjoint-Paths takes $O(4nm) = O(nm)$, reason illustrate in lecture slide. Restore takes $O(2m) = O(m)$ because it will check each edge at most once. In total $O(nm)$

Correctness

We can assume Edge-Disjoint-Paths is correct. For every nodes(except s , t) who has more than one income edge and outcome edge, it can be in more than one edge-disjoint path, but only one node-disjoint path. After the modification, there is edge (v, v') , so for all path pass v , it must path (v, v') . So, there can be only one edge-disjoint path, same as node-disjoint path. For other nodes whose number of income path or outcome path is 1, the number of edge-disjoint path and node-disjoint path pass it is 1, the same.

Therefore the number of elements in max edge-disjoint path set of G' is the same as number of elements in max node-disjoint path set of G . Hence correct.