CSC165H1F Problem Set 4

December 4, 2019

1. (a) *Proof.* Let $n \in \mathbb{N}$. Let $x \in \mathcal{I}_{has\_odd,n}$. By definition of input $\mathcal{I}_{has\_odd}$, x is an arbitrary list of length n containing $0, 1, 2, 3, 4$ only.

   We want to show this function's worst case has an upper bound $U(n) = n + 1$ basic operations.

   Line 3-4 is a basic operation contained in the loop defined in line 2-4. Because the loop in line 2-4 runs at most n times, the loop takes at most n basic operations.

   In addition, the return statement takes 1 basic operation.

   Therefore, the function takes at most $n + 1$ basic operations and $U(n) = n + 1$ is an upper bound for its worst case, and we have $WC_{has\_odd}(n) \in \mathcal{O}(n)$.

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   (b) *Proof.* Let $n \in \mathbb{N}$. Let x be a list of length n containing only 0. This x is possible to satisfy the requirement that $x \in \mathcal{I}_{has\_odd,n}$.

   We want to show this function's worst case has an lower bound $L(n) = n + 1$ basic operations.

   The loop in line 2-4 runs at least n times, while each time it loops it takes 1 basic operation. Therefore, the loop takes at least n basic operations.

   The return statement in line 5 takes 1 basic operations.

   Therefore, the function takes at least $n + 1$ basic operations in its worst case and $L(n) = n + 1$ is an lower bound for its worst case.

   Then we have $WC_{has\_odd}(n) \in \Omega(n)$.

   In conclusion, since $WC_{has\_odd}(n) \in \mathcal{O}(n)$ as proved in 1(a) and $WC_{has\_odd}(n) \in \Omega(n)$, we have $WC_{has\_odd}(n) \in \Theta(n)$.

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(c) *Proof.* The average run time for *has_odd* is the following expression.

$$Avg_{has\_odd}(n) = \frac{1}{|\mathcal{I}_{has\_odd,n}|} \sum_{number\_list \in \mathcal{I}_{has\_odd,n}} \text{running time of } has\_odd(number\_list) \tag{1}$$

Let $n = 2$. Because the function *has_odd* only considers the parity of each element in its *number_list* input list, we can assume $\mathcal{I}_{has\_odd,n}$ be the set of all sequence only containing $0, 1, 2, 3, 4$ with length n without loss of generality.

Therefore, the set of all input $\mathcal{I}_{has\_odd,2}$ is

$$\begin{gathered} \{0,0\}, \{0,1\}, \{0,2\}, \{0,3\}, \{0,4\}, \\ \{1,0\}, \{1,1\}, \{1,2\}, \{1,3\}, \{1,4\}, \\ \{2,0\}, \{2,1\}, \{2,2\}, \{2,3\}, \{2,4\}, \\ \{3,0\}, \{3,1\}, \{3,2\}, \{3,3\}, \{3,4\}, \\ \{4,0\}, \{4,1\}, \{4,2\}, \{4,3\}, \{4,4\} \end{gathered} \tag{2}$$

Therefore, the size of input $|\mathcal{I}_{has\_odd,2}|$ is 25.

For each $number\_list \in \mathcal{I}_{has\_odd,n}$, we have their running time

$$\begin{aligned} \{0,0\}, \{0,2\}, \{0,4\}, \{2,0\}, \{2,2\}, \{2,4\}, \{4,0\}, \{4,2\}, \{4,4\} &: 3 \text{ steps} \\ \{0,1\}, \{0,3\}, \{2,1\}, \{2,3\}, \{4,1\}, \{4,3\} &: 2 \text{ steps} \\ \{1,0\}, \{1,1\}, \{1,2\}, \{1,3\}, \{1,4\}, \{3,0\}, \{3,1\}, \{3,2\}, \{3,3\}, \{3,4\} &: 1 \text{ steps} \end{aligned} \tag{3}$$

Therefore, the average steps for all inputs of length 2 is

$$Avg_{has\_odd}(n) = \frac{1}{25}(3 \cdot 9 + 2 \cdot 6 + 1 \cdot 10) = \frac{27 + 12 + 10}{25} = \frac{49}{25} \tag{4}$$

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(d) *Proof.* The average run time for *has_odd* is the following expression.

$$Avg_{has\_odd}(n) = \frac{1}{|\mathcal{I}_{has\_odd,n}|} \sum_{number\_list \in \mathcal{I}_{has\_odd,n}} \text{running time of } has\_odd(number\_list) \tag{5}$$

We assume $\mathcal{I}_{has\_odd,n}$ be the set of all sequence only containing $0, 1, 2, 3, 4$ with length n.

Therefore, the size of $\mathcal{I}_{has\_odd,n}$ is 5^n.

$$Avg_{has\_odd}(n) = \frac{1}{5^n} \sum_{number\_list \in \mathcal{I}_{has\_odd,n}} \text{running time of } has\_odd(number\_list) \tag{6}$$

Because the function exits immediately after it encounters an 1 in its loop, we have

$$Avg_{has\_odd}(n) = \frac{1}{5^n} \sum_{i=0}^{n-1} \sum_{\substack{number\_list \in \mathcal{I}_{has\_odd,n} \\ \text{first 1 appears on index i}}} \text{running time of } has\_odd(number\_list) \tag{7}$$

The running time of *has_odd* is $i+1$ if the i is the first index where 1 or 3 appears for all inputs containing 1 or 3, plus the runtime that list of $0, 2, 4$ only which have running time of $n+1$ for each, then we have

$$Avg_{has\_odd}(n) = \frac{1}{5^n}\left(\left(\sum_{i=0}^{n-1}\sum_{\substack{number\_list \in \mathcal{I}_{has\_odd,n} \\ \text{first 1 appears on index i}}} i+1\right) + (n+1)\cdot \text{Number of list contains only 0, 2, 4}\right) \tag{8}$$

For inner summation, we have $2\cdot 3^i \cdot 5^{n-i-1}$ lists with first 1 or 3 appears in index i. Therefore, it can be rewrite as

$$Avg_{has\_odd}(n) = \frac{1}{5^n}\left(\left(\sum_{i=0}^{n-1} 2\cdot 3^i\cdot 5^{n-i-1}\cdot (i+1)\right) + 3^n\cdot(n+1)\right) \tag{9}$$

Therefore, the final expression for average running time of function *has_odd* is

$$\begin{aligned}
Avg_{has_odd}(n) =& \frac{1}{5^n}\left(\left(\sum_{i=0}^{n-1} 2\cdot 3^i\cdot 5^{n-i-1}\cdot (i+1)\right) + 3^n\cdot(n+1)\right)\\
Avg_{has_odd}(n) =& \frac{1}{5^n}\left(2\cdot 5^{n-1}\left(\sum_{i=0}^{n-1} 3^i\cdot 5^{-i}\cdot (i+1)\right) + 3^n\cdot(n+1)\right)\\
Avg_{has_odd}(n) =& \frac{1}{5^n}\left(2\cdot 5^{n-1}\left(\sum_{i=0}^{n-1} \frac{3}{5}^i\cdot (i+1)\right) + 3^n\cdot(n+1)\right)\\
Avg_{has_odd}(n) =& \frac{1}{5^n}\left(2\cdot 5^{n-1}\left(\sum_{i=0}^{n-1} \frac{3}{5}^i\cdot i + \sum_{i=0}^{n-1}\frac{3}{5}^i\right) + 3^n\cdot(n+1)\right)\\
Avg_{has_odd}(n) =& \frac{1}{5^n}\left(2\cdot 5^{n-1}\left(\frac{n\frac{3}{5}^n}{\frac{3}{5}-1} + \frac{\frac{3}{5}-\frac{3}{5}^{n+1}}{(\frac{3}{5}-1)^2} + \frac{1-\frac{3}{5}^n}{1-\frac{3}{5}}\right) + 3^n\cdot(n+1)\right)\\
Avg_{has_odd}(n) =& \frac{2}{5}\left(\frac{n\frac{3}{5}^n}{\frac{3}{5}-1} + \frac{\frac{3}{5}-\frac{3}{5}^{n+1}}{(\frac{3}{5}-1)^2} + \frac{\frac{3}{5}^n-1}{\frac{3}{5}-1}\right) + \frac{3}{5}^n\cdot(n+1)\\
Avg_{has_odd}(n) =& \frac{2}{5}\left(\frac{(n+1)\frac{3}{5}^n-1}{-\frac{2}{5}} + \frac{\frac{3}{5}-\frac{3}{5}^{n+1}}{\frac{4}{25}}\right) + \frac{3}{5}^n\cdot(n+1)\\
Avg_{has_odd}(n) =& -\left((n+1)\frac{3}{5}^n-1\right) + \frac{5}{2}\left(\frac{3}{5}-\frac{3}{5}^{n+1}\right) + \frac{3}{5}^n\cdot(n+1)\\
Avg_{has_odd}(n) =& -(n+1)\frac{3}{5}^n + 1 + \frac{3}{2} - \frac{5}{2}\cdot\frac{3}{5}^{n+1} + \frac{3}{5}^n\cdot(n+1)\\
Avg_{has_odd}(n) =& \frac{5}{2} - \frac{5}{2}\cdot\left(\frac{3}{5}\right)^{n+1}
\end{aligned} \tag{10}$$

Therefore, the average steps for all inputs of length n is $\frac{5}{2}-\frac{5}{2}\cdot\left(\frac{3}{5}\right)^{n+1}$.

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2. (a) *Proof.* Let $G = (V, E)$. Let C be a cycle $v_0, v_1, ...v_k$ where $v_0 = v_k$. Let e be an edge in C.
We want to show removing e leave C connected.

Let v_i, v_j be arbitrary distinct vertices within the cycle C.

Due to the fact that v_i, v_j is in cycle C, there exist a path P_1 from v_i to v_j, which is $v_i, v_{i+1}, ..., v_{j-1}, v_j$ and there exist another path P_2 from v_i to v_j, which is $v_i, v_{i-1}, ..., v_1, v_0(v_k), v_{k-1}, ..., v_{j+1}, v_j$.

Therefore, there are 2 distinct paths between v_i, v_j.

Because P_1 and P_2 do not contain duplicate vertices except v_i, v_j, they contains distinct edges.

Therefore, removing an edge e on the cycle C at most destroy one path within P_1 and P_2.

Since at least one path between v_i, v_j is preserved, v_i, v_j is still connected.

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(b) *Proof.* Let $G = (V, E)$. Assume for all $v_0 \in V$, $deg(v_0) \geq \lfloor \frac{|V|}{3} \rfloor$.
We want to show for all distinct $u, v, w \in V$, there exists a path of length no more than 2 from u to v, or from v to w, or from w to u.
Let $u, v, w \in V$.

We divide our proof into 2 cases.

Case 1. Assume there exists a path of length less or equal to 2 from u to v OR from u to w.
Then we have shown there exists a path of length no more than 2 from u to v, or from v to w, or from w to u for this case.

Case 2. Assume there does not exist a path of length less or equal to 2 from u to v AND from u to w.
Then we want to show there must exists a path of length no more than 2 from v to w for this case.

We further divide our proof into 2 sub cases.

Case 2.1. Assume v and w is adjacent, then we have shown there exists a path of length no more than 2 from v to w for this case.

Case 2.2. Assume v and w is NOT adjacent.

Let V' be the set of vertices which is adjacent to u. By the assumption that $deg(u) \geq \lfloor \frac{|V|}{3} \rfloor$, $|V'| \geq \lfloor \frac{|V|}{3} \rfloor$.

Since v, w is not adjacent to all vertices in V' and u, and v is not adjacent to w, we know that maximum number of vertices v and w can adjacent to is total number of vertices minus 1 (which is u) and minus 2 (which is v and w), and minus $\lfloor \frac{|V|}{3} \rfloor$ (which is all vertices adjacent to u).

In equation, we have the maximum number of vertices v and w can adjacent to is $|V| - 1 - 2 - \lfloor \frac{|V|}{3} \rfloor$.

Due to $deg(v) \geq \lfloor \frac{|V|}{3} \rfloor$ and $deg(w) \geq \lfloor \frac{|V|}{3} \rfloor$, v and w must be adjacent to at least $2 \cdot \lfloor \frac{|V|}{3} \rfloor$ vertices in total, and since the vertices they adjacent to must be selected from at most $|V| - 1 - 2 - \lfloor \frac{|V|}{3} \rfloor$ vertices, which satisfy $2 \cdot \lfloor \frac{|V|}{3} \rfloor > |V| - 1 - 2 - \lfloor \frac{|V|}{3} \rfloor$.

Therefore, there exist at least 1 vertice x such that it is adjacent to both v and w. Let path P be the path containing vertice v, x, w, because v, x, w are distinct vertices and v is adjacent to x and x is adjacent to w, and the length of path P is 2, we have shown there exists a path of length no more than 2 from v to w for this case.

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(c) *Proof.* Let $G = (V, E)$.
Assume it has an odd number of vertices with even degree.
Let V' be the set of vertices that has even degree.
By assumption, we know that $|V'|$ is odd.
We want to show $|V|$ is odd.

Let $V_0 = V \setminus V'$. Let E' be the set of edges connecting vertices within V'.
Let E'_0 be the set of edges between vertices in V' and V_0.
Let E_0 be the set of edges connecting vertices within V_0.

Let $e = (u, v)$ be arbitrary edge in E'.

Because $u, v \in V'$, the presence of e occupies 1 degree for u and 1 degree for v, thus it occupies 2 degrees within V' in total. Therefore, the number of edges between V' and V_0, $|E'_0|$ is equal to the sum of degrees of vertices within V' minus total number of degrees used within V', which is 2 times the number of edges within the V'.

Since the sum of degrees of vertices within V' is even due to V' is the set of vertices that all has even degree, we conclude that $|E'_0|$ is also even.

Let $e_0 = (u_0, v_0)$ be arbitrary edge in E_0.

Because $u_0, v_0 \in V_0$, the presence of e_0 occupies 1 degree for u_0 and 1 degree for v_0, thus it occupies 2 degrees within V_0 in total. Therefore, the number of edges between V' and V_0, $|E'_0|$ is also equal to the sum of degrees of vertices within V_0 minus total number of degrees used within V_0, which is 2 times the number of edges within the V_0.

Since $|E'_0|$ is also even as we proved before, we conclude that the sum of degrees of vertices within V_0 is also even.

Because all vertices in V_0 must have odd degree by assumption (vertices with even degree already belongs to V'), and the sum of degrees of vertices within V_0 is even, we conclude number of vertices in V_0 can only be even, because the sum of odd numbers of odd number is also odd while the sum of even numbers of odd number is even.

Since V' and V_0 is a partition of V, $|V| = |V'| + |V_0|$, which is an odd number plus an even number, hence the result is odd.
Therefore, we have shown $|V|$ is odd.

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(d) *Proof.* We want to prove $\forall G = (V, E), |V| \geq 13 \wedge (\forall v \in V, deg(v) \geq |V| - 7) \implies G$ is connected.

Let $P(n)$ be the statement: $\forall G = (V, E), |V| = n \wedge (\forall v \in V, deg(v) \geq |V| - 7) \implies G$ is connected.

We want to prove $\forall n \in \mathbb{N}, b \geq 13 \implies P(n)$ by induction.

Base Case. Let $n = 13$. Let $G = (V, E)$ where $|V| = 13$. Assume $\forall v \in V, deg(v) \geq |V| - 7$. We want to show G is connected.

Let $u, v \in V$. We divide our proof into two cases.

Case 1. Assume u, v are adjacent.
Then, u, v is connected.

Case 2. Assume u, v are not adjacent.
Then, of all vertices u are adjacent to must not be v.
For the same reason, of all vertices v are adjacent to must not be u.
In addition, u must not be adjacent to itself and v must not be adjacent to itself.
Therefore, both u, v are not adjacent to either u, v.

Because of the assumption $\forall v \in V, deg(v) \geq |V| - 7$, $deg(u) \geq 6$ and $deg(v) \geq 6$.
u, v are adjacent to totally at least 12 vertices other than u, v, but there are only $13 - 2 = 11$ vertices which is not either u, v.

In conclusion, u, v must be both adjacent to at least 1 vertices.
By transistivity of connectedness, u, v are connected and the graph G is connected.
We verified the base case $P(13)$ is true.

Inductive Steps. Let $k \in \mathbb{N}$. Assume $k \geq 13$. Assume $P(k)$ is true. We want to show that $P(k+1)$ is also true.

Let $G = (V, E)$ where $|V| = k + 1$. Assume $\forall v \in V, deg(v) \geq |V| - 7$, that is $\forall v \in V, deg(v) \geq k + 1 - 7$. We want to show G is connected.

Let $w \in V$. We remove w from G and obtained a new graph $G' = (V', E')$, which have $|V| - 1$ vertices.

Because we only remove 1 vertice, the degree of G' decrease at most by 1.

Therefore, by assumption $\forall v \in V, deg(v) \geq k + 1 - 7$, we have $\forall v \in V', deg(v) \geq k + 1 - 7 - 1$, equivalently $\forall v \in V', deg(v) \geq |V'| - 7$. Together with the fact $|V'| = k$, we have G' is connected.

Because G' is connected, we know the any two vertices not equal to w in G is connected.

Because $\forall v \in V, deg(v) \geq k+1-7$ and $w \in V$, there exists an vertice w' is adjacent to w.

Because w' is connected to all other vertices, and w is connected to w' due to they are adjacent, by transistivity of connectedness, w is connected to all other vertices in G. Finally, G is also connected.

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(e) *Proof.* We want to prove $\forall G = (V, E), \forall v \in V, deg(v) \geq 5 \implies G$ has a cycle by contradiction.

Assume $\forall G = (V, E), \forall v \in V, deg(v) \geq 5 \wedge G$ does not have a cycle.

Then, for each connected component of G, the component must be a tree.

However, all tree have at least 1 node whose degree is exactly 1.

By assumption, because all vertices in the connected component also belongs to graph G, their degree must be at least 5. Contradiction!

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(f) *Proof.* Let $G = (V, E)$. Assume $\forall v \in V, deg(v) \geq 4$. Let $u \in V$. We divide our proof into 5 steps, where each step except the first one relies on the result from previous ones.

Step 1. We want to show there are at least 1 path of length 0 starting from u.
The path $P_0 = u$ is a path of length 0 and starts from u.

Step 2. We want to show there are at least 4 path of length 1 starting from u.
Because $deg(u) \geq 4$, u is adjacent to at least 4 vertices other than itself.
Let u_1 be an arbitrary vertice that are adjacent to u.
There are at least 4 paths $P_1 = u, u_1$, which starts at u and have length of 1.

Step 3. We want to show there are at least 12 path of length 2 starting from u.
Because $deg(u_1) \geq 4$, u_1 is adjacent to at least 4 vertices other than itself.
But in P_1, we already have u.
There are at least 3 vertices which does not duplicate with u and adjacent to u_1.

Let u_2 be an arbitrary vertice that are adjacent to u_1 and is not u.
There are at least 3 vertices we can append to path P_1 to form a new, distinct path P_2.
Therefore, there are at least $3 \times 4 = 12$ paths $P_2 = u, u_1, u_2$, which starts at u and have length of 2.

Step 4. We want to show there are at least 24 path of length 3 starting from u.
Because $deg(u_2) \geq 4$, u_2 is adjacent to at least 4 vertices other than itself.
But in P_2, we already have u, u_1.
There are at least 2 vertices which does not duplicate with u, u_1 and adjacent to u_2.
Let u_3 be an arbitrary vertice that are adjacent to u_2 and is not u, u_1.
There are at least 2 vertices we can append to path P_2 to form a new, distinct path P_3.
Therefore, there are at least $12 \times 2 = 24$ paths $P_3 = u, u_1, u_2, u_3$, which starts at u and have length of 3.

Step 5. We want to show there are at least 24 path of length 4 starting from u.
Because $deg(u_3) \geq 4$, u_3 is adjacent to at least 4 vertices other than itself.
But in P_3, we already have u, u_1, u_2.
There are at least 1 vertices which does not duplicate with u, u_1, u_2 and adjacent to u_3.
Let u_4 be an arbitrary vertice that are adjacent to u_3 and is not u, u_1, u_2.
There are at least 1 vertices we can append to path P_3 to form a new, distinct path P_4.
Therefore, there are at least $24 \times 1 = 24$ paths $P_4 = u, u_1, u_2, u_3, u_4$, which starts at u and have length of 4.

Adding up all possible paths of length $0, 1, 2, 3, 4$, we have at least $1+4+12+24+24 = 65$ paths.

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