(4th Edition)

```
DIJKSTRA(G, w, s)
1  INITIALIZE-SINGLE-SOURCE(G, s)
2  S = ∅
3  Q = ∅
4  for each vertex u ∈ G.V
5      INSERT(Q, u)
6  while Q ≠ ∅
7      u = EXTRACT-MIN(Q)
8      S = S ∪ {u}
9      for each vertex v in G.Adj[u]
10         RELAX(u, v, w)
11         if the call of RELAX decreased v.d
12             DECREASE-KEY(Q, v, v.d)
```

Exercise 22.3-3

22.3-3

Suppose that you change line 6 of Dijkstra's algorithm to read

6 **while** $|Q| > 1$

This change causes the **while** loop to execute $|V| - 1$ times instead of $|V|$ times. Is this proposed algorithm correct?

Sol)

Yes, the algorithm still works. Let u be the leftover vertex that does not get extracted from the priority queue Q. If u is not reachable from s, then $u.d = \delta(s, u) = \infty$. If u is reachable from s, then there is a shortest path $p = s \leadsto x \to u$. When the vertex x was extracted, $x.d = \delta(s, x)$ and then the edge (x, u) was relaxed; thus, $u.d = \delta(s, u)$.

Exercise 22.3-4

22.3-4

Modify the DIJKSTRA procedure so that the priority queue Q is more like the queue in the BFS procedure in that it contains only vertices that have been reached from source s so far: $Q \subseteq V - S$ and $v \in Q$ implies $v.d \neq \infty$.

Sol)

```
DIJKSTRA(G, w, s)
  INITIALIZE-SINGLE-SOURCE(G, s)
  S = ∅
  Q = ∅
  INSERT(Q, s)
  while Q ≠ ∅
      u = EXTRACT-MIN(Q)
      S = S ∪ {u}
      for each vertex v in G.Adj[u]
          if v.d == ∞
              INSERT(Q, v)
          RELAX(u, v, w)
          if the call of RELAX decreased v.d
              DECREASE-KEY(Q, v, v.d)
```

Exercise 22.3-11

22.3-11

Suppose that you are given a weighted, directed graph $G = (V, E)$ in which edges that leave the source vertex s may have negative weights, all other edge weights are nonnegative, and there are no negative-weight cycles. Argue that Dijkstra's algorithm correctly finds shortest paths from s in this graph.

Sol)

Two of the properties that the proof of Theorem 22.6 relies on are that after the first iteration of the **while** loop, set S is nonempty and that $\delta(s, y) \leq \delta(s, u)$, where u is a vertex in $V - S$ with the smallest d value and y is the first vertex on a shortest path from s to u that is in $V - S$. Once the source s has been placed into set S and the edges leaving s—which may have negative weights—have been relaxed, all edges between vertices in $V - S$ have nonnegative weights, so that the key inequality $\delta(s, y) \leq \delta(s, u)$ still holds and the proof goes through.

Exercise 24.1-1

24.1-1

Show that splitting an edge in a flow network yields an equivalent network. More formally, suppose that flow network G contains edge (u, v), and define a new flow network G' by creating a new vertex x and replacing (u, v) by new edges (u, x) and (x, v) with $c(u, x) = c(x, v) = c(u, v)$. Show that a maximum flow in G' has the same value as a maximum flow in G.

Sol)

We will prove that for every flow in $G = (V, E)$, we can construct a flow in $G' = (V', E')$ that has the same value as that of the flow in G. The required result follows since a maximum flow in G is also a flow. Let f be a flow in G. By construction, $V' = V \cup \{x\}$ and $E' = (E - \{(u, v)\}) \cup \{(u, x), (x, v)\}$. Construct f' in G' as follows:

$$f'(y, z) = \begin{cases} f(y, z) & \text{if } (y, z) \neq (u, x) \text{ and } (y, z) \neq (x, v) , \\ f(u, v) & \text{if } (y, z) = (u, x) \text{ or } (y, z) = (x, v) . \end{cases}$$

Informally, f' is the same as f, except that the flow $f(u, v)$ now passes through an intermediate vertex x. The vertex x has incoming flow (if any) only from u, and has outgoing flow (if any) only to vertex v.

We first prove that f' satisfies the required properties of a flow. It is obvious that the capacity constraint is satisfied for every edge in $E' - \{(u, x), (x, v)\}$ and that every vertex in $V' - \{u, v, x\}$ obeys flow conservation.

To show that edges (u, x) and (x, v) obey the capacity constraint, we have

$$\begin{aligned} f(u, x) &= f(u, v) \leq c(u, v) = c(u, x) , \\ f(x, v) &= f(u, v) \leq c(u, v) = c(x, v) . \end{aligned}$$

We now prove flow conservation for u. Assuming that $u \notin \{s, t\}$, we have

$$\begin{aligned} \sum_{y \in V'} f'(u, y) &= \sum_{y \in V' - \{x\}} f'(u, y) + f'(u, x) \\ &= \sum_{y \in V - \{v\}} f(u, y) + f(u, v) \\ &= \sum_{y \in V} f(u, y) \\ &= \sum_{y \in V} f(y, u) \qquad \text{(because } f \text{ obeys flow conservation)} \\ &= \sum_{y \in V'} f'(y, u) . \end{aligned}$$

For vertex v, a symmetric argument proves flow conservation.

For vertex x, we have

$$\begin{aligned}
\sum_{y \in V'} f'(y,x) &= f'(u,x) \\
&= f'(x,v) \\
&= \sum_{y \in V'} f'(x,y) .
\end{aligned}$$

Thus, f' is a valid flow in G'.

We now prove that the values of the flow in both cases are equal. If the source s is neither u nor v, the proof is trivial, since our construction assigns the same flows to incoming and outgoing edges of s. If $s = u$, then

$$\begin{aligned}
|f'| &= \sum_{y \in V'} f'(u,y) - \sum_{y \in V'} f'(y,u) \\
&= \sum_{y \in V' - \{x\}} f'(u,y) - \sum_{y \in V'} f'(y,u) + f'(u,x) \\
&= \sum_{y \in V - \{v\}} f(u,y) - \sum_{y \in V} f(y,u) + f(u,v) \\
&= \sum_{y \in V} f(u,y) - \sum_{y \in V} f(y,u) \\
&= |f| .
\end{aligned}$$

The case when $s = v$ is symmetric. We conclude that f' is a valid flow in G' with $|f'| = |f|$.

Exercise 24.1-4

24.1-4

Let f be a flow in a network, and let α be a real number. The ***scalar flow product***, denoted αf, is a function from $V \times V$ to $\mathbb{R}$ defined by

$$(\alpha f)(u,v) = \alpha \cdot f(u,v) .$$

Prove that the flows in a network form a ***convex set***. That is, show that if f_1 and f_2 are flows, then so is $\alpha f_1 + (1 - \alpha) f_2$ for all α in the range $0 \le \alpha \le 1$.

Sol)

To see that the flows form a convex set, we show that if f_1 and f_2 are flows, then so is $\alpha f_1 + (1-\alpha) f_2$ for all α such that $0 \le \alpha \le 1$.

For the capacity constraint, first observe that $\alpha \le 1$ implies that $1 - \alpha \ge 0$. Thus, for any $u, v \in V$, we have

$$\begin{aligned}
\alpha f_1(u,v) + (1-\alpha) f_2(u,v) &\ge 0 \cdot f_1(u,v) + 0 \cdot (1-\alpha) f_2(u,v) \\
&= 0 .
\end{aligned}$$

Since $f_1(u,v) \le c(u,v)$ and $f_2(u,v) \le c(u,v)$, we also have

$$\begin{aligned}
\alpha f_1(u,v) + (1-\alpha) f_2(u,v) &\le \alpha c(u,v) + (1-\alpha) c(u,v) \\
&= (\alpha + (1-\alpha)) c(u,v) \\
&= c(u,v) .
\end{aligned}$$

For flow conservation, observe that since f_1 and f_2 obey flow conservation, we have $\sum_{v \in V} f_1(v,u) = \sum_{v \in V} f_1(u,v)$ and $\sum_{v \in V} f_1(v,u) = \sum_{v \in V} f_1(u,v)$ for any $u \in V - \{s,t\}$. We need to show that

$$\sum_{v \in V} (\alpha f_1(v,u) + (1-\alpha) f_2(v,u)) = \sum_{v \in V} (\alpha f_1(u,v) + (1-\alpha) f_2(u,v))$$

for any $u \in V - \{s,t\}$. We multiply both sides of the equality for f_1 by α, multiply both sides of the equality for f_2 by $1-\alpha$, and add the left-hand and right-hand sides of the resulting equalities to get

$$\alpha \sum_{v \in V} f_1(v,u) + (1-\alpha) \sum_{v \in V} f_2(v,u) = \alpha \sum_{v \in V} f_1(u,v) + (1-\alpha) \sum_{v \in V} f_2(u,v) .$$

Observing that

$$\begin{aligned}
\alpha \sum_{v \in V} f_1(v,u) + (1-\alpha) \sum_{v \in V} f_2(v,u) &= \sum_{v \in V} \alpha f_1(v,u) + \sum_{v \in V} (1-\alpha) f_2(v,u) \\
&= \sum_{v \in V} (\alpha f_1(v,u) + (1-\alpha) f_2(v,u))
\end{aligned}$$

and, likewise, that

$$\alpha \sum_{v \in V} f_1(u,v) + (1-\alpha) \sum_{v \in V} f_2(u,v) = \sum_{v \in V} (\alpha f_1(u,v) + (1-\alpha) f_2(u,v))$$

completes the proof that flow conservation holds, and thus that flows form a convex set.

Exercise 24.1-6

24.1-6

Professor Adam has two children who, unfortunately, dislike each other. The problem is so severe that not only do they refuse to walk to school together, but in fact each one refuses to walk on any block that the other child has stepped on that day. The children have no problem with their paths crossing at a corner. Fortunately both the professor's house and the school are on corners, but beyond that he is not sure if it is going to be possible to send both of his children to the same school. The professor has a map of his town. Show how to formulate the problem of determining whether both his children can go to the same school as a maximum-flow problem.

Sol)

Create a vertex for each corner, and if there is a street between corners u and v, create directed edges (u, v), (v, v_u), and (v_u, u), where v_u is a unique vertex created for only this street between corners u and v. (We need vertex v_u to avoid antiparallel edges. Note that if there is a street between corners u and v and between corners x and v, then the vertices v_u and v_x are distinct.) Set the capacity of each edge to 1. Let the source be the corner on which the professor's house sits, and let the sink be the corner on which the school is located. We wish to find a flow of value 2 that also has the property that $f(u, v)$ is an integer for all vertices u and v. Such a flow represents two edge-disjoint paths from the house to the school.