Exercise 4.3-7

4.3-7

Using the master method in Section 4.5, you can show that the solution to the recurrence $T(n) = 4T(n/3) + n$ is $T(n) = \Theta(n^{\log_3 4})$. Show that a substitution proof with the assumption $T(n) \le cn^{\log_3 4}$ fails. Then show how to subtract off a lower-order term to make a substitution proof work.

Solution to Exercise 4.3-7

If we were to try a straight substitution proof, assuming that $T(n) \le cn^{\log_3 4}$, we would get stuck:

$$\begin{aligned} T(n) &\le 4(c(n/3)^{\log_3 4}) + n \\ &= 4c\left(\frac{n^{\log_3 4}}{4}\right) + n \\ &= cn^{\log_3 4} + n , \end{aligned}$$

which is greater than $cn^{\log_3 4}$. Instead, we subtract off a lower-order term and assume that $T(n) \le cn^{\log_3 4} - dn$. Now we have

$$\begin{aligned} T(n) &\le 4(c(n/3)^{\log_3 4} - dn/3) + n \\ &= 4\left(\frac{cn^{\log_3 4}}{4} - \frac{dn}{3}\right) + n \\ &= cn^{\log_3 4} - \frac{4}{3}dn + n , \end{aligned}$$

which is less than or equal to $cn^{\log_3 4} - dn$ if $d \ge 3$.

Problem 4-1

Problems

4-1 Recurrence examples

Give asymptotic upper and lower bounds for $T(n)$ in each of the following recurrences. Assume that $T(n)$ is constant for $n \le 2$. Make your bounds as tight as possible, and justify your answers.

a. $T(n) = 2T(n/2) + n^4$.

b. $T(n) = T(7n/10) + n$.

c. $T(n) = 16T(n/4) + n^2$.

d. $T(n) = 7T(n/3) + n^2$.

e. $T(n) = 7T(n/2) + n^2$.

f. $T(n) = 2T(n/4) + \sqrt{n}$.

g. $T(n) = T(n-2) + n^2$.

Solution to Problem 4-1

Note: In parts (a), (b), and (d) below, we are applying case 3 of the master theorem, which requires the regularity condition that $af(n/b) \le cf(n)$ for some constant $c < 1$. In each of these parts, $f(n)$ has the form n^k. The regularity condition is satisfied because $af(n/b) = an^k/b^k = (a/b^k)n^k = (a/b^k)f(n)$, and in each of the cases below, a/b^k is a constant strictly less than 1.

a. $T(n) = 2T(n/2) + n^3 = \Theta(n^3)$. This is a divide-and-conquer recurrence with $a = 2$, $b = 2$, $f(n) = n^3$, and $n^{\log_b a} = n^{\log_2 2} = n$. Since $n^3 = \Omega(n^{\log_2 2+2})$ and $a/b^k = 2/2^3 = 1/4 < 1$, case 3 of the master theorem applies, and $T(n) = \Theta(n^3)$.

b. $T(n) = T(9n/10) + n = \Theta(n)$. This is a divide-and-conquer recurrence with $a = 1$, $b = 10/9$, $f(n) = n$, and $n^{\log_b a} = n^{\log_{10/9} 1} = n^0 = 1$. Since $n = \Omega(n^{\log_{10/9} 1+1})$ and $a/b^k = 1/(10/9)^1 = 9/10 < 1$, case 3 of the master theorem applies, and $T(n) = \Theta(n)$.

c. $T(n) = 16T(n/4) + n^2 = \Theta(n^2 \lg n)$. This is another divide-and-conquer recurrence with $a = 16$, $b = 4$, $f(n) = n^2$, and $n^{\log_b a} = n^{\log_4 16} = n^2$. Since $n^2 = \Theta(n^{\log_4 16})$, case 2 of the master theorem applies, and $T(n) = \Theta(n^2 \lg n)$.

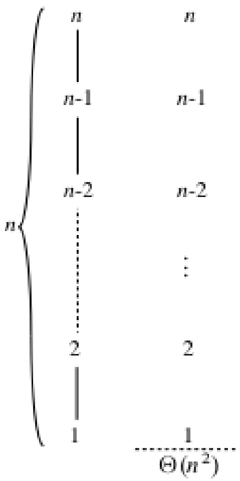
d. $T(n) = 7T(n/3) + n^2 = \Theta(n^2)$. This is a divide-and-conquer recurrence with $a = 7$, $b = 3$, $f(n) = n^2$, and $n^{\log_b a} = n^{\log_3 7}$. Since $1 < \log_3 7 < 2$, we have that $n^2 = \Omega(n^{\log_3 7+\epsilon})$ for some constant $\epsilon > 0$. We also have $a/b^k = 7/3^2 = 7/9 < 1$, so that case 3 of the master theorem applies, and $T(n) = \Theta(n^2)$.

e. $T(n) = 7T(n/2) + n^2 = O(n^{\lg 7})$. This is a divide-and-conquer recurrence with $a = 7$, $b = 2$, $f(n) = n^2$, and $n^{\log_b a} = n^{\log_2 7}$. Since $2 < \lg 7 < 3$, we have that $n^2 = O(n^{\log_2 7-\epsilon})$ for some constant $\epsilon > 0$. Thus, case 1 of the master theorem applies, and $T(n) = \Theta(n^{\lg 7})$.

f. $T(n) = 2T(n/4) + \sqrt{n} = \Theta(\sqrt{n}\lg n)$. This is another divide-and-conquer recurrence with $a = 2$, $b = 4$, $f(n) = \sqrt{n}$, and $n^{\log_b a} = n^{\log_4 2} = \sqrt{n}$. Since $\sqrt{n} = \Theta(n^{\log_4 2})$, case 2 of the master theorem applies, and $T(n) = \Theta(\sqrt{n}\lg n)$.

g. $T(n) = T(n-1) + n$

Using the recursion tree shown below, we get a guess of $T(n) = \Theta(n^2)$.



First, we prove the $T(n) = \Omega(n^2)$ part by induction. The inductive hypothesis is $T(n) \geq cn^2$ for some constant $c > 0$.

$$\begin{aligned} T(n) &= T(n-1) + n \\ &\geq c(n-1)^2 + n \\ &= cn^2 - 2cn + c + n \\ &\geq cn^2 \end{aligned}$$

if $-2cn + n + c \geq 0$ or, equivalently, $n(1-2c) + c \geq 0$. This condition holds when $n \geq 0$ and $0 < c \leq 1/2$.

For the upper bound, $T(n) = O(n^2)$, we use the inductive hypothesis that $T(n) \leq cn^2$ for some constant $c > 0$. By a similar derivation, we get that $T(n) \leq cn^2$ if $-2cn + n + c \leq 0$ or, equivalently, $n(1-2c) + c \leq 0$. This condition holds for $c = 1$ and $n \geq 1$.

Thus, $T(n) = \Omega(n^2)$ and $T(n) = O(n^2)$, so we conclude that $T(n) = \Theta(n^2)$.

Problem 4-3 (a,c,e,f,g,h,i)

4-3 More recurrence examples

Give asymptotic upper and lower bounds for $T(n)$ in each of the following recurrences. Assume that $T(n)$ is constant for sufficiently small n. Make your bounds as tight as possible, and justify your answers.

a. $T(n) = 4T(n/3) + n \lg n$.

b. $T(n) = 3T(n/3) + n/\lg n$.

c. $T(n) = 4T(n/2) + n^2\sqrt{n}$.

d. $T(n) = 3T(n/3 - 2) + n/2$.

e. $T(n) = 2T(n/2) + n/\lg n$.

f. $T(n) = T(n/2) + T(n/4) + T(n/8) + n$.

g. $T(n) = T(n-1) + 1/n$.

h. $T(n) = T(n-1) + \lg n$.

i. $T(n) = T(n-2) + 1/\lg n$.

j. $T(n) = \sqrt{n}T(\sqrt{n}) + n$.

Solution to Problem 4-3

[This problem is solved only for parts a, c, e, f, g, h, and i.]

a. $T(n) = 3T(n/2) + n \lg n$

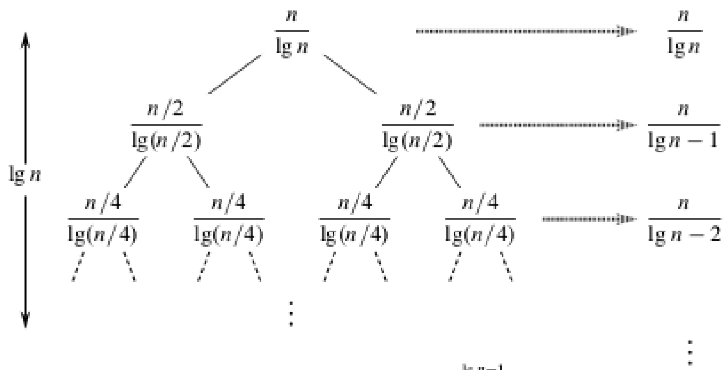
We have $f(n) = n \lg n$ and $n^{\log_b a} = n^{\lg 3} \approx n^{1.585}$. Since $n \lg n = O(n^{\lg 3 - \epsilon})$ for any $0 < \epsilon \le 0.58$, by case 1 of the master theorem, we have $T(n) = \Theta(n^{\lg 3})$.

c. $T(n) = 4T(n/2) + n^2\sqrt{n}$

We have $f(n) = n^2\sqrt{n} = n^{5/2}$ and $n^{\log_b a} = n^{\log_2 4} = n^2$. Since $n^{5/2} = \Omega(n^{2+\epsilon})$ for $\epsilon = 1/2$, we look at the regularity condition in case 3 of the master theorem. We have $af(n/b) = 4(n/2)^2\sqrt{n/2} = n^{5/2}/\sqrt{2} \le cn^{5/2}$ for $1/\sqrt{2} \le c < 1$. Case 3 applies, and we have $T(n) = \Theta(n^2\sqrt{n})$.

e. $T(n) = 2T(n/2) + n/\lg n$

We can get a guess by means of a recursion tree:



$$\sum_{i=0}^{\lg n - 1} \frac{n}{\lg n - i} = \Theta(n \lg \lg n)$$

We get the sum on each level by observing that at depth i, we have 2^i nodes, each with a numerator of $n/2^i$ and a denominator of $\lg(n/2^i) = \lg n - i$, so that the cost at depth i is

$$2^i \cdot \frac{n/2^i}{\lg n - i} = \frac{n}{\lg n - i} \; .$$

The sum for all levels is

$$\begin{aligned}
\sum_{i=0}^{\lg n-1} \frac{n}{\lg n - i} &= n \sum_{i=1}^{\lg n} \frac{n}{i} \\
&= n \sum_{i=1}^{\lg n} 1/i \\
&= n \cdot \Theta(\lg \lg n) \quad \text{(by equation (A.7), the harmonic series)} \\
&= \Theta(n \lg \lg n) .
\end{aligned}$$

We can use this analysis as a guess that $T(n) = \Theta(n \lg \lg n)$. If we were to do a straight substitution proof, it would be rather involved. Instead, we will show by substitution that $T(n) \le n(1 + H_{\lfloor \lg n \rfloor})$ and $T(n) \ge n \cdot H_{\lceil \lg n \rceil}$, where H_k is the kth harmonic number: $H_k = 1/1 + 1/2 + 1/3 + \cdots + 1/k$. We also define $H_0 = 0$. Since $H_k = \Theta(\lg k)$, we have that $H_{\lfloor \lg n \rfloor} = \Theta(\lg \lfloor \lg n \rfloor) = \Theta(\lg \lg n)$ and $H_{\lceil \lg n \rceil} = \Theta(\lg \lceil \lg n \rceil) = \Theta(\lg \lg n)$. Thus, we will have that $T(n) = \Theta(n \lg \lg n)$.

The base case for the proof is for $n = 1$, and we use $T(1) = 1$. Here, $\lg n = 0$, so that $\lg n = \lfloor \lg n \rfloor = \lceil \lg n \rceil$. Since $H_0 = 0$, we have $T(1) = 1 \le 1(1 + H_0)$ and $T(1) = 1 \ge 0 = 1 \cdot H_0$.

For the upper bound of $T(n) \le n(1 + H_{\lfloor \lg n \rfloor})$, we have

$$\begin{aligned}
T(n) &= 2T(n/2) + n/\lg n \\
&\le 2((n/2)(1 + H_{\lfloor \lg(n/2) \rfloor})) + n/\lg n \\
&= n(1 + H_{\lfloor \lg n - 1 \rfloor}) + n/\lg n \\
&= n(1 + H_{\lfloor \lg n \rfloor - 1} + 1/\lg n) \\
&\le n(1 + H_{\lfloor \lg n \rfloor - 1} + 1/\lfloor \lg n \rfloor) \\
&= n(1 + H_{\lfloor \lg n \rfloor}) ,
\end{aligned}$$

where the last line follows from the identity $H_k = H_{k-1} + 1/k$.

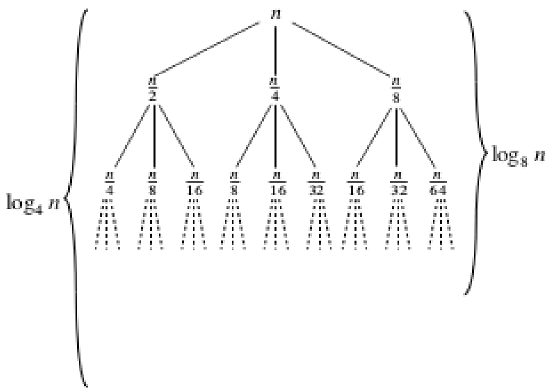
The upper bound of $T(n) \ge n \cdot H_{\lceil \lg n \rceil}$ is similar:

$$\begin{aligned}
T(n) &= 2T(n/2) + n/\lg n \\
&\ge 2((n/2) \cdot H_{\lceil \lg(n/2) \rceil}) + n/\lg n \\
&= n \cdot H_{\lceil \lg n - 1 \rceil} + n/\lg n \\
&= n \cdot (H_{\lceil \lg n \rceil - 1} + 1/\lg n) \\
&\ge n \cdot (H_{\lceil \lg n \rceil - 1} + 1/\lceil \lg n \rceil) \\
&= n \cdot H_{\lceil \lg n \rceil} .
\end{aligned}$$

Thus, $T(n) = \Theta(n \lg \lg n)$.

f. $T(n) = T(n/2) + T(n/4) + T(n/8) + n$

Using the recursion tree shown below, we get a guess of $T(n) = \Theta(n)$.



$$n$$

$$n(\tfrac{4+2+1}{8}) = \tfrac{7}{8}n$$

$$\begin{aligned} & n(\tfrac{1}{4} + \tfrac{2}{8} + \tfrac{3}{16} + \tfrac{2}{32} + \tfrac{1}{64}) \\ &= n\tfrac{16+16+12+4+1}{64} \\ &= n\tfrac{49}{64} = \tfrac{7}{8}^2 n \end{aligned}$$

$$\vdots$$

$$\sum_{i=1}^{\log n} \left(\frac{7}{8}\right)^i n = \Theta(n)$$

We use the substitution method to prove that $T(n) = O(n)$. Our inductive hypothesis is that $T(n) \le cn$ for some constant $c > 0$. We have

$$\begin{aligned} T(n) &= T(n/2) + T(n/4) + T(n/8) + n \\ &\le cn/2 + cn/4 + cn/8 + n \\ &= 7cn/8 + n \\ &= (1 + 7c/8)n \\ &\le cn \qquad \text{if } c \ge 8 . \end{aligned}$$

Therefore, $T(n) = O(n)$.

Showing that $T(n) = \Omega(n)$ is easy:

$$T(n) = T(n/2) + T(n/4) + T(n/8) + n \ge n .$$

Since $T(n) = O(n)$ and $T(n) = \Omega(n)$, we have that $T(n) = \Theta(n)$.

g. $T(n) = T(n-1) + 1/n$

This recurrence corresponds to the harmonic series, so that $T(n) = H_n$, where $H_n = 1/1 + 1/2 + 1/3 + \cdots + 1/n$. For the base case, we have $T(1) = 1 = H_1$. For the inductive step, we assume that $T(n-1) = H_{n-1}$, and we have

$$\begin{aligned} T(n) &= T(n-1) + 1/n \\ &= H_{n-1} + 1/n \\ &= H_n . \end{aligned}$$

Since $H_n = \Theta(\lg n)$ by equation (A.7), we have that $T(n) = \Theta(\lg n)$.

h. $T(n) = T(n-1) + \lg n$

We guess that $T(n) = \Theta(n \lg n)$. To prove the upper bound, we will show that $T(n) = O(n \lg n)$. Our inductive hypothesis is that $T(n) \le cn \lg n$ for some constant c. We have

$$
\begin{aligned}
T(n) &= T(n-1) + \lg n \\
&\le c(n-1)\lg(n-1) + \lg n \\
&= cn\lg(n-1) - c\lg(n-1) + \lg n \\
&\le cn\lg(n-1) - c\lg(n/2) + \lg n \\
&\qquad \text{(since } \lg(n-1) \ge \lg(n/2) \text{ for } n \ge 2) \\
&= cn\lg(n-1) - c\lg n + c + \lg n \\
&< cn\lg n - c\lg n + c + \lg n \\
&\le cn\lg n\ ,
\end{aligned}
$$

if $-c\lg n + c + \lg n \le 0$. Equivalently,

$$
\begin{aligned}
-c\lg n + c + \lg n &\le 0 \\
c &\le (c-1)\lg n \\
\lg n &\ge c/(c-1)\ .
\end{aligned}
$$

This works for $c = 2$ and all $n \ge 4$.

To prove the lower bound, we will show that $T(n) = \Omega(n\lg n)$. Our inductive hypothesis is that $T(n) \ge cn\lg n + dn$ for constants c and d. We have

$$
\begin{aligned}
T(n) &= T(n-1) + \lg n \\
&\ge c(n-1)\lg(n-1) + d(n-1) + \lg n \\
&= cn\lg(n-1) - c\lg(n-1) + dn - d + \lg n \\
&\ge cn\lg(n/2) - c\lg(n-1) + dn - d + \lg n \\
&\qquad \text{(since } \lg(n-1) \ge \lg(n/2) \text{ for } n \ge 2) \\
&= cn\lg n - cn - c\lg(n-1) + dn - d + \lg n \\
&\ge cn\lg n\ ,
\end{aligned}
$$

if $-cn - c\lg(n-1) + dn - d + \lg n \ge 0$. Since

$$
\begin{aligned}
&-cn - c\lg(n-1) + dn - d + \lg n > \\
&\qquad -cn - c\lg(n-1) + dn - d + \lg(n-1)\ ,
\end{aligned}
$$

it suffices to find conditions in which $-cn - c\lg(n-1) + dn - d + \lg(n-1) \ge 0$. Equivalently,

$$
\begin{aligned}
-cn - c\lg(n-1) + dn - d + \lg(n-1) &\ge 0 \\
(d-c)n &\ge (c-1)\lg(n-1) + d\ .
\end{aligned}
$$

This works for $c = 1$, $d = 2$, and all $n \ge 2$.

Since $T(n) = O(n\lg n)$ and $T(n) = \Omega(n\lg n)$, we conclude that $T(n) = \Theta(n\lg n)$.

i. $T(n) = T(n-2) + 2\lg n$

We guess that $T(n) = \Theta(n\lg n)$. We show the upper bound of $T(n) = O(n\lg n)$ by means of the inductive hypothesis $T(n) \le cn\lg n$ for some constant $c > 0$. We have

$$
\begin{aligned}
T(n) &= T(n-2) + 2\lg n \\
&\le c(n-2)\lg(n-2) + 2\lg n \\
&\le c(n-2)\lg n + 2\lg n \\
&= (cn - 2c + 2)\lg n
\end{aligned}
$$

$$
\begin{aligned}
&= cn \lg n + (2 - 2c) \lg n \\
&\le cn \lg n \qquad \text{if } c > 1 .
\end{aligned}
$$

Therefore, $T(n) = O(n \lg n)$.

For the lower bound of $T(n) = \Omega(n \lg n)$, we'll show that $T(n) \ge cn \lg n + dn$, for constants $c, d > 0$ to be chosen. We assume that $n \ge 4$, which implies that

1. $\lg(n - 2) \ge \lg(n/2)$,
2. $n/2 \ge \lg n$, and
3. $n/2 \ge 2$.

(We'll use these inequalities as we go along.) We have

$$
\begin{aligned}
T(n) &\ge c(n-2)\lg(n-2) + d(n-2) + 2\lg n \\
&= cn \lg(n-2) - 2c \lg(n-2) + dn - 2d + 2\lg n \\
&> cn \lg(n-2) - 2c \lg n + dn - 2d + 2\lg n \\
&\qquad (\text{since } -\lg n < -\lg(n-2)) \\
&= cn \lg(n-2) - 2(c-1)\lg n + dn - 2d \\
&\ge cn \lg(n/2) - 2(c-1)\lg n + dn - 2d \quad \text{(by inequality (1) above)} \\
&= cn \lg n - cn - 2(c-1)\lg n + dn - 2d \\
&\ge cn \lg n ,
\end{aligned}
$$

if $-cn - 2(c-1)\lg n + dn - 2d \ge 0$ or, equivalently, $dn \ge cn + 2(c-1)\lg n + 2d$. Pick any constant $c > 1/2$, and then pick any constant d such that

$$d \ge 2(2c - 1) .$$

(The requirement that $c > 1/2$ means that d is positive.) Then

$$d/2 \ge 2c - 1 = c + (c - 1) ,$$

and adding $d/2$ to both sides, we have

$$d \ge c + (c - 1) + d/2 .$$

Multiplying by n yields

$$dn \ge cn + (c - 1)n + dn/2 ,$$

and then both multiplying and dividing the middle term by 2 gives

$$dn \ge cn + 2(c - 1)n/2 + dn/2 .$$

Using inequalities (2) and (3) above, we get

$$dn \ge cn + 2(c - 1)\lg n + 2d ,$$

which is what we needed to show. Thus $T(n) = \Omega(n \lg n)$. Since $T(n) = O(n \lg n)$ and $T(n) = \Omega(n \lg n)$, we conclude that $T(n) = \Theta(n \lg n)$.