The $\mathbb{T}$-invariant Trace via 1-dimensional Factorization Homology

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February 26, 2023

Abstract

Notes for Topics in Algebraic Topology 22-23. Before we introduced the cobordism hypothesis and some (higher) trace methods. Here we introduce factorization homology to connect the above two topics. Section 1 comes from [AF1] and Section 2 and 3 come from [AF2].

Contents

1 Factorization homology

Definition 1.1. A smooth M manifold is **finitary** if it admits a finite open cover $\mathcal{U} := \{U \subseteq M\}$ such that for each finite subset $\mathcal{S} \subseteq \mathcal{U}$, the intersection $\bigcap_{U \in \mathcal{S}} U$ is either empty or diffeomorphic to an Euclidean space.

Convention 1.2. In this article, a manifold is always smooth and finitary unless otherwise stated.

Definition 1.3. Given an n-dimensional manifold M, its tangent bundle classified by a map $M \to BO(n)$, and a map $B \to BO(n)$ between spaces, a **B-framing** on M is the following homotopy commutative diagram

$$\begin{tikzcd} & B \arrow[d] \\ M \arrow[ur] \arrow[r] & BO(n) \end{tikzcd}.$$

Definition 1.4. The symmetric monoidal topological category Mfd_n^B consists of n-dimensional B-framed manifolds as objects and the morphism space $Mor_{Mfd_n^B}(M,N) := Emb(M,N)$ with the compact-open topology for any two objects M and N. The symmetric monoidal structure is disjoint union.

Definition 1.5. The symmetric monoidal topological full subcategory $Disk_n^B \subseteq Mfd_n^B$ consists of disjoint unions of n-dimensional B-framed Euclidean spaces as objects and the morphism space $Mor_{Disk_n^B}(D,D') := Emb(D,D')$ for any two objects D and D'. The symmetric monoidal structure is disjoint union.

Definition 1.6. A symmetric monoidal ∞-category $\mathfrak{X}$ is **$\otimes$-presentable** if $\mathfrak{X}$ is presentable and tis symmetric monoidal structure distributes over colimits.

Convention 1.7. The symmetric monoidal ∞-category $\mathfrak{X}$ we use later will be $\otimes$-presentable. In fact, we may only need $\mathfrak{X}$ to be $\otimes$-Δ^{op} cocomplete, but in this note we want to simplify our discussion. Moreover, when we write a functor $Mfd_n^B \to \mathfrak{X}$, we mean the functor $\mathcal{N}(Mfd_n^B) \to \mathfrak{X}$, where $\mathcal{N}$ is the topological nerve. This also works for $Disk_n^B$. Generally, this works for all other topological categories we mentioned later. (Regard the ordinary categories as topological categories with discrete topology on morphism sets.)

Definition 1.8. The ∞-category of **$Disk_n^B$-algebras** in $\mathfrak{X}$ is that of symmetric monoidal functors from $Disk_n^B$ to $\mathfrak{X}$:

$$Alg_{Disk_n^B}(\mathfrak{X}) := Fun^{\otimes}(Disk_n^B, \mathfrak{X}) \quad .$$

Remark 1.9. When $B = fr$, then $Alg_{Disk_n^{fr}}(\mathfrak{X})$ is equivalent to the ∞-category of E_n-algebras in $\mathfrak{X}$.

Definition 1.10. Let M be an object in Mfd_n^B and A an object in $Alg_{Disk_n^B}(\mathfrak{X})$ for a given ∞-category $\mathfrak{X}$. **Factorization homology** with coefficient in A is the left Kan extension

$$\begin{tikzcd} Disk_n^B \arrow[r, "A"] \arrow[d, hook] & \mathfrak{X} \\ Mfd_n^B \arrow[ur, dashed, "\int A"'] & \end{tikzcd} \quad .$$

Then factorization homology of M with coefficients in A is an object in $\mathfrak{X}$ given by a colimit:

$$\int_M A := colim(Disk_{n/M}^B \to Disk_n^B \xrightarrow{A} \mathfrak{X}). \tag{1.1}$$

Definition 1.11. Let M be an object in Mfd_n^B. A **collar-gluing** of M is a continuous map

$$f : M \to [-1,1]$$

to the closed interval for which the restriction on the subtarget $(-1,1)$ is a smooth fiber bundle.

We will often denote a collar-gluing $M \xrightarrow{f} [-1,1]$ as the open cover

$$M_- \bigcup_{M_0 \times \mathbb{R}} M_+ \cong M\,,$$

where $M_- := f^{-1}([-1,1))$, $M_+ := f^{-1}((-1,1])$ and $M_0 := f^{-1}(0)$.

With the help of [AF1, Construction 2.21, Corollary 3.12], given a collar-gluing $M \xrightarrow{f} [-1,1]$ for $M \in Mfd_n^B$ and a symmetric monoidal functor $Mfd_n^B \xrightarrow{\mathcal{F}} \mathfrak{X}$ with $\otimes$-presentable $\mathfrak{X}$, there is a canonical morphism in $\mathfrak{X}$

$$\mathcal{F}(M_-) \bigotimes_{\mathcal{F}(M_0 \times \mathbb{R})} \mathcal{F}(M_+) \to \mathcal{F}(M)\,. \tag{1.2}$$

Definition 1.12. A symmetric monoidal functor $\mathcal{F} : Mfd_n^B \to \mathfrak{X}$ satisfies **⊗-excision** if for each collar-gluing $M_- \bigcup_{M_0 \times \mathbb{R}} M_+ \cong M$ for $M \in Mfd_n^B$, the canonical morphism (1.2) is an equivalence in $\mathfrak{X}$. The ∞-category of homology theories for B-framed n-manifolds valued in $\mathfrak{X}$ is the full ∞-subcategory

$$H(Mfd_n^B, \mathfrak{X}) \subset Fun^{\otimes}(Mfd_n^B, \mathfrak{X})$$

consisting of symmetric monoidal functors that satisfy ⊗-excision.

Lemma 1.13. [AF1, Lemma 3.18] For any $A \in Alg_{Disk_n^B}(\mathfrak{X})$, factorization homology with coefficient in A satisfies ⊗-excision.

Example 1.14. For $A \in Alg_{Disk_1^{fr}}(\mathfrak{X})$, i.e. an E_1-algebra in $\mathfrak{X}$, we have

$$\int_{\mathbb{S}^1} A \simeq \int_{\mathbb{R}} A \bigotimes_{\int_{\mathbb{S}^0 \times \mathbb{R}} A} \int_{\mathbb{R}} A \simeq A \bigotimes_{A \otimes A^{op}} A \simeq HH(A),$$

where $HH(A)$ is the Hochshild complex of A in $\mathfrak{X}$. Moreover, factorization homology gives a $\mathbb{S}^1$-action on $HH(A)$ as follows:

$$\mathbb{S}^1 \simeq Emb(\mathbb{S}^1, \mathbb{S}^1) \xrightarrow{\int A} End_{\mathfrak{X}}(\int_{\mathbb{S}^1} A) \simeq End_{\mathfrak{X}}(HH(A)).$$

We have the following characterization for factorization homology:

Theorem 1.15. [AF1, Theorem 3.24] There is an equivalence

$$\int : Alg_{Disk_n^B}(\mathfrak{X}) \rightleftarrows H(Mfd_n^B, \mathfrak{X}) : ev_{\mathbb{R}^n},$$

where $ev_{\mathbb{R}^n}$ is the functor of evaluation on $\mathbb{R}^n$.

Finally, considering an E_∞-algebra in $\mathfrak{X}$ as a symmetric monoidal functor $Fin \to \mathfrak{X}$, we can define a forgetful functor

$$fgt : CAlg(\mathfrak{X}) \to Alg_{Disk_n^B}(\mathfrak{X})$$

via the restriction along the connected components functor $[-] : Disk_n^B \to Disk_n \xrightarrow{[-]} Fin$ for any $n \geq 0$. Next, define the following functor:

$$An \times CAlg(\mathfrak{X}) \xrightarrow{\otimes} CAlg(\mathfrak{X}), \quad (M, A) \mapsto A^{\otimes M} := colim(M \to * \xrightarrow{A} CAlg(\mathfrak{X})).$$

We then have the following result:

Proposition 1.16. [AF1, Proposition 5.1] The following diagram among ∞-categories commutes:

$$\begin{tikzcd}
Mfd_n^B \times CAlg(\mathfrak{X}) \arrow[r, "U \times id"] \arrow[d, "id \times fgt"'] & An \times CAlg(\mathfrak{X}) \arrow[r, "\otimes"] & CAlg(\mathfrak{X}) \arrow[d] \\
Mfd_n^B \times Alg_{Disk_n^B}(\mathfrak{X}) \arrow[rr, "\int"'] & & \mathfrak{X}
\end{tikzcd},$$

where U is the underlying space functor and the right downward arrow is the standard forgetful functor. In particular, there is a natural equivalence

$$\int_M A \simeq A^{\otimes M}$$

between the factorization homology of M with coefficients in A and the tensor of the commutative algebra A with the underlying space of M.

2 The $\mathbb{T}$-invariant trace

Remember that in this section, we still work uner Convention 1.2 and Convention 1.7.

Definition 2.1. For a group G, G-actions of objects in $\mathfrak{X}$ can be encoded into the functor ∞-category $Fun(BG, \mathfrak{X})$, whose 1-morphisms are called G-equivariant maps. Given two objects $F_A, F_B \in Fun(BG, \mathfrak{X})$ whose images of the object in BG are $F_A(*) := A$, $F_B(*) := B$, respectively, a 1-morphism $\eta : F_A \to F_B$ is called BG**-invariant** if for any 1-morphism $g \in BG$, the following diagram commutes up to coherent homotopy:

$$\begin{tikzcd} A \arrow[r, "\eta(*)"] \arrow[d, "F_A(g)"'] \arrow[dr, "\eta(*)"] & B \arrow[d, "F_B(g)"] \\ A \arrow[r, "\eta(*)"'] & B \end{tikzcd} .$$

Then an object F_A is called BG**-invariant** if the identity morphism $\mathrm{id}_A : F_A \to F_A$ is BG-invariant.

Notice that if either F_A or F_B is BG-invariant, then η is automatically G-invariant.

2.1 Construction I. The $\mathbb{T}$-invariant $unit : \mathbf{1} \to HH(A)$

Remark 2.2. Even though we can obtain such a map using the functor $\int A$ on the unique morphism $\emptyset \to \mathbb{S}^1$, this section contains the first step to construct the $\mathbb{T} \cong B\mathbb{Z}$-invariant map $HH(A) \to \mathbf{1}$ in the next subsection.

Definition 2.3. A **paracyclic category** $\Delta_{\circlearrowleft}$ consists of:

- Objects: Nonempty linearly ordered sets with $\mathbb{Z}$-action that satisfy two conditions:
  - (a) For an object Λ and each $\lambda \in \Lambda$, we have $\lambda < \lambda + 1$, where we indicate the $\mathbb{Z}$-action on Λ by $+$.
  - (b) For every pair of elements $\lambda, \lambda' \in \Lambda$, the set $\{\mu \in \Lambda : \lambda \leq \mu \leq \lambda'\}$ is finite.
- Morphisms: $\mathbb{Z}$-equivariant and nondecreasing maps between objects.

Definition 2.4. The **walking monad** is the monoidal category O in which an object is a linearly ordered finite sets, a morphism is an order preserving map, and whose monoidal structure is join of linearly ordered sets.

Notice that the initial object in O is the empty set. Then we have the equivalences

$$\Delta^{\triangleleft} \simeq O, \qquad (\Delta^{op})^{\triangleright} \simeq O^{op},$$

where $\Delta^{\triangleleft}$ is the simplex category Δ adding an initial object.

Remark 2.5. [AF2, Observation A.15] For each monoidal ∞-category $\mathfrak{Y}$, we have an equivalence of ∞-categories:

$$Fun^{E_1}(O, \mathfrak{Y}) \xrightarrow{\simeq} Alg_{E_1}(\mathfrak{Y}), \quad F \mapsto F(*),$$

where $* \in O$ is the final object.

Lemma 2.6. [AF2, Observation 1.3] We have the equivalences:

$$Disk^{fr}_{1/\mathbb{R}} \simeq O, \quad Disk^{fr}_{1/\mathbb{S}^1} \simeq \Delta^{\triangleleft}_{\circlearrowleft}.$$

Proof. In general, for each framed 1-manifold M, consider the composite functor:

$$Disk^{fr}_{1/M} \xrightarrow{forget} Disk^{fr}_1 \xrightarrow{\pi_0} Set.$$

For $M = \mathbb{R}$, we have the equivalence:

$$Disk^{fr}_{1/\mathbb{R}} \to O, \quad (U \xrightarrow{emb} \mathbb{R}) \mapsto \pi_0(U).$$

For $M = \mathbb{S}^1$, consider the universal covering map $\mathbb{R} \xrightarrow{exp} \mathbb{S}^1$. The linear order on $\pi_0\, exp^{-1}(U)$ inherited from an embedding $exp^{-1}(U) \to \mathbb{R}$, together with the $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$-action by deck-transformations, determines a lift of this composite functor to $\Delta^{\triangleleft}_{\circlearrowleft}$. This lift is an equivalence of $B\mathbb{Z}$-module categories:

$$Disk^{fr}_{1/\mathbb{S}^1} \to \Delta^{\triangleleft}_{\circlearrowleft}, \quad (U \to \mathbb{S}^1) \mapsto \pi_0\, exp^{-1}(U).$$

$\square$

Lemma 2.7. [AF2, Lemma 1.4] Define a symmetric monoidal functor:

$$Disk^{fr}_{1/(-)} : Mfd^{fr}_1 \to Cat_1, \quad M \to Disk^{fr}_{1/M}.$$

Then there is an equivalence $\int O \simeq Disk^{fr}_{1/(-)}$ between symmetric monoidal functors.

Proof. First, Lemma 2.6 tells us these two functors evaluate identically on $\mathbb{R}$. Second, Lemma 2.4.1 in [AF3] verifies that this symmetric monoidal functor $Disk^{fr}_{1/(-)}$ satisfies the $\otimes$-excision condition. Finally, the characterization Theorem 1.15 gives the equivalence we want. $\square$

Therefore, for any $A \in Alg_{E_1}(\mathfrak{X})$ that corresponds to the monoidal functor $\langle A\rangle \in Fun^{E_1}(O, \mathfrak{X})$ by Remark 2.5, we have the following diagram in $\mathfrak{X}$:

$$\Delta^{\triangleleft}_{\circlearrowleft} \xrightarrow{\simeq} Disk^{fr}_{1/\mathbb{S}^1} \xrightarrow{\simeq} \int_{\mathbb{S}^1} O \xrightarrow{\int_{\mathbb{S}^1}\langle A\rangle} \int_{\mathbb{S}^1} \mathfrak{X} \xrightarrow{\int_! \mathfrak{X}} \mathfrak{X}, \tag{2.1}$$

where the first and second equivalences are given by Lemma 2.6 and Lemma 2.7 respectively, and the last functor is given by the unique map $! : \mathbb{S}^1 \to *$ via Proposition 1.16.

Lemma 2.8. The colimit of the diagram (2.1) is equivalent to $\int_{\mathbb{S}^1} A$.

Proof. Given an inclusion $\mathbb{R} \to \mathbb{S}^1$, we can construct the following commutative diagram up to equivalence:

$$\begin{tikzcd}
& Disk^{fr}_{1/*} \arrow[rr, "\simeq"] & & Disk^{fr}_1 \arrow[rr, "\langle A\rangle"] & & \int_* \mathfrak{X} \simeq \mathfrak{X} \\
Disk^{fr}_{1/\mathbb{R}} \arrow[ur] \arrow[rr, "\simeq"] \arrow[d] & & \int_{\mathbb{R}} O \arrow[ur] \arrow[rr, "\int_{\mathbb{R}}\langle A\rangle"] \arrow[d] & & \int_{\mathbb{R}} \mathfrak{X} \arrow[ur, "\int_! \mathfrak{X}"] \arrow[d] & \\
Disk^{fr}_{1/\mathbb{S}^1} \arrow[uur, dashed] \arrow[rr, "\simeq"] & & \int_{\mathbb{S}^1} O \arrow[uur, dashed] \arrow[rr, "\int_{\mathbb{S}^1}\langle A\rangle"] & & \int_{\mathbb{S}^1} \mathfrak{X} \arrow[uur, "\int_! \mathfrak{X}"'] &
\end{tikzcd}$$

From the definition of factorization homology (1.1), we know that

$$\int_{\mathbb{S}^1} A = colim(Disk^{fr}_{1/\mathbb{S}^1} \to Disk^{fr}_1 \xrightarrow{A} \mathfrak{X}) \simeq colim(Disk^{fr}_{1/\mathbb{S}^1} \xrightarrow{\simeq} \int_{\mathbb{S}^1} O \xrightarrow{\int_{\mathbb{S}^1}\langle A\rangle} \int_{\mathbb{S}^1} \mathfrak{X} \xrightarrow{\int_! \mathfrak{X}} \mathfrak{X}).$$

$\square$

The initial object $\triangleleft \in \Delta^{\triangleleft}_{\circlearrowleft}$ will be sent to the unit object $\mathbf{1} \in \mathfrak{X}$ due to the E_1-structure of A. Then we have the following canonical morphism:

$$unit : \mathbf{1} \to colim(\Delta^{\triangleleft}_{\circlearrowleft} \to \mathfrak{X}) \xrightarrow{\simeq (Lemma\, 2.8)} \int_{\mathbb{S}^1} A \simeq HH(A). \tag{2.2}$$

Since the initial object $\triangleleft$ is $\mathbb{T}$-invariant, then the morphism *unit* is $\mathbb{T}$-equivariant.

2.2 Construction II. The $\mathbb{T}$-invariant $trace : HH(A) \to \mathbf{1}$

Convention 2.9. From now on, we only focus on $A := \underline{End}(V) := V^{\vee} \otimes V \in Alg_{E_1}(\mathfrak{X})$, where V is a dualizable object in $\mathfrak{X}$. This implies that $\underline{End}(V)$ has the structure of a Frobenius algebra. See the discussion on [AF2, (31)].

Definition 2.10. The **walking adjunction** is the $(\infty, 2)$-category Adj that consists of two objects $-$ and $+$, and morphisms are generated by identities and a pair of adjunction $- \xrightarrow{L} +$ and $+ \xrightarrow{R} -$. Its monoidal categories of endomorphisms of each of its two objects are canonically identified as the walking monad and the walking comonad:

$$O \xrightarrow{I \mapsto (R \circ L)^{\circ I}} End_{Adj}(-), \quad O^{op} \xrightarrow{I \mapsto (L \circ R)^{\circ I}} End_{Adj}(+).$$

Remark 2.11. In particular, there are fully-faithful functors between $(\infty, 2)$-categories

$$\mathfrak{B}O \hookrightarrow Adj \hookleftarrow \mathfrak{B}O^{op},$$

where $\mathfrak{B}$ means the deloop of categories.

We have the following characterization for Adj:

Remark 2.12. For each $(\infty, 2)$-category $\mathcal{C}$, the evaluation map

$$ev_L : Mor_{Cat_2}(Adj, \mathcal{C}) \xrightarrow{\simeq} Mor(\mathcal{C})^{l.adj}, \quad F \mapsto F(L)$$

is an equivalence to the subspace of those 1-morphisms in $\mathcal{C}$ that are left adjoints.

Definition 2.13. [AF2, Appendix B] For the ∞-category $\mathfrak{X}$ mentioned before, the ∞-category of **category-objects internal to** $\mathfrak{X}$ is the full ∞-subcategory

$$fCat_1[\mathfrak{X}] \subset Fun(\Delta^{op}, \mathfrak{X})$$

consisting of those functors that satisfy Segal conditions. Then we can define the **beta-version of factorization homology**:

$$\int^{\beta} : fCat_1[\mathfrak{X}] \to Fun(\mathcal{M}, \mathfrak{X}), \quad \mathcal{C} \mapsto (M \mapsto \int^{\beta}_M \mathcal{C}),$$

where $\mathcal{M}$ is the ∞-category of compact solidly 1-framed stratified spaces.

Remark 2.14. Let $\mathfrak{X}$ be Cat_1. Note the standard fully-faithful embedding

$$Cat_2 \subset fCat_2 := fCat_1[Cat_1] = Fun^{Segal}(\Delta^{op}, Cat_1).$$

Then we can define the factorization homology of $(\infty, 2)$-categories:

$$\int^{\beta} : Cat_2 \subset fCat_2 \to Fun(\mathcal{M}, Cat_1), \quad \mathcal{C} \mapsto (M \mapsto \int^{\beta}_M \mathcal{C}).$$

Proposition 2.15. [AF2, Proposition B.12] For any associative algebra $A \in Alg_{E_1}(\mathfrak{X})$, there is a canonical $\mathbb{T}$-equivariant identification

$$\int_{\mathbb{S}^1} A \simeq \int_{\mathbb{S}^1}^{\beta} \mathfrak{B}A.$$

Remark 2.12 and [AF2, Observation A.24] imply the identification $Mor_{fCat_2}(Adj, \mathfrak{B}\mathfrak{X}) \simeq Obj(\mathfrak{X}^{duals})$ of ∞-groupoids. Then for the dualizable object $A := End_{\mathfrak{X}}(V)$, we have the resulting composite functor:

$$\mathfrak{B}O \hookrightarrow Adj \xrightarrow{\langle End_{\mathfrak{X}}(V)\rangle} \mathfrak{B}\mathfrak{X}.$$

The following theorem we will use right now is the main and most technical part in the paper [AF2]. We only state the result below:

Theorem 2.16. [AF2, Theorem 1.1] There are canonical $\mathbb{T}$-equivariant equivalences

$$\int_{\mathbb{S}^1}^{\beta} Adj \simeq \Delta_{\circlearrowleft}^{\triangleleft\triangleright}, \qquad \int_{\mathbb{S}^1} End_{Adj}(-) \simeq \Delta_{\circlearrowleft}^{\triangleleft}.$$

Now we construct the following commutative diagram up to equivalence in $\mathfrak{X}$:

$$\begin{tikzcd}
\int_{\mathbb{S}^1}^{\beta} \mathfrak{B}O \arrow[rr, "Proposition\ 2.15"] \arrow[d] & & \int_{\mathbb{S}^1} O \arrow[d, "\int_{\mathbb{S}^1}\langle End_{\mathfrak{X}}(V)\rangle"] & \\
\int_{\mathbb{S}^1}^{\beta} Adj \arrow[r, "\int_{\mathbb{S}^1}^{\beta}\langle End_{\mathfrak{X}}(V)\rangle"] & \int_{\mathbb{S}^1}^{\beta} \mathfrak{B}\mathfrak{X} \arrow[r, "Proposition 2.15"] & \int_{\mathbb{S}^1} \mathfrak{X} \arrow[r, "\int_! \mathfrak{X}"] & \mathfrak{X}
\end{tikzcd}$$

We simplify this diagram by Theorem 2.16:

$$\begin{tikzcd}
\Delta_{\circlearrowleft}^{\triangleleft} \arrow[d, hook] \arrow[dr] & \\
\Delta_{\circlearrowleft}^{\triangleleft\triangleright} \arrow[r] & \mathfrak{X}
\end{tikzcd} \tag{2.3}$$

Notice the right-down arrow is equivalent to the diagram (2.1). Then we have $HH(End_{\mathfrak{X}}(V)) \simeq colim(\Delta_{\circlearrowleft}^{\triangleleft} \to \mathfrak{X})$ here. The final point $\triangleright \in \Delta_{\circlearrowleft}^{\triangleleft\triangleright}$ is sent to the unit $\mathbf{1} \in \mathfrak{X}$ due to the coalgebra structure in $End_{\mathfrak{X}}(V)$, and the universal property of this colimit determines the following morphism:

$$trace : HH(End_{\mathfrak{X}}(V)) \to \mathbf{1}.$$

Since the final object $\triangleright$ is $\mathbb{T}$-invariant, then the morphism *trace* is $\mathbb{T}$-equivariant.

2.3 The composite morphism *trace* ∘ *unit*

Finally, we compose the morphisms *unit* and *trace* of $End_{\mathfrak{X}}(V)$ for any dualizable object $V \in \mathfrak{X}$ to obtain a $\mathbb{T}$-invariant endomorphism in $End_{\mathfrak{X}}(\mathbf{1})$.

Theorem 2.17. [AF2, Theorem 2.6.(3)] Forgetting $\mathbb{T}$-invariance, this composite map $trace \circ unit$ is equivalent up to coherent homotopy with the composite

$$\mathbf{1} \xrightarrow{\eta} V^{\vee} \otimes V \xrightarrow{\simeq} V \otimes V^{\vee} \xrightarrow{\epsilon} \mathbf{1}\,,$$

where η and ϵ are the respective unit and counit of the duality between V and $V^{\vee}$.

Proof. In the diagram (2.3), we use the (nerve of) ordinary categories $\Delta_{\circlearrowleft}^{\lhd\rhd}$ and $\Delta_{\circlearrowleft}^{\lhd}$. Thus the map $(\lhd \to \rhd)$ in $\Delta_{\circlearrowleft}^{\lhd\rhd}$ is unique, and it uniquely factors through any object in $\Delta_{\circlearrowleft}^{\lhd\rhd}$. From Lemma 2.6, we know that the object $\mathbb{Z} \circlearrowleft \mathbb{Z} \in \Delta_{\circlearrowleft}^{\lhd\rhd}$ is mapped to $End_{\mathfrak{X}}(V) \in \mathfrak{X}$ in the diagram (2.3). Therefore the unique factorization $(\lhd \to \mathbb{Z} \circlearrowleft \mathbb{Z} \to \rhd) \in \Delta_{\circlearrowleft}^{\lhd\rhd}$ is carried to

$$\mathbf{1} \xrightarrow{\eta} V^{\vee} \otimes V \xrightarrow{\simeq} V \otimes V^{\vee} \xrightarrow{\epsilon} \mathbf{1}\ .$$

Then using the universal property of colimits in $\mathfrak{X}$, we see the composite *trace* $\circ$ *unit* is equivalent up to coherent homotopy with the composite morphism mentioned above. □

Remark 2.18. Here we omit the $\mathbb{T}$-invariance because the equivalence $V^{\vee} \otimes V \xrightarrow{\simeq} V \otimes V^{\vee}$ is not $\mathbb{T}$-invariant.

3 Epilogue

In this section we mention slightly some applications of the $\mathbb{T}$-invariant trace.

3.1 A conjecture of Töen–Vezzosi

The $\mathbb{T}$-invariant trace we constructed last section can be an essential part of the proof of Conjecture 5.1 of [TV]. The proof also needs 1-dimensional non-abelian Poincaré duality ([AF2, Proposition B.7]) and other skills that we do not mention here, so we only give a statement below.

Definition 3.1. We define the full ∞-subcatgory $CAlg(Cat_1)^{rigid} \subset CAlg(Cat_1)$ of the rigid symmetric monoidal ∞-categories (i,e,. those symmetric monoidal ∞-categories in which each object is dualizable).Then we define the **moduli space of objects** functor to be

$$Obj : CAlg(Cat_1)^{rigid} \to An, \quad \mathfrak{X} \mapsto Obj(\mathfrak{X}),$$

and the **free loop** functor to be

$$L : CAlg(Cat_1)^{rigid} \to Fun(B\mathbb{T}, An), \quad \mathfrak{X} \mapsto L\mathfrak{X},$$

where $L\mathfrak{X}$ is the space $Obj(\mathfrak{X})$ with a $\mathbb{T}$-action. Precomposing by the unique map $\mathbb{T} \xrightarrow{!} *$ determines a natural transformation

$$Obj \xrightarrow{constant} L$$

in which the domain is regarded as taking in $Fun(B\mathbb{T}, An)$ via the functor $An \xrightarrow{trivial} Fun(B\mathbb{T}, An)$.

Finally define the **categorical based loop** functor to be

$$End(\mathbf{1}) : CAlg(Cat_1)^{rigid} \xrightarrow{End(\mathbf{1})} Anx \to trivialFun(B\mathbb{T}, An), \quad \mathfrak{X} \mapsto End_{\mathfrak{X}}(\mathbf{1}).$$

So each $\mathbb{T}$-space $End_{\mathfrak{X}}(\mathbf{1})$ is endowed with the trivial $\mathbb{T}$-action.

Corollary 3.2. [AF2, Corollary 2.9] There is a natural transformation between functors from $CAlg(Cat_1)^{rigid}$ to $Fun(B\mathbb{T}, An)$,

$$L \xrightarrow{trace} End(\mathbf{1}),$$

with the property that the composite natural transformation evaluates as

$$Obj \xrightarrow{constant} L \xrightarrow{trace} End(\mathbf{1}), \quad (V \in Obj(\mathfrak{X})) \mapsto ((\mathbf{1} \xrightarrow{\eta} V^{\vee} \otimes V \xrightarrow{\simeq} V \otimes V^{\vee} \xrightarrow{\epsilon} \mathbf{1}) \in End_{\mathfrak{X}}(\mathbf{1})).$$

3.2 1-dimensional cobordism hypothesis

From the 1-dimensional cobordism hypothesis [**?**], we have the equivalence

$$Fun^{\otimes}(Bord_1^{fr}, \mathfrak{X}) \simeq \mathfrak{X}^{dual}$$

of ∞-groupoids of 1-dimensional fully-extended topological field theories(TFTs) and dualizable objects in $\mathfrak{X}$. To check the dualizable object $V \in \mathfrak{X}$ determines a TFT Z_V, we need to identify the value $Z_V(\mathbb{S}^1)$. We can witness $\mathbb{S}^1$ as a union of two hemispherical 1-disks to determine an identification

$$(\mathbf{1} \xrightarrow{Z_V(\mathbb{S}^1)}) \simeq (\mathbf{1} \xrightarrow{\eta} V^{\vee} \otimes V \xrightarrow{\simeq} V \otimes V^{\vee} \xrightarrow{\epsilon} \mathbf{1}).$$

In this way, one can easily identify all values of the sought TFT Z_V, via combinatorial presentations of 1-manifolds as gluings of disjoint unions of 1-disks along boundary 0-spheres. The key difficulty in proving the 1-dimensional cobordism hypothesis is to verify coherent compatibilities among each value of Z_V determined by a combinatorial presentation. Each combinatorial presentation of $\mathbb{S}^1$ sullpies a natural cyclic group action with some order $r \in \mathbb{N}$ on $Z_V(\mathbb{S}^1)$. These actions should be invariant on $Z_V(\mathbb{S}^1)$ for any $r \in \mathbb{N}$ and then extended to a $Diff^{fr}(\mathbb{S}^1) \simeq \mathbb{T}$-invariant action on $Z_V(\mathbb{S}^1)$, which is not obvious. The construction of $\mathbb{T}$-invariant map *trace* $\circ$ *unit* in the last section solves the key difficulty mentioned above, therefore finding a possible way to prove the 1-dimensional cobordism hypothesis.

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