∞-operads

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1 Colored operads

1.1 Symmetric monoidal categories

Before introducing colored operads, let us see an example first, that is, the symmetric monoidal category. By MacLane's definition, roughly we need a unit, a binary operation, a symmetric braiding, and some coherent conditions(unit, pentagon and hexagon axioms). When it comes to higher categories, those coherent conditions will become much more complicated. Thus we need to find some more efficient way to encode those infomation. In this section, we first consider how to give a more gentle definition of symmetric monoidal categories, and then to generalize it into symmetric monoidal ∞-categories.

To begin with, we recall MacLane's definition of the symmetric monoidal category.

Definition 1.1. A **monoidal category** $(\mathcal{C}, \otimes, \mathbf{1}, \alpha, l, r)$ consists of the following data:

1. a category $\mathcal{C}$.
2. a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.
3. an object $\mathbf{1} \in \mathcal{C}$.
4. a natural isomorphism $\alpha : \otimes \circ (\otimes \times \mathrm{id}_{\mathcal{C}}) \Rightarrow \otimes \circ (\mathrm{id}_{\mathcal{C}} \times \otimes)$
5. a natural isomorphism $l : (\mathbf{1} \otimes -) \Rightarrow \mathrm{id}_{\mathcal{C}}$
6. a natural isomorphism $r : (- \otimes \mathbf{1}) \Rightarrow \mathrm{id}_{\mathcal{C}}$

satisfying the following conditions:

1. (Unity/Triangle Axiom) For any objects $X, Y \in \mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
(X \otimes \mathbf{1}) \otimes Y \arrow[rr, "\alpha_{X,\mathbf{1},Y}"] \arrow[dr, "r_X \otimes \mathrm{id}_Y"'] & & X \otimes (\mathbf{1} \otimes Y) \arrow[dl, "\mathrm{id}_X \otimes l_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

2. (Pentagon Axiom) For any objects $X, Y, Z, W \in \mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
& ((X \otimes Y) \otimes Z) \otimes W \arrow[dl, "\alpha_{X,Y,Z} \otimes \mathrm{id}_W"'] \arrow[dr, "\alpha_{X \otimes Y, Z, W}"] & \\
(X \otimes (Y \otimes Z)) \otimes W \arrow[d, "\alpha_{X, Y \otimes Z, W}"'] & & (X \otimes Y) \otimes (Z \otimes W) \arrow[d, "\alpha_{X,Y,Z \otimes W}"] \\
X \otimes ((Y \otimes Z) \otimes W) \arrow[rr, "id_X \otimes \alpha_{Y,Z,W}"] & & X \otimes (Y \otimes (Z \otimes W))
\end{tikzcd}$$

Definition 1.2. A **braiding** in a monoidal category $\mathcal{C}$ is a natural isomorphism $B : \otimes \Rightarrow \otimes^{rev} := \otimes \circ \tau$, where $\tau : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$, $(X, Y) \mapsto (Y, X)$, $\forall X, Y, Z \in \mathcal{C}$, satisfying the following diagrams commute: (Hexagon axiom)

$$\begin{tikzcd}
& (X \otimes Y) \otimes Z \arrow[r, "B_{X\otimes Y,Z}"] & Z \otimes (X \otimes Y) \arrow[dr, "\alpha^{-1}_{Z,X,Y}"] & \\
X \otimes (Y \otimes Z) \arrow[ur, "\alpha^{-1}_{X,Y,Z}"] \arrow[dr, "\mathrm{id}_X \otimes B_{Y,Z}"'] & & & (Z \otimes X) \otimes Y \\
& X \otimes (Z \otimes Y) \arrow[r, "\alpha^{-1}_{X,Z,Y}"] & (X \otimes Z) \otimes Y \arrow[ur, "B_{X,Z} \otimes \mathrm{id}_Y"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& X \otimes (Y \otimes Z) \arrow[r, "B_{X,Y\otimes Z}"] & (Y \otimes Z) \otimes X \arrow[dr, "\alpha_{Y,Z,X}"] & \\
(X \otimes Y) \otimes Z \arrow[ur, "\alpha_{X,Y,Z}"] \arrow[dr, "B_{X,Y} \otimes \mathrm{id}_Z"'] & & & Y \otimes (Z \otimes X) \\
& (Y \otimes X) \otimes Z \arrow[r, "\alpha_{Y,X,Z}"] & Y \otimes (X \otimes Z) \arrow[ur, "\mathrm{id}_Y \otimes B_{X,Z}"'] &
\end{tikzcd}$$

We call the monoidal category equipped with a braiding the **braided monoidal category**. If the braiding satisfies $B_{Y,X} \circ B_{X,Y} = \mathrm{id}_{X \otimes Y}$, $\forall X, Y \in \mathcal{C}$, or in other words, the following diagram commutes:

$$\begin{tikzcd}
X \otimes Y \arrow[r, "B_{X,Y}"] \arrow[dr, "\mathrm{id}_{X\otimes Y}"'] & Y \otimes X \arrow[d, "B_{Y,X}"] \\
& X \otimes Y
\end{tikzcd}$$

then we call this braided monoidal category the **symmetric monoidal category**.

Construction 1.3. Given a symmetric monoidal category $(\mathcal{C}, \otimes)$, we define a new category $\mathcal{C}^{\otimes}$ as follows:

1. An object of $\mathcal{C}^{\otimes}$ is a finite (possibly empty) sequence of objects of $\mathcal{C}$, which we will denote by $[C_1, ..., C_n]$.

2. A morphism from $[C_1, ..., C_n]$ to $[C'_1, ..., C'_m]$ is a triple $(S, \alpha, \{f_j : \otimes_{\alpha(i)=j} C_i \to C'_j\}_{1 \le j \le m})$, where S is a subset of $\{1, ..., n\}$ (possibly empty), a map of sets $\alpha : S \to \{1, ..., m\}$, and every f_j is a morphism in $\mathcal{C}$.

3. Given morphisms $f : [C_1, ..., C_n] \to [C'_1, ..., C'_m]$ and $g : [C'_1, ..., C'_m] \to [C''_1, ..., C''_l]$, where $f = (S, \alpha, \{f_j\}_{1 \le j \le m})$ and $g = (T, \beta, \{g_j\}_{1 \le j \le l})$, we define the composition of f and g to be the map $g \circ f := (\alpha^{-1}T, \beta \circ \alpha, \{h_k\}_{1 \le k \le l})$, where the morphism h_k is the composition of the canonical isomorphism specified by the symmetric monoidal structure on $\mathcal{C}$ and the maps specified by the maps f and g as follows:

$$h_k : \bigotimes_{(\beta\circ\alpha)(i)=k} C_i \simeq \bigotimes_{\beta(j)=k} \bigotimes_{\alpha(i)=j} C_i \to \bigotimes_{\beta(j)=k} C'_j \to C''_k$$

Definition 1.4. The category of pointed finite sets $\mathcal{F}in_*$ consists of

1. objects are the sets $\langle n \rangle = \{0, 1, ..., n\}$, $n \ge 0$, which can be regarded as the pointed set of $\langle n \rangle^\circ := \{1, ..., n\}$.

2. a morphism $\alpha : \langle n \rangle \to \langle m \rangle$ is a map of sets such that $\alpha(0) = 0$.

Specially, we can find a set of morphisms $\rho^i : \langle n \rangle \to \langle 1 \rangle,\ 1 \leq i \leq n$ by the formula:

$$\rho^i(j) := \begin{cases} 1 & \text{if i=j} \\ 0 & \text{others} \end{cases}$$

For every symmetric monoidal category $\mathcal{C}$, we obtain a category $C^{\otimes}$ from 1.3. There is a forgetful functor $p : \mathcal{C}^{\otimes} \to \mathcal{F}in_*$, which sends an object $[C_1, .., C_n]$ to $\langle n \rangle$. Then we can check that p satisfies two conditions:

(M1) The functor p is a coCartesian fibration of categories.(See the definition 1.9)

(M2) Under the assumption of (M1), the morphism below induced by $\{\rho^i\}_{1 \leq i \leq n}$ is an equivalence:(See the construction 1.11)

$$\prod_{i=1}^{n} (\rho^i_!) : \mathcal{C}^{\otimes}_{\langle n \rangle} \xrightarrow{\simeq} \prod_{i=1}^{n} (\mathcal{C}^{\otimes}_{\langle 1 \rangle})$$

When $n = 0$, $\mathcal{C}^{\otimes}_{\langle 0 \rangle}$ has a single object and a morphism.

Definition 1.5. Given a functor $p : \mathcal{C} \to \mathcal{D}$ of categories $\mathcal{C}$ and $\mathcal{D}$, a morphsim $f : x \to y$ in $\mathcal{C}$ is a **p-Cartesian** if given any maps $g : z \to y$ in $\mathcal{C}$, and $u : pz \to px$ in $\mathcal{D}$ such that $pg = pf \circ u$, there exists a unique morphism $v : z \to x$ in $\mathcal{C}$ s.t. $pv = u$ and $f \circ v = g$.

Remark 1.6. The above definition is equivalent to the following map is bijective:

$$Mor_{\mathcal{C}}(z, x) \xrightarrow{\simeq} Mor_{\mathcal{C}}(z, y) \times_{Mor_{\mathcal{D}}(pz,py)} Mor_{\mathcal{D}}(pz, px), \quad v \mapsto (f \circ v, pv)$$

Definition 1.7. Given a functor $p : \mathcal{C} \to \mathcal{D}$ of categories $\mathcal{C}$ and $\mathcal{D}$, a morphsim $f : x \to y$ in $\mathcal{C}$ is a **p-coCartesian** if given any maps $g : x \to z$ in $\mathcal{C}$ and $u : py \to pz$ in $\mathcal{D}$ such that $pg = u \circ pf$, there exists a unique morphism $v : y \to z$ in $\mathcal{C}$ s.t. $pv = u$ and $v \circ f = g$.

Remark 1.8. The above definition is equivalent to the following map is bijective:

$$Mor_{\mathcal{C}}(y, z) \xrightarrow{\simeq} Mor_{\mathcal{C}}(x, z) \times_{Mor_{\mathcal{D}}(px,pz)} Mor_{\mathcal{D}}(py, pz), \quad v \mapsto (v \circ f, pv)$$

Definition 1.9. Given a functor $p : \mathcal{C} \to \mathcal{D}$ of categories $\mathcal{C}$ and $\mathcal{D}$, p is called a **coCartesian fibration** if for any $g : a \to b$ in $\mathcal{D}$, and $x \in \mathcal{C}$ such that $p(x) = a$, there exists a p-coCartesian morphism $f : x \to y$ s.t. $P(f) = g$.

Definition 1.10. Given a functor $p : \mathcal{C} \to \mathcal{D}$ of categories $\mathcal{C}$ and $\mathcal{D}$, p is called a **Cartesian fibration** if $p^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$ is a coCartesian fibration.

Construction 1.11. Let the functor $p : \mathcal{C} \to \mathcal{D}$ be a coCartesian fibration for some category $\mathcal{C}$ and $\mathcal{D}$. Given an object $a \in \mathcal{D}$, we define $\mathcal{C}_{\langle a \rangle} \subset \mathcal{C}$ to be the (non-full) subcategory, whose objects are $c \in \mathcal{C}$, s.t. $p(c) = a$ and morphisms $f : c \to c'$ s.t. $pf = id_a$, as the fiber of p over the object $a \in \mathcal{D}$.

Now given any map $g : a \to b$ in $\mathcal{D}$, for any $x \in \mathcal{C}_{\langle a \rangle}$, we have a p-coCartesian morphism $f : x \to y$ in $\mathcal{C}$ s.t. $pf = g$, since p is a coCartesian fibration. Then we can induce a morphism $g_! : \mathcal{C}_{\langle a \rangle} \to \mathcal{C}_{\langle b \rangle}$ s.t. $g_!(x) = y$.

If there is another p-coCartesian morphism $f' : x \to y'$, then we have a unique map $h : y \to y'$ s.t. $h \circ f = f'$, and a unique map $h' : y' \to y$ s.t. $h' \circ f' = f$, respectively, which implies y is isomorphic to y'. Thus the choice of y is unique up to a unique isomorphism. Then the functor $g_!$ is unique up to a unique natural isomorphism.

On the other hand, for some category $\mathcal{C}$, if we have a functor $p : \mathcal{C} \to \mathcal{F}in_*$ that satisfies (M1) and (M2), then we can equip the fiber $\mathcal{C}_{\langle 1\rangle}$ a symmetric monoidal structure as below:

1. (Unit object $\mathbf{1}$) The unique map $\langle 0\rangle \to \langle 1\rangle$ lifts a p-coCartesian fibration $\mathcal{C}_{\langle 0\rangle} \to \mathcal{C}_{\langle 1\rangle}$. Define the image of the unique object in $\mathcal{C}_{\langle 0\rangle}$ to be the unit $\mathbf{1} \in \mathcal{C}_{\langle 1\rangle}$

2. (Tensor product $\otimes$) Given a morphism $\beta : \langle 2\rangle \to \langle 1\rangle$, s.t. $\beta(1) = \beta(2) = 1$ and $\beta(0) = 0$, we can lift a p-coCartesian fibration $\beta_! : \mathcal{C}_{\langle 2\rangle} \to \mathcal{C}_{\langle 1\rangle}$. By (M2), we have an equivalence $\rho^1_! \times \rho^2_! : \mathcal{C}_{\langle 2\rangle} \to \mathcal{C}_{\langle 1\rangle} \times \mathcal{C}_{\langle 1\rangle}$. Difine the tensor product $\otimes$ as the composition of $(\rho^1_! \times \rho^2_!)^{-1}$ and $\beta_!$ as follows:
$$\otimes := \beta_! \circ (\rho^1_! \times \rho^2_!)^{-1} : \mathcal{C}_{\langle 1\rangle} \times \mathcal{C}_{\langle 1\rangle} \to \mathcal{C}_{\langle 1\rangle}.$$

3. (Natural isomorphism α) For $1 \le i \le n$, let $\tau^n_i : \langle n\rangle \to \langle n-1\rangle$ denote the map given by the formula:
$$\tau^n_i(j) := \begin{cases} j & \text{if } 1 \le j \le i \\ j-1 & \text{if } i < j \le n \\ 0 & \text{if } j = 0 \end{cases}$$

   Then the commutative diagram ont the left determines a diagram on the right below.
$$\begin{tikzcd} \langle 3\rangle \arrow[r, "\tau^3_1"] \arrow[d, "\tau^3_2"] & \langle 2\rangle \arrow[d, "\tau^2_1"] \\ \langle 2\rangle \arrow[r, "\tau^2_1"] & \langle 1\rangle \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}_{\langle 3\rangle} \arrow[r, "(\tau^3_1)_!"] \arrow[d, "(\tau^3_2)_!"] & \mathcal{C}_{\langle 2\rangle} \arrow[d, "(\tau^2_1)_!"] \\ \mathcal{C}_{\langle 2\rangle} \arrow[r, "(\tau^2_1)_!"] & \mathcal{C}_{\langle 1\rangle} \end{tikzcd}$$

   By (M1), we know that the induced map $(\tau^2_1 \circ \tau^3_2)_!$ is unique up to a unique natural isomorphism, and so is $(\tau^2_1 \circ \tau^3_1)_!$. Then we can find a unique natural isomorphism $(\tau^2_1 \circ \tau^3_1)_! \Rightarrow (\tau^2_1 \circ \tau^3_2)_!$. We can also observe that this unique natural isomorphism can induce the unique natural isomorphism $\alpha : \otimes \circ (\otimes \times \mathrm{id}_{\mathcal{C}}) \Rightarrow \otimes \circ (\mathrm{id}_{\mathcal{C}} \times \otimes)$ as follows: (the front and back faces are connected by equivalences induced by (M2))
$$\begin{tikzcd} & & (\mathcal{C}_{\langle 1\rangle})^3 \arrow[r, "\otimes\times\mathrm{id}_{\mathcal{C}}"] \arrow[d, "\mathrm{id}_{\mathcal{C}}\times\otimes"', "\simeq"] & (\mathcal{C}_{\langle 1\rangle})^2 \arrow[d, "\otimes"] \\ \mathcal{C}_{\langle 3\rangle} \arrow[urr, "\simeq"] \arrow[r, "(\tau^3_1)_!"'] \arrow[d, "(\tau^3_2)_!"'] & \mathcal{C}_{\langle 2\rangle} \arrow[urr] \arrow[d, "\simeq"', "(\tau^1_2)_!"] & (\mathcal{C}_{\langle 1\rangle})^2 \arrow[r, "\otimes"] & \mathcal{C}_{\langle 1\rangle} \\ \mathcal{C}_{\langle 2\rangle} \arrow[urr] \arrow[r, "(\tau^1_2)_!"'] & \mathcal{C}_{\langle 1\rangle} \arrow[urr, "="'] & & \end{tikzcd}$$

4. (Natural isomorphisms l, r) Similar with above, we first construct a commutative diagram in $\mathcal{F}in_*$, and deduce the unique natural isomorphisms we want.

   The commutative diagram is as follows:
$$\begin{tikzcd} \langle 0\rangle \wedge \langle 1\rangle \arrow[d] & \langle 1\rangle \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d, "="] & \langle 1\rangle \wedge \langle 0\rangle \arrow[d] \\ \langle 1\rangle \wedge \langle 1\rangle \arrow[r] \arrow[dr, "\simeq"'] & \langle 1\rangle & \langle 1\rangle \wedge \langle 1\rangle \arrow[l] \arrow[dl, "\simeq"] \\ & \langle 2\rangle \arrow[u, "\beta"] & \end{tikzcd}$$

Then we can induce the unique natural isomorphisms $l : (\mathbf{1}\otimes -) \Rightarrow \mathrm{id}_{\mathcal{C}}$ and $r : (-\otimes \mathbf{1}) \Rightarrow \mathrm{id}_{\mathcal{C}}$ by the follow diagram:

$$\begin{tikzcd}
\mathcal{C}_{\langle 0\rangle} \times \mathcal{C}_{\langle 1\rangle} \arrow[d] & \mathcal{C}_{\langle 1\rangle} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d, "="] & \mathcal{C}_{\langle 1\rangle} \times \mathcal{C}_{\langle 0\rangle} \arrow[d] \\
\mathcal{C}_{\langle 1\rangle} \times \mathcal{C}_{\langle 1\rangle} \arrow[r, "\otimes"] & \mathcal{C}_{\langle 1\rangle} & \mathcal{C}_{\langle 1\rangle} \times \mathcal{C}_{\langle 1\rangle} \arrow[l, "\otimes"']
\end{tikzcd}$$

5. (Braiding B) For the morphism $\sigma : \langle 2\rangle \to \langle 2\rangle$ that exchanges the elements 1 and 2, we have $\beta \circ \sigma = \beta$ and $\rho^i \circ \sigma = \rho^{\sigma(i)}$. Then we have the following commutative diagram and the induced diagram

$$\begin{tikzcd}
\langle 1\rangle \wedge \langle 1\rangle \arrow[dr] & \langle 2\rangle \arrow[l, "\rho^1\times\rho^2"'] \arrow[r, "\rho^2\times\rho^1"] \arrow[d, dashed, "\beta=\beta\circ\sigma"] & \langle 1\rangle \wedge \langle 1\rangle \arrow[dl] \\
 & \langle 1\rangle &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}_{\langle 1\rangle} \times \mathcal{C}_{\langle 1\rangle} \arrow[dr, "\otimes"'] & \mathcal{C}_{\langle 2\rangle} \arrow[l, "\rho^1_!\times\rho^2_!"'] \arrow[r, "\rho^2_!\times\rho^1_!"] \arrow[d, dashed, "\beta_!=(\beta\circ\sigma)_!"] & \mathcal{C}_{\langle 1\rangle} \times \mathcal{C}_{\langle 1\rangle} \arrow[dl, "\otimes"] \\
 & \mathcal{C}_{\langle 1\rangle} &
\end{tikzcd}$$

Define $\tau := (\rho^2_! \times \rho^1_!) \circ (\rho^1_! \times \rho^2_!)^{-1}$, and we can construct the unique natural isomorphism $B : \otimes \Rightarrow \otimes^{rev} := \otimes \circ \tau$.

Since these natural isomorphisms are all unique, the unit, pentagon, and hexagon axioms are all satisfied, and the braiding is symmetric. Therefore we can define a symmetric monoidal structure on $\mathcal{C}_{\langle 1\rangle}$.

Define the (large) category $CAlg(Cat)$, whose objects are (small) coCartesian fibrations over $\mathcal{F}in_*$ satisfying (M1) and (M2), and functors over $\mathcal{F}in_*$ are preserving coCartesian edges, and the (large) category $CAlg(Cat)'$, whose objects are (small) symmetric monoidal categories and functors are symmetric monoidal functors. From the former discussion, we can construct a functor

$$F : CAlg(Cat)' \to CAlg(Cat),\ (\mathcal{A}, \otimes) \mapsto \mathcal{A}^{\otimes},$$

$$((\mathcal{A}, \otimes) \xrightarrow{f} (\mathcal{B}, \otimes)) \mapsto (\mathcal{A}^{\otimes} \xrightarrow{F(f)} \mathcal{B}^{\otimes}),\ s.t.\ F(f)([A_1, ..., A_m]) = [f(A_1), ..., f(A_m)].$$

Also, we can define a functor

$$G : CAlg(Cat) \to CAlg(Cat)',\ \mathcal{C} \mapsto \mathcal{C}_{\langle 1\rangle}, G(g : \mathcal{C} \to \mathcal{D}) = g_{\langle 1\rangle}, \text{which is induced by } g.$$

We can check that we can construct natural isomorphisms $G \circ F \Rightarrow \mathrm{id}_{CAlg(Cat)'}$ and $F \circ G \Rightarrow \mathrm{id}_{CAlg(Cat)}$, which means these two categories are equivalent. Therefore we have shown that it is equivalent to define the symmetric monoidal category as a functor $p : \mathcal{C}^{\otimes} \to \mathcal{F}in_*$ that satisfies (M1) and (M2).

In general, we can simply give a definition of symmetric monoidal ∞-categories as follows, and more details are remained in the next section.

Definition 1.12. Let $p : X \to S$ be an inner fibration of simplical sets, and $f : x \to y$ an edge in X. We say f is **p-coCartesian** if for every $n \geq 2$ there is a lifting for every commutative diagram below:

$$\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[d, hook] \arrow[dr, "f"] & \\
\Lambda^n_0 \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

Definition 1.13. We say that p defined above is a **coCartesian fibration** of simplical sets if for every edge $g : a \to b$ in S and every vertex $x \in X$ s.t. $p(x) = a$, there exists a p-coCartesian edge $f : x \to y$ s.t. $p(f) = g$.

Construction 1.14. For the map of simplicial sets $p : X \to S$, and a vertex $s \in S$, we can define a new simplicial set X_s as a pullback of $\Delta^0 \xrightarrow{s} S \xleftarrow{p} X$.

If p is a coCartesian fibration, given an edge $g : a \to b$ in S, and any vertex $x \in X_a$, there exists a p-coCartesian edge $f_x : x \to y_x$ s.t. $pf_x = g$. Then we can define an induced map $g_! : X_a \to X_b$ s.t. $g_!(x) = y_x$, and the higher edges can be extended via the lifting property of the coCartesian fibration. Moreover, the choice of $g_!$ is unique up to contractible ambiguity.

Definition 1.15. A **symmetric monoidal ∞-category** is a coCartesian fibration of simplicial sets $p : \mathcal{C}^{\otimes} \to \mathcal{N}(\mathcal{F}in_*)$ such that the map:

$$\prod_{i=1}^{n}(\rho^i_!) : \mathcal{C}^{\otimes}_{\langle n \rangle} \xrightarrow{\simeq} \prod_{i=1}^{n}(\mathcal{C}^{\otimes}_{\langle 1 \rangle}),\ \text{for maps } \{\rho^i : \langle n \rangle \to \langle 1 \rangle\}_{1 \le i \le n} \text{ defined earlier,}$$

is an equivalence.

1.2 Definition of colored operads

For symmetric monoidal categories, we can see that one of the core part is to construct the "n-to-1" operation, and the corresponding coherent condition of this operation. Now we introduce colored operads, which is a generalization of symmetry monoidal categories that focus on the "n-to-1" operation.

Definition 1.16. A **colored operad** $\mathcal{O}$ consists of the following data:

- a collection of objects/colors $\{X, Y, Z, ...\}$ of $\mathcal{O}$.
- for any finite set I, I-indexed collection of objects $\{X_i\}_{i \in I} \in \mathcal{O}$, and $Y \in \mathcal{O}$, a set $Mul_{\mathcal{O}}(\{X_i\}_{i\in I}, Y)$ of morphisms from $\{X_i\}_{i \in I}$ to Y.
- for any map of finite sets $I \to J$ with fibers $\{I_j\}_{j \in J}$, given any finite collection of objects $\{X_i\}_{i \in I}$, $\{Y_j\}_{j \in J}$ and Z, a composition map:

$$\prod_{j \in J} Mul_{\mathcal{O}}(\{X_i\}_{i \in I_j}, Y_j) \times Mul_{\mathcal{O}}(\{Y_j\}_{j \in J}, Z) \to Mul_{\mathcal{O}}(\{X_i\}_{i \in I}, Z)$$

- a collection of morphsims $\{id_X \in Mul_{\mathcal{O}}(\{X\}, X)\}_{X \in \mathcal{O}}$ that are both left and right units for composition:

$$Mul_{\mathcal{O}}(\{X_i\}_{i \in I}, Y) \simeq Mul_{\mathcal{O}}(\{X_i\}_{i \in I}, Y) \times \{id_Y\} \simeq (\prod_{i \in I}\{id_{X_i}\}) \times Mul_{\mathcal{O}}(\{X_i\}_{i \in I}, Y)$$

- for any map of finite sets $I \to J \to K$ with $\{W_i\}_{i \in I}, \{X_j\}_{j \in J}, \{Y_k\}_{k \in K}$, and Z, the diagram:

$$\begin{tikzcd}
\prod_{k\in K} Mul_{\mathcal{O}}(\{W_i\}_{i\in I_k}, Y_k) \times Mul_{\mathcal{O}}(\{Y_k\}_{k\in K}, Z) \arrow[dr] & \\
\prod_{j\in J} Mul_{\mathcal{O}}(\{W_i\}_{i\in I_j}, X_j) \times \prod_{k\in K} Mul_{\mathcal{O}}(\{X_j\}_{j\in J_k}, Y_k) \times Mul_{\mathcal{O}}(\{Y_k\}_{k\in K}, Z) \arrow[u] \arrow[d] & Mul_{\mathcal{O}}(\{W_i\}_{i\in I}, Z) \\
\prod_{j\in J} Mul_{\mathcal{O}}(\{W_i\}_{i\in I_j}, X_j) \times Mul_{\mathcal{O}}(\{X_j\}_{j\in J}, Z) \arrow[ur] &
\end{tikzcd}$$

  is commutative.

Remark 1.17. Colored operads can be regarded as a category-like thing, or a category with additional data, that is, a category with those colors as objects, $Mor_{\mathcal{O}}(X,Y) := Mul_{\mathcal{O}}(\{X\},Y)$ as morphisms, and other sets $Mul_{\mathcal{O}}(\{X_j\}_{j\in J},Y)$ as additional data. There is a forgetful functor from the category $\mathcal{CO}p$ of (small) colored operads to the category $\mathcal{C}at$ of (small) categories. Moreover, this forgetful functor has a left adjoint, which maps objects of a category to colors, and $Mor(X,Y)$ to $Mul(\{X\},Y)$ without additional data, that is, $Mul(\{X_j\}_{j\in J},Y) = \emptyset$ if the index set J has more than one element.

Example 1.18. Every symmetric monoidal category $(\mathcal{C},\otimes)$ can be regarded as a colored operad, with objects as colors, and $Mul(\{X_j\}_{j\in J},Y) := Mor_{\mathcal{C}}(\otimes_{j\in J}X_j,Y)$ for every finite set J.

Example 1.19. An operad $\mathcal{O}$ by May's definition [1] is a colored oeprad with only one color $\mathbf{1}$, and n-ary operations $\{\mathcal{O}_n := Mul_{\mathcal{O}}(\{\mathbf{1}\}_{1\le i\le n},\mathbf{1})\}_{n\ge 0}$ with an action of symmetry group Σ_n on $\mathcal{O}_n$ for each n and a composition map $\mathcal{O}_m\times(\prod_{1\le i\le m}\mathcal{O}_{n_i})\to\mathcal{O}_{n_1+\ldots+n_m}$ given by the composition law of colored operads.

Inspired by the alternative definition of symmetric monoidal categories, we can also make a construction of colored operads like 1.3 as follows:

Construction 1.20. For a colored oeprad $\mathcal{O}$, define a category $\mathcal{O}^{\otimes}$ consisting of

1. objects are finite sequence of colors $\mathcal{O}_1,\ldots,\mathcal{O}_n$ in $\mathcal{O}$.
2. Given two sequences of objects $[X_1,\ldots,X_m]$ and $[Y_1,\ldots,Y_n]$, a morphism $(\alpha,\{\phi_j\}_{1\le j\le n})$ between them is determined by a map of sets $\alpha:\langle m\rangle\to\langle n\rangle$ in $\mathcal{F}in_*$ and $\phi_j\in Mul_{\mathcal{O}}(\{X_i\}_{i\in\alpha^{-1}(j)},Y_j)$ for each $j=1,\ldots,n$.
3. Composition of morphisms is determined by the composition laws on $\mathcal{F}in_*$ and $\mathcal{O}$.

If we equip a forgetful functor $\pi:\mathcal{O}^{\otimes}\to\mathcal{F}in_*$, then we can reconstruct $\mathcal{O}$ up to canonical equivalence, with $\mathcal{O}^{\otimes}_{\langle 1\rangle}=\pi^{-1}(\langle 1\rangle)$. Using those morphisms $\{\rho^i\}_{1\le i\le n}$ appeared before, we can induce an equivalence of categories $\mathcal{O}^{\otimes}_{\langle n\rangle}\simeq(\mathcal{O}^{\otimes}_{\langle 1\rangle})^n$, and the morphism sets $Mul_{\mathcal{O}}(\{X_i\}_{1\le i\le n},Y)$ can be determined by the map $\langle n\rangle\to\langle 1\rangle$, $i\mapsto 1$ when $i\ne 0$. Thus we can write down an alternative definition of colored operads:

Definition 1.21. A colored operad is a functor of categories $\pi:\mathcal{O}\to\mathcal{F}in_*$, such that $\mathcal{O}_{\langle n\rangle}\simeq(\mathcal{O}_{\langle 1\rangle})^n$ induced by $\{\rho^i\}_{1\le i\le n}$, and the fiber $\mathcal{O}_{\langle 1\rangle}$ consists of the data in the definition 1.16.

This definition gives us a way to move to ∞-categorical setting more conveniently. We will start discuss about the ∞-categorical cases in the next section.

2 ∞-operads

2.1 Definition

To begin with, we first introduce two kinds of special maps in $\mathcal{F}in_*$:

Definition 2.1. In the category $\mathcal{F}in_*$, given a map $f:\langle m\rangle\to\langle n\rangle$, we say:

- f is **inert** if for each $i\in\langle n\rangle^{\circ}$, the inverse image $f^{-1}(i)$ has exactly one element.
- f is **active** if the inverse image $f^{-1}(0)=0$.

Definition 2.2. An ∞**-operad** is a functor is a functor $p:\mathcal{O}^{\otimes}\to\mathcal{N}(\mathcal{F}in_*)$ between ∞-categories satisfying:

1. for any inert morphism $f : \langle m\rangle \to \langle n\rangle$ in $\mathcal{N}(\mathcal{F}in_*)$, and any object $C \in \mathcal{O}^{\otimes}_{<m>}$, there is a p-coCartesian morphism $\overline{f} : C \to C'$ in $\mathcal{O}^{\otimes}$ lifting f. In particular, f induces a functor $f_! : \mathcal{O}^{\otimes}_{\langle m\rangle} \to \mathcal{O}^{\otimes}_{\langle n\rangle}$.

2. Choose $C \in \mathcal{O}^{\otimes}_{\langle m\rangle}$, $C' \in \mathcal{O}^{\otimes}_{\langle n\rangle}$, and a map $f : \langle m\rangle \to \langle n\rangle$, define $Mor^{f}_{\mathcal{O}^{\otimes}}(C, C')$ to be the union of those connected components of $Mor_{\mathcal{O}^{\otimes}}(C, C')$ which lie over f.

   Choosing p-coCartesian morphisms $\{C' \to C'_i\}_{1\leq i\leq n}$ lying over the inert morphism $\rho^i : \langle n\rangle \to \langle 1\rangle$ for $1 \leq i \leq n$, they can deduce a homotopy equivalence:

$$Mor^{f}_{\mathcal{O}^{\otimes}}(C, C') \to \prod_{1\leq i\leq n} Mor^{\rho^i\circ f}_{\mathcal{O}^{\otimes}}(C, C'_i)$$

3. For every finite collection of objects $C_1, ..., C_n \in \mathcal{O}^{\otimes}_{<1>}$, there exists an object $C \in \mathcal{O}^{\otimes}_{<n>}$ and a collection of p-coCartesian morphisms $C \to C_i$ covering $\rho^i : \langle n\rangle \to \langle 1\rangle$.

Remark 2.3. By abuse terminology, we often refer to $\mathcal{O}^{\otimes}$ as an ∞-operad. Besides, we usually define $\mathcal{O} := \mathcal{O}^{\otimes}_{\langle 1\rangle} \simeq p^{-1}(\langle 1\rangle)$ and call it the **underlying ∞-category** of $\mathcal{O}^{\otimes}$.

Remark 2.4. From HTT 2.3.1.5., we see that p is a inner fibration. Moreover, p is also a categorical fibration. We list some useful lemmas and prove it below.

Definition 2.5. A map $q : S \to S'$ of simplicial sets is a **categorical fibration** if q has the right lifting property with respect to all maps which are cofibrations and categorical equivalences.

Lemma 2.6. Followed from the above definition, if S′ is an ∞-category, then q is a categorical fibration if and only if q is an inner fibration and for any equivalence $f : y \to y'$ in S', and any object $x \in S$ s.t. $q(x) = y$, there is an equivalence $\overline{f} : x \to x'$ in S s.t. $q(\overline{f}) = f$.

Proof. HTT 2.4.6.5. □

Lemma 2.7. Let q from the above definition be an inner fibration between ∞-categories and $g : s \to s'$ a morphism in S. Then the following conditions are equivalent:

1. the morphism f is an equivalence in S

2. the morphism f is q-coCartesian, and q(f) is an equivalence in S'.

Proof. Dual of HTT 2.4.1.5. □

Since our ∞-operad p is an inner fibration, then using the above two lemmas, we can deduce that p is a categorical fibration.

Remark 2.8. We can observe that for each $n \geq 0$, the functors $\{\rho^i_! : \mathcal{O}^{\otimes}_{\langle n\rangle} \to \mathcal{O}\}_{1\leq i\leq n}$ can deduce an equivalence of ∞-categories $\phi : \mathcal{O}^{\otimes}_{\langle n\rangle} \to (\mathcal{O})^n$, because the definition 2.2, (2.) ensures that ϕ is fully faithful, and (3.) says that ϕ is essentially surjective.

Since we have the equivalence $\mathcal{O}^{\otimes}_{\langle n\rangle} \to (\mathcal{O})^n$ now, we can identify objects in $\mathcal{O}^{\otimes}_{\langle n\rangle}$ as a sequence $(X_1, ..., X_n)$, for each $X_i \in \mathcal{O}$. After choosing an object $Y \in \mathcal{O}$, we can define $Mul_{\mathcal{O}}(\{X_i\}_{1\leq i\leq n}, Y)$ to be the union of components of $Mor_{\mathcal{O}^{\otimes}}((X_i)_{1\leq i\leq n}, Y)$ which lie over the active map $\langle n\rangle \to \langle 1\rangle$. We can regard $Mul_{\mathcal{O}}(\{X_i\}_{1\leq i\leq n}, Y)$ as an object in the homotopy category of spaces, which is well-defined up to canonical isomorphism.

Given an ∞-operad $\mathcal{O}^{\otimes}$, we can regard it as an ∞-category $\mathcal{O}$ with a series of morphsim spaces $Mul_{\mathcal{O}}(\{X_i\}_{1\leq i\leq n}, Y)$ for any $n \geq 0$. These morphism spaces can be composed satisfying the composition law of the definition 1.16 up to coherent homotopy. Thus it is indeed an ∞-version of colored operads.

Remark 2.9. More specially, if we want to describe $\infty-$version operads in 1.19 rather only colored operads, we can consider the ∞-operad $p : \mathcal{O}^{\otimes} \to \mathcal{N}(\mathcal{F}in_*)$ equipped with an essentially surjective functor $\Delta^0 \to \mathcal{O}$, since there is only one color in any operads.

Example 2.10. By definition, any symmetric monoidal ∞-category is a special case of ∞-operads.

Example 2.11. The identity map $\mathcal{N}(\mathcal{F}in_*) \to \mathcal{N}(\mathcal{F}in_*)$ is an ∞-operad, whose underlying ∞-category is isomorphic to Δ^0.

Example 2.12. Given a colored operad $\mathcal{O}$, by the construction 1.20, we can obtain an ∞-operad $\mathcal{N}(\mathcal{C}^{\otimes}) \to \mathcal{N}(\mathcal{F}in_*)$ after taking the nerve functor on both sides.

2.2 Maps of ∞-Operads

Definition 2.13. Given an ∞-operad $p : \mathcal{O}^{\otimes} \to \mathcal{N}(\mathcal{F}in_*)$ and a morphism f in $\mathcal{O}^{\otimes}$, we say

- the map f is **inert** if $p(f)$ is inert and p-coCartesian.
- the map f is **active** if $p(f)$ is active.

Definition 2.14. Given two ∞-operads $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$, an ∞**-operad map** from $\mathcal{O}^{\otimes}$ to $\mathcal{O}'^{\otimes}$ is a map of simplical sets $f : \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ such that:

1. the following diagram commutes: $$\begin{tikzcd} \mathcal{O}^{\otimes} \arrow[rr, "f"] \arrow[dr] & & \mathcal{O}'^{\otimes} \arrow[dl] \\ & \mathcal{N}(\mathcal{F}in_*) & \end{tikzcd}$$
2. f maps inert morphisms in $\mathcal{O}^{\otimes}$ to inert morphisms $\mathcal{O}'^{\otimes}$.

Then we can define $Alg_{\mathcal{O}}(\mathcal{O}')$ to be the full subcategory of $Fun_{\mathcal{N}(\mathcal{F}in_*)}(\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes})$ spanned by all the ∞-operad maps.

Definition 2.15. A map of ∞-operads $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a **fibration** of ∞-operads if q is a categorical fibration.

There is a special case of fibrations of ∞-operads: coCartesian fibrations, with very good properties:

Proposition 2.16. Given an ∞-operad $\mathcal{O}^{\otimes}$, and a coCartesian fibration of simplical sets $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$, then the following conditions are equivalent:

1. The composition map $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes} \to \mathcal{N}(\mathcal{F}in_*)$ exhibits $\mathcal{C}^{\otimes}$ as an ∞-operad.
2. For any object $T \simeq T_1 \oplus T_2 \oplus ... \oplus T_n \in \mathcal{C}^{\otimes}_{<n>}$, the inert morphisms $\{T \to T_i\}_{1 \le i \le n}$ induce an equivalence of ∞-categories $\mathcal{C}^{\otimes}_T \to \prod_{1 \le i \le n} \mathcal{C}^{\otimes}_{T_i}$.

Proof. HA 2.1.2.12. □

Definition 2.17. Given an ∞-operad $\mathcal{O}^{\otimes}$, we call a map $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a **coCartesian fibration** of ∞-operads, if p satisfies the proposition 2.16. In this case we also say that p exhibits $\mathcal{C}^{\otimes}$ as an $\mathcal{O}$**-monoidal** ∞**-category**.

Remark 2.18. Under the above definition, we will generally denote the fiber product $\mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{O}$ by $\mathcal{C}$, and sometimes abuse terminology by saying that $\mathcal{C}$ is an $\mathcal{O}$-monoidal ∞-category.

For a coCartesian fibration of ∞-operads $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$, given a object $X \in \mathcal{O}^{\otimes}_{\langle n \rangle}$ that can be identified by a sequence of objects $\{X_i \in \mathcal{O}\}_{1 \le i \le n}$, by the proposition 2.16, we have a canonical equivalence $\mathcal{C}^{\otimes}_X \simeq \prod_{1 \le i \le n} \mathcal{C}_{X_i}$, where $\mathcal{C}^{\otimes}_X$ and $\mathcal{C}_{X_i}$ are defined as in the construction 1.14.

Given any morphism $f \in Mul_{\mathcal{O}}(\{X_i\}_{1 \le i \le n}, Y)$ of $\mathcal{O}$, the coCartesian fibration p determines a functor well-defined up to equivalence as follows:

$$\otimes_f : \prod_{1 \le i \le n} \mathcal{C}_{X_i} \simeq \mathcal{C}^{\otimes}_X \to \mathcal{C}_Y$$

If we give a coCartesian fibration of ∞-operads $p : \mathcal{C}^{\otimes} \to \mathcal{N}(\mathcal{F}in_*)$, we can observe from above that it exactly fits the definition of symmetric monoidal ∞-category 1.15.

Taking two active morphisms $\alpha : \langle 0 \rangle \to \langle 1 \rangle$ and $\beta : \langle 2 \rangle \to \langle 1 \rangle$, we can induce functors

$$\Delta^0 \to \mathcal{C} \qquad \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

which indicate the unit object and tensor product in the underlying ∞-category $\mathcal{C}$ respectively, and they are well-defined up to a contractible space of a choice. Then we can check that these two functors satisfy the unit, pentagon and hexagon axioms of original symmetric monoidal category up to homotopy. In particular, these two functors give the homotopy category $h\mathcal{C}$ a symmetric monoidal structure.

Finally, there is a useful criterion for detecting ∞-operad fibrations:

Proposition 2.19. Given a map of ∞-operads $q : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$, then the following conditions are equivalent:

1. The map q is a fibration of ∞-operads.
2. For every object $C \in \mathcal{C}^{\otimes}$, and every inert morphism $f : q(C) \to X$ in $\mathcal{O}^{\otimes}$, there is an inert morphism $\overline{f} : C \to \overline{X}$ in $\mathcal{C}^{\otimes}$ s.t. $f = q(\overline{f})$.

Moreover, if these conditions are satisfied, then for any morphism $g \in \mathcal{C}^{\otimes}$, we have that g is inert and q-coCartesian if and only if $q(g)$ is inert.

Proof. HA 2.1.2.22. □

2.3 Algebra objects

Definition 2.20. Given a fibration of ∞-operads $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$, and a map of ∞-operads $\alpha : \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$, let $Alg_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ be the full subcategory of $Fun_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})$, spanned by the maps of ∞-operads.

Equivalently, $Alg_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ can be described as the fiber over the vertex α of the categorical fibration $Alg_{\mathcal{O}}(\mathcal{C}) \to Alg_{\mathcal{O}'}(\mathcal{O})$ given by the composition with p.

$$\begin{tikzcd} \mathcal{O}'^{\otimes} \arrow[rr, dashed] \arrow[dr, "\alpha"'] & & \mathcal{C}^{\otimes} \arrow[dl, "p"] \\ & \mathcal{O}^{\otimes} & \end{tikzcd}$$

When $\mathcal{O}'^{\otimes} = \mathcal{O}^{\otimes}$ and α is an identity, we can denote $Alg_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ by $Alg_{/\mathcal{O}}(\mathcal{C})$.

Remark 2.21. The ∞-category of algebra objects $Alg_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ is generally not equivalent to $Alg_{\mathcal{O}'}(\mathcal{C})$ defined as in 2.13, except that $\mathcal{O}^{\otimes} = \mathcal{N}(\mathcal{F}in_*)$.

Definition 2.22. Given an ∞-operads $\mathcal{O}^{\otimes}$, and two coCartesian fibrations of ∞-operads $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ and $q : \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$, we say that an ∞-operad map $f \in Alg_{\mathcal{C}}(\mathcal{D})$ is an **$\mathcal{O}$-monoidal functor** if it maps all p-coCartesian morphisms to q-coCartesian morphisms. Define $Fun^{\otimes}_{\mathcal{O}}(\mathcal{C}, \mathcal{D})$ to be the full subcategory of $Fun_{\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ spanned by the $\mathcal{O}$-monoidal functors.

Remark 2.23. Given a functor F in $\in Fun^{\otimes}_{\mathcal{O}}(\mathcal{C},\mathcal{D})$ between $\mathcal{O}$-monoidal ∞-categories, by HTT2.4.4.4., there are several equivalent conditions:

1. The functor F is an equivalence.

2. The underlying map of ∞-categories $\mathcal{C} \to \mathcal{D}$ is an equivalence.

3. $\forall X \in \mathcal{O}$, the induced map of fibers $\mathcal{C}_X \to \mathcal{D}_X$ is an equivalence.

Notice that this is not generally ture when $F \in Alg_{\mathcal{C}/\mathcal{O}}(\mathcal{D})$.

Remark 2.24. When $\mathcal{O}^{\otimes} = \mathcal{N}(\mathcal{F}in_*)$, we can observe that p and q are symmetric monoidal ∞-categories. Then we write $Fun^{\otimes}(\mathcal{C},\mathcal{D}) := Fun^{\otimes}_{\mathcal{O}}(\mathcal{C},\mathcal{D})$ and its objects are called **symmetric monoidal functors** from $\mathcal{C}^{\otimes}$ to $\mathcal{D}^{\otimes}$.

Pay attention that $Fun^{\otimes}(\mathcal{C},\mathcal{D})$ is generally not equivalent to $Alg_{\mathcal{C}}(\mathcal{D})$, whose objects map p-coCartesian morphisms over inert morphisms to q-coCartesian morphisms. We call objects in $Alg_{\mathcal{C}}(\mathcal{D})$ **lax symmetric monoidal functors**.

Given a lax symmetric monoidal functor $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$, any object $x \in \mathcal{C}^{\otimes}_{\langle 2\rangle}$ and the active morphism $\alpha : \langle 2\rangle \to \langle 1\rangle$, we can obtain a p-coCartesian morphism $f_x : x \to y_x$ in $\mathcal{C}^{\otimes}$ and a q-coCartesian morphism $g_{F(x)} : F(x) \to y_{F(x)}$ in $\mathcal{D}^{\otimes}$. We can choose a sequence $(x_1, x_2) \in \mathcal{C}^{\otimes}_{\langle 1\rangle} \times \mathcal{C}^{\otimes}_{\langle 1\rangle}$ to identify $x \in \mathcal{C}^{\otimes}_{\langle 2\rangle}$, and $(F(x_1), F(x_2)) \in \mathcal{D}^{\otimes}_{\langle 1\rangle} \times \mathcal{D}^{\otimes}_{\langle 1\rangle}$ to identify $F(x) \in \mathcal{D}^{\otimes}_{\langle 2\rangle}$. Assume that α induces the tensor products $\otimes_{\mathcal{C}}$ and $\otimes_{\mathcal{D}}$ respectively, define $y_x := x_1 \otimes_{\mathcal{C}} x_2$ and $y_{F(x)} := F(x_1) \otimes_{\mathcal{D}} F(x_2)$, and we have horn-like morphisms in $\mathcal{D}^{\otimes}$:

$$\begin{tikzcd} & F(x) \arrow[dl, "g_{F(x)}"'] \arrow[dr, "F(f_x)"] & \\ F(x_1) \otimes_{\mathcal{D}} F(x_2) & & F(x_1 \otimes_{\mathcal{C}} x_2) \end{tikzcd}$$

Since $g_{F(x)}$ is q-coCartesian, then we have a morphism $u_x : F(x_1) \otimes_{\mathcal{D}} F(x_2) \to F(x_1 \otimes_{\mathcal{C}} x_2)$, for all $x \in \mathcal{C}^{\otimes}_{\langle 2\rangle}$, and we can obtain a natural transformation $u : \alpha_! \circ F \Rightarrow F \circ \alpha_!$. More generally, we can also check by the same way that any morphism $\beta : \langle m\rangle \to \langle n\rangle$ can induce a natural transformation $u_\beta : \beta_! \circ F \Rightarrow F \circ \beta_!$. When taking $\langle 0\rangle \to \langle 1\rangle$, we can obtain a morphism $\mathbf{1}_{\mathcal{D}} \to F(\mathbf{1}_{\mathcal{C}})$. Finally we can check that these maps above are compatible with unitality, commutativity and associativity properties of $\otimes_{\mathcal{C}}$ and $\otimes_{\mathcal{D}}$.

If F given above is a symmetric monoidal functor, then $F(f_x)$ is also q-coCartesian, which implies there is an inverse of the morphism u_x, that is, u_x gives an equivalence between $F(x_1) \otimes_{\mathcal{D}} F(x_2)$ and $F(x_1 \otimes_{\mathcal{C}} x_2)$.

2.4 Examples

Now we will give some examples of ∞-operads and see what their algebra objects look like.

Example 2.25. Define the **commutative ∞-operad** $Comm^{\otimes}$ to be $\mathcal{N}(\mathcal{F}in_*)$. When $\mathcal{O}'^{\otimes} = \mathcal{O}^{\otimes} = Comm^{\otimes}$, the ∞-category of commutative algebra objects of $\mathcal{C}$ $CAlg(\mathcal{C}) := Alg_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) = Alg_{\mathcal{O}}(\mathcal{C})$ consists of sections of the functor $p : \mathcal{C}^{\otimes} \to \mathcal{N}(\mathcal{F}in_*)$.

Example 2.26. Let $Triv$ be the subcategory of $\mathcal{F}in_*$ having the same objects, but only containing inert morphisms. The inclusion $Triv \hookrightarrow \mathcal{F}in_*$ induces a functor $\mathcal{T}riv^{\otimes} := \mathcal{N}(Triv) \to \mathcal{N}(\mathcal{F}in_*)$, which exhibits $\mathcal{T}riv^{\otimes}$ as an ∞-operad. We call it **trivial ∞-operad**. By HA 2.1.3.6., given an ∞-operad $\mathcal{O}^{\otimes}$, we can construct a trivial Kan fibration $Alg_{\mathcal{T}riv}(\mathcal{O}) \to \mathcal{O}$.

Example 2.27. Let $\mathcal{F}in_*^{inj}$ be the subcategory of $\mathcal{F}in_*$ having the same objects, but having morphisms $f : \langle m\rangle \to \langle n\rangle$ s.t. the set $f^{-1}(i)$ has at most one element for $1 \leq i \leq n$. Thus $Triv$ defined above is a subcategory of $\mathcal{F}in_*^{inj}$. Taking its nerve and then we define an ∞-operad $\mathbb{E}_0^{\otimes} := \mathcal{N}(\mathcal{F}in_*^{inj})$. Then by HA 2.1.3.9., taking a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ with unit object $\mathbf{1}$, we can identify $Alg_{/\mathbb{E}_0}(\mathcal{C})$ with the ∞-category $\mathcal{C}_{\mathbf{1}/}$. In this situation, every such algebra object is equipped with a unit, but no other structure.

Example 2.28. Define a colored oeprad **Assoc** called the **associative operad**, which has only one color $\mathfrak{a}$, and for any finite set I, the set of operations $Mul_{\mathbf{Assoc}}(\{\mathfrak{a}\}_{i\in I}, \mathfrak{a})$ is the set of linear orderings of I. Define $\mathbf{Assoc}^{\otimes}$ to be the category obtained after the construction 1.20, and define $Assoc^{\otimes}$ to be $\mathcal{N}(\mathbf{Assoc}^{\otimes})$ as the **associative ∞-operad**.

Given a fibration of ∞-operads $q : \mathcal{C}^{\otimes} \to Assoc^{\otimes}$, define $Alg(\mathcal{C})$ to be the $Alg_{/Assoc}(\mathcal{C})$ of ∞-operad sections of q. We call $Alg(\mathcal{C})$ as the ∞-category of associative algebra objects of $\mathcal{C}$.

Definition 2.29. A **monoidal ∞-category** is a coCartesian fibration of ∞-operads $\mathcal{C}^{\otimes} \to Assoc^{\otimes}$.

Example 2.30. For $k \geq 0$, define an open cube of dimension k to be $\square^k := (-1,1)^k$, and we can define a rectilinear embedding $f : \square^k \to \square^k$, $f(x_1, ..., x_k) = (a_1x_1 + b_1, ..., a_kx_k + b_k)$, for some proper $b_i \in (-1,1), a_i \in (0,1), 1 \leq i \leq k$. We can regard the collection of finite disjoint rectilinear embeddings $Rect(\square^k \times S, \square^k) \subset (\mathbb{R}^{2k})^S$ as a topological space, for some finite set S, and define ${}^t\mathbb{E}_k^{\otimes}$ to be the **little k-cubes operad** with n-ary operations $\{Rect(\square^k \times \{1,2,...,n\}, \square^k)\}_{n\in\mathbb{N}}$.

In this situation, we can define a topological category ${}^t\mathbb{E}_k^{\otimes}$ as follows:

1. objects are the same objects in $\mathcal{F}in_*$.
2. morphisms from $\langle m\rangle$ to $\langle n\rangle$ are collection pairs $(\alpha, \{f_j\}_{1\leq j\leq n})$, where α is a map of sets $\langle m\rangle \to \langle n\rangle$, and $f_j : \square^k \times \alpha^{-1}(j) \to \square^k$ for each j=1,...,n. In other words, the set of morphisms from $\langle m\rangle$ to $\langle n\rangle$ can be written as

$$Mor_{{}^t\mathbb{E}_k^{\otimes}}(\langle m\rangle, \langle n\rangle) := \coprod_{\alpha:\langle m\rangle\to\langle n\rangle} \prod_{1\leq j\leq n} Rect(\square^k \times \alpha^{-1}(j), \square^k)$$

which is a topological space.

Construct a forgetful functor ${}^t\mathbb{E}_k^{\otimes} \to \mathcal{F}in_*$, take the coherent nerve functor, and we can obtain an ∞-operad $\mathbb{E}_k^{\otimes} := \mathcal{N}({}^t\mathbb{E}_k^{\otimes}) \to \mathcal{N}(\mathcal{F}in_*)$.

When k=0, it is indeed isomorphic to the ∞-operad $\mathbb{E}_0^{\otimes}$ mentioned in the example 2.27.

When k=1, we can obtain a weak homotopy equivalence of topological categories ${}^t\mathbb{E}_k^{\otimes} \to \mathbf{Assov}^{\otimes}$. After taking the coherent nerve functor, we then have an equivalence of ∞-operads $\mathbb{E}_k^{\otimes} \xrightarrow{\simeq} Assov^{\otimes}$. See the details in HA 5.1.0.7.

When $k = \infty$, define the colimit of the sequence of ∞-operads $(\mathbb{E}_0^{\otimes} \to \mathbb{E}_1^{\otimes} \to \mathbb{E}_2^{\otimes} \to ...)$ to be $\mathbb{E}_{\infty}^{\otimes}$. By HA 5.1.1.4., $\mathbb{E}_{\infty}^{\otimes}$ is equivalent to the commutative ∞-operad $Comm^{\otimes}$.

There are some additional materials to show why we can claim $\mathbb{E}_k^{\otimes}$ is an ∞-operad:

Definition 2.31. A **simplicial colored operad** is a colored operad such that every morphism set is a simplical set.

Construction 2.32. For any simplical colored operad $\mathcal{O}$, we can construct a simplicial category $\mathcal{O}^{\otimes}$ as follows:

1. objects are pairs $(\langle n\rangle, (C_1, ..., C_n))$, where $\langle n\rangle \in \mathcal{F}in_*$ and $C_i \in \mathcal{O}$.

2. for objects $C := (\langle m\rangle, (C_1, ..., C_m))$ and $C' := (\langle n\rangle, (C'_1, ..., C'_n))$, the morphism set

$$Mor_{\mathcal{O}^{\otimes}}(C, C') := \coprod_{\alpha:\langle m\rangle \to \langle n\rangle} \prod_{1\le j\le n} Mul_{\mathcal{O}}(\{C_i\}_{\alpha(i)=j}, C'_j),$$

is also a simplical set.

We can take the coherent nerve of the simplicial category $\mathcal{O}^{\otimes}$ to be $\mathcal{N}^{\otimes}(\mathcal{O})$, which is also called the **operadic nerve** of $\mathcal{O}$.

Definition 2.33. A simplical colored operad $\mathcal{O}$ is **fibrant** if for any finite set I, and $X_i, Y \in \mathcal{O}$, $i \in I$, the simplical set $Mul_{\mathcal{O}}(\{X_i\}_{i\in I}, Y)$ is a Kan complex.

Proposition 2.34. If $\mathcal{O}$ is a fibrant simplical colored operad, then the operadic nerve $\mathcal{N}^{\otimes}(\mathcal{O})$ is an ∞-operad.

Proof. HA 2.1.1.27. □

We can observe that we have a canonical isomorphism $\mathbb{E}_k^{\otimes} \simeq \mathcal{N}^{\otimes}(\mathcal{O})$, where $\mathcal{O}$ is the simplical operad with only one color $\square^k$ and the morphism set $Mul_{\mathcal{O}}(\{\square^k\}_{i\in I}, \square^k) := SingRect(\square^k \times I, \square^k)$ is a Kan complex for any finite set I. Thus we can see $\mathcal{O}$ is fibrant, which means $\mathbb{E}_k^{\otimes} \simeq \mathcal{N}^{\otimes}(\mathcal{O})$ is an ∞-operad by 2.34.

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