UNIVERSITY OF COPENHAGEN
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Master thesis

Braid Group Representations and Orbifoldization

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Abstract

Let G be a finite group. By evaluation of a once-extended 3-2-1-dimensional G-equivariant topological field theory Z on the circle one obtains a G-ribbon category $\mathcal{C}$. From a homotopy G-fixed point in this G-ribbon category $\mathcal{C}$, we can construct two types of representations of framed braid groups $f\mathcal{B}_n$ with n strands for any $n \geq 1$: 1) We can construct a representation of $f\mathcal{B}_n$ by evaluating Z on genus zero surfaces with G-bundle decoration. 2) We can orbifoldize the G-ribbon category $\mathcal{C}$ and obtain the $f\mathcal{B}_n$ representation algebraically by using the braiding and twist of the orbifold category. We prove that these two representations are equivalent.

Contents

1 Introduction and summary

Topological field theories (TFTs) of dimension n, $n \geq 0$, as defined by Atiyah in [A], are symmetric monoidal functors from bordism categories to arbitrary arbitrary symmetric monoidal categories. For example, in the 3-dimensional case, a TFT is a symmetric monoidal functor

$$Z : \mathbf{Cob}(3) \to \mathbf{Vect}.$$

The bordism category $\mathbf{Cob}(3)$ consists of 2-dimensional closed oriented manifolds, or surfaces in short, as objects. A morphism from one surface Σ_0 to another surface Σ_1 is an (equivalence class of a) 3-dimensional compact oriented bordism from Σ_0 to Σ_1, i.e. an (equivalence class of a) 3-dimensional oriented manifold M with boundary equipped with an orientation-preserving diffeomorphism $\overline{\Sigma_0} \coprod \Sigma_1 \to \partial M$, where the overline indicates reversing the orientation of Σ_0. The composition of morphisms is the gluing of bordisms. This category $\mathbf{Cob}(3)$ is symmetric monoidal with the disjoint union as the monoidal product and the empty set as the unit object. The category $\mathbf{Vect}$ is the category of $\mathbb{C}$-vector spaces with the tensor product as the monoidal product.

By definition a 3-dimensional TFT $Z : \mathbf{Cob}(3) \to \mathbf{Vect}$ assigns to a surface a $\mathbb{C}$-vector space and to a 3-dimensional bordism $M : \Sigma_0 \to \Sigma_1$ a linear map $Z(M) : Z(\Sigma_0) \to Z(\Sigma_1)$. For a closed 3-dimensional bordism, we then obtain a manifold invariant.

Considering an orientation-preserving diffeomorphism of a surface $f : \Sigma \to \Sigma$, we can construct a bordism $M_f := \Sigma \times [0, 1]$ with two boundary embeddings

$$\Sigma \times 0 \xrightarrow{id \times 0} \Sigma \times [0, 1] \xleftarrow{f \times 1} \Sigma \times 1.$$

If there is another bordism M_g constructed by another diffeomorphism $g : \Sigma \to \Sigma$, then these two bordisms are equivalent if and only if these two diffeomorphisms f, g are isotopic, see [BK, Theorem 4.2.3.]. Thus we can obtain a representation ρ of the mapping class group $MCG(\Sigma)$ of the surface Σ through a TFT Z as follows:

$$\rho : MCG(\Sigma) \to Aut(Z(\Sigma)), \quad [f] \mapsto Z(M_f) \tag{1.1}$$

Topological field theories are by definition compatible with gluing along manifolds of codimension one. Asking for compatibility with gluing along manifolds of higher codimension naturally leads to extended topological field theories. For example, we can have fully extended TFTs as defined in [L]. For us, gluing along manifolds of codimension two will be sufficient. Therefore, we will treat *once*-extended 3-dimensional TFTs.

As developed in [SP, BDSPV], a once-extended 3-dimensional TFT, i.e. a 3-2-1-dimensional TFT is a symmetric monoidal functor:

$$Z : \mathbf{Cob}(3, 2, 1) \to \mathbf{2Vect}.$$

The symmetric monoidal bicategory $\mathbf{Cob}$(3,2,1) consists of 1-dimensional closed oriented manifolds as objects, bordisms between objects as 1-morphisms, and equivalence classes of bordisms of 1-morphisms as 2-morphisms with additional coherence conditions. The symmetric monoidal bicategory $\mathbf{2Vect}$ consists of abelian $\mathbb{C}$-linear finitely semisimple categories as objects, linear functors as 1-morphisms and natural transformations as 2-morphisms.

Similar as the construction of the representation (1.1) above, we can regard a surface Σ_∂ with boundary as a 1-morphism. For a given orientation-preserving diffeomorphism $f : \Sigma_\partial \to \Sigma_\partial$ fixing the boundary pointwise, we can construct a cylinder $M_f := \Sigma_\partial \times [0, 1]$ as a 2-morphism including two boundary embeddings

$$\Sigma_\partial \times 0 \xrightarrow{id \times 0} \Sigma_\partial \times [0, 1] \xleftarrow{f \times 1} \Sigma_\partial \times 1.$$

For another 2-morphism M_g constructed by another diffeomorphism $g : \Sigma_\partial \to \Sigma_\partial$, two 2-morphisms M_f and M_g are equivalent if and only if f, g are isotopic. By evaluation of Z on M_f, we will get a natural isomorphism $Z(M_f) : Z(\Sigma_\partial) \Rightarrow Z(\Sigma_\partial)$, where $Z(\Sigma_\partial)$ is the functor associated to the surface Σ_∂. Since a 1-dimensional closed oriented manifold must be a disjoint union of circles, we can assume the boundary of Σ_∂ is diffeomorphic to $(\overline{\coprod_{i=1}^{p} \mathbb{S}^1}) \coprod (\coprod_{i=1}^{q} \mathbb{S}^1)$, $p, q \geq 0$. Then we can write down the functor $Z(\Sigma_\partial) : Z(\mathbb{S}^1)^{\boxtimes p} \to Z(\mathbb{S}^1)^{\boxtimes q}$. Taking any object X in $Z(\mathbb{S}^1)^{\boxtimes p}$, we obtain a representation ρ_X of the mapping class group $MCG(\Sigma_\partial)$ of the surface Σ_∂ as below:

$$\rho_X : MCG(\Sigma_\partial) \to Aut(Z(\Sigma_\partial)(X)), \quad [f] \mapsto Z(M_f)(X)$$

Therefore we construct a mapping class group representation of a surface with boundary.

For a finite group G, there is a natural generalization: One can define G-**Cob**(3,2,1) by adding the decoration of a G-bundle: A map to the classifying space BG on every object, 1-morphism, 2-morphism in **Cob**(3,2,1) with additional coherence conditions. On the circle, a G-bundle can be described as an element in G, since the groupoid of G-bundles on $\mathbb{S}^1$ is equivalent to the action groupoid $G//G$ of the action of G on itself by conjugation.

Now we consider once-extended equivariant topological field theories developed in [T], see [SW2] for the extended case. Briefly, a 3-2-1-dimensional once-extended G-equivariant topological field theory (3-2-1-G-TFT) for some finite group G is a symmetric monoidal functor:

$$Z : G - \mathbf{Cob}(3,2,1) \to \mathbf{2Vect}.$$

Taking evaluation on $(\mathbb{S}^1, g)$ for all elements $g \in G$, we will obtain a G-ribbon category $\mathcal{C} := \bigoplus_{g \in G} \mathcal{C}_g$ as shown in [SW2, Theorem 4.1.]. This G-ribbon category $\mathcal{C}$ is a monoidal G-category, where the G-action is a family of functors $\phi_g : \mathcal{C} \to \mathcal{C}$ labeled by elements in $g \in G$ with coherence conditions given in [K, Definition 2.1.]. Every functor ϕ_g maps $Y \in \mathcal{C}_h$ to $g.Y := \phi_g(Y) \in \mathcal{C}_{ghg^{-1}}$ for every $g, h \in G$. Then we have G-braidings $\sigma_{X,Y} : X \otimes Y \to g.Y \otimes X$ and G-twists $\theta_X : X \to g.X$ for every $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h$ for any $g, h \in G$ satisfying some coherence conditions, see Definition 2.13 and Definition 2.14. Also, for the G-action on $\mathcal{C}$, we can consider homotopy G-fixed points as $(X, \{\varphi_h\}_{h \in G})$ in $\mathcal{C}$, where X is an object in $\mathcal{C}$ with a family of coherence isomorphisms $\varphi_h : h.X \to X$, see Definition 3.5.

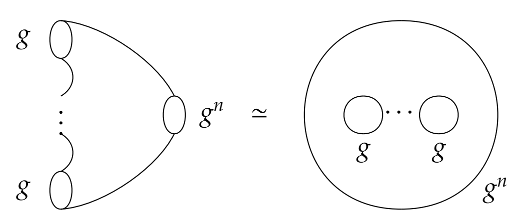
For a given 3-2-1-G-TFT Z, there are two ways to extract representations of framed braid groups, and the main result of this thesis is a proof that these agree. We will construct the first way precisely in Section 4.1. The second way is a corollary of [K, Theorem 3.9.], giving in Section 4.2. Briefly, these two constructions of framed braid group representations from Z are the following:

1. Evaluation of Z on decorated genus zero surfaces and homotopy fixed point data:

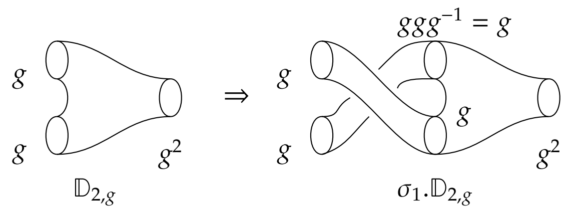
   First, we can consider evaluation of Z on the decorated genus zero surface $\mathbb{D}_{n,g}$ with n ingoing boundary circles $(\mathbb{S}^1, g)$, $g \in G$ and one outgoing boundary circle $(\mathbb{S}^1, g^n)$. Then we can obtain an n-linear functor, i.e. one 1-morphism in **2Vect** as follows:

   $$Z(\mathbb{D}_{n,g}) : \mathcal{C}_g^{\boxtimes n} \to \mathcal{C}_{g^n},\ X^{\boxtimes n} \mapsto X^{\otimes n}$$

   It is important that this surface is homeomorphic to the disk with n holes, whose mapping class group is isomorphic to $f\mathcal{B}_n$, as we draw below:

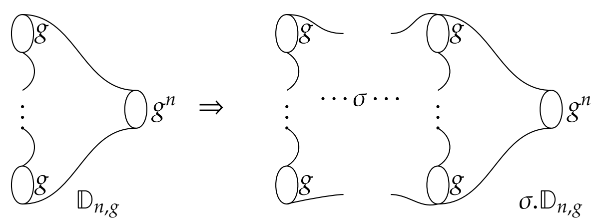


The elements of $f\mathcal{B}_n$ correspond to the braidings and framings of n strands, which can be also used to describe the braidings and twists of the n ingoing boundary circles of $\mathbb{D}_{n,g}$. For example, when n=2, we have the following 2-morphism:



In general, for a 1-morphism (Σ, φ) in $G - \mathbf{Cob}(3,2,1)$ and an element $\sigma \in MCG(\Sigma)$, we define a new 1-morphism $\sigma.(\Sigma, \varphi) := (\Sigma, \varphi \circ \sigma)$. Here we can write $\sigma.\mathbb{D}_{n,g}$ to be the new 1-morphism for the disk $\mathbb{D}_{n,g}$ and an element $\sigma \in f\mathcal{B}_n$.

We can also take a composition of braidings and twists on some of the n ingoing boundary circles in $\mathbb{D}_{n,g}$, which corresponds to an element $\sigma \in f\mathcal{B}_n$. Then we will obtain a 2-morphism



Taking evaluation of Z on $\sigma.\mathbb{D}_{n,g}$, we can obtain an n-linear functor:

$$Z(\sigma.\mathbb{D}_{n,g}) : \mathcal{C}_g^{\boxtimes n} \to \mathcal{C}_{g^n},\ X^{\boxtimes n} \mapsto \sigma.X^{\otimes n}.$$

Therefore we have a natural isomorphism

$$\eta_\sigma : Z(\mathbb{D}_{n,g}) \Rightarrow Z(\sigma.\mathbb{D}_{n,g}), \quad \eta_\sigma(X^{\boxtimes n}) : X^{\otimes n} \to \sigma.X^{\otimes n}. \tag{4.1}$$

The objects $X^{\otimes n}$ and $\sigma.X^{\otimes n}$ will generally not coincide. Now we consider a homotopy G-fixed point $(X, \{\varphi_h\}_{h\in G})$ and use the coherence isomorphisms on X to form an isomorphism $\sigma.X^{\otimes n} \to X^{\otimes n}$, for every $\sigma \in f\mathcal{B}_n$, which leads to the representation:

$$\rho_{X,n}^{\varphi_g} : f\mathcal{B}_n \to Aut(X^{\otimes n}) \tag{4.4}$$

In the above discussion, we only dealed with the object $X \in \mathcal{C}_g$ for some element $g \in G$. In general, we can consider the homotopy G-fixed point $Y = (X, \{\varphi_h\}_{h\in G})$ for arbitrary $X \in \mathcal{C}$, and construct the representation

$$\rho_{X,n}^{\varphi} : f\mathcal{B}_n \to Aut(X^{\otimes n}) \tag{4.7}$$

that can be determined by the representations $\rho_{X,n}^{\varphi_g}$ in (4.4) for all $g \in G$.

2. Construction via orbifoldization:

We can consider the $f\mathcal{B}_n$ representation via orbifoldization. The orbifold category $\mathcal{C}/G$ is a ribbon category and homotopy G-fixed points are its objects. Then we can take one object $Y := (X, \{\varphi_h\}_{h\in G})$ in $\mathcal{C}/G$ and use the braiding and twist in $\mathcal{C}/G$ to construct the $f\mathcal{B}_n$ representation

$$\rho_{Y,n} : f\mathcal{B}_n \to Aut(Y^{\otimes' n}) \tag{4.16}$$

Its underlying representation can be written as:

$$\rho^{Y}_{X,n} : f\mathcal{B}_n \to Aut(X^{\otimes n}) \tag{4.19}$$

There is now the natural question whether the two representations (4.19) and (4.7) are same. In this thesis, we answer this question affirmatively:

Theorem 4.5. *Let Z be a once-extended 3-2-1-dimensional G-equivariant topological field theory, $\mathcal{C}$ the G-ribbon category obtained by evaluation of Z on the circle with varying bundle decoration, and $Y = (X, \{\varphi_h\}_{h\in G})$ a homotopy G-fixed point, i.e. an object $X \in \mathcal{C}$ with a family of coherence isomorphisms*

$$\varphi_h : h.X \to X, \forall h \in G.$$

Then the framed braid group representations (4.7) obtained by evaluating of Z on genus zero surfaces with boundary labels constructed from X and its homotopy fixed data are equivalent to the framed braid group representations (4.19) on $X^{\otimes n}$ obtained from Y as an object in the orbifold category $\mathcal{C}/G$.

Moreover, an object $X \in \mathcal{C}$ can possibly be the underlying object of two different homotopy G-fixed points Y and Y', because being a homotopy G-fixed point is structure and not just a property. In this situation, we have two a priori different framed braid group representations $\rho^{Y}_{X,n}$ and $\rho^{Y'}_{X,n}$. By Theorem 4.5, it does not matter whether these are constructed from the equivariant topological field theories or orbifoldization. It is now a natural question whether $\rho^{Y}_{X,n}$ and $\rho^{Y'}_{X,n}$ agree. For the simple object $X \in \mathcal{C}$, we prove that they are equivalent at least up to invertible scalars, i.e. as projective representations.

Theorem 4.6. *Consider the framed braid group representations $\rho^{Y}_{X,n}$ and $\rho^{Y'}_{X,n}$ associated to two homotopy G-fixed points $Y, Y' \in \mathcal{C}/G$ with the same underlying simple object $X \in \mathcal{C}$. Then there is an equivalence $\rho^{Y}_{X,n} \cong \rho^{Y'}_{X,n}$ as projective framed braid group representations.*

The Theorems 4.5 and 4.6 prove a conjecture of Rowell and Wang [RW, Conjecture 7.2.].

2 Extended equivariant topological field theories and equivariant ribbon categories

In this section, we will introduce the basic definitions. First, we introduce the general once-extended homotopy quantum field theories based on the ordinary case in [T]. We will focus mostly later on the 3-2-1-dimensional case with the aspherical target, and see their relation with G-ribbon monoidal categories. For the details, we refer to [SW2].

2.1 Extended equivariant topological field theories

In [T], homotopy quantum field theories have been defined in an axiomatic way. Also, we can use the categorical language to package those axioms into a symmetric monoidal functor $Z : T - \mathbf{Cob}(n, n-1) \to S$ for $n \geq 1$, an arbitrary target topological space T and symmetric monoidal category S. Here we construct the bordism category $T - \mathbf{Cob}(n, n-1)$ with 1-morphisms as equivalent classes of bordisms between objects.

In the once extented case, in order to extend the ordinary category $T - \mathbf{Cob}(n, n-1)$ into the bicategory $T - \mathbf{Cob}(n+1, n, n-1)$ with extra coherence conditions, we need to consider the bordisms between bordisms, i.e. 2-bordisms, to define the 2-morphisms. Therefore manifolds with corners are unavoidable. We give a brief review of manifolds with corners below (for details, see [SP, Section 3.1.1.])

Definition 2.1. An n-dimensional manifold with corners of codimension 2 is a second countable Hausdorff space M with a maximal atlas of charts of the form

$$M \supseteq U \xrightarrow{\varphi} V \subset \mathbb{R}^{n-2} \times (\mathbb{R}_{\geq 0})^2.$$

Definition 2.2. ([T], [SW2], Bordism bicategory for arbitrary target space) For a fixed integer $n \geq 2$ and a non-empty topological space T as the target space, the bicategory $T - \mathbf{Cob}(n, n-1, n-2)$ of bordisms with maps to T is defined as follows:

- Objects (0-cells): the pairs (S, ξ), where S is an $(n-2)$-dimensional oriented closed manifold, and $\xi : S \to T$ is a continuous map.
- 1-morphisms (1-cells): $(\Sigma, \varphi) : (S_0, \xi_0) \to (S_1, \xi_1)$, where
  - (a) Σ is a compact oriented $(n-1)$-dimensional manifold with boundary. Define $\Sigma_- \cup \Sigma_+$ to be the collar of $\partial\Sigma$, and $(\Sigma, \chi_-, \chi_+) : S_0 \to S_1$ to be an oriented compact collared bordism with orientation preserving diffeomorphisms $\chi_- : S_0 \times [0, 1) \to \Sigma_-$ and $\chi_+ : S_1 \times (-1, 0] \to \Sigma_+$.
  - (b) $\varphi : \Sigma \to T$ is a continuous map such that the following diagram commutes:

$$\begin{tikzcd} & \Sigma \arrow[dd, "\varphi"] & \\ S_0 \times \{0\} \arrow[ur, "\chi_-"] \arrow[dr, "\xi_0"'] & & S_1 \times \{0\} \arrow[ul, "\chi_+"'] \arrow[dl, "\xi_1"] \\ & T & \end{tikzcd}$$

- 2-morphisms (2-cells): equivalence classes of pairs $(M, \psi) : (\Sigma, \varphi) \Rightarrow (\Sigma', \varphi')$ between 1-morphisms $(S_0, \xi_0) \to (S_1, \xi_1)$, where
  - (a) $M : \Sigma \to \Sigma'$ is an n-dimensional collared compact oriented bordism with corners, i.e. a $\langle 2 \rangle$-manifold with the coherence conditions about faces and corners in [SW2, Definition 2.1.(2)]. The equivalence of two 2-morphisms is also defined there.
  - (b) $\psi : M \to T$ is a continuous map with the coherence conditions in [SW2, Definition 2.1.(2)].
- Vertical and horizontal compositions are defined in [SW2, Definition 2.1.].
- Disjoint union endows the bicategory $T - \mathbf{Cob}(n, n-1, n-2)$ with the structure of a symmetric monoidal bicategory with duals.

Definition 2.3. ([T], [SW2], Extended homotopy quantum field theory) An n-dimensional extended homotopy quantum field theory with target space T taking values in a symmetric monoidal bicategory S is a symmetric monoidal functor $Z : T - \mathbf{Cob}(n, n-1, n-2) \to S$ satisfying the homotopy invariance property:

For any two 2-morphisms $(M,\psi),(M,\psi') : (\Sigma_a,\varphi_a) \Rightarrow (\Sigma_b,\varphi_b)$ between 1-morphisms $(\Sigma_a,\varphi_a),(\Sigma_b,\varphi_b) : (S_0,\xi_0) \to (S_1,\xi_1)$ with $\psi \simeq \psi'$ relative ∂M, we have the equality of 2-morphisms:

$$\begin{tikzcd} Z(S_0,\xi_0) \arrow[rr, bend left, "Z(\Sigma_a{,}\varphi_a)"] \arrow[rr, bend right, "Z(\Sigma_b{,}\varphi_b)"'] \arrow[rr, phantom, "Z(M{,}\psi)\Downarrow"] & & Z(S_1,\xi_1) \end{tikzcd} = \begin{tikzcd} Z(S_0,\xi_0) \arrow[rr, bend left, "Z(\Sigma_a{,}\varphi_a)"] \arrow[rr, bend right, "Z(\Sigma_b{,}\varphi_b)"'] \arrow[rr, phantom, "Z(M{,}\psi')\Downarrow"] & & Z(S_1,\xi_1) \end{tikzcd}$$

We denote by $HSym(T-\mathbf{Cob}(n,n-1,n-2),S)$ the bicategory of n-dimensional extended homotopy quantum field theories, which is a 2-groupoid.

Now we will see the extended homotopy quantum field theories with aspherical targets, which are called extended equivariant topological field theories:

Definition 2.4 (Extended equivariant topological field theory)**.** Given a group G, we define

$$G-\mathbf{Cob}(n,n-1,n-2) := BG-\mathbf{Cob}(n,n-1,n-2)$$

where BG is the classifying space of G. Then an n-dimensional extented G-equivariant topological field theory with values in a symmetric bicategory S is a homotopy quantum field theory $Z : G-\mathbf{Cob}(n,n-1,n-2) \to S$.

Remark 2.5. Generally, we can define extended equivariant topological field theories with an arbitrary group G, but we will only use finite groups later. From now on, we assume the given group G to be finite.

Next we will have a look at a very special and important example, which we will focus on in our main result.

Example 2.6 (The 3-2-1-dimensional case)**.** Now consider the 3-dimensional extented G-equivariant topological field theory $Z : G-\mathbf{Cob}(3,2,1) \to S$. Since the only kind of oriented connected closed 1-manifolds is the circle, the objects of $G-\mathbf{Cob}(3,2,1)$ are $(\mathbb{S}^1, \xi : \mathbb{S}^1 \to BG)$ and disjoint unions thereof.

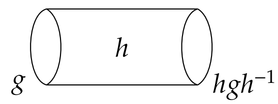
On the one hand, if we take a basepoint x on BG, a basepoint y and an orientation on $\mathbb{S}^1$, by [W, V.4.3.], we have an isomorphism

$$[(\mathbb{S}^1,y),(BG,x)] \cong Hom(\pi_1(\mathbb{S}^1,y),\ \pi_1(BG,x)) \cong G$$

Thus in the free homotopy case, we have the isomorphism $[\mathbb{S}^1,BG] \cong G/G$ with the equivalence relation $g \sim hgh^{-1}$ for arbitrary $g,h \in G$. We can construct an action groupoid $G//G$, whose objects are elements in G and morphisms are adjoint actions.

On the other hand, collecting principal G-bundles over $\mathbb{S}^1$ as objects, and isomorphisms between them as morphisms, we can form a groupoid $\mathbf{PBun}_G(\mathbb{S}^1)$. Also, we know that $[\mathbb{S}^1,BG]$ is bijective to the isomorphism classes of principal G-bundles over $\mathbb{S}^1$. Therefore we have a canonical equivalence $\mathbf{PBun}_G(\mathbb{S}^1) \simeq G//G$, once we have chosen the basepoints and orientation.

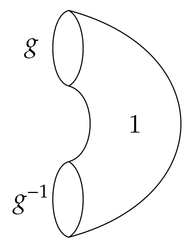
In this situation, given a decorated map $\xi : (\mathbb{S}^1,y) \to (BG,x)$, we can write $(\mathbb{S}^1,\xi)$ as $(\mathbb{S}^1,g)$, where the element g means the holonomy with respect to the map ξ. Then the adjoint action h, i.e. the morphism $g \to hgh^{-1}$ in the action groupoid $G//G$ can be regarded as a homotopy between the principal G-bundles characterized by the holonomies g and hgh^{-1}, which is described as a 1-morphism:



Taking evaluation of Z on this 1-morphism, we can obtain an isomorphism between objects in S:

$$\phi_h : Z(\mathbb{S}^1, g) \to Z(\mathbb{S}^1, hgh^{-1})$$

We also see that the orientation of $\mathbb{S}^1$ indeed matters. Taking $h = 1$ and bending the above bordism, we will obtain the the bent cylinder as follows:



The lower copy of the circle carries the opposite orientation with the upper one, which leads to the inverse holonomy values.

Remark 2.7. Actually, the above construction can be encoded into a 2-functor:

$$G//G \to S,\ g \mapsto Z(\mathbb{S}^1, g),\ (h : g \to hgh^{-1}) \mapsto \phi_h,$$

where we now regard $G//G$ as a 2-groupoid with trivial 2-morphisms.

Notation 2.8. In the later cases, we are only interested in the case $S = \mathbf{2Vect}$, that is, the symmetric monoidal bicategory of 2-vector spaces, whose objects are abelian $\mathbb{C}$-linear finitely semisimple categories, 1-morphisms are linear functors and 2-morphisms are natural transformations. For details, we refer to [BDSPV]. We will call a homotopy quantum field theory $Z : G - \mathbf{Cob}(n, n-1, n-2) \to \mathbf{2Vect}$ as (n,n-1,n-2)-G-TFT Z in short. We will concentrate on the case $n = 3$.

2.2 Equivariant ribbon categories

In this section, we will introduce the basic categorical notions we will need. For more details, we refer to [K, Section 2], [T, Chapter VI.].

Definition 2.9. Let G be a finite group and K an algebraically closed field with $Char\, K = 0$. A G-equivariant monoidal category $\mathcal{C}$ over K is a K-linear abelian category with left duality with the following additional structure:

- G-grading: there is a decomposition

$$\mathcal{C} := \bigoplus_{g \in G} \mathcal{C}_g,$$

  where each $\mathcal{C}_g$ is a full subcatgory in $\mathcal{C}$. For any $X \in \mathcal{C}_g$, $Y \in \mathcal{C}_h$, $g, h \in G$, if $g \neq h$, then we have $Mor_{\mathcal{C}}(X, Y) = 0$.

- $\mathbf{1} \in \mathcal{C}_1$ and if $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h$, then $X \otimes Y \in \mathcal{C}_{gh}$.

- If $X \in \mathcal{C}_g$, then $X^* \in \mathcal{C}_{g^{-1}}$, where X^* is the dual of X.

- G-action: A family of functors $\{\phi_g : \mathcal{C} \to \mathcal{C}\}_{g\in G}$ and functorial isomorphisms $\{\alpha_{g,h} : \phi_g \circ \phi_h \xrightarrow{\sim} \phi_{gh}\}_{g,h\in G}$ such that $\phi_1 \simeq \mathbf{id}_{\mathcal{C}}$, $\alpha_{ij,k} \circ \alpha_{i,j} = \alpha_{i,jk} \circ \alpha_{j,k} : \phi_i \circ \phi_j \circ \phi_k \xrightarrow{\sim} \phi_{ijk}$, $\forall i, j, k \in G$.

  For any elements $g, h \in G$ and object $X \in \mathcal{C}_h$, define $g.X := \phi_g(X) \in \mathcal{C}_{ghg^{-1}}$.

Notation 2.10. From now on, we always ask the given field K is an algebraically closed field with $Char\, K = 0$.

Definition 2.11. Let $\mathcal{C}$ be a G-equivariant monoidal category over K. An object X of $\mathcal{C}$ is simple if $End_{\mathcal{C}}(X) = K$. An object of $\mathcal{C}$ is (finitely) semisimple if it is the (finite) direct sum of simple objects. If every object is (finitely) semisimple, then we call this category $\mathcal{C}$ is (finitely) semisimple.

Definition 2.12. We call a G-equivariant monoidal category complex finitely semisimple if it is a $\mathbb{C}-$linear (i.e. $K = \mathbb{C}$), abelian, finitely semisimple category such that the monoidal product is $\mathbb{C}-$bilinear.

Definition 2.13. A G-braiding in a G-equivariant monoidal category $\mathcal{C}$ is a family of natural isomorphisms

$$c_{X,Y} : X \otimes Y \to g.Y \otimes X,\ \forall X \in \mathcal{C}_g, Y \in \mathcal{C}_h,\ g, h \in G.$$

satisfying the following conditions:

1. For any $f : X \to X'$, $f' : Y \to Y'$ such that $X, X' \in \mathcal{C}_g$, then

$$c_{X',Y'} \circ (f \otimes f') = (g.f' \otimes f) \circ c_{X,Y}$$

2. The equivariant hexagon axioms, see [T, VI.2.2.(2.2.2)],
3. For any $g \in G$, $X, Y \in \mathcal{C}$,

$$g.c_{X,Y} = c_{g.X,g.Y}$$

We call a G-equivariant monoidal category with a G-braiding a G-braided monoidal category.

Definition 2.14. A G-twist in a G-braided monoidal category $\mathcal{C}$ is a family of natural isomorphisms

$$\theta_X : X \to g.X,\ X \in \mathcal{C}_g,\ g \in G.$$

satisfying the following conditions:

1. For any $f : X \to Y$ in $\mathcal{C}_g$, $g \in G$,

$$\theta_Y \circ f = g.f \circ \theta_X$$

2. For any $X \in \mathcal{C}_g$, $g \in G$,

$$(\theta_X)^* = \theta_{g.(X^*)}$$

3. For any $X \in \mathcal{C}_g$, $Y \in \mathcal{C}_h$, $g, h \in G$,

$$\theta_{X\otimes Y} = c_{gh.Y,g.X} \circ c_{g.X,h.Y} \circ (\theta_X \otimes \theta_Y)$$

4. For any $X \in \mathcal{C}_g$, $g \in G$,

$$g.\theta_X = \theta_{g.X}$$

We call a G-braided monoidal category with a G-twist a G-ribbon monoidal category.

Remark 2.15. Here we do not require the monoidal unit **1** to be simple, but in [K, Section 2], the monoidal unit is simple in G-equivariant fusion categories.

Example 2.16. (The representation category of a G-ribbon Hopf algebra, [MNS, Section 4.2.])

A G-Hopf algebra over $\mathbb{C}$ ([MNS, Definition 4.13]) is a Hopf algebra A with

- a G-grading: $A = \bigoplus_{g\in G} A_g$,
- a weak G-action: A family of algebra automorphisms $\{\varphi_g : A \to A\}_{g\in G}$ and invertible elements $\{c_{g,h} \in A\}_{g,h\in G}$ with additional conditions, see [MNS, Definition 4.10.].

such that:

- The algebra structure of A restricts to the structure of an associative algebra on each homogeneous component so that A is the direct sum of the components A_g as an algebra.
- G acts by homomorphisms of Hopf algebras.
- The G-action is compatible with the grading, i.e. $\varphi_g(A_h) \subset A_{ghg^{-1}}$, $g,h \in G$.
- The coproduct $\Delta : A \to A \otimes A$ respects the grading, i.e. $\Delta(A_g) \subset \bigoplus_{pq=g} A_p \otimes A_q$.
- The elements $\{c_{g,h} \in A\}_{g,h\in G}$ are group-like, i.e. $\Delta(c_{g,h}) = c_{g,h} \otimes c_{g,h}$.

Then by [MNS, Lemma 4,15], the category A-mod of finite dimensional modules over a G-Hopf algebra A is a complex finitely semisimple G-equivariant monoidal category. Moreover, A is called a G-ribbon algebra if A has

1. a G-equivariant R-matrix: An element $R = R_{(1)} \otimes R_{(2)} \in A \otimes A$ such that for $V \in (A\text{-mod})_g$, $W \in A$-mod, the map

$$c_{V,W} : V \otimes W \to \varphi_g(W) \otimes V, \quad v \otimes w \mapsto R_{(2)}.w \otimes R_{(1)}.v$$

   is a G-braiding on the category A-mod according to Definition 2.13.

2. a G-twist: An invertible element $\theta \in A$ such that for every object $V \in (A\text{-mod})_g$, the induced map

$$\theta_V : V \to \varphi_g(V), \quad v \mapsto \theta^{-1}.v$$

   is a G-twist on the category A-mod according to Definition 2.14.

By [MNS, Proposition 4.18.], the category A-mod is a G-ribbon category.

2.3 The 3-2-1-dimensional case

Given a 3-2-1-G-TFT Z, if we take evaluation on the circle decorated with an element in G, we will obtain an object in **2Vect**, that is, an additive $\mathbb{C}$-linear finitely semisimple categories. From the result in Example (2.6), we know that for any $g,h \in G$, we have the 2-linear equivalence between 2-vector spaces:

$$\phi_h : \mathcal{C}_g^Z := Z(\mathbb{S}^1, g) \to \mathcal{C}_{hgh^{-1}}^Z := (\mathbb{S}^1, hgh^{-1}) \tag{2.1}$$

Then we can collect these data as follows:

- The category $\mathcal{C}^Z := \bigoplus\limits_{g\in G} \mathcal{C}_g^Z$

- The equivalences $\phi_h : \mathcal{C}^Z \to \mathcal{C}^Z$ obtained from the equivalences (2.1).
- The natural isomorphisms $\alpha_{g,h} : \phi_g \circ \phi_h \cong \phi_{gh}$ and $\phi_1 \cong \mathbf{id}_{\mathcal{C}^Z}$ and other coherence data from the functor $G//G \to \mathbf{2Vect}$.

This category $\mathcal{C}^Z$ carries lots of structures, which can be summarized as follows:

Theorem 2.17. ([SW2, Theorem 4.1.]) *The evaluation of a 3-2-1-G-TFT on the circle is naturally a complex finitely semisimple G-ribbon category.*

3 Topological and algebraic orbifoldizations

In this section, we will explain how to take topological and algebraic orbifoldization on (n,n-1,n-2)-G-TFTs and G-ribbon categories, respectively. In particular, we will see the topological and algebraic orbifoldizations in the 3-2-1-dimensional case.

3.1 Topological orbifoldizations

Notation 3.1. For any two topological spaces X, Y, we can define the mapping space $Y^X := Mor_{Top}(X, Y)$ with compact-open topology. We can also regard these topological spaces as Kan complexes when necessary. In this situation, we define the fundamental groupoid and the fundamental 2-groupoid $\Pi_j(X)$, $j = 1, 2$ for any Kan complex X, and we define $\Pi_j(X, Y) := \Pi_j(Y^X)$. Using these notations, we can obtain the following result:

Proposition 3.2. ([SW2, Proposition 2.5.]) For any extended homotopy quantum field theory $Z : T - \mathbf{Cob}(n, n-1, n-2) \to S$ and any closed oriented (n-2)-dimensional manifold X, we have a representation:

$$\hat{Z}(X) := Z(X, ?) : \Pi_2(X, Y) \to S, \quad (\xi : X \to T) \mapsto Z(X, \xi).$$

Given an (n,n-1,n-2)-G-TFT $Z : G - \mathbf{Cob}(n, n-1, n-2) \to \mathbf{2Vect}$, we can produce an extended equivariant topological field theory with the trivial group

$$\hat{Z} : \mathbf{Cob}(n, n-1, n-2) \to \mathbf{2VecBunGrpd}$$

where the symmetric monoidal bicategory **2VecBunGrpd** consists of the following data briefly:

- Objects: pairs (Γ, ρ), where Γ is an essentially finite groupoid and $\rho : \Gamma \to \mathbf{2Vect}$ a 2-vector bundle over Γ, i.e. a representation of Γ.
- 1-morphisms: $(\Lambda, r_0, r_1; \lambda) : (\Gamma_0, \rho_0) \to (\Gamma_1, \rho_1)$ that can be written as a span:

  $$\Gamma_0 \xleftarrow{r_0} \Lambda \xrightarrow{r_1} \Gamma_1$$

  and an intertwiner:

  $$\lambda : r_0^* \rho_0 \to r_1^* \rho_1$$

- 2-morphisms: spans of spans and a natural morphisms of intertwiners shown in 1-morphisms. The details can be found in [SW2, Section 3.1.].

In this situation, we have the following results:

Theorem 3.3. ([SW2, Theorem 3.1.]) *For any finite group G, the assignment $Z \mapsto \hat{Z}$ from Proposition 3.2 naturally extends to a functor*

$$\hat{?} : HSym(G - \boldsymbol{Cob}(n, n-1, n-2), \boldsymbol{2Vect}) \to Sym(\boldsymbol{Cob}(n, n-1, n-2), \boldsymbol{2VecBunGrpd}) \quad (3.1)$$

One calls (3.1) the change to equivariant coefficients.

After changing to the equivariant coefficients, we have obtained the non-equivariant version of the extended topological field theories. In fact, we hide the equivariant information into the target **2VecBunGrpd**. Now we need to change our target back to **2Vect**, which needs the help of the parallel section functor:

$$Par : \mathbf{2VecBunGrpd} \to \mathbf{2Vect}$$

which is symmetric monoidal and has been defined in [SW1, Theorem 4.9.].

Definition 3.4. (Topological orbifoldization, [SW2, Definition 3.2.]) Let G be a finite group. Then the topological orbifoldization for the (n,n-1,n-2)-G-TFT is the functor:

$$\frac{?}{G} : HSym(G - \mathbf{Cob}(n, n-1, n-2), \mathbf{2Vect}) \to Sym(\mathbf{Cob}(n, n-1, n-2), \mathbf{2Vect})$$

defined as the concatenation:

$$\begin{tikzcd} HSym(G - \mathbf{Cob}(n, n-1, n-2), \mathbf{2Vect}) \arrow[r, "\hat{?}"] \arrow[dr, "\frac{?}{G}"'] & Sym(\mathbf{Cob}(n, n-1, n-2), \mathbf{2VecBunGrpd}) \arrow[d, "Par \circ -"] \\ & Sym(\mathbf{Cob}(n, n-1, n-2), \mathbf{2Vect}) \end{tikzcd}$$

For a concrete description of $\frac{Z}{G}$ for any $Z \in HSym(G - \mathbf{Cob}(n, n-1, n-2), \mathbf{2Vect})$, we refer to [SW2, Proposition 3.3.].

3.2 Algebraic orbifoldizations

We have introduced G-ribbon categories in Section 2.2. Now we will see the construction of the algebraic orbifoldization. Our notion is based on [K, Definition 3.1.], where the algebraic orbifoldization is defined as the orbifold category.

Definition 3.5 (Algebraic orbifoldization)**.** Let $\mathcal{C}$ be a complex finitely semisimple G-ribbon monoidal category. The **orbifold category** $\mathcal{C}/G$ is a category with

- objects: $(X, \{\varphi_g\}_{g \in G})$, where $X \in \mathcal{C}$, with isomorphisms $\varphi_g : g.X \xrightarrow{\sim} X$ such that the coherence data $\varphi_1 : 1.X \xrightarrow{\sim} X$ concides with the identity and

$$\varphi_g \circ g.\varphi_h = \varphi_{gh} \quad \text{i.e. the diagram commutes:} \quad \begin{tikzcd} g.h.X \arrow[r, "g.\varphi_h"] \arrow[d, "\cong"] & g.X \arrow[d, "\varphi_g"] \\ (gh).X \arrow[r, "\varphi_{gh}"] & X \end{tikzcd}$$

- morphisms: $Mor_{\mathcal{C}}((X, \{\varphi_g\}_{g \in G}), (Y, \{\psi_g\}_{g \in G}))$ is the set of $\mathcal{C}-$morphisms $f : X \to Y$ such that

$$\psi_g \circ g.f = f \circ \varphi_g \quad \text{i.e. the diagram commutes:} \quad \begin{tikzcd} g.X \arrow[r, "g.f"] \arrow[d, "\varphi_g"'] & g.Y \arrow[d, "\psi_g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Remark 3.6. We call the objects in $\mathcal{C}/G$ homotopy G-fixed points. Given an object $X \in \mathcal{C}_g$ for some g, if X is a homotopy G-fixed point, i.e. there exists a set $\{\varphi_h : h.X \xrightarrow{\sim} X\}_{h\in G}$ of coherence isomorphisms in $\mathcal{C}$ for all $h \in G$, then the element g is in the center of G.

Not every object in $\mathcal{C}$ can be equipped with the structure of a homotopy G-fixed point. If we can not find coherence isomorphisms $\varphi_h : h.X \xrightarrow{\sim} X$ for every $h \in G$, we say X is not homotopy G-fixed, and there is no its corresponding object in $\mathcal{C}/G$.

Remark 3.7. Being a homotopy fixed point is a structure, not just a property. If there are two different sets of isomorphisms $\{\varphi_g\}_{g\in G}$ and $\{\varphi'_g\}_{g\in G}$, then we may have two non-isomorphic objects $(X, \{\varphi_g\}_{g\in G})$ and $(X, \{\varphi'_g\}_{g\in G})$ in $\mathcal{C}/G$.

Theorem 3.8. ([K, Theorem 3.9.]) *Let $\mathcal{C}$ be a complex finitely semisimple G-ribbon monoidal category. Then the orbifold category $\mathcal{C}/G$ is a complex finitely semisimple ribbon monoidal category with:*

1. *(Unit)* $\mathbf{1}_{\mathcal{C}/G} = (\mathbf{1}, \{\mathrm{id}\}_{g\in G})$
2. *(Duals)* $(X, \{\varphi_g\}_{g\in G})^* = (X^*, \{(\varphi_g^*)^{-1}\}_{g\in G})$
3. *(Tensor product $\otimes'$)* $(X, \{\varphi_g\}_{g\in G}) \otimes' (Y, \{\psi_g\}_{g\in G}) := (X \otimes Y, \{\varphi_g \otimes \psi_g\}_{g\in G})$, $\forall (X, \{\varphi_g\}_{g\in G}), (Y, \{\psi_g\}_{g\in G}) \in \mathcal{C}/G$.
4. *(Braiding c') Given $(X, \{\varphi_g\}_{g\in G})$, $(Y, \{\psi_g\}_{g\in G})$ such that $X \in \mathcal{C}_g$, $Y \in \mathcal{C}_h$, we have a family of natural isomorphisms*

$$c'_{(X,\{\varphi_g\}_{g\in G}),(Y,\{\psi_g\}_{g\in G})} : (X \otimes Y, \{\varphi_g \otimes \psi_g\}_{g\in G}) \xrightarrow{\sim} (Y \otimes X, \{\psi_g \otimes \varphi_g\}_{g\in G})$$

   Define its underlying braiding as follows:

$$c'_{X,Y} : X \otimes' Y \xrightarrow{c_{X,Y}} g.Y \otimes X \xrightarrow{\psi_g \otimes id_X} Y \otimes X$$

5. *(Twist θ') Given $(X, \{\varphi_g\}_{g\in G})$, $X \in \mathcal{C}_g$, we have a family of natural isomorphisms*

$$\theta'_{(X,\{\varphi_g\}_{g\in G})} : (X, \{\varphi_g\}_{g\in G}) \xrightarrow{\sim} (X, \{\varphi_g\}_{g\in G})$$

   Define its underlying twist as follows:

$$\theta'_X : X \xrightarrow{\theta_X} g.X \xrightarrow{\varphi_g} X$$

3.3 Review: the 3-2-1-dimensional case

In the last two sections, we have recalled two kinds of orbifoldization. Now we will review their connection in the 3-2-1-dimensional case.

In Section 2.3, we have mentioned that we can obtain a complex finitely semisimple G-ribbon monoidal category $\mathcal{C}$ by taking evaluation of the given 3-2-1-G-TFT Z on the circle. If we take algebraic orbifoldization on $\mathcal{C}$, we will obtain a complex finitely semisimple ribbon monoidal category $\mathcal{C}/G$ by Theorem 3.8.

On the other hand, if we take topological orbifoldization on Z first, we will obtain a 3-2-1-TFT Z/G. At this time, taking evaluation of Z/G on the circle, we can also obtain a complex finitely semisimple ribbon monoidal category $Z/G(\mathbb{S}^1)$ shown in [BDSPV]. The next theorem says the categories $\mathcal{C}/G$ and $Z/G(\mathbb{S}^1)$ concide.

Theorem 3.9. ([SW2, Theorem 4.17.]) *For any 3-2-1-G-TFT Z, the evaluation of its topological orbifoldization Z/G on $\mathbb{S}^1$ yields an equivalence*

$$\frac{Z}{G}(\mathbb{S}^1) \simeq \mathcal{C}/G \tag{3.2}$$

as 2-vector spaces. Both categories carry the structure of a complex finitely semisimple ribbon monoidal category, and both structures agree.

Diagrammatically, this theorem means the following diagram commutes up to natural isomorphism:

$$\begin{tikzcd}
\text{3-2-1-G-TFTs} \arrow[r, "evaluation\ on\ \mathbb{S}^1"] \arrow[d, "Top.Orbi\ \frac{?}{G}"] & \text{complex finitely semisimple G-ribbon categories} \arrow[d, "Alg.Orbi."] \\
\text{3-2-1-TFTs} \arrow[r, "evaluation\ on\ \mathbb{S}^1"] & \text{complex finitely semisimple ribbon categories}
\end{tikzcd}$$

4 Main results

Here we first introduce the definitions of (framed) braid groups and mapping class groups:

Definition 4.1. The braid group $\mathcal{B}_n$ for some positive integer n has $(n-1)$ generators $\{\sigma_i\}_{i=1}^{n-1}$ with the following relations:

(R1) $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| > 1$

(R2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

The framed braid group $f\mathcal{B}_n$ for some positive integer n consists not only the generators and relations of $\mathcal{B}_n$ above, but also n generators $\{t_i\}_{i=1}^{n}$ and relations:

(R3) $t_it_j = t_jt_i$, $\forall i, j = 1, ..., n$

(R4) $\sigma_i t_j = t_{\sigma_i(j)}\sigma_i$, $\forall i = 1, ..., n-1$; $j = 1, ..., n$.

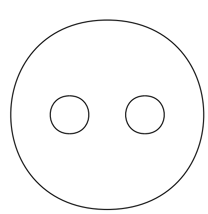
Remark 4.2. We can also write $f\mathcal{B}_n$ as a semiproduct $\mathbb{Z}^n \rtimes \mathcal{B}_n$, where the element $\sigma \in \mathcal{B}_n$ acts $(t_1, ..., t_n) \in \mathbb{Z}^n$ into $(t_{\sigma(1)}, ..., t_{\sigma(n)}) \in \mathbb{Z}^n$.

Definition 4.3. Let S be a surface possibly with boundary. Let $Homeo^+(S, \partial S)$ be the group of orientation-preserving homeomorphisms of S that restrict to the identity on ∂S. This group can be endowed with the compact-open topology. Define the mapping class group $MCG(S)$ of S to be the group

$$MCG(S) := \pi_0(Homeo^+(S, \partial S)).$$

Warn that in general, the definition of the mapping class group of a surface possibly with boundary has many variations. We refer to [FM, Section 2.1.] for example.

In our thesis, we are concerning the mapping class group of a disk with n holes. We will name a disk with n holes as $\mathbb{D}_n$. For example, when $n = 2$, we can see the following figure:



Now we start to define the mapping class group $MCG(\mathbb{D}_n)$. We call the n+1 parametrised boundary circles $\partial_0, \partial_1, ..., \partial_n$ in $\mathbb{D}_n$, and the gorup $MCG(\mathbb{D}_n)$ is the same as $\pi_0(Homeo^+(\mathbb{D}_n, \partial\mathbb{D}_n))$, but with only one variation:

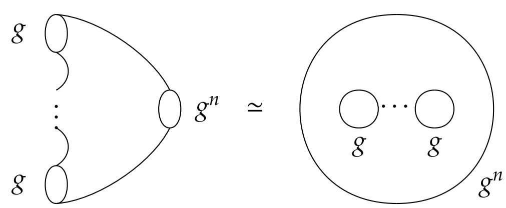
- We redefine the group $Homeo^+(\mathbb{D}_n, \partial\mathbb{D}_n)$ to be the group of orientation-preserving homeomorphisms that fix ∂_0 pointwise, but may permute the other n boundary circles $\partial_1, ..., \partial_n$ as long as they preserve the parametrisation of each.

In this situation, the mapping class group $MCG(\mathbb{D}_n)$ of a disk with n holes is isomorphic to the framed braid group $f\mathcal{B}_n$ with n strands. Now we start to construct framed braid group representations in two different ways: topologically and algebraically, and to give a comparison later.

4.1 Construction I. Evaluation of a 3-2-1-G-TFT Z on decorated genus zero surfaces

Given a 3-2-1-G-TFT Z for some finite group G, taking evaluation on $(\mathbb{S}^1, g)$ for all different elements $g \in G$, we can obtain a complex finitely semisimple G-ribbon category $\mathcal{C} := \bigoplus_{g \in G} \mathcal{C}_g$ as in Theorem 2.17.

Now we consider taking evaluation on a genus zero surface with n ingoing boundary circles decorated with $g \in G$ and one outgoing boundary circle decorated with $g^n \in G$. Recall that the group elements g and g^n give the holonomies of every ingoing boundary circle and the outgoing boundary circle, respectively. This surface is homeomorphic to a disk with n holes $\mathbb{D}_n$. Considering the decorations g and g^n for ingoing boundary circles and the outgoing boundary circle, we call this disk $\mathbb{D}_{n,g}$.



In this situation, we can write the evaluation of Z on $\mathbb{D}_{n,g}$ as an n-linear functor:

$$Z(\mathbb{D}_{n,g}) : \mathcal{C}_g^{\boxtimes n} \to \mathcal{C}_{g^n},\ X^{\boxtimes n} \mapsto X^{\otimes n},$$

where $\boxtimes$ is the Deligne tensor product.

Definition 4.4. Given a 1-morphism $(\Sigma, \varphi) : (S_0, \xi_0) \to (S_1, \xi_1)$ in $G-\mathbf{Cob}(3,2,1)$ (see Definition 2.2), and an element $\sigma \in MCG(\Sigma)$, we define a new 1-morphism $\sigma.(\Sigma, \varphi) := (\Sigma, \varphi \circ \sigma)$: $(S_0, \xi_0 \circ \sigma \mid_{S_0}) \to (S_1, \xi_1 \sigma \mid_{S_1})$. In particular, we can write $\sigma.\mathbb{D}_{n,g}$ to be the new 1-morphism for the disk $\mathbb{D}_{n,g}$ and an element $\sigma \in MCG(\mathbb{D}_{n,g})$.

Recall that the mapping class group of the disk with n holes is isomorphic to the framed braid group with n strands $f\mathcal{B}_n$. Any element in $f\mathcal{B}_n$ is a composition of generators σ_i and t_j, $i = 1, ..., n-1; j = 1, ..., n$. The generator σ_i corresponds to the braiding of the i^{th} strand and the $(i+1)^{th}$ strand, and the generator t_j corresponds to the framing on the j^{th} strand. Therefore we can use the elements in $f\mathcal{B}_n$ to present the corresponding compositions of braidings and twists of the n ingoing boundary circles of $\mathbb{D}_{n,g}$.

We can see some examples below:

When n=2, we have the 2-morphism shown in Figure 1. (see [SW2, Section 4.3.] in general)

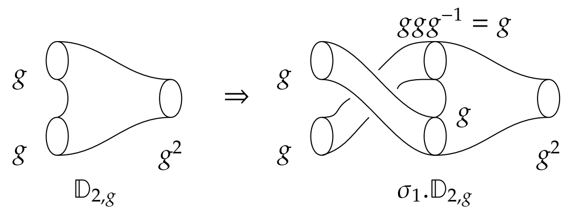


Figure 1: Braiding in $\mathbb{D}_{2,g}$

Taking evaluation of Z on this 2-morphism, we can obtain two functors

$$Z(\mathbb{D}_{2,g}) : \mathcal{C}_g^{\boxtimes 2} \to \mathcal{C}_{g^2},\ X^{\boxtimes 2} \mapsto X^{\otimes 2};\quad Z(\sigma_1.\mathbb{D}_{2,g}) : \mathcal{C}_g^{\boxtimes 2} \to \mathcal{C}_{g^2},\ X^{\boxtimes 2} \mapsto g.X \otimes X,$$

and a natural isomorphism between them

$$\eta_{\sigma_1} : Z(\mathbb{D}_{2,g}) \Rightarrow Z(\sigma_1.\mathbb{D}_{2,g}),\ \eta_{\sigma_1}(X^{\boxtimes 2}) = c_{X,X} : X \otimes X \to g.X \otimes X,$$

where c is the G-braiding in $\mathcal{C}$.

When n=1, we have the 2-morphism shown in Figure 2. (see [SW2, Section 4.4.] in general)

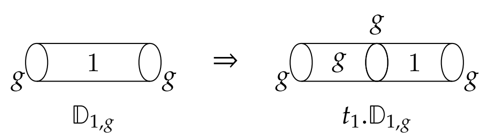


Figure 2: Twist in $\mathbb{D}_{1,g}$

Taking evaluation of Z on this 2-morphism, we can obtain two functors

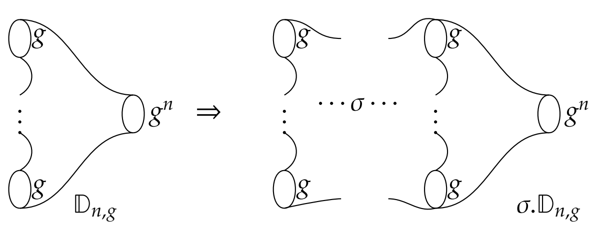
$$Z(\mathbb{D}_{1,g}) : \mathcal{C}_g \to \mathcal{C}_g,\ X \mapsto X;\quad Z(t_1.\mathbb{D}_{1,g}) : \mathcal{C}_g \to \mathcal{C}_g,\ X \mapsto g.X,$$

and a natural isomorphism between them

$$\eta_{t_1} : Z(\mathbb{D}_{1,g}) \Rightarrow Z(t_1.\mathbb{D}_{1,g}),\ \eta_{t_1}(X) = \theta_X : X \to g.X,$$

where θ is the G-twist in $\mathcal{C}$.

Now let's return back to the general n case. We can also take a composition of braidings and twists on some of the n ingoing boundary circles in $\mathbb{D}_{n,g}$ as we do in Figure 1 and Figure 2, which corresponds to an element $\sigma \in f\mathcal{B}_n$. Then we will obtain a 2-morphism



Taking evaluation of Z on $\sigma.\mathbb{D}_{n,g}$, we can obtain an n-linear functor:

$$Z(\sigma.\mathbb{D}_{n,g}) : \mathcal{C}_g^{\boxtimes n} \to \mathcal{C}_{g^n},\ X^{\boxtimes n} \mapsto \sigma.X^{\otimes n}.$$

Therefore we have a natural isomorphism

$$\eta_\sigma : Z(\mathbb{D}_{n,g}) \Rightarrow Z(\sigma.\mathbb{D}_{n,g}), \quad \eta_\sigma(X^{\boxtimes n}) : X^{\otimes n} \to \sigma.X^{\otimes n}. \tag{4.1}$$

In particular, taking σ as generators σ_i, t_j, $i = 1, \ldots, n-1$, $j = 1, \ldots, n$ of $f\mathcal{B}_n$, then we know the natural isomorphisms

$$\eta_{\sigma_i}(X^{\boxtimes n}) = \mathrm{id}_X^{\otimes i-1} \otimes c_{X,X} \otimes \mathrm{id}_X^{\otimes n-i-1} : X^{\otimes n} \to \sigma_i.X^{\otimes n} := X^{\otimes(i-1)} \otimes g.X \otimes X^{\otimes(n-i)},\ i = 1, \ldots, n-1 \tag{4.2}$$

$$\eta_{t_j}(X^{\boxtimes n}) = \mathrm{id}_X^{\otimes j-1} \otimes \theta_X \otimes \mathrm{id}_X^{\otimes n-j} : X^{\otimes n} \to t_j.X^{\otimes n} := X^{\otimes(j-1)} \otimes g.X \otimes X^{\otimes(n-j)},\ j = 1, \ldots, n \tag{4.3}$$

where c and θ are the G-braiding and G-twist in $\mathcal{C}$.

From the above discussion, for every $\sigma \in f\mathcal{B}_n$, we can obtain a natural isomorphism $\eta_\sigma(X^{\boxtimes n}) : X^{\otimes n} \to \sigma.X^{\otimes n}$, where the objects $X^{\otimes n}$ and $\sigma.X^{\otimes n}$ will generally not coincide. In the thesis, our solution is to take our representation on the (tensor product of) homotopy G-fixed points in $\mathcal{C}$.

Recall that an object $Y = (X, \{\varphi_h\}_{h\in G}) \in \mathcal{C}/G$ is a homotopy G-fixed point with a family of coherence isomorphisms

$$\varphi_h : h.X \to X,\ \forall h \in G.$$

Then we can construct the representation of $f\mathcal{B}_n$ on $X^{\otimes n}$ in $\mathcal{C}$ with the help of these coherence isomorphisms

$$\rho_{X,n}^{\varphi_g} : f\mathcal{B}_n \to Aut(X^{\otimes n}) \tag{4.4}$$

such that

$$\rho_{X,n}^{\varphi_g}(\sigma_i) := (\mathrm{id}_X^{\otimes i-1} \otimes \varphi_g \otimes \mathrm{id}_X^{\otimes n-i}) \circ \eta_{\sigma_i}(X^{\boxtimes n}),\ i = 1, \ldots, n-1; \tag{4.5}$$

$$\rho_{X,n}^{\varphi_g}(t_j) := (\mathrm{id}_X^{\otimes j-1} \otimes \varphi_g \otimes \mathrm{id}_X^{\otimes n-j}) \circ \eta_{t_j}(X^{\boxtimes n}),\ j = 1, \ldots, n. \tag{4.6}$$

Notice that the representation (4.4) only treats with the homotopy G-fixed point $Y = (X, \{\varphi_h\}_{h\in G})$ with the object $X \in \mathcal{C}_g$ for some element $g \in G$. Now let us discuss the general case, that is, the homotopy G-fixed point $Y = (X, \{\varphi_h\}_{h\in G})$ for arbitrary object $X \in \mathcal{C}$. Since the category $\mathcal{C}$ is G-graded, we can decompose the homotopy G-fixed point $Y = (X, \{\varphi_h\}_{h\in G})$ as a direct sum of homotopy G-fixed points $\bigoplus_{g\in G} Y_g$, where each object $Y_g = (X_g, \{\varphi_h^g\}_{h\in G})$ contains the object $X_g \in \mathcal{C}_g$, $X = \bigoplus_{g\in G} X_g$ and a family of coherence isomorphisms

$$\varphi_h^g : h.X_g \to X_g,\ \varphi_h = \bigoplus_{g\in G} \varphi_h^g,\ \forall h \in G.$$

Expanding the term $X^{\otimes n} = (\bigoplus_{l=1}^{m} X_l)^{\otimes n}$, the automorphism group $Aut(X^{\otimes n})$ is isomorphic to $\left(\bigoplus_{g\in G} Aut(X_g^{\otimes n})\right) \oplus XOTHERS$, where the part $XOTHERS$ is the direct sum of all terms of the form $Aut(\bigotimes_{g\in G} X_g^{\otimes p_g})$ with $0 \le p_g \le n-1$ and $\sum_{g\in G} p_g = n$ or 0. Remark that $\sum_{g\in G} p_g = 0$ if and only if we step back to the case $X \in \mathcal{C}_g$ for some $g \in G$. Therefore for the general homotopy G-fixed point $Y = (X, \{\varphi_h\}_{h\in G})$, we can use the automorphisms (4.5) and (4.6) to construct the representation of $f\mathcal{B}_n$ on the general object $X^{\otimes n}$

$$\rho_{X,n}^{\varphi} : f\mathcal{B}_n \to Aut(X^{\otimes n}) \tag{4.7}$$

such that

$$\rho^{\varphi}_{X,n}(\sigma_i) := \Big(\bigoplus_{g\in G} \rho^{\varphi^g_g}_{X_g,n}(\sigma_i)\Big) \oplus \mathrm{id}_{XOTHERS},\ i = 1, ..., n-1; \tag{4.8}$$

$$\rho^{\varphi}_{X,n}(t_j) := \Big(\bigoplus_{g\in G} \rho^{\varphi^g_g}_{X_g,n}(t_j)\Big) \oplus \mathrm{id}_{XOTHERS},\ j = 1, ..., n. \tag{4.9}$$

4.2 Construction II. Via orbifoldization

Taking the topological orbifoldization of the given 3-2-1-G-TFT Z as Definition 3.4, we can obtain the 3-2-1-TFT Z/G. By [BDSPV], we know that $Z/G(\mathbb{S}^1)$ is a complex finitely semisimple ribbon category. Then we can obtain a $f\mathcal{B}_n$ representation on some object in $Z/G(\mathbb{S}^1)$ shown as the equations (4.2) and (4.3).

Theorem 3.9 says that the category $Z/G(\mathbb{S}^1)$ obtained after topological orbifoldization is equivalent to the algebraic orbifoldization $\mathcal{C}/G$ of the category $\mathcal{C}$ as 2-vector spaces, and $\mathcal{C}/G$ still carries the structure of a complex finitely semisimple ribbon category.

Now we can consider the well-defined representation of $f\mathcal{B}_n$ mentioned above in $\mathcal{C}/G$. Given an object $Y := (X, \{\varphi_h\}_{h\in G}) \in \mathcal{C}/G$ for some $X \in \mathcal{C}_g$, $g \in G$, we have a tensor product of n copies of Y and obtain an object $Y^{\otimes' n} := (X^{\otimes n}, \{\varphi_h^{\otimes n}\})_{h\in G}$. Define a representation of the framed braid group $f\mathcal{B}_n$ with n strands:

$$\rho_{Y,n} : f\mathcal{B}_n \to Aut(Y^{\otimes' n}) \tag{4.10}$$

as follows

$$\rho_{Y,n}(\sigma_i) := \mathrm{id}_Y^{\otimes' i-1} \otimes' c'_{Y,Y} \otimes' \mathrm{id}_Y^{\otimes' n-i-1},\ i = 1, ..., n-1; \tag{4.11}$$

$$\rho_{Y,n}(t_j) := \mathrm{id}_Y^{\otimes' i-1} \otimes' \theta'_Y \otimes' \mathrm{id}_Y^{\otimes' n-i},\ j = 1, ..., n. \tag{4.12}$$

By Theorem 3.8, we can obtain the underlying representation of $f\mathcal{B}_n$ on $X^{\otimes n}$:

$$\rho^{Y}_{X,n} : f\mathcal{B}_n \to Aut(X^{\otimes n}) \tag{4.13}$$

with

$$\rho^{Y}_{X,n}(\sigma_i) = (\mathrm{id}_X^{\otimes i-1} \otimes \varphi_g \otimes \mathrm{id}_X^{\otimes n-i}) \circ (\mathrm{id}_X^{\otimes i-1} \otimes c_{X,X} \otimes \mathrm{id}_X^{\otimes n-i-1}),\ i = 1, ..., n-1; \tag{4.14}$$

$$\rho^{Y}_{X,n}(t_j) = (\mathrm{id}_X^{\otimes j-1} \otimes \varphi_g \otimes \mathrm{id}_X^{\otimes n-j}) \circ (\mathrm{id}_X^{\otimes j-1} \otimes \theta_X \otimes \mathrm{id}_X^{\otimes n-j}),\ j = 1, ..., n. \tag{4.15}$$

Recall the general homotopy G-fixed point $Y = (X, \{\varphi_h\}_{h\in G})$ we used in Section 4.1. We know the object $Y = \bigoplus_{g\in G} Y_g = \bigoplus_{g\in G} (X_g, \{\varphi^g_h\}_{h\in G})$ contains the object $X_g \in \mathcal{C}_g$, $X = \bigoplus_{g\in G} X_g$ and a family of coherence isomorphisms

$$\varphi^g_h : h.X_g \to X_g,\ \varphi_h = \bigoplus_{g\in G} \varphi^g_h,\ \forall h \in G.$$

Here we write the automorphism group $Aut(Y^{\otimes' n})$ as $\Big(\bigoplus_{g\in G} Aut(Y_g^{\otimes' n})\Big) \oplus YOTHERS$, where the part $YOTHERS$ is the direct sum of all terms of the form $Aut(\bigotimes_{g\in G} Y_g^{\otimes q_g})$ with $0 \le q_g \le n-1$ and $\sum_{g\in G} q_g = n$ or 0. Remark that $\sum_{g\in G} q_g = 0$ if and only if we step back to the case $X \in \mathcal{C}_g$ for some $g \in G$. Then we define the representation of $f\mathcal{B}_n$ on $Y^{\otimes' n}$

$$\rho_{Y,n} : f\mathcal{B}_n \to Aut(Y^{\otimes' n}) \tag{4.16}$$

as follows

$$\rho_{Y,n}(\sigma_i) := \Big(\bigoplus_{g\in G} \rho_{Y_g,n}(\sigma_i)\Big) \oplus \mathrm{id}_{YOTHERS},\ i = 1, \ldots, n-1; \tag{4.17}$$

$$\rho_{Y,n}(t_j) := \Big(\bigoplus_{g\in G} \rho_{Y_g,n}(t_j)\Big) \oplus \mathrm{id}_{YOTHERS},\ j = 1, \ldots, n. \tag{4.18}$$

Using Theorem 3.8 and automorphisms (4.14) and (4.15), we can obtain the underlying representation of $f\mathcal{B}_n$ on $X^{\otimes n}$:

$$\rho^{Y}_{X,n} : f\mathcal{B}_n \to Aut(X^{\otimes n}) \tag{4.19}$$

with

$$\rho^{Y}_{X,n}(\sigma_i) := \Big(\bigoplus_{g\in G} \rho^{Y_g}_{X_g,n}(\sigma_i)\Big) \oplus \mathrm{id}_{XOTHERS},\ i = 1, \ldots, n-1; \tag{4.20}$$

$$\rho^{Y}_{X,n}(t_j) := \Big(\bigoplus_{g\in G} \rho^{Y_g}_{X_g,n}(t_j)\Big) \oplus \mathrm{id}_{XOTHERS},\ j = 1, \ldots, n. \tag{4.21}$$

4.3 Main theorem

Theorem 4.5. *Let Z be a once-extended 3-2-1-dimensional G-equivariant topological field theory, $\mathcal{C}$ the G-ribbon category obtained by evaluation of Z on the circle with varying bundle decoration, and $Y = (X, \{\varphi_h\}_{h\in G})$ a homotopy G-fixed point, i.e. an object $X \in \mathcal{C}$ with a family of coherence isomorphisms*

$$\varphi_h : h.X \to X,\ \forall h \in G.$$

Then the framed braid group representations (4.7) obtained by evaluating of Z on genus zero surfaces with boundary labels constructed from X and its homotopy fixed data are equivalent to the framed braid group representations (4.19) on $X^{\otimes n}$ obtained from Y as an object in the orbifold category $\mathcal{C}/G$.

Proof. We first consider the homotopy G-fixed point $Y = (X, \{\varphi_h\}_{h\in G})$ with $X \in \mathcal{C}_g$ for some element $g \in G$. Substituting $\eta_{\sigma_i}(X^{\boxtimes n})$ and $\eta_{t_j}(X^{\boxtimes n})$ shown in (4.2) and (4.3) into (4.5) and (4.6) respectively and then comparing them with (4.14) and (4.15), we can obtain:

$$\rho^{Y}_{X,n}(\sigma_i) = \rho^{\varphi_g}_{X,n}(\sigma_i), \quad i = 1, \ldots, n-1;$$

$$\rho^{Y}_{X,n}(t_j) = \rho^{\varphi_g}_{X,n}(t_j), \quad j = 1, \ldots, n.$$

Since the elements σ_i, t_j, $i = 1, \ldots, n-1; j = 1, \ldots, n$ are generators of $f\mathcal{B}_n$, we can finally get $\rho^{Y}_{X,n}(\sigma) = \rho^{\varphi_g}_{X,n}(\sigma)$, $\forall \sigma \in f\mathcal{B}_n$, which means the two representations $\rho^{Y}_{X,n}$ and $\rho^{\varphi_g}_{X,n}$ are equivalent.

Now let us compare the representations (4.7) and (4.19) for the general homotopy G-fixed point object $Y = (X, \{\varphi_h\}_{h\in G}) = \bigoplus_{g\in G} Y_g = \bigoplus_{g\in G} (X_g, \{\varphi^g_h\}_{h\in G})$ containing the object $X_g \in \mathcal{C}_g$.

For each element $g \in G$ and the object $Y_g = (X_g, \{\varphi^g_h\}_{h\in G})$, we have

$$\rho^{Y_g}_{X_g,n}(\sigma_i) = \rho^{\varphi^g_g}_{X_g,n}(\sigma_i), \quad i = 1, \ldots, n-1;$$

$$\rho^{Y_g}_{X_g,n}(t_j) = \rho^{\varphi^g_g}_{X_g,n}(t_j), \quad j = 1, \ldots, n.$$

as we just proved. Then we can deduce that the automorphisms $\rho^{\varphi}_{X,n}(\sigma_i)$ in (4.8) and $\rho^{\varphi}_{X,n}(t_j)$ in (4.9) are equal to the automorphisms $\rho^{Y}_{X,n}(\sigma_i)$ in (4.20) and $\rho^{Y}_{X,n}(t_j)$ in (4.21), respectively. Therefore the representations $\rho^{\varphi}_{X,n}$ in (4.7) and $\rho^{Y}_{X,n}$ in (4.19) are equivalent.

□

4.4 Representations on homotopy G-fixed points with the same underlying simple object

Taking algebraic orbifoldization $\mathcal{C}/G$ of $\mathcal{C}$, there may be some homotopy G-fixed points with the same underlying object but with different coherence isomorphisms, such as $Y = (X, \{\varphi_h\}_{h\in G}) \in \mathcal{C}/G$ and $Y' = (X, \{\psi_h\}_{h\in G}) \in \mathcal{C}/G$, where X belongs to $\mathcal{C}_g$, and the isomorphisms φ_g and ψ_g may be different.

In this section, we consider the simple object X in $\mathcal{C}_g$ and we will see two representations $\rho_{Y,n}$ and $\rho_{Y',n}$ defined in (4.10) are isomorphic in the projective sense. This is the same to say their underlying representations $\rho^Y_{X,n}$ and $\rho^{Y'}_{X,n}$ defined as (4.13) are isomorphic in the projective sense.

Theorem 4.6. *Consider the framed braid group representations $\rho^Y_{X,n}$ and $\rho^{Y'}_{X,n}$ associated to two homotopy G-fixed points $Y, Y' \in \mathcal{C}/G$ with the same underlying simple object $X \in \mathcal{C}$. Then there is an equivalence $\rho^Y_{X,n} \cong \rho^{Y'}_{X,n}$ as projective framed braid group representations.*

Proof. To begin with, we first show the following diagram commutes up to an invertible scalar:

$$\begin{tikzcd} g.X \arrow[r, "\varphi_g"] \arrow[d, "g.\mathrm{id}_X"'] & X \arrow[d, "\mathrm{id}_X"] \\ g.X \arrow[r, "\psi_g"] & X \end{tikzcd} . \tag{4.22}$$

Since X is a simple object in the $\mathbb{C}$-linear category $\mathcal{C}$, we then know that $End(X) \cong \mathbb{C}$. Since there are coherence isomorphisms $\varphi_g, \psi_g : g.X \to X$, then the object $g.X$ is simple and lies in the isomorphism classes of the simple object X. Thus we have the isomorphism group $Iso(g.X, X) \cong Aut(X) \cong \mathbb{C}^\times$.

The key point is to correspond every endomorphism in $End(X)$ to a scalar in $\mathbb{C}$, and the composition of endomorphisms to the multiplication of scalars. Therefore we correspond isomorphisms $\mathrm{id}_X, \varphi_g, \psi_g$ to scalars $1, k_\varphi, k_\psi \in \mathbb{C}^\times$, respectively. Since the G-action preserves all identity morphisms in $\mathcal{C}$, we have $g.\mathrm{id}_X = \mathrm{id}_{g.X}$. Thus the isomorphism $g.\mathrm{id}_X$ also corresponds to 1.

Then showing that the diagram (4.22) commutes up to an invertible scalar called $k \in \mathbb{C}^\times$ is equivalent to check the existence of k such that the equality

$$1 \cdot k_\varphi = k_\psi \cdot 1 \cdot k.$$

works. Since the scalars k_φ and k_ψ are invertible, then the scalar k exists and we obtain $k = \frac{k_\varphi}{k_\psi} \in \mathbb{C}^\times$.

Next, since θ and c are natural isomorphisms, the following diagrams commute up to an invertible scalar:

$$\begin{tikzcd} X \arrow[r, "\theta_X"] \arrow[d, "\mathrm{id}_X"'] & g.X \arrow[r, "\varphi_g"] & X \arrow[d, "\mathrm{id}_X"] \\ X \arrow[r, "\theta_X"] & g.X \arrow[r, "\psi_g"] & X \end{tikzcd} \qquad \begin{tikzcd} X\otimes X \arrow[r, "c_{X,X}"] \arrow[d, "\mathrm{id}_X\otimes\mathrm{id}_X"'] & g.X\otimes X \arrow[r, "\varphi_g\otimes\mathrm{id}_X"] & X\otimes X \arrow[d, "\mathrm{id}_X\otimes\mathrm{id}_X"] \\ X\otimes X \arrow[r, "c_{X,X}"] & g.X\otimes X \arrow[r, "\psi_g\otimes\mathrm{id}_X"] & X\otimes X \end{tikzcd}$$

Therefore for any $\sigma \in f\mathcal{B}_n$, the following diagram commutes up to an invertible scalar:

$$\begin{tikzcd} X^{\otimes n} \arrow[r, "\rho^Y_{X,n}(\sigma)"] \arrow[d, "\mathrm{id}_X^{\otimes n}"'] & X^{\otimes n} \arrow[d, "\mathrm{id}_X^{\otimes n}"] \\ X^{\otimes n} \arrow[r, "\rho^{Y'}_{X,n}(\sigma)"] & X^{\otimes n} \end{tikzcd} ,$$

which finishes our proof. ∎

Remark 4.7. In the proof of Theorem 4.6, we showed the existence of the scalar $k = \frac{k_\varphi}{k_\psi} \in \mathbb{C}^\times$. Here we even more show that the choices of this scalar k are finite.

Recall that g has the finite order $|g|$, since G is a finite group, and $Y = (X, \{\varphi_h\}_{h\in G}) \in \mathcal{C}/G$ is a homotopy G-fixed point. Then the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "="] \arrow[rrrrr, bend left, "\varphi_1 = \varphi_{g^{|g|}}"] & g^{|g|}.X \arrow[r] & \cdots \arrow[r] & g^2.X \arrow[r, "g.\varphi_g"] & g.X \arrow[r, "\varphi_g"] & X \end{tikzcd} \tag{4.23}$$

Then we can obtain $k_\varphi^{|g|} = 1$, i.e.

$$k_\varphi = e^{(2\pi i)\cdot \frac{c_1}{|g|}},\ c_1 = 1, \dots, |g|.$$

For the same reason, we can obtain

$$k_\psi = e^{(2\pi i)\cdot \frac{c_2}{|g|}},\ c_2 = 1, \dots, |g|,$$

and therefore

$$k = \frac{k_\varphi}{k_\psi} = e^{(2\pi i)\cdot \frac{c_1 - c_2}{|g|}} \in \mathbb{C}^\times,\ c_1, c_2 = 1, \dots, |g|.$$

This fact also tells us that the set of isomorphism classes of homotopy G-fixed points with the same underlying object has finite elements.

Remark 4.8. In topological quantum computing (TQC), representations of braid groups $\mathcal{B}_n \subset f\mathcal{B}_n$ on some unitary group $U(m)$ for some m (m may be infinite) can be regarded to be the statistics of n particles called anyons ([RW, Section 2.3.4.]), which stand in an important position in 2D physics. Using the language of category theory, we can define anyons as simple objects in unitary modular categories (UMC), see [RW, Section 2.5.2.].

For the generalization of the equivariant case, we also have the equivariant version of UMC, called the modular equivariant category, which is a ribbon equivariant category with an additional property, see [K, Definition 10.1.]. Anyons can be generalized to be simple objects in modular equivariant categories, which are called symmetric defects, or called defects in short.

Given a modular G-equivariant category $\mathcal{C}$ for some finite group G, we will focus on one kind of special defects, which are fixed by the G action. This implies that these defects are actually homotopy G-fixed points described in Remark 3.6.

In [RW, Section 7.2.], Rowell and Wang constructed arbitrary braid group representation from one homotopy G-fixed point, and asked if this representation is equivalent to the representation after equivariantization in the projective sense, i.e. [RW, Conjecture 7.2.]. Note that the term "equivariantization" appears in [DGNO, Section 4.1.3.], which is the same as the algebraic orbifoldization in Definition 3.4.

Our Theorems 4.5 and 4.6 prove [RW, Conjecture 7.2.].

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