- The identity: 1.

- $2 \cdot 4 = 8$ rotations by 120°.

- 3 rotations of 180°.

So we have a bijection $r : \{B, P, W, Y\} \to \{B, P, W, Y\}$ where

$$\begin{tikzcd}[row sep=tiny]
B \arrow[r] & B \\
P \arrow[rdd] & P \\
W \arrow[ru] & W \\
Y \arrow[ru] & Y
\end{tikzcd}$$

Symmetric Group

Let S be a set (e.g. $E = \{B, P, W, Y\}$). The Symmetric Group $\mathrm{Sym}(E)$ is the set of bijections $f : E \to E$ equipped with the binary operation $\circ$ (composition).

October 3, 2023

Homework

First homework should be released this Thursday, October 5th.
Next lecture will be on group actions.

Symmetric Group

Let X be a set.
When $|X| = n$ denote the elements $\{1, 2, \ldots, n\}$.
$\mathrm{Sym}(X) = \{f : X \to X | f \text{ is bijective}\}$.
With $\circ$ (composition of functions) as a binary operation, $\mathrm{Sym}(X)$ is a group.

Symmetric Group Order

If $|X| = n$, then $|\mathrm{Sym}(X)| = n!$

- Proof Let $X = \{1, 2, \ldots, n\}$. A bijection f consists of $f(1), f(2), \ldots, f(n)$.
  For $f(1)$, we have n choices; for $f(2)$ we have $n - 1$ choices. This continues until only 1 choice remains for $f(n)$.
  Therefore the choices are $(n)(n-1)\cdots(1) = n!$

Example

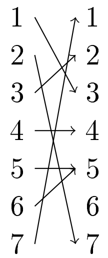
For the symmetric group on four letters $\{a, b, c, d\}$, $|\mathrm{Sym}(4)| = 4! = 24$

Cycles

Let $x = \{1, \ldots, n\}$, $m \geq 1$ be an integer and $a_1, a_2, \ldots, a_m$ distinct elements in X.
Then the m-cycle denoted by $(a_1\ a_2 \cdots a_m)$ is the element of $\mathrm{Sym}(X)$ which maps a_1 to a_2, a_2 to a_3,…, a_{m-1} to a_m, and a_m to a_1.

Example

Let $n = 7$ and $m = 4$. Then $(2\ 7\ 1\ 3)$ is a bijection.



Degenerate Case

$m = 1$ gives Id_X.

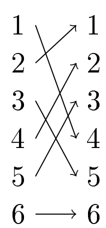
First Non-Degenerate Case

A transposition is, by definition a 2-cycle: $(a_1\ a_2)$.

Symmetric Group as Cycle Composition

Every elenent in $\mathrm{Sym}(X)$ is the product (using $\circ$) of m-cycles, where m can vary.

- Proof Consider Sym(6).

  This gives a bijection $\pi = (1\ 4\ 2)(3\ 5)(6)$ which is the composition of cycles.
  We say that this π has cycle type $3 + 2 + 1$.

- Cycle Type If instead $\pi = (1\ 4\ 2)(3\ 5\ 6)$ then the cycle type is given as $3 + 3$.

Finite Symmetric Groups

For $n = 2$, $\mathrm{Sym}(X) = \{\mathrm{Id}, (1\ 2)\}$.
This gives cylce types $1 + 1$ and 2.
For $n = 3$, $\mathrm{Sym}(X) = \{\mathrm{Id}, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$.
This gives cycle types $1 + 1$, $2 + 1$ and 3.

Symmetric Group for Tetrahedron

For $n = 4$ let $X = \{B, P, W, Y\}$.
Partitions of $n = 4$ are

$$\begin{array}{llllllll}
4 = 4 & 6 & (B\ P\ W\ Y) & \cdots & & & & \text{sign} = -1 \\
\ \ = 3+1 & 4 \cdot 2 = 8 & (P\ W\ Y) & \cdots & & & & \text{sign} = +1 \\
\ \ = 2+2 & \frac{\binom{4}{2}}{2} = 3 & (B\ P)(W\ Y) & (B\ W)(P\ Y) & (B\ Y)(P\ W) & & & \text{sign} = +1 \\
\ \ = 2+1+1 & \binom{4}{2} = 6 & (B\ P) & (B\ W) & (B\ Y) & (P\ W) \quad (P\ Y) & (W\ Y) & \text{sign} = -1 \\
\ \ = 1+1+1+1 & 1 & \mathrm{Id}_X & & & & & \text{sign} = +1
\end{array}$$

Rotation Group for Tetrahedron

$$\begin{aligned} A &= \{\text{Rotational Symmetries}\} \\ &= \{\mathrm{Id}_X, 8 \text{ 3-cycles}, 3 \text{ of type } 2{+}2\} \end{aligned}$$

Note, from the sign, that $A \leq \mathrm{Sym}(4)$.

Symmetries Not in Rotation

Why, for example, is $(B\ P)$ not in the rotation group?
If it were, it should be possible to swap vertices and then undo the switch with only rotation.
However, the two tetrahedra are mirror images across a plane.
Observe that the right hand rule with respect to P, W and Y will give opposite, orthogonal vectors.

Rotation as a Subgroup of Symmetry

Q: Is A a subgroup of $\mathrm{Sym}(4)$?
Following the definition, it would be necessary to veryify

- $\mathrm{Id} \in A$
- A is closed under inverse.
- A is closed under composition.

Group Homomorphism

Let G and H be groups (whose binary operations are denoted by $g_1 \cdot g_2$).
A (group) homomorphism from G to H is a function $\phi : G \to H$ such that

- $\phi(g_1 \underset{G}{\cdot} g_2) = \phi(g_1) \underset{H}{\cdot} \phi(g_2)$

Properties of Group Homomorphism

1. $\phi(1_G) = 1_H$

2. $\phi(g^{-1}) = [\phi(g)]^{-1}, \ \forall g \in G$

- Proof By definition, $\phi(1_G \cdot 1_G) = \phi(1_G) \cdot \phi(1_G)$.
  Letting $e = \phi(1_G)$, we get $e = e \cdot e$.
  By multiplying both sides by e^{-1}, we get $1_H = e$.
  Part two is left as an exercise.

Example 1

Let $n \geq 1$ and $G = \mathrm{GL}_n(\mathbb{R}) = \{A \in M_{n \times n}(\mathbb{R}) | \det(A) \neq 0\}$.
In particular, when $n = 1$, $\mathrm{GL}_1(\mathbb{R}) = \mathbb{R}^* = \{r \in \mathbb{R} | r \neq 0\}$ (with multiplication as the binary operation).
Then $\det : G \to H$ is a group homomorphism.
That is $\det(AB) = \det(A)\det(B)$ (as learned in MATH 21).

Example 2

Let $n \geq 1$, $G = \mathrm{Sym}(n)$, $H = \mathrm{GL}_n(\mathbb{R})$.
Construct a group homomorphism $\rho : G \to H$.

Recall that a linear transformation $A \in H$ is completely determined by $Ae_1, \ Ae_2, \ \ldots \ , Ae_n$ where $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$.

For $\pi \in G = \mathrm{Sym}(n)$, $\rho(\pi)$ is the linear transformation that maps e_i to e_j whenever π maps i to j.
This is a surjective linear transformation on a vector space and, therefore, invertible.

- Example For $n = 4$ and $\pi = (2\ 3\ 4)$

$$\begin{tikzcd}[row sep=tiny]
1 \arrow[r] & 1 \\
2 \arrow[dr] & 2 \\
3 \arrow[dr] & 3 \\
4 \arrow[uur] & 4
\end{tikzcd} \quad \rho(\pi)$$

$$\begin{tikzcd}[row sep=tiny]
e_1 \arrow[r] & e_1 \\
e_2 \arrow[dr] & e_2 \\
e_3 \arrow[dr] & e_3 \\
e_4 \arrow[uur] & e_4
\end{tikzcd}$$

Therefore

$$\rho(\pi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

  - Is this a group homomorphism? Let $\pi_1, \pi_2 \in G$ be arbitrary elements.

Need to show: $\rho(\pi_1 \circ \pi_2) = \rho(\pi_1) \circ \rho(\pi_2)$.
Both sides are linear transformations and, hence, determined by their actions on e_i for $i = 1, \dots, n$.

$$\begin{aligned}\rho(\pi_1 \circ \pi_2)e_i &= e_{\pi(i)} \\ &= e_{\pi_1(\pi_2(i))} \\ \rho(\pi_1)(\rho(\pi_2)e_i) &= \rho(\pi_1)(e_{\pi_2(i)})\end{aligned}$$

Composition of Group Homomorphisms

Let G, H and K be groups and $G \xrightarrow{\phi} H$ and $H \xrightarrow{\psi} K$ be homomorphisms.
Then the composite $\psi \circ \phi : G \to K$ is a group homomorphism.

Proof

Let $g_1, g_2 \in G$ be arbitrary.

$$\begin{aligned}(\psi \circ \phi)(g_1 g_2) &= \psi(\phi(g_1 g_2)) && \text{by definition of } \circ \\ &= \psi(\phi(g_1 \phi(g_2)) && \text{since } \phi \text{ is a group homomorphism} \\ &= \psi(\phi(g_1))\psi(\phi(g_2)) && \text{since } \psi \text{ is a group homomorphism} \\ &= (\psi \circ \phi)(g_1) \circ (\psi \circ \phi)(g_2) && \text{by definition of } \circ\end{aligned}$$

Sign Homomorphism

Let $n \geq 1$ and $G = \mathrm{Sym}(n)$.
Ths sign homomorphism is the composition sign: $G \xrightarrow{\rho} \mathrm{GL}_n(\mathbb{R}) \xrightarrow{\det} \mathbb{R}^*$

Sign of Symmetric Group

$\mathrm{sign}(\mathrm{sym}(n)) \subseteq \{1, -1\} \leq \mathbb{R}^*$

- Lemma Let $a_1, \dots, a_m$ be distinct numbers between 1 and n.
  Then $(a_1 \cdots a_m)$ is equal to $(a_1 \cdots a_{m-1})(a_{m-1}\ a_m)$.
  This will be proven on homework.

- Corollary Any m cycle is the composition of $m - 1$ transpositions.
  Namely, $(a_1, \dots, a_m) = (a_1\ a_2)(a_2\ a_3)\cdots(a_{m-1}\ a_m)$.
  Easily check: $\mathrm{sign}((a_i\ a_{i+1})) = -1$.
  Now any $g \in \mathrm{Sym}(n)$ allows a cycle decomposition.

Kernel of a Homomorphism

Let $G \xrightarrow{\phi} H$ be a group homomorphism.
The kernel of ϕ is $\ker(\phi) := \{g \in G | \phi(g) = 1_H\}$.

The Kernel is a Subgroup

Let $g_1, g_2 \in \ker(\phi)$. Then

$$\begin{aligned} \phi(g_1 g_2) &= \phi(g_1)\phi(g_2) && \phi \text{ is a homomorphism} \\ &= 1_H 1_H && g_1, g_2 \in \ker(\phi) \\ &= 1_H && g_1, g_2 \in \ker(\phi) \end{aligned}$$

Similarly, $1_G \in \ker(\phi)$ and $g^{-1} \in \ker(\phi)$ if $g \in \ker(\phi)$. ■

Alternating Group

Let X be a set, $|X| = n \leq \infty$.
The alternating group on X is the $\mathrm{Alt}(X) = \ker(\mathrm{sign} : \mathrm{Sym}(X) \to \{\pm 1\})$.

October 5, 2023

Group Action

Let G be a group and X a set.
A (left) action of G on X is a function $\alpha : G \times X \to X$ which satisfies two conditions:

1. $\alpha(1_G, x) = x$ for all $x \in X$.

2. $\alpha(g_1, \alpha(g_2, x)) = \alpha(g_1 g_2, x)$ for all $g_1, g_2 \in G$ and $x \in X$.

Notation

Write $\alpha(g, x) = g * x = g \cdot x = gx$.

Example A

Let X be any set, and let $G = \mathrm{Sym}(X) = \{f : X \to X \text{ bijections}\}$ where the group operation $\circ$ is the composition of functions.
Then G acts (on the left) on X by $f * x \underset{\text{def}}{=} f(x)$.
Then the features

1. $\mathrm{Id}_X(x) = x, \ \forall x \in X$

2. $g_1 * (g_2 * x) = (g_1 \circ g_2) * x, \ \forall g_1, g_2 \in G, \ \forall x \in X$

    - Or $g_1(g_2(x)) = (g_1 \circ g_2)(x)$

are satisfied.