

# Manifolds I

September 26, 2024

## Class Organization

1 Takehome Midterm

1 Takehome Final

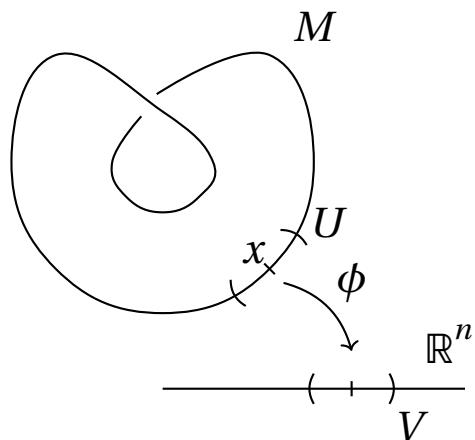
Homeworks assigned, but not graded.

<https://ginzburg.math.ucsc.edu/teaching/208manifolds1-2024/syl.html>

## Definition: Topological Manifolds

For  $M$  a topological space,  $M$  is a topological manifold if  $\forall x \in M, \exists M \ni U \ni x$  and homeomorphism  $\phi: U \rightarrow V \subset \mathbb{R}^n$  for  $V$  open.

To avoid problems (see below), further assume that  $M$  is Hausdorff and second countable.

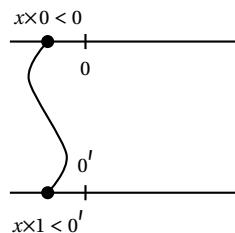


## Exercise

We can require  $V$  to be an open ball.

## Problems

- $M$  need not be Hausdorff.



With  $(\mathbb{R} \times 0 \coprod \mathbb{R} \times 1) / \sim$ .

- $M$  need not be second countable.

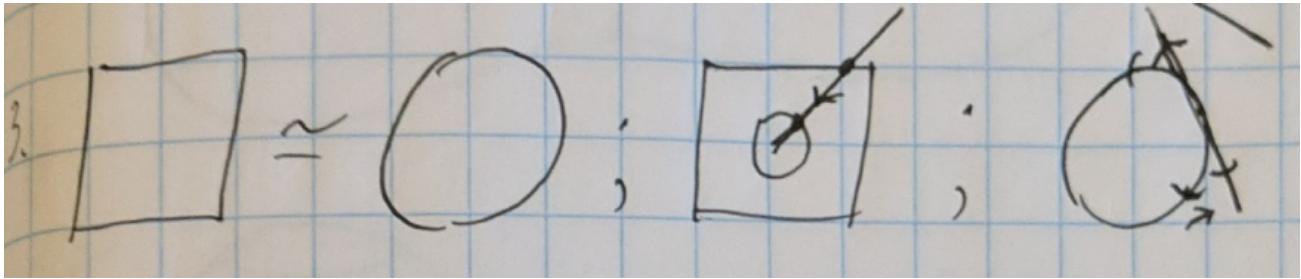
Take  $\coprod_S \mathbb{R}_S$  where  $S$  is an uncountable index.

## Examples

### Example 1

If  $N \underset{\text{homeo}}{\simeq} M$ , this implies  $N$  is a manifold.

### Example 2



### Example 3

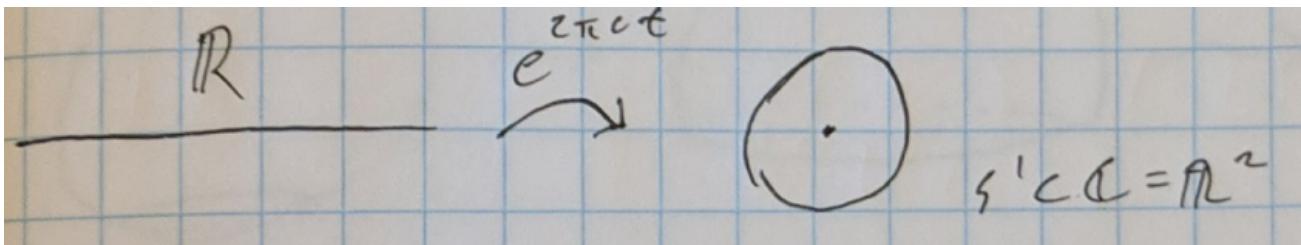
An open subset of a manifold is a manifold.

### Example 4

$M, N$  manifolds implies  $M \times N$  is a manifold.

### Example 5

Take  $\mathbb{R}/\mathbb{Z}$  by the equivalence relation  $t \sim t'$  iff  $t' - t \in \mathbb{Z}$ .



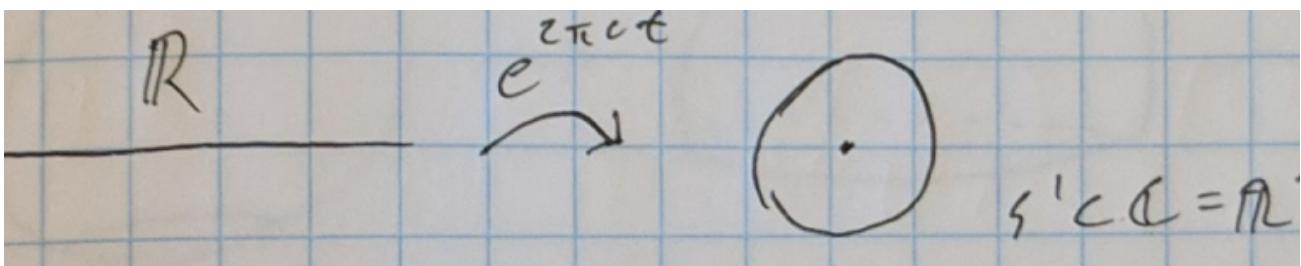
Then  $C^0(S^1)$  relates to periodic functions with period 1.

### Example 6

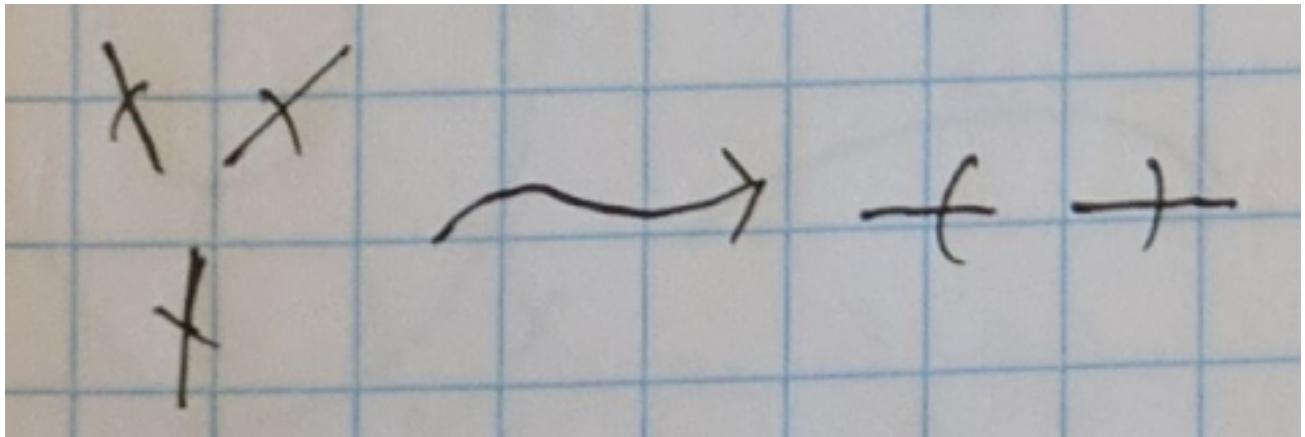
$$\mathbb{T}^n = S^1 \times \cdots \times S^1.$$

### Counterexample 1

$[0, 1]$  is not a manifold.

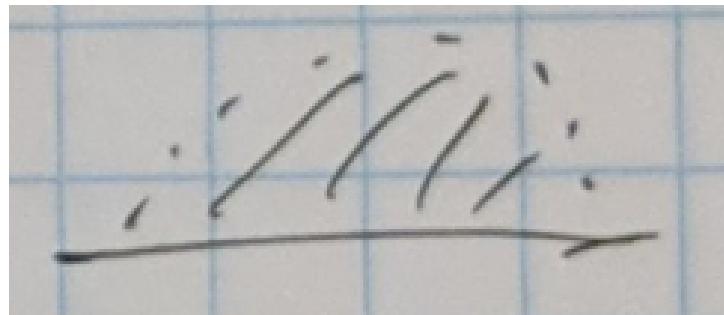


Since 0 must map somewhere in the open interval, its deletion results in a connected space in the former case but a disconnected one in the latter. Similarly, the following breaks into three and two connected components respectively.



## Definition: Manifold with Boundary

There exists a neighborhood  $\forall x \in M$  homeomorphic to either the open ball or the half-closed half-ball.



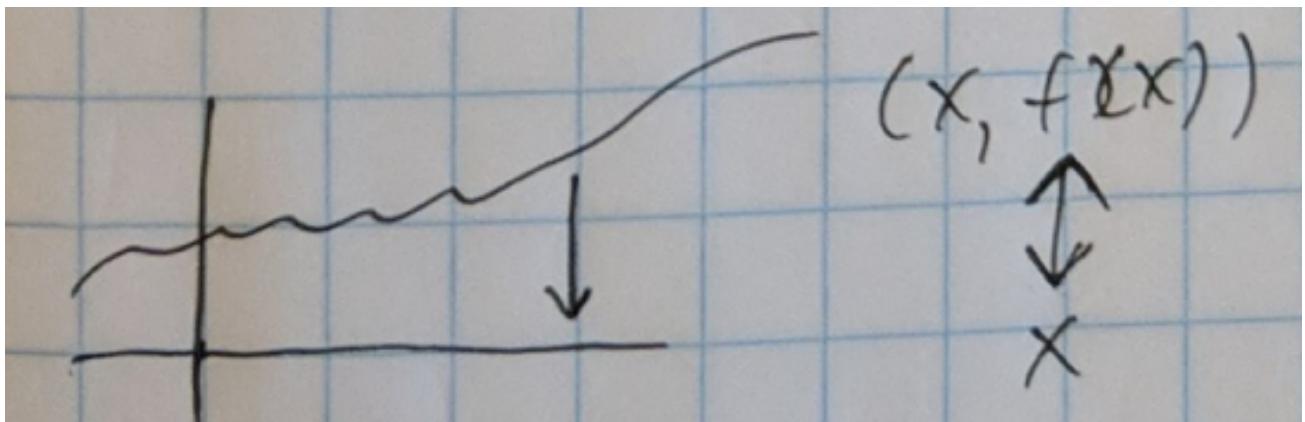
## Exercise

A connected manifold is path-connected.

## Examples

### Example 7

Take  $f : \mathbb{R}^n \xrightarrow{C^0} \mathbb{R}$  with graph  $\Gamma_f = \{(x, f(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^n \times \mathbb{R}$ .

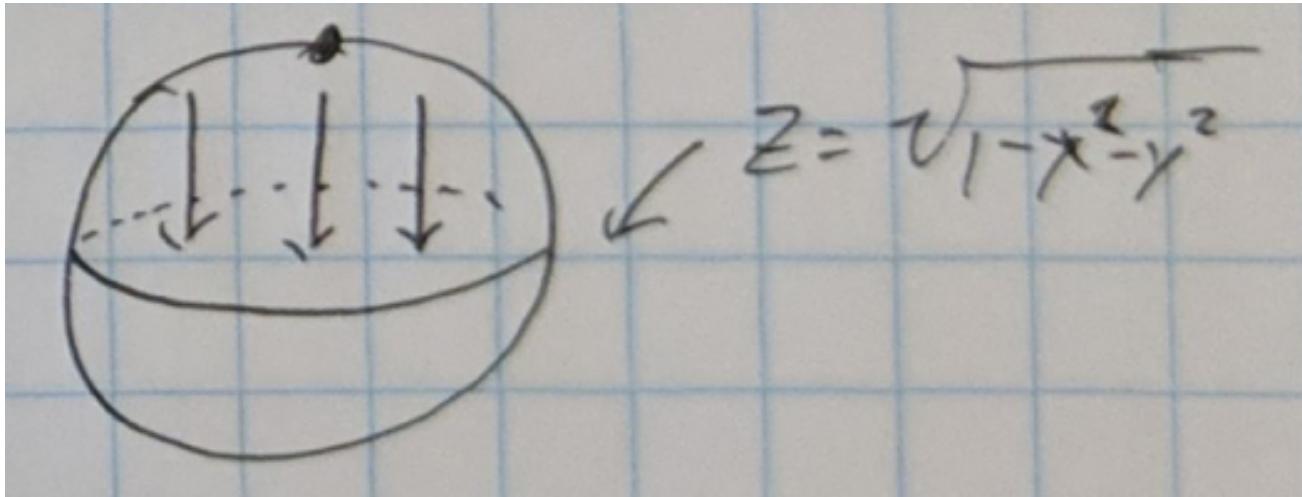


### Example 8

Take  $f : M \rightarrow N$  between manifolds, then  $M \simeq \Gamma_f \subseteq M \times N$ .

### Example 9

$$S^n \subset \mathbb{R}^{n+1}.$$



### Definition: Real Projective Spaces

Take  $\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim$  where  $x \sim y \iff x = \lambda y$  for  $\lambda \neq 0$ .

Informally, the collection of lines through the origin.

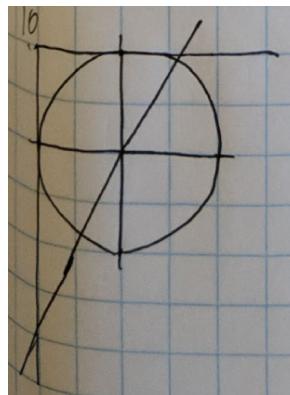
Alternatively,  $\mathbb{RP}^n = S^n / \sim$  where  $x \sim -x$ .

That is, identifying the antipodal points of the unit sphere.

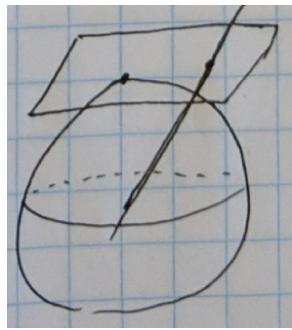
We may also consider  $\mathbb{RP}^n = SO(n+1)/SO(n)$ .

### Claim

$\mathbb{RP}^n$  is a manifold.



$$\mathbb{RP}^1 \setminus \{x\text{-axis}\} \xrightarrow{\text{homeo}} \mathbb{R}.$$



$$\mathbb{RP}^2 \setminus \mathbb{RP}^1 \xrightarrow{\text{homeo}} \mathbb{R}^2$$

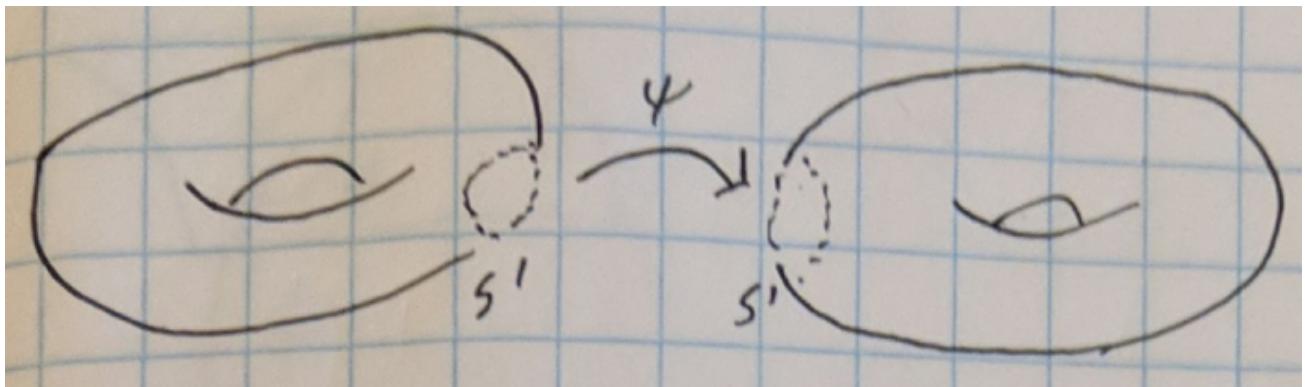
We have that  $\mathbb{RP}^1$  is homeomorphic to the circle, and  $\mathbb{RP}^n = \mathbb{RP}^{n-1} \cup B^n$ .

Take  $x = (x_0, \dots, x_n)$ ,  $y = (y_0, \dots, y_n) = (\lambda x_0, \dots, \lambda x_n)$  and  $[x] = [x_0 : x_1 : \dots : x_n]$ .

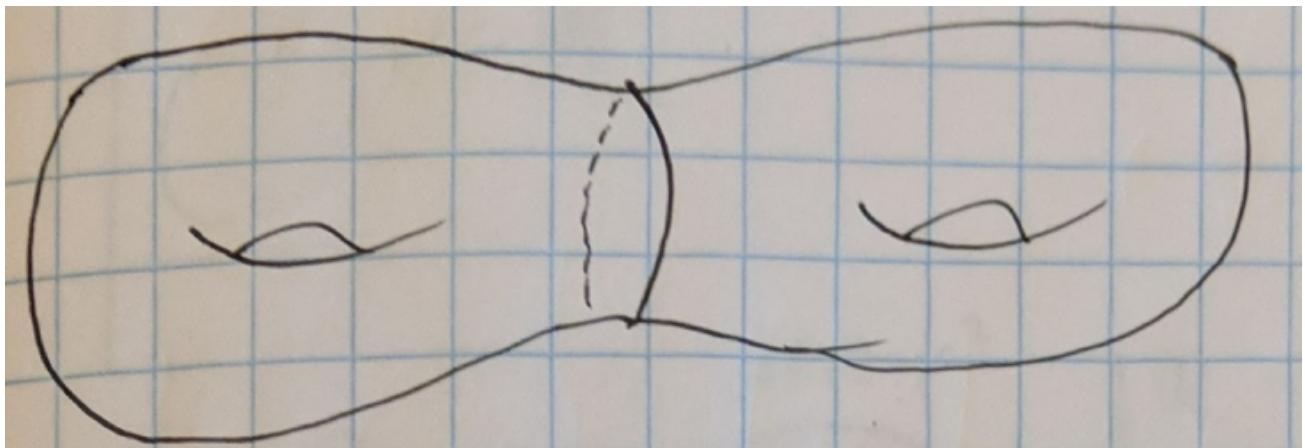
Then for  $U_k \subset \mathbb{RP}^n$  with  $U_k = \{[x] : x_k \neq 0\}$ , we have that  $U_0, \dots, U_n$  covers  $\mathbb{RP}^n$ .

Then define  $U_k \rightarrow \mathbb{R}^n$  by  $[x_0 : \dots : x_n] \mapsto \left(\frac{x_0}{x_k}, \dots, \frac{x_{k-1}}{x_k}, \frac{x_{k+1}}{x_k}, \dots, \frac{x_n}{x_k}\right)$ .

## Connected Sum of Manifolds

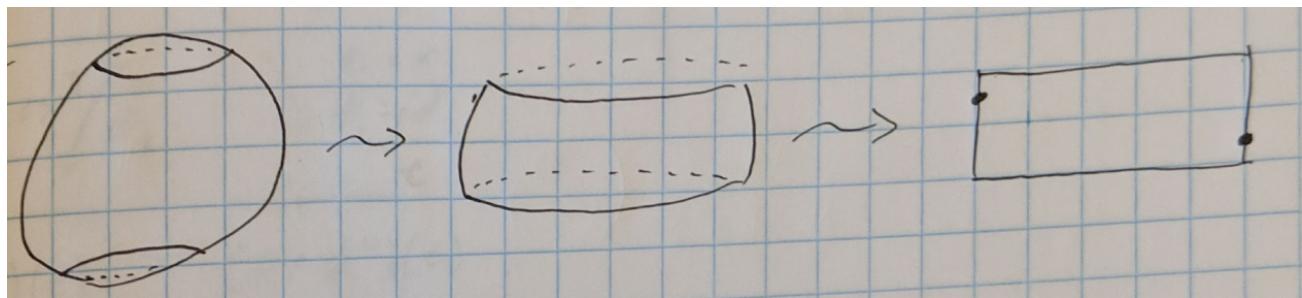


$$M \setminus B^n \coprod N \setminus B^n$$



$$M \# N.$$

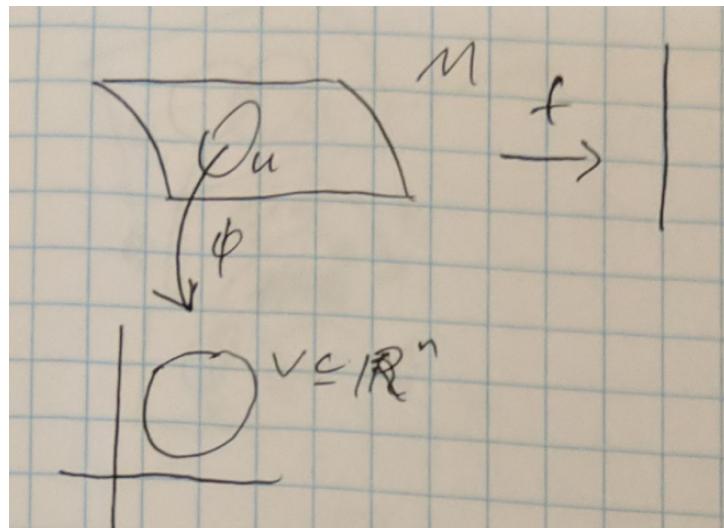
## Möbius Band



October 1, 2024

## A Failed Definition

$$f \in C^{r \geq 1}; f \circ \phi^{-1} : V \xrightarrow{C^r} \mathbb{R}.$$



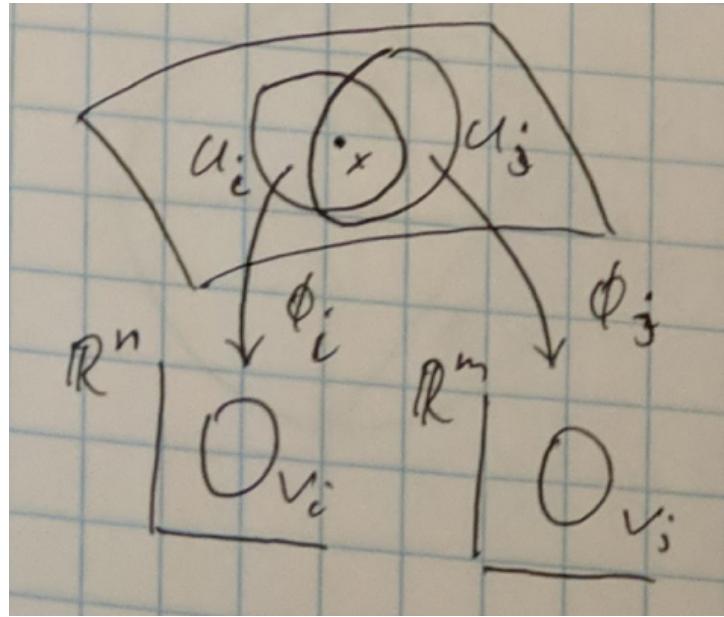
## Example

$$\begin{array}{c}
 2. \quad M = \mathbb{R} \xrightarrow{x} \mathbb{R} \\
 \downarrow \begin{array}{l} t = x^3 \\ \phi_2 \end{array} \quad \downarrow \begin{array}{l} t = x \\ \phi_1 = id \end{array} \quad \xrightarrow{f(x) = x^2} \\
 t \quad t \quad t \\
 \hline
 (f \circ \phi_2^{-1})(t) = t^2 \quad (f \circ \phi_1^{-1})(t) = t^2 \\
 \text{Not } C^1 \quad \text{in } C^\infty
 \end{array}$$

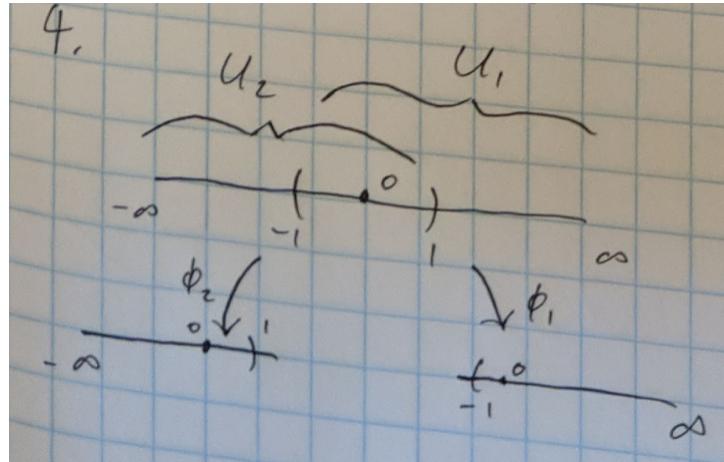
## Definition: Charts

Say there exists a cover  $U_i$  by open sets and  $U_i \xrightarrow{\phi_i} V_i \subseteq \mathbb{R}^n$  fixed.  
Then the pair  $(U_i, \phi_i)$  is a chart.

**What if a point belongs to two charts?**



With  $f$  smooth at  $x$ ,  $f \circ \phi_i^{-1}$  smooth at  $\phi_i(x)$  and  $f \circ \phi_j^{-1}$  smooth at  $\phi_j(x)$ .

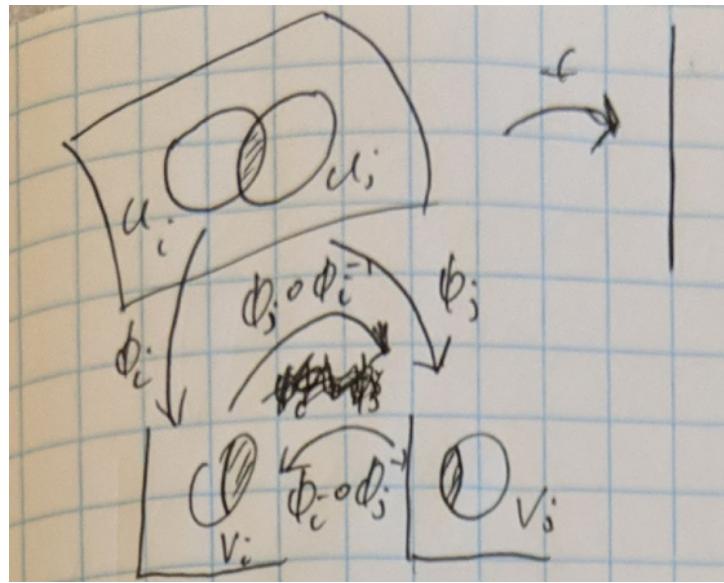


## Notation

The notation  $C^r$  will be used interchangably with the term smooth.

## Definition: Smooth Atlas

Let  $M$  be a topological manifold. A smooth atlas on  $M$  is a cover  $(U_i, \phi_i : U_i \xrightarrow{\sim} V_i \subset \mathbb{R}^n)$  where  $\phi_j \circ \phi_i^{-1}$  and  $\phi_i \circ \phi_j^{-1}$  are smooth for every  $i$  and  $j$ .



Say that the charts are (smooth) compatible.

## Definition: Smooth Function

Say that  $f$  is smooth at  $x \in M$  if there exists a chart  $U_i \ni x$  such that  $f \circ \phi_i$  is smooth at  $\phi_i(x)$ . Equivalently, if for every chart  $U_i \ni x$  we have that  $f \circ \phi_i$  is smooth at  $\phi_i(x)$ .

- Proof

$$f \circ \phi_j^{-1} = (f \circ \phi_i^{-1}) \circ \underbrace{(\phi_i \circ \phi_j^{-1})}_{C^r}$$

## Definition: Compatibility (Equivalence) of Atlases

Atlases  $A_1$  and  $A_2$  are compatible or equivalent if every chart in  $A_1$  is compatible with every chart in  $A_2$ . Equivalently,  $A_1 \cup A_2$  is also an atlas.

- Claim: This is an equivalence relation.

## Example

Consider  $\mathbb{R}$ .

Atlas 1:  $U = \mathbb{R}$  and  $\phi = \text{id}$ .

Atlas 2:  $U_1 = (1, \infty)$ ,  $\phi_1(x) = x^2$ ,  $U_2 = (-\infty, 2)$  and  $\phi_2(x) = x$ .

## Definition: Diffeomorphism

$\mathbb{R}^n \supset V \xrightarrow{F} W \subset \mathbb{R}^n$  is a diffeomorphism if

- $F$  is  $C^r$ ,
- $F$  is invertible, and
- $F^{-1}$  is  $C^r$

## Counterexample

$y = x^3$  is a smooth homeomorphism but not a diffeomorphism.

## Definition: Smooth Structure / Maximal Atlas

Given an atlas, we may take all compatible atlases and define a smooth structure by the union of all such objects (i.e. the maximal atlas).

### Lemma:

Every smooth manifold has a countable, locally finite atlas of precompact charts.

## Examples

- Zero dimensional manifolds (i.e. a point).
- $\mathbb{R}^n$  and open subsets of  $\mathbb{R}^n$ .
- If  $M, N$  are smooth manifolds, then  $M \times N$  is a smooth manifold.

That is, if we have atlases  $(U_i, \phi_i)$  and  $(W_j, \psi_j)$ , we may generate  $(U_i \times W_j, \phi_i \times \psi_j)$ .

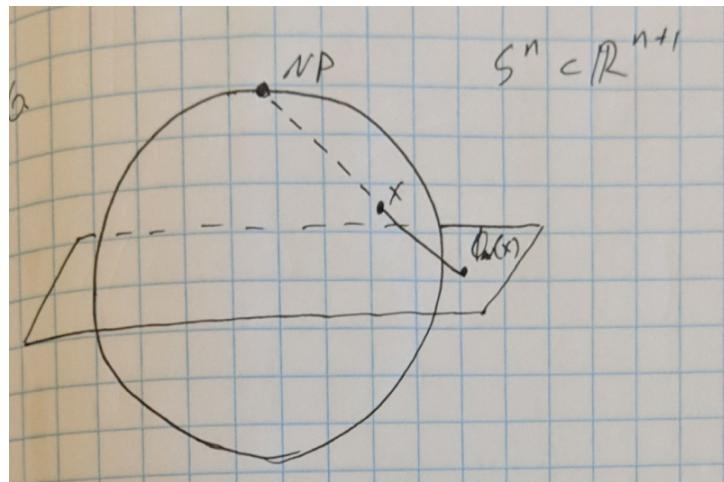
- Take  $F: M \xrightarrow{\text{homeo}} N$  with  $N$  a smooth manifold. Then  $M$  is smooth.

Take an atlas  $A$  on  $N$  and the pullback  $F^{-1}A = \{(F^{-1}(U_i), \phi_i \circ F)\}$ .

- An open subset of a smooth  $M$  is a smooth manifold.
- $\text{GL}(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$ .

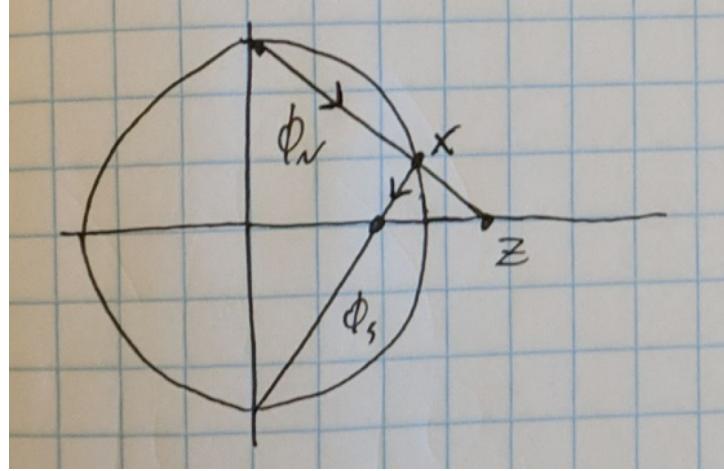
## The n-Sphere

- $S^n$  is a manifold



$$U_N = S^n \setminus NP \xrightarrow{\phi_N} \mathbb{R}^n$$

$$U_S = S^n \setminus SP \xrightarrow{\phi_S} \mathbb{R}^n$$



$$\phi_S \phi_N^{-1}(z) = \frac{z}{|z|^2}.$$

– A different construction for  $S^n$ .

Take hemispheres  $U \xrightarrow{\text{orthogonal projection}} B^n$ .

## Projective Space

$$\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus 0 / \sim \text{ where } x \sim \lambda x \text{ for } \lambda \neq 0.$$

$$[x] = [x_0 : x_1 : \dots : x_n] = [\lambda x_0 : \lambda x_1 : \dots : \lambda x_n].$$

Take  $U_i = \{x_i \neq 0\}$  and open cover, and maps  $U_i \rightarrow \mathbb{R}^n$  given by  $[x_0 : \dots : x_n] \mapsto \left( \frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, 1, \frac{x_i}{x_i}, \dots, \frac{x_n}{x_i} \right)$ . Then for  $j < i$  take

$$\phi_j \phi_i^{-1}(y_1, \dots, y_n) = \left( \frac{y_0}{y_j}, \dots, \frac{y_{j-1}}{y_j}, \frac{y_{j+1}}{y_j}, \dots, \frac{y_{i-1}}{y_1}, 1, \frac{y_i}{y_i}, \dots, \frac{y_n}{y_1} \right)$$

## Definition: Diffeomorphism

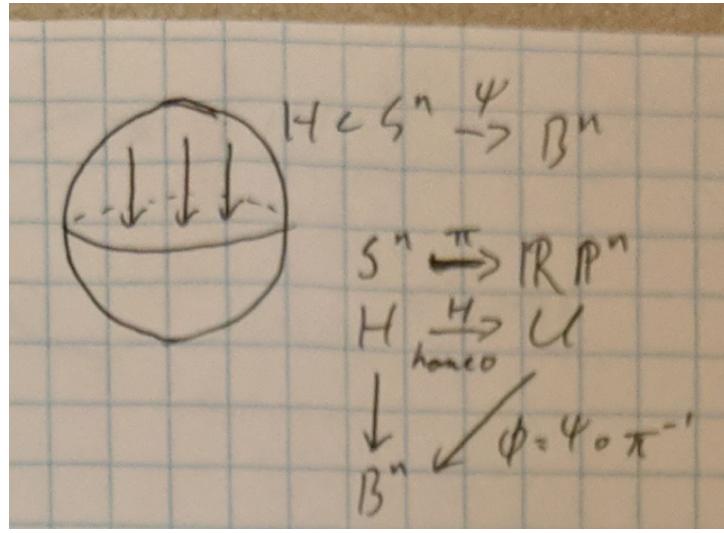
$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ B \subset B_{\max} & & A \supset A_{\max} \end{array}$$

$F$  is a diffeomorphism if  $F$  is a homoeomorphism and  $F^{-1} A_{\max} = B_{\max}$  ( $F^{-1} A \sim B$ ).

**October 3, 2024**

## Recall

$$\mathbb{RP}^n = \begin{cases} \mathbb{R}^{n+1} \setminus 0 / \sim & x \mapsto \lambda x \\ S^n / x \sim -x \end{cases}$$



## Note

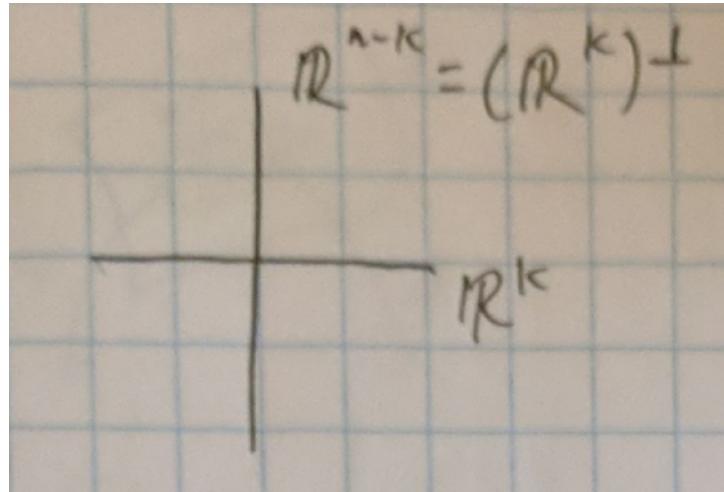
Given a manifold  $M$  and  $A$  a smooth atlas, we generate a continuum of smooth atlases not equivalent to each other. That is, given  $M \xrightarrow[\text{homeo}]{} M$ ,  $F^{-1}A \neq A$ .

## Confer With Groups

$$G \xrightarrow{F} G, a * b = F^{-1}(F(a)F(b)).$$

## Definition: Grassmannians

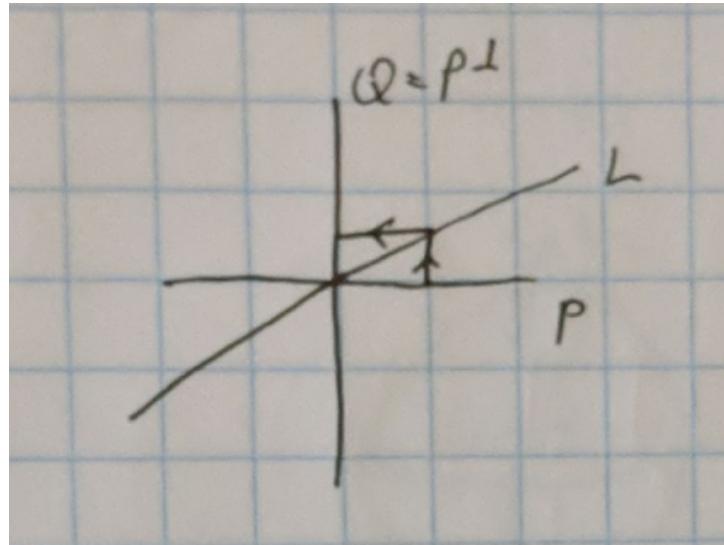
Write  $G_k(n)$ , the collection of all  $k$ -dimensional subspace  $L$  in  $\mathbb{R}^n$ .



Observe that if  $O(i)$  is the collection of orthogonal transformations in dimension  $i$ ,

$$G_k(n) = \frac{O(n)}{O(k) \times O(n-k)}$$

with  $X \sim Y$  when  $Y = XA = X(O(K) \times O(n-k))$ .

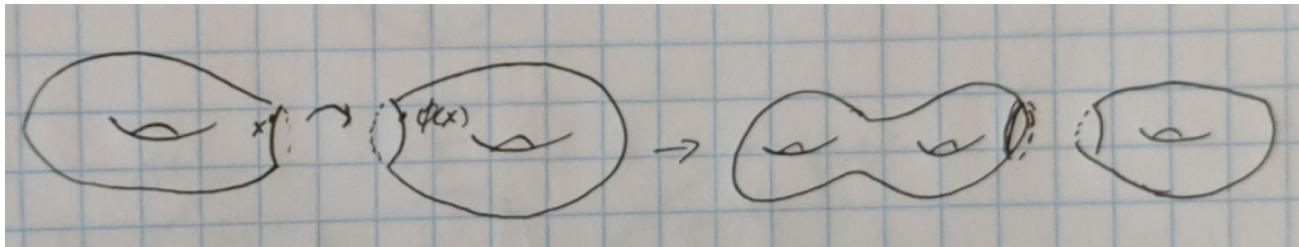


Where  $\dim(L) = k$ ,  $U_p = \{L : L \cap Q = \{0\}\}$ ,  $L = \text{graph}(A : P \rightarrow Q)$ , and we have a homeomorphism

$$U_p \xrightarrow{\phi} \underbrace{\{\text{linear maps } P \rightarrow Q\}}_{\mathbb{R}^{k \times (n-k)}}.$$

## Surfaces

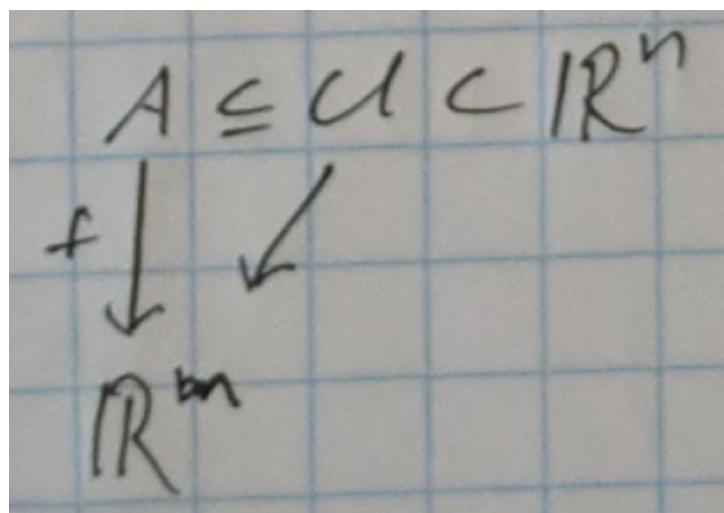
We have explored  $S^2$ ,  $\mathbb{RP}^2$ ,  $\mathbb{T}^2 = S^1 \times S^1$ . We have also connected sums.



## Terminological Remark

Let  $\mathbb{R}^N \ni A \xrightarrow{f} \mathbb{R}^m$ .

Then  $f$  is smooth if it extends to a smooth map



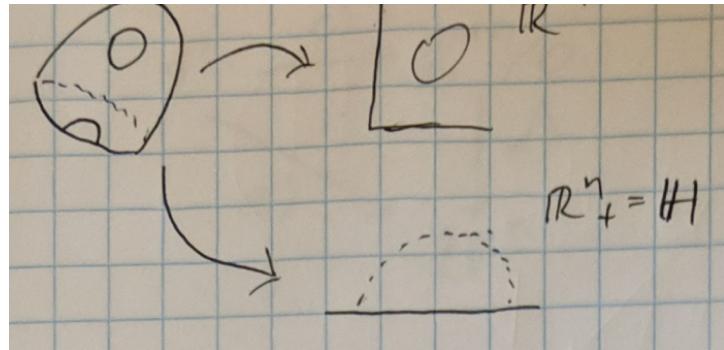
## Exercise

Let  $A = [0, \infty) \subset \mathbb{R}$ .  $f : A \rightarrow \mathbb{R}$  is smooth if and only if it is infinitely differentiable.  
Construct  $(-\varepsilon, \infty)$ .

## Definition: Smooth Manifold with Boundary

A smooth manifold with boundary is a topological space along with an atlas  $\mathcal{A}$  with charts of two types

$$\begin{aligned}\phi : U \rightarrow B^n &\quad (\text{open ball}) \\ \phi : U \rightarrow B^n \cap H\end{aligned}$$



As before,  $\phi_i \circ \phi_j^{-1}$  must be smooth.

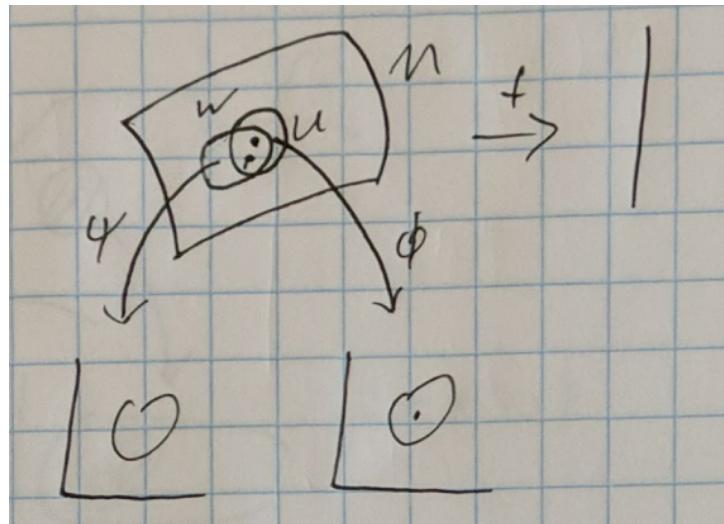
## Examples

- $M \setminus \text{open ball}.$
- The upper half space.

## Definition-Lemma: Smooth Function

A function  $f : M \rightarrow \mathbb{R}$  is smooth at  $p \in M$  if either of the following equivalent conditions is satisfied

1.  $\exists$  a chart  $(U, \phi) \ni p$  such that  $f \circ \phi^{-1}$  is smooth at  $\phi(p)$ .
2.  $\forall$  a chart  $(U, \phi) \ni p$  such that  $f \circ \phi^{-1}$  is smooth at  $\phi(p)$ .



Where  $f \circ \phi^{-1} = f \circ \psi^{-1}(\psi \circ \phi^{-1})$ .

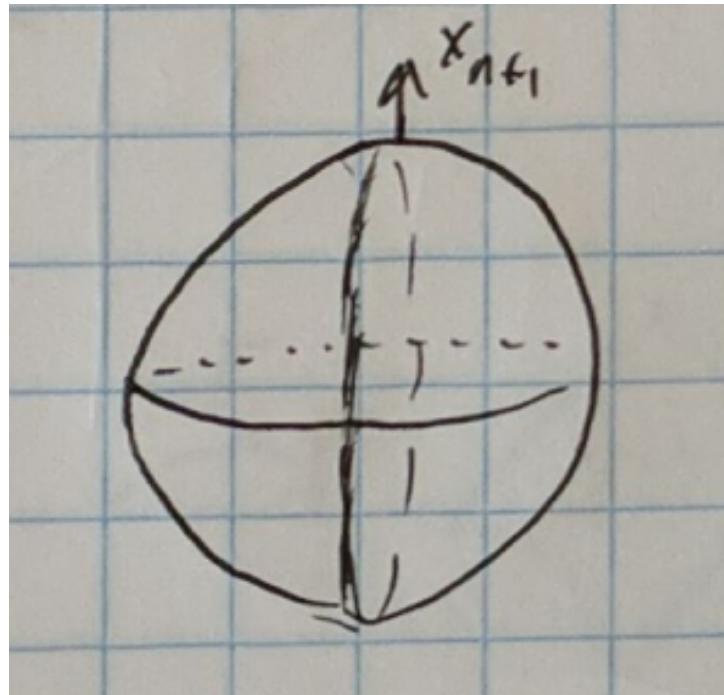
If the above hold for each  $p \in M$ , then  $f$  is smooth.

### Remark

$f$  smooth implies  $f$  is  $C^0$

### Exercise / Sketch

The height function on  $S^n$  is smooth.

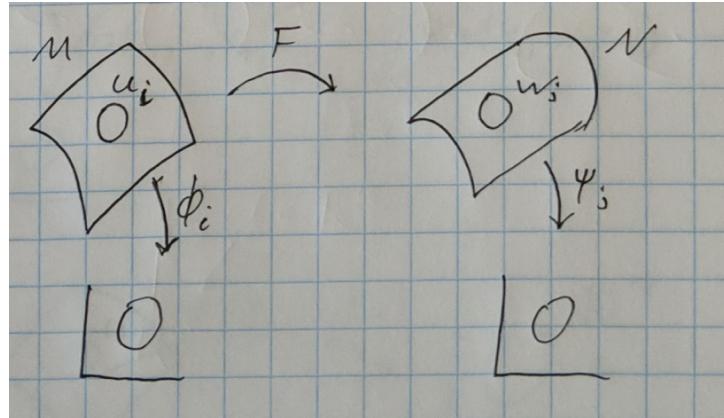


$$\phi : (x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_n).$$

$$f \circ \phi^{-1} = \pm \sqrt{1 - x_1^2 - \dots - x_n^2}.$$

Note that handling the equator requires examining the Eastern and Western hemisphere.  
The stereographic projection leads to a simpler proof.

## Definition: Smooth Function Between Manifolds



$F : M \rightarrow N$  is smooth if  $F$  is  $C^0$  and one of the following equivalent conditions is satisfied

1.  $\exists$  an atlas  $A \subset A_{\max}$  on  $M$  and an atlas  $B \subset B_{\max}$  on  $N$  such that  $\psi_j \circ F \circ \phi_i^{-1}$  is smooth on  $F^{-1}(W_j) \cap U_i$ .
2. The same as a., but for  $A_{\max}$  and  $B_{\max}$ .

Consider as an example  $S^n \rightarrow \mathbb{RP}^n$ .

### Properties of Smooth Maps

$$C^\omega \implies C^\infty \implies C^r \implies C^{r-1} \implies C^1 \implies C^0.$$

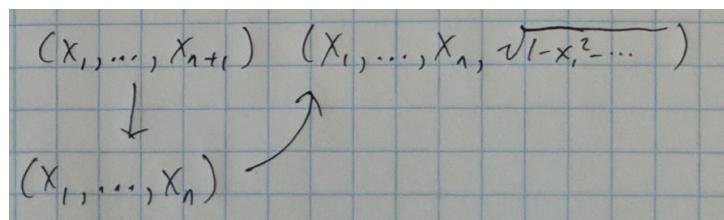
The sum and product of smooth functions is smooth.

### Exercise

The composition of smooth maps is smooth.

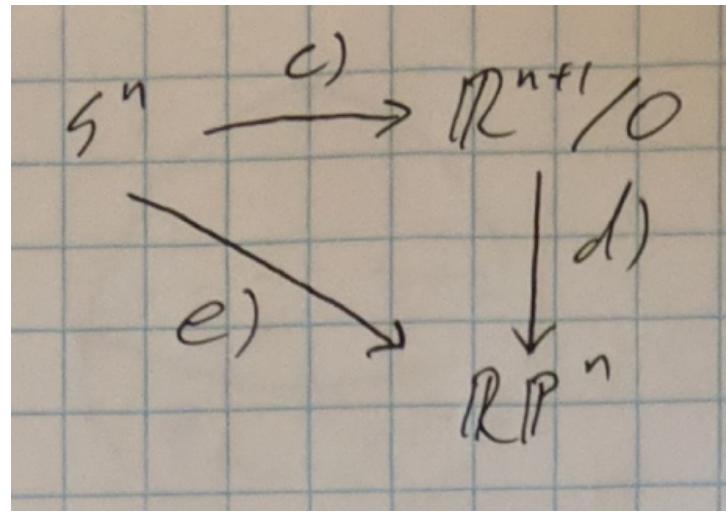
### Examples

1.  $M \times N \xrightarrow{pr} M$  is smooth.
2.  $\underbrace{M \xrightarrow{(F_1, F_2)} N_1 \times N_2}_{\text{smooth}}$  if and only if  $F_1$  and  $F_2$  are smooth.
3.  $S^n \hookrightarrow \mathbb{R}^{n+1}$  is smooth.



$$1. \mathbb{R}^{n+1} \setminus 0 \rightarrow \mathbb{RP}^n \text{ is smooth with } [x_0 : \dots : x_n] \mapsto \left( \frac{x_0}{x_i}, \dots, \frac{\hat{x}_i}{x_i}, \dots, \frac{x_n}{x_i} \right).$$

$$2. S^n \rightarrow \mathbb{RP}^n.$$



## Definition: Diffeomorphism

$$F: M \xrightarrow[A_{\max}]^{\text{diffeo}} N \text{ if } B_{\max}$$

- $F$  is smooth.
- $F$  is invertible.
- $F^{-1}$  is smooth.

## Previous Definition

$$F^{-1}(B_{\max}) = A_{\max}.$$

## Exercise

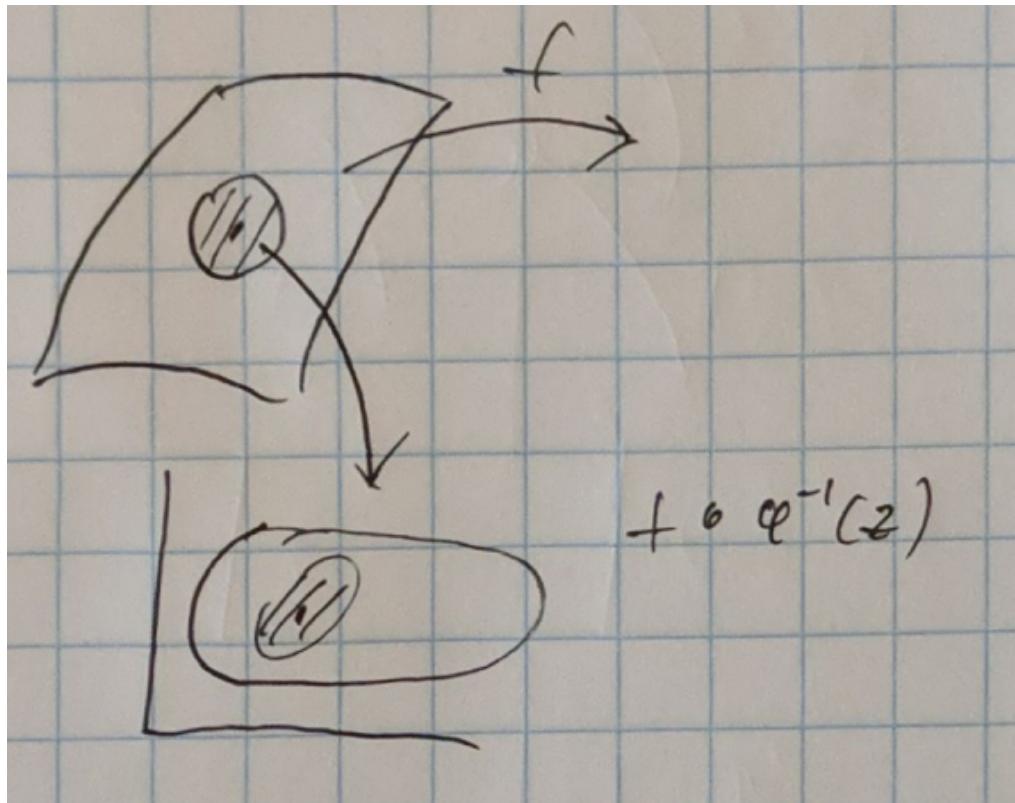
Prove that the definitions are equivalent.

## Examples

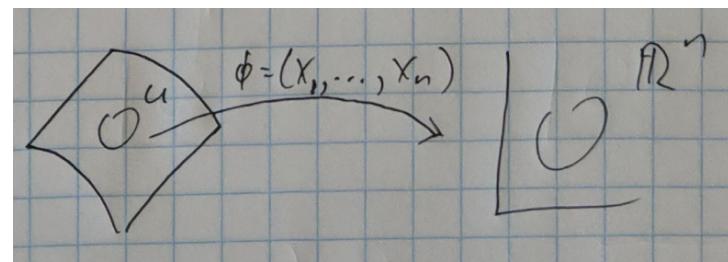
1.  $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \xrightarrow{\text{diffeo}} \mathbb{R}$ .
2.  $x \mapsto x^3$ ,  $\mathbb{R} \rightarrow \mathbb{R}$  is not a diffeomorphism.
3.  $G_k(n) \leftrightarrow G_{n-k}(n)$  with  $P \leftrightarrow P^\perp$ .

## Example 4

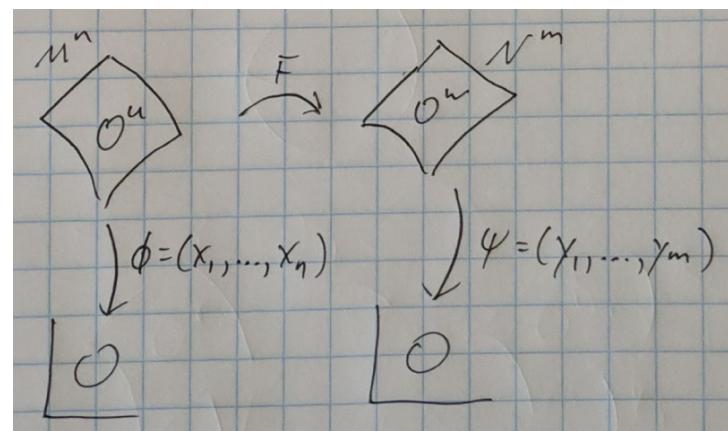
A compact, analytic manifold admits only constant smooth functions by the maximum modulus principle.



### Example 5

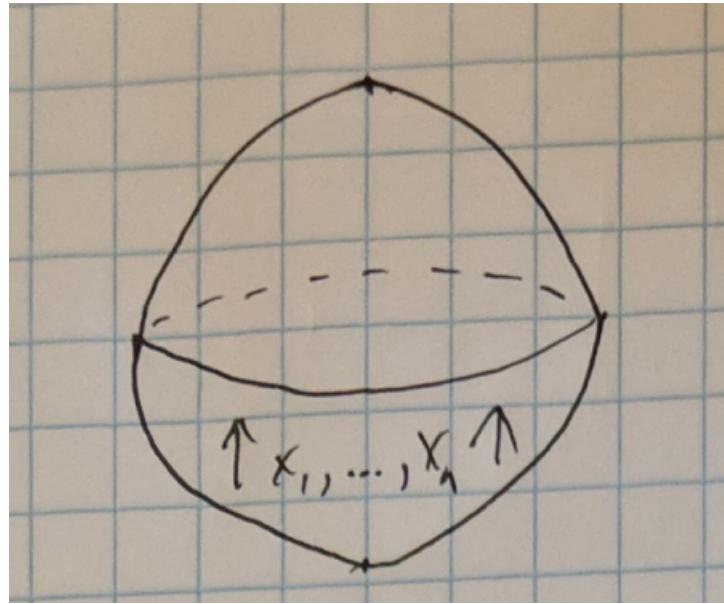


Where  $\phi = (x_1, \dots, x_n)$  and each  $x_i$  is a real-valued function.



$$\psi \circ F \circ \phi^{-1} = (y_1(x_1, \dots, x_n), \dots, y_m(x_1, \dots, x_n)).$$

Then  $S^n \hookrightarrow \mathbb{R}^{n+1}$



where  $y_1 = x_1, \dots, y_n = x_n$ , and  $y_{n+1} = -\sqrt{1 - x_1^2 - \dots - x_n^2}$ .

### Example 6

$$\mathbb{R}^{n+1} \setminus 0 \xrightarrow{F} \mathbb{RP}^n.$$

Need to check that  $\psi_j \circ F$  is smooth.

$$[t_0 : \dots : t_n] \xrightarrow{\psi_0} \left( \frac{t_1}{t_0}, \dots, \frac{t_n}{t_0} \right) \text{ with } U_0 : t_0 \neq 0$$

**October 8, 2024**

## Questions

### Question 1

Given  $M$  smooth and  $x \neq y \in M$ , does there exist  $f \in C^r(M)$  such that  $f(x) = 0$  and  $f(y) = 1$ .

### Question 2

Given  $K \subset U \subset M$  with  $K$  compact and  $U$  open and  $g : K \xrightarrow{C^r} \mathbb{R}$ , does there exist a  $C^r$  extension  $f$  of  $g$  on  $M$  such that  $\text{supp } f \subset U$ .

## Definition: Partitions of Unity

Let  $W_i$  be a locally finite open cover.

A partition of unity subordinated to  $W_i$ , is a collection of functions  $f_i : M \xrightarrow{C^r} \mathbb{R}$  satisfying

- $0 \leq f_i \leq 1$
- $\text{supp } f_i \subset W_i$
- $\sum f_i \equiv 1$

## Definition: Refinement

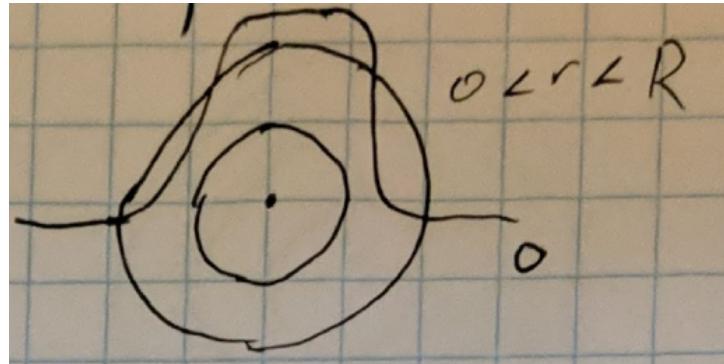
Given covers  $U_j$  and  $W_i$ ,  $U_j$  is a refinement if for each  $j$  we may find  $i$  such that  $U_j \subset W_i$ .

## Theorem

There exists a partition of unity subordinated to  $W_i$ .

### Lemma 1

Take  $B(r) \subset B(R) \subset \mathbb{R}^n$ .



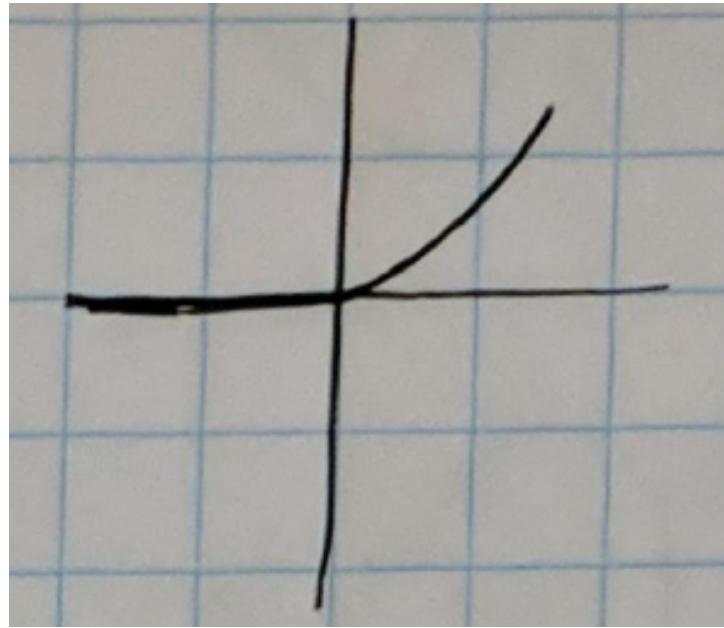
Then there exists  $g \in C^\infty$  such that

- $0 \leq g \leq 1$
- $g|_{\overline{B(r)}} \equiv 1$
- $\text{supp } g \subset B(R)$

## Proof

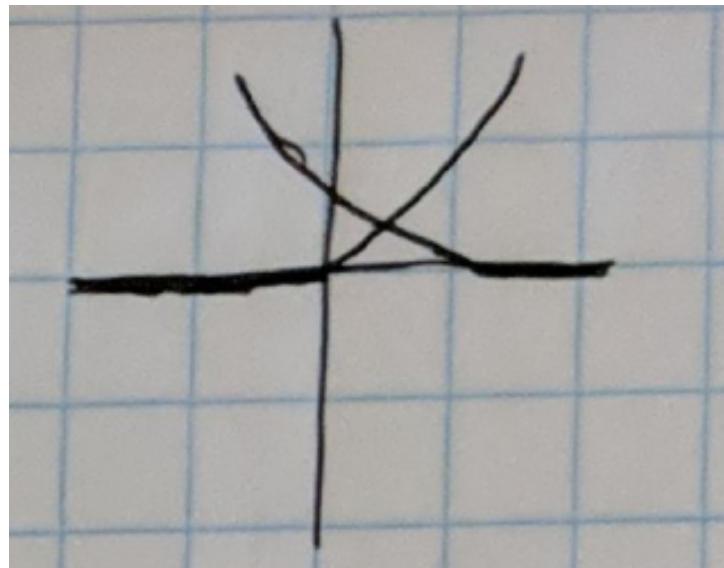
Take

$$h_0(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$



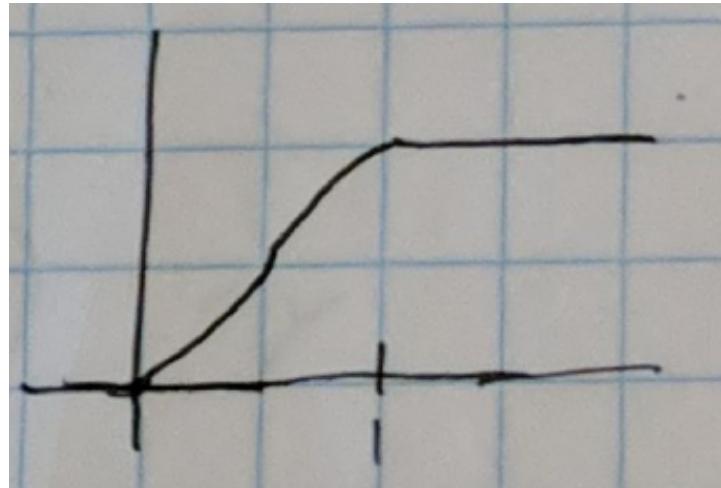
- Exercise: show that  $h_0 \in C^\infty(\mathbb{R})$  (Hint: show that derivatives agree at zero from the right.)

Then take  $h_1(x) = h_0(x) \cdot h_0(1-x)$ .

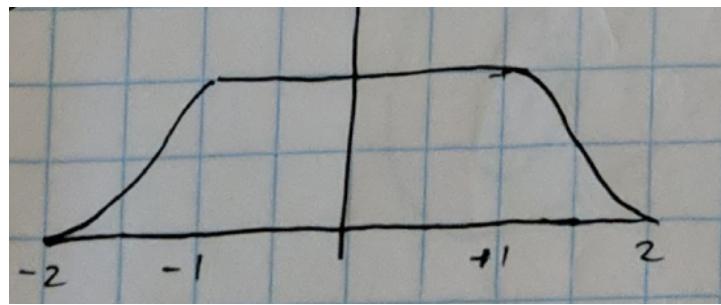


Then define

$$h_2(x) = \frac{1}{\int_{-\infty}^{\infty} h_1(t) dt} \int_{-\infty}^x h_1(t) dt$$



Finally, we define  $h(x) = h_2(x+2) \cdot h_2(2-x)$ .



Then  $g(x) = h(||x||)$  satisfies our requirements.

## Lemma 2

There exists a refinement  $(U_j, \phi_j)$  by coordinate charts such that

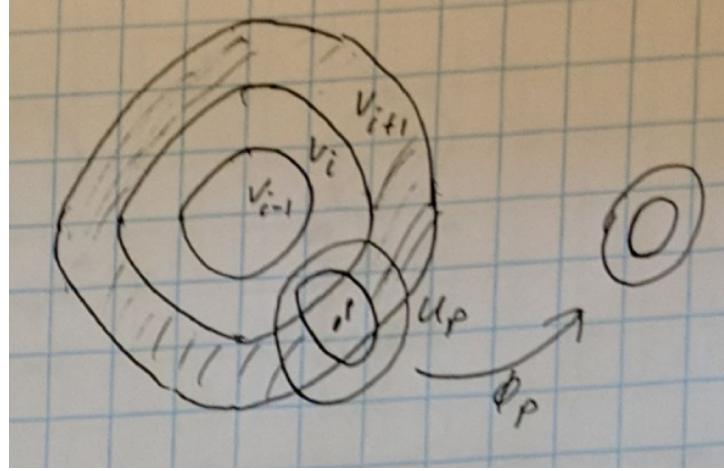
1.  $U_j$  is locally finite.
2.  $\phi_i : U_j \xrightarrow{\text{diffeo}} B^n(2)$ .
3.  $\phi_j^{-1}(B(1))$  is also a cover.

## Proof

There exists a compact exhaustion of  $M$ ,  $C_1 \subset C_2 \subset C_3 \subset \dots$  where  $\bigcup C_i = M$ .

There exists also an open exhaustion by precompact open sets  $\emptyset = V_0 \subset V_1 \subset V_2 \subset \dots$  where  $\overline{V_i} \subset V_{i+1}$ .

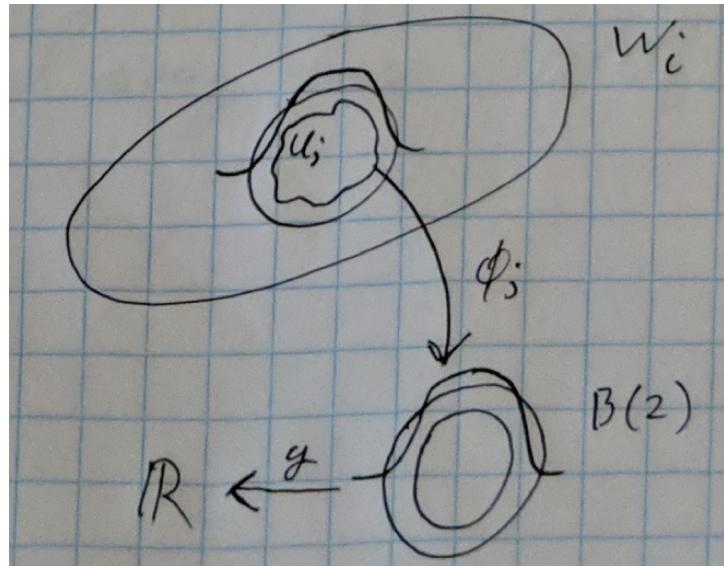
Then define  $V_i := \text{nbhd}(C_i \cup \overline{V}_{i-1})$ .



Take  $A_i = \overline{V}_{i+1} \setminus V_i$  compact,  $p \in A_i$ . Then we have a map  $U_p \xrightarrow{\phi_p} B(2)$ .

- $U_p \subset W_i$  for some  $i$ . (Refinement)
- $U_p \cap V_{i-1} = \emptyset$ . (Locally Finite)
- There exists a finite subcover such that  $\phi_p^{-1}(B(1))$  is also an open cover.

### Proof of Theorem



Set  $g_j = g \circ \phi_j \in C^r$ , extended to 0 outside of  $U_j$ .

- $\text{supp } g_j \subset U_j$
- $\forall p \in M, \exists g_j(p) = 1 \neq 0 \implies \sum g_j > 0$
- $\text{supp } g_j$  is locally finite.

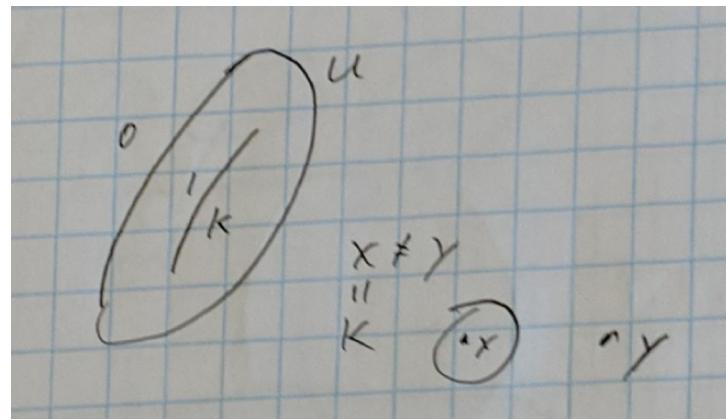
Then we have

$$f_j = \frac{g_j}{\sum g_j}$$

### Corollary 1

For  $K \subset U$ ,  $K$  compact and  $U$  open, there exists  $h \in C^r(M)$  such that

- $h|_K \equiv 1$
- $\text{supp } h \subset U$



### Proof

Take  $U_0 = U$  and  $U_1 = M \setminus K$ .

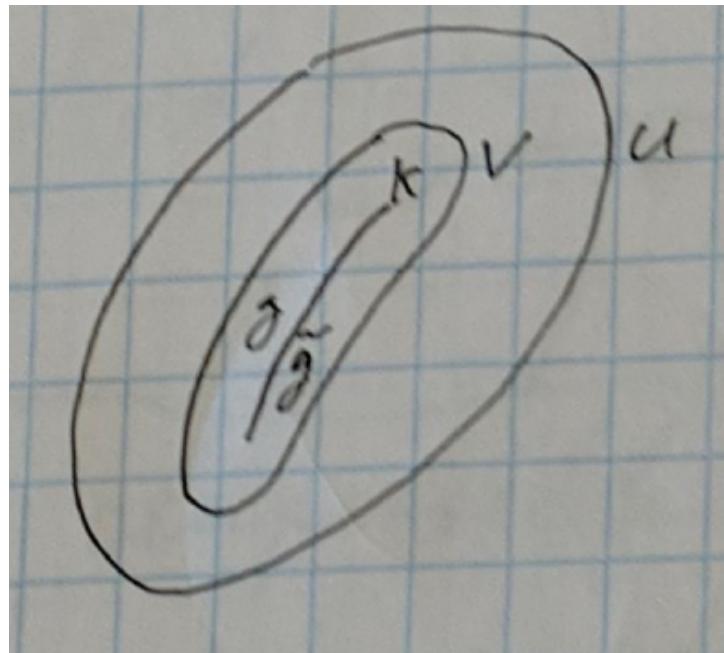
Then there exists a partition of unity  $f_0$  and  $f_1$  where  $f_0 + f_1 = 1$ . Therefore  $f_0$  has support in  $U$  and  $f_1$  has support in  $M \setminus K$  this occurs if and only if  $f_1|_K = 0$ .

### Corollary 2

For  $K \subset U$ ,  $K$  compact and  $U$  open, and  $g : K \xrightarrow{C^r} \mathbb{R}$ , there exists an extension  $f : M \rightarrow \mathbb{R}$  of  $g$ :  $f|_K = g$  such that  $\text{supp } f \subset U$ .

### Proof

For  $g \in C^r(K)$ , there exists a neighborhood  $V$  of  $K$  and a function  $\tilde{g} : V \rightarrow \mathbb{R}$  such that  $\tilde{g}|_K = g$ .



By corollary 1, there exists  $h \in C^r(M)$  such that

- $\text{supp } h \subset V$
- $h|_K = 1$

Therefore  $f = h \cdot \tilde{g}$ .

### Corollary 3

There exists  $f : M \xrightarrow{C^r} \mathbb{R}$  bounded from below and proper.

#### Definition: Proper Function

$f$  is proper if and only if  $f^{-1}(K)$  is compact for  $K$  compact.

#### Proof

See Textbook.

#### Consequence

Take  $\{x : f(x) \leq C_i\} =: E_i$  as  $i \rightarrow \infty$ . Then we get a compact exhaustion  $E_1 \subset E_2 \subset E_3 \subset \dots$   
e.g.  $f(x) = \|x\|^2$ .

**October 10, 2024**

### Algebra = Analysis (Geometry)

Take  $M$  to be either a compact metrizable space ( $A = C^0(M)$ ) or a copact manifold ( $A = C^\infty(M)$ ).

$$M \leftrightarrow A$$

Let  $I \subset A$  be an ideal and take  $V(I) = \{x : f(x) = 0, \forall f^n \in I\}$ .

Take also  $Y \subset M$  closed and consider  $I_Y = \{f : f|_Y = 0\}$  which is also an ideal.

$$\begin{array}{c} Y \rightarrow I_y \\ V(I) \leftarrow I \end{array}$$

Denote  $I_x = \{f : f(x) = 0\}$ .

## Theorem

- $I_x$  is a maximal ideal.
- Every maximal ideal is of this form, and  $x$  is unique.

$$M \leftrightarrow \text{Maximal Ideals of } A$$

## Proof

### Maximal Ideal

Take  $I_x$  and  $g \notin I_x$ .

We want to show that  $g$  with  $I_x$  generates  $A$ .

Then take  $f \in A$  defined as  $f = h + ag$  for some  $h \in I_x$ .

Since  $g \notin I_x$ ,  $g(x) \neq 0$ . Take  $a = \frac{f(x)}{g(x)}$  and define  $h := f - ag$ .

Then  $f(x) - \frac{f(x)}{g(x)}g(x) = 0 \in I_x$ .

### Every Maximal Ideal

Let  $I$  be a proper ideal such that  $V(I) \neq 0$ ,  $\exists x$  such that  $f(x) = 0, \forall f \in I$ .

Maximal  $\implies V(I) = \{x\}$ .

By contradiction, assume not:  $\forall x \in M, \exists f_x \in I, f_x(x) \neq 0$ .

It follows that  $f_x|_{U_x} \neq 0$  where  $U_x \ni x$  is a neighborhood from an open cover.

Then, by compactness, we have a finite subcover

$I \ni f_i = f_{x_i} \neq 0$  on  $U_i$ .

$I \ni \underbrace{\sum f_i^2}_{g>0} > 0$  on  $M$ . But then  $1 = g^{-1}g \in I$ .

### Uniqueness

$$I_{x_1} = I_{x_2} \iff x_1 = x_2$$

( $\iff$ ) is obvious.

( $\implies$ )  $x_1 \neq x_2$  implies that there exists  $f \in A$  such that  $f(x_1) = 1$  and  $f(x_2) = 0$ .

Then  $f \in I_{x_2}$  while  $f \notin I_{x_1}$  implies that  $I_{x_1} \neq I_{x_2}$ .

### Reading the Topology / Smooth Structure

We have a correspondence

$$\text{closed sets} \leftrightarrow \text{ideals}$$

## Algebra Homomorphisms

Take  $\phi : A \rightarrow \mathbb{R}$ . Then  $\ker \phi$  is a maximal ideal, and

$$\begin{aligned} A/I_x &\xrightarrow{\delta_x} \mathbb{R} \\ x &\mapsto \pm f(x) \end{aligned}$$

Then

$$\begin{aligned} M &\leftrightarrow \text{Alg. Hom. } A \rightarrow \mathbb{R} \\ 1 &\mapsto 1 \\ A > 0 &\mapsto \text{pos. \#} \end{aligned}$$

## Counterexample

Take instead  $M = \mathbb{R}$ . Claim: there exist a maximal ideals other than  $I_x$ .

### Proof

Take  $J$  to be the ideal of all compactly supported functions such that  $J \subset M$  for some maximal ideal  $M$ . However,  $\forall x \in \mathbb{R}$  there exists a compactly supported function  $f(x) \neq 0$ . So  $M \neq I_x$ .

## Functorial Considerations

Take  $F : M \rightarrow N$ ,  $M$  and  $N$  compact. Then

$$\begin{aligned} F : M &\rightarrow N \\ C^{0/\infty}(M) &\xleftarrow[F^A]{} C^0(N) \\ f \circ F &\leftrightarrow f \end{aligned}$$

## Definition: Germ

Take  $M \ni p$ .

IMAGE 3

Where  $(U, f) \sim (V, g)$  if  $f \equiv g$  on some neighborhood of  $p$ .

## Definition: Tangent Spaces

IMAGE 1

IMAGE 2

With  $\gamma_1 : (-\varepsilon, \varepsilon) \rightarrow M$ ,  $\gamma_2 : (-\varepsilon, \varepsilon) \rightarrow M$ , and  $\gamma_1 \sim \gamma_2 \iff \gamma'_1(0) = \gamma'_2(0)$ .

$\gamma(t) = ((x(t), y(t), z(t)))$  and  $\gamma'(0) = (x'(0), y'(0), z'(0))$ .

The tangent space to  $M$  at  $p$ , written  $T_p M$ , is the set of equivalence classes.

## Remarks: Tangent Vectors

Take  $V \ni p$  a finite-dimensional vector space ( $= \mathbb{R}^n$ ).

$$\begin{aligned} 0 &\mapsto p \\ \gamma_1, \gamma_2 : (-\varepsilon, \varepsilon) &\rightarrow V \quad (\text{germs}) \\ \gamma'_1 \sim \gamma'_2 : \gamma'_1(0) &= \gamma'_2(0) \end{aligned}$$

IMAGE 4

Write

$$\gamma'(0) = \lim_{t \rightarrow 0} \frac{\gamma(t) - \gamma(0)}{t}$$

$\gamma = (x_1, \dots, x_n)$  and  $\gamma'(0) = (x'_1(0), \dots, x'_n(0))$ .

A tangent vector to  $V$  at  $p$  is an equivalence class  $T_p V$ .

Claim:  $T_p V$  is a vector space.

## Operations

Take  $p = 0$ . Write

$$\begin{aligned} [\gamma_1] + [\gamma_2] &= [\gamma_1 + \gamma_2] \\ \lambda[\gamma] &= [\lambda\gamma] \end{aligned}$$

When  $p \neq 0$ , instead

$$\begin{aligned} [\gamma_1] + [\gamma_2] &= [\gamma_1 + \gamma_2 - p] \\ \lambda[\gamma] &= [\lambda\gamma + (1 - \lambda)p] \end{aligned}$$

Claim:  $T_p V$  is canonically isomorphic to  $V$

$$\begin{aligned} [\gamma] &\mapsto \gamma'(0) \\ [\gamma = (x_1, \dots, x_n)] &\mapsto (x'_1(0), \dots, x'_n(0)) \\ T_p V &\rightarrow V \\ p + t\gamma &\leftrightarrow \gamma \end{aligned}$$

IMAGE 5

## Proposition

Take

$$\begin{aligned} V &\xrightarrow{\mathbb{C}^r} W \\ \mathbb{R}^n &\xrightarrow{\mathbb{R}^m} \\ p &\mapsto q \end{aligned}$$

Then

$$\begin{aligned} V &\xrightarrow{\sim} W \\ DF_p = F_* : T_p V &\rightarrow T_q W \\ [\gamma] &\mapsto [F \circ \gamma] \end{aligned}$$

## IMAGE 6

Then  $F$  is well-defined and linear.

$F = (F_1, \dots, F_m)$  and  $F \circ \gamma(F_1(\gamma_1, \dots, \gamma_n), \dots, F_m(\gamma_1, \dots, \gamma_n))$ .  
 We have that  $[\gamma] = \gamma'(0)$  and  $[F \circ \gamma] = \frac{d}{dt} F \cdot \gamma(t)|_{t=0}$ . By chain rule,

$$\frac{d}{dt}(F \circ \gamma)|_{t=0} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{pmatrix} \begin{pmatrix} \gamma'_1(0) \\ \vdots \\ \gamma'_n(0) \end{pmatrix}$$

## Tangent Space

## IMAGE 7

$\gamma_1 \sim \gamma_2 \iff \phi \circ \gamma_1 \sim \phi \circ \gamma_2$  and  $\gamma_1 \sim \gamma_2 \iff \psi \circ \gamma_1 \sim \psi \circ \gamma_2$ , so

$$(\psi \circ \phi^{-1})(\phi \circ \gamma_1) \sim (\psi \circ \phi^{-1})(\phi \circ \gamma_2)$$

Now, take  $\{[\gamma]\} = T_p M$ . Claim: this is a vector space.

$$\begin{array}{ccc} T_p M & \xrightarrow{D\phi} & T_{\phi(p)} \mathbb{R}^n \\ & \searrow D\psi & \downarrow D(\underbrace{\psi \circ \phi^{-1}}_F) \\ & & T_{\psi(p)} \mathbb{R}^n \end{array}$$

$$[\gamma_1] + [\gamma_2] = [\phi^{-1}(\phi \circ \gamma_1 + \phi \circ \gamma_2)].$$

**October 15, 2024**

## Recall: Tangent Space by Equivalence Classes

$\gamma: (-\varepsilon, \varepsilon) \rightarrow M, \gamma(0) = p, \gamma_1 \sim \gamma_2 \iff \phi \circ \gamma'_1 \sim \phi \circ \gamma'_2 \iff (\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0).$   
 $T_p M = \{[\gamma]\} \xrightarrow{D\phi = \phi_*} T_{\phi(p)} \mathbb{R}^n = \mathbb{R}^n$ .

## IMAGE 1

Then for  $(F_1, \dots, F_m)$ , we have  $D(\psi F \phi^{-1}): \underbrace{T_{\phi(p)} \mathbb{R}^n}_{\mathbb{R}^n} \rightarrow \underbrace{T_{\psi(q)} \mathbb{R}^m}_{\mathbb{R}^m}$  where

$$D(\psi F \phi^{-1}) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

## IMAGE 2

## Chain Rule

We have that  $D(FG) = DF \circ DG$  since  $(FG) \circ \gamma = F(G \circ \gamma)$ .

## Example

Take

$$f: \underbrace{M}_{(x_1, \dots, x_n)} \rightarrow \mathbb{R}$$

and  $Df = \left( \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$ .

## Definition: Directional Derivatives

Take  $v = [\gamma] \in T_p M$  and  $f \in C^r(M)$ .

The directional derivative is given by

$$L_v f = \frac{d}{dt} f(\gamma(t))|_{t=0} = \frac{d}{dt} \underbrace{(f \circ \phi^{-1})}_{g} \underbrace{(\phi \circ \gamma)}_{\eta}(t)|_{t=0}$$

Then

$$\frac{d}{dt} g(\eta(t))|_{t=0} = \sum \frac{\partial g}{\partial x_i}(\phi) \eta'_i(0)$$

which is determined by  $(\eta'_1(0), \dots, \eta'_n(0)) = \eta'(0)$ .

## Properties

$$L_v : C^\infty(M) \rightarrow \mathbb{R}$$

1. Linear over  $\mathbb{R}$

2.  $L_v(fg) = (L_v f)g(\phi) + f(\phi)(L_v g)$  (product rule)

3. Linear in  $v$ .

## Derivations

The collection  $\mathcal{D}_p = \{C^r(M) \rightarrow \mathbb{R} : (1) \text{ and } (2) \text{ hold}\}$  is called the derivations at  $p$ .

## Algebraic Aside

$\delta_p : A \rightarrow \mathbb{R}$  given by  $f \mapsto f(p)$  yields

$$D(fg) = Df\delta_p(g) + \delta_p(f)Dg$$

## Theorem

$T_p M \rightarrow \mathcal{D}_p$  given by  $v = [\gamma] \mapsto L_v$  is a linear isomorphism.

## Recall: Germ

Take  $(f, U) \ni p$  and  $(g, V) \ni p$ . Then  $(f, U) \sim (g, V)$  if and only iff  $f \equiv g$  on  $W \subset U \cap V$ . The equivalence classes of this relation are germs.

## Hadamard's Lemma

Take  $f \in C^r(\mathbb{R}^n, 0)$  on  $\mathbb{R}^n$  with  $f(0) = 0$ .

There exists  $g_1, \dots, g_n \in C^{r-1}(\mathbb{R}^n, 0)$  such that

$$f(x) = \sum x_i g_i(x)$$

and  $g_i(0) = \frac{\partial f}{\partial x_i}(0)$ .

### Example

For  $n = 1$  and  $f : \mathbb{R} \xrightarrow{C^r} \mathbb{R}$  with  $f(0) = 0$ . Then we have  $f(x) = \overbrace{x g(x)}^{C^{r-1}}$  given by

$$g(x) = \begin{cases} \frac{f(x)}{x} & x \neq 0 \\ f'(0) & x = 0 \end{cases}.$$

### Proof of Example

IMAGE 3

Take

$$\begin{aligned} \int_0^1 \underbrace{\frac{d}{dt} f(tx) dt}_{f'(tx) \cdot x} &= x \underbrace{\int_0^1 f'(tx) dt}_{g(x)} \\ &= f(1 \cdot x) - f(0 \cdot x) \end{aligned}$$

### Proof of Lemma

$$\int_0^1 \underbrace{\frac{d}{dt} f(tx) dt}_{\sum \frac{\partial f}{\partial x_i}(tx) \cdot x_i} = \sum x_i \underbrace{\int_0^1 \frac{\partial f}{\partial x_i}(tx) dt}_{g_i}$$

## Lemma

For  $D \in \mathcal{D}_p$ ,  $Df$  depends only on the germ of  $f$ .

### Proof

Need to show that if  $f \equiv 0$  near  $p$ ,  $Df = 0$ .

IMAGE 4

Where  $\text{supp } h \subset \{x : f(x) = 0\}$ ,  $h(p) = 1$  and, consequently,  $f \circ h = 0$ .

So  $D(0 = f \circ h) = Df \cdot h(p) + f(p)Dh$  and  $0 = Df$ .

## Lemma

$D(k) = 0$  for  $k$  constant.

### Proof

$$1 \cdot 1 = 1 \implies D(1) = D(1 \cdot 1) = D(1) \cdot 1 + 1 \cdot D(1) = 2 \cdot D(1), \text{ so } D(1) = 0.$$

Arbitrary constants follow from the fact that  $D$  is linear.

## Proof: One-to-One

Let  $L_v f = L_w f$  for all  $f = x_i$ , then

$$\sum v_i \frac{\partial f}{\partial x_i} = \sum w_i \frac{\partial f}{\partial x_i}$$

for each  $f$  and  $v_i = w_i$ .

## Lemma

$$\text{Der}(C^r(m)@p) = \text{Der}(C^r(M, p))$$

### Proof

Take  $M = \mathbb{R}^n$  and  $p = 0$ .

Given  $D$ , we need  $v = \sum v_i \frac{\partial}{\partial x_i}$  with

$$L_v f = \sum v_i \frac{\partial f}{\partial x_i} = Df$$

for every  $f$ .

By Hadamard's lemma, write  $f = \sum x_i g_i(x)$ . Then

$$\begin{aligned} Df &= \sum (Dx_i) g_i(0) + \overbrace{x_i(0)}^{=0} Dg_i \\ &= \sum \underbrace{(Dx_i)}_{v_i} \frac{\partial f}{\partial x_i}(0) \end{aligned}$$

That is, since  $v_i = L_v x_i$ , we have  $v_i = Dx_i$ .

## Definition: Tangent Bundle

IMAGE 5

For every  $p \in M$ , we have  $T_p M$ .

The tangent bundle  $TM = \coprod_{p \in M} T_p M$ .

IMAGE 6

IMAGE 7

When  $M$  is  $C^r$ ,  $TM$  is  $C^{r-1}$ .