Topics in Analysis (F24)

September 30, 2024

Chapter 1: Banach Algebras

1.1: Definitions and Basic Properties

Definition: Banach Space

A Banach space X (over $\mathbb{C}$) is a normed vector space with algebraic operations

$$\begin{aligned} (x, y) &\mapsto x + y && \text{addition} \\ (\lambda, y) &\mapsto \lambda y && \text{scalar multiplication} \end{aligned}$$

and a norm

$$x \mapsto ||x||$$

which is complete (i.e. every Cauchy sequence converges).

Definition: (Complex) Banach Algebra

A (complex) Banach algebra B is a Banach space in which there is multiplication

$$(x, y) \in B \times B \mapsto xy \in B$$

such that

1. $x(yz) = (xy)z$
2. $(x + y)z = xz + yz$ and $x(y + z) = xy + xz$
3. $\lambda(xy) = (\lambda x)y = x(\lambda y)$
4. $||xy|| \leq ||x|| \cdot ||y||$

Definition: Unital Banach Algebra

B is called a unital Banach algebra if $\exists e \in B$ such that

$$xe = ex = x \quad \text{and} \quad ||e|| = 1.$$

If e exists, it is unique.

1.2: Examples

Example 1

If X is a Banach space, then $B = \mathcal{L}(X)$ (the set of all bounded inear operators $A : X \to X$) equipped with algebraic operations

$$
\begin{aligned}
(A+B)x &= Ax + Bx \\
(\lambda A)x &= \lambda(Ax) \\
(AB)x &= A(Bx)
\end{aligned}
$$

and the operator norm

$$||A||_{\mathcal{L}(X)} = \sup_{x \neq 0} \frac{||Ax||_X}{||x||_X}.$$

$B = \mathcal{L}(X)$ is complete because X is complete.
The unit element is given by $I_X x = x$.

Example 2

If $X = \mathbb{C}^n$, then $B = \mathcal{L}(\mathbb{C}^n) \cong \mathbb{C}^{n\times n}$.

$$A = (a_{ij})_{i,j=1}^n \qquad Ax = y \qquad \sum_{j=1}^n a_{ij}x_j = y_i.$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

The norm in $\mathbb{C}^n$ leadsto a norm in $\mathbb{C}^{n\times n}$

$$
\begin{aligned}
||(x_i)|| &= \left(\sum |x_i|^2\right)^{1/2} & ||A|| &= \\
||(x_i)|| &= \sum |x_i| & ||A|| &= \max_j \sum_i |a_{ij}| \\
||(x_i)|| &= \max |x_i| & ||A|| &= \max_i \sum_j |a_{ij}|
\end{aligned}
$$

All norms are quivalent.

Example 3

Take $B = C(K)$ with K a compact Hausdorff space, $f : K \to \mathbb{C}$ continuous and $||f|| = \max_{t\in K} |f(t)|$.

Example 4

Take $B = A(K)$, $K \subseteq \mathbb{C}$ compact with $\operatorname{int}(K) \neq 0$, $f : K \to \mathbb{C}$ continuous where f is holomorphic on $\operatorname{int}(K)$ and

$$||f|| = \max_{t\in K} |f(t)| = \max_{t\in K\setminus \operatorname{int}(K)} |f(t)|$$

e.g. $K = \overline{\mathbb{D}} = \{t \in \mathbb{C} : |t| \leq 1\}$. Then $A(K) \subseteq C(K)$.

Example 5

Take $B = \ell^\infty(\mathbb{N})$ or $B = L^\infty(S,\sigma,\mu)$ with (S,σ,μ) a measure space, $f : S \to \mathbb{C}$ essentially bounded functions and

$$||f|| = \text{ess sup}_{t\in S}|f(t)| = \inf_{\substack{N\subseteq S\\ \mu(N)}}\left(\sup_{t\in S\setminus N}|f(t)|\right)$$

Example 6

Take $B = \ell^1(\mathbb{Z})$ or $B = L^1(\mathbb{R}^d)$ with $||\{x_n\}|| = \sum|x_n|$ and $||f|| = \int_{\mathbb{R}^d}|f(t)|\,dt$ respectively.
Multiplication is given by the convolution. e.g.

$$fg = (f*g)(x) = \int_{\mathbb{R}^d} f(x-t)g(t)\,dt$$

$\ell^1(\mathbb{Z})$ is unital, but $L^1(\mathbb{R}^d)$ is non-unital (since the unit of convolution is the Dirac delta; see Example 7).

Example 7

Take $B = M(\mathbb{R}^d)$ the complex measures on $\mathbb{R}^d$ with bounded variation.
Then multiplications is given as

$$(\mu*\nu)(A) = \int_{\mathbb{R}^d}\mu(A-x)d\nu(x)$$

and norm

$$||\mu|| = \sup_{\substack{\mathbb{R}^d=\bigcup A_i\\ \text{disjoint}}}\sum_{i=1}^{n}|\mu(A_i)| < +\infty.$$

Then, $f\,dm = d\mu$ gives $L^1(\mathbb{R}^d)\to M(\mathbb{R}^d)$.

Example 8

Take $B = C^{n\times n}[K]$ with K compcat and Hausdorff, continuous functions $f : K \to \mathbb{C}^{n\times n}$ and norm

$$||f||_B = \max_{t\in k}||f(t)||_{C^{n\times n}}.$$

Then $B \cong (C(K))^{n\times n}$ the $n\times n$ matrices with entries from $C(K)$.

1.3: Remarks

- If B does not have a unit element, consider $B_1 = B\times\mathbb{C}$ with operations

$$\begin{aligned}(b_1,\lambda_1)+(b_2,\lambda_2) &= (b_1+b_2,\lambda_1+\lambda_2)\\ \alpha(b,\lambda) &= (\alpha b,\alpha\lambda)\\ (b_1,\lambda_1)(b_2,\lambda_2) &= b_1b_2+\lambda_1b_2+\lambda_2b_1,\lambda_1\lambda_2)\end{aligned}$$

and norm

$$||(b,\lambda)|| = ||b|| + |\lambda|.$$

Then B_1 is a unital Banach algebra with $e = (0,1)$. One writes $(b,\lambda) = (b,0) + \lambda(0,1) = b + \lambda \cdot e$.
In some sense, $B \subseteq B_1$ where $b \in B \mapsto (b,0) \in B_1$.

1.4: Definitions

Definition: Commutative Banach Algebra

B is called commutative if $xy = yx$.

Definition: Banach Subalgebra

A subset B_0 of a B-algebra is called a subalgebra if it is closed with respect to the algebraic operations

$$x, y \in B_0,\ \lambda \in \mathbb{C} \rightsquigarrow x + y, xy, \lambda x \in B$$

Definition: Closed Subalgebra

B_0 is a closed subalgebra or Banach subalgebra if it is norm-closed.

- Proposition: B_0 is a Banach algebra.

Definition: Generated Subalgebra

Let $M \neq \emptyset$ be a subset of a Banach algebra B.
The Banach subalgebra generated by M is the smallest closed subalgebra containing M.

$$\operatorname{alg} M = (\operatorname{clos} \operatorname{alg}_B M)$$

- Remark

  $\operatorname{alg} M$ is the intersection of all closed subalgebras containing M.
  $\operatorname{alg} M = \operatorname{clos}\left\{\sum_{i=1}^{N} \lambda_i a_1^{(i)} a_2^{(i)} \cdots a_{n_i}^{(i)}\right\}$ is the norm-closure of finite linear combinations of finite products of $a_j^{(i)} \in M$.

1.5: Examples

Exammple 1

Take B unital, $b \in B$. Then

$$\operatorname{alg}\{e, b\} = \operatorname{clos}_B \left\{ \sum_{i=0}^{N} \lambda_i b^i \ :\ \lambda_i \in \mathbb{C},\ N \in \mathbb{N} \right\}$$

where $b^0 = e$.

1.6 Definitions

Definition: Banach Algebra Homomorphism

A Banach algebra homomorphism is a map $\phi : B_1 \to B_2$ between Banach algebras B_1 and B_2 such that

- ϕ is linear
- ϕ is bounded (continuous)
- ϕ is multiplicative

  $\phi(b_1 b_2) = \phi(b_1) \cdot \phi(b_2)$
- ϕ is unital if both B_1, B_2 have units and $\phi(e_{B_1}) = e_{B_2}$.

Definition: Banach Algebra Isomorphism

A Banach algebra homomorphism which is bijective is called a Banach algebra isomorphism.
Then $\phi^{-1} : B_2 \to B_1$ is an isomorphism as well.

Definition: Banach Algebra Isometry

ϕ is an isometry if $||\phi(x)|| = ||x||$.

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Recall

Given $M \subseteq \mathcal{L}(X)$ with X a Banach space (and $\mathcal{L}(X)$ itself a Banach algebra), we may construct $B = \mathrm{alg}_{\mathcal{L}(X)} M$.

1.7 Proposition

Let B be a unital Banach algebra. Then the map

$$\phi : B \ni x \to L_x \in \mathcal{L}(B)$$

is an isometric isomorphism onto a closed subalgebra of $\mathcal{L}(B)$ where

$$L_x : B \ni z \mapsto xz \in B$$

is the left-representation of x.

Proof

L_x is in $\mathcal{L}(B)$ since $L_x z = xz$

- is linear in z and
- $||L_x z|| = ||xz|| \leq ||x|| \cdot ||z||$ implies $||L_x|| \leq ||x||$ (i.e. L_x is a bounded).

The map $\phi : x \mapsto L_x$ is linear

$$L_{x_1+x_2}z = (x_1 + x_2)z = x_1 z + x_2 z = L_{x_1}z + L_{x_2}z = (L_{x_1} + L_{x_2})z$$

ϕ is multiplicative

$$L_{x_1 x_2}z = (x_1 x_2)z = x_1(x_2 z) = L_{x_1}(L_{x_2}z)$$

From the above, we conclude that ϕ is a homomorphism.
To show that ϕ is an isometry,

$$||L_x|| = \sup_{z \neq 0} \frac{||L_x z||}{||z||} \geq \frac{||L_x e}{||e||} = \frac{||x||}{1} = ||x||.$$

Then also ϕ is injective and $\mathrm{im}\,\phi$ is closed. Since $\mathrm{im}\,\phi$ is a Banach algebra, it is therefore a closed subalgebra.

1.7 Remark: Right-Regular Representation

Every unital Banach algebra is isometrically isomorphic to a Banach algebra of operators.
Right-regular representation:

$$R_x = z \mapsto zx$$

Chapter 2: Group of Invertible Elements in a Banach Algebra

2.1 Definition: Invertible Element

Let B be a unital Banach algebra. An element $x \in B$ (in B) if there exists $y \in B$ such that $xy = yx = e$.
Note that $y = x^{-1}$ is uniquely determined.
Write GB for the set of all invertible elements of B.

Remark

GB is a (multiplicative group).

- $x, y \in GB \implies xy \in GB$ and $(xy)^{-1} = y^{-1}x^{-1}$,
- $x \in GB \implies x^{-1} \in GB$ and $(x^{-1})^{-1} = x$, and
- $e \in GB$.

2.2 Lemma

If $x \in B$ and $||x|| < 1$, then $e - x \in GB$.

Proof

Take the Neumann series

$$e + x + x^2 + x^3 + \cdots$$

which converges to some $s \in B$

$$s_n = e + x + \cdots + x^n$$

where s_n are Cauchy:

$$||s_{n+k} - s_n|| = ||x^{n+1} + \cdots x^{n+k}|| \leq ||x||^{n+1} + ||x||^{n+2} + \cdots = \frac{||x||^{n+1}}{1 - ||x||}.$$

So $s_n \to S$,

$$(e - x)s_n = s_n(e - x)e - x^{n+1}.$$

Taking $n \to \infty$

$$(e - x)s = s(e - x) = e.$$

2.3 Proposition

The group GB is open in B and the map $\Lambda : GB \ni x \mapsto x^{-1} \in GB$ is continuous (in the norm).

Proof

Take $x \in GB$ and consider $y \in B$ with $||y|| < \frac{1}{||x^{-1}||} = \varepsilon$.
Then $x + y \in B_\varepsilon(x)$ is invertible,

$$x + y = x(e + x^{-1}y),$$

and

$$||x^{-1}y|| \leq ||x^{-1}|| \cdot ||x|| < 1.$$

Therefore GB is open, since $B_\varepsilon(X) \subseteq GB$. The inverse

$$(x + y)^{-1} = (e + x^{-1}y)^{-1}x^{-1} = \sum_{n=0}^{\infty}(-x^{-1}y)^n x^{-1} = x^{-1} + \sum_{n=1}^{\infty}(-x^{-1}y)^n x^{-1}$$

so

$$||(x + y)^{-1} - x^{-1}|| \leq \sum_{n=1}^{\infty} ||x^{-1}||^{n+1}||y||^n = \frac{||x^{-1}||^2||y||}{1 - ||x^{-1}|| \cdot ||y||}.$$

This converges to zero as $||y|| \to 0$.

2.4 Examples

Example 1

$B = C(K)$, K compact Hausdroff, $f : K \to \mathbb{C}$ continuous.
$GB = \{f \in C(K) \,:\, f(t) \neq 0,\ \forall\, t \in K\}$.

Example 2

$B = C^{n \times n}$.
$GB = \{A \in \mathbb{C}^{n \times n} \,:\, \det A \neq 0\}$.

2.5 Definition:

Let $G_0 B$ stand for the connected componet of GB containing e.

Remarks

- the ε-neighborhoods $B_\varepsilon(x) \subseteq B$ are (path-)connected.

$$B_\varepsilon(x) = \{y \in B \,:\, ||x - y|| < \varepsilon\}$$

  For $y_1, y_2 \in B_\varepsilon(x)$, there is a continuous path

$$\sigma : [0,1] \ni \lambda \mapsto y_1 \lambda + y_2(1 - \lambda) \in B_\varepsilon(x)$$

- Because GB is open and $B_\varepsilon(x)$ is path-connected, GB is locally (path-)connected (i.e. every $x \in GB$ has a (path-)connected open neighborhood in GB).

- In this context, connectedness and path-connectedness are equivalent. Therefore the components of GB are the path-components of GB.

- GB is the union of disjoint (path-)components where each component is both open and closed in GB.

- $x, y \in GB$ belong to the same path-component if there exists a continuous path $\gamma : [0,1] \to GB$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Here, $x \sim y$ is an equivalence relation.

- $G_0 B = \{x \in GB \,:\, \exists$ a path in GB connecting e and $x\}$.

2.6 Examples

Example 1

Take $B = C(\mathbb{T})$ with $\mathbb{T} = \{z \in \mathbb{C} \,:\, |z| = 1\}$ and continuous functions $f : \mathbb{T} \to \mathbb{C}$.
GB is the non-vanishing continuous functions $f : \mathbb{T} \to \mathbb{C}$ ($f(t) \neq 0,\ \forall\, t \in \mathbb{T}$).
For $f \in GB$ one can define a winding number.

IMAGE 1

We have $\frac{1}{2\pi} \arg f(e^{ix})$ a continuous function with

$$\mathrm{wind}(t) = \left[\frac{1}{2\pi} \arg f(e^{ix})\right]_{x=0}^{2\pi} = \phi(2\pi) - \phi(0)$$

and $\mathrm{wind}(t) \in \mathbb{Z}$.
The map $GB \ni f \mapsto \mathrm{wind}(t) \in \mathbb{Z}$ is continuous, hence locally constant (i.e. constant on each connected component).
Therefore $G_0 C(\mathbb{T}) \subseteq \{f \in GC(\mathbb{T}) \,:\, \mathrm{wind}(f) = 0\}$. In fact, we will see that we have equality.
That is, f can be contracted (in GB) to the constant function $e(t) = 1$.

2.7 Proposition

G_0B is a normal subgroup of GB.

Proof

- G_0B is a group.

  For any $x, y \in G_0B$, there exist paths $\gamma_1 : [0,1] \to GB$ and $\gamma_2 : [0,1] \to GB$ with $\gamma_1(0) = \gamma_2(0) = e$, $\gamma_1(1) = x$ and $\gamma_2(1) = y$.
  Define $\gamma(t) = \gamma_1(t)\gamma_2(t)$ a path in GB such that $\gamma(0) = e$ and $\gamma(1) = xy$. Then $xy \in G_0B$.
  Following from Lemma 2.2, $\hat{\gamma} = (\gamma_1(t))^{-1}$ is a continuous path with $\hat{\gamma}_1(0) = e$, $\hat{\gamma}_1(1) = x^{-1}$ and $x^{-1} \in GB$.

- G_0B is a normal subgroup of GB.

  For every $y \in GB$, $yG_0By^{-1} \subseteq G_0B$ if and only if $yG_0B = G_0By$.
  Take $x \in G_0B$ with path γ, then

$$\delta(t) = y\gamma(t)y^{-1}, \quad \delta(0) = yey^{-1} = e, \quad \text{and} \quad \delta(1)yxy^{-1} \in G_0B.$$

2.8 Definition: Abstract Index Group

The quotient group GB/G_0B is called the abstract index group of B.

Remark

GB/G_0B is in 1-to-1 correspondence with the set of connected components of GB.
Indeed, the (path-)connected components of GB are given by $yG_0B = G_0By$ (for $y \in GB$).

$$y_1G_0B = y_2G_0B \iff y_2^{-1}y_1G_0B = G_0B \iff y_2^{-1}y_1 \in G_0B \iff [y_2] = [y_1] \text{ in } GB/G_0B.$$

2.9 Definition: Exponential Map

For $x \in B$, we define the exponential map $B \ni x \mapsto \exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

2.10 Lemma

The exponential map $B \ni x \mapsto \exp(x) \in GB$ is well-defined and continuous.
For $xy = yx$, we have $\exp(x+y) = \exp(x)\exp(y)$.
In particular, $(\exp(x))^{-1} = \exp(-x)$.

Proof

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergent.

$$\sum_{n=0}^{\infty} \frac{||x||^n}{n!} < +\infty.$$

It follows that $s_n = \sum_{n=0}^{k} \frac{x^k}{k!}$ is a Cauchy sequence and therefore converges.
Continuity left as an exercise. Need to show:

$$\left|\left|\sum \frac{x^n}{n!} - \sum \frac{y^n}{n!}\right|\right| \leq ||x-y|| \cdot M_{x,y}$$

The fact that $\exp(x+y) = \exp(x)\exp(y)$ follows from multiplying terms and the binomial formula.