

Analysis I

October 2, 2023

Lecture Notes

Class will not have dedicated lecture notes. Many are available already.

Undergraduate notes are available on Canvas.

Lecture 1 overview available on Canvas (lecture1.pdf).

Tentative Office Hours

Mondays 2-3pm and Tuesday 1-2pm.

Homework

Nominally due at beginning of class; ask for leeway if needed.

First week homework will be review of undergraduate proofs.

First homework due Wednesday, October 11.

Notation

Natural Numbers: $\mathbb{N} = \{1, 2, 3, \dots\}$

Non Negative Integers: $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$

Integers: $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$

Rationals: $\mathbb{Q} = \left\{ \frac{p}{q}, p \in \mathbb{Z}, q \in \mathbb{Z} \right\} = \mathbb{Z} \times \mathbb{N} / \sim$

- Equivalent representation of rationals: $(p_1, q_1) \sim (p_2, q_2)$ iff $p_1 q_2 = p_2 q_1$

Sequence of Rationals: $\{u_n\}_{n \in \mathbb{N}}, u_n \in \mathbb{Q}, \forall n$.

Properties of the Rationals

$(\mathbb{Q}, +, \cdot)$ is a (ii) totally ordered (i) field satisfying the (iii) Archimedean property.

(i) Field

1. $+$ is associative: $(a + b) + c = a + (b + c)$

2. $+$ is commutative: $a + b = b + a$

3. \cdot is associative and commutative.
4. $\exists 0 \in \mathbb{Q}$ such that $\forall a \in \mathbb{Q}, 0 + a = a + 0$
5. $\exists 1 \in \mathbb{Q} \setminus \{0\}$ such that $\forall a \in \mathbb{Q}, 1 \cdot a = a \cdot 1 = a$
6. $\forall a \in \mathbb{Q} \setminus \{0\} \exists b \in \mathbb{Q}, a \cdot b = b \cdot a = 1$

- $b = a^{-1} = \frac{1}{a}$

(ii) Totally Ordered

\exists a set $\mathbb{Q}_+ \subseteq \mathbb{Q}$ of “Positive Numbers” stable under $+$ and \cdot such that $\forall A \in \mathbb{Q}$ either $a > 0$ ($a \in \mathbb{Q}_+$), $-a > 0$ (also $a < 0$) or $a = 0$.

- Ordering: $\forall a, b \in \mathbb{Q}, a < b$ if and only if $b - a > -0$.
- Trichotomy: $\forall a, b \in \mathbb{Q}$ either $a < b$, $a > b$, or $a = b$.
- $\max(a, b) = \begin{cases} a & \text{if } a > b \\ b & \text{otherwise} \end{cases}$.
- $|a| = \max(a, -a)$ (helps measure distance in \mathbb{Q}).
- $\text{dist}(a, b) := |b - a|$
- Triangle Inequality: $|u \pm v| \leq |u| + |v|$
- Observe also: $||u| - |v|| \leq |u \pm v|$. The triangle inequality may be used to prove this.
- Proof of Triangle Inequality $-|u| \leq u \leq |u|$ and $-|v| \leq v \leq |v|$, therefore $-|u| - |v| \leq u + v \leq |u| + |v|$.
Therefore $u + v \leq |u| + |v|$ and $-(u + v) \leq |u| + |v|$ implies $|u + v| \leq |u| + |v|$.

(iii) Archimedian Property:

$$\forall \epsilon > 0, \exists N, \forall n \geq N, \frac{1}{n} < \epsilon.$$

Bounded Sequence of Rationals

$\{u_n\}_{n \in \mathbb{N}}$ is bounded if $\exists m \in \mathbb{Q}_+$ such that $|u_n| \leq m, \forall n$.

$\{u_n\}_{n \in \mathbb{N}}$ converges to $a \in \mathbb{Q}$ ($\lim_{n \rightarrow \infty} u_n = a$) if $\forall \epsilon > 0, \exists N, \forall n \geq N, |u_n - a| < \epsilon$.

Famous Limits

Decaying Rational

1. $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$

- $\forall \epsilon \in \mathbb{Q}_+, \exists n \in \mathbb{N}, 0 < \frac{1}{n} < \epsilon$

- $\forall n \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq N$

– b. and c. are equivalent.

Decaying Exponential Rational

$r \in \mathbb{Q}, 0 < r < 1, \lim_{n \rightarrow \infty} r^n = 0.$

- Proof: Write $r = \frac{1}{1+k}$ for some $k > 0$. Then $r^n = \frac{1}{(1+k)^n} \stackrel{\text{Bernoulli}}{\leq} \frac{1}{1+nk}.$

Geometric

1. $r \in \mathbb{Q}, 0 < r < 1, u_n = 1 + r + \dots r^n = \frac{1-r^{n+1}}{1-r} \rightarrow \frac{1}{1-r}$

Features of Limits

Limits are Unique

If the limit of a sequence exists, it is unique.

Squeezing Lemma

If $\{a_n\}, \{b_n\}$ are such that $0 \leq a_n \leq b_n$, and $b_n \rightarrow 0$ as $n \rightarrow \infty$, then $a_n \rightarrow 0$.

Limits Preserve Order

If $a_n \leq b_n \forall n$ and a_n and b_n converge, then $\lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n$.

Limit Algebraic Rules

$\lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} (a_n + b_n)$ when a_n and b_n converge.

If $\lim_{n \rightarrow \infty} b_n \neq 0$, then $\frac{a_n}{b_n} \rightarrow \frac{\lim a_n}{\lim b_n}.$

Peculiarity of the Rationals

\mathbb{Q} lacks completeness.

Examples

Consider $u_1 = 1$ and $u_{n+1} = \frac{1}{2}(u_n + \frac{2}{u_n})$.

Then $u_n \in \mathbb{Q}$, $\forall n \in \mathbb{N}$.

It can further be proven, by induction, that $u_n \geq 1$, $\forall n$. $\left(u_{n+1} - 1 = \frac{1}{2}(u_n + \frac{1}{u_n}) - 1 = \frac{1}{2u_n}((u_n - 1)^2 + 1)\right)$.
 $\lim_{n \rightarrow \infty} u_n^2 = 2$.

$$\begin{aligned} u_{n+1}^2 - 2 &= \left(\frac{1}{2}\left(u_n + \frac{2}{u_n}\right)\right)^2 - 2 \\ &= \left(1 \frac{1}{2u_n}(u_n^2 + 2)^2 - 4u_n\right) \\ &= 1 \frac{4}{u_n^2}(u_n^2 - 2)^2 \\ &\leq \frac{1}{4}(u_n^2 - 2)^2 \end{aligned}$$

If u_n converged in \mathbb{Q} to L , by algebraic limit rules, $2 = \lim u_n^2 = (\lim u_n)^2 = L^2$, yet $\sqrt{2} \notin \mathbb{Q}$.

Cauchy Criterion

A sequence $\{u_n\}_{n \in \mathbb{N}}$ of rationals is Cauchy if $\forall \epsilon > 0$, $\exists n \in \mathbb{N}$, $\forall p, q \geq n$, $|u_p - u_q| < \epsilon$.

Visual Justification



Example 1

The sequence from before is Cauchy.

$$|u_p - u_q| = \frac{|u_p^2 - u_q^2|}{|u_p + u_q|} \leq \frac{1}{2}|u_p^2 - u_q^2|$$

Example 2

$$u_n = \sum_{k=0}^n \frac{1}{k!} \in \mathbb{Q}.$$

- This is increasing.
- It is bounded above by 3:

$$\begin{aligned} 1 + 1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{2 \cdots n} &\leq 1 + 1 + \cdots \frac{1}{2^{n-1}} \\ &\leq 1 + \frac{1 - 2^{-n}}{1 - \frac{1}{2}} \\ &\leq 3 \end{aligned}$$

Convergence, Cauchy and Boundedness.

Given a sequence $\{u_n\}_{n \in \mathbb{N}}$,

$\{u_n\}$ converges $\implies \{u_n\}$ is Cauchy $\implies \{u_n\}$ is bounded.

Note that in \mathbb{Q} none of these implications may be reversed.

Construction of the Real Numbers

Short version: If the limit of a sequence isn't in the set, then define the limit as the sequence itself.

Let $C_{\mathbb{Q}} = \{\text{Cauchy sequences of rationals.}\}$.

Two Operations

- Termwise Addition $\{u_n\}_n + \{v_n\}_n := \{u_n + v_n\}_{n \in \mathbb{N}}$
- Termwise Multiplication $\{u_n\}_n \cdot \{v_n\}_n := \{u_n \cdot v_n\}_{n \in \mathbb{N}}$

Closure of Cauchy Sequence

If $\{u_n\}_n, \{v_n\}_n \in C_{\mathbb{Q}}$, then $\{u_n\}_n + \{v_n\}_n \in C_{\mathbb{Q}}$ and $\{u_n\}_n \cdot \{v_n\}_n \in C_{\mathbb{Q}}$.

Example

Infinite decimal expansion.

Fix $N \in \mathbb{Z}$, $a_1 \cdots a_n \in \{0, \dots, 9\}$.

Then let $u_n = N + \sum_{k=1}^n a_k (10)^{-k}$ (that is the number $N.a_1 a_2 \dots a_n$).

This is always increasing and bounded above by $N + \sum_{k=1}^n 9 \cdot (10)^{-k} = N + \frac{9}{10} \cdot \sum_{k=1}^n (10)^{-(k+1)} \leq N + 1$.

Hence, it is Cauchy.

Increasing and Bounded Above Implies Cauchy

By contrapositive, increasing and not Cauchy implies not bounded.

By the negation of Cauchy and letting $p \geq q$ without loss of generality, we can force $u_p > u_q + \epsilon$.

Negation of Cauchy

$\exists \epsilon > 0, \forall N, \exists p, q \geq N, |u_p - u_q| > \epsilon$.

Real Numbers as Equivalence Classes of Cauchy Sequences

On $C_{\mathbb{Q}}$ define the relation $\{x_n\}_n \sim \{y_n\}_n$ if and only if $\lim_{n \rightarrow \infty} |x_n - y_n| = 0$.

Equivalence Relation

Reflexive: $x_n - x_n = 0$

Transitive: Uses algebraic limit rules. $x_n - z_n = x_n - y_n + y_n - z_n$.

Symmetric.

Definition of the Reals

$\mathbb{R} := C_{\mathbb{Q}} / \sim$

Then $x \in \mathbb{R}$, $x = [\{x_n\}_n]$.

Addition and Multiplication of Reals

- Addition $x + y := [\{x_n + y_n\}_n]$.
- Multiplication $x \cdot y := [\{x_n \cdot y_n\}_n]$.

Operations Do Not Depend on Choice of Representative

If $\{x_n\}_n \sim \{x'_n\}_n$ and $\{y_n\}_n \sim \{y'_n\}_n$, then $\{x_n\}_n + \{y_n\}_n \sim \{x'_n\}_n + \{y'_n\}_n$.

If $\{x_n\}_n \sim \{x'_n\}_n$ and $\{y_n\}_n \sim \{y'_n\}_n$, then $\{x_n\}_n \cdot \{y_n\}_n \sim \{x'_n\}_n \cdot \{y'_n\}_n$.

The Reals are a Field

There are nine properties to check, eight of which are “obvious”:

Commutativity of Addition (and Other “Obvious” Features)

$$[\{x_n\}_n] + [\{y_n\}_n] = [\{x_n + y_n\}_n] = [\{y_n + x_n\}_n] = [\{y_n\}_n] + [\{x_n\}_n]$$

That is, the Reals inherit most field features from the Rationals.

- Zero Element $0_{\mathbb{R}} = [\{0_{\mathbb{Q}}\}_n]$
- One Element $1_{\mathbb{R}} = [\{1_{\mathbb{Q}}\}_n]$

Multiplicative Inverses

How to define x^{-1} for $x \in \mathbb{R}$ where $x \neq 0$?

- Idea If $x = [\{x_n\}_n]$ choose $x^{-1} = [\{\frac{1}{x_n}\}_n]$.
If $x \in \mathbb{R}$, $x \neq 0$ then

1. $\exists \{x_n\}_n \in C_{\mathbb{Q}}$ representing x with non zero entries.
 2. $\{\frac{1}{x_n}\}_n$ is Cauchy.
- Proof of 1 Pick any $\{x_n\}_n$ representing x .

* $x \neq 0$, so NOT $(\lim_{n \rightarrow \infty} x_n = 0: \exists \epsilon_0 > 0, \forall N, \exists n \geq N, |x_n| > \epsilon_0)$.

* $\{x_n\}$ is Cauchy: $\forall \epsilon > 0, \exists N, \forall p, q \geq N, |x_p - x_q| < \epsilon$.

Therefore, $\exists N$ such that $\forall p, q \geq N_1, |x_p - x_q| < \frac{\epsilon_0}{2}$

And $\exists N_2 \geq N, |x_{N_2}| > \epsilon_0$.

For $q \geq N_2$, the Cauchy Criterion states that $|x_q| = |x_q - x_{N_2} + x_{N_2}| \geq |x_{N_2}| - |x_{N_2} - x_q| \geq \epsilon_0 - \frac{\epsilon_0}{2} \geq \frac{\epsilon_0}{2}$.

Therefore, the sought sequence is $\{x_{N_2} + k\}_{k \in \mathbb{N}}$.

$$- \text{Proof of } 2 \left| \frac{1}{x_p} - \frac{1}{x_q} \right| = \frac{|x_p - x_q|}{|x_p||x_q|} \leq \frac{4}{\epsilon_0^2} |x_p - x_q|.$$

Order on the Reals

Let $x \neq 0$, $\exists \{x_n\}_{n \in \mathbb{N}}$ be a representation of x and $\epsilon_0 > 0$.

Then for $|x_n| > \epsilon_0$, $\forall n \in \mathbb{N}$, there is a dichotomy:

- Either $\exists N \in \mathbb{N}$, $x_n > \epsilon_0$, $\forall n \geq N$ (in which case we write $x > 0$)
- Or $\exists N \in \mathbb{N}$, $x_n < -\epsilon_0$, $\forall n \geq N$ (in which case we write $x < 0$)

Thus the Reals are totally ordered.

October 4, 2023

Overview

Completeness of \mathbb{R} .

Topology of the Real Line.

Non-zero Reals Are Either Positive or Negative

Given $x \in \mathbb{R} \setminus \{0\}$, $\exists \delta \in \mathbb{Q}_+$ such that $\forall \{x_n\}_n$ representing x , $\exists N \in \mathbb{N}$ such that $|x_n| > \delta$, $\forall n \geq N$.

Moreover, one of the following (but not both) holds:

1. $\forall \{x_n\}_n \in x$, $\exists, x_n > \delta$, $\forall n \geq N$ (i.e. $x > 0$)
2. $\forall \{x_n\}_n \in x$, $\exists, x_n < -\delta$, $\forall n \geq N$ (i.e. $x < 0$)

Recall that $x \in \mathbb{R} \setminus \{0\}$ is an equivalence class of Cauchy sequences.

Total Ordering of the Reals

$x > 0$ produces a total ordering of \mathbb{R} where $x < y$ if and only if $y - x > 0$.

$$\leadsto \max(x, y) = \begin{cases} x & \text{if } x > y \\ y & \text{otherwise} \end{cases}$$

$|x| = \max(x, -x)$ (which satisfies the triangle inequality)

Lemma A

Let $x, y \in \mathbb{R}$. If $\{x_n\}_n, \{y_n\}_n$ represent x, y and satisfy $x_n < y_n$, $\exists N \in \mathbb{N}$, $\forall n \geq N$, then $x \leq y$.

- Proof By contradiction, suppose $x > y$ and $\exists \{x_n\}_n, \{y_n\}_n$ representing x, y such that $x_n \leq y_n$, $\forall n \geq N_1$. Then, by definition, $x - y > 0 \implies \exists \delta > 0$, $\exists N_2$, $x_n - y_n > \delta$ for $n \geq N_2$. But $x_n \leq y_n$ contradicts $x_n - y_n > \delta$.

Sequences of Reals

$\{x_n\}_n$, $x_n \in \mathbb{R}$

The definition of bounded, convergent and Cauchy sequences are the same as in \mathbb{Q} .

Injection of Rationals

$\iota : \mathbb{Q} \rightarrow \mathbb{R}$ such that $r \mapsto [\{u_n = r\}_n]$

This is isometric in the sense that $|\iota(r) - \iota(s)|_{\mathbb{R}} = |r - s|_{\mathbb{Q}}$

Theorem (Completeness 1)

Let $\{x_n\}_n \in C_{\mathbb{Q}}$ and $x = [\{x_n\}_n]$, then $\{\iota(x_n)\}_n$ converges to x .

Proof

What to show: $\forall \epsilon > 0$, $\exists N$, $\forall n \geq N$, $|\iota(x_n) - x| < \epsilon$.

Let $\epsilon \in \mathbb{Q}_+$. By the Cauchy criterion, $\exists N$, $\forall q, p \geq N$, $|x_p - x_q| < \epsilon$.

This is equivalent to $x_q - \epsilon \leq x_p \leq x_q + \epsilon$ where p is frozen.

Then by Lemma A, $x - \epsilon \leq \iota(x_p) \leq x + \epsilon$.

It follows that $\forall p \geq N$, $|\iota(x_p) - x| \leq \epsilon$.

Corollary

$\mathbb{Q} \cong \iota(\mathbb{Q})$ is dense in \mathbb{R} . That is, $\forall \epsilon > 0$, $\forall x \in \mathbb{R}$, $\exists r \in \mathbb{Q}$, $|\iota(r) - x| < \epsilon$.

The Isometric Copy of Rationals

For brevity, the ι notation will be dropped and the \mathbb{Q} will be understood as $\iota(\mathbb{Q})$.

Completeness of the Real Numbers

A sequence of real numbers converges in \mathbb{R} if and only if it is Cauchy.

Proof

(\implies) This is clear.

(\impliedby) Take a Cauchy sequence of reals $\{x_n\}_n$. Then $\forall \epsilon > 0$, $\exists N$, $\forall p, q \geq N$, $|x_p - x_q| < \epsilon$.

Using the density of \mathbb{Q} , $\forall n \in \mathbb{N}$, $\exists r_n \in \mathbb{Q}$ such that $|x_n - r_n| < \frac{1}{n}$.

Claim: $\{r_n\}_n$ is Cauchy. Indeed,

$$\begin{aligned} |r_p - r_q| &= |r_p - x_p + x_p - x_q + x_q - r_q| \\ &\leq |r_p - x_p| + |x_p - x_q| + |x_q - r_q| \\ &\leq \frac{1}{p} + |x_p - x_q| + \frac{1}{q} \end{aligned}$$

Take $\epsilon > 0$. $\{x_n\}$ cauchy implies $\exists N_1, \forall p, q \geq N, |x_p - x_q| \leq \frac{\epsilon}{3}$ and $\exists N_2, \forall p, q \geq N_2, \frac{1}{p} \leq \frac{\epsilon}{3}, \frac{1}{q} \leq \frac{\epsilon}{3}$ for $p, q \geq \max(N_1, N_2)$ $|r_p - r_q| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$.

Then, for Cauchy $\{r_n\}_n$, call $r = [\{r_n\}_n]$, then $\lim_{n \rightarrow \infty} r_n = r$ by the above theorem.

Then my algebraic limit rules, $x_n(x_n - r_n) + r_n$ where $(x_n - r_n) \rightarrow 0$ and $r_n \rightarrow r$ as $n \rightarrow \infty$. So $\{x_n\}$ converges.

Example

Let $x_1 = 1, x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n})$.

Then $\{x_n\}_n \in C_{\mathbb{Q}}$, and it converges to $L \in \mathbb{R}$.

By algebraic limit rules, $L^2(\lim x_n)^2 = \lim x_n^2 = 2$.

Subsets of the Reals, Infimum and Supremum

Notation

Subset: $S \subseteq \mathbb{R}$

Inclusion: $x \in S$

Open Interval: $(a, b) = \{x \in \mathbb{R} | a < x < b\}$

Semiclosed Interval: $(a, b] = \{x \in \mathbb{R} | a < x \leq b\}$

Closed Interval: $[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$

Unbounded Semiclosed Interval: $(-\infty, a] = \{x \in \mathbb{R} | x \leq a\}$

Unbounded Open: $(-\infty, a) = \{x \in \mathbb{R} | x < a\}$

Supremum

$S \subseteq \mathbb{R}$ is bounded above (respectively below) if $\exists M \in \mathbb{R}, \forall x \in S, x \leq M$ (respectively $\exists L \in \mathbb{R}, \forall x \in S, L \leq x$)

S admits a least upper bound, LUB, supremum or $\sup M$ if

1. $\forall x \in S, x \leq M$

2. $\forall M' \in \mathbb{R}, \text{upper bound of } S, M \leq M'$

If $\sup S$ exists, it is unique.

If $x \in S$ and x is an upper bound for S , then $x = \sup S$.

Example 1

$$\sup(0, 1) = \sup[0, 1] = 1$$

Example 2

$S = \{x \in \mathbb{Q}, x^2 < 2\}$ does not have a greatest element in \mathbb{Q} , nor a least upper bound in \mathbb{Q} .

Theorem (Completeness 2)

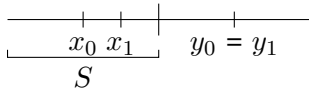
Every subset $S \subseteq \mathbb{R}$, nonempty and bounded above, has a supremum in \mathbb{R} .

Proof

By dichotomy.

$S \neq \emptyset \implies \exists x_0 \in S$ and S bounded above implies $\exists y_0 \in \mathbb{R}, \forall x \in S, x \leq y_0$ (in particular $x_0 \leq y_0$).

If $x_0 = y_0$, done. Otherwise, consider $m_0 = \frac{x_0 + y_0}{2}$.



Two options exist: if m_0 is an upper bound for S , set $y_1 = m_0$ and $x_1 = x_0$.

Otherwise, $\exists x_1 \in S$, such that $m_0 < x_1$ so set $y_1 = y_0$.

Repeat this process forever to construct two sequences x_n, y_n .

$\forall n, x_n \in S, y_n$ is an upper bound for S .

- $x_n \leq y_n$
- x_n is increasing and bounded above by y_0 , so it must be Cauchy and converging to x .
- y_n is decreasing and bounded below by x_0 , so it must be Cauchy and converging to y .
- $|x_{n+1} - y_{n+1}| \leq \frac{|x_n - y_n|}{2}$ which implies $|x_n - y_n| \leq \frac{1}{2^n} |x_0 - y_0|$ and $x = y = z$.

Therefore, the process may be understood as $x_0 \leq \dots \leq x_n \leq x_{n+1} \leq y_{n+1} \leq y_n \leq \dots \leq y_0$.

There remain two things to check: (1) z is an upper bound for S and (2) z is no larger than any other upper bound for S .

1. Take $x \in S, \forall n, x \leq y_n \xrightarrow{n \rightarrow \infty} x \leq z$.
2. Take upper bound for $S, z', x_n \leq z', \forall n \xrightarrow{n \rightarrow \infty} z \leq z'$.

So $z = \sup S$.

Monotone Convergence Theorem (Completeness 3)

An increasing sequence of reals, $\{x_n\}_n$, that is bounded above, converges to $\sup X = \sup\{x_n | n \in \mathbb{N}\}$.

To prove that this converges, since it is monotone and bounded above it is Cauchy and therefore must be convergent.

Proof

Call x the limit, then $\forall n, x_n \leq x$. To see this, suppose $\exists n_0, x < x_{n_0}$ then $\forall m \geq m_0, x < x_{m_0} \leq x_m \implies |x_m - x| \geq |x_{n_0} - x| > 0, \forall m \geq n_0$ is a contradiction.

Let M be an upper bound of X . Then $x_n \leq M, \forall n \xrightarrow{n \rightarrow \infty} x \leq M \implies x = \sup X$.

Theorem (Existence of Roots)

$\forall x \in \mathbb{R}$ where $x > 0, p \in \{2, 3, \dots\}, \exists! y > 0$ such that $y^p = x$.

Proof

Left as an exercise.

Either by dichotomy or consider $S = \{y \in \mathbb{R} | y^p < x\}$, show: $S \neq \emptyset$, bounded above and $(\sup S)^p = x$.

For uniqueness, show $y_1^p = y_2^p = x \iff 0 = y_1^p - y_2^p = (y_1 - y_2)(\dots \neq 0) \implies y_1 = y_2$.

Topological Properties

$S \subseteq \mathbb{R}$ is open if $\forall x \in S, \exists a, b \in \mathbb{R}, x \in (a, b) \subset S$.

x is an accumulation or limit point of S if $\forall \epsilon > 0, \exists y \in S, 0 < |x - y| < \epsilon$.

$S \subseteq \mathbb{R}$ is closed if it contains all its limit points.

A set may be both open closed, just open, just closed or neither.

Given $S \subseteq \mathbb{R}$, the interior of S is $\bigcup_{S' \text{ open} \subset S} S' = S^{\text{int}} = S^0$.

The closure is $\bigcap_{F \text{ closed} \supseteq S} F = \overline{S} \stackrel{\text{wts}}{=} S \cup \{\text{limit points of } S\}$.

Example

$\{x\}$ is not open, but, since the limit points of x are \emptyset , it is closed.

Propositions

1. Arbitrary unions and finite intersections of open sets are open.
2. S is open if and only the complement $S^c = \mathbb{R} \setminus S$ is closed.
3. Arbitrary intersections and finite unions of closed sets are closed.

Bolzano-Weierstrass Theorem

A bounded sequence in \mathbb{R} admits a convergent (Cauchy) subsequence. $\exists M, |x_n| \leq M, \forall n$

Proof by Dichotomy

Suppose $I_0 = [a, b]$ contains the sequence.

Construct a sequence of intervals by indicators: if $\left[a, \frac{a+b}{2}\right]$ contains infinitely terms of $\{x_n\}_n$, choose n such that $x_{n_1} \in \left[a, \frac{a+b}{2}\right]$ and call $I_1 = \left[a, \frac{a+b}{2}\right]$.

Otherwise, $\left[\frac{a+b}{2}, b\right]$ must contain infinitely many terms. Choose n in a similar fashion as above such that $I_1 = \left[\frac{a+b}{2}, b\right]$.

This process may be repeated to create a sequence of intervals such that $I_k \supseteq I_{k+1} \supseteq I_{k+2}$ and $l(I_k) = \frac{b-a}{2^k}$. A subsequence $\{u_{n_k}\}_k$ such that $u_{n_k} \in I_l$ for $k \geq l$.

Exercise

Extract a Cauchy criterion out of the above.

October 9, 2023

Overview

- Topology of \mathbb{R} continued.
- Numerical series.

Next Wednesday

- Absolute and Conditional Convergence.
- Rearrangement theorem.

Last Time

Finished with Bolzano-Weierstrass.

Limits

Limit Point

We say $x \in \mathbb{R}$ is a limit point of $\{x_n\}_n$ if a subsequence of $\{x_n\}_n$ converges to x .

Equivalently, $\forall \epsilon > 0, \forall n_0 \in \mathbb{N}, \exists n \geq n_0, |x_n - x| < \epsilon$.

That is, the sequence revisits an epsilon neighborhood of x infinitely many times.

Limit Set

The limit set of $\{x_n\}_n$: $LS(\{x_n\}_n)$ = the set of limit points of $\{x_n\}_n$.

- Comments
 - if $\lim_{n \rightarrow \infty} \{x_n\} = x$, then $LS(\{x_n\}_n) = \{x\}$.
 - The limit set can be as big as \mathbb{R} !

- Open Cover $\{O_\alpha\}_{\alpha \in I}$, with O_α open, is an open cover of A if $A \subseteq \bigcup_{\alpha \in I} O_\alpha$.
- Finite Subcover $O_1, \dots, O_n, n \in \mathbb{N}$.

Heine-Borel Theorem

Let $A \subseteq \mathbb{R}$.

The following are equivalent

1. A is Heine-Borel compact.
2. A is closed and bounded.
3. A is sequentially compact.

Proof

(1) \implies (2) \implies (3) \implies (1)

- Heine-Borel Compact Implies Closed and Bounded Suppose A satisfies the Heine-Borel property.
Consider $\{(-n, n)\}_{n \in \mathbb{N}}$. Clearly $\bigcup_n (-n, n) = \mathbb{R} \supseteq A$.
By Heine-Borel, $\exists n_0, \dots, n_p$ such that $A \subseteq \bigcup_{j=0}^p (-n_j, n_j) = (-N, N)$, $N = \max(n_0, \dots, n_p)$. So A is bounded.
 A is closed if $y \notin A \implies y$ is not a limit point of A .
Take $y \in A^c$, then $A \subseteq \mathbb{R} \setminus \{y\} = \bigcup_{n \in \mathbb{N}} (-\infty, y - \frac{1}{n}) \cup (y + \frac{1}{n}, \infty)$.
By the Heine-Borel property,

$$\begin{aligned} A &\subseteq \bigcup_{n_0, \dots, n_p} (-\infty, y - \frac{1}{n}) \cup (y + \frac{1}{n}, \infty) \\ &= (-\infty, y - \frac{1}{N}) \cup (y + \frac{1}{N}, \infty) \end{aligned}$$

Which implies $A \cap [y - \frac{1}{N}, y + \frac{1}{N}] = \emptyset$ and y is not a limit point of A .
That is, A contains its limit points.

- Closed and Bounded Implies Sequential Compactness Suppose A is both closed and bounded.
Let $\{x_n\}_n \in A$. Then $\{x_n\}_n$ is bounded. By Bolzano-Weierstrass, it has a limit point x and a subsequence $\{x_{n_k}\}_k$ converging to x .
Since A is closed, $\lim_{k \rightarrow \infty} x_{n_k} = x \in A$. ■
- Sequential Compactness Implies Heine-Borel Suppose $A \subseteq \mathbb{R}$ is sequentially compact.
Consider an open cover of A , $\{O_\alpha | \alpha \in I\}$.
First, turn it into a countable cover:

$$- \forall \alpha \in I, O_\alpha \subseteq (r_\alpha^1, r_\alpha^2), r_\alpha^1, r_\alpha^2 \in \mathbb{Q}$$

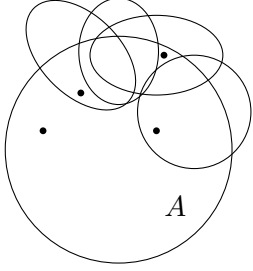
Assume that $\{O_\alpha\}_\alpha$ can be made countable (O_1, \dots, O_n)

By contradiction, suppose $\forall n \in \mathbb{N}, A \setminus \left(\bigcup_{j=1}^n O_j\right) \neq \emptyset$.

Take $x_n \in A \setminus \left(\bigcup_{j=1}^n O_j\right)$. Since A is sequentially compact, $\exists \{x_{n_k}\}_k$ subsequence of $\{x_n\}_n$ converging to $x \in A$.

Since $A \subset \bigcup_{j \in \mathbb{N}} O_j$, $\exists j_0, x \in O_{j_0}$, O_{j_0} is open: $\exists \delta > 0, (x - \delta, x + \delta) \subseteq O_{j_0}$.

Then $\exists N, k \geq N \implies x_{n_k} \in (x - \delta, x + \delta) \subseteq O_{j_0}$. But if k is such that $n_k > j_0$, we also have $x_{n_k} \notin O_{j_0}$ which is a contradiction!



Structure of Open and Closed Sets

A is open in \mathbb{R} if and only if it can be written as an at most countable, disjoint union of open intervals.

TODO Proof

For $x \in A$, $\exists (a, b)$, such that $x \in (a, b) \subseteq A$.

Let $I_x = \bigcup_{\text{open interval containing } x, I \subseteq A} I$. This is the maximal interval containing x in A .

Then, $A \subseteq \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} I_x \subseteq A$.

That is, $A = \bigcup_{x \in A} I_x$ (*).

Next, if $x, y \in A$, then $\begin{cases} \text{either } I_x = I_y \\ \text{or } I_x \cap I_y = \emptyset \end{cases}$

IMAGE HERE

The union (*), as a disjoint union, is at most countable because each distinct one must contain a distinct rational and \mathbb{Q} is countable.

Long Story Short

The topology of the reals avoids exotic sets. Closed sets, on the other hand, can be quite complicated.

TODO Cantor Set

$C := \bigcap_{k \in \mathbb{N}_0} I_k$. I_{k+1} is obtained by removing the middle open third of each interval making I_k .

IMAGE HERE - CANTOR

$I_0 = [0, 1]$. One interval of length 1.

$I_1 = [0, 1/3] \cup [2/3, 1]$. Two intervals of length $2/3$.

$I_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$. Four intervals of $(2/3)^2$

I_k is 2^k intervals of length $(2/3)^k$.

$I_{k+1} \subseteq I_k \implies C \subseteq I_k, \forall k \implies l(C) \leq l(I_k) = (2/3)^k \implies l(C) = 0$.

TODO Triadic Expansions

Goal:

1. C is perfect (i.e. every point in C is a limit point of C).
2. C contains no open intervals.

Property 2 is easy because $C \subseteq I_k$, which does contain interval of length greater than $(1/3)^k$.

1. C is uncountable.

Every $x \in [0, 1]$ can be written in the form $x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$, $a_k \in \{0, 1, 2\}$.

That is, $x = 0.a_1a_2\dots$ in base 3. This is not always unique (e.g. $1/3 = 0.100\dots = 0.022\dots$).

IMAGE HERE - THIRDS OF INTERVAL

Note that the Cantor set is removing all decimal expansions with leading 1s. That is, $x \in C$ if and only if it has a triadic expansion only made of 0s and 2s.

- Proof of 1 If $x \in C$, $x = \sum_{k \geq 1} \frac{a_k}{3^k} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{a_k}{3^k}$, then $x_n \in C$, $\forall n$ and $x_n = 0.a_1\dots a_n0000\dots$ where $a_1, a_n \in \{0, 2\}$.

Unique representation can be maintained by forcing the behavior of the $n + 1$ th digit.

- Proof of 3 Every point in $[0, 1]$ can also be written as $x = \sum_{n=1}^{\infty} \frac{b_n}{2^n}$, $b_n \in \{0, 1\}$ (i.e. a binary expansion). Then $C \mapsto [0, 1]$ gives $x = \sum \frac{a_k}{3^k} \mapsto \sum \frac{b_k}{2^k}$, $b_k = \frac{a_k}{2}$ for $a_k \in \{0, 2\}$ is a bijection!

October 11, 2023

Overview: Numeric Series

- Series with non-negative terms.
- Series with general terms.
- Convergence criteria.
- Algebraic rules.
- Rearrangements.

General Notation

Sequence $\{x_n\}_{n \geq n_0}$ (often $n_0 \in \{0, 1\}$)

Definition: Partial Sum

$$S_n = \sum_{k=n_0}^n x_k \quad (x_n = S_n - S_{n-1})$$

We say $\sum_n x_n$ converges if $\lim_{n \rightarrow \infty} S_n$ exists.

We denote $\sum_{k=n_0}^{\infty} x_k = \lim_{n \rightarrow \infty} S_n$

- Example: Geometric Series $\sum_{k=0}^n r^k = S_n$, $r \in (0, 1)$
 $\frac{1-r^{n+1}}{1-r} \rightarrow \frac{1}{1-r}$
- Example: P Series $\sum_{k=1}^n \frac{1}{k^p}$, $p > 0$
- Example: Exponential $\sum_{k=0}^n \frac{1}{k!}$

Series without Non-negative Terms

The series has non-negative terms if $x_n \geq 0$, $\forall n$.

Obvious Algebraic Limit Rules

If $\sum_{n \geq n_0} a_n$ and $\sum_{n \geq n_0} b_n$ converge and $\alpha \in \mathbb{R}$, then $\sum_{n \geq n_0} (a_n + \alpha b_n)$ converges to

$$\sum_{n=n_0}^{\infty} a_n + \alpha \sum_{n=n_0}^{\infty} b_n = \sum_{n=n_0}^{\infty} (a_n + \alpha b_n)$$

- Proof (Sketch) Reason on the partial sums; use algebraic limit rules on sequences.

Proposition

If $\sum_n x_n$ converges in \mathbb{R} , then $\lim_{n \rightarrow \infty} x_n = 0$.

- Proof $x_n = S_n - S_{n-1} \xrightarrow{n \rightarrow \infty} S - S = 0$
 Since $S_n \xrightarrow{n \rightarrow \infty} S$ and $S_{n-1} \xrightarrow{n \rightarrow \infty} S = \sum_{n=n_0}^{\infty} x_n$.

Series with Non-negative Terms

If $x_n \geq 0$, $\forall n$, $S_n = \sum_{k=n_0}^n x_k$ is non-decreasing.

By monotone convergence theorem, S_n is either bounded, and therefore converges, or unbounded from above where

$$\forall m > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, S_n \geq m$$

This is “diverging to $+\infty$.”

Theorem: Convergence Criteria

- Term Test If $0 \leq a_n \leq b_n$, $\forall n \geq n_0$ and $\sum_n b_n$ converges, then $\sum_n a_n$ converges.

- Proof Suppose $0 \leq a_n \leq b_n$, and $t_n = \sum_{k=n_0}^n b_k$ converges and, therefore, is bounded above by $B = \sum_{k=n_0}^{\infty} b_k$.
 Then $\forall n$, $\sum_{k=n_0}^n a_k \leq \sum_{k=n_0}^n b_k \leq B$.
 Thus, by monotone convergence theorem, $\sum_{k=n_0}^{\infty} a_k$ converges.

- Ratio Test If $a_n > 0$, $\forall n$ and $\exists n_0 \in \mathbb{R}$ such that $\frac{a_{n+1}}{a_n} \leq r < 1$, $\forall n \geq n_0$, then $\sum_n a_n$ converges.
 - Clarification The harmonic series has ratio $\frac{k}{k+1} < 1$ but since $\frac{k}{k+1} \xrightarrow{k \rightarrow \infty} 1$, there is no r which satisfies the ratio test.
 - Proof Suppose $a_{n+1} \leq r a_n$ for $n \geq n_0$.
Then $a_{m_0+p} \leq a_{m_0+(p-1)} r \leq a_{m_0+(p-2)} r^2 \leq \dots \leq a_{m_0} r^p$.
Then for $n \geq n_0$,

$$\sum_{k=n_0}^n a_k = \sum_{k=n_0}^{m_0} a_k + \sum_{k=m_0+1}^n a_k \leq \sum_{k=m_0}^{m_0+(n-m_0)} a_{m_0} r^{n-m_0} \leq a_{m_0} \sum_{k=m_0}^{n-m_0} r^{n-m_0} \leq \frac{1}{1-r}$$
 - Rate of Convergnce The above proof shows that the ratio test implies a geometric rate of convergence.
- Root Test If $\exists n_0 \in \mathbb{N}$ and $r \in (0, 1)$ such that $a_n^{1/n} \leq r$, then $\sum_n a_n$ converges.
 - Proof (Sketch) Same story as the ratio test: $a_n^{1/n} \leq r \implies a_n \leq r^n$.
- Rejection of Ratio/Root If $\exists n_0 \in \mathbb{N}$ such that either $\frac{a_{n+1}}{a_n} \geq 1$ for $n \geq n_0$ or $a_n^{1/n} \geq 1$ for $n \geq n_0$, then $\sum_n a_n$ diverges to $+\infty$.
 - Proof (Sketch) In either case, a_n cannot converge to zero. Therefore the series cannot converge.

Prototype Scales

Geometric Rates

$\sum_{n \geq 1} \frac{1}{n^\alpha}$ converges if and only if $\alpha > 1$ (to $\zeta(\alpha)$)

$$a_k = \frac{1}{k^\alpha} \rightsquigarrow 2^k a_{2^k} = \frac{2^k}{2^{k\alpha}} = \left(\frac{1}{2^{\alpha-1}}\right)^k \implies t_n = \sum_{k=1}^n 2^k a_{2^k} \text{ converges if and only if } \frac{1}{2^{\alpha-1}} < 1 \text{ if and only if } \alpha > 1.$$

Log Geometric Case

$\sum_{n \geq 1} \frac{1}{n(\log(n))^\beta}$ converges if and only if $\beta > 1$.

$$a_k = \frac{1}{k(\log(k))^\beta} \rightsquigarrow 2^k a_{2^k} = \frac{2^k}{2^k (\log(2^k))^\beta} = \frac{1}{(\log(2))^\beta k^\beta} \text{ converges if and only if } \beta > 1.$$

Lemma:

Suppose a_n decreases to 0.

Then the sequence $S_n = \sum_{k=1}^n a_k$ converges if and only if $t_n = \sum_{k=1}^n 2^k a_{2^k}$ converges.

$$S_{2^n} = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \dots$$

• Proof

$$a_3 + a_3 \leq \underbrace{\quad} \leq a_2 + a_3$$

$$S_n = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \dots$$

$$= a_1 + \sum_{k=1}^n \sum_{p=1}^{2^k-1} a_{2^k+p}$$

$$\leq a_1 + 2^k a_{2^{k+1}} +$$

This gives

$$\frac{1}{2}(t_n - a_1) \leq S_{2^n} - a_1 \leq t_{n-1}$$

Therefore S_{2^n} converges, which implies that t_n converges, and, since S_n is monotone, S_n itself converges.

Series with General Terms

General term is signed.

Trick

Write $a_n = a_n^+ - a_n^-$ and $a_n^\pm = \max(0, \pm a)$. Then

$$S_n = \sum_{k=n_0}^n a_k = \left(\sum_{k=n_0}^n a_k^+ \right) - \left(\sum_{k=n_0}^n a_k^- \right)$$

Convergence Outcomes

	$\sum_{k=n_0}^\infty a_k^+ < \infty$	$\sum_{k=n_0}^\infty a_k^+ = \infty$	If
$\sum_{k=n_0}^\infty a_k^- < \infty$	absolute convergence	$\lim S_n = +\infty$	
$\sum_{k=n_0}^\infty a_k^+ = \infty$	$\lim S_n = -\infty$	lots of things can happen; divergence, convergence, any limit sequence	

S_n^+ and S_n^- converge, we can return to algebraic limit rules.

S_n converges to $\lim_{n \rightarrow \infty} S_n^+ - \lim_{n \rightarrow \infty} S_n^-$

Definition: Absolute Convergence

We say $\sum_n a_n$ converges absolutely if and only if $\sum_n |a_n|$ converges.

Note

$$|a_n| = a_n^+ + a_n^-$$

Proposition: Absolute Convergence Implies Convergence

Proof

Absolute convergence $\implies \sum |a_n|$ converges $\implies \sum a_n^+$ and $\sum a_n^-$ converges $\implies \sum (a_n^+ - a_n^-)$ converges.

Definition: Conditional Convergence

$\sum_n a_n$ converges conditionally if and only if $\sum_n a_n$ converges while $\sum_n |a_n|$ diverges.

Criteria for Convergence

For absolute convergence, run root/ratio/term test on $\sum_n |a_n|$.
Other criteria which might indicate conditional convergence.

Alternating Series Test

If $a_n(-1)^n b_n$, $b_n \geq 0$ decreases to zero, the series is conditionally convergent.

—++•++—

Dirichlet Test

If $a_n = b_n c_n$, where b_n decreases to zero and c_n satisfies $|c_0 + c_1 + \dots + c_n| \leq C$, $\exists C \in \mathbb{R}, \forall n \in \mathbb{N}$, then $\sum_{n \geq 0} a_n$ converges conditionally.

- Applications $\sum_{n \geq 1} \frac{(-1)^n}{n}$
 $\sum_{n \geq 1} \frac{\cos(n)}{n}$
- Proof Write $C_n = c_0 + c_1 + \dots + c_n$, such that $|C_n| \leq C, \forall n$.
Then $c_n = C_n - C_{n-1}$, and

$$\sum_{k=0}^n b_k c_k = \sum_{k=0}^n b_k (C_k - C_{k-1}) = \sum_{k=0}^n b_k C_k - \sum_{k=0}^n b_k C_{k-1} \stackrel{l=k-1}{=} \sum_{k=0}^n b_k C_k - \sum_{l=0}^{n-1} b_{l+1} C_l = b_n C_n + \sum_{k=0}^{n-1} (b_k - b_{k+1}) C_k$$

Then, since $b_n C_n \xrightarrow{n \rightarrow \infty} 0$, we only need to show that the final term converges absolutely. Consider

$$\sum_{k=0}^{n-1} |b_k - b_{k+1}| |C_k| \leq C \sum_{k=0}^{n-1} (b_k - b_{k+1}) = C(b_0 - b_n) \leq C(b_0)$$

independent of n . Hence, $\sum_{k=0}^n b_k c_k$ converges.

Definition: Rearrangement

Take $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ a bijection and $\sum_{n \geq 1} a_n$ a series such that $S_n = \sum_{k=1}^n a_k$.

Then define a rearranged sum $S_n^{(\sigma)} = \sum_{k=1}^n a_{\sigma(k)}$.

Q: When does the rearranged sum converge; to where?

- Theorem: Rearrangement of Absolute Convergence If $\sum a_n$ converges absolutely, then $\forall \sigma, \lim_{n \rightarrow \infty} S_n^{(\sigma)} = \lim_{n \rightarrow \infty} S_n$.
- Theorem: Rearrangement of Conditional Convergence If $\sum a_n$ converges conditionally, then $\forall x \in \mathbb{R}, \exists \sigma$ such that $\lim_{n \rightarrow \infty} S_n^{(\sigma)} = x$.

October 16, 2023

Overview

Sequences and Series of Functions

Things that will be glossed over for time

- Limits
- Continuity
- Differentiability
- Integrability

Why care about sequences and series?

Extending features of functions.

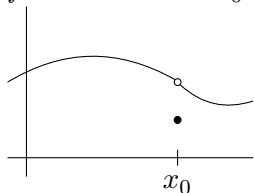
Approximations.

Limits and Continuity

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \rightarrow \mathbb{R}$, $x_0 \in I$.

Definition: Limit

f has a limit at x_0 if $\exists \ell \in \mathbb{R}, \forall \epsilon > 0, \exists \delta > 0, 0 < |x - x_0| < \delta \implies |f(x) - \ell| < \epsilon$

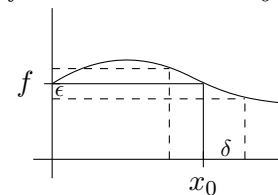


- Equivalently

For every sequence $\{x_n\}_n$ in I converging to x (but distinct to x), $\lim_{n \rightarrow \infty} f(x_n) = \ell$.

Definition: Continuous

f is continuous at x_0 if $\lim_{x \rightarrow x_0} f(x) = f(x_0)$.



- Modulus of Continuity $\forall \epsilon > 0, \exists \delta > 0, \forall x \in I, |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$
Then $\delta(x_0, \epsilon)$ is the modulus of continuity.

Definition: Uniform Continuity on I

f is uniformly continuous on I if $\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in I, |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$.
Where δ is $\delta(\epsilon)$. That is, the modulus of continuity does not depend on the points.

Special Types of Uniform Continuity

Hölder Continuous

f is α -Hölder continuous on I for $\alpha \in (0, 1]$, if $\exists c > 0$ such that $\forall x, y \in I, |f(x) - f(y)| \leq c|x - y|^\alpha$
 $\alpha = 1$ implies that f is “Lipschitz-continuous”

- Example

If f' exists and is bounded on $[a, b]$ by M , then by the Mean Value Theorem:
 $|f(x) - f(y)| = |f'(\xi)||x - y| \leq M|x - y|$, where $x \leq \xi \leq y$.

Continuity on Compact Sets

Let $K \subseteq \mathbb{R}$ be a compact set and $f : K \rightarrow \mathbb{R}$ be continuous.
Then

1. $f(K)$ is compact. In particular, f is bounded on K .
2. f achieves its extrema on K . (e.g. $\exists M \in K$ such that $f(M) = \sup\{f(x) \mid x \in K\}$).
3. f is uniformly continuous on K .

Note: the proofs of these features are good practice. In particular, proofs that exploit the Heine-Borel property.

Proof 1: Compact

Let y_n be a sequence in $f(K)$.

Then, $\forall n, y_n = f(x_n)$ for $x_n \in K$.

It follows that there exists a subsequence $\{x_{n_k}\}_k$ converging to x in K .

By continuity, $y_{n_k} = f(x_{n_k}) \xrightarrow{k \rightarrow \infty} f(x) \in f(K)$.

Proof 2: Achieves Its Extrema

Construct M .

By the supremum property, $S = \sup\{f(x) \mid x \in \mathbb{R}\}$, $\forall n, \exists x_n \in K$ such that $S - \frac{1}{n} \leq f(x_n) < S$.

Since K is compact, there exists a subsequence $\{x_{n_k}\}_k$ converging to $x \in K$.

Since f is continuous at x , $f(x_{n_k}) \xrightarrow{k \rightarrow \infty} f(x)$, and also $S - \frac{1}{n_k} \leq f(x_{n_k}) \leq S \xrightarrow{k \rightarrow \infty} S = f(x)$.

Proof 3: Uniformly Continuous

Suppose, for sake of contradiction, that $\exists \epsilon > 0, \forall \delta > 0, \exists x_\delta, y_\delta \in K, |x_\delta - y_\delta| < \delta$ and $|f(x_\delta) - f(y_\delta)| \geq \epsilon$.

Letting $\delta = \frac{1}{n}$, we may write $x_n, y_n \in K, |x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \epsilon$.

Since K is compact, there exists a subsequence $\{x_{n_k}\}_k$ which converges to $x \in K$.

Since $|x_{n_k} - y_{n_k}| < \frac{1}{n_k}$, then $\{y_{n_k}\}_k$ also converges to x .

By continuity of f at x , $\lim_{k \rightarrow \infty} f(x_{n_k}) - f(y_{n_k}) = 0$. However, this contradicts the established fact that $|f(x_n) - f(y_n)| \geq \epsilon$ for $\epsilon > 0$.

Notation

Let $I \subseteq \mathbb{R}$ be an interval.

Sequence of Functions

$$f_n(x), x \in I, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$$

Series of Functions

$$S_n(x) = \sum_{k=0}^n f_k(x)$$

Definition: Pointwise Convergence

A sequence or series of functions converges pointwise on I if and only if $\forall x \in I, \{f_n(x)\}_n$ is convergent.

Call $f(x)$ the limit.

Q: Under what conditions do properties of a sequence (e.g. continuity, differentiability, integrability) propagate to the limit?

Power Series

$$S_n(x) = \sum_{k=0}^n a_k (x - x_0)^k$$

$\underbrace{\hspace{10em}}_{x_0}$

Fourier Series

$$S_n = a_0 + \sum_{n=1}^N a_n \cos(nx) + b_n \sin(nx) \text{ is } 2\pi\text{-periodic.}$$

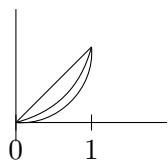
Approximation

For purposes of approximation, it is useful to know if, for example, the integral may be approximated by taking integrals of the partial sums.

Deficiencies of Pointwise Convergence

Example 1

On $[0, 1]$, $f_n(x) = x^n \xrightarrow{n \rightarrow \infty} \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases}$,



f_n is continuous on $[0, 1]$, $\forall n$, but f is not.

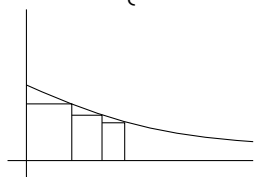
- Exercise

Show that there is no uniform convergence here.

Hint: negate uniform convergence and prove the negation.

Example 2

$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$ is not Riemann-integrable on $[0, 1]$.



If r_n denotes a denumeration of rationals in $[0, 1]$, define $f_n(x) = \begin{cases} 1 & \text{if } x \in \{r_0, \dots, r_n\} \\ 0 & \text{otherwise} \end{cases}$.

So f_n converges pointwise on $\chi_{\mathbb{Q}}$.

Yet, $\forall n$, f_n is Riemann-integrable and $\int_0^1 f_n(x) dx = 0$.

Definition: Uniform Convergence

We say $f_n : D \rightarrow \mathbb{R}$ (e.g. D an interval) converges uniformly to f on D (notation $f_n \rightrightarrows f$ on D) if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, n \geq N \implies \begin{cases} |f_n(x) - f(x)| < \epsilon, \forall x \in D \\ \sup_D |f_n - f| < \epsilon \end{cases}$$

Compare with Pointwise Convergence

Compare to $f_n \rightarrow f$ pointwise on D .

$$\forall x \in D, \forall \epsilon > 0, \exists N \in \mathbb{N}, n \geq N \implies |f_n(x) - f(x)| < \epsilon.$$

In this case, the behavior is primarily contingent upon the choice of x . That is $N(x, \epsilon)$ is dependent on x .

Theorem: Weierstrass M-Test

Let $f_n : D \rightarrow \mathbb{R}$ be bounded by M_n on D .

If $\sum_{n=1}^{\infty} M_n < \infty$, then the series $S_n(x) = \sum_{k=1}^n f_k(x)$ converges uniformly to $S(x)$

Proof

$\forall x \in D, |S_n(x) - S(x)| = |\sum_{k=n+1}^{\infty} f_k(x)| \stackrel{\text{triangle inequality}}{\leq} \sum_{k=n+1}^{\infty} |f_k(x)| \leq \sum_{k=n+1}^{\infty} M_k$, where $\sum_{k=n+1}^{\infty} M_k$ is a uniform bound in x .

Let $\epsilon > 0, \exists n, n \geq N \implies \sum_{k=n+1}^{\infty} M_k < \epsilon$.

Then $\forall x \in D, n \geq N, |S_n(x) - S(x)| \leq \sum_{k=n+1}^{\infty} M_k < \epsilon$. ■

Theorem: Continuity and Uniform Limits

Let $f_n : D \rightarrow \mathbb{R}$ be continuous on D for all n and $f_n \rightarrow f$ on D ($\lim_{n \rightarrow \infty} \sup_D |f_n - f| = 0$).

Then f is continuous on D .

Proof

Fix $x \in D$, with x_n converging to x in D .

What To Show: $f(x_n) \xrightarrow{n \rightarrow \infty} f(x)$.

Scratch: $f(x_n) - f(x) = (f(x_n) - f_p(x_n)) + (f_p(x_n) - f_p(x)) + (f_p(x) - f(x))$.

Let $\epsilon > 0$ be given.

$f_n \rightarrow f : \exists N, n \geq N \implies |f_n(y) - f(y)| < \frac{\epsilon}{3}, \forall y \in D$.

For $p \geq N, |f_p(y) - f(y)| < \frac{\epsilon}{3}, \forall y \in D \implies \forall n \in \mathbb{N}, |f(x_n) - f(x)| \stackrel{\text{triangle inequality}}{\leq} \frac{2\epsilon}{3} + |f_p(x_n) - f_p(x)|$.

With $p = N$, since f_p is continuous at x , $\exists N_1, n \geq N_1 \implies |f_p(x_n) - f_p(x)| < \frac{\epsilon}{3}$.

Hence, for $n \geq N_1, |f(x_n) - f(x)| \leq \epsilon$. ■

Riemann-Integrability

Fix $D = [a, b]$ and $g : [a, b] \rightarrow \mathbb{R}$ bounded by $|g(x)| \leq M, \forall x$.

Definition: Subdivision

$$\sigma = \{a = x_0 < x_1 < x_2 < \dots < x_n = b\}$$

Definition: Upper and Lower Riemann Sums

$S^+(g, \sigma) = \sum_{k=1}^n (x_k - x_{k-1})M_k$ is the upper sum.

$S^-(g, \sigma) = \sum_{k=1}^n (x_k - x_{k-1})m_k$ is the lower sum.

Where $M_k = \sup_{[x_{k-1}, x_k]} g$ and $m_k = \inf_{[x_{k-1}, x_k]} g$.

This gives $-M(b-a) \leq S^-(g, \sigma) \leq S^+(g, \sigma) \leq (b-a)M$.

If $\mathfrak{S}[a, b] = \{\text{subdivisions of } [a, b]\}$, then

$I^-(g) = \sup_{\sigma \in \mathfrak{S}[a, b]} S^-(g, \sigma)$ and $I^+(g) = \inf_{\sigma \in \mathfrak{S}[a, b]} S^+(g, \sigma)$.

Definition: Riemann Integrable

g is Riemann integrable if $I^+(g) = I^-(g)$ and we denote $\int_a^b g(t) dt = I^+(g)$.

Lemma

g is Riemann integrable if and only if $\forall \epsilon > 0, \exists \sigma \in \mathfrak{S}[a, b]$ such that $S^+(g, \sigma) - S^-(g, \sigma) < \epsilon$.

Properties

1. Continuous functions and monotone functions are Riemann Integrable.
2. $f \mapsto \int_a^b f(t) dt$ is linear.
3. If f, g are Riemann Integrable and $f(x) \leq g(x), \forall x \in [a, b]$, then $\int_a^b f(t) dt \leq \int_a^b g(t) dt$.

Theorem:

If $f_n \Rightarrow f$ on $[a, b]$ and f_n is Riemann Integrable for all n , then f is Riemann Integrable on $[a, b]$ and $\lim_{n \rightarrow \infty} \int_a^b f_n(t) dt = \int_a^b \lim_{n \rightarrow \infty} f_n(t) dt = \int_a^b f(t) dt$.

Proof

$\forall n, \forall x \in [a, b], f_n(x) - \epsilon \leq f(x) \leq f_n(x) + \epsilon$ where $\epsilon_n = \sup_{x \in [a, b]} |f_n(x) - f(x)|$ (by hypothesis $\epsilon_n \xrightarrow{n \rightarrow \infty} 0$)

Then, for any $\sigma \in \mathfrak{S}[a, b]$, $S^-(f_n, \sigma) - \epsilon_n(b-a) \leq S^-(f, \sigma) \leq S^+(f, \sigma) \leq S^+(f_n, \sigma) + \epsilon_n(b-a)$.

It follows that $S^+(f, \sigma) - S^-(f, \sigma) \leq S^+(f_n, \sigma) - S^-(f_n, \sigma) + 2\epsilon_n(b-a)$.

Finishing the proof is left as an exercise.

October 18, 2023

Overview

- Sequences/Series
- Power Series
- Exponential and Logarithms

Fundamental Theorems of Calculus

Full proofs in 105A lecture notes.

Differentiation of the Integral

$f : [a, b] \rightarrow \mathbb{R}$ continuous.

$\forall x \in [a, b]$, can define $F(x) = \int_a^x f(t) dt$.

Then F is continuously differentiable on $[a, b]$

$F'(x) = f(x)$ for $x \in [a, b]$.

Integration of the Derivative

$f \in C^1[a, b]$ with one-sided derivatives at a and b well defined. (e.g. $\frac{f(a+h)-f(a)}{h} \xrightarrow{h>0; h \rightarrow 0} f'(a)$).

Then $\forall x, y, a \leq x \leq y \leq b$, $f(y) - f(x) = \int_x^y f'(t) dt$.

Theorem: Differentiability of Uniform Limits

Let $f_n : (a, b) \rightarrow \mathbb{R}$ be a sequence in $C^1[a, b]$, and assume $f_n(x) \rightarrow f(x)$ pointwise while $f'_n(x) \rightrightarrows g(x)$ uniformly. Then $f \in C^1(a, b)$ and $f' = g$.

Proof

Fix $a_0 \in (a, b)$.

Then $\forall x \in (a, b)$, by the Fundamental Theorem of Calculus,

$$f_n(x) - f_n(a_0) = \int_{a_0}^x f'_n(t) dt$$

Observe that $f_n(x) \xrightarrow{n \rightarrow \infty} f(x)$ and $f_n(a_0) \xrightarrow{n \rightarrow \infty} f(a_0)$ pointwise, and $\int_{a_0}^x f'_n(t) dt \rightarrow \int_{a_0}^x g(t) dt$ by the integrability of uniform limits. Then

$$f(x) - f(a_0) = \int_{a_0}^x g(t) dt, \quad \forall x \in (a, b)$$

which implies $f \in C^1$ and $f' = g$. ■

Interesting Applications

$$S_n(x) = \sum_{k=0}^n f_k(x).$$

Suppose pointwise convergence, that $S'_n(x) = \sum_{k=0}^n f'_k(x)$ is continuous, $|f'_k(x)| \leq M_k$ and $\sum_{k=0}^{\infty} M_k < \infty$. Long story short, this implies

$$\left(\sum_{k=0}^{\infty} f_k(x) \right)' = \sum_{k=0}^{\infty} f'_k(x)$$

Example

$$f(x) = \sum_{n=0}^{\infty} \frac{\cos(nx)}{n^3}$$

Call $u_n(x) = \frac{\cos(nx)}{n^3}$, then $|u_n(x)| \leq \frac{1}{n^3}$ summable and $|u'_n(x)| = \left| \frac{-\sin(nx)}{n^2} \right| \leq \frac{1}{n^2}$ summable.

This implies $f'(x) = -\sum_{n=0}^{\infty} \frac{\sin(nx)}{n^2}$.

Repetition of this process informs us that $f \in C^2$.

Power Series

$S_n(x) = \sum_{k=1}^n a_k(x - x_0)^k$ for, $x_0 \in \mathbb{R}$ fixed, is 'centered at x_0 .' Note that each term is $C^\infty(\mathbb{R})$.

Example 1

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \text{ for } |x| < 1.$$

Example 2

$\exp(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges $\forall x \in \mathbb{R}$.

- Why?
Ratio Test.

$$\frac{a_{k+1}}{a_k} = \frac{x^{k+1}}{(k+1)!} \cdot \frac{k!}{x^k} = \frac{x}{k+1}$$

$$\text{So } \left| \frac{a_{k+1}}{a_k} \right| \xrightarrow[k \rightarrow \infty]{} 0$$

Lemma: Radius of Convergence

Suppose a power series $\sum_{n \geq 0} a_n x^n$ converges at $b \in \mathbb{R}$.

1. Converges absolutely $\forall x, |x| < |b|$.
2. $\forall a \in (0, b)$ converges uniformly on $[-a, a]$.

- Proof of 1
Suppose $\sum_{n \geq 0} a_n b^n$ converges.
Then $a_n b^n \rightarrow 0$.
Let x such that $|x| < b$, then

$$|a_n x^n| = \left| a_n b^n \left(\frac{x}{b} \right)^n \right| \leq M \left(\frac{|x|}{b} \right)^n$$

By term test, $\sum_{n=0}^{\infty} |a_n x^n| < \infty \implies \sum a_n x^n$ converges absolutely.

- Proof of 2
If $|x| \leq a < b$,

$$|a_n x^n| \leq M \left(\frac{|x|}{b} \right)^n \leq M \left(\frac{a}{b} \right)^n$$

Thus, by M -test for $x \in [-a, a]$, the series converges uniformly on $[-a, a]$.

- Upshot
The set where a power series converges is an interval centered at x_0 .

Theorem: Radius of Convergence

Given a power series, define R to be such that $\frac{1}{R} = \limsup_n |a_n|^{1/n}$. Then

1. $\forall a \in (0, R)$, the series converges uniformly on $[-a, a]$.
2. If $|x| > R$, the series diverges.

Proof

IMAGE HERE - RADIUS OF CONVERGENCE

Fix x . As an exercise, $\limsup_n |a_n x^n|^{1/n} = |x| \cdot \limsup_n |a_n|^{1/n} = \frac{|x|}{R}$.

Recall that $\limsup_n |a_n x^n|^{1/n} = \lim_{n \rightarrow \infty} y_n$ where $y_n = \sup_{k > n} \{|a_k x^k|^{1/k}\}$.

If $\frac{|x|}{R} < 1$, then $\exists N_0, n \geq N_0 \implies y_n < \frac{1 + \frac{|x|}{R}}{2} < 1$.

This implies $\forall k \geq N_0, |a_k x^k|^{1/k} \leq \frac{1 + \frac{|x|}{R}}{2} < 1$ and, by the root test, the series converges.

If $\frac{|x|}{R} > 1$, $\forall n, \sup_{k > n} \{|a_k x^k|^{1/k}\} \geq \frac{|x|}{R}$.

By the properties of the supremum with $\epsilon = \left(\frac{|x|}{R} - 1\right)/2 > 0$,

$$\forall n, \exists k, 1 \leq \frac{\frac{|x|}{R} + 1}{2} \leq y_n - \epsilon \leq |a_k x^k|^{1/k} \leq y_n$$

Therefore $\forall n, \exists k > n, |a_k x^k|^{1/k} \geq 1$. ■

Observation: Behavior at Endpoints

At the endpoints of $(-R, R)$, a series might

Converge Absolutely

e.g. $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$, $R = 1$, $\frac{1}{R} = \limsup_n \left(\frac{1}{n^2}\right)^{1/n} \xrightarrow{n \rightarrow \infty} 1$

Converge Conditionally

e.g. $\sum_{k=1}^{\infty} \frac{x^k}{k}$, $R = 1 \implies \frac{1}{R} = \limsup_n \left(\frac{1}{n}\right)^{1/n} = 1$
Converges conditionally at $x = -1$.

Diverge

e.g. $\sum_{k=0}^{\infty} x^k$, $R = 1$

Theorem: Power Series Differentiation

Let $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ converge on $(x_0 - R, x_0 + R)$.

Then $\forall k > 0, f \in C^k(x_0 - R, x_0 + R)$ and $f^{(k)}(x) = \sum_{n=k}^{\infty} a_n n(n-1) \cdots (n-k+1) (x - x_0)^{n-k}$, $\forall x \in (x_0 - R, x_0 + R)$

Exercise

Show that if $a_n \rightarrow a > 0$, then $\limsup a_n b_n = a \limsup b_n$.

Proof (by Induction)

Consider the series $S_n(x) = \sum_{n=1}^{\infty} a_n n(x - x_0)^{n-1} = \sum_{n=0}^{\infty} a_{n+1}(n+1)(x - x_0)^n$.
Then

$$(x - x_0) \frac{1}{R \text{ of series of derivatives}} = \limsup_{n \rightarrow \infty} (a_n n)^{1/n} \limsup_{n \rightarrow \infty} a_n^{1/n} n^{1/n} = \limsup_{n \rightarrow \infty} a_n^{1/n} = \frac{1}{R}$$

This implies $\sum_{k=0}^{\infty} \frac{d}{dx} (a_k(x - x_0)^k)$ converges uniformly on $[x_0 - a, x_0 + a]$, $\forall a \in (0, R)$.

By the Theorem on Differentiability of Uniform Limits, $f'(x)$ exists and $\forall x \in (x_0 - R, x_0 + R)$

$$f'(x) = \sum_{n=1}^{\infty} a_n n(x - x_0)^{n-1}$$

Repeat to get higher derivatives.

Integration

It is similarly possible to integrate term by term.

Famous Power Series

- $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, $|x| < 1$
- PSE of $\frac{1}{x}$ centered at $x_0 > 0$

IMAGE HERE - GRAPH

$$\frac{1}{x} = \frac{1}{x - x_0 + x_0} = \frac{1}{x_0} \cdot \frac{1}{1 + \frac{x-x_0}{x_0}} = \frac{1}{x_0} \sum_{k=0}^{\infty} \left(-\frac{x-x_0}{x_0} \right)^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{x_0^{k+1}} (x - x_0)^k \text{ if } |x - x_0| < |x_0|, x \in (0, 2x_0)$$

- $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
- $\exp(0) = 1$
- $\exp'(x) = \sum_{k=1}^{\infty} \frac{kx^{k-1}}{k!} = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} = \exp(x)$

Law of Exponents

$\exp(a) \exp(b) = \exp(a + b)$, $\forall a, b \in \mathbb{R}$

Proof

Special case of the “Cauchy product of convergent series.”

If $\sum_{n \geq 0} a_n$ converges absolutely to A and $\sum_{n \geq 0} b_n$ converges to B , then $\sum_{n \geq 0} c_n$ converges to AB , where

$$c_n = \sum_{k=0}^n a_k b_{n-k} = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$$

- Heuristics

$$\left(\sum_{p=0}^{\infty} a_p x^p \right) \left(\sum_{l=0}^{\infty} b_l x^l \right) = \sum_{p,l \in \mathbb{N}_0^2} a_p b_l x^{p+l}$$

IMAGE HERE - CIRCLES FROM L TO P

$$\{(p, l) : p + l = n, p, l \in \mathbb{N}_0\} = \{(0, n), (1, n-1), \dots, (n, 0)\}$$

Proof Continued

$\text{Aexp}(a) = \sum_{k=0}^{\infty} \frac{a^k}{k!}$ and $\exp(b) = \sum_{l=0}^{\infty} \frac{b^l}{l!}$, thus $\exp(a) \exp(b) = \sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} \frac{(a+b)^n}{n!} = \exp(a+b)$ since

$$c_n = \frac{1}{n!} \sum_{k=0}^n \frac{a^k}{k!} \cdot \frac{b^{n-k}}{(n-k)!} \text{ and } n! = \frac{1}{n!} (a+b)^n$$

Power Series Expansion of Exponential

Centered at x_0 , we have

$$\exp(x) = \exp(x - x_0) \exp(x_0) = \exp(x_0) \sum_{k=0}^{\infty} \frac{(x - x_0)^k}{k!}$$

Observation:

\exp is the only $C^1(\mathbb{R})$ solution of $\begin{cases} \exp'(x) = \exp(x) \\ \exp(0) = 1 \end{cases}$

- Proof If f solves the above, then for some constant c

$$\frac{d}{dx} (f(x) \exp(-x)) = f'(x) \exp(-x) - f(x) \exp(-x) = 0 \xRightarrow{\text{MVT}} f(x) \exp(-x) = c = f(0) \exp(-0) = 1$$

this implies

$$f(x) = \exp(x) f(x) \exp(-x) = \exp(x)$$

Exponential Features

$$\exp(x) > 0, \forall x \in \mathbb{R} \implies \begin{cases} \text{if } x \geq 0, \exp(x) \geq 1 > 0 \\ \text{if } x < 0, \exp(x) = \frac{1}{\exp(-x)} > 0 \end{cases}$$

Theorem: Exponential and e

$\exp(x) = (\exp(1))^x \forall x \in \mathbb{R}$ and $e = \exp(1)$

Proof

Using law of exponents for

$$x \in \mathbb{N} : \quad \exp(n) = \exp(1 + (n-1)) = e \cdot \exp(n-1) = \cdots = e^n \exp(0)$$

$$x = \frac{1}{q}, q \in \mathbb{N} : \quad \left(\exp\left(\frac{1}{q}\right) \right)^q = \exp\left(\frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}\right) = \exp(1) = e$$
$$\therefore \exp\left(\frac{1}{q}\right) = e^{1/q}$$

$$x = \frac{p}{q}, p, q \in \mathbb{N} : \quad \exp\left(\frac{p}{q}\right) = \exp\left(\overbrace{\frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}}^p\right) = \left(e^{1/q}\right)^p = e^{p/q}$$

$$x \in -\mathbb{N}, \mathbb{Q} < 0 : \quad \text{left as an exercise}$$

Therefore, the functions $x \mapsto \begin{cases} \exp(x) \\ e^x \end{cases}$ are continuous on \mathbb{R} and agree on \mathbb{Q} . This implies that they must be equal everywhere.

October 23, 2023

Today

Exp and log.

Real-analytic functions. (Newest bit of information.)

Trig functions.

Wednesday, October 25, 2023

Analytic vs C^∞

Approximation by polynomials.

Next Week

Fourier series.

Exponential and Log

Covered Last Lecture

Law of Exponents

$$\exp(x) = e^x \text{ and } e = \exp(1) = \sum_{k=0}^{\infty} \frac{1}{k!}$$

Error Estimate

$e = \lim_{n \rightarrow \infty} S_n$ where $S_n = \sum_{k=0}^{\infty} \frac{1}{k!}$ (increases).

$$e - S_n = \sum_{k=n+1}^{\infty} \frac{1}{k!}$$

For $k = n + 1 + p$, $p \geq 0$, $e - S_n = \sum_{p=0}^{\infty} \frac{1}{(n+1+p)!}$.

Then,

$$\begin{aligned} \frac{1}{(n+1+p)!} &= \frac{1}{(n+1)!} \cdot \frac{1}{\underbrace{(n+2)(n+3)\cdots(n+p+1)}_{p \text{ factors}}} \\ &\leq \frac{1}{(n+1)!} \cdot \frac{1}{(n+1)^p} \end{aligned}$$

and

$$\begin{aligned} e - S_n &= \sum_{k=n+1}^{\infty} \frac{1}{k!} \\ &= \sum_{p=0}^{\infty} \frac{1}{(n+1+p)!} \\ &\leq \frac{1}{(n+1)!} \cdot \sum_{p=0}^{\infty} \left(\frac{1}{n+1} \right)^p \\ &= \frac{1}{(n+1)!} \cdot \frac{1}{1 - \frac{1}{n+1}} \\ &= \frac{1}{(n+1)!} \cdot \frac{n+1}{n} \end{aligned}$$

Therefore,

$$0 \leq e - S_n \leq \frac{1}{n!} \cdot \frac{1}{n}$$

Theorem: e is Irrational

Proof

Suppose $e = \frac{p}{q}$, $q > 2$, and p and q coprime. Consider

$$\begin{aligned} 0 &< e - S_q \leq \frac{1}{q!} \cdot \frac{1}{q} \\ 0 &< q!(e - S_q) \leq \frac{1}{q} \\ 0 &< q! \left(\frac{p}{q} - \sum_{k=0}^q \frac{1}{k!} \right) \leq \frac{1}{q} < \frac{1}{2} \end{aligned}$$

where $q! \left(\frac{p}{q} - \sum_{k=0}^q \frac{1}{k!} \right) \in \mathbb{N}$.

This is a contradiction. Thus, e must be irrational. ■

Exponential Decay

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$
$$\lim_{x \rightarrow +\infty} x^k e^{-x} = 0, \forall k \in \mathbb{N}$$

For $x > 0$, $\exp(x) \geq \frac{x^{k+1}}{(k+1)!}$ if and only if $x^k \exp(-x) \leq \frac{(k+1)!}{x} \xrightarrow{x \rightarrow +\infty} 0$.

Exponential Strictly Positive Over Reals

$$\exp(x) > 0, \forall x \in \mathbb{R}$$

$x > 0$ is obvious.

$$x \leq 0, \exp(x) = \frac{1}{\exp(-x)} > 0$$

$$\lim_{x \rightarrow -\infty} \exp(x) = \lim_{x \rightarrow -\infty} \frac{1}{\exp(-x)} = 0.$$

Proposition: Exponential is a Bijection

$\exp : \mathbb{R} \rightarrow (0, \infty)$ is a C^∞ ($\exp' = \exp$) bijection (diffeomorphism in the sense that $\exp'(x) > 0, \forall x \in \mathbb{R}$).

By Inverse Function Theorem then, define $\log : (0, \infty) \rightarrow \mathbb{R}$ such that $\exp(\log(x)) = x$.

$$\text{By MATH 105A, } \frac{d}{dx}(\log(x)) = \frac{d}{dx}(\exp^{-1}(x)) = \frac{1}{\exp'(\exp^{-1}(x))} = \frac{1}{\exp(\log(x))} = \frac{1}{x}.$$

$\log(1) = 0$ (since $\exp(0) = 1$) which implies, by the Fundamental Theorem of Calculus, that $\log(x) - \log(1) = \int_1^x \frac{dt}{t}$.

Properties (left as an exercise)

- $\log(xy) = \log(x) + \log(y), x, y > 0$
- Power Series Expansion: $\log(1-x) = -\sum_{k=0}^{\infty} \frac{x^k}{k}, x \text{ near } 0, \text{ radius of convergence: } 1.$
- $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = \exp(x)$

Definition: Real-Analytic Functions

A function $f : (a, b) \rightarrow \mathbb{R}$ is real-analytic on (a, b) if $\forall x_0 \in (a, b), \exists r > 0$ and a power series $\sum_{n \geq 0} (x - x_0)^n$ converging to f on $(x_0 - r, x_0 + r)$.

When such a power series exists, $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$, then

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

The radius of convergence is related by $\frac{1}{R} = \limsup_n |a_n|^{1/n}$ which provides a constraint on rate of divergence.

Example 1: Polynomial

For every polynomial, $p : \mathbb{R} \rightarrow \mathbb{R}$, and $\forall x_0 \in \mathbb{R}$,

$$p(x) = \sum_{k=0}^{\infty} \frac{p^{(k)}(x_0)}{k!} (x - x_0)^k, \forall x \in \mathbb{R}$$

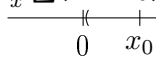
Example 2: Exponential

$$\exp(x) = \exp(x - x_0 + x_0) = \sum_{k=0}^{\infty} \frac{e^{x_0}}{k!} (x - x_0)^k$$

and the radius of convergence, $R = \infty$.

Example 3: $1/x$

$$\frac{1}{x} \text{ analytic on } (0, \infty)$$

$$\frac{1}{x} \sum_{k=0}^{\infty} (x - x_0)^k \text{ and } R = |x_0|.$$


Remark: Analyticity Implies Smoothness

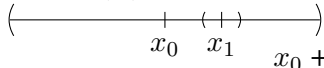
f analytic on $(a, b) \implies f$ smooth (C^∞) on (a, b)

The converse is not true. (Example Wednesday)

Proposition:

Suppose $\sum_{n \geq 0} a_n (x - x_0)^n$ converges to $f(x)$ on $(x_0 - R, x_0 + R)$.

Then $f(x)$ is analytic on $(x_0 - R, x_0 + R)$.



That is to say, $\forall x_1 \in (x_0 - R, x_0 + R)$, there exists some power series expansion for f , centered at x_1 , with positive radius of convergence.

Proof

Let $x_0 = 0$ for simplicity and $x_1 \in (-R, R)$.

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n (x - x_1 + x_1)^n = \sum_{n=0}^{\infty} a_n \sum_{k=0}^n \binom{n}{k} (x - x_1)^k x_1^{n-k}$$

Assuming that rearrangement is possible, this is

$$\sum_{n, k, n \geq 0} a_n \binom{n}{k} (x - x_1)^k x_1^{n-k} = \sum_{k=0}^{\infty} \underbrace{\left(\sum_{n=k}^{\infty} a_n \binom{n}{k} x_1^{n-k} \right)}_{b_k} (x - x_1)^k$$

Need to prove two things:

1. b_k is well-defined
2. Interchange of sums valid.

- Proof of 1

For k fixed, $\binom{n}{k}$ is a degree k (degree k) polynomial in n .

Letting

$$b_k = \sum_{p=0}^{\infty} a_{p+k} \binom{p+k}{k} x_1^p$$

where $p = n - k$, we have

$$\limsup_{p \rightarrow \infty} \left(|a_{p+k}| \binom{p+k}{k} \right)^{1/p} = \limsup_{p \rightarrow \infty} |a_p|^{1/p}$$

since $x_1 \in (-R, R)$, $b_k < \infty$, $\forall k$.

- Proof of 2

The proof requires invoking Fubini's Theorem to allow rearrangement.

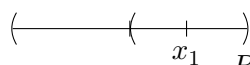
Need to check that

$$\sum_{n,k, n \geq k} |a_n| \binom{n}{k} |(x - x_1)^k x_1^{n-k}|$$

converges.

Consider

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |a_n| \binom{n}{k} |x - x_1|^k |x_1|^{n-k} = \sum_{n=0}^{\infty} |a_n| (|x - x_1| + |x_1|)^n$$



Make sure that $|x - x_1| = r < R - |x_1|$, then

$$\sum_{n=0}^{\infty} |a_n| r^n$$

where $r < R$ which, by absolute convergence of the original power series, is finite. ■

Remark: Analytic Continuation

The process of recentering a power series is also called “analytic continuation.”

The radius of convergence of the new series might actually be larger and allow the original function.

Example

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

IMAGE HERE - Decaying curve.

Facts: Analytic Functions

- If f, g are analytic on (a, b) , then so is $f \cdot g$.
- If f, g are analytic and g does not vanish on (a, b) , then $\frac{f}{g}$ is analytic.
- If f is analytic on $(x_0 - R, x_0 + R)$ and g is analytic on $(f(x_0) - \delta, f(x_0) + \delta)$, then $g \circ f$ is analytic on a neighborhood of x_0 . (Proof in ; page number in lecture notes).

Remark: No Analytic Bump Functions

IMAGE HERE - BUMP FUNCTION $-|x|$

Trig Functions

IMAGE HERE – UNIT CIRCLE

We want $(\cos(\theta), \sin(\theta))$ to be the point on the unit circle making an arclength θ from $(1, 0)$.

For x in the right-half plane, $\cos(\theta) \geq 0$.

For x in top right quadrant,

$$\theta = \int_0^{\sin(\theta)} \sqrt{1 + (f'(y))^2} dy$$

Then, $y \mapsto (\underbrace{\sqrt{1 - y^2}}_{f(y)}, y)$, $y \in (-1, 1)$. It follows that

$$\theta = \lim_{y \rightarrow 0}^{\sin(\theta)} \frac{dy}{\sqrt{1 - y^2}} \xrightarrow{\text{FTC}} \arcsin'(x) = \frac{1}{\sqrt{1 - x^2}} \in C^\infty((-1, 1))$$

and

$$\arcsin(x) = \lim_{y \rightarrow 0}^x \frac{dy}{\sqrt{1 - y^2}}$$

Therefore, \arcsin is a diffeomorphism from $(-1, 1) \rightarrow (\lim_{x \rightarrow -1} \arcsin(x), \lim_{x \rightarrow 1} \arcsin(x))$.

Since $\frac{1}{\sqrt{1 - x^2}}$ is integrable near ± 1 , these limits are finite.

Definition: Pi

$$\pi = 2 \lim_{x \rightarrow 1} \arcsin(x)$$

Inverse Function Theorem

$\sin : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow (-1, 1)$ exists as a C^1 inverse of \arcsin .

On $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, define $\cos(\theta) = +\sqrt{1 - \sin^2(\theta)}$. Then

$$\sin'(\theta) = \frac{1}{\arcsin'(\sin(\theta))} = \sqrt{1 - \sin^2(\theta)} = \cos(\theta).$$

Similarly, $\cos'(\theta) = -\sin(\theta) \rightsquigarrow \sin, \cos$ are C^∞ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Extension to the Reals

By graphical or geometric arguments, for $\theta \in (0, \frac{\pi}{2})$,

$$\begin{aligned}\cos(\theta) &= -\sin\left(\theta - \frac{\pi}{2}\right) \\ \sin(\theta) &= -\cos\left(\theta - \frac{\pi}{2}\right)\end{aligned}$$

This helps extend to \mathbb{R} , with 2π -periodicity such that

$$\begin{cases} \cos' &= -\sin \\ \sin' &= \cos \\ \cos(0) &= 1 \\ \sin(0) &= 0 \end{cases}$$

Therefore, you get all derivatives at $x = 0$ and the corresponding Taylor expansion looks like

$$C(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \qquad S(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

We find that $R = \infty$ for both, and

$$C(0) = 1, \qquad S(0) = 0, \qquad C'(x) = -S(x), \qquad S'(x) = C(x).$$

Take

$$\epsilon(x) = (C(x) - \cos(x))^2 + (S(x) - \sin(x))^2$$

with $\epsilon(0) = 0$. Then, finally,

$$\epsilon'(x) = 0 \implies \epsilon = \text{some constant} = 0.$$

October 25, 2023

Today

Analytic vs C^∞

Approximation by Polynomials

Definition: Real Analytic

f is real analytic on (a, b) if $\forall x_0 \in (a, b), \exists \delta > 0, \{a_n\}_n$ such that $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \forall x \in (x_0 - \delta, x_0 + \delta)$.

Proposition: Analyticity Implies Smoothness

Analytic on $(a, b) \implies C^\infty$ smooth on (a, b) .

$$\sum_{n=0}^{\infty} (x - x_0)^n \rightsquigarrow a_n - \frac{f^n(x_0)}{n!}$$

Note: $C^w(a, b) \not\subseteq C^\infty(a, b)$

The converse is not true.

Example

Let $x \in \mathbb{R}$ and $f(x) = \begin{cases} 0 & x < 0 \\ \exp\left(\frac{-1}{x^2}\right) & x > 0 \end{cases}$

IMAGE HERE - FUNCTION

$x \neq 0, f \in C^\infty(\mathbb{R} \setminus \{0\})$.

- What about at $x = 0$?

$$\lim_{x \rightarrow 0; x < 0} f(x) = 0 = \lim_{x \rightarrow 0; x > 0} e^{-\frac{1}{x^2}}$$

So we can define $f(0) = 0$, the resulting function is continuous on \mathbb{R} .

- What about higher derivatives?

Claim: $\forall k > 0, \lim_{x \rightarrow 0; x > 0} \frac{d^k}{dx^k} \left(e^{-\frac{1}{x^2}} \right) = 0$

- Proof (Sketch)

$$\frac{d}{dx} \left(e^{-x^2} \right) = 2x^{-3} e^{-x^{-2}}$$

$$\lim_{x \rightarrow 0} \frac{e^{-\frac{1}{x^2}}}{x^3} \stackrel{y=\frac{1}{x^2}}{=} \lim_{y \rightarrow \infty} y^3 e^{-y} = 0$$

Claim by induction:

$$\frac{d^k}{dx^k} \left(e^{-\frac{1}{x^2}} \right) = p_k(1/x) e^{-\frac{1}{x^2}}$$

for some polynomial p_k .

If the claim is true, then

$$\lim_{x \rightarrow 0} p_k \left(\frac{1}{x} \right) e^{-\frac{1}{x^2}} = \lim_{y \rightarrow +\infty} p_k(y) e^{-y^2} = 0 \quad \blacksquare$$

Then we can extend $f^{(k)}$ as a continuous function on \mathbb{R} such that $f^{(k)}(0) = 0$.

- Claim

$f(x)$ is not analytic on any neighborhood of $x_0 = 0$.

If it were, it would equal $\sum_{n=0}^{\infty} a_n x^n$ on $(-\delta, \delta)$ for some a_k s. But,

$$a_k = \frac{f^{(k)}(0)}{k!} = 0 \quad \text{then} \quad \sum_{n=0}^{\infty} a_n x^n = 0, \forall x \in (-\delta, \delta)$$

which is impossible, since $f(x) \neq 0$ whenever $x > 0$. \blacksquare

Remark: Contraposition Can Disprove Analyticity

The existence of a non-zero radius of convergence for $\sum a_k(x - x_0)^k$ means

$$\frac{1}{R} = \limsup_n |a_n|^{1/n} = \left(\frac{f^{(n)}(x_0)}{n!} \right)^{1/n} < \infty$$

and

$$\left(\frac{f^{(n)}(x_0)}{n!} \right)^{1/n} \rightsquigarrow f^{(n)}(x_0) \leq n! \left(\frac{c}{R} \right)^n$$

for some constant c .

Remark: Analyticity is Not Guaranteed

The conditions

$$\begin{cases} h \in C^\infty(R) \\ \limsup_n \left(\frac{h^{(n)}(0)}{n!} \right)^{1/n} < \infty \end{cases}$$

are not sufficient to claim h is analytic on any neighborhood of 0.

Indeed, if h is analytic then $h(x) + f(x)$ will not be for otherwise

$$f(x) = -(h(x) + f(x)) - h(x)$$

would also be analytic, which it isn't.

Definition: Exponential Blip Function

Let $g(x) = \frac{f(x+1)f(1-x)}{f(1)^2}$, where f is the “exponential glue” function.

IMAGE HERE - FUNCTION

Smooth on \mathbb{R} ; $g(x) \geq 0$.

TODO Theorem: Borel

TODO - Name for theorem?

Given any sequence $\{a_n\}_n$ of reals and any $\begin{cases} x_0 \in \mathbb{R} \\ \lambda > 0 \end{cases}$, $\exists f \in C^\infty(\mathbb{R})$ such that

$$\begin{cases} f^{(k)}(x_0) = a_k & \forall k \\ f(x) = 0 & \text{if } |x - x_0| > \lambda \end{cases}$$

IMAGE HERE - BUMPY MOUNTAIN CLOSE TO X0

Proof

Reductions: $x_0 = 0$ and $\lambda = 1$.

Ansatz: $f(x) = \sum_{k=0}^{\infty} b_k x^k g\left(\frac{x}{\lambda_k}\right)$ where b_k s and λ_k s need to be tuned.

IMAGE HERE - G(X) and G(X/LAMBDA K)

$$g(x) = 0 \iff |x| \geq 1 \text{ and } g\left(\frac{x}{\lambda_k}\right) = 0 \iff \left|\frac{x}{\lambda_k}\right| \geq 1 \iff |x| \geq \lambda_k$$

Observations: if $\lambda_k \xrightarrow[k \rightarrow \infty]{} 0$, then $\forall x \neq 0$ the series is actually finite!

Since $g\left(\frac{x}{\lambda_k}\right) = 0$ once $\lambda_k < |x|$.

Therefore, convergent and C^∞ on $\mathbb{R} \setminus \{0\}$.

Constraints:

$$\begin{aligned} a_0 &= f(0) = b_0 \\ a_1 &= f'(0) = \frac{d}{dx} \left(b_0 g\left(\frac{x}{\lambda_0}\right) \right) \Big|_{x=0} + b_1 \end{aligned}$$

Generally,

$$a_n = \sum_{k=0}^{n-1} \frac{d^n}{dx^n} \left(b_k x^k g\left(\frac{x}{\lambda_k}\right) \right) \Big|_{x=0} + n! b_n$$

If λ_n are chosen, these constraints uniquely determine the b_n s.

How to Choose Lambdas?

Want to enforce

$$\max_{0 \leq k \leq n-1} \sup_{x \in \mathbb{R}} \left| \frac{d^k}{dx^k} \left(b_n x^n g\left(\frac{x}{\lambda_n}\right) \right) \right| \leq 2^{-n}$$

- Example

Determine λ_2 :

$$\begin{aligned} k=0 : \left| b_n x^n g\left(\frac{x}{\lambda_n}\right) \right| &\leq |b_n| \lambda_n^n 2^{-n} \\ k=1 : \left| b_n \left(n x^{n-1} g\left(\frac{x}{\lambda_n}\right) \right) + b_n x^n \frac{1}{\lambda_n} g'\left(\frac{x}{\lambda_n}\right) \right| &\leq |b_n| \lambda_n^{n-1} (n + \|g'\|_\infty) \leq 2^{-n} \end{aligned}$$

In general,

$$a \lambda_n^p < 2^{-n}$$

for $p > 0$.

So we construct b_0 , then λ_0 , then b_1 , then λ_1, \dots

Claim: Produces Uniform Convergence

When

$$\max_{0 \leq k \leq n-1} \sup_{x \in \mathbb{R}} \left| \frac{d^k}{dx^k} \left(b_n x^n g\left(\frac{x}{\lambda_n}\right) \right) \right| \leq 2^{-n}$$

is satisfied, $\forall k \in \mathbb{N}$

$$\sum_{n=0}^{\infty} \frac{d^k}{dx^k} \left(b_n x^n g\left(\frac{x}{\lambda_n}\right) \right)$$

satisfies the Weierstrass M-Test. Therefore it is uniformly convergent. Because

$$\sum_{n=0}^{\infty} \left| \frac{d^k}{dx^k} \left(b_n x^n g\left(\frac{x}{\lambda_n}\right) \right) \right| \leq \underbrace{\sum_{n=0}^k \left| \frac{d^k}{dx^k} \left(b_n x^n g\left(\frac{x}{\lambda_n}\right) \right) \right|}_{\text{finite sum, uniformly bounded}} + \sum_{n=k+1}^{\infty} 2^{-n}$$

Approximation by Polynomials

Goal (Weierstrass Approximation Theorem):

If $f : [a, b] \rightarrow \mathbb{R}$ is continuous on the compact set $[a, b]$, then there exists a sequence of polynomials p_n such that $\lim_{n \rightarrow \infty} \sup_{x \in [a, b]} |f(x) - p_n(x)| = 0$.

That is, polynomials are dense in $(C([a, b]), \|\cdot\|_{\infty})$, where $\|f\|_{\infty} := \sup_{x \in [a, b]} |f(x)|$.

How to do this?

Lagrange Interpolation

Give $f \in C([a, b])$.

Idea: subdivide $[a, b]$ with $a = x_0 < x_1 < \dots < x_n < b$ where $x_k = x_0 + k \left(\frac{b-a}{n} \right)$.

IMAGE HERE - UNIFORM SUBDIVISION

Let $p_n(x) = \sum_{k=0}^n f(x_k) \prod_{j \neq k} \frac{x - x_j}{x_k - x_j}$.

Problem: the Runge phenomenon.

IMAGE HERE - SMOOTHEST FUNCTION I CAN THINK OF (use the bump again) $1/(1+25x^2)$

Definition: Convolution

Take $f, g : \mathbb{R} \rightarrow \mathbb{R}$, define

$$h(x) = f * g(x) = \int_{\mathbb{R}} f(t)g(x-t) dt \underset{y=x-t}{=} \int_{\mathbb{R}} f(x-y)g(y) dy = g * f(x)$$

Take $f, g \in C(\mathbb{R})$ with compact support ($C_C(\mathbb{R})$). That is, they vanish outside a compact set.

IMAGE HERE - F AND G CONVOLVED

Definition: Approximation of Identity

An approximation of the identity is a sequence $\{g_n\}_n$, all piecewise continuous, defined on \mathbb{R} such that

$$\begin{cases} g_n(x) \geq 0 & \forall x \\ \int_{\mathbb{R}} g_n(x) dx = 1 \\ \forall \delta > 0, & \lim_{n \rightarrow \infty} \int_{|x| > \delta} g_n(x) dx = 0 \end{cases}$$

IMAGE HERE - CONVOLUTION ACCUMULATING BETWEEN -DELTA AND DELTA

Example

Let $g_n(x) = \frac{n \cdot g(nx)}{\int_{\mathbb{R}} g(x) dx}$.

Lemma:

If $\{g_n\}_n$ is an approximation of identity, then $\forall f \in C_C(\mathbb{R})$

$$g_n * f \rightrightarrows f$$

on \mathbb{R} .

October 30, 2023

Today

Approximation by polynomials.
Fourier Series.

Recall: Convolution

$f, g \in C_C(\mathbb{R})$, $f * g(x) = \int_{\mathbb{R}} f(x-y)g(y) dy$.

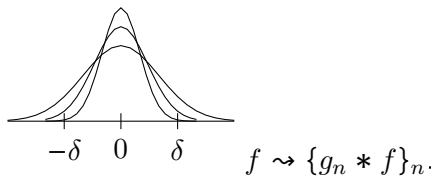
Recall: Approximation of Identity

$\{g_n\}_n$ where $g_n : \mathbb{R} \rightarrow \mathbb{R}$ is piecewise continuous (this is overkill but sufficient).

1. $\int g_n dx = 1$.

2. $g_n(x) \geq 0$.

3. $\forall \delta > 0, \lim_{n \rightarrow \infty} \int_{|x| > \delta} g_n(x) dx = 0$.



Example

Take any $g(x) \geq 0$ (piecewise continuous) with $\int_{\mathbb{R}} g(x) dx = 1$.

Define $g_n(x) = n \cdot g(nx)$.

Claim: this defined an approximation of identity.

Lemma: Convolution of Approximation of Identity Converges Uniformly

Suppose $\{g_n\}_n$ is an approximation of identity.

Then, for any $f \in C_C(\mathbb{R})$,

$$g_n * f \quad \text{converges uniformly to } f \text{ on } \mathbb{R}$$

That is to say, $\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} |g_n * f(x) - f(x)| = 0$.

Proof

Since $\int_{\mathbb{R}} g_n(y) dy = 1$,

$$\begin{aligned} g_n * f(x) - f(x) &= \int_{\mathbb{R}} g_n(y) f(x-y) dy - f(x) \cdot \int_{\mathbb{R}} g_n(y) dy \\ &= \int_{\mathbb{R}} g_n(y) (f(x-y) - f(x)) dy \\ &= \int_{|y| \geq \delta} g_n(y) \overbrace{(f(x-y) - f(x))}^{\leq 2M} dy + \int_{|y| < \delta} g_n(y) \overbrace{(f(x-y) - f(x))}^{\leq \epsilon} dy \end{aligned}$$

By assumption, $f \in C_C(\mathbb{R})$ so f is bounded by M on \mathbb{R} .

f is continuous on $\text{supp}(f)$, which is compact, so f is uniformly continuous on \mathbb{R} .

Let $\epsilon > 0$ be given.

By uniform continuity, $\exists \delta > 0, \forall x, y \in \mathbb{R}, |x-y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{2}$.

By the Approximation of Identity property, $\exists N, \forall n \geq N, \int_{|y| \geq \delta} g_n(y) dy < \frac{\epsilon}{4M}$.

For $n \geq N$,

$$\begin{aligned} |g_n * f(x) - f(x)| &= \left| \int_{\mathbb{R}} g_n(y) (f(x-y) - f(x)) dy \right| \\ &\leq \int_{|y| \geq \delta} g_n(y) \overbrace{|f(x-y) - f(x)|}^{\leq 2M} dy + \int_{|y| < \delta} g_n(y) \overbrace{|f(x-y) - f(x)|}^{\leq \frac{\epsilon}{2} \text{ since } |x-y-x|=|y| < \delta} dy \\ &\leq 2M \frac{\epsilon}{4M} + \frac{\epsilon}{2} \overbrace{\int_{|y| < \delta} g_n(y) dy}^{\leq 1} \\ &\leq \epsilon, \quad \forall x \in \mathbb{R} \quad \blacksquare \end{aligned}$$

Recall: Riemann Integral Properties

If f is Riemann integrable, then

$$\left| \int f dx \right| \leq \int |f| dx$$

$$\left| \sum_{n=1}^{\infty} S_n \right| \leq \sum_{n=1}^{\infty} |S_n|$$

$$\left| \int f^+ dx - \int f^- dx \right| \leq \int f^+ dx + \int f^- dx = \int (f^+ + f^-) dx$$

Theorem: Weierstrass Approximation Theorem

If $[a, b]$ is compact, then $\forall f \in C([a, b])$, there exists a sequence of polynomials $p_n(x)$ converging uniformly to f .

Step 1

Extend f into $F \in C_c(\mathbb{R})$.

IMAGE HERE - EXTEND FUNCTION

$$F(x) = \begin{cases} 0 & \text{on } (-\infty, a-1] \cup [b+1, \infty) \\ f(x) & \text{on } [a, b] \\ f(a)(x - (a-1)) & \text{on } [a-1, a] \\ f(b)(b+1-x) & \text{on } [b, b+1] \end{cases}$$

Step 2

Note: $\forall \{g_n\}_n$ Approximation of Identity, $g_n * f \rightrightarrows F(x)$ on \mathbb{R} (by previous lemma),

and $\sup_{x \in [a, b]} |g_n * F(x) - f(x)| \leq \sup_{x \in \mathbb{R}} |g_n * F(x) - F(x)|$.

Trick: Construct g_n such that $g_n * F$ is a polynomial on $[a, b]$.

Answer:

$$g_n(x) = \begin{cases} a_n \left(1 - \frac{x^2}{(b-a+1)^2}\right)^n & \text{if } x \in [-(b-a+1), b-a+1] \\ 0 & \text{otherwise} \end{cases}$$

where a_n is chosen such that $\int_{\mathbb{R}} g_n(x) dx = 1$.

IMAGE HERE - NARROWING GAUSSIAN WITH PEAK AT (0,1) BETWEEN -(b-a+1) and b-a+1

If $x \in [a, b]$ and $y \in [a-1, b+1]$, then

$$-b-1 \leq -y \leq -a+1 \implies -(b-a+1) \leq x-y \leq b-a+1$$

Then

$$\begin{aligned} g_n * F(x) &= \int_{a-1}^{b+1} F(y) \underbrace{g_n(x-y)}_{a_n \left(1 - \frac{(x-y)^2}{(b-a+1)^2}\right)^n = \sum_{p=0}^{2n} x^p a_{p,n(y)}} dy \\ &= \sum_{p=0}^{2n} x^p \int_{a-1}^{b+1} F(y) a_{p,n(y)} dy \quad \blacksquare \end{aligned}$$

Background: Fourier Series

Historical Perspective

In Strichartz.

Associated with solving the wave equation on $[0, L]_x \times [0, T]_t$ (Bernoulli) and the heat equation (Fourier).

Wave Equation

On $[0, L]_x \times [0, T]_t$, $u(x, t)$ displacement field.

IMAGE HERE - WAVE FROM 0 to L PEAK OF FIRST OSCILLATION AT U(X,T)

$$\frac{\partial^2 u}{\partial t^2}(x, t) = c^2 \frac{\partial^2 u}{\partial x^2}(x, t)$$

where c is a fixed coefficient.

Plus Initial Conditions and Boundary Conditions

$$\text{Initial Condition : } u|_{t=0}(x) = f(x)$$

$$\frac{\partial u}{\partial t}|_{t=0}(x) = 0$$

$$\text{Boundary Conditions : } u(0, t) = u(L, t) = 0$$

Observation: if $f(x) = \sin\left(\frac{k\pi x}{L}\right)$,

IMAGE HERE - THREE SINUSOIDAL WAVES OVERLAPPING

Ansatz: $u(x, t) = \sin\left(\frac{k\pi x}{L}\right)g(t)$.

Plug into the PDE:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \sin\left(\frac{k\pi x}{L}\right)g''(t) \\ c^2 \frac{\partial^2 u}{\partial x^2} &= -\frac{k^2 \pi^2}{L^2} c^2 \sin\left(\frac{k\pi x}{L}\right)g(t) \end{aligned}$$

Setting

$$\sin\left(\frac{k\pi x}{L}\right)g''(t) = -\frac{k^2 \pi^2}{L^2} c^2 \sin\left(\frac{k\pi x}{L}\right)g(t) \xrightarrow[g]{\text{ode for}} g'' = -\frac{k^2 \pi^2}{L^2} c^2 g$$

Which gives a general solution

$$g(t) = A \cos\left(\frac{k\pi ct}{L}\right) + B \sin\left(\frac{k\pi ct}{L}\right).$$

Initial conditions imply that $g(0) = 1$ and $g'(0) = 0$ which gives

$$g(t) = \cos\left(\frac{k\pi ct}{L}\right).$$

Thus

$$u(x, t) = \sin\left(\frac{k\pi x}{L}\right) \cos\left(\frac{k\pi ct}{L}\right)$$

Solves the PDE!

Wave Equation Superposition

Consider instead

$$f(x) = \sum_{k=0}^n \sin\left(\frac{k\pi x}{L}\right) a_k$$

Then

$$u(x, t) = \sum_{k=0}^n a_k \sin\left(\frac{k\pi x}{L}\right) \cos\left(c \frac{k\pi x}{L}\right)$$

Next Question:

What if f is more general?

\implies existence of Fourier series?

In what sense do they converge?

Definition: Fourier Series

Context: $f : [-\pi, \pi) \rightarrow \mathbb{R}$ Riemann-Integrable or

$f : \mathbb{R} \rightarrow \mathbb{R}$ 2π -periodic. ($f(x + 2\pi) = f(x)$, $\forall x$).

The Fourier series of f :

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^n a_k[f] \cos(kx) + b_k[f] \sin(kx)$$

where $a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx$ and $b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx$.

Alternatively,

$$S_n(x) = \sum_{k=-n}^n c_k e^{ikx}$$

where $c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$.

As an exercise: relate c_k s to a_k s and b_k s and prove that these are equivalent.

Question:

In what sense does $S_n(x)$ converge to $f(x)$? That is

- For what topology?

– Uniform Convergence: $\sup_{x \in [-\pi, \pi)} |S_n(x) - f(x)| \xrightarrow{n \rightarrow \infty} 0$

– L^2 Convergence: $\int_{-\pi}^{\pi} |S_n(x) - f(x)|^2 dx \xrightarrow{n \rightarrow \infty} 0$

- What are the (smoothness) requirements on f ?

– Observation: if $f(x) = \sum_{k=-N}^N f_k e^{ikx}$ is a trigonometric polynomial, then, for $n \geq N$, $S_n(x) = f(x)$.

Lemma: The Kronecker Delta

Fix $N \in \mathbb{N}$

If $\sum_{k=-N}^N f_k e^{ikx} = \sum_{k=-N}^N c_k e^{ikx}$, then $f_k = c_k, \forall k$.

Note

$$\int_{-\pi}^{\pi} e^{ikx} e^{-imx} dx = \int_{-\pi}^{\pi} e^{i(k-m)x} dx = \begin{cases} 2\pi & \text{if } k = m \\ \left[\frac{1}{i(k-m)} e^{i(k-m)x} \right]_{-\pi}^{\pi} = 0 & \text{otherwise} \end{cases}$$

Why $-imx$?

$$\begin{aligned} \langle if, g \rangle &= i \langle f, g \rangle \\ \langle f, ig \rangle &= -i \langle f, g \rangle \end{aligned}$$

and

$$\int_{-\pi}^{\pi} f(x) \overline{g(x)} dx$$

November 1, 2023

Fourier Series

For f Riemann-integrable on $(-\pi, \pi)$, define

$$S_n(x) = \sum_{k=-n}^n c_k e^{ikx}$$

with

$$c_k := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$$

Then $f : [-\pi, \pi) \rightarrow \mathbb{R}$.

Recall

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} e^{-ilx} dx = \begin{cases} 0 & \text{if } k \neq l \\ 1 & \text{if } k = l \end{cases} = \delta_{kl} \text{ (the Kronecker delta)}$$

Definition: Norm

$|| \cdot || : E \rightarrow \mathbb{R}_{\geq 0}$ is a “norm” on E if

1. $||f|| = 0 \iff f \equiv 0$
2. $||\lambda f|| = |\lambda| \cdot ||f||, \forall \lambda \in \mathbb{R}, f \in E$
3. $||f + g|| \leq ||f|| + ||g||$

Definition: Normed Space

$(E, || \cdot ||)$ is a normed space.
 e.g. $(\mathbb{R}, | \cdot |)$ or $(\mathbb{Q}, | \cdot |)$

Definition: Complete Space

$(E, || \cdot ||)$ is complete if every cauchy sequence in E converges in E .

In what sense does a Fourier series converge?

Depends on regularity of f and the topology used.

Note

On $C([-\pi, \pi])$, can put 2 norms.

- $||f||_{\infty} = \sup_{x \in [-\pi, \pi]} |f(x)|$

$d(f, g) = ||f - g||_{\infty}$: “ f_n converges uniformly to f ” $\leftrightarrow \lim_{n \rightarrow \infty} ||f_n - f||_{\infty} = 0$.
 $(C([-\pi, \pi]), || \cdot ||_{\infty})$ is complete.

- $||f||_2 := \left(\int_{-\pi}^{\pi} f^2(x) dx \right)^{1/2}$

“ f_n converges to f in L^2 ” $\leftrightarrow \lim_{n \rightarrow \infty} ||f_n - f||_2 = 0$.
 $(C([-\pi, \pi]), || \cdot ||_2)$ is not complete.

Example

Take $f(x) = \begin{cases} 1 & \text{if } |x| \leq \pi/2 \\ 0 & \text{if } |x| > \pi/2 \end{cases}$

IMAGE HERE - BOX FUNCTION FROM $-\pi/2$ to $\pi/2$



$$c_k = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{-ikx} dx = \frac{1}{2\pi} \left[\frac{1}{-ik} e^{-ikx} \right]_{-\pi/2}^{\pi/2} = \frac{1}{2\pi} \frac{1}{-ik} \left[e^{-ik(\pi/2)} - e^{ik(\pi/2)} \right] = \frac{1}{k\pi} \sin(k(\pi/2))$$

So $c_k = 0$ and for $k = 2p + 1$: $c_{2p+1} = \frac{(-1)^p}{\pi(2p+1)}$.

IMAGE HERE - BOX FUNCTION WITH SINUSOIDALS APPROXIMATING

However, the approximation will over and undershoot at the boundaries. This is the “Gibbs Phenomenon”, and the discrepancy is roughly 12%.

For $k < 0$:

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{ikx} dx = \overline{c_{-k}} = c_k$$

In the end,

$$S_{2p+1}(x) = \sum_{l=1}^p \frac{(-1)^p}{\pi(2p+1)} \underbrace{\left(e^{i(2p+1)x} + e^{-i(2p+1)x} \right)}_{2 \cos((2p+1)x)}$$

Theorem: Uniform Convergence of Continuously Differentiable Continuous Functions

1. If f is C^2 , 2π -periodic, then $S_n \Rightarrow f$ on $[-\pi, \pi)$.

Moreover, $\|S_n - f\|_\infty \leq \frac{c}{n}$ for some $c > 0$.

1. If $f \in C^1$, 2π -periodic, same conclusion with $\|S_n - f\|_\infty \leq \frac{c}{\sqrt{n}}$ for some $c > 0$.

Proof of Part 1

Write

$$S_n(x) = \sum_{k=-n}^n c_k e^{ikx} = \sum_{k=-n}^n \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iky} dy e^{ikx} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \underbrace{\left[\sum_{k=-n}^n e^{ik(x-y)} \right]}_{D_n(x-y)} dy$$

Where $D_n(t) = \sum_{k=-n}^n e^{ikt}$ is the “Dirichlet kernel.”

That is S_n is a convolution of $f(y)$ with some kernel.

$$e^{it} \cdot D_n(t) = \sum_{k=-n}^n e^{i(k+1)t} = \sum_{l=k+1}^{n+1} e^{ilt} = D_n(t) + e^{i(n+1)t} - e^{-int}$$

Therefore

$$D_n(t) = \frac{e^{i(n+1)t} - e^{-int}}{e^{it} - 1} = \frac{e^{(it)/2} \left(e^{i(n+(1/2))t} - e^{-i(n+(1/2))t} \right)}{e^{(it)/2} \left(e^{(it)/2} - e^{-(it)/2} \right)} = \frac{\sin((n+1/2)t)}{\sin(t/2)}$$

IMAGE HERE - DN(T) OSCILLATING WITH MANY ZEROS THEN PEAKING TO $2N+1$ at $X=0$
 _____ So

$$\int_{-\pi}^{\pi} D_n(t) dt = 2\pi$$

Then

$$\begin{aligned} S_n(x) - f(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) D_n(x-y) dy - f(x) \\ &= \frac{1}{2\pi} \int_{z=x-y}^{\pi} f(x-z) D_n(z) dz - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) D_n(z) dz \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x-z) - f(x)) D_n(z) dz \end{aligned}$$

and

$$S_n(x) \cdot f(x) = \frac{1}{2\pi} \underbrace{\frac{(f(x-y) - f(x))}{\sin(y/2)}}_{\text{call } g_x(y) = \frac{f(x-y) - f(x)}{\sin(y/2)}} \sin((n + (1/2)y) dy$$

If $g_x(y)$ was differentiable (in fact C^1), then integrating by parts

$$S_n(x) - f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_x(y) \sin((n + (1/2)y) dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} g'_x(y) \frac{\cos((n + (1/2)y)}{n + (1/2)} dy$$

Then

$$|S_n(x) - f(x)| \leq \sup_{y \in [-\pi, \pi]} |g'_x(y)| \frac{1}{n + (1/2)}$$

- Claim

If $f \in C^2$, 2π -periodic, then $\sup_{x \in [-\pi, \pi]} |g'_x(y)| < \infty$. Then the first part of the theorem is proved.

– Proof of Claim

$f \in C^2 \implies g_x \in C^2$ away from $y = 0$. ($g''_x(y)$ = differentiation rules).

At $y = 0$, write

$$f(x-y) - f(x) = \int_x^{x-y} f'(t) dt$$

Changing variables such that $t = x + u(x - y - x) = x - uy$ for $u \in [0, 1]$ gives $dt = -y du$

$$= -y \int_0^1 f'(x - uy) du$$

Therefore

$$g_x(y) = \underbrace{\left(\frac{-y}{\sin(y/2)} \right)}_{\text{smooth near } y=0} \int_0^1 f'(x - uy) du$$

Calling the smooth piece $h(y)$,

$$g_x(y) = h(y) \int_0^1 f'(x - yu) du$$

is differentiable at 0 if and only if $\frac{d}{dy} \left(\int_0^1 f'(x - yu) du \right) = \int_0^1 f''(x - yu)(-u) du$ exists. ■

Proof of Part 2 (Sketch)

$$S_n(x) - f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_x(y) \sin(n + (1/2)y) dy$$

If f is only C^1 , then g is C^1 away from 0, so it is unclear near $y = 0$. So, for some δ to be chosen later

$$S_n(x) - f(x) = \underbrace{\frac{1}{2\pi} \int_{[-\delta, \delta]} g_x(y) \sin((n + (1/2))y) dy}_{\leq \frac{2\delta}{2\pi} (\|f'\|_\infty + \|f\|_\infty)} + \underbrace{\frac{1}{2\pi} \int_{[-\pi, \pi] \setminus [-\delta, \delta]} g_x(y) \sin((n + (1/2))y) dy}_{\text{integrate by parts}}$$

Study \int_δ^π (study of $\int_{-\pi}^{-\delta}$ is similar)

$$\begin{aligned} \int_\delta^\pi g_x(y) \sin((n + (1/2))y) dy &= \int_\delta^\pi g_x(y) \frac{d}{dy} \left(\frac{-\cos((n + (1/2))y)}{n + (1/2)} \right) dy \\ &= \int_\delta^\pi \frac{d}{dy} \left(g_x(y) \frac{-\cos((n + (1/2))y)}{n + (1/2)} \right) - \int_\delta^\pi g'_x(y) \frac{\cos((n + (1/2))y)}{n + (1/2)} dy \\ &= -g_x(\pi) \frac{\cos((n + (1/2))\pi)}{n + (1/2)} + g_x(\delta) \frac{\cos((n + (1/2))\delta)}{n + (1/2)} - \int_\delta^\pi g'_x(y) \frac{\cos((n + (1/2))y)}{n + (1/2)} dy \end{aligned}$$

Problem:

$$g'_x(y) = \frac{-f'(x - y) \sin(y/2) - (1/2) \cos(y/2) (f(x - y) - f(x))}{(\sin(y/2))^2} \approx \frac{c}{y} \text{ near } y = 0$$

So

$$\left| \int_\delta^\pi g_x(y) \sin((n + (1/2))y) dy \right| \leq \frac{1}{n + (1/2)} \cdot \frac{1}{\delta}$$

Combining all estimates, for $\delta > 0$

$$|S_n(x) - f(x)| \leq C_1 \delta + C_2 \frac{1}{n\delta}$$

Since we are free to choose δ , we may optimize over δ .

Balancing out the terms is done by choosing $\delta = \delta(n)$ such that

$$\delta \stackrel{n \rightarrow \infty}{\sim} \frac{1}{n\delta} \iff n\delta^2 \sim 1 \iff \delta \sim \frac{1}{\sqrt{n}}$$

which gives

$$|S_n(x) - f(x)| \leq C_1 \delta + C_2 \frac{1}{n\delta} = \frac{C_1}{\sqrt{n}} + C_2 \frac{1}{n \frac{1}{\sqrt{n}}} \leq \frac{C_1 + c_2}{\sqrt{n}}$$

- Comment on the Sketch

Morally, we want $|g'_x(y)| \leq \frac{c}{y}$ for some constant c .

Numerator:

$$\left| -f'(x - y) \sin(y/2) - (1/2) \cos(y/2) (f(x - y) - f(x)) \right| \leq \|f'\|_\infty (y/2) + (\dots)y \leq Cy$$

Since $|\sin(y/2)| \leq (y/2)$,

$$\begin{aligned} |\sin(x) - \sin(0)| &= |\cos(\xi)| |x - 0| \\ &= 1|x| \end{aligned}$$

Denominator

$$(\sin(y/2))^2 \geq \left(\frac{2y}{2\pi}\right)^2 = \frac{y^2}{\pi}$$

So,

$$\left|g'_x(y)\right| \leq \frac{Cy}{\left(\frac{y}{\pi}\right)^2} \leq \frac{C^1}{y}$$

Theorem: Continuous, Periodic Functions Converge in L2

If f is continuous, 2π -periodic, then $\lim_{n \rightarrow \infty} \|S_n - f\|_2 = 0$.

That is, $\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} |S_n - f(x)|^2 dx = 0$.

IMAGE HERE - PERIODIZE $f(x)=x$ THEN APPROXIMATE WITH FOURIER

November 6, 2023

Recall: Fourier Series

$$f : [-\pi, \pi] \rightarrow \mathbb{R} \text{ or } \mathbb{C}$$

Fourier Coefficient:

$$c_k(f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{ikx} dx, \quad k \in \mathbb{Z}$$

$$s_n(x) = \sum_{k=-n}^n c_k e^{ikx} = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(x-y) f(y) dy$$

Dirichlet Kernel:

$$D_n(y) := \frac{\sin((n+1/2)y)}{\sin((1/2)y)}$$

Theorem: L2 Convergence of S_n to N

If f is C^0 , 2π -periodic, then

$$\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} |s_n(x) - f(x)|^2 dx = 0$$

Recall: Kronecker Delta

For $m, n \in \mathbb{Z}$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{imx} e^{-inx} dx = \delta_{m,n} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

That is $\{e^{inx}\}_{n \in \mathbb{Z}}$ is an orthonormal system for the inner product

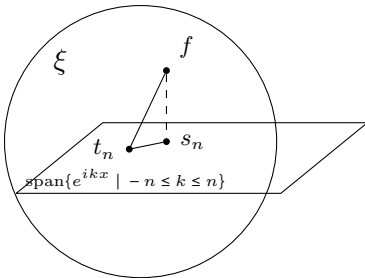
$$\begin{aligned} \xi \times \xi &\rightarrow \mathbb{C} \\ (f, g) &\mapsto \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx \end{aligned}$$

where $\xi = \{f : \mathbb{R} \rightarrow \mathbb{C}, 2\pi\text{-periodic, continuous}\}$.

Example

For $f \in \xi$, fixing $n \in \mathbb{N}_0$, consider the map

$$\begin{aligned} \mathbb{C}^{2n+1} &\rightarrow \mathbb{R} \\ (d_{-n}, \dots, d_n) &\mapsto \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - \sum_{k=-n}^n d_k e^{ikx}|^2 dx \end{aligned}$$



- Claim:

F_n is minimal if and only if $\lambda_k = c_k(f)$, $\forall -n \leq k \leq n$.

- Proof:

Take any $\lambda_n, \lambda_{n+1}, \dots, \lambda_n$ and set $t_n(x) = \sum_{k=-n}^n \lambda_k e^{ikx}$. Then

$$\int_{-\pi}^{\pi} |f(x) - t_n(x)|^2 dx = \int_{-\pi}^{\pi} |f(x) - s_n(x) + s_n(x) - t_n(x)|^2 dx$$

Then, since

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) = |z_1|^2 + |z_2|^2 + 2 \cdot \Re(z_1 \overline{z_2})$$

$$\begin{aligned} &\int_{-\pi}^{\pi} |f(x) - t_n(x)|^2 dx \\ &= \int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 dx + \int_{-\pi}^{\pi} |s_n(x) - t_n(x)|^2 dx + 2 \cdot \Re \int_{-\pi}^{\pi} (f(x) - s_n(x)) \overline{(s_n(x) - t_n(x))} dx \end{aligned}$$

What to Show: Integral on real part is zero.

$$\begin{aligned}
A &= \int_{-\pi}^{\pi} (f(x) - s_n(x)) \overline{\sum_{k=-n}^n (c_k - \lambda_k) e^{ikx}} dx \\
&= \sum_{k=-n}^n \overline{(c_k - \lambda_k)} \underbrace{\int_{-\pi}^{\pi} (f(x) - s_n(x)) e^{-ikx} dx}_{2\pi(c_k - c_k)=0}
\end{aligned}$$

Since

$$\int_{-\pi}^{\pi} s_n(x) e^{-ikx} dx = \int_{-\pi}^{\pi} \sum_{p=-n}^n c_p e^{ipx} e^{-ikx} dx = 2\pi c_k$$

It follows that

$$\begin{aligned}
\underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - t_n(x)|^2 dx}_{F_n(\lambda_{-n}, \dots, \lambda_n)} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 dx + \frac{1}{2\pi} \int_{-\pi}^{\pi} |t_n(x) - s_n(x)|^2 dx \\
&\geq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_n(x)| dx \\
&\geq F_n(c_{-n}, \dots, c_n)
\end{aligned}$$

Moreover:

$$\begin{aligned}
\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\underbrace{t_n(x) - s_n(x)}_{\underbrace{(t_n - s_n)}_{\sum_{p=-n}^n (\lambda_p - c_p) e^{ipx}}} \right|^2 dx &= \frac{1}{2\pi} \sum_{p,l=-n}^n (\lambda_p - c_p) \overline{(\lambda_l - c_l)} \underbrace{\int_{-\pi}^{\pi} e^{ipx} e^{-ilx} dx}_{\delta_{p,l}} \\
&= \frac{1}{2\pi} \sum_{p=-n}^n |\lambda_p - c_p|^2
\end{aligned}$$

Conclusion:

- * $\forall (\lambda_{-n}, \dots, \lambda_n \neq (c_{-n}, \dots, c_n), F_n(\lambda_{-n}, \dots, \lambda_n) > F_n(c_{-n}, \dots, c_n)$
- * $F_n(c_{-n}, \dots, c_n) = F_n(c_{-n}, \dots, c_n)$
- * Lemma

For all trigonometric polynomials of degree at most n , of the form $\sum_{k=-n}^n \lambda_k e^{ikx} = t_n(x)$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 dx \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - t_n(x)|^2 dx$$

and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |t_n(x)|^2 dx = \sum_{k=-n}^n |\lambda_k|^2$$

Apply this to $(\lambda_{-n}, \dots, \lambda_n) = (0, \dots, 0)$:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 dx + \sum_{k=-n}^n |c_k|^2$$

As a consequence, for all n ,

$$\sum_{k=-n}^n |c_k|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$$

which implies that $\sum_{k=-n}^n |c_k|^2$ converges absolutely and, in particular, $c_k \rightarrow 0$ as $k \rightarrow \infty$.

Riemann-Lebesgue Lemma

The above proves that if $f \in \xi$ (more generally, if f is Riemann-integrable), then

$$\lim_{k \rightarrow \infty} \int_{-\pi}^{\pi} f(x) e^{\pm i k x} dx = 0$$

Moreover, sending $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} \sum_{k=-n}^n |c_k|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$$

Importantly, there is equality whenever $\lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 dx = 0$.
When does that happen?

Theorem:

If $f \in \xi$, then

$$\lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 dx = 0$$

Proof

For $n \geq 0$, define $\sigma_n = \frac{1}{n+1} \sum_{k=0}^n s_k$ (the ‘‘Cesano sum’’).

Then

$$\sigma_n \in \text{span}\langle e^{-inx}, \dots, e^{inx} \rangle.$$

In particular,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 dx \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - \sigma_n(x)|^2 dx \leq \left(\sup_{[-\pi, \pi]} |f - \sigma_n| \right)^2$$

What to show: $\sigma_n \rightrightarrows f$ on $[-\pi, \pi]$.

Recall that

$$s_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(x-y) f(y) dy$$

$$\begin{aligned} \sigma_n(x) &= \frac{1}{n+1} \sum_{k=0}^n s_k(x) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(x-y) f(y) dy \end{aligned}$$

Where

$$\begin{aligned} K_n(y) &= \frac{1}{n+1} \sum_{k=0}^n D_k(y) \\ &= \frac{1}{n+1} \frac{1}{\sin(y/2)} \sum_{k=0}^n \sin((k+1/2)y) \end{aligned}$$

Using $2 \sin((k+1/2)y) \sin(y/2) = \cos(ky) - \cos((k+1)y)$.

$$\begin{aligned} &= \frac{1}{n+1} \frac{1}{(\sin(y/2))^2} \frac{1}{2} \sum_{k=0}^n \underbrace{\cos(ky) - \cos((k+1)y)}_{\substack{\frac{1 - \cos((n+1)y)}{2} \\ \underbrace{\qquad\qquad\qquad}_{\sin^2((\frac{n+1}{2})y)}}} \\ &= \frac{1}{n+1} \left(\frac{\sin\left(\left(\frac{n+1}{2}\right)y\right)}{\sin(y/2)} \right)^2 \end{aligned}$$

This is the Féjer kernel.

IMAGE HERE - FÉJER KERNEL

Claims:

1. $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(y) dy = 1$
2. $K_n(y) \geq 0$ on $[-\pi, \pi]$ (obvious)
3. $\forall \delta > 0, K_n \rightrightarrows 0$

- Proof of 1

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(y) dy = \frac{1}{2\pi} \frac{1}{n+1} \sum_{k=0}^n \underbrace{\int_{-\pi}^{\pi} D_k(y) dy}_{2\pi} = 1$$

- Proof of 3 If $|y| \geq \delta$,

$$|K_n(y)| = \frac{1}{n+1} \frac{\overbrace{|\sin((n+1)y/2)|^2}^{\leq 1}}{|\sin(y/2)|^2}$$

Recall $|\sin(x)| \geq \frac{2|x|}{\pi}$

$$\begin{aligned} &\leq \frac{1}{n+1} \frac{1}{(|y|/\pi)^2} \\ &\leq \frac{1}{n+1} \frac{1}{(\delta/\pi)^2} \end{aligned}$$

Which goes uniformly to 0 on $[-\pi, \pi] \setminus [-\delta, \delta]$ as $n \rightarrow \infty$.

What to show: $K_n * f \rightrightarrows f$ on $[-\pi, \pi]$.

The proof scheme is identical to: if $f \in C_c(\mathbb{R})$ and K_n is an approximation of identity, then $K_n * f \rightrightarrows f$ on \mathbb{R} .

Left as an exercise.

Corollary: Parseval's Equality

$\forall \delta \in \xi$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \lim_{n \rightarrow \infty} \sum_{k=-n}^n |c_k|^2$$

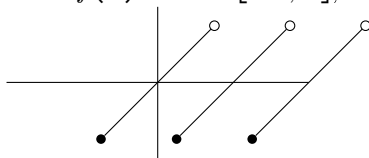
Remark:

This should hold for a larger class of function.

- Piecewise Continuous
- L^2 functions

Example

Take $f(x) = x$ on $[-\pi, \pi]$, 2π -periodized



Then $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

Application to Solving the Heat Equation

On $[0, L]_x \times \mathbb{R}_+$, $u(x, t)$ is the “heat distribution”

IMAGE HERE - ONE DIMENSIONAL ROD HEAT EQUATION YADA YADA

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right)$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left(-\frac{\partial u}{\partial x} \right) = 0$$

Problem

$$\begin{array}{lll}
 \text{PDE} & \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} & \text{on } [0, L] \times [0, T] \\
 \text{Boundary Conditions} & u(0, t) = u(L, t) = 0 & \\
 \text{Initial Conditions} & u(x, 0) = f(x) & f \text{ continuous, } f(0) = f(L) = 0
 \end{array}$$

IMAGE HERE - POSITION TIME PLANE

- Step 1: Separation of Variables

Seek an ansatz of the form

$$u(x, t) = g(x)h(t)$$

Where

$$\begin{aligned}
 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} &\iff g(x)h'(t) = g''(x)h(t) \\
 &\iff \frac{h'(t)}{h(t)} = \frac{g''(x)}{g(x)} = c
 \end{aligned}$$

Left Solving:

$$g''(x) = cg(x) \quad g(0) = 0 = g(L)$$

$$h'(t) = ch(t) \rightsquigarrow h(t) = h(0)e^{ct}$$

Then

$$\begin{aligned}
 g''(x) - cg(x) = 0 &\rightsquigarrow c = 0. \quad g(x) = a + bx \\
 c > 0. \quad g(x) &= ae^{\sqrt{c}x} + be^{-\sqrt{c}x} \\
 c < 0. \quad g(x) &= a \cos(\sqrt{-c}x) + b \sin(\sqrt{-c}x)
 \end{aligned}$$

and

$$g(0) = 0 = g(L) \implies \begin{cases} c = 0 : & g \equiv 0 & \text{(no solution)} \\ c > 0 : & g \equiv 0 & \text{(no solution)} \\ c < 0 : & a = 0. \quad g(x) = b \sin(\sqrt{-c}x) \end{cases}$$

$$\begin{aligned}
 g(L) = 0 &\implies \sin(\sqrt{-c}L) = 0 \\
 &\implies L\sqrt{-c} = k\pi \\
 &\implies c = -\left(\frac{k\pi}{L}\right)^2, k \in \mathbb{N}_0
 \end{aligned}$$

$$\text{For } c = -\left(\frac{k\pi}{L}\right)^2 = \lambda_k,$$

$$g_k(x) = \sin\left(\frac{k\pi x}{L}\right)$$

$$h_k(x) = h_k(0) \exp\left(-\left(\frac{k\pi}{L}\right)^2 t\right)$$

For all $k \in \mathbb{N}_0$,

$$u_k(x, t) = g_k(x)h_k(t)$$

solves the heat equation with boundary conditions.

Initial conditions $g_k(x)$, fix $h_k(0) = 1$.

Ansatz for a solution:

$$u(x, t) = \sum_{k=0}^{\infty} a_k g_k(x) h_k(t) \implies u(x, 0) = \sum_{k=0}^{\infty} a_k \sin\left(\frac{k\pi x}{L}\right) = f(x)$$

Thus, the left hand side is the solution.

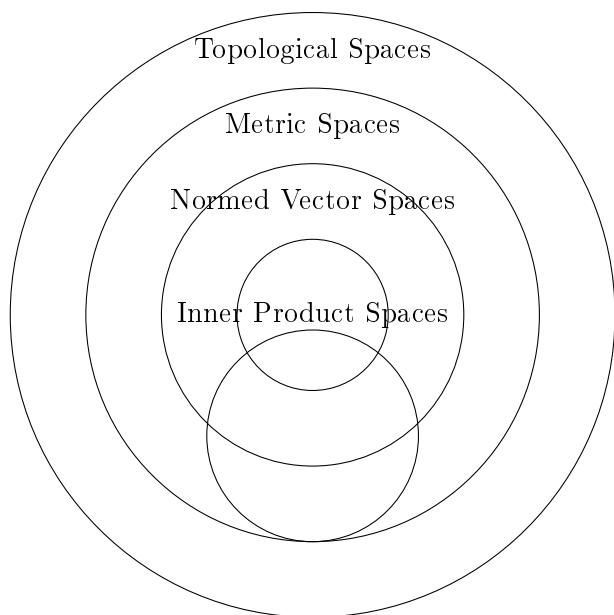
November 8, 2023

Topology of Metric Spaces

Definition: Topological Space

A pair (X, τ) is a topological space if

- X is a set.
 - $\tau \subseteq \mathcal{P}(X)$ and satisfies
 1. $\emptyset, X \in \tau$
 2. τ is stable under arbitrary unions and finite intersections.
 - Elements of τ are called “open sets”.
- IMAGE HERE - ADD COMPLETE BANACH HILBERT



Definition: Vector Space

$(E, +, \cdot)$ is a vector space (over \mathbb{R}) if

There are two operations $+: E \times E \rightarrow E$ and $\cdot: \mathbb{R} \times E \rightarrow E$ ($(\lambda, x) \mapsto \lambda x$) such that

$$(E, +) \text{ is a commutative group } \begin{cases} x + y = y + x & \forall x, y \in E \\ (x + y) + z = x + (y + z) & \forall x, y, z \in E \\ \exists 0 \in E \text{ such that } x + 0 = 0 + x = x & \forall x \in E \\ \forall x \in E, \exists -x \in E, \text{ such that } x + (-x) = (-x) + x = 0 \end{cases}$$

$$\text{and } \begin{cases} \lambda(x + y) = \lambda \cdot x + \lambda \cdot y & \forall \lambda \in \mathbb{R}, x, y \in E \\ (\lambda \cdot \mu) \cdot x = \lambda \cdot (\mu \cdot x) & \forall \lambda, \mu \in \mathbb{R}, x \in E \\ (\lambda + \mu) \cdot x = \lambda \cdot x + \mu \cdot x & \forall \lambda, \mu \in \mathbb{R}, x \in E \end{cases}$$

Example 1

$$(\mathbb{R}, +, \cdot)$$

Example 2

$$\begin{aligned} \mathbb{R}^n : x &= (x_1, \dots, x_n) \\ x + y &= (x_1 + y_1, \dots, x_n + y_n) \\ \lambda \cdot x &= (\lambda x_1, \dots, \lambda x_n) \end{aligned}$$

Example 3

Functions from $\mathbb{R} \rightarrow \mathbb{R}$

$$\begin{aligned} (f + g)(x) &= f(x) + g(x) \\ \lambda \cdot f(x) &= \lambda f(x) \end{aligned}$$

Sequences $\mathbb{N}_0 \rightarrow \mathbb{R}$
 $C(\mathbb{R}); C^k(\mathbb{R}), \forall k; C^\infty(\mathbb{R}); C^w(\mathbb{R})$ (real-analytic functions).

Definition: Normed Vector Space

A norm on a vector space E is a map $||\cdot|| : E \rightarrow \mathbb{R}$ such that

1. $||x|| \geq 0, \forall x \in E$, with equality if and only if $x = 0$.
2. $||\lambda x|| = |\lambda| ||x||, \forall \lambda \in \mathbb{R}, \forall x \in E$.
3. $||x + y|| \leq ||x|| + ||y||, \forall x, y \in E$ (triangle inequality).

$(E, ||\cdot||)$ is a normed vector space.

Example 1

$$\text{On } \mathbb{R}^n : ||x||_\infty = \max_{1 \leq i \leq n} |x_i|$$

$$1 \leq p \leq \infty ||x||_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}$$

Example 2

$$\text{On } \mathbb{C}([a, b]) : ||f||_\infty = \sup_{x \in [a, b]} |f(x)|$$

$$[a, b] \text{ compact } ||f||_p = \left(\int_a^b |f(x)|^p dx \right)^{1/p}$$

Definition: Inner Product Space

An inner product on \mathbb{R} -vector space E is a map $\langle \cdot, \cdot \rangle : E \times E \rightarrow \mathbb{R}$ such that

1. It is bilinear: $\langle \lambda f + \mu g, h \rangle = \lambda \langle f, h \rangle + \mu \langle g, h \rangle$, for all $\lambda, \mu \in \mathbb{R}, f, g, h \in E$
2. It is symmetric: $\langle f, g \rangle = \langle g, f \rangle, \forall f, g \in E$
3. It is positive definite: $\langle f, f \rangle \geq 0$, with equality if and only if $f = 0$.

The pair $(E, \langle \cdot, \cdot \rangle)$ is called an inner product space (or a pre-hilber space).

Example 1

$$\text{On } \mathbb{R}^n : \langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n \\ x = (x_1, \dots, x_n)$$

- Proof
 1. Satisfied by properties of \mathbb{R} .
 2. Satisfied by multiplicative commutativity.
 3. $\langle x, x \rangle = \sum_{i=1}^n x_i^2 = 0$ if and only if $x_i = 0, \forall i$.

Example 2

On $\mathbb{R}^n : A(a_{ij})_{i,j=1}^n$ symmetric positive definite matrix.
Then $\langle x, y \rangle_A := \langle x, Ay \rangle$ is an inner product.
Notice $\langle x, x \rangle = ||x||_2^2$.

Example 3

On $C([a, b])$, $\langle f, g \rangle = \int_a^b f(t)g(t) dt$.

Fact: Every Inner Product Gives Rise to a Norm

If the inner product is $\langle \cdot, \cdot \rangle$, then the norm is $||x|| = \langle x, x \rangle^{1/2}$.
But not every norm comes from an inner product.

Proposition:

Let $(E, \langle \cdot, \cdot \rangle)$ an inner product space.
Denote $||x|| := \langle x, x \rangle^{1/2}$ for $x \in E$.
Then

1. $\forall x, y \in E, |\langle x, y \rangle| \leq ||x|| ||y||$ (Cauchy-Schwarz)
2. $\forall x, y \in E, ||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2)$ (Parallelogram Identity)
3. $\forall x, y \in E, ||x + y|| \leq ||x|| + ||y|| \implies ||\cdot||$ is a norm.

- Proof of 1
$$\forall t \in \mathbb{R}, \langle x + ty, x + ty \rangle \geq 0$$

$$0 \leq \langle x + ty, x + ty \rangle = \langle x, x \rangle + 2t\langle x, y \rangle + t^2\langle y, y \rangle$$

Therefore the discriminant is less than 0 and

$$(2\langle x, y \rangle)^2 - 4\langle x, x \rangle \langle y, y \rangle \leq 0$$

implies that $\langle x, y \rangle^2 \leq ||x||^2 ||y||^2$.

- Proof of 2

$$\begin{aligned} ||x+y||^2 + ||x-y||^2 &= \langle x+y, x+y \rangle + \langle x-y, x-y \rangle \\ &= ||x||^2 + 2\langle x, y \rangle + ||y||^2 + ||x||^2 - 2\langle x, y \rangle + ||y||^2 \\ &= 2(||x||^2 + ||y||^2) \end{aligned}$$

- Proof of 3

$$\begin{aligned} || &= ||x||^2 + 2\langle x, y \rangle + ||y||^2 \\ &\stackrel{\text{CS}}{\leq} ||x||^2 + 2||x|| ||y|| + ||y||^2 \\ &\leq (||x|| + ||y||)^2 \\ &\xrightarrow{\sqrt{}} ||x+y|| \leq ||x|| + ||y|| \end{aligned}$$

Proposition:

Let $(E, || \cdot ||)$ be a normed space such that $|| \cdot ||$ satisfies the parallelogram law, then

$$\langle x, y \rangle := \underbrace{\frac{1}{4} (||x+y||^2 - ||x-y||^2)}_{\text{"polarization identity"}}$$

is an inner product on E .

Definition: Metric Space

A pair (X, d) is a metric space if

- X is a set.
- $d : X \times X \rightarrow \mathbb{R}$ such that

1. $d(x, y) \geq 0$, $\forall x, y \in X$ with equality if $x = y$.
2. $d(x, y) = d(y, x)$, $\forall x, y \in X$.
3. $d(x, y) \leq d(x, z) + d(z, y)$, $\forall x, y, z \in X$.

d is a "distance function."

Example 1

On \mathbb{R} , $d(x, y) = |x - y|$.

Example 2

On $(E, \|\cdot\|)$, $d(x, y) = \|x - y\|$ is a distance function.

Note that $d(x + z, y + z) = \|x + z - y - z\| = \|x - y\|$ (translation-invariance).

$\|x - y\| = \|x - z + z - y\| \leq \|x - z\| + \|z - y\|$.

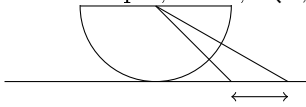
Therefore, every normed space gives rise to a metric space.

Proposition:

Not every metric space is a normed space.

If E is a vector space, might be interested in non-translation-invariant distances.

For example, on \mathbb{R} , $d(x, y) = |\tan^{-1}(x) - \tan^{-1}(y)|$.

**Proposition:**

Also, E might not be a vector space.

For example S^1 , manifolds, graphs, etc.

Definition: Open Ball

Let (X, d) be a metric space.

$x \in X, \delta > 0$ define $B_\delta(x) := \{y \in X, d(x, y) \leq \delta\}$ (and “open ball”).

We say $A \subseteq X$ is open if and only if $\forall x \in A, \exists \delta > 0, B_\delta(x) \subseteq A$.

Definition: Open Neighborhood

An open neighborhood of $x \in X$ is any open set A containing x .

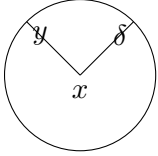
Proposition:

Let (X, d) be a metric space.

1. Open balls in X are open sets.
2. Arbitrary unions and finite intersections of open sets are open.
3. \emptyset, X are open.

Proof of 1

Take $B_\delta(x)$, $x \in X$, $\delta > 0$.



Take $y \in B_\delta(x)$, then $d(x, y) < \delta$. Set $\epsilon = \frac{\delta - d(x, y)}{2}$

Consider $B_\epsilon(y)$: if $z \in B_\epsilon(y)$, $d(z, y) < \frac{\delta - d(x, y)}{2}$, then

$$d(x, z) \leq d(x, y) + \underbrace{d(y, z)}_{\frac{\delta - d(x, y)}{2}} < \frac{d(x, y) + \delta}{2} < \delta$$

Hence $B_\epsilon(y) \subseteq B_\delta(x)$.

Proof of 2

Arbitrary Union:

Suppose A_α , $\alpha \in I$ are all open.

$x \in \bigcup_\alpha A_\alpha$, $\exists \alpha_0$, $x \in A_{\alpha_0}$.

$\exists \delta > 0$, $B_\delta(x) \subseteq A_0 \subseteq \bigcup_\alpha A_\alpha$

Finite Intersection:

Suppose $A_1 \cdot A_n$ are open.

$$x \in \bigcap_{j=1}^n A_j, \quad \forall j, \quad \exists \delta_j > 0$$

$B_{\delta_j}(x) \subseteq A_j$.

Take $\delta = \min(\delta_1, \dots, \delta_n)$, then

$$B_\delta \subseteq \bigcap_{j=1}^n A_j$$

Definitions that Generalize on a Metric Space

Limits of sequences: x_n converges to x if $\lim_{n \rightarrow \infty} d(x_n, x) = 0$.

Cauchy sequences.

Boundedness

Definition: Limit Points of Sets

$a \in X$ is a limit point of $A \subseteq X$ if $\forall \epsilon > 0$, $\exists b \in A \setminus \{a\}$, $d(a, b) < \epsilon$.

Equivalently, every open neighborhood of a has a point in $A \setminus \{a\}$.

Definition: Closed Set in Metric Space

A is closed if and only if it contains all its limit points.

Proposition:

On a metric space (X, d) ,

1. $A \subseteq X$ is closed if and only if A^C is open ($A^C = X \setminus A$).
2. Finite unions and countable intersections of closed sets are closed.

Proof of 1

Suppose A is open. Want to show that A^C is closed.

If $x \notin A^C$, then $x \in A$, then $\exists \delta > 0$ such that $B_\delta(x) \subseteq A$ (i.e. $B_\delta(x) \cap A^C = \emptyset$).

Therefore x is not a limit point of A^C .

Suppose A not open.

Then $\exists x \in A$ such that $\forall \delta > 0, B_\delta(x) \cap A^C \neq \emptyset$.

Therefore $x \in A^C$ and x is a limit point of A^C , so A^C is not closed.

Proof of 2 (Finite Union)

Take F_1, \dots, F_n closed sets and consider

$$\bigcup_{j=1}^n F_j = \left(\bigcup_{j=1}^n F_j \right)^{CC} = \left(\bigcap_{j=1}^n \underbrace{F_j^C}_{\text{open by 1}} \right)^C$$

is closed.

Definition: Completeness

A metric space (X, d) is complete if and only if every Cauchy sequence converges to a point in X .

If (X, d) comes from a normed vector space $(X, || \cdot ||)$, it is called Banach.

If (X, d) comes from an inner product space, it is called Hilbert.

Examples

These are complete.

$(\mathbb{R}, || \cdot ||)$

$(\mathbb{R}^n, || \cdot ||_2)$

$(C([a, b]), ||f - g||_\infty)$

Counter Examples

This is not complete.

$(C([a, b]), ||f - g||_2)$ where $||f - g||_2 = \left(\int_a^b (f(t) - g(t))^2 dt \right)^{1/2}$.

Consider x^n on $[0, 1]$.

$$\left(\int_0^1 (x^n - 0)^2 dx \right)^{1/2} = \frac{1}{\sqrt{2n+1}}.$$

Theorem:

$\forall n \in \mathbb{N}$, $(\mathbb{R}^n, || \cdot ||_2)$ is complete.

Let x_p be a cauchy sequence in \mathbb{R}^n , $x_p = (x_{p,1}, \dots, x_{p,n})$.

Note for $1 \leq j \leq n$, $|x_{p,j} - x_{q,j}| \leq ||x_p - x_q||_2$.

Therefore $\forall 1 \leq j \leq n$, $\{x_{p,j}\}_p$ is Cauchy in \mathbb{R} .

November 13, 2023

Topology of metric spaces.

- Induced Topology
- Continuous Maps
- Connectedness
- Compact Sets / Separability

Induced Topology

Let (X, d) be a metric space, and $M \subseteq X$.

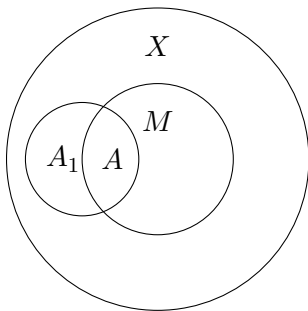
Can restrict the distance function to $M \times M$, then $(M, d|_{M \times M})$ is a metric space.

$U \subseteq M$ is open if $\forall a \in U$, $\exists \delta > 0$, $\{x \in M \mid d(a, x) < \delta\} = B_\delta(a) \subset U$.

Note: if $a \in M \subseteq X$,

$$B_\delta^M(a) = \{x \in M \mid d(a, x) < \delta\} = B_\delta^X(a) \cap M$$

$$B_\delta^X(a) = \{x \in X \mid d(a, x) < \delta\}$$



Open sets for both topologies may differ!

Note: M is always open for $(M, d|_{M \times M})$.

e.g. $X = \mathbb{R}$ and $M = [0, 1]$.

Proposition:

In the setting above, $A \subset M$ is open in (M, d) if and only if $\exists A_1 \subseteq X$ open in (X, d) such that $A = A_1 \cap M$.

- Proof

(\implies) Suppose A open in (M, d) , $\forall a \in A$, $\exists \delta_a > 0$, $B_{\delta_a}^M(A) \subset A$.

$$A = \bigcup_{a \in A} B_{\delta_a}^M(a) = \bigcup_{a \in A} (B_{\delta_a}^X(a) \cap M) = \underbrace{\left(\bigcup_{a \in A} B_{\delta_a}^X(a) \right)}_{A_1} \cap M$$

(\impliedby) Suppose $A = A_1 \cap M$, A_1 open in (X, d) . Let $a \in A_1$.
 $a \in A_1 : \exists \delta > 0$, $B_\delta^X(a) \subseteq A_1$.

Then

$$B_\delta^M(a) = \underbrace{B_\delta^X(a)}_{\subseteq A_1} \cap M \subseteq A_1 \cap M = A \quad \blacksquare$$

Proposition:

A closed subspace M of a complete metric space (X, d) is also complete.

- Proof

Take a Cauchy sequence in M , $\{x_k\}_k$, then it is also Cauchy in X .

Therefore it converges to $x \in X$.

Since M contains its limit points, $x \in M$.

Theorem:

Let $f : (M, d_M) \rightarrow (N, d_N)$, where (M, d_m) and (N, d_n) are metric spaces. The following are equivalent.

1. $\forall x \in M$, $\forall \epsilon > 0$, $\exists \delta > 0$

$$d_m(x, y) < \delta \implies d_N(f(x), f(y)) < \epsilon$$

(in short: $\forall \epsilon > 0$, $\exists \delta > 0$, $f(B_\delta^M(x)) \subset B_\epsilon^N(f(x))$)

2. $\forall x \in M$ and $\{x_n\}_n$ convergin to $x \in M$, $f(x_n)$ converges to $f(x)$ in N .

3. $\forall O$ open in N , $f^{-1}(O)$ is open in M .

Definition: Continuity

When (1), (2) or (3) is satisfied, we say “ f is continuous on M ”.

Proof that 1 Implies 2

Let $x \in M$, $\{x_n\}_n$ converges to x .

What to show: $\lim_{n \rightarrow \infty} d_N(f(x_n), f(x)) = 0$.

Let $\epsilon > 0$, by (1), $\exists \delta > 0$, $f(B_\delta^M(x)) \subseteq B_\epsilon^N(f(x))$ (*).

Since $x_n \rightarrow x$, $\exists n_0, n \geq n_0 \implies d_M(x_n, x) < \delta$ (i.e. $x_n \in B_\delta^M(x)$).

Then, by (*), $f(x_n) \in B_\epsilon^N(f(x))$ (i.e. $d_N(f(x_n), f(x)) < \epsilon$, $\forall n \geq n_0$)

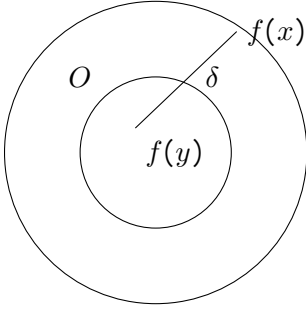
Proof that 2 Implies 3

Assume (2) and let $\exists O$ open in N such that $f^{-1}(O)$ is not open:

$$\exists y \in f^{-1}(O), \forall \delta = \frac{1}{n} > 0, \exists x_n, d_M(x_n, y) < \frac{1}{n} \text{ and } f(x_n) \notin O$$

Then x_n converges to y , but $f(x_n)$ cannot converge to $f(y)$.

Completion of proof left as an exercise.



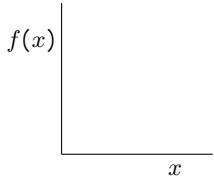
Proof that 3 Implies 1

Let $x \in M$ and $\epsilon > 0$. $B_\epsilon^N(f(x))$ is open.

By (3), $f^{-1}(B_\epsilon^N(f(x)))$ is open and contains x .

Then $\exists \delta > 0$, $B_\delta^M(x) \subseteq f^{-1}(B_\epsilon^N(f(x)))$, which implies

$$f(B_\delta^M(x)) \subseteq B_\epsilon^N(f(x))$$



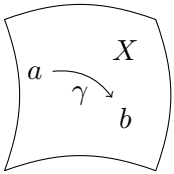
Definition: Path Connected

Let (X, d) be a metric space.

We say X is path-connected if $\forall a, b \in X$,

$$\exists \gamma : [0, 1] \rightarrow X$$

continuous with $\gamma(0) = a$ and $\gamma(1) = b$.



Definition: Connected

We say X is connected if the only subsets of X that are both open and closed are \emptyset and X .

Alternatively, $X = A_1 \cup A_2$, $A_1 \neq \emptyset$, $A_2 \neq \emptyset$, $A_1 \cap A_2 \neq \emptyset$, A_1, A_2 open.

Proposition: Path-connectedness Implies Connectedness

Let (X, d) be a metric space. If (X, d) is path-connected, then it is connected.

Remarks

If $(X, d) = (\mathbb{R}, |\cdot|)$, $A \subseteq \mathbb{R}$ is connected if and only if it is path-connected if and only if it is an interval. In general, connectedness does not imply path connectedness.

- Counterexample

In \mathbb{R}^2 , the topologist's sine wave

$$A = \left\{ (0, 0) \right\} \cup \left\{ \left(x, \sin \frac{1}{x} \right) \mid x \in \mathbb{R} \setminus \{0\} \right\}$$

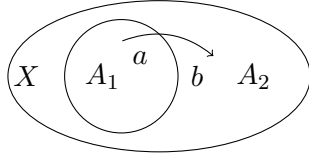
IMAGE HERE - TOPOLOGISTS SINE WAVE

Proof

Suppose X is path-connected but not connected. That is

$X = A_1 \cup A_2$, A_1, A_2 open and nonempty, $A_1 \cap A_2 = \emptyset$.

Pick $a \in A_1$, $b \in A_2$ and consider $\gamma : [0, 1] \rightarrow X$ continuous with $\gamma(0) = a$ and $\gamma(1) = b$.



$a \in A_1$, which is open, so $\exists \delta > 0$, $B_\delta(a) \subseteq A_1$.

Then $\gamma^{-1}(B_\delta(a))$ is open and contains 0, therefore it contains $[0, \epsilon)$ for some $\epsilon > 0$.

Let $l = \sup\{\epsilon > 0 \mid \gamma([0, \epsilon)) \subset A_1\}$.

Then $\gamma([0, l]) \subset A_1$.

If $l < 1$, $\gamma(l) \in A_1$ open. Then $\exists \delta' > 0$, $B_{\delta'}(\gamma(l)) \subseteq A_1$, and $\gamma^{-1}(B_{\delta'}(\gamma(l)))$ is open in $[0, 1]$, of the form $(l - \epsilon', l + \epsilon')$.

This contradicts the supremum property of l , so $l = 1$.

But then $\gamma(l) \in A_1$ and $\gamma(l) = \gamma(1) = b \in A_2$, which is a contradiction.

Definition: Compact

Let (X, d) be a metric space.

1. We say that $A \subseteq X$ is compact if every sequence in A has a limit point in A .
2. We say $A \subseteq X$ satisfies Heine-Borel property if every open cover of A has a finite subcover, still covering A .

Definition: Dense

We say A is dense in X if $\forall x \in X, \forall \epsilon > 0, \exists a \in A, d(x, a) < \epsilon$.

Definition: Separable

We say B is separable if B has a countable dense subset.

i.e. $\exists \{x_n\}_n \in B$ such that every point in B is a limit point of $\{x_n\}_n$.

Example 1

$(\mathbb{R}, |\cdot|)$, with dense subset \mathbb{Q} .

Example 2

$(C([a, b]), \|\cdot\|_\infty)$, with dense subset polynomials with rational coefficients.

Proof left as an exercise.

Theorem:

Suppose (X, d) is a compact metric space. Then it is separable.

Distance Between Sets.

Given a finite collection $\{x_1, \dots, x_n\}$, write $d(x, \{x_1, \dots, x_n\}) = \min_{1 \leq j \leq n} d(x, x_j)$.

Proof

Pick $x_1 \in X$.

Look at $R_1 := \sup\{d(x, x_1) \mid x \in X\}$.

Claim: $R_1 < \infty$. Otherwise, construct a sequence with no convergent subsequence.

Then, pick x_2 such that $d(x_1, x_2) > \frac{1}{2}R_1$.

Look at $R_2 := \sup\{d(x, \{x_1, x_2\}) \mid x \in X\} < \infty$.

Pick x_3 such that $d(x_3, \{x_1, x_2\}) > \frac{1}{2}R_2$.

Repeat: if x_1, \dots, x_k are constructed, look at $R_k := \sup\{d(x, \{x_1, \dots, x_k\}) \mid x \in X\} < \infty$ and set x_{k+1} such that $d(x_{k+1}, \{x_1, \dots, x_k\}) > \frac{1}{2}R_k$.

Claim: $R_k \xrightarrow[k \rightarrow \infty]{} 0$, otherwise $\{x_n\}_n$ has no convergent subsequences.

But then, $\forall x \in X, d(x, \{x_1, \dots, x_k\}) \leq R_k$. Hence $\lim_{k \rightarrow \infty} d(x, \{x_1, \dots, x_k\}) = 0$.

Then for $\epsilon > 0, \exists k$ such that $d(x, \{x_1, \dots, x_k\}) < \epsilon$.

i.e. $\exists k_0 \in \{1, \dots, k\}, d(x, x_{k_0}) < \epsilon$. ■

Theorem:

A subsets A of a metric space (X, d) is compact if and only if it satisfies the Heine-Borel property.

Proof

(\Leftarrow) (Note: true even if A is not separable)

Take a sequence $\{x_n\}_n$ in A and argue by contradiction.

Case 1: $\{x_n\}_n$ has a limit point $b \notin A$.

Then $U_k = \{x \in X \mid d(x, b) > \frac{1}{k}\}$.

IMAGE HERE - COVERS AROUND B WITH X AND A

$\bigcup_k U_k = X \setminus \{b\}$ covers A , but no finite subcover covers A since b is a limit point of A .

Case 2: $\{x_n\}_n$ has no limit points at all.

$V_k = X \setminus \{x_k, x_{k+1}, \dots\}$ is open and $\bigcup_k V_k$ covers A , but no finite subcover covers A . ■

(\implies) Sketch

To do: assume compactness show that it leads to Heine-Borel.

Take an arbitrary cover.

Separability implies one can extract a countable subcover of \mathcal{O} covering A .

November 15, 2023

Contraction Mapping Theorem

Application 1: Local Existence Theorem for ODEs

Application 2: Implicit Function Theorem

Definition: Contraction

$f : M \rightarrow N$, (M, d_M) , (N, d_N) two metric spaces.

f is a contraction if $\exists C \in [0, 1)$ such that

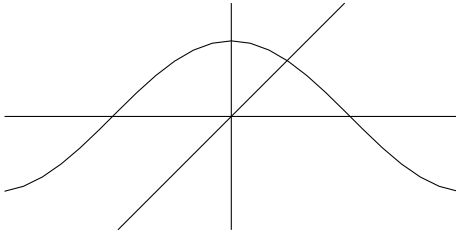
$$\forall x, y \in M \quad d_N(f(x), f(y)) \leq C d_M(x, y)$$

Example:

$$f(x) = \frac{1}{2} \cos(x), \quad f : \mathbb{R} \rightarrow \mathbb{R}.$$

$$|f(x) - f(y)| \stackrel{\substack{\text{Mean Value Theorem} \\ \underbrace{\quad}}}{=} \underbrace{|f'(\xi)|}_{|\frac{1}{2} \sin(\xi)|} |x - y| \leq \frac{1}{2} |x - y|$$

for ξ between x and y .



Theorem: Contraction Mapping Theorem

Suppose (M, d) is a complete metric space and $f : M \rightarrow M$ a contraction. Then

$$\exists! x \in M, f(x) = x$$

(i.e. f has a unique fixed point).

Proof

- Existence.

Pick any $x_0 \in M$. Consider the sequence $x_{k+1} = f(x_k)$.

Claim 1: $\{x_n\}_n$ is Cauchy (then, by completeness, it converges to some $x \in M$)

Claim 2: If Claim 1 is true, by continuity of f at x ,

$$f(x) = f\left(\lim_{k \rightarrow \infty} x_k\right) \stackrel{\text{continuity @ } x}{=} \lim_{k \rightarrow \infty} \underbrace{f(x_k)}_{x_{k+1}} = \lim_{k \rightarrow \infty} x_{k+1} = x$$

What to show: $\{x_n\}_n$ is Cauchy.

$\forall \epsilon > 0, \exists N, \forall p \geq N, k \geq 0, d(x_{p+k}, x_p) < \epsilon$

– Scratch Work

$$\begin{aligned} d(x_{p+k}, x_p) &\leq d(x_{p+k}, x_{p+k-1}) + d(x_{p+k-1}, x_{p+k-2}) + \cdots + d(x_{p+1}, x_p) \\ &\leq \sum_{q=0}^{k-1} \underbrace{d(x_{p+q+1}, x_{p+q})}_{C^q d(x_{p+1}, x_p)} \\ &\leq \underbrace{d(x_{p+1}, x_p)}_{C^p d(x_1, x_0)} \cdot \underbrace{\sum_{q=0}^{k-1} C^q}_{\leq \frac{1}{1-C}} \end{aligned}$$

$$d(x_{p+2}, x_{p+1}) = d(f(x_{p+1}), f(x_p)) \leq C d(x_{p+1}, x_p)$$

$$d(x_2, x_1) = d(f(x_1), f(x_0)) \leq C d(x_1, x_0)$$

$$d(x_3, x_2) \leq C d(x_2, x_1) \leq C C d(x_1, x_0)$$

Ultimately,

$$(*) \quad d(x_{p+k}, x_p) \leq \frac{d(x_1, x_0)}{1-C} C^p$$

- Proof of Cauchy

Let $\epsilon > 0$, since $\lim_{p \rightarrow \infty} \frac{d(x_1, x_0)}{1-C} C^p = 0$,

$$\exists N, \forall p \geq N, \frac{d(x_1, x_0)}{1-C} C^p < \epsilon$$

Then, for $p \geq N$ and $k \geq 0$,

$$d(x_{p+k}, x_p) \stackrel{(*)}{\leq} \frac{d(x_1, x_0)}{1-C} C^p < \epsilon \quad \blacksquare$$

- Uniqueness

If x, y satisfy $f(x) = x$ and $f(y) = y$

$$d(x, y) = d(f(x), f(y)) \leq C d(x, y)$$

If $d(x, y) \neq 0$, then $1 \leq C$ is a contradiction.

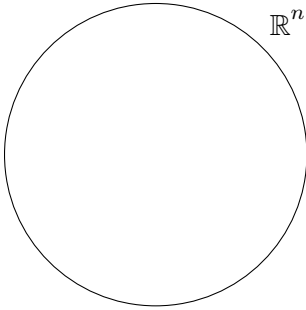
Application to ODEs

A system of 1st order Ordinary Differential Equations takes the form

$$\begin{cases} \frac{dx}{dt} = f(t, x(t)) & (*) \\ x(0) = x_0 \end{cases}$$

$$x : [0, b]_t \rightarrow \mathbb{R}^n$$

$$f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$$



Under what conditions on f does $(*)$ have unique solution $x(t)$ on some interval $[0, \epsilon]$?

Example

$$\frac{d^2 x}{dt^2} = \sin(x(t))$$

IMAGE HERE - PENDULUM WITH GRAVITY VECTOR, LENGTH AND SIN(T) PERIOD

Meta-Principle:

An ODE of order k in \mathbb{R}^n : $\frac{d^k x}{dt^k} x(t) = f(t, x(t), x^1(t), \dots, x^{k-1}(t))$ can be rephrased as a 1st-order system in \mathbb{R}^{nk} upon introducing variables

$$\begin{aligned} x_1(t) &= \frac{dx}{dt}(t) \\ x_2(t) &= \frac{dx_1}{dt}(t) \\ &\vdots \\ x_{k-1}(t) &= \frac{dx_{k-2}}{dt}(t) \end{aligned}$$

The ODE becomes

$$\frac{d}{dt} \underbrace{\begin{bmatrix} x(t) \\ x_1(t) \\ \vdots \\ x_{k-1}(t) \end{bmatrix}}_{U(t)} = \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{k-1}(t) \\ f(t, x(t), \dots, x_{k-1}(t)) \end{bmatrix}}_{F(U(t))}$$

Theorem: Local Existence Theorem of ODEs

Take $x_0 \in \mathbb{R}^n$. Suppose $\exists \delta > 0$ such that f is continuous on $\mathbb{R} \times \overline{B_{\delta}(x_0)}$.

Where $\overline{B_{\delta}(x_0)} = \{x \in \mathbb{R}^n \mid ||x - x_0||_2 < \delta\}$

And f is Lipschitz with respect to $x : |f(t, x) - f(t, y)| \leq C|x - y|, \forall t \in \mathbb{R}, \forall x, y \in \overline{B_\delta(x_0)}$.

Notation: $|x| = ||x||_2$.

Then $\exists > 0$ and $\exists! x \in C^1([0, \epsilon]; \mathbb{R}^n)$ solution to $(*)$.

Proof

If x solves $(*)$,

$$\int_0^t \rightsquigarrow x(t) - x_0 = \int_0^t f(u, x(u)) \, du \iff x(t) = x_0 + \underbrace{\int_0^t f(u, x(u)) \, du}_{Tx(t)}$$

This is a fixed point problem.

Goal:

1. find (E, d) a complete metric space
2. such that $T(E) \subseteq E$ and T is a contraction.
3. Then, by the Contraction Mapping Theorem, $\exists! x \in E$.
4. Finally, show that x is actually C^1 .

• Part A

$\overline{B_\delta(x_0)}$ is closed in $(\mathbb{R}^n, ||\cdot||_2)$ which is complete $\implies (\overline{B_\delta(x_0)}, ||\cdot||_2)$ is complete.

Then $\forall \epsilon > 0, C([0, \epsilon], \overline{B_\delta(x_0)})$, with norm $||x(t)||_\infty = \sup_{t \in [0, \epsilon]} ||x(t)||_2$, is complete.

Set

$$E = C([0, \epsilon], \overline{B_\delta(x_0)})$$

• Part B

When is $T(E) \subseteq E$?

Suppose $x \in E$, i.e. $||x(t) - x_0||_2 \leq \delta, \forall t \in [0, \epsilon]$.

$$||x(t) - x_0||_2 = \left\| \int_0^t f(u, x(u)) \, du \right\|_2 \leq \int_0^t \underbrace{||f(u, x(u))||_2}_{\text{Note}} \, du \leq tM \leq \epsilon M \quad \forall t \in [0, \epsilon]$$

Note: since f is continuous on $\mathbb{R} \times \overline{B_\delta(x_0)} \implies f$ bounded by M .

Upon making $\epsilon M \leq \delta$, then $\sup_{t \in [0, \epsilon]} ||Tx(t) - x_0||_2 \leq \delta$

$\implies T(E) \subseteq E$ if $\epsilon \leq \frac{\delta}{M}$.

Is T continuous on E ?

$$\begin{aligned}
\|Tx(t) - Ty(t)\|_2 &= \left\| \int_0^t (f(u, x(u)) - f(u, y(u))) \, du \right\|_2 \\
&\leq \underbrace{\int_0^t \|f(u, x(u)) - f(u, y(u))\|_2 \, du}_{\substack{\leq L \|x(u) - y(u)\|_2 \\ L \sup_{u \in [0, \epsilon]} \|x(u) - y(u)\|_2 \\ \|x - y\|_E}} \\
&\leq tL \|x - y\|_E \leq \epsilon L \|x - y\|_E
\end{aligned}$$

Therefore $\|Tx - Ty\|_E \leq \epsilon L \|x - y\|_E$ is a contraction if $\epsilon L < 1$, i.e. $\epsilon < \frac{1}{L}$.

- Part C

By *CMT*, $\exists! x \in E(\epsilon)$ as long as $\epsilon \leq \frac{\delta}{M}$ and $\epsilon < \frac{1}{L}$.

- Part D

$x(t) \in C([0, t], \overline{B_\delta(x_0)})$, but also notice

$$\begin{aligned}
x(t) &= \underbrace{x_0 + \int_0^t f(u, x(u)) \, du}_{\text{differentiable with derivative } f(t, x(t))}
\end{aligned}$$

which is continuous.

Therefore x is C^1 , $x'(t) = f(t, x(t))$, and $x(0) = x_0 + \int_0^0 = x_0$.

Remark: Lipschitz Requirement for Uniqueness

f being locally Lipschitz is necessary for uniqueness purposes.

Example

$$\begin{cases} \frac{dx}{dt} = 3x^{2/3} \\ x(0) = 0 \end{cases}$$

This ODE has two distinct solutions:

$$\begin{cases} x(t) = 0 \\ x(t) = t^3 \rightsquigarrow \frac{dx}{dt} = 3t^2 = 3x^{2/3} \end{cases}$$

Remark:

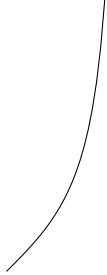
The time of existence is not sharply controlled (could be infinite, finite, very small)

Example

$$\begin{cases} \frac{dx}{dt} = \overbrace{x^2}^{\text{Lipschitz}} \leadsto \frac{1}{x^2} \frac{dx}{dt} = 1 \implies -\frac{d}{dt} \left(\frac{1}{x} \right) \implies \frac{-1}{x(t)} + \frac{1}{x_0} = t \implies x(t) = \frac{x_0}{1-x_0 t} \\ x(0) = x_0 \end{cases}$$

Lifetime: $t^* = \frac{1}{x_0}$.

IMAGE HERE - FROM X0 EXPLODING AT 1/X0; FASTER FOR 2X0



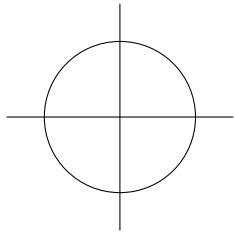
Theorem: Implicit Function Theorem / Inverse Function Theorem

Example

$$F : \mathbb{R}^2 \rightarrow \mathbb{R}, F(x, y) = x^2 + y^2.$$

Question: Can the set $F(x, y) = 1$ be described by equations “ $x(y)$ ” or “ $y(x)$ ” ?

IMAGE HERE - CIRCLE AS TWO EQUATIONS WITH VERTICAL LINE TEST



Implicit Function Theorem gives situations where it is possible.

Setting

$F : \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathbb{R}^q$ continuously differentiable.

Fix $(a, b) \in \mathbb{R}^p \times \mathbb{R}^q$ and set $F(a, b) = c$.

If the $q \times q$ matrix

$$\frac{\partial F}{\partial y}(a, b) = \begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} & \dots & \frac{\partial F_1}{\partial y_q} \\ \vdots & \ddots & & \vdots \\ \frac{\partial F_q}{\partial y_1} & \dots & \dots & \frac{\partial F_q}{\partial y_q} \end{bmatrix}$$

is invertible, then $\exists \Omega$ neighborhood of a and a function $y : \Omega \rightarrow \mathbb{R}^q$ such that $\forall x \in \Omega$,

$$F(x, y(x)) = c$$

Moreover:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial x} = 0 \implies \frac{\partial y}{\partial x} = \left[\frac{\partial F}{\partial y}(x, y(x)) \right]^{-1} \frac{\partial F}{\partial x}(x, y(x))$$

Parallel: solve $f(x) = 0$ by Newton's iteration.

x_{n+1} = the zero of the line $(f(x_n) + (x - x_n)f'(x_n) = 0) = x_n - (f'(x_n))^{-1}f(x_n)$.

Left iterating

$$\phi(x) = x - (f'(x))^{-1}f(x)$$

Claim: ϕ is a contraction near fixed points x^* whenever $f'(x^*) \neq 0$.

- Proof (Sketch)

Uses CMT.

Freeze x ; approximately solve for y the equation $F(x, y) - c = 0$.

Iterate $\phi_x(y) = y - \frac{\partial F}{\partial Y}^{-1}(a, b)(F(x, y) - c)$.

November 20, 2023

Continuity and compactness.

Uniform equicontinuity and Arzela-Ascoli.

Bolzano's example of a continuous, nowhere differentiable function.

Continuity on Compact Domains

Context:

(M, d_m) , (N, d_n) two metric spaces.

$f : M \rightarrow N$ continuous.

M compact.

Proposition:

In the context above, we can deduce:

1. f is uniformly continuous in the sense that

$$\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in M, d_M(x, y) < \delta \implies d_n(f(x), f(y)) < \epsilon$$

2. $f(M)$ is compact.

- Corollary:

If (M, d_M) is a compact metric space and $f(M) \rightarrow (\mathbb{R}, |\cdot|)$ continuous, then f is bounded and achieves its extrema.

$$\exists x_{\pm} \in M \text{ such that } f(x_+) = \sup\{|f(x)| \mid x \in M\} \text{ and } f(x_-) = \inf\{|f(x)| \mid x \in M\}$$

- Proof of 1

Alternative proof in lecture notes.

By contradiction, suppose

$$\exists \epsilon > 0, \forall \underbrace{\delta}_{1/n} > 0, \exists \underbrace{x, y}_{x_n, y_n} \in M, \underbrace{d_M(x, y) < \delta}_{d_M(x_n, y_n) < 1/n} \text{ and } d_N(f(x), f(y)) \geq \epsilon$$

x_n lives in M compact, therefore a subsequence x_{n_k} converges to x in M .

$$\begin{aligned} d_M(x_{n_k}, y_{n_k}) < \frac{1}{n_k} &\implies d_M(y_{n_k}, x) \leq d_M(y_{n_k}, x_{n_k}) + d_M(x_{n_k}, x) \\ &\leq d_M(x_{n_k}, x) + \frac{1}{n_k} \xrightarrow{k \rightarrow \infty} 0 \end{aligned}$$

So y_{n_k} converges to x .

Since f is continuous at x , $f(x_{n_k}) \rightarrow f(x)$ and $f(y_{n_k}) \rightarrow f(x)$ as $k \rightarrow \infty$.

Hence $d_N(f(x_{n_k}), f(y_{n_k})) \xrightarrow{k \rightarrow \infty} d_N(f(x), f(y)) = 0$ (cf. homework problem).

This contradicts the assumption.

- Proof of 2

What to show: $\forall \mathcal{O}$ open covers of $f(M)$, there exists a finite subcover $\mathcal{O}' \subset \mathcal{O}$ still covering $f(M)$.

Take $\mathcal{O} = \{O_\alpha \mid \alpha \in I\}$ covering $f(M)$.

Set $V_\alpha = f^{-1}(O_\alpha)$ open by the continuity assumption.

Claim: $\mathcal{V} = \{V_\alpha \mid \alpha \in I\}$ covers M : let $x \in M$, $f(x) \in f(M)$, then $f(x) \in O_\alpha$ for some $\alpha \in I$.

By Heine-Borel Compactness,

$$\exists \{ \underbrace{V_1}_{f^{-1}(O_1)}, \dots, \underbrace{V_n}_{f^{-1}(O_n)} \}$$

subcollection of \mathcal{V} covering M .

Claim: (O_1, \dots, O_n) covers $f(M)$: Pick $y \in f(M)$, $y = f(x)$ for $x \in M$

M covered by

$$V_1, \dots, V_n \implies \exists j, x \in V_j \implies \underbrace{f(x)}_y \in O_j$$

Arzela-Ascoli

Q: What are the compact sets of $(C([a, b]), \|\cdot\|_\infty)$?

Context:

$[a, b] \subset \mathbb{R}$ a compact interval.

$C([a, b])$ continuous functions on $[a, b]$.

$\|f\|_\infty := \sup_{x \in [a, b]} |f(x)|$ is a norm.

$(C([a, b]), \|\cdot\|_\infty)$ is complete.

Definition: Uniform Boundedness / Uniform Equicontinuity

For $\mathcal{F} \subset C([a, b])$, we say

1. \mathcal{F} is uniformly bounded if $\exists M \geq 0, \forall x \in [a, b], \forall f \in \mathcal{F}, |f(x)| \leq M$

2. \mathcal{F} is uniformly equicontinuous (UEC) if $\forall \epsilon > 0, \exists \delta > 0,$

$$\forall x, y \in [a, b], |x - y| < \delta \implies |f(x) - f(y)| < \epsilon, \quad \forall f \in \mathcal{F}$$

- Example 1

Any finite $\mathcal{F} = \{f_1, \dots, f_n\}$ is uniformly bounded and uniformly equicontinuous.

$$M = \max\{M_1, \dots, M_n\}$$

$$\delta(\epsilon) = \min\{\delta_1(\epsilon), \dots, \delta_n(\epsilon)\}$$

- Example 2

Sets satisfying a uniform Lipschitz/Hölder criterion, i.e. $\exists L > 0, \alpha \in (0, 1]$,

$$|f(x) - f(y)| \leq L|x - y|^\alpha, x, y \in [a, b], f \in \mathcal{F}, \left(\delta(\epsilon) = \left(\frac{\epsilon}{L}\right)^{1/\alpha}\right)$$

- Example 3

Sets satisfying a uniform bound on first derivatives (assuming they exist).

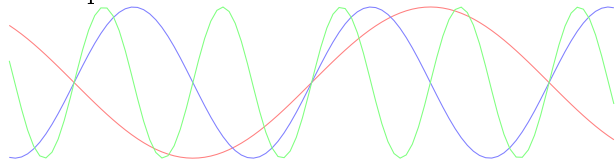
$$\mathcal{F} = \{f \in C^1([a, b]) \mid |f'(x)| \leq M, x \in [a, b]\}$$

- Non-example

$$\mathcal{F} = \{\sin(nx) \mid x \in [0, \pi], n \in \mathbb{N}\}$$

Uniformly bounded by 1.

Not equicontinuous.



Proposition:

If $f_n \in C([a, b])$ converges uniformly to $f \in C([a, b])$, then $\{f_n \mid n \in \mathbb{N}\}$ is uniformly bounded and UEC.

Proof

- Uniform Boundedness

Since $\|f_n - f\|_\infty \xrightarrow{n \rightarrow \infty} 0$ and f is continuous on $[a, b]$ compact, then f bounded implies $\|f\|_\infty < \infty$.

For

$$\begin{aligned} &\implies \|f_n - f\|_\infty < 1 \\ \implies \|f_n\|_\infty &= \|f_n - f + f\|_\infty \end{aligned}$$

Then $M = \max\{\|f_1\|_\infty, \dots, \|f_n\|_\infty, \|f\|_\infty + 1\}$ is a uniform bound.

- Uniform Equicontinuity

$$\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in [a, b], |x - y| < \delta \implies |f_n(x) - f_n(y)| < \epsilon \quad \forall n \in \mathbb{N}$$

f is continuous, hence uniform continuous on $[a, b]$ with modulus of continuity $d_f(\epsilon)$.

Let $\epsilon > 0$, by uniform convergence, (a) $\exists N, n \geq N \implies |f_n(x) - f(x)| \leq \epsilon/3 \quad \forall x \in [a, b]$.

For $x, y, |x - y| < \delta_f$, and $n \geq N$,

$$|f_n(x) - f_n(y)| = \underbrace{|f_n(x) - f(x)|}_{< \frac{\epsilon}{3} \text{ by (a)}} + \underbrace{|f(x) - f(y)|}_{< \frac{\epsilon}{3} \text{ by uniform convergence of } f} + \underbrace{|f(y) - f_n(y)|}_{< \frac{\epsilon}{3} \text{ by (a)}}$$

Therefore $\mathcal{F} = \{f_n\}_{n \geq N}$ is UEC.

$$\delta(\epsilon) = \min\{d_f(\epsilon/3), \delta_1(\epsilon), \dots, \delta_N(\epsilon)\}.$$

Theorem: Arzela-Ascoli

Let $[a, b] \subset \mathbb{R}$ be compact and $f_k \in C([a, b])$ be uniformly bounded and uniformly equicontinuous. Then a subsequence of f_k converges uniformly.

Proof

Step A: construct a subsequence converging at all rationals in $[a, b]$.

(uniform boundedness is enough)

Step B: show that the subsequence is uniformly Cauchy.

$$\forall \epsilon > 0, \exists N, \forall p, q \geq N, \|f_p - f_q\|_\infty < \epsilon$$

(uses UEC)

- Step A

Let r_k be a denumeration of all rationals in $[a, b]$.

$\{f_n(r_1)\}_n$ has a convergent subsequence (Bolzano-Weierstrass), $\{f_{1,n}(r_1)\}_n$.

$\{f_{1,n}(x)\}_n$ converges at r_1 ; a subsequence $f_{2,n}$ converges at $r_2 \implies f_{2,n}$ converges at r_1, r_2 .

Repeat $\{f_{2,n}(r_3)\}_n$ is bounded \implies a subsequence $\{f_{3,n}(r_3)\}_n$ converges $\implies f_{3,n}$ converges at r_1, r_2, r_3 .

$\forall p, \{f_{p+1,n}\}_n$ subsequence of $\{f_{p,n}\}_n$ converges at r_1, \dots, r_p, r_{p+1} .

Consider $\{f_{k,k}\}_{k \in \mathbb{N}}$ subsequence of $\{f_n\}_n$.

$\forall p, f_{k,k}(r_p)$ converges because $\{f_{k,k}(r_p)\}_{k \geq p}$ is a subsequence of $\{f_{p,n}\}_n$.

- Step B

Simplified statement: if f_n is uniformly bounded, uniformly equicontinuous and converges at all rationals in $[a, b]$, then f_n is uniformly Cauchy.

Let $\epsilon > 0$. By UEC, $\exists \delta > 0, \forall x, y \in [a, b], |x - y| < \delta \implies |f_n(x) - f_n(y)| < \delta \quad \forall n \in \mathbb{N}$.

$\exists r_{i_1}, \dots, r_{i_l}$ rationals such that $\forall x \in [a, b], \min_{1 \leq j \leq l} |x - r_{i_j}| < \delta$.

Since $f_n(r_{i_j})$ converges $\forall j$,

$$\exists N, p, q \geq N, |f_p(r_{i_j}) - f_q(r_{i_j})| < \frac{\epsilon}{3}$$

Now for any $x \in [a, b]$ and $p, q \geq N$,

$$|f_p(x) - f_q(x)| \leq \underbrace{|f_p(x) - f_p(r_{i_j})|}_{< \frac{\epsilon}{3} \text{ by UEC}} + \underbrace{|f_p(r_{i_j}) - f_q(r_{i_j})|}_{< \frac{\epsilon}{3}} + \underbrace{|f_q(r_{i_j}) - f_q(x)|}_{< \frac{\epsilon}{3} \text{ by UEC}}$$

We showed $\forall \epsilon > 0, \exists N, p, q \geq N \implies \|f_p - f_q\|_\infty < \epsilon$. ■

Consequence

$\mathcal{F} \subseteq C([a, b])$ is compact if and only if it is closed, uniformly bounded and uniformly equicontinuous.

Interesting Functional Analytic Consequence

Consider $(C^1([a, b]), f \mapsto \|f\|_\infty + \|f'\|_\infty)$

One can show that it is complete.

Arzela-Ascoli \implies bounded sets in $C^1([a, b])$ are precompact (i.e. have compact closures) in $(C([a, b]), \|\cdot\|_\infty)$.

We say the injection

$$C^1([a, b]) \hookrightarrow C([a, b])$$

is compact.

November 22, 2023

Topological spaces.

- Definitions
- Hausdorff
- Pathological Cases
- Base for a Topology

Definition:

Given X a set and $\tau \subset \mathcal{P}(X)$, we say that τ is a topology if

1. $\emptyset, X \in \tau$
2. τ is stable under finite intersection and arbitrary union.

- $O_1, \dots, O_n \in \tau \implies \bigcap_{j=1}^n O_j \in \tau$
- $\{O_\alpha\}_\alpha \in \tau \implies \bigcup_\alpha O_\alpha \in \tau$

We say (X, τ) is a topological space and elements of τ are called open sets.

Examples

1. (X, d) a metric space, then $\tau = \{\text{open sets defined by } \forall x \in O, \exists \delta > 0, B_{\delta(x)} \subset O\}$
2. $(X, \{\emptyset, X\})$
3. $(X, \mathcal{P}(X))$

Topological Definitions

Fix (X, τ) a topological space.

Definition: Open Neighborhood

An open neighborhood of $x \in X$ is an open set $U \ni x$.

Definition: Interior Point

$A \subseteq X$, then $a \in A$ is an interior point of A if $\exists U$ an open neighborhood of x such that $x \in U \subseteq A$.

Definition: Interior

The interior of $A \subseteq X$, denoted A° , is the set of all interior points.

Definition: Convergence

x_n converges to x if and only if for every neighborhood U of x , $\exists N, n \geq N \implies x_n \in U$.

Definition: Limit Point

x is a limit point of A if for every neighborhood U of x , $A \cap (U \setminus \{x\}) \neq \emptyset$.

Definition: Closed Set

A is closed if it contains all its limit points.

Definition: Closure

The closure of A , called \overline{A} , is $A \cup \{\text{limit points of } A\}$.

Proposition: Induced Topology

Given (X, τ) a topological space and $A \subseteq X$, then $\tau_A = \{U \cap A \mid U \in \tau\}$ is a topology on A called the induced topology.

Topological Propositions

Take (X, τ) a topological space (fixed) and $A \subseteq X$.

1. $A \in \tau$ if and only if every point in A is an interior point of A . (Then A deserves to be called open)
2. A is closed if and only if A^c is open
3. Arbitrary intersection and finite union of closed sets are closed.
4. A° is open and $A^\circ = \bigcup_{\substack{O \subseteq A \\ O \in \tau}} O$ (which implies A open if and only if $A = A^\circ$).
5. \overline{A} is closed and $\overline{A} = \bigcap_{\substack{F \supseteq A \\ F^c \in \tau}} F$ (which implies A is closed if and only if $\overline{A} = A$)

Proof of 1

(\Leftarrow) Take A .

$$\begin{aligned} \forall x \in A, \exists U_x \in \tau, x \in U_x \subseteq A &\implies A = \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} U_x \subseteq A \\ &\implies A = \bigcup_{x \in A} U_x \in \tau \end{aligned}$$

(\Rightarrow) $A \in \tau$, $x \in A$, take $U \in \tau$, containing x , then $A \cap U \in \tau$, $x \in A \cap U \subseteq A$.

Proof of 2

A open $\implies A^c$ closed.

If $x \notin A^c$, then $x \in A$.

Therefore $\exists U \in \tau, x \in U \subseteq A \implies U \cap A^c = \emptyset \implies x$ not a limit point of A^c .

So A^c contains its limit points.

Converse left as an exercise.

Proof of 3 (Technique)

De Morgan's Laws.

Proof of 4

Take $x \in A^\circ : \exists U \in \tau, x \in U \subseteq A$.

Claim $U \subseteq A^\circ$. Why?

If $y \in U$, $U \in \tau$ so $\exists V \in \tau, y \in V \subseteq U \subseteq A \implies y \in A^\circ$.

Therefore A° is open.

$$A^\circ = \bigcup_{\substack{O \subseteq A \\ O \in \tau}} O:$$

$$(\subset) A^\circ \subseteq A, \text{ so } A^\circ \subseteq \bigcup_{\substack{O \subseteq A \\ O \in \tau}} O.$$

(^{\supset}) Claim: if $O \in \tau, O \subseteq A$, then $O \subseteq A^\circ$.

Pick $x \in O, \exists U \in \tau, x \in U \subseteq A \implies x \in A^\circ$.

Therefore $\bigcup_{\substack{O \subseteq A \\ O \in \tau}} O \subseteq A^\circ$.

Proof of 5 (Exercise)

$(\bar{A})^c = (A^c)^\circ$? Prove or disprove.

Definition: Continuous Maps

Fix $(X, \tau), (Y, \sigma)$ topological spaces.

$f : X \rightarrow Y$ is continuous if $\forall O \in \sigma, f^{-1}(O) \in \tau$.

Can also define continuity at $x \in X$:

$\forall V$ open neighborhood of $f(x), \exists U$ open neighborhood of x such that $f(U) \subseteq V$.

IMAGE HERE - EPSILON DELTA NEIGHBORHOODS ON \mathbb{R}^2

Proposition:

f is continuous on X if and only if it is continuous at every $x \in X$.

Proposition:

f continuous at x if and only if $\forall x_n$ such that $\lim_{n \rightarrow \infty} x_n = x$, then $\lim_{n \rightarrow \infty} f(x_n) = f(x)$.

Definition: Homeomorphism

$f : (X, \tau) \rightarrow (Y, \sigma)$ is a homeomorphism if and only if f is continuous, bijective and $f^{-1} : (Y, \sigma) \rightarrow (X, \tau)$ is continuous.

A homeomorphism induces a one to one correspondence between τ and σ .

$$O \longrightarrow f(O) = (f^{-1})^{-1}(O)$$

i.e. $(X, \tau) \sim (Y, \sigma)$ if and only if $\exists f : X \rightarrow Y$ homomorphic is an equivalence relation.

Examples

1. $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), |\cdot| \rightarrow (\mathbb{R}, |\cdot|)$ is a homeomorphism.

2. X, d_1, d_2 two metrics such that $\exists C_1, C_2 > 0, \forall x, y \in X, C_1 d_2(x, y) \leq d_1(x, y) \leq C_2 d_2(x, y)$, then

$$\text{id} : (X, \tau(d_1)) \rightarrow (X, \tau(d_2))$$

is a homeomorphism.

Topological Connectedness

Definition: Connected

(X, τ) is connected if $X = A \cup B$, $A \cap B = \emptyset$, and $A, B \in \tau$ implies $A = \emptyset$ or $B = \emptyset$.
Equivalently, the only two subsets of X that are open and closed are \emptyset and X .

Definition: Path Connected

(X, τ) is path-connected under the same definition as before.

Definition: Connected Subspace

A is connected if and only if (A, τ_A) is connected where τ_A is the induced topology).

Proposition:

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be continuous.

1. If X is connected, so is $f(X)$.
2. If X is path-connected, so is $f(X)$.

Topological Compactness

Keep the Heine-Borel definition only.

Definition: Compact

A set $A \subseteq X$ is (HB)-compact if \forall open covers \mathcal{O} of A , a finite subcollection of \mathcal{O} still covers A .

Proposition:

If $K \subset X$ is compact and $F \subseteq K$ is closed, then F is compact.

- Proof

Take $\{O_\alpha\}_\alpha$ open cover of F . Then $\{O_\alpha\}_\alpha, F^c$ covers K .

By HBP, a finite subcover (e.g. $\{O_\alpha, \dots, O_{\alpha_n}, F^c\}$) covers K , in particular F .

And since $F \cap F^c = \emptyset$, $O_\alpha, \dots, O_{\alpha_n}$ covers F . ■

Proposition:

If $f : X \rightarrow Y$ continuous and K compact in X , then $f(K)$ compact.

Definition: Hausdorff Property

We say that a topological space (X, τ) is Hausdorff if and only if “it separates points”:

$$\forall x, y \in X, x \neq y, \exists U_x \ni x, \exists U_y \ni y, U_x, U_y \in \tau, U_x \cap U_y = \emptyset$$

IMAGE HERE - HAUSDORFF SEPARABILITY DRAWING

Example 1

Any $(X, \tau(d))$ induced by a metric d is Hausdorff.

- Proof

IF $x \neq y$, let $\delta = d(x, y) > 0$, take $U_x = B_{\delta/3}(x)$ and $U_y = B_{\delta/3}(y)$.

IMAGE HERE – POINTS X Y WITH DELTA/3 BALLS

Non-example 2

$(X = \{0, 1\}, \tau = \{\emptyset, X\})$ not Hausdorff.

Example 3

$(X, \mathcal{P}(X))$ is Hausdorff.

Example 4

If X is not Hausdorff, the singleton $\{x\}$ need not be closed.

In Non-example 2, $\{0\}$ is neither open nor closed.

Theorem:

If (X, τ) is Hausdorff, then for $K \subseteq X$ compact and $x \notin K$, $\exists U, V$ open sets such that $U \cap V = \emptyset$, $x \in U$, $K \subset V$.

Proof

$\forall y \in K, \exists U_y, V_y \in \tau, x \in U_y, y \in V_y, U_y \cap V_y = \emptyset$.

Then $\{V_y \mid y \in K\}$ is an open cover of K , so $\exists y_1, \dots, y_n$ such that K is covered by V_{y_1}, \dots, V_{y_n} .

$\tau \ni \bigcup_{j=1}^n V_{y_j} \supseteq K$ and $U = \bigcap_{j=1}^n U_{y_j} \in \tau$, contains x .

$$\forall j, U \cap V_{y_j} = \bigcap_{p=1}^n U_{y_p} \cap V_{y_j} \subseteq U_{y_j} \cap V_{y_j} = \emptyset$$

Therefore $U \cap \bigcup_{j=1}^n V_{y_j} = \emptyset$,

Corollary

Let (X, τ) be Hausdorff.

1. $K \subseteq X$ compact implies K closed.
2. K compact and F closed implies $F \cap K$ compact (i.e. K^c contains no limit points of K).

Theorem:

Let (X, τ) be Hausdorff.

Suppose that $\{K_\alpha\}_\alpha$ is a collection of compact sets in X such that $\bigcap_\alpha K_\alpha = \emptyset$.

Then $\exists \alpha_1, \dots, \alpha_n, \bigcap_{j=1}^n K_{\alpha_j} = \emptyset$.

Proof

Single out K_{α_1} .

Then $K_{\alpha_1} \cap \bigcap_{\alpha \neq \alpha_1} K_\alpha = \emptyset$ i.e. $K_{\alpha_1} \subset \left(\bigcap_{\alpha \neq \alpha_1} K_\alpha\right)^c = \bigcup_{\alpha \neq \alpha_1} K_\alpha^c$.

Therefore $\{K_\alpha^c\}_{\alpha \neq \alpha_1}$ is an open cover of K_{α_1} .

Take a finite subcover $K_{\alpha_2}, \dots, K_{\alpha_n}$, then $K_{\alpha_1} \subseteq \bigcup_{j=2}^n K_{\alpha_j}^c = \left(\bigcap_{j=2}^n K_{\alpha_j}\right)^c$.

Therefore $\bigcap_{j=1}^n K_{\alpha_j} = \emptyset$. ■

Some Heuristic Statements

One usually seeks a topology that is not too small (coarse) and not too large (fine).

On X , given τ_1 and τ_2 , we say τ_1 is finer (contains more open sets) than τ_2 if $\tau_2 \subseteq \tau_1$.

The finest of all is $\mathcal{P}(X)$.

- Hausdorff.
- Completely disconnected (every set is open and closed).
- Very few compact sets (only the finite sets).

The coarser sets of all is $\{\emptyset, X\}$.

- Very few open sets.
- Not Hausdorff (as soon as X has more than one element)
- Compact
- Connected

In general, due to Heine-Borel, more open sets means fewer compact sets and vice versa.

IMAGE HERE - PRODUCT AND QUOTIENT TOPOLOGIES

November 27, 2023

Quotient Topology

Base for a topology.

Topology on finite products.

Definition: Equivalence Relation on X

Let X be a set and \sim an equivalence relation on X :

- $\sim \in \mathcal{P}(X \times X)$.
- Reflexive: $\forall x \in X, x \sim x$.
- Symmetric: $\forall x \in X, y \in X, x \sim y \implies y \sim x$
- Transitive $\forall x, y, z \in X, x \sim y \wedge y \sim z \implies x \sim z$

Definition: Equivalence Class on X

$[x] := \{y \in X \mid y \sim x\}$

Lemma

Two equivalence classes are either disjoint or equal.

Definition: Quotient Space

The equivalence relation induces a partitioning of X into equivalence classes.

Define $X/\sim = \{[x] \mid x \in X\}$ (the quotient space), then there exists a natural projection map $\pi : X \rightarrow X/\sim$, $x \mapsto [x]$.

Question:

If X carries a topology, τ , can we induce one on X/\sim ?

Answer: Yes. We say U is open (i.e. $U \in \tau_\sim$) in X/\sim if and only if $\pi^{-1}(U)$ is open in X .

- Claim:
 τ_\sim is a topology.

– Proof

$$\pi^{-1}(\emptyset) = \emptyset \in \tau \text{ so } \emptyset \in \tau_\sim.$$

$$\pi^{-1}(X/\sim) = X \in \tau \text{ so } X/\sim \in \tau_\sim.$$

Stability under finite intersection and arbitrary union:

$$\pi^{-1}\left(\bigcup_\alpha U_\alpha\right) = \bigcup_\alpha \pi^{-1}(U_\alpha).$$

$$\pi^{-1}\left(\bigcap_{n=1}^k V_n\right) = \bigcap_{n=1}^k \pi^{-1}(V_n).$$

- Claims:
 τ_{\sim} makes π continuous.
 τ_{\sim} is the finest topology making π continuous.

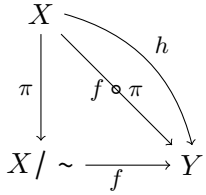
Obvious Corollaries

If X is compact/connected/path-connected, so is X/\sim .

- Proof:

$\backslash (X/\sim = \pi(x) \backslash), \pi$ continuous.

IMAGE HERE - COMMUTATIVE DIAGRAM



Proposition

Let X, τ, \sim be as above.

Let $f : X/\sim \rightarrow (Y, \tau_Y)$.

Then f is continuous if and only if $f \circ \pi$ is continuous.

Proof

(\implies) Obvious (composition of continuous maps).

(\impliedby) Suppose $f \circ \pi$ is continuous.

Take $U \in \tau_Y$, then $(f \circ \pi)^{-1}(U) = \pi^{-1}(f^{-1}(U))$ is open.

Then $f^{-1}(U)$ is open. Therefore f is continuous.

Proposition:

Let $X, \sim, \tau, (Y, \tau_Y)$ as above and $h : X \rightarrow Y$.

Then $\exists f : X/\sim \rightarrow Y$ such that $h = f \circ \pi$ if and only if $\forall [a] \in X/\sim, \forall x \in [a], h(x) = h(a)$.

Moreover (by previous propositions), f continuous if and only if h is continuous.

Proof:

(\implies) If $h = f \circ \pi, [a] \in X/\sim, x \in [a]$

$$h(x) = f(\pi(x)) = f([x]) = f([a]) = f(\pi(a)) = h(a)$$

(\impliedby) If $\forall [a] \in X/\sim, \forall x \in [a], h(x) = h(a)(*)$, define $f([a]) := h(a)$ ro h (any representative of $[a]$).
 f is well defined thanks to $(*)$.

Note:

Hausdorff property can be lost in a quotient construction.

IMAGE HERE - ON TWO UNIT INTERVALS SEND ALL POSITIVE VALUES BUT NOT ZERO

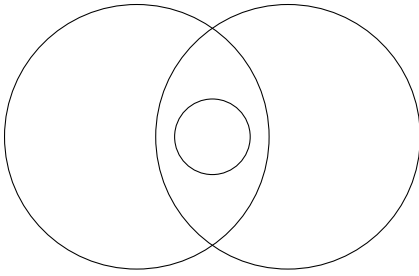
Definition: Base for a Topology

Take X a set.

Suppose $\sigma \subset \mathcal{P}(X)$ satisfies

1. σ covers X (i.e. $X \subseteq \bigcup_{A \in \sigma} A$)
2. $\forall A, B \in \sigma, x \in A \cap B, \exists C \in \sigma$ such that $x \in C \subset A \cap B$.

IMAGE HERE - VENN DIAGRAM WITH A, B, INTERSECT SIGMA AND C IN INTERSECT



Then $\tau := \{\text{textarbitrary unionsofelementin}\sigma\} \cup \{\emptyset\} = \mathcal{T}(\sigma)$ is a topology.

$\mathcal{T} : \mathcal{P}(\mathcal{P}(X)) \rightarrow \mathcal{P}(\mathcal{P}(X))$ if τ is a topology, $\mathcal{T}(\tau) = \tau$.

σ is called a base for τ .

Proof

τ is stable under arbitrary unions.

τ is stable under intersection: let $A, B \in \sigma$.

Then $A \cap B = \bigcup_{x \in A \cap B} C_x$ where C_x is given by (2), so $A \cap B \in \tau$.

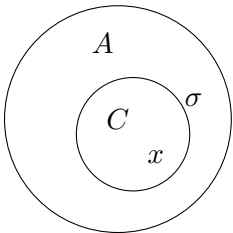
If $A, B \in \tau$, $A = \bigcup_{\alpha} A_{\alpha} \in \sigma$ and $B = \bigcup_{\beta} B_{\beta} \in \sigma$, so

$$A \cap B = \left(\bigcup_{\alpha} A_{\alpha} \right) \cap \left(\bigcup_{\beta} B_{\beta} \right) = \bigcup_{\alpha, \beta} \underbrace{(A_{\alpha} \cap B_{\beta})}_{\in \tau} \underbrace{\hspace{1cm}}_{\in \tau}$$

Proposition: Criterion for a Basis

(X, τ) topological space and $\sigma \subset \tau$.

Then σ is a base for τ if and only if $\forall A \in \tau, \forall x \in A, \exists C \in \sigma, x \in C \subset A$.



Proof

(\implies) Let $A \in \tau$, then $A = \bigcup_{\alpha} C_{\alpha}$, $C_{\alpha} \in \sigma$. If $x \in A$, then $x \in C_{\alpha_0}$ for some α_0 .

(\impliedby) Let $A \in \tau$,

$$A = \bigcup_{x \in A} C_x$$

Where $C_x \in \sigma$ comes from the hypothesis.

Proposition:

Let $\sigma_1, \sigma_2 \in \mathcal{P}(\mathcal{P}(X))$, then $\mathcal{T}(\sigma_1) \subset \mathcal{T}(\sigma_2)$ if and only if $\forall A \in \sigma_1, \forall x \in A, \exists B \in \sigma_2, x \in B \subset A$.

This helps give a criterion for when $\mathcal{T}(\sigma_1) = \mathcal{T}(\sigma_2)$.

Example

A base for the standard topology on \mathbb{R} is

$$\sigma = \left\{ \left(r - \frac{1}{n}, r + \frac{1}{n} \right) \mid r \in \mathbb{Q}, n \in \mathbb{N} \right\}$$

Note: it's countable.

Definition: Dense Space

Take (X, τ) a topological space. We say $A \subset X$ is dense in X if $\forall x \in X, \forall U$ open neighborhood of x , $U \cap A \neq \emptyset$.

Definition: Separable Space

Take (X, τ) a topological space. We say X is separable if it has a countable dense subset.

Proposition:

Given (X, τ) a topological space, if τ has a countable base then

1. X is separable.
2. Any cover of X has a countable subcover.

Proof of 1

Call $\sigma = \{C_k \mid k \in \mathbb{N}\}$ a countable base.

$\forall k$, let $x_k \in C_k$.

Claim: $A := \{x_k\}_k$ is dense in X .

Indeed, if $x \in X$, U open neighborhood of x , $U \in \tau$, $U = \bigcup_{j \in \mathbb{N}} C_{k_j}$ then $x_{k_j} \in U$ for all j .

Proof of 2

Write $\mathcal{O} = \{O_\alpha\}_\alpha$ a cover of X .

Construct \mathcal{O}' as follows:

$\forall k \in \mathbb{N}$, if $C_k \subset O_\alpha$ for some α_k , adjoin O_{α_k} to \mathcal{O}' .

That is at most countably many.

Let $x \in X$.

Since \mathcal{O} covers X , $\exists O_\alpha$ such that $x \in O_\alpha$.

$$O_\alpha \in \tau \implies O_\alpha = \bigcup_{j \in \mathbb{N}} C_{k_j}^\alpha \implies \exists j, x \in C_{k_j}^\alpha$$

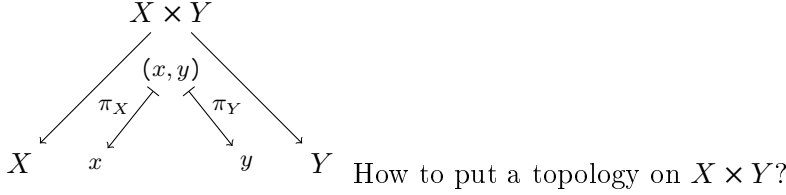
Then either $O_\alpha \in \mathcal{O}'$ or another $x \in O_{\alpha_k} \in \mathcal{O}'$.

Either way, $\exists V \in \mathcal{O}', x \in V$.

Topology of Finite Products

Setup: (X, τ_X) and (Y, τ_Y) , $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$.

Two natural projections π_X and π_Y .



It should make π_X, π_Y continuous.

That is, $\forall U \in \tau_X, \pi_X^{-1}(U) = U \times Y \in \tau$ and $\forall V \in \tau_Y, \pi_Y^{-1}(V) = X \times V \in \tau$.

By stability under intersection, τ should contain

$$(U \times Y) \cap (X \times V) = (U \cap X) \times (Y \cap V) = U \times V$$

Now define $\sigma := \{U \times V \mid U \in \tau_X, V \in \tau_Y\}$. σ satisfies

1. It covers $X \times Y$, since $X \times Y \in \sigma$.

2. $A, B \in \sigma \implies A \cap B \in \sigma$. Indeed,

$$U_1 \times V_1 \cap U_2 \times V_2 = \underbrace{(U_1 \cap U_2)}_{\in \tau_X} \times \underbrace{(V_1 \cap V_2)}_{\in \tau_Y}$$

Definition: Product Topology

The product topology on $X \times Y$ is $\mathcal{T}_{X \times Y}$.

It is the coarsest topology making π_X, π_Y continuous (i.e. if τ is a topology on $X \times Y$ such that π_X, π_Y continuous, then $\tau \supset \mathcal{T}(\sigma)$).

Proposition:

Given (Z, τ_Z) and $f : Z \rightarrow (X \times Y, \mathcal{T}_{X \times Y})$, f is continuous if and only if $\pi_X \circ f$ and $\pi_Y \circ f$ are continuous.

Proposition

(\implies) Clear: composition of continuous functions.

(\impliedby) Take $U \in \mathcal{T}_{X \times Y}$, write $U = \bigcup_{\alpha} U_{\alpha} \times V_{\alpha}$, then

$$\begin{aligned} f^{-1}(U) &= f^{-1}\left(\bigcup_{\alpha} U_{\alpha} \times V_{\alpha}\right) \\ &= \bigcup_{\alpha} f^{-1}(U_{\alpha} \times V_{\alpha}) \\ &= \bigcup_{\alpha} \underbrace{(\pi_X \circ f)(U_{\alpha})}_{\in \tau_Z} \cap \underbrace{(\pi_Y \circ f)(V_{\alpha})}_{\in \tau_Z} \end{aligned}$$

where

$$(U_{\alpha} \times V_{\alpha}) = \{z \in Z \mid f(z) \in U_{\alpha} \times V_{\alpha}\} = \{z \in Z \mid \pi_X \circ f(z) \in U_{\alpha}, \pi_Y \circ f(z) \in V_{\alpha}\}$$

Proposition:

Let X, Y be two topological spaces and $X \times Y$ with the product topology.

1. If X, Y are Hausdorff, so is $X \times Y$.
2. If X, Y are connected, so is $X \times Y$.
3. If X, Y are compact, so is $X \times Y$.

Proof of 1

Pick $(x_1, y_1), (x_2, y_2) \in X \times Y$.

If $x_1 \neq x_2$, then $\exists U_1, U_2 \in \tau_X, U_1 \cap U_2 = \emptyset, x_1 \in U_1, x_2 \in U_2$.

Then $U_1 \times Y$ and $U_2 \times Y$ separate (x_1, y_1) and (x_2, y_2) .

IMAGE HERE - X Y PLANE WITH VERTICAL LINES DEMONSTRATING

If $x_1 = x_2$, then $y_1 \neq y_2$ by a similar construction.

Proof of 2

Suppose $A \neq \emptyset$ both open and closed in $X \times Y$.

What to show: $A = X \times Y$.

Will follow from: $(x, y) \in A \implies \{x\} \times Y \subset A$ and $X \times \{y\} \in A$.

Claim: $\{x\} \times Y \cap A$ is both open and closed in $\{x\} \times Y$.

Since $\{x\} \times Y$ is connected, then $\{x\} \times Y \cap A = \{x\} \times Y$.

Therefore $\{x\} \times Y \subset A, \forall x \in A$.

Similarly, $X \times \{y\} \subset A, \forall y \in A$.