Analysis II

January 9, 2024

(Real) Analysis

- Calculus
  - Differential
  - Integral (Riemann)
- Functions and Maps
  - Measure Theory
  - (Lebesgue) Integration
- Topology
  - Completeness (as a metric space)
  - Compactness (Bolzano-Weierstrass theorem [real]) (Arzela-Ascoli)
  - Paracompactness / Metrizable / Baire Category Theorem
  - Algebraic / Combinatoric (continuous maps or functions)

Definition: Cardinality

For sets A, B, $\mathrm{Card}(A) = \mathrm{Card}(B)$ if there exists a one-to-one correspondence $q : A \leftrightarrow B$.
Counting, labelling, indexing, etc.
$\mathrm{Card}(A) \leq \mathrm{Card}(B)$ if $A \subset B$ or there exists a one-to-one mapping $A \to B$.

Definition: Countable

If $A \hookrightarrow \mathbb{N}$, then A is countable.

Theorem

The countable union of countable sets is countable.

Proof

Let $A_i = \{a_j\}_{j=1}^{\infty}$, $i = 1, 2, \ldots$.

$$\begin{matrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ \vdots & & & \\ a_{k1} & a_{k2} & a_{k3} & \cdots \end{matrix}$$

Index by diagonalization.

Theorem

The cartesian product of countable sets is countable.

Proof

$X \times Y = \{(x_i, y_j \mid x_i \in X, y_j \in Y\}$

$$\begin{array}{cccc} (x_1, y_1) & (x_1, y_2) & (x_1, y_3) & \cdots \\ (x_2, y_1) & (x_2, y_2) & (x_2, y_3) & \cdots \\ \vdots & & & \\ (x_k, y_1) & (x_k, y_2) & (x_k, y_3) & \cdots \end{array}$$

Theorem

$\mathrm{Card}\left(2^X\right) > \mathrm{Card}(X)$, where $2^X = \{A \subset X\}$ is the power set of X.

Proof

For all $x \in X$, $\{x\} \subset 2^X$, so $\mathrm{Card}(X) \leq \mathrm{Card}\left(2^X\right)$.
Assume, for sake of contradiction, that $\mathrm{Card}(X) = \mathrm{Card}\left(2^X\right)$.
Then, by definition, there exists a one-to-one correspondence $\phi : X \leftrightarrow 2^X$.
Set $A = \{x \subset X \mid x \notin \phi(x)\}$, and let $a = \phi^{-1}(A)$ (i.e. $A = \phi(a)$).
If $a \in A$, then $a \notin A \subset \phi(a)$; but if $a \notin A$, then $a \in A$, a contradiction.

Theorem

$\mathrm{Card}(\mathbb{R}) = \mathrm{Card}\left(2^{\mathbb{N}}\right)$.

Topology of the Real Line

Completeness (as a metric space)

$d(a, b) = |a - b|, \quad \forall a, b \in \mathbb{R}$.

1. $x_i \to x$ if $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ such that $|x_i - x| < \varepsilon$, $\forall i \geq n$.

2. $\{x_i\}$ is Cauchy if $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ such that $|x_i - x_j| < \varepsilon$, $\forall i, j \geq n$.

Definition: Open Inteval

(a, b) is an open set on the real line.
There exist interior points for any subset A of real numbers.
$\forall x \in A$, x is interior if $\exists (a, b)$ such that (1) $x \in (a, b)$ and (2) $(a, b) \subset A$.

- Theorem
  The union of open sets is open.
  The intersection of finitely many open sets is open.
  $\varnothing$ and $\mathbb{R}$ are open.

Definition: Limit Point

A limit point $x \in \mathbb{R}$ of a subset A is a limit point in A if for every open neighborhood U of X, $(U \setminus \{x\}) \cap A \neq \varnothing$.

Definition: Closed

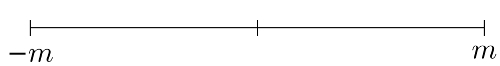
A is closed if A contains all of its limit points.

- Theorem
  A is closed if and only if $A^c = \mathbb{R} \setminus A$ is open.
  - Proof
    A closed $\implies$ A^c open.
    Otherwise, $\exists x \in A^c$ such that for every neighborhood U of X, $(U \setminus \{x\}) \cap A = \varnothing$ which would make it a limit point of A not in A. By assumption, A contains all its limit points so this is a contradiction.
    A^c open $\implies$ A closed.
    For any x a limit point of A, assume otherwise that $x \in A^c$.
    Then there exists some neighborhood U of x such that $U \subset A^c$ (since A^c is open).
    It follows that $(U \setminus \{x\}) \cap A = \varnothing$ and x is not a limit point of A, which is a contradiction.

Definition: Sequential Compactness

A is compact if $\forall \{x_i\}$, $x_i \in A$ there exists a convergent subsequence $\{x_{i_k}\}$ and $x_{i_k} \to x \in A$.

- Theorem: Bolzano-Weierstrass
  For $A \subseteq \mathbb{R}$, A is compact if and only if A is closed and bounded.
  - Proof
    A compact $\implies$ A closed and bounded.
    Assume that A is not bounded from abvove.
    Then there exists a sequence $\{x_i\}$, $x_i \in A$ where $x_{i+1} > x_i + 1$ and $\{x_i\}$ has no convergent subsequences.
    Then compactness implies closedness.
    A closed and bounded $\implies$ A (sequentially) compact.
    Let any $\{x_i\}$, $x_i \in A$.
    Claim: $\forall \{x_i\}$ of reals, if there exists $m \in \mathbb{R}$ such that $|x_i| \leq m$, $\forall m$ then there is some convergent subsequence.

    Divide and conquer: dividing the interval in half necessitates that at least one half contains infinitely many points. Repeat indefinitely.

- Theorem: Heine-Borel)
  $A \subseteq \mathbb{R}$ is (sequentially) compact if and only if any open cover has a finite subcover.
  - Proof
    Heine-Borel Property $\implies$ closed and bounded.
    Assume that A is unbounded, $U_n = (-n, n)$ and $\{U_n\}_{n=1}^{\infty}$ an open cover for $A \subseteq \mathbb{R}$ has no finite subcover.
    Assume A is not closed, then $x \in \dot{A}$ (where $\dot{A}$ is the limit set of A) and $x \notin A$, $U_n \left\{\left(-\infty, x - \frac{1}{n}\right) \cup \left(x + \frac{1}{n}, +\infty\right)\right\}$.
    Then $\{U_n\}$ covers $\mathbb{R} \setminus \{x\} \supset A$ has no finite subcover of A.
    A is bounded and closed $\implies$ A is Heine-Borel

Divide and conquer: using open sets with respect to open covers.

Definition: Cantor Set

$C = \{x \in [0,1] \mid \text{the ternary expansion of } x \text{ has only the digits } \{0,2\}\}$.
Equivalenetly, let $C_0 = [0,1]$, $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$.
Then $C_n = \bigcup_{k=1}^{2^n} C_n^k$ and $C = \bigcap_{n=1}^{\infty} C_n$.
$|C_n| = 2^n \left(\frac{1}{3}\right)^n \to 0$.

Definition: Perfectly Symmetric Sets

Let $\{\xi_n\}$ where $\xi_n \in \left(0, \frac{1}{2}\right)$.
$E_0 = [0,1]$, $E_1 = [0, \xi_1] \cup [1 - \xi_1, 1]$, $E_2 = [0, \xi_1\xi_2] \cup [\xi_1 - \xi_1\xi_2, \xi_1] \cup [1 - \xi_1, 1 - \xi_1 + \xi_1\xi_2] \cup [1 - \xi_1\xi_2, 1]$.
Then the cantor set is given by $\xi_n = \frac{1}{3}$.
$E_n = \bigcup_{k=1}^{2^n} E_n^k$, $|E_n^k| = \xi_1\xi_2\cdots\xi_n$, and $|E_n| = \sum |E_n^k| = 2^n \xi_1\xi_2\cdots\xi_n$.
Therefore, $E = \bigcap_{n=1}^{\infty} E_n$ and we define $|E| = \lim_{n\to\infty} |E_n| = \lim_{n\to\infty} \left(2^n \xi_1\xi_2\cdots\xi_n\right) = \lambda$ where $\lambda \in [0,1)$.
Let

$$2\xi_n = \frac{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n-1}\right)^{n-1}}{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n}\right)^{n}} < 1$$

, then

$$2^n \xi_1\cdots\xi_n = \frac{1}{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n}\right)^{n}} \to \lambda.$$

Proof

$\lim_{n\to\infty} \left(\left(1 + \frac{x}{n}\right)^{n/x}\right)^x = e^x$, then $\lim_{y\to 0} (1+y)^{1/y} = e$, $\log(1+y)^{1/y} = \frac{\log(1+y)}{y} \underset{y\to 0}{\to} 1$.
Observe that

$$\left(\frac{\log(1+y)}{y}\right)' = \frac{\frac{y}{1+y} - \log(1+y)}{y^2} = \left(1 + \frac{1}{1+y} - \log(1+y)\right)' = \frac{1}{(1+y)^2} - \frac{1}{1+y} = -\frac{y}{(1+y)^2} < 0$$

Theorem

Cantor sets and perfect symmetric sets are closed, perfect, uncountable, and nowhere dense.