Manifolds II

January 6, 2025

Recall: Tangent Bundle

Given a chart (U,ϕ) about a point p, we have coordinates $(x^1,\dots,x^n)$ and a basis for T_qM of $\left(\frac{\partial}{\partial x^1}|_q,\dots,\frac{\partial}{\partial x^n}|_q\right)$ for $q \in U$.
Then given $TM \xrightarrow{\pi} M$, we may write $v_q = v^i \frac{\partial}{\partial x^i}|_q$.

Definition:

For M a topological manifold. A (real) vector bndle of rank k over M is a topological space E with a surjective continuous map $\pi : E \to M$ such that

1. $\forall p \in M$, the fiber $\pi^{-1}(p) =: E_p$ is endowed with the structure of a (real) vector space of dimension k.
2. $\forall p \in M$, there exists a neighborhood U of p in M and a homeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ called a local trivialization.

$$\begin{tikzcd} \Phi : \pi^{-1}(U) \arrow[rr] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^k \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd}$$

and $\Phi|_{E_q} : E_q \to \{q\} \times \mathbb{R}^k$ is a linear isometry.

Examples

1. $TM \xrightarrow{\pi} M$
2. $E = M \times \mathbb{R}^k$ with a global trivialization.
3. The Mobius bundle over S^1. $\gamma : \mathbb{R}^2 \to \mathbb{R}^2$ by $(x,y) \mapsto (x+1, (-1)\cdot y)$. Then $\langle\gamma\rangle \cong \mathbb{Z}$ a subgroup acting freely and isometrically on $\mathbb{R}^2$. Then $E = \mathbb{R}^2/\langle\gamma\rangle \xrightarrow{\pi} S^1 = \mathbb{R}/\mathbb{Z}$ by $\overline{(x,y)} \mapsto \overline{x}$ is a vector bundle.

IMAGE 1

- We want to show that $\pi^{-1}(U) \cong U \times \mathbb{R}$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "q"] \arrow[d, "\pi"] & E \arrow[d, "\pi"] & (x,y) \arrow[r, mapsto] \arrow[d, mapsto] & \overline{(x,y)} \arrow[d, mapsto] \\ \mathbb{R} \arrow[r, "\varepsilon"] & S^1 & x \arrow[r, mapsto] & e^{(2\pi i)x} \end{tikzcd}$$

Then let $p \in S^1$. We choose U a neighborhood of p such that U is evenly covered by ε. This means $\varepsilon^{-1}(U)$ is a disjoint union of open sets difeomorphic to U.

IMAGE 2

Let $\tilde{U}$ be a component in $\pi^{-1}(U)$. Then $\pi_1^{-1}(\tilde{U}) \cong \tilde{U} \times \mathbb{R}$ and $\pi^{-1}(U)$ is diffeomorphic to $U \times \mathbb{R}$.

Definition: Transition Function

Take $E \xrightarrow{\pi} M$ with $U, V \subseteq M$ admitting trivializations $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ and $\Psi : \pi^{-1}(V) \to V \times \mathbb{R}^k$. Let $w = U \cap V (\neq \varnothing)$.

$$\Phi \circ \Psi^{-1} : \quad \begin{tikzcd} W \times \mathbb{R}^k \arrow[r] \arrow[dr] & \pi^{-1}(W) \arrow[r] \arrow[d] & W \times \mathbb{R}^k \arrow[dl] \\ & W & \end{tikzcd}$$

Then $\Phi \circ \Psi^{-1}|_{\{p\}\times\mathbb{R}^k}$ by $\{p\} \times \mathbb{R}^k \to \{p\} \times \mathbb{R}^k$ is a linear isomorphism.
$\Phi \circ \Psi^{-1}(p, v) = (p, \tau(p)v)$ by $\tau : p \mapsto \tau(p)$ and $\tau(p) \in GL(k, \mathbb{R})$ gives a smooth map $W \to GL(k, \mathbb{R})$.

Definition:

Let $\{E_1, \ldots, E_k\}$ be a basis of $\mathbb{R}^k$. Then

$$\tau(p) \cdot E_i = \sum_j \tau(p)_i^j E_j$$

with $\tau(p) = (\tau(p)_i^j)$ and $\tau(p)_j^i \in \mathbb{R}$. It suffices to show each $\tau(*)_i^j$ mapping $W \to \mathbb{R}$ and $p \mapsto (\tau(p)_i^j)$ is smooth. Then if $\sigma(p, v) := \Phi \circ \Psi^{-1}(p, v)$, $\tau(p)_i^j = \pi_j(\sigma(p, E_i))$ and π_j is a projection to the j-th component in $\mathbb{R}^k$.

Lemma 10.6 (Vector Bundle Chart Lemma)

Given M a smooth manifold, suppose that $\forall p \in M$ we are given a vector space E_p of dimension k. Let $E = \coprod_{p \in M} E_p$ (as a set) and $\pi : E \to M$ a mapping E_p to p. Suppose also that we have

1. $\{U_\alpha\}_{\alpha \in A}$ an open cover of M with a countable subcover.
2. $\forall \alpha \in A$ we hav ea bijection $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ such that $\Phi_\alpha|_{E_p} : E_p \to \{p\} \times \mathbb{R}^k$ is a linear isomorphism.
3. $\forall \alpha, \beta \in A$ with $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \varnothing$ we have a smooth map $\tau_{\alpha\beta} : U_{\alpha\beta} \to GL(k, \mathbb{R})$ such that $\Phi_\alpha \circ \phi_\beta^{-1} : U_{\alpha\beta} \times \mathbb{R}^k \to U_{\alpha\beta} \times \mathbb{R}^k$ by $(p, v) \mapsto (p, \tau(p)v)$.

Then $E \xrightarrow{\pi} M$ is a vector bundle.

Example (Whitney Sum):

Suppose we have $E' \xrightarrow{\pi'} M$ and $E'' \xrightarrow{\pi''} M$ two vector bundles over M.
Define $E = E' \oplus E''$ a new vector bundle over M by $E_p = E'_p \oplus E''_p$. Let $\{U_\alpha\}_{\alpha \in A}$ be a countable open cover of M such that each U_α admits trivializations for E' and E''. Then for $\pi : E \to M$, define $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{k'} \times \mathbb{R}^{k''}$ by $(v', v'')_p \mapsto (p, \pi_2 \circ \Phi_\alpha^{-1}(v'), \pi_2 \circ \Phi''_\alpha(v''))$ where

$$\pi'(U_\alpha) \xrightarrow{\Phi'_\alpha} U_\alpha \times \mathbb{R}^{k'} \xrightarrow{\pi_2} \mathbb{R}^{k'}$$

Note that π_2 is the projection into the second component. Then $\tau : U_{\alpha\beta} \to G(k' + k'', \mathbb{R})$ by

$$p \mapsto \begin{pmatrix} \tau'(p) & 0 \\ 0 & \tau''(p) \end{pmatrix}$$

Example

For $\tau_{\alpha\beta} : U_{\alpha\beta} \to GL(k,\mathbb{R})$ by $p \mapsto \tau_{\alpha\beta}(p)$, we can write $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cup U_\gamma (\neq \varnothing)$ and get $\tau_{\alpha\beta} \cdot \tau_{\beta\gamma} = \tau_{\alpha\gamma}$.
Note that this is $\Phi_\alpha \circ (\phi_\beta^{-1} \circ \phi_\beta) \circ \Phi_\gamma^{-1}$.
Without loss of generality, we assume each U_α is a chart for M. Then we want to show that we satisfy Lemma 1.35 from Lee

$$\pi^{-1}(U_\alpha) \xrightarrow{\Phi_\alpha} U_{\alpha \times \mathbb{R}^k} \xrightarrow{\phi_\alpha \times \mathrm{id}} \phi_\alpha(U_\alpha) \times \mathbb{R}^k \subseteq \mathbb{R}^{n+k}$$

$(\pi^{-1}(U_\alpha) \cdot \tilde{\phi}_\alpha = (\phi_\alpha \times \mathrm{id}) \circ \Phi_\alpha)_{\alpha \in A}$ which satisfies (1).
Since

$$\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta) = \pi^{-1}(U_\alpha \cap U_\beta) \to \phi_\alpha(U_{\alpha\beta}) \times \mathbb{R}^K$$

we have that (2) is satisfied.
Finally, for (3),

$$\tilde{\phi_\beta} \circ \tilde{\phi_\alpha}^{-1} = (\Phi_\beta \circ (\phi_\beta \times \mathrm{id})) \circ ((\phi_\alpha \times \mathrm{id})^{-1} \circ \Phi_\alpha^{-1}) = \Phi_\beta \circ ((\phi_\beta \circ \phi_\alpha) \times \mathrm{id}) \circ \Phi_\alpha^{-1}$$

gives $(x,c) \mapsto ((\phi_\beta \circ \phi_\alpha^{-1})x, (\Phi_\beta \circ \Phi_\alpha^{-1})v)$ a diffeomorphism.
(4) and (5) are trivial, and this is indeed a smooth manifold. Now we wish to show that it is a vector bundle. To show that $\pi : E \to M$ is smooth,

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[r, "\pi"] & U_\alpha \arrow[d, "\phi_\alpha"] \\
\phi_\alpha(U_\alpha) \times \mathbb{R}^k \arrow[u, "\tilde{\phi}_\alpha^{-1}"] & \phi_\alpha(U_\alpha)
\end{tikzcd}$$

We have $\tilde{\phi}_\alpha^{-1} = (\phi_\alpha \times \mathrm{id})^{-1} \circ \Phi_\alpha^{-1}$.

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[r, "\Phi_\alpha"] & U_\alpha \times \mathbb{R}^k \arrow[d, "\phi_\alpha \times \mathrm{id}"] \\
\phi_\alpha(U_\alpha) \times \mathbb{R}^k \arrow[u, "\tilde{\phi}_\alpha^{-1}"] & \phi_\alpha(U_\alpha \times \mathbb{R}^k)
\end{tikzcd}$$

Definition: Section of a Bundle

A (smooth) section of $E \xrightarrow{\pi} M$ is a (smooth) map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{id}_M$.
$\Gamma(E) = \{$smooth sections of $E \xrightarrow{\pi} M\}$ and $\Gamma(E)$ is a $C^\infty(M)$-module.
The zero section $Z : M \to E$ is given by $p \mapsto 0_p \in E_p$.
If U has a local trivialization, $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$.

$$\begin{tikzcd}
\Phi : & \pi^{-1}(U) \arrow[rr] & & U \times \mathbb{R}^k & & (p, e_i) \arrow[ll, "\Phi^{-1}"'] \\
& & U \arrow[ul, dashed] \arrow[ur] & & p \arrow[ur, mapsto, "\tilde{e}_i"] &
\end{tikzcd}$$

Define $\sigma_i : U \to \pi^{-1}(U)$ by $\sigma_i = \Phi^{-1} \circ \tilde{e}_i$ gives a local section that is non-zero on U.
$\{\sigma_1, \ldots, \sigma_n\}$ form a local frame on U (i.e. form a basis in E_p, $\forall\, p \in U$).