Algebra II

January 8, 2024

How To Prove a Big Theorem

1. Reduce to a linear algebra problem.
2. Solve the linear algebra problem.

Grades

- Weekly Homework
  - For completion, graded by peers or presented. Survey to follow.
- Midterm
- Final
  - March 18, 2024
  - 4:00 PM to 7:00 PM

Office Hours

McHenry 4174
Monday / Wednesday from 1:05 PM to 2:05 PM.
E-mail ahead if arriving promptly at 1:05 PM.

Definition: Module

Let R be a ring.
A (left) R-module is a set M with binary operations $\cdot : R \times M \to M$ and $+ : M \times M \to M$ such that

1. $(M, +)$ is an Abelian group.
   (a) $\exists 0 \in M$ such that $\forall m \in M$, $m + 0 = m = 0 + m$.
   (b) $\forall m \in M$, $\exists n \in M$ such that $m + n = 0 = n + m$.
   (c) $\forall m_1, m_2, m_3 \in M$, $(m_1 + m_2) + m_3 = m_1 + (m_2 + m_3)$.
   (d) $\forall m_1, m_2 \in M$, $m_1 + m_2 = m_2 + m_1$.
2. Distibution.

$$\begin{aligned} (r_1 + r_2) \cdot m &= r_1 \cdot m + r_2 \cdot m \\ r \cdot (m_1 + m_2) &= r \cdot m_1 + r \cdot m_2 \end{aligned}$$

3. $1 \cdot m = m$ where $1 \in R$ is the multiplicative identity.

4. $(r_1 \cdot r_2) \cdot m = r_1 \cdot (r_2 \cdot m)$

- Note that $\cdot$ may represent scalar multiplication or multiplication in the ring.

Example 1

$n \in \mathbb{Z}$, $n = 1, 2, 3, \ldots$, $R = \mathbb{R}$, $M = \mathbb{R}^n$, equipped with $+$ vector addition and $\cdot$ scalar multiplication.

Example 2

Let R be your favorite field $\mathbb{Z}/p$, $\mathbb{Q}$, $\mathbb{C}$, $\mathbb{F}_q$, $\mathbb{Q}_p$, and $M = \mathbb{R}^n$.
Similarly with rings $R = \mathbb{Z}$, $R = \mathbb{Z}[x]$, etc.

Example 3

Let $R = \mathbb{Z}$ and M be your favorite Abelian group.

Example 4

Let R be any ring (e.g. $\mathbb{Z}[x]$) and M be any left ideal (e.g. $R \cdot x + R \cdot 3$).

Example 5

Fix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$.
Let $R = \mathbb{R}[x]$, the polynomial ring, and $M = \mathbb{R}^2$ where $+$ is standard addition, and $\cdot$ is matrix multiplication.

$$x \cdot m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot m$$

Example 6

Let R be any ring and M be functions $R \to R$ where $+$ and $\cdot$ are pointwise operations.

Example 6'

Let $R = \mathbb{R}$ and have M require that f is continuous, differntiable, etc.