Analysis III

Homework

Exercises (homework) can be found in the script at the end of each Chapter.
Chapter I: # 3 (only for convex sets), # 4 due Th 4-25
Chapter II: # 2,4,5,6 due Th 5-2
Chapter III: # 3c, 4 due Th 5-9
Chapter IV: # 2b, 3, 4, 6 due Th 5-16
Chapter V: # 2,4,6 due Th 5-25
Chapter VI: # 2,3,4 due Th 6-1

Key Dates

Instruction begins: Mo, April 1
Instruction ends: Fr, June 7
Final's week: June 10-13 (Mo-Th)
Holiday: Mo, May 27

April 2, 2024

No class Thursday, April 04.
Makeup class (tentatively) on Friday, April 12 at 10:30.
Discussion sections on Fridays (tentatively) at 11:40.

Topological Vector Spaces

Definition: Vector Spaces

V over a field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$).

Definition: Topological Spaces

(X, τ) where $\tau \subseteq \mathcal{P}(X)$ satisfying

1. $\varnothing, X \in \tau$
2. $A, B \in \tau \implies A \cap B \in \tau$
3. $A_\omega \in \tau \implies \bigcup_\omega A_\omega \in \tau$

Recall: $A \in \tau \iff A$ open $\iff X \setminus A$ closed.
$A^\circ = \bigcup_{\substack{U \subseteq A \\ U \text{ open}}} U$ the set of interior points of A.
$\overline{A} = \bigcap_{\substack{F \supseteq A \\ F \text{ closed}}}$ the closure of A.
A' limit points of A.
Compact sets.
Locally compact sets.
Recall: X is Hausdorff iff $\forall x, y \in X$, $\exists U, V \in \tau$ such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$.

Definition: Bases for Topological Spaces

Definition: Let (X, τ) be a topological space. $\sigma \subseteq \tau$ is called a base for topology τ if $\forall x \in X$, $\forall U \in \tau$, $x \in U$, $\exists W \in \sigma$ such that $x \in W \subseteq U$.

Proposition

$\sigma \subseteq \tau$ is a base for τ if and only if every $U \in \tau$ is the union of certain sets taken from σ.

$$(*) \quad \tau = \left\{ \bigcup_{\omega \subset \Omega} W_\omega \,:\, \{W_\omega\}_{\omega \in \Omega} \subseteq \sigma,\, \Omega \right\}$$

Proof

$(\Longleftarrow)$ ✓
$(\Longrightarrow)$ Take $U \in \tau$ and let $x \in U$, $\rightsquigarrow$ find $W_x \in \sigma$, $x \in W_x \subseteq U$.

$$U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} W_x \subseteq U$$

Therefore $\bigcup_{x \in U} W_x = U$.

Proposition

If σ is a base for some topology τ on X, then

1. $\forall x \in X$, $\exists W \in \sigma$ such that $x \in W$.
2. $\forall U, V \in \sigma$, $\forall x \in U \cap V$, $\exists W \in \sigma$ such that $x \in W \subseteq U \cap V$.

Conversely, if $\sigma \in \mathcal{P}(X)$ ($\varnothing \notin \sigma$) satisfies (1) and (2), then σ is the base for a topology τ (and τ is given by $(*)$).
Note that $U, V \in \tau \implies U \cap V \in \tau$ (requires (2)).
If $U = \bigcup U_\alpha$ and $V = \bigcup V_\beta$, then $U \cap V = \bigcup_{\alpha,\beta}(U_\alpha \cap V_\beta) = \bigcup_{\alpha,\beta} \bigcup_{x \in U} W_{\alpha,\beta,x}$.

Example: Metric Spaces

(X, d) is a metric space if $d : X \times X \to [0, +\infty)$ satisfies

1. $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

Definition: Epsilon Neighborhoods

$$B_\varepsilon(x) = \{y \in x \,:\, d(x, y) < \varepsilon\}$$

$A \subseteq X$ is open if and only if $\forall x \in A$, $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A$. $x \in B_\varepsilon(x)$.

τ = set of all open sets.

$$\sigma_1 = \{B_\varepsilon(x) \,:\, x \in X, ; \varepsilon > 0\}$$

is a base for the topology τ.

$$\sigma_2 = \left\{B_{1/n}(x) \,:\, x \in X,\ n \in \mathbb{N}\right\}$$

is also a base for τ.

Definition: Direct Product - Product Topology

Let (X_1, τ_1) and (X_2, τ_2) be topological spaces.
Consider $X = X_1 \times X_2$. The product topology τ on X is given by the base

$$\sigma = \{U_1 \times U_2 \,:\, U_1 \in \tau_1,\ U_2 \in \tau_2\}$$

More explicitly,

$$\tau = \left\{\bigcup_{\omega} U_{1,\omega} \times U_{2,\omega} \,:\, U_{i,\omega} \in \tau_i\right\}$$

(X_ω, τ_ω) topological spaces ($\omega \in \Omega$)

$$X = \prod_{\omega \in \Omega} = \{(x_\omega)_{\omega \in \Omega} \,:\, x_\omega \in X_\omega\}$$

Formally, $f \cong (x_\omega)_{\omega \in \Omega}$, $x_\omega = f(\omega)$, $f : \Omega \to \bigcup_{\omega \in \Omega} X_\omega$ such that $f(\omega) \in X_\omega$.
[$x \neq \varnothing \Longleftarrow X_\omega \neq \varnothing$ axiom of choice]

$$\sigma = \left\{\prod_{\omega \in \Omega} U_\omega \,:\, U_\omega \in \tau_\omega \text{ and all but finitely many } U_\omega = X_\omega\right\}$$

Definition: Subspace Topology

Given (X, τ) and $Y \subseteq X$, then (Y, τ_Y) is also a topological space where

$$\tau_Y \{U \cap Y \,:\, U \in \tau\}$$

Definition: Local Bases for Topological Spaces

A collection $\gamma \subseteq \tau$ is called a local base at $x \in X$ if

1. $\forall U \in \tau$, $x \in U$, $\exists W \in \gamma$ such that $x \in W \subseteq U$.
2. $\forall W \in \gamma$, $x \in W$

Example

Let (X, d) be a metric space.

$$\gamma_x = \{B_\varepsilon(x) \,:\, \varepsilon > 0\}$$

is a local base at x. Similarly,

$$\tilde{\gamma}_x = \left\{B_{1/n} \,:\, n \in \mathbb{N}\right\}$$

is a countable local base.

Proposition

If γ_x $(x \in X)$ are local bases for τ at X, then

$$\sigma = \bigcup_{x \in X} \gamma_x$$

is a bse for τ.

Proposition

$\{\gamma_x\}_{x \in X}$ are local bases at x for some topology τ if and only if

1. $\forall x \in X$, γ_x is a non-empty collection of subsets containing x.
2. If $U \in \gamma_x$, $V \in \gamma_y$, and $z \in U \cap V$, then $\exists W \in \gamma_z$ such that $z \in W \subseteq U \cap V$.

Definition: Topological Vector Spaces

Suppose V is a vector space over $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ and let τ be a topology on V.
Then V is a topological vector space (TVS) if

1. $\forall x \in V$, $\{x\}$ is closed.
2. The functions f, g (i.e. algebraic operations) are continuous.

   $f : V \times V \to V,\ f(x, y) = x + y$
   $g : \mathbb{F} \times V \to V,\ g(\lambda, x) = \lambda \cdot x$

Notation

For $A_1, A_2 \subseteq V$ and $B \subseteq \mathbb{F}$,

$$\begin{aligned} A_1 + A_2 &= \{a_1 + a_2 \,:\, a_1 \in A_1,\ a_2 \in A_2\} \\ a + A_1 &= \{a + \alpha \,:\, \alpha \in A\} \\ B \cdot A &= \{\beta \cdot a \,:\, \beta \in B,\ a \in A\} \\ \alpha \cdot A &= \{\alpha \cdot a \,:\, a \in A\} \end{aligned}$$

Lemma

Let V be a TVS. Then

1. $\forall x, y \in V$, $\forall$ open $U_{x+y} \ni x+y$, $\exists$ open $U_x \ni x$, open $U_y \ni y$ such that $U_x + U_y \subseteq U_{x+y}$.
2. $\forall x \in V$, $\alpha \in \mathbb{F}$, $\forall$ open $U_{\alpha x} \ni \alpha x$, $\exists$ open $U_x \ni x$, $U_\alpha \ni \alpha$ such that $U_\alpha \cdot U_x \subseteq U_{\alpha x}$.

Proof of 1

Given $x, y \in X$, $x + y \in U_{x+y}$ open.

$$f(x, y) = x + y \in U_{x+y}$$

and $(x, y) \in f^{-1}(U_{x+y})$ open. In the product topology

$$(x, y) \in U_x \times U_y \subseteq f^{-1}(U_{x+y})$$

which implies $x \in U_x$ and $y \in U_y$, both open, and $U_x + U_y \leq U_{x+y}$.

April 9, 2024

Office Hours: Th 11:30 AM - 1:00 PM
Makeup Class: Friday, April 12 at 11:00 AM.

Lemma 1

Let V be a TVS

1. $\forall x, y \in V$, $\forall U_{x+y} \ni x+y$ open, $\exists U_x \ni x, U_y \ni y$ such that $U_x + U_y \subseteq U_{x+y}$.
2. $\forall \alpha \in F$, $\forall U_{\alpha x} \ni \alpha x$ open, $\exists U_\alpha \ni \alpha$ open in F, $U_x \ni x$ such that $U_\alpha \cdot U_x \subseteq U_{\alpha x}$.

For 2. with $\alpha = 0$, $\forall x \in X$, $\forall U \ni 0$ open, $\exists \delta > 0$, $U_x \ni x$ open such that $B_\delta(0) \cdot U_x \subseteq U$. That is, $\beta U_x \subseteq U$, $\forall |\beta| < \delta$.

Proposition

In a TVS, the maps

1. Translation: $T_a : x \in V \mapsto X + a \in V \ (a \in V)$
2. Multiplication: $M_\lambda : x \in V \mapsto \lambda \cdot x \in V \ (\lambda \in \mathbb{F},\ \lambda \neq 0)$

are continuous (in fact, homeomorphic).

Proof

We know $(x, y) \mapsto x + y$ and $(\lambda, x) \mapsto \lambda \cdot x$ are continuous.

Inversions

$T_a \circ T_{-a} = \text{id}$, $T_{-a} \circ T_a = \text{id}$, $M_\lambda \circ M_{1/\lambda} = \text{id}$, and $M_{1/\lambda} \circ M_\lambda = \text{id}$.
Therefore they are bijective and the inverses are continuous.

Remark

If U is open, then $a + U$ is also open.
If γ_0 is a local base at 0, then $\gamma_x = \{x + U \,:\, U \in \gamma_0\} = x + \gamma_0$ is a local base at x.
Recall that γ_x is a local base at x if $\forall W \ni x$ open, $\exists U \in \gamma_x$ such that $x \in U \subseteq W$.
That is, in a TVS only local bses at 0 are needed. We may interpret “local base” as “local base at 0”.

$$\sigma = \bigcup_{x \in V} (x + \gamma_0)$$

is a base for the TVS.

Types of Topologial Vector Spaces

Normed Spaces / Banach Spaces

A normed space is a vector space over $\mathbb{F}$ together with a norm $||\cdot||$, i.e. a map $||\cdot|| : x \in V \mapsto ||x|| \in [0, \infty)$ such that

1. $||x|| = 0 \iff x = 0$.
2. $||x + y|| \leq ||x|| + ||y||$.
3. $||\lambda x|| = |\lambda| \cdot ||x||$.

Remarks

A normed space is a metric space with $d(x, y) = ||x - y||$.
A local base (at 0) is given by ε-neighborhoods:

$$\gamma_0 = \{B_\varepsilon(0) \,:\, \varepsilon > 0\}$$

where

$$B_\varepsilon(0) = \{x \in V \,:\, ||x|| < \varepsilon\}$$

(open ball with radius $\varepsilon > 0$).

Convergence in Normed Space

A sequence $\{x_n\}$ ($x_n \in V$) converges to $\lambda \in V$ if $\lim_{n\to\infty} ||x_n - x|| = 0$.
A sequence $\{x_n\}$ is Cauchy if $\forall \varepsilon > 0$, $\exists N$ such that $\forall j, k \geq N$, $||x_j - x_k|| < \varepsilon$.
A normed space is complete if $\{x_n\}$ Cauchy implies $\exists x \in V$ such that $x_n \to x$.
Complete normed spaces are called Banach spaces.

Example 1

$\ell^p(\mathbb{N})$, $1 \le p < \infty$, the set of all sequences $\{x_n\}_{n=1}^{\infty} = x$ such that

$$||x|| = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} < +\infty$$

Recall $\{x_n\} + \{y_n\} = \{x_n + y_n\}$ and $\lambda\{x_n\} = \{\lambda x_n\}$.
ℓ^p spaces are complete and therefore Banach.
If $\{x_n\} \in \ell^p$ and $\{y_n\} \in \ell^q$, then $\{x_n y_n\} \in \ell^r$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \in [0,1]$ (e.g. $\ell^2 \cdot \ell^2 \le \ell^1$)

Example 2

$\ell^\infty(\mathbb{N})$, the set of all bounded sequences

$$||x|| = \sup_{n\in\mathbb{N}} |x_n| < +\infty$$

Example 3

$C_0(\mathbb{N}) \subseteq \ell^p(\mathbb{N})$, the set of all sequences $\{x_n\}$

$$\lim_{n\to\infty} x_n = 0$$

C_0 is a closed subspace, and both are Banach.

Example 4

$L^p(\Omega)$, $1 \le p < \infty$, $\Omega \subseteq \mathbb{R}^d$ a Lebesgue measurable set with $m(\Omega) > 0$, the space of all equivalence classes of Lebesgue measurable functions $f : \Omega \to \mathbb{F}$ such that

$$||f|| = \left(\int_\Omega |f(x)|^p \, dx\right)^{1/p} < +\infty$$

Example 5

$L^\infty(\Omega)$, the measurable and essentially bounded functions

$$\begin{aligned} ||f|| &= \inf_{\substack{N\subseteq\Omega \\ m(N)=0}} \sup_{X\in\Omega\setminus N} |f(x)| < +\infty \\ &= \text{ess sup}_{x\in\Omega} |f(x)| \end{aligned}$$

$L^p(\Omega)$ spaces, $1 \le p \le \infty$, are Banach.

Example 6

For $\Omega \ne \varnothing$, let $B(\Omega)$ the set of all bounded functions $f : \Omega \to \mathbb{F}$ with

$$||f|| = \sup_{x\in\Omega} |f(x)|$$

is a Banach space.
$f_n \to f$ in $B(\Omega)$ if and only if f_n converges uniformly on Ω to f.

Example 7

Let Ω be a topological space and $BC(\Omega)$ the set of all bounded, continuous functions $f : \Omega \to \mathbb{F}$.
Then $BC(\Omega) \subseteq B(\Omega)$ is a closed Banach subspace under the same norm.
That is, the uniform limit of continuous functions is a continuous function.

$$\underset{f_n \in BC(\Omega)}{f_n \to f} \implies f \in B(\Omega)$$

Example 8

Let K be a compact, Hausdorff space.
Then $C(K)$ is the set of all continous functions $f : K \to \mathbb{F}$ and $C(K) = BC(K)$.

F Spaces / pre-F Spaces

A pre-F-space is a TVS where the topology is given by some invariant metric $d(x+z, y+z) = d(x,y)$ or $d(x,y) = d(x-y, 0)$.
An F-space is a complete pre-F-space.
A local base (at 0) is given by

$$\gamma_0 = \{B_\varepsilon(0) \ : \ \varepsilon > 0\}, \quad B_\varepsilon(x) = \{y \in V \ : \ d(x,y) < \varepsilon\}$$

Example 1

$\ell^p(\mathbb{N})$, $0 < p < 1$, the set of all $\{x_n\}_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} |x_n|^p < +\infty$$

with

$$d(x,y) = \sum_{n=1}^{\infty} |x_n - y_n|^p$$

Note that this obeys the triangle inequality specifically because it has not been raised to $1/p$.

$$d(\lambda x, \lambda y) = |\lambda|^p \cdot d(x,y)$$

means that $d(z, 0)$ is not a norm.
Here, $B_\varepsilon(x)$ are not convex sets.

Side Remark

Given $\mathbb{R}^2$, the ℓ^p norm for $1 \leq p \leq \infty$ is given by

$$||(x_1, x_2)|| = \left(|x_1|^p + |x_2|^p\right)^{1/p}$$

and the distance for $0 < p < 1$ by

$$d((x_1, x_2))) = |x_1|^p + |x_2|^p$$

The ε neighborhoods for $p=1$ are diamonds, $p=2$ circles, $p=\infty$ squares with smooth transition between them.
However, for $0<p<1$, we have concave diamond shapes.
These norms and metrics are all equivalent on $\mathbb{R}^2$ in the sense that they give the same topology.

Locally Convex TVS

A TVS which has a local base γ at 0 consisting of open neighborhoods of 0 which are all convex.

Definition: Convex Set

A set $A \subseteq V$ is convex if $\forall x, y \in A$, $\lambda \in [0,1]$, then $\lambda x + (1-\lambda) y \in A$
Alternatively, the line segment between x and y is contained in A ($[x,y] \subseteq A$).

Fréchet Spaces and pre-Fréchet Spaces

A pre-Fréchet space is locally convex.
A Fréchet space is a locally convex F-space.

April 11, 2024

Fréchet Spaces

Example

$\mathcal{S} = \{\{\{x_n\}_{n=1}^{\infty}$ the space of all sequences $x_n \in \mathbb{F}$.

$$d(\{x_n\},\{y_n\}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|x_n - y_n|}{1+|x_n-y_n|} < +\infty$$

Triangle inequality:

$$\frac{a+b}{1+a+b} \le \frac{a}{1+a} + \frac{b}{1+b}, \quad a,b \ge 0$$

invariant metric, complete.
$\gamma_0 = \{B_\varepsilon(0) : \varepsilon > 0$ is a local base.
$\hat{\gamma}_0 = \{U_{\varepsilon,N} : \varepsilon > 0, N \in \mathbb{N}\}$.
$U_{\varepsilon,N} = \{\{x_n\}_{n=1}^{\infty} : |x|_n < \varepsilon, \forall n = 1,\dots,n\}$.
$\forall \varepsilon > 0$, $\exists \hat{\varepsilon} > 0, N$ such that $U_{\hat{\varepsilon},N} \subseteq B_\varepsilon(0)$.
$\forall \hat{\varepsilon} > 0, N$, $\exists \varepsilon > 0$ such that $B_\varepsilon(0) \subseteq U_{\hat{\varepsilon},N}$.
$x^{(m)} \to x$ in metric of $\mathcal{S}$ as $m \to \infty$.
$x^{(m)} = \{x_n^{(m)}\}_{n=1}^{\infty}$, $x = \{x_n\}_{n=1}^{\infty}$ if and only if $\forall n \in \mathbb{N}$, $x_n^{(m)} \to x_n$ as $m \to \infty$ (pointwise, componentwise convergence).

Example

$C(\mathbb{R}^d)$, the set of continuous functions $f : \mathbb{R}^d \to \mathbb{F}$.

$$|||f|||_N = \sup_{\substack{x \in \mathbb{R}^d \\ ||x|| \le N}} |f(x)|$$

$$d(f,g) = \sum_{N=1}^{\infty} \frac{1}{2^N} \cdot \frac{|||f-g|||_N}{1+|||f-g|||_N}$$

Complete Fréchet space.
“Locally uniform congergence” such that $f_n \to f$ in metric of $C(\mathbb{R}^d)$ if and only if $\forall$ compact set $K \subseteq \mathbb{R}^d$, f_n converges to f uniformly on K.

Example

$C^\infty[0,1]$ the set of infinitely differentiable functions $f : [0,1] \to \mathbb{F}$.

$$|||f|||_n = \sup_{x \in [0,1]} |f^{(n)}(x)|$$

$$d(f,g) = \sum_{n=0}^{\infty} \frac{1}{2^n} \cdot \frac{|||f-g|||_n}{1-|||f-g|||_n}$$

Fréchet space.
$f_m \to f$ in $C^\infty[0,1]$ as $m \to \infty$ if and only if for every $m \in \{0,1,\dots\}$, $f_m^{(n)} \to f^{(n)}$ uniformly on $[0,1]$ as $m \to \infty$.

Proposition

Every TVS is Huasdorff.

Proof

Let $x, y \in V$, $x \neq y$.
For $U = V \setminus \{0\}$, and open set, $x - y \in U$.
Using the continuity of $(x^2, y^2) \mapsto x^2 - y^2$ and Lemma 1, there exist $U_x \ni x$ and $U_y \ni y$ open such that $U_x - U_y \subseteq U$.
Note that $U_x \cap U_y = \varnothing$, otherwise there would exist $z \in U_x \cap U_y$ such that $0 = z - z \in U_x - U_y \subseteq U$ a contradiction.

Definition: Balancedness

A subset U of a vector space V is called balanced if $\forall \lambda \in \mathbb{F}$, $|\lambda| \leq 1$, $\lambda U \subseteq U$.

Example

For $V = \mathbb{R}^2$, $\mathbb{F} = \mathbb{R}$, an ellipse is convex and balanced.
Note that since $\lambda = 0$ is a valid choice, 0 is always in a balanced set.
A retangle, offset from the origin, is convex but not balanced.
A concave diamond centered at 0 may be balanced.
An annulus is neither.

Exercise

Show that for $V = \mathbb{C}$, $\mathbb{F} = \mathbb{C}$, the balanced, convex sets are the open and closed disks along with the entire plane.

Proposition

1. Every TVS has a balanced, local base.

2. Every locally convex TVS has a balanced and convex local base.

Proof of A

e.g. $\gamma = \{U : U \text{ open}, 0 \in U\}$.
For every $U \in \gamma$, construct another $\hat{U}$ open, $0 \in \hat{U} \subseteq U$ balanced.
Then $\hat{\gamma} = \{\hat{U} : U \text{ taken from } \gamma\}$ is a local base.
Use Lemma 1 again and the continuity of $(\lambda, x') \mapsto \lambda \cdot x'$ at $\lambda = 0$, $x' = 0$.
Given open $U \ni 0$, find $\delta > 0$ and open $U_0 \ni 0$ such that $B_{2\delta}(0) \cdot U_0 \subseteq U$.
Then for $\alpha \in \mathbb{F}$, $|\alpha| \leq \delta$, $\alpha \cdot U_0 \subseteq U$. Take

$$\hat{U} = \bigcup_{\substack{\alpha \in \mathbb{F} \\ |\alpha| \leq \delta}} \alpha \cdot U_0$$

Therefore $\hat{U}$ is a union of open sets and $0 \in \hat{U} \subseteq U$. Finally, for $|\lambda| \leq 1$,

$$\lambda \cdot \hat{U} = \bigcup_{\substack{\alpha \in \mathbb{F} \\ |\alpha| < \delta}} \lambda \cdot \alpha \cdot U_0 = \bigcup_{\substack{\beta \\ |\beta| \leq |\lambda| \cdot \delta \leq \delta}} \beta U_0 = \hat{U}$$

Proof of B

We have a local base $\gamma = \{U_\omega\}$, $U_\omega \ni 0$ open and convex.
We want to construct $\hat{\gamma} = \{\hat{U}_\omega\}$, $\hat{U}_\omega \ni 0$ open, convex and balanced.
Given U convex, define

$$\hat{U} = \bigcap_{|\alpha| \leq \delta} \alpha U$$

convex and balanced.
Need to show that $\hat{U} \ni 0$ is an open neighbrhood.
Rest of the owl left to the reader.

Recall

The intrinsic characterization of a base for a topological space or a local base for a topological space X, $\{\gamma_x\}_{x \in X}$.

- $\forall x \in X, U \in \gamma_x, x \in U$
- $\forall x, y \in X, U \in \gamma_x, V \in \gamma_y, z \in U \cap V, \exists W \in \gamma_z, W \subseteq U \cap V$.

Proposition

A balanced, local base γ (at 0) of a TVS V has the following properties:

1. γ is a nonempty collection of subsets of V containing 0.
2. $\forall U_1, U_2 \in \gamma$, $\exists U \in \gamma$ such that $U \subseteq U_1 \cap U_2$.
3. $\forall U \in \gamma$, $x \in U$, $\exists W \in \gamma$ such that $x + W \subseteq U$.

4. $\forall U \in \gamma$, $\exists W \in \gamma$ such that $W + W \subseteq U$ (continuity of $(x, y) \mapsto x + y$ at $(x = y = 0)$).

5. $\forall U \in \gamma$, $\forall x \in V$, $\exists t > 0$, $x \in t \cdot U$ (continuty of scalar multiplication $(\lambda, x') \mapsto \lambda x'$ at $\lambda = 0$, $x' = x$).

   $\frac{1}{t} \cdot x \in U$, $\frac{\delta}{2} \cdot x \subset B_\delta(0) \cdot \hat{U} \subseteq U$.

6. $\forall x \in V$, $x \neq 0$, $\exists U \in \gamma$, $x \notin U$ ($\{x\}$ closed; $0 \in V \setminus \{x\}$ open; $0 \in U \subseteq V \setminus \{x\}$). (Hausdorff)

Converse

Conversely, if γ satisfies properties 1-6, then there exists a unique topology on V such that γ is a balanced, local base for V and V with this topology is a TVS.

Theorem:

Any two TVS of finite dimension d (over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) are homeomorphic to eachother.

Proof

Let V be a TVS with $\dim(V) = d$.
We want to show that $V \cong \mathbb{F}^d$. We have

$$V = \text{lin}\{v_1, \ldots, v_d\}$$

a basis and

$$f : (\lambda_1, \ldots, \lambda_n) \in \mathbb{F}^d \mapsto \sum_{i=1}^{d} \lambda_i v_i \in V$$

an isomorphism between $\mathbb{F}^d$ and V as vector spaces. Further, f is continuous. Consider $\mathbb{F}^d$ equipped with the product topology and the continuity of addition and scalar multiplication.
We need only that f^{-1} is continuous at 0 which is equivalent to $\forall U \ni 0$ open in $\mathbb{F}^d$, $\exists W \ni 0$ open in V such that $W \subseteq f(U)$ ($(f^{-1})^{-1}(U)$).

April 12, 2024

Lemma

$\forall U \ni 0$ open in $\mathbb{F}^d$, $\exists W \ni 0$ open such that $f(U) \supseteq W$.
That is, 0 is an interior point of $f(U)$.

Proof

$f : \mathbb{F}^d \to V$, continuous.
We may assume without loss of generality that $U = B_1(0)$.
Let $S = \{\lambda \in \mathbb{F}^d \, : \, ||\lambda|| = 1\}$, a compact set.
Since f continuous, $f(S)$ is compact in V. Since V is Hausdorff, $f(S)$ is closed.
Take $\hat{U} = V \setminus f(S) \ni 0$ open (because $0 \notin f(S)$ else $f(\lambda) = 0$ would imply $||\lambda|| = 1$)
Now, there exists a balanced, open set $0 \in W \subseteq \hat{U}$. Therefore, $W \subseteq f(U)$.
Otherwise, $x \in W$, $x \notin f(U)$, $x = f(\lambda)$, $\lambda \notin U$, $||\lambda|| \geq 1$ would give $\frac{x}{||\lambda||} = \frac{1}{||\lambda||} \cdot f(\lambda) = f\left(\frac{\lambda}{||\lambda||}\right) \in f(S)$.
But, $\frac{x}{||\lambda||} \in W \subseteq \hat{U}$ because $x \in W$, $\frac{1}{||\lambda} \in [0, 1]$ and W is balanced shows a contradiction.

Theorem

Any finite-dimensional subspace in a TVS is closed.

Theorem

Every locally compact TVS is finite-dimensional.

Definition: Locally Compact

V is locally compact if $\forall x \in V$, $\exists U \ni x$ open and $K \subseteq V$ such that $U \subseteq K$.
For Hausdorff spaces, $\forall x \in V$, $\exists U \ni x$ open such that $\overline{U}$ compact.

Example

Let V be a normed space, $\dim(V) = +\infty$.
Then $\overline{B_1(0)}\{x \in V \,:\, ||x|| \leq 1\}$ is not compact.

Definition: Semi-norm

A semi-norm on a metric space V (over $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$) is a map

$$p : V \to [0, +\infty)$$

such that

1. $p(x + y) \leq p(x) + p(y)$
2. $p(\lambda x) = |\lambda| \cdot p(x)$.

Note that $p(0) = 0$ and $(p(x - y) \geq |p(x) - p(y)|$.

$$N_p = \{x \in V \,:\, p(x) = 0\}$$

is a linear subspace of V: $x, y \in N$ such that $p(x + y) \leq p(x) + p(y) = 0,\ p(\lambda x) = 0$.
A semi-norm on V induces a norm on the quotient space V/N_p.

$$||[x]_{N_p}|| = p(x) \quad [x]_N = \{x + z \,:\, z \in N\} = x + N$$

Definition: Absorbing

A set $A \subseteq V$ is called absorbing if $\forall x \in V$, $\exists \lambda > 0$ such that $\lambda x \in A$.
Equivalently, $\bigcup_{\lambda > 0} \frac{1}{\lambda} A = V$.
There is a relationhip between semi-norms on V and balanced, convex and absorbing subsets of V.

Proposition

If p is a semi-norm on a vector space V, then

$$A = \{x \in V \,:\, p(x) < 1\}$$

is balanced, convex and absorbing.

Proof

Convex: $x, y \in A,\ p(x) < 1,\ p(y) < 1,$

$$p(\lambda x + (1-\lambda)y) \leq \lambda \cdot p(x) + (1-\lambda)p(y) < 1$$

Balanced: $x \in A,\ |\lambda| \leq 1,\ p(x) < 1,$

$$p(\lambda x) = |\lambda| \cdot p(x) < 1$$

Absorbing: $x \in V$. If $p(x) = 0$, then $x \in A$ $(\lambda = 1)$.
If $p(x) > 0$, $\lambda = \frac{1}{2p(x)}$ gives $p(\lambda x) = |\lambda| \cdot p(x) = \frac{1}{2} < 1.$

Example

Let $V = \mathbb{R}^2$ and $\mathbb{F} = \mathbb{R}$.
An ellipse is balanced, convex and absorbing.
A band is also balanced, convex and absorbing.
An open interval along the axis is balanced and convex, but it is not absorbing.

Proposition

Each open neighborhood of 0 in a TVS is absorbing.

Proof

Continuity of the map $(\lambda, x) \mapsto \lambda x'$ at $\lambda = 0$ and $x' = x$.
Given $x \in V$, $U \ni 0$ open, $\exists \delta > 0,\ W \ni x$ such that $B_r(0) \cdot W \subseteq U$ and $\frac{\delta}{2} \cdot x \in U$.

Definition: Minkowski Functional

Let A be a subset in a vector space V.
If A is absorbing, one can define the Minkowski functional

$$\mu_A(x) = \inf\left\{\lambda > 0 \,:\, \frac{x}{\lambda} \in A\right\} = \inf\{\lambda > 0 \,:\, x \in \lambda \cdot A\}$$

Proposition

If A is convex, balanced and absorbing, then μ_A is a semi-norm.

Proof

Absorbing $\rightsquigarrow \mu_A$ is well defined, $\mu_A(x) \in [0, +\infty)$. For $\alpha \neq 0$,

$$\begin{aligned}
\mu_A(\lambda x) &= \inf\left\{\lambda > 0 \,:\, \frac{\alpha \cdot x}{\lambda} \in A\right\} \\
&= \inf\left\{|\alpha| \cdot \lambda > 0 \,:\, \frac{\alpha \cdot x}{|\alpha| \cdot \lambda} \in A\right\} \quad (\lambda \mapsto |\alpha| \cdot \lambda) \\
&= |\alpha| \cdot \inf\left\{\lambda > 0 \,:\, \frac{x}{\lambda} \in \frac{|\alpha|}{\alpha} A\right\} \\
&= |\alpha| \cdot \inf\left\{\lambda > 0 \,:\, \frac{x}{\lambda} \in A\right\} \\
&= |\alpha| \cdot \mu_A(x)
\end{aligned}$$

since A is balanced, $\frac{\alpha}{|\alpha|}A = A$.
Note that $\mu_A(0) = 0$ since $0 \in A$ balanced.
Given $x, y \in V$ and $\varepsilon > 0$, let $s = \mu_A(x) + \varepsilon$ and $t = \mu_A(y) + \varepsilon$. Then, since A is balanced, $\frac{x}{s}, \frac{y}{t} \in A$. By convexity,

$$\frac{x+y}{t+s} = \underbrace{\frac{s}{t+s}}_{(1-\sigma)} \cdot \frac{\overbrace{x}^{\in A}}{s} + \underbrace{\frac{t}{t+s}}_{\sigma} \cdot \frac{\overbrace{y}^{\in A}}{t} \in A$$

Therefore, $\mu_A(x+y) \le t+s$ which implies $\mu_A(x+y) \le \mu_A(x) + \mu_A(y) + 2\varepsilon$ for all $\varepsilon > 0$.

Equivalence between Semi-norm and ABC Sets

$p \rightsquigarrow A = \{x : p(x) < 1\} \rightsquigarrow \mu_a = p$.
A bounded, convex, absorbing $\rightsquigarrow \mu_A \rightsquigarrow \tilde{A} = \{x : \mu_A(x) < 1\}$ where $\tilde{A} \subseteq A$ differing possibly by the boundary.

Question: which TVS are normable?

That is a norm such that the topology is vien by this norm.

Definition: Bounded Sets

A subset A in a TVS is bounded if $\forall U \ni 0$ open, $\exists \delta > 0$ such that $A \subseteq t \cdot U$, $\forall t > \delta$.

Theorem:

A TVS is normable if and only if there exists a bounded, convex, balanced, open neighborhood of 0.

Proof (Sketch)

Suppose V is a normed space with norm $||\cdot||$.

$$B = \{x \in V : ||x|| < 1\} = B_1(0)$$

is convex, balanced, and an open neighborhood of 0.
B is bounded, since given $U \ni 0$ open, $B_\varepsilon(0) \subseteq U$, so $B = \frac{1}{\varepsilon} \cdot B_\varepsilon(0) \subseteq \lambda B_\varepsilon(0) \subseteq \lambda \cdot U$ for $\lambda \ge \frac{1}{\varepsilon}$.
Now, let B be a bounded, convex, balanced, open neighborhood of 0 in a TVS.
Therefore B is absorbing (as an open neighborhood of 0).
It follows that the semi-norm $\mu_B(x)$ may be defined.
Then $\mu_B(x) = 0 \implies x = 0$ since B is bounded, otherwise $0 \in U = V \setminus \{x\}$ open gives $B \subseteq t \cdot U$, $\forall t > \delta$ and $\frac{1}{t}B \subseteq U$, $\forall t > \delta$.
Thus, $||x|| = \mu_B(x)$ is a norm on V.
One need only demonstrate that the norm topology is the same as the original topology on V.
That is, $\forall U \ni 0$ open, $\exists \varepsilon > 0$ such that $\varepsilon \cdot B \subseteq U$.
$\forall \varepsilon > 0$, $\exists \hat{U} \ni 0$ open such that $\hat{U} \subseteq \varepsilon B$.

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Recall

Given p a semi-norm,

$$A = \{x \in V : p(x) < 1\}$$

a convex, balanced absorbing set gives the Minkowski formula for a seminorm μ_a.
The TVS V is normable if and only if there exist bounded, convex, balanced, open $U \ni 0$.

Definition: Separating Family of Semi-norms

Let V be a vector space.
A family of semi-norms $\{p_\omega\}_{\omega\in\Omega}$ is called separating if $\forall x \in V$, $x \neq 0$, $\exists \omega \in \Omega$ such that $p_\omega(x) \neq 0$.
Equivalently,

$$\{x \in V : \forall \omega \in \Omega,\ p_\omega(x) = 0\} = \{0\}$$

or

$$\bigcap_{\omega\in\Omega} N_{p_\omega} = \bigcap_{\omega\in\Omega} \{x \in V : p(x) = 0\} = \{0\}.$$

For a given separating family of semi-norms, define

$$U_{n,\omega} = \left\{x \in V : p_\omega(x) < \frac{1}{n}\right\}$$
$$U_{n,\omega_1,\ldots,\omega_N} = \left\{x \in V : p_{\omega_i}(x) < \frac{1}{n\ i = 1, \ldots, N}\right\} = \bigcap_{i=1}^{N} U_{n,\omega_i}$$

and put

$$\gamma = \{U_{n,\omega_1,\ldots,\omega_N} : n \in \mathbb{N},\ \omega_1, \ldots, \omega_N \in \Omega,\ N \in \mathbb{N}\}.$$

One can show by earlier propositions that γ is a local base at at 0 for some topology τ.
Perhaps unsurprisingly, if $\{p_\omega\}$ is separating, then this locally convex TVS is Hausdorff.

Theorem:

Let $\{p_\omega\}$ be a separating family of semi-norms on a vector space V. Then with local base γ defined above, V becomes a locally convex TVS, and all $p_\omega : V \to [0, +\infty)$ continuous.

Example

$$\mathcal{S} = \{\{x_n\}_{n=1}^{\infty} \text{ all sequences}\}$$

with $p_n(x) = |x_n|$, $x = \{x_n\}_{n=1}^{\infty}$, $d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1+|x_n - y_n|}$.

Remark

Local base at x

$$\gamma_x = x + \gamma = \{U_{n,\omega_1,\ldots,\omega_N}[x] : n, N \in \mathbb{N},\ \omega_1, \ldots, \omega_N \in \Omega\}$$

$$U_{n,\omega_1,\ldots,\omega_N}[x] = \left\{y \in V : p_{\omega_i}(x - y) < \frac{1}{n},\ i = 1, \ldots, N\right\}$$

Theorem:

Let V be a locally convex TVS. Then there exists a separating family of semi-norms $\{p_\omega\}_{\omega\in\Omega}$ on V such that the topology defined by $\{p_\omega\}$ coincides with the original toplogy.

Proof (Sketch)

V is locally convex, so it has a convex, balanced, local base at 0

$$\hat{\gamma} = \{U_\omega\}_{\omega\in\Omega}$$

where $U_\omega \ni 0$ are open, convex, balanced, and absorbing.
Put $p_\omega = \mu_{U_\omega}$ (i.e. set the Minkowski functional to be the semi-norm).
Then we need to show that these are separating.
Then define $U_{n,\omega_1,\dots,\omega_N}$, $\gamma = \{U_{n,\omega_1,\dots,\omega_N}\}$, $U_\omega = U_{1,\omega}$, $\hat{\gamma} \subseteq \gamma$ and show that γ and $\hat{\gamma}$ induce the same topology.

Theorem:

A TVS V is a pre-Fréchet space if and only if V has a countable, convex, balanced local base.

Proof

($\Longrightarrow$) Assume that V is a pre-Fréchet space.
Then we have an invariant metric d and

$$B_\varepsilon(x) = \{y \in V \,:\, d(x,y) < \varepsilon\}.$$

It follows that $\gamma_1 = \{B_{1/n}(0) \,:\, n \in \mathbb{N}\}$ is a local base.
The fact that V is locally convex means that $\gamma_2 = \{U_\omega \,:\, \omega \in \Omega\}$ with $U_\omega \ni 0$ open, convex and balanced is a convex, balanced local base.
To every $n \in \mathbb{N}$, $B_{1/n}(0)$ is an open neighborhood of 0, and there exists $\omega_n \in \Omega$, $0 \in U_{\omega_n} \subseteq B_{1/n}(0)$. Put

$$\gamma_3 = \left\{U_{\omega_n} \,:\, n \in \mathbb{N}\right\}$$

a countable, convex, balanced collection.
Then, for any $U \ni 0$ open, $\exists n$ such that $U_{\omega_n} \subseteq B_{1/n}(0) \subseteq U$. So γ_3 is a local base.
($\Longleftarrow$) Assume a TVS V has a countable, convex, balanced local base:

$$\gamma = \{U_n \,:\, n \in \mathbb{N}\}$$

Without loss of generality, we may assume that $U_{n+1} \subseteq U_n$. Otherwise, we may take $\hat{U}_n = U_1 \cap \cdots \cap U_n \subseteq U_n$ such that $\{\hat{U}_n \,:\, n \in \mathbb{N}\}$ is also a local base where $\hat{U}_{n+1} \subseteq \hat{U}_n$.
Then, since U_n are open, they are absorbing and $p_n = \mu_{U_n}$ gives a separating semi-norm from the Minkowski functional.
Define a metric

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x-y)}{1+p_n(x-y)}$$

where $d(x,y) = 0 \implies x = y$ since $\{p_n\}$ are separating.
Claim: the metric topology (local base $\tilde{\gamma}$) is the same as the original topology (local base γ). Write

$$\tilde{\gamma} = \{B_{1/2^m}(0)\ :\ m \in \mathbb{N}\}$$

Then for all $m \in \mathbb{N}$,

$$\frac{1}{2^{m+1}} U_{m+1} \subseteq B_{1/2^m}(0)$$

there exists $n \in \mathbb{N}$ such that $U_n \subseteq \frac{1}{2^{m+1}} U_{m+1}$.
Also, $\forall n$, $B_{1/2^{n+1}}(0) \subseteq U_n$. Then V is locally convex (γ) and has an invariant metric ($\tilde{\gamma}$). That is, V is pre-Fréchet space.

Corollary

In a pre-Fréchet space, the metric can always be defined by

$$\forall n, \quad B_{1/2^{n+1}}(0) \subseteq U_n$$

where $\{p_n\}$ are separating family of semi-norms.
A separating family of semi-norms leads to a locally convex TVS.
A countable, separating family of semi-norms leads to a pre-Fréchet space.
A finite, separating family of semi-norms leads to a normed space.

Quotient Spaces

For a vector space X and a linear subspace $N \subseteq X$, $X/N = \{[x]_N\ :\ x \in X\}$, $[x]_N = x + N$.
$\pi : X \to X/N$ is the quotient map to the vector space X/N.
For a TVS X, $N \subseteq X$ a subspace, $\pi : X \to X/N$ where τ is the topology of X and $\hat{\tau}$ is the topology of X/N given by

$$\hat{\tau} = \{\pi(U)\ :\ U \in \tau\}.$$

N is closed if and only if X/N is Hausdorff.

Thoerem:

For X a TVS and $N \subseteq X$ a linear subspace, X/N is a TVS and $\pi : X \to X/N$ is open and continuous.

Normed / Banach

For X a normed (Banach) space, X/N is a normed (Banach) space where $||[x]||_{X/N} = \inf_{z \in N} ||x + z||$.

Pre-Fréchet / Fréchet

For X a (pre-)Fréchet space, X/N is a (pre-)Fréchet space where $d_{X/N}(x, y) = \inf_{z \in N} d(x + z, y) = \inf_{z_1, z_2} d(x + z_1, y + z_2)$.

Definition: Linear Operator

A map $T : V \to W$ between vector spaces V, W is linear (or a linear operator) if

$$T(x + y) = Tx + Ty \quad \text{and} \quad T(\alpha x) = \alpha(Tx)$$

Notation

$M(V,W)$ is the set of all linear operators.
$M(V,V) = M(V)$.
$V' = M(V,\mathbb{F})$ (linear functionals) is the algebraic dual of V.
Note that $M(V,W)$ is a vector space.

$$(T_1 + T_2)(x) := T_1 x + T_2 x \quad \text{and} \quad (\lambda T)(x) := \lambda(Tx)$$

If T_1, T_2 are linear, then $T_1 + T_2$ is linear; likewise, λT is linear precisely when T is linear.

Definition: Continuous Linear Operator

For V, W TVS, T is a continuous linear operator if $T \in M(V,W)$ and T is continuous with respect to the topologies.

Notation

$L(V,W)$ is the set of all continuous linear operators.
$L(V,V) = L(V)$.
$V^* = L(V,\mathbb{F})$, the set of continuous linear functionals on V, is the dual space of V.

Example

Let $V = \mathbb{R}^n$, $W = \mathbb{R}^m$.
$M(V,W) = L(V,W)$.
To an $m \times n$ matrix $A = (a_{ij})_{i=1,j=1}^{m,n}$, one associates the linear operator T_A

$$T_A : (x_j)_{j=1}^n \mapsto (y_i)_{i=1}^m, \quad y_i = \sum_{j=1}^n a_{ij} x_j$$

$V' = V^*$. Given $\phi = (\phi_1, \ldots, \phi_n) \in \mathbb{R}^n$ (a row vector),

$$\phi(x) = \phi \cdot x = \sum_{j=1}^n \phi_j x_j$$

In this case, $V^* \cong \mathbb{R}^n$.

Defiition: Image or Range

For $T \in M(V,W)$, $T : V \to W$,

$$\text{im} T = R(t) = \{Tx \,:\, x \in V\}$$

Definition: Kernel or Nullspace

$$\ker T = N(T) = \{x \in V \,:\, Tx = 0\}$$

Remarks

$R(T)$ is a linear subspace of W while $N(T)$ is a linear subspace of V.
T is injective if and only if $N(t) = \{0\}$.
If T is inective, then one has an inverse map $T^{-1} : R(T) \to V$. T^{-1} is linear.
T is invertible if and only if T is injective and surjective if and only if $N(T) = \{0\}$ and $R(T) = W$.

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Proposition

Let V, W be TVS.

1. a linear operator $T : V \to W$ is continuous if and only if T is continuous at some $x_0 \in V$.

2. if T is a continuous linear operator, then $N(T) = \ker(T)$ is a closed, linear subspace of V.

Proof of A

$(\Longrightarrow)$ continuous at all points imply continuous at x_0.
$(\Longleftarrow)$ Write $f(x) = T(x + x_0 - x_1) - T(x_0 - x_1)$ and assume T is continuous at $x = x_0$.
Then $T(x + x_0 - x_1)$ is continuous at $x = x_1$.

Proof of B

We have that $\ker(T) = \{x \in V \ : \ Tx = 0\} = T^{-1}(\{0\})$ where $\{0\}$ is closed and so must be its preimage.

Definition: Bounded Linear Operator

Let V, W be normed spaces with norms $||\cdot||_V$, $||\cdot||_W$.
A linear operator $T : V \to W$ is called bounded if there exists some $c \geq 0$ such that

$$||Tx||_W < c \cdot ||x||_V, \quad \forall x \in V$$

Proposition:

A linear operator $T : V \to W$ (V, W normed spaces) is continuous if and only if it is bounded.

Proof

$(\Longleftarrow)$ We know that $||Tx||_W \leq c \cdot ||x||_V$, $\forall x$.
Consider $\{x_n\}$, $x_n \to a$ in V. Then

$$\lim_{n\to\infty} ||x_n - a|| = 0$$

so $||Tx_n - Ta||_W \leq c \cdot ||x_n - a||_V$, $||Tx_n - Ta||_W = 0$, and $Tx_n \to Ta$ in W.
$(\Longrightarrow)$ For every $n \in \mathbb{N}$, find $x_n \in W$ such that

$$||Tx_n||_W > n \cdot ||x_n||_V$$

Then $y_n = \frac{x_n}{||Tx_n||}$, since $||y_n|| = \frac{||x_n||}{||Tx_n||} < \frac{1}{n}$ it must be $y_n \to 0$.
Hence, $Ty_n \to T0 = 0$ (T continuous) $\implies Ty_n = \frac{Tx_n}{||Tx_n||}$.
But $||Ty_n|| = 1$, so $Ty_n \not\to 0$ a contradiction.

Remark

The following statements are equivalent

- T is continuous.
- T is bounded.
- $Tx_n \to 0$ whenever $x_n \to 0$.
- $\{Tx_n\}$ is bounded whenever $\{x_n\}$ is bounded.

Definition: Operator Norm

For V, W normed spaces.
For $T : V \to W$ a bounded linear operator, we define

$$||T|| = \sup_{\substack{x \in V \\ x \neq 0}} \frac{||Tx||_W}{||x||_V}$$

the operator norm of T.

Remark

$||T|| \in [0, +\infty)$ and it is equal to the smallest $c \geq 0$ such that $||Tx||_W \leq c \cdot ||x||_V$, $\forall x \in V$.
Indeed, if this holds for some $c \geq 0$, then $||T|| \leq c$.
Conversely, from the definition $||Tx||_W \leq ||T|| \cdot ||x||_V$.
That is, $||T|| = \min\{c \geq 0 \,:\, ||Tx||_W \leq c \cdot ||x||_V, \, \forall x\}$.

Remark

$$||T|| = \sup_{\substack{x \in V \\ ||x|| = 1}} ||Tx|| = \sup_{\substack{x \in V \\ ||x|| \leq 1}} ||Tx||$$

Note that

$$\sup_{x \neq 0} \frac{||Tx||_W}{||x||_V} = \sup_{x \neq 0} \left|\left| T\left(\frac{x}{||x||_V} \right) \right|\right|_W = \sup_{||z||_V = 1} ||Tz||_W$$

Remark

$M(V, W)$ and $L(V, W)$ are linear spaces,

$$(T + S)(x) = Tx + TS$$
$$(\lambda T)(x) = \lambda(Tx)$$

If T, S are continuous, linear operators, then $T + S$ and λT are continuous linear operators.

Further Properties

- $||T|| = 0$ if and only if $T = 0$ (i.e. $Tx = 0,\ \forall x \in V$).
- $||T + S|| \le ||T|| + ||S||$, because

$$||(T+S)x||_W = ||Tx + Ts||_W \le ||Tx||_W + ||Sx||_W \le ||T|| \cdot ||x||_V + ||S|| \cdot ||x||_V \le (\underbrace{||T|| + ||S||}_{c}) \cdot ||x||_V$$

Since $T + S$ is bounded. $\frac{||(T+S)x||_W}{||x||_V} \le ||T|| + ||S||$, etc.

- $||\alpha T|| = |\alpha| \cdot ||T||$.
- if $T \in L(U, V)$ and $S \in L(V, W)$, then $ST \in L(U, W)$ and

$$||ST|| \le ||S|| \cdot ||T||$$

Proposition

Let $V,\ W$ be normed spaces.
Then $L(V, W)$ is a normed space with the operator norm.
If, in addition, W is Banach, then $L(V, W)$ is also Banach.

Proof

Part A

$||\cdot||$ is a norm.

Part B

Let W be a Banach space, and let $T_n \in L(V, W)$ be such that $\{T_n\}$ is a Cauchy sequence in the operator norm.
Then, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall j, k \ge N$, $||T_j - T_k|| < \varepsilon$.
So $\forall x \in V$, $\{T_n x\}$ is Cauchy in W.

$$||T_j x - T_k x|| = ||(T_j - T_k)x|| \le ||T_j - T_k|| \cdot ||x|| \le \varepsilon \cdot ||x||$$

By completeness, for every $x \in V$, $T_n x$ converges in W. Define

$$Tx = \lim_{n\to\infty} T_n x$$

such that $||Tx - T_n x|| \to 0$ as $n \to \infty$.
We need to show that T is a linear operator:
$T(x + y) = \lim_{n\to\infty} T_n(x + y) = \lim_{n\to\infty} T_n x + \lim_{n\to\infty} T_n y = Tx + Ty$.
$T(\lambda x) = \lambda \cdot Tx$.
We need also show that T is bounded:

$$\frac{||Tx||_W}{||x||_V} = \lim_{n\to\infty} \frac{||T_n x||_W}{||x||_V} = \liminf_{n\to\infty} ||T_n||$$

Since $\{T_n\}$ is Cauchy, it is bounded and $\liminf_{n\to\infty} ||T_n|| \leq c$ for some c.
We have that $\lim_{n\to\infty} ||Tx - T_n x|| = 0$ such that T_n converges pointwise.
We need that $\lim_{n\to\infty} ||T - T_n|| = 0$.
For given $\varepsilon > 0$, we find N such that $\forall\, j, k \geq N,\ x \in V$:

$$||T_j x - T_k x|| \leq \varepsilon \cdot ||x||$$

Then

$$||T_j x - Tx|| = ||T_j x - T_k x + T_k x - Tx|| \leq \varepsilon \cdot ||x|| + ||T_k x - Tx||$$

and sending $k \to 0$ sends $T_k x - Tx$ to 0.
Therefore, $||T_j x - Tx|| \leq \varepsilon \cdot ||x||$, $\forall\, j \geq N$, $\forall\, x \in V$. It follows that

$$\frac{||T_j x - Tx||}{||x||} \leq \varepsilon$$

and, taking the supremum over x, that $||T_j - T|| \leq \varepsilon$, $\forall\, j \geq N$, $\forall\, x \in V$.
Hence, $\lim_{n\to\infty} ||T_n - T|| = 0$.
That is, $L(V, W)$ is complete.

Corollary

The dual space of a normed space is a Banach space. Recall $V^* = L(V, \mathbb{F})$, and both $\mathbb{R}$ and $\mathbb{C}$ are complete.

Notation

Read $\dotplus$ as a direct sum implied to be between components of a larger space.
Read $\mathrm{lin}\{v_1, \ldots, v_n\}$ as the linear combinations of $v_1, \ldots, v_n$.

Definition: Codimension

If V is a vector space and W is a subspace, we say that W has codimension n in V if there exists a subspace $\hat{W} \subseteq V$ such that

$$V = W \dotplus \hat{W}$$

and $\dim(\hat{W}) = n$.
Equivalently, $\dim(V/W) = n$, $V/W = \mathrm{lin}\{[e_1], \ldots [e_n]\}$ basis and $\hat{W} = \mathrm{lin}\{e_1, \ldots, e_n\}$ implies $V = W \dotplus \hat{W}$.

Proposition:

Let V be a vector space and $\phi \neq V'$, $\phi \neq 0$. Then $\ker(\phi)$ is a subspace of V of codimension 1.

Proof

$\phi \neq 0$. Find $x_0 \in V$ such that $\phi(x_0) = 1$.
Claim: $V = \ker(\phi) \dotplus \mathrm{lin}\{x_0\}$.
Indeed, for $x \in V$ write

$$x = \underbrace{x - \phi(x) \cdot x_0}_{ker(\phi)} + \underbrace{\phi(x)}_{\in \text{lin}\{x_0\}} \cdot x_0$$

so

$$\phi(x - \phi(x) \cdot x_0) = \phi(x) - \phi(\phi(x) \cdot x_0) = \phi(x) - \phi(x) \cdot \phi(x_0) = 0$$

and
$\ker(\phi) \cap \text{lin}\{x_0\} = \{0\}$ which means $z = \lambda \cdot x_0 \in \ker(\phi)$. Therefore

$$0 = \phi(\lambda x_0) = \lambda \cdot 1$$

so $\lambda = 0$ and $z = 0$.

Proposition:

Let V be a normed space and $\phi \in V'$.
Then ϕ is bounded if and only if $\ker(\phi)$ is closed in V.

Proof

($\Longrightarrow$) ϕ continuous, as a linear operator, implies $\ker(\phi) = \phi^{-1}(\{0\})$ is closed.
($\Longleftarrow$) assume that $\ker(\phi)$ is closed. Then

$$V = \ker(\phi) \dot{+} \text{lin}\{x_0\}$$

for some $x_0 \in V$ and $x_0 \notin \ker(\phi)$.
Without loss of generality, we may assume $\phi(x_0) = 1$.
Claim: $\inf_{x \in \ker(\phi)} ||x_0 - x|| = \text{dist}(\ker(\phi), x_0) > 0$.
Otherwise, there would exist some sequence $\{x_n\} \subseteq \ker(\phi)$ such that $||x_0 - x_n|| \to 0$.
From the assumption of closure, this would mean $x_0 \in \ker(\phi)$ a contradiction.
Therefore, $\exists c > 0$ such that $||x_0 - x|| \geq c$, $\forall x \in \ker(\phi)$. So

$$\begin{aligned} ||\lambda x_0 - \lambda x|| &\geq c \cdot |\lambda| \\ ||\lambda x_0 - u|| &\geq c \cdot |\lambda|, \quad \forall u \in \ker(\phi) \end{aligned}$$

Write $y \in V$ as $y = \underbrace{-u}_{\in \ker(\phi)} + \underbrace{\lambda x_0}_{\in \text{lim}\{x_0\}}$. So $\phi(y) = 0 + \lambda \cdot \phi(x_0) = \lambda$.
Thus, $\forall x \in V$, $||x|| \geq c \cdot |\phi(x)|$ and $|\phi(x)| \leq \frac{1}{c} \cdot ||x||$ and ϕ is bounded.

April 23, 2024

Proposition:

A linear functional ϕ on a TVS V is continuous if and only if $\ker(\phi)$ is closed in V.

Proof

($\Longleftarrow$) Difficult.
($\Longrightarrow$) $\ker(\phi) = \phi^{-1}(\{0\})$.

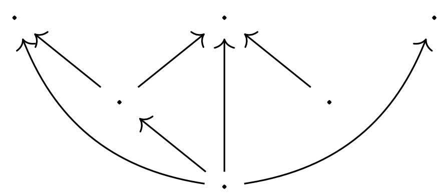
Recall:

V' is the set of linear functionals on V $\phi : V \to \mathbb{F}$ linear.
V^* is the set of continuous linear functionals on V $\phi : V \to \mathbb{F}$ linear and continuous.
On a normed V, continuous and bounded are equivalent.

Zorn's Lemma

A non-empty partially ordered set $(S, \leq)$ has a maximal element if every totally ordered subset has an upper bound.

- $(S, \leq)$ reflexive, transitive and anti-symmetric.
- $S_0 \subseteq S$ is totally (or linearly) ordered if $\forall a, b \in S$ either $a \leq b$ or $b \leq a$.
- S_0 has an upper bound if $\exists b \in S$ such that $\forall x \in S_0$, $x \leq b$.
- m is a maximal element of S is $\forall x \geq m$, $x = m$.



Theorem:

Let V be a vector space, $W_0 \subseteq V$ a subspace, and a linear functional ϕ_0 on W_0 (i.e. $\phi_0 \in W_0'$).
Then there exists an extension, i.e. a linear functional, $\phi \in V'$ such that $\phi|_{W_0} = \phi_0$.

Proof

Let S be the set of all pairs (W, ϕ) such that

- $W_0 \subseteq W \subseteq V$ is a linear subspace and
- $\phi \in W'$, $\phi|_{W_0} = \phi_0$.

Say that $(W_1, \phi_1) \leq (W_2, \phi_2)$ if and only if $W_1 \subseteq W_2$ and $\phi_2|_{W_1} = \phi_1$.
Since $\leq$ is reflexive, transitive and anti-symmetric, it is an order relation.
A totally ordered subset has an upper bound. Given

$$S_0 = \{(W_\omega, \phi_\omega)\}$$

totally ordered, the upper bound is given by (W, ϕ) where

$$W = \bigcup_\omega W_\omega$$
$$\phi(x) = \phi_\omega(x) \quad \text{if } x \in W_\omega$$

such that for $x \in W_{\omega_1} \cap W_{\omega_2}$ we have $\phi_{\omega_1}(x) = \phi_{\omega_2}(x)$ and consequently $(W_{\omega_1}, \phi_{\omega_1}) \leq (W_{\omega_2}, \phi_{\omega_2})$.
Then, by Zorn's Lemma, we have that S has a maximal element $(\hat{W}, \hat{\phi})$.
Claim: $\hat{W} = V$, $\hat{\phi} \in V'$, and $\hat{\phi}|_{W_0} = \phi_0$.
Otherwise, there exists $(\hat{\hat{W}}, \hat{\hat{\phi}}) > (\hat{W}, \hat{V})$.
Namely, $\hat{\hat{W}} = \hat{W} \dot{+} \lim\{x_0\} = \{\hat{w} + \lambda x_0 \,:\, \hat{w} \in \hat{W},\, \lambda \in \mathbb{F}\}$, $x_0 \in V \setminus \hat{W}$ with $\hat{W} \underset{\neq}{\subset} V$.
Then $\hat{W} \underset{\neq}{\subset} \hat{\hat{W}} \subseteq V$.
Define $\hat{\hat{\phi}}$ on $\hat{\hat{W}}$ as

$$\hat{\hat{\phi}}(\hat{W} + \lambda x_0) = \hat{\phi}(\hat{W}) + \lambda \cdot \hat{\hat{\phi}}(x_0) = \hat{\phi}(\hat{W}) + \lambda \cdot c$$

with c an arbitrary choice. Then $\hat{\hat{\phi}}$ is linear.

Conclusion

Each infinite dimensional, normed space has an unbounded linear functional.
For $(V, ||\cdot||)$ a normed space, there exist $\{e_1, e_2, \ldots\}$ linearly independent and

$$W_0 = \operatorname{lin}\{e_1, e_2, \ldots\}$$

is the set of all finite linear combinations. So

$$\phi_0\left(\sum \lambda_k e_k\right) = \sum \lambda_k \cdot k \cdot ||e_k||$$

where $\phi_0 \in W_0'$ and ϕ_0 is unbounded. Take $\phi_0(e_k) = k \cdot ||e_k||$. Then

$$\sup_{\substack{x \in W_0 \\ x \neq 0}} \frac{|\phi_0(x)|}{||x||} \geq \sup \frac{k||e_k||}{||e_k||} = +\infty$$

Then extend ϕ_0 to a linear functional on V, $\phi|_{W_0} = \phi_0$, $\phi \in V'$, ϕ unbounded.

Preliminaries: Hahn-Banach

On normed space, given $\phi_0 \in W_0^*$ bounded we have a bounded extension $\phi \in V^*$ where $||\phi|| = ||\phi - 0||$.
On locally convex TVS, continuous $\phi_0 \in W^*$ implies a continuous extension $\phi \in V^*$.
Equivalently, given $p(x)$ a seminorm, $|\phi_0(x)| \leq p(x)$ implies $|\phi(x)| \leq p(x)$.

Lemma:

Let V be a vector space and p a seminorm on V.
Let W be a subspace of codimension 1,

$$V = W \dot{+} \operatorname{lin}\{x_0\}$$

Let ϕ be a real linear functional on W such that

$$\phi(x) \leq p(x) \quad \forall x \in W$$

Then there exists an extension $\hat{\phi}$ (a real linear functional on V) such that

$$\hat{\phi}(x) \leq p(x) \quad \forall x \in V$$

Proof

Write $V = W \dotplus \mathrm{lin}\{x_0\}$ such that

$$\hat{\phi}(W + \lambda x_0) := \phi(W) + \lambda \cdot c$$

with a suitable choice c.
We know already that $\hat{\phi} \in V'$. For $u, v \in W$,

$$\begin{aligned}
\phi(u) - \phi(v) &= \phi(u - v) \\
&\leq p(u - v) \\
&= p((u + x_0) - (v + x_0)) \\
&\leq p(u + x_0) + p(v + x_0)
\end{aligned}$$

Therefore

$$-p(v + x_0) - \phi(v) \leq p(u + x_0) - \phi(u)$$

and $\exists c \in \mathbb{R}$ such that

$$-p(v + x_0) - \phi(v) \leq c \leq p(u + x_0) - \phi(u)$$

(e.g. take $\inf$ or $\sup$). So

$$\begin{aligned}
-p(v + x_0) &\leq \phi(v) + c & \phi(u) + c &\leq p(u + x_0) \\
-p(v + x_0) &\leq \hat{\phi}(v + x_0) & \hat{\phi}(u + x_0) &\leq p(u + x_0) \\
v = \frac{w}{\lambda}, &\ \lambda < 0 & u = \frac{w}{\lambda}, &\ \lambda > 0 \\
p(w + \lambda x_0) &\geq \hat{\phi}(w + \lambda x_0) & \hat{\phi}(w + \lambda x_0) &\leq p(w + \lambda x_0)
\end{aligned}$$

and

$$\hat{\phi}(w + \lambda x_0) \leq p(w + \lambda x_0) \quad \forall \lambda \in \mathbb{R},\ w \in W$$

Lemma

Take $\mathbb{F} = \mathbb{C}$, let W be a subspace of V and

$$V = W \dotplus \mathrm{lin}\{e_0\}$$

such that $\phi \in W'$

$$|\phi(x)| \leq p(x) \quad \forall x \in W$$

Then there exists an extension $\hat{\phi} \in V'$ on, $\hat{\phi}|_W = \phi$ such that

$$|\hat{\phi}(x)| \leq p(x) \quad \forall x \in V$$

Proof

Given ϕ on W, define the real linear functional

$$\psi(x) = \Re(\phi(x))$$

Note that

$$\psi(ix) = \Re(i\phi(x)) = -\Im(\phi(x))$$

Therefore

$$\phi(x) = \psi(x) - i\psi(ix)$$

So by extending $\hat{\psi}$ on V we can construct an extension $\hat{\phi}$ on V. We know

$$\psi(x) = |\phi(x)| \leq p(x) \quad \forall x \in W$$

therefore $\hat{\psi}(x) \leq p(x)$ for all $x \in V$.
Now define $\hat{\phi}$ on V by

$$\hat{\phi}(x) := \hat{\psi}(x) - i\hat{\psi}(ix)$$

1. $\hat{\phi}$ is a real linear functional on V

$$\hat{\phi}|_W = \phi$$

1. $\hat{\phi}$ is a complex linear functional on V

$$\begin{aligned}
\hat{\phi}(\alpha x) &= \alpha\hat{\phi}(x) \\
\alpha &= \alpha_1 + i\alpha_2 \\
\hat{\phi}(ix) &= i\hat{\phi}(x) \\
\hat{\psi}(ix) - i\hat{\psi}(i^2 x) &= i(\hat{\psi}(x) - i\hat{\psi}(ix))
\end{aligned}$$

1. $|\hat{\phi}(x)| \leq p(x)$, $\forall \lambda \in V$

We know that $\hat{\psi}(x) \leq p(x)$.
For any $x \in V$, find $\alpha \in \mathbb{C}$, $|\alpha| = 1$ such that $0 \leq \alpha\hat{\phi}(x)$. Then

$$\begin{aligned}
0 \leq \alpha\hat{\phi}(x) &= \hat{\phi}(\alpha x) \\
&= \underbrace{\hat{\psi}(\alpha x)}_{\text{real}} - \underbrace{i\hat{\psi}(i\alpha x)}_{\substack{\text{imaginary} \\ =0}} \\
&= \hat{\psi}(\alpha x) \leq p(\alpha x) = |\alpha| p(x) = p(x)
\end{aligned}$$

Therefore $0 \leq \alpha\hat{\phi}(x) \leq p(x)$ and $|\hat{\phi}(x)| \leq p(x)$.

Corollary

Let V be a normed space with the seminorm p and $W_0 \subseteq V$ a subspace with $\phi_0 \in W_0'$ such that

$$|\phi_0(x)| \le p(x), \quad x \in W_0$$

Then there exists $\hat{\phi} \in V'$ such that $\hat{\phi}|_{W_0} = \phi_0$ and

$$|\hat{\phi}(x)| \le p(x), \quad x \in V$$

Proof

Apply the two lemmas and Zorn's lemma.

April 25, 2024

Recall:

Take $W_0 \subseteq V$, p a seminorm, and $\phi_0 \in W_0'$ such that

$$|\phi_0(x)| \le p(x),\, x \in W$$

Then there exists an extension $\hat{\phi} \in V'$, $\hat{\phi}|_{W_0} = \phi_0$ where

$$|\hat{\phi}(x)| \le p(x),\, d \in V$$

Theorem: Hahn-Banach for Normed Spaces

Let V be a normed space, $W_0 \subseteq V$ a linear subspace, and $\phi_0 \in (W_0)^*$.
Then there exist $\hat{\phi} \in (V)^*$ such that $\hat{\phi}|_{W_0} = \phi_0$ and

$$||\hat{\phi}|| = ||\phi_0||$$

Proof:

From the previous result with

$$p(x) = ||x|| \cdot ||\phi_0||$$

it is obvious that $|\phi_0(x)| \le p(x)$, $x \in W_0$.
Then there is an extension $\hat{\phi} \in V'$ where

$$|\hat{\phi}(x)| \le p(x) = ||x|| \cdot ||\phi_0||,\, x \in V$$

It follows that $\hat{\phi} \in V^*$ is bounded and

$$\sup \frac{|\hat{\phi}(x)|}{||x||} \le ||\phi_0||$$

Consequently $||\hat{\phi}|| \le ||\phi_0||$.
We have also that $||\hat{\phi}|| \ge ||\phi_0||$ because $\hat{\phi}$ is an extension of ϕ_0.

Corollary

$\forall x_0 \in V$, V a normed space, $x_0 = 0$, $\exists \hat{\phi} \in V^*$ such that $\hat{\phi}(x_0) = ||x_0||$ and $||\hat{\phi}|| = 1$.

Definition:

For $\mathcal{F} \subseteq V'$, we say that $\mathcal{F}$ separates the points of V is

$$\forall x_0 \in V,\ x_0 \neq 0,\ \exists \phi \in \mathcal{F} \ :\ \phi(x_0) \neq 0$$

Remark

- V' separates the points of V on any vector space V.
- V^* separates the points of V on any normed space.

Theorem: Hahn-Banach for Locally Convex TVS

Let V be a locally convex TVS, $W_0 \subseteq V$ a linear subspace, and $\phi_0 \in (W_0)^*$ a continuous linear functional.
Then there exist $\hat{\phi} \in V^*$ continuous linear functionals such that $\hat{\phi}|_{W_0} = \phi_0$.
Consequently, V^* separates the points of V.

Proof

$\phi_0 : W_0 \to \mathbb{F}$ continuous gives

$$U = \{x \in W_0 \ :\ |\phi_0(x)| < 1\}$$

open with respect to the subspace topology in W_0.
That is, $U = \hat{U} \cap W_0$ with $\hat{U}$ open in V and $0 \in \hat{U}$.
Therefore, there exists some $\tilde{U}$ convex, balanced, and open such that $0 \in \tilde{U} \subseteq \hat{U}$.
Let $p(x) = \mu_{\tilde{U}}(x)$, the Minkowski Functional and a seminorm on V.
It follows that $|\phi_0(x)| \leq p(x)$, $x \in W_0$.
Equivalently, $p(x) < 1 \implies |\phi_0(x)| < 1$, $x \in W_0$.

$$\begin{tikzcd}
p(x) < 1 \arrow[rr, Rightarrow] \arrow[d] & & \mid \phi_0 \mid < 1 \\
x \in \tilde{U} \arrow[r, Rightarrow] & x \in \hat{U} \arrow[r, Rightarrow] & x \in U \arrow[u]
\end{tikzcd}$$

Therefore there exists an extension $\hat{\phi} \in V'$ such that

$$|\hat{\phi}(x)| \leq p(x),\ x \in V$$

We have

$$\underbrace{\{x \in V \ :\ p(x) < 1\}}_{\tilde{U} \ni 0 \text{ open}} \subseteq \underbrace{\{x \in V \ :\ |\hat{\phi}(x)| < 1\}}_{\hat{\phi}^{-1}(B,(0))}$$

Therefore $\hat{\phi}$ is continuous at $x_0 = 0$ and $\hat{\phi}$ is continuous.

Theorem:

Let $0 < p < 1$, $V = L^p[0,1]$. Then $V^* = \{0\}$.

Remark

The F-space $L^p[0,1]$ is not a locally convex TVS.

Definition: (Nowhere) Dense Subset

Let X be a topological space and $A \subseteq X$.
Then A is called dense in X if $\operatorname{clos}(A) = X$.
A is called nowhere dense in X if $\operatorname{int}(\operatorname{clos}(A)) = \varnothing$.
One can say A is dense at $x_0 \in X$ if $x_0 \in \operatorname{int}(\operatorname{clos}(A))$.

Examples

$X = \mathbb{R}$ and $A = \mathbb{Q}$, then A is dense in $\mathbb{R}$.
$X = \mathbb{R}^n$ and A a proper linear subspace, then A is nowhere dense.
$X = \mathbb{R}$ and $A = [0,1] \cap \mathbb{Q}$, then A is dense at points in $(0,1)$.

Lemma:

If A is open: A is dense if and only if $X \setminus A$ is nowhere dense.
If B is closed: $X \setminus B$ is dense if and only if B is nowhere dense.

$$\begin{aligned} B \text{ nowhere dense} &\iff \operatorname{int}(\operatorname{clos}(B)) = \varnothing \\ &\iff \operatorname{int}(B) = \varnothing \\ &\iff X \setminus \operatorname{int}(B) = \varnothing \\ &\iff \operatorname{clos}(X \setminus B) = \varnothing \\ &\iff X \setminus B \text{ dense in } X \end{aligned}$$

Proposition:

Any closed proper linear subspace W of a TVS V is nowhere dense in V.

Proof

Let $\operatorname{clos}(W) = W$, $W \subsetneq V$.
Find $x_0 \in V$, $x_0 \neq 0$

$$V \supseteq V_1 = W \dotplus \operatorname{lin}\{x_0\}$$

To show: $\operatorname{int}(W) = \varnothing$.
Otherwise, $v \in \operatorname{int}(W)$, U open, $V \in U \subseteq W$.
Now $\lambda \in \mathbb{F} \mapsto v + \lambda x_0$ continuous, $\lambda = 0 \mapsto v \in U$.
Then there exists some $\delta > 0$ such that $|\lambda| < \delta \implies v + \lambda x_0 \in U$.
For some $\lambda \neq 0$, $v + \lambda x_0 \in U \subseteq W$, $v \in U \subseteq W$ linear.
Then $\lambda x_0 \in W$ and $x_0 \in W$ a contradiction.

Definition: First and Second Category (Meager)

A topological space X is called of

- first category (meager) if X is the countable union of nowhere dense subsets.
- second category (nonmeager) otherwise.

Examples

$X = \mathbb{Q}$ is first category. $\mathbb{Q} = \bigcup_{q\in\mathbb{Q}}\{q\}$.
$X = \ell^1 = \{\{x_k\}_{k=1}^{\infty} : \sum|x_k| < +\infty\}$ is Banach of second category.
$X_n = \{\{x_k\}_{k=1}^{\infty} = x : x = \{x_1, x_2, \dots, x_n, 0, 0, \dots\}\} \subseteq X$ an n-dimensional subspace. Take

$$\hat{X} = \bigcup_{n=1}^{\infty} X_{nj}$$

Then $\hat{X}$ is of first category. $X_n \subseteq \hat{X}$ a closed, proper subspace which is nowhere dense.

Theorem: Baire Category Theorem

Every complete metric space is of second category.
All Banach spaces or F-spaces (Fréchet spaces) are of second category.

Remark: Uniform Bounded Principle

For normed spaces / Banach spaces (more general; see notes for F-spaces).

Theorem: (Uniform Bounded Norm)

Let X, Y be normed spaces and let $\{T_\omega\}_{\omega\in\Omega}$ be a collection of bounded linear operators $T_\omega \in L(X, Y)$.
Suppose that the set E of all $x \in X$ such that

1. $\sup_{\omega\in\Omega} ||T_\omega x|| < +\infty$ is of second category.

   Then

2. $\sup_{\omega\in\Omega} ||T_\omega|| < +\infty$.

Remark

If (2) holds, then (1) holds for all $x \in X$.

$$||T_\omega x|| \leq ||T_\omega|| \cdot ||x||$$

so $\sup ||T_\omega x|| \leq \sup ||T_\omega|| \cdot ||x||$ and $E = X$.

Proof

Define

$$E_n := \{x \in X : \sup_{\omega \in \Omega} ||T_\omega x|| \leq n\}$$

Then $E = \bigcup_{n=1}^{\infty} E_n$.
If E is of second category, then there exists n_0 such that E_{n_0} is not nowhere dense.
We know that E_n is closed since

$$E_n = \bigcap_{\omega \in \Omega} \{x \in X : ||T_\omega x|| \leq n\}$$

which are preimages with respect to T_ω of closed balls $\overline{B_n(0)} \subseteq Y$ and therefore closed in X.
Then $\operatorname{int}(\operatorname{clos}(E_n)) = \operatorname{int}(E_n) \neq \varnothing$, so there exists $x_0 \in X$, $\varepsilon > 0$ such that

$$B_\varepsilon(x_0) \subseteq E_{n_0}$$

Consider $x \in X$, $||x|| \leq 1$. Then $x_0 + \frac{\varepsilon}{2}x \in B_\varepsilon(x_0) \subseteq E_{n_0}$ and $x_0 \in B_\varepsilon(x_0) \subseteq E_{n_0}$.
It follows that

$$\begin{aligned} \left|\left|T_\omega\left(x_0 + \frac{\varepsilon}{2}x\right)\right|\right| &\leq n, \ \forall \omega \\ ||T_\omega(x_0)|| &\leq n, \ \forall \omega \end{aligned}$$

and

$$\begin{aligned} \left|\left|T_\omega\left(\frac{\varepsilon}{2}x\right)\right|\right| &\leq \left|\left|T_\omega\left(x_0 + \frac{\varepsilon}{2}x\right)\right|\right| + ||T_\omega x_0|| \\ ||T_\omega x|| &\leq \frac{4n_0}{\varepsilon} = C \end{aligned}$$

holds for all x with $||x|| < 1$. Therefore

$$||T_\omega|| = \sup_{x \neq 0} \frac{||T_\omega x||}{||x||} = \sup_{x \neq 0} \left|\left|T_\omega \frac{x}{||x||}\right|\right| = \sup_{||x||=1} ||T_\omega x|| \leq C$$

April 30, 2024

Recall: Uniform Boundedness Principle

X, Y normed spaces.
$\{T_\omega\}$, $T_\omega \in L(X, Y)$ bounded.
If the set E of all $x \in X$

1. $\sup ||T_\omega x|| < +\infty$ is of second category, then

2. $\sup ||T_\omega|| < +\infty$.

Theorem: Banach-Steinhaus

Let X, Y be Banach spaces and $\{T_\omega\}$ a collection of bounded linear operators ($T_\omega \in L(X,Y)$).
If

1. $\forall x \in X$: $\sup_\omega ||T_\omega x|| < +\infty$, then
2. $\sup_\omega ||T_\omega|| < +\infty$.

Proof

$E = X$ a Banach space, which is complete and therefore second category by Baire Category Theorem.

Remark

If X is not complete, then the conclusion may fail.

Example

Let $\hat{X} = \ell^1(\mathbb{N})$ (sequences $\{x_n\}_{n=1}^\infty$ such that $\sum |x_n| < +\infty$).
Take $X = \{x \in \{x_n\}_{n=1}^\infty \in \hat{X} \,:\, \exists N, \forall n \geq N, x_n = 0\}$.

$$X = \underbrace{\bigcup_{N=1}^{\infty} X_N}_{\text{1st Category}} \quad \text{and} \quad X_N = \{\{x_1, \dots, x_N, 0, 0 \dots\}\}$$

Then for $T_n \in L(X, \mathbb{F}) = X^*$, $T_n x = n \cdot x_n$ for $x = \{x_n\}$.
T_n linear and bounded, since

$$|T_n x| = n \cdot |x_n| \leq n \cdot \sum_{k=1}^{\infty} |x_k| = n \cdot ||x||$$

and therefore $||T_n|| \leq n$. In fact $||T_n|| = n$ because $x = \{0, \dots, 0, \underbrace{1}_{n\text{th}}, 0, \dots\}$ gives $T_n x = n$, $||x|| = 1$.

Therefore, 2 fails $\sup_n ||T_n|| = +\infty$.
However, 1 holds for all $x \in X$. For $x = \{x_1, \dots, x_N, 0, \dots\}$ take

$$\sup_n |T_n x| = \sup_n n \cdot |x_n| = \max_{1 \leq n \leq N} n \cdot |x_n| < +\infty$$

Definition: Strong Convergence

Let X and Y be normed spaces and $T_n, T \in L(X,Y)$.

1. T_n is said to converge strongly on X to T if $\forall x \in X$: $\lim_{n\to\infty} ||T_n x - Tx|| = 0$.
2. T_n is said to be strongly convergent on X if $\forall x \in X$, $\exists y \in Y$: $\lim_{n\to\infty} ||T_n x - y|| = 0$.

Obviously $(1) \implies (2)$.
Suppose (2) holds. Then one can define

$$Tx := \lim_{n\to\infty} T_n x$$

such that $||T_n x - Tx|| \to 0$.
One can show that T is a linear operator, but T does not need to be bounded.

Example

$\hat{X} \subseteq \ell^1$, $X = \{x \in \{x_n\}_{n=1}^{\infty} \in \hat{X} \, : \, \exists N, \forall n \geq N, x_n = 0\}$. Take

$$S_n x = \{1 \cdot x_1, 2 \cdot x_2, 3 \cdot x_3, \ldots, n \cdot x_n, 0, 0, \ldots\}$$

then $S_n : X \to X$ is linear, and bounded where

$$||S_n x|| = \sum_{k=1}^{n} k \cdot |x_k| = n \cdot \sum_{k=1}^{n} |x_k| \leq n \cdot ||x||_{\ell^1}$$

implies $||S_n|| = n$. Define

$$Sx = \{1 \cdot x_1, 2 \cdot x_2, \ldots, k \cdot x_k, \ldots\}$$

which is a linear operator $S : X \to X$ but is not bounded since

$$x = e_k = \{0, \ldots, \underbrace{1}_{k\text{th}}, 0, \ldots\}$$

gives $Se_k = k \cdot e_k$ implies $\frac{||Se_k||}{||e_k||} = k$ so $\sup \frac{||Sx||}{||x||} = +\infty$.
Yet $||S_n x - Sx|| \to 0$, $\forall x \in X$ since for

$$x = \{x_1, \ldots, x_N, 0, 0, \ldots\}$$

we have that $S_n x = Sx$ for $n \geq N$.
We conclude that S_n is strongly convergent on X; it converges to S but S is not bounded.
Note X not of second category.

Theorem:

Let X and Y be Banach spaces and $T_n \in L(X, Y)$. If T_n converges strongly on X, then

$$\sup_n ||T_n|| < +\infty$$

and there exists an operator $T \in L(X, Y)$ such that $Tx = \lim_{n\to\infty} T_n x$ (i.e. $\lim_{n\to\infty} ||T_n x - Tx|| = 0$, $\forall x \in X$).
Moreover,

$$||T|| \leq \liminf_{n\to\infty} ||T_n|| \leq \sup_n ||T_n|| < +\infty$$

Proof

For all $x \in X$, $T_n x$ converges to some $y \in Y$.
Since convergent sequences are bounded in normed spaces, this implies $\sup_n ||T_n x|| < +\infty$.
By the Banach-Steinhaus theorem, $C = \sup_n ||T_n|| < +\infty$.
Now define $Tx = \lim_{n\to\infty} T_n x = y$. So $T : X \to Y$ is a linear map

$$\lim_{n\to\infty} ||T_n x - Tx|| = 0, \ \forall x \in X$$

Then T is bounded since

$$||T_n x|| \le ||T_n|| \cdot ||x|| \le C||x||$$

or equivalently, taking the limit,

$$\lim_{n\to\infty} ||Tx|| \le \lim_{n\to\infty} ||-T_n x + Tx|| + ||T_n x|| \le \lim_{n\to\infty} ||Tx - T_n x|| + C||x||$$

implies that $||Tx|| < C||x||$.
Take $\alpha = \liminf_{n\to\infty}(||T_n||)$ and find $\{T_{n_k}\}$ such that $\alpha = \lim_{k\to\infty} T_{n_k}$. Then

$$||Tx|| \le \underbrace{||Tx - T_{n_k} x||}_{\to 0} + \underbrace{||T_{n_k}||}_{\to\alpha} \cdot ||x||$$

implies that $||Tx|| \le \alpha \cdot ||x||$ and $||T|| \le \alpha$.

Remark

For X and Y normed spaces and $T_n \in L(X, Y)$,
Convergence in the operator norm: $||T_n - T||_{L(X,Y)} \to 0$.
Strong convergence of operators: $\forall x \in X : ||T_n x - Tx||_Y \to 0$.
The former implies the latter, but not vice versa.
Strong convergence of operators is analogous to pointwise convergence.

Example

$Q_n : \ell^p \to \ell^p$, $1 \le p < \infty$.
$Q_n : \{x_k\} \mapsto \{0, \ldots, 0, x_{n+1}, x_{n+2}, \ldots\}$.
$||Q_n x|| \le ||x||$ implies that $||Q_n|| \le 1$ and, for

$$e_{n+1} = \{0, \ldots, 0, \underbrace{1}_{n+1}, 0, \ldots\}$$

we have $||Q_n e_{n+1}|| = 1$and $||e_{n+1}|| = 1$ which implies $||Q_n|| = 1$.
Therefore $Q_n \not\to 0$ in operator norm. But $Q_n \to 0$ strongly.
For $x \in \ell^p$,

$$||Q_n x|| = \left(\sum_{k=n+1}^{\infty} |x_k|^p \right)^{1/p} \underset{n\to\infty}{\to} 0$$

because $\sum_{k=1}^{\infty} |x_k|^p < +\infty$.

Divergence of Fourier Series

$X = C_{\text{per}}[-\pi, \pi] \ni f$ (continuous, periodic functions)
$f : [-\pi, \pi] \to \mathbb{C}$ continuous, $f(-\pi) = f(\pi)$.
Define Fourier coefficients

$$f_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx,\ n \in \mathbb{Z}$$

and consider the formal Fourier series

$$\sum_{n=-\infty}^{\infty} f_n e^{inx}$$

Consider the partial sums

$$F_n(x) = \sum_{k=-n}^{n} f_k e^{-ikx}$$

Theorem

There exists an $f \in X = C_{\text{per}}[-\pi, \pi]$ such that $f_n(0)$ does not converge (i.e. we do not even have pointwise convergence).

Proof

Write

$$\begin{aligned}
F_n(x) &= \sum_{k=-n}^{n} f_k e^{ikx} \\
&= \sum_{k=-n}^{n} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} f(t) e^{-itx}\, dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{\left(\sum_{k=-n}^{n} e^{i(x-t)k} \right)}_{D_n(x-t)} f(t)\, dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(x-t) f(t)\, dt
\end{aligned}$$

where

$$D_n(t) = \sum_{k=-n}^{n} e^{itx} = \frac{\sin(n+1/2)t}{\sin(t/2)}$$

is the Dirichlet kernel. Note that $D_n(t) = D_n(-t)$.
Define a map $L_n : f \in X \to \mathbb{C}$ as

$$L_n(f) = F_n(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(t) f(t)\, dt$$

By contradiction, assume that $F_n(0) = L_n(f)$ converges for every $f \in X$.
We have that L_n is a linear operator (as an integral).
Then given

$$|L_n(f)| \leq \sup_{t\in[-\pi,\pi)} |f(t)| \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)\, dt \leq ||D_n||_{L^1} \cdot ||f||_X$$

since $D_n(t)$ is continuous on $[-\pi, \pi]$ we have that L_n is bounded and $||L_n||_{X^*} \leq ||D_n||_{L^1}$.
Therefore, L_n is strongly convergent on X and $L_n \in L(X, \mathbb{C}) = X^{*j}$.
So, by Banach-Steinhaus $\sup_{n\in\mathbb{N}} ||L_n|| < +\infty$.
But $||L_n||_{X^*} = ||D_n||_{L^1}$ and $||D_n||_{L^1} \to +\infty$. (See below)
We have that $D_n(0) = 2n + 1$ and that the Dirichlet kernel oscillates as a sinusoidal. We want to find $f \in C_{\text{per}}[-\pi, \pi]$ such that

$$|L_n(f)| = ||D_n||_{L^1} \cdot ||f||_{C(-\pi,\pi)}$$

That is

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(t) f(t)\, dt \right| = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)|\, dt \cdot \sup_t |f(t)|$$

which is satisfied by

$$g = \begin{cases} +1 & \text{if } D_n(t) > 0 \\ -1 & \text{if } D_n(t) < 0 \end{cases}.$$

If we approximate $g(t)$ by suitable continuous functions, calling that function f_ε, then

$$|L_n(g - f_\varepsilon) = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(t)(g - f_\varepsilon)\, dt \right| \leq ||D_n||_{C[-\pi,\pi]} \cdot ||g - f_\varepsilon||_{L^1}$$

We can show (see lecture notes) that

$$\int_{-\pi}^{\pi} \left| \frac{\sin(n + 1/2)t}{\sin(t/2)} \right| \geq \alpha_n$$

where $\alpha_n \to +\infty$.

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Recall:

$f \in C_{\text{per}}[-\pi, \pi]$
$f_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx$

$$F_n(x) = \sum_{k=-n}^{n} f_k e^{ikx} \xrightarrow[\times]{?} f(x)$$

$$F_n(x) = \int_{-\pi}^{\pi} f(x - t) D_n(t)\, dt$$

with

$$D_n(t) = \sum_{k=-n}^{n} e^{-nt} = \frac{\sin(n+1/2)t}{\sin(t/2)}$$

the Dirichlet kernel.

Fejér-Cesàro Means

$$\begin{aligned}\sigma_n &= \frac{1}{n}\sum_{k=0}^{n-1} F_k(x) = \sum_{k=-n}^{n}\left(1-\frac{|k|}{n}\right) f_k e^{ikx} \\ &= \int_{-\pi}^{\pi} f(x-t)s_n(t)\,dA\end{aligned}$$

with

$$s_n(t) = \sum_{k=-n}^{n}\left(1-\frac{|k|}{n}\right)e^{ikt}\frac{1}{n}\left(\frac{\sin(nt/2)}{\sin(t/2)}\right)^2$$

the Fejér kernel.
Note that $\int_{-\pi}^{\pi} s_n(t)\,dt = 1$ and $s_n(t) \rightrightarrows 0$ for $\delta \le |t| \le \pi$.

Theorem

For $f \in C_{\text{per}}[-\pi,\pi]$, $\sigma_n(x) \rightrightarrows f(x)$ uniformly on $[-\pi,\pi]$ as $n \to \infty$.

Proof (Sketch)

$$\begin{aligned}\sigma_n(x) - f(x) &= \int_{-\pi}^{\pi}(f(x-t)-f(x))s_n(t)\,dt \\ &= \int_{|t|<\delta}(f(x-t)-f(x))s_n(t)\,dt + \int_{\pi\ge|t|\ge\delta}(f(x-t)-f(x))s_n(t)\,dt \\ |\sigma_n(x)-f(x)| &= \sup_{|t|\le\delta}|f(x-t)-f(x)|\cdot||s_n||_{L^1} + 2||f||_\infty \cdot 2\pi \cdot \sup_{\pi\ge|t|\ge\delta}|s_n(t)|\end{aligned}$$

Given ε, by the uniform continuity of f, find $\delta > 0$ such that

$$\sup_x \sup_{|t|\le\delta}|f(x-t)-f(x)| < \varepsilon$$

Then $||s_n||_{L^1} = 1,\ s_n(t) \ge 0,\ \int_{-\pi}^{\pi} s_n(t)\,dt = 1$ and, for fixed δ,

$$\lim_{n\to\infty}\sup_{\pi\ge|t|\ge\delta}|s_n(t)| = 0$$

It follows that

$$\sup_x|\sigma_n(x)-f(x)| \le \varepsilon + c\cdot\sup_{|t|\ge\delta}|s_n(t)|$$

Taking $n \to \infty$,

$$\limsup_{n\to\infty} \sup |\sigma_n(x) - f(x)| \le \varepsilon,\ \forall \varepsilon > 0$$

and

$$\lim_{n\to\infty} \sup_x |\sigma_n(x) - f(x)| = 0$$

Operator Interpretation

One can define $A_n : f \in C_{\text{per}}[-\pi,\pi] \to \sigma_n(x) \in C_{\text{per}}[-\pi,\pi]$ where $\sigma_n \rightrightarrows f$ means $A_n \to I$ strongly on $C_{\text{per}}[-\pi,\pi]$.
Since $\forall f \in C_{\text{per}}[-\pi,\pi]$, we have $\sigma_n = A_n f \to f$ in the norm of $C_{\text{per}}[-\pi,\pi]$.

Theorem: Open Mapping Theorem

Let V be an F-space, W be a TVS, and let $T : V \to W$ be a continuous linear operator such that $\operatorname{im} V$ is of 2nd category in W.
Then T is open, $\operatorname{im} T = W$ and W is an F-space.

Remark

$\operatorname{im} T$ is of 2nd category in W means $\operatorname{im} T$ is not a countable union of nowhere dense subsets in W.

Definition: Open Map

T open means T maps open sets into open sets.

Proof

Have to show: for each open neighborhood $U \ni 0$ in V, $T(U)$ contains an open neighborhood of 0.
Consider $V_n = \{x \in V \,:\, d(x,0) < r/2^n\}$ and $r > 0$ such that $V_0 \subseteq U$.
Idea: $\overline{TV_1} \subseteq TV_0 \subseteq TU$ and $\overline{TV_1}$ contains an open neighborhood of 0.

Step 1

$\overline{TV_n}$ contains an open neighborhood of 0.
Note that $d(x,0) < r/2^{n+1}$ and $d(y,0) < r/2^{n+1}$ implies

$$d(x-y,0) = d(x,y) \le d(x,0) + d(0,y) < 2 \cdot r/2^{n+1}$$

Take $V_n \supseteq V_{n+1} - V_{n+1}$ such that $TV_n \subseteq T(V_{n+1} - V_{n+1}) = TV_{n+1} - TV_{n+1}$. Then

$$\overline{TV_{n+1}} \supseteq \overline{TV_{n+1} - TV_{n+1}} \supseteq \overline{TV_{n+1}} - \overline{TV_{n+1}}$$

Obviously,

$$V = \bigcup_{k=1}^{\infty} k \cdot V_{n+1}$$

because V_{n+1} is an open neighborhood of zero and absorbing. Hence

$$TV = \bigcup_{k=1}^{\infty} kTv_{n+1} \quad \text{and} \quad TV \subseteq \bigcup_{k=1}^{\infty} k\overline{TV_{n+1}}$$

Since TV is of second category, there exists some k such that kTV_{n+1} is not nowhere dense.
Then $\operatorname{int}(k\overline{TV_{n+1}}) \neq \varnothing$ which implies $\operatorname{int}(\overline{TV_{n+1}}) \neq \varnothing$. That is, $\overline{TV_{n+1}}$ contains an interior point, say x_0.
Then there exists an open neighrborhood $\hat{U} \ni 0$ such that $x_0 + \hat{U} \subseteq \overline{TV_{n+1}}$.

$$\hat{U} = (x_0 + \hat{U}) - x_0 \subseteq \overline{TV_{n+1}} - \overline{TV_{n+1}} \subseteq \overline{TV_n}$$

Step 2

$\overline{TV_1} \subseteq TV_0$.
Let $y_1 \in \overline{TV_1}$, $y_1 - \overline{TV_2}$ contains some neighborhood of y_1.
Then $(y_1 - \overline{TV_2}) \cap TV_1 \neq \varnothing$. Choose $w_1 = y_1 - y_2$, $y_2 \in \overline{TV_2}$, $w_1 = Tx_1$, $x_1 \in V_1$.
By the same argument, choose $w_2 = y_2 - y_3$, $y_3 \in \overline{TV_3}$, $w_2 = Tx_2$, $x_2 \in V_2$.
Continuing gives $y_1, y_2, y_3, \ldots$, $x_1, x_2, x_3, \ldots$, $w_1, w_2, w_3, \ldots$
Where $x_n \in V_n$, $y_n \in \overline{TV_n}$, $w_n = y_n - y_{n+1} = Tx_n$ or, equivalently, $y_{n+1} = y_n - Tx_n$.
It follows that $y_{n+1} = y_1 - T(x_1 + \cdots + x_n)$.
Because $x_n \in V_n$ $(d(x_n, 0) < r/2^n)$, $x_1 + \cdots + x_n$ is a Cauchy sequence.
That is, by completeness, $v = \sum_{n=1}^{\infty} x_n$ with $d(V, 0) \leq \sum_{k=1}^{\infty} d(x_k, 0) < r$ and $v \in V_0 \subseteq V$.
Taking $n \to \infty$, $\lim_{n\to\infty} y_n = y_1 - Tv$.
Claim: $y := \lim_{n\to\infty} y_n = 0$. Otherwise, $y \neq 0$, $y \in W$ where W is Hausdorff, there exists open neighborhoods of 0 and y where

$$W_0 \cap W_y = \varnothing$$

But as a continuous linear operator, $T^{-1}(W_0)$ has an open neighborhood of 0.
So there exists some n such that $V_n \subseteq T^{-1}(W_0)$ which implies that $TV_n \subseteq W_0 \subseteq W \setminus W_y$ closed.
Then $\overline{TV_n} \subseteq W \setminus W_y$ but $W \setminus W_y$ which implies $y \notin \overline{TV_n}$.
For $N \geq n$, $y_n \in \overline{TV_N} \subseteq \overline{TV_n}$. So $y_n \to y$, $y_n \in \overline{TV_n}$ $(N \geq n)$, $y \notin \overline{TV_n}$ a contradiction.
Therefore $y = 0$, $y_1 = TV$, $v \in V_0$, $y_1 \in TV_0$ and finally $\overline{TV_1} \subseteq TV_0$.

To Show

The above demonsrates that T is open.
We still need that $\operatorname{im} T = W$ and W is an F-space.

Part 3

We have that

$$\operatorname{im} T = T(V)$$

open in W. Since open neighborhoods of 0 are absorbing,

$$\bigcup_{k=1}^{\infty} kTV = W = \bigcup_{k=1}^{\infty} T(kV) = \bigcup_{k=1}^{\infty} TV = TV$$

so $TV = W$.

Part 4 (Sketch)

We have that $T : V \to W$ open, surjective, and continuous.
Define $\hat{T} : V/\ker(T) \to W$ as

$$\begin{aligned} \hat{T} :& [x] \to Tx \\ & [x] = x + \ker(T) \end{aligned}$$

a continuous linear operator with $\ker(T)$ a closed subspace. Then

$$\begin{tikzcd} V \arrow[r, "\pi"] \arrow[dr, "T"'] & V/\ker(T) \arrow[d, "\hat{T}"] \\ & W \end{tikzcd}$$

We have that $V/\ker(T) = F$-space, $\hat{d}([x],[y]) = \inf_{z\in\ker(T)} d(x+z,y)$.
Per the commutative diagram, $\hat{T}$ is open and continuous (a linear homeomorphism). Take

$$\begin{aligned} \hat{T} : V/\ker(T) &\to W \\ \hat{d} &\rightsquigarrow d_W \end{aligned}$$

With $d_W(Tx_1, Tx_2) = \hat{d}([x_1],[x_2]) = \inf_z d(x_1+z, x_2)$.
Then $\hat{T}$ is an isometry and, with d_W, an F-space.
The topology induced by d_W is equivalent to the original topology.

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Theorem: Open Mapping Theorem

Let V and W be F-spaces, and let $T : V \to W$ be a continuous linear operator which is surjective.
Then T is open.

Proof

$\operatorname{im} T = W$ is of second category since it is an F-space.

Corollary: Banach's Theorem About the Inverse Operator

Let V, W be F-spaces, and let $T : V \to W$ be a continuous linear operator which is bijective (invertible).
Then the inverse $T^{-1} : W \to V$ is continuous.

Remark

This result implies:

- Each (pre-)F-space of dimension n is topologically isomorphic to $\mathbb{F}^n$.

Proof

For V a pre-F-space, $T : \mathbb{F}^n \to V$ a linear bijection and V complete, T^{-1} is continuous.

Corollary

Let V be a vector space with two topologies τ_1, τ_2 such taht (V, τ_1) and (V, τ_2) become F-spaces. If $\tau_1 \subseteq \tau_2$, then $\tau_1 = \tau_2$.

Proof

For $I : \underset{(\tau_2)}{V} \to \underset{(\tau_1)}{V}$ the identity map $Ix = x$, T is continuous.
Then $I^{-1} : \underset{(\tau_1)}{V} \to \underset{(\tau_2)}{V}$ is continuous and $\tau_2 \subseteq \tau_1$.

Corollary

Let V, W be Banach spaces and $T : V \to W$ be a bounded linear operator which is bijective (invertible).
Then $\exists a, b > 0$ such that

$$a \cdot ||x||_V \leq ||Tx||_W \leq b \cdot ||x||_V$$

Proof

Since $T : V \to W$ is bounded (continuous),

$$||Tx|| \leq \underbrace{||T||_{L(V,W)}}_{b} \cdot ||x||$$

and since $T^{-1} : W \to V$ is bounded

$$||x|| = ||T^{-1}Tx|| \leq \underbrace{||T^{-1}_{L(W,V)}}_{1/a} \cdot ||Tx||$$

Corollary

Let V be a vector space with two norms $||\cdot||_1$ and $||\cdot||_2$ such that both $(V, ||\cdot||_1)$ and $(V, ||\cdot||_2)$ are Banach spaces.
Assume that there exists some M such that (1) $||x||_1 \leq M \cdot ||x||_2$, $\forall x \in V$.
Then both norms are equivalent, and there exists $m > 0$ such that

$$(2) \quad ||x||_2 \leq m \cdot ||x||_1, \quad \forall x \in V$$

Proof

For I the identity operator, $I : (V, ||\cdot||_2) \to (V, ||\cdot||_1)$,
(1) implies that I is bounded which implies I^{-1} is bounded which finally implies (2).

Examples

Counter-Example 1

For $\ell^1 \subseteq \ell^\infty$, take $I : \ell^1 \to \ell^\infty$ the identity map $Ix = x$.
Take $V = (\ell^1, ||\cdot||_1)$ where $||x||_1 = \sum_{n=1}^{\infty} |x_n|$ and $W = (\ell^1, ||\cdot||_\infty)$ where $||x||_\infty = \sup_{n \geq 1} ||x_n|$.
V is complete while W is not complete (completion $c_0 = \{x \in \ell^\infty : \lim_{n\to\infty} x_n = 0\}$.
I is bounded, so

$$||Ix||_\infty = ||x||_\infty = \sup_{n\geq 1}|x_n| \leq \sum_{n=1}^{\infty}|x_n| = ||x||_1$$

However, I^{-1} is not bounded otherwise for some constant $b > 0$,

$$||x||_1 \leq b||x||_\infty, \quad \forall x \in \ell^1$$

and

$$\sum_{n=1}^{\infty}|x_n| \leq b \cdot \sup_{n\geq 1}|x_n|$$

If we choose

$$x = (\underbrace{1, 1, \ldots, 1}_{n \text{ times}}, 0, 0, 0, \ldots\}$$

Then $n \geq b \cdot 1$ and sending $n \to \infty$ causes a contradiction.

Counter-Example 2

Let V be an infinite dimensional Banach space with norm $||\cdot||$.
Choose an unbounded linear functional $\phi \in V'$ ($\phi \notin V^*$), $\phi : V \to \mathbb{F}$.
Define a new norm $||x||_* = ||x|| + |\phi(x)|$. Then take the identity map

$$I : \underset{\text{not complete}}{(V, ||\cdot||_*)} \to \underset{\text{complete}}{(V, ||\cdot||)}$$

Obviously $||x|| \leq ||x||_*$, so I is bounded. But it is not true that $||x||_* \leq C \cdot ||x||$, $\forall x \in V$.
Otherwise we would have that $|\phi(x)| \leq C||x||$ which would make ϕ bounded, a contradiction.
By previous corollary, this implies that $(V, ||\cdot||_*)$ is not complete.

Definition: Graph of a Function

Given $f : X \to Y$, the graph of f: $G(f) = \{(x, f(x)) : x \in X\} \subseteq X \times Y$.
Simetimes, $f : D(f) \subseteq X \to Y$ where $D(f)$ is the domain and $G(f) = \{(x, f(x)) : x \in D(f)\} \subseteq X \times Y$.

Definition: Closed Graph of a Function

Let x, Y be topological spaces and f be a function from X (or $D(f) \subseteq X$) into Y.
Then f is of closed graph if $G(f)$ is a closed subset in $X \times Y$.

Examples

$f(x) = \frac{1}{x}$, $D = \mathbb{R} \setminus \{0\}$, $X = Y = \mathbb{R}$ is continuous on D and has a closed graph. Contrarily

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$D(f) = X = Y = \mathbb{R}$ is of closed graph but not continuous. Finally

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

is neither continuous nor of closed graph.

Lemma:

Let X, Y be metric spaces and $f : D(f) \subseteq X \to Y$.
Then f is of closed graph if and only if whenever $x_n \to x$ with $x_n \in D(f)$ and $f(x_n) \to y$, $x \in x$, $y \in y$, then $f(x) = y$ and $x \in D(f)$.

Proof

For $G(f)$ closed in $X \times Y$, we have that whenever $(x_n, f(x_n)) \in G(f)$ converges $(x_n, f(x_n)) \to (x, y)$, then $(x, y) \in G(f)$.
Then whenever $x_n \in D(f)$ converges $x_n \to x$ and $f(x_n) \to y$, then $x \in D(f)$ and $y = f(x)$.

Proposition:

If $f : X \to Y$ is continuous, X a topological space and Y Hausdorff, then f is of closed graph.

Proof

Take $U = (X \times Y) \setminus G(f)$, $(x_0, y_0) \in U$.
Then $(x_0, y_0) \notin G(f)$, so $y_0 \neq f(x_0)$. Since Y is Hausdorff, there exist open sets $U_{f(x_0)} \ni f(x_0)$ and $U_{y_0} \ni y_0$ with $U_{y_0} \cap U_{f(x_0)} = \emptyset$.
$U_{x_0} = f^{-1}(U_{f(x_0)})$ is open in X with $x_0 \in U_{x_0}$.
Claim: $U_{x_0} \times U_{y_0} \subseteq U$ a neighborhood of (x_0, y_0) so (x_0, y_0) is an interior point of U.
We have that $(U_{x_0} \times U_{y_0}) \cap G(f) = \emptyset$ with $(x, y) \in G(f)$.
But $y = f(x) \in U_{y_0}$, $x \in U_{x_0} = f^{-1}(U_{f(x_0)})$, and $f(x) \in U_{f(x_0)}$ contradicts the fact that they are disjoint.

Theorem: Closed Graph Theorem

Let X, Y be F-spaces and $A : X \to Y$ be a linear operator which is of closed graph.
Then A is continuous.

Proof

$X \times Y$ is an F-spaces equipped with a metric $d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$.
Then $\{(x, Ax) : x \in X\} = G(A) \subseteq X \times Y$ is a linear subspace and closed by assumption.

$$\begin{gathered} (x_1, Ax_1) + (x_2, Ax_2) = (x_1 + x_2, A(x_1 + x_2)) \\ \lambda(x, Ax) = (\lambda x, A(\lambda x)) \end{gathered}$$

Further, $G(A)$ is an F-space (complete). Take the projection

$$\begin{aligned} \pi : G(A) &\to X \\ (x, Ax) &\mapsto X \end{aligned}$$

a continuous linear operator since

$$(x_n, Ax_n) \to (x, Ax) \implies x_n \to x$$

We have also that π is bijective, since

$$\pi((x, Ax)) = x \quad \text{and} \quad \pi(x, Ax)) = 0 \implies x = 0 \implies Ax = 0$$

Applying the open mapping theorem and the Banach theorem for inverse operators,

$$\begin{aligned} \pi^{-1} : X &\to G(A) \\ x &\mapsto (x, Ax) \end{aligned}$$

is also continuous. If $x_n \to x$, then $\pi^{-1}(x_n) \to \pi^{-1}(x)$ (($(x_n, Ax_n) \to (x, Ax)$) gives $Ax_n \to Ax$ and A is continuous.

For Banach Spaces

X, $||x||$.

$$||x||_* = ||\pi^{-1}(x)|| = ||x|| + ||Ax||$$

$||x|| \neq |\phi(x)|$.

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Example

Consider $X = C^1[0,1] \subseteq C[0,1]$ and $Y = C[0,1]$ both with the norm $||f|| = \sup_{x\in[0,1]} |f(x)|$.
Note that X is not complete but Y is complete. Take

$$T : f \mapsto f'$$

where $T : X \to Y$ is closed but not bounded.
Given $f_n = \sin(nt)$, for n sufficiently large, $||f_n|| = 1$.
However, $Tf_n = f_n' = n \cdot \cos(nt)$ and $||Tf_n|| = n$. Therefore $||Tf|| \leq C||f||$ cannot hold for all $f \in X$.
Now, given $f_n \in C^1[0,1]$ where $f_n \to f \in C[0,1]$ and, consquently, that $Tf_n \to g \in C[0,1]$.
Since $f_n \rightrightarrows f$ and $f_n' \rightrightarrows g$ uniformly on $[0,1]$. Then

$$\int_0^x f_n(t)\,dt \rightrightarrows \int_0^x g(t)\,dt$$

unfiromly on $x \in [0,1]$. So

$$f_n(x) - f_n(0) \rightrightarrows f(x) - f(0) = \int_0^x g(t)\,dt$$

and $\frac{d}{dx}\int_0^x g(t)\,dt = g(x)$ so f is differentiable. It follows that $f' = Tf = g$.

Example

Take $X = Y = L^1[0,1]$ and $D(T) = \left\{ f \in L^1[0,1] \,:\, f = c + \int_0^x g(t)\, dt,\ g \in L^1[0,1] \right\}$ with $T : f \to f'$.
T is closed graph $(T : D(T) \subseteq X \to Y)$; T is not bounded.

Proposition:

Let $X,\ Y$ be pre-F-spaces (or even TVS), and let $T : D(T) \subseteq X \to R(T) \subseteq Y$ be a linear operator which has an inverse.
Then $T^{-1} : R(T) \subseteq Y \to D(T) \subseteq X$ and T is closed graph if and only if T^{-1} is closed graph.

Proof

$G(T) = \{(x, Tx) \,:\, x \in D(T)\} \subseteq X \times Y.$
$G(T^{-1}) = \{(y, T^{-1}y) \,:\, y \in R(T)\} = \{(Tx, x) \,:\, x \in D(T)\} \subseteq Y \times X.$

Remark

The inverse of a bijective continuous operator between two TVS is closed graph.

Proof

$T : X \to Y$ bijective, linear and continuous is of closed graph.
Then $T^{-1} : Y \to X$ is of closed graph.

Definition: Closable Operator

Let $X,\ Y$ be F-spaces, $X_0 \subseteq X$ a subspace.
$T : X_0 \subseteq X \to Y$ is closed graph if $G(T)$ is closed in $X \times Y$.
$T : X_0 \subseteq X \to Y$ is closable if there exists an operator $\hat{T} : X_1 \subseteq X \to Y$ such that $G(\hat{T}) = \overline{G(T)}$ where $x_1 \supseteq x_0$.

Remark

T is closed if and only if $x_n \in X_0,\ x_n \to x,\ Tx_n \to y$ implies that $x \in X$ and $Tx = y$.
T is closable if and only if $x_n \in X_0,\ x_n \to x,\ Tx_n \to y$ implies that $y = 0$.

Construction

Take $X_1 = \{x \in D(T) \,:\, \exists\{x_n\} \subseteq X_0,\ x_n \to x,\ Tx_n$ converges$\}$.
$\hat{T}x = \lim_{n\to\infty} Tx_n$ where $x_n \to x$ and Tx_n also converges.

Example

Take $X = Y = L^2[0,1]$.
For $X_0 = D(T) = C^1[0,1]$, $T : f \to f'$ is a closable (but not closed) operator.

Applications of Closed Graph Theorem

Projections and Direct Sums

Given a direct sum $X = X_1 \dot{+} X_2$ where every $x \in X$ is the sum $x = x_1 + x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$. One can define

$$P_1 : x = x_1 + x_2 \in X \mapsto x_1 \text{ the projection of } X \text{ onto } X_1 \text{ along } X_2$$
$$P_2 : x = x_1 + x_2 \in X \mapsto x_2 \text{ the projection of } X \text{ onto } X_2 \text{ along } X_1$$

Then $P_1 P_1 = P_1$, $P_2 P_2 = P_2$, and $I = P_1 + P_2$.

$$x = x_1 + x_2 \overset{P_1}{\mapsto} x_1 = x_1 + 0 \overset{P_1}{\mapsto} x_1$$

Note that $R(P_1) = X_1$, $N(P_1) = X_2$, $N(P_2) = X_1$ and $R(P_2) = X_2$.
Conversely, given $P : X \to X$ a linear operator sastisfying $P^2 = P$, we can define $X_1 := R(P) = N(I - P)$ and $X_2 := N(P) = R(I - P)$.
Then $X = X_1 + X_2$ and $P : x = x_1 + x_2 \mapsto x_1$.

Theorem

Let X be an F-space, $X = X_1 \dot{+} X_2$ and P be the projection of X onto X_1 along X_2.
Then P is continuous if and only if X_1, X_2 are closed.

Proof

($\Longrightarrow$) For P continuous, $X_1 = N(I - P)$ and $X_2 = N(P)$ are both closed (as they are the preimage of $\{0\}$).
($\Longleftarrow$) By the closed graph theorem, if P is of closed graph then P is continuous.
Take $x_n \to x$, $Px_n \to y$. We want to show that $Px = y$.
Then $x_n = x_n^{(1)} + x_n^{(2)} \to x$, $Px_n = x_n^{(1)} \to y$. Since X_1 is closed, $y \in X_1$. It follows that

$$x_n^{(2)} \to (x^{(1)} - y) + x^{(2)}$$

and, since X_2 is closed, $(x^{(1)} - y) + x^{(2)} \in X_2$ which implies that $x^{(1)} - y = 0$.
Therefore $y = x^{(1)} = Px$.

Alternative Proof (Sketch)

Consider a linear map $\pi : X_1 \times X_2 \to X_1 + X_2 = X$ $((x_1, x_2) \mapsto x_1 + x_2)$.
Then $X_1, X_2 \subseteq X$ a complete space. It follows that X_1, X_2, and importantly $X_1 \times X_2$ are F-spaces.
Then π is continuous, since

$$||x_1 + x_2|| \leq ||x_1|| + ||x_2|| = ||(x_1, x_2)||_{X \times Y}$$

or for F-spaces

$$(x_1^{(n)}, x_2^{(n)}) \mapsto (x_1, x_2)$$

implies that $x_1^{(n)} + x_2^{(n)} \to x_1 + x_2$.
Since π is bijective, Banachs' theorem about inverse operators states that

$$\pi^{-1} : X = X_1 \dot{+} X_2 \to X_1 \times X_2$$

is continuous. Then

$$\begin{tikzcd} x_1 + x_2 \in X \arrow[r, "\pi^{-1}"] \arrow[dr, "P"'] & X_1 \times X_2 \ni (x_1, x_2) \arrow[d, "\pi_1"] \\ & X_1 \end{tikzcd}$$

So $P = \pi_1 \circ \pi^{-1}$ is continuous.

Applications Continued

Fourier Series

Consider the Fourier coefficients on $L^1[-\pi, \pi]$-functions. Take

$$T : f \in L^1 \mapsto \{f_n\}_{n=-\infty}^{\infty} \in \ell^{\infty}(\mathbb{Z})$$

where $f_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int}\, dt$ for $n \in \mathbb{Z}$. We have that $|f_n| \leq ||f||_{L^{-1}}$ and

$$||\{f_n\}||_{\ell^\infty} = \sup_n |f_n| \leq ||f||_{L^1}$$

Actually,

$$\lim_{|n| \to \infty} |f_n| = 0$$

so $T : f \in L^1 \to C_0(\mathbb{Z}) \subseteq \ell^{\infty}(\mathbb{Z})$ where $C_0(\mathbb{Z})$ is the set of all $\{f_n\}_{n=-\infty}^{\infty}$ such that $\lim_{|n| \to +\infty} |f_n| = 0$.
Claim: $\operatorname{im} T$ is of first category in C_0. In particular, $\operatorname{im} T \neq C_0$.
Otherwise, $T : L^1 \to C_0$ is open. We state without proof that $N(T) = 0$ (Fourier coefficients of L^1-functions are unique).
This would imply that T^{-1} is continuous. However

$$f^{(N)} = \sum_{n=-N}^{N} e^{inx}$$

where

$$Tf^{(N)} = \{\dots, 0, 0, \underbrace{1}_{-N}, 1, \dots, 1, \underbrace{1}_{N}, 0, 0, \dots\} = \{f_n^{(N)}\}$$

with $||Tf^{(N)}|| = 1$, $||f^{(N)}||_{L^1} \to +\infty$ as $N \to \infty$. This would mean

$$||f^{(N)}|| \leq ||T^{-1}|| \cdot ||Tf^{(N)}|| \leq ||T^{-1}|| \cdot 1$$

which is a contradiction.

Reflexive Spaces

Consider V a normed space.
$V^* = L(V, \mathbb{F})$, the dual space, is Banach.

$$||f||_{V^*} = \sup_{\substack{x \in V \\ x \neq 0}} \frac{|f(x)|}{||x||_V}$$

$(f_1 + f_2)(x) := f_1(x) + f_2(x)$ and $(\lambda f)(x) := \lambda f(x)$.
$(V^*)^* = L(V^*, \mathbb{F})$, the bidual or second dual of V.
V can be identified wiht a subset of $(V^*)^* = V^{**}$. Define

$$\tau : x \in V \mapsto \phi_x \in V^{**}$$

where $\phi_x(f) = f(x)$, $f \in V^*$.

Proposition

$\phi_x \in V^{**}$.

Proof

$\phi_x : V^* \to \mathbb{F}$ a map.
Linearity:

$$\phi_x(f_1 + f_2) = (f_1 + f_2)(x) = f_1(x) + f_2(x) = \phi_x(f_1) + \phi_x(f_2)$$

$$\phi_x(\lambda f) = (\lambda f)(x) = \lambda f(x) = \lambda \phi_x(f)$$

Boundedness:

$$|\phi_x(f)| = |f(x)| \leq ||f||_{V^*} ||x||_V, \quad \forall f \in V^*$$

so ϕ_x is bounded and

$$\frac{|\phi_x(f)|}{||f||_{V^*}} \leq ||x||$$

Taking the supremum over $f \in V^*$ gives

$$||\phi_x|| \leq ||x||$$

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Recall: Reflexive Banach Spaces

For V a normed space, take V^* the dual, and V^{**} the bidual.
We have $\tau : x \in V \mapsto \phi_x \in V^{**}$ with $\phi_x(f) = f(x)$, $f \in V^*$.

Theorem:

τ is an isometric isomorphism from V onto $\operatorname{im} \tau \subseteq V^{**}$.

Proof

τ is linear, since $\tau(x+y) = \phi_{x+y}$ and

$$\begin{aligned}
\phi_{x+y}(f) &= f(x+y) && f \in V^* \\
&= f(x) + f(y) \\
&= \phi_x(f) + \phi_y(f) && \text{addition in } V^{**} \\
&= (\phi_x + \phi_y)(f) \\
\phi_{x+y} &= \phi_x + \phi_y = \tau(x) + \tau(y)
\end{aligned}$$

Isometric means $||\tau(x)|| = ||x||$, $||\phi_x|| = ||x||$.
We know that $||\phi_x|| \leq ||x||$. For $x \neq 0$, define $f_0 \in (\text{lin}\{x\})^*$ by $f_0(\lambda x) = \lambda ||x||$. Then

$$||f_0|| = \sup_{\lambda \neq 0} \frac{|f(\lambda x)|}{||\lambda x||} = 1$$

and we may extend f_0 by Hahn-Banach to $\hat{f} \in V^*$ with the same norm $||\hat{f}|| = 1$. We have that

$$||\phi_x|| = \sup_{\substack{f \in V^* \\ f \neq 0}} \frac{|\phi_x(f)|}{||f||} \geq \frac{|\phi_x(\hat{f})|}{||\hat{f}||} = \frac{|\hat{f}(x)|}{1} = |f_0(x)| = ||x||$$

τ is injective (because it is isometric).
We see, since $\tau(x) = 0 \implies ||\tau(x)|| = 0 = ||x|| \implies x = 0$, the kernel is trivial.
Therefore we conclude that τ is an isomorphism $\tau : V \to \text{im}(\tau) \subseteq V^{**}$.

Remark

τ need not be surjective $(\text{im}(\tau) \underset{\neq}{\subset} V^{**})$.

Definition: Reflexive Space

V is called reflexive if τ is surjective (i.e. $\text{im}(\tau) = V^{**}$)

Proposition:

A reflexive normed space is Banach.

Proof

Assume $\tau : V \to V^{**}$ is a surjective isometry.
V is complete, since $V^{**} = (V^*)^*$ is complete.
Take $\{x_n\}$ Cauchy in V, then $\tau(x_n)$ is Cauchy in V^{**} hence $\tau(x_n) \to y$.
Since τ is surjective, $y = \tau(x)$, for some $x \in V$. Then

$$||x_n - x|| = ||\tau(x_n) - \tau(x)|| = ||\tau(x_n) - y||$$

so $x_n \to x$.

Remark:

τ can be used to construct a completion of a normed space.
$\tau : V \to \operatorname{im}(\tau) \subseteq \overline{\operatorname{im}(\tau)} = W \subseteq V^{**}$.
Then W is complete and $\operatorname{im}(\tau)$ is dense in W.

Remark:

For reflexive space V, $V \cong V^{**}$ (isomorphically isometric).
Converse is not true. There exist examples where $V \cong V^{**}$ but τ is not surjective.

Theorem:

Let V be a Banach space.
Then V is reflexive if and only if V^* is reflexive.

$$\begin{tikzcd}
V \arrow[rr, bend right, "\cong"'] & V^* \arrow[rr, bend left, "\cong"] & V^{**} \arrow[rr, bend right, "\cong"'] & V^{3*} \arrow[rr, bend left, "\cong"] & V^{4*} & V^{5*} \cdots
\end{tikzcd}$$

Proof

Informally, $V \cong V^{**}$ if and only if $V^* \cong V^{3*}$.

$$\begin{aligned}
\tau &: V \to V^{**} & \tau(x) &= \phi_x & \phi_x(f) &= f(x) & f &\in V^* \\
\hat{\tau} &: V^* \to V^{3*} & \hat{\tau}(x) &= \psi_x & \psi_x(f) &= \phi(f) & \phi &\in V^{**}
\end{aligned}$$

Then ($\Longrightarrow$)

$$V \cong V^{**} \implies V^* \cong V^{3*}$$

$$\begin{aligned}
\tau &: V \to V^{**} & \tau^{-1} &: V^{**} \to V \\
\tau^* &: V^{3*} \to V^* & (\tau^*)^{-1} &: V^* \to V^{3*}
\end{aligned}$$

Taking the adjoint, $\hat{\tau} = (\tau^*)^{-1} = (\tau^{-1})^*$ is bijective.
($\Longleftarrow$) Assume taht $\hat{\tau}$ is surjective and V Banach.
For a contradiction, assume that τ is not surjective. Then $\operatorname{im}(\tau) \subsetneq V^{**}$.
But $\operatorname{im}(\tau)$ is complete and closed ($\operatorname{im}(\tau) \cong V$ an isometry).
Then there exists some $\phi_0 \notin \operatorname{im}(\tau)$. By Hahn-Banach (and the closer of the image) this means there exists some $\psi_0 \in (V^{**})^*$ where

$$\psi_0(\phi_0) = 1 \qquad\qquad \psi_0|_{\operatorname{im}(\tau)} = 0$$

By assumption, V^* is reflexive so $\hat{\tau} : V^* \to V^{3*}$ is surjective.
Then there exists some $f_0 \in V^*$ where $\hat{\tau}(f_0) = \psi_0$. But $\psi_0 \neq 0$ implies that $f_0 \neq 0$.
Now $0 = \psi_0(\tau(x)) = \psi_0(\phi_x) = (\hat{\tau}(f_0))(\phi_x) = \phi_x(f_0) = f_0(x)$, so $f_0(x) \equiv 0$ for any x which is a contradiction.

Theorem:

A closed subspace of a reflexive space is reflexive.

Remark

For V reflexive, $V \cong V^{**} \cong V^{4*} \cong \cdots$ and $V^* \cong V^{3*} \cong V^{5*} \cdots$.
For V Banach but not reflexive, $V \underset{\neq}{\subset} V^{**} \underset{\neq}{\subset} V^{4*} \underset{\neq}{\subset} \cdots$ and $V^* \underset{\neq}{\subset} V^{3*} \underset{\neq}{\subset} V^{5*} \underset{\neq}{\subset} \cdots$.

Examples

$\ell^p \left\{\{x_n\}_{n=1}^{\infty} : ||x||_p = \left(\sum |x_n|^p\right)^{1/p} < \infty\right\}$, $1 \leq p < \infty$.
$\ell^\infty \{\{x_n\}_{n=1}^{\infty} : ||x||_\infty = \sup_n |x_n|\}$
$C_0 \{\{x_n\}_{n=1}^{\infty} \in \ell^\infty : \lim x_n = 0\}$
C_0 is a closed subspace of ℓ^∞.

Example 1

For $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$

$$(\ell^p)^* \cong \ell^q$$

These spaces are reflexive.

Example 2

$(C_0)^* \cong \ell^1$, $(\ell^1)^* \cong \ell^\infty$, $(\ell^\infty)^* \cong ?$.
$(C_0)^{**} \cong \ell^\infty$ and $C_0 \subseteq \ell^\infty$.
These spaces are not reflexive.

Theorem:

Take $1 \leq p < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and

$$\Lambda : \ell^q \to (\ell^p)^*$$
$$y = \{y_n\}_{n=1}^{\infty} \mapsto \phi_y$$

with $\phi_y(\{x_n\}) = \sum_{n=1}^{\infty} y_n x_n$.
Then $\Lambda : \ell^q \to (\ell^p)^*$ is an isometric isomorphism.

Hölder's Inequality

$$\sum |x_n y_n| \leq \left(\sum |x_n|^p\right)^{1/p} \left(\sum |y_n|^q\right)^{1/q}$$

Prooof (Sketch)

If $x \in \ell^p$ and $y \in \ell^q$, then $\phi_y(x)$ is well defined, and $|\phi_y(x)| \leq ||x||_p \cdot ||y||_q$.
We have also that ϕ_y is linear in x_n and bounded, since

$$||\phi_y|| = \sup \frac{|\phi_y(x)|}{||x||_p} \leq ||y||_q$$

It follows that $\phi_y \in (\ell^p)^*$, $\forall\, y \in \ell^q$.
$\Lambda : y \to \phi_y$ is linear in y_n and bounded, since

$$||\phi_y|| \leq ||y||, \ \forall\, y \in \ell^q$$

Now, given $y = \{y_n\}$, put $x_n = \frac{\overline{y}_n}{|y_n|} \cdot |y_n|^{q/p}$. Then $|x_n|^p = |y_n|^q$ and $x_n y_n = |y_n|^{1+q/p} = |y_n|^q$. Therefore

$$\phi_y(x) = \left(\sum x_n y_n\right)^{1/p} \left(\sum x_n y_n\right)^{1/q} = ||x||_p \cdot ||y||_q$$

If $y = 0$, we simply set $\phi_0(x) = 0$. So

$$||\phi_y|| = \sup_{x \neq 0} \frac{|\phi_y(x')|}{||x'||} \geq \frac{|\phi_y(x)|}{||x||} = ||y||$$

and Λ is an isometry.
Note that for $p = \infty$ and $q = 1$, we may define $\Lambda : \ell^1 \to (\ell^\infty)^*$ but it is not surjective.
Instead, we have that $\Lambda : \ell^1 \to (C_0)^*$ as surjective.
For $1 \leq p < \infty$, for $\phi \in (\ell^p)^*$ find $y \in \ell^q$ such that $\phi = \phi_y$. Take

$$e_n = \{0, \ldots, 0, \underbrace{1}_{n}, 0, \ldots\}$$

and put $y_n = \phi(e_n)$. Now, we want to show that $y = \{y_n\}_{n=1}^\infty \in \ell^q$ and that $\phi = \phi_y$.
Define x and $x_n = \frac{\overline{y}_n}{|y_n|} \cdot |y_n|^{q/p}$ where $|x_n y_n| = |y_n|^q$. Then

$$\left(\sum_{n=1}^N |x_n|^p\right)^{1/p} \left(\sum_{n=1}^N |y_n|^q\right)^{1/q} = \sum_{n=1}^N |y_n|^q = \sum_{n=1}^N x_n y_n = \sum_{n=1}^N x_n \phi(e_n) = \phi\left(\sum_{n=1}^N x_n e_n\right) \leq ||\phi|| \cdot ||\sum_{n=1}^N x_n e_n||_p = ||\phi|| \left(\sum_{n=1}^N |x_n|^p\right)^{1/p}$$

Finally, we want to show that $\phi = \phi_y$.
By density, we can restrict to $x = \sum_{n=1}^N x_n e_n$ (except in ℓ^∞. Take

$$\phi(x) = \sum_{n=1}^N \phi(x_n e_n) = \sum_{n=1}^N x_n \phi(e_n) = \sum_{n=1}^N x_n y_n = \phi_y(x)$$

where $x = \{x_1, x_2, \ldots, x_N, 0, 0, \ldots\}$.
By continuity, this caries to the closure and then the whole space so $\phi(x) = \phi_y(x)$, $\forall\, x \in \ell^p$.
Therefore $\phi = \phi_y$.