

Advanced Analysis

September 25, 2025

Suppose we have some function of the form $-\Delta + q \in \mathbb{L}(H)$ satisfying $R_A(\lambda)(A - \lambda I)^{-1}$ bounded on $\text{Im}(\lambda) > 0$ and not surjective for $\text{Im}(\lambda) = 0$.

IMAGE 1

Waves: solutions to $\partial_{tt}u + Au = 0$ on \mathbb{R}^n .

Mathematical Tools

- Spectral theory of unbounded operators
- Complex analysis
- Functional analysis
- Microlocal analysis
- Semiclassical analysis

Classical Resonances in ODEs

IMAGE 2

A harmonic oscillator assuming no friction.

We have an acceleration force, $m\ddot{x}(t) = -kx(t)$ which gives $\ddot{x} + \omega_0^2 x = 0$ with $\omega_0 = \sqrt{\frac{k}{m}}$ and has solution $x(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t)$.

With forcing, i.e. $m\ddot{x}(t) = -kx(t) + A \sin(\omega t)$, we have $\ddot{x} + \omega_0^2 x = A' \sin(\omega t)$.

If $|\omega| \neq |\omega_0|$, then $x(t) \sim \text{trig}\left(\left(\frac{\omega - \omega_0}{2}\right)t\right) \left(\left(\frac{\omega + \omega_0}{2}\right)t\right)$ the low and high frequencies respectively.

IMAGE 3

Beats (non-amplified)

If instead $|\omega| = |\omega_0|$, then $x(t) \propto \text{trig}(\omega t)t$.

IMAGE 4

In general, $\dot{x} + Ax = 0$ for $x \in \mathbb{R}^n$, $x(t) = \exp(-tA) + x(0)$.

In the case where A is skew-adjoint, i.e. $\text{sp}(A) \subseteq i\mathbb{R}$, $(x, Ax) = 0 \forall x \in \mathbb{R}^n$, then

$$\frac{d}{dt}(x, x) = (\dot{x}, x) + (x, \dot{x}) = (-Ax, x) - (x, Ax) = 0$$

Which implies that $\|x(t)\|$ is constant and the dynamics are norm perserving.

To generate resonant solutions, if $(i\omega, v)$ is an eigenpair of A ($\omega \in \mathbb{R}$), consider $\dot{x} + Ax = e^{-i\omega t}v$. As an ansatz, we look for a solution of the form $x(t) = a(t)v$ and the equation becomes $(\dot{a}(t) + i\omega a)v = e^{-i\omega t}v$. Then

$$\begin{aligned} e^{-i\omega t} \frac{d}{dt}(e^{i\omega t} a) &= e^{-i\omega t} \\ \frac{d}{dt}(e^{i\omega t} a) &= 1 \\ a(t) &= te^{-i\omega t}. \end{aligned}$$

Resonances in PDEs

Consider one-dimensional waves on $[0, L]$, $L > 0$.

$$\begin{cases} \partial_{tt}u + \partial_{xx}u = 0 \\ u|_{t=0} = f & x \in [0, L] \\ \partial_t u|_{t=0} = g & x \in [0, L] \\ u(0, t) = u(L, t) = 0 & \forall t \geq 0 \end{cases}$$

We want to think about this as $\partial_{tt}u = Au = 0$ where A is the Dirichlet Laplacian $Au = -\partial_{xx}u$ with Dirichlet boundary conditions. We then want to find the spectral decomposition of A , $Au - \lambda u = 0 = -\partial_x^2 u - \lambda u$.

$$\begin{aligned} \lambda = 0. \quad u(x) = A + Bx &\implies A = B = 0 \\ \lambda = -p^2. \quad u(x) = Ae^{px} + be^{-px} &\implies A = B = 0 \\ \lambda = p^2. \quad u(x) = A\cos(px) + B\sin(px) &\implies 0 = u(0) = A \quad 0 = u(L) = B\sin(pl) \implies p = k\pi, k \in \mathbb{N} \end{aligned}$$

Therefore there are infinitely many eigenpairs $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $\phi_n(x) = \sin\left(\frac{k\pi x}{L}\right)$.

IMAGE 5

The family $\{\phi_n, n \in \mathbb{N}\}$ is dense in $L^2([0, L])$ where the unbounded operator $(-\partial_x^2)$ with Dirichlet boundary conditions is self-adjoint.

Other Prototypes

(of unbounded self-adjoint operators with discrete spectrum)

- Laplace-Beltrami operators on compact manifolds without boundary.

IMAGE 6

- On compact domains with boundary there is the Laplacian with Dirichlet boundary conditions.

The (Quantum) Harmonic Oscillator

$H = -\frac{d^2}{dx^2} + x^2$ on \mathbb{R} , on $L^2(\mathbb{R})$ with $(f, g) = \int_{\mathbb{R}} f(x) \overline{g(x)} dx$.

H acts on the Schwarz space $\mathcal{S}(\mathbb{R}) := \left\{ f \in C^\infty(\mathbb{R}), \forall k, \ell \geq 0, \sup_{x \in \mathbb{R}} \left| x^k \left(\frac{d}{dx} \right)^\ell f(x) \right| < \infty \right\}$.

- The action of $H : \mathcal{S}(\mathbb{R})$ is continuous.
- H is L^2 -symmetric: $\int_{\mathbb{R}} -f'' \bar{g} + x^2 f \bar{g} dx = (Hf, g) = (f, Hg) = \int_{\mathbb{R}} -\bar{g}'' f + x^2 f \bar{g} dx$ (integrating by parts).

We seek eigenvalues $Hu = \lambda u$. If (u, λ) and (v, μ) are eigenpairs, then

$$0 = (Hu, v) - (u, Hv) = (\lambda u, v) - (u, \mu v) = (\lambda - \bar{\mu})(u, v)$$

Where if the difference is nonzero then $(u, v) = 0$.

We can write $H = L^+ L^- + I$ where $L^+ = -\frac{d}{dx} + x$ and $L^- = \frac{d}{dx} + x$ and also $[H, L^+] = 2L^+$ and $[H, L^-] = -2L^-$.

Note that H is a non-negative operators

$$(Hf, f) = \int_{\mathbb{R}} ((f')^2 + x^2 f^2) dx > 0$$

for $f \neq 0$ and $f \in \mathcal{S}(\mathbb{R})$. Thus $\text{sp}(H) \subseteq (0, \infty)$. If $Hv = \lambda v$, then $H(L^+ v) = [H, L^+]v + L^+(Hv) = (\lambda + 2)L^+ v$. Similarly $H(L^- v) = (\lambda - 2)L^- v$.

Now we want to solve $L^- \phi_0 = 0$. $\frac{d}{dx} \phi_0 + x \phi_0 = 0$ tells us that $\phi_0(x) = \frac{1}{\sqrt{\pi}} e^{-x^2/2}$ (L^2 -normalized). Therefore $H\phi_0 = \phi_0$ and the we have an eigenvalues of one. So we may construct $\phi_n = \frac{(L^+)^n \phi_0}{|| (L^+)^n \phi_0 ||}$ which gives an eigenvector of H with eigenvalues $2n + 1$. Note that $|| (L^+)^n \phi_0 || = \sqrt{2^n n!}$.

Fact: $\phi_n = p_n(x) e^{-x^2/2}$ where p_n is the Hermite polynomial of degree n .

$$\delta_{nq} = (\phi_n, \phi_q) = \int_{\mathbb{R}} p_n(x) p_q(x) e^{-x^2} dx$$

Theorem

$\{\phi_n\}_{n \geq 0}$ is dense in $L^2(\mathbb{R})$ (if $\int_{\mathbb{R}} g \phi_n dx = 0$ for all n , then $g = 0$).

Proof (Sketch)

For $g \in L^2$, $\xi \in \mathbb{R}$, $F_g(\xi) = \int_{\mathbb{R}} e^{ix\xi} g(x) \phi_0(x) dx = \widehat{g\phi_0}(\xi)$. We observe that

- F_g is real-analytic in ξ .
- $F_g^{(k)}(0) = \int_{\mathbb{R}} (-ix)^k g(x) \phi_0(x) dx = 0$ by assumption.

So we have a real-analytic function where all derivatives vanish at a point. So $F_g \equiv 0$, $g\phi_0 = 0$, and $g = 0$.

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One of the overarching goals is to obtain large time asymptotics of the solution $v(x, t)$ ($x \in \mathbb{R}$, $t > 0$) to

$$\begin{cases} -\partial_{tt} v - P_V v = F(x, t) & \text{on } \mathbb{R}_x \times (0, \infty)_t \\ v(x, 0) = \partial_t v(x, 0) = 0, & F \in C_C^\infty(\mathbb{R}_x \times (0, \infty)_t) \end{cases}$$

where $P_V = D_x^2 + V(x) = -\left(\frac{\partial}{\partial x}\right)^2 + V(x)$ and $D_x = \frac{1}{i} \frac{\partial}{\partial x}$. The operator D_x is symmetric and self-adjoint on appropriately chosen domains. For $f(x)$ and $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx$, $\widehat{D_x f} = \xi \hat{f}(\xi)$. $V \in L_{\text{comp}}^\infty(\mathbb{R})$ (i.e. compactly supported L^∞) is the potential. If $f, g \in \mathcal{S}(\mathbb{R})$, then $(P_V f, g)_{L^2(\mathbb{R})} = (f, P_V g)_{L^2(\mathbb{R})}$.

IMAGE 1

Another way to look at this assuming v exists, we can consider $u(x, \lambda) := \int_0^\infty e^{it\lambda} v(x, t) dt$ (the Fourier-Laplace transform of v) with $\lambda \in \mathbb{C}$, $\text{Im}(\lambda) > 0$. Write $\lambda = \xi + ic$, $c > 0$, such that $u(x, \xi + ic) = \int_0^\infty e^{it\xi} e^{-ct} v(x, t) dt = \mathcal{F}_{t \mapsto \xi}(t \mapsto$

$e^{-ct} v(x, t))(x, -\xi)$. Then $u(x, \lambda)$ solves

$$\begin{aligned} \int_0^\infty e^{it\lambda} (-\partial_{tt} v - P_V v) dt &= \int_0^\infty e^{it\lambda} F(x, t) dt = \hat{F}(x, \lambda) \\ (\lambda^2 - P_V) \underbrace{\int_0^\infty e^{it\lambda} v(x, t) dt}_{u(x, \lambda)} &= \hat{F}(x, \lambda) \end{aligned}$$

which is an entire function in λ .

To Do:

- Study solvability of $(\lambda^2 - P_V)u = \hat{F}(x, \lambda)$.
- Return to v .

For frozen c , we can get $v(x, t)$ back by Fourier inversion.

$$\begin{aligned} e^{-ct} v(x, t) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it\xi} u(x, \xi + ic) d\xi \\ v(x, t) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it(\xi + ic)} u(x, \xi + ic) d\xi \\ v(x, t) &= \frac{1}{2\pi} \int_{\text{Im}(\lambda)=c} e^{-it\lambda} u(x, \lambda) d\lambda \end{aligned}$$

IMAGE 2

where the spectral problem is invertible.

1D Waves in the Time Domain

Suppose $R > 0$ is such that $\text{supp } V \subset [-R, R]$ and $\text{supp } F \subset [-R, R] \times (0, \infty)$. If $|x| > R$, the PDE looks like $\partial_{tt} v - \partial_{xx} v = 0 = (\partial_t + \partial_x)(\partial_t - \partial_x)v$. Setting $\xi = x + t$ and $\mu = x - t$, then it follows that

$$\partial_\xi \partial_\mu v = 0 \implies v = F(\xi) + G(\mu) = F(x + t) + G(x - t)$$

IMAGE 3

On $x > R$, we can expect $v(x, t) = F_+(x + t) + G_+(x - t)$; on $x < -R$, we expect $v(x, t) = F_-(x + t) + G_-(x - t)$. The terms G_+ and F_- are outgoing whereas the terms F_+ and G_- are incoming and, given that we assumed a source, we expect to be zero.

What does incoming/outgoing look like on the spectral side? $(\lambda^2 - P_V)u = \hat{F}(x, \lambda)$ supported in $|x| \leq R$. For $|x| > R$, $(\lambda^2 + \partial_x^2)u = 0$ leads to $u = Ae^{ix\lambda} + Be^{-ix\lambda}$. For $x > R$, $u(x) = a_+ e^{i\lambda|x|} + b_+ e^{-i\lambda|x|}$ for $x < -R$, $u(x) = a_- e^{i\lambda|x|} + b_- e^{-i\lambda|x|}$. u is outgoing if and only if $b_\pm = 0$ and incoming if and only if $a_\pm = 0$.

P_V is an unbounded, symmetric operator on a Hilbert space. For $z \in \mathbb{C}$, $\text{sp}(P_V)$ is the set on the complement of which $(P_V - Z)$ is boundedly invertible. That is, $\forall f, \exists ! u$ such that $(P_V - z)u = f$ and $\|u\| \lesssim \|f\|$.

Waves in the Time Domain [Evans, §2.4]

Goal: if v solves

$$\begin{aligned}\partial_{tt}v - \partial_{xx}v &= f(x, t) \quad x \in \mathbb{R}, \quad t > 0, \quad f \in C_C^\infty(\mathbb{R} \times (0, \infty)) \\ v(x, 0) &= \partial_t v(x, 0) = 0 \quad x \in \mathbb{R}\end{aligned}$$

then $v(x, t) = \frac{1}{2} \int_0^t \int_{x-s}^{x+s} f(y, t-s) dy ds$. We look at

$$\begin{cases} \partial_{tt}v - \partial_{xx}v = 0 \rightsquigarrow v(x, t) = F(x+t) + G(x-t) \\ v(x, 0) = g(x), \quad \partial_t v(x, 0) = h(x) \end{cases}$$

Initial conditions gives us

$$\begin{cases} F(x) + G(x) = g(x) \\ F'(x) - G'(x) = h(x) \end{cases} \quad \begin{cases} G'(x) = \frac{1}{2}(g'(x) - h(x)) \\ F'(x) = \frac{1}{2}(g'(x) + h(x)) \end{cases}$$

So

$$\begin{aligned}F(x) &= \frac{1}{2} \left(g(x) + \int_0^x h(s) ds \right) + C_1 \\ G(x) &= \frac{1}{2} \left(g(x) - \int_0^x h(s) ds \right) + C_2 \\ v(x, t) &= \frac{1}{2} (g(x+t) + g(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} h(s) ds + C\end{aligned}$$

IMAGE 4

This has a finite speed of propagation in the sense that if we suppose $\text{supp}(g, h) \subset [-R, R]$ then $v(x, t) = 0$ whenever $x > R+t$ or $x < -R-t$.

Now we want to go from the homogeneous problem to the inhomogeneous problem. The idea is to think about $v(x, t) = \int_0^t v(x, t; s) ds$ where $v(x, t; s)$ solves the homogeneous problem

$$\begin{cases} \partial_{tt}v(\cdot, \cdot; s) - \partial_{xx}v(\cdot, \cdot; s) = 0 \\ v(\cdot, s; s) = 0, \quad \partial_t v(\cdot, s; s) = f(x, s) \end{cases}$$

Then

$$\partial_{tt}v - \partial_{xx}v = 0 \iff \partial_t \begin{pmatrix} v \\ \partial_t v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \partial_{xx} & 0 \end{pmatrix} \begin{pmatrix} v \\ \partial_t v \end{pmatrix} \quad \text{and} \quad \begin{bmatrix} v \\ \partial_t v \end{bmatrix}_{t=s} = \begin{bmatrix} * \\ * \end{bmatrix}$$

So $v(x, t; s) = \frac{1}{2} \int_{x-(t-s)}^{x+(t-s)} f(y, s) dy$ and $v(x, t) = \frac{1}{2} \int_0^t \int_{x-s}^{x+s} f(y, t-s) dy ds$ follows.

Going back to the original PDE, $(-\partial_{tt} - P_V)v = F$ is equivalent to $(\partial_{tt} - \partial_{xx})v = -(Vv + F)$ which leads to the conclusion that $v(x, t) = -\frac{1}{2} \int_0^t \int_{x-s}^{x+s} (Vv + F)(y, t-s) dy ds$. For $|x| > R$, v is outgoing.

IMAGE 5

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Take some complex vector space and consider the Hilbert space $(\mathcal{H}, (\cdot, \cdot))$ with (\cdot, \cdot) satisfying

$$\begin{cases} (\lambda f, g) = \lambda(f, g) \\ (f, \lambda g) = \overline{\lambda}(f, g) \\ (f, g) = \overline{(g, f)} \\ f \mapsto (f, g) =: \|f\|^2 \text{ a norm} \\ (\mathcal{H}, \|\cdot\|) \text{ complete with respect to the norm} \end{cases}$$

• Examples

- $(\mathbb{C}^n, (a, b) = \sum_{j=1}^n a_j \overline{b_j})$, $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$
- $L^2(x, \mu)$ (e.g. $[0, 1]$ and the Lebesgue measure), $(f, g) = \int_X f \overline{g} d\mu$.

Bounded Operators: $T : \mathcal{H} \rightarrow \mathcal{H}$ bounded if and only if $\overbrace{\sup_{\|x\|=1} \|Tx\|}^{\|T\|} < \infty$ satisfying

$$\begin{cases} \mathcal{B}(\mathcal{H}) \text{ the space of bounded operators on } \mathcal{H} \text{ (a complex vector space)} \\ \|\cdot\| \text{ is a norm on } \mathcal{B}(\mathcal{H}), \text{ making it complete} \\ \text{There is a multiplication, } \mathcal{B}(\mathcal{H}) \ni A, B \mapsto AB \text{ and } \|AB\| \leq \|A\| \|B\| \end{cases}$$

Adjoint: if $A \in \mathcal{B}(\mathcal{H})$, $\exists! A^* \in \mathcal{B}(\mathcal{H})$ such that $\forall f, g \in \mathcal{H}$, $(Af, g) = (f, A^*g)$ where A is symmetric/self-adjoint if $A = A^*$. These notions are different in the world of unbounded operators.

• Example

- $\mathcal{H} = \mathbb{C}^n$: $T \in M_n(\mathbb{C})$ symmetric if and only if T is Hermitian. $t_{ij} = \overline{t_{ji}}$.
- $\mathcal{H} = L^2([0, 1])$, $Tf(t) = tf(t)$. $(Tf, g) = \int_0^1 tf(t) \overline{g(t)} dt = \int_0^1 f(t) \overline{tg(t)} dt = (f, Tg)$.
- $\mathcal{H} = L^2(\mathbb{R})$ with the Fourier transform. $\|f(x)\|^2 = c \|\hat{f}(\xi)\|^2$ (Parseval's Equality).

Finite Dimensional Spectral Theorem

If $A \in M_n(\mathbb{C})$ is Hermitian, there exists an orthonormal basis (ϕ_1, \dots, ϕ_n) of \mathbb{C}^n and real eigenvalues $\lambda_1, \dots, \lambda_n$ such that $A\phi_j = \lambda_j\phi_j$.

Important observation: if A is Hermitian, then λ_j is real for each j , and $\overline{(A\phi_j, \phi_j)} = (\phi_j, A\phi_j) = (A\phi_j, \phi_j) = \lambda_j \|\phi_j\|^2$.

So $\lambda_j = \frac{(A\phi_j, \phi_j)}{\|\phi_j\|^2}$ is real. If $\lambda_j \neq \lambda_k$, then $(\phi_j, \phi_k) = 0$ since $(A\phi_j, \phi_k) - (\phi_j, A\phi_k) = (\lambda_j - \overline{\lambda_k})(\phi_j, \phi_k)$.

Notation: Let $u, v \in \mathbb{C}^n$, denote $u \otimes \overline{v}$ the operator $(u \otimes \overline{v})w = (w, v)u$.

With A as in the theorem, we can write $A = \sum_{j=1}^n \lambda_j \phi_j \otimes \overline{\phi_j}$ ($I = \sum_{j=1}^n \phi_j \otimes \overline{\phi_j}$). A second way of writing this is

IMAGE 1

Where $U^* = U^{-1}$ and $A = U\Lambda U^*$. This allows us to construct a functional calculus for A where

$$\begin{cases} A^2 = U\Lambda U^* U\Lambda U^* = U\Lambda^2 U^* \\ A^n = U\Lambda^n U^* \\ p(A) = U \cdot p(\Lambda) \cdot U^*, \text{ } p \text{ a polynomial} \end{cases}$$

Defining $f(A) := U \cdot f(\Lambda) \cdot U^*$, we obtain a Banach algebra homomorphism. Then $f \in C([-||A||, ||A||])$ is also a Banach algebra with sup norm and pointwise multiplication.

IMAGE 2

Then we can map $C([-||A||, ||A||]) \ni f \mapsto f(A) \in \mathcal{B}(\mathcal{H})$. This is useful for solving ODEs.

• Prototypes

- Heat equation: $\partial_t u + Au = 0$, $u|_{t=0} = u_0$, $u(t) = e^{-tA}u_0$.
- Schrödinger equation: $i\partial_t u + Au = 0$, $u|_{t=0} = u_0$, $u(t) = e^{-itA}u_0$.
- Wave equation: $\partial_{tt} u + Au = 0$, $u|_{t=0} = u_0$, $\partial_t u|_{t=0} = u_1$.

Write $u(t) := \sum_{j=1}^n u_j(t)\phi_j$ with the PDE $\sum_{j=1}^n (u_j'' + \lambda_j u_j)\phi_j = 0$. Then $u_j'' + \lambda_j u_j = 0$, $u_j(0) = u_{j,0}$, and $u_j'(0) = u_{j,1}$. Suppose $\lambda_j > 0$ for all j . Then $u_j(t) = u_{j,0} \cos(\sqrt{\lambda_j}t) + \frac{u_{j,1}}{\sqrt{\lambda_j}} \sin(\sqrt{\lambda_j}t)$. So

$$u(t) = \sum_{j=1}^n \cos(\sqrt{\lambda_j}t) u_{j,0} \phi_j + \frac{1}{\sqrt{\lambda_j}} \sin(\sqrt{\lambda_j}t) u_{j,1} \phi_j$$

Therefore $u = \cos(t\sqrt{A})u_0 + A^{-1/2} \sin(t\sqrt{A})u_1$.

Spectrum of a Bounded Operator

Take $T \in \mathcal{B}(\mathcal{H})$. We say that T is invertible (within $\mathcal{B}(\mathcal{H})$) if and only if $\exists S \in \mathcal{B}(\mathcal{H})$ such that $TS = ST = I$.

Counterexample: take $\mathcal{H} = \ell^2(\mathbb{N}_0) = \{u = (u_n)_{n \geq 0} : \sum |u_n|^2 < \infty\}$ and $Au = \left(\frac{1}{n}u_n\right)_{n \geq 0}$. Then the proxy for $A^{-1}u = (nu_n)_{n \geq 0}$ is not bounded.

Given $T \in \mathcal{B}(\mathcal{H})$, the resolvent set of T is $\rho(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \text{ is invertible}\}$. Invertibility is equivalent to $\forall y \in \mathcal{H}$, $\exists! x$ such that $Tx - \lambda x = y$ with an estimate $||x|| \leq ||y||$.

For $\lambda \in \rho(T)$, denote $R(\lambda)$ or $R_T(\lambda) = (T - \lambda I)^{-1} \in \mathcal{B}(\mathcal{H})$ the resolvent of T . Properties of the resolvent set:

1. $\rho(T) \neq \emptyset$ (in fact, if $||\lambda| > ||T||$ then $\lambda \in \rho(T)$).
2. $\rho(T)$ is open.
3. the map $\rho(T) \ni \lambda \mapsto R_T(\lambda) \in \mathcal{B}(\mathcal{H})$ is holomorphic in the sense that $\forall \lambda_0 \in \rho(T)$, $\exists R_T'(\lambda_0) \in \mathcal{B}(\mathcal{H})$ such that $\lim_{\lambda \rightarrow \lambda_0} \left\| \frac{R_T(\lambda) - R_T(\lambda_0)}{\lambda - \lambda_0} - R_T'(\lambda_0) \right\| = 0$.

For a., if $||\lambda| > ||T||$, $Tx - \lambda x = y \iff \left(I - \frac{T}{\lambda}\right)x = -\frac{y}{\lambda} \iff x = \sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}}y$. Then $R_T(\lambda) = \frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{T}{\lambda}\right)^k$ and

$$||R_T(\lambda)|| \leq \frac{1}{||\lambda||} \frac{1}{1 - ||T/\lambda||} \leq \frac{1}{||\lambda| - ||T||}$$

For b., pick $\lambda_0 \in \rho(T)$ and find $r > 0$ such that $||\lambda - \lambda_0| < r \implies \lambda \in \rho(T)$. Then $Tx - \lambda x = y \iff (T - \lambda_0)x - (\lambda - \lambda_0)x = y \iff x - (\lambda - \lambda_0)R_T(\lambda_0)x = R_T(\lambda_0)y$ where if $||(\lambda - \lambda_0)R_T(\lambda_0)|| < 1$ it is boundedly solvable by Neumann series.

For c.,

$$\begin{aligned}
R_T(\lambda) - R_T(\lambda_0) &= (T - \lambda I)^{-1} - (T - \lambda_0 I)^{-1} \\
(T - \lambda I)(R_T(\lambda) - R_T(\lambda_0)) &= I - (T - \lambda_0 I + (\lambda_0 - \lambda)I)(T - \lambda_0 I)^{-1} \\
(T - \lambda I)(R_T(\lambda) - R_T(\lambda_0)) &= I - I + (\lambda - \lambda_0)R_T(\lambda_0) \\
R_T(\lambda) - R_T(\lambda_0) &= (\lambda - \lambda_0)R_T(\lambda)R_T(\lambda_0)
\end{aligned}$$

So $\frac{R_T(\lambda) - R_T(\lambda_0)}{\lambda - \lambda_0} - R_T(\lambda_0)^2 = o(\lambda - \lambda_0)$.

Then we define the spectrum $\sigma(T) := \mathbb{C} \setminus \rho(T)$ which is closed since $\rho(T)$ is open.

Lemma

If $T \in \mathcal{B}(\mathcal{H})$ is self-adjoint, then $\sigma(T) \subseteq [-\|T\|, \|T\|]$.

• Proof

First we know $\sigma(T) \subseteq \{|\lambda| \leq \|T\|\}$. We want to show that it is real, and that if $\lambda = a + ib$ and $b \neq 0$ then $T - (a + ib)I$ is invertible.

$T - (a + bi)I$ is injective.

$$\begin{aligned}
\|(T - (a + ib))x\|^2 &= (Tx - (a + ib)x, Tx - (a + ib)x) \\
&= \|Tx\|^2 + (a^2 + b^2)\|x\|^2 - (Tx, (a + ib)x) - ((a + ib)x, Tx) \\
&= \|Tx\|^2 + (a^2 + b^2)\|x\|^2 - (a - ib)(Tx, x) - (a + ib)(x, Tx) \\
&= \|Tx\|^2 + a^2\|x\|^2 - 2a(x, Tx) + b^2\|x\|^2 \geq b^2\|x\|^2
\end{aligned}$$

since $\|Tx\|^2 + a^2\|x\|^2 - 2a(x, Tx) \geq 0$ by Cauchy-Schwarz. Therefore $T - (a + ib)$ is injective and, by the open mapping theorem, $(T - (a + ib))^* = T - (a - ib)$ is surjective. Similarly for $T - (a - ib)$, and the norm estimate is $\|(T - (a + ib))^{-1}\| \leq \frac{1}{b}$. Note that $\frac{1}{b} = \frac{1}{\text{dist}(a + ib, \mathbb{R})}$.

Note that the spectrum of T may no longer be made of eigenvalues in the non-finite case. There may exist λ such that $T - \lambda I$ is not injective, $\exists v \neq 0$ $Tv = \lambda v$. Recall the example $Tf(t) = tf(t)$ with $f \in L^2((\cdot, \cdot), dt)$. T is self-adjoint, $\|T\| \leq 1$, and $(Tf, f) = \int_0^1 t|f(t)|^2 dt \geq 0$. So $\sigma(T) \subseteq [0, 1]$. For $\lambda \in [0, 1]$ is $T - \lambda I$ injective? $Tf = \lambda f \iff tf(t) = \lambda f(t) \iff (t - \lambda)f(t) = 0$ which implies $f \equiv 0$ in $L^2([0, 1])$. Is $T - \lambda I$ surjective? $(t - \lambda)f(t) = g(t) \iff f(t) = \frac{g(t)}{t - \lambda}$, so $g(t) \equiv 1 \in L^2([0, 1])$ which implies $f(t) = \frac{1}{t - \lambda}$ is not $L^2([0, 1])$ and $\sigma(T) = [0, 1]$.