Random Matrix Theory

April 1, 2025

Preliminaries

Let ξ_{ij}, η_{ij} be normal random variables (i.e. Gaussian, mean 0, variance 1).
e.g. $\mathbb{P}(\xi_{11} < s) = \int_{-\infty}^{s} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$.
$\int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$ is the variance.
$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is the Probability Density Function (PDF).
$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$ is the probability measure on our probability space (i.e. totally finite measure space).
We build matrices

$$\begin{bmatrix} \xi_{11} & \frac{\xi_{12}+i\eta_{12}}{\sqrt{2}} & \frac{\xi_{13}+i\eta_{13}}{\sqrt{2}} & \cdots \\ \frac{\xi_{21}+i\eta_{21}}{\sqrt{2}} & \xi_{22} & \frac{\xi_{22}+i\eta_{22}}{\sqrt{2}} & \\ \frac{\xi_{31}+i\eta_{31}}{\sqrt{2}} & \frac{\xi_{32}+i\eta_{32}}{\sqrt{2}} & \xi_{33} & \\ \vdots & & & \ddots \end{bmatrix}$$

Computing Random Matrices in Matlab

Gassuain, real valued 1x1 matrix.

```
randn
```

Gaussian, real valued 2x2 matrix.

```
randn(2)
```

Gaussian, complex valued 2x2 matrix.

```
randn(2)+sqrt(-1)*randn(2)
```

Gaussian, complex valued, self-adjoint 2x2 matrix.
Note that appending ' to a matrix takes the conjugate transpose, and matlab reserves i for the imaginary unit.

```
m = randn(2)+i*randn(2)
(m+m')/2
```

Producing eigenvalues.

```
m = randn(2)+i*randn(2);
l=(m+m')/2;
eig(l)
```

Running tests to see how many hits we get within the interval $[0, 2]$.

```
edges=[0,2];
H=zeros(1,length(edges)-1);
trials=10;
for j=1:trials
m = randn(2)+i*randn(2);
l=(m+m')/2;
ev=eig(l);
H=H+histcount(ev,edges)
end
```

Homework

Is the PDF of $\frac{a+b}{2}$ the same as $\frac{\xi_{12}}{\sqrt{2}}$ for normal RVs a, b, ξ_{12}?
i.e. $\mathbb{P}\left(\frac{a+b}{2} < s\right) \stackrel{?}{=} \mathbb{P}\left(\frac{\xi_{12}}{\sqrt{2}} < s\right)$

2x2 Random Matrix

Our matrix L corresponds to eigvenvalues λ_1, λ_2 which are random variables determined by $\{\xi_{ij}, \eta_{ij}\}$.
Then the number of evaulations in the interval B is given by $\sum_{j=1}^{2} \chi_B(\lambda_j)$. We may take the average by

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{j=1}^{2} \chi_B(\lambda_j) \frac{1}{\sqrt{2\pi}} e^{-\xi_{11}^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\xi_{22}^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\xi_{12}^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\eta_{12}^2} \, d\xi_{11} d\xi_{22} d\xi_{12} d\eta_{12}.$$

Expected Evaluations

We have that the expecation of the number of evaluations in the interval (a, b) is given by $\int_a^b G(s)\, ds$ where

$$G(s) = e^{-\frac{s^2}{2}} \sum_{\ell=0}^{2} P_\ell(s)^2$$

and $P_\ell(s)$ is the Hermite polynomial of degree d.

April 3, 2025

Differntiability

```
delta = 0.05;
edges=-6:delta:6;
dimensions = 3;
trials = 1000000;

H=zeros(dimensions,trials);

for j=1:trials
m=randn(dimensions)+1i*randn(dimensions);
L=(m+m')/2;
ev=eig(L);
H(:,j) = ev;
end

G = histcounts(H,edges);
plot(edges(1:end-1),G/(trials*delta),'*')
```

IMAGE 1

Observe that each * in the graph corresponds to the average number of eigenvalues in the interaval (a, b). Therefore, they correspond to $\int_a^b C(\lambda)\, d\lambda$. We may consider the limit of the expectation of hits in each interval

$$\lim_{\Delta\to 0} \frac{\mathbb{E}(\#(a, a+\Delta))}{\Delta}.$$

```
delta = 0.01;
edges=-6:delta:6;
dimensions = 3;
trials = 1000000;

H=zeros(dimensions,trials);

for j=1:trials
m=randn(dimensions)+1i*randn(dimensions);
L=(m+m')/2;
ev=eig(L);
H(:,j) = ev;
end

G = histcounts(H,edges);
plot(edges(1:end-1),G/(trials*delta),'*')
```

As dimension grows large, we observe that the plot tends to a semi-circle with endpoints about $\pm 2\sqrt{\text{dimension}}$. We therefore want a rescaling by $\sqrt{N}$ where $\dim = N$. Then if $G(\alpha) = \frac{d}{d\alpha}\mathbb{E}(\#$ of evals in $(a, \alpha))$, we want

$$\int_{-\infty}^{\infty} G(\alpha)\, d\alpha = N.$$

Guess: $G(\alpha) \approx cN^{1/2} \cdot \sqrt{A^2 - \alpha^2/N} \cdot \chi_{(-A\sqrt{N}, A\sqrt{N})}(\alpha)$. We compute

$$\int_{-A\sqrt{N}}^{A\sqrt{N}} cN^{1/2}\sqrt{A^2 - \alpha^2/N}\, d\alpha \overset{\alpha=\sqrt{N}t}{=} cN \int_{-A}^{A} \sqrt{A^2 - t^2}\, dt = \frac{c\pi N A^2}{2}.$$

Choosing $A = 2$ and c such that $\frac{\pi A^2 c}{2} = 1$, we get

$$\int_{-\infty}^{\infty} G(\alpha)\, d\alpha \approx \frac{N^{1/2}}{2\pi} \int_{-\infty}^{\infty} \sqrt{4 - \alpha^2/N}\, d\alpha = N.$$

Number of Eigenvalues in an Interval

Let B be a subset of $\mathbb{R}$ (typically an interval). Write $n(B) = \#\{$evaluations in $B\}$, a random variable.
Recall that variance is given by the expectation of the square minus the square of the expectation. That is

$$\operatorname{var}(n(B)) = \mathbb{E}(n(B)^2) - (\mathbb{E}(n(B))^2.$$

Our ultimate goal is to understand PDF and $\mathbb{P}(n(B)) = \ell)$ as (the dimension) $N \to \infty$.

Smallest Scale of Interest

Suppose $B = (0, S)$ and N is large (i.e. $N \to \infty$). How large should we choose s such that $\mathbb{E}(n(B)) = 1$? We compute

$$\int_0^S cN^{1/2}\sqrt{4-\alpha^2/N}\,d\alpha \overset{\alpha=\sqrt{N}t}{=} \int_0^{\frac{S}{\sqrt{N}}} cN\sqrt{4-t^2}\,dt \approx cN\cdot 2\frac{S}{\sqrt{N}} = 2cS\sqrt{N}.$$

Sets of size $N^{-1/2}$, the smallest interesting scale, are called the "microscopic scaling regime".

Homework: Largest Scale of Interest

How large should B be to see a fraction of the eigenvalues (on average)? That is, how should we scale a and b such that $\mathbb{E}(n((a,b))) = r\cdot N$ for $0 < r < 1$?

Level Repulsion

```
m=randn(2)+sqrt(-1)*randn(2);
L=(m+m')/2;
ev=eig(L);
subplot(2,1,2),plot(real(ev),imag(ev))
xlim([edges(1),edges(end)])
```

April 8, 2025

Macroscopic Scaling Regime for Random Matrices

Suppose $a = \alpha\sqrt{N}$ and $b = \beta\sqrt{N}$ such that $\alpha < \beta$, $-2 < \alpha$ and $\beta < 2$. Then

$$\lim_{n\to\infty} \frac{\mathbb{E}(\#\text{ of evaluations in }(\alpha\sqrt{N},\beta\sqrt{N}))}{N} = \kappa > 0.$$

Recall that we defined $G(b) = \frac{d}{db}\mathbb{E}(\#\text{ of evaluations in }(a,b))$ and

$$G(b) \approx cN^{1/2}\sqrt{A^2 - x^2/N}\chi_{[-A\sqrt{N},A\sqrt{n}]}(x).$$

We want that $\int_a^b G(x)\,dx = \kappa N$.

Spacings

Suppose we have eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_N = \lambda_{\max}$. We can take the spacing $s_j = \lambda_{j+1} - \lambda_j$.

```
m=randn(2)+sqrt(-1)*randn(2);
L=(m+m')/2;
ev=sort(eig(L));
spacing=diff(ev)
```

0.4839

Summary So Far

Given ξ_{ij} and η_{ij} iid RVs with distribution $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, we have explored

- The behavior of average $n_N(B)$.
- Microscopic, macroscopic (and mesoscopic) scaling.
- That $\lambda_{\max} \sim 2\sqrt{N}$ Tracy-Widom distribution.
- Eigenvalue repulsion.

Induced Distribution

Let M be our matrix built using random variables. Then $M = F\Lambda F^T$ where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \\ \vdots & & \ddots \end{pmatrix}, \quad F = \begin{pmatrix} | & | & & | \\ f_{\lambda_1} & f_{\lambda_2} & \cdots & f_{\lambda_N} \\ | & | & & | \end{pmatrix},$$

and $Mf_{\lambda_j} = \lambda_j f_{\lambda_j}$. What we are interested in is the induced joint PDF on $\{\lambda_1, \dots, \lambda_N\}$. We may write explicitly

$$\frac{1}{Z^n} e^{-\frac{1}{2}\sum_{j=1}^N \lambda_j^2} \prod_{1 \le j < k \le N} (\lambda_k - \lambda_j)^2.$$

Example

Let $N = 2$ and, suppressing the constant term, write

$$\rho = e^{-\frac{1}{2}(x^2+y^2)}(x-y)^2.$$

Taking partial derivatives, we have that

$$\rho_x = e^{-\frac{1}{2}(x^2+y^2)}(x-y)^2(-x + \frac{2}{x-y})$$
$$\rho_y = e^{-\frac{1}{2}(x^2+y^2)}(x-y)^2(-x + \frac{2}{y-x})$$

which implies maxima at $x = \pm 1$ and $y = -x$.

Example

If $N = 3$,

$$\rho = e^{-\frac{1}{2}(x^2+y^2+z^2)}(x-y)^2(x-z)^2(y-z)^2.$$

We may visualize the maxima here by level surfaces (homework).

April 15, 2025

Recall: Spectral Theorem

Let $M = F\Lambda F^\dagger$ where $F^\dagger F = I = FF^\dagger$

$$\Lambda = \begin{pmatrix} \lambda_N & 0 & \cdots & \\ 0 & \lambda_{N-1} & & \\ \vdots & & \ddots & \\ & & & \lambda_1 \end{pmatrix}, \quad F = \begin{pmatrix} | & | & & | \\ f_{\lambda_1} & f_{\lambda_2} & \cdots & f_{\lambda_N} \\ | & | & & | \end{pmatrix},$$

for $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$.

Deriving the Joint PDF

Let $n = 2$. If

$$F = \begin{pmatrix} | & | \\ V & W \\ | & | \end{pmatrix},$$

then the expectation of eigenvalues may be computed by

$$\begin{aligned} \mathbb{E}(\mathcal{G}(M)) &= \frac{1}{Z_2^4} \int \cdots \int \mathcal{G}(M(\xi_{11}, \xi_{12}, \xi_{22}, \eta_{12}))x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\xi_{11}^2 + \xi_{12}^2 + \xi_{22}^2 + \eta_{12}^2)}\, d\eta_{12} d\xi_{22} d\xi_{12} d\xi_{11} \\ &= \int \mathcal{G}(M(\lambda_1, \lambda_2, V_1, \phi))x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\xi_{11}^2 + \xi_{12}^2 + \xi_{22}^2 + \eta_{12}^2)}\, d\eta_{12} d\xi_{22} d\xi_{12} d\xi_{11}. \end{aligned}$$

So we need the Jacobian, and therefore a reparameterization using spectral theorem. We want a collection of independent variables which will produce all 2×2 Hermitian matrices. Consider $Mv = \lambda_2 v$ and $||v||^2 = |v_1|^2 + |v_2|^2 = 1$, then multiply by $e^{i\eta}$ such that $v_1 \in \mathbb{R}_+$. Then $v_2 = \sqrt{1 - v_1^2} e^{i\theta}$. That is, $0 \leq v_1 \leq 1$ and $v_2 = \sqrt{1 - v_1^2}(\cos\theta + i \sin\theta)$.
We want that $|w_1|^2 + |w_2|^2 = 1$ and know that $w \perp v$, so $v_1 w_1 + \overline{v}_2 w_2 = 0$. As before, we can choose w such that $w_2 \in \mathbb{R}_+$. This implies that w_1 and $\overline{v}_2$ have the same argument, and $w_1 = -|w_1| e^{-i\theta}$. Therefore $e^{-i\theta}(-v_1|w_1| + |v_2| w_2) = 0$, and $v_1|w_1| - |v_2| w_2 = 0$. It follows that

$$v_1^2(1 - w_2^2) = w_2^2(1 - v_1^2) \quad \iff \quad v_1 = w_2.$$

Therefore, the entire system may be parameterized by v_1 and θ. We write

$$F = \begin{pmatrix} v_1 & -\sqrt{1 - v_1^2} e^{-i\theta} \\ \sqrt{1 - v_1^2} e^{i\theta} & v_1 \end{pmatrix}$$

and

$$M = F\Lambda F^\dagger = \begin{pmatrix} v_1 & -\sqrt{1 - v_1^2} e^{-i\theta} \\ \sqrt{1 - v_1^2} e^{i\theta} & v_1 \end{pmatrix} \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix} \begin{pmatrix} v_1 & \sqrt{1 - v_1^2} e^{-i\theta} \\ -\sqrt{1 - v_1^2} e^{i\theta} & v_1 \end{pmatrix}.$$

Therefore

$$M = \begin{pmatrix} \lambda_2 v_1^2 + \lambda_1(1 - v_1^2) & v_1\sqrt{1 - v_1^2} e^{-i\theta}(\lambda_2 - \lambda_1) \\ v_1\sqrt{1 - v_1^2} e^{-i\theta}(\lambda_2 - \lambda_1) & \lambda_2(1 - v_1^2) + \lambda_1 v_1^2 \end{pmatrix}.$$

Recall, we want $\mathcal{G}(M(\xi)) \rightsquigarrow \mathcal{G}(M(\lambda_2, \lambda_1, v_1, \theta))$ and the Jacobian of $M = M(\lambda_2, \lambda_1, v_1, \theta)$. After computation, write

$$|\det J = (\lambda_2 - \lambda_1)^2 \det J' = (\lambda_2 - \lambda_1)^2 Q(v_1, \theta).$$

We integrate

$$\int \cdots \int \mathcal{G}(M(\xi, \eta_{12})) e^{-\frac{1}{2}(\xi_{11}^2 + \xi_{12}^2 + \xi_{22}^2 + \eta_{12}^2)} \frac{1}{(2\pi)^4} \, d\xi_{11} d\xi_{12} d\xi_{22} d\eta_{12}$$

which we may think of uas a function of λ_1 and λ_2 alone. So

$$\frac{1}{(2\pi)^2} \int \cdots \int \mathcal{G}(\lambda_1, \lambda_2) e^{-\frac{1}{2}[M_{11}^2 + M_{22}^2 + 2 \cdot \mathrm{Re}(M_{12})^2 + 2 \cdot \mathrm{Im}(M_{12})^2]} \, d\xi_{11} d\xi_{12} d\xi_{22} d\eta_{12}$$

where we observe that $M_{11}^2 + M_{22}^2 + 2 \cdot \mathrm{Re}(M_{12})^2 + 2 \cdot \mathrm{Im}(M_{12})^2 = \mathrm{Tr}(M^2)$. It follows that we have

$$\begin{aligned} \frac{1}{(2\pi)^2} \int \cdots \int \mathcal{G}(\lambda_1, \lambda_2) e^{-\frac{1}{2}(\lambda_1^2 + \lambda_2^2} \, d\xi_{11} d\xi_{12} d\xi_{22} d\eta_{12} &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^1 \int \int_{-\infty < \lambda_1 \le \lambda_2 < \infty} \mathcal{G}(\lambda_1, \lambda_2) e^{-\frac{1}{2}(\lambda_1^2 + \lambda_2^2} (\lambda_2 - \lambda_1)^2 Q(v, \theta) \, d\xi_1 \\ &= \int \int_{-\infty < \lambda_1 \le \lambda_2 < \infty} \mathcal{G}(\lambda_1, \lambda_2) e^{-\frac{1}{2}(\lambda_1^2 + \lambda_2^2)} (\lambda_2 - \lambda_1)^2 \int_0^{2\pi} \int_0^1 \frac{Q(v, \theta)}{(2\pi)^2} \, dv_1 d\theta d \\ &= c \int \int \mathcal{G}(\lambda_1, \lambda_2) e^{-\frac{1}{2}(\lambda_1^2 + \lambda_2^2)} (\lambda_2 - \lambda_1)^2 \, d\lambda_1 d\lambda_2 \end{aligned}$$