Advanced Analysis

September 25, 2025

Suppose we have some function of the form $-\Delta + q \in \mathbb{L}(H)$ satisfying $R_A(\lambda)(A-\lambda I)^{-1}$ bounded on $\mathrm{Im}(\lambda) > 0$ and not surjective for $\mathrm{Im}(\lambda) = 0$.

IMAGE 1

Waves: solutions to $\partial_{tt} u + Au = 0$ on $\mathbb{R}^n$.
Mathematical Tools

- Spectral theory of unbounded operators
- Complex analysis
- Functional analysis
- Microlocal analysis
- Semiclassical analysis

Classical Resonances in ODEs

IMAGE 2

A harmonic oscillator assuming no friction.
We have an acceleration force, $m\ddot{x}(t) = -kx(t)$ which gives $\ddot{x} + \omega_0^2 x = 0$ with $w_0 = \sqrt{\frac{k}{m}}$ and has solution $x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$.
With forcing, i.e. $m\ddot{x}(t) = -kx(t) + A\sin(\omega t)$, we have $\ddot{x} + \omega_0^2 A' \sin(\omega t)$.
If $|\omega| \neq |\omega_0|$, then $x(t) \sim \mathrm{trig}\left(\left(\frac{\omega-\omega_0}{2}\right)t\right)\left(\left(\frac{\omega-\omega_0}{2}\right)t\right)$ the low and high frequencies respectively.

IMAGE 3

Beats (non-amplified)
If isntead $|\omega| = |\omega_0|$, then $x(t) \propto \mathrm{trig}(\omega t)t$.

IMAGE 4

In general, $\dot{x} + Ax = 0$ for $x \in \mathbb{R}^n$, $x(t) = \exp(-tA) + x(0)$.
In the case where A is skew-adjoint, $i.e.$ $\mathrm{sp}(A) \subseteq i\mathbb{R}$, $(x, Ax) = 0$ $\forall x \in \mathbb{R}^n$, then

$$\frac{d}{dt}(x,x) = (\dot{x}, x) + (x, \dot{x}) = (-Ax, x) - (x, Ax) = 0$$

Which implies that $||x(t)||$ is constant and the dynamics are norm perserving.
To generate resonant solutions, if (iw, v) is an eigenpair of A ($\omega \in \mathbb{R}$), consider $\dot{x} + Ax = e^{-i\omega t}v$. As an ansatz, we look for a solution of the form $x(t) = a(t)v$ and the equation becomes $(\dot{a}(t) + i\omega a)v = e^{-i\omega t}v$. Then

$$e^{-i\omega t}\frac{d}{dt}(e^{i\omega t}a) = e^{-i\omega t}$$
$$\frac{d}{dt}(e^{i\omega t}a) = 1$$
$$a(t) = te^{-i\omega t}.$$

Resonances in PDEs

Consider one-dimensional waves on $[0, L]$, $L > 0$.

$$\begin{cases} \partial_{tt}u + \partial_{xx}u = 0 & \\ u|_{t=0} = f & x \in [0, L] \\ \partial_t u|_{t=0} = g & x \in [0, L] \\ u(0, t) = u(L, t) = 0 & \forall\, t \geq 0 \end{cases}$$

We want to think about this as $\partial_{tt}u = Au = 0$ where A is the Dirichlet Laplacian $Au = -\partial_{xx}u$ with Dirichlet boundary conditions. We then want to find the spectral decomposition of A, $Au - \lambda u = 0 = -\partial_x^2 u - \lambda u$.

$$\begin{aligned} \lambda = 0. &\quad u(x) = A + Bx \implies A = B = 0 \\ \lambda = -p^2. &\quad u(x) = Ae^{px} + be^{-px} \implies A = B = 0 \\ \lambda = p^2. &\quad u(x) = A\cos(px) + B\sin(px) \implies 0 = u(0) = A \quad 0 = u(L) = B\sin(pl) \implies p = k\pi,\ k \in \mathbb{N} \end{aligned}$$

Therefore there are infinitely many eigenpairs $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $\phi_n(x) = \sin\left(\frac{k\pi x}{L}\right)$.

IMAGE 5

The family $\{\phi_n, n \in \mathbb{N}\}$ is dense in $L^2([0, L])$ where the unbounded operator $(-\partial_x^2)$ with Dirichlet boundary conditions is self-adjoint.

Other Prototypes

(of unbounded self-adjoint operators with discrete spectrum)

- Laplace-Beltrami operators on compact manifolds without boundary.

IMAGE 6

- On compact domains with boundary there is the Laplacian with Dirichlet boundary conditions.

The (Quantum) Harmonic Oscillator

$H = -\frac{d^2}{dx^2} + x^2$ on $\mathbb{R}$, on $L^2(\mathbb{R})$ with $(f, g) = \int_{\mathbb{R}} f(x)\overline{g(x)}\, dx$.
H acts on the Schwarz space $\mathcal{S}(\mathbb{R}) := \left\{ f \in C^\infty(\mathbb{R}),\ \forall k, \ell \geq 0,\ \sup_{x \in \mathbb{R}} \left| x^k \left(\frac{d}{dx}\right)^\ell f(x) \right| < \infty \right\}$.

- The action of $H : \mathcal{S}(\mathbb{R})$ is continuous.
- H is L^2-symmetric: $\int_{\mathbb{R}} -f''\overline{g} + x^2 f\overline{g}\, dx = (Hf, g) = (f, Hg) = \int_{\mathbb{R}} -\overline{g}'' f + x^2 f\overline{g}\, dx$ (integrating by parts).

We seek eigenvalues $Hu = \lambda u$. If (u, λ) and (v, μ) are eigenpairs, then

$$0 = (Hu, v) - (u, Hv) = (\lambda u, v) - (u, \mu v) = (\lambda - \overline{\mu})(u, v)$$

Where if the difference is nonzero then $(u, v) = 0$.
We can write $H = L^+L^- + I$ where $L^+ = -\frac{d}{dx} + x$ and $L^- = \frac{d}{dx} + x$ and also $[H, L^+] = 2L^+$ and $[H, L^-] = -2L^-$.
Note that H is a non-negative operators

$$(Hf, f) = \int_{\mathbb{R}} ((f')^2 + x^2 f^2)\, dx > 0$$

for $f \neq 0$ and $f \in \mathcal{S}(\mathbb{R})$. Thus $\mathrm{sp}(H) \subseteq (0, \infty)$. If $Hv = \lambda v$, then $H(L^+v) = [H, L^+]v + L^+(Hv) = (\lambda + 2)L^+v$. Similarly $H(L^-v) = (\lambda - 2)L^-v$.
Now we want to solve $L^-\phi_0 = 0$. $\frac{d}{dx}\phi_0 + x\phi_0 = 0$ tells us that $\phi_0(x) = \frac{1}{\sqrt{\pi}}e^{-x^2/2}$ (L^2-normalized). Therefore $H\phi_0 = \phi_0$ and the we have an eigenvalues of one. So we may construct $\phi_n = \frac{(L^+)^n\phi_0}{||(L^+)^n\phi_0||}$ which gives an eigenvector of H with eigenvalues $2n + 1$. Note that $||(L^+)^n\phi_0|| = \sqrt{2^n n!}$.
Fact: $\phi_n = p_n(x)e^{-x^2/2}$ where p_n is the Hermite polynomial of degree n.

$$\delta_{nq} = (\phi_n, \phi_q) = \int_{\mathbb{R}} p_n(x)p_q(x)e^{-x^2}\, dx$$

Theorem

$\{\phi_n\}_{n\geq 0}$ is dense in $L^2(\mathbb{R})$ (if $\int_{\mathbb{R}} g\phi_n\, dx = 0$ for all n, then $g = 0$).

Proof (Sketch)

For $g \in L^2$, $\xi \in \mathbb{R}$, $F_g(\xi) = \int_{\mathbb{R}} e^{ix\xi}g(x)\phi_0(x)\, dx = \widehat{g\phi_0}(\xi)$. We observe that

- F_g is real-analytic in ξ.
- $F_g^{(k)}(0) = \int_{\mathbb{R}}(-ix)^k g(x)\phi_0(x)\, dx = 0$ by assumption.

So we have a real-analytic function where all derivatives vanish at a point. So $F_g \equiv 0$, $g\phi_0 = 0$, and $g = 0$.

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One of the overarching goals is to obtain large time asymptotics of the solution $v(x, t)$ ($x \in \mathbb{R}$, $t > 0$) to

$$\begin{cases} -\partial_{tt}v - P_V v = F(x, t) & \text{on } \mathbb{R}_x \times (0, \infty)_t \\ v(x, 0) = \partial_t v(x, 0) = 0, & F \in C_C^\infty(R_x \times (0, \infty)_t) \end{cases}$$

where $P_V = D_x^2 + V(x) = -\left(\frac{\partial}{\partial}\right)^2 + V(x)$ and $D_x = \frac{1}{i}\frac{\partial}{\partial x}$. The operator D_x is symmetric and self-adjoint on appropriately chosen domains. For $f(x)$ and $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi}f(x)\, dx$, $\widehat{D_x f} = \xi\hat{f}(\xi)$. $V \in L^\infty_{\text{comp.}}(\mathbb{R})$ (i.e. compactly supported L^∞) is the potential. If $f, g \in \mathcal{S}(\mathbb{R})$, then $(P_V f, g)_{L^2(\mathbb{R})} = (f, P_V g)_{L^2(\mathbb{R})}$.

IMAGE 1

Another way to look at this assuming v exists, we can consider $u(x, \lambda) := \int_0^\infty e^{it\lambda}v(x, t)\, dt$ (the Fourier-Laplace transform of v) with $\lambda \in \mathbb{C}$, $\mathrm{Im}(\lambda) > 0$. Write $\lambda = \xi + ic$, $c > 0$, such that $u(x, \xi + ic) = \int_0^\infty e^{it\xi}e^{-ct}v(x, t)\, dt = \mathcal{F}_{t\mapsto\xi}(t \mapsto$

$e^{-ct}v(x,t))(x,-\xi)$. Then $u(x,\lambda)$ solves

$$\int_0^\infty e^{it\lambda}(-\partial_{tt}v - P_V v)\,dt = \int_0^\infty e^{it\lambda}F(x,t) = \hat{F}(x,\lambda)$$
$$(\lambda^2 - P_V)\underbrace{\int_0^\infty e^{it\lambda}v(x,t)\,dt}_{u(x,\lambda)} = \hat{F}(x,\lambda)$$

which is an entire function in λ.
To Do:

- Study solvability of $(\lambda^2 - P_V)u = \hat{F}(x,\lambda)$.
- Return to v.

For frozen c, we can get $v(x,t)$ back by Fourier inversion.

$$e^{-ct}v(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-it\xi}u(x,\xi+ic)\,d\xi$$
$$v(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-it(\xi+ic)}u(x,\xi+ic)\,d\xi$$
$$v(x,t) = \frac{1}{2\pi}\int_{\mathrm{Im}(\lambda)=\mathrm{c}} e^{-it\lambda}u(x,\lambda)\,d\lambda$$

IMAGE 2

where the spectral problem is invertible.

1D Waves in the Time Domain

Suppose $R > 0$ is such that $\operatorname{supp} V \subset [-R,R]$ and $\operatorname{supp} F \subset [-R,R]\times(0,\infty)$. If $|x| > R$, the PDE looks like $\partial_{tt}v - \partial_{xx}v = 0 = (\partial_t + \partial_x)(\partial_t - \partial_x)v$. Setting $\xi = x+t$ and $\mu = x-t$, then it follows that

$$\partial_\xi\partial_\mu v = 0 \implies v = F(\xi) + G(\mu) = F(x+t) + G(x-t)$$

IMAGE 3

On $x > R$, we can expect $v(x,t) = F_+(x+t) + G_+(x-t)$; on $x < R$, we expect $v(x,t) = F_-(x+t) + G_-(x-t)$. The terms G_+ and F_- are outgoing whereas the terms F_+ and G_- are incoming and, given that we assumed a source, we expect to be zero.
What does incoming/outgoing look like on the spectral side? $(\lambda^2 - P_v)u = \hat{F}(x,\lambda)$ supported in $|x| \le R$. For $|x| > R$, $(\lambda^2 + \partial_x^2)u = 0$ leads to $u = Ae^{ix\lambda} + Be^{-ix\lambda}$. For $x > R$, $u(x) = a_+e^{i\lambda|x|} + b_+e^{-i\lambda|x|}$ for $x < -R$, $u(x) = a_-e^{i\lambda|x|} + b_-e^{-i\lambda|x|}$. u is outgoing if and only if $b_\pm = 0$ and incomming if and only if $a_\pm = 0$.
P_V is an unbounded, symmetric operator on a Hilbert space. For $z \in \mathbb{C}$, $\mathrm{sp}(P_V)$ is the set on the complement of which $(P_V - Z)$ is boundedly invertible. That is, $\forall f$, $\exists! u$ such that $(P_V - z)u = f$ and $||u|| \lesssim ||f|$.

Waves in the Time Domain [Evans, ğ2.4]

Goal: if v solves

$$\begin{aligned} \partial_{tt}v - \partial_{xx}v &= f(x,t) x \in \mathbb{R},\ t>0,\ f \in C_C^\infty(\mathbb{R}\times(0,\infty)) \\ v(x,0) = \partial_t v(x,0) &= 0 x \in \mathbb{R} \end{aligned}$$

then $v(x,t) = \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(y,t-s)\,dyds$. We look at

$$\begin{cases} \partial_{tt}v - \partial_{xx}v = 0 \rightsquigarrow v(x,t) = F(x+t) + G(x-t) \\ v(x,0) = g(x),\ \partial_t v(x,0) = h(x) \end{cases}$$

Initial conditions gives us

$$\begin{cases} F(x) + G(x) = g(x) \\ F'(x) - G'(x) = h(x) \end{cases} \qquad \begin{cases} G'(x) = \frac{1}{2}(g'(x) - h(x)) \\ F'(x) = \frac{1}{2}(g'(x) + h(x)) \end{cases}$$

So

$$\begin{aligned} F(x) &= \frac{1}{2}\left(g(x) + \int_0^x h(s)\,ds\right) + C_1 \\ G(x) &= \frac{1}{2}\left(g(x) - \int_0^x h(s)\,ds\right) + C_2 \\ v(x,t) &= \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(s)\,ds + C \end{aligned}$$

IMAGE 4

This has a finite speed of propogation in the sense that if we suppose $\operatorname{supp}(g,h) \subset [-R,R]$ then $v(x,t) = 0$ whenever $x > R+t$ or $x < -R-t$.

Now we want to go from the homogeneous problem to the inhomogeneous problem. The idea is to think about $v(x,t) = \int_0^t v(x,t;s)\,ds$ where $v(x,t;s)$ solves the homogeneous problem

$$\begin{cases} \partial_{tt}v(\cdot,\cdot;s) - \partial_{xx}v(\cdot,\cdot;s) = 0 \\ v(\cdot,s;s) = 0,\ \partial_t v(\cdot,s;s) = f(x,s) \end{cases}$$

Then

$$\partial_{tt}v - \partial_{xx}v = 0 \iff \partial_t \begin{pmatrix} v \\ \partial_t v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \partial_{xx} & 0 \end{pmatrix}\begin{pmatrix} v \\ \partial_t v \end{pmatrix} \quad \text{and} \quad \begin{bmatrix} v \\ \partial_t v \end{bmatrix}_{t=s} = \begin{bmatrix} * \\ * \end{bmatrix}$$

So $v(x,t;s) = \frac{1}{2}\int_{x-(t-s)}^{x+(t-s)} f(y,s)\,ds$ and $v(x,t) = \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(y,t-s)\,dyds$ follows.

Going back to the original PDE, $(-\partial_{tt} - P_V)v = F$ is equivalent to $(\partial_{tt} - \partial_{xx})v = -(Vv+F)$ which leads to the conclusion that $v(x,t) = -\frac{1}{2}\int_0^t \int_{x-s}^{x+s} (Vv+F)(y,t-s)\,dyds$. For $|x| > R$, v is outgoing.

IMAGE 5

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Take come complex vector space and consider the Hilbert space $(\mathcal{H}, (\cdot,\cdot))$ with $(\cdot,\cdot)$ satisfying

$$\begin{cases} (\lambda f, g) = \lambda(f,g) \\ (f, \lambda g) = \overline{\lambda}(f,g) \\ (f,g) = \overline{(g,f)} \\ f \mapsto (f,g) =: ||f||^2 \text{ a norm} \\ (\mathcal{H}, ||\cdot||) \text{ complete with respect to the norm} \end{cases}$$

- Examples
  - $\left(\mathbb{C}^n, (a,b) = \sum_{j=1}^n a_j \overline{b}_j\right)$, $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$
  - $L^2(x, \mu)$ (e.g. $[0,1]$ and the Lebesgue measure), $(f,g) = \int_X f\overline{g}\, d\mu$.

Bounded Operators: $T : \mathcal{H} \to \mathcal{H}$ bounded if and only if $\overbrace{\sup_{||x||=1} ||Tx||}^{||T||} < \infty$ satisfying

$$\begin{cases} \mathcal{B}(\mathcal{H}) \text{ the space of bunded operators on } \mathcal{H} \text{ (a complex vector space)} \\ ||\cdot|| \text{ is a norm on } \mathcal{B}(\mathcal{H}) \text{, making it complete} \\ \text{There is a multiplication, } \mathcal{B}(\mathcal{H}) \ni A, B \mapsto AB \text{ and } ||AB|| \leq ||A||||B|| \end{cases}$$

Adjoint: if $A \in \mathcal{B}(\mathcal{H})$, $\exists! A^* \in \mathcal{B}(\mathcal{H})$ such that $\forall f, g \in \mathcal{H}$, $(Af, g) = (f, A^* g)$ where A is symmetric/self-adjoint if $A = A^*$. Thes notions are different in the world of unbounded operators.

- Example
  - $\mathcal{H} = \mathbb{C}^n$: $T \in M_n(\mathbb{C})$ symmetric if and only if T is Hermitian. $t_{ij} = \overline{t_{ji}}$.
  - $\mathcal{H} = L^2([0,1])$, $Tf(t) = tf(t)$. $(Tf, g) = \int_0^1 tf(t)\overline{g(t)}\, dt = \int_0^1 f(t)\overline{tg(t)}\, dt = (f, Tg)$.
  - $\mathcal{H} = L^2(\mathbb{R})$ with the Fourier transform. $||f(x)||^2 = c||\hat{f}(\xi)||^2$ (Parseval's Equality).

Finite Dimensional Spectral Theorem

If $A \in M_n(\mathbb{C})$ is Hermitian, there exists an orthonormal basis $(\phi_1, \dots, \phi_n)$ of $\mathbb{C}^n$ and real eigenvalues $\lambda_1, \dots, \lambda_n$ such that $A\phi_j = \lambda_j \phi_j$.
Important observation: if A is Hermitian, then λ_j is real for each j, and $\overline{(A\phi_j, \phi_j)} = (\phi_j, A\phi_j) = (A\phi_j, \phi_j) = \lambda_j ||\phi_j||^2$.
So $\lambda_j = \frac{(A\phi_j, \phi_j)}{||\phi_j||^2}$ is real. If $\lambda_j \neq \lambda_k$, then $(\phi_j, \phi_k) = 0$ since $(A\phi_j, \phi_k) - (\phi_j A\phi_k) = (\lambda_j - \overline{\lambda}_k)(\phi_j, \phi_k)$.
Notation: Let $u, v \in \mathbb{C}^n$, denote $u \otimes \overline{v}$ the operator $(u \otimes \overline{v})w = (w, v)u$.
With A as in the theorem, we can write $A = \sum_{j=1}^n \lambda_j \phi_j \otimes \overline{\phi}_j$ $\left(I = \sum_{j=1}^n \phi_j \otimes \overline{\phi}_j\right)$. A second way of writing this is

IMAGE 1

Where $U^* = U^{-1}$ and $A = U\Lambda U^*$. This allows us to construct a functional calculus for A where

$$\begin{cases} A^2 = U\Lambda U^* U\Lambda U^* = U\Lambda^2 U^* \\ A^n = U\Lambda^n U^* \\ p(A) = U \cdot p(\Lambda) \cdot U^*,\ p \text{ a polynomial} \end{cases}$$

Defining $f(A) := U \cdot f(\Lambda) \cdot U^*$, we obtain a Banach algebra homomorphism. Then $f \in C([-||A||, ||A||])$ is also a Banach algebra with sup norm and pointwise multiplication.

IMAGE 2

Then we can map $C([-||A||, ||A||]) \ni f \mapsto f(A) \in \mathcal{B}(\mathcal{H})$. This is useful for solving ODEs.

- Prototypes
  - Heat equation: $\partial_t u + Au = 0$, $u|_{t=0} = u_0$, $u(t) = e^{-tA} u_0$.
  - Schrödinger equation: $i\partial_t u + Au = 0$, $u|_{t=0} = u_0$, $u(t) = e^{-itA} u_0$.
  - Wave equation: $\partial_{tt} u + Au = 0$, $u|_{t=0} = u_0$, $\partial_t u|_{t=0} = u_1$.

Write $u(t) := \sum_{j=1}^n u_j(t)\phi_j$ with the PDE $\sum_{j=1}^n (u_j'' + \lambda_j u_j)\phi_j = 0$. Then $u_j'' + \lambda_j u_j = 0$, $u_j(0) = u_{j,0}$, and $u_j'(0) = u_{j,1}$. Suppose $\lambda_j > 0$ for all j. Then $u_j(t) = u_{j,0} \cos(\sqrt{\lambda_j} t) + \frac{u_{j,1}}{\sqrt{\lambda_j}} \sin(\sqrt{\lambda_j} t)$. So

$$u(t) = \sum_{j=1}^n \cos(\sqrt{\lambda_j} t) u_{j,0} \phi_j + \frac{1}{\sqrt{\lambda_j}} \sin(\sqrt{\lambda_j} t) u_{j,1} \phi_j$$

Therefore $u = \cos(t\sqrt{A}) u_0 + A^{-1/2} \sin(t\sqrt{A}) u_1$.

Spectrum of a Bounded Operator

Take $T \in \mathcal{B}(\mathcal{H})$. We say that T is invertible (within $\mathcal{B}(\mathcal{H})$) if and only if $\exists S \in \mathcal{B}(\mathcal{H})$ such that $TS = ST = I$.
Counterexample: take $\mathcal{H} = \ell^2(\mathbb{N}_0) = \{u = (u_n)_{n\geq 0} : \sum |u_n|^2 < \infty\}$ and $Au = \left(\frac{1}{n} u_n\right)_{n \geq 0}$. Then the proxy for $A^{-1}u = (nu_n)_{n\geq 0}$ is not bounded.
Given $T \in \mathcal{B}(\mathcal{H})$, the resolvent set of T is $\rho(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \text{ is invertible}\}$. Invertibility is equivalent to $\forall y \in \mathcal{H}$, $\exists! x$ such that $Tx - \lambda x = y$ with an estimate $||x|| \lesssim ||y|$.
For $\lambda \in \rho(T)$, denote $R(\lambda)$ or $R_T(\lambda) = (T - \lambda I)^{-1} \in \mathcal{B}(\mathcal{H})$ the resolvent of T. Properties of the resolvent set:

1. $\rho(t) \neq 0$ (in fact, if $|\lambda| > ||T||$ then $\lambda \in \rho(T)$.
2. $\rho(t)$ is open.
3. the map $\rho(t) \ni \lambda \mapsto R_T(\lambda) \in \mathcal{B}(\mathcal{H})$ is holomorphic in the sense that $\forall \lambda_0 \in \rho(t)$, $\exists R_T'(\lambda_0) \in \mathcal{B}(\mathcal{H})$ such that $\lim_{\lambda\to\lambda_0} \left|\left| \frac{R_T(\lambda) - R_T(\lambda_0)}{\lambda - \lambda_0} - R_T'(\lambda_0) \right|\right| = 0$.

For a., if $|\lambda| > ||T||$, $Tx - \lambda x = y \iff \left(I - \frac{T}{\lambda}\right)x = -\frac{y}{\lambda} \iff x = \sum_{k=0}^\infty \frac{T^k}{\lambda^{k+1}} y$. Then $R_T(\lambda) = \frac{1}{\lambda} \sum_{k=0}^\infty \left(\frac{T}{\lambda}\right)^k$ and

$$||R_T(\lambda)|| \leq \frac{1}{||\lambda||} \frac{1}{1 - ||T/\lambda||} \leq \frac{1}{|\lambda| - ||T|||}$$

For b., pick $\lambda_0 \in \rho(T)$ and find $r > 0$ such that $|\lambda - \lambda_0| < r \implies \lambda \in \rho(T)$. Then $Tx - \lambda x = y \iff (T - \lambda_0)x - (\lambda - \lambda_0)x = y \iff x - (\lambda - \lambda_0) R_T(\lambda_0) x = R_T(\lambda_0) y$ where if $||(\lambda - \lambda_0) R_T(\lambda_0)|| < 1$ it is boundedly solvable by Neumann series.

For c.,

$$\begin{aligned}R_T(\lambda)-R_T(\lambda_0)&=(T-\lambda I)^{-1}-(T-\lambda_0 I)^{-1}\\(T-\lambda I)(R_T(\lambda)-R_T(\lambda_0))&=I-(T-\lambda_0 I+(\lambda_0-\lambda)I)(T-\lambda_0 I)^{-1}\\(T-\lambda I)(R_T(\lambda)-R_T(\lambda_0))&=I-I+(\lambda-\lambda_0)R_T(\lambda_0)\\R_T(\lambda)-R_T(\lambda_0)&=(\lambda-\lambda_0)R_T(\lambda)R_T(\lambda_0)\end{aligned}$$

So $\frac{R_T(\lambda)-R_T(\lambda_0)}{\lambda-\lambda_0}-R_T(\lambda_0)^2=o(\lambda-\lambda_0)$.
Then we define the spectrum $\sigma(T):=\mathbb{C}\setminus\rho(T)$ which is closed since $\rho(T)$ is open.

Lemma

If $T\in\mathcal{B}(\mathcal{H})$ is self-adjoint, then $\sigma(T)\subseteq[-||T||,||T||]$.

- Proof

First we know $\sigma(T)\subseteq\{|\lambda|\leq||T||\}$. We want to show that it is real, and that if $\lambda=a+ib$ and $b\neq 0$ then $T-(a+ib)I$ is invertible.
$T-(a+bi)I$ is injective.

$$\begin{aligned}||(T-(a+ib))x||^2&=(Tx-(a+ib)x,Tx-(a+ib)x)\\&=||Tx||^2+(a^2+b^2)||x||^2-(Tx,(a+ib)x)-((a+ib)x,Tx)\\&=||Tx||^2+(a^2+b^2)||x||^2-(a-ib)(Tx,x)-(a+ib)(x,Tx)\\&=||Tx||^2+a^2||x||^2-2a(x,Tx)+b^2||x||^2\geq b^2||x||^2\end{aligned}$$

since $||Tx||^2+a^2||x||^2-2a(x,Tx)\geq 0$by Cauchy-Schwarz. Therefore $T-(a+ib)$ is injective and, by the open mapping theorem, $(T-(a+ib))^*=T-(a-ib)$ is surjective. Similarly for $T-(a-ib)$, and the norm estimate is $||(T-(a+ib))^{-1}||\leq\frac{1}{b}$. Note that $\frac{1}{b}=\frac{1}{\mathrm{dist}(a+ib,\mathbb{R})}$.

Note that the spectrum of T may no longer be made of eigenvalues in the non-finite case. There may exist λ such that $T-\lambda I$ nont injective, $\exists v\neq 0$ $Tv=\lambda v$. Recall the example $Tf(t)=tf(t)$ with $f\in L^2((\cdot,\cdot),dt)$. T is self-adjoint, $||T||\leq 1$, and $(Tf,f)=\int_0^1 t|f(t)|^2\,dt\geq 0$. So $\sigma(T)\subseteq[0,1]$. For $\lambda\in[0,1]$ is $T-\lambda I$ injective? $Tf=\lambda f\iff tf(t)=\lambda f(t)\iff(t-\lambda)f(t)=0$ which implies $f\equiv 0$ in $L^2([0,1])$. Is $T-\lambda I$ surjective? $(t-\lambda)f(t)=g(t)\iff f(t)=\frac{g(t)}{t-\lambda}$, so $g(t)\equiv 1\in L^2([0,1])$ which implies $f(t)=\frac{1}{t-\lambda}$ is not $L^2([0,1])$ and $\sigma(T)=[0,1]$.

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Spectral Resolution

Take $\mathcal{H}$ a hlibert space, and say that $P\in\mathcal{B}(\mathcal{H})$ is an orthogonal projection if $P^2=P$ and $P^*=P$. Then let $\mathcal{P}(\mathcal{H})=\{P\in\mathcal{B}(\mathcal{H}),\ P$ is an orthogonal projection$\}$.

- Examples
  - $\phi\in\mathcal{H}$, $||\phi||=1$, $P:=\phi\otimes\overline{\phi}$. Then $P\psi=(\psi,\phi)\phi$ and $P^2\psi=P(\psi,\phi)\phi=(\psi,\phi)(\phi,\phi)\phi=P\psi$.

- $\phi_1, \ldots, \phi_n$ an orthonormal family with $P = \sum_{k=1}^{n} \phi_k \otimes \overline{\phi}_k$.
- $\mathcal{H} = \ell^2(\mathbb{N})$, $e_j = (0, 0, \ldots, 0, 1, 0, \ldots)$ and $I = \sum_{j=1}^{\infty} e_j \otimes \overline{e}_j$.
- $\mathcal{H} = L^2(\mathbb{R})$. Fix I an interval with χ_I the characteristic function for I. Then take $Pf := \chi_I f$.
  * $PPf = \chi_I \chi_I f = \chi_I f = Pf$.
  * $\int_{\mathbb{R}} \chi_I f \overline{g}\, dx = \int_{\mathbb{R}} f \overline{\chi_I g}\, dx$.
  * If I has a nonempty interior, then $\mathrm{Range}(P) = \{f \in L^2(\mathbb{R}),\ \operatorname{supp} f \subset I\} \simeq L^2(I)$.

Definition: Spectral Resolution

A spectral resolution is a map $\mathbb{R} \ni \lambda \mapsto E(\lambda) \in \mathcal{P}(\mathcal{H})$ satisfying

1. $\forall f \in \mathcal{H}$, $||E(\lambda)f||$ is increasing.
2. $\exists [a, b]$ such that $E(\lambda) = 0$ if $\lambda < a$ and $E(\lambda) = \mathrm{Id}$ if $\lambda \geq b$.
3. $E(\lambda)$ is right continuous. That is, $\forall f \in \mathcal{H}$, $\lambda \in \mathbb{R}$,

$$\lim_{\substack{\mu \to \lambda \\ \mu > \lambda}} ||E(\mu)f - E(\lambda)f|| = 0$$

Alternatively, we can require $E(\lambda)E(\mu) = E(\min\{\mu, \lambda\})$.

Long story short: the collection of self-adjoint bounded operators is in one-to-one correspondence with the collection of spectral resolutions.

- Examples
  - $A \in M_n(\mathbb{C})$, $A^* = A$, with eigencouples $(\lambda_1, \phi_1), \ldots, (\lambda_n, \phi_n)$ and simple spectrum $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. Define $E(\lambda) := \sum_{j : \lambda_j \leq \lambda} \phi_j \otimes \overline{\phi}_j$.
    * $A = I$ gives $E(\lambda) = 0$ for $\lambda < 1$ and $E(\lambda) = \mathrm{Id}$ for $\lambda \geq 1$.
    * $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ gives $E(\lambda) = \mathrm{Id}_{\lambda \geq 1}\, e_1 \otimes e_1 + \mathrm{Id}_{\lambda \geq 2}\, e_2 \otimes e_2$.
      · If $f = f_1 e_1 + f_2 e_2$, then $||E(\lambda)f||^2 = \mathrm{Id}_{\lambda \geq 1}(\lambda)||f_1||^2 + \mathrm{Id}_{\lambda \geq 2}(\lambda)|f_2|^2$.

Spectral Measures

A spectral resolution gives rise to spectral measures

$$f, g \in \mathcal{H} \quad \lambda \mapsto (E(\lambda)f, g) = F(\lambda) \in \mathbb{C}$$

This defines a Lebesgue-Stieljes measure

$$\mu_F : \mu_F((a, b]) = F(b) - F(a), \quad \forall a, b \in \mathbb{R}$$

We can construct this as follows:

- When $f = g$, $\lambda \mapsto (E(\lambda)f, f) = (E(\lambda)^2 f, f) = ||E(\lambda)f||^2$ (increasing).
- When $f \neq g$,

$$(E(\lambda)f, g) = (E\lambda(f), E(\lambda)g) = \frac{1}{4}\left(||E(\lambda)(f+g)||^2 - ||E(\lambda)(f-g)||^2 - i||E(\lambda)(f+ig)||^2 + i||E(\lambda)(f-ig)||^2\right)$$

$\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ defines a projection-valued measure E_Ω, $\Omega \subset \mathbb{R}$ a Borel set. Start with $E_{(a,b]} = E(b) - E(a)$. We would like for E_Ω to satisfy $E_{\Omega_1}E_{\Omega_2} = E_{\Omega_1\cap\Omega_2}$, $E_\varnothing = 0$, $E_\mathbb{R} = \mathrm{Id}$.

Theorem: Spectral Theorem

For $A \in \mathcal{B}(\mathcal{H})$ self-adjoint, there exist $a, b \in \mathbb{R}$ and an A-dependent spectral resolution $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ such that

$$A = \int_{a_-}^{b} \lambda \, dE(\lambda)$$

in the sense that $(Af, g) = \int_{a_-}^{b} \lambda \, d(E(\lambda)f, g)$ for all $f, g \in \mathcal{H}$. This is amenable to creating a functional calculus

$$C([-||A||, ||A||]) \to \text{bounded self-adjoint operators that commute with } A$$
$$h \mapsto h(A) := \int_{a_-}^{b} h(\lambda)\, dE(\lambda)$$

The idea is that $E(\lambda) = \chi_{(-\infty,\lambda]}(A)$. Once E is constructed, this leads to E_Ω for Ω Borel. We say that a measure μ is supported in G (Borel) if for every Ω Borel, $\mu(\Omega) = \mu(\Omega \cap G)$. Then $\operatorname{supp} E \subset \sigma(A)$.

Functional Calculus

We want to make sense of $h(A)$ for h in a large enough class. If p is a polynomial, we can make sense of $p(A) = \sum_{k=0}^{n} a_k A^k$ which is self-adjoint and bounded.
For $h \in C(-||A||, ||A||)$, h is uniformly approximated by polynomials. We want to show that p_n is uniformly Cauchy which implies that $p_n(A)$ converges to some $h(A)$.
For $h = \chi_{(-\infty,\lambda]}$, we proceed by approximation by tent functions.

IMAGE 1

Definition: Positive Operator

If S is a self-adjoint, bounded operator on $\mathcal{H}$, we say that S is positive ($S \geq 0$) if $(Sf, f) \geq 0$, $\forall f \in \mathcal{H}$. For S_1, S_2 self-adjoint and bounded, we say that $S_1 \geq S_2$ if and only if $S_1 - S_2 \geq 0$.
For $T \in \mathcal{B}(\mathcal{H})$, self-adjoint, set $a := \inf_{||f||=1}(Tf, f)$ and $b := \sup_{||f||=1}(Tf, f)$. Then $a\mathrm{Id} \leq T \leq b\mathrm{Id}$.

$$((T - a\mathrm{Id})f, f) = (Tf, f) - a(f, f) = (f, f)\left(\left(T\frac{f}{||f||}, \frac{f}{||f||}\right) - a\right) \geq 0$$

We want to show that if p is a polynomial on $[-||A||, ||A||]$, then $(\inf_{[-||A||,||A||} p)\mathrm{Id} \leq p(A) \leq (\sup_{[-||A||,||A||} p)\mathrm{Id}$.

Lemma

If T_1 and T_2 are postive and commute, then $T_1T_2 \geq 0$.

Square Root Lemma

If $A \geq 0$ (i.e. bounded, self-adjoint, and positive), then $\exists! B \geq 0$ such that $B^2 = A$ and B comutes with any operator that commutes with A.

- Proof
  Use the power series of $z \mapsto \sqrt{1-z}$ at $z = 0$.

$$1 + \sum_{k=1}^{\infty} c_k z^k$$

We can find that $c_k < 0$ for all $k \geq 1$ and that the series converges uniformly on $\{|z| \leq 1\}$.
Now let $A \geq 0$ which implies that $0\,\mathrm{Id} \leq I - A \leq 1\,\mathrm{Id}$. Without loss of generality, suppose $\operatorname{supp} ||A|| \leq 1$. The idea is to write

$$B = \sqrt{A} = \sqrt{I - (I - A)} = I + \sum_{k=1}^{\infty} c_k (I - A)^k$$

which converges strongly because the series converges uniformly. Then $B^2 = A$. We see that $B \geq 0$ using the fact that $\operatorname{sign}(c_k) < 0$ which implies $\sum_{k \geq 1} c_k \geq -1$. The proof of uniqueness can be found in the text.

Proof of Lemma

Assuming the square root lemma, write $T_2 = B^2$. Then since $[T_1, T_2] = 0$, $[B, T_1] = 0$. Then

$$(T_1 T_2 f, f) = (T_1 B^2 f, f) = (B T_1 B f, f) = (T_1(Bf), Bf) \geq 0$$

Weaker Version

Instead of $(\inf_{[-||A||,||A||} p)\,\mathrm{Id} \leq p(A) \leq (\sup_{[-||A||,||A||} p)\,\mathrm{Id}$, we have that if $\min_{[-||A||,||A||]} p \geq 0$, then $p(A) \geq 0$.

Proof

If $p \geq 0$ on $[-||A||, ||A||]$, we can factor it as a product of positive pieces

$$\begin{aligned} p(x) &= \prod \overbrace{(x - r_j)}^{r_j < -||A||} \overbrace{(s_j - x)}^{s_j \geq ||A||} ((x - a_j)^2 + b_j^2) \\ p(A) &= \prod \underbrace{(A - r_j)}_{\geq 0} \underbrace{(s_j - A)}_{\geq 0} \underbrace{((A - a_j)^2 b_j^2}_{\geq 0} \end{aligned}$$

Using the previous lemma, we have that $P(A) \geq 0$.

Proof

Finally, to show that $(\inf_{[-||A||,||A||} p)\,\mathrm{Id} \leq p(A) \leq (\sup_{[-||A||,||A||} p)\,\mathrm{Id}$, we see that $p - \inf p$ and $\operatorname{supp} f - p$ are positive polynomials and apply the weaker version to them.

Definition

We can define $h(A)$ for $h \in C(-||A||, ||A||$ by Weierstrass approximation. There exist p_n polynomials such that $\sup_{[-||A||,||A||]} |p_n - h| \underset{n\to\infty}{\longrightarrow} 0$. Then p_n is uniformly Cauchy, so

$$\inf(p_n - p_m)\,\mathrm{Id} \leq p_n(A) - p_m(A) \leq \sup(p_n - p_m)\,\mathrm{Id}$$

which implies that

$$||p_n(A) - p_{n-1}(A)|| \leq \max(\sup(p_n - p_m), -\inf(p_n - p_m)) \underset{n,m\to\infty}{\longrightarrow} 0.$$

So $p_n(A)$ is Cauchy in $(\mathcal{B}(\mathcal{H}), ||\cdot||)$ which means it converges. We call $h(A) = \lim_{n\to\infty} p_n(A)$. We still want to show that $h(A)$ is bounded and self-adjoint.