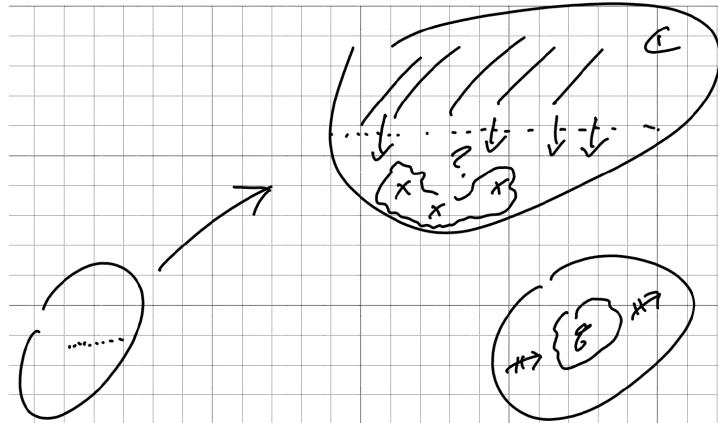


# Advanced Analysis

September 25, 2025

Suppose we have some function of the form  $-\Delta + q \in \mathbb{L}(H)$  satisfying  $R_A(\lambda)(A - \lambda I)^{-1}$  bounded on  $\text{Im}(\lambda) > 0$  and not surjective for  $\text{Im}(\lambda) = 0$ .

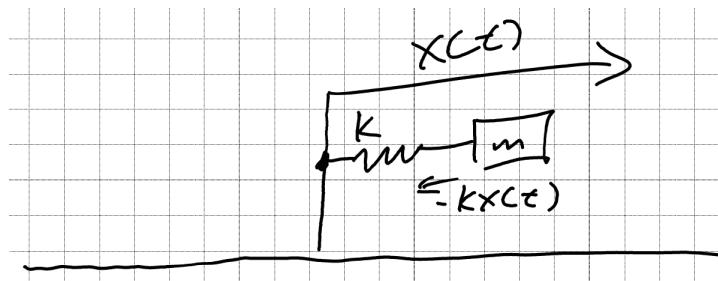


Waves: solutions to  $\partial_{tt}u + Au = 0$  on  $\mathbb{R}^n$ .

Mathematical Tools

- Spectral theory of unbounded operators
- Complex analysis
- Functional analysis
- Microlocal analysis
- Semiclassical analysis

## Classical Resonances in ODEs

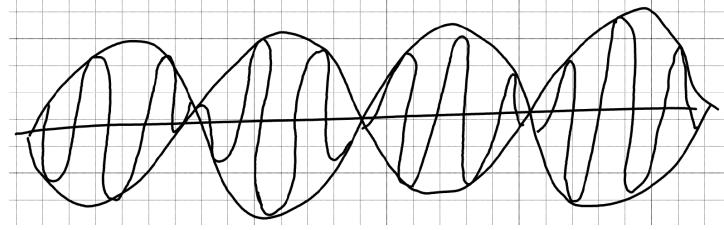


A harmonic oscillator assuming no friction.

We have an acceleration force,  $m\ddot{x}(t) = -kx(t)$  which gives  $\ddot{x} + \omega_0^2 x = 0$  with  $\omega_0 = \sqrt{\frac{k}{m}}$  and has solution  $x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$ .

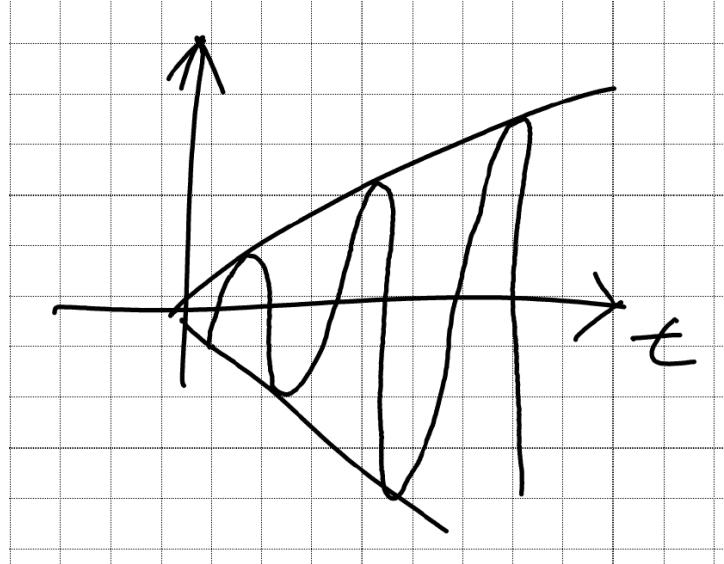
With forcing, i.e.  $m\ddot{x}(t) = -kx(t) + A\sin(\omega t)$ , we have  $\ddot{x} + \omega_0^2 x' = A\sin(\omega t)$ .

If  $|\omega| \neq |\omega_0|$ , then  $x(t) \sim \text{trig}\left(\left(\frac{\omega-\omega_0}{2}\right)t\right)\left(\left(\frac{\omega+\omega_0}{2}\right)t\right)$  the low and high frequencies respectively.



Beats (non-amplified)

If instead  $|\omega| = |\omega_0|$ , then  $x(t) \propto \text{trig}(\omega t) t$ .



In general,  $\dot{x} + Ax = 0$  for  $x \in \mathbb{R}^n$ ,  $x(t) = \exp(-tA)x(0)$ .

In the case where  $A$  is skew-adjoint, i.e.  $\text{sp}(A) \subseteq i\mathbb{R}$ ,  $(x, Ax) = 0 \forall x \in \mathbb{R}^n$ , then

$$\frac{d}{dt}(x, x) = (\dot{x}, x) + (x, \dot{x}) = (-Ax, x) - (x, Ax) = 0$$

Which implies that  $\|x(t)\|$  is constant and the dynamics are norm preserving.

To generate resonant solutions, if  $(i\omega, v)$  is an eigenpair of  $A$  ( $\omega \in \mathbb{R}$ ), consider  $\dot{x} + Ax = e^{-i\omega t}v$ . As an ansatz, we look for a solution of the form  $x(t) = a(t)v$  and the equation becomes  $(\dot{a}(t) + i\omega a)v = e^{-i\omega t}v$ . Then

$$\begin{aligned} e^{-i\omega t} \frac{d}{dt}(e^{i\omega t}a) &= e^{-i\omega t} \\ \frac{d}{dt}(e^{i\omega t}a) &= 1 \\ a(t) &= te^{-i\omega t}. \end{aligned}$$

## Resonances in PDEs

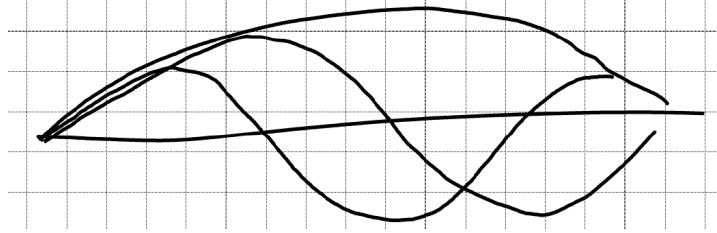
Consider one-dimensional waves on  $[0, L]$ ,  $L > 0$ .

$$\begin{cases} \partial_{tt}u + \partial_{xx}u = 0 \\ u|_{t=0} = f & x \in [0, L] \\ \partial_tu|_{t=0} = g & x \in [0, L] \\ u(0, t) = u(L, t) = 0 & \forall t \geq 0 \end{cases}$$

We want to think about this as  $\partial_{tt}u = Au = 0$  where  $A$  is the Dirichlet Laplacian  $Au = -\partial_{xx}u$  with Dirichlet boundary conditions. We then want to find the spectral decomposition of  $A$ ,  $Au - \lambda u = 0 = -\partial_x^2 u - \lambda u$ .

$$\begin{aligned}\lambda = 0. \quad u(x) &= A + Bx \implies A = B = 0 \\ \lambda = -p^2. \quad u(x) &= Ae^{px} + be^{-px} \implies A = B = 0 \\ \lambda = p^2. \quad u(x) &= A\cos(px) + B\sin(px) \implies 0 = u(0) = A \quad 0 = u(L) = B\sin(pl) \implies p = k\pi, k \in \mathbb{N}\end{aligned}$$

Therefore there are infinitely many eigenpairs  $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ ,  $\phi_n(x) = \sin\left(\frac{k\pi x}{L}\right)$ .

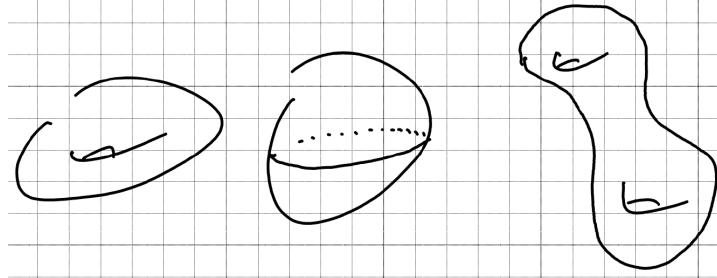


The family  $\{\phi_n, n \in \mathbb{N}\}$  is dense in  $L^2([0, L])$  where the unbounded operator  $(-\partial_x^2)$  with Dirichlet boundary conditions is self-adjoint.

## Other Prototypes

(of unbounded self-adjoint operators with discrete spectrum)

- Laplace-Beltrami operators on compact manifolds without boundary.



- On compact domains with boundary there is the Laplacian with Dirichlet boundary conditions.

## The (Quantum) Harmonic Oscillator

$H = -\frac{d^2}{dx^2} + x^2$  on  $\mathbb{R}$ , on  $L^2(\mathbb{R})$  with  $(f, g) = \int_{\mathbb{R}} f(x)\overline{g(x)} dx$ .

$H$  acts on the Schwarz space  $\mathcal{S}(\mathbb{R}) := \left\{ f \in C^\infty(\mathbb{R}), \forall k, \ell \geq 0, \sup_{x \in \mathbb{R}} \left| x^k \left( \frac{d}{dx} \right)^\ell f(x) \right| < \infty \right\}$ .

- The action of  $H : \mathcal{S}(\mathbb{R})$  is continuous.
- $H$  is  $L^2$ -symmetric:  $\int_{\mathbb{R}} -f''\bar{g} + x^2 f\bar{g} dx = (Hf, g) = (f, Hg) = \int_{\mathbb{R}} -\bar{g}''f + x^2 f\bar{g} dx$  (integrating by parts).

We seek eigenvalues  $Hu = \lambda u$ . If  $(u, \lambda)$  and  $(v, \mu)$  are eigenpairs, then

$$0 = (Hu, v) - (u, Hv) = (\lambda u, v) - (u, \mu v) = (\lambda - \mu)(u, v)$$

Where if the difference is nonzero then  $(u, v) = 0$ .

We can write  $H = L^+ L^- + I$  where  $L^+ = -\frac{d}{dx} + x$  and  $L^- = \frac{d}{dx} + x$  and also  $[H, L^+] = 2L^+$  and  $[H, L^-] = -2L^-$ .

Note that  $H$  is a non-negative operators

$$(Hf, f) = \int_{\mathbb{R}} ((f')^2 + x^2 f^2) dx > 0$$

for  $f \neq 0$  and  $f \in \mathcal{S}(\mathbb{R})$ . Thus  $\text{sp}(H) \subseteq (0, \infty)$ . If  $Hv = \lambda v$ , then  $H(L^+ v) = [H, L^+]v + L^+(Hv) = (\lambda + 2)L^+ v$ . Similarly  $H(L^- v) = (\lambda - 2)L^- v$ .

Now we want to solve  $L^- \phi_0 = 0$ .  $\frac{d}{dx} \phi_0 + x \phi_0 = 0$  tells us that  $\phi_0(x) = \frac{1}{\sqrt{\pi}} e^{-x^2/2}$  ( $L^2$ -normalized). Therefore  $H\phi_0 = \phi_0$  and we have an eigenvalues of one. So we may construct  $\phi_n = \frac{(L^+)^n \phi_0}{\|(L^+)^n \phi_0\|}$  which gives an eigenvector of  $H$  with eigenvalues  $2n + 1$ . Note that  $\|(L^+)^n \phi_0\| = \sqrt{2^n n!}$ .

Fact:  $\phi_n = p_n(x) e^{-x^2/2}$  where  $p_n$  is the Hermite polynomial of degree  $n$ .

$$\delta_{nq} = (\phi_n, \phi_q) = \int_{\mathbb{R}} p_n(x) p_q(x) e^{-x^2} dx$$

## Theorem

$\{\phi_n\}_{n \geq 0}$  is dense in  $L^2(\mathbb{R})$  (if  $\int_{\mathbb{R}} g \phi_n dx = 0$  for all  $n$ , then  $g = 0$ ).

### Proof (Sketch)

For  $g \in L^2$ ,  $\xi \in \mathbb{R}$ ,  $F_g(\xi) = \int_{\mathbb{R}} e^{ix\xi} g(x) \phi_0(x) dx = \widehat{g\phi_0}(\xi)$ . We observe that

- $F_g$  is real-analytic in  $\xi$ .
- $F_g^{(k)}(0) = \int_{\mathbb{R}} (-ix)^k g(x) \phi_0(x) dx = 0$  by assumption.

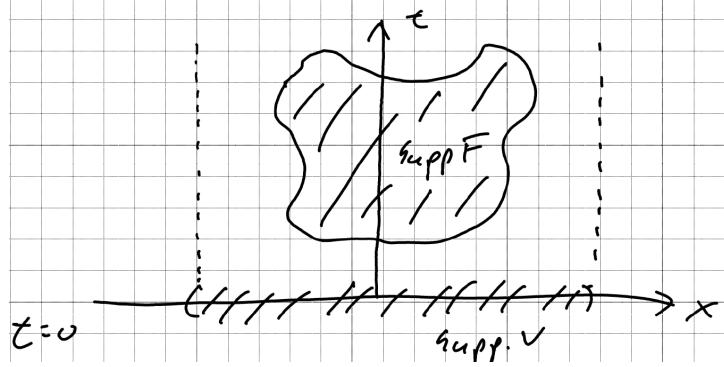
So we have a real-analytic function where all derivatives vanish at a point. So  $F_g \equiv 0$ ,  $g\phi_0 = 0$ , and  $g = 0$ .

## September 30, 2025

One of the overarching goals is to obtain large time asymptotics of the solution  $v(x, t)$  ( $x \in \mathbb{R}$ ,  $t > 0$ ) to

$$\begin{cases} -\partial_{tt} v - P_V v = F(x, t) & \text{on } \mathbb{R}_x \times (0, \infty)_t \\ v(x, 0) = \partial_t v(x, 0) = 0, & F \in C_C^\infty(\mathbb{R}_x \times (0, \infty)_t) \end{cases}$$

where  $P_V = D_x^2 + V(x) = -\left(\frac{\partial}{\partial}\right)^2 + V(x)$  and  $D_x = \frac{1}{i} \frac{\partial}{\partial x}$ . The operator  $D_x$  is symmetric and self-adjoint on appropriately chosen domains. For  $f(x)$  and  $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx$ ,  $\widehat{D_x f} = \xi \hat{f}(\xi)$ .  $V \in L_{\text{comp.}}^\infty(\mathbb{R})$  (i.e. compactly supported  $L^\infty$ ) is the potential. If  $f, g \in \mathcal{S}(\mathbb{R})$ , then  $(P_V f, g)_{L^2(\mathbb{R})} = (f, P_V g)_{L^2(\mathbb{R})}$ .



Another way to look at this assuming  $v$  exists, we can consider  $u(x, \lambda) := \int_0^\infty e^{it\lambda} v(x, t) dt$  (the Fourier-Laplace transform of  $v$ ) with  $\lambda \in \mathbb{C}$ ,  $\text{Im}(\lambda) > 0$ . Write  $\lambda = \xi + ic$ ,  $c > 0$ , such that  $u(x, \xi + ic) = \int_0^\infty e^{it\xi} e^{-ct} v(x, t) dt = \mathcal{F}_{t \mapsto \xi}(t \mapsto e^{-ct} v(x, t))(x, -\xi)$ . Then  $u(x, \lambda)$  solves

$$\begin{aligned} \int_0^\infty e^{it\lambda} (-\partial_{tt} v - P_V v) dt &= \int_0^\infty e^{it\lambda} F(x, t) = \hat{F}(x, \lambda) \\ (\lambda^2 - P_V) \underbrace{\int_0^\infty e^{it\lambda} v(x, t) dt}_{u(x, \lambda)} &= \hat{F}(x, \lambda) \end{aligned}$$

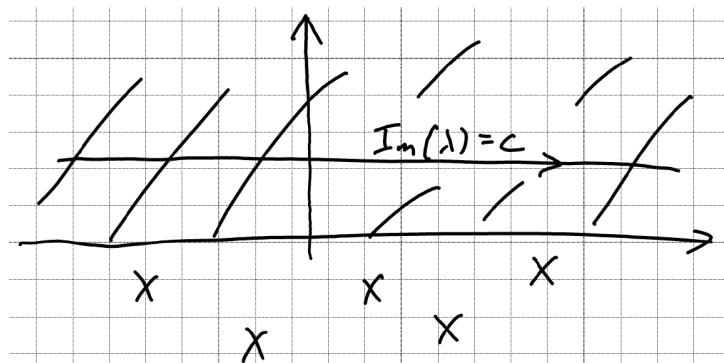
which is an entire function in  $\lambda$ .

To Do:

- Study solvability of  $(\lambda^2 - P_V)u = \hat{F}(x, \lambda)$ .
- Return to  $v$ .

For frozen  $c$ , we can get  $v(x, t)$  back by Fourier inversion.

$$\begin{aligned} e^{-ct} v(x, t) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it\xi} u(x, \xi + ic) d\xi \\ v(x, t) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it(\xi+ic)} u(x, \xi + ic) d\xi \\ v(x, t) &= \frac{1}{2\pi} \int_{\text{Im}(\lambda)=c} e^{-it\lambda} u(x, \lambda) d\lambda \end{aligned}$$

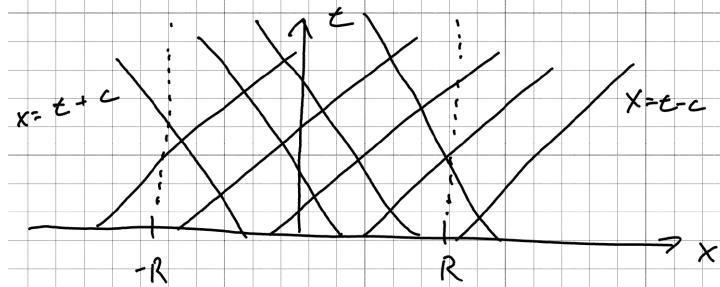


where the spectral problem is invertible.

## 1D Waves in the Time Domain

Suppose  $R > 0$  is such that  $\text{supp } V \subset [-R, R]$  and  $\text{supp } F \subset [-R, R] \times (0, \infty)$ . If  $|x| > R$ , the PDE looks like  $\partial_{tt}v - \partial_{xx}v = 0 = (\partial_t + \partial_x)(\partial_t - \partial_x)v$ . Setting  $\xi = x + t$  and  $\mu = x - t$ , then it follows that

$$\partial_\xi \partial_\mu v = 0 \implies v = F(\xi) + G(\mu) = F(x+t) + G(x-t)$$



On  $x > R$ , we can expect  $v(x, t) = F_+(x+t) + G_-(x-t)$ ; on  $x < -R$ , we expect  $v(x, t) = F_-(x+t) + G_-(x-t)$ . The terms  $G_+$  and  $F_-$  are outgoing whereas the terms  $F_+$  and  $G_-$  are incoming and, given that we assumed a source, we expect to be zero.

What does incoming/outgoing look like on the spectral side?  $(\lambda^2 - P_v)u = \hat{F}(x, \lambda)$  supported in  $|x| \leq R$ . For  $|x| > R$ ,  $(\lambda^2 + \partial_x^2)u = 0$  leads to  $u = Ae^{ix\lambda} + Be^{-ix\lambda}$ . For  $x > R$ ,  $u(x) = a_+e^{i\lambda|x|} + b_+e^{-i\lambda|x|}$  for  $x < -R$ ,  $u(x) = a_-e^{i\lambda|x|} + b_-e^{-i\lambda|x|}$ .  $u$  is outgoing if and only if  $b_\pm = 0$  and incoming if and only if  $a_\pm = 0$ .

$P_V$  is an unbounded, symmetric operator on a Hilbert space. For  $z \in \mathbb{C}$ ,  $\text{sp}(P_V)$  is the set on the complement of which  $(P_V - z)$  is boundedly invertible. That is,  $\forall f, \exists! u$  such that  $(P_V - z)u = f$  and  $\|u\| \lesssim \|f\|$ .

## Waves in the Time Domain [Evans, §2.4]

Goal: if  $v$  solves

$$\begin{aligned} \partial_{tt}v - \partial_{xx}v &= f(x, t) \quad x \in \mathbb{R}, t > 0, f \in C_C^\infty(\mathbb{R} \times (0, \infty)) \\ v(x, 0) &= \partial_t v(x, 0) = 0 \end{aligned}$$

then  $v(x, t) = \frac{1}{2} \int_0^t \int_{x-s}^{x+s} f(y, t-s) dy ds$ . We look at

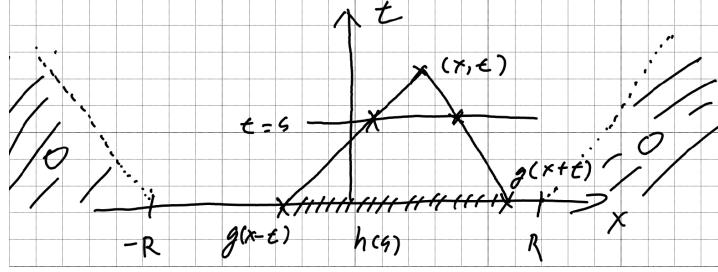
$$\begin{cases} \partial_{tt}v - \partial_{xx}v = 0 \rightsquigarrow v(x, t) = F(x+t) + G(x-t) \\ v(x, 0) = g(x), \partial_t v(x, 0) = h(x) \end{cases}$$

Initial conditions gives us

$$\begin{cases} F(x) + G(x) = g(x) \\ F'(x) - G'(x) = h(x) \end{cases} \quad \begin{cases} G'(x) = \frac{1}{2}(g'(x) - h(x)) \\ F'(x) = \frac{1}{2}(g'(x) + h(x)) \end{cases}$$

So

$$\begin{aligned} F(x) &= \frac{1}{2} \left( g(x) + \int_0^x h(s) ds \right) + C_1 \\ G(x) &= \frac{1}{2} \left( g(x) - \int_0^x h(s) ds \right) + C_2 \\ v(x, t) &= \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} h(s) ds + C \end{aligned}$$



This has a finite speed of propagation in the sense that if we suppose  $\text{supp}(g, h) \subset [-R, R]$  then  $v(x, t) = 0$  whenever  $x > R + t$  or  $x < -R - t$ .

Now we want to go from the homogeneous problem to the inhomogeneous problem. The idea is to think about  $v(x, t) = \int_0^t v(x, t; s) ds$  where  $v(x, t; s)$  solves the homogeneous problem

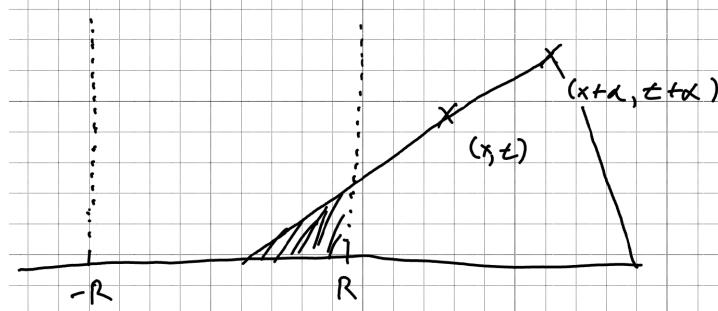
$$\begin{cases} \partial_{tt}v(\cdot, \cdot; s) - \partial_{xx}v(\cdot, \cdot; s) = 0 \\ v(\cdot, s; s) = 0, \partial_t v(\cdot, s; s) = f(x, s) \end{cases}$$

Then

$$\partial_{tt}v - \partial_{xx}v = 0 \iff \partial_t \begin{pmatrix} v \\ \partial_t v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \partial_{xx} & 0 \end{pmatrix} \begin{pmatrix} v \\ \partial_t v \end{pmatrix} \quad \text{and} \quad \begin{bmatrix} v \\ \partial_t v \end{bmatrix}_{t=s} = \begin{bmatrix} * \\ * \end{bmatrix}$$

So  $v(x, t; s) = \frac{1}{2} \int_{x-(t-s)}^{x+(t-s)} f(y, s) dy$  and  $v(x, t) = \frac{1}{2} \int_0^t \int_{x-s}^{x+s} f(y, t-s) dy ds$  follows.

Going back to the original PDE,  $(-\partial_{tt} - P_V)v = F$  is equivalent to  $(\partial_{tt} - \partial_{xx})v = -(Vv + F)$  which leads to the conclusion that  $v(x, t) = -\frac{1}{2} \int_0^t \int_{x-s}^{x+s} (Vv + F)(y, t-s) dy ds$ . For  $|x| > R$ ,  $v$  is outgoing.



## October 2, 2025

Take some complex vector space and consider the Hilbert space  $(\mathcal{H}, (\cdot, \cdot))$  with  $(\cdot, \cdot)$  satisfying

$$\begin{cases} (\lambda f, g) = \lambda(f, g) \\ (f, \lambda g) = \overline{\lambda}(f, g) \\ (f, g) = \overline{(g, f)} \\ f \mapsto (f, f) =: \|f\|^2 \text{ a norm} \\ (\mathcal{H}, \|\cdot\|) \text{ complete with respect to the norm} \end{cases}$$

- Examples

-  $(\mathbb{C}^n, (a, b) = \sum_{j=1}^n a_j \bar{b}_j)$ ,  $a = (a_1, \dots, a_n)$ ,  $b = (b_1, \dots, b_n)$

- $L^2(x, \mu)$  (e.g.  $[0, 1]$  and the Lebesgue measure),  $(f, g) = \int_X f\bar{g} d\mu$ .

Bounded Operators:  $T : \mathcal{H} \rightarrow \mathcal{H}$  bounded if and only if  $\sup_{||x||=1} \overbrace{||T||}^{||Tx||} < \infty$  satisfying

$$\begin{cases} \mathcal{B}(\mathcal{H}) \text{ the space of bounded operators on } \mathcal{H} \text{ (a complex vector space)} \\ ||\cdot|| \text{ is a norm on } \mathcal{B}(\mathcal{H}), \text{ making it complete} \\ \text{There is a multiplication, } \mathcal{B}(\mathcal{H}) \ni A, B \mapsto AB \text{ and } ||AB|| \leq ||A|| ||B|| \end{cases}$$

Adjoint: if  $A \in \mathcal{B}(\mathcal{H})$ ,  $\exists! A^* \in \mathcal{B}(\mathcal{H})$  such that  $\forall f, g \in \mathcal{H}$ ,  $(Af, g) = (f, A^*g)$  where  $A$  is symmetric/self-adjoint if  $A = A^*$ .  
These notions are different in the world of unbounded operators.

- Example

- $\mathcal{H} = \mathbb{C}^n$ :  $T \in M_n(\mathbb{C})$  symmetric if and only if  $T$  is Hermitian.  $t_{ij} = \overline{t_{ji}}$ .
- $\mathcal{H} = L^2([0, 1])$ ,  $Tf(t) = tf(t)$ .  $(Tf, g) = \int_0^1 tf(t)\overline{g(t)} dt = \int_0^1 f(t)\overline{tg(t)} dt = (f, Tg)$ .
- $\mathcal{H} = L^2(\mathbb{R})$  with the Fourier transform.  $||f(x)||^2 = c||\hat{f}(\xi)||^2$  (Parseval's Equality).

## Finite Dimensional Spectral Theorem

If  $A \in M_n(\mathbb{C})$  is Hermitian, there exists an orthonormal basis  $(\phi_1, \dots, \phi_n)$  of  $\mathbb{C}^n$  and real eigenvalues  $\lambda_1, \dots, \lambda_n$  such that  $A\phi_j = \lambda_j\phi_j$ .

Important observation: if  $A$  is Hermitian, then  $\lambda_j$  is real for each  $j$ , and  $\overline{(A\phi_j, \phi_j)} = (\phi_j, A\phi_j) = (A\phi_j, \phi_j) = \lambda_j||\phi_j||^2$ .  
So  $\lambda_j = \frac{(A\phi_j, \phi_j)}{||\phi_j||^2}$  is real. If  $\lambda_j \neq \lambda_k$ , then  $(\phi_j, \phi_k) = 0$  since  $(A\phi_j, \phi_k) - (\phi_j, A\phi_k) = (\lambda_j - \lambda_k)(\phi_j, \phi_k)$ .

Notation: Let  $u, v \in \mathbb{C}^n$ , denote  $u \otimes \bar{v}$  the operator  $(u \otimes \bar{v})w = (w, v)u$ .

With  $A$  as in the theorem, we can write  $A = \sum_{j=1}^n \lambda_j \phi_j \otimes \bar{\phi}_j$  ( $I = \sum_{j=1}^n \phi_j \otimes \bar{\phi}_j$ ). A second way of writing this is

$$A \left[ \underbrace{\phi_1 | \dots | \phi_n}_{U} \right] = \left[ \underbrace{\phi_1 | \dots | \phi_n}_{U} \right] \left[ \underbrace{\lambda_1 \dots \lambda_n}_{\Lambda} \right]$$

Where  $U^* = U^{-1}$  and  $A = U\Lambda U^*$ . This allows us to construct a functional calculus for  $A$  where

$$\begin{cases} A^2 = U\Lambda U^* U\Lambda U^* = U\Lambda^2 U^* \\ A^n = U\Lambda^n U^* \\ p(A) = U \cdot p(\Lambda) \cdot U^*, \text{ } p \text{ a polynomial} \end{cases}$$

Defining  $f(A) := U \cdot f(\Lambda) \cdot U^*$ , we obtain a Banach algebra homomorphism. Then  $f \in C([-||A||, ||A||])$  is also a Banach algebra with sup norm and pointwise multiplication.

$$f(A) := U \cdot \begin{pmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n) \end{pmatrix} \cdot U^*$$

Then we can map  $C([-||A||, ||A||]) \ni f \mapsto f(A) \in \mathcal{B}(\mathcal{H})$ . This is useful for solving ODEs.

- Prototypes

- Heat equation:  $\partial_t u + Au = 0$ ,  $u|_{t=0} = u_0$ ,  $u(t) = e^{-tA}u_0$ .
- Schrödinger equation:  $i\partial_t u + Au = 0$ ,  $u|_{t=0} = u_0$ ,  $u(t) = e^{-itA}u_0$ .
- Wave equation:  $\partial_{tt} u + Au = 0$ ,  $u|_{t=0} = u_0$ ,  $\partial_t u|_{t=0} = u_1$ .

Write  $u(t) := \sum_{j=1}^n u_j(t)\phi_j$  with the PDE  $\sum_{j=1}^n (u_j'' + \lambda_j u_j)\phi_j = 0$ . Then  $u_j'' + \lambda_j u_j = 0$ ,  $u_j(0) = u_{j,0}$ , and  $u_j'(0) = u_{j,1}$ . Suppose  $\lambda_j > 0$  for all  $j$ . Then  $u_j(t) = u_{j,0} \cos(\sqrt{\lambda_j}t) + \frac{u_{j,1}}{\sqrt{\lambda_j}} \sin(\sqrt{\lambda_j}t)$ . So

$$u(t) = \sum_{j=1}^n \cos(\sqrt{\lambda_j}t)u_{j,0}\phi_j + \frac{1}{\sqrt{\lambda_j}} \sin(\sqrt{\lambda_j}t)u_{j,1}\phi_j$$

Therefore  $u = \cos(t\sqrt{A})u_0 + A^{-1/2} \sin(t\sqrt{A})u_1$ .

## Spectrum of a Bounded Operator

Take  $T \in \mathcal{B}(\mathcal{H})$ . We say that  $T$  is invertible (within  $\mathcal{B}(\mathcal{H})$ ) if and only if  $\exists S \in \mathcal{B}(\mathcal{H})$  such that  $TS = ST = I$ .

Counterexample: take  $\mathcal{H} = \ell^2(\mathbb{N}_0) = \{u = (u_n)_{n \geq 0} : \sum |u_n|^2 < \infty\}$  and  $Au = \left(\frac{1}{n}u_n\right)_{n \geq 0}$ . Then the proxy for  $A^{-1}u = (nu_n)_{n \geq 0}$  is not bounded.

Given  $T \in \mathcal{B}(\mathcal{H})$ , the resolvent set of  $T$  is  $\rho(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \text{ is invertible}\}$ . Invertibility is equivalent to  $\forall y \in \mathcal{H}$ ,  $\exists!x$  such that  $Tx - \lambda x = y$  with an estimate  $||x|| \lesssim ||y||$ .

For  $\lambda \in \rho(T)$ , denote  $R(\lambda)$  or  $R_T(\lambda) = (T - \lambda I)^{-1} \in \mathcal{B}(\mathcal{H})$  the resolvent of  $T$ . Properties of the resolvent set:

1.  $\rho(T) \neq \emptyset$  (in fact, if  $|\lambda| > ||T||$  then  $\lambda \in \rho(T)$ ).
2.  $\rho(T)$  is open.
3. the map  $\rho(T) \ni \lambda \mapsto R_T(\lambda) \in \mathcal{B}(\mathcal{H})$  is holomorphic in the sense that  $\forall \lambda_0 \in \rho(T)$ ,  $\exists R_T^l(\lambda_0) \in \mathcal{B}(\mathcal{H})$  such that  $\lim_{\lambda \rightarrow \lambda_0} \left| \left| \frac{R_T(\lambda) - R_T(\lambda_0)}{\lambda - \lambda_0} - R_T^l(\lambda_0) \right| \right| = 0$ .

For a., if  $|\lambda| > ||T||$ ,  $Tx - \lambda x = y \iff (I - \frac{T}{\lambda})x = -\frac{y}{\lambda} \iff x = \sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}}y$ . Then  $R_T(\lambda) = \frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{T}{\lambda}\right)^k$  and

$$||R_T(\lambda)|| \leq \frac{1}{||\lambda||} \frac{1}{1 - ||T/\lambda||} \leq \frac{1}{|\lambda| - ||T||}$$

For b., pick  $\lambda_0 \in \rho(T)$  and find  $r > 0$  such that  $|\lambda - \lambda_0| < r \implies \lambda \in \rho(T)$ . Then  $Tx - \lambda x = y \iff (T - \lambda_0)x - (\lambda - \lambda_0)x = y \iff x - (\lambda - \lambda_0)R_T(\lambda_0)x = R_T(\lambda_0)y$  where if  $||(\lambda - \lambda_0)R_T(\lambda_0)|| < 1$  it is boundedly solvable by Neumann series.

For c.,

$$\begin{aligned} R_T(\lambda) - R_T(\lambda_0) &= (T - \lambda I)^{-1} - (T - \lambda_0 I)^{-1} \\ (T - \lambda I)(R_T(\lambda) - R_T(\lambda_0)) &= I - (T - \lambda_0 I + (\lambda_0 - \lambda)I)(T - \lambda_0 I)^{-1} \\ (T - \lambda I)(R_T(\lambda) - R_T(\lambda_0)) &= I - I + (\lambda - \lambda_0)R_T(\lambda_0) \\ R_T(\lambda) - R_T(\lambda_0) &= (\lambda - \lambda_0)R_T(\lambda)R_T(\lambda_0) \end{aligned}$$

So  $\frac{R_T(\lambda) - R_T(\lambda_0)}{\lambda - \lambda_0} - R_T(\lambda_0)^2 = o(\lambda - \lambda_0)$ .

Then we define the spectrum  $\sigma(T) := \mathbb{C} \setminus \rho(T)$  which is closed since  $\rho(T)$  is open.

## Lemma

If  $T \in \mathcal{B}(\mathcal{H})$  is self-adjoint, then  $\sigma(T) \subseteq [-\|T\|, \|T\|]$ .

- Proof

First we know  $\sigma(T) \subseteq \{|\lambda| \leq \|T\|\}$ . We want to show that it is real, and that if  $\lambda = a+ib$  and  $b \neq 0$  then  $T - (a+ib)I$  is invertible.

$T - (a+ib)I$  is injective.

$$\begin{aligned} \|(T - (a+ib))x\|^2 &= (Tx - (a+ib)x, Tx - (a+ib)x) \\ &= \|Tx\|^2 + (a^2 + b^2)\|x\|^2 - (Tx, (a+ib)x) - ((a+ib)x, Tx) \\ &= \|Tx\|^2 + (a^2 + b^2)\|x\|^2 - (a-ib)(Tx, x) - (a+ib)(x, Tx) \\ &= \|Tx\|^2 + a^2\|x\|^2 - 2a(x, Tx) + b^2\|x\|^2 \geq b^2\|x\|^2 \end{aligned}$$

since  $\|Tx\|^2 + a^2\|x\|^2 - 2a(x, Tx) \geq 0$  by Cauchy-Schwarz. Therefore  $T - (a+ib)$  is injective and, by the open mapping theorem,  $(T - (a+ib))^* = T - (a-ib)$  is surjective. Similarly for  $T - (a-ib)$ , and the norm estimate is  $\|(T - (a+ib))^{-1}\| \leq \frac{1}{b}$ . Note that  $\frac{1}{b} = \frac{1}{\text{dist}(a+ib, \mathbb{R})}$ .

Note that the spectrum of  $T$  may no longer be made of eigenvalues in the non-finite case. There may exist  $\lambda$  such that  $T - \lambda I$  is not injective,  $\exists v \neq 0$   $Tv = \lambda v$ . Recall the example  $Tf(t) = tf(t)$  with  $f \in L^2((\cdot, \cdot), dt)$ .  $T$  is self-adjoint,  $\|T\| \leq 1$ , and  $(Tf, f) = \int_0^1 t|f(t)|^2 dt \geq 0$ . So  $\sigma(T) \subseteq [0, 1]$ . For  $\lambda \in [0, 1]$  is  $T - \lambda I$  injective?  $Tf = \lambda f \iff tf(t) = \lambda f(t) \iff (t-\lambda)f(t) = 0$  which implies  $f \equiv 0$  in  $L^2([0, 1])$ . Is  $T - \lambda I$  surjective?  $(t-\lambda)f(t) = g(t) \iff f(t) = \frac{g(t)}{t-\lambda}$ , so  $g(t) \equiv 1 \in L^2([0, 1])$  which implies  $f(t) = \frac{1}{t-\lambda}$  is not  $L^2([0, 1])$  and  $\sigma(T) = [0, 1]$ .

## October 9, 2025

### Spectral Resolution

Take  $\mathcal{H}$  a Hilbert space, and say that  $P \in \mathcal{B}(\mathcal{H})$  is an orthogonal projection if  $P^2 = P$  and  $P^* = P$ . Then let  $\mathcal{P}(\mathcal{H}) = \{P \in \mathcal{B}(\mathcal{H}), P \text{ is an orthogonal projection}\}$ .

- Examples

- $\phi \in \mathcal{H}$ ,  $\|\phi\| = 1$ ,  $P := \phi \otimes \bar{\phi}$ . Then  $P\psi = (\psi, \phi)\phi$  and  $P^2\psi = P(\psi, \phi)\phi = (\psi, \phi)(\phi, \phi)\phi = P\psi$ .
- $\phi_1, \dots, \phi_n$  an orthonormal family with  $P = \sum_{k=1}^n \phi_k \otimes \bar{\phi}_k$ .
- $\mathcal{H} = \ell^2(\mathbb{N})$ ,  $e_j = (0, 0, \dots, 0, 1, 0, \dots)$  and  $I = \sum_{j=1}^{\infty} e_j \otimes \bar{e}_j$ .
- $\mathcal{H} = L^2(\mathbb{R})$ . Fix  $I$  an interval with  $\chi_I$  the characteristic function for  $I$ . Then take  $Pf := \chi_I f$ .
  - \*  $PPf = \chi_I \chi_I f = \chi_I f = Pf$ .
  - \*  $\int_{\mathbb{R}} \chi_I f \bar{g} dx = \int_{\mathbb{R}} f \bar{\chi_I g} dx$ .
  - \* If  $I$  has a nonempty interior, then  $\text{Range}(P) = \{f \in L^2(\mathbb{R}), \text{supp } f \subset I\} \simeq L^2(I)$ .

## Definition: Spectral Resolution

A spectral resolution is a map  $\mathbb{R} \ni \lambda \mapsto E(\lambda) \in \mathcal{P}(\mathcal{H})$  satisfying

1.  $\forall f \in \mathcal{H}, ||E(\lambda)f||$  is increasing.
2.  $\exists [a, b]$  such that  $E(\lambda) = 0$  if  $\lambda < a$  and  $E(\lambda) = \text{Id}$  if  $\lambda \geq b$ .
3.  $E(\lambda)$  is right continuous. That is,  $\forall f \in \mathcal{H}, \lambda \in \mathbb{R}$ ,

$$\lim_{\substack{\mu \rightarrow \lambda \\ \mu > \lambda}} ||E(\mu)f - E(\lambda)f|| = 0$$

Alternatively, we can require  $E(\lambda)E(\mu) = E(\min\{\mu, \lambda\})$ .

Long story short: the collection of self-adjoint bounded operators is in one-to-one correspondence with the collection of spectral resolutions.

- Examples

- $A \in M_n(\mathbb{C}), A^* = A$ , with eigencouples  $(\lambda_1, \phi_1), \dots, (\lambda_n, \phi_n)$  and simple spectrum  $\lambda_1 < \lambda_2 < \dots < \lambda_n$ . Define  $E(\lambda) := \sum_{j: \lambda_j \leq \lambda} \phi_j \otimes \bar{\phi}_j$ .
  - \*  $A = I$  gives  $E(\lambda) = 0$  for  $\lambda < 1$  and  $E(\lambda) = \text{Id}$  for  $\lambda \geq 1$ .
  - \*  $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$  gives  $E(\lambda) = \text{Id}_{\lambda \geq 1} e_1 \otimes e_1 + \text{Id}_{\lambda \geq 2} e_2 \otimes e_2$ .
    - If  $f = f_1 e_1 + f_2 e_2$ , then  $||E(\lambda)f||^2 = \text{Id}_{\lambda \geq 1}(\lambda) ||f_1||^2 + \text{Id}_{\lambda \geq 2}(\lambda) ||f_2||^2$ .

## Spectral Measures

A spectral resolution gives rise to spectral measures

$$f, g \in \mathcal{H} \quad \lambda \mapsto (E(\lambda)f, g) = F(\lambda) \in \mathbb{C}$$

This defines a Lebesgue-Stieltjes measure

$$\mu_F : \mu_F((a, b]) = F(b) - F(a), \quad \forall a, b \in \mathbb{R}$$

We can construct this as follows:

- When  $f = g$ ,  $\lambda \mapsto (E(\lambda)f, f) = (E(\lambda)^2 f, f) = ||E(\lambda)f||^2$  (increasing).
- When  $f \neq g$ ,

$$(E(\lambda)f, g) = (E\lambda(f), E\lambda(g)) = \frac{1}{4} (||E(\lambda)(f+g)||^2 - ||E(\lambda)(f-g)||^2 - i||E(\lambda)(f+ig)||^2 + i||E(\lambda)(f-ig)||^2)$$

$\{E(\lambda)\}_{\lambda \in \mathbb{R}}$  defines a projection-valued measure  $E_\Omega$ ,  $\Omega \subset \mathbb{R}$  a Borel set. Start with  $E_{(a, b]} = E(b) - E(a)$ . We would like for  $E_\Omega$  to satisfy  $E_{\Omega_1} E_{\Omega_2} = E_{\Omega_1 \cap \Omega_2}$ ,  $E_\emptyset = 0$ ,  $E_{\mathbb{R}} = \text{Id}$ .

## Theorem: Spectral Theorem

For  $A \in \mathcal{B}(\mathcal{H})$  self-adjoint, there exist  $a, b \in \mathbb{R}$  and an  $A$ -dependent spectral resolution  $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$  such that

$$A = \int_{a_-}^b \lambda dE(\lambda)$$

in the sense that  $(Af, g) = \int_{a_-}^b \lambda d(E(\lambda)f, g)$  for all  $f, g \in \mathcal{H}$ . This is amenable to creating a functional calculus

$C([-||A||, ||A||]) \rightarrow$  bounded self-adjoint operators that commute with  $A$

$$h \mapsto h(A) := \int_{a_-}^b h(\lambda) dE(\lambda)$$

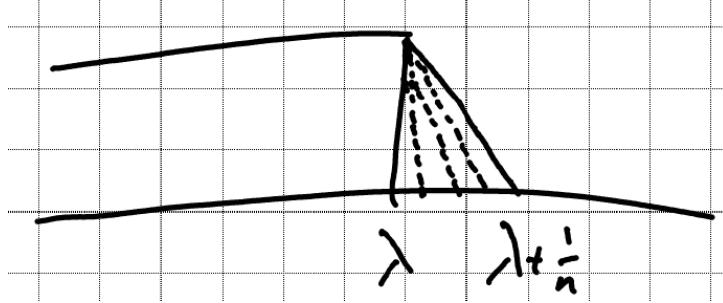
The idea is that  $E(\lambda) = \chi_{(-\infty, \lambda]}(A)$ . Once  $E$  is constructed, this leads to  $E_\Omega$  for  $\Omega$  Borel. We say that a measure  $\mu$  is supported in  $G$  (Borel) if for every  $\Omega$  Borel,  $\mu(\Omega) = \mu(\Omega \cap G)$ . Then  $\text{supp } E \subset \sigma(A)$ .

## Functional Calculus

We want to make sense of  $h(A)$  for  $h$  in a large enough class. If  $p$  is a polynomial, we can make sense of  $p(A) = \sum_{k=0}^n a_k A^k$  which is self-adjoint and bounded.

For  $h \in C(-||A||, ||A||)$ ,  $h$  is uniformly approximated by polynomials. We want to show that  $p_n$  is uniformly Cauchy which implies that  $p_n(A)$  converges to some  $h(A)$ .

For  $h = \chi_{(-\infty, \lambda]}$ , we proceed by approximation by tent functions.



## Definition: Positive Operator

If  $S$  is a self-adjoint, bounded operator on  $\mathcal{H}$ , we say that  $S$  is positive ( $S \geq 0$ ) if  $(Sf, f) \geq 0$ ,  $\forall f \in \mathcal{H}$ . For  $S_1, S_2$  self-adjoint and bounded, we say that  $S_1 \geq S_2$  if and only if  $S_1 - S_2 \geq 0$ .

For  $T \in \mathcal{B}(\mathcal{H})$ , self-adjoint, set  $a := \inf_{\|f\|=1} (Tf, f)$  and  $b := \sup_{\|f\|=1} (Tf, f)$ . Then  $a \text{Id} \leq T \leq b \text{Id}$ .

$$((T - a \text{Id})f, f) = (Tf, f) - a(f, f) = (f, f) \left( \left( T \frac{f}{\|f\|}, \frac{f}{\|f\|} \right) - a \right) \geq 0$$

We want to show that if  $p$  is a polynomial on  $[-||A||, ||A||]$ , then  $(\inf_{[-||A||, ||A||]} p) \text{Id} \leq p(A) \leq (\sup_{[-||A||, ||A||]} p) \text{Id}$ .

## Lemma

If  $T_1$  and  $T_2$  are positive and commute, then  $T_1 T_2 \geq 0$ .

## Square Root Lemma

If  $A \geq 0$  (i.e. bounded, self-adjoint, and positive), then  $\exists! B \geq 0$  such that  $B^2 = A$  and  $B$  commutes with any operator that commutes with  $A$ .

- Proof

Use the power series of  $z \mapsto \sqrt{1-z}$  at  $z=0$ .

$$1 + \sum_{k=1}^{\infty} c_k z^k$$

We can find that  $c_k < 0$  for all  $k \geq 1$  and that the series converges uniformly on  $\{|z| \leq 1\}$ .

Now let  $A \geq 0$  which implies that  $0 \text{Id} \leq I - A \leq 1 \text{Id}$ . Without loss of generality, suppose  $\text{supp } ||A|| \leq 1$ . The idea is to write

$$B = \sqrt{A} = \sqrt{I - (I - A)} = I + \sum_{k=1}^{\infty} c_k (I - A)^k$$

which converges strongly because the series converges uniformly. Then  $B^2 = A$ . We see that  $B \geq 0$  using the fact that  $\text{sign}(c_k) < 0$  which implies  $\sum_{k \geq 1} c_k \geq -1$ . The proof of uniqueness can be found in the text.

## Proof of Lemma

Assuming the square root lemma, write  $T_2 = B^2$ . Then since  $[T_1, T_2] = 0$ ,  $[B, T_1] = 0$ . Then

$$(T_1 T_2 f, f) = (T_1 B^2 f, f) = (B T_1 B f, f) = (T_1 (B f), B f) \geq 0$$

## Weaker Version

Instead of  $(\inf_{[-||A||, ||A||]} p) \text{Id} \leq p(A) \leq (\sup_{[-||A||, ||A||]} p) \text{Id}$ , we have that if  $\min_{[-||A||, ||A||]} p \geq 0$ , then  $p(A) \geq 0$ .

## Proof

If  $p \geq 0$  on  $[-||A||, ||A||]$ , we can factor it as a product of positive pieces

$$\begin{aligned} p(x) &= \prod_{j=1}^{r_j < -||A||} \underbrace{(x - r_j)}_{\geq 0} \underbrace{(s_j - x)}_{\geq 0} ((x - a_j)^2 + b_j^2) \\ p(A) &= \prod_{j=1}^{r_j < -||A||} \underbrace{(A - r_j)}_{\geq 0} \underbrace{(s_j - A)}_{\geq 0} \underbrace{((A - a_j)^2 + b_j^2)}_{\geq 0} \end{aligned}$$

Using the previous lemma, we have that  $P(A) \geq 0$ .

## Proof

Finally, to show that  $(\inf_{[-||A||, ||A||]} p) \text{Id} \leq p(A) \leq (\sup_{[-||A||, ||A||]} p) \text{Id}$ , we see that  $p - \inf p$  and  $\text{supp } f - p$  are positive polynomials and apply the weaker version to them.

## Definition

We can define  $h(A)$  for  $h \in C(-||A||, ||A||)$  by Weierstrass approximation. There exist  $p_n$  polynomials such that  $\sup_{[-||A||, ||A||]} |p_n - h| \xrightarrow{n \rightarrow \infty} 0$ . Then  $p_n$  is uniformly Cauchy, so

$$\inf(p_n - p_m) \text{Id} \leq p_n(A) - p_m(A) \leq \sup(p_n - p_m) \text{Id}$$

which implies that

$$||p_n(A) - p_{n-1}(A)|| \leq \max(\sup(p_n - p_m), -\inf(p_n - p_m)) \xrightarrow{n, m \rightarrow \infty} 0.$$

So  $p_n(A)$  is Cauchy in  $(\mathcal{B}(\mathcal{H}), || \cdot ||)$  which means it converges. We call  $h(A) = \lim_{n \rightarrow \infty} p_n(A)$ . We still want to show that  $h(A)$  is bounded and self-adjoint.

**October 14, 2025**

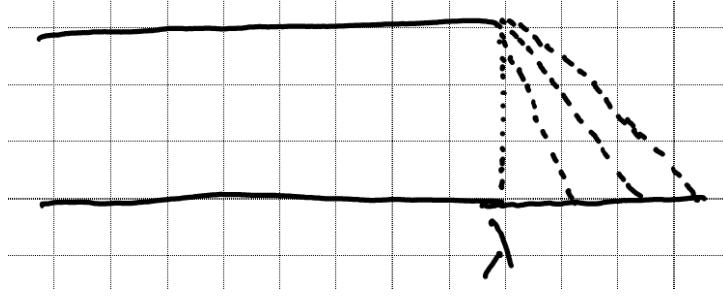
## Spectral Theorem for Bounded Self-Adjoint Operators

If  $A \in \mathcal{B}(\mathcal{H})$  is self-adjoint such that  $a||f||^2 \leq (Af, f) \leq b||f||^2 \forall f \in \mathcal{H}$ , then there exists a spectral resolution  $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$  such that  $A = \int_{a^-}^b \lambda dE(\lambda)$ .

### Proof (Continued)

We would like that  $E(\lambda) = \phi^\lambda(A)$  where  $\phi^\lambda := \chi_{(-\infty, \lambda]}$ , however  $\phi^\lambda$  is not continuous so instead we can approximate it as

$$\phi_n^\lambda := \begin{cases} 1, & x \leq \lambda \\ \text{linear on } [\lambda, \lambda + \frac{1}{n}] \\ 0, & x \geq \lambda + \frac{1}{n} \end{cases}$$



To demonstrate this, we need the following proposition.

### Proposition

If  $T_n$  is a sequence of positive operators and  $T_n \geq T_{n+1} \geq 0$ , then there exists some  $T \geq 0$  such that  $T_n f \rightarrow T f, \forall f \in \mathcal{H}$ .

### Proof

Fix  $f \in \mathcal{H}$ , and consider  $(T_n f, f)$  which is decreasing, bounded from below, and therefore converges and is Cauchy. Now as an estimate, we can say that if  $S \in \mathcal{B}(\mathcal{H})$  is self-adjoint where  $0 \leq S \leq MI$ , then  $\forall f \in \mathcal{H}$  we know that

$$||Sf||^2 \leq (Sf, f)^{1/2} M^{3/2} ||f||.$$

To see this, we look at  $t \mapsto (S(S+tI)f, (S+tI)f) \geq 0$  since  $S \geq 0$ . Then

$$(S(S+tI)f, (S+tI)f) = (S^2f, Sf) + 2t||Sf||^2 + t^2(Sf, f)$$

So  $\Delta < 0$  if and only if

$$\begin{aligned} ||Sf||^4 - (S^2f, Sf)(Sf, f) &< 0 \\ ||Sf||^4 &\leq (Sf, f)(S^2f, Sf) \\ &\leq (Sf, f) \underbrace{||S^2f|| ||Sf||}_{M^3 ||f||^2} \end{aligned}$$

We apply this to  $T_n - T_m = S$  where  $T_n - T_m \leq T_0$  for all  $n \leq m$ . Then

$$||(T_n - T_m)f||^2 \leq ((T_n - T_m)f, f)^{1/2} ||f|| ||T_0||^{3/2}$$

Since  $(T_n f, f)$  is Cauchy,  $T_n f$  is Cauchy and therefore converges. It remains to check that  $T$  is linear, positive, satisfies,  $T \leq T_0$ , etc.

### Spectral Theorem Proof Continued

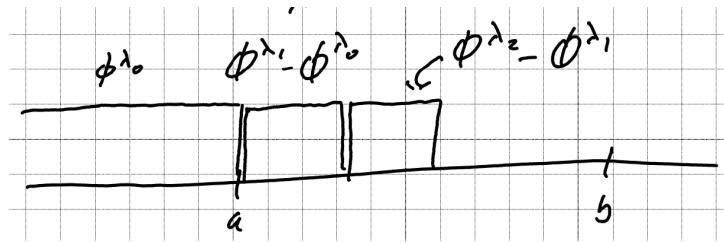
For each  $n$ ,  $\phi_n^\lambda(A)$  makes sense and is positive since  $\phi_n^\lambda(t) \geq 0$ . Since  $\phi_n^\lambda \geq \phi_{n+1}^\lambda$ ,  $\phi_n^\lambda(A) \geq \phi_{n+1}^\lambda(A)$ . Thus, by the preceding proposition,  $\phi^\lambda(A) := \lim_{n \rightarrow \infty} \phi_n^\lambda(A)$  exists as a bounded, self-adjoint, positive operator.

It remains to check the spectral resolution properties.

The final property to check is that  $\forall f \in \mathcal{H}, (Af, f) = \int_a^b \lambda d(E(\lambda)f, f)$ . The idea is to approximate multiplication by  $t$  with piecewise constant functions. We fix a partition  $a = \lambda_0 \leq \dots \leq \lambda_k = b$  such that  $\sup(\lambda_{j+1} - \lambda_j) < \delta$ . Then

$$t\phi^{\lambda_k}(t) = t = \phi^{\lambda_0}(t) + \sum_{j=1}^k t(\phi^{\lambda_j}(t) - \phi^{\lambda_{j-1}}(t))$$

for all  $a \leq t \leq b$ .

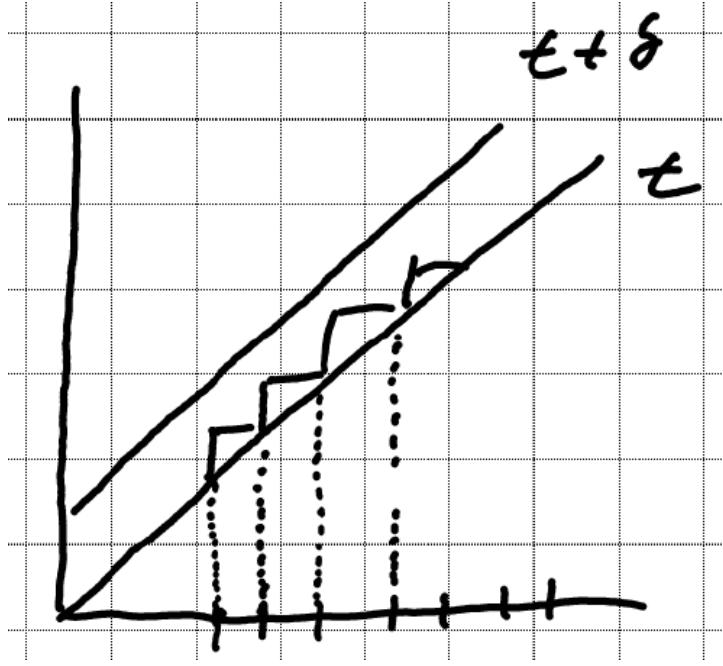


Then

$$\begin{aligned} \lambda^{j-1} &\leq t \leq \lambda^j \\ \lambda^{j-1}(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) &\leq t(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \leq \lambda^j(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \\ \sum_{j=1}^k \lambda^{j-1}(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) &\leq t \sum_{j=1}^k (\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \leq \sum_{j=1}^k \lambda^j(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \\ &\leq \sum_{j=1}^k (\lambda^j - \delta)(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \end{aligned}$$

Adding  $\lambda_0 \phi^{\lambda_0}$ , we see that

$$t \leq \lambda_0 \phi^{\lambda_0} + \sum_{j=1}^k \lambda^j (\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \leq t + \delta$$



When we apply this to  $A$ ,

$$A \leq \lambda_0 E(\lambda_0) + \sum_{j=1}^k \lambda^j (E(\lambda_j) - E(\lambda_{j-1})) \leq A + \delta I$$

Finally, we see that

As one refines the partition,  $\delta \rightarrow 0$  and  $(A, f, f) = \int_{a^-}^b \lambda d(E(\lambda)f, f)$ .

Then if  $\phi \in C([-||A||, ||A||])$ ,  $\phi(A) = \int_{a^-}^b \phi(\lambda) dE(\lambda)$ .

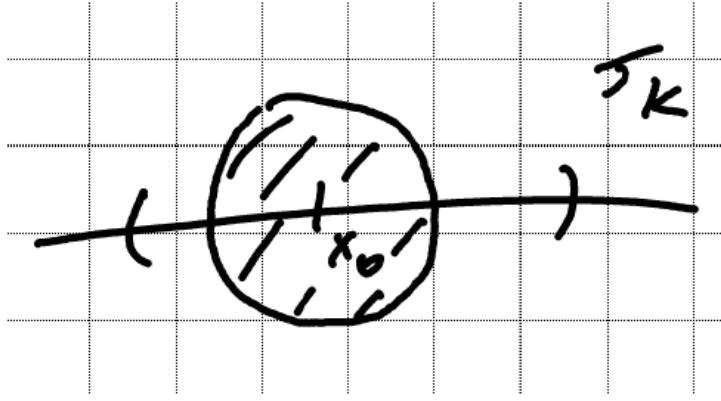
$$\left| (Af, f) - \lambda_0 (E(\lambda_0)f, f) - \sum_{j=1}^k \lambda^j (E(\lambda_j)f, f) - (E(\lambda_{j-1}f, f)) \right| \leq \delta ||I||^2$$

## Functional Calculus

We observe that for  $g \in C_C^\infty(\mathbb{C})$ ,  $g(w) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{w-z} \partial_{\bar{z}} g(z) d^2 z$ . Then  $f(A) := \frac{i}{\pi} \int_{\mathbb{C}} (A - z)^{-1} \partial_{\bar{z}} \tilde{f}(z) d^2 z$  where  $\partial_{\bar{z}} \tilde{f} = O((\text{Im}(z))^n)$  for  $n \geq 3$  as  $\text{Im}(z) \rightarrow 0$ .

## Proposition

For every  $f \in \mathcal{H}$ , the Lebesgue-Stieltjes measure corresponding to  $F(\lambda) = (E(\lambda f, f))$  is supported on  $\sigma(A)$ . Since  $\sigma(A) \subseteq [a, b]$  is closed,  $[a, b] \setminus \sigma(A)$  is open (i.e.  $\bigcup_{k \in \mathbb{N}} J_k$  for open intervals  $J_k$ ). We want to show that  $F(\lambda)$  is constant on each  $J_k$  (equivalently:  $\int_{J_k} dF(\lambda) = 0$ ).



Fix  $J_k \ni x_0$ . Then  $R_A(x_0) = (A - x_0 I)^{-1}$  exists, and we can pick  $\varepsilon > 0$  such that  $\forall z \in \overline{B_\varepsilon(x_0)}$ ,  $\|R_A(z)\| \leq M$ . Then for  $z \in B_\varepsilon(x_0) + \text{Im}(z) \neq 0$ ,

$$R_A(z) = (A - zI)^{-1} = \phi_z(A)$$

and  $\phi_z(t) = \frac{1}{t-z} \in C([a, b])$ . Consider

$$R_A(z)R_A(\bar{z}) = \psi_z(A) = \int \frac{1}{|\lambda - z|^2} dE(\lambda)$$

with  $\psi_z(t) = \frac{1}{|t-z|^2}$ . It follows that for all  $z \in \overline{B_\varepsilon(x_0)} \setminus \mathbb{R}$ ,

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{1}{|\lambda - z|^2} dF(\lambda) \leq \int \frac{1}{|\lambda - z|^2} dF(\lambda) = (R_A(z)R_A(\bar{z})f, f) \leq M^2 \|f\|^2$$

which stays true for all  $z \in \overline{B_\varepsilon(x_0)}$ . In particular for  $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ ,

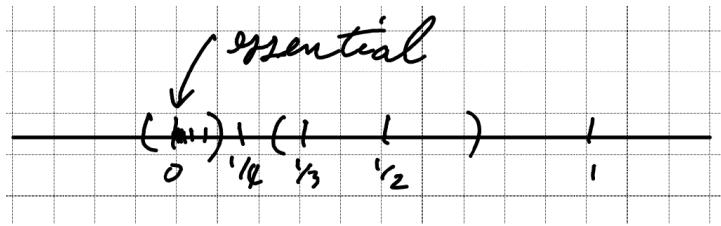
$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} dx \int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{1}{|\lambda - z|^2} dF(\lambda) \leq 2\varepsilon M^2 \|f\|^2$$

Since Fubini holds, we observe that  $\int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{1}{|\lambda-x|^2} dx = \infty$ , so it must be the case that  $\int_{x_0-\varepsilon}^{x_0+\varepsilon} dF(\lambda) = 0$ .

## Discrete Spectrum vs Essential Spectrum

Recall the spectral measure of  $A$ ,  $E_\Omega(A)$  for  $\Omega \subset \mathbb{R}$  Borel where  $E_{(a,b]} = E(b) - E(a)$ . We say that  $\lambda \in \sigma_d(A)$  (the discrete spectrum of  $A$ ) if there exists  $\varepsilon > 0$  such that  $\dim(\text{range}(E_{(\lambda-\varepsilon, \lambda+\varepsilon)})) < \infty$ . Likewise,  $\lambda \in \sigma_{\text{ess}}(A)$  (the essential spectrum) if  $\forall \varepsilon > 0$ ,  $\dim(\text{range}(E_{(\lambda-\varepsilon, \lambda+\varepsilon)})) = \infty$ .

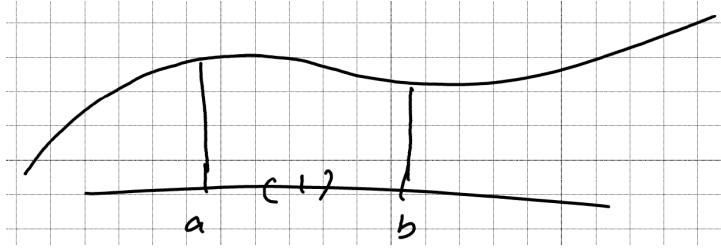
As an example, take  $\mathcal{H} = \ell^2(\mathbb{N})$  with  $(Au)_n = \frac{u_n}{n}$  and  $A = \sum_{j=1}^{\infty} \frac{1}{j} e_j \otimes e_j$ .



Discrete spectra include eigenvalues of finite multiplicity.

Essential spectra include accumulation points of eigenvalues, eigenvalues of infinite multiplicity, absolutely continuous spectrum, s.c. spectrum.

Another example if  $Af(t) = tf(t)$  on  $L^2([0, 1])$ . Then  $E(\lambda)f(t) = \chi_{(-\infty, \lambda]}f(t)$ , and  $E_{(a,b]}f(t) = \chi_{(a,b]}f(t)$ .  $\forall x_0 \in [0, 1]$ , we have that  $\text{range}(E_{(x_0-\varepsilon, x_0+\varepsilon)}) = L^2((x_0 - \varepsilon, x_0 + \varepsilon))$ .



**October 16, 2025**

## Compact Operators and Analytic Fredholm Theorem

### Definition: Spectral Radius

For  $A \in \mathcal{B}(\mathcal{H})$ , we say that the spectral radius of  $A$  is  $r(A) = \sup_{\lambda \in \sigma(A)} |\lambda| < \infty$

### Theorem

1. if  $A \in \mathcal{B}(\mathcal{H})$ , then  $r(A) = \limsup_{n \rightarrow \infty} \|A^n\|^{1/n}$ .

2. If  $A$  is, in addition, self-adjoint, then  $r(A) = \|A\|$ .

As a non-example, consider  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ . Then  $\sigma(A) = \{0\}$ , but  $r(A) = 0 \neq \|A\|$ .

### Proof

Recall Hadamard's Formula:  $\sum_{k=0}^{\infty} a_k z^k$  has radius of convergence  $R$  computed by  $\frac{1}{R} = \limsup_{n \rightarrow \infty} |a_n|^{1/n}$ . This holds even when  $a_k$  are members of a Banach algebra (e.g.  $\mathcal{B}(\mathcal{H})$ ).

Set  $z = \frac{1}{\lambda}$ , such that  $0 < |z| < \frac{1}{r(A)}$  and implies the existence of

$$R_A\left(\frac{1}{z}\right) = \left(A - \frac{1}{z}I\right)^{-1} = -(I - zA)^{-1} = -z \sum_{k=0}^{\infty} A^k z^k.$$

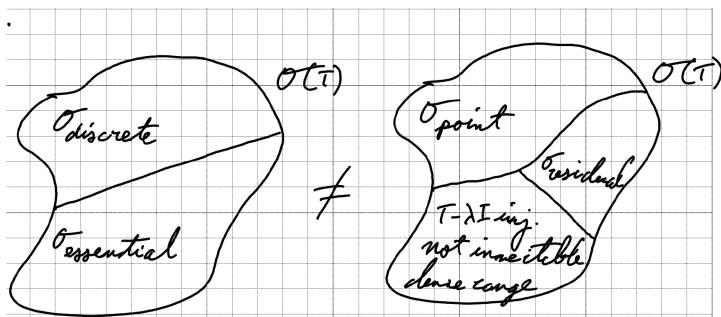
So we have that  $r(A) = \frac{1}{R} = \limsup_{k \rightarrow \infty} \|A^k\|^{1/k}$ .

Now if  $A$  is self-adjoint,  $\|A^2\| = \|A\|^2$  since  $\|A^2\| \leq \|A\|^2$  by submultiplicativity and  $\|A^2\| \geq \sup_{\|x\|=1} (x, A^2 x) = \sup_{\|x\|=1} \|Ax\|^2 = \|A\|^2$  using Cauchy-Schwarz.

By induction,  $\|A^{2^k}\|^{1/2^k} = \|A\|$  which implies that  $r(A) = \|A\|$ .

### Another Spectral Decomposition

For  $T : \mathcal{H} \rightarrow \mathcal{H}$  bounded, we say that  $\lambda$  is in the point spectrum of  $T$  if  $T - \lambda I$  is not-injective. We say that  $\lambda$  is in the residual spectrum of  $T$  if  $T - \lambda I$  is injective but does not have dense range.



Self-adjoint operators have no residual spectrum (RS, Thm VI.8).

## Definition: Compact Operators

$K \in \mathcal{B}(\mathcal{H})$  is compact if  $K$  maps bounded sequences to sequences with a limit point. Equivalently, if  $B_{\mathcal{H}} = \{x \in H : \|x\| \leq 1\}$  then  $K$  is compact if  $K(B_{\mathcal{H}})$  has compact closure (i.e. is precompact).

$\mathcal{K}(\mathcal{H})$ , the collection of compact operators on  $\mathcal{H}$ , is a closed linear subspace of  $\mathcal{B}(\mathcal{H})$  since if  $T, S \in \mathcal{K}(\mathcal{H})$ , then  $T + S \in \mathcal{K}(\mathcal{H})$ . We have also that  $\mathcal{K}(\mathcal{H})$  is a 2-sided ideal of  $\mathcal{B}(\mathcal{H})$  since when  $T$  is compact and  $S$  is bounded,  $ST$  and  $TS$  are compact.

Examples

- For a finite-rank operator  $A : \mathcal{H} \rightarrow \mathcal{H}$ ,  $\text{range}(A) < \infty$ . The general form of  $A$  is  $A = \sum_{j=1}^n \phi_j \otimes \bar{\psi}_j$  for  $\phi_j, \psi_j \in \mathcal{H}$ .
- Strong limits of finite-rank operators.
- $(Au)_n = \frac{1}{n} u_n$  on  $\ell^2(\mathbb{N})$ . Note that  $A_n = \sum_{j=1}^n \frac{1}{j} e_j \otimes e_j$  shows that  $\|A - A_n\| \leq \frac{1}{n+1}$ .
- The inclusion  $h^1 \hookrightarrow \ell^2$  where  $h^1 = \{u \in \mathbb{C}^{\mathbb{N}} : \sum_{j=1}^{\infty} j^2 |u_j|^2 < \infty\}$ .

## Proposition

If  $\mathcal{H}$  is separable, all compact operators arise as limits of finite-rank operators.

### Proof

We want to show that for  $A \in \mathcal{K}(\mathcal{H})$ ,  $\forall \varepsilon > 0$ ,  $\exists A_\varepsilon$  finite-rank such that  $\|A - A_\varepsilon\| < \varepsilon$ .

Let  $\varepsilon > 0$  be given. Since  $A(B_{\mathcal{H}})$  is precompact, it is totally bounded. That is  $\exists y_1, \dots, y_n \in A(B_{\mathcal{H}})$  such that  $\forall x \in A(B_{\mathcal{H}})$ ,  $\min_{1 \leq j \leq n} \|x - y_j\| < \varepsilon$ .

Let  $P_\varepsilon$  be the orthogonal projection onto  $\text{span}\{y_1, \dots, y_n\}$ , and set  $A_\varepsilon = P_\varepsilon A$ . Then for  $f \in B_{\mathcal{H}}$  and  $x = Af \in A(B_{\mathcal{H}})$ ,

$$\|Af - A_\varepsilon f\| = \|x - P_\varepsilon x\| \leq \min_{1 \leq j \leq n} \|x - y_j\| < \varepsilon$$

So  $\|A - A_\varepsilon\| < \varepsilon$ .

Exercise: confirm whether this argument needs separability.

## Theorem: Analytic Fredholm Theorem

Let  $D \subset \mathbb{C}$  be open and connected. Let  $f : D \rightarrow \mathcal{B}(\mathcal{H})$  be an analytic, operator-valued function such that  $f(z) \in \mathcal{K}(\mathcal{H})$  for all  $z \in D$ . Then

1. either  $(I - f(z))^{-1}$  exists for no  $z \in D$
2. or  $(I - f(z))^{-1}$  exists for all  $z \in D \setminus S$ , where  $S$  is a discrete set (finite-rank, no accumulation points) in  $D$ .

Then  $(I - f(z))^{-1}$  is meromorphic in  $D$ , analytic in  $D \setminus S$ , residues at poles are finite-rank, and if  $z \in S$ ,  $f(z)\psi = \psi$  has a nonzero solution for  $\psi \in \mathcal{H}$ .

## Application

If  $K$  is compact, consider  $f(z) = zK$ . At  $z = 0$ ,  $I - f(z) = I$  so this is invertible which implies that the theorem holds for  $\frac{1}{z} \in \mathbb{C}$  (taking  $D = \mathbb{C} \setminus \{0\}$ ). We have  $R_K(\lambda) = -\frac{1}{\lambda} (I - f(\frac{1}{\lambda}))^{-1}$ . Note that  $K$  is not necessarily self-adjoint. Note that  $K$

is not necessarily self-adjoint.

### Proof

We want to prove that either (a) or (b) hold locally for any  $z_0 \in D$ .

Fix  $z_0 \in D$ ,  $r > 0$  such that  $\|f(z) - f(z_0)\| < \frac{1}{2}$  for  $z \in D_r(z_0)$ . Choose  $F = \sum_{j=1}^n \phi_j \otimes \bar{\psi}_j$  a finite-rank operator such that  $\|f(z_0) - F\| < \frac{1}{2}$ . Then  $\forall z \in D_r(z_0)$ ,

$$\|f(z) - F\| \leq \|f(z) - f(z_0)\| + \|f(z_0) - F\| < 1$$

which implies that  $(I - (f(z) - F))^{-1}$  exists and is holomorphic on  $D_r(z_0)$ .

Define  $g(z) = F(I - f(z) + F)^{-1}$ , and observe that  $(I - f(z)) = (I - g(z))(I - f(z) + F) = I - f(z) + F - \overbrace{g(z)(I - f(z) + F)}^F$ . Write

$$g(z) = \sum_{j=1}^n \phi_j \otimes \bar{\psi}_j \cdot (I - f(z) + F)^{-1} = \sum_{j=1}^n \phi_j \otimes \overline{\psi_j(z)}$$

where  $\psi_j(z) = ((I - f(z) + F)^{-1})^* \psi_j$  is holomorphic in  $z$ . Then  $I - f(z)$  is invertible if and only if  $I - g(z)$  is invertible. We claim that this holds if and only if  $d(z) \neq 0$  for some holomorphic function  $d$ .

When is  $I - g(z)$  invertible?

Injectivity: if  $g(z)\phi = \phi$ , we expect  $\phi = \sum_{j=1}^n \beta_j \phi_j$ . So  $g(z)\phi = \phi$  if and only if

$$\sum_{j=1}^n \phi_j \left( \sum_{k=1}^n \beta_k \phi_k, \psi_j \right) = \sum_{j=1}^n \beta_j \phi_j$$

where  $\beta_j = \sum_{k=1}^n \beta_k (\phi_k, \psi_j)$ . If  $A_{jk}(z) := (\phi_k, \psi_j(z))$ , then this has a solution if and only if  $\det(I - A(z)) = 0$ . Call  $d := \det(I - A(z))$ . Moreover, if  $d(z) \neq 0$  then  $I - g(z)$  is invertible. We can solve  $(I - g(z))\phi = \psi$  for  $\phi$  given  $\psi \in \mathcal{H}$ . So  $\phi = \psi + g(z)\phi$  which motivates an ansatz  $\phi = \psi + \sum_{j=1}^n \beta_j \phi_j$ . Then

$$(I - g(z))\phi = (I - g(z))\psi + \sum_{j=1}^n (\beta_j - A_{jk}\beta_k) \phi_j = \psi$$

If and only if

$$\sum_{j=1}^n (\beta_j - A_{jk}\beta_k) \phi_j = \sum_{j=1}^n (\psi, \psi_j(z)) \phi_j$$

which is boundedly invertible as long as  $d(z) \neq 0$ .

## October 21, 2025

### Definition: Unbounded Operator

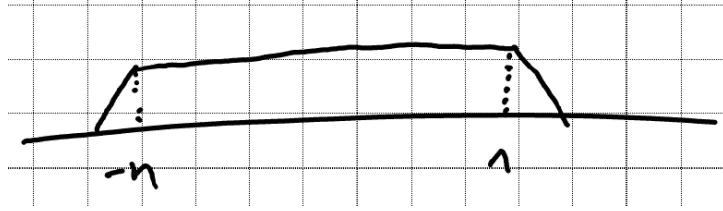
Let  $(\mathcal{H}, (\cdot, \cdot))$  a Hilbert space. An unbounded operator is a linear map  $A : \mathcal{D}(A) \rightarrow \mathcal{H}$ ,  $\mathcal{D}(A) \subsetneq \mathcal{H}$  ( $\mathcal{D}(A)$  the domain of  $A$ ).

Hypothesis:  $\overline{\mathcal{D}(A)} = \mathcal{H}$ .

Why “unbounded?” BEcause, generally,  $\sup_{f \in \mathcal{D}(A) \setminus \{0\}} \frac{\|Af\|}{\|f\|} = \infty$ .

Examples

1.  $\mathcal{H} = L^2(\mathbb{R})$ ,  $Af(t) = tf(t)$ . Then  $A\left(\frac{1}{|t|^2}\right) \notin L^2(\mathbb{R})$ , and  $\mathcal{D}(A) = C_C(\mathbb{R})$ . Consider  $f_n(t)$  like



1.  $\mathcal{H} = L^2(\mathbb{R})$ ,  $Af(x) = D_x f(x) = \frac{1}{i} \frac{\partial}{\partial x} f(x)$ . Then  $\mathcal{D}(A) = \mathcal{S}(\mathbb{R})$ , and we can see unboundedness from  $f(x) = e^{-\frac{x^2}{\sigma}}$  for  $\sigma > 0$ .

$$\begin{aligned} \int f^2(x) dx &= \int e^{-\frac{x^2}{\sigma}} dx \stackrel{y=\frac{x}{\sqrt{\sigma}}}{=} \sqrt{\sigma} \int e^{-2y^2} dy \\ \int (f')^2(x) dx &= \int \frac{4x^2}{\sigma^2} e^{-\frac{x^2}{\sigma}} dx \stackrel{y=\frac{x}{\sqrt{\sigma}}}{=} \frac{1}{\sqrt{\sigma}} \int 4y^2 e^{-2y^2} dy \end{aligned}$$

$$\text{So } \frac{\|Af\|^2}{\|f\|^2} = \frac{C}{\sigma} \xrightarrow[\sigma \rightarrow 0]{} \infty.$$

### Definition: Extension of an Unbounded Operator

We say that  $B$  extends  $A$  (and write  $A \subseteq B$ ) if  $\mathcal{D}(A) \subset \mathcal{D}(B)$  and  $B|_{\mathcal{D}(A)} = A$ . We have that  $A = B$  when  $A \subset B$  and  $B \subseteq A$ .

### Definition: Closed Operator

To recover a notion of continuity, we look at closed operators.

We say that  $(A, \mathcal{D}(A))$  is closed if and only if  $\forall u_n \in \mathcal{D}(A)$  such that  $u_n \rightarrow u \in \mathcal{H}$  and  $Au_n \rightarrow v \in \mathcal{H}$ ,  $u \in \mathcal{D}(A)$  and  $v = Au$ .

Equivalently,  $A$  is closed if the graph of  $A$ ,  $\Gamma(A) = \{(u, Au) : u \in \mathcal{D}(A)\}$ , is closed in  $\mathcal{H} \times \mathcal{H}$ .

We say that  $A$  is closable if there exists some  $B$  such that  $A \subseteq B$ .

Example

- $(P, \mathcal{D}(P), \mathcal{H}) = (D_x, C_C^\infty(\mathbb{R}), L^2(\mathbb{R}))$ .

- $P$  is not closed, because there exists  $u_n \in C_C^\infty(\mathbb{R})$  such that  $u_n \rightarrow u \in \mathcal{H}$ ,  $D_x u_n \rightarrow v \in \mathcal{H}$ , yet  $u \notin C_C^\infty(\mathbb{R})$ .
- Take  $u \in H^1 \setminus C_C^\infty(\mathbb{R})$ ,  $u_n \in C_C^\infty(\mathbb{R})$  converging to  $u$  in  $H^1$ .

- Recall that

- $H^1(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : f'(x) \in L^2(\mathbb{R})\}$ ,  $\|f\|_{H^1}^2 = \|f\|_{L^2(\mathbb{R})}^2 + \|f'\|_{L^2(\mathbb{R})}^2$ .
- $C_C^\infty(\mathbb{R})$  is dense in  $H^1(\mathbb{R})$ .
- $H^1(\mathbb{R})$  is complete.

Exercise: prove that if  $\mathcal{D}(B) := H^1(\mathbb{R})$  and  $Bf = D_x f$ , then  $B$  is closed and  $B|_{C_C^\infty(\mathbb{R})} = P$ .

## Adjoints of Unbounded Operators

In the bounded case, we have  $A \in \mathcal{B}(\mathcal{H})$ ,  $u \in \mathcal{H}$  and define  $\ell_u(v) = (u, Av) \in \mathcal{H}'$  ( $\mathcal{H}'$  the dual space). By Riesz Representation Theorem,  $\exists! w := A^* u$  such that  $\forall v \in \mathcal{H}$ ,  $(A^* u, v) = (u, A^* v)$ .

In the unbounded case, we need to define  $A^*$  and  $\mathcal{D}(A^*)$ .

For  $u \in \mathcal{H}$ , set  $\ell_u = (u, Av)$ ,  $v \in \mathcal{D}(A)$ . If  $\ell_u$  extends to an element of  $\mathcal{H}'$  (i.e. if we can prove an estimate  $|\ell_u(v)| \leq c\|v\|_{\mathcal{H}}$ ), then by Riesz Representation Theorem  $\exists! w \in \mathcal{H}$ , called  $A^* u$ , such that  $\forall v \in \mathcal{D}(A)$ ,  $(A^* u, v) = (u, A^* v)$ . Then

$$\mathcal{D}(A^*) = \{u \in \mathcal{H} : \begin{matrix} \mathcal{D}(A) \rightarrow \mathbb{C} \\ v \mapsto (u, Av) \end{matrix} \text{ extends to an element of } \mathcal{H}'\}.$$

Example

- $(P, \mathcal{D}(P), \mathcal{H}) = (D_x, C_C^\infty(\mathbb{R}), L^2(\mathbb{R}))$ .

- Observe that  $\forall f, g \in C_C^\infty(\mathbb{R})$ ,  $(D_x f, g) = (f, D_x g)$ .
- Take  $f \in \mathcal{H}$  and  $g \in C_C^\infty(\mathbb{R})$ . Then if  $f \in C_C^\infty$  or  $f \in H^1(\mathbb{R})$ ,  $g \mapsto (f, D_x g) = (D_x f, g)$  and

$$|(D_x f, g)| \leq \|D_x f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})} \leq \|f\|_{H^1(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}$$

- Then  $B(f, g) = D_x f, g - (f, D_x g)$ .
- So  $\mathcal{D}(P^*) \supseteq C_C^\infty(\mathbb{R})$  and, in fact,  $\mathcal{D}(P^*) = H^1(\mathbb{R})$ .
- $P^*$  is closed (this is always true of any adjoint).
- $P \subset P^*$ , we say that  $P$  is symmetric.

Generally, if  $(A, \mathcal{D}(A))$  is an unbounded operator, then  $\Gamma(A^*) = J((\Gamma(A))^\perp)$  where  $J : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H} \times \mathcal{H}$  by  $J(u, v) = (-v, u)$ . Recall that  $(\Gamma(A))^\perp$  is always closed, so  $A^*$  is always closed.

### Definitions:

We say that  $P$  is symmetric if  $P \subseteq P^*$ .

We say that  $P$  is self-adjoint if  $P = P^*$ .

We say that  $P$  is essentially self-adjoint if  $P^* = \overline{P}$ .

### Important Questions

1. Given a symmetric operator  $(P \subseteq P^*)$ , to find a self-adjoint operator  $A$  such that  $P \subseteq A = A^* \subseteq P^*$ . We call  $A$  a self-adjoint extension of  $P$ . If  $P^* = \overline{P}$  (essentially self-adjoint), then it must be the unique extension.
2. How to find or parameterize all self-adjoint extensions? (This problem is hard.)

Example

- $P = (D_x, H_0^1([0, 1]), L^2(0, 1))$  (where elements of  $H_0^1([0, 1])$  vanish on the boundary).

- $P^* = (D_x, H^1([0,1])) \supsetneq P$ .
- There exists a “circle” of self-adjoint extensions, parameterized by any point on the unit circle.

## General Principle

If  $\Omega \subset \mathbb{R}^n$  is open, bounded and with smooth boundary  $\partial\Omega$  ( $\mathcal{H} = L^2(\Omega)$ ), and  $P$  is some differential operator (e.g.  $P = \Delta$ ) such that  $\forall f, g \in C_C^\infty(\Omega)$ .  $(Pf, g) = (f, Pg)$ . By Green’s Identity,,

$$\begin{aligned}\int_{\Omega} (\Delta f g - f \Delta g) dx &= \int_{\Omega} \nabla((\nabla f)g(-f(\nabla g))) dx \\ &= \int_{\partial\Omega} (g \nabla f - f \nabla g) \cdot \nu ds \\ &= \int_{\partial\Omega} \left( g \frac{\partial f}{\partial \nu} - f \frac{\partial g}{\partial \nu} \right) ds\end{aligned}$$

### Examples

- $P = (P = \Delta, C_C^\infty(\Omega), L^2(\Omega))$ .
    - $P$  symmetric.
    - $\mathcal{D}(P^*) = \{u \in L^2(\Omega) : \Delta u \in L^2\}$ .
    - Let  $P_{\min} = \overline{(P, C_C^\infty(\Omega))}$  and define  $P_{\max} = P_{\min}^*$ . Then  $\mathcal{D}(P_{\max}) = \{u \in L^2(\Omega) : Pu \in L^2(\Omega)\}$ .
    - The self-adjoint extensions lie inbetween.
  - $(D_x, H_0^1([0,1]) = \{u \in H^1([0,1]) : u(0) = u(1) = 0\}, L^2([0,1]))$ .
    - Observe that the evaluation map  $C^\infty([0,1]) \ni f \mapsto f(0)$  extends boundedly to  $\tau : H^1([0,1]) \rightarrow \mathbb{C}$ .
    - For  $f, g \in C^\infty([0,1])$ ,
- $$(D_x f, g) - (f, D_x g) = \int_0^1 \frac{1}{i} f' \bar{g} - f \left( \overline{\frac{1}{i} g'} \right) = \frac{1}{i} \int_0^1 (f \bar{g})' = \frac{1}{i} (f(1) \bar{g}(1) - f(0) \bar{g}(0))$$
- Then if  $g \in \mathcal{D}(P_{\min})$ ,
- $$\begin{aligned}|(f, D_x g)| &= |(D_x f, g) + \frac{1}{i} (\overbrace{f(1) \bar{g}(1)}^{=0} - \overbrace{f(0) \bar{g}(0)}^{=0})| \\ &\leq \|f\|_{H^1} \|g\|_{L^2}\end{aligned}$$
- Therefore  $\mathcal{D}(P_{\min}^*) = \mathcal{D}(P_{\max}) = H^1([0,1])$ , and we note that  $P_{\max}^* = P_{\min}$ .

Now suppose that  $T$  is a self-adjoint extension of  $P$  ( $P_{\min} \subseteq T \subseteq P_{\max}$ ). If  $T$  is self-adjoint, then  $\forall f, g \in \mathcal{D}(T) \subseteq H^1([0,1])$ ,  $f(0) \bar{g}(0) - f(1) \bar{g}(1) = 0$ . For  $f = g$ ,  $|f(0)|^2 = |f(1)|^2$  for some fixed  $f$ , then  $\exists \alpha \in S^1$  such that  $f(1) = \alpha f(0)$ . Then for any other  $g$ ,

$$0 = f(0)(\bar{g}(0) - \alpha \bar{g}(1)) = \alpha f(0)(\overline{\alpha g(0) - g(1)})$$

Therefore  $g(1) = \alpha g(0)$  for the same fixed  $\alpha$ .

## October 23, 2025

### Definition: Resolvent Set and Spectrum of an Unbounded Operator

For  $(A, \mathcal{D}(A), \mathcal{H})$  unbounded and closed, the resolvent set of  $A$ ,  $\rho(A)$ , is the set of  $\lambda \in \mathbb{C}$  such that  $(A - \lambda) : \mathcal{D}(A) \rightarrow \mathcal{H}$  is bijective with bounded inverse  $R_A(\lambda) := (A - \lambda)^{-1}$ .

The spectrum of  $A$ ,  $\sigma(A)$ , is  $\sigma(A) = \mathbb{C} \setminus \rho(A)$ . If  $\rho(A)$  is open, then  $\sigma(A)$  is closed.

Example

- Take  $P = (D_x, H_0^1([0, 1]), L^2(0, 1))$  and define two closed extensions  $P_0$  and  $P_1$

$$- \mathcal{D}(P_0) = \{u \in H^1([0, 1]) : u(0) = 0\}$$

\* For  $\lambda \in \mathbb{C}$ ,  $f \in L^1([0, 1])$ , we solve  $\begin{cases} D_x u + \lambda u = f \\ u(0) = 0 \end{cases}$

\* So  $u(x) = i e^{-\lambda x} \int_0^x e^{i \lambda t} f(t) dt$  and  $\sigma(P_0) = \emptyset$ .

$$- \mathcal{D}(P_1) = H^1([0, 1])$$

\* For  $\lambda \in \mathbb{C}$ ,  $\ker(D_x - \lambda) \cap L^2([0, 1]) \neq \emptyset$ . So  $\sigma(P_1) = \mathbb{C}$ .

### Proposition

If  $(A, \mathcal{D}(A))$  is self-adjoint, then  $\sigma(A) \subseteq \mathbb{R}$ ,  $\sigma(A) \neq \emptyset$ , and we have a resolvent estimate  $\|R_A(z)\| \leq \frac{1}{|\operatorname{Im}(z)|}$  for  $\operatorname{Im}(z) \neq 0$ .

The proof of real value and the estimate are the same as in the bounded case.

To show that  $\sigma(A) \neq \emptyset$ , by contradiction if  $\sigma(A) = \emptyset$  then  $A^{-1} = R_A(0)$  exists and is bounded. We claim that this requires  $\sigma(A^{-1}) = \{0\}$  and pick  $\lambda \neq 0$ , such that

$$\begin{aligned} (A^{-1} - \lambda)u &= f \\ \left(\frac{1}{\lambda} - A\right)u &= \frac{1}{\lambda}Af \end{aligned}$$

But  $\frac{1}{\lambda} \notin \sigma(A)$  implies  $u = -R_A\left(\frac{1}{\lambda}\right)\frac{1}{\lambda}Af$ . Therefore  $R_{A^{-1}}(\lambda) = -\frac{1}{\lambda}AR_A\left(\frac{1}{\lambda}\right)$  and  $\lambda \notin \sigma(A^{-1})$ . Therefore  $A^{-1} = 0$  which contradicts the assumption that  $A^{-1}A = \operatorname{Id}_{\mathcal{D}(A)}$ .

### Generalizing Spectral Resolution

We can extend  $\{E_\lambda\}_{a \leq \lambda \leq b}$  to  $a = -\infty$  and  $b = \infty$  by setting strong limits  $E_{-\infty} = 0$  and  $E_\infty = I$ .

Example

- On  $L^2(\mathbb{R})$ ,  $E_\lambda f(t) = \chi_{(-\infty, \lambda)}(t)f(t)$ .

## Theorem: Spectral Theorem for Unbounded Self-Adjoint Operators

Theorem 8.15 Helffer.

Any self-adjoint operator  $A$  on a Hilbert space  $\mathcal{H}$  admits a spectral decomposition  $\{E_\lambda\}_\lambda$  such that  $\forall x, y \in \mathcal{H}$ ,

$$(Ax, y) = \int_{\mathbb{R}} \lambda d(E_\lambda x, y)$$

$$Ax = \int_{\mathbb{R}} \lambda d(E_\lambda x)$$

### Proof

Largely, we will use the proof for the bounded case.

Suppose  $A$  is semibounded from below (i.e.  $\exists \mu \in \mathbb{R}, \forall x \in \mathcal{D}(A), (Ax, x) \geq \mu ||x||^2$ ) then  $R_A(\lambda)$  exists for  $\lambda_0 < \mu$  ( $((A - \lambda)x, x) \geq (\mu - \lambda_0)||x||^2$  coercive in the sense that  $(\mu - \lambda_0) > 0$ ). Then it is also a self-adjoint bounded operator which implies that  $R_A(\lambda_0)$  has a spectral representation. Then  $A = f(R_A(\lambda_0))$ ,  $f(x) = \lambda_0 + \frac{1}{x}$ . We can work with a general case  $(A - i)^{-1}$ .

Example

- Take  $(D_x, H^1(\mathbb{R}), L^2(\mathbb{R}))$ , and define the Fourier transform  $f(x) \mapsto \mathcal{F}f = \hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx$ . Then  $\widehat{D_x f}(\xi) = \xi \hat{f}(\xi)$ , and we have the following commutative diagram

$$\begin{array}{ccc} H^1(\mathbb{R}) & \xrightarrow{D_x} & L^2(\mathbb{R}) \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\ H^1(\mathbb{R}) & \xrightarrow[f(t) \mapsto t f(t)]{} & L^2(\mathbb{R}) \end{array}$$

- So  $E_\lambda f = \mathcal{F}^{-1} \chi_{(-\infty, \lambda)} \mathcal{F}f$ .
- $(D_x^2, H^2(\mathbb{R}), L^2(\mathbb{R}))$ . Then  $\widehat{D_x^2 f}(\xi) = \xi^2 \hat{f}(\xi)$ ,  $\sigma(D_x^2) = [0, \infty)$  and  $D_x^2 f = \mathcal{F}^{-1} \xi^2 \mathcal{F}f$ .
  - So  $E_\lambda f = 0$  for  $\lambda < 0$ ;  $E_\lambda f = \mathcal{F}^{-1} \chi_{\xi^2 < \lambda}(\xi) \mathcal{F}f$ .
- $(D_x^2 + V(x), C_C^\infty(\mathbb{R}), L^2(\mathbb{R}))$ ,  $V \in C(\mathbb{R}; \mathbb{R})$  bounded from below. Then  $D_x^2 + V$  is semibounded from below and the spectrum depends drastically on  $V$ .
  - $V = 0$  gives  $\sigma = [0, \infty)$ .
  - $V = x^2$  gives  $\sigma = \{2n + 1 : n \geq 0\}$
  - $V \in C_C(\mathbb{R})$  gives  $\sigma = [0, \infty)$

### Proposition

If for some  $z_0 \in \rho(A) \cap \mathbb{R}$   $R_A(z_0)$  is compact, then  $A$  has purely discrete spectrum  $(z_n)_{n \geq 1}$ ,  $|z_n| \rightarrow \infty$  as  $n \rightarrow \infty$ .

## Proof

$R_A(z_0)$  is compact and self-adjoint, so there exists a basis for  $\mathcal{H}$  made of normalized eigenvectors  $(\phi_n)_{n \in \mathbb{N}}$  with eigenvalues  $|\lambda_n| \searrow 0$ .

$$\begin{aligned} (A - z_0)^{-1}\phi_n &= \lambda_n \phi_n \\ A\phi_n &= \left(z_0 + \frac{1}{\lambda_n}\right)\phi_n \\ \mathcal{H} &= \left\{ u = \sum u_n \phi_n : \sum |u_n|^2 < \infty \right\} \\ \mathcal{D}(A) &= \left\{ u = \sum u_n \phi_n : \sum |u_n|^2 \left(z_0 + \frac{1}{\lambda_n}\right)^2 < \infty \right\} \end{aligned}$$

## Applications

- Laplacian (Leplace-Beltrami Operator) on closed Riemann manifolds or Dirichlet Laplace-Beltrami if  $\partial M \neq 0$ .
- Dirichlet Laplacian ( $\Delta = -\sum_{k=1}^n \partial_{x^k}^2$ ) on bounded domains in  $\mathbb{R}^n$  ( $\Omega$  open in  $\mathbb{R}^n$ , regular,  $\partial\Omega$  smooth).
  - $\Delta_D = (\Delta, H_0^1(\Omega) \cap H^2(\Omega), L^2(\Omega))$  where  $H_0^1 = \{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\}$  ( $u|_{\partial\Omega}$  makes sense due to a trace theorem from Evans 5.5) and  $H^2(\Omega) = \{u \in L^2(\Omega) : \partial_i \partial_j u \in L^2, \partial_i u \in L^2, \forall i, j\}$ .
  - $\int_{\Omega} |\nabla f|^2 dx = (\Delta f, f) \geq 0$  (Green's Theorem)
  - Claim:  $\Delta_D + 1$  has compact inverse.
- Solvability of  $\Delta_D + 1$

Our goal is, given  $f \in L^2$ , to find  $u \in ?$  such that  $\begin{cases} \Delta u + u = f \\ u|_{\partial\Omega} = 0 \end{cases}$ . We have a weak formulation, if  $v \in H_0^1$ , we multiply by  $v$  and get an IBP

$$\underbrace{\int \nabla u \cdot \nabla v + \int uv}_{B(u,v)} = \int fv$$

where  $|\int fv| \leq \|f\|_{L^2} \|v\|_{L^2} \leq \|f\|_{L^2} (B(v, v))^{1/2}$ . By Riesz Representation Theorem, there exists some unique  $u \in H_0^1$  such that  $\forall v \in H_0^1$ ,  $B(v, v) = \int fv$ . Moreover, we can recover  $H^2$  regularity on  $u$ . Also,  $|B(u, u)| = |\int fu| \leq \|f\|_{L^2} (B(u, u))^{1/2}$  if and only if  $\|(\Delta_D + 1)^{-1} f\|_{H_0^1(\Omega)} = \|u\|_{H_0^1(\Omega)} \leq \|f\|_{L^2}$ . Therefore  $(\Delta_D + 1)^{-1} : L^2(\Omega) \rightarrow H_0^1(\Omega)$  is bounded. By Rellich Compactness,  $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$  compactly. Hence  $(\Delta_D + 1)^{-1} : L^2 \rightarrow L^2$  is compact.

## Theorem

On  $\mathbb{R}$ ,  $H^1(\mathbb{R})$  does not embed compactly into  $L^2(\mathbb{R})$ . For example,  $f \in H^1$  with  $\text{supp } f \subset [0, 1]$  defined by  $f_n(x) = f(x - n)$ .

So if  $H \subset L^2(\mathbb{R})$  where  $\|u\|_H^2 = \|u'\|_L^2 + \|a(x)^{-1} u\|_{L^2}^2$  for  $a(x) > 0$  and  $\lim_{|x| \rightarrow \infty} a(x) = 0$ , then  $H$  embeds compactly in  $L^2$ .

It follows that  $D_x^2 + x^2$  has compact resolvent and discrete spectrum.  $\|u\|_H^2 = \|d_x u\|_{L^2}^2 + \|xu\|_{L^2}^2$ .

**October 30, 2025**

## Friedrichs Extension

Let  $(A, \mathcal{D}(A), \mathcal{H})$  be symmetric, bounded from below in the sense that there exists  $\alpha \in \mathbb{R}$  such that  $\forall x \in \mathcal{D}(A)$  we have  $(Ax, x)_{\mathcal{H}} \geq \alpha \|x\|_{\mathcal{H}}^2$ .

### Theorem:

If  $(A, \mathcal{D}(A), \mathcal{H})$  is densely defined, symmetric, and bounded below, then  $A$  admits a self-adjoint extension  $(S, \mathcal{D}(S))$ . The procedure for producing this extension is the Friedrichs Extensions.

### Proof

Assume that  $\alpha = 1$ , otherwise consider  $A - \lambda I$  for some well-chosen  $\lambda$ . Define the symmetric quadratic form

$$q_A(u, v) = (Au, v), \quad (u, v) \in \mathcal{D}(A) \times \mathcal{D}(A).$$

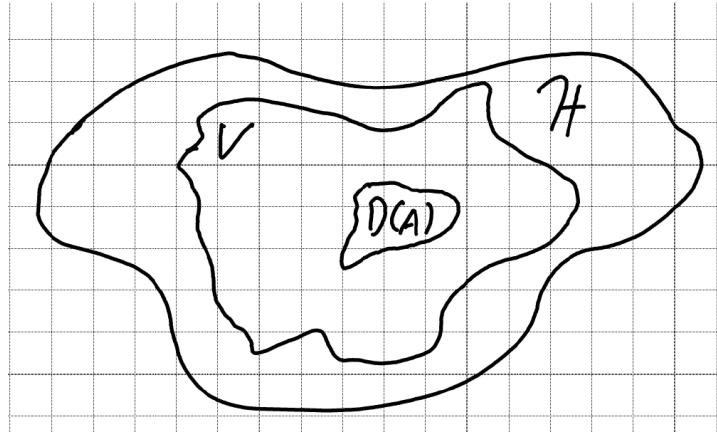
Note that  $q_A(u, u) = (Au, u) \geq \|u\|_{\mathcal{H}}^2$ , so this form is coercive. Next, we construct  $V \subset \mathcal{H}$  by taking  $p(u) = (q_A(u, u))^{1/2}$  and letting  $u \in \mathcal{D}(A)$  belong to  $V$  if  $\exists u_n \in \mathcal{H}$  such that  $\|u_n - u\|_{\mathcal{H}} \xrightarrow{n \rightarrow \infty} 0$  and  $\{u_n\}$  is  $p$ -cauchy (i.e.  $\forall \varepsilon > 0$ ,  $\exists N, m, n \geq N \implies p(u_n - u_m) < \varepsilon$ ).

Then we can define a norm on  $V$ :  $\|u\|_V := \lim_{n \rightarrow \infty} p(u_n)$  where  $u_n \xrightarrow{\mathcal{H}} u$  and  $u_n$  is  $p$ -Cauchy.

Exercise: show that this definition does not depend on  $\{u_n\}$ .

We also define  $(u, v)_V := \lim_{n \rightarrow \infty} q_A(u_n, v_n)$  for  $u_n$  and  $v_n$  similarly converging and  $p$ -cauchy.

So  $(V, \|\cdot\|_V)$  is complete and  $p(u_n) \geq \|u_n\|_{\mathcal{H}}$  becomes  $\|u\|_V \geq \|u\|_{\mathcal{H}}$  as  $n \rightarrow \infty$ . Hence  $V \hookrightarrow \mathcal{H}$  is continuous.



We now construct  $S$  starting with a domain  $D(S) = \{u \in V : V \ni v \mapsto q_A(u, v) \text{ satisfies an continuity estimate of the form } \leq C\|v\|_{\mathcal{H}}\}$ . For  $u \in \mathcal{D}(S)$ , the map  $v \mapsto q_A(u, v)$  extends to an element of  $\mathcal{H}'$  (bilinear functionals on  $\mathcal{H}$ ). By Riesz-Representation Theorem, there exists some  $w := Su$  such that  $\forall v \in V$ ,  $q_A(u, v) = (Su, v)_{\mathcal{H}}$ .

It remains to show that  $S$  is self-adjoint,  $\mathcal{D}(S) \supset \mathcal{D}(A)$ ,  $S|_{\mathcal{D}(A)} = A$ . Self-adjointness is left as an exercise. If  $u \in \mathcal{D}(A)$ , then

$$|q_A(u, v)| = |(Au, v)| \leq \|Au\|_{\mathcal{H}} \|v\|_{\mathcal{H}}$$

so  $u \in \mathcal{D}(S)$  and  $Su = Au$ .

### Remark

We could start with a quadratic form  $q(u, v)$  which is sesquilinear, coercive and  $\mathcal{D}(q) \subset \mathcal{H}$  densely. This construction produces a self-adjoint operator as well.

### Example (Dirichlet Laplacian)

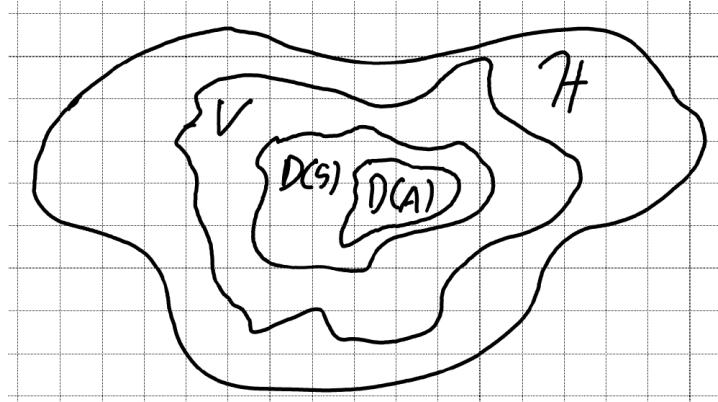
$\Delta = \sum_{j=1}^n \partial_{x_j}^2$  with  $\Omega = \mathbb{R}^n$  open, bounded, and regular.  
 $(A = -\Delta, \mathcal{D}(A) = C_C^\infty(\Omega), \mathcal{H} = L^2(\Omega))$ . For  $u \in \mathcal{D}(A)$ ,

$$\begin{aligned} (Au, u) &= \int_{\Omega} -\Delta u \cdot \bar{u} dx \\ &= \int_{\Omega} -\nabla(\nabla u \cdot \bar{u}) + \nabla u \cdot \nabla \bar{u} dx \\ &= \underbrace{\int_{\partial\Omega} -\partial_{\nu} u ds}_{=0} + \int_{\Omega} \nabla u \cdot \nabla \bar{u} dx \geq 0 \end{aligned}$$

So Friedrich's procedure produces a self-adjoint extension

$$q_A(u, v) = \int_{\Omega} \nabla u \cdot \nabla \bar{u} dx, \quad u, v \in C_C^\infty(\Omega)$$

which leads to  $V = \overline{C_C^\infty(\Omega)}^{H^1(\Omega)} = H_0^1(\Omega)$  (that is the  $H^1$  closure of  $C_C^\infty(\Omega)$ ). What about  $\mathcal{D}(S)$ ?



For  $v \in C_C^\infty(\Omega)$ ,  $\int_{\Omega} \nabla u \cdot \nabla \bar{v} dx = \int_{\Omega} -\Delta u \cdot v \leq C \|v\|_{L^2}$ . Then we have

$$\mathcal{D}(S) = \{u \in H_0^1(\Omega) : \Delta u \in L^2(\Omega)\} = H_0^1(\Omega) \cap H^2(\Omega)$$

### Example (Neumann Laplacian)

Start from a quadratic form  $q(u, v) = \int_{\Omega} \nabla u \cdot \nabla \bar{v} + u \bar{v} dx$  with  $u, v \in C^\infty(\overline{\Omega})$ . Then  $V = H^1(\Omega)$ . We compute  $\mathcal{D}(S)$  by first recalling that for  $u, v \in C^\infty(\overline{\Omega})$ ,

$$\int_{\Omega} -\Delta u \bar{v} dx = \int_{\Omega} \nabla u \cdot \nabla \bar{v} dx - \int_{\partial\Omega} \partial_{\nu} u \bar{v} dx$$

We sense that  $\Delta u \in L^2(\Omega)$ , so we at least need  $u \in W(\Omega) = \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\}$ . How do we make sense of the Neumann trace on  $W(\Omega)$ ?

- Lemma

- The restriction map  $C(\bar{\Omega}) \rightarrow C(\partial\Omega)$  extends to a bounded surjective “Dirichlet trace”  $\tau_D : H^1(\Omega) \rightarrow H^{1/2}(\partial\Omega)$  with bounded right inverse  $R$ .
- Example

- if  $\Omega = \mathbb{D}$ ,  $\partial\Omega = S^1$ ,  $f \in C^\infty(S^1) \leftrightarrow f(\theta) \sum_{k \in \mathbb{Z}} f_k e^{ik\theta}$  where  $f_k = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-ik\theta} d\theta$ , we can for each  $s \in \mathbb{R}$  define

$$H^S(S^1) = \{f \in \mathcal{D}'(S^1) : \sum_{k \in \mathbb{Z}} (1+k^2)^s |f_k|^2 < \infty\} = \mathcal{D}((-d_\theta^2)^{s/2})$$

- Fact

- For  $s \geq 0$ ,  $H^{-s}(\partial\Omega) \cong (H^S(\partial\Omega))'$ .

- If  $f \in H^{-s}$ ,  $g \in C^\infty(S^1)$ ,

$$\langle f, g \rangle_{\mathcal{D}', \mathcal{D}} = \int (f, g) = c \sum_{k \in \mathbb{Z}} f_k \overline{g_k} \frac{(1+k^2)^{s/2}}{(1+k^2)^{s/2}} \leq c \left( \sum_{k \in \mathbb{Z}} (1+k^2)^{-s} |f_k|^2 \right) \left( \sum_{k \in \mathbb{Z}} (1+k^2)^s |g_k|^2 \right)$$

Returning to the Neumann Laplacian, the Neumann trace  $C^1(\bar{\Omega}) \rightarrow C(\partial\Omega)$  by  $f \mapsto \partial_\nu f$  extends to a bounded operator  $\tau_N : W(\Omega) \rightarrow H^{-1/2}(\partial\Omega)$ .

- Proof

If  $u \in W(\Omega)$ , set

$$\begin{aligned} \Phi_u(v) &:= \int_{\Omega} -\Delta u \bar{v} dx - \int_{\Omega} \nabla u \cdot \nabla \bar{v} dx, \quad v \in H^1(\Omega) \\ &\leq \|\Delta u\|_{L^2} \|v\|_{L^2} + \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} \end{aligned}$$

Using  $R : H^{1/2}(\partial\Omega) \rightarrow H^1(\Omega)$ ,  $w \in H^{1/2}(\partial\Omega)$ ,

$$|\Phi_u(Rw)| \leq (\|\Delta u\|_{L^2} + \|\nabla u\|_{L^2}) \|Rw\|_{H^1} \leq (\|\Delta u\|_{L^2} + \|\nabla u\|_{L^2}) \|w\|_{H^{1/2}(\Omega)}$$

which implies that there exists some  $h \in H^{-1/2}(\partial\Omega)$  such that  $\Phi_u(Rw) = \langle h, w \rangle_{H^{-1/2}, H^{1/2}}$ .

We want to extend

$$\int_{\Omega} -\Delta u \bar{v} dx = \int_{\Omega} \nabla u \cdot \nabla \bar{v} dx - \int_{\partial\Omega} \partial_\nu u \bar{v} dx$$

to  $u \in W(\Omega)$  and  $v \in H^1(\Omega)$ . First, if  $v \in H_0^1(\Omega)$  then

$$\int_{\Omega} -\Delta u \bar{v} dx = \int_{\Omega} \nabla u \cdot \nabla \bar{v} dx$$

(by passing to the limit with a sequence  $v_n \in C_C^\infty(\Omega)$  converging to  $v$  in  $H^1$ ).

Next, if  $v \in H^1$ , we write  $v = v_0 + R\tau_D v$  with  $v_0 \in H_0^1$ . Then

$$\int_{\Omega} -\Delta u \bar{v} - \nabla u \cdot \nabla \bar{v} dx = \overbrace{\int_{\Omega} -\Delta u \bar{v}_0 - \nabla u \cdot \nabla \bar{v}_0 dx}^{=0} + \Phi_u(R\tau_D v) = \langle \tau_N u, \tau_D v \rangle_{H^{-1/2}, H^{1/2}}$$

We conclude that for each  $u \in W(\Omega)$ ,  $v \in H^1(\Omega)$

$$\int_{\Omega} -\Delta u \cdot \bar{v} dx = \int_{\Omega} \nabla u \cdot \nabla \bar{v} dx + \langle \tau_N u, \tau_D v \rangle_{H^{-1/2}, H^{1/2}}$$

Therefore we have

$$q(u, v) = \int_{\Omega} (-\Delta u + u) \bar{v} dx + \langle \tau_N u, \tau_D v \rangle_{H^{-1/2}, H^{1/2}}$$

and  $v \mapsto q(u, v)$  is  $L^2$ -bounded if and only if  $v \mapsto \langle \tau_N u, \tau_D v \rangle_{H^{-1/2}, H^{1/2}}$  is  $L^2$ -bounded which holds if and only if  $\tau_N u = 0$ . So

$$\mathcal{D}(S) = \{u \in H^1(\Omega) : -\Delta u \in L^2, \tau_N u = 0\} = \{u \in H^2(\Omega) : \tau_N u = 0\}$$

### Observation

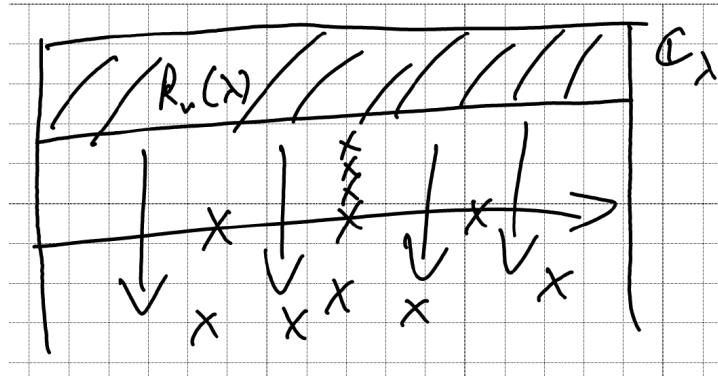
The form domains satisfy  $\mathcal{D}(q_D^{-\Delta}) \subseteq \mathcal{D}(q_N^{-\Delta})$ .

## November 04 and 06, 2025

See class notes on variational principles for eigenvalues and Weyl's criterion for essential spectrum.

## November 13, 2025

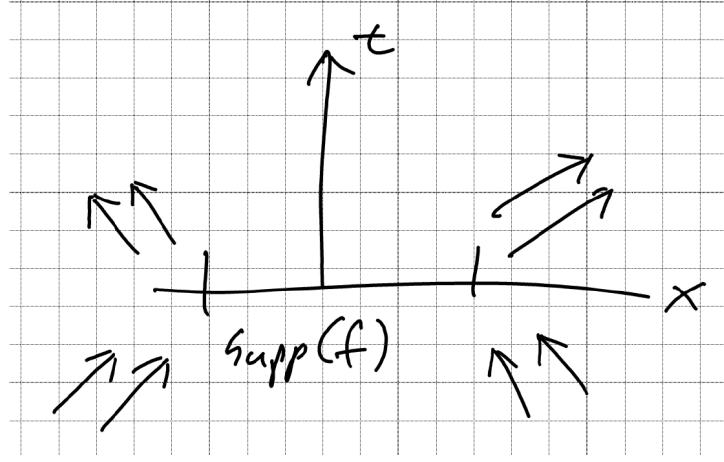
Consider  $(D_x^2 - \lambda^2)^{-1} = R_0(\lambda)$ , the “outgoing” resolvent and let  $V$  be in  $C_C^\infty(\mathbb{R}, \mathbb{R})$  or  $L_{\text{comp}}^\infty(\mathbb{R}, \mathbb{R})$ . Then we may define  $R_V(\lambda) = (D_x^2 + V - \lambda^2)^{-1}$ . We want to show that  $-R_V(\lambda)$  makes sense for  $\text{Im}(\lambda) \gg 0$  as an  $L^2 \rightarrow L^2$  operator and extends meromorphically to either  $\{\text{Im}(\lambda) > 0\}$  as an  $L^2 \rightarrow L^2$  bounded operator or to  $\mathbb{C}$  as a  $L_{\text{comp}}^2 \rightarrow L_{\text{loc}}^2$  operator. Recall that  $L_{\text{comp}}^2 = \{f \in L^2 : \text{supp } f \text{ compact}\}$  and  $L_{\text{loc}}^2 = \{f : \mathbb{R} \rightarrow \mathbb{R} : \forall K \text{ compact}, \int_K |f|^2 < \infty\}$ .



We will further see that there are no poles (embedded eigenvalues) on  $\mathbb{R} \setminus \{0\}$ .

### Outgoing/Incoming Solution

We have that  $(D_x^2 - \lambda^2)u = 0$  if and only if  $u(x) = Ae^{i\lambda x} + Be^{-i\lambda x}$ . A solution to  $(D_x^2 - \lambda^2)u = f \in C_C^\infty(\mathbb{R})$  is said to be outgoing if  $x \gg 0$  implies that  $u(x) = A_+ e^{i\lambda x}$  and  $x \ll 0$  implies that  $u(x) = B_+ e^{-i\lambda x}$ .



Likewise, the solution is incoming if  $x \gg 0$  implies  $u(x) = B_- e^{-i\lambda x}$  and  $x \ll 0$  implies  $u(x) = A_- e^{i\lambda x}$ . We have seen previously that the D'Alembert Solution to

$$\begin{cases} (\partial_t^2 - \partial_x^2 + V)u = f \\ u|_{t=0} = \partial_t u|_{t=0} = 0 \end{cases}$$

is outgoing.

We will find that an outgoing solution to  $(D_x^2 - \lambda^2)u = f$  is  $R_0(\lambda)f := u_+ = \frac{i}{2\lambda} \int_{\mathbb{R}} f(y) e^{i\lambda|x-y|} dy$ . The incoming solution, then, is  $u_-(x, \lambda) = -\frac{i}{2\lambda} \int_{\mathbb{R}} f(y) e^{-i\lambda|x-y|} dy$ .

Note that  $u_+(x, \lambda) = u_-(x, -\lambda) = \overline{u_-(x, \lambda)}$ . Note also that the outgoing solution is in  $L^2(\mathbb{R})$  if and only if  $\text{Im}(\lambda) > 0$ , and the incoming solution if and only if  $\text{Im}(\lambda) < 0$ .

The above solution may be derived via Fourier transform. Namely

$$(\xi^2 - \lambda^2)\hat{u}(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx$$

which implies that  $\hat{u}(\xi) = \frac{\hat{f}(\xi)}{\xi^2 - \lambda^2}$  if and only if  $\text{dist}(\lambda^2, [0, \infty)) > 0$ . Considering the outgoing solution, this occurs only when  $\text{Im}(\lambda) > 0$ .

Note, that with respect to Plancherel Theorem giving  $\|u\|_{L^2} = c\|\hat{u}(\xi)\|_{L^2}$ ,

$$\frac{1}{c}\|u\|_{L^2} = \|\hat{u}(\xi)\|_{L^2} \leq \frac{1}{\text{dist}(\lambda^2, [0, \infty))} \|\hat{f}(\xi)\|_{L^2} = \frac{1}{c}\|f\|_{L^2}$$

This gives continuity estimates for  $R_0(\lambda)$ . Then  $u(x) = h_\lambda(x) * f(x)$  where  $h_\lambda(x) = \int_{\mathbb{R}} e^{ix\xi} \frac{d\xi}{\xi^2 - \lambda^2}$ . We claim, computing by residue theorem, that  $h_\lambda(x) = \frac{i}{2\lambda} e^{i\lambda|x|}$ .

Exercise: do the residue computation.

So we have  $R_0(\lambda) = \frac{i}{2\lambda} \int_{\mathbb{R}} f(y) e^{i\lambda|x-y|} dy$  with Schwarz kernel  $R_0(x, y, \lambda) = \frac{i}{2\lambda} e^{i\lambda|x-y|}$ . When  $x$  and  $y$  are frozen, this looks meromorphic in  $\lambda$  with simple pole at  $\lambda = 0$  and residue  $\frac{i}{2}$ . This still makes sense as a map  $C_c^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})$  or  $L^2_{\text{comp}}(\mathbb{R}) \rightarrow L^2_{\text{loc}}(\mathbb{R})$ .

Alternatively, we can formulate this as  $\forall \rho \in C_C^\infty(\mathbb{R})$ ,  $\rho R_0 \rho : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  extends meromorphically to  $\mathbb{C}$ . Then  $\rho R_0(\lambda) \rho$  has Schwarz kernel  $\frac{i}{2\lambda} \rho(x) \rho(y) e^{i\lambda|x-y|}$ .

Exercise: if  $\lambda \neq 0$ , this is  $L^2 \rightarrow L^2$  bounded.

Then  $R_0(\lambda) = \frac{P}{\lambda} + Q(\lambda)$  where  $Pf(x) = \frac{i}{2} \int_{\mathbb{R}} f(y) dy$ ,  $P = \phi \otimes \bar{\phi}$  and  $\phi(x) = \frac{e^{i\pi/4}}{\sqrt{2}}$ . Here  $Q : L^2_{\text{comp}}(\mathbb{R}) \rightarrow L^2_{\text{loc}}(\mathbb{R})$  is entire.

## Compactly Supported Potential

Take  $V \in L_{\text{comp}}^\infty(\mathbb{R})$ , where  $\sigma_{\text{ess}}(D_x^2 + V) = [0, \infty) + (\text{possibly negative eigenvalues}) + (\text{embedded eigenvalues})$ .

### Theorem:

The outgoing resolvent  $(D_x^2 + V - \lambda^2)^{-1}$  extends meromorphically to  $\mathbb{C}$  with no poles at  $\lambda \in \mathbb{R} \setminus \{0\}$ .

### Proof

Goal: when can we solve  $(D_x^2 + V - \lambda^2)u = f$ ?

We start with the identity  $(D_x^2 + V - \lambda^2)R_0(\lambda) = I + VR_0(\lambda)$ . For  $\text{Im}(\lambda) > 0$ ,  $\|VR_0(\lambda)\|_{L^2 \rightarrow L^2} \leq \frac{\|V\|_\infty}{\text{dist}(\lambda^2, [0, \infty))}$ . So if  $\text{Im}(\lambda) \gg 1$ ,  $\|VR_0(\lambda)\|_{L^2 \rightarrow L^2} < 1$ . Then

$$(D_x^2 + V - \lambda^2)R_0(\lambda) \overbrace{\sum_{p=0}^{\infty} (-VR_0(\lambda))^p}^{R_V(\lambda)} = I.$$

So  $R_V(\lambda)$  is an inverse for  $D_x^2 + V - \lambda^2$  on  $\text{Im} \gg 1$ , bounded with operator norm below  $\|R_0(\lambda)\| \frac{1}{1 - \|VR_0(\lambda)\|}$ . It produced outgoing solutions due to its factored form.

Now, to extend  $R_V(\lambda)$  meromorphically as  $L_{\text{comp}}^2 \rightarrow L_{\text{loc}}^2$ , we look at  $\rho R_V(\lambda) \rho$  for  $\rho \in C_C^\infty(\mathbb{R})$  such that  $\rho V = V$ . Then for  $\text{Im}(\lambda) \gg 0$ ,

$$R_V(\lambda)\rho = R_0(\lambda)(I + VR_0(\lambda))^{-1}\rho = R_0(\lambda)\rho \sum_{p=0}^{\infty} (VR_0(\lambda)\rho)^p = R_0(\lambda)\rho(I + VR_0(\lambda)\rho)^{-1}$$

Therefore

$$\rho R_V(\lambda)\rho = \rho R_0(\lambda)\rho(I + \rho VR_0(\lambda)\rho)^{-1}$$

We claim that  $\lambda \mapsto \rho VR_0(\lambda)\rho$  is a meromorphic on  $\mathbb{C} \setminus \{0\}$  family of compact operators (at least on  $V \in C_C^\infty(\mathbb{R})$ ). To see this, we use  $R_0(\lambda) : L^2(\mathbb{R}) \rightarrow H^2(\mathbb{R})$ , then  $\rho VR_0(\lambda)\rho : L^2(\mathbb{R}) \rightarrow \{u \in H^2(\mathbb{R}) : \text{supp } u \subset \text{supp } V\}$  embeds compactly into  $L^2(\mathbb{R})$ .

By analytic Fredholm theory, since  $I + \rho VR_0(\lambda)\rho$  is invertible for  $\text{Im}(\lambda) \gg 1$ , it is invertible on  $\mathbb{C} \setminus \{0\}$  outside of a discrete set with finite-dimensional obstructions at that discrete set.

To upgrade this to a meromorphic extension of  $R_V(\lambda) : L_{\text{comp}}^2 \rightarrow L_{\text{loc}}^2$ , pick an increasing sequence of cutoff functions. If  $\rho_1$  is such that  $\rho_1\rho = \rho$ , then  $\rho_1 R_V(\lambda) \rho_1$  extends  $\rho R_V(\lambda) \rho$  in the sense that  $R_V(\lambda)\rho_1 f = R_V(\lambda)\rho f$  when  $\text{supp } f \subset \{\rho = 1\}$ .

Claim: if  $\lambda \in \mathbb{R} \setminus \{0\}$ , then  $\lambda$  is not a pole of  $R_V(\lambda)$ .

- Step 1: if  $\lambda$  is a pole, then there exists an outgoing solution to  $(D_x^2 + V - \lambda^2)u = 0$ .
- Step 2: if  $\lambda \in \mathbb{R} \setminus \{0\}$ , and  $u$  is an outgoing solution to  $(D_x^2 + V - \lambda^2)u = 0$ , then  $u$  has compact support.
- Step 3: if  $u$  is a compactly supported solution of  $(D_x^2 + V - \lambda^2)u = 0$ , then  $u = 0$ .

Step 1. If  $\lambda = \lambda_0$  is a pole, we may write  $R_V(\lambda) = \frac{P_N}{(\lambda - \lambda_0)^N} + \dots + \frac{P_1}{\lambda - \lambda_0} + Q(\lambda)$  near  $\lambda = \lambda_0$  where each term is  $L_{\text{comp}}^2 \rightarrow L_{\text{loc}}^2$ . If  $\lambda \neq \lambda_0$ , then

$$(\lambda - \lambda_0)^N \overbrace{(D_x^2 + V - \lambda^2)R_V(\lambda)}^I = (D_x^2 + V - \lambda^2)P_N + (\lambda - \lambda_0)(\dots)$$

Then as  $\lambda \rightarrow \lambda_0$ ,  $(D_x^2 + V - \lambda^2)P_n = 0$ . So  $P_N$  produces outgoing solutions in  $L_{\text{loc}}^2$ .

Step 2. If  $\lambda \in \mathbb{R}$  and  $u$  solves  $(D_x^2 + V - \lambda^2)u = 0$ , then  $\bar{u}$  also solves  $(D_x^2 + V - \lambda^2)\bar{u} = 0$ .

Since  $u$  has compact support, write  $u(x) = A_+ e^{i\lambda x} + B_- e^{-i\lambda x}$  when  $x \gg 0$  and  $u(x) = A_- e^{i\lambda x} + B_+ e^{-i\lambda x}$  when  $x \ll 0$ . Compute the Wronskian,  $W(u, \bar{u}) = u\bar{u}' - u' \bar{u}$  and  $\frac{d}{dx}W(u, \bar{u}) = u\bar{u}'' - u'' \bar{u} = 0$ . We find that this is  $|A_+|^2 - |B_-|^2$  for  $x \gg 0$  and  $|A_-|^2 - |B_+|^2$  for  $x \ll 0$ .

Therefore  $|A_+|^2 + |B_+|^2 = |A_-|^2 + |B_-|^2$ . That is, if  $u$  is outgoing then  $A_- = B_- = 0$  and  $A_+ = B_+ = 0$ .

Step 3 is insane.

- Lemma

If we take  $u \in L^\infty(\mathbb{R})$ ,  $(D_x^2 + W)u = 0$  where  $W \in L^\infty(\mathbb{R})$ , then if  $u = 0$  on  $(-\infty, 0)$  then  $u \equiv 0$ .

Fix  $h > 0$  and let  $v = e^{-x/h}u$ .

$$\begin{aligned} \|e^{-x/h}(hD_x)^2 e^{x/h}v\|_{L^2} &= \|(h^2 D_x^2 - 2ihD_x - 1)v\|_{L^2} \\ &= \|(h\xi - i)^2 \hat{v}\|_{L^2} \\ &\geq \|\hat{v}\|_{L^2} \\ &= \|v\|_L^2 \end{aligned}$$

Equivalently

$$\|e^{-x/h}u\|_{L^2} \leq h^2 \|e^{-x/h}D_x^2 u\| \leq h^2 \|W\|_{L^\infty} \|e^{-x/h}u\|_{L^2}$$

## November 25, 2025

### Recall

For  $V \in C_C^\infty(\mathbb{R})$ ,  $R_V(\lambda)$  is the scattering resolvent ( $\text{Im}(\lambda) \gg 0$ ,  $R_V(\lambda) = (D_x^2 + V - \lambda^2)^{-1}$ ).

This is meromorphic on  $\{\text{Im}(\lambda) > 0\}$  as a  $L^2 \rightarrow L^2$  bounded family. Similarly, it is meromorphic on  $\mathbb{C} \setminus \{0\}$  as a  $L_{\text{comp}}^2 \rightarrow L_{\text{loc}}^2$  family.

Then  $e_\pm(x, \lambda) = e^{\pm ix\lambda} - R_V(\lambda)(Ve^{\pm i\lambda x})$  are two solutions of  $(D_x^2 + V - \lambda^2)u = 0$ .

### IMAGE 1

Then we take  $S(\lambda)$ , the scattering matrix, defined wherever possible such that

$$S(\lambda) \begin{bmatrix} A_- \\ B_- \end{bmatrix} = \begin{bmatrix} A_+ \\ B_+ \end{bmatrix} \quad S(\lambda) = \begin{bmatrix} T(\lambda) & R_-(\lambda) \\ R_+(\lambda) & T(\lambda) \end{bmatrix}$$

Computing Wronskians gives

$$\begin{aligned} R_-(\lambda) &= -R_+(-\lambda)T(\lambda)/T(-\lambda) \\ S(-\lambda) &= JS(\bar{\lambda})^*J, \quad J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ S(\lambda)JS(-\lambda)J &= I \end{aligned}$$

Last time, we defined  $\phi_+(x, \lambda) = \frac{1}{T(-\lambda)}e_+(x, -\lambda)$  and  $\phi_-(x, \lambda) = \frac{1}{T(\lambda)}e_-(x, \lambda)$ .

## IMAGE 2

We have that ehse solve  $D_x^2 + V - \lambda = 0$ , so  $\forall x, \mathbb{R} \ni \lambda \mapsto \phi_{\pm}(x, \lambda)$  is tempered. Therefore, we may take its Fourier transform in  $\lambda$ .

$$A_{\pm}(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} \phi_{\pm}(x, \lambda) e^{i\lambda y} d\lambda$$

makes sense as a tempered distribution.

### **Claim**

$A_{\pm}$  is a distributional solution of  $(D_x^2 + V)A_{\pm}(x, y) = D_y^2 A_{\pm}(x, y)$ . That is, if we define  $\mathcal{A}_{\pm}f(x) = \int_{\mathbb{R}} A_{\pm}(x, y)f(y) dy$ , then  $(D_x^2 + V) \circ \mathcal{A}_{\pm} = \mathcal{A}_{\pm} \circ D_x^2$ . We compute

$$\begin{aligned} D_y^2 A_{\pm}(x, y) &= \frac{1}{2\pi} \int_{\mathbb{R}} \phi_{\pm}(x, \lambda) \lambda^2 e^{i\lambda y} d\lambda \\ (D_x^2 + V)A_{\pm}(x, y) &= \frac{1}{2\pi} \int_{\mathbb{R}} \lambda^2 \phi_{\pm}(x, \lambda) e^{i\lambda y} d\lambda \end{aligned}$$

In addition,  $A_{\pm}$  satisfies  $A_{\pm}(x, y) = \delta(x - y)$  for  $\pm x \gg 0$  where we can think of the Dirac delta as  $\delta(x - y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\lambda(y-x)} d\lambda$ . Now, our goal is to get an alternative expression for  $S(\lambda)$ . We can achieve this by looking at  $\phi_{-}(x, \lambda)$ .

## IMAGE 3

## IMAGE 4

We have that  $\text{clos}(\text{supp}(V)) \subset [a, b]$ . By pictures, for  $x \geq b$ ,

$$\partial_y A_{-}(x, y) = x(y - x) + y(x + y) = \frac{1}{2\pi} \int_{\mathbb{R}} \phi_{-}(x, \lambda) i\lambda e^{i\lambda y} d\lambda$$

for  $x$  and  $y$  distributions. Via Fourier transform in  $y$ , this implies that

$$i\lambda \phi_{-}(x, \lambda) = \int_{\mathbb{R}} e^{-i\lambda y} (X(y - x) + Y(x + y)) dy = e^{-i\lambda x} \hat{X}(\lambda) + e^{i\lambda x} \hat{Y}(\lambda)$$

Exercise: show that  $\text{supp}(X) \subset [-2(b-1), 0]$  and  $\text{supp}(Y) \subset [2a, 2b]$ .

Copying expressions, we have that

$$\begin{aligned} \frac{1}{T(\lambda)} &= \frac{\hat{X}(\lambda)}{i\lambda} \iff T(\lambda) = \frac{i\lambda}{\hat{X}(\lambda)} \\ \frac{R_{-}(\lambda)}{T(\lambda)} &= \frac{\hat{Y}(\lambda)}{i\lambda} \iff R_{-}(\lambda) = \frac{\hat{Y}(\lambda)}{\hat{X}(\lambda)} \end{aligned}$$

It follows that

$$R_{+}(\lambda) = -R_{-}(-\lambda) \frac{T(\lambda)}{T(-\lambda)} = -\frac{\hat{Y}(\lambda)}{\hat{X}(\lambda)} \cdot \frac{i\lambda}{\hat{X}(\lambda)} \cdot \frac{\hat{X}(-\lambda)}{-i\lambda} = \frac{\hat{Y}(-\lambda)}{\hat{X}(\lambda)}$$

The bottom line, then, is that we may write

$$S(\lambda) = \frac{1}{\hat{X}(\lambda)} \begin{bmatrix} i\lambda & \hat{Y}(\lambda) \\ \hat{Y}(-\lambda) & i\lambda \end{bmatrix}$$

Notation: recall that  $\mathcal{D}'$  are distributions,  $\mathcal{E}'$  are distributions with compact support, and  $\mathcal{S}'$  are tempered distribution. Then, since  $X, Y \in \mathcal{E}'(\mathbb{R})$ ,  $\hat{X}$  and  $\hat{Y}$  are entire on  $\mathbb{C}$ . It follows that  $\hat{X}(\lambda) = \int_{-2(b-a)}^0 X(x)e^{-i\lambda x} dx = \langle X, e^{-i\lambda} \rangle$ .

If  $\hat{X}(\lambda) = 0$ , then  $i\lambda\phi_-(x, \lambda) = e^{ix\lambda}\hat{Y}(\lambda)$ . Then also  $\phi_-(x, \lambda) \in L^2(\mathbb{R})$  if  $\text{Im}(\lambda) > 0$  which implies an eigenfunction.

## Wave Operators

If  $H$  is a self-adjoint operator, for  $t \in \mathbb{R}$ ,  $e^{-itH}$  makes sense.

$$(e^{-itH})^* = e^{(-itH)^*} = e^{itH} = (e^{-itH})^{-1}$$

so  $e^{-itH}$  is unitary. For  $u \in L^2(\mathbb{R})$ ,  $v(t) := e^{-itH}u$  solves  $(i\partial_t - H)v = 0$ ,  $v|_{t=0} = u$ .

Given  $H_0 = D_x^2$  and  $H_V = D_x^2 + V$ ,  $V \in L_{\text{comp}}^\infty(\mathbb{R})$ , the idea is that  $\forall u \in L^2(\mathbb{R})$   $u \perp \{\text{eigenfunctions of } H\}$ . There exists  $u_\pm \in L^2(\mathbb{R})$  where  $e^{-itH_V}u \approx e^{-itH_0}u_\pm$

## IMAGE 5

as  $t \rightarrow \pm\infty$ .

## Theorem

With the above setting, if  $u \in L^2(\mathbb{R})$ , the following strong limits  $W_\pm u = \lim_{t \rightarrow \pm\infty} e^{itH_V} e^{-itH_0} u$ .

Also, we have  $W_\pm H_0 = H_V W_\pm$  and  $\|W_\pm u\| = \|u\|$ .  $W_\pm$  are partial isometries intertwining  $H_0$  and  $H_V$ .

Then define the scattering operator  $S : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  by  $S = W_+^* W_-$ .

The connection between scattering/vertex operators comes from defining

$$\Phi : L^2(\mathbb{R}) \rightarrow (L^2([0, \infty]))^2$$

$$u(x) \mapsto \begin{pmatrix} \hat{u}(\lambda) \\ \hat{u}(-\lambda) \end{pmatrix}, \quad \lambda \geq 0$$

## Theorem

$$S := \Phi^* S(\lambda) \Phi = W_+^* W_-.$$

### Proof: Existence

For  $u \in L^2$ ,  $v(t)u = e^{itH_V} e^{-itH_0} u$ .

$$\frac{d}{dt}(v(t)u) = i(e^{itH_V} H_V e^{-itH_0} u - e^{itH_V} H_0 e^{-itH_0} u) = i e^{itH_V} V e^{-itH_0} u$$

So

$$v(t)u = u + i \int_0^t e^{isH_V} V e^{-isH_0} u ds$$

We want to show that

$$\int_{\mathbb{R}} \frac{\|Ve^{-isH_0}u\|_{L^2}}{\|e^{isH_v}Ve^{-isH_0}u\|_{L^2}} ds < \infty$$

If  $u_t(x) = e^{-itH_0}u$ , we want to show that  $u_t(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(x\xi - t\xi^2)} \hat{u}(\xi) d\xi$ . Note that  $e^{i(x\xi - t\xi^2)} = \left( \frac{1}{i(x-2t\xi)} \frac{d}{d\xi} \right)^2 e^{i(x\xi - t\xi^2)}$ .

### Proof of Quantum-Classical

If  $v \in C_c^\infty(\mathbb{R})$ , define  $u := \Phi^* S(\lambda) \Phi v$  and we want to show  $u = W_+^* W_- v$  if and only if  $W_+ u = W_- v$ . Write

$$w(x) = \frac{1}{2\pi} \int_0^\infty (e_+(x, \lambda) \hat{v}(\lambda) + e_-(x, \lambda) \hat{v}(-\lambda)) d\lambda$$

$$\tilde{w}(x) = \frac{1}{2\pi} \int_0^\infty (e_+(x, -\lambda) \hat{u}(-\lambda) + e_-(x, -\lambda) \hat{u}(\lambda)) d\lambda$$

$$\begin{cases} w = W_- v \\ \tilde{w} = W_+ u \\ w = \tilde{w} \end{cases}$$

where the last equality implies that

$$\tilde{w}(x) = \frac{1}{2\pi} \int_0^\infty \frac{(e_+(x, \lambda), e_-(x, \lambda S(\lambda)^{-1} J)}{(e_+(x, -\lambda), e_-(x, -\lambda))} \begin{bmatrix} \hat{u}(-\lambda) \\ \hat{u}(\lambda) \end{bmatrix} d\lambda = w(x)$$

So

$$e^{-itH_V} w = \frac{1}{2\pi} \int_0^{+\infty} e^{-it\lambda^2} (e_+(x, \lambda) \hat{v}(\lambda) + e_-(x, \lambda) \hat{v}(-\lambda)) d\lambda$$

$$= e^{itH_0} v + \overbrace{\frac{1}{2\pi} \int_0^{+\infty} \left( \frac{-R_V(\lambda)(Ve^{i\lambda})}{f_+(x, \lambda)} \hat{v}(\lambda) + \frac{-R_V(\lambda)Ve^{-i\lambda}}{f_-(x, \lambda)} \hat{v}(-\lambda) \right) d\lambda}^{:= I}$$

To show that  $\lim_{t \rightarrow \infty} I(t) = 0$ , deform  $\lambda \in [0, \infty)$  to  $\{\mu + i\mu : \mu \geq 0\}$ . Then

$$e^{-it\lambda^2} = e^{-it(\mu+i\mu)^2} = e^{2t\mu^2}$$

**December 2, 2025**

IMAGE 1

### Theorem:

Let  $V \in L_{\text{comp}}^\infty(\mathbb{R}, \mathbb{R})$ , and suppose  $w(x, t)$  solves

$$\begin{cases} (D_t^2 - P_V)w(x, t) = 0 & \text{on } \mathbb{R} \times \mathbb{R} \\ w|_{t=0} = w_0 \in H_{\text{comp}}^1(\mathbb{R}) \\ \partial_t w|_{t=0} = w_1 \in L_{\text{comp}}^2(\mathbb{R}) \end{cases}$$

## IMAGE 2

Then, for any  $A > 0$ ,

$$w(x, t) = \sum_{\operatorname{Im}(\lambda_j) > -A} \underbrace{\sum_{\ell=0}^{m_R(\lambda_j)-1} t^\ell e^{-i\lambda_j t} f_{j,\ell}(x)}_{-\operatorname{Res}_{\mu=\lambda_j}((iR_V(\mu)w_1 + \mu R_V(\mu)w_0)e^{i\mu t})} + E_A(t)$$

where  $m_R(\lambda_j) = \operatorname{rank} \oint_\lambda R_V(\xi) d\xi$  and the integral is about a small contour isolating  $\lambda$ . Note that  $\sum_{\operatorname{Im}(\lambda_j) > -A}$  is a finite sum.

We also have that  $(P_V - \lambda_j^2)f_{j,\ell} = 0$ . Then  $\forall k > 0$ ,  $\operatorname{supp}(w_j) \subset (-k, k)$ ,  $\exists C_{k,A}, T_{k,A}$  such that

$$\|E_A(t)\|_{H^2([-k,k])} \leq C_{k,A} e^{-tA} (\|w_0\|_{H^1} + \|w_1\|_{L^2})$$

## IMAGE 3

### **Returning to the Meromorphic Continuation $\rho b$ for $\operatorname{Im}(\Lambda) \gg 1$**

We have

$$R_V(\lambda) = R_0(\lambda)(I + VR_0(\lambda))^{-1}$$

which is ok via Neumann series, since  $\|R_0\|_{L^2 \rightarrow L^2} \rightarrow 0$  as  $\operatorname{Im}(\lambda) \rightarrow \infty$ . Fix  $\rho \in C_C^\infty(\mathbb{R})$ ,  $\rho V = V$  which implies  $(1 - \rho)V = 0$ . Then

$$I + VR_0(\lambda) = I + VR_0(\lambda)\rho + VR_0(\lambda)(1 - \rho) = (I + VR_0(\lambda)(1 - \rho))(I + VR_0(\lambda)\rho)$$

Notice also that  $(I - VR_0(\lambda)(1 - \rho))(I + VR_0(\lambda)(1 - \rho)) = I$ . Therefore

$$R_V(\lambda) = R_0(\lambda)(I + VR_0(\lambda)\rho)^{-1}(I - VR_0(\lambda)(1 - \rho))$$

and poles of  $R_V(\lambda)$  occur when  $I + VR_0(\lambda)\rho$  is not invertible. We have an estimate on  $VR_0(\lambda)\rho = V\rho R_0(\lambda)\rho$  of

$$\|VR_0(\lambda)\rho\|_{L^2 \rightarrow L^2} \leq \|V\|_\infty \|\rho R_0(\lambda)\rho\|_{L^2 \rightarrow L^2}$$

Write  $R_0(\lambda)f(x) = \frac{i}{2\lambda} \int_{\mathbb{R}} f(y) e^{i|x-y|\lambda} dy$  and, by Cauchy-Schwarz,

$$|\rho R_0(\lambda)\rho f(x)|^2 = \left| \frac{i}{2\lambda} \rho(x) \int \rho(y) f(y) e^{i|x-y|\lambda} dy \right|^2 \leq \frac{\rho^2(x)}{4|\lambda|^2} \|f\|_{L^2}^2 \int \rho^2(y) e^{-2\operatorname{Im}(\lambda)|x-y|} dy$$

where  $\operatorname{supp} \rho \subset [-R, R]$  and  $\rho \leq 1$ . Computing, we may find that

$$\|\rho R_0(\lambda)\rho f\|_{L^2} \leq \frac{C_1}{|\lambda|} e^{T(\operatorname{Im}(\lambda))_-} \|f\|_{L^2}$$

where  $x_- = \max\{0, -x\}$ .

## Question:

When is  $\|VR_0(\lambda)\rho\|_{L^2 \rightarrow L^2} \leq \frac{1}{2}$ ?

This happens if and only if  $\frac{C}{|\lambda|} e^{T(\text{Im}(\lambda))_-} \leq \frac{1}{2}$ , if and only if  $x^2 + y^2 = Ce^{\pm Ty}$ . If  $\text{Im}(\lambda) \geq 0$ , then  $|\lambda| \geq 2C$ ; if  $\text{Im}(\lambda) < 0$ , then  $e^{-T\text{Im}(\lambda)} \leq \frac{1}{2C} |\lambda|$  if and only if  $\text{Im}(\lambda) \geq -A - \delta \log(1 + |\lambda|)$  with  $A, \delta > 0$ .

## Claim (Exercise)

For every  $A' > 0$ , the above estimate implies that  $\Omega_{A'} = \{\lambda : \text{Im}(\lambda) > -A' - \delta \log(1 + |\lambda|)\}$  has finitely many resonances.  
Claim:  $\Omega_{A'} \cap \Omega_A$  is compact?

## Left-Right Symmetry of Resonances

Since  $P_V$  is self-adjoint,  $(P_V - \bar{z}) = \overline{P_V - z}$ , hence it is involutively stable under  $z \mapsto \bar{z}$ . If  $z = \lambda^2$ ,  $\bar{z} = (-\bar{\lambda})^2$  and  $R_V(-\bar{\lambda}) = (R_V(\lambda))^*$  is true for  $\text{Im} \gg 0$ . Hence  $\lambda$  is not a pole if and only if  $-\bar{\lambda}$  is not a pole.

## Stone's Formula

If  $E = E(P)$  is the spectral measure of some self-adjoint operator  $P$ , then

$$\frac{1}{2} [E((a, b)) + E([a, b])] = \int_a^b (P - (t + i\epsilon))^{-1} - (P - (t - i\epsilon))^{-1} dt$$

In  $\sigma_{\text{ess}}$ ,  $dE_\lambda = \frac{1}{i\pi} (R_V(\lambda) - R_V(-\lambda)) \lambda d\lambda$ .

## Expression of the Wave Solution

$P_V$  is self-adjoint, so it admits a functional calculus. So  $w = \cos(tP_V)w_0 + \frac{\sin(tP_V)}{P_V}w_1$ . Write

$$P_V = \sum_{k=1}^K -\mu_k^2 v_k \otimes v_k + \int_0^\infty z dE_z$$

where we observe that the summands are multiplicity 1 (by ODE theory and outgoing conditions), genuinely  $L^2$  bounded and normalized states. We can restrict to the case where  $w_0 = 0$  and  $w_1 \in H^2$ . In this case

$$w = \frac{\sin(tP_V)}{P_V} w_1 = \sum_{k=1}^K \frac{\sin(t\mu_k)}{\mu_k} (w_1, v_k) v_k + \int_0^\infty \frac{\sin(t\lambda)}{\lambda} dE_\lambda(w_1)$$

Then, applying Stone's Formula,

$$\begin{aligned} \int_0^\infty \frac{\sin(t\lambda)}{\lambda} dE_\lambda(w_1) &= \frac{1}{i\pi} \int_0^\infty \frac{e^{it\lambda} - e^{-it\lambda}}{2i} (R_V(\lambda) - R_V(-\lambda)) w_1 d\lambda \\ &= \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi} \int_{\mathbb{R} \setminus (-\epsilon, \epsilon)} e^{-it\lambda} (R_V(\lambda) - R_V(-\lambda)) w_1 d\lambda \end{aligned}$$

We write  $\int_{\mathbb{R} \setminus (-\epsilon, \epsilon)} = \int_{\Sigma_\epsilon} + \int_{\sigma_\epsilon}$  and compute

IMAGE 4

$$\int_{\Sigma_\epsilon} = \frac{1}{2\pi} \lim_{\epsilon \rightarrow 0} \int_{\sigma_\epsilon} e^{it\lambda} \frac{2\pi_0}{\lambda} w_1 d\lambda = -i \overbrace{\pi_0 w_1}^{(w_1, v_0) v_0}$$

Then define  $\Gamma = \{\lambda - i(A + \epsilon + \delta \log(1 + |\lambda|)) : \lambda \in \mathbb{R}\}$  and fix  $R > 0$ .

### IMAGE 5

Then  $\pi_A(t) := -i \sum_{\lambda_j \in \Omega_A} \operatorname{Res}_{\lambda=\lambda_j} (\rho R_V(\lambda) \rho e^{-i\lambda t})$ . By residue theorem,  $\rho V(t) \rho = -im_R(0)\pi_0\rho + \pi_A(t) + E_{\Gamma_R}(t) + E_{\gamma_R^\pm}(t) + E_{\gamma_R^\infty}(t)$ . So

$$E_\gamma(t) := \frac{1}{2\pi} \int_{\gamma} e^{-it\lambda} \rho(R_V(\lambda) - R_V(-\lambda)) \rho w_1 d\lambda$$

As  $R \rightarrow \infty$ ,  $E_{\gamma_R^\pm}(t), E_{\gamma_R^\infty}(t) \rightarrow 0$  and  $E_{\Gamma_R}(t) \rightarrow E_\Gamma(t) = O(e^{-at})$ .

## December 4, 2025

### Operators of Interest on Riemannian Manifolds

- Laplace-Beltrami:  $\Delta_g \geq 0$  (in the Euclidean case  $\Delta_g = -\sum_{k=1}^n \partial_x^2$ )
- Shrödinger:  $\Delta_g + V$  (where  $V$  is the potential)
- Dirac (prototypes of 1st order, elliptic, differential operators)

If  $M$  is closed, then  $\Delta_g$  has one self-adjoint realization with compact resolvent. This implies that  $\sigma(\Delta_g) = \{\lambda_0 \leq \lambda_1 \leq \dots \rightarrow \infty\}$ .

If  $\partial M \neq \emptyset$ , then  $\Delta_g^D$  (Dirichlet boundary conditions) we can define a counting function  $N(\lambda) = \#\{j : \lambda_j \leq \lambda\}$ .

### Trace Class Operators

Let  $\mathcal{H}$  separable and Hilbert, and  $\mathcal{Q} \in \mathcal{B}(\mathcal{H})$ .

We say  $\mathcal{Q}$  is trace class if  $\exists \{e_n\}_{n \geq 0}$  on orthonormal basis of  $\mathcal{H}$  such that

$$\underbrace{\sum_{n=0}^{\infty} \langle |\mathcal{Q}| e_n, e_n \rangle_{\mathcal{H}}}_{\text{call this } \operatorname{tr}(\mathcal{Q})} < \infty, \quad |\mathcal{Q}| = \sqrt{\mathcal{Q}\mathcal{Q}^*}$$

When  $\operatorname{tr} \mathcal{Q}$  exists, it does not depend on choice of basis.

### Example: Matrices

If  $A \in M_n(\mathbb{C})$ ,  $\operatorname{tr}(A) = \sum_j A_{jj} = \sum_{j=1}^n \langle Ae_j, e_j \rangle$ .

If  $A$  is a self-adjoint, nonnegative, compact operator, then denote  $(e_j, \lambda_j)_{j \geq 0}$  a complete eigenset. Then  $A$  is traceless if and only if  $\sum_{j=0}^{\infty} \lambda_j < \infty$ .

If  $\sum_{j=0}^{\infty} \lambda_j^2 < \infty$ , this is Hilbert-Schmidt; if  $\sum_{j=0}^{\infty} \lambda_j^p < \infty$ , it is  $p$ -Schatten class.

## Example

Computing these can extract analytic information.

If  $\mathcal{H} = L^2(M, \mu)$  and  $\mathcal{Q}$  is self-adjoint, compact, nonnegative, and  $(e_j(x), \lambda_j)$  is an eigenset, then

$$\mathcal{Q}f(x) = \int_M \underbrace{\mathcal{K}(x, y)}_{\text{Schwartz kernel}} f(y) d\mu(y) = \sum_{j=0}^{\infty} \lambda_j \overbrace{\left( \int_M f(y) \overline{e_j} d\mu \right)}^{\langle f, e_j \rangle} e_j(x)$$

Recall that  $\mathcal{K}(x, y) = \sum_{j=0}^{\infty} \lambda_j e_j(x) \overline{e_j(y)}$ , and notice that

$$\sum_{j=0}^{\infty} \lambda_j = \int_M \mathcal{K}(x, y) d\mu(x) = h\mathcal{Q}$$

## How to generate trace class operators out of our favorite ones?

Take  $P$  self-adjoint, nonnegative with compact resolvent ( $\lambda_0 \leq \lambda_1 \leq \dots \rightarrow \infty$ ). Usually  $P = \Delta$ .

### Example: Heat Trace

$$\begin{cases} \partial_t u + Pu = 0 & (\text{heat equation}) \\ u|_{t=0} = u_0 \end{cases}$$

Set  $u(t) = e^{-tP} u_0$  with eigenvalues  $e^{-t\lambda_0}, \dots, e^{-t\lambda_n}, \dots$ . Then  $\text{tr}(e^{-tP}) = \sum_{j=0}^{\infty} e^{-t\lambda_j}$ ,  $t > 0$ . This is trace class thanks to Weyl's law ( $N(\lambda) \sim C\lambda^{\frac{\dim(M)}{2}}$ ).

### Example: "Half-Wave" Trace

$\frac{1}{i}\partial_t u - Pu = 0$ . Any solution solves  $\partial_{tt} = \partial_t(iPu) = iP(\partial_t u) = -P^2 u$  or  $\partial_{tt} u + \Delta u = 0$  where  $\Delta u = -\sum \partial_{x_j}^2$ . Then  $u(t) = e^{itP} u_0$  which leads to  $\text{tr}(e^{itP}) = \sum_{j=0}^{\infty} e^{it\lambda_j}$  for  $t \in \mathbb{R}$ .

## Counting Functions

Consider  $H = -\partial_x^2 + V(x)$  on  $[a, b]$ .

Then  $N(\lambda) \sim \frac{1}{2\pi} \int_a^b \sqrt{\lambda - V(x)} dx$  as  $\lambda \rightarrow \infty$ . Writing  $\sqrt{\lambda - V(x)} = \lambda^{1/2} \left(1 - \frac{V(x)}{\lambda}\right)^{1/2}$ , we also see that  $N(\lambda) \sim \frac{\lambda^{1/2}}{2\pi} (b-a)$  as  $\lambda$  grows large.

We can consider an "open" analogue  $H = -\partial_x^2 + V(x)$  on  $\mathbb{R}$ . Taking  $R_V(\lambda)$  a meromorphic continuation of the outgoing resolvent for  $\lambda \in \mathbb{C}$  and defining  $m_R(\lambda) = \text{rank} \int_{\lambda} R_V(\xi) d\xi$ . By Proposition 1.9 in the Tang-Zworski notes,

$m_R(\lambda) = \frac{1}{2\pi i} \oint_{\lambda} \frac{\hat{X}'(\xi)}{\hat{X}(\xi)} d\xi$  which is the order of vanishing of  $\hat{X}$  at  $\lambda$ .

We have a counting function

$$N(r) = \sum \{m_R(\lambda) : |\lambda| \leq r\} \underset{r \rightarrow \infty}{=} \frac{2|\text{chsupp } V|}{\pi} r(1 + O(1))$$

where  $\text{chsupp } V$  is the convex hull of the support of  $V$ .

## Titchmarsh's Theorem

If  $f \in L^1_{\text{comp}}(\mathbb{R})$ , then  $|\text{chsupp } f| = b - a$ . Tang-Zworski upgrades this to  $f \in \mathcal{E}'(\mathbb{R})$ . Then  $N_{\hat{f}}(r) := \#\{\text{zeros of } \hat{f} \text{ in } D_R(0) \text{ with multiplicity}\} = \frac{b-a}{\pi}(r + O(r))$ .

## Poisson Summation Formula

Let  $\mathcal{H} = L^2(S_T^1) = \{f : \mathbb{R} \rightarrow \mathbb{C} : T\text{-periodic, } \int_0^T |f(x)|^2 dx < \infty\}$  and  $P = D_x = \frac{1}{i}\partial_x$  with eigen decomposition  $e_j(x) = \frac{1}{\sqrt{T}}e^{ijwx}$ ,  $w = \frac{2\pi}{T}$ ,  $\lambda_j = jw$  for  $j \in \mathbb{Z}$ . Then  $e^{-itP}f(x) = f(x-t)$ . For any  $\varphi \in \mathcal{S}(\mathbb{R})$ ,

$$\overbrace{\sum_{j \in \mathbb{Z}} \varphi(jw)}^{\text{tr}(\varphi(P))} = T \sum_{k \in \mathbb{Z}} \check{\varphi}(kT)$$

where  $\check{\varphi}(\xi) = \frac{1}{2\pi} \int e^{ix\xi} \varphi(x) dx$ . Then the left hand side is the spectral story while the right is the Schwarz kernel side. We can compute

$$\begin{aligned} \varphi(P)f(x) &= \left[ \int_{-\infty}^{\infty} e^{-itP} \check{\varphi}(t) dt \right] f(x) \\ &= \int_{-\infty}^{\infty} f(x-t) \check{\varphi}(t) dt \\ &= \sum_{k \in \mathbb{Z}} \int_{kT}^{(k+1)T} f(x-t) \check{\varphi}(t) dt \\ &= \int_0^T f(x-t) \underbrace{\sum_{k \in \mathbb{Z}} \check{\varphi}(t+kT)}_{:=\tilde{\varphi}(t)} dt \\ &= \int_0^T f(s) \underbrace{\tilde{\varphi}(x-s)}_{\mathcal{K}(x,s)} ds \end{aligned}$$

Therefore

$$\overbrace{\sum_{j \in \mathbb{Z}} \varphi(jw)}^{\text{tr}(\varphi(P))} = \int_0^T \mathcal{K}(x,x) dx = T\tilde{\varphi}(0)$$

At the distributional level,

$$\sum_{j \in \mathbb{Z}} \delta_{jw} = \frac{T}{2\pi} \sum_{k \in \mathbb{Z}} e^{ikTx}$$

where  $\delta_{jw}$  is the Dirac comb.

In 1975 Duistermaat and Guilleman generalized this to manifolds in the study of singular supports and the wave trace.

## An Open Analogue

If  $V \in L^\infty_{\text{comp}}(\mathbb{R}, \mathbb{R})$ ,  $P_V = -\partial_x^2 + V$ ,  $P_0 = -\partial_x^2$ , we have for  $f \in \mathcal{S}(\mathbb{R})$  that  $f(P_V) - f(P_0)$  is trace class (!) and

$$\text{tr}(f(P_V) - f(P_0)) = \int_0^\infty f(\lambda^2) \frac{\text{tr}(S(\lambda)^{-1} \partial_\lambda S(\lambda))}{2\pi i} d\lambda + \sum_{k=1}^K f(E_k) + \frac{1}{2}(m_R(0) - 1)f(0)$$

where  $S(\lambda)$  is the scattering matrix  $S(\lambda) = \begin{bmatrix} T(\lambda) & R_-(\lambda) \\ R_+(\lambda) & T(\lambda) \end{bmatrix}$  and we have the Breit-Wigner approximation for  $\frac{\text{tr}(S(\lambda)^{-1} \partial_\lambda S(\lambda))}{2\pi i}$  of

$$-\frac{1}{\pi} |\text{chsupp } V| - \frac{1}{\pi} \sum_j \frac{\text{Im } \lambda_j}{|\lambda - \lambda_j|^2}$$

with  $j$  the resonances quantity.

### Theorem (DZ, 2.21)

Let  $f_t(s) \cos(t + \sqrt{s})$ . Then for any  $\varphi \in C_C^\infty(\mathbb{R})$ ,

$$\int_{\mathbb{R}} \varphi(t) (\cos(t\sqrt{P_V}) - \cos(t\sqrt{P_0})) dt$$

is trace class and

$$2 \text{tr}((\cos(t\sqrt{P_V}) - \cos(t\sqrt{P_0})) = \text{p.v.} \sum_{\lambda \text{ resonances}} m_R(\lambda) e^{-i\lambda|t|} - 2|\text{chsupp } V|\delta_0 - 1$$

where

$$\left\langle \text{p.v.} \sum_j e^{-i\lambda_j|t|}, \varphi \right\rangle := \lim_{R \rightarrow \infty} \sum_{|\lambda_j| \leq R} \int_{\mathbb{R}} \varphi(t) e^{-i\lambda_j|t|} dt$$

## December 11, 2025

### Inversion of the Fourier Transform via Riemann-Hilbert Problems

For  $q, \frac{dq}{dx} \in L^1(\mathbb{R})$ ,  $\hat{q}(k) := \int_{\mathbb{R}} e^{-ikx} q(x) dx$ , we want to prove that  $q(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx} \hat{q}(k) dk$ . Consider the differential equation

$$\begin{cases} \frac{d\mu}{dx} - ik\mu = q(x) \\ \mu \rightarrow 0, & |x| \rightarrow \infty \end{cases}$$

By integrating factors,  $\frac{d}{dx}(e^{-ikx} \mu(x)) = e^{-ikx} q(x)$ . So if  $\mu(-\infty) = 0$ , we get

$$\begin{aligned} e^{-ikx} \mu(x) &= \int_{-\infty}^x e^{-iky} q(y) dy \\ \mu_+(x, k) &= \int_{-\infty}^x e^{ik(x-y)} q(y) dy \end{aligned}$$

IMAGE 1

If instead  $\mu(+\infty) = 0$ ,

$$\mu_-(x, k) = - \int_x^\infty e^{ik(x-y)} q(y) dy$$

It follows that for  $k \in \mathbb{R}$

$$\hat{q}(x) = \lim_{x \rightarrow \infty} e^{-ikx} \mu_+(x, k) = \lim_{x \rightarrow -\infty} e^{-ikx} \mu_-(x, k)$$

For  $\text{Im}(k) \geq 0$ , write  $k = \text{Im}(k) + i\text{Im}(k) = k_R + ik_I$  and notice

$$e^{ik(x-y)} = e^{ik_R(x-y)} e^{-k_I(x-y)}$$

where  $e^{-k_I(x-y)} \leq 1$  if  $x - y \geq 0$  and  $k_I \geq 0$ . It follows that  $\mu_+(x, k)$  makes sense on  $\mathbb{C}_+$  as a holomorphic function on  $\{k_I > 0\}$  with continuous limits at  $k \in \mathbb{R}$ . Likewise,  $\mu_-(x, k)$  is the same on  $\mathbb{C}_-$  and  $\{k_I < 0\}$  respectively.

Notice also that  $\hat{q}(k)e^{ikx} = \mu_+(x, k) - \mu_-(x, k)$ ,  $k \in \mathbb{R}$ . With  $x$  frozen,  $\mu(x, k) := \mu_\pm(x, k)$  on  $\mathbb{C}_\pm^\circ$  satisfies

- $\mu$  is sectionally analytic on  $\mathbb{C}$  with left and right limits as  $k \rightarrow \mathbb{R}$ .
- $\mu_+(x, k) - \mu_-(x, k) = \hat{q}(k)e^{ikx} := \phi_x(k)$  (prescribed jump condition)
- $\lim_{|k| \rightarrow \infty} \mu_\pm(x, k) = 0$

To see the third point,

$$k\mu_+(x, k) = \int_{-\infty}^x \overbrace{\frac{\frac{d}{dy}ie^{ik(x-y)}}{ke^{ik(x-y)}}}^{\frac{d}{dy}ie^{ik(x-y)}} q(y) dy = \overbrace{\left[ q(y)ie^{ik(x-y)} \right]_{-\infty}^x}^{iq(x)} - \overbrace{\int_{-\infty}^x ie^{ik(x-y)} q'(y) dy}^{\text{bounded on } \mathbb{C}}$$

So  $\mu_+(x, k) \leq \frac{iq(x)+C}{k}$ . We find that the solution to the Riemann-Hilbert Problem is unique and given by

$$\mu(k) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\phi(\ell)}{\ell - k} d\ell = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\hat{\mu}(\ell)e^{i\ell x}}{\ell - k} d\ell, \quad k \in \mathbb{C} \setminus \mathbb{R}$$

Then

$$k\mu(k) = \frac{1}{2\pi i} \int_{\mathbb{R}} \hat{q}(\ell) e^{i\ell x} \frac{1}{\ell - k} d\ell \rightarrow -\frac{1}{2\pi i} \int_{\mathbb{R}} \hat{q}(\ell) e^{i\ell x} d\ell$$

## Inverse Scattering

$Lu = -\frac{d^2u}{dx^2} - V(x)u$  with  $V \in C_C^\infty(\mathbb{R})$ ,  $\sigma(L) = \sigma_{\text{ess}}(L) = [0, +\infty)$ , and  $\lambda = s^2$ .

Define four solutions to  $Lu = s^2 u$  for  $s \in \mathbb{R}$ ,

$\psi_p(x; s) = e^{isx}$ , $x \gg 0$	$\xrightarrow{*} \xleftarrow{*}   V   \xrightarrow{1} \xleftarrow{0}$
$\phi_p(x; s) = e^{isx}$ , $x \ll 0$	$\xrightarrow{1} \xleftarrow{0}   V   \xrightarrow{*} \xleftarrow{*}$
$\psi_m(x; s) = e^{-isx}$ , $x \gg 0$	$\xrightarrow{*} \xleftarrow{*}   V   \xrightarrow{0} \xleftarrow{1}$
$\phi_m(x; s) = e^{-isx}$ , $x \gg 0$	$\xrightarrow{0} \xleftarrow{1}   V   \xrightarrow{*} \xleftarrow{*}$

Since they are four solutions of a second order ODE, they must be related.

$$\begin{bmatrix} \psi_p(x; s) & \psi'_p(x; s) \\ \psi_m(x; s) & \psi'_m(x; s) \end{bmatrix} = S(s) \begin{bmatrix} \phi_p(x; s) & \phi'_p(x; s) \\ \phi_m(x; s) & \phi'_m(x; s) \end{bmatrix}$$

where

$$\begin{aligned} S(s) &= \begin{bmatrix} \psi_p(x; s) & \psi'_p(x; s) \\ \psi_m(x; s) & \psi'_m(x; s) \end{bmatrix} \frac{1}{W(\phi_p(x; s), \phi_m(x; s))} \begin{bmatrix} \phi'_m(x; s) & -\phi'_p(x; s) \\ -\phi_m(x; s) & \phi_p(x; s) \end{bmatrix} \\ &= -\frac{1}{2is} \begin{bmatrix} W(\psi_p(x; s), \phi_m(x; s)) & W(\phi_p(x; s), \psi_p(x; s)) \\ W(\psi_m(x; s), \phi_m(x; s)) & W(\phi_p(x; s), \psi_m(x; s)) \end{bmatrix} \\ &= \begin{bmatrix} A(s) & B(s) \\ b(s) & a(s) \end{bmatrix} \end{aligned}$$

Note also that  $\psi_p = \overline{\psi_m}$ ,  $\phi_p = \overline{\phi_m}$ , so  $a(s) = \overline{A(s)}$  and  $b(s) = \overline{B(s)}$ . We have a reflection coefficient  $\rho(s) := \frac{b(s)}{a(s)}$ . We observe that  $\psi_p$  and  $\phi_m$  can be analytically continued to  $\mathbb{C}_-$  while  $\phi_p$  and  $\psi_m$  extend to  $\mathbb{C}_+$ .

- Example

$$\frac{\Phi_1^-(x; z)}{\phi_m(x; z) e^{izx}} = 1 - \frac{1}{2iz} \int_{-\infty}^x (1 - e^{2iz(x-\tau)}) v(z) \frac{\Phi_1^-(\tau; z)}{\phi_m(\tau; z) e^{iz\tau}} d\tau$$

which is still solvable for  $z \in \mathbb{C}_+$ .

So  $[\psi_m \ \phi_p] = [\phi_m \ \psi_p] \tilde{G}(s)$  where

$$\tilde{G}(s) = \begin{bmatrix} a(s) - \frac{B(s)b(s)}{A(s)} & -\frac{B(s)}{A(s)} \\ \frac{b(s)}{A(s)} & \frac{1}{A(s)} \end{bmatrix}$$

but we have a problem:  $\tilde{G}(s)$  has monodromy (i.e.  $\det \tilde{G}(s)$  has nonzero winding number). We renormalize

$$\begin{aligned} [\psi_m &\ \phi_p] \begin{bmatrix} e^{izx} & 0 \\ 0 & e^{-izx} \end{bmatrix} = [1 &\ 1] + \mathcal{O}\left(\frac{1}{z}\right), \quad |z| \rightarrow \infty \\ [\phi_m &\ \psi_p] \begin{bmatrix} e^{izx} & 0 \\ 0 & e^{-izx} \end{bmatrix} = [1 &\ 1] + \mathcal{O}\left(\frac{1}{z}\right), \quad |z| \rightarrow \infty \end{aligned}$$

and instead look at

$$\begin{bmatrix} \frac{\psi_m}{a(s)} & \phi_p \end{bmatrix} = \begin{bmatrix} \phi_m & \frac{\psi_p}{A(s)} \end{bmatrix} G(s)$$

for  $s \in \mathbb{R}$  where

$$G(s) = \begin{bmatrix} 1 - \rho(s)\bar{\rho}(s) & -\bar{\rho}(s) \\ \rho(s) & 1 \end{bmatrix}$$

and  $\det G(s) = 1$ . Now we define

$$\Phi(z) := \begin{cases} \Phi^- := \begin{bmatrix} \frac{\psi_m}{a} & \phi_p \end{bmatrix} \begin{bmatrix} e^{izx} & 0 \\ 0 & e^{-izx} \end{bmatrix}, & \mathbb{C}_+ \\ \Phi^+ := \begin{bmatrix} \phi_m & \frac{\psi_p}{A} \end{bmatrix} \begin{bmatrix} e^{izx} & 0 \\ 0 & e^{-izx} \end{bmatrix}, & \mathbb{C}_- \end{cases}$$

Normalizing  $\Phi(\infty) = [1 \ 1]$ , we have a jump

$$\Phi^+(s) = \Phi^-(s) \overbrace{\begin{bmatrix} e^{-isx} & 0 \\ 0 & e^{isx} \end{bmatrix}}^{\det=1} G(s) \begin{bmatrix} e^{isx} & 0 \\ 0 & e^{-isx} \end{bmatrix}$$

Riemann-Hilbert theory tells us that there exists a unique solution  $\Phi$  to this problem and gives us a way to compute it

$$\partial_x \Phi^-(z)_1 = e^{-2izx} \int_{-\infty}^x e^{-2iz\tau} V(\tau) \Phi_1^-(\tau; z) d\tau \sim \overbrace{\frac{1}{2iz} \Phi_1^-(x; z)}^1 V(x)$$