

Topics in Analysis (F24)

September 30, 2024

Chapter 1: Banach Algebras

1.1: Definitions and Basic Properties

Definition: Banach Space

A Banach space X (over \mathbb{C}) is a normed vector space with algebraic operations

$$\begin{aligned}(x, y) &\mapsto x + y && \text{addition} \\ (\lambda, y) &\mapsto \lambda y && \text{scalar multiplication}\end{aligned}$$

and a norm

$$x \mapsto \|x\|$$

which is complete (i.e. every Cauchy sequence converges).

Definition: (Complex) Banach Algebra

A (complex) Banach algebra B is a Banach space in which there is multiplication

$$B \times B \ni (x, y) \mapsto xy \in B$$

such that

1. $x(yz) = (xy)z$
2. $(x+y)z = xz + yz$ and $x(y+z) = xy + xz$
3. $\lambda(xy) = (\lambda x)y = x(\lambda y)$
4. $\|xy\| \leq \|x\| \cdot \|y\|$

Definition: Unital Banach Algebra

B is called a unital Banach algebra if $\exists e \in B$ such that

$$xe = ex = x \quad \text{and} \quad \|e\| = 1.$$

If e exists, it is unique.

1.2: Examples

Example 1

If X is a Banach space, then $B = \mathcal{L}(X)$ (the set of all bounded linear operators $A : X \rightarrow X$) equipped with algebraic operations

$$\begin{aligned}
(A+B)x &= Ax + Bx \\
(\lambda A)x &= \lambda(Ax) \\
(AB)x &= A(Bx)
\end{aligned}$$

and the operator norm

$$\|A\|_{\mathcal{L}(X)} = \sup_{x \neq 0} \frac{\|Ax\|_X}{\|x\|_X}.$$

$B = \mathcal{L}(X)$ is complete because X is complete.

The unit element is given by $I_X x = x$.

Example 2

If $X = \mathbb{C}^n$, then $B = \mathcal{L}(\mathbb{C}^n) \cong \mathbb{C}^{n \times n}$.

$$A = (a_{ij})_{i,j=1}^n \quad Ax = y \quad \sum_{j=1}^n a_{ij} x_j = y_i.$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

The norm in \mathbb{C}^n leads to a norm in $\mathbb{C}^{n \times n}$

$$\begin{aligned}
\|(x_i)\| &= \left(\sum |x_i|^2 \right)^{1/2} & \|A\| &= \\
\|(x_i)\| &= \sum |x_i| & \|A\| &= \max_j \sum_i |a_{ij}| \\
\|(x_i)\| &= \max |x_i| & \|A\| &= \max_i \sum_j |a_{ij}|
\end{aligned}$$

All norms are equivalent.

Example 3

Take $B = C(K)$ with K a compact Hausdorff space, $f : K \rightarrow \mathbb{C}$ continuous and $\|f\| = \max_{t \in K} |f(t)|$.

Example 4

Take $B = A(K)$, $K \subseteq \mathbb{C}$ compact with $\text{int}(K) \neq \emptyset$, $f : K \rightarrow \mathbb{C}$ continuous where f is holomorphic on $\text{int}(K)$ and

$$\|f\| = \max_{t \in K} |f(t)| = \max_{t \in K \setminus \text{int}(K)} |f(t)|$$

e.g. $K = \overline{\mathbb{D}} = \{t \in \mathbb{C} : |t| \leq 1\}$. Then $A(K) \subseteq C(K)$.

Example 5

Take $B = \ell^\infty(\mathbb{N})$ or $B = L^\infty(S, \sigma, \mu)$ with (S, σ, μ) a measure space, $f : S \rightarrow \mathbb{C}$ essentially bounded functions and

$$\|f\| = \text{ess sup}_{t \in S} |f(t)| = \inf_{\substack{N \subseteq S \\ \mu(N)}} \left(\sup_{t \in S \setminus N} |f(t)| \right)$$

Example 6

Take $B = \ell^1(\mathbb{Z})$ or $B = L^1(\mathbb{R}^d)$ with $\|\{x_n\}\| = \sum |x_n|$ and $\|f\| = \int_{\mathbb{R}^d} |f(t)| dt$ respectively. Multiplication is given by the convolution. e.g.

$$fg = (f * g)(x) = \int_{\mathbb{R}^d} f(x-t)g(t) dt$$

$\ell^1(\mathbb{Z})$ is unital, but $L^1(\mathbb{R}^d)$ is non-unital (since the unit of convolution is the Dirac delta; see Example 7).

Example 7

Take $B = M(\mathbb{R}^d)$ the complex measures on \mathbb{R}^d with bounded variation.

Then multiplication is given as

$$(\mu * \nu)(A) = \int_{\mathbb{R}^d} \mu(A-x) d\nu(x)$$

and norm

$$\|\mu\| = \sup_{\substack{\mathbb{R}^d = \bigcup A_i \\ \text{disjoint}}} \sum_{i=1}^n |\mu(A_i)| < +\infty.$$

Then, $f dm = d\mu$ gives $L^1(\mathbb{R}^d) \rightarrow M(\mathbb{R}^d)$.

Example 8

Take $B = C^{n \times n}[K]$ with K compact and Hausdorff, continuous functions $f : K \rightarrow \mathbb{C}^{n \times n}$ and norm

$$\|f\|_B = \max_{t \in K} \|f(t)\|_{C^{n \times n}}.$$

Then $B \cong (C(K))^{n \times n}$ the $n \times n$ matrices with entries from $C(K)$.

1.3: Remarks

- If B does not have a unit element, consider $B_1 = B \times \mathbb{C}$ with operations

$$\begin{aligned} (b_1, \lambda_1) + (b_2, \lambda_2) &= (b_1 + b_2, \lambda_1 + \lambda_2) \\ \alpha(b, \lambda) &= (\alpha b, \alpha \lambda) \\ (b_1, \lambda_1)(b_2, \lambda_2) &= b_1 b_2 + \lambda_1 b_2 + \lambda_2 b_1, \lambda_1 \lambda_2) \end{aligned}$$

and norm

$$||(b, \lambda)|| = ||b|| + |\lambda|.$$

Then B_1 is a unital Banach algebra with $e = (0, 1)$. One writes $(b, \lambda) = (b, 0) + \lambda(0, 1) = b + \lambda \cdot e$. In some sense, $B \subseteq B_1$ where $b \in B \mapsto (b, 0) \in B_1$.

1.4: Definitions

Definition: Commutative Banach Algebra

B is called commutative if $xy = yx$.

Definition: Banach Subalgebra

A subset B_0 of a B -algebra is called a subalgebra if it is closed with respect to the algebraic operations

$$x, y \in B_0, \lambda \in \mathbb{C} \rightsquigarrow x + y, xy, \lambda x \in B$$

Definition: Closed Subalgebra

B_0 is a closed subalgebra or Banach subalgebra if it is norm-closed.

- Proposition: B_0 is a Banach algebra.

Definition: Generated Subalgebra

Let $M \neq \emptyset$ be a subset of a Banach algebra B .

The Banach subalgebra generated by M is the smallest closed subalgebra containing M .

$$\text{alg } M = (\text{clos alg}_B M)$$

- Remark

$\text{alg } M$ is the intersection of all closed subalgebras containing M .

$\text{alg } M = \text{clos} \left\{ \sum_{i=1}^N \lambda_i a_1^{(i)} a_2^{(i)} \cdots a_{n_i}^{(i)} \right\}$ is the norm-closure of finite linear combinations of finite products of $a_j^{(i)} \in M$.

1.5: Examples

Example 1

Take B unital, $b \in B$. Then

$$\text{alg}\{e, b\} = \text{clos}_B \left\{ \sum_{i=0}^N \lambda_i b^i : \lambda_i \in \mathbb{C}, N \in \mathbb{N} \right\}$$

where $b^0 = e$.

1.6 Definitions

Definition: Banach Algebra Homomorphism

A Banach algebra homomorphism is a map $\phi : B_1 \rightarrow B_2$ between Banach algebras B_1 and B_2 such that

- ϕ is linear
- ϕ is bounded (continuous)
- ϕ is multiplicative

$$\phi(b_1 b_2) = \phi(b_1) \cdot \phi(b_2)$$

- ϕ is unital if both B_1, B_2 have units and $\phi(e_{B_1}) = e_{B_2}$.

Definition: Banach Algebra Isomorphism

A Banach algebra homomorphism which is bijective is called a Banach algebra isomorphism.

Then $\phi^{-1} : B_2 \rightarrow B_1$ is an isomorphism as well.

Definition: Banach Algebra Isometry

ϕ is an isometry if $||\phi(x)|| = ||x||$.

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Recall

Given $M \subseteq \mathcal{L}(X)$ with X a Banach space (and $\mathcal{L}(X)$ itself a Banach algebra), we may construct $B = \text{alg}_{\mathcal{L}(X)} M$.

1.7 Proposition

Let B be a unital Banach algebra. Then the map

$$\phi : B \ni x \rightarrow L_x \in \mathcal{L}(B)$$

is an isometric isomorphism onto a closed subalgebra of $\mathcal{L}(B)$ where

$$L_x : B \ni z \mapsto xz \in B$$

is the left-representation of x .

Proof

L_x is in $\mathcal{L}(B)$ since $L_x z = xz$

- is linear in z and
- $||L_x z|| = ||xz|| \leq ||x|| \cdot ||z||$ implies $||L_x|| \leq ||x||$ (i.e. L_x is a bounded).

The map $\phi : x \mapsto L_x$ is linear

$$L_{x_1+x_2}z = (x_1 + x_2)z = x_1z + x_2z = L_{x_1}z + L_{x_2}z = (L_{x_1} + L_{x_2})z$$

ϕ is multiplicative

$$L_{x_1x_2}z = (x_1x_2)z = x_1(x_2z) = L_{x_1}(L_{x_2}z)$$

From the above, we conclude that ϕ is a homomorphism.

To show that ϕ is an isometry,

$$\|L_x\| = \sup_{z \neq 0} \frac{\|L_x z\|}{\|z\|} \geq \frac{\|L_x e\|}{\|e\|} = \frac{\|x\|}{1} = \|x\|.$$

Then also ϕ is injective and $\text{im } \phi$ is closed. Since $\text{im } \phi$ is a Banach algebra, it is therefore a closed subalgebra.

1.7 Remark: Right-Regular Representation

Every unital Banach algebra is isometrically isomorphic to a Banach algebra of operators.

Right-regular representation:

$$R_x = z \mapsto zx$$

Section 1.2: Group of Invertible Elements in a Banach Algebra

2.1 Definition: Invertible Element

Let B be a unital Banach algebra. An element $x \in B$ (in B) if there exists $y \in B$ such that $xy = yx = e$.

Note that $y = x^{-1}$ is uniquely determined.

Write GB for the set of all invertible elements of B .

Remark

GB is a (multiplicative group).

- $x, y \in GB \implies xy \in GB$ and $(xy)^{-1} = y^{-1}x^{-1}$,
- $x \in GB \implies x^{-1} \in GB$ and $(x^{-1})^{-1} = x$, and
- $e \in GB$.

2.2 Lemma

If $x \in B$ and $\|x\| < 1$, then $e - x \in GB$.

Proof

Take the Neumann series

$$e + x + x^2 + x^3 + \dots$$

which converges to some $s \in B$

$$s_n = e + x + \cdots + x^n$$

where s_n are Cauchy:

$$||s_{n+k} - s_n|| = ||x^{n+1} + \cdots + x^{n+k}|| \leq ||x||^{n+1} + ||x||^{n+2} + \cdots = \frac{||x||^{n+1}}{1 - ||x||}.$$

So $s_n \rightarrow S$,

$$(e - x)s_n = s_n(e - x)e - x^{n+1}.$$

Taking $n \rightarrow \infty$

$$(e - x)s = s(e - x) = e.$$

2.3 Proposition

The group GB is open in B and the map $\Lambda : GB \ni x \mapsto x^{-1} \in GB$ is continuous (in the norm).

Proof

Take $x \in GB$ and consider $y \in B$ with $||y|| < \frac{1}{||x^{-1}||} = \varepsilon$.

Then $x + y \in B_\varepsilon(x)$ is invertible,

$$x + y = x(e + x^{-1}y),$$

and

$$||x^{-1}y|| \leq ||x^{-1}|| \cdot ||x|| < 1.$$

Therefore GB is open, since $B_\varepsilon(X) \subseteq GB$. The inverse

$$(x + y)^{-1} = (e + x^{-1}y)^{-1}x^{-1} = \sum_{n=0}^{\infty} (-x^{-1}y)^n x^{-1} = x^{-1} + \sum_{n=1}^{\infty} (-x^{-1}y)^n x^{-1}$$

so

$$||(x + y)^{-1} - x^{-1}|| \leq \sum_{n=1}^{\infty} ||x^{-1}||^{n+1} ||y||^n = \frac{||x^{-1}||^2 ||y||}{1 - ||x^{-1}|| \cdot ||y||}.$$

This converges to zero as $||y|| \rightarrow 0$.

2.4 Examples

Example 1

$B = C(K)$, K compact Hausdorff, $f : K \rightarrow \mathbb{C}$ continuous.

$GB = \{f \in C(K) : f(t) \neq 0, \forall t \in K\}$.

Example 2

$$B = \mathbb{C}^{n \times n}.$$

$$GB = \{A \in \mathbb{C}^{n \times n} : \det A \neq 0\}.$$

2.5 Definition:

Let $G_0 B$ stand for the connected component of GB containing e .

Remarks

- the ε -neighborhoods $B_\varepsilon(x) \subseteq B$ are (path-)connected.

$$B_\varepsilon(x) = \{y \in B : ||x - y|| < \varepsilon\}$$

For $y_1, y_2 \in B_\varepsilon(x)$, there is a continuous path

$$\sigma : [0, 1] \ni \lambda \mapsto y_1 \lambda + y_2 (1 - \lambda) \in B_\varepsilon(x)$$

- Because GB is open and $B_\varepsilon(x)$ is path-connected, GB is locally (path-)connected (i.e. every $x \in GB$ has a (path-)connected open neighborhood in GB).
- In this context, connectedness and path-connectedness are equivalent. Therefore the components of GB are the path-components of GB .
- GB is the union of disjoint (path-)components where each component is both open and closed in GB .
- $x, y \in GB$ belong to the same path-component if there exists a continuous path $\gamma : [0, 1] \rightarrow GB$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Here, $x \sim y$ is an equivalence relation.
- $G_0 B = \{x \in GB : \exists \text{ a path in } GB \text{ connecting } e \text{ and } x\}$.

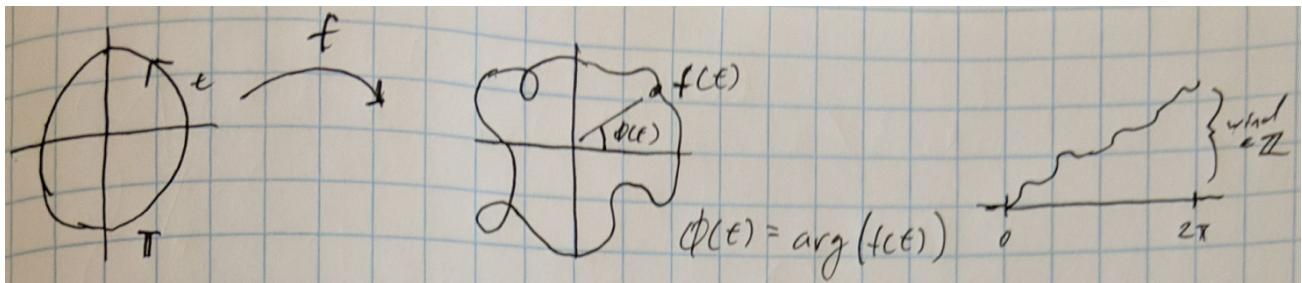
2.6 Examples

Example 1

Take $B = C(\mathbb{T})$ with $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and continuous functions $f : \mathbb{T} \rightarrow \mathbb{C}$.

GB is the non-vanishing continuous functions $f : \mathbb{T} \rightarrow \mathbb{C}$ ($f(t) \neq 0, \forall t \in \mathbb{T}$).

For $f \in GB$ one can define a winding number.



We have $\frac{1}{2\pi} \arg f(e^{ix})$ a continuous function with

$$\text{wind}(t) = \left[\frac{1}{2\pi} \arg f(e^{ix}) \right]_{x=0}^{2\pi} = \phi(2\pi) - \phi(0)$$

and $\text{wind}(t) \in \mathbb{Z}$.

The map $GB \ni f \mapsto \text{wind}(f) \in \mathbb{Z}$ is continuous, hence locally constant (i.e. constant on each connected component).

Therefore $G_0 C(\mathbb{T}) \subseteq \{f \in GC(\mathbb{T}) : \text{wind}(f) = 0\}$. In fact, we will see that we have equality.

That is, f can be contracted (in GB) to the constant function $e(t) = 1$.

2.7 Proposition

$G_0 B$ is a normal subgroup of GB .

Proof

- $G_0 B$ is a group.

For any $x, y \in G_0 B$, there exist paths $\gamma_1 : [0, 1] \rightarrow GB$ and $\gamma_2 : [0, 1] \rightarrow GB$ with $\gamma_1(0) = \gamma_2(0) = e$, $\gamma_1(1) = x$ and $\gamma_2(1) = y$.

Define $\gamma(t) = \gamma_1(t)\gamma_2(t)$ a path in GB such that $\gamma(0) = e$ and $\gamma(1) = xy$. Then $xy \in G_0 B$.

Following from Lemma 2.2, $\hat{\gamma} = (\gamma_1(t))^{-1}$ is a continuous path with $\hat{\gamma}_1(0) = e$, $\hat{\gamma}_1(1) = x^{-1}$ and $x^{-1} \in GB$.

- $G_0 B$ is a normal subgroup of GB .

For every $y \in GB$, $yG_0By^{-1} \subseteq G_0B$ if and only if $yG_0B = G_0By$.

Take $x \in G_0 B$ with path γ , then

$$\delta(t) = y\gamma(t)y^{-1}, \quad \delta(0) = yey^{-1} = e, \quad \text{and} \quad \delta(1)yxy^{-1} \in G_0 B.$$

2.8 Definition: Abstract Index Group

The quotient group $GB/G_0 B$ is called the abstract index group of B .

Remark

$GB/G_0 B$ is in 1-to-1 correspondence with the set of connected components of GB .

Indeed, the (path-)connected components of GB are given by $yG_0 B = G_0 B y$ (for $y \in GB$).

$$y_1 G_0 B = y_2 G_0 B \iff y_2^{-1} y_1 G_0 B = G_0 B \iff y_2^{-1} y_1 \in G_0 B \iff [y_2] = [y_1] \text{ in } GB/G_0 B.$$

2.9 Definition: Exponential Map

For $x \in B$, we define the exponential map $B \ni x \mapsto \exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

2.10 Lemma

The exponential map $B \ni x \mapsto \exp(x) \in GB$ is well-defined and continuous.

For $xy = yx$, we have $\exp(x+y) = \exp(x)\exp(y)$.

In particular, $(\exp(x))^{-1} = \exp(-x)$.

Proof

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergent.

$$\sum_{n=0}^{\infty} \frac{||x||^n}{n!} < +\infty.$$

It follows that $s_n = \sum_{k=0}^n \frac{x^k}{k!}$ is a Cauchy sequence and therefore converges.
 Continuity left as an exercise. Need to show:

$$\left| \left| \sum \frac{x^n}{n!} - \sum \frac{y^n}{n!} \right| \right| \leq ||x - y|| \cdot M_{x,y}$$

The fact that $\exp(x + y) = \exp(x)\exp(y)$ follows from multiplying terms and the binomial formula.

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Recall

GB $e + x$.

G_0B connected component of GB containing e .

GB/G_0B is the abstract index group.

$B = C(\mathbb{T}) \rightsquigarrow f \in GC(\mathbb{T}) \rightsquigarrow \text{ind}(f)$.

Definition: Exponential Map

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \in GB$$

Lemma:

For $y \in B$, $||y|| < 1$, there exists $x \in B$ such that $\exp(x) = e + y$.

Proof

Define

$$\log(e + y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \dots \in B.$$

This converges absolutely ($||y|| < 1$), therefore it converges in B by completeness.

Identities

$$\exp(\log(e + y)) = \sum_{n=0}^{\infty} \frac{\left(\sum_k \frac{y^k}{k} (-1)^{k-1} \right)^n}{n!} = e + y$$

Proof

G_0B is equal to the set of all finite products of exponentials of elements in B .

$$G_0B = \bigcup_{n=0}^{\infty} \Gamma_n = \bigcup_{n=0}^{\infty} \{ \exp(a_1) \exp(a_2) \cdots \exp_{a_n} \in B \}$$

Proof

Call $\Gamma = \bigcup_{n=0}^{\infty} \Gamma^n$.

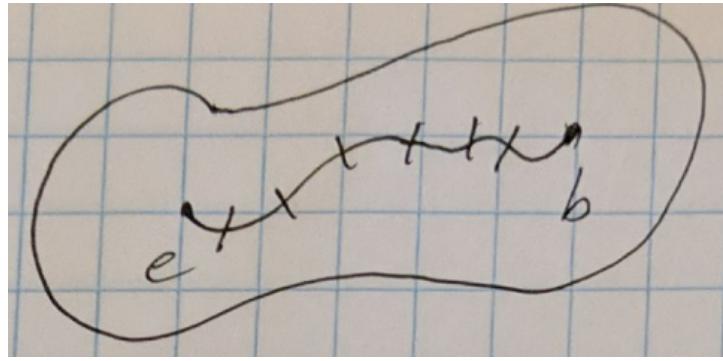
Then observe that each Γ_n is path-connected and contains e .

For $b = \exp(a_1) \cdots \exp(a_n) \in \Gamma_n$, define a path

- $\sigma : [0, 1] \rightarrow \Gamma_n$
- $\sigma(t) = \exp(ta_1) \cdots \exp(ta_n)$ is continuous with $\sigma(0) = e$ and $\sigma(1) = b$.

Therefore, Γ is path-connected and contains e . It follows that $\Gamma \subseteq G_0 B$.

To prove that $G_0 B \subseteq \Gamma$, take $b \in G_0 B$ and show that there exists a path in GB $\gamma : [0, 1] \rightarrow GB$ continuous with $\gamma(0) = e$ and $\gamma(1) = b$.



We have that $(\gamma(t))^{-1}$ is continuous and bounded in the norm. Then $\gamma(t)$ is uniformly continuous.

$$\|\gamma^{-1}(t)\| \leq M.$$

$$(\exists N) : |t - s| \leq \frac{1}{N} \implies \|\gamma(t) - \gamma(s)\| \leq \frac{1}{M} \cdot \frac{1}{2}. \text{ Write}$$

$$b = \gamma(1) \cdot \gamma^{-1}(0) = \gamma(1) \gamma^{-1}\left(\frac{N-1}{N}\right) \gamma\left(\frac{N-1}{N}\right) \gamma^{-1}\left(\frac{N-2}{2}\right) \cdots \gamma\left(\frac{1}{N}\right) \gamma^{-1}\left(\frac{1}{N}\right) \gamma(0) = \prod_{k=1}^N \gamma^{-1}\left(\frac{k}{N}\right) \gamma\left(\frac{k-1}{N}\right).$$

Therefore, with $s_k = \gamma^{-1}\left(\frac{k}{N}\right) \gamma\left(\frac{k-1}{N}\right)$, $b = \prod_{k=1}^N \exp(\log(s_k))$.

$$\|s_k - e\| \leq \|\gamma^{-1}\left(\frac{k}{N}\right)\| \cdot \|\gamma\left(\frac{k-1}{N}\right) - \gamma\left(\frac{k}{N}\right)\| \leq M \cdot \frac{1}{2M} \leq \frac{1}{2}.$$

Corollary

If B is commutative, $G_0 B = \{\exp(a) : a \in B\}$.

Remark

Special case: $B = C(K)$ (K compact Hausdorff space).

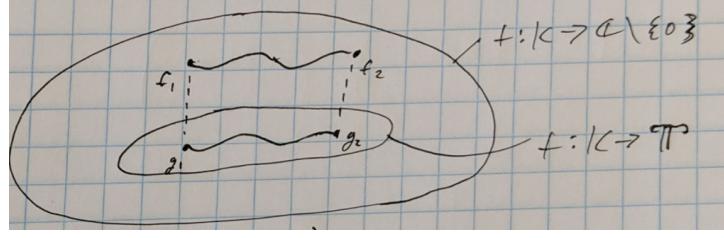
$$G_0 B = \{\exp(a) : a \in C(K)\}.$$

$GB/G_0 B$ is an equivalence class of functions $f : K \rightarrow \mathbb{C} \setminus \{0\}$ with respect to path-connectedness.

That is, $f_1 \sim f_2$ if and only if there exists continuous $F(t, x) : [0, 1] \times K \rightarrow \mathbb{C} \setminus \{0\}$ with $F(0, x) = f_1(x)$ and $F(1, x) = f_2(x)$.

These are the homotopy classes of continuous functions $f : K \rightarrow \mathbb{C} \setminus \{0\}$.

This corresponds to homotopy classes of continuous functions $f : K \rightarrow \mathbb{T}$ (with $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$) called the 1st co-homotopy group of K $\pi_1^*(K)$.



$f : K \rightarrow \mathbb{C} \setminus \{0\}$ and $\frac{f}{|f|} : K \rightarrow \mathbb{C} \setminus \{0\}$ are path-connected by $\sigma(s) = \frac{f}{|f|^s}$, $s \in [0, 1]$.

$f_1 \sim f_2$ in $K \rightarrow \mathbb{C} \setminus \{0\}$ implies that $\frac{f_1}{\|f_1\|} \sim \frac{f_2}{\|f_2\|}$ in $K \rightarrow \mathbb{T}$ by $F(s, x)$ and $\frac{F(s, x)}{|F(s, x)|}$.

We conclude that $\pi^1(K) \cong GC(K)/G_0C(K)$.

Example

Let $B = C(\mathbb{T})$.

$$G_0B = \{\exp(a) : a \in C(\mathbb{T}) = \{f \in GC(\mathbb{T}) : \text{wind}(f) = 0\}\}$$

For $f \in GC(\mathbb{T})$, $\text{wind}(f) = 0$ implies that $f = \exp(a)$ has a logarithm.

This implies that $f \in G_0B$ which itself implies that $\text{wind}(f) = 0$, since $\text{wind}(f)$ is continuous on $GC(\mathbb{T})$ and therefore constant on the component.

Therefore, $GB/G_0B \cong \mathbb{Z}$ via the winding number.

For connected components of GB , define $\chi_n(t) = t^n$, $|t| = 1$, where $\text{wind}(\chi_n) = n$.

Remark: Closed Subalgebras and Invertibility

Let A be a closed subalgebra of B (both being unital, $e \in A$, $e \in B$).

Obviously, if $a \in A$ is invertible in A (i.e. $a^{-1} \in A$) then a is invertible in B . Then $GA \subseteq GB \cap A \subseteq GB$.

Example

Take $B = C(\mathbb{T})$ and $A = \{f \in C(\mathbb{T}) : f_n = 0, \forall n < 0\} = C_+(\mathbb{T})$ where $f_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{ix}) e^{-inx} dx$ is the n th Fourier coefficient.

Formally: $f(t) \cong \sum_{n=-\infty}^{\infty} f_n t^n$ in $B = C(\mathbb{T})$, $|t| = 1$.

$f \in A : f(t) = \sum_{n=0}^{\infty} f_n t^n$, $|t| = 1$ has an analytic extension into the unit disk $|t| < 1$.

More precisely, $\phi : A(\overline{\mathbb{D}}) \rightarrow C_+(\mathbb{T}) \subseteq C(\mathbb{T})$ by $f \mapsto f|_{\mathbb{T}}$.

Where $A(\overline{\mathbb{D}}) = \{f \in \overline{\mathbb{D}} \rightarrow \mathbb{C} \text{ continuous, holomorphic on } \mathbb{D}\}$ and $\mathbb{D} = \{t \in \mathbb{C} : |t| \leq 1\}$.

Then, for $f \in A(\overline{\mathbb{D}})$ with $n \in \{-1, -2, -3, \dots\}$,

$$f_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{ix}) e^{-inx} dx = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(z)}{z^{n+1}} dz = \lim_{r \rightarrow 1^-} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(rz)}{z^{n+1}} dz = 0$$

- In fact, ϕ is an isometry.

$$\|f\|_{A(\overline{\mathbb{D}})} = \sup_{|z| \leq 1} |f(z)| = \max_{|z|=1} |f(z)| = \|f\|_{\mathbb{T}} \|f\|_{C(\mathbb{T})}$$

By maximum modulus principle of holomorphic functions, since ϕ is not constant.

- ϕ is linear and multiplicative.

- $C_+(\mathbb{T})$ is a closed subset of $C(\mathbb{T})$.

$$\Lambda_n : C(\mathbb{T}) \ni f \mapsto f_n \in \mathbb{C}$$

is a continuous linear functional.

$$C_+(\mathbb{T}) = \bigcap_{n=0} \ker \Lambda_n$$

- Less trivially, ϕ is surjective and $C_+(\mathbb{T})$ is an algebra.

Example

$\chi_1(t) = t$ is invertible in $C(\mathbb{T}) = B$.
 $\chi_1^{-1}(t) = \frac{1}{t} = x_{-1}(t) \notin C_+(\mathbb{T})$ while $\chi_1(t) \in C_+(\mathbb{T})$.
Therefore $GA \subseteq GB \cap A$ may not be equal.

Definition: Boundary

The boundary of a subset U of a topological space X is $\partial U = \overline{U} \setminus \text{int}(U)$.

Remark

For $U \subseteq X$, $X = \text{int}(U) \cup \partial U \cup \text{int}(X \setminus U)$ a union of disjoint sets.

Lemma:

1. if $a \in \partial GA$, then $a \notin GA$ and there exists a sequence $a_n \in GA$ such that $a_n \rightarrow a$.
2. if $a \in \partial a$ and $a_n \in GA$ such that $a_n \rightarrow a$, then $\|a_n^{-1}\| \rightarrow +\infty$.

Proof of 1

$a \in GA$ would imply $a \in \text{int}(GA)$ and not a boundary point.

Proof of 2

Otherwise, there would exist a bounded subsequence $\|a_{n_i}^{-1}\| \leq M$.

$$\|a_{n_i}^{-1} - a_{n_j}^{-1}\| \leq \|a_{n_i}^{-1}\| \cdot \|a_{n_j} - a_{n_i}\| \cdot \|a_{n_j}^{-1}\| \leq M^2 \|a_{n_i} - a_{n_j}\|$$

Since a_n converges, $\{a_n\}$ is Cauchy which implies $a_{n_i}^{-1}$ is Cauchy.

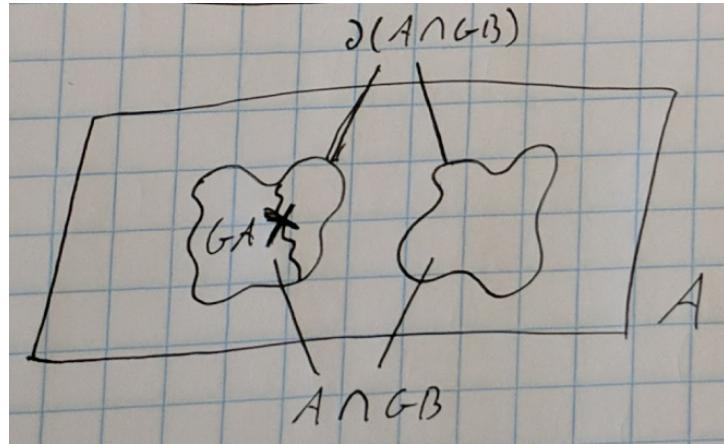
Then $a_{n_i}^{-1} \rightarrow b \in A$. $e = a_{n_i} a_{n_i}^{-1} \rightarrow ab$ implies $a^{-1} = b$ and $a \in GA$. However $a \notin GA$.

Proposition

Let A be a closed subalgebra of B ($e \in A$, $e \in B$). Then $\partial GA \subseteq \partial(A \cap GB)$ (both boundaries are considered in A).

Remark

Both GA and $A \cap GB$ are open subsets of A .



Proof

Take $a \in \partial GA$ and suppose $a \notin \partial(A \cap GB)$.

Take $a \in \partial GA$: $a_n \in GA$, $a \notin GA$, $a_n \rightarrow a$, $\|a_n^{-1}\| \rightarrow +\infty$.

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Recall

$A \subseteq B$, $GA \subseteq A \cap GB$.

If $A = C_+(\mathbb{T}) \cong A(\overline{\mathbb{D}})$ and $B = C(\mathbb{T})$.

Recall: Theorem

For GA , $A \cap GB$ open sets in A , $U \subseteq X$, $\partial U = \overline{U} \setminus \text{int } U$, we have that $\partial GA \subseteq \partial(A \cap GB)$.

Proof

Take $a \in \partial GA$, $a_n \rightarrow a$, $a \notin GA$, $a \in A$.

Since $a_n \in GA$, $\|a_n^{-1}\| \rightarrow +\infty$.

However, $a \notin GB$ otherwise $a \in GB$, $a_n \rightarrow a$ implies $a_n^{-1} \rightarrow a^{-1}$ (in GB) and, consequently, $\sup \|a_n^{-1}\| < +\infty$, a contradiction.

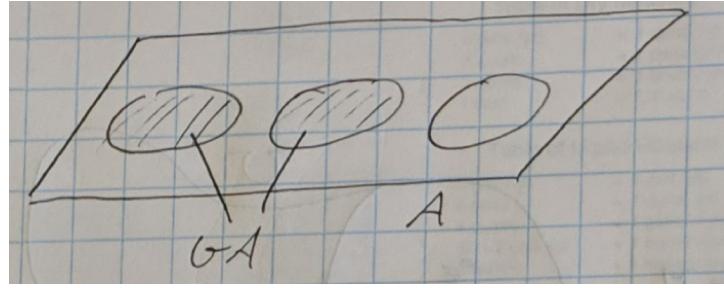
Therefore $a \notin A \cap GB$ and, consequently, $a \in \partial(A \cap GB) = \overline{(A \cap GB)} \setminus (A \cap GB)$.

Theorem

Let A be a closed subalgebra of B .

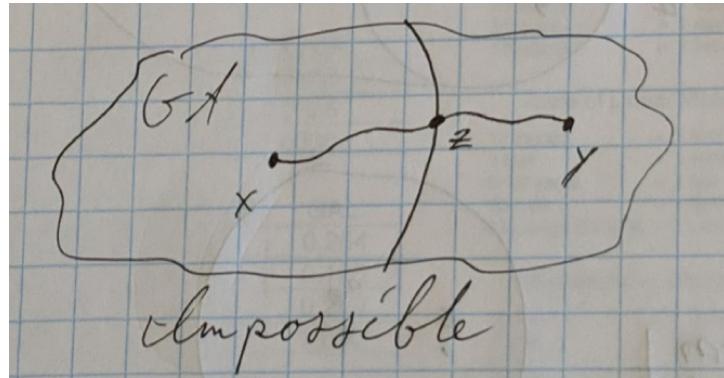
GA is equal to the union of some components of $A \cap GB$.

Proof



Let U be a component of $A \cap GB$.

We want to show that either $U \cap GA \neq \emptyset$ or $U \subseteq GA$.



The above cannot occur since, by path-connectedness, for $x, y \in U$, $x \in GA$, $y \notin GA$, there would need to be some $z \in \partial GA$ with $z \notin A \cap GB$ a contradiction.

Alternatively, take $A \cap GB$ open in A .

Then $A \cap GB \cap \partial(A \cap GB) = \emptyset$ and $(A \cap GB) \cap \partial GA = \emptyset$ by the previous theorem.

Write $A = GA \cup \partial GA \cup \text{int}(A \setminus GA)$. Then

$$A \cap GB = GA \cup \emptyset \cup \text{int}(A \setminus GA) \cap (A \cap GB)$$

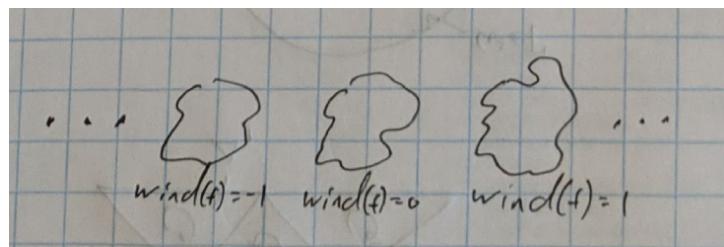
and $U = (GA \cap U) \cup \text{int}(A \setminus GB) \cap U$ where $(GA \cap U) \cap \text{int}(A \setminus GA) = \emptyset$ and open in U .

Therefore either $GA \cap U = \emptyset$ or $GA \cap U = U$ which implies that $U \subseteq GA$.

Example

Take $B(\mathbb{T})$ and $A = C_+(\mathbb{T}) \cong A(\overline{D})$.

Then $GB = \{f: \mathbb{T} \rightarrow \mathbb{C} : f(t) \neq 0\}$.



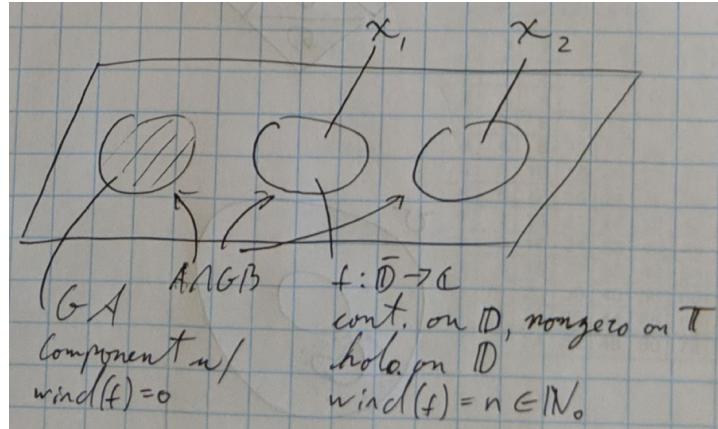
Then take

$$A \cap GB = \{f: \mathbb{T} \rightarrow \mathbb{C} \text{ continuous, } f(t) \neq 0, |t| = 1 \text{ with analytic continuation into } |t| < 1\}$$

such that $f \in A \cap GB$ which implies $\text{wind}(f) \in \{0, 1, 2, 3, \dots\}$ gives the number of zeroes of f inside \mathbb{D} .

$$\begin{aligned}\text{wind}(f) &= \frac{1}{2\pi i} \left[\log f(e^{ix}) \right]_{x=0}^{2\pi} \\ &= \frac{1}{2\pi i} \lim_{r \rightarrow 1^-} \left[\log f(re^{ix}) \right]_{x=0}^{2\pi} \\ &= \frac{1}{2\pi i} \lim_{r \rightarrow 1} \int_0^\pi \frac{f'(re^{ix})}{f(re^{ix})} ire^{ix} dx \\ &= \frac{1}{2\pi i} \lim_{z=re^{ix}} \int_{|z|=r} \frac{f'(z)}{f(z)} dz\end{aligned}$$

Which gives the number of zeros of $f(z)$ inside $|z| < 1$



Section 1.3: Holomorphic Vector-Valued Functions

Goal

Define the notion of holomorphic/analytic functions $f : \Omega \rightarrow X$ where $\Omega \subseteq \mathbb{C}$ open and X a (complex) Banach space.

Summary

- Basically all classical results remain true.
- There is a strong and a weak version of holomorphy, but they are equivalent.

Theorem

For a function $f : \Omega \rightarrow X$, $\Omega \subseteq \mathbb{C}$ open and X Banach, the following are equivalent

1. f is differentiable at every $z_0 \in \Omega$, i.e. there exists $f'(z_0) \in X$ such that

$$\lim_{z \rightarrow z_0} \left\| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right\|_X = 0$$

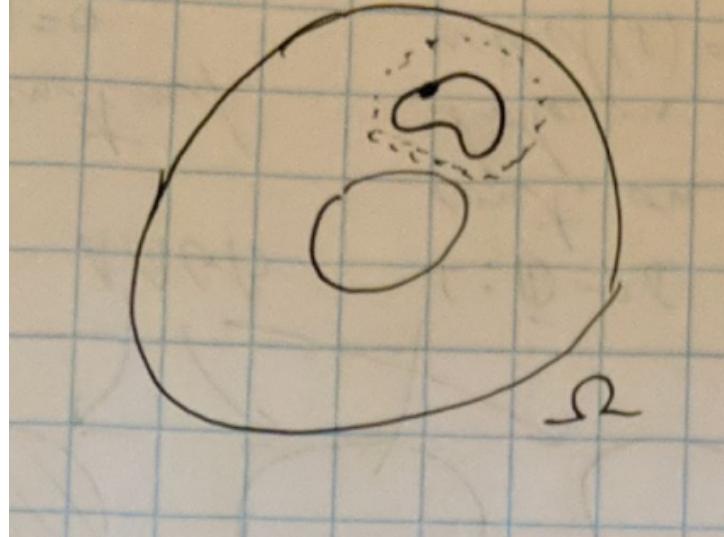
2. f is analytic at each point $z_0 \in \Omega$, i.e. f has a convergent power series at z_0 with radius of convergence $R_{z_0} > 0$.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, |z - z_0| < R_{z_0}, a_n \in X$$

which converges in the norm of X .

3. $f : \Omega \rightarrow X$ is continuous (in the norm) and for every piecewise smooth closed contour Γ contained in a disk D ($\Gamma \subseteq D \subseteq \Omega$).

$$\int_{\Gamma} f(z) dz = 0$$



Definition: (Strongly) Holomorphic Function

If (1)-(3) hold, then f is (strongly)-holomorphic.

Remarks: Integration of Vector-Valued Functions

A piecewise smooth contour Γ can be parameterized by $\sigma : [0, 1] \rightarrow \Omega$.

$$\int_{\Gamma} f(z) dz = \int_0^1 \underbrace{f(\sigma(t))\sigma'(t)}_{h(t) \text{ continuous}} dt$$

This is independent of the choice of parameterization.

Now $I = \int_0^1 h(t) dt$ can be defined via Riemann sums. Given a partition P , $h : [0, 1] \rightarrow X$ continuous.

$$\lim_{\text{mesh}(P) \rightarrow 0} \|S(h, P, \xi) - I\|_X = 0$$

where $S(h, P, \xi) = \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1})$, $P = \{x_0, x_1, \dots, x_n\}$, $\xi_i \in [x_{i-1}, x_i]$.

Note that h is uniformly continuous and $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\text{mesh}(P_1) < \delta$, $\text{mesh}(P_2) < \delta$ implies

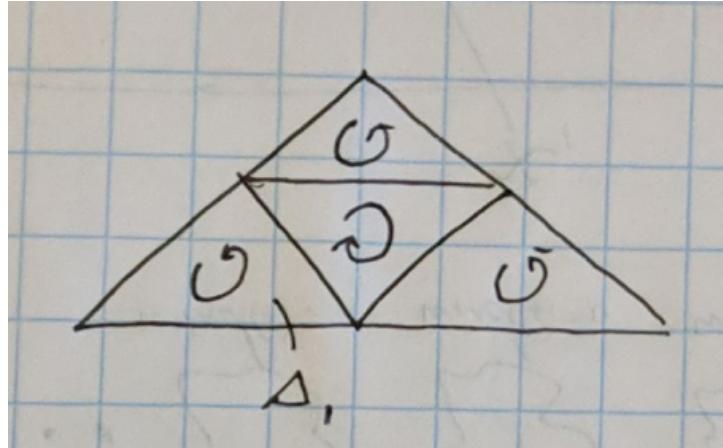
$$\|S(f, P_1, \xi^{(1)}) - S(f, P_2, \xi^{(2)})\| < \varepsilon$$

All usual properties of integrals hold.

- linear in integrand
- $\|\int_{\Gamma} f(z) dz\| \leq \int_{\Gamma} \|f(z)\| |dz| \leq (\text{length}(\Gamma)) \sup_{z \in \Gamma} \|f(z)\|$.

Sketch of Proof (1) to (3)

To show: $\int_{\Delta} f(z) dz = x_0 = 0$ by contradiction that $x_0 \neq 0$.

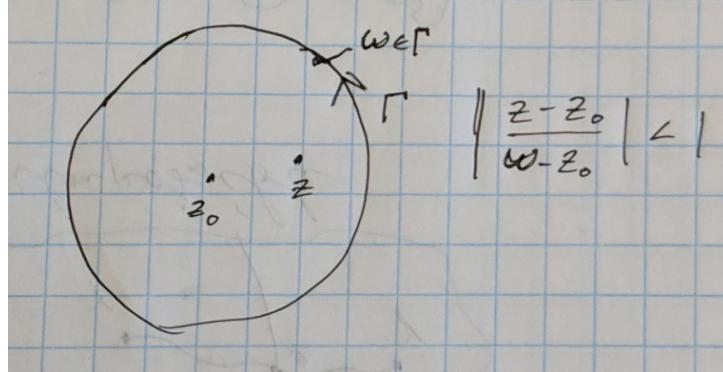


We have $\left| \int_{\Delta_1} f dz \right| \geq \frac{\|x_0\|}{4}$, $\left| \int_{\Delta_n} f dz \right| \geq \frac{\|x_0\|}{4^n}$.

Sketch of Proof (3) to (2)

$\int_{\Gamma} f dz = 0$ implies the Cauchy integral formula. Take

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\omega)}{\omega - z} d\omega$$



$$\frac{1}{\omega - z} = \frac{1}{(\omega - z_0) - (z - z_0)} = \frac{1}{w - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - t} \right)^n$$

Therefore

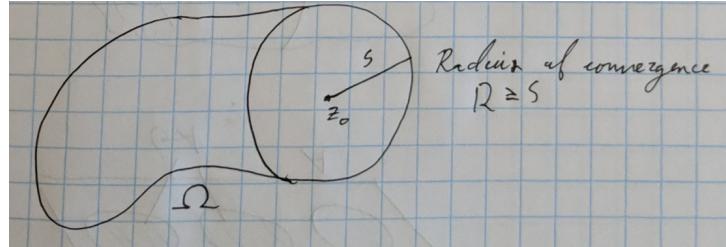
$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} f(\omega) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\omega - z)^{n+1}} d\omega = \sum_{n=0}^{\infty} (z - z_0)^n \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\omega)}{(\omega - z)^{n+1}} d\omega = \sum_{n=0}^{\infty} (z - z_0)^n a_n$$

with the sequence converging (in X) on $|z - z_0| < |\omega - z_0|$.

- Radius of Convergence

$$R^{-1} = \limsup_{n \rightarrow \infty} \|a_n\|^{1/n}$$

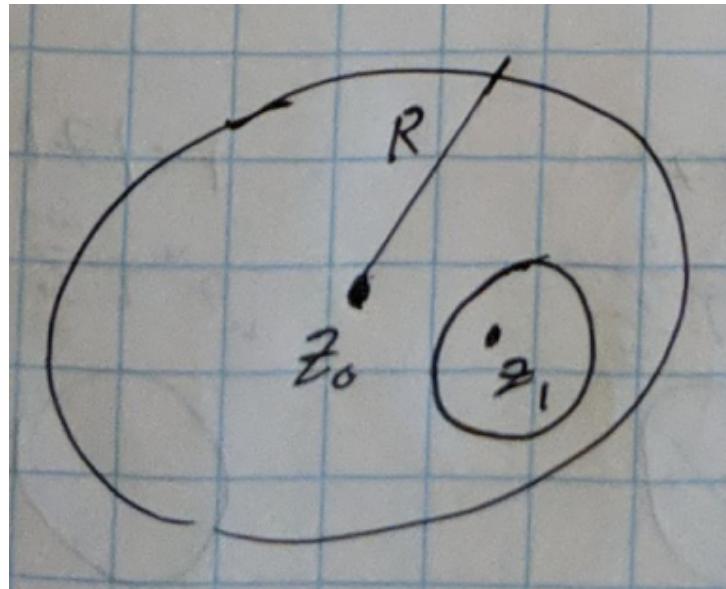
(Root Test: $|z - z_0| < R$ convergence; $|z - z_0| > R$ divergence)



Sketch of Proof (2) to (1)

One can show that a function defined by convergent power series is differentiable, $f(z) = \sum a_n(z - z_0)^n$, then $f'(z) = \sum a_n \cdot n(z - z_0)^{n-1}$.

The radius of convergence is the same. This also implies that f is infinitely differentiable.



Take $z - z_0 = (z - z_1) + (z_1 - z_0)$ and, by the binomial theorem,

$$f(z) = \sum_{k=0}^{\infty} (z - z_1)^k \left(\sum_{n=k}^{\infty} a_n \binom{n}{k} (z_1 - z_0) \right)$$

which converges for at least $|z - z_1| < R - |z_1 - z_0|$.

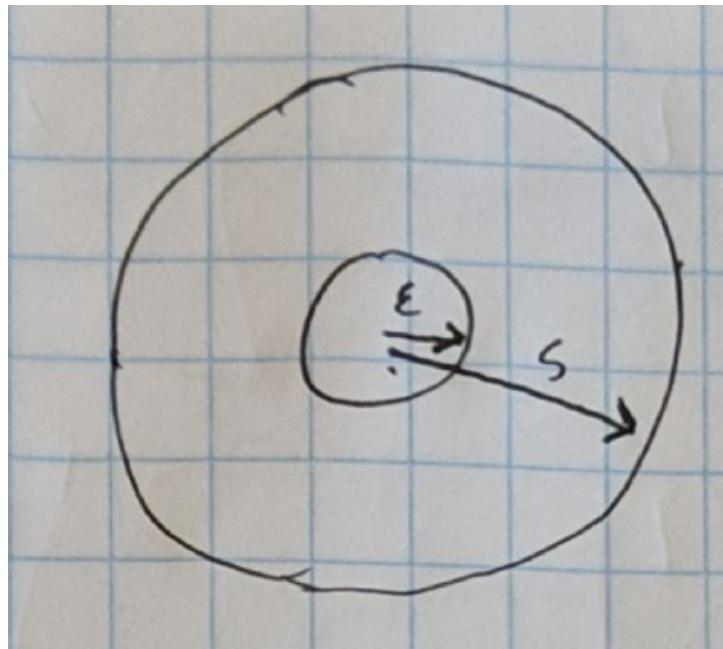
October 14, 2024

Theorem

Let $f : D_\varepsilon(z_0) \rightarrow X$ ($D_\varepsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$) be holomorphic.

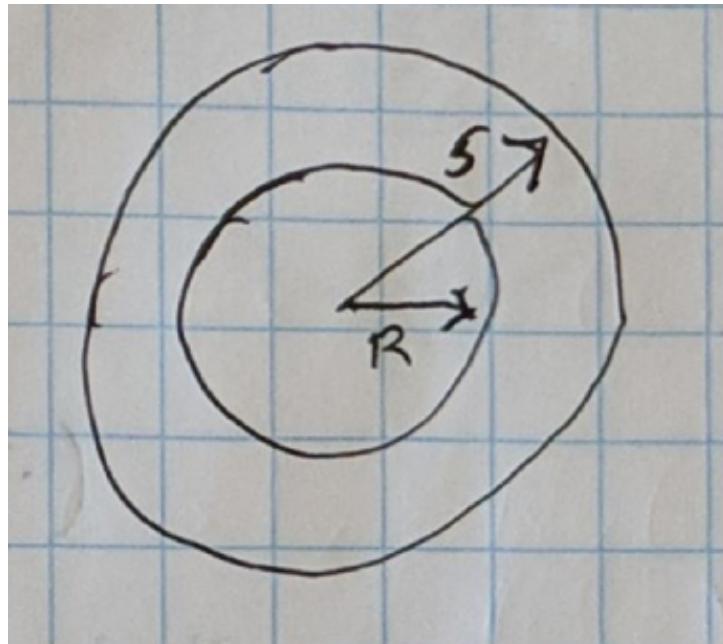
Then $R = S$ where

1. R is the radius of convergence of $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ ($R^{-1} = \limsup_{n \rightarrow \infty} \|a_n\|^{\frac{1}{n}}$).
2. S is the radius of the largest open disk $D_S(z_0)$ such that there exists an analytic extension of f from $D_\varepsilon(z_0)$ to $D_S(z_0)$.



Proof

By definition, $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for $|z-z_0| < R$. Then $|z-z_0| < R$ if and only if $\limsup_{n \rightarrow \infty} ||a_n(z-z_0)^n||^{\frac{1}{n}} < 1$ if and only if $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges. Therefore, it converges to a holomorphic function on $R \leq S$. If $f(z)$ has an analytic extension to $D_S(z_0)$, see step (3) \implies (2) of previous theorem.



Then $f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\omega)}{\omega - z} d\omega = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for $|z-z_0| < r < S$ with $a_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\omega)}{(z-\omega)^{n+1}} d\omega$. From this, we conclude $R \geq S$.

Definition: (Weakly) Holomorphic Function

A function $f : \Omega \rightarrow X$ ($\Omega \subseteq \mathbb{C}$ open, X Banach) is called weakly holomorphic if $\phi \circ f : \Omega \rightarrow \mathbb{C}$ is holomorphic, $\forall \phi \in X^* = \mathcal{L}(X; \mathbb{C})$ bounded linear functionals.

A function $f : \Omega \rightarrow \mathcal{L}(X, Y)$ (X, Y Banach) is weakly-operator holomorphic if $h_{\phi, X} : \Omega \rightarrow \mathbb{C}$ is holomorphic for all $\phi \in Y^*$, $x \in X$ where $h_{\phi, X}(z) = \phi(f(z)x)$.

Remarks

Obviously: f strongly holomorphic $\implies f$ weakly holomorphic.

$$\left\| \frac{\phi(f(z+h)) - \phi(f(z))}{h} - \phi(f'(z)) \right\| \leq \|\phi\| \cdot \left\| \frac{f(z+h) - f(z)}{h} - f'(z) \right\|$$

For $f : \Omega \rightarrow \mathcal{L}(X, Y)$: f strongly holomorphic $\implies f$ weakly holomorphic $\implies f$ weakly operator holomorphic.

For $x \in X$, $\phi \in Y^*$, $\Lambda_{x,\phi} : \mathcal{L}(X, y) \ni A \mapsto \phi(Ax) \in \mathbb{C}$ and $\Lambda_{x,\phi} \in (\mathcal{L}(X, y))^*$.

All the converses are also true.

Theorem (Dunford)

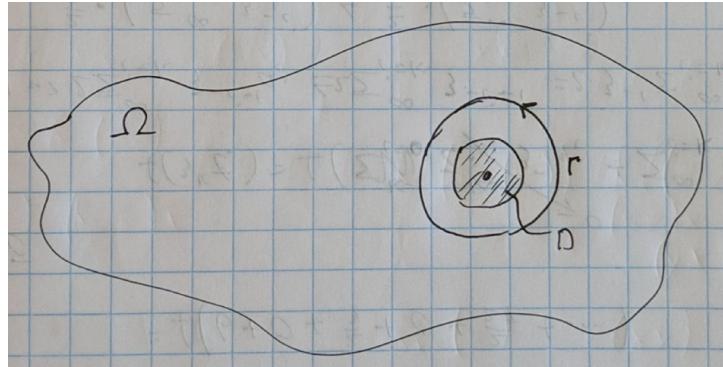
Take X Banach, $\Omega \subseteq \mathbb{C}$ open.

If $f : \Omega \rightarrow X$ is weakly holomorphic, then it is strongly holomorphic.

Proof

We want to show that for any $z_0 \in \Omega$, $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists in X .

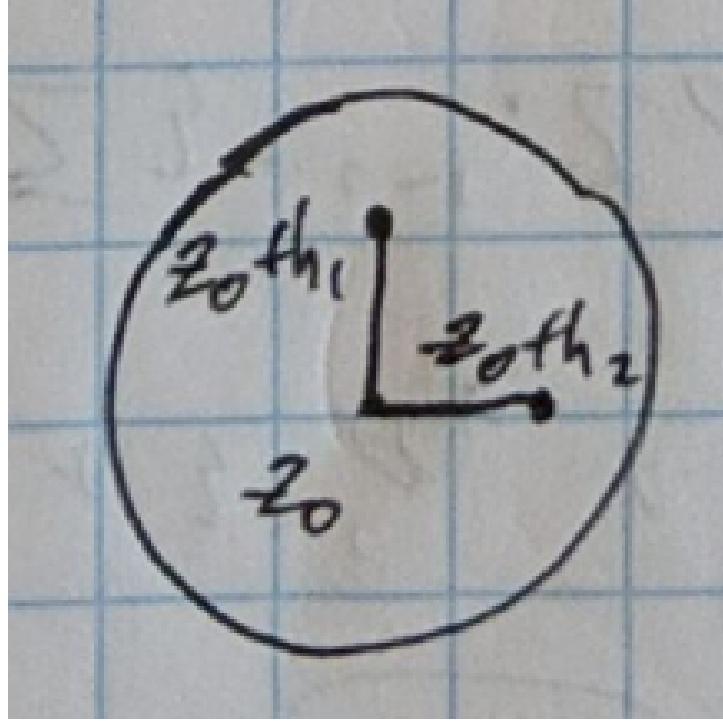
Choose $\varepsilon > 0$ such that the disk $D_\varepsilon(z_0)$ and circle $C_{2\varepsilon}(z_0) = \Gamma$ are in Ω .



For $\phi \in X^*$, $\phi(f(z))$ is holomorphic in Ω .

$$\phi(f(z)) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\phi(f(\omega))}{z - \omega} d\omega, z \in D$$

Apply this to $z = z_0$, $z = z_0 + h_1$ and $z = z_0 + h_2$ with $0 < |h_1| < \varepsilon$, $0 < |h_2| < \varepsilon$, $h_1 \neq h_2$.



$$\begin{aligned}
A_{h_1, h_2} &= \frac{1}{h_1 - h_2} \left\{ \frac{f(z_0 + h_1) - f(z_0)}{h_1} - \frac{f(z_0 + h_2) - f(z_0)}{h_2} \right\} \\
\phi(A_{h_1, h_2}) &= \frac{1}{h_1 - h_2} \left\{ \frac{\phi(f(z_0 + h_1)) - \phi(f(z_0))}{h_1} - \frac{\phi(f(z_0 + h_2)) - \phi(f(z_0))}{h_2} \right\} \\
&= \frac{1}{2\pi i} \int_{\Gamma} \phi(f(\omega)) \frac{1}{h_1 - h_2} \left\{ \frac{1}{h_1} \left(\frac{1}{z_0 + h_1 - \omega} - \frac{1}{z_0 - \omega} \right) - \frac{1}{h_2} \left(\frac{1}{z_0 + h_2 - \omega} - \frac{1}{z_0 - \omega} \right) \right\} d\omega \\
&= \frac{1}{2\pi i} \int_{\Gamma} \phi(f(\omega)) \frac{1}{h_1 - h_2} \left\{ \frac{1}{(z + h_1 - \omega)(z_0 - \omega)} - \frac{1}{(z + h_2 - \omega)(z_0 - \omega)} \right\} d\omega \\
&= \frac{1}{2\pi i} \int_{\Gamma} \phi(f(\omega)) \frac{1}{(z_0 + h_1 - \omega)(z_0 + h_2 - \omega)(z_0 - \omega)} d\omega
\end{aligned}$$

Observe that the denominator is at least ε^3 , therefore $|\phi(A_{h_1, h_2})| \leq \frac{\varepsilon^3}{2\pi} \sup_{\omega \in \Gamma} |f(\omega)| \cdot |\phi|$ (so long as f continuous, which will be proven).

Therefore $\forall \phi \in X^*$,

$$\sup_{\substack{0 < |h_1| < \varepsilon \\ 0 < |h_2| < \varepsilon \\ h_1 \neq h_2}} |\phi(A_{h_1, h_2})| < +\infty.$$

By the uniform boundedness principle, identify $A_{h_1, h_2} \in X$ with $X^{**} = \mathcal{L}(X^*, \mathbb{C})$.

Then $\sup_{h_1, h_2} \|A_{h_1, h_2}\| < +\infty$ and

$$\left\| \frac{f(z_0 + h_1) - f(z_0)}{h_1} - \frac{f(z_0 + h_2) - f(z_0)}{h_2} \right\| \leq C \cdot |h_1 - h_2|.$$

Now, for any sequence $\{h_n\}_{n=3}^{\infty}$, $0 < |h_n| < \varepsilon$, $h_n \rightarrow 0$,

$$\frac{f(z_0 + h_n) - f(z_0)}{h_n}$$

is a cauchy sequence. Therefore $\lim_{n \rightarrow \infty} \frac{f(z_{0+h_n}) - f(z_0)}{h_n}$ exists in X independent of choice of $\{h_n\}$. That is

$$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists in X .

Section 1.4: Spectrum and Resolvent

Consider a unital Banach algebra B .

Definition: Spectrum

For $b \in B$, the spectrum of b in B $\sigma_B(b) = \{\lambda \in \mathbb{C} : \lambda e - b \text{ is not invertible in } B\}$.

Definition: Resolvent

The resolvent is a function $R(b; \lambda) = (\lambda e - b)^{-1}$. $R(b, \cdot) : \mathbb{C} \setminus \sigma_B(b) \rightarrow B$.
 $\mathbb{C} \setminus \sigma_B(b)$ is the resolvent set.

Theorem

1. The spectrum $\sigma_B(b)$ is a non-empty, compact subset of \mathbb{C} .
2. The resolvent $R(b, \lambda)$ is an analytic, Banach valued function on $\mathbb{C} \setminus \sigma_B(b)$.

Proof of (a)

$\sigma_B(b)$ is bounded, because $\lambda e - b$ is invertible for $|\lambda| > \|b\|$.

$$\lambda e - b = \lambda \left(e - \frac{1}{\lambda} b \right)$$

has $\left\| \frac{1}{\lambda} b \right\| < 1$ for sufficiently large λ . Therefore, $\sigma_B(b) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|b\|\}$.

To show that $\sigma_B(b)$ is closed, if $\lambda \notin \sigma_B(b)$ then $\forall \mu$ such that $|\lambda - \mu| < \varepsilon$ we have that $\mu \notin \sigma_B(b)$.

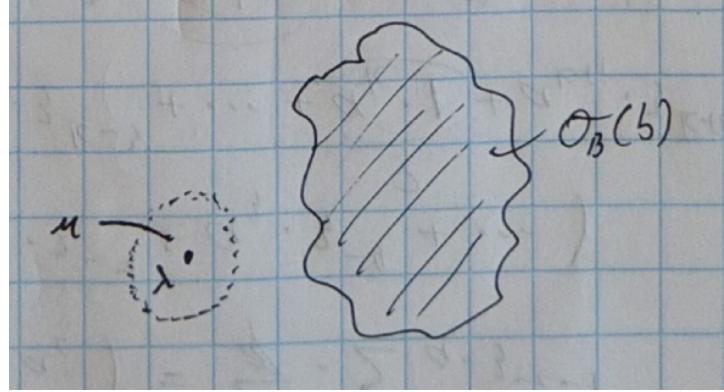
$$\mu e - b = \lambda e - b + (\mu - \lambda)e = (\lambda e - b) \underbrace{\left[e + \frac{(\mu - \lambda)(\lambda e - b)^{-1}}{\|\cdot\| < 1} \right]}_{\text{closed set}}$$

when $|\mu - \lambda| < \frac{1}{\|(\lambda e - b)^{-1}\|}$.

Therefore $\mathbb{C} \setminus \sigma_B(b)$ is open.

Proof of (b)

Take $\lambda \notin \sigma_B(b)$



$$\begin{aligned}
 \frac{R(b, \mu) - R(b, \lambda)}{\mu - \lambda} &= \frac{1}{\mu - \lambda} \left((\mu e - b)^{-1} - (\lambda e - b)^{-1} \right) \\
 &= \frac{1}{-\mu - \lambda} (\mu e - b)^{-1} \{(\lambda e - b) - (\mu e - b)\} (\lambda e - b)^{-1} \\
 &= -(\mu e - b)^{-1} (\lambda e - b)^{-1}
 \end{aligned}$$

Using continuity with $GB \ni a \mapsto a^{-1} \in GB$ in the norm, $-((\mu e - b)^{-1})(\lambda e - b)^{-1} \rightarrow -((\lambda e - b)^{-1})^2$ as $\mu \rightarrow \lambda$. Therefore $R^1(b, \lambda) = -(R(b, \lambda))^2$ and $R(b, \lambda)$ is analytic.

Proof of non-empty in (a)

Take $\sigma_B(b) \neq 0$, otherwise $R(b, \lambda)$ is analytic on \mathbb{C} and bounded

$$(\lambda e - b)^{-1} = \frac{1}{\lambda} \left(e - \frac{1}{\lambda} b \right)^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\lambda^{n+1}} b^n$$

We can estimate

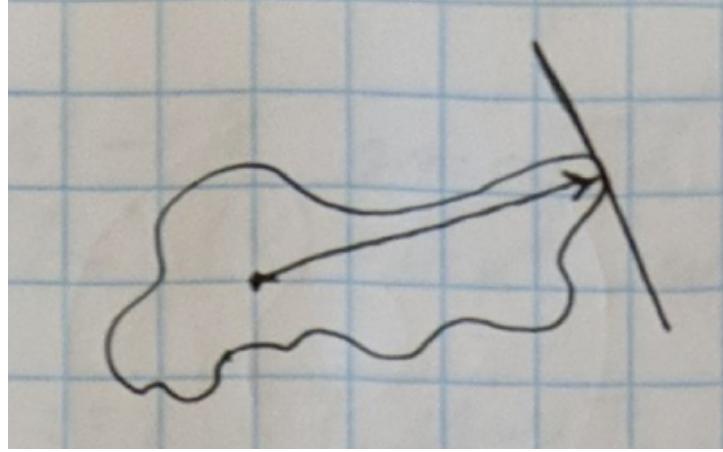
$$\|\cdot\| \leq \frac{1}{|\lambda| \left(1 - \frac{\|b\|}{|\lambda|} \right)} = \frac{1}{|\lambda| - \|b\|}$$

so $\lim_{\lambda \rightarrow \infty} \|(\lambda e - b)^{-1}\| = 0$.

By Liouville's theorem, bounded and entire functions are constant. But we may also proceed by weak analyticity. If $\phi(R(b, \lambda))$ is analytic and bounded on \mathbb{C} , $\forall \phi \in B^*$, it follows that $\phi(R(b, \lambda)) \equiv 0$, $\forall \lambda$, $\forall \phi \in B^*$ and that $R(b, \lambda) \equiv 0$ for any λ a contradiction.

Definition: Spectral Radius

For $b \in B$, the spectral radius $r(b) = \max\{|\lambda| : \lambda \in \sigma_B(b)\}$.



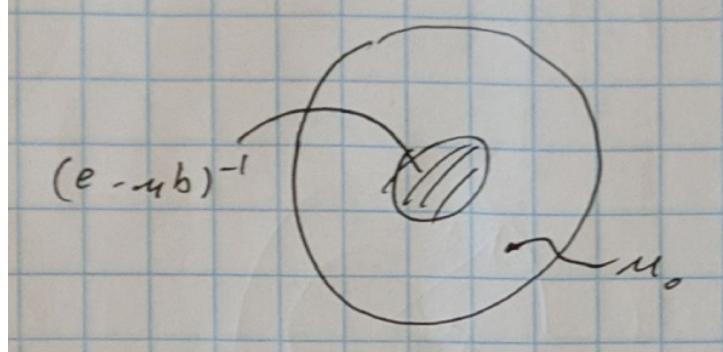
Remark

Write $\frac{1}{r(b)} = \min\{|\lambda|^{-1} : \lambda e - b \text{ is not invertible}\} = \min\{|\mu| : e - \mu b \text{ is not invertible}\}$ with $\mu = \frac{1}{\lambda}$.

$$\underbrace{(e - \mu b)^{-1}}_{\text{analytic in } |\mu| < \frac{1}{||b||}} = \sum_{n=0}^{\infty} \mu^n b^n$$

converges for $|\mu| < \frac{1}{||b||}$.

Then the radius of convergence $R^{-1} = \limsup_{n \rightarrow \infty} ||b^n||^{\frac{1}{n}}$ gives us that R is equal to the largest disk where $(e - \mu b)^{-1}$ has an analytic extension. Therefore $S = \frac{1}{r(b)}$.



Suppose we have an analytic extension $f(\mu)$ beyond S .

$$f(\mu)(e - \mu b) = (e - \mu b)f(\mu) = e$$

implies that and, if $(e - \mu_0 b)$ not invertible, $f(\mu_0)(e - \mu_0 b) = \dots = e$ a contradiction.

Theorem

$$r(b) = \lim_{n \rightarrow \infty} ||b^n||^{\frac{1}{n}} = \inf_{n \in \mathbb{N}} ||b^n||^{\frac{1}{n}}$$

Proof

To demonstrate existence, fix $n_0 \in \mathbb{N}$, $n = q \cdot n_0 + r$, $0 \leq r < n_0$.

$$\begin{aligned} ||b^n|| &\leq ||b^{n_0}||^q \cdot ||b||^r \\ ||b^n||^{\frac{1}{n}} &\leq ||b^{n_0}||^{\frac{q}{n}} \cdot ||b||^{\frac{r}{n}} \\ \limsup_{n \rightarrow \infty} ||b^n||^{\frac{1}{n}} &\leq ||b^{n_0}||^{\frac{1}{n_0}} \cdot 1 \end{aligned}$$

Since $1 = \frac{q}{n} \cdot n_0 + \frac{r}{n}$. Take $n \rightarrow \infty$. Write

$$\limsup_{n \rightarrow \infty} ||b^n||^{\frac{1}{n}} \leq \inf_{n_0 \in \mathbb{N}} ||b^{n_0}||^{\frac{1}{n_0}} \leq \liminf_{n \rightarrow \infty} ||b^n||^{\frac{1}{n}}$$

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Note: Closed Subalgebras

Assume A is a closed subalgebra of B ($e \in A \subseteq B$).

Take $b \in A \subseteq B$.

Obviously, $b - \lambda e$ being invertible in A implies $b - \lambda e$ is invertible in B . We also have

$$\mathbb{C} \setminus \text{sp}_A(b) \subseteq \mathbb{C} \setminus \text{sp}_B(b)$$

(confer. $GA \subseteq GB$ with $\partial GA = \partial(A \cap GB)$) and, equivalently,

$$\text{sp}_B(b) \subseteq \text{sp}_A(b).$$

One can show similarly that

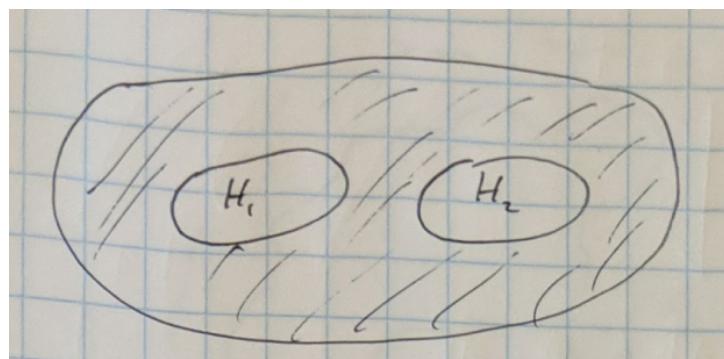
$$\begin{aligned} \partial(\mathbb{C} \setminus \text{sp}_A(b)) &\subseteq \partial(\mathbb{C} \setminus \text{sp}_B(b)) \\ &= \dots = \\ \partial \text{sp}_A(b) &\subseteq \partial \text{sp}_B(b) \end{aligned}$$

Proposition

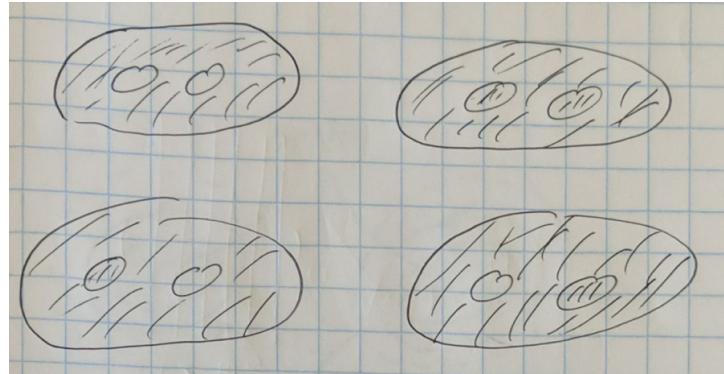
1. $\mathbb{C} \setminus \text{sp}_A(b)$ is the union of some components of $\mathbb{C} \setminus \text{sp}_B(b)$.
2. $\text{sp}_A(b) = \text{sp}_B(b) \cup \bigcup_{\omega} H_{\omega}$ where H_{ω} are some components of $\mathbb{C} \setminus \text{sp}_B(b)$.

Example 1

Suppose $\text{sp}_B(b)$ looks like

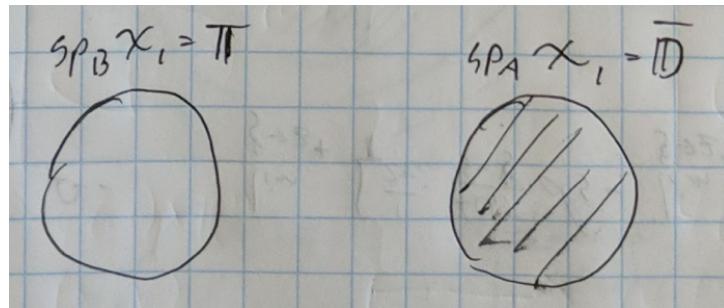


Now $\text{sp}_A(b)$ can only be one of the 4 possibilities.



Example 2

$B = C(\mathbb{T})$, $A = C_+(\mathbb{T}) \simeq A(\mathbb{D})$, $\chi_1(t) = t$, $\text{sp}_B \chi_1 = \mathbb{T}$.



Theorem: Spectral Mapping Theorem (Simple Version)

For a polynomial $p(z) = \sum_{n=0}^N p_n z^n$ we define $p(b) = \sum_{n=0}^N p_n b^n$ for $b \in B$ where $b^0 = e$.

Let p be a polynomial and $b \in B$ with B a unital Banach algebra, then $\text{sp}(p(b)) = p(\text{sp}(b)) := \{p(z) : z \in \text{sp}(b)\}$.

Proof

For $\lambda \in \mathbb{C}$, consider $q(z) = p(z) - \lambda = c \prod_{i=1}^N (z - \gamma_i)$.

Now, $q(b) = p(b) - \lambda e = c \prod_{i=1}^N (b - \gamma_i e)$. It follows that

$$\begin{aligned}
 \lambda \notin \text{sp}(p(b)) &\iff p(b) - \lambda e \text{ is invertible.} \\
 &\iff \underbrace{\prod_{i=1}^N (b - \gamma_i e)}_{\text{a commuting product}} \text{ is invertible.} \\
 &\iff \forall i, b - \gamma_i e \text{ is invertible.} \\
 &\iff \forall i, \gamma_i \notin \text{sp}(b) \\
 &\iff \forall z \in \text{sp}(b), q(z) = c \prod_{i=1}^N (z - \gamma_i) \neq 0 \\
 &\iff \forall z \in \text{sp}(b), p(z) \neq \lambda \\
 &\iff \lambda \notin p(\text{sp}(b))
 \end{aligned}$$

Applications

If $p(b) = 0$, then $\text{sp}(b) \subseteq \{z \in \mathbb{C} : p(z) = 0\}$, because

$$\{0\} = \text{sp } 0 = \text{sp } p(b) \stackrel{\text{SMT}}{=} p(\text{sp } b).$$

It follows that if b is nilpotent, such that $b^n = 0$ for some n ($p(z) = z^2$), then $\text{sp}(b) = \{0\}$.

If b is idempotent, such that $b^2 = b$ ($p(z) = z^2 - z$), then $\text{sp}(b) \subseteq \{0, 1\}$.

If b is unipotent (or flip), such that $b^2 = e$, then $\text{sp}(b) = \{\pm 1\}$.

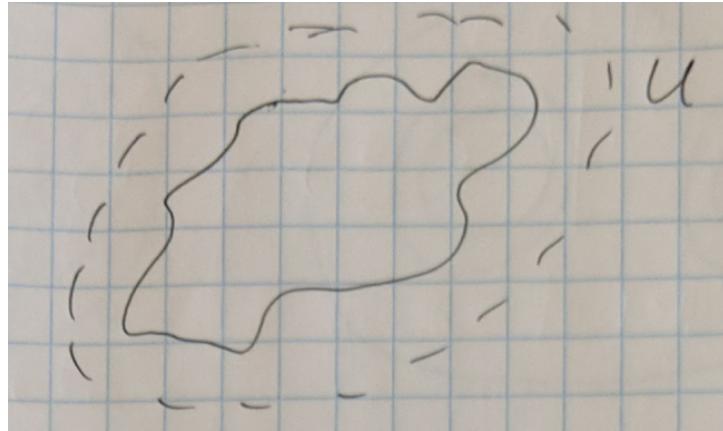
Section 1.5: Riesz Functional Calculus

Question:

Can one define $f(b)$ for $b \in B$ a unital Banach algebra for more general functions f ?

Definition: Set of Functions Holomorphic on the Spectrum

For a unital Banach algebra B and $b \in B$, let $A[\text{sp}(b)]$ stand for the set of all functions f which are holomorphic on some open neighborhood U of $\text{sp}(b)$.



Lemma

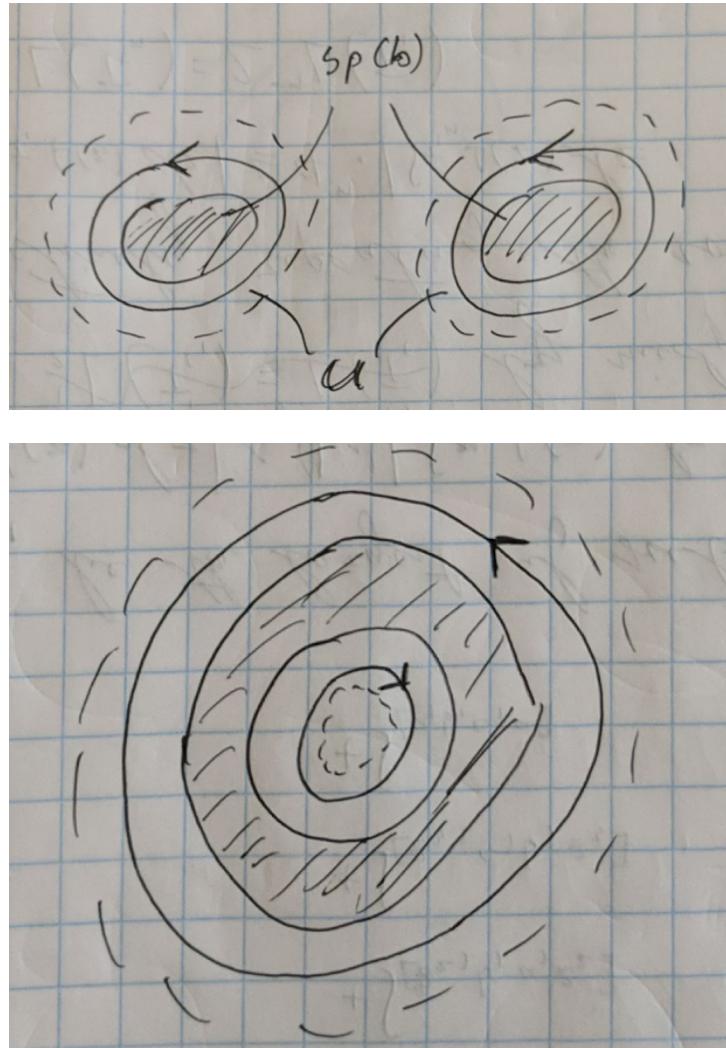
Let $f \in A[\text{sp}(b)]$, i.e. $f : U \rightarrow \mathbb{C}$ holomorphic. Then there exists an open set W with (piece-)smooth boundary such that

$$\text{sp}(f) \subseteq W \subseteq \overline{W} \subseteq U$$

(i.e. $\partial \overline{W} \subseteq U \setminus \text{sp}(b)$) and

$$\frac{1}{2\pi} \int_{\partial W} \frac{d\omega}{\omega - z} = \begin{cases} 1 & z \in \text{sp}(b) \\ 0 & z \notin U \end{cases}.$$

Example



- Proof

IMAGE 7

SQUARES

Definition:

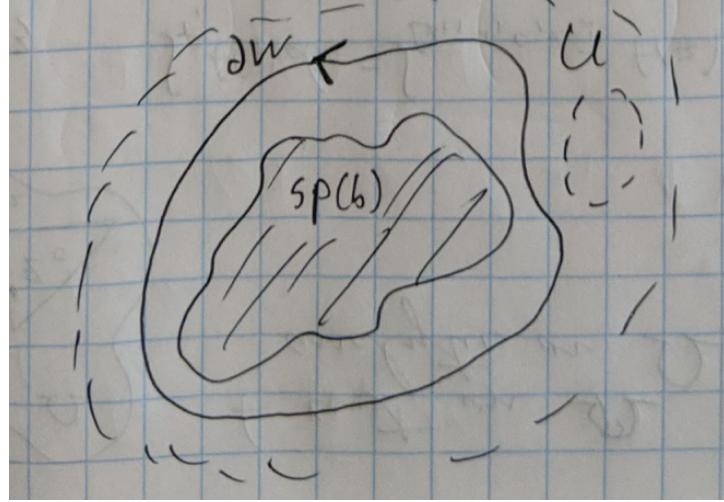
Using the lemma, we define for $f \in A[\text{sp}(b)]$

$$f(b) := \frac{1}{2\pi i} \int_{\partial W} f(\lambda)(\lambda e - b)^{-1} d\lambda$$

(where $\text{sp}(b) \subseteq W \subseteq \overline{W} \subseteq U$).

One can show that this is independent of choice of W (and also of U).

Note $f(\lambda)(\lambda e - b)^{-1}$ is holomorphic on $U \setminus \text{sp}(b)$.



Remark

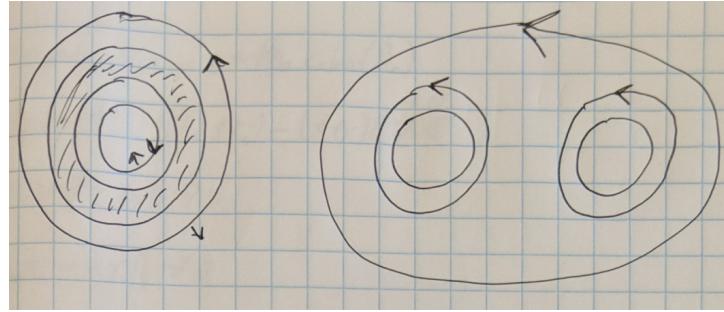
$f_1, f_2 \in A[\text{sp}(b)]$ implies $f_1 + f_2 \in A[\text{sp}(b)]$ and $(f_1 + f_2)(b) = f_1(b) + f_2(b)$.

Proposition

For a polynomial $f(z) = p(z) = \sum p_i z^i$, we get $f(b) = p(b) = \sum p_i b^i$.

Proof

$$\frac{1}{2\pi i} \int_{\partial W} (\lambda e - b)^{-1} d\lambda = \frac{1}{2\pi i} \int_{|\lambda|=R} (\lambda e - b)^{-1} d\lambda = \frac{1}{2\pi i} \int_{|\lambda|=R} \sum_{n=0}^{\infty} \frac{b^n}{\lambda^{n+1}} d\lambda = e$$



Therefore, $p(b) = \frac{1}{2\pi i} \int_{\partial W} p(b)(\lambda e - b)^{-1} d\lambda$, and

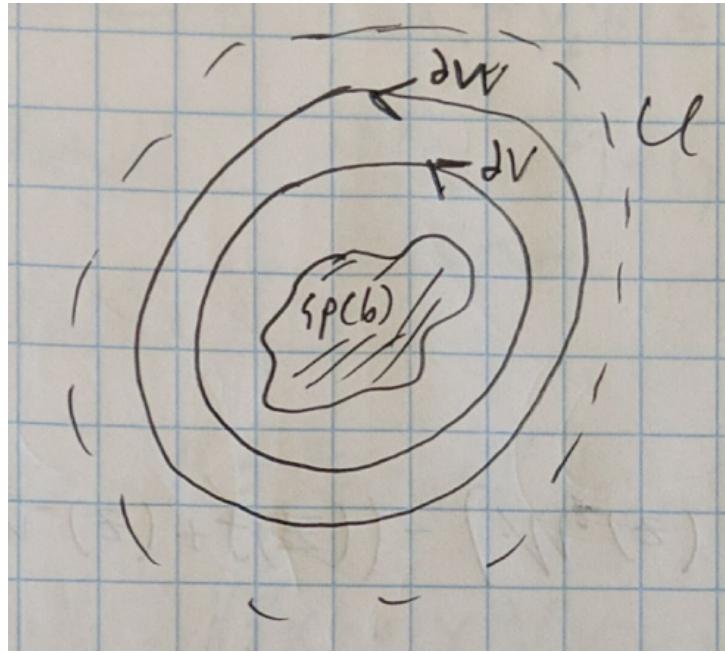
$$\begin{aligned} f(b) - p(b) &= \frac{1}{2\pi i} \int_{\partial W} (f(\lambda)e - p(b))(\lambda e - b)^{-1} d\lambda \\ &= \frac{1}{2\pi i} \int_{\partial W} \sum_{n=0}^N \underbrace{(\lambda^n e - b^n)}_{(\lambda^{n-1}e + \dots + b^{n-1})(\lambda e - b)} (\lambda e - b)^{-1} d\lambda \\ &= \frac{1}{2\pi i} \int_{\partial W} \text{"polynomial in } \lambda, b \text{" } d\lambda = 0 \end{aligned}$$

Proposition

If $f_1, f_2 \in A[\text{sp}(b)]$, then $f_1 f_2 \in A[\text{sp}(b)]$.

$$(f_1 f_2)(b) = f_1(b) \cdot f_2(b)$$

Proof



We assume ∂V is inside ∂W .

$$f_1(b) = \frac{1}{2\pi i} \int_{\partial W} f_1(\lambda)(\lambda e - b)^{-1} d\lambda$$

$$f_2(b) = \frac{1}{2\pi i} \int_{\partial V} f_2(\xi)(\xi e - b)^{-1} d\xi$$

Then

$$f_1(b)f_2(b) = \frac{1}{(2\pi i)^2} \int_{\partial W} \int_{\partial V} f_1(\lambda)f_2(\xi)(\lambda e - b)^{-1}(\xi e - b)^{-1} d\xi d\lambda$$

Recall that

$$(\lambda e - b)^{-1}(\xi e - b)^{-1} = (\lambda e - b)^{-1} \left[\frac{(\lambda e - b) - (\xi e - b)}{\lambda - \xi} \right] (\xi e - b)^{-1} = \frac{(\xi e - b)^{-1} - (\lambda e - b)^{-1}}{\lambda - \xi}$$

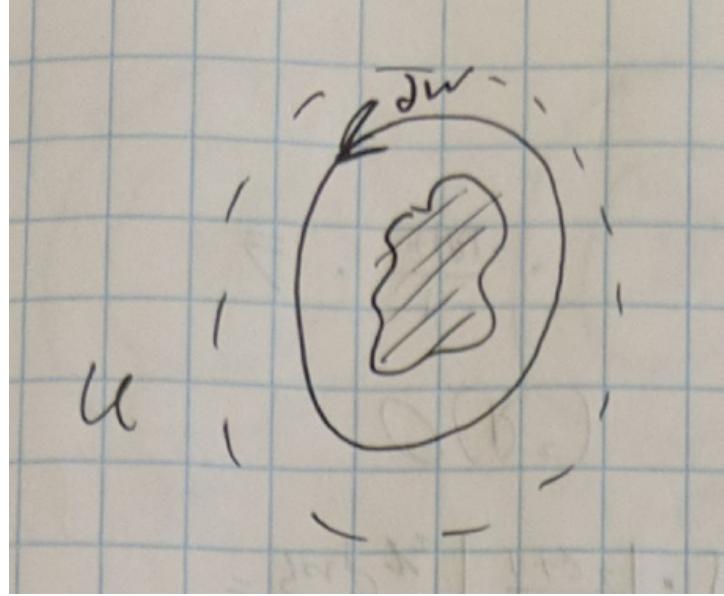
Therefore

$$\begin{aligned} f_1(b)f_2(b) &= \frac{1}{(2\pi i)^2} \int_{\partial V} \int_{\partial W} f_1(\lambda)f_2(\xi)(\xi e - b)^{-1} \frac{1}{\lambda - \xi} d\lambda d\xi - \frac{1}{(2\pi i)^2} \int_{\partial W} \int_{\partial V} f_1(\lambda)f_2(\xi)(\lambda e - b)^{-1} \frac{1}{\lambda - \xi} d\xi d\lambda \\ &= \frac{1}{(2\pi i)^2} \int_{\partial V} f_2(\xi)(\xi e - b)^{-1} \underbrace{\int_{\partial W} \frac{f_1(\lambda)}{\lambda - \xi} d\lambda}_{\equiv f_2(\xi)} d\xi - \frac{1}{(2\pi i)^2} \int_{\partial W} f_1(\lambda)(\lambda e - b)^{-1} \underbrace{\int_{\partial V} \frac{f_2(\xi)}{\lambda - \xi} d\xi}_{\equiv 0} d\lambda \\ &= \frac{1}{2\pi i} \int_{\partial V} f_2(\xi)f_1(\xi)(\xi e - b)^{-1} d\xi \\ &= (f_1f_2)(b) \end{aligned}$$

Recall

$f \in A[\text{sp } b]$, $f : U \rightarrow \mathbb{C}$, $\text{sp}(b) \subseteq U$ open. Define

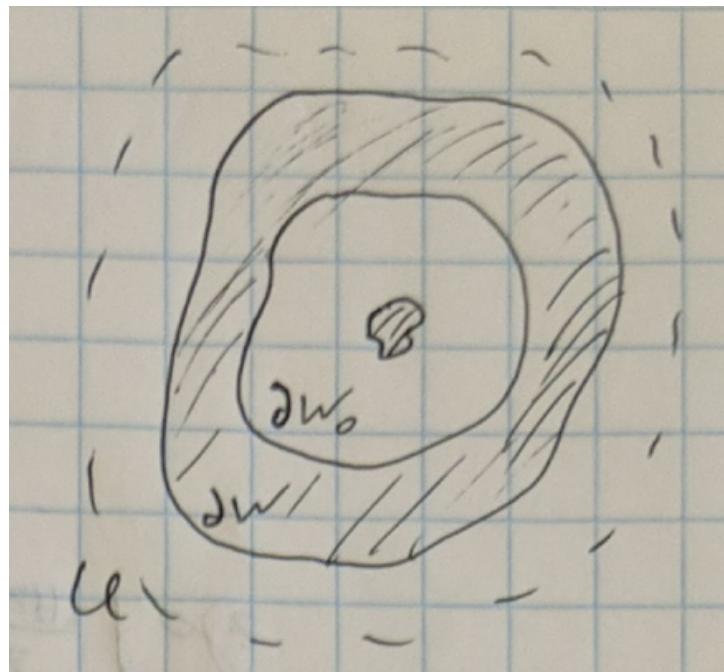
$$(1) \quad f(b) = \frac{1}{2\pi i} \int_{\partial W} \underbrace{f(\lambda)(\lambda e - b)^{-1}}_{\text{analytic in } U \setminus \text{sp } b} dz$$



with $\text{sp } b \subseteq W \subseteq \overline{W} \subseteq U$ and ∂W piecewise smooth.

From the above lemma, applied to W , we get W_0 such that $\text{sp } b \subseteq W_0 \subseteq \overline{W}_0 \subseteq W \subseteq \overline{W} \subseteq U$. Then

$$(2) \quad \frac{1}{2\pi i} \int_{\partial W_0} f(\lambda)(\lambda e - b)^{-1} dz$$



with $V = W \setminus W_0$, $\partial V = \partial W \cup \partial W_0$.

$$(1) - (2) = \frac{1}{2\pi i} \int_{\partial V} \underbrace{\frac{f(\lambda)(\lambda e - b))^{-1}}{\text{holomorphic on } V}} dz = 0$$

and $V \subseteq \overline{V} \subseteq U \setminus \text{sp}(b)$.

Results

$$f_1, f_2 \in A[\text{sp } b] \implies f_1 + f_2 \in A[\text{sp } b]$$

$$f_1(b) + f_2(b) = (f_1 + f_2)(b).$$

For f polynomial, $\sum_{n=0}^N f_n t^n$, $f(b) = \sum_{n=0}^N f_n b^n$.

Proposition

$$f_1(b)f_2(b) = (f_1 f_2)(b).$$

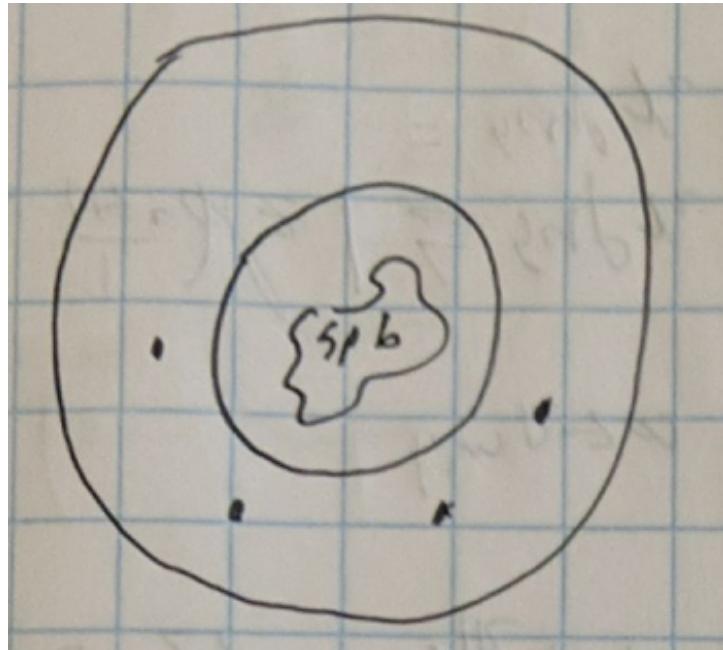
Theorem: Spectral Mapping Theorem

Let $b \in B$ and $f \in A[\text{sp } b]$. Then $\text{sp}(f(b)) = f(\text{sp } b) := \{f(z) : z \in \text{sp } b\}$.

Proof

1. take $\mu \notin f(\text{sp } b)$.

Then $\mu \notin f(z)$, $\forall z \in \text{sp } b$ and $\mu - f(z) \neq 0$.



Therefore, there exist an open $U_1 \ni \text{sp}(b)$, $U_1 \subseteq U$, such that $\mu - f(z) \neq 0$, $\forall z \in U_1$.

Define $g(z) = \frac{1}{\mu - f(z)}$ holomorphic on U_1 , and

$$g(z) \cdot (\mu - f(z)) = 1 \implies g(b) \cdot (\mu e - f(b)) = e$$

by the previous proposition and the polynomial result. So $\mu e - f(b)$ is invertible, and $\mu \notin \text{sp}(f(b))$.

- Remark

$$(\mu e - f(b))^{-1} = \frac{1}{2\pi i} \int_{\partial W_1} \frac{1}{\mu - f(z)} (ze - b)^{-1} dz$$

for $\text{sp } b \subseteq W_1 \subseteq \overline{W}_1 \subseteq U_1$.

- take $\mu \notin \text{sp}(f(b))$ and, for contradiction, assume $\mu \in f(\text{sp } b)$.

Then $\mu e - f(b)$ is invertible, $\mu = f(\lambda)$ for some $\lambda \in \text{sp } b$.

- Idea

$$\mu e - f(b) = f(\lambda)e - f(b) = (\lambda e - b) \cdot g_\lambda(b)$$

We define

$$g_\lambda(z) = \begin{cases} \frac{f(\lambda)e - f(z)}{\lambda - z} & z \in U \supseteq \text{sp}(b) \\ f'(\lambda) & z = \lambda \end{cases}$$

such that $g_\lambda(z)$ is holomorphic on U . Therefore $g_\lambda(b) \in B$,

$$(\lambda - z)g_\lambda(z) = f(\lambda) - f(z), \quad \forall z \in U$$

and $(\lambda e - b)g_\lambda(b) = f(\lambda)e - f(b) = g_\lambda(b)(\lambda e - b)$. Since this is invertible, $(\lambda e - b)$ is left and right invertible.

Remark

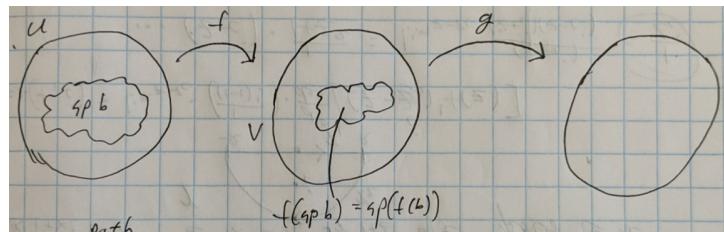
$$g_\lambda(b) = \frac{1}{2\pi i} \int_{\partial W} \frac{f(\lambda) - f(z)}{\lambda - z} (ze - b)^{-1} dz$$

Theorem: Composition of Functions

Let $b \in B$ unital, $f \in A[\text{sp } b]$, and $g \in A[\text{sp}(f(b))] = A[f(\text{sp } b)]$.

Then $h = g \circ f \in A[\text{sp } b]$ and $h(b) = g(f(b))$.

Remark



f is an open mapping and maps U to the open set $V \supseteq \text{sp}(f(b))$.

Applications

- Exponentials

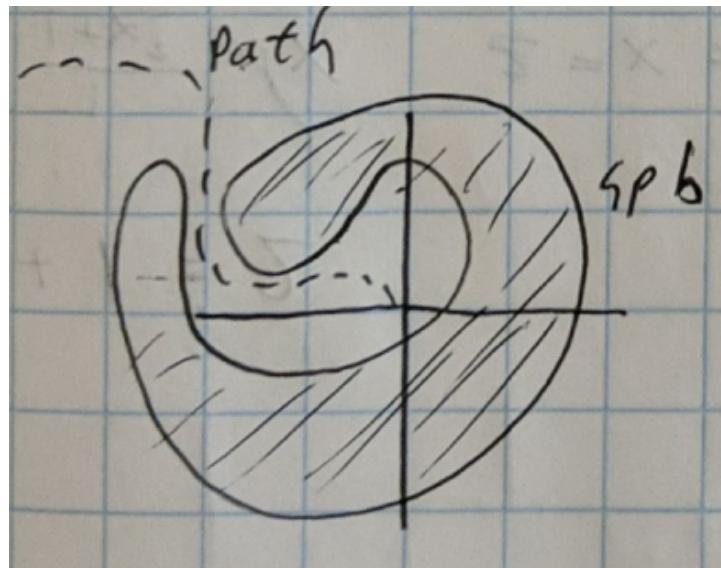
$$\exp(b) = \sum_{n=0}^{\infty} \frac{b^n}{n!} = \frac{1}{2\pi i} \int_{|z|=R} e^z (ze - b)^{-1} dz$$

- Logarithms

$\log b, b \in B$ under the assumption that

- $0 \notin \text{sp } b$
- There exists a path connecting 0 to ∞ in $\mathbb{C} \setminus \text{sp } b$.

This gives us that $\log z$ is analytic on $U \supseteq \text{sp } b$.



$\mathbb{C} \setminus \text{path}$ is simply connected, so there exists an analytic $\log z$ on $\mathbb{C} \setminus \text{path}$.

- if $\log b$ is well-defined, then $\exp(\log b)) = b$ (via composition)
- likewise, one can define powers $f(z) = z^\alpha$ ($\alpha \in \mathbb{C}$)

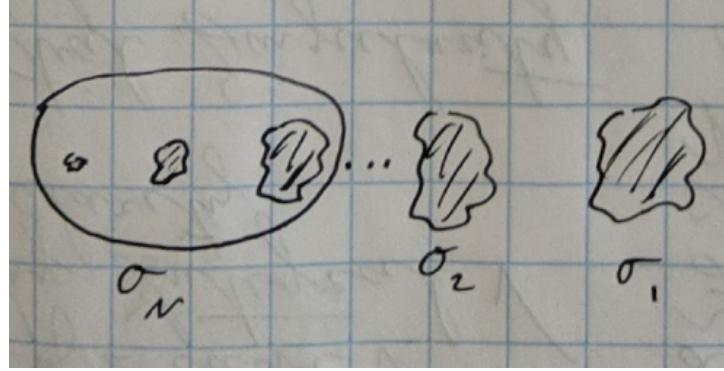
Application: Spectral Idempotents (Riesz Idempotents)

p is idempotent if $p^2 = p$.

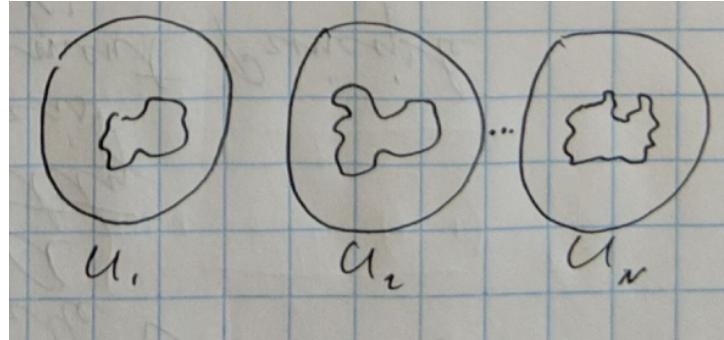
Assume that $b \in B$ and that $\text{sp } b$ is not connected.

$$\text{sp } b = \sigma_1 \cup \sigma_2 \cup \dots \cup \sigma_N$$

with σ_i closed and disjoint subsets of $\text{sp } b$.



Now let U_1, \dots, U_n be open neighborhoods of $\sigma_1, \dots, \sigma_n$ which are themselves disjoint.



Write $U = U_1 \cup \dots \cup U_n \supseteq \text{sp } b$, and consider

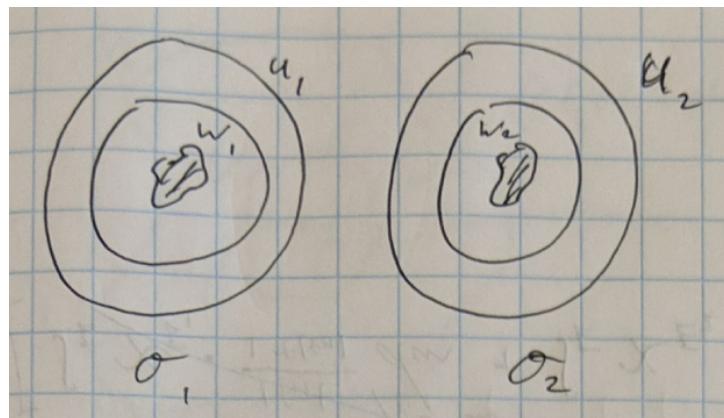
$$\chi_i(x) = \begin{cases} 1 & x \in U_i \\ 0 & x \in U_j, j \neq i \end{cases}$$

Then χ_i is analytic on $U \supseteq \text{sp}(b)$.

Put $p_i = \chi_i(b)$ the spectral or Riesz idempotents.

Properties / Remarks

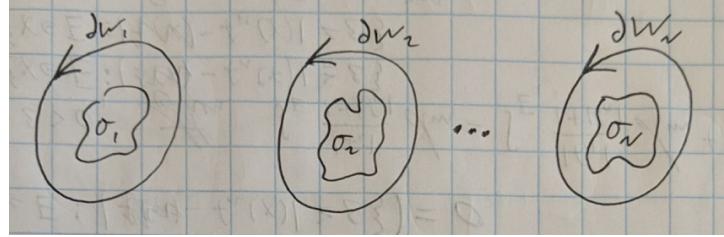
- $p_i^2 = p_i$ because $(\chi_i)^2 = \chi_i$.
- $e = p_1 + \dots + p_N$, mutually orthogonal such that $p_i p_j = 0$, $\forall i \neq j$, because $\chi_1 + \dots + \chi_N = 1$ and $\chi_i \chi_j = 0$.
- $p_i b = b p_i$, because $\chi_i f = f \chi_i$ for $f(z) = z$.
- $p_i = \frac{1}{2\pi i} \int_{\partial W} \chi_i(z)(ze - b)^{-1} dz$ where $\text{sp } b \subseteq W \subseteq \overline{W} \subseteq U$.



$W_i = W \cap U_i$, $W_1 \cup \dots \cup W_N = W$. Therefore

$$p_i = \frac{1}{2\pi i} \sum_{j \neq i}^N \int_{\partial W_j} \chi_i(z) (ze - b)^{-1} dz = 0, \quad i \neq j$$

Then $p_i = \frac{1}{2\pi i} \int_{\partial W_i} (ze - b)^{-1} dz$.



- Write

$$b = (p_1 + p_2 + \dots + p_N)b(p_1 + p_2 + \dots + p_N) = p_1bp_1 + p_2bp_2 + \dots + p_Nbp_N$$

since $p_i bp_j = bp_i p_j = 0$.

- For an idempotent $p \neq 0$,

$$B_p = \{pap : a \in B\}$$

and, therefore, B_p has a unit element p .

Lemma

Assume $b \in B$ with Riesz idempotents $p_1, \dots, p_N \neq 0$.

Then b is invertible if and only if $p_i bp_i$ is invertible in B_{p_i} for all i .

Proof

$b^{-1} = c$, $bc = e$, then

$$\begin{aligned} (p_1 + \dots + p_N)b(p_1 + \dots + p_N)c &= e \\ \sum p_i b(p_i p_i)c &= e \\ (p_i b_i)(p_i c p_i) &= p_i \end{aligned}$$

Suppose $p_i bp_i$ invertible in B_{p_i} . Then $p_i bp_i c = p_i$, $c_i = p_i cp_i$ and $b^{-1} = c = \sum_{i=1}^N p_i c_i p_i$.

Remark

$$B = \begin{pmatrix} B_1 & & 0 \\ & B_2 & \\ 0 & & \ddots & B_N \end{pmatrix} \quad P_1 = \begin{pmatrix} I & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \quad P_2 = \begin{pmatrix} 0 & & & \\ & I & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

Therefore B invertible if and only if B_i are invertible. $B_i \cong P_i B P_i$.

$$\begin{pmatrix} 0 & & & \\ & B_i & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

October 23, 2024

Lemma

Let $b \in B$ and $p_1, \dots, p_n \in B$ satisfying $p_i^2 = p_i$, $p_i p_j = 0$ ($i \neq j$), $p_1 + \dots + p_n = e$, $b p_i = p_i b$. Then b is invertible in B if and only if for each i , $p_i b p_i$ are invertible in B_{p_i} and

$$\text{sp}_B(b) = \bigcup_{i=1}^N \text{sp}_{B_{p_i}}(p_i b p_i)$$

where $B_{p_i} = \{p_i a p_i : a \in B\}$ is a unital Banach algebra with unit p_i .

Theorem

Let p_1, \dots, p_N be spectral idempotents of b with respect to $\text{sp}(b) = \sigma_1 \cup \sigma_2 \cup \dots \cup \sigma_N$ (closed and disjoint). If $\sigma_1, \dots, \sigma_N \neq \emptyset$, then $p_1, \dots, p_N \neq 0$ and $\text{sp}_{B_{p_i}}(p_i b p_i) = \sigma_i$.

Note: if $\sigma_i = \emptyset$ then

$$p_i = \chi_{U_i}(b) = \frac{1}{2\pi i} \int_{\partial W_i} \underbrace{|ze - b|}_{\text{analytic}}^{-1} dz.$$

Proof

Without loss of generality, we may assume $p_1, \dots, p_M \neq 0$ ($M \geq 1$) and $p_{M+1} = \dots = p_N = 0$. Then by the above lemma $p_1 + \dots + p_M = e$ and

$$\text{sp}_B(b) = \bigcup_{i=1}^M \text{sp}_{B_{p_i}}(p_i b p_i)$$

Assuming $\text{sp}_{B_{p_i}}(p_i b p_i) = \sigma_i$ is proven for $j = 1, \dots, M$, then

$$\text{sp}_B(b) = \bigcup_{i=1}^M \sigma_i = \bigcup_{i=1}^N \sigma_i$$

and therefore that $M = N$.

To prove that $\text{sp}_{B_{p_i}}(p_i b p_i) = \sigma_i$ for each $p_i \neq 0$,

$$\text{sp}_{B_{p_i}}(p_i b p_i) = \{\lambda \in \mathbb{C} : p_i(b - \lambda)p_i + e - p_i \text{ not invertible in } B\}$$

For fixed λ , $f_\lambda(z) = \chi_i(z)(z - \lambda)\chi_i(z) + (1 - \chi_i(z))$ is analytic in a neighborhood of $\text{sp}(b)$.

$$f_\lambda(b) = p_i(b - \lambda e)p_i + (1 - p_i)$$

Then $\lambda \in \text{sp}_{B_{p_i}}(p_i b p_i)$ if and only if $f_\lambda(b)$ is not invertible in B .

Equivalently that $0 \in \text{sp}(f_\lambda(b))$ or, by spectral mapping theorem, $0 \in f_\lambda(\text{sp } b)$.

This is further equivalent to there existing some $\xi \in \text{sp } b : 0 = f_\lambda(\xi)$

$$f_\lambda(z) = \begin{cases} 1 & z \in \sigma_j \subseteq U_j, i \neq j \\ z - \lambda & z \in \sigma_i \end{cases}$$

That is, if $\xi \in \text{sp } b : \xi \in \sigma_i$ and $\xi = \lambda$ or, simply, $\lambda \in \sigma_i$.

Chapter 2: Commutative Banach Algebras

Section 2.1: Homomorphisms, Ideals and Quotient Algebras.

A need not be commutative.

Definition: Banach Algebra Homomorphisms

$\phi : A \rightarrow B$ is a Banach algebra homomorphism if it is linear, multiplicative and bounded.

Definition: Banach Algebra Ideal

A (two-sided) ideal J of a Banach algebra is a linear subspace $J \subseteq A$ such that $\forall a \in A, \forall j \in J, aj, ja \in J$.

Remark

If $\phi : A \rightarrow B$ is a Banach algebra homomorphism then $\ker \phi$ is a closed two-sided ideal of A .

Proof

Put $J \in \ker \phi$, $a \in A, j \in J$. Then $\phi(j) = 0, \phi(aj) = \phi(a)\phi(j) = 0 = \phi(j)\phi(a) = \phi(ja)$ and $aj, ja \in J$.

Definition: Quotient Algebra

If J is a closed, two-sided ideal of A ($J \neq A$), then A/J is a Banach algebra. A/J is a Banach algebra $[a] = a + J$.

A/J is a vector space, a normed space (J closed) with $\|[a]_J\| = \inf_{j \in J} \|a + j\|$, and a Banach space because A is complete.

$$[a_1] + [a_2] = [a_1 + a_2] \text{ and } [a_1] \cdot [a_2] = [a_1 \cdot a_2]$$

$$(a_1 + j_1)(a_2 + j_2) = a_1 a_2 + \underbrace{a_1 j_2 + a_2 j_1 + j_1 j_2}_{\in J}$$

Definition: Quotient Map

Take $\pi : A \rightarrow A/J$ by $a \mapsto [a]$.

This is a Banach algebra homomorphism which is surjective with $\ker \pi = J$.

Proposition

Let $\phi : A \rightarrow B$ be a Banach algebra homomorphism and $J \subseteq \ker \phi$ a closed, two-sided ideal of A .

Then there exists a Banach algebra homomorphism $\phi^J : A/J \rightarrow B$ such that $\phi = \phi^J \circ \pi$

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ & \searrow \pi & \swarrow \phi^J \\ & A/J & \end{array}$$

Write $\phi^J([a]_J) = \phi(a)$, and $[a_1] = [a_2]$ implies $a_1 - a_2 \in J \subseteq \ker \phi$ and subsequently that $\phi(a_1) = \phi(a_2)$.

Remark

$J = \{0\}$ and $J = A$ are always closed, two-sided ideals of A .

Examples

- $A = \mathbb{C}^{n \times n}$. Only ideals are $\{0\}$ and A .
- $A = L(X)$ (continuous operators) for X a Banach space. Then at least $\{0\}$, $K(X)$ (compact operators), and A are ideals.
- X a separable hilbert space. Only $\{0\}$, $K(X)$ and A .
- $A = \mathbb{C}_{\text{upper}}^{n \times n}$ upper triangular matrices. Then there are many (one sided) ideals for $n = 2$.
- $A = C(X)$ for X compact Hausdorff spaces. Then every closed set $E \subseteq X$ generates a closed ideal

$$J_E = \{f \in C(X) : f|_E \equiv 0\}$$

In particular, $E = \{x_0\}$, $J_{x_0} = \{f \in C(X) : f(x_0) = 0\}$, $\dim(A/J_{x_0}) = 1$ implies $A/J_{x_0} \cong \mathbb{C}$.

Remark

Every closed (2-sided) ideal is a closed subalgebra of A but not vice versa.

For a set $S \subseteq A$, let $J = \text{clos id}_A(S)$ be the smallest closed 2-sided ideal containing S (i.e. the ideal generated by S or the intersection of all ideals containing S). One can show that

$$J = \text{clos}_A \left\{ \sum_{i=1}^N a_i j_i b_i : a_i, b_i \in A, j_i \in S \right\}$$

Section 2.2: Maximal Ideals and Multiplicative Linear Functionals

From now on, B is a unital, commutative Banach algebra.

Definition: Multiplicative Linear Functional

A multiplicative linear functional on B is a linear map $\phi : B \rightarrow \mathbb{C}$ such that $\phi(ab) = \phi(a)\phi(b)$ ($\phi \neq 0$).

Proposition

A multiplicative linear functional on B is bounded. In fact $\phi \in B^*$, $\|\phi\| = 1$, $\phi(e) = 1$.

Proof

$\phi \neq 0$ means that there exists $a \in B$ such that $\phi(a) \neq 0$.

Then $\phi(e)\phi(a) = \phi(ea) = \phi(a)$ so $\phi(e) = 1$ and consequently that $\|\phi\| \geq 1$.

If $|\phi(a)| \leq ||a||$, then $||\phi|| \leq 1$. If this were not the case,

$$|\phi(a)| > ||a|| \iff \left\| \frac{a}{\phi(a)} \right\| < 1$$

and $e - \frac{a}{\phi(a)}$ is invertible. Call the inverse b . Then

$$\begin{aligned} b \left(e - \frac{a}{\phi(a)} \right) = e &\implies \underbrace{\phi(b) \phi \left(e - \frac{a}{\phi(a)} \right)}_{= \phi(e) - \frac{1}{\phi(a)} \phi(a) = 0} = \phi(e) = 1 \end{aligned}$$

which is a clear contradiction.

Definition: Maximal Ideal

A (two-sided) ideal I of B is called maximal if

- $I \neq B$ (I is a proper ideal)
- if J is another ideal of B such that $I \subseteq J \subseteq B$, then either $I = J$ or $J = B$.

Proposition

A maximal ideal I is closed and B/I is a field.

Proof (Closed)

We have $I \subseteq \bar{I} \subseteq B$ with \bar{I} an ideal. Since I is maximal, either $I = \bar{I}$ or $\bar{I} = B$. But then $e \in \bar{I}$, and there exists $a \in I$ such that $||a - e|| < 1$. Then $a = e + (a - e)$ is invertible and for each $b \in B$, $b = ba^{-1}a \in I$ and $I = B$ a contradiction.

October 28, 2024

Recall: Multiplicative Linear Functionals

$$\phi : B \rightarrow \mathbb{C}$$

- linear
- $\phi(ab) = \phi(a)\phi(b)$, $\phi \neq 0$.

Then $\phi(e) = 1$, $||\phi|| = 1$

Recall: Maximal Ideals

I is a maximal ideal of B if

- $I \subsetneq B$
- If J is an ideal with $I \subseteq J \subseteq B$, then either $I = J$ or $J = B$.

Proposition

Every maximal ideal is closed and, in a commutative Banach algebra B , B/I is a field.

Proof (Field)

We know that B/I is a Banach algebra. We need to show that every nonzero $[a] \in B/I$ is invertible.

Consider $[a] \in B/I$ with $[a] \neq 0$. Then $a \notin I$. Define

$$J = \{i + ax : i \in I, x \in B\}$$

Then J is a linear subspace and an ideal in B since for any $y \in B$

$$\begin{aligned} y(i + ax) &= yia(yx) \in J \\ (i + ax)y &= \underbrace{iy}_{\in I} + a(xy) \in J \end{aligned}$$

Since $I \subseteq J \subseteq B$ and I is maximal, it cannot be that $I = J$ since $0 + a \cdot e \in J$ implies $a \in I$ a contradiction.

If $J = B$, then $e \in J$ and $e = i + ax$ for some $i \in I$ and $x \in B$. Therefore, in the quotient,

$$[e] = [a] \cdot [x] = [x] \cdot [a]$$

and $[a]$ is invertible in B/I .

Theorem: Gelfand/Mazur

Any (complex) Banach algebra which is a field is isomorphic to \mathbb{C} .

Proof

Let A be a Banach algebra which is a field with unit $e \in A$, and consider the map

$$\Lambda : \mathbb{C} \ni \lambda \mapsto \lambda e \in A$$

This Banach algebra homomorphism is isometric ($\|\lambda e\| = |\lambda|$) and injective.

If Λ is surjective, then it is a Banach algebra isomorphism.

Take $a \in A$. We know that $\text{sp}(a) \neq \emptyset$, therefore $\exists \lambda \in \mathbb{C}$ such that $\lambda e - a$ is not invertible.

It follows that $\lambda e - a = 0$ and, consequently, $a = \lambda e = \Lambda(\lambda)$.

Corollary

Let B be a unital, commutative Banach algebra and I be a maximal ideal. Then $B/I \cong \mathbb{C}$.

Theorem: 1-1 Correspondence Between Maximal Ideals and Multiplicative Linear Functionals

Let B be a unital, commutative Banach algebra.

1. If ϕ is a multiplicative linear functional on B , then $\ker \phi$ is a maximal ideal in B .
2. If I is a maximal ideal in B , then there exists a unique multiplicative linear functional ϕ such that $\ker \phi = I$.

Proof of 1

If ϕ is a multiplicative linear functional, it is bounded and $\ker \phi$ is a closed, two-sided ideal.

We know $I \subseteq B$ because $\phi \not\equiv 0$ ($\phi(e) = 1$).

We have that $B = \ker \phi + \mathbb{C} \cdot e$ because

$$b = \underbrace{b - \phi(b) \cdot e}_{\in \ker \phi} + \phi(b) \cdot e$$

$$\phi(b - \phi(b) \cdot e) = \phi(b) - \phi(b) \cdot \phi(e) = 0$$

and therefore $\dim B/I = 1$.

If $I \subseteq J \subseteq B$, then $J/I \subseteq B/I$ and either $\dim(J/I) = 0$ ($J = I$) or $\dim(J/I) = 1$ ($J = B$).

Proof of 2

Let I be a maximal ideal in B .

By Gelfand/Mazur (corollary) $B/I \cong \mathbb{C}$, so there exists a Banach algebra isomorphism $\psi : B/I \rightarrow \mathbb{C}$.

$$\begin{array}{ccc} B & \xrightarrow{\phi} & \mathbb{C} \\ & \searrow \pi & \nearrow \psi \\ & B/I & \end{array}$$

where π and ψ are Banach algebra homomorphisms.

Put $\phi = \psi \circ \pi : B \rightarrow \mathbb{C}$ by $\phi(b) = \psi([b])$. Then ϕ is a Banach algebra homomorphism that is also a multiplicative linear functional with $\phi(e) = 1$. Since ψ is injective,

$$\ker \phi = \ker \pi = I$$

Suppose $\ker \phi_1 = \ker \phi_2 = I$. Then

$$b - \phi_1(b) \cdot e \in \ker \phi_1 = I$$

$$b - \phi_2(b) \cdot e \in \ker \phi_2 = I$$

and $(\phi_1(b) - \phi_2(b))e \in I$. But a proper ideal cannot contain invertible elements. Therefore $\phi_1(b) = \phi_2(b)$. Since this holds for all b , $\phi_1 \equiv \phi_2$.

Definition: Maximal Ideal Space

For a unital (complex) commutative Banach algebra B , $M(B)$ denotes the maximal ideal space (i.e. the set of all multiplicative linear functionals on B).

Remark

$$M(B) \subseteq B^*$$

$$M(B) \subseteq \{\phi \in B^* : \|\phi\| \leq 1\}$$

The multiplicative linear functionals are a proper subset of bounded linear functionals.

Example

$B = C(X)$ (the set of continuous functions $f : X \rightarrow \mathbb{C}$) for X a compact Hausdorff space.

Every $x_0 \in X$ determines a multiplicative linear functional $\phi_{x_0} : B \rightarrow \mathbb{C}$ by $\phi_{x_0}(f) = f(x_0)$ for $f \in B$.

The corresponding maximal ideals are of the form $I_{x_0} = \ker \phi_{x_0} = \{f \in C(X) : f(x_0) = 0\}$. Then

$$\begin{aligned} C(X) &= I_{x_0} + \mathbb{C} \cdot e, \quad e(x) \equiv 1 \\ C(X)/I_{x_0} &\cong \mathbb{C} \end{aligned}$$

Therefore $\phi_{x_0} \in M(C(X))$. In fact one can show that $M(C(X)) = \{\phi_x : x \in X\}$. So $M(C(X)) \cong X$.

Remark

$$M(C(X)) \subseteq C(X)^*$$

$C(X)^*$ is isomorphic to the set of all complex Borel measures on X by

$$\phi(f) = \int_X f(x) d\mu(x)$$

with bounded linear functional $\mu = \delta_{x_0} \rightsquigarrow \delta_{x_0} \in M(C(X))$.

Section 2.3: Gelfand Theory and Gelfand Transform

Setting

Unital commutative (complex) Banach algebra B .

Lemma

Every proper (two-sided) ideal I_0 of B is contained in some maximal ideal of B .

Proof

Zorn's lemma applied to the collection S of all proper ideals.

Lemma

Every non-invertible element $a \in B$ is contained in at least one maximal ideal.

Proof

Consider $a \in B$ and $I_0 \in \{ax : x \in B\}$ an ideal ($y(ax) = a(yx) \in I_0$ and $(ax)y = a(xy) \in I_0$).

I_0 is proper, otherwise $I_0 = B$, $e \in I$ and $ax = xa = e$ a contradiction.

Then, by the previous lemma, $a = ae \in I_0 \subset I$ for some maximal ideal I .

Theorem: Gelfand Theory

Let B be a unital commutative Banach algebra and $b \in B$ arbitrary. Then b is invertible in B if and only if $\phi(b) \neq 0$, $\forall \phi \in M(B)$.

Proof

(\implies) if b is invertible,

$$1 = \phi(e) = \phi(b^{-1}b) = \phi(b^{-1}) \cdot \phi(b)$$

so $\phi(b) \neq 0$ for all ϕ .

(\impliedby) if b is not invertible, then there exists $\phi \in M(B)$ such that $\phi(b) = 0$.

If b is not invertible, then b is contained in some maximal ideal I and $I = \ker \phi$ for some $\phi \in M(B)$. Therefore $b \in I = \ker \phi$ implies $\phi(b) = 0$.

Definition/Notation: Gelfand Transform

The Gelfand transform of an element $b \in B$ is the function $\hat{b} : M(B) \rightarrow \mathbb{C}$ defined by $\hat{b}(\phi) := \phi(b)$.

Remark

$$\begin{aligned}\widehat{a+b} &= \hat{a} + \hat{b} \\ \widehat{ab} &= \hat{a} \cdot \hat{b}\end{aligned}$$

$$\sup_{\phi \in M(B)} |\hat{a}(\phi)| = \sup_{\phi \in M(B)} |\phi(a)| \leq \|a\|.$$

Later we will consider $M(B)$ with topology (a compact Hausdorff space).

Definition: Gelfand Transform of B

$$\Lambda : B \ni b \mapsto \hat{b} \in C(M(B))$$

It is a Banach algebra homomorphism.

Gelfand's theorem states that b is invertible in B if and only if $\Lambda(b)$ is invertible in $C(M(B))$.

Note

b is invertible if and only if $\phi(b) = \hat{b}(\phi) \neq 0, \forall \phi \in M(B)$.

Equivalently, \hat{b} is invertible in $C(M(B))$.

A continuous functional is invertible within the set of continuous functions if and only if it is non-zero everywhere.

Purpose of Gelfand Theory

Invertibility in B corresponds to invertibility in $C(M(B))$.

We need to determine $M(B)$.

Remark

If $B = C(X)$ (X compact Hausdorff), then $M(B) = M(C(X))$ is homeomorphic to X .

$M(B) \cong X$ (homeomorphic) implies that $C(M(B)) \cong C(X) = B$ isometric Banach algebra isomorphisms.

October 30, 2024

Recall

$M(B)$ the multiplicative linear functionals or the maximal ideal space.

b invertible if and only if $\phi(b) \neq 0, \forall \phi \in M(B)$.

$\hat{b} : M(B) \rightarrow \mathbb{C}$ where $\hat{b}(\phi) = \phi(b)$ the Gelfand transform of b .

$\Lambda : B \rightarrow C(M(B))$ where $b \mapsto \hat{b}$ is the Gelfand transform of B .

Section 2.4: The Topology of the Maximal Ideal Space

Since $M(B) \subseteq \{\phi \in B^* : \|\phi\| \leq 1\} \subseteq B^*$, $M(B)$ is a topological space with the subspace topology with respect to the weak*-topology of B^* .

A base for the topology in $M(B)$ is given by

$$U_{\varepsilon; b_1, \dots, b_n}[\phi] = \{\psi \in M(B) : |\psi(b_i) - \phi(b_i)| < \varepsilon, i = 1, \dots, n\}$$

with $\varepsilon > 0$, $b_1, \dots, b_n \in B$ and $\phi \in M(B)$.

Theorem

$M(B)$ is a compact Hausdorff space.

Proof

$M(B)$ is Hausdorff because it is a subspace of the Hausdorff space B^* .

By Banach-Alaoglu, the unit ball is compact in the weak*-topology. We need that $M(B)$ is a closed subset of the unit ball.

Let ϕ be in the closure of $M(B)$ with respect to the unit ball such that $\phi \in B^*$ and $\|\phi\| \leq 1$. To show that $\phi(ab) = \phi(a)\phi(b)$, consider

$$U_{\varepsilon;a,b,ab}[\phi] = \{\psi \in B^* : |\psi(b_i) - \phi(b_i)| < \varepsilon, i = 1, \dots, n\}$$

Then $\psi \in M(B) \cap U_{\varepsilon;a,b,ab}[\phi]$, so we have

$$|\psi(a) - \phi(a)| < \varepsilon, \quad |\psi(b) - \phi(b)| < \varepsilon, \quad \text{and} \quad |\psi(ab) - \phi(ab)| < \varepsilon.$$

We know that $\psi(ab) = \psi(a)\psi(b)$. Therefore

$$\begin{aligned} |\phi(ab) - \phi(a)\phi(b)| &= |\phi(ab) - \psi(ab) - \phi(a)\phi(b) + \psi(a)\psi(b)| \\ &\leq |\phi(ab) - \psi(ab)| + |\phi(a) - \psi(a)| \cdot |\phi(b)| + |\psi(a)| \cdot |\phi(b) - \psi(b)| \\ &\leq \varepsilon \|b\| + \varepsilon \|a\| \end{aligned}$$

Taking $\varepsilon \rightarrow 0$, $\phi(ab) = \phi(a)\phi(b)$, $\forall a, b \in B$. Similarly, $\phi(e) = 1$.

Thus $\phi \in M(B)$ and $M(B)$ is closed in the unit ball of B^* .

Proposition

For $b \in B$, the Gelfand transform $\hat{b} : M(B) \rightarrow \mathbb{C}$ is continuous and $\|\hat{b}\| := \max_{\phi \in M(B)} |\hat{b}(\phi)| \leq \|b\|$. In other words, $\hat{b} \in C(M(B))$.

Proof

We need to show that \hat{b} is continuous at each $\phi_0 \in M(B)$.

Consider $U = B_\varepsilon(\hat{b}(\phi_0))$ an ε -neighborhood in \mathbb{C} . Then the preimage is

$$\hat{b}^{-1}(U) = \{\phi \in M(B) : \hat{b}(\phi) \in B_\varepsilon(\hat{b}(\phi_0))\} = \{\phi \in M(B) : |\hat{b}(\phi) - \hat{b}(\phi_0)| < \varepsilon\} = \{\phi \in M(B) : |\phi(b) - \phi_0(b)| < \varepsilon\} = U_{\varepsilon;b}[\phi_0]$$

with $[\phi_0]$ open in $M(B)$.

Also note that

$$|\hat{b}(\phi)| = |\phi(b)| \leq \|\phi\| \cdot \|b\| = \|b\|$$

Corollary

The Gelfand transform of B , $\Lambda : B \rightarrow C(M(B))$ by $b \mapsto \hat{b}$ is a Banach algebra homomorphism with $\|\Lambda\| = 1$

Proof

Λ is linear and multiplicative with

$$\begin{aligned}\widehat{a+b} &= \hat{a} + \hat{b} \\ \widehat{ab} &= \hat{a}\hat{b}\end{aligned}$$

$||\Lambda|| = 1$ because $||\hat{b}|| \leq ||b||$ and $||\hat{e}|| = ||e|| = 1$.

Then $\hat{e}(\phi) = \phi(e) = 1$.

It follows also that

$$(\widehat{ab})(\phi) = \phi(ab) = \phi(a)\phi(b) = \hat{a}(\phi) \cdot \hat{b}(\phi) = (\hat{a} \cdot \hat{b})(\phi)$$

Corollary

For $b \in B$, $b \in GB$ if and only if $\hat{b} = \Lambda(b) \in GC(M(B))$.

As a consequence, $\text{sp}_B(b) = \text{sb}_{C(M(B))} \Lambda(b)$ (Λ preserves spectrum).

Proof

$b \in GB$ implies that $ab = e$ and that $\hat{a} \cdot \hat{b} = \hat{e} = 1$. Therefore $(\hat{b})^{-1} = \hat{a} = \widehat{b^{-1}}$.

We have also that $\hat{b} \in GC(M(B))$ implies $\hat{b}(\phi) \neq 0$, $\forall \phi \in M(B)$ and $\phi(b) \neq 0$ similarly. Therefore $b \in GB$.

Gelfand Theory

Two problems for a given Banach algebra B :

1. How to determine $M(B)$.
2. How to determine its topology.

Theorem

Let B be a commutative Banach algebra with maximal ideal space $M(B) = X$ and topology τ on X .

Now assume we have another topology ρ on X such that

1. (X, ρ) is a compact topological space.
2. $\forall b \in B$, $\hat{b} : X \rightarrow \mathbb{C}$ is continuous in the (X, ρ) topology.

Then $\tau = \rho$.

Proof

First show that $\tau \subseteq \rho$.

Take $U \in \tau$ from the base of the topology

$$U = U_{\varepsilon; b_1, \dots, b_n}[\phi] = \{\psi \in X : |\psi(b_i) - \phi(b_i)| < \varepsilon, \forall i\}$$

Then

$$U = \bigcap_{i=1}^n \{\psi \in X : |\hat{b}_i(\psi) - \hat{b}_i(\phi)| < \varepsilon\} = \bigcap_{i=1}^n (\hat{b}_i)^{-1}(B_\varepsilon(\hat{b}_i(\phi)))$$

which is open in the (X, ρ) topology because \hat{b}_i is continuous. Therefore $U \in \rho$.

We have that $\text{id} : (X, \rho) \rightarrow (X, \tau)$ where (X, ρ) is compact and (X, τ) is Hausdorff is continuous ($\tau \leq \rho$).

Then as an open map, we map closed sets to closed sets.

$A \subset X$ closed $\implies A$ compact $\implies \text{i}(A)$ compact $\implies \text{id}(A)$ closed.

Therefore $(\text{id})^{-1}$ is continuous and $\rho = \tau$.

Theorem

Let X be a compact Hausdorff space and $B = C(X)$.

Then $M(B)$ is homeomorphic to X by the map

$$\tau : X \ni x \mapsto \phi_x \in M(B)$$

where $\phi_x(b) = b(x)$ is the point evaluation for $b \in B = C(X)$.

Proof

We have that ϕ_x is indeed in $M(B)$ easily.

Then to show τ is injective, $\phi_x = \phi_y$ implies that $\phi_x(b) = \phi_y(b)$, $\forall b \in B$. Then $b(x) = b(y)$, $\forall b \in C(X)$ and $x = y$.

Because $x \neq y$ implies that there exists $b \in C(X)$ such that $b(x) \neq b(y)$.

Since X is a normed space, $\{x\}$ and $\{y\}$ are closed. By Urysohn's Lemma, there exists a continuous b such that $b|_{\{x\}} = 0$ and $b|_{\{y\}} = 1$.

IMAGE 1

To see that τ is surjective, otherwise there would exist $\phi \in M(B)$ such that $\phi \neq \phi_x$, $\forall x \in X$.

That implies that there exists $b_x \in B$ such that $\phi(b_x) \neq \phi_x(b_x) = b_x(x)$.

Put $a_x = b_x - \phi(b_x) \cdot e$. Then $\phi(a_x) = 0$. So

$$\phi_x(a_x) = \phi_x(b_x) - \phi(b_x) \neq 0$$

IMAGE 2

With a_x continuous, we find a neighborhood $U_x \ni x$ where $a_x(t) \neq 0$, $\forall t \in U_x$.

For $\{U_x\}_{x \in X}$ an open cover of X , there exists a finite subcover $\{U_{x_i}\}_{i=1,\dots,N}$. Consider

$$a(t) = \sum_{i=1}^N |a_{x_i}(t)|^2$$

which is itself continuous and $a(t) > 0$ for each $t \in X$.

So $a \in B = C(X)$, a is invertible in B and

$$\phi(a) = \sum_{i=1}^N \phi(\overline{a_{x_i}}) \underbrace{\phi(a_{x_i})}_{=0}$$

which would imply that a is not invertible, a contradiction.

To show that $\tau : X \ni x \mapsto \phi_x \in M(B)$ is continuous, take $X \ni x_0 \mapsto \phi_{x_0}$ and consider

$$U = U_{\varepsilon; b_1, \dots, b_N}[\phi_{x_0}] = \{\phi_x : |\phi_x(b_i) - \phi_{x_0}(b_i)| < \varepsilon\} = \{\phi_x : |b_i(x) - b_i(x_0)| < \varepsilon\}$$

Then

$$\tau^{-1}(U) = \{x \in X : |b_i(x) - b_i(x_0)| < \varepsilon\}$$

which is the open preimage of $b_i : X \rightarrow \mathbb{C}$ with b_i continuous.

Then $\tau : X \rightarrow M(B)$ is a continuous map between compact, Hausdorff spaces and τ^{-1} is continuous.

Section 2.5: Commutative Banach Algebras Generated by Single Elements

Consider the Banach algebra $A \ni B$ and

$$B = \text{alg}_A\{e, b\} = \text{clos}_A \left\{ \sum_{n=0}^N \lambda_n b^n \right\}$$

Theorem

The maximal ideal space $M(B)$ of $B = \text{alg}_A\{e, b\}$ is homeomorphic to $\text{sp}_B(b)$.

November 4, 2024

Recall

Let A be a unital Banach algebra, $b \in A$ and $B = \text{alg}_A\{e, b\} = \text{clos}_A\{\sum \lambda_i b^i : \lambda_i \in \mathbb{C}, N \in \mathbb{N}\}$ the smallest closed subalgebra containing e and b .

Then B is a commutative Banach algebra and the closure of all polynomials in b , $p(b)$.

Theorem

The maximal ideal space of $B = \text{alg}\{e, b\}$ is homeomorphic to $\text{sp}_B(b)$. The map

$$\tau : M(B) \ni \phi \mapsto \phi(b) \in \text{sp}_B(b)$$

is a homeomorphism.

Proof

- $\tau : M(B) \rightarrow \mathbb{C}$ maps into $\text{sp}_B(b)$.

Otherwise, $z = \phi(b) \notin \text{sp}_B(b)$ (for some ϕ) and $b - z \cdot e$ is invertible in B . So

$$\phi(b - z \cdot e) = \phi(b) - z \cdot \phi(e) = 0$$

and, therefore,

$$\phi((b - ze)^{-1}(b - ze)) = \phi(e) = 1$$

a contradiction.

- τ is injective.

Assume that $\phi_1(b) = \phi_2(b)$.
Then $\phi_1(b^n) = \phi_2(b^n)$ for $n = 0, 1, \dots$ and, consequently,

$$\phi_1\left(\sum_{n=0}^N \lambda_n b^n\right) = \phi_2\left(\sum_{n=0}^N \lambda_n b^n\right)$$

Because $B = \text{clos}\{\sum \lambda_n b^n\}$ and ϕ_i is continuous, $\phi_1(a) = \phi_2(a)$ for each $a \in B$ and $\phi_1 = \phi_2$.

- τ is surjective.

Take $z \in \text{sp}_B(b)$. Then $b - z \cdot e$ is not invertible in B .
It follows from Gelfand theorem that there exists some $\phi \in M(B)$ such that $\phi(b - ze) = 0$ and $\phi(b) = z$.

- We know $\tau : M(B) \rightarrow \text{sp}_B(b)$ is an injection with the natural topology (weak*-topology of B^*) on $M(B)$ and $\text{sp}_B(b)$ Hausdorff compact.

Define (another) topology on $M(B)$ via τ .
To show that both topologies are the same, we need that for each $b \in B$, $\hat{b} : M(B) \rightarrow \mathbb{C}$ is a continuous function.

$$\text{sp}_B(b) \xrightarrow{\tau^{-1}} M(B) \xrightarrow{\hat{a}} \mathbb{C}$$

equivalently, $\tilde{a} = \hat{a} \circ \tau^{-1} : \text{sp}_B(b) \rightarrow \mathbb{C}$ is continuous.

Let $\phi = \tau^{-1}(z)$, $\tau(\phi) = z$ and $\hat{b}(\phi) = \phi(b) = z$.

Then if $a = e$, $\hat{e}(\phi) = \phi(e) = 1$, $\tilde{e}(z) = 1$

If $a = b$, $\hat{b}(\phi) = \phi(b) = z$ and $\tilde{b}(z) = z$ continuous.

For $a = b^n$, $\hat{b}^n = (\hat{b})^n$ and $\tilde{b}^n = (\tilde{b})^n$ so $\tilde{b}^n(z) = z^n$.

When $a = p(b)$, $p(z) = \sum_{i=1}^N \lambda_i z^i$ a polynomial,

$$\begin{aligned} \hat{a} &= \sum_{i=1}^N \widehat{\lambda_i b^i} = \sum_{i=1}^N \lambda_i \hat{b}_i \\ \tilde{a} &= \sum_{i=1}^N \lambda_i \tilde{b}^i \\ \tilde{a}(t) &= \sum_{i=1}^N \lambda_i t^i = p(z) \end{aligned}$$

For $a \in B$, $\|a - a_n\| \rightarrow 0$, $a_n = p_n(b)$, we have $\hat{a} - \hat{a}_n = \widehat{a - a_n}$ and take

$$\max_{\phi \in M(B)} |\hat{a}(\phi) - \hat{a}_n(\phi)| \leq \|a - a_n\|_B = \max_{z \in \text{sp}_B(b)} |\tilde{a}(z) - \tilde{a}_n(z)|$$

Therefore $\tilde{a}_n \Rightarrow \tilde{a}$ uniformly on $\text{sp}_B(b)$ with \tilde{a} continuous.

Both topologies on $M(B)$ coincide. Therefore $\tau : \text{sp}_B(b) \rightarrow M(B)$ is a homeomorphism.

Theorem

Let A be a unital Banach algebra, and $b \in A$ an invertible element. Then for

$$B = \text{alg}_A\{e, b, b^{-1}\} = \text{clos}_A \left\{ \sum_{i=-N}^N \lambda_i b^i : \lambda_i \in \mathbb{C}, N \in \mathbb{N} \right\}$$

This is the closure of all trigonometric polynomials. The map

$$\tau : M(B) \ni \phi \mapsto \phi(b) \in \text{sp}_B(b)$$

is a homomorphism.

The proof follows similarly to that for the previous theorem.

Example: Wiener Algebra

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ with $C(\mathbb{T})$ continuous functions $\mathbb{T} \rightarrow \mathbb{C}$.

$W \subseteq C(\mathbb{T})$ is the set of all functions with an absolutely convergent Fourier series.

$$a(t) = \sum_{n=-\infty}^{\infty} a_n t^n$$

such that $\|a\|_W := \sum_{n=-\infty}^{\infty} |a_n| < +\infty$.

This is a Banach algebra with addition and multiplication defined pointwise and $\|ab\| \leq \|a\| \cdot \|b\|$ (verify!).

Further, for $a \in W$, $\|a\|_{C(\mathbb{T})} \leq \|a\|_W$ and $W \subseteq C(\mathbb{T})$ is a continuous embedding.

Remark

W is a Banach algebra.

W is isometrically isomorphic to $\ell^1(\mathbb{Z})$ (as a Banach algebra)

$$\begin{aligned} c(t) &= a(t) \cdot b(t) & \{c_n\} &= \{a_n\} * \{b_n\} \\ (\sum c_n t^n) &= (\sum a_n t^n)(\sum b_n t^n) & c_n &= \sum_{k \in \mathbb{Z}} a_{n-k} b_k \end{aligned}$$

Consider $\chi_n(t) = t^n$ for $n \in \mathbb{Z}$. $\chi_0 = e$, $\chi_n = (\chi_1)^n$, $(\chi_1)^{-1} = \chi_{-1}$. Note that

$$W = \text{alg}_W\{X_0, X_1, X_{-1}\}$$

Write

$$W = \text{clos}_W \left\{ p(\chi_1) = \sum_{i=-N}^N \lambda_i (\chi_1)^i \right\}$$

and, to show that the trigonometric polynomials are dense in W , for $a(t) = \sum_{n=-\infty}^{\infty} a_n t^n$, consider $a^{(N)}(t) = \sum_{n=-N}^N a_n t^n$ and

$$\|a(t) - a^{(N)}(t)\| = \sum_{|n|>N} |a_n|$$

which converges to 0 as $N \rightarrow +\infty$ because $\sum_{n \in \mathbb{N}} |a_n| < +\infty$.

Claim: $\text{sp}_W(\chi_1) = \mathbb{T}$.

Indeed, for $|z| > 1$,

$$(\chi_1 - z\chi_0)^{-1} = \frac{1}{t-z} = -\frac{1}{z} \frac{1}{1-t/z} = -\frac{1}{z} \sum_{n=0}^{\infty} \left(-\frac{t}{z}\right)^n \in W$$

For $|z| < 1$

$$(\chi_1 - z\chi_0)^{-1} = \frac{1}{t-z} = \frac{1}{t} \frac{1}{1-z/t} = \frac{1}{t} \sum_{n=-\infty}^0 \left(-\frac{t}{z}\right)^n \in W$$

However, for $|z| = 1$, $\chi_1 - z\chi_0 = t - z$ vanishes at $t = z \in \mathbb{T}$. Thus $\chi_1 - z\chi_0 \notin GW$.

Therefore, the maximal ideal space of W is homeomorphic to \mathbb{T} .

If we identify $M(W) \cong \mathbb{T}$ via τ from above,

$$\phi(\chi_0) = t_0 \in \mathbb{T}$$

$$\mathbb{T} \xrightarrow{\tau^{-1}} M(W) \xrightarrow{\hat{a}} \mathbb{C}$$

$$\text{For } a = \sum a_n t^n = \sum a_n \chi_n,$$

$$\phi(a) = \sum a_n \phi(\chi_n) = \sum a_n \phi(\chi_1)^n = \sum a_n t_0^n$$

Then $\hat{a}(\phi) = \sum a_n t_0^n$ and $\hat{a}(t^{-1}(t_0)) = \sum a_n t_0^n$.

Finally $\tilde{a}(t_0) = \sum a_n t_0^n$ implies that $\tilde{a} = a$.

Theorem (Wiener)

Let $a \in W$. Then a is invertible in W if and only if $a(t) \neq 0$, $\forall t \in \mathbb{T}$ (i.e. if a is invertible in $C(\mathbb{T})$).

Remark

$a \in W$ and $a(t) \neq 0$, $\forall t \in \mathbb{T}$ implies that $\frac{1}{a} \in W$.

That is, if a has an absolutely convergent Fourier series (under these conditions) then $\frac{1}{a}$ has an absolutely convergent Fourier series.

Example

Let $A = C(\mathbb{T})$ and $B = \text{alg}_A\{X_0, X_1\} = \text{clos}_{C(\mathbb{T})} \{p(t) = \sum_{n=0}^N \lambda_n t^n\}$.

One can show that $B = C_+(\mathbb{T}) = \iota(A(\mathbb{D}))$ where

$$C_+(\mathbb{T}) = \left\{ a \in C(\mathbb{T}) : \int_0^{2\pi} e^{inx} a(e^{ix}) dx = 0, \forall n < 0 \right\}$$

and

$$\begin{aligned} A(\mathbb{D}) &= \{a \in C(\overline{\mathbb{D}}) : a|_{\mathbb{D}} \text{ holomorphic}\} \\ \iota : A(\mathbb{D}) &\rightarrow a|_{\mathbb{T}} \in C(\mathbb{T}) \end{aligned}$$

Claim

$\text{sp}_B(\chi_1) = \overline{\mathbb{D}}$.
For $|z| < 1$,

$$\frac{1}{\chi_1 - z} = \frac{1}{t - z} \in C(\mathbb{T})$$

and (for $|t| \leq 1$), $\frac{1}{t - z} \notin A(\mathbb{D})$.

So $M(B) \cong \overline{\mathbb{D}}$. For $|t_0| = 1$, $\phi_{t_0}(a) = a(t_0)$ ($a \in B$).

For $|t_0| < 1$,

$$\phi_{t_0}(a) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{a(t)}{t - t_0} dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{a(e^{ix})}{1 - e^{-ix} t_0} dx$$

or $\phi_{t_0}(a) = a(t_0)$ (if a is holomorphically extended into \mathbb{D}).

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Theorem

Let A be a unital Banach algebra and $b_1, \dots, b_n \in A$ be commuting elements.

Then the maximal ideal space of $B = \text{alg}_A\{e, b_1, \dots, b_n\}$ is homeomorphic to some compact subset of

$$K \subseteq \text{sp}_B(b_1) \times \cdots \times \text{sp}_B(b_n) \subseteq \mathbb{C}^n$$

That is, the map

$$\tau : M(B) \ni \phi \mapsto (\phi(b_1), \dots, \phi(b_n)) \in K \subseteq \mathbb{C}^n$$

is a homeomorphism.

Remark

$$K = \text{im } \tau = \tau(M(\phi)).$$

$$\phi(b_i) \subseteq \text{sp}_B(b_i).$$

$$\text{For } a \in B, \hat{A} : M(B) \rightarrow \mathbb{C}, \tilde{a} = \hat{a} \circ \tau^{-1} : K \rightarrow \mathbb{C},$$

$$\begin{aligned} \tilde{e}(x) &= 1 & x &= (x_1, \dots, x_n) \\ \tilde{b}_i(x) &= x_i \end{aligned}$$

$$\Lambda : b \mapsto \hat{b} \in C(M(B)), \tilde{b} \in C(K).$$

Section 2.6: Shilov Idempotent and Arens-Royden Theorems

For B a unital, commutative Banach algebra,

$$\Lambda : B \rightarrow \underbrace{C(M(B))}_{M(B)}$$

Obviously, if $p \in B$ is idempotent then $(\hat{p})^2 = \hat{p}$.

Theorem: Shilov

Let $\chi \in C(M(B))$ be an idempotent.

Then there exists a unique idempotent $p \in B$ such that $\hat{p} = \chi$.

Note

Idempotents $\chi \in C(M(B))$ correspond (uniquely) to clopen (closed and open) subsets of $K \subseteq M(B)$.

$$\chi(\phi) = \begin{cases} 0 & \emptyset \notin K \\ 1 & \emptyset \in K \end{cases}$$

Theorem: Arens-Royden

The abstract index group $\kappa(B) = GB/G_0B$ is group-isomorphic to the abstract index group $\kappa(C(M(B)))$ via the map

$$\iota : \kappa(B) \ni [b] \mapsto [\hat{b}] \in \kappa(C(M(B)))$$

Remark

For $\hat{B} = C(M(B))$ and $\Lambda : B \ni b \mapsto \hat{b} \in \hat{B}$,

- $GB \ni b \mapsto \hat{b} \in G\hat{B}$.
- $G_0B \ni b \mapsto \hat{b} \in G_0\hat{B}$

$$G_0B = \{\exp(a) : a \in B\}$$

$$\exp(a) = b \mapsto \hat{b} = \exp(\hat{a}) \in G_0\hat{B}$$

- $\iota[b] \mapsto [\hat{b}]$ is well defined and

$$[b] = \{b \exp(a) : a \in B\}$$

$$[\hat{b}] = \{\hat{b} \exp(\alpha) : \alpha \in \hat{B}\}$$

- Easy: ι is a group homomorphism.
- ι is injective and surjective (non-trivial)
 - ι injective means that if $b \in GB$ is such that \hat{b} has a logarithm in \hat{B} , $\hat{b} = \exp(\alpha)$, $\alpha \in \hat{B}$, then b has a logarithm in B , $b = \exp(a)$ for some $a \in B$ (and $\hat{a} = \alpha$). Or if $b_1, b_2 \in GB$ such that $\hat{b}_1, \hat{b}_2 \in \hat{B}$ are homotopic, continuous functions $M(B) \rightarrow \mathbb{C} \setminus \{0\}$. Then b_1 and b_2 are connected by a path in GB (i.e. $b_1 = b_2 \exp(a)$)
 - ι surjective means that if $\gamma \in G\hat{B}$ (i.e. $\gamma : M(B) \rightarrow \mathbb{C} \setminus \{0\}$) is continuous, then there exist a $c \in GB$ such that $\hat{c} = \gamma \cdot \exp(\alpha)$ with $\alpha \in \hat{B}$.

Chapter 3: C*-Algebras

Section 3.1: Operators on Hilbert Space

- Inner-product space: a complex vector space H with inner product $H \times H \ni (x, y) \mapsto \langle x, y \rangle \in \mathbb{C}$ such that

1. $\langle x, y \rangle$ is linear in x and anti-linear in y

2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

3. $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \iff x = 0$.

- Norm: $\|x\| = \sqrt{\langle x, x \rangle}$

- Cauchy-Schwarz Inequality: $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$.

- Triangle Inequality: $\|x + y\| \leq \|x\| + \|y\|$.

Definition: Hilbert Space

A Hilbert space is a complete inner-product space.

IMAGE 1

Examples

$$H = \mathbb{C}^n, \langle x, y \rangle = \sum_{i=1}^n x_i \bar{y}_i$$

$$H = \ell^2(\Omega) (\Omega \text{ possibly uncountable}), \langle x, y \rangle = \sum_{\omega \in \Omega} x_\omega \bar{y}_\omega \text{ where } \|x\|^2 = \sum_{\omega \in \Omega} |x_\omega|^2 < +\infty$$

$$H = L^2(S, d\mu) \text{ with } (S, B_s, d\mu) \text{ a measure space.}$$

Theorem: Riesz Representation

For every $\phi \in H^*$, there exists a unique $y \in H$ such that $\phi(x) = \langle x, y \rangle$.

The map $\tau : H \ni y \mapsto \phi \in H^*$ is an isometric antilinear (almost) isomorphism.

$$\tau(\lambda y) = \bar{\lambda} \tau(y)$$

$$\langle x, \lambda y \rangle = \bar{\lambda} \langle x, y \rangle$$

We will consider $L(H)$ the Banach algebra of bounded linear operators $A : H \rightarrow H$ equipped with the norm

$$\|A\|_{L(H)} = \sup_{\substack{x \in H \\ x \neq 0}} \frac{\|Ax\|}{\|x\|}$$

Definition: Adjoint Operator (for Hilbert Spaces)

For $A \in L(H)$, its adjoint $A^* \in L(H)$ is given by $\langle Ax, y \rangle = \langle x, A^* y \rangle$ for any $x, y \in H$.

Remark: Well-Defined

A^* is well-defined. For $y \in H$, consider $\phi(x) = \langle Ax, y \rangle$ which is a bounded, linear functional ($\phi \in H^*$).

By Cauchy-Schwarz, $|\langle Ax, y \rangle| \leq \|Ax\| \cdot \|y\| \leq \|A\| \cdot \|x\| \cdot \|y\|$, so $\|\phi\| \leq \|A\| \cdot \|y\|$.

By the Riesz Representation Theorem, there exists $z_y \in H$ such that $\phi(x) = \langle x, z_y \rangle$.

Put $A^*(y) = z_y$, $A^* : H \ni y \mapsto z_y \in H$ such that $\langle Ax, y \rangle = \langle x, A^* y \rangle$.

Remark: Linearity

A^* is linear.

Remark: Boundedness

A^* is bounded, $\|A^*y\| = \|z_y\| = \|\phi\| \leq \|A\| \cdot \|y\|$.
 Therefore, $A^* \in L(H)$.

Properties

- $(A^*)^* = A$.
- $\|A^*\| = \|A\|$
- $(A+B)^* = A^* + B^*$ and $(\lambda A)^* = \bar{\lambda} A^*$.
- $(AB)^* = B^* A^*$
- A is invertible if and only if A^* is invertible and $(A^{-1})^* = (A^*)^{-1}$.
- $\|A^*A\| = \|A\|^2$.

$$\begin{aligned}\langle A^*Ax, x \rangle &= \langle Ax, Ax \rangle = \|Ax\|^2 \\ \|Ax\|^2 &\leq \|A^*Ax\| \cdot \|x\| \leq \|A^*A\| \cdot \|x\|^2 \\ \|A\|^2 &= \sup_{x \neq 0} \frac{\|Ax\|^2}{\|x\|^2} \leq \|A^*A\| \\ \|A^*A\| &\leq \|A^*\| \cdot \|A\| = \|A\|^2\end{aligned}$$

Definitions

$A \in L(H)$ is called

- Self-adjoint if $A^* = A$.
- Unitary if $A^*A = AA^* = I$
- Normal if $A^*A = AA^*$
- Positive ($A \geq 0$) if $\langle Ax, x \rangle \geq 0, \forall x \in H$

Positive implies self-adjoint which implies normal.

Unitary implies normal.

Proposition

$A \in L(H)$ is self-adjoint if and only if $\langle Ax, x \rangle \in \mathbb{R}$.

Proof

$A = A^*$ implies $\langle Ax, x \rangle = \langle x, A^*x \rangle = \langle x, Ax \rangle = \overline{\langle Ax, x \rangle}$. Therefore $\langle Ax, x \rangle \in \mathbb{R}$.
 $\langle Ax, x \rangle \in \mathbb{R}$ implies $\langle Ax, x \rangle = \overline{\langle x, Ax \rangle} = \langle x, Ax \rangle = \langle A^*x, x \rangle$. Therefore $\langle (A - A^*)x, x \rangle = 0$ so (exercise) $\langle (A - A^*)x, y \rangle = 0$.
 Use with $\phi(x, y) = \langle (A - A^*)x, y \rangle$, $\phi(x, y) = \frac{1}{4}(\phi(x+y, x+y) - \phi(x-y, x-y) + i\phi(x+iy, x+iy) - i\phi(x-iy, x-iy))$.

Proposition

For $A \in L(H)$, $A^* A \geq 0$.

Proof

$$\langle A^* Ax, x \rangle = \langle Ax, Ax \rangle = \|Ax\|^2 \geq 0.$$

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Recall: Propositions

A is self adjoint if and only if $\langle Ax, x \rangle \in \mathbb{R}$.

$A^* A$ is positive (i.e. $\langle A^* Ax, x \rangle \geq 0$).

Theorem

Let $A \in L(H)$

1. if $A = A^*$, then $\text{sp}(A) \subseteq \mathbb{R}$.
2. if $A \geq 0$, then $\text{sp}(A) \subseteq [0, +\infty)$.

Lemma

If $K \in L(H)$, $\|Kx\| \geq \delta \|x\|$, and $\|K^* x\| \geq \delta \|x\|$ for $\delta > 0$, then K is invertible.

- Proof

K is injective ($\ker K = \{0\}$).

The image $\text{im } K$ is closed (use cauchy sequences with $Kx_n \rightarrow y$, $\{Kx_n\}$ Cauchy, $\{x_n\}$ Cauchy, $x_n \rightarrow x$. Then $Kx_n \rightarrow Kx \implies Kx = y$).

$\ker K^* = \{0\}$ since

$$\begin{aligned} (\ker K^*)^\perp &= (\text{im } K)^\perp \\ &= \{y \in H : \langle y, z \rangle = 0, \forall z \in \text{im } K\} \\ &= \{y \in H : \langle y, Kx \rangle = 0, \forall x \in H\} \\ &= \{y \in H : \langle K^* y, x \rangle = 0, \forall x \in H\} \\ &= \{y \in H : Ky^* = 0\} \end{aligned}$$

and $\{0\} = (\text{im } K)^\perp$ implies that $\text{im } K$ is dense.

Therefore, $\text{im } K = H$, K is surjective and injective, and ultimately K is invertible.

Proof of a

Take $\lambda = \alpha + i\beta \in \mathbb{C} \setminus \mathbb{R}$ ($\beta \neq 0$), and write

$$A - \lambda I = A - \alpha I - i\beta I = \beta \left(\frac{\overbrace{A - \alpha I}^{\stackrel{\text{:=} T}{}}}{\beta} - i \right) = \beta \underbrace{(T - iI)}_{\stackrel{\text{:=} K}}$$

Then $K = T - iI$ and $K^* = T + iI$. So

$$\|(T - iI)x\|^2 \langle (T - iI)x, (T - iI)x \rangle = \|Tx\|^2 + \|x\|^2 - i\langle x, Tx \rangle + i\langle Tx, x \rangle$$

Since $\overline{\langle x, Tx \rangle} = \langle Tx, x \rangle$, we get $\|Tx\|^2 + \|x\|^2 \geq \|x\|^2$.

Likewise, $\|(T + iI)x\|^2 \geq \|x\|^2$ so $T - iI$ is invertible and $A - \lambda I$ is invertible.

Proof of b

Recall that positive implies self-adjoint.

For $\lambda \in (-\infty, 0)$, $\lambda = -\alpha$, $\alpha > 0$, we have $A - \lambda I = A + \alpha I =: K$. Then

$$\|Kx\|^2 = \langle (A + \alpha I)x, (A + \alpha I)x \rangle = \|Ax\|^2 + \alpha^2 \|x\|^2 + \alpha(\langle Ax, x \rangle + \langle x, Ax \rangle)$$

By assumption, $\langle Ax, x \rangle \geq 0$ and $\langle x, Ax \rangle \geq 0$, so $\|Kx\|^2 \geq \alpha^2 \|x\|^2$.

Remark

For $A \in L(H)$ where $A = A^*$, the following are equivalent

1. A is positive.
2. $\text{sp}(A) \subseteq [0, +\infty)$.
3. $A = B^*B$ for some $B \in L(H)$.

Section 3.2: C* Algebras

Definition:

A C^* -algebra is a Banach algebra B which has a map

$$*: B \ni a \mapsto a^* \in B$$

(called an involution) such that

- $(a^*)^* = a$, $(ab)^* = b^*a^*$, and $e^* = e$.
- $(a+b)^* = a^* + b^*$, $(\lambda a)^* = \bar{\lambda}a^*$.
- $\|a^*a\| = \|a\|^2$.

Remark

If B has a unit, $e^* = e$.

For invertible elements, b invertible if and only if b^* is invertible and $(b^*)^{-1} = (b^{-1})^*$.

$\|a^*\| = \|a\|$. Indeed $\|a\|^2 = \|a^*a\| \leq \|a\| \cdot \|a^*\|$ so $\|a\| \leq \|a^*\|$ (for $a \neq 0$).

Since $a \mapsto a^*$, $\|a^*\| \leq \|a^{**}\| = \|a\|$.

Examples

- $B = L(H)$ bounded linear operators on Hilbert spaces.
- $B = C(X)$ with X a compact Hausdorff space given by $a : X \rightarrow \mathbb{C}$ continuous and $a^*(x) := \overline{a(x)}$ (complex conjugate).
- $B = L^\infty(S, d\mu)$ essentially bounded functions on a measure space (S, \mathcal{B}_S, μ) again with $a^*(x) := \overline{a(x)}$.

Non-examples

$B = W = \{\sum_{n \in \mathbb{Z}} a_n t^n : \sum |a_n| < +\infty\}$. We have $\|a\| = \sum |a_n|$ and $\|a^*\| = \|a\|$ but not $\|a^* a\| = \|a\|^2$.
 $B = C^1[0, 1]$.

Definitions:

An element $b \in B$ is called

- self adjoint if $b^* = b$
- unitary if $b^* b = b b^* = e$
- normal if $b^* b = b b^*$
- positive if $b^* = b$ and $\text{sp}(b) \subseteq [0, +\infty]$.

Proposition

For $b \in B$ normal, the spectral radius $r(b) := \max\{|\lambda| : \lambda \in \text{sp}(b)\} = \|b\|$.

Proof

We know that $r(b) = \lim_{n \rightarrow \infty} \|b^n\|^{1/n} = \inf_{n \in \mathbb{N}} \|b^n\|^{1/n}$. Therefore, $r(b) \leq \|b\|$.

For any element a which is self adjoint, we have $\|a^* a\| = \|a^2\| = \|a\|^2$. By induction, $\|a^{2^k}\| = \|a\|^{2^k}$ so $\|a^n\|^{1/n} = \|a\|$ for $n = 2^k$.

For a normal element b ,

$$\|b^n\|^2 = \|(b^n)^* b^n\| = \|(b^* b)^n\|$$

Then, since $b^* b$ is self-adjoint, $\|b^n\|^2 = \|b^* b\|^n$ for $n = 2^k$ and $\|b^n\|^{1/n} = \|b^* b\|^{1/2}$.

Therefore $r(b) = \|b^* b\|^{1/2} = \|b\|$.

Corollary

The norm in a C^* -algebra is uniquely determined (i.e. there are no different equivalent norms).

Proof

Let $a \in B$ be arbitrary. Then $r(a^* a) = \|a^* a\| = \|a\|^2$.

Therefore $\|a\| = \sqrt{r(a^* a)}$. The spectral radius (and the spectrum) are determined in terms of algebraic properties.

Proposition

The spectrum of

1. a unitary element is contained in $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.
2. a self-adjoint element is contained in \mathbb{R} (in fact, it is a subset of $[-||a||, ||a||]$).

Proof of 1

For a unitary element, $||a||^2 = ||a^*a|| = ||e|| = 1$, so $||a|| = 1$ and $\text{sp}(a) \subseteq \mathbb{T}$. Since $a^{-1} = a^*$, $(a^{-1})^*a^{-1} = a^{-1}(a^{-1})^* = e$, a^{-1} is also unitary and $\text{sp}(a^{-1}) \subseteq \mathbb{T}$. Observe that $\text{sp}(a) = \left\{ \frac{1}{\lambda} : \lambda \in \text{sp}(a^{-1}) \right\}$, then $\text{sp}(a) \subseteq \mathbb{T}$ and $\text{sp}(a^{-1}) \subseteq \mathbb{T}$.

Proof of 2

Recall that $\exp(c) = \sum_{n=0}^{\infty} \frac{c^n}{n!}$. Then $(\exp(c))^* = \exp(c^*)$.

Let a be a self adjoint element, then $u = \exp(ia)$ is unitary since $u^* = \exp((ia)^*) = \exp(-ia) = u^{-1}$. By the spectral mapping theorem,

$$\exp(i\text{sp}(a)) = \exp(\text{sp}(ia)) = \text{sp}(\exp(ia)) \subseteq \mathbb{T}$$

Therefore $\text{sp}(a) \subseteq \mathbb{R}$.

Proposition

Each $b \in B$ can be written uniquely as $b = b_1 + ib_2$ with b_1 and b_2 self-adjoint.

Proof

Define $b_1 = \frac{b+b^*}{2}$ and $b_2 = \frac{b-b^*}{2i}$ which are self-adjoint and compute

$$b_1 + ib_2 = b \quad \text{and} \quad b_1 - ib_2 = b^*$$

For uniqueness, assume that $0 = b_1 + ib_2$ with b_1 and b_2 self-adjoint. Then $b_1 = -ib_2$ and, since $\text{sp}(b_1) \subseteq \mathbb{R}$ and $\text{sp}(b_2) \subseteq \mathbb{R}$. Then $\text{sp}(b_1) = \text{sp}(b_2) = \{0\}$. Therefore $||b_1|| = r(b_1) = 0$, and, since b_1 is self-adjoint, $b_1 = 0$ (similarly for b_2).

Theorem

Let $B \subseteq A$ be a unital, C^* -subalgebra of a C^* subalgebra A . Then B is inverse closed in A (i.e. $GA \cap B = GB$).

Remarks

B is inverse-closed if and only if $\forall b \in B$, if b invertible in A then $b^{-1} \in B$.

Equivalently, $\text{sp}_A(b) = \text{sp}_B(b)$ for every $b \in B$.

Proof

Let $b \in B$ be invertible in A . Then $c = b^*b$ is invertible in A ($c^{-1} = b^{-1}(b^{-1})^* \in A$). Now c is a self-adjoint, therefore $\text{sp}_A(c) \subseteq [-||c||, ||c||] \subseteq \mathbb{R}$.

For subalgebras, $\text{sp}_B(c) \supseteq \text{sp}_A(c)$ and $\partial \text{sp}_B(c) \subseteq \partial \text{sp}_A(c)$. In fact, $\text{sp}_B(c) = \text{sp}_A(c) \cup \bigcup_{\omega \in \Omega} H_\omega$ where H_ω are (bounded) connected components of $\mathbb{C} \setminus \text{sp}_A(c)$.

However, because $\text{sp}_A(c) \subseteq \mathbb{R}$ there are no holes. Therefore $\text{sp}_B(c) = \text{sp}_A(c)$.

We know that $0 \notin \text{sp}_A(c)$ so $0 \notin \text{sp}_B(c)$. So c is invertible in B , so b is left invertible in B since

$$e = c^{-1}c = \underbrace{c^{-1}b^*b}_{\in B}$$

Similarly, b is right invertible by repeating the argument with $d = bb^*$. Therefore b is invertible in B .

November 18, 2024

Section 3.3: Commutative C*-Algebras

Proposition

For a unital commutative C^* -algebra B and $\phi \in M(B)$, we have $\phi(b^*) = \overline{\phi(b)}$, $\forall b \in B$.

Proof

Write $b = b_1 + ib_2$ with b_1 and b_2 self-adjoint. Then $b^* = b_1 - ib_2$.

Then $\text{sp}(b_i) \subseteq \mathbb{R}$ implies that $\phi(b_i) \in \mathbb{R}$, otherwise $\phi(b_i) = z \in \mathbb{C} \setminus \mathbb{R}$ gives $\phi(b_1 - ze) = 0$ which implies $b_1 - ze$ is not invertible and $z \in \text{sp}(b_i)$.

Then $\phi(b) = \phi(b_1) + i\phi(b_2)$ and $\phi(b^*) = \phi(b_1) - i\phi(b_2)$ with $\phi(b_i)$ real.

Remarks

A map $\phi : B \rightarrow \mathbb{C}$ is a $*$ -homomorphism if $\phi(b^*) = \phi(b)^* = \overline{\phi(b)}$ (involution in \mathbb{C}).

The Gelfand transform is also a $*$ -homomorphism

$$\Lambda : B \ni b \mapsto \hat{b} \in C(M(B))$$

with $\Lambda(b^*) = (\Lambda(b))^*$ and $\widehat{b^*} = (\hat{b})^*$. So

$$\widehat{b^*}(\phi) = \phi(b^*) = \overline{\phi(b)} = \overline{\hat{b}(\phi)} = (\hat{b})^*(\phi)$$

Theorem: Stone-Weierstrass

Let X be a compact Hausdorff space and $\mathfrak{a} \subseteq C(X)$ a subalgebra of $C(X)$ such that

- \mathfrak{a} is norm-closed in $C(X)$
- $I \in \mathfrak{a}$
- if $f \in \mathfrak{a}$ then $\bar{f} \in \mathfrak{a}$
- $\forall x, y \in X$ with $x \neq y$, there exists $f \in \mathfrak{a}$ such that $f(x) \neq f(y)$.

$C_r(X)$ the set of all continuous functions $f : X \rightarrow \mathbb{R}$ and $\mathfrak{a}_r = \mathfrak{a} \cap C_r(X)$.

Proof

It is enough to show that $\mathfrak{a}_r = c_R(X)$ for $f \in C(X)$, $f = \operatorname{Re} f + i \operatorname{Im} f$ then $f \in \mathfrak{a}$.
If $f \in \mathfrak{a}_r$, then $|f| \in \mathfrak{a}_r$. Without loss of generality, $\|f\| \leq 1$, $|f(x)| \leq 1$. Note that

$$\phi(t) = \sqrt{1-t} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} t^n$$

which converges uniformly on $[-1, 1]$. Write $|f| = \sqrt{|f|^2} = \sqrt{f^2} = \sqrt{1 - (1 - f^2)}$. Then for $t = 1 - f^2(x) \in [0, 1]$,

$$|f(x)| = \left| \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (1 - f^2(x))^n \right| = \sum_{n=0}^N (\dots) + R_N(1 - f^2(x))$$

where the first term is polynomial in $f^2 = \bar{f}f \in \mathfrak{a}_r$ and we may estimate the remainder

$$\sup_{x \in X} |R_n(\dots)| \leq \sup_{t \in [0, 1]} |R_N(t)|$$

which converges to 0 as $N \rightarrow \infty$. Therefore $|f|$ may be approximated by elements in \mathfrak{a}_r .
Then \mathfrak{a}_r is a lattice. Given $f, g \in \mathfrak{a}_r$, we have

$$f \vee g = \max\{f, g\} = \frac{1}{2}\{f + g + |f - g|\} \quad \text{and} \quad f \wedge g = \min\{f, g\} = \frac{1}{2}\{f + g - |f - g|\}$$

and $f \vee g, f \wedge g \in \mathfrak{a}_r$.

Now, $\forall \alpha, \beta \in \mathbb{R}$, there exists $f \in \mathfrak{a}_r$ such that $f(x) = \alpha$ and $f(y) = \beta$.

First, we obtain a function $h \in \mathfrak{a}$ with $h(x) \neq h(y)$ and $1 \in \mathfrak{a}$, then put $f = \gamma 1 + \delta h$. We need

$$\alpha = \gamma + \delta h(x) \quad \text{and} \quad \beta = \gamma + \delta h(y)$$

which can be found by solving $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & h(x) \\ 1 & h(y) \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$.

Now we replace f by $\operatorname{Re} f = \frac{1}{2}(f + \bar{f}) \in \mathfrak{a}$ so $\operatorname{Re} f \in \mathfrak{a}_r$.

Let $f \in C_r(X)$. We want to show that f can be approximated by functions in \mathfrak{a}_r .

Let $\varepsilon > 0$, fix $x_0 \in X$ and take $x \in X$ arbitrary. Find $g_x \in \mathfrak{a}_r$ such that $g_x(x_0) = f(x_0)$ and $g_x(x) = f(x)$.

Since these functions are continuous, we have that $g_x(y) \leq f(y) + \varepsilon$ in some open neighborhood $U_x \ni x$. Then U_x covers X as we range across $x \in X$. We have a finite subcover U_{x_1}, \dots, U_{x_N} and $h_{x_0} = g_{x_1} \wedge g_{x_2} \wedge \dots \wedge g_{x_N}$.

Then $h_{x_0}(y) \leq f(y) + \varepsilon$ for every $y \in X$ where $h_{x_0}(x_0) = f(x_0)$.

Doing this for all $x_0 \in X$ gives a collection $\{h_{x_0}\}_{x_0 \in X}$. For some $V_{x_0} \ni x_0$, $h_{x_0}(y) \geq f(y) - \varepsilon$.

Then $\{V_{x_0}\}_{x_0 \in X}$ covers X and admits a finite subcover V_{x_1}, \dots, V_{x_n} which gives rise to $h = h_{x_1} \vee h_{x_2} \vee \dots \vee h_{x_n}$.

Therefore $f(y) - \varepsilon \leq h(y) \leq f(y) + \varepsilon$. Therefore $|f(y) - h(y)| \leq \varepsilon$ for every $y \in X$ and $h \in \mathfrak{a}_r$ (by the lattice property).

Theorem

Let B be a unital C^* -algebra.

Then the Gelfand transform $\Lambda : B \ni b \mapsto \hat{b} \in C(M(B))$ is an isometric $*$ -isomorphism.

Proof

We know that Λ is a Banach algebra homomorphism (i.e. multiplicative, linear and $\|\Lambda\| = 1$).
We know also that Λ is a $*$ -homomorphism with $\Lambda(b^*) = (\Lambda(b))^*$.

The Gelfand transform also preserves spectrum where $\text{sp}_B(b) = \text{sp}_{C(M(B))}(\Lambda(b))$ because $b \in B$ is invertible if and only if $\phi(b) = \hat{b}(\phi)$ is non-zero for each $\phi \in M(B)$ and, equivalently, $\hat{b} = \Lambda(b)$ is invertible in $C(M(B))$. Therefore the spectral radius is the same $r(b) = r(\Lambda(b))$.

Take $b = a^*a$ for arbitrary $a \in B$. Then $\|a^*a\| = \|\Lambda(a^*a)\| = \|\Lambda(a)^*\Lambda(a)\|$, so $\|a\|^2 = \|\Lambda(a)\|^2$ and $\|a\| = \|\Lambda(a)\|$ an isometry. It follows that Λ is injective ($\Lambda(a) = 0 \implies a = 0$).

To show that Λ is surjective ($\text{im } \Lambda = C(M(B))$), put $\mathfrak{a} = \text{im } \Lambda$. Then

- \mathfrak{a} is an algebra
- \mathfrak{a} is isometrically isomorphic to B ($\Lambda : B \rightarrow \mathfrak{a} \subseteq C(M(B))$)
- \mathfrak{a} is norm closed (via Cauchy sequences)
- $1 \in \mathfrak{a}$
- $f \in \mathfrak{a}$ implies $\bar{f} \in \mathfrak{a}$ since $f = \Lambda(b)$ gives $\bar{f} = \Lambda(b)^* = \Lambda(b^*)$.
- for $\phi_1, \phi_2 \in M(B)$, $\phi_1 \neq \phi_2$ implies there exists $b \in B$ such that $\phi_1(b) \neq \phi_2(b)$, so $\hat{b}(\phi_1) \neq \hat{b}(\phi_2)$.

Therefore, we may apply Stone-Weierstrass to see that $\mathfrak{a} = C(M(B))$ implies Λ is surjective.

Examples

- $B = \ell^\infty(\mathbb{N}) = \{\{x_n\}_{n=0}^\infty : \|x\| = \sup_n |x_n| < +\infty\}$ is a commutative C^* -algebra. Therefore ℓ^∞ is isometrically $*$ -isomorphic to $C(M(\ell^\infty))$. Some maximal ideals can be identified with \mathbb{N} .

$$\begin{aligned}\phi_N : x = \{x_n\}_{n=0}^\infty &\mapsto X_N \\ \mathbb{N} \cong \{\phi_N\}_{N=0}^\infty &\subseteq M(\ell^\infty(\mathbb{N})) = X\end{aligned}$$

where ϕ_N are dense in X . All other functionals are given by axiom of choice. X may be identified with the ech-compcatification of \mathbb{N} which is uncountable.

- $B = L^\infty([0,1])$. $X = M(B)$ is an exotic space. It is "totally disconnected" and uncountable, but $B \cong C(X)$.
- $B = PC(\mathbb{T})$ gives a maximal ideal space that can be roughly identified as $X \cong \mathbb{T} \times \{0,1\}$ with a non-trivial topology.
- Finitely generated algebras. Caution: if \mathcal{A} is a C^* -algebra and $a \in \mathcal{A}$, $B = \text{alg}\{e, a\}$ may not be a C^* -algebra since a^* may not be in B . If $B = \text{alg}\{e, a, a^*\}$, then it is a C^* -algebra but may not be commutative.
- Take \mathcal{A} a C^* -algebra, $a \in A$ with $B = \text{alg}\{e, a\}$ with $a = a^*$ or $B = \text{alg}\{e, a, a^*\}$ with $aa^* = a^*a$. Then these are both unital, commutative C^* -algebras. In these cases, $B \cong C(X)$ where $X \cong \text{sp}_{\mathcal{A}}(a) \text{sp}_B(a)$. We have a homeomorphism

$$\tau : M(B) \ni \phi \mapsto \phi(a) \in \text{sp}(a)$$

with $\phi(a^*) = \overline{\phi(a)}$.

November 22, 2024

Recall

Given a unital, commutative C^* -algebra B , $B \cong C(X)$ where $X = M(B)$.

Proposition

Let B be a C^* -algebra such that $B = \text{alg}\{e, b, b^*\}$ where $b^*b = bb^*$ (i.e. b is normal). Then B is commutative and $M(B)$ is homeomorphic to $X = \text{sp } b$ by

$$\tau : M(B) \rightarrow \phi(b) \in X = \text{sp } b.$$

In particular, B is isometrically $*$ -isomorphic to $C(x)$ by

$$\tilde{\Lambda} : B \ni a \mapsto \hat{a} \circ \tau^{-1} = \tilde{a} \in C(X)$$

Remarks

- $\text{sp } b^* = \overline{\text{sp } b}$; $\phi(b^*) = \overline{\phi(b)}$.
- $a \in B \xrightarrow{\Lambda} \hat{a} \in C(M(B)) \rightarrow \tilde{a} = (\hat{a} \circ \tau^{-1}) \in C(X)$
- $\tilde{b}(x) = \hat{b}(\tau^{-1}(x)) = \hat{b}(\phi) = \phi(b) = x$.
- $b \mapsto \tilde{b}(x) = x$, $\tilde{b} \in C(\text{sp } b)$.
- $\tilde{b}^*(x) = \widehat{b^*}(\tau^{-1}(x)) = \hat{b}^*(\phi) = \phi(b^*) = \overline{\phi(b)} = \overline{x}$.

Section 3.3.5: Functional Calculus for Normal Elements in C^* -Algebras

For a continuous function $g \in C(X)$ where $X = \text{sp}(b)$, $b \in B$ normal (i.e. $b^*b = bb^*$, define $\widetilde{g}(b)$ such that $\widetilde{g}(b)(x) = g(x)$ for $g(b) \in \text{alg}\{e, b, b^*\}$ (i.e. $g(b) = \tilde{\Lambda}^{-1}(g)$ where $\tilde{\Lambda} : B = \text{alg}\{e, b, b^*\} \rightarrow C(\text{sp } b)$).

Then $b \mapsto \hat{b}(x) = x$, $b^* \mapsto \widehat{b^*} = \overline{x}$ and $g(b) \mapsto g(x)$.

In particular, if $p(x) = \sum p_i x^i$, then $p(b) = \sum p_i b^i$.

$q(x) = \sum_{i,j} p_{ij} x^i \overline{x}^j$ and $q(b) = \sum_{i,h} p_{ij} b^i (b^*)^j$.

Spectral Mapping Theorem

If b is normal and $g \in C(\text{sp } b)$, then $\text{sp}_B(g(b)) = g(\text{sp}_B b)$ ($= \text{im } g$).

Section 3.4: Positive Elements in C^* -Algebras

Recall

$a \geq 0$ (a positive) if and only if $a^* = a$ and $\text{sp } a \subseteq [0, +\infty)$.

This generalizes to $A \in L(H)$ where $A \geq 0$ if and only if $\forall x \in H$, $\langle Ax, x \rangle \geq 0$.

For $a \in C(X)$, $a \geq 0$ if and only if $a(x) \geq 0$, $\forall x \in X$.

Proposition

If $a \geq 0$ and $b \geq 0$, then $a + b \geq 0$.

Proof

If $a \geq 0$ and $b \geq 0$, then both a and b are self-adjoint and $(a + b)^* = (a + b)$.

Then $\text{sp } a \subseteq [0, \|a\|]$ and $\text{sp } b \subseteq [0, \|b\|]$ implies that $\text{sp}(a - \frac{\|a\|}{2}e) \subseteq [-\frac{\|a\|}{2}, \frac{\|a\|}{2}]$ and $\text{sp}(b - \frac{\|b\|}{2}e) \subseteq [-\frac{\|b\|}{2}, \frac{\|b\|}{2}]$. Since they are self adjoint,

$$\left\| a - \frac{\|a\|}{2}e \right\| = r\left(a - \frac{\|a\|}{2}e\right) \leq \frac{\|a\|}{2} \quad \text{and} \quad \left\| b - \frac{\|b\|}{2}e \right\| = r\left(b - \frac{\|b\|}{2}e\right) \leq \frac{\|b\|}{2}$$

So

$$\left\| a + b - \frac{\|a\| + \|b\|}{2}e \right\| \leq \left\| a - \frac{\|a\|}{2}e \right\| + \left\| b - \frac{\|b\|}{2}e \right\| \leq \frac{\|a\| + \|b\|}{2}$$

Therefore $r(a + b - \frac{\|a\| + \|b\|}{2}e) \leq \frac{\|a\| + \|b\|}{2}$ and

$$\text{sp}\left(a + b - \frac{\|a\| + \|b\|}{2}e\right) \subseteq \left[-\frac{\|a\| + \|b\|}{2}, \frac{\|a\| + \|b\|}{2}\right]$$

We conclude that $\text{sp}(a + b) \subseteq \left[0, \frac{\|a\| + \|b\|}{2}\right] \subseteq [0, +\infty)$.

Theorem: Square Roots of Positive Operators

Let $a \geq 0$. Then there exists a unique element $b \geq 0$ such that $a = b^2$ (notation: $b = \sqrt{a}$).

Proof: Existence

(Using functional calculus)

Consider $B = \text{alg}\{e, a\}$ with $a = a^*$ a C^* -algebra. Then $B \cong C(X)$ for $X = \text{sp } a \subseteq [0, +\infty)$.

$\tilde{a}(x)$ is continuous and positive, so $\sqrt{\tilde{a}(x)}$ is also continuous and positive and therefore an element of $C(X)$.

Then there must exist $b \in B$ such that $\tilde{b}(x) = \sqrt{\tilde{a}(x)}$. So $(\tilde{b})^2 = a$, $\tilde{b}(x) \geq 0$ and $b^2 = a$ with $b \geq 0$.

Proof: Uniqueness

Assume $a = b^2 = c^2$ where $b = \sqrt{a}$ as above and $c \geq 0$. We have $b \in \text{alg}\{e, a\}$.

Obviously, $ca = ac$ implies $ca^n = a^n c$ and $cx = xc$ for all $x \in \text{alg}\{e, a\}$. So $cb = bc$.

Now consider $B_0 = \text{alg}\{e, a, c\}$, $a^* = a$, $c^* = c$ and $ac = ca$ a commutative C^* -algebra. Then $b = \sqrt{a} \in B_0$, $a = b^2 = c^2$, $b \geq 0$ and $c \geq 0$ implies that $\hat{a} = (\hat{b})^2 = (\hat{c})^2$, $\hat{b} \geq 0$, $\hat{c} \geq 0$ (continuous functions on $M(\cdots)$). Therefore $\hat{b} = \hat{c}$ and $b = c$.

Lemma

For $a = a^*$, there exist $a_+, a_- \geq 0$ such that $a = a_+ - a_-$, $a_+ a_- = a_- a_+ = 0$.

Proof

For $B = \text{alg}\{e, a\}$ a commutitve C^* -algebra, we apply Gelfand Theory / Functional Calculus. For $x \in M(B)$, define

$$\hat{a}_+ = \begin{cases} \hat{a}(x) & \hat{a}(x) \geq 0 \\ 0 & \hat{a}(x) < 0 \end{cases} \quad \text{and} \quad \hat{a}_- = \begin{cases} -\hat{a}(x) & \hat{a}(x) \leq 0 \\ 0 & \hat{a}(x) > 0 \end{cases}$$

So $a_+, a_- \in B$ and $\hat{a}_\pm C(M(B))$.

Theorem:

Let $b \in B$ be an arbitrary element in a unital C^* -algebra. Then $b^* b \geq 0$.

Remarks

- Obviously true for $B = L(H)$ or $B = C(X)$.
- It was open for quite some time for general C^* -algebras. It would be trivial if every C^* -algebra were isomorphic to some $*$ -subalgebra of $L(H)$. This turns out to be true.
- Recall that $b^* b \geq 0$ boils down to whether $e + b^* b$ is invertible.

Proof

$b^* b$ is self-adjoint. Therefore $b^* b = c - d$ for $c \geq 0$ and $d \geq 0$ with $cd = dc = 0$.

To show: $d = 0$. Consider $(bd)^*(bd) = d(b^* b)d = -d^3 \leq 0$ (i.e. $d^3 \geq 0$).

Then take $bd = s + it$ where s and t are self-adjoint. Since s and t are self-adjoint, $s^2 \geq 0$, $t^2 \geq 0$ and $(s^2 + t^2) \geq 0$. So $(bd)^*(bd) + (bd)(bd)^* = 2(s^2 + t^2) \geq 0$.

Therefore $(bd)(bd)^* = 2(s^2 + t^2) + d^3 \geq 0$. We know that $(bd)(bd)^*$ and $(bd)^*(bd)$ have the same spectrum (except possibly $\{0\}$). Recall from the homework that for $\lambda \neq 0$, $\lambda - xy$ is invertible if and only if $\lambda - yx$ is invertible.

Therefore the spectrum of $(bd)(bd)^*$ and $(bd)^*(bd)$ is $\{0\}$ and

$$\|bd\|^2 = \|(bd)^*(bd)\| = r((bd)^*(bd)) = 0$$

so $bd = 0$ implies $d^3 = 0$, $0 = r(d^3) = (r(d))^3 = \|d\|^3$ and therefore $d = 0$.

Corollary

An element $a \in B$ is positive ($a \geq 0$) if and only if $a = b^* b$ for some $b \in B$.

Proof

(\Leftarrow) by previous theorem.

(\Rightarrow) $a \geq 0$ (a self-adjoint) means $b = \sqrt{a}$. Therefore $b \geq 0$ and $b^2 = a$ which implies $b = b^*$ and $a = b^* b$.

Theorem: Polar Decomposition

Let $a \in B$ be invertible. Then there exists a unique unitary element $u \in B$ and positive element $r \in B$ ($r \geq 0$) such that $a = u \cdot r$.

Proof: Existence

Define $r = \sqrt{a^* a}$. Since a and a^* are invertible, $\text{sp}(a^* a) \subseteq [\delta, +\infty)$ and r is also invertible.

Put $u = a \cdot r^{-1}$ such that $u^* u = r^{-*} a a^* r^{-1} = r^{-1} r^2 r^{-1} = e$ and $u u^* = a r^{-1} a^{-*} a^* = a r^{-2} a^* = a (a^* a)^{-1} a = e$.

Proof: Uniqueness

Write $a = u_1 r_1 = u_2 r_2$ and consider $a^* a = (r_1)^2 = (r_2)^2$ for $r_1, r_2 \geq 0$ which implies $r_1 = r_2$. It follows that $u_i = a r_i^{-1}$ is likewise unique.

Remarks

- left / right polar decompositions $a = ur = sv$ for $r, s \geq 0$ and u, v unitary.
- for $A \in L(H)$ (not in general, not even for $C(X)$) (but it does work for $L^p(S)$), $A = U \cdot R$ for $R \geq 0$ and U a partial isometry ($UU^*U = U$ or equivalently that $(U^*U)^2 = U^*U$). $\ker A$ and $\ker A^*$ may not be trivial.

Corollary

If two Hilbert spaces are isomorphic as Banach spaces, then they are isomorphic as Hilbert spaces.

Proof

Take H_1 and H_2 Hilbert spaces with $A : H_1 \rightarrow H_2$ an invertible bounded linear operator.

Define $R = (A^*A)^{1/2} : H_1 \rightarrow H_1$. Then $A = U \cdot R$ for $U : H_1 \rightarrow H_2$ unitary gives $U^*U = I_{H_1}$ and $UU^* = I_{H_2}$. So

$$H_1 \xrightarrow{R} H_1 \xrightarrow{U} H_2$$

where U is a Hilbert space isomorphism and isometry. $U^*U = I_{H_1}$ gives

$$\langle x, y \rangle_{H_1} = \langle U^*Ux, y \rangle_{H_1} = \langle UX, Uy \rangle_{H_2}$$

so U preserves $\langle \cdot, \cdot \rangle$ and $\|x\| = \|UX\|$.

November 25, 2024

Section 3.5: -ideals, *-homomorphisms and Quotients in C-algebras

Definition: *-homomorphism

A homomorphism $\phi : A \rightarrow B$ such that $\phi(a^*) = \phi(a)^*$.

Definition: *-ideal

An ideal $J \subseteq A$ such that $a \in J$ implies that $a^* \in J$.

Definition: *-subalgebra

$A \subseteq B$ is a subalgebra if it is an algebra and $a \in A$ implies $a^* \in A$.

Proposition

If $\phi : A \rightarrow B$ is a *-homomorphism between C^* -algebras, then $\ker \phi$ is a *-ideal and $\text{im } \phi$ is a *-subalgebra.

Theorem

Let $I \subseteq A$ be a closed, two-sided ideal in a C^* -algebra. Then I is a *-ideal.

Proof

Take $a \in I$, consider $b = a^*a \in I$. Consider $u_n(t) = \frac{t}{1/n+t}$ for $t \geq 0$.

IMAGE 1

Then $u_n(b) = b \cdot \left(\frac{1}{n}e + b\right)^{-1} \in I$.

IMAGE 2

For $0 \leq u_n(t) \leq 1$,

$$0 \leq (u_n(t) - 1)^2 \cdot t = \left(\frac{\frac{1}{n}}{\frac{1}{n} + t} \right)^2 \cdot t \leq \frac{1}{4n}$$

With $u_n \in C([0, ||b||])$ and $\text{sp } b \subseteq [0, ||b||]$, $||(u_n(t) - 1)^2 t||_{C([0, ||b||])} \rightarrow 0$ and, equivalently, $||(u_n(b) - e)^2 b||_A \rightarrow 0$. Since $(u_n(b) - e)^* = u_n(b) - e$, it follows that

$$\begin{aligned} ||(u_n(t) - 1)^2 t||_{C([0, ||b||])} &= ||(u_n(b) - e)^2 b||_A \\ &= ||(u_n(b) - e)b(u_n(b) - e)|| \\ &= ||(u_n(b) - e)^* a^* a(u_n(b) - e)|| \\ &= ||a(u_n(b) - e)||^2 \\ &= ||(u_n(b) - e)a^*||^2 \rightarrow 0 \end{aligned}$$

Since $u_n(b) \in I$, $u_n(b)a^* \in I$ and since I is closed and $u_n(b)a^* \rightarrow a^*$, $a^* \in I$.

Theorem

Let A be a C^* -algebra and $I \subseteq A$ be a closed $*$ -ideal. Then A/I is a C^* -algebra.

Proof

A/I is a Banach algebra with operations $[a] + [b] = [a + b]$ and $[a] \cdot [b] = [a \cdot b]$ where $[a] = a + I = \{a + i : i \in I\}$. We may further define $[a]^* = [a^*]$ and a norm $||[a]|| = \inf_{i \in I} ||a + i||_A$. We want to show that $||[a]||^2 = ||[a]^*[a]||^2$.

$$||[a]_I|| = \inf_{i \in I} ||a + i|| = \inf_{i \in I} ||a^* + i^*|| = \inf_{i \in I} ||a^* + i|| = ||[a^*]|| = ||[a]^*||$$

By the sub-multiplicativity of the norm, $||[a]^*[a]|| \leq ||[a^*]|| \cdot ||[a]|| \leq ||[a]||^2$. In the other direction, write $0 \leq z \leq e$ to mean $z \geq 0$ and $e - z \geq 0$ (i.e. $z = z^*$ and $\text{sp } z \subseteq [0, 1]$). Then for $z \in I$,

$$||[a]|| = \inf_{i \in I} ||a + i|| \geq \inf_{0 \leq z \leq e} ||a + z|| \geq ||[a]||$$

Since $||e - z|| \leq 1$,

$$\begin{aligned} ||[a]||^2 &= \inf \{ ||a - az||^2 : 0 \leq z \leq e, z \in I \} \\ &= \inf \{ ||(e - z)a^* a(e - z)|| : 0 \leq z \leq e, z \in I \} \\ &\leq \{ ||a^* a(e - z)|| : 0 \leq z \leq e, z \in I \} \\ &= ||[a^* a]|| \end{aligned}$$

Theorem

Let A, B be C^* -algebras and $\phi : A \rightarrow B$ a $*$ -homomorphism. Then $\text{im } \phi = \phi(A)$ is a closed $*$ -subalgebra of B . If ϕ is injective, then it is an isometric $*$ -isomorphism to its image.

Proof

Assume ϕ is injective.

Obviously, $\text{sp}(\phi(a)) \subseteq \text{sp } a$ since $a - \lambda e$ invertible implies $\phi(a) - \lambda e$ is invertible.

Claim: for $a = a^*$, $\text{sp}(\phi(a)) = \text{sp}(a)$. Otherwise, there would exist $\lambda_0 \in \text{sp}(a)$ with $\lambda_0 \notin \text{sp}(\phi(a))$.

IMAGE 3

So there must exist $f : \text{sp}(a) \rightarrow [0, 1]$ continuous with $f|_{\text{sp}(\phi(a))} = 0$ and $f(\lambda_0) = 1$.

IMAGE 4

Therefore $\text{sp } f(\phi(a)) = f(\text{sp}(\phi(a))) = 0$ and, since f is real valued and $\phi(a)$ is self adjoint $f(\phi(a))$ suffices. It follows that $f(\phi(a)) = 0 = \phi(f(a))$ and that $f(a) = 0$. However, this contradicts our construction of f which set $f(\lambda_0) = 1$. Hence, $\text{sp}(\phi(a)) = \text{sp}(a)$ for any $a = a^*$. Further, $\|\phi(a)\| = r(\phi(a)) = r(a) = \|a\|$ for any such element. For an arbitrary element b ,

$$\|\phi(b)\|^2 = \|\phi(b)^* \phi(b)\| = \|\phi(b^* b)\| = \|b^* b\| = \|b\|^2$$

We conclude that ϕ is an isometry. Therefore, its image is closed since it is an isometry.

In the general case, assume that ϕ is not injective.

Consider the quotient algebra with $I = \ker \phi$ and $\psi([a]) = \phi(a)$.

$$\begin{array}{ccc} A & \xrightarrow{\phi} & \phi(A) \subseteq B \\ & \searrow \pi & \swarrow \psi \\ & A/I & \end{array}$$

We have that π is a surjective $*$ -homomorphism, ψ is an injective $*$ -homomorphism, and A/I is a C^* -algebra. Therefore $A/I \cong \phi(A)$ is an isometric isomorphism as above.

Corollary

If $B \subseteq A$ are C^* -algebras with I a closed $*$ -ideal of A , then $B + I$ is a $*$ -subalgebra of A .

$$(B + I)/I \cong B/(B \cap I)$$

Proof

Consider the quotient map $\pi : A \rightarrow A/I$ restricted to B , $\pi|_B : B \rightarrow (B + I)/I$ which is closed in A/I . Therefore $B + I$ is closed in A because if $b_n + i_n \rightarrow a$, $[b_n] \rightarrow [a]$. Therefore $[a] \in (B + I)/I$ and $a = b + i \in B + I$.

We have that $B + I$ is a $*$ -subalgebra of A where $b \mapsto [b]$ is surjective ($[b + i] = [b]$) $*$ -homomorphism.

Then $\ker \pi|_B = B \cap I$, and $\hat{\pi} : B/(B \cap I) \rightarrow (B + I)/I$ is an isometric $*$ -isomorphism.

Section 3.6: Positive Linear Functionals

Definition: Positive Linear Functional

A linear functional ϕ (on a C^* -algebra) is called positive ($\phi \geq 0$) if for all positive ($a \geq 0$) $a \in A$, $\phi(a) \geq 0$. If, in addition, $\phi(e) = 1$, then ϕ is called a state.

Remark

Positive functionals satisfy $\phi(a^*) = \overline{\phi(a)}$.

Proof

For $a = a^*$, $a = a_+ - a_-$, $a_{\pm} \geq 0$. Therefore $\phi(a) = \phi(a_+) - \phi(a_-) \in \mathbb{R}$.

For general $a = s + it$, s and t self-adjoint,

$$\phi(a^*) = \phi(s - it) = \phi(s) - i\phi(t) = \overline{\phi(s) + i\phi(t)} = \overline{\phi(a)}$$

Remarks

- $\phi_1, \phi_2 \geq 0$ implies $\phi_1 + \phi_2 \geq 0$
- $\phi_1 \geq \phi_2$ if and only if $\phi_1 - \phi_2 \geq 0$

Examples

$B = C(X)$ with X compact Hausdorff. $M(X)$ regular borel measures.

- All bounded linear functionals, μ a complex regular borel measure.

$$\phi_\mu(f) = \int_X f d\mu$$

- Positive linear functionals, μ a (positive) regular Borel measure

$$\phi_\mu(f) = \int_X f d\mu$$

- State $\phi(e) = 1$ if and only if $\mu(x) = \int_X d\mu = 1$.
- Multiplicative linear functionals, δ_{x_0} the Dirac point measure.

$$\phi_\mu(f) = f(x_0) = \int f d\delta_{x_0}$$

- $[f, g] = \int_X \overline{f}g d\mu$ semi-inner product (not a norm since $[f, f] = 0$ does not imply $f = 0$, and no completeness).

November 27, 2024

Recall: Positive Linear Functionals

C^* -algebra A , $\phi \geq 0$ if $\phi(a) \geq 0$, $\forall a \in A$, $a \geq 0$.

- $\phi \geq 0$ gives $\overline{\phi(a)} = \phi(a^*)$
- $\phi_1, \phi_2 \geq 0$ implies $\phi_1 + \phi_2 \geq 0$
- $\phi_1 \geq \phi_2$ if and only if $\phi_1 - \phi_2 \geq 0$

Example

$A = C(X)$ positive linear functions, $\phi(f) = \int_X f d\mu$, μ regular Borel measure.
 ϕ is a state if $\phi(e) = 1$

Example

$A = L(\mathbb{C}^n) = \mathbb{C}^{n \times n}$ linear functionals

$$\phi_B(A) = \text{trace}(B^T A) = \sum_{i,j=1}^n b_{ij} a_{ij}$$

Positive linear functionals correspond to $B \geq 0$. For $A \geq 0$, this implies

$$\phi_B(A) = \text{trace}(A^{1/2} B^T A^{1/2}) = \text{trace}(B^T A)$$

where $A = A^{1/2} A^{1/2}$. $B \geq 0$ implies that

$$\phi_B(A) = \text{trace}(A^{1/2} (B^T)^{1/2} (B^T)^{1/2} A^{1/2}) = \text{trace}(C^* C) = \sum_{j,k} \overline{C_{jk}} C_{jk} \geq 0$$

Conversely, assume $\phi_B \geq 0$ and choose

$$A = xx^* = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} \overline{x_1} & 7 \cdots & \overline{x_n} \end{pmatrix}$$

Then

$$\phi_B(A) = \text{trace}(B^T xx^*) = x^* B^T x = \langle B^T x, x \rangle$$

for any $x \in \mathbb{C}^n$. Therefore $B^T \geq 0$ and $B \geq 0$.

Example

$A = L(H)$. A subset of positive functionals is described by positive trace class operators $\mathcal{C}_1(H)$.
For $K \in \mathcal{C}_1(H)$, $\phi_K(A) = \text{trace}(KA)$.

Finite Rank Operators

Finite rank operators are a subset of trace class operators. For $x \in H$, $K = x \otimes x^*$.
 $kz = x \cdot \langle z, x \rangle$.

Proposition: Schwarz Inequality

For a positive linear functional $\phi \geq 0$, $|\phi(x^* y)| \leq \phi(x^* x) \cdot \phi(y^* y)$, $\forall x, y \in A$.

Proof

$x^*x \geq 0$ and $y^*y \geq 0$, so $\phi(x^*x) \geq 0$ and $\phi(y^*y) \geq 0$.

Define $[x, y] = \phi(y^*x)$ which has

- linearity in x and anti-linearity in y .
- $[y, x] = \overline{[x, y]}$ since $\phi(x^*y) = \phi((y^*x)^*) = \overline{\phi(y^*x)}$.
- $[x, x] = \phi(x^*x) \geq 0$.

Then $[x - \lambda y, x - \lambda y] \geq 0$ for all λ . Assume, without loss of generality, that $[x, y]$ is real (otherwise we may replace $([\tau x, y]$ for $|\tau| = 1$) If $[x, x] = 0$ or $[y, y] = 0$, then $[x, y] = 0$. Otherwise, put $\lambda = -\frac{[x, y]}{[y, y]}$.

If $[x, x] = [y, y] = 0$, then $-\lambda[x, y] + \bar{\lambda}[y, x] = 0$ implies that $[x, y] = 0$.

Rmeark

$p(x) = \sqrt{[x, x]}$ is only a seminorm, not a norm. That is $p(x) = 0 / x = 0$.

Proposition:

If $\phi \geq 0$, then ϕ is bounded and $||\phi|| = \phi(e)$.

Proof

$$|\phi(e^*x)|^2 \leq \phi(x^*x) \cdot \phi(e^*e)$$

We know that $r(x^*x) = ||x^*x|| = ||x||^2$ and $\text{sp}(x^*x) \subseteq [0, ||x||^2]$.
So $||x||^2 \cdot e - x^*x \geq 0$ ($x^*x \leq ||x||^2 \cdot e$) and $\phi(||x||^2 \cdot e - x^*x) \geq 0$.

$$\phi(e) \cdot ||x||^2 \geq \phi(x^*x)$$

Therefore $|\phi(x)|^2 \leq ||x||^2 \cdot \phi(e)^2$ implies that $\phi(e) \leq ||\phi|| \leq \phi(e)$ which shows also that ϕ is bounded.

Remark

$\phi \geq 0$ and $\phi(e) = 0$ implies that $\phi \equiv 0$.

Proposition

If ϕ is a bounded linear functional on A and $\phi(e) = ||\phi||$ then $\phi \geq 0$.

Proof

Without loss of generality, consider $\phi(e) = 1$.

Assume that $a \geq 0$ and $\phi(a) = \alpha + i\beta < 0$.

If $\beta \neq 0$, them consider $\phi(a + i\lambda e) = \alpha + i(\beta + \lambda)$ for $\lambda \in \mathbb{R}$.

$$\begin{aligned}
|\beta + \lambda| &\leq |\phi(a + i\lambda e)| \\
&\leq ||a + i\lambda e|| \\
&= ||(a + i\lambda e)^*(a + i\lambda e)||^{1/2} \\
&= ||a^*a + a^*i\lambda - ai\lambda + \lambda^2 e||^{1/2} \\
&\leq (||a||^2 + \lambda^2)^{1/2}
\end{aligned}$$

So $(\beta + \lambda)^2 \leq ||a||^2 + \lambda^2$ which means $\beta^2 + 2\beta\lambda \leq ||a||^2$ which cannot hold for $\lambda \rightarrow \pm\infty$.
If $\beta = 0$ and $\alpha < 0$, then

$$\phi(||a||e - a) = ||a|| - \alpha > ||a||$$

On the other hand $\phi(||a||e - a) \leq ||||a||e - a|| \leq ||a||$ since $\text{sp}(a) \subseteq [0, ||a||]$ implies that $\text{sp}(||a||e - a) \subseteq [0, ||a||]$ and $||||a||e - a|| = r(||a||e - a||) \leq ||a||$ which is also a contradiction.

Theorem:

Let B be a C^* -subalgebra of a C^* -algebra A (all unital) and ϕ be a positive linear functional on B . Then ϕ has an extension to a positive linear functional on A .

Proof

$\phi \geq 0$ (on B), so $\phi \in B^*$ and $||\phi|| = \phi(e)$.

Hahn-Banach tells us that there exists an extension $\psi \in A^*$ onto A with $\psi|_B = \phi$ such that $||\psi|| = ||\phi||$. So $||\psi|| = \phi(e) = \psi(e)$. By the preceding proposition, $\psi \geq 0$ on A .

Section 3.7: Representations of C^* -Algebras and GNS-Construction

Definition: Representation of a C^* -Algebra

A representation (π, H) of a C^* -algebra A is a $*$ -homomorphism $\pi : A \rightarrow L(H)$ such that $\pi(e) = I$.

Example

$A = L(H)$, $\pi = \text{id}$ is trivial $[\text{id}, H]$.

Definition: Inflation

If (π, H) is a representation of A , the inflation π^n is given by the block matrix

$$\pi^n(a) = \begin{pmatrix} \pi(A) & & & \\ & \pi(A) & & \\ & & \ddots & \\ & & & \pi(A) \end{pmatrix}$$

with $H_n = H \oplus H \oplus \cdots \oplus H (= H \times H \times \cdots \times H)$.

Example

$A = C(X)$ (arbitrary μ Borel measure) or $A = L^\infty(X, d\mu)$.

Take $H = L^2(X, d\mu)$, then $\pi : C(X) \ni f \mapsto M_f \in L(H)$ by $M_f g = fg$ where $g \in L^2(X, d\mu)$.

Definition: Direct Sum of Representations

The direct sum of representations (π_n, H_n) by $\pi = \bigoplus_n \pi_n$ and $H = \bigoplus_n H_n$. For a finite or countable direct sum, we may think of this as

$$\pi(a) = \begin{pmatrix} \pi_1(a) & & & \\ & \pi_2(a) & & \\ & & \ddots & \\ & & & \pi_n(a) \end{pmatrix}$$

Definition: Equivalence of Representations

Two representations (π_1, H_1) and (π_2, H_2) are said to be equivalent if there exists unitary $U : H_1 \rightarrow H_2$ ($U^* U = I_{H_1}$ and $U U^* = I_{H_2}$) such that $U\pi_1(a)U^* = \pi_2(a)$.

Definition: Cyclic Representation

A representation (π, H) is cyclic if there exists some $v \in H$ such that $\text{clos}(\pi(A)v) = H$ (i.e. $\text{clos}\{\pi(a)v : a \in A\} = H$).

Example

Take

$$A = \left\{ \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \right\}$$

with $\pi(a) = a \in L(\mathbb{C}^2)$ since by $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ for $v_1, v_2 \neq 0$.

If instead we take

$$A = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \right\}$$

and $\pi(a) = a$ then this is not cyclic since $\pi(A)v = \text{lin}\{v\}$.

Theorem

Each representation is equivalent to a direct sum $\bigoplus_n \pi_n$ of cyclic representations.

Proof

If π is not cyclic, show that $\pi \cong \pi_1 \oplus \pi_2$.

Apply Zorn's lemma.

December 2, 2024

Recall

- Representation (π, H)
 - $\pi : A \rightarrow L(H)$ a $*$ -homomorphism.

- Direct Sums
 - $\pi = \bigoplus \pi_\omega$ and $H = \bigoplus H_\omega$
- Cyclic (π, H)
 - $\exists v \in H$ such that $\text{clos}(\pi(A)v) = H$.
- Theorem: Every representation is equivalent to a direct sum of cyclic representations.
 - $(\pi_1, H_1) \sim (\pi_2, H_2)$ if $\pi_1(a) = U\pi_2(a)U^*$ for $U : H_2 \rightarrow H_1$.

Proof

Let $\mathcal{E} = \{E \subseteq H : \pi(A)v_1 \perp \pi(A)v_2, \forall v_1, v_2 \in E, v_1 \neq v_2\}$ which may be ordered by inclusion to apply Zorn's lemma. Then \mathcal{E} admits a maximal element $E \in \mathcal{E}$. Define $H_0 = \bigoplus_{v \in E} \text{clos}\{\pi(A)v\} \subseteq H$. Note that H_0 is closed, since it is an orthogonal sum.

Claim: $H_0 = H$, otherwise $H = H_0 \oplus H_1$ for $H_1 \neq \{0\}$. Then for every $v \in H_0$, $h \in H_1$ and $a, b \in A$, $\langle \pi(A)v, h \rangle = \langle \pi(b^*a)v, h \rangle = 0$. Therefore

$$\begin{aligned} 0 &= \langle \pi(a)v, \pi(b)h \rangle \\ \pi(a)v &\perp \pi(b)h \\ \pi(A)v &\perp \pi(A)h \end{aligned}$$

This means that $E \cup \{h\} \in \mathcal{E}$, but E was maximal by assumption so $h \in E$. This implies that $H_1 = \{0\}$, a contradiction. Now $H_v = \text{clos}(\pi(a)v)$ for $v \in E$, so $H_{v_1} \perp H_{v_2}$ for $v_1 \neq v_2$, $H = \bigoplus_{v \in E} H_v$, and $\pi_v(a) = \pi(a)|_{H_v}$ is a representation. $\pi(a)H_v \subseteq H_v$ and $\pi(a^*)H_v \subseteq H_v$.

For $H = H_{v_1} \oplus H_{v_2}$, with $\begin{pmatrix} * \\ 0 \end{pmatrix} = H_{v_1}$,

$$\pi(a) = \begin{pmatrix} * & * \\ * & * \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

Then, since $\pi(a^*) = \pi(a)^* = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$,

$$\pi(a) = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

with respect to $H_{v_1} \oplus H_{v_2}$. Therefore $\pi(a) \cong \bigoplus_{v \in E} \pi_v(a)$ and $H = (+)H_v \cong \bigoplus H_v$ (where the former is a literal sum). Write

$$v = \sum_{b_v \in H_v} b_v \rightarrow (b_v)_{v \in E}$$

$$\text{and } ||v|| = \left| \left| \sum b_v \right| \right| = \left(\sum \left| \left| b_v \right| \right|^2 \right)^{1/2} = \left| \left| (b_v) \right| \right|_{\bigoplus H_v}.$$

GNS-construction

Gelfand-Naimark-Segal

Theorem

For a unital C^* -algebra A :

- for a positive linear functional ϕ , there exists a cyclic representation (π_ϕ, H_ϕ) and a cyclic vector v_ϕ such that

$$\phi(a) = \langle \pi_\phi(a)v_\phi, v_\phi \rangle$$

- For a cyclic representation (π, H) with cyclic vector $v \in H$ and positive linear functional $\phi(a) := \langle \pi(a)v, v \rangle$, we have that $(\pi, H) \sim (\pi_\phi, H_\phi)$.

Remark

Positive linear functionals are in one to one correspondence with equivalence classes of cyclic representations.

Proof of A

Write $[a, b] = \phi(b^* a)$, a semi-inner product ($[a, a] = 0 \implies a = 0$).

Take $L = \{a \in A : \phi(a^* a) = 0\}$ a closed linear subspace. Since $|\phi(b^* a)|^2 \leq \phi(a^* a) \cdot \phi(b^* b)$, for each $a, b \in L$ we have that $[a, b] = 0$ implies that $[a + b, a + b] = 0$.

L is a left-ideal. That is, $a \in A$ and $x \in L$ implies $ax \in L$. Write

$$\phi((ax)^* ax)^2 = \phi(x^* a^* ax)^2 \leq \phi((x^* a^* a)^* x^* a^* a) \cdot \underbrace{\phi(x^* x)}_{=0}$$

For $H_0 = A/L$ a vector space with well-defined inner product $\langle a + L, b + L \rangle = [a, b] = \phi(b^* a)$, we have an induced norm $\|a + L\|^2 = \phi(a^* a)$ and can take H_ϕ to be the completion of H_0 . Then

$$\|ax + L\|^2 = \phi(x^* a^* ax) \leq \|a\|^2 \cdot \|x + L\|^2$$

Write $\pi_\phi(a) : H_0 \rightarrow H_0$ by $x + L \mapsto ax + L$ (which is well-defined because L is a left-ideal). This map is linear, bounded (by previous computation), and may be extended by continuity to $\pi_\phi(a) : H_\phi \rightarrow H_\phi$.

It remains to show that (π_ϕ, H_ϕ) is a representation. It is multiplicative $\pi_\phi(ab) = \pi_\phi(a)\pi_\phi(b)$ from the definition. We have also that $\pi_\phi(a^*) = \pi_\phi(a)^*$ since

$$\begin{aligned} \langle \pi_\phi(a)^*(x + L), y + L \rangle_{H_0} &= \langle x + L, \pi_\phi(a)(y + L) \rangle \\ &= \langle x + L, ay + L \rangle \\ &= \phi((ay)^* x) \\ &= \phi(y^*(a^* x)) \\ &= \langle a^* x + L, y + L \rangle \\ &= \langle \pi_\phi(a^*)(x + L), y + L \rangle_{H_0} \end{aligned}$$

Finally, we have cyclic vector $v_\phi = e + L$ since $\pi_\phi(a)(e + L) = ae + L = a + L$. So $\pi_\phi(A)(e + L) = A/L = H_0$ and the closure is H_ϕ .

Definition: State space of a C^* -Algebra

The state space of an algebra A is $\mathcal{S} = \{\phi \geq 0 : \phi(e) = 1\}$ for positive linear functionals ϕ .

Remark

$\|\phi\| = \phi(e) = 1$, so \mathcal{S}_A is a subset of the unit ball of the dual space A' of A . This is convex and compact in the weak*-topology.

If A is commutative, then \mathcal{S}_A is a maximal ideal space.

Definition: Extreme Points of a Convex Set

The point $a \in C$, for C convex, is extreme if $a = \lambda a_1 + (1 - \lambda) a_2$, $a_1, a_2 \in C$, $\lambda \in (0, 1)$ implies that $a_1 = a_2 = a$. Write $\text{ex}(C)$ for the set of extreme points.

IMAGE 1

The collection $\text{ex}(\mathcal{S}_A)$ are called the pure states.

Theorem: Krein-Milman

For a compact, convex set C of a (Hausdorff, locally compact) topological vector space such as $C = \mathcal{S}_A$, $\text{ex}(\mathcal{S}_A) \neq \emptyset$ and $\mathcal{S}_A = \text{clos}(\text{conv}(\text{ex}(\mathcal{S}_A)))$ (i.e. the closure of the convex hull of the extreme points).

Proposition:

$$\|a\| = \max\{\phi(a) : \phi \in \mathcal{S}_A\} \text{ for all } a \geq 0.$$

Proof

Case $A = C(X)$ for X a compact, Hausdorff space, \mathcal{S}_A can be identified with the positive regular Borel measures μ on X with $\mu(X) = 1$.

$$\phi_\mu(a) = \int_X a(x) d\mu(x)$$

The extreme points are given by the Dirac measures $\text{ex}(\mathcal{S}_A) \cong \{\delta_{x_0} : x_0 \in X\}$. Then for all $\phi \in \mathcal{S}_A$

$$|\phi(a)| \leq \underbrace{\|\phi\|}_{=\phi(e)=1} = \|a\|$$

So

$$\|a\|_A = \max_{x \in X} \{a(x)\} = a(x_0) = \phi_{\delta_{x_0}}(a)$$

for some $x_0 \in X$.

In the general case, consider $a \geq 0$ and $A_0 = \text{alg}\{e, a\}$ a commutative C^* -algebra. By what we have just proved, for some $\phi_0 \in \mathcal{S}_{A_0}$,

$$\|a\| = \phi_0(a)$$

We may extend $\phi_0 \in \mathcal{S}_{A_0}$ to $\phi \in \mathcal{S}_A$ by Hahn-Banach for positive linear functionals such that

$$\|a\| = \phi_0(a) = \phi(a)$$

Theorem

Let A be a unital C^* -algebra. Then there exists a representation (π, H) such that $\pi : A \rightarrow L(H)$ is a $*$ -isometry. If A is separable, then H can be chosen to be separable.

Proof

$F \subseteq \mathcal{S}_A$ weak*-dense (A separable implies the existence of a countable F).

Using the GNS-construction, $\phi \in F$ gives (π_ϕ, H_ϕ) . Take $\pi = \bigoplus_{\phi \in F} \pi_\phi$ and $H = \bigoplus_{\phi \in F} H_\phi$. $\pi : A \rightarrow L(H)$ is a $*$ -homomorphism, since it is the direct sum of $*$ -homomorphisms.

- $||\pi(a)|| \leq ||a||$
- $||\pi_\phi(a)||^2 \geq ||\pi_\phi(a)v_\phi||^2 = \langle \pi_\phi(a^*a)v_\phi, v_\phi \rangle = [a^*a + L, e + L] = \phi(a^*a).$

$$||\pi(a)||^2 = \sup_{\phi \in F} ||\pi_\phi(a)||^2 \geq \sup_{\phi \in F} \phi(a^*a) = \max_{\phi \in \mathcal{S}_A} \phi(a^*a) = ||a^*a|| = ||a||^2$$

so $||\pi(a)|| = ||a||$ and π is an isometry.

Corollary

Every unital C^* -algebra is isometrically $*$ -isomorphic to some closed $*$ -subalgebra of some $L(H)$.