

Partial Differential Equations I

January 8, 2024

Homework

Assigned exercises and concept maps. Graded by completion.

Presentations

Assigned topics; responsible for giving a class.

Definition: Partial Differential Equation(s) (PDE)

An identity relating an unknown function, its partial derivatives and its variables.

$$F(D^k u, \dots, D^2 u, Du, u, x) = 0, \quad x \in U \subseteq \mathbb{R}^n$$

where U is an open subset of \mathbb{R}^n , $u : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$, $Du = (\partial_{x_1} u, \dots, \partial_{x_n} u)$.

Then $F : \mathbb{R}^{n^k} \times \dots \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$, where F is given.

$x = (x_1, \dots, x_n)$ is (are) the independent variable(s).

u is the unknown function or dependent variable.

k is the order of the PDE.

Goal

Given a PDE, we consider

- Existence
- Uniqueness
- Stability

Recall: Multiindex Notation

$\alpha = (\alpha_1, \dots, \alpha_n)$ a vector such that $\alpha_i \in \mathbb{Z}_{\geq 0}$.

Then we say that α is a multiindex with order $|\alpha| = \alpha_1 + \dots + \alpha_n$.

Notation

$u : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$, $\alpha = (\alpha_1, \dots, \alpha_n)$.

$u^\alpha := D^\alpha u = \partial_{x_n}^{\alpha_n} \dots \partial_{x_1}^{\alpha_1} u$, where $\partial^0 u = u$.

Definition: Linear Partial Differential Equation

A linear PDE of order k is of the form

$$(*) \sum_{|\alpha|=k} a_\alpha(x) D^\alpha u = f(x)$$

Remark

This means that F is multilinear in the first $n^k + n^{k-1} + \dots$ variables.

Definition: Homogeneous Linear Partial Differential Equation

A linear given by $(*)$ is homogeneous if $f(x) \equiv 0$.
Otherwise, it is non-homogeneous.

Example 1: Linear Transport Equation

$$\nabla u \cdot (1, b) = u_t + b \cdot Du = f(t, x)$$

This is a linear PDE of order 1 on $\mathbb{R} \times \mathbb{R}^n \equiv \mathbb{R}^{n+1}$ where (t, x) are independent variables and u is dependent. Here, x is the spatial variable while t is time and Du represents the gradient.
 $\nabla u = (\partial_t u, \nabla u)$, $b \cdot Du = \sum_{i=1}^n b_i \partial_{x_i} u$, $(b_1, \dots, b_n) \in \mathbb{R}^n$ is fixed.

Example 2: Laplace Equation

$$\Delta u := \sum_{i=1}^n \partial_{x_i}^2 u = 0$$

This is a linear, homogeneous PDE of order 2.

Example 3: Poisson Equation

$$-\Delta u := f(u)$$

This is a nonlinear PDE of order 2.

Consider $f(u) = u^2$.

Example 4: Heat Equation (Diffusion Equation)

$$u_t - \Delta u = 0$$

This is a linear, homogeneous PDE of order 2.

Example 5: Wave Equation

$$u_{tt} - \Delta u = 0$$

This is a linear, homogeneous PDE of order 2.

Transport Equation

$u : \mathbb{R}^n(0, \infty) \rightarrow \mathbb{R}$ given by

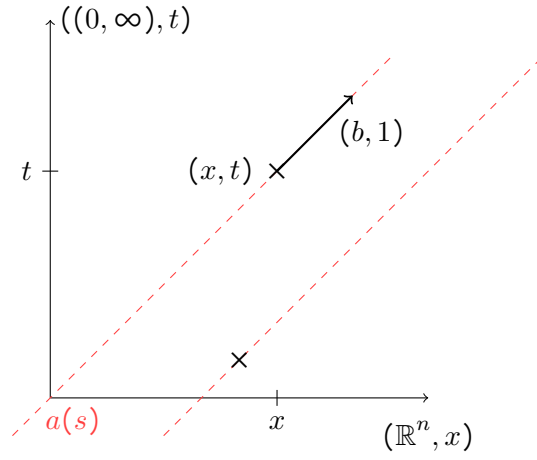
$$u_t + b \cdot Du = 0, \quad b \in \mathbb{R}^n$$

In order to get a solution, first assume that there exists a “nice” (e.g. smooth, C^1 , differentiable, etc.) solution.

Step 1

Notice that the PDE is equivalent to

$$\nabla u \cdot (b, 1) = 0$$



Step 2

Consider a curve on \mathbb{R}^{n+1} with velocity $(1, b)$ which passes through (x, t) . i.e.

$$\alpha(s) = (x + sb, t + s)$$

Notice $\alpha'(s) = (b, 1)$.

Then, let us study u along the curve $\alpha(s)$.

$$z(s) := u(\alpha(s))$$

Taking the derivative with respect to s ,

$$z'(s) := \frac{d}{ds}(u \circ \alpha(s)) = \nabla u|_{\alpha(s)} \cdot \alpha'(s) = \nabla u|_{\alpha(s)} \cdot (b, 1) = 0$$

That is $z'(s) = 0$, $z(s)$ is constant, and u along $\alpha(s)$ is constant.

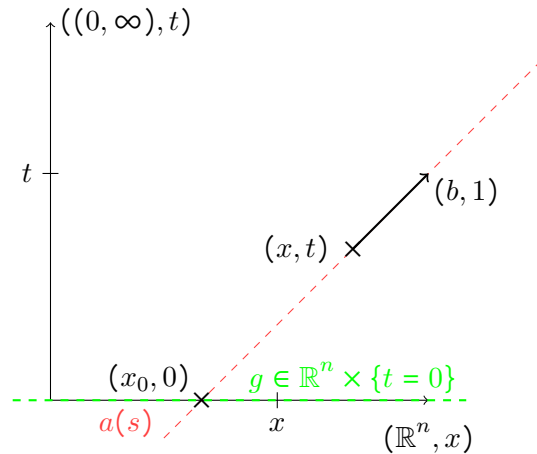
Conclusion

If we know some value of u along $\alpha(s)$, then we know all values along $\alpha(s)$.

If we have some value of u along every $\alpha(s)$, then we know u on $\mathbb{R}^n \times (0, \infty)$.

Transport Equation - Homogeneous Initial Value Problem

$$(*) \begin{cases} \nabla u \cdot (b, 1) = 0, & \mathbb{R}^n \times (0, \infty) \\ u = g, & \mathbb{R}^n \times \{t = 0\} \end{cases}$$



Here, $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is given.

Consider (x, t) ; we want to find $(x_0, 0)$.

We know $\alpha(s) = (x + sb, t + s) = (x_0, 0)$, therefore

$$\begin{cases} x + sb = x_0 & (1) \\ t + s = 0 & \implies s = -t \end{cases} \quad (2)$$

Then, by replacing (2) in (1),

$$x_0 = x - tb$$

Then from the conclusion

$$u(x, t) = u(x_0, 0) = g(x_0) = g(x - tb)$$

Therefore, $u(x, t) := g(x - tb)$ (♥).

Remark

1. If there exists a regular (differentiable or C^1) solution u for $*$, then u should look like ♥.
2. If g is (differentiable or C^1), then u defined by ♥ is a (differentiable or C^1) solution for my problem.

Homework

Show that ♥ satisfies $*$.

Transport Equation - Non-homogeneous Initial Value Problem

$$(*) \begin{cases} \nabla u \cdot (b, 1) = f(x, t), & \mathbb{R}^n \times (0, \infty) \\ u = g, & \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Where $g : \mathbb{R}^n \rightarrow \mathbb{R}$ and $f : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}$ are given.

Solution

Notice that the PDE is equivalent to

$$\nabla u \cdot (b, 1) = f(x, t)$$

Define the “characteristic curve”

$$\alpha(s) = (x + sb, t + s)$$

and

$$z(s) := u(\alpha(s))$$

Taking $\frac{d}{ds}$,

$$z'(s) = \nabla u|_{\alpha(s)} \cdot (b, 1) = f(\alpha(s)) \implies z'(s) = f(x + sb, t + s) \quad (c)$$

Notice that c is an ordinary differential equation. Integrating from $-t$ to 0.

$$\begin{aligned} \int_{-t}^0 z'(s) \, ds &= \int_{-t}^0 f(x + sb, t + s) \, ds \\ z(0) - z(-t) &= \int_{-t}^0 f(x + sb, t + s) \, ds \end{aligned}$$

Notice that $z(0) = u(x, t)$ and $z(-t) = u(\alpha(-t)) = u(x - tb, 0)$.

$$u(x, t) = u(x - tb, 0) + \int_{-t}^0 f(x + sb, t + s) \, ds$$

Then

$$\begin{aligned} u(x, t) &= g(x - tb, 0) + \int_{-t}^0 f(x + sb, t + s) \, ds \\ &\stackrel{\bar{s}=s+t}{=} g(x - tb, 0) + \int_0^t f(x + (\bar{s} - t)b, \bar{s}) \, d\bar{s} \\ &= g(x - tb, 0) + \int_0^t f(x + (s - t)b, s) \, ds \end{aligned}$$

Remark: Method of Characteristics

Try to vert the PDE into an ODE and solve using characteristic curves.

January 10, 2024

Definition: Harmonic Function

If $u \in C^2$ such that $\Delta u = 0$, then u is a harmonic function.

Laplace Equation

Consider $u : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ with U open, then the homogeneous (Laplace) form is given by

$$\Delta u = 0$$

and the non-homogeneous (Poisson) form is

$$-\Delta u = f$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is given.

Remark / Exercise

The Laplace equation is invariant under translation and rotation.

That is, if $\Delta u(x) = 0$ and $v(x) = u(x - y)$, then $\Delta v = 0$.

Similarly, if $w(x) = u(O(x))$ then $\Delta w = 0$ where O is an orthogonal matrix.

Fundamental Solution of the Laplace Equation

Remark: since the Laplace equation is invariant under rotation, we can consider a function in terms of the radius $v(x) = v(|x|)$.

Recall $r = |x| = (x_1^2 + \dots + x_n^2)^{1/2}$.

Because of this remark, assume that $u(x) = v(|x|) = v(r(x))$ (*) where $v : (0, \infty) \rightarrow \mathbb{R}$.

Therefore, we need

$$\frac{\partial r}{\partial x_i} = \frac{1}{2} \cdot \frac{2x_i}{(x_1^2 + \dots + x_n^2)^{1/2}} = \frac{x_i}{r}$$

Replace (*) in the PDE

$$\frac{\partial u}{\partial x_i} = \frac{\partial}{\partial x_i} v(r(x)) = v'(r(x)) \cdot \frac{\partial r}{\partial x_i} = v'(r(x)) \cdot \frac{x_i}{r}$$

and

$$\begin{aligned} \frac{\partial^2 u}{\partial x_i^2} &= \frac{\partial}{\partial x_i} \left(v'(r(x)) \cdot \frac{x_i}{r} \right) \\ &= \frac{\partial}{\partial x_i} (v'(r(x))) \cdot \frac{x_i}{r} + v'(r(x)) \cdot \frac{\partial}{\partial x_i} \left(\frac{x_i}{r} \right) \\ &= v''(r(x)) \cdot \frac{x_i^2}{r^2} + v'(r(x)) \left[\frac{1}{r} + x_i \frac{\partial}{\partial x_i} \left(\frac{1}{r} \right) \right] \\ &= v'' \frac{x_i^2}{r^2} + v' \left[\frac{1}{r} - \frac{x_i^2}{r^3} \right] \end{aligned}$$

Then, summing across i ,

$$\Delta u = v'' + v' \left[\frac{n}{r} + \frac{1}{r} \right] = 0$$

Then the PDE is equivalent to

$$v''(r) + \frac{v'}{r}(n+1) = 0 \quad (\square)$$

We need to find a solution for \square .

$$v''(r) = -\frac{(n+1)v'}{r}$$

Assume, without loss of generality, that $v' \neq 0$ such that

$$\frac{v''(r)}{v'(r)} = -\frac{n+1}{r} \implies (\log(|v'|))' = -\frac{n+1}{r}$$

Then, integrating,

$$\log(|v'|) = -(n+1) \log(r) + C = \log(r^{-(n+1)}) + C$$

such that

$$|v'| = Cr^{-(n+1)} \implies v' = Cr^{-(n+1)} \implies v(r) = Cr^{-n} + D = Cr^{2-n} + D$$

Definition: Fundamental Solution of the Laplace Equation

The function Φ given by

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log |x|, & n = 2 \\ \frac{1}{n(n-2)\alpha(n)} \cdot \frac{1}{|x|^{n-2}}, & n \geq 3 \end{cases}$$

where $\alpha(n)$ is the volume of the unit ball in \mathbb{R}^n is called the fundamental solution.

Remark

Φ solves the Laplace equation away from 0.

Lemma: Estimates for the Fundamental Solution

- First Estimate

$$|D\Phi(x)| \leq \frac{C}{|x|^{n-1}}, \text{ for } x \neq 0.$$

$$\frac{\partial \Phi}{\partial x_i} = C \frac{\partial}{\partial x_i} (|x|^{2-n}) = \frac{C(2-n)}{1-n} |x|^{2-n-1} \frac{\partial |x|}{\partial x_i} = |x|^{1-n} \cdot \frac{x_i}{|x|} = C x_i |x|^{-n}$$

Therefore

$$|D\Phi(x)| \leq C|x||x|^{-n} \implies |D\Phi(x)| \leq C|X|^{1-n}$$

– Exercise

Compute for $n = 2$.

- Second Estimate

$$|D^2\Phi(x)| \leq \frac{C}{|x|^n}, \text{ for } x \neq 0.$$

$$\begin{aligned} \frac{\partial^2}{\partial x_J \partial x_i} \Phi &= C \frac{\partial}{\partial x_J} (x_i |x|^{-n}) \\ &= C \left[\delta_{iJ} |x|^{-n} + x_i \frac{\partial}{\partial x_J} |x|^{-n} \right] \\ &= C \left[\delta_{iJ} |x|^{-n} + (-n) \cdot \frac{x_i |x|^{-n-1} x_J}{|x|} \right] \\ &= C \left[\frac{\delta_{iJ} |x|}{|x|^n} + \frac{C x_i x_J}{|x|^{n+1}} \right] \end{aligned}$$

Then

$$\left| \frac{\partial \Phi}{\partial x_i \partial x_J} \right| \leq \frac{C}{|x|^n} + \frac{C|x_i||x_J|}{|x|^{n+2}} \leq \frac{2C}{|x|^n} = \frac{C}{|x|^n}$$

Then, we are done since

$$|D^2\Phi(x)| = \sqrt{\sum_J \left(\frac{\partial \Phi}{\partial x_i \partial x_J} \right)^2}$$

Poisson Equation

Motivation

Suppose we have $\Phi(x)$, the fundamental solution.

Then for an arbitrary, fixed element $y \in \mathbb{R}^n$, then we have $x \rightarrow \Phi(x - y)$ harmonic for $x \neq y$.

Consider $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $y \rightarrow f(y)$ then $x \rightarrow f(y)\Phi(x - y)$ is similarly harmonic for $x \neq y$.

Now, if given $\{y_1, \dots, y_m\}$ where $y_i \in \mathbb{R}^n$, then $x \rightarrow \sum_{i=1}^m f(y_i)\Phi(x - y_i)$ is harmonic $\forall x \neq \{y_1, \dots, y_m\}$.

Then, what happens if we consider

$$u(x) := \int_{\mathbb{R}^n} f(y)\Phi(x - y) dy \quad (\square_3)$$

Is u harmonic? No, since $\Delta\Phi(x - y)$ is not summable in \mathbb{R}^n we may not pass the limit into the integral.

(to be covered later) However, since $\Delta\Phi(x - y)$ acts as δ_{xy} in distribution, this may solve the Poisson equation.

Remark / Exercise

Assume that $f \in C_c^2(\mathbb{R}^n)$ (i.e f is twice continuously differentiable with compact support on \mathbb{R}^n).

The function Φ is integrable near the singularity on compact sets.

Prove using spherical coordinates.

Therefore, u defined by \square_3 is well defined

$$|u| = \left| \int_{\mathbb{R}^n} f(y)\Phi(x - y) dy \right| = \left| \int_K \Phi(x - y) dy \right| < \infty$$

Theorem: Solving the Poisson Equation

If $f \in C_c^2(\mathbb{R}^n)$ and u is defined by \square_3 , then

1. $u \in C^2(\mathbb{R}^n)$
2. $-\Delta u = f$, in \mathbb{R}^n

• Proof of 1

Since Φ presents a problem at $x = y$ but f is well behaved, we will change variables such that $\bar{y} = x - y$, $y = x - \bar{y}$, and $\frac{dy}{d\bar{y}}(-1)I_{m \times m}$ and then redefine $\bar{y} = y$.

$$u(x) = \int_{\mathbb{R}^n} f(y)\Phi(x - y) dy = \int_{\mathbb{R}^n} f(x - \bar{y})\Phi(\bar{y}) d\bar{y} = \int_{\mathbb{R}^n} f(x - y)\Phi(y) dy$$

In short, we have sent the problem from Φ to f .

Now, let us consider $e_i = (0, \dots, 1, \dots, 0)$.

Then for $h > 0$,

$$\frac{u(x + he_i) - u(x)}{h} = \frac{1}{h} \int_{\mathbb{R}^n} \Phi(y) [f(x + he_i - y) - f(x - y)] dy$$

Now, the limit as $h \rightarrow 0$

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{u(x + he_i) - u(x)}{h} &= \lim_{h \rightarrow 0} \int_{\mathbb{R}^n} \overbrace{\Phi(y) \left[\frac{f(x + he_i - y) - f(x - y)}{h} \right]}^{H(h,y)} dy \\ &= \int_{\mathbb{R}^n} \Phi(y) \cdot \frac{\partial f(x - y)}{\partial x_i} dy \end{aligned}$$

To justify passing the limit into the integral, take an arbitrary sequence $h_m \xrightarrow{0} 0$ and consider

$$f_m(y) := H(h_m, y)$$

We want to consider

$$\begin{aligned} |H(h_m, y)| &\leq \Phi(y) \left[\frac{f(x + h_m e_i - y) - f(x - y)}{h} \right] \\ &\leq \Phi(y) f'(c) \end{aligned}$$

Where c is along the curve between $f(x + h_m e_i - y)$ and $f(x - y)$ and chosen by mean value theorem.

– Exercise

$$|H(h_m, y)| \leq \Phi(y) \|f'\|_{L^\infty \chi_{B(x, R)}}(y)$$

Note that

$$C \int_{\mathbb{R}^n} |\Phi(y)| \chi_{B(x, R)}(y) dy = \int_{B(x, R)} |\Phi(y)| dy < \infty$$

– Exercise

Using the fact that a continuous function is uniformly continuous on a compact set, show that $u \in C^2(\mathbb{R}^n)$.

Dominated Convergence Theorem

If $f_m(x)$ such that $f_m(x) \xrightarrow[m \rightarrow \infty]{\text{pointwise}} f(x)$, and $|f_m(x)| \leq g(x)$ for $g \in L^1$, then f is integrable and

$$\lim_{m \rightarrow \infty} \int f_m(x) dx = \int f(x) dx$$