Topics in Analysis (F24)

Chapter 1: Banach Algebras

Section 1.1.1: Definitions and Basic Properties

Definition: Banach Space

A Banach space X (over $\mathbb{C}$) is a normed vector space with algebraic operations

$$\begin{aligned} (x,y) &\mapsto x+y && \text{addition} \\ (\lambda,y) &\mapsto \lambda y && \text{scalar multiplication} \end{aligned}$$

and a norm

$$x \mapsto ||x||$$

which is complete (i.e. every Cauchy sequence converges).

Definition: (Complex) Banach Algebra

A (complex) Banach algebra B is a Banach space in which there is multiplication

$$(x,y) \in B \times B \mapsto xy \in B$$

such that

1. $x(yz) = (xy)z$
2. $(x+y)z = xz + yz$ and $x(y+z) = xy + xz$
3. $\lambda(xy) = (\lambda x)y = x(\lambda y)$
4. $||xy|| \leq ||x|| \cdot ||y||$

Definition: Unital Banach Algebra

B is called a unital Banach algebra if $\exists e \in B$ such that

$$xe = ex = x \quad \text{and} \quad ||e|| = 1.$$

If e exists, it is unique.

Section 1.1.2: Examples

Example 1

If X is a Banach space, then $B = \mathcal{L}(X)$ (the set of all bounded inear operators $A : X \to X$) equipped with algebraic operations

$$\begin{aligned} (A+B)x &= Ax + Bx \\ (\lambda A)x &= \lambda(Ax) \\ (AB)x &= A(Bx) \end{aligned}$$

and the operator norm

$$||A||_{\mathcal{L}(X)} = \sup_{x\neq 0} \frac{||Ax||_X}{||x||_X}.$$

$B = \mathcal{L}(X)$ is complete because X is complete.
The unit element is given by $I_X x = x$.

Example 2

If $X = \mathbb{C}^n$, then $B = \mathcal{L}(\mathbb{C}^n) \cong \mathbb{C}^{n\times n}$.

$$A = (a_{ij})_{i,j=1}^n \qquad Ax = y \qquad \sum_{j=1}^n a_{ij}x_j = y_i.$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

The norm in $\mathbb{C}^n$ leadsto a norm in $\mathbb{C}^{n\times n}$

$$||(x_i)|| = \left(\sum |x_i|^2\right)^{1/2} \qquad ||A|| =$$

$$||(x_i)|| = \sum |x_i| \qquad ||A|| = \max_j \sum_i |a_{ij}|$$

$$||(x_i)|| = \max |x_i| \qquad ||A|| = \max_i \sum_j |a_{ij}|$$

All norms are quivalent.

Example 3

Take $B = C(K)$ with K a compact Hausdorff space, $f : K \to \mathbb{C}$ continuous and $||f|| = \max_{t\in K} |f(t)|$.

Example 4

Take $B = A(K)$, $K \subseteq \mathbb{C}$ compact with $\mathrm{int}(K) \neq 0$, $f : K \to \mathbb{C}$ continuous where f is holomorphic on $\mathrm{int}(K)$ and

$$||f|| = \max_{t\in K} |f(t)| = \max_{t\in K\setminus \mathrm{int}(K)} |f(t)|$$

e.g. $K = \overline{\mathbb{D}} = \{t \in \mathbb{C} \,:\, |t| \leq 1\}$. Then $A(K) \subseteq C(K)$.

Example 5

Take $B = \ell^\infty(\mathbb{N})$ or $B = L^\infty(S,\sigma,\mu)$ with (S,σ,μ) a measure space, $f : S \to \mathbb{C}$ essentially bounded functions and

$$||f|| = \operatorname{ess\,sup}_{t\in S} |f(t)| = \inf_{\substack{N\subseteq S \\ \mu(N)}} \left(\sup_{t\in S\setminus N} |f(t)| \right)$$

Example 6

Take $B = \ell^1(\mathbb{Z})$ or $B = L^1(\mathbb{R}^d)$ with $||\{x_n\}|| = \sum |x_n|$ and $||f|| = \int_{\mathbb{R}^d} |f(t)|\, dt$ respectively.
Multiplication is given by the convolution. e.g.

$$fg = (f * g)(x) = \int_{\mathbb{R}^d} f(x-t)g(t)\, dt$$

$\ell^1(\mathbb{Z})$ is unital, but $L^1(\mathbb{R}^d)$ is non-unital (since the unit of convolution is the Dirac delta; see Example 7).

Example 7

Take $B = M(\mathbb{R}^d)$ the complex measures on $\mathbb{R}^d$ with bounded variation.
Then multiplications is given as

$$(\mu * \nu)(A) = \int_{\mathbb{R}^d} \mu(A-x) d\nu(x)$$

and norm

$$||\mu|| = \sup_{\substack{\mathbb{R}^d = \bigcup A_i \\ \text{disjoint}}} \sum_{i=1}^{n} |\mu(A_i)| < +\infty.$$

Then, $f\, dm = d\mu$ gives $L^1(\mathbb{R}^d) \to M(\mathbb{R}^d)$.

Example 8

Take $B = C^{n\times n}[K]$ with K compcat and Hausdorff, continuous functions $f : K \to \mathbb{C}^{n\times n}$ and norm

$$||f||_B = \max_{t\in k} ||f(t)||_{C^{n\times n}}.$$

Then $B \cong (C(K))^{n\times n}$ the $n \times n$ matrices with entries from $C(K)$.

Section 1.1.3: Remarks

- If B does not have a unit element, consider $B_1 = B \times \mathbb{C}$ with operations

$$\begin{aligned}(b_1, \lambda_1) + (b_2, \lambda_2) &= (b_1 + b_2, \lambda_1 + \lambda_2) \\ \alpha(b, \lambda) &= (\alpha b, \alpha\lambda) \\ (b_1, \lambda_1)(b_2, \lambda_2) &= b_1 b_2 + \lambda_1 b_2 + \lambda_2 b_1, \lambda_1\lambda_2)\end{aligned}$$

and norm

$$||(b, \lambda)|| = ||b|| + |\lambda|.$$

Then B_1 is a unital Banach algebra with $e = (0, 1)$. One writes $(b, \lambda) = (b, 0) + \lambda(0, 1) = b + \lambda \cdot e$.
In some sense, $B \subseteq B_1$ where $b \in B \mapsto (b, 0) \in B_1$.

Section 1.1.4: Definitions

Definition: Commutative Banach Algebra

B is called commutative if $xy = yx$.

Definition: Banach Subalgebra

A subset B_0 of a B-algebra is called a subalgebra if it is closed with respect to the algebraic operations

$$x, y \in B_0,\ \lambda \in \mathbb{C} \rightsquigarrow x + y, xy, \lambda x \in B$$

Definition: Closed Subalgebra

B_0 is a closed subalgebra or Banach subalgebra if it is norm-closed.

- Proposition: B_0 is a Banach algebra.

Definition: Generated Subalgebra

Let $M \neq \varnothing$ be a subset of a Banach algebra B.
The Banach subalgebra generated by M is the smallest closed subalgebra containing M.

$$\operatorname{alg} M = (\operatorname{clos} \operatorname{alg}_B M)$$

- Remark

  $\operatorname{alg} M$ is the intersection of all closed subalgebras containing M.
  $\operatorname{alg} M = \operatorname{clos}\left\{\sum_{i=1}^{N} \lambda_i a_1^{(i)} a_2^{(i)} \cdots a_{n_i}^{(i)}\right\}$ is the norm-closure of finite linear combinations of finite products of $a_j^{(i)} \in M$.

Section 1.1.5: Examples

Exammple 1

Take B unital, $b \in B$. Then

$$\operatorname{alg}\{e, b\} = \operatorname{clos}_B \left\{ \sum_{i=0}^{N} \lambda_i b^i \ :\ \lambda_i \in \mathbb{C},\ N \in \mathbb{N} \right\}$$

where $b^0 = e$.

1.1.6 Definitions

Definition: Banach Algebra Homomorphism

A Banach algebra homomorphism is a map $\phi : B_1 \to B_2$ between Banach algebras B_1 and B_2 such that

- ϕ is linear

- ϕ is bounded (continuous)
- ϕ is multiplicative

  $\phi(b_1 b_2) = \phi(b_1) \cdot \phi(b_2)$

- ϕ is unital if both B_1, B_2 have units and $\phi(e_{B_1}) = e_{B_2}$.

Definition: Banach Algebra Isomorphism

A Banach algebra homomorphism which is bijective is called a Banach algebra isomorphism.
Then $\phi^{-1} : B_2 \to B_1$ is an isomorphism as well.

Definition: Banach Algebra Isometry

ϕ is an isometry if $||\phi(x)|| = ||x||$.