

Analysis II

January 9, 2024

(Real) Analysis

- Calculus
 - Differential
 - Integral (Riemann)
- Functions and Maps
 - Measure Theory
 - (Lebesgue) Integration
- Topology
 - Completeness (as a metric space)
 - Compactness (Bolzano-Weierstrass theorem [real]) (Arzela-Ascoli)
 - Paracompactness / Metrizable / Baire Category Theorem
 - Algebraic / Combinatoric (continuous maps or functions)

Definition: Cardinality

For sets A, B , $\text{Card}(A) = \text{Card}(B)$ if there exists a one-to-one correspondence $q : A \leftrightarrow B$.

Counting, labelling, indexing, etc.

$\text{Card}(A) \leq \text{Card}(B)$ if $A \subset B$ or there exists a one-to-one mapping $A \rightarrow B$.

Definition: Countable

If $A \hookrightarrow \mathbb{N}$, then A is countable.

Theorem

The countable union of countable sets is countable.

Proof

Let $A_i = \{a_j\}_{j=1}^{\infty}$, $i = 1, 2, \dots$

$$\begin{array}{cccc} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ \vdots & & & \\ a_{k1} & a_{k2} & a_{k3} & \cdots \end{array}$$

Index by diagonalization.

Theorem

The cartesian product of countable sets is countable.

Proof

$$X \times Y = \{(x_i, y_j) \mid x_i \in X, y_j \in Y\}$$

$$\begin{array}{cccc}
(x_1, y_1) & (x_1, y_2) & (x_1, y_3) & \cdots \\
(x_2, y_1) & (x_2, y_2) & (x_2, y_3) & \cdots \\
\vdots & & & \\
(x_k, y_1) & (x_k, y_2) & (x_k, y_3) & \cdots
\end{array}$$

Theorem

$\text{Card}(2^X) > \text{Card}(X)$, where $2^X = \{A \subset X\}$ is the power set of X .

Proof

For all $x \in X$, $\{x\} \subset 2^X$, so $\text{Card}(X) \leq \text{Card}(2^X)$.

Assume, for sake of contradiction, that $\text{Card}(X) = \text{Card}(2^X)$.

Then, by definition, there exists a one-to-one correspondence $\phi : X \leftrightarrow 2^X$.

Set $A = \{x \in X \mid x \notin \phi(x)\}$, and let $a = \phi^{-1}(A)$ (i.e. $A = \phi(a)$).

If $a \in A$, then $a \notin A \subset \phi(a)$; but if $a \notin A$, then $a \in A$, a contradiction.

Theorem

$$\text{Card}(\mathbb{R}) = \text{Card}(2^{\mathbb{N}}).$$

Topology of the Real Line

Completeness (as a metric space)

$$d(a, b) = |a - b|, \quad \forall a, b \in \mathbb{R}.$$

1. $x_i \rightarrow x$ if $\forall \varepsilon > 0, \exists n \in \mathbb{N}$ such that $|x_i - x| < \varepsilon, \forall i \geq n$.
2. $\{x_i\}$ is Cauchy if $\forall \varepsilon > 0, \exists n \in \mathbb{N}$ such that $|x_i - x_j| < \varepsilon, \forall i, j \geq n$.

Definition: Open Interval

(a, b) is an open set on the real line.

There exist interior points for any subset A of real numbers.

$\forall x \in A$, x is interior if $\exists (a, b)$ such that (1) $x \in (a, b)$ and (2) $(a, b) \subset A$.

- Theorem

The union of open sets is open.

The intersection of finitely many open sets is open.

\emptyset and \mathbb{R} are open.

Definition: Limit Point

A limit point $x \in \mathbb{R}$ of a subset A is a limit point in A if for every open neighborhood U of x , $(U \setminus \{x\}) \cap A \neq \emptyset$.

Definition: Closed

A is closed if A contains all of its limit points.

- Theorem

A is closed if and only if $A^c = \mathbb{R} \setminus A$ is open.

- Proof

A closed $\implies A^c$ open.

Otherwise, $\exists x \in A^c$ such that for every neighborhood U of x , $(U \setminus \{x\}) \cap A \neq \emptyset$ which would make it a limit point of A not in A . By assumption, A contains all its limit points so this is a contradiction.

A^c open $\implies A$ closed.

For any x a limit point of A , assume otherwise that $x \in A^c$.

Then there exists some neighborhood U of x such that $U \subset A^c$ (since A^c is open).

It follows that $(U \setminus \{x\}) \cap A = \emptyset$ and x is not a limit point of A , which is a contradiction.

Definition: Sequential Compactness

A is compact if $\forall \{x_i\}$, $x_i \in A$ there exists a convergent subsequence $\{x_{i_k}\}$ and $x_{i_k} \rightarrow x \in A$.

- Theorem: Bolzano-Weierstrass

For $A \subseteq \mathbb{R}$, A is compact if and only if A is closed and bounded.

- Proof

A compact $\implies A$ closed and bounded.

Assume that A is not bounded from above.

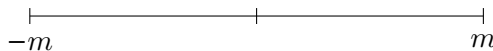
Then there exists a sequence $\{x_i\}$, $x_i \in A$ where $x_{i+1} > x_i + 1$ and $\{x_i\}$ has no convergent subsequences.

Then compactness implies closedness.

A closed and bounded $\implies A$ (sequentially) compact.

Let any $\{x_i\}$, $x_i \in A$.

Claim: $\forall \{x_i\}$ of reals, if there exists $m \in \mathbb{R}$ such that $|x_i| \leq m$, $\forall m$ then there is some convergent subsequence.



Divide and conquer: dividing the interval in half necessitates that at least one half contains infinitely many points. Repeat indefinitely.

- Theorem: Heine-Borel

$A \subseteq \mathbb{R}$ is (sequentially) compact if and only if any open cover has a finite subcover.

- Proof

Heine-Borel Property \implies closed and bounded.

Assume that A is unbounded, $U_n = (-n, n)$ and $\{U_n\}_{n=1}^{\infty}$ an open cover for $A \subseteq \mathbb{R}$ has no finite subcover.

Assume A is not closed, then $x \in \dot{A}$ (where \dot{A} is the limit set of A) and $x \notin A$, $U_n \left\{ \left(-\infty, x - \frac{1}{n} \right) \cup \left(x + \frac{1}{n}, +\infty \right) \right\}$.

Then $\{U_n\}$ covers $\mathbb{R} \setminus \{x\} \supset A$ has no finite subcover of A .

A is bounded and closed $\implies A$ is Heine-Borel

Divide and conquer: using open sets with respect to open covers.

Definition: Cantor Set

$C = \{x \in [0, 1] \mid \text{the ternary expansion of } x \text{ has only the digits } \{0, 2\}\}.$

Equivalently, let $C_0 = [0, 1]$, $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$.

Then $C_n = \bigcup_{k=1}^{2^n} C_n^k$ and $C = \bigcap_{n=1}^{\infty} C_n$.

$|C_n| = 2^n \left(\frac{1}{3}\right)^n \rightarrow 0.$

Definition: Perfectly Symmetric Sets

Let $\{\xi_n\}$ where $\xi_n \in \left(0, \frac{1}{2}\right)$.

$E_0 = [0, 1]$, $E_1 = [0, \xi_1] \cup [1 - \xi_1, 1]$, $E_2 = [0, \xi_1 \xi_2] \cup [\xi_1 - \xi_1 \xi_2, \xi_1] \cup [1 - \xi_1, 1 - \xi_1 + \xi_1 \xi_2] \cup [1 - \xi_1 \xi_2, 1]$.

Then the cantor set is given by $\xi_n = \frac{1}{3}$.

$E_n = \bigcup_{k=1}^{2^n} E_n^k$, $|E_n^k| = \xi_1 \xi_2 \cdots \xi_n$, and $|E_n| = \sum |E_n^k| = 2^n \xi_1 \xi_2 \cdots \xi_n$.

Therefore, $E = \bigcap_{n=1}^{\infty} E_n$ and we define $|E| = \lim_{n \rightarrow \infty} |E_n| = \lim_{n \rightarrow \infty} (2^n \xi_1 \xi_2 \cdots \xi_n) = \lambda$ where $\lambda \in [0, 1)$.

Let

$$2\xi_n = \frac{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n-1}\right)^{n-1}}{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n}\right)^n} < 1$$

, then

$$2^n \xi_1 \cdots \xi_n = \frac{1}{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n}\right)^n} \rightarrow \lambda.$$

Proof

$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^{n/x} = e^x$, then $\lim_{y \rightarrow 0} (1 + y)^{1/y} = e$, $\log(1 + y)^{1/y} = \frac{\log(1+y)}{y} \xrightarrow{y \rightarrow 0} 1$.

Observe that

$$\left(\frac{\log(1 + y)}{y}\right)' = \frac{\frac{y}{1+y} - \log(1 + y)}{y^2} = \left(1 + \frac{1}{1 + y} - \log(1 + y)\right)' = \frac{1}{(1 + y)^2} - \frac{1}{1 + y} = -\frac{y}{(1 + y)^2} < 0$$

Theorem

Cantor sets and perfect symmetric sets are closed, perfect, uncountable, and nowhere dense.

January 11, 2024

Last Week

Cardinality.

Topology of the reals.

- Cantor (perfect symmetric sets)

$$\begin{aligned}
C_0 &= [0, 1] \\
C_1 &= [0, 1/3] \cup [2/3, 1] \\
C_2 &= [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 1] \\
C_n &= \bigcup_{k=1}^{2^n} C_n^k \\
|C_n^k| &= \left(\frac{1}{3}\right)^n \\
C &= \bigcap_{n=1}^{\infty} C_n \\
|C_n| &= 2^n \frac{1}{3^n} = \left(\frac{2}{3}\right)^n \implies |C| = \lim_{n \rightarrow \infty} |C_n| = 0 \\
&\text{Closed, no interior points and uncountable.}
\end{aligned}$$

- Perfect Symmetric Sets

$$\begin{aligned}
\{\xi_k\} &\in \left(0, \frac{1}{2}\right) \\
E_0 &= [0, 1] \\
E_1 &= [0, \xi_1] \cup [1 - \xi_1, 1] \\
E_2 &= [0, \xi_1 \xi_2] \cup [\xi_1 - \xi_1 \xi_2, \xi_1] \cup [1 - \xi_1, 1 - \xi_1 + \xi_1 \xi_2] \cup [1 - \xi_1 \xi_2, 1] \\
E_n &= \bigcup_{k=1}^{2^n} E_n^k \\
|E_n^k| &= \xi_1 \xi_2 \cdots \xi_n \\
|E_n| &= 2^n \xi_1 \xi_2 \cdots \xi_n \\
&= \left(1 + \frac{\log\left(\frac{1}{n}\right)}{n-1}\right)^{n-1} \\
2\xi_n &= \frac{1}{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n}\right)^n} < 1 \\
|E_n| &= \frac{1}{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n}\right)^n} \\
|E| &= \lim_{n \rightarrow \infty} |E_n| = \frac{1}{e^{\log\left(\frac{1}{n}\right)}} = \lambda, \quad \lambda \in (0, 1)
\end{aligned}$$

Volterra's Function

$$\phi(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

IMAGE HERE - graph of phi(x)

$$\phi'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$f(x) = \begin{cases} 0 & x \in E \\ \phi(x-a) & x \in (a, a+y) \\ -\phi(b-x) & x \in (b-y, b) \\ \phi(y) & x \in (a+y, b-y) \end{cases}, \quad (a, b) \in E^c$$

IMAGE HERE - f interval (a,b)

Propositions

$$1. \quad f'(x) = 0 \text{ for } x \in E.$$

2. $f'(x)$ discontinuous on E .
3. f' exists on $[0, 1]$ and is bounded.

Since $|E| > 0$, $f'(x)$ is not Riemann integrable and, therefore, the fundamental theorem of calculus does not apply.

Lebesgue Outer Measure

$$|(a, b)| = b - a.$$

Let $A \subseteq \mathbb{R}$, then $m^*(A) = \inf \left\{ \sum_{n=1}^{\infty} |I_n| \mid A \subseteq \bigcup_{n=1}^{\infty} I_n \right\}$

Question: $m^*(A \cup B) \stackrel{?}{=} m^*(A) + m^*(B)$ for $A \cap B \neq \emptyset$?

Properties

1. $A \subseteq B \implies m^*(A) \leq m^*(B)$.
2. $m^*(\emptyset) = 0$.
3. If I is an interval, then $m^*(I) = |I|$.
4. If $\{A_i\}$ is countable, $m^*\left(\bigcup A_i\right) \leq \sum m^*(A_i)$.

- Proof of 4

$\forall A_i, \exists \{I_n\}$ open intervals such that $\sum_n |I_n| < m^*(A_i) + \frac{\varepsilon}{2^i}$.

Then $\bigcup_i \bigcup_n I_n^i \supset \bigcup_i A_i$, and $\sum_{n,i} |I_n^i| = \sum_i \left(\sum_n |I_n^i| \right) \leq \sum_i \left(m^*(A_i) + \frac{\varepsilon}{2^i} \right)$.

– Corollary

If A is countable, then $m^*(A) = 0$.

Thus, by contraposition, every interval is uncountable.

Proposition

For $A \subseteq \mathbb{R}$, $\forall \varepsilon > 0$, $\exists U$ open such that $A \subseteq U$ and $m^*(U) \leq m^*(A) + \varepsilon$.

Corollary

There exists G in the intersection of countable open sets such that $m^*(G) = m^*(A)$ and $G \supseteq A$.

Caratheodory Criteria

If $\forall E, m^*(E \cap A) + m^*(E \cap A^c) = m^*(E)$, then A is Lebesgue measurable.

- Remark: $m^*(E \cap A) + m^*(E \cap A^c) \leq m^*(E) \leq +\infty$

Propositions

1. If A is measurable, then A^c is measurable.

2. $m^*(A) = 0$, then A is measurable.
3. If A, B are measurable, then $A \cup B, A \cap B, A \setminus B$ are measurable.
4. If $\{A_i\}_{i=1}^k$ are disjoint and measurable, then $m^*\left(\bigcup_{i=1}^k A_i\right) = \sum_{i=1}^k m^*(A_i)$.

• Proof of 3

$$\begin{aligned}
m^*(E \cap (A \cup B)) + m^*(E \cap (A \cup B)^c) &= m^*((E \cap A) \cup (E \cap B)) + m^*(E \cap A^c \cap B^c) \\
&= m^*(E \cap A) + m^*((E \cap A^c) \cap B) + m^*((E \cap A^c) \cap B^c) \\
&\leq m^*(E)
\end{aligned}$$

Since $(A \cap B)^C = A^c \cup B^c$, this holds from before; similarly, $A \setminus B = A \cap B^c = A^c \cup B$.
If A, B disjoint, then

$$\begin{aligned}
m^*(A \cup B) &= m^*(E \cap A) + m^*(E \cap A^c) \\
&= m^*(A) + m^*(B)
\end{aligned}$$

Theorem

If $\{A_i\}$ is a countable collection of disjoint and measurable sets, then

1. $\bigcup_i A_i$ is measurable.
2. $m^*\left(\bigcup_i A_i\right) = \sum_i m^*(A_i)$.

Proof of 1

Want to show:

$$m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) + m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right) \leq m^*(E)$$

By assumption, since the measure of E is finite, $m^*(E \cap \bigcup_{i=1}^{\infty} A_i) < +\infty$.

Claim: $\forall \varepsilon > 0, \exists k$ such that

Therefore $m^*\left(E \cap \bigcup_{i=1}^k A_i\right) \geq m^*(E \cap \bigcup_{i=1}^{\infty} A_i) - \varepsilon$.

$$m^*(E) \leq m^*\left(E \cap \bigcup_{i=1}^k A_i\right) + \varepsilon + m^*\left(E \cap \left(\bigcup_{i=1}^k A_i\right)^c\right) \leq m^*(E) + \varepsilon$$

Proof of 2

We have shown $m^*\left(\bigcup_i A_i\right) \leq \sum_{i=1}^{\infty} m^*(A_i)$.

Assume $m^*\left(\bigcup_i A_i\right) < +\infty$, then

$$\sum_{i=1}^k m^*(A_i) = m^*\left(\bigcup_{i=1}^k A_i\right) \leq m^*\left(\bigcup_i A_i\right) \implies \sum_{i=1}^{\infty} m^*(A_i) \leq m^*\left(\bigcup_i A_i\right)$$

January 16, 2024

Office Hours Tuesday / Thursday 10 AM - 11:30 AM

A note on notation: Latin characters are to be understood as countable indices; greek as possible uncountable.

Lebesgue Outer Measure

$A \subset \mathbb{R}$

$$m^*(A) = \inf \left\{ \sum_{i=1}^{\infty} |I_i| \mid \bigcup_{i=1}^{\infty} I_i \supset A, I_i \text{ open intervals} \right\}$$

Properties

1. $A \subset B \implies m^*(A) \leq m^*(B)$.
2. $m^*(\emptyset) = 0$.
3. $m^*(I) = |I|$ for I an interval.
4. Countable Subadditivity: $\{A_i\}_{i=1}^{\infty} \implies m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} m^*(A_i)$.
5. $\forall A \subset \mathbb{R}, \forall \varepsilon > 0, \exists$ open neighborhood $U \supseteq A$ such that $m^*(U) \leq m^*(A) + \varepsilon$.
6. $\exists G \in \bigcap_{n=1}^{\infty} U_n, U_n \text{ open}, U_n \supseteq A \implies G \supseteq A$, such that $m^*(G) = m^*(A)$.

Measurable (Caratheodory Criterion)

$\forall A \subseteq \mathbb{R}$ is Lebesgue measurable if

$$m^*(A) = m^*(E \cap A) + m^*(E \cap A^c)$$

Essentially, $m^*(E \cap A) + m^*(E \cap A^c) \leq m^*(E) \leq +\infty$.

• Propositions

1. A measurable $\implies A^c$ measurable.
2. $m^*(A) = 0 \implies A$ measurable.
3. $\{A_i\}_{i=1}^{\infty}$ countable with A_i measurable, then
 - (a) $\bigcap_{i=1}^{\infty} A_i$ are measurable.
 - (b) Moreover, $A_i \cap A_j = \emptyset \implies m^*\left(\bigcup_{i=1}^{\infty} (A_i)\right) = \sum_{i=1}^{\infty} m^*(A_i)$.
 - (c) A, B measurable $\implies A \cup B, A \cap B, A \setminus B$ measurable.
 - (d) $A \cap B = \emptyset \implies m^*(A \cup B) = m^*(A) + m^*(B)$.
 - (e) $\{A_i\}_{i=1}^{\infty}$ with A_i measurable, then $\bigcup_{i=1}^{\infty} A_i$ is measurable and $A_i \cap A_j = \emptyset \implies m^*\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m^*(A_i)$.
- Proof of e $\forall E \subset \mathbb{R}, m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) + m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right)$.

Claim: $m^*(E \cap (\bigcup_{i=1}^{\infty} A_i)) = \sum_{i=1}^{\infty} m^*(E \cap A_i)$ for $A_i \cap A_j = \emptyset$.
Then, $\forall \varepsilon > 0, \exists n \in \mathbb{N}$,

$$\begin{aligned} m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) &= \sum_{i=1}^{\infty} m^*(E \cap A_i) \leq \sum_{i=1}^n m^*(E \cap A_i) + \varepsilon \\ \implies m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) + m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right) &\leq m^*\left(E \cap \left(\bigcup_{i=1}^n A_i\right)\right) + m^*\left(E \cap \left(\bigcup_{i=1}^n A_i\right)^c\right) + \varepsilon \leq m^*(E) + \varepsilon \\ &\implies \bigcup_{i=1}^{\infty} A_i \text{ measurable} \end{aligned}$$

Proof of Claim:

Step 1: A, B measurable and $A \cap B = \emptyset$. Since A is measurable,

$$\begin{aligned} m^*(E \cap (A \cup B)) &= m^*((E \cap (A \cup B)) \cap A) + m^*((E \cap (A \cup B)) \cap A^c) \\ &= m^*(E \cap A) + m^*(E \cap A^c) \end{aligned}$$

For $\{A_i\}_{i=1}^{\infty}, \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} A'_i$ with $A_1 = A'_1$ and $A'_i = A_i \setminus \bigcup_{k=1}^{i-1} A_k, \forall i \geq 2$.
Therefore $A'_i \cap A'_j = \emptyset$ and A'_i is measurable.

$$\begin{aligned} m^*\left(\bigcup_{i=1}^n A_i\right) &\leq m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} m^*(A_i) \\ m^*\left(\bigcup_{i=1}^n A_i\right) &= \sum_{i=1}^n m^*(A_i) \leq m^*\left(\bigcup_{k=1}^{\infty} A_k\right) < +\infty \implies \sum_{i=1}^{\infty} m^*(A_i) \leq m^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{i=1}^{\infty} m^*(A_i) \end{aligned}$$

Sigma Algebra and Borel Sets

Definition: Sigma Algebra

Let $S \subset 2^X$ for some set X . Then S is said to be a σ -algebra if

1. $\emptyset \in S$.
2. $A^c \in S$ if $A \in S$.
3. $\bigcup_{i=1}^{\infty} A_i \in S$ if $A_i \in S$.

- Equivalently, $\bigcap_{i=1}^{\infty} A_i \in S$ if $A_i \in S$.

Theorem:

The collection \mathcal{L} of all Lebesgue measurable sets is a σ -algebra.

Definition: Borel Set

Let B be the σ -algebra generated by open sets of reals (i.e. the smallest σ -algebra containing all open sets of reals).
Then $b \in B$ is called a Borel set.

Remark

B is generated by $\{(a, +\infty) \mid a \in \mathbb{R}\}$.

1. $(a, +\infty)^c = (-\infty, a]$.
2. $\bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, +\infty\right) = [a, +\infty)$.
3. $[a, +\infty)^c = (-\infty, a)$.
4. $(-\infty, b) \cap (a, +\infty) = (a, b)$.
5. $(-\infty, b] \cap [a, +\infty) = [a, b]$.

Theorem:

Any Borel set is Lebesgue measurable.

Proof

It suffices to demonstrate that $(a, +\infty)$ is measurable $\forall a \in \mathbb{R}$.

$\forall E \subset \mathbb{R}$, we want to show that $m^*(E \cap (a, +\infty)) + m^*(-\infty, a] \leq m^*(E)$.

Then, $\forall \varepsilon > 0$, $\exists \mathcal{C} = \{I_i\}$ with I_i open intervals such that $\sum_{I_i \in \mathcal{C}} |I_i| \leq m^*(E) + \varepsilon/2$. Set

$$\begin{aligned}\mathcal{C}^\ell &= \{I \in \mathcal{C} \mid x < a, \forall x \in I\} \\ \mathcal{C}^r &= \{I \in \mathcal{C} \mid x > a, \forall x \in I\} \\ \mathcal{C}^m &= \{I \in \mathcal{C} \mid a \in I\} = \{I_k\}\end{aligned}$$

Then $\mathcal{AC} = \mathcal{C}^\ell \cup \mathcal{C}^r \cup \mathcal{C}^m$.

$\forall I_k \in \mathcal{C}^m = \{I_k\}$, $I_k = (c_k, d_k)$ for some $c_k, d_k \in \mathbb{R}$, define

$$\begin{aligned}I_k^\ell &= \left(c_k, a + \frac{\varepsilon}{2^{k+1}}\right) \\ I_k^r &= (a, d_k)\end{aligned}$$

Let $\mathcal{C}^m = \{I_k^\ell\} \cup \{I_k^r\} = \overline{\mathcal{C}}^{m\ell} \cup \overline{\mathcal{C}}^{mr}$. Then

$$\begin{aligned}\mathcal{C}^\ell \cup \overline{\mathcal{C}}^{m\ell} &\text{ covers } E \cap (-\infty, k] \\ \mathcal{C}^r \cup \overline{\mathcal{C}}^{mr} &\text{ covers } E \cap (k, +\infty) \\ \mathcal{C}^\ell \cup \mathcal{C}^r \cup \mathcal{C}^m &\text{ covers } E\end{aligned}$$

Observe that

$$|I_k^\ell| + |I_k^r| \leq |I_k| + \frac{\varepsilon}{2^{k+1}}$$

Therefore

$$m^*(E \cap (a, +\infty)) \leq \sum_{I \in \mathcal{C}^R + \bar{\mathcal{C}}^{mr}} |I|$$

$$m^*(E \cap [-\infty, a]) \leq \sum_{I \in \mathcal{C}^\ell + \bar{\mathcal{C}}^{m\ell}} |I|$$

Therefore

$$\begin{aligned}
m^*(E \cap (a, +\infty)) + m^*(E \cap (-\infty, a]) &\leq \sum_{I \in \mathcal{C}^r \cup \bar{\mathcal{C}}^{mr}} |I| + \sum_{I \in \mathcal{C}^\ell \cup \bar{\mathcal{C}}^{m\ell}} |I| \\
&= \sum_{I \in \mathcal{C}^r} |I| + \sum_{I \in \mathcal{C}^\ell} |I| + \sum_k (|I_k^\ell| + |I_k^r|) \\
&\leq \sum_{I \in \mathcal{C}} |I| + \sum_{k=1}^{+\infty} \frac{\varepsilon}{2^{k+1}} \\
&\leq m^*(E) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&\leq m^*(E) + \varepsilon
\end{aligned}$$

Lebesgue Measurable vs Borel

Theorem

The following statements are equivalent

1. A is measurable.
2. $\forall \varepsilon > 0, \exists U$ open, $U \supset A$ such that $m(U \setminus A) < \varepsilon$.
3. $\forall \varepsilon > 0, \exists C$ closed, $C \subset A$ such that $m(A \setminus C) < \varepsilon$.
4. $\forall A \in \mathbb{R}, \exists \bigcap_{n=1}^{\infty} U_n = F \in B, U_n$ open, $U_n \supset A$ such that $F \supset A$ and $m(F \setminus A) = 0$.
5. $\exists \{C_n\}, C_n$ closed and $C_n \subset A$ such that $G = \bigcup_{n=1}^{\infty} C_n \subset A$ and $m(A \setminus G) = 0$.

Corollary

Every measurable set is the union of a Borel set and a measure zero set.

Proof 1 Implies 2

Step 1: if $m(A) < \infty$, then for $\varepsilon > 0, \exists U$ open and $U \supset A$, then

$$m(U) \leq m(A) + \varepsilon \iff m(U \setminus A) = m(U) - m(A) \leq \varepsilon$$

Step 2: let $A_n = A \cap (-n, n), n \in \mathbb{N}$.

Then $m(A_n) \leq 2n < +\infty$.

For each $A_n, \exists U_n$ open with $U_n \supset A_n$ and $m(U_n \setminus A_n) < \frac{\varepsilon}{2^n}$

Let $U = \bigcup_{n=1}^{\infty} U_n$ and $A = \bigcup_{n=1}^{\infty} A_n$.

Now verify that

$$m(U \setminus A) = m\left(\bigcup_{n=1}^{\infty} U_n \setminus \bigcup_{n=1}^{\infty} A_n\right) = m\left(\bigcup_{n=1}^{\infty} U_n \setminus A_n\right) \leq \sum_{n=1}^{\infty} m(U_n \setminus A_n) \leq \varepsilon$$

Proof 2 Implies 3

Write

$$A \setminus C = A \cap C^c = C^c \cap A = C^c \setminus A^c$$

Apply (2).

Proof 3 Implies 4

U_n comes from 2.

Proof 4 Implies 5

Follows from 4.

Proof 5 Implies 1

$A = G \cup (A \setminus G) \implies A$ is measurable.

Example: Non-measurable Set

Define $x \sim y$ if $x - y \in \mathbb{Q}$, $\forall x, y \in \mathbb{R}$.

Let $A = \{x \in (0, 1) \mid x \text{ is a representative of each class } \mathbb{R}/\sim\} \subset (0, 1) \subset \mathbb{R}$.

Claim: A is not Lebesgue measurable.

Let $(-1, 1) \supset S = \bigcup_{r \in \mathbb{Q} \cap (0, 1)} (A + r) \supset (0, 1)$, and observe that $\mathbb{Q} \cap (0, 1)$ is countable.

So $(A + r) \cap (A + s) = \emptyset$ for $s \neq r$.

Then $1 < m(S) < 2$, so $m(A) = 0$ and $m(A) > 0$ are both contradictions.

January 18, 2024

Abstract measure theory.

Definition: Topological Space

A set X equipped with a collection of subsets $\tau \subset 2^X$ where τ is a topology if

1. $\emptyset, X \in \tau$
2. Union of subsets in τ remains in τ .
3. Intersection of finitely many subsets in τ remains in τ .

Any subset of τ is called an open set of X .

Definition: Measure Space

For a set X with $\Lambda \subset 2^X$ a σ -algebra such that

1. $\emptyset \in \Lambda$
2. $A^c \in \Lambda$ if $A \in \Lambda$.
3. $\bigcup_{i=1}^{\infty} A_i \in \Lambda$ if $A_i \in \Lambda$.
4. Remark: Borel Sigma Algebra

The σ -algebra generated by τ for a topological space (X, τ) .

The measure space (X, Λ, μ) , $\Lambda \subset 2^X$ a σ -algebra equipped with set function $\mu : \Lambda \rightarrow [0, +\infty]$ such that

1. $\mu(\emptyset) = 0$
2. $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$ for $A_i \in \Lambda$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$ (countable additivity).

Proposition: Monotonicity

$$A, B \in \Lambda, A \subseteq B \implies \mu(A) \leq \mu(B).$$

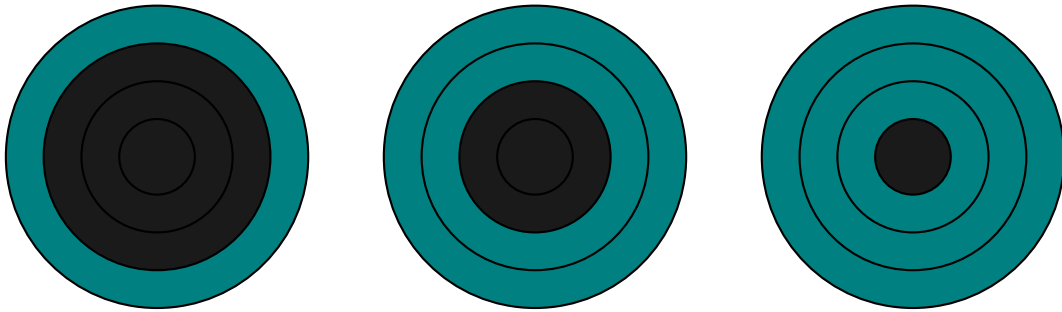
Proposition: Countable Subadditivity

$$\mu\left(\bigcup A_i\right) \leq \sum \mu(A_i) \text{ if } A_i \in \Lambda$$

Proposition: Monotone Convergence

Given $A_i \in \Lambda$ such that $A_i \subset A_{i+1}$ where $A = \bigcup_{i=1}^{\infty} A_i$, then $\mu(A_i) \rightarrow \mu(A)$.

Similarly, if $A_i \supset A_{i+1}$ such that $A = \bigcap_{i=1}^{\infty} A_i$, then $\mu(A_i) \rightarrow \mu(A)$ if $\mu(A_k) < +\infty$ for some $k = 1, 2, 3, \dots$



$$\text{Given } A'_i = \begin{cases} A_1 & i = 1 \\ A_i \setminus \bigcup_{j=1}^{i-1} A_j & i > 1 \end{cases}, \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} A'_i \text{ and}$$

$$\mu(A) \sum_{i=1}^{\infty} A'_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(A'_i)$$

and

$$\sum_{i=1}^n \mu(A'_i) = \mu(A_1) + (\mu(A_2) - \mu(A_1)) + (\mu(A_3) - \mu(A_2)) + \dots + (\mu(A_n) - \mu(A_{n-1})) = \mu(A_n)$$

Similarly, $A_1 \setminus A = \bigcup_{i=2}^{\infty} (A_i \setminus A_{i-1})$ where $\mu(A_1) < +\infty$ gives

$$\mu(A_1) - \mu(A) = \mu(A_1) + \sum_{i=2}^{\infty} (\mu(A_i) - \mu(A_{i-1})) = \lim_{n \rightarrow \infty} \mu(A_n)$$

Definition: Complete Measure Space

A measure space (X, Λ, μ) is complete if $\forall A \in \Lambda$ with $\mu(A) = 0$, then $\forall B \subset A$ and $B \in \Lambda$.

Example

The Lebesgue measure space on the reals $(\mathbb{R}, \mathcal{L}, m)$ is complete.

Theorem: Completion of a Measure Space

Given a measure space (X, Λ, μ) , then there exists $(X, \overline{\Lambda}, \overline{\mu})$ such that

1. $\Lambda \subset \overline{\Lambda}$.
2. If $A \in \Lambda$, then $\overline{\mu}(A) = \mu(A)$.
3. $(X, \overline{\Lambda}, \overline{\mu})$ is complete.

Proof (Construction)

Let $\overline{\Lambda} = \{A \cup Z \mid A \in \Lambda, \exists D \in \Lambda, \mu(D) = 0, Z \subset D\}$ and $\overline{\mu}(A \cup Z) := \mu(A)$.

Verify:

1. $\overline{\Lambda}$ is a σ -Algebra.
 - (a) If $A \cup Z \in \overline{\Lambda}$, then $(A \cup Z)^c \in \overline{\Lambda}$.
 - (b) If $A_i \cup Z_i \in \overline{\Lambda}$, then $\bigcup (A_i \cup Z_i) \in \overline{\Lambda}$.

2. $\overline{\mu}$ is a well-defined measure on $\overline{\Lambda}$.

3. $(X, \overline{\Lambda}, \overline{\mu})$ is complete.

• Proof of 1

Given $A \in \Lambda$ and $Z \subset D$ where $\mu(D) = 0$ and $D \in \Lambda$, we know $D^c \subset Z^c$ and $Z^c = D^c \cup (Z^c \cap D)$. Therefore

$$(A \cup Z)^c = A^c \cap Z^c = A^c \cap (D^c \cup (Z^c \cap D)) = (A^c \cap D^c) \cup (A^c \cap Z^c \cap D) \in \overline{\Lambda}$$

Since $A^c \cap D^c \in \Lambda$ and $A^c \cap Z^c \cap D \in D$

Since $\bigcup A_i \in \Lambda$ and $\bigcup Z_i \subset \bigcup D_i$,

$$\bigcup_{i=1}^{\infty} (A_i \cup Z_i) = \left(\bigcup_{i=1}^{\infty} A_i \right) \cup \left(\bigcup_{i=1}^{\infty} Z_i \right) \in \overline{\Lambda}$$

- Proof of 2

Given $A_1 \cup Z_1 = A_2 \cup Z_2$, $A_1 \subset A_2 \cup Z_2 \subset A_2 \cup D_2$ implies $\mu(A_1) \leq \mu(A_2)$.

Then, $\mu(A_2) \leq \mu(A_1) \implies \mu(A_1) = \mu(A_2)$. So $\bar{\mu}$ is well defined.

Given $\{A_i \cup Z_i\}$ with $(A_i \cup Z_i) \cap (A_j \cup Z_j) = \emptyset$ for all $i \neq j$,

$$\bar{\mu}\left(\bigcup_{i=1}^{\infty} (A_i \cup Z_i)\right) = \bar{\mu}\left(\left(\bigcup_{i=1}^{\infty} A_i\right) \cup \bigcup_{i=1}^{\infty} Z_i\right) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty} \bar{\mu}(A_i \cup Z_i)$$

So $\bar{\mu}$ is countably additive and therefore a measure.

Borel Measure and Radon Measure

Given a measure space (X, Λ, μ) and an underlying topology (X, τ) ,

Definition: Borel Measure

μ is a Borel measure if all borel sets $\tau \subset \Lambda$.

Definition: Locally Finite Measure

μ is locally finite if $\forall x \in X, \exists U \subset X$ a neighborhood such that $\mu(U) < +\infty$.

Definition: Borel Regularity

μ is Borel regular if $\forall A \in \Lambda, \exists B$ a Borel set such that $B \supseteq A$ and $\mu(B) = \mu(A)$.

Definition: Radon Measure

μ is a Radon measure if

1. it is a Borel measure.
2. $\mu(K) \leq +\infty$ for K compact.
3. $\mu(V) = \sup\{\mu(K) \mid K \subset V, K \text{ compact}\}$, V open.
4. $\mu(A) = \inf\{\mu(V) \mid A \subset V, V \text{ open}\}$, $\forall A \in \Lambda$.

- Example 1

Lebesgue measure.

- Example 2

Point charge: $\mu(\{x\}) = 1$ and $\mu(A) = 0$ if $x \notin A$.

Theorem:

Let (X, Λ, μ) be a Borel regular measure space where the underlying topology (X, τ) is a metric space. Then

1. For $A \in \Lambda$ with $\mu(A) < +\infty$, $\forall \varepsilon > 0$, $\exists C \subseteq A$ closed such that $\mu(A \setminus C) < \varepsilon$.
2. For $A \in \Lambda$, $\exists \{V_i\}$ open sets such that $A \subset \bigcup_{i=1}^{\infty} V_i$ and $\mu(V_i) < +\infty$. Then $\forall \varepsilon > 0$, $\exists U$ open with $A \subset U$ and $\mu(U \setminus A) < \varepsilon$.

Proof

Given $\mu(A) < +\infty$, $\nu(B) = \mu(B \cap A) < +\infty$, $\forall B \in \Lambda$ and (X, Λ, ν) .

Let $F = \{B \in \Lambda \mid \forall \varepsilon > 0, \exists C \subset B \text{ closed, with } \nu(B \setminus C) < \varepsilon\}$.

Note that closed sets are in F .

Claim 1: the Borel σ -algebra is in F .

Claim 2: if $A_i \in F$, $\bigcup A_i, \bigcap A_i \in F$.

Given claim 2, $\forall U$ open, U^c is closed. Then $U_\varepsilon = \{x \in U \mid \text{dist}(x, U^c) \leq \varepsilon\}$ is closed and, therefore, $U = \bigcup_{i=1}^{\infty} U_{1/i}$.

So, given $A_i \in F$, $\exists C_i \subset A_i$ closed where $\nu(A_i \setminus C_i) < \varepsilon/2^{i+1}$. We want to show that $\nu(\bigcap A_i \setminus \bigcap C_i) < \varepsilon$.

Then, for $x \in \bigcap A_i \setminus \bigcap C_i$, $x \in A_i$ for all i and $x \notin C_{i_0}$ for some i_0 .

Therefore $x \in A_{i_0}$, $x \notin C_{i_0}$, and $x \in A_{i_0} \setminus C_{i_0}$. It follows that

$$\begin{aligned} \bigcap_{i=1}^{\infty} A_i \setminus \bigcap_{i=1}^{\infty} C_i &\subset \bigcup_{i=1}^{\infty} (A_i \setminus C_i) \\ \nu\left(\bigcap_{i=1}^{\infty} A_i \setminus \bigcap_{i=1}^{\infty} C_i\right) &\leq \sum_{i=1}^{\infty} \nu(A_i \setminus C_i) < \varepsilon \end{aligned}$$

Therefore

$$\nu\left(\bigcup_{i=1}^{\infty} A_i \setminus \bigcup_{i=1}^n C_i\right) \rightarrow \nu\left(\bigcup_{i=1}^{\infty} A_i \setminus \bigcup_{i=1}^{\infty} C_i\right) \leq \nu\left(\bigcup_{i=1}^{\infty} (A_i \setminus C_i)\right) < \frac{\varepsilon}{2}$$

so $\exists N \gg 1$ such that $\nu\left(\bigcup_{i=1}^{\infty} \setminus \bigcup_{i=1}^N C_i\right) < \varepsilon$ with $\bigcup_{i=1}^N C_i$ closed.

Restatement

For A Borel,

$$\varepsilon > \nu(A \setminus C) = \mu((A \setminus C) \cap A) = \mu(A \setminus C)$$

January 23, 2024

Review - Abstract Measure

Given (X, Λ, μ) where $\Lambda \subseteq 2^X$ is a σ -algebra, $\mu : \Lambda \rightarrow [0, +\infty]$

1. $\mu(\emptyset) = 0$.
2. $m(\bigcup A_i) = \sum \mu(A_i)$, $A_i \cap A_j = \emptyset$.

Properties of a Measure

Monotonicity

$$\mu(A) \leq \mu(B), A, B \in \Lambda, A \subseteq B$$

Countable Subadditivity

$$\mu\left(\bigcup A_i\right) \leq \sum \mu(A_i)$$

Monotone Convergence

$$\begin{aligned} A_i \subset A_{i+1}, A_i &\rightarrow \bigcup A_i \implies \mu(A) = \mu\left(\bigcup A_i\right). \\ A_i \supset A_{i+1}, A_i &\rightarrow \bigcap A_i \implies \mu(A_i) \rightarrow \mu\left(\bigcap A_i\right) \text{ if } \mu(A_1) < \infty \end{aligned}$$

- Example

$$A_n = (n, +\infty) \text{ gives } \bigcap A_n = \emptyset$$

Completeness of a Measure

(X, Λ, μ) is complete if $\forall A \in \Lambda$ with $\mu(A) = 0$, then $\forall B \in \Lambda$ if $B \subseteq A$.

Theorem:

Given (X, Λ, μ) , there exists $(X, \bar{\Lambda}, \bar{\mu})$ such that $\Lambda \subset \bar{\Lambda}$ and $\bar{\mu}(A) = \mu(A)$ if $A \in \Lambda$.

$$\bar{\Lambda} = \{A \cup Z \mid A \in \Lambda, Z \subset D, D \in \Lambda \text{ with } \mu(D) = 0\}$$

$$\bar{\mu}(A \cup Z) = \mu(A)$$

$(X, \bar{\Lambda}, \bar{\mu})$ is complete.

Measure Space with Topology

Given a topological space (X, τ) , a measure space (X, Λ, μ)

Definition: Locally Finite

The measure μ is locally finite if $\forall x \in X$, there exists an open neighborhood U of x such that $U \in \Lambda$ and $\mu(U) < +\infty$.

Definition: Borel Measure

μ is a Borel measure if the Borel σ -algebra generated by τ , \mathcal{B} , is a subset of Λ .

Definition: Borel Regular

$$\forall A \in \Lambda, \exists B \in \mathcal{B} \text{ such that } B \supset A \text{ and } \mu(B) = \mu(A).$$

Definition: Radon Measure

1. Borel.
2. $\mu(K) < +\infty$ for K compact.
3. $\mu(V) = \sup\{\mu(K) \mid K \text{ compact}, K \subset V\}$, $\forall V$ open.
4. $\mu(A) = \inf\{\mu(V) \mid V \text{ open}, A \subset V\}$, $\forall A \in \Lambda$.

Theorem:

If X is a metric space equipped with a Borel regular (X, Λ, μ) , then

1. $\forall A \in \Lambda, \mu(A) < +\infty, \forall \varepsilon > 0, \exists C$ closed where $C \subset A$ and $\mu(C \setminus A) < \varepsilon$.
2. If $\exists \{V_i\}$, V_i open and $\mu(V_i) < +\infty$, and $A \in \Lambda$ with $A \subset \bigcup V_i$, then $\exists U$ open such that $A \subset U$ and $\mu(U \setminus A) < \varepsilon$.

Proof of 1

Define $\nu(B) = \mu(B \cap A)$ such that (X, Λ, ν) is a new measure space.

Define $F = \{B \in \Lambda \mid \forall \varepsilon > 0, \exists C \text{ closed}, C \subset B, \nu(B \setminus C) < \varepsilon\}$, all closed sets in F .

Claim 1: $\bigcap A_i, \bigcup A_i \in F$ if $A_i \in F$.

Claim 2: U is open.

$U = \bigcup U_i, U_i = \{x \in U \mid \text{dist}(x, U^c) \leq \frac{1}{i}\}$, therefore $B \subset F$.

IMAGE HERE - 1

If A is Borel, then $\forall \varepsilon > 0, \exists C$ closed with $C \subset A$ and $\mu(A \setminus C) < \varepsilon$.

To finish, $\forall A \in \Lambda$ by Borel Regularity of μ , $\exists B \in \mathcal{B}$ such that $B \supset A$ and $\mu(B) = \mu(A)$.

Note also that this requires $\mu(B \setminus A) = 0$ since $\mu(A) < +\infty$.

IMAGE HERE - 2

Then $B \setminus A \in \Lambda$, $\exists D \in \mathcal{B}$ such that $DB \setminus A$ and $\mu(D) = \mu(B \setminus A) = 0$. Then

$$\begin{aligned} B \cap A^c &= B \setminus A \subset D \\ (B \cap A^c)^c &\supset D^c \\ B \cap (B^c \cup A) &\supset D^c \cap B \\ A &\supset B \setminus D \end{aligned}$$

$$A \setminus (B \setminus D) = A \cap (B \cap D^c)^c = A \cap (B^c \cup D) = \overbrace{(A \cap B^c)}^{\emptyset} \cup A \cap D = A \cap D \subset D$$

Therefore $B \setminus D \subset A$, and $\mu(A \setminus (B \setminus D)) = 0$.

$B \setminus D \in \mathcal{B}$, $\forall \varepsilon > 0, \exists C$ closed such that $C \subset B \setminus D \subset A$, $\mu((B \setminus D) \setminus C) < \varepsilon$.

This implies that $\mu(A \setminus C) = \mu(A \setminus (B \setminus D)) + \mu((B \setminus D) \setminus C) < \varepsilon$.

Proof of 2

Consider $V_i \setminus A$ where $\mu(V_i \setminus A) \leq \mu(V_i) < +\infty$.

By (1), $\exists C_i$ closed with $C_i \subset V_i \setminus A$ and $\mu((V_i \setminus A) \setminus C_i) < \varepsilon/2^{i+1}$. Write

$$(V_i \setminus A) \setminus C_i = (V_i \setminus A) \cap C_i^c = V_i \cap A^c \cap C_i^c = (V_i \cap C_i^c) \cap A^c = (V_i \setminus C_i) \setminus A$$

Note that $V_i \setminus C_i$ is open, since C_i is closed.

Define $U = \bigcup (V_i \setminus C_i) \supset A$. Then,

$$U \setminus A = \left(\bigcup (V_i \setminus C_i) \right) \setminus A = \bigcup ((V_i \setminus C_i) \setminus A)$$

Therefore $\mu(U \setminus A) \leq \varepsilon \sum_{i=1}^{\infty} \frac{1}{2^{i+1}} = \varepsilon$.

Remark

$X = \bigcup V_i$, V_i open and $\mu(V_i) < +\infty$.

Then $\forall A \in \Lambda$, $\forall \varepsilon > 0$, $\exists U$ open such that $U \supset A$ and $\mu(U \setminus A) < \varepsilon$.

For A^c , $\exists U \supset A^c$ ($\implies U^c \subset A$), $\mu(U \setminus A^c) < \varepsilon$. So

$$U \cap A = U \setminus A^c = A \setminus U^c = A \cap U$$

and $\mu(A \setminus U^c) < \varepsilon$, $U^c \subset A$ with U^c closed.

Corollary

For \mathbb{R}^n , a measure is Radon if and only if it is locally finite and Borel regular.

- Proof

(\implies)

Let $B(r, x_0) = \{x \in \mathbb{R}^n \mid |x - x_0| < r\}$ and $\overline{B(r, x_0)} = \{x \in \mathbb{R}^n \mid |x - x_0| \leq r, \text{ compact}\}$.

Then $\mu(B(r, x_0)) \leq \mu(\overline{B(r, x_0)}) < +\infty$. So μ is locally finite.

For $A \in \Lambda$, we may assume without loss of generality that $\mu(A) < +\infty$.

Then $\forall i, \exists U_i$ open where $U_i \supset A$ and $\mu(A) \leq \mu(U_i) \leq \mu(A) + \frac{1}{i} < +\infty$.

Set $G = \bigcap U_i \in \mathcal{B}$, then $\mu(G) = \mu(A)$.

(\impliedby)

1. Borel regular implies Borel.

2. For K compact, $\forall x \in K \ni U_x$ open where $\mu(U_x) < +\infty$.

$\{U_\lambda\}_{\lambda \in k}$ is an open cover. Therefore there is a finite subcover $\{U_{\lambda_i}\}_{i=1}^\lambda$ where

$$\mu(K) \leq \mu\left(\bigcup_{i=1}^k U_{\lambda_i}\right) \leq \sum_{i=1}^k \mu(U_{\lambda_i}) < +\infty$$

3. $\forall V$ open, $B(i) = B(i, 0)$, $V \cap B(i)$, $\mu(V \cap B(i)) < +\infty$, $\exists C_i$ closed where $C_i \subset V \cap B(i)$ so C_i is bounded and therefore compact.

So $\mu(C_i) \leq \mu((V \cap B(i)) \setminus C_i) < \frac{1}{i}$ and $\mu(V \cap B(i)) \leq \mu(C_i) + \frac{1}{i}$.

Then $\mu(V) = \lim_{i \rightarrow \infty} \mu(V \cap B(i)) = \lim_{i \rightarrow \infty} \mu(C_i)$, and $C_i \subset V \cap B(i) \subset V$ compact.

Therefore $\mu(V) = \sup\{\mu(K) \mid K \text{ compact}, K \subset V\}$.

4. $\forall A \in \Lambda$, $\forall i, \exists U_i$ open where $U_i \supset A$ and $\mu(U_i \setminus A) < \frac{1}{i}$

This implies that $\mu(A) \leq \mu(U_i) \leq \mu(A) + \frac{1}{i}$ and therefore $\mu(A) = \inf\{\mu(U) \mid U \supset A, U \text{ open}\}$.

Caratheodory Construction

Definition: Outer Measure

$\mu^*(A), \forall A \in 2^X$

1. $\mu^*(\emptyset) = 0$.
2. $\mu^*(A) \leq \mu^*(B)$ if $A \subseteq B$.
3. $\mu^*\left(\bigcup A_i\right) \leq \sum \mu^*(A_i), \forall A_i \in 2^X$ (countable subadditivity)

Define $\Lambda = \{A \in 2^X \mid \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c), \forall E \in 2^X\}$.

Then $\mu(A) = \mu^*(A)$ if $A \in \Lambda$.

(X, Λ, μ) is complete.

January 25, 2024

Theorem: Caratheodory Construction

Outer Measure

$$\mu^* : 2^X \rightarrow [0, +\infty].$$

1. $\mu^*(\emptyset) = 0$
2. Monotonicity: $\mu^*(A) \leq \mu^*(B), A \subseteq B$
3. Countable Subadditivity: $\mu^*\left(\bigcup_i A_i\right) \leq \sum_i \mu^*(A_i).$

Caratheodory Criterion

$A \subset X$ is measurable if $\forall E \in X,$

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

Theorem

The collection Λ of all measurable sets is a σ -algebra.

(X, Λ, μ) is a complete measure space (cf. proof of Lebesgue completeness).

Hausdorff Measure

$\forall A \subseteq \mathbb{R}^n, \forall s \geq 0, H_s^\delta(A) = \inf \left\{ \sum_i (d(E_i))^s \mid \bigcup_i E_i \supset A, d(E_i) \leq \delta \right\}$ where $d(E_i)$ is the diameter of E_i .

Notice that $H_s^{\delta_1}(A) \leq H_s^{\delta_2}(A)$ if $\delta_2 \leq \delta_1$.

Let $H_s^*(A) = \lim_{\delta \rightarrow 0} H_s^\delta(A), \forall A \in 2^{\mathbb{R}^n}$.

Claim: H_s^* is an outer measure.

- Verify

1. $H_s^*(\emptyset) = 0.$
2. $H_s^*(A) \leq H_s^*(B), \forall A \subseteq B \subseteq \mathbb{R}^n.$
3. Given $A_i \subset \mathbb{R}^N,$

$\exists \delta_0 > 0$ such that $\forall \delta < \delta_0, H_s^*\left(\bigcup_i A_i\right) \leq H_s^\delta\left(\bigcup_i A_i\right) + \frac{\varepsilon}{2}.$

Then $\forall \delta < \delta_0$ fixed, $\forall A_i, \exists \{E_i^j\}$ such that $\bigcup_j E_i^j \supset A_i, \sum_j (d(E_i^j))^s \leq H_s^\delta(A_i) + \frac{\varepsilon}{2^{i+1}},$ and $d(E_i^j) \leq \delta.$ So

$$\begin{aligned}
H_s^\delta \left(\bigcup_i A_i \right) &\leq \sum_{i,j} (d(E_i^j))^s \\
&= \sum_i \left(\sum_j (d(E_i^j))^s \right) \\
&= \sum_i \left(H_s^\delta(A_i) + \frac{\varepsilon}{2^{i+1}} \right) \\
&= \sum_i H_s^\delta(A_i) + \frac{\varepsilon}{2}
\end{aligned}$$

and

$$H_s^* \left(\bigcup_i A_i \right) \leq \sum_i H_s^\delta(A_i) + \varepsilon \leq \sum_i H_s^*(A_i) + \varepsilon, \quad \forall \varepsilon > 0.$$

Then, since H_s^* is an outer measure, it is a measure by the Caratheodory construction.

Definition: Hausdorff Measure

The Hausdroff Measure $H_s : \Lambda \rightarrow [0, +\infty)$ on a σ -algebra $\Lambda \subset 2^{\mathbb{R}^n}$.

Not Locally Finite

Consider $B(0, 1) = \{x \mid |x| < 1\}$.

Then $H_s(B(0, 1)) = \infty$ for $s < n$.

That is, the Hausdorff measure is not locally finite for $s < n$.

Complete

The Hausdorff measure, by the Caratheodory construction, is complete.

Symmetry

1. Translation Invariance: $H_s(A + x) = H_s(A)$.
2. Rotation Invariance: $H_s(RA) = H_s(A)$.
3. Scaling: $H_s(\lambda A) = \lambda^s H_s(A)$.

Open Balls Measurable

What about $B(0, 1) \subset \mathbb{R}^n$. For $\delta > 0$,

$$H_s^*(E \cap B(0, 1)) + H_s^*(E \cap B(0, 1)^c) \leq H_s^*(E \cap B(0, 1 - \delta)) + H_s^*(E \cap (B(0, 1) \setminus B(0, 1 - \delta))) + H_s^*(E \cap B(0, 1)^c)$$

Want to show that for all $\varepsilon > 0$, this is $\leq H_s^*(E) + \varepsilon$.

- Lemma 1

$$\begin{aligned}
H_s^*(E \cap B(0, 1 - \delta)) + H_s^*(E \cap B(0, 1)^c) &= H_s^*(E \cap (B(0, 1 - \delta) \cup B(0, 1)^c)) \\
&\leq H_s^*(E)
\end{aligned}$$

- Lemma 2

$$H_s^*(E \cap (B(0, 1) \setminus B(0, 1 - \delta))) < \varepsilon.$$

- Lemma 1'

If $A, B \subset \mathbb{R}^n$, $\text{dist}(A, B) > 0$, then $H_s^*(A \cup B) = H_s^*(A) + H_s^*(B)$.
 Since $\{E_i\}$ covering $A \cup B$, $d(E_i) < \frac{1}{4}\text{dist}(A, B)$ gives

$$\delta < \frac{1}{4}\text{dist}(A, B) \iff \{E_j^A\} \cup \{E_k^B\}$$

if and only if $\{E_j^A\}$ covers A and $\{E_k^B\}$ covers B . Therefore,

$$\begin{aligned} \sum_i (d(E_i))^s &= \sum_j (d(E_j^A))^s + \sum_k (d(E_k^B))^s \\ \inf \left\{ \sum_i (d(E_i))^s \right\} &= \inf \left\{ \sum_j (d(E_j^A))^s \right\} + \inf \left\{ \sum_k (d(E_k^B))^s \right\} \end{aligned}$$

and $H_s^\delta(A \cup B) = H_s^\delta(A) + H_s^\delta(B)$.
 Thus $H_s^*(A \cup B) = H_s^*(A) + H_s^*(B)$.

Let $T_i = E \cap \left(B\left(0, 1 - \frac{1}{i+1}\right) \setminus B\left(0, 1 - \frac{1}{i}\right) \right)$.
 IMAGE HERE - 1 CONCENTRIC RINGS

We want to show that $H_s^*(E \cap (B(0, 1) \setminus B(0, \frac{1}{i}))) < \varepsilon$ for $i \gg 1$. Then

$$\begin{aligned} \bigcup_{k=1} T_k &= (B(0, 1) \setminus \{0\}) \cap E \\ \bigcup_{k=i} T_k &= \left(B(0, 1) \setminus B\left(0, 1 - \frac{1}{i}\right) \right) \cap E \end{aligned}$$

Claim: $\sum_i H_s^*(T_i) < +\infty$. It suffices to prove this claim.

$$\sum_{i \text{ even}}^{2k} H_s^*(T_i) = H_s^*\left(\bigcup_{i \text{ even}}^{2k}\right) \leq H_s^*(E) < +\infty$$

$$\sum_{i \text{ odd}}^{2k+1} H_s^*(T_i) = H_s^*\left(\bigcup_{i \text{ odd}}^{2k+1}\right) \leq H_s^*(E) < +\infty$$

Then $\sum_i^k H_s^*(T_i) < \infty$.

Borel

Take a countable, dense set $\{q_i\} \subset \mathbb{R}^n$ and $\left\{B\left(q_i, \frac{1}{k}\right)\right\}_{i,k}$.

Claim: $\forall V \subseteq \mathbb{R}^n$ open, then $V = \bigcup_l B\left(q_{i_l}, \frac{1}{k_l}\right)$.

Then $\mathcal{B} \subseteq \Lambda$ and the Hausdorff measure is Borel.

Borel Regular

$\forall A \subset \Lambda, \exists B \in \mathcal{B}$ such that $B \supset A$ and $H_s(B) = H_s(A)$.

$\forall \delta = \frac{1}{j}, \{E_i^j\}$ closed balls with $d(E_i^j) < \frac{1}{j}$,

$$\sum_i (d(E_i))^s \leq H_s^{\frac{1}{j}}(A) + \frac{1}{j}$$

Take $B = \bigcap_j \left(\bigcup_i E_i^j \right) \in \mathcal{B}$ since $B = \bigcap_j \bigcup_i E_i^j \supset A$. Then

$$\begin{aligned} H_s^{\frac{1}{j}}(B) &\leq H_s^{\frac{1}{j}}\left(\bigcup_i E_i^j\right) \\ &\leq \sum_i H_s^{\frac{1}{j}}(E_i^j) \\ &\leq \sum_i (d(E_i^j))^s \\ &\leq H_s^{\frac{1}{j}}(A) + \frac{1}{j} \end{aligned}$$

and in the limit as $j \rightarrow \infty$

$$H_s^*(A) \leq H_s^*(B) \leq H_s^*(A)$$

Fractional or Hausdorff Dimension

Theorem:

1. $H_s^*(A) < +\infty \implies H_t^*(A) = 0, \forall t > s \geq 0.$
2. $H_t^s > 0 \implies H_s(A) = \infty, \forall 0 \leq s < t$

Proof

$$\begin{aligned} H_s^\delta(A) &\sim \sum_i (d(E_i))^s \\ &= \sum_i (d(E_i))^t (d(E_i))^{s-t} \end{aligned}$$

So $s < t$ gives $\geq \delta^{s-t}$.

In the other direction, when $s < t$

$$\begin{aligned} \sum_i (d(E_i))^t &= \sum_i (d(E_i))^s (d(E_i))^{t-s} \\ &\leq \delta^{t-s} \sum_i (d(E_i))^s \end{aligned}$$

Definition: Hausdorff Dimension

Given $A \subset \mathbb{R}^n$,

$$\begin{aligned}
\dim_H(A) &= \sup \{s \mid H_s^*(A) = \infty\} \\
&= \sup \{s \mid H_s^*(A) > 0\} \\
&= \inf \{s \mid H_s^*(A) = 0\} \\
&= \inf \{s \mid H_s^*(A) < +\infty\}
\end{aligned}$$

Example 1

\mathbb{R}^n has n Hausdorff dimension.

Consider the n -cube with sides d , $C(d)$. Then

$$H_s(C(d)) = C(n, s)d^s$$

So $C(n, s) = C(n, s)2^{nk} \frac{1}{(2^k)^s} = C(n, s)2^{(n-1)k}$.

If $s < n$, this tends to infinity as $k \rightarrow \infty$.

Is $s > n$ it tends to 0.

Example 2

Cantor set has Hausdorff dimension $\frac{\log(2)}{\log(3)}$.

$$\bigcup_{k=1}^{2^n} C_n^k = \frac{\log(2)}{\log(3)}$$

where $|C_n^k| = \frac{1}{3^n}$, so $H_s^\delta(C^n) \sim \frac{2^n}{(3^n)^s} = \left(\frac{2}{3^s}\right)^n$.

Example 3

The Koch snowflake has dimension $\frac{\log(4)}{\log(3)}$.

January 30, 2024

Lemma:

Given a measure space (X, Λ, μ) and an extended real-valued function $f : X \rightarrow [-\infty, +\infty]$, the following are equivalent

1. $\forall \alpha \in \mathbb{R}, \{x \in X \mid f(x) > \alpha\} \in \Lambda$.
2. $\forall \alpha \in \mathbb{R}, \{x \in X \mid f(x) \geq \alpha\} \in \Lambda$.
3. $\forall \alpha \in \mathbb{R}, \{x \in X \mid f(x) < \alpha\} \in \Lambda$.
4. $\forall \alpha \in \mathbb{R}, \{x \in X \mid f(x) \leq \alpha\} \in \Lambda$.
5. $\forall U \subset \mathbb{R}$ open, $f^{-1}(U) \in \Lambda$ and $f^{-1}(\pm\infty) \in \Lambda$.

Proof 1 Implies 2

$$\{x \in X \mid f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \left\{x \in X \mid f(x) > \alpha - \frac{1}{n}\right\}.$$

Proof 2 Implies 3

$$\{x \in X \mid f(x) < \alpha\} = \{x \in X \mid f(x) \geq \alpha\}^c$$

Proof 3 Implies 4

$$\{x \in X \mid f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty} \left\{x \in X \mid f(x) < \alpha + \frac{1}{n}\right\}$$

Proof 4 Implies 1

$$\{x \in X \mid f(x) > \alpha\} = \{x \in X \mid f(x) \leq \alpha\}^c$$

Proof of 5

$\forall U \subset \mathbb{R}$ open, $V = \bigcup_i I_i$ disjoint open intervals.

Therefore $f^{-1}((a, b)) = \{x \in X \mid f(x) > a\} \cap \{x \in X \mid f(x) < b\}$.

Similarly, $f^{-1}(-\infty) = \bigcap_n \{x \in X \mid f(x) < -n\}$ and $f^{-1}(\infty) = \bigcap_n \{x \in X \mid f(x) > n\}$.

Proof 5 Implies 1

$$\{x \in X \mid f(x) > \alpha\} = f^{-1}((\alpha, +\infty)) \cup f^{-1}(+\infty) \in \Lambda.$$

Definition: Measurable Function

For a measure space (X, Λ, μ) , an extended real-valued function $f : X \rightarrow [-\infty, +\infty]$ is said to be measurable if one or all of (1)-(5) hold.

Remark:

If (X, Λ, μ) is Borel, then continuous functions are always measurable.

Remark:

The characteristic function

$$\chi_A = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

is measurable if $A \in \Lambda$.

Definition: Simple Functions

The function ϕ is simple if

$$\phi(x) = \sum_{i=1}^k \lambda_i \chi_{A_i}, \quad \lambda_i \in \mathbb{R}, \quad A_i \in \Lambda$$

Proposition:

Given a measure space (X, Λ, μ) and measurable, real-valued f, g ,

- $f \pm g$ is measurable.

$$\{x \in X \mid f(x) + g(x) < \alpha\} = \bigcup_{r \in \mathbb{Q}} (\{x \in X \mid f(x) < r\} \cup \{x \in X \mid g(x) < \alpha - r\}).$$

- f^2 is measurable

$$\forall \alpha \geq 0, \{x \in X \mid f^2(x) < \alpha\} = \{x \in X \mid f(x) < \sqrt{\alpha}\} \cap \{x \in X \mid f(x) > -\sqrt{\alpha}\}.$$

- $f \cdot g$ is measurable

$$f(x) \cdot g(x) = \frac{1}{2} \left((f+g)^2 - f^2 - g^2 \right).$$

Definition: Almost Everywhere Equality

Measurable functions f and g on the space (X, Λ, μ) are the same almost everywhere with respect to μ (written μ -a.e.) if

$$\mu(\{x \in X \mid f(x) \neq g(x)\}) = 0$$

Propositon:

For a complete measure space (X, Λ, μ) , if f and g are equal μ -a.e., then f is measurable if and only if g is measurable.

Proof

$$\begin{aligned} \{x \in X \mid f(x) > \alpha\} &= (\{x \in X \mid f(x) > \alpha\} \cap \{x \in X \mid f(x) = g(x)\}) \cup \{x \in X \mid f(x) > \alpha\} \cap \underbrace{\{x \in X \mid f(x) \neq g(x)\}}_{\mu=0} \\ &= (\{x \in X \mid g(x) > \alpha\} \cap \{x \in X \mid f(x) = g(x)\}) \cup \{x \in X \mid f(x) > \alpha\} \cap \underbrace{\{x \in X \mid f(x) \neq g(x)\}}_{\mu=0} \end{aligned}$$

Proppsotion:

Given $\{f_k(x)\}$ measurable.

1. $g_n(x) = \sup\{f_1(x), f_2(x), \dots, f_n(x)\}$ and $h_n(x) = \inf\{f_1(x), f_2(x), \dots, f_n(x)\}$ measurable.
2. $g(x) = \sup\{f_n(x)\}$ and $h(x) = \inf\{f_n(x)\}$ measurable.
3. $\limsup_{n \rightarrow +\infty} f_n(x) = \inf_n \sup\{f_n(x), f_{n+1}(x), \dots\}$ and $\liminf_{n \rightarrow +\infty} f_n(x) = \sup_n \inf\{f_n(x), f_{n+1}(x), \dots\}$ measurable.
4. $f_n(x) \rightarrow f(x)$ pointwise $\implies f$ measurable.

Proof of A

$$\begin{aligned} \{x \in X \mid g_n(x) > \alpha\} &= \bigcup_{k=1}^n \{x \in X \mid f_k(x) > \alpha\} \\ \{x \in X \mid h_n(x) < \alpha\} &= \bigcup_{k=1}^n \{x \in X \mid f_k(x) < \alpha\} \end{aligned}$$

Proof of B

$$\begin{aligned} \{x \in X \mid g(x) > \alpha\} &= \bigcup_n \{x \in X \mid f_n(x) > \alpha\} \\ \{x \in X \mid h(x) < \alpha\} &= \bigcup_n \{x \in X \mid f_n(x) < \alpha\} \end{aligned}$$

Definition: Almost Everywhere Convergence

For $f_n(x)$ measurable, $f_n(x) \rightarrow f(x)$ μ -a.e. in X if $f_n(x) \rightarrow f(x)$ in $A \subset X$ pointwise where $\mu(X \setminus A) = 0$.

Proposition:

On a complete measure space (X, Λ, μ) with f_n measurable and $f_n(x) \rightarrow f(x)$ μ -a.e. in X , $f(x)$ is measurable.

Proof

$f_n(x) \rightarrow f(x)$ pointwise in A and $\mu(A^c) = 0$.

$\{x \in X \mid f(x) > \alpha\} = (\{x \in X \mid f(x) > \alpha\} \cap A) \cup (\{x \in X \mid f(x) > \alpha\} \cap A^c)$.

Theorem:

With (X, Λ, μ) a measure space and f measurable, there exist simple functions ϕ_n such that

1. $|\phi_n(x)| \leq |\phi_{n+1}(x)|$.
2. $\phi_n(x) \rightarrow f(x)$ pointwise in X .
3. If f is bounded, then $\phi_n(x) \rightrightarrows f(x)$ in X .

Proof

Consider $(-\infty, -n] \cup (-n, n) \cup [n, +\infty)$, and define $N_n = \{x \in X \mid f(x) \leq -n\}$ and $P_n = \{x \in X \mid f(x) \geq n\}$. Then $\bigcap_n (N_n \cup P_n) = \emptyset$.

Define

$$\begin{aligned} A_{n,k} &= \left\{ x \in X \mid \frac{k-1}{2^n} < f(x) \leq \frac{k}{2^n} \right\}_{k=-1, -2, \dots, -n2^n+1} \\ A_{n,0} &= \left\{ x \in X \mid \frac{-1}{2^n} < f(x) < 0 \right\} \\ A_{n,1} &= \left\{ x \in X \mid 0 < f(x) < \frac{1}{2^n} \right\} \\ A_{n,k} &= \left\{ x \in X \mid \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n} \right\}_{k=2, 3, \dots, n2^n} \end{aligned}$$

and set

$$\phi_n(x) = -n\chi_{N_n} + \sum_{k=0}^{-n2^n+1} \frac{k}{2^n} \chi_{A_{n,k}} + \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \chi_{A_{n,k}} + n\chi_{P_n}$$

Claim:

1. $\forall x \in X, \phi_n(x) \rightarrow f(x)$.
2. if $\exists N \in \mathbb{N}$ such that $|f(x)| < N \implies \phi_n(x) \rightrightarrows f(x)$ in X .

Proof

$|\phi_n(x) - f(x)| \leq \frac{1}{2^n}, \forall x \in X \setminus (U_n \cup P_n)$

Note $\forall x \in X, \exists m \in \mathbb{N}$ such that $x \notin N_m \cup P_m$. So $|f(x)| < m$.

Then boundedness implies $\exists N$ such that $N_N \cup P_N = \emptyset$.

Therefore $\forall x \in X, |\phi_n(x) - f(x)| < \frac{1}{2^n}, \forall n \geq N$.

Theorem: Egoroff

Given a measure space (X, Λ, μ) , $\mu(x) < +\infty$ and $f_n \rightarrow f$ μ -a.e. in X , then $\forall \delta > 0$, $\exists A \in \Lambda$ such that $\mu(X \setminus A) < \delta$ and $f_n(x) \rightarrow f(x)$ in A .

Recall: Pointwise Convergence

$\forall x \in X$, $f_n(x) \rightarrow f(x)$ if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$, $\forall n \geq N$.

$B_{N,\varepsilon} = \{x \in X \mid \exists N \in \mathbb{N}, |f_n(x) - f(x)| < \varepsilon, \forall n \geq N\}$

In negation, $\exists \varepsilon > 0$ such that $\forall N \in \mathbb{N}$, $\exists m \geq N$ such that $|f_m(x) - f(x)| \geq \varepsilon$.

$A_{N,\varepsilon} = B_{N,\varepsilon}^c = \{x \in X \mid \exists m \geq N, |f_m(x) - f(x)| \geq \varepsilon\}$

Then $\{x \in X \mid f_n(x) \rightarrow f(x)\} = \bigcap_{\varepsilon > 0} \bigcup_N B_{N,\varepsilon} = \bigcap_{\varepsilon_i \rightarrow 0} \bigcup_i B_{N_i, \varepsilon_i}$

and $\{x \in X \mid f_n(x) \not\rightarrow f(x)\} = \bigcup_{\varepsilon > 0} \bigcap_N A_{N,\varepsilon} = \bigcup_{\varepsilon_i \rightarrow 0} \bigcap_i A_{N_i, \varepsilon_i}$ where $\varepsilon_i = \frac{1}{i}$.

February 2, 2024

Review: Measurable Function

An extended, real-valued function $f : X \rightarrow [-\infty, +\infty]$ is measurable if one or all of the following hold

1. $\forall \alpha \in \mathbb{R}$, $\{x \mid f(x) > \alpha\} \in \Lambda$.
2. $\forall \alpha \in \mathbb{R}$, $\{x \mid f(x) \geq \alpha\} \in \Lambda$.
3. $\forall \alpha \in \mathbb{R}$, $\{x \mid f(x) < \alpha\} \in \Lambda$.
4. $\forall \alpha \in \mathbb{R}$, $\{x \mid f(x) \leq \alpha\} \in \Lambda$.
5. $\forall V \subseteq \mathbb{R}$ open, $f^{-1}(V) = \{x \mid f(x) \in V\}$ and $f^{-1}(-\infty)$, $f^{-1}(+\infty) \in \Lambda$.

Properties

1. For $f = g$ μ -a.e., f is measurable if and only if g is measurable.
2. For f, g measurable, $f + g$ and $f \cdot g$ are measurable.
3. For $\{f_n\}$ measurable,
 - (a) $\sup_{n \leq k} \{f_n\}$ and $\inf_{n \leq k} \{f_n\}$ are measurable.
 - (b) $\sup_n \{f_n\}$ and $\inf_n \{f_n\}$ are measurable.
 - (c) $\limsup_{n \rightarrow \infty} f_n$ and $\liminf_{n \rightarrow \infty} f_n$ are measurable.
 - (d) if $f_n \rightarrow f$ μ -a.e. in X , then f is measurable.

Examples

Characteristic Functions

$$\chi_A = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}, \quad A \in \Lambda$$

Simple Functions

$$\sum_{i=1}^k \alpha_i \chi_{A_i}, \quad \alpha_i \in \mathbb{R}, \quad A_i \in \Lambda, \quad A_j \cap A_k = \emptyset$$

Step Functions

$$\sum_{i=1}^k \alpha_i \chi_{I_i}, \quad I_i \text{ interval}$$

Theorem:

On a measure space (X, Λ, μ) , suppose f is measurable.

There exists a sequence of simple functions $\{\phi_n\}$ such that

1. $\phi_n \rightarrow f$ pointwise.
2. $\phi_n \rightrightarrows f$ for f bounded.

Proof

Let $N_n = \{x \mid f(x) \leq -n\}$ and $A_{n,k} = \left\{x \mid \frac{k-1}{2^n} < f(x) \leq \frac{k}{2^n}\right\}$. Then

$$\begin{aligned} A_{n,0} &= \left\{x \mid -\frac{1}{2^n} < f(x) < 0\right\} \\ A_{n,1} &= \left\{x \mid 0 < f(x) < \frac{1}{2^n}\right\} \\ A_{n,k} &= \left\{x \mid \frac{k-1}{2^n} < f(x) < \frac{k}{2^n}\right\} \\ P_n &= \{x \mid f(x) \geq n\} \end{aligned}$$

and

$$\phi_n = -n\chi_{N_n} + \sum_{k=-n2^n+1}^D \frac{k}{2^n} \chi_{A_{n,k}} + \sum_1^{n2^n} \chi_{A_{n,k}} + n\chi_{P_n}$$

So

$$|\phi_n(x) - f(x)| \leq \frac{1}{2^n}, \quad x \in X \setminus (N_n \cup P_n), \quad \bigcap_n (N_n \cap P_n) = \emptyset$$

Egoroff Theorem

Given (X, Λ, μ) where $\mu(X) < +\infty$, if

1. $f_n(x) \rightarrow f(x)$ μ -a.e. in X and
2. f_n, f μ -a.e. finite.

Then, $\forall \delta > 0, \exists A \in \Lambda$ with $\mu(A) < \delta$ such that $f_n(x) \rightrightarrows f(x)$ on A^c .

Proof

Define $D = \{x \mid f_n(x) \rightarrow f(x)\} = X$.

Then $\forall \varepsilon > 0, \exists m \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon, \forall n \geq m$.

Say that the universal quantifier \forall is equivalent to grand intersection and the existential quantifier \exists is equivalent to grand union. Then

$$D_{m,\varepsilon} = \{x \mid f_n(x) - f(x) < \varepsilon, \forall n \geq m\}$$

and

$$\bigcap_{\varepsilon > 0} \bigcup_m D_{m,\varepsilon} = X.$$

The negation is

$$D_{n,\varepsilon}^c = \{x \mid \exists n \geq m, |f_n(x) - f(x)| \geq \varepsilon\}$$

Then injection is equivalent to the complement.

Set $\varepsilon_i = \frac{1}{i}$ such that

$$D = \bigcap_i \bigcup_{m_i} D_{m_i, 1/i}$$

$$\emptyset = D^c = \bigcup_i \bigcap_m D_{m, 1/i}^c$$

So $\bigcap_m D_{m, 1/i}^c = \emptyset$,

$$D_{m, 1/i}^c = A_{m, 1/i} = \left\{x \mid \exists n \geq m, |f_n(x) - f(x)| \geq \frac{1}{i}\right\}$$

and $A_{n, 1/i} \supset A_{n+1, 1/i} \supset \dots$. Therefore

$$\mu(A_{n, 1/i}) \rightarrow \mu\left(\bigcap_m A_{m, 1/i}\right) = 0$$

for $\mu(X) < +\infty$.

Thus, $\forall i, \exists m_i$ such that $\mu(A_{m_i, 1/i}) < \frac{\delta}{2^{i+1}}$. It follows that $A = \bigcup_i (A_{m_i, 1/i})$,

$$\mu(A) \leq \sum \mu(A_{m_i, 1/i}) < \delta$$

and

$$x \in A^c = \bigcap_i A_{m_i, 1/i}^c = \bigcap_i D_{m_i, 1/i} = \bigcap_i \left\{x \mid |f_n(x) - f(x)| < \frac{1}{i}, \forall n \geq m_i\right\}$$

Finally, this implies $f_n(x) \rightrightarrows f(x)$ in A^c .

Example

Take $f_n = \chi_{[n, n+1]}$ on \mathbb{R} , then $f_n(x) \rightarrow 0$ in \mathbb{R} but $A \subset \mathbb{R}$, $\mu(A) < \frac{1}{2}$, $A^c \cap [n, n+1] \neq \emptyset$, $\forall n$.
That is, $\forall n$, $\exists x \in A^c$ such that $f_n(x) = 1$ but $f(x) = 0$.
Therefore $f_n(x) \not\rightarrow f(x)$ on \mathbb{R} .

Definition: Essential Bounds

On a measure space (X, Λ, μ) with f measurable, define $\|f\|_\infty = \inf\{M \mid \mu(\{x \mid |f(x)| > M\}) = 0\}$.
This is the L^∞ -norm.

Proposition:

$f_n \Rightarrow f$ on A where $\mu(A^c) = 0$ if and only if $\|f_n - f\|_\infty \rightarrow 0$.

Proof

(\Rightarrow)

$\forall \varepsilon > 0$, $\exists m \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$, $\forall x \in A$.

Claim: $\|f_n(x) - f(x)\|_\infty < \varepsilon$, $\forall n \geq m$.

$$\|f_n(x) - f(x)\|_\infty = \inf\{M \mid \mu(\{x \mid |f_n(x) - f(x)| > M\}) = 0\}$$

Where $\{x \mid |f_n(x) - f(x)| > n\} \subset A^c$ and $n \geq m$ and $M \geq \varepsilon/2$.

(\Leftarrow)

Recall: Urysohn's Lemma

For X locally compact and Hausdorff, $K \subset U$ for K compact and U open, $\exists \phi$ continuous such that $\phi = \begin{cases} 1 & K \\ 0 & U^c \end{cases}$.

Theorem: Vitali-Lusin

On measure space (X, Λ, μ) with X locally compact and Hausdorff and μ a Radon measure.

For f measurable, μ -a.e. finite and vanishing outside A where $\mu(A) < +\infty$,

$\forall \varepsilon > 0$, $\exists g$ continuous with compact support such that $\mu(\{x \mid f(x) \neq g(x)\}) < \varepsilon$.

Proof

1. $\exists C \subset A$ compact with $\mu(A \setminus C) < \varepsilon$.
2. For A compact with $\mu(A) < +\infty$, $\exists U \supset A$ open neighborhood with compact closure and $\mu(U \setminus A) < \varepsilon$.
3. $\phi_n = -n\chi_{N_n} + \sum_{-n2^n+1}^0 \frac{k}{2^n} \chi_{A_{n,k}} + \sum_1^{n2^n} \frac{k-1}{2^n} \chi_{A_{n,k}} + n\chi_{P_n}$

Since we may minimize $\mu(N_n \cup P_n) < \varepsilon$,

$$\phi_n = \sum_{-n2^n+1}^0 \frac{k}{2^n} \chi_{A_{n,k}} + \sum_1^{n2^n} \frac{k-1}{2^n} \chi_{A_{n,k}}$$

Take $C_{1,k} \subset A_{1,k}$ compact with $\mu(C_{1,k}) \geq \mu(A_{1,k}) - 2^{-1}2^{-|k|+1}\varepsilon$. Write

$$C_1 = \bigcup_k C_{1,k}$$

and inductively define $C_{n-1,k}$ and $C_{n-1} = \bigcup_k C_{n-1,k}$ such that $C_{n,k} \subset A_{n,k} \cap C_{n-1}$ compact and

$$\mu(C_{n,k}) \geq \mu(A_{n,k} \cap C_{n-1}) - 2^{-1} 2^{-|k|+1} \varepsilon$$

Define, by Urysohn's Lemma,

$$\tilde{\chi}_{A_{n,k}} := \begin{cases} 1 & C_{n,k} \\ 0 & U^c \cup \bigcup_{l \neq k} C_{n,l} \end{cases}$$

where $C_n \subset C_{n-1}$, $C = \bigcap C_n$, $C_n = \bigcup_k C_{n,k}$.

Then define

$$g_n := \sum_{-n2^n+1}^0 \frac{k}{2^n} \tilde{\chi}_{A_{n,k}} + \sum_1^{n2^n} \frac{k-1}{2^n} \tilde{\chi}_{A_{n,k}}$$

Then $g_n = \phi_n$ on C for all n .

Therefore $g_n = \phi_n \rightrightarrows \hat{g} = f$ on C .

By uniform convergence, \hat{g} is continuous on C .

So, again by Urysohn's Lemma, $g = \phi \hat{g}$ and $\{x \mid g \neq f\} = U \setminus C$.

February 8, 2024

Midterm Review

Problem 2

Given a finite measure space (X, Λ, μ) , $\mu(X) < +\infty$ and a function f which is μ -a.e. finite.

Monotone Convergence Theorem:

1. $A_1 \subset A_2 \subset \dots$, then $\mu\left(\bigcup_i A_i\right) = \lim_{i \rightarrow \infty} \mu(A_i)$.
2. $A_1 \supset A_2 \supset \dots$, then $\mu\left(\bigcap_i A_i\right) = \lim_{i \rightarrow \infty} \mu(A_i)$ for $\mu(A_1) < +\infty$.

If $A_k = \{x \mid |f(x)| > k\}$ and

$$F = \bigcap_{k=1}^{\infty} A_k$$

then $\mu(F) = \lim_{k \rightarrow \infty} \mu(A_k) = 0$ since $\mu(X) < +\infty$.

If instead we consider A_k^c , then

$$\bigcup_k A_k^c = X \setminus F$$

Problem 3

1. Borel

Given $(\alpha, +\infty)$, we want $\forall E \subset \mathbb{R}$

$$m^*(E \cap (\alpha, +\infty)) + m^*(E \cap (-\infty, \alpha]) \leq m^*(E)$$

$\forall \varepsilon > 0, \exists \{I_i\}$ pen intervals

$$\bigcup_i I_i \supset E \quad \sum_i |I_i| \leq m^*(E) + \varepsilon/2$$

Divide $\{I_i\}$ into 3 groups,

$$\begin{aligned} C^\ell &= \{I \in \{I_i\} \mid I \text{ is to the left of } \alpha\} \\ C^r &= \{I \in \{I_i\} \mid I \text{ is to the right of } \alpha\} \\ C^m &= \{I \in \{I_i\} \mid \alpha \in I\} \end{aligned}$$

Then, $\forall I_k^m \in C^m = \{I_k^m\}$, and

$$\begin{aligned} {}^\ell I_k^n &= \left(a_k, \alpha + \frac{2}{2^{k+2}}\right) \\ {}^r I_k^n &= \left(\alpha - \frac{2}{2^{k+2}}, b_k\right) \\ {}^m I_k^n &= (a_k, b_k) \end{aligned}$$

where also

$$\begin{aligned} A_n \supset (\alpha, +\infty)^c \quad A_n &= \left(-\infty, \alpha + \frac{1}{2^n}\right) \\ B_n \supset (\alpha, +\infty) \quad B_n &= \left(\alpha + \frac{1}{2^n}, +\infty\right) \\ A_n \cap B_n &= \left(\alpha - \frac{1}{2^n}, \alpha + \frac{1}{2^n}\right) \end{aligned}$$

So ${}^\ell I_k^n \cup {}^r I_k^n = I_k^n$, and $|{}^\ell I_k^n| + |{}^r I_k^n| = |I_k^n| + \frac{\varepsilon}{2^{k+1}}$.

Finally

$$\begin{aligned} m^*(E \cap (\alpha, +\infty)) + m^*(E \cap (-\infty, \alpha]) &\leq \sum_{I \in C^r} |I| + \sum_k |{}^r I_k^n| + \sum_{I \in C^\ell} |I| + \sum_k |{}^\ell I_k^n| \\ &\leq \sum_{I \in C^r} |I| + \sum_{I \in C^\ell} |I| + \sum_k |I_k^n| + \frac{\varepsilon}{2} \\ &\leq m^*(E) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \end{aligned}$$

2. $\mu(K) < +\infty$ for $K \subset \mathbb{R}$ compact.

K is bounded, $k \subset (-M, M)$ for large M .

Therefore $\mu(K) \leq 2M < +\infty$.

3. $\forall U \subset \mathbb{R}$ open, we want to show $\exists K_n$ compact such that $K_n \subset U$ and $\mu(K_n) \rightarrow \mu(U)$.

Let $U = \bigcup_i I_i$ a union of countably many disjoint open intervals (e.g. $I_i = (a_i, b_i)$).

Then $m(U) = \sum_i m(I_i)$.

Set $I_i^n = \left[a_i + \frac{1}{n2^{i+1}}, b_i - \frac{1}{n2^{i+1}} \right]$. Then

$$\sum_i^k |I_i^n| \geq \sum_{i=1}^k |I_i| - \frac{1}{n}, \quad \forall k$$

It follows that

$$\sum_{i=1}^k |I_i| \rightarrow \sum_{i=1}^{\infty} |I_i|, \quad \text{as } k \rightarrow +\infty$$

and

$$K_k^n = \bigcup_{i=1}^k I_i^n \subset U \quad \text{compact}$$

$$m(U) \geq m(K_k^n) = \sum_{i=1}^n |I_i^n| \geq \underbrace{\sum_{i=1}^{\infty} |I_i|}_{m(U)} - \frac{1}{n}$$

Alternatively, we have the theorem that if X is a metric space and μ is Borel regular on (X, Λ) , then

(a) $A \in \Lambda$, $\mu(A) < +\infty$, $\forall \varepsilon > 0$, $\exists C$ closed with $C \subset A$ such that $\mu(A \setminus C) < \varepsilon$.

(b) $\exists \{U_i\}$, $\mu(U_i) < +\infty$, U_i open where $A \subset \bigcup_i U_i$, $\forall \varepsilon > 0$ there exists V open such that $V \supset A$ and $\mu(V \setminus A) < \varepsilon$.

With the corollary that for μ on \mathbb{R}^n , μ is Radon if and only if it is locally finite and Borel regular.

4. For $A \in \Lambda$, $m(A) = \inf\{m(V) \mid V \supset A, V \text{ open}\}$

Recall Borel regularity: $\forall A \in \Lambda$, there is some Borel set $B \supset A$ with $m(B) = m(A)$.

We may assume $m(A) < +\infty$. Then, $\forall \varepsilon > 0$, there is some collection of open intervals $\{I_i^n\}$ containing A where

$$\sum_i |I_i^n| \leq m(A) + \varepsilon$$

Set $\varepsilon = \frac{1}{n}$ and let $U^n = \bigcup_i I_i^n \supset A$ open. Then

$$m(A) \leq m(U^n) \leq \sum_i |I_i^n| \leq m(A) + \frac{1}{n}$$

If $B = \bigcap_n U^n$, then $\lim_{n \rightarrow \infty} m(U^n) = m(A)$ and $m(B) = m(A)$.

Problem 4

Given $f : \mathbb{R} \rightarrow \mathbb{R}$, continuous outside a measure zero set D .

That is, $\bar{f} : \mathbb{R} \setminus D \rightarrow \mathbb{R}$ is continuous.

$\forall V \subset \mathbb{R}, f^{-1}(V) = (f^{-1})V \cap (\mathbb{R} \setminus D) \cup (f^{-1}(V) \cap D)$.

By measure completeness, we are automatically safe on $f^{-1}(V) \cap D$.

Claim: $f^{-1}(V) \cap (\mathbb{R} \setminus D) = \bar{f}^{-1}(V)$.

Claim: \bar{f}^{-1} is measurable.

Claim: $\bar{f}^{-1}(V) = U \cap (\mathbb{R} \setminus D)$ where $U \subset \mathbb{R}$ open.

Since $U \cap (\mathbb{R} \setminus D)$ is open in the subspace topology, we are done.

Alternatively (similary to Problem 8 below), for D such that $m(D) = 0$, $\forall n, \exists U^n$ such that $m(U^n) \leq 2^{-n}$, $U^n \supset D$ and $U^n = \bigcup_i (a_i, b_i)$ where $(a_j, b_j) \cap (a_k, b_k) = \emptyset$ and $a_i, b_i \in \mathbb{R} \setminus D$. So

$$f_n = \begin{cases} f(x), & x \in (U^n)^c \\ f(a_i) + \frac{f(b_i) - f(a_i)}{b_i - a_i}(x - a_i), & x \in (a_i, b_i) \subset U^n \end{cases}$$

Then $\{x \mid f_n(x) \neq f(x)\} \subset U^n$ and $m(\{x \mid f_n(x) \neq f(x)\}) \leq 2^{-n}$.

Homework 4 Problem 8

Assume $f(x)$ is decreasing.

1. Discontinuities are limited to jump discontinuities.
2. Discontinuities are countable.
3. $D = \{x_i\}_i$, $\forall n$ there exists an open cover $\{I_i^n = (a_i, b_i)\}$ where $\bigcup_i I_i^n = C^n \supset \{x_i\}_i$ and $m(C^n) \leq 2^{-n}$.

Then $\{x \mid f_n(x) \neq f(x)\} \subset C^n$ and $\mu(\{x \mid f_n(x) \neq f(x)\}) \leq 2^{-n}$.

Claim: $f_n(x) \rightarrow f(x)$ on $\mathbb{R} \setminus G$ where $G = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{x \mid f_k(x) \neq f(x)\}$.

By monotone convergence, $\mu(g) = \lim_{n \rightarrow +\infty} \mu(\bigcup_{k=n}^{\infty} \{x \mid f_k(x) \neq f(x)\}) = \lim_{n \rightarrow +\infty} \left(\sum_{k=n}^{\infty} 2^{-k} \right) = 0$.

Consider the complement, $G^c = \bigcap_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \{x \mid f_k(x) = f(x)\}$.

Then $\forall x \in G^c, x \in \bigcap_{k=n_0}^{\infty} \{x \mid f_k(x) = f(x)\}$, so $f_n(x) = f(x) \forall n \geq n_0$.

Riemann Integration

Given a function $f : [a, b] \rightarrow \mathbb{R}$ bounded and P a partiation of $[a, b]$ where

$$a = x_0 < x_1 < \dots < x_n = b$$

The Cauchy sum

$$C(P, [a, b]) = \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}), \quad \xi_i \in [x_{i-1}, x_i)$$

alternatively

$$\phi(P, [a, b]) = \sum_i f(\xi_i) \chi_{[x_i, x_{i+1})}$$

Consider the upper Riemann sum

$$S(P, [a, b]) = \sum_i M_i(x_i, x_{i+1}), \quad M_i = \sup_{[x_i, x_{i+1}]} f(x)$$

and the lower Riemann sum

$$s(P, [a, b]) = \sum_i m_i(x_i, x_{i+1}), \quad m_i = \inf_{[x_i, x_{i+1}]} f(x)$$

then define

$$S = \inf_P S(P, [a, b]) = s = \sup_P s(P, [a, b]) \implies \int_a^b f(x) dx = \lim_{l(P) \rightarrow 0} C(P, [a, b])$$

Theorem:

f is Riemann integrable on $[a, b]$ if and only if f is continuous m -a.e. (w.r.t Lebesgue measure) on $[a, b]$.

Proof

(\implies) Let f be Riemann integrable on $[a, b]$.

Define the oscillation

$$\begin{aligned} \text{Osc}_I(f) &= \sup_I f(x) - \inf_I f(x) \\ \text{Osc}_x(f) &= \lim_{\delta \rightarrow 0} \text{Osc}_{(x-\delta, x+\delta)}(f) \end{aligned}$$

and observe that f is continuous at x if and only if $\text{Osc}_x(f) = 0$.

Let $D = \{x \mid \text{Osc}_x(f) > 0\}$ and $D_k = \left\{x \mid \text{Osc}_x(f) > \frac{1}{k}\right\}$ such that $D_k \subset D_{k+1}$ and $D = \bigcup_k D_k$.

Therefore $m(D_k) \rightarrow m(D)$.

To show that $m(D) = 0$, assume otherwise that $m(D) > 0$.

Therefore, $\exists k$ such that $m(D_k) > d_{k_0}$ for any $k \geq k_0$.

Then, for any partition P we may examine

$$S(P, [a, b]) - s(P, [a, b]) = \sum_{I_i} (M_i - m_i) |I_i|$$

We want to show that this is $\geq \delta > 0$ for any P .