Topics in Analysis (F24)

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Chapter 1: Banach Algebras

1.1: Definitions and Basic Properties

Definition: Banach Space

A Banach space X (over $\mathbb{C}$) is a normed vector space with algebraic operations

$$\begin{aligned} (x, y) &\mapsto x + y && \text{addition} \\ (\lambda, y) &\mapsto \lambda y && \text{scalar multiplication} \end{aligned}$$

and a norm

$$x \mapsto ||x||$$

which is complete (i.e. every Cauchy sequence converges).

Definition: (Complex) Banach Algebra

A (complex) Banach algebra B is a Banach space in which there is multiplication

$$B \times B \ni (x, y) \mapsto xy \in B$$

such that

1. $x(yz) = (xy)z$
2. $(x + y)z = xz + yz$ and $x(y + z) = xy + xz$
3. $\lambda(xy) = (\lambda x)y = x(\lambda y)$
4. $||xy|| \leq ||x|| \cdot ||y||$

Definition: Unital Banach Algebra

B is called a unital Banach algebra if $\exists e \in B$ such that

$$xe = ex = x \quad \text{and} \quad ||e|| = 1.$$

If e exists, it is unique.

1.2: Examples

Example 1

If X is a Banach space, then $B = \mathcal{L}(X)$ (the set of all bounded inear operators $A : X \to X$) equipped with algebraic operations

$$
\begin{aligned}
(A+B)x &= Ax + Bx \\
(\lambda A)x &= \lambda(Ax) \\
(AB)x &= A(Bx)
\end{aligned}
$$

and the operator norm

$$||A||_{\mathcal{L}(X)} = \sup_{x \neq 0} \frac{||Ax||_X}{||x||_X}.$$

$B = \mathcal{L}(X)$ is complete because X is complete.
The unit element is given by $I_X x = x$.

Example 2

If $X = \mathbb{C}^n$, then $B = \mathcal{L}(\mathbb{C}^n) \cong \mathbb{C}^{n \times n}$.

$$A = (a_{ij})_{i,j=1}^n \qquad Ax = y \qquad \sum_{j=1}^n a_{ij} x_j = y_i.$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

The norm in $\mathbb{C}^n$ leadsto a norm in $\mathbb{C}^{n \times n}$

$$
\begin{aligned}
||(x_i)|| &= \left(\sum |x_i|^2\right)^{1/2} & ||A|| &= \\
||(x_i)|| &= \sum |x_i| & ||A|| &= \max_j \sum_i |a_{ij}| \\
||(x_i)|| &= \max |x_i| & ||A|| &= \max_i \sum_j |a_{ij}|
\end{aligned}
$$

All norms are quivalent.

Example 3

Take $B = C(K)$ with K a compact Hausdorff space, $f : K \to \mathbb{C}$ continuous and $||f|| = \max_{t \in K} |f(t)|$.

Example 4

Take $B = A(K)$, $K \subseteq \mathbb{C}$ compact with $\operatorname{int}(K) \neq 0$, $f : K \to \mathbb{C}$ continuous where f is holomorphic on $\operatorname{int}(K)$ and

$$||f|| = \max_{t \in K} |f(t)| = \max_{t \in K \setminus \operatorname{int}(K)} |f(t)|$$

e.g. $K = \overline{\mathbb{D}} = \{t \in \mathbb{C} : |t| \leq 1\}$. Then $A(K) \subseteq C(K)$.

Example 5

Take $B = \ell^\infty(\mathbb{N})$ or $B = L^\infty(S, \sigma, \mu)$ with (S, σ, μ) a measure space, $f : S \to \mathbb{C}$ essentially bounded functions and

$$||f|| = \text{ess sup}_{t\in S} |f(t)| = \inf_{\substack{N\subseteq S \\ \mu(N)}} \left(\sup_{t\in S\setminus N} |f(t)| \right)$$

Example 6

Take $B = \ell^1(\mathbb{Z})$ or $B = L^1(\mathbb{R}^d)$ with $||\{x_n\}|| = \sum |x_n|$ and $||f|| = \int_{\mathbb{R}^d} |f(t)|\, dt$ respectively.
Multiplication is given by the convolution. e.g.

$$fg = (f * g)(x) = \int_{\mathbb{R}^d} f(x-t)g(t)\, dt$$

$\ell^1(\mathbb{Z})$ is unital, but $L^1(\mathbb{R}^d)$ is non-unital (since the unit of convolution is the Dirac delta; see Example 7).

Example 7

Take $B = M(\mathbb{R}^d)$ the complex measures on $\mathbb{R}^d$ with bounded variation.
Then multiplications is given as

$$(\mu * \nu)(A) = \int_{\mathbb{R}^d} \mu(A-x) d\nu(x)$$

and norm

$$||\mu|| = \sup_{\substack{\mathbb{R}^d = \bigcup A_i \\ \text{disjoint}}} \sum_{i=1}^{n} |\mu(A_i)| < +\infty.$$

Then, $f\, dm = d\mu$ gives $L^1(\mathbb{R}^d) \to M(\mathbb{R}^d)$.

Example 8

Take $B = C^{n\times n}[K]$ with K compcat and Hausdorff, continuous functions $f : K \to \mathbb{C}^{n\times n}$ and norm

$$||f||_B = \max_{t\in k} ||f(t)||_{C^{n\times n}}.$$

Then $B \cong (C(K))^{n\times n}$ the $n \times n$ matrices with entries from $C(K)$.

1.3: Remarks

- If B does not have a unit element, consider $B_1 = B \times \mathbb{C}$ with operations

$$\begin{aligned}
(b_1, \lambda_1) + (b_2, \lambda_2) &= (b_1 + b_2, \lambda_1 + \lambda_2) \\
\alpha(b, \lambda) &= (\alpha b, \alpha\lambda) \\
(b_1, \lambda_1)(b_2, \lambda_2) &= b_1 b_2 + \lambda_1 b_2 + \lambda_2 b_1, \lambda_1\lambda_2)
\end{aligned}$$

and norm

$$||(b,\lambda)|| = ||b|| + |\lambda|.$$

Then B_1 is a unital Banach algebra with $e = (0,1)$. One writes $(b,\lambda) = (b,0) + \lambda(0,1) = b + \lambda \cdot e$.
In some sense, $B \subseteq B_1$ where $b \in B \mapsto (b,0) \in B_1$.

1.4: Definitions

Definition: Commutative Banach Algebra

B is called commutative if $xy = yx$.

Definition: Banach Subalgebra

A subset B_0 of a B-algebra is called a subalgebra if it is closed with respect to the algebraic operations

$$x, y \in B_0,\ \lambda \in \mathbb{C} \rightsquigarrow x + y, xy, \lambda x \in B$$

Definition: Closed Subalgebra

B_0 is a closed subalgebra or Banach subalgebra if it is norm-closed.

- Proposition: B_0 is a Banach algebra.

Definition: Generated Subalgebra

Let $M \neq \varnothing$ be a subset of a Banach algebra B.
The Banach subalgebra generated by M is the smallest closed subalgebra containing M.

$$\operatorname{alg} M = (\operatorname{clos} \operatorname{alg}_B M)$$

- Remark

  $\operatorname{alg} M$ is the intersection of all closed subalgebras containing M.
  $\operatorname{alg} M = \operatorname{clos}\left\{\sum_{i=1}^{N} \lambda_i a_1^{(i)} a_2^{(i)} \cdots a_{n_i}^{(i)}\right\}$ is the norm-closure of finite linear combinations of finite products of $a_j^{(i)} \in M$.

1.5: Examples

Exammple 1

Take B unital, $b \in B$. Then

$$\operatorname{alg}\{e, b\} = \operatorname{clos}_B \left\{ \sum_{i=0}^{N} \lambda_i b^i \ :\ \lambda_i \in \mathbb{C},\ N \in \mathbb{N} \right\}$$

where $b^0 = e$.

1.6 Definitions

Definition: Banach Algebra Homomorphism

A Banach algebra homomorphism is a map $\phi : B_1 \to B_2$ between Banach algebras B_1 and B_2 such that

- ϕ is linear
- ϕ is bounded (continuous)
- ϕ is multiplicative

  $\phi(b_1 b_2) = \phi(b_1) \cdot \phi(b_2)$
- ϕ is unital if both B_1, B_2 have units and $\phi(e_{B_1}) = e_{B_2}$.

Definition: Banach Algebra Isomorphism

A Banach algebra homomorphism which is bijective is called a Banach algebra isomorphism.
Then $\phi^{-1} : B_2 \to B_1$ is an isomorphism as well.

Definition: Banach Algebra Isometry

ϕ is an isometry if $||\phi(x)|| = ||x||$.

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Recall

Given $M \subseteq \mathcal{L}(X)$ with X a Banach space (and $\mathcal{L}(X)$ itself a Banach algebra), we may construct $B = \operatorname{alg}_{\mathcal{L}(X)} M$.

1.7 Proposition

Let B be a unital Banach algebra. Then the map

$$\phi : B \ni x \to L_x \in \mathcal{L}(B)$$

is an isometric isomorphism onto a closed subalgebra of $\mathcal{L}(B)$ where

$$L_x : B \ni z \mapsto xz \in B$$

is the left-representation of x.

Proof

L_x is in $\mathcal{L}(B)$ since $L_x z = xz$

- is linear in z and
- $||L_x z|| = ||xz|| \leq ||x|| \cdot ||z||$ implies $||L_x|| \leq ||x||$ (i.e. L_x is a bounded).

The map $\phi : x \mapsto L_x$ is linear

$$L_{x_1+x_2}z = (x_1 + x_2)z = x_1 z + x_2 z = L_{x_1} z + L_{x_2} z = (L_{x_1} + L_{x_2})z$$

ϕ is multiplicative

$$L_{x_1 x_2} z = (x_1 x_2)z = x_1(x_2 z) = L_{x_1}(L_{x_2} z)$$

From the above, we conclude that ϕ is a homomorphism.
To show that ϕ is an isometry,

$$||L_x|| = \sup_{z \neq 0} \frac{||L_x z||}{||z||} \geq \frac{||L_x e||}{||e||} = \frac{||x||}{1} = ||x||.$$

Then also ϕ is injective and $\mathrm{im}\,\phi$ is closed. Since $\mathrm{im}\,\phi$ is a Banach algebra, it is therefore a closed subalgebra.

1.7 Remark: Right-Regular Representation

Every unital Banach algebra is isometrically isomorphic to a Banach algebra of operators.
Right-regular representation:

$$R_x = z \mapsto zx$$

Chapter 2: Group of Invertible Elements in a Banach Algebra

2.1 Definition: Invertible Element

Let B be a unital Banach algebra. An element $x \in B$ (in B) if there exists $y \in B$ such that $xy = yx = e$.
Note that $y = x^{-1}$ is uniquely determined.
Write GB for the set of all invertible elements of B.

Remark

GB is a (multiplicative group).

- $x, y \in GB \implies xy \in GB$ and $(xy)^{-1} = y^{-1}x^{-1}$,
- $x \in GB \implies x^{-1} \in GB$ and $(x^{-1})^{-1} = x$, and
- $e \in GB$.

2.2 Lemma

If $x \in B$ and $||x|| < 1$, then $e - x \in GB$.

Proof

Take the Neumann series

$$e + x + x^2 + x^3 + \cdots$$

which converges to some $s \in B$

$$s_n = e + x + \cdots + x^n$$

where s_n are Cauchy:

$$||s_{n+k} - s_n|| = ||x^{n+1} + \cdots x^{n+k}|| \leq ||x||^{n+1} + ||x||^{n+2} + \cdots = \frac{||x||^{n+1}}{1 - ||x||}.$$

So $s_n \to S$,

$$(e - x)s_n = s_n(e - x)e - x^{n+1}.$$

Taking $n \to \infty$

$$(e - x)s = s(e - x) = e.$$

2.3 Proposition

The group GB is open in B and the map $\Lambda : GB \ni x \mapsto x^{-1} \in GB$ is continuous (in the norm).

Proof

Take $x \in GB$ and consider $y \in B$ with $||y|| < \frac{1}{||x^{-1}||} = \varepsilon$.
Then $x + y \in B_\varepsilon(x)$ is invertible,

$$x + y = x(e + x^{-1}y),$$

and

$$||x^{-1}y|| \leq ||x^{-1}|| \cdot ||x|| < 1.$$

Therefore GB is open, since $B_\varepsilon(X) \subseteq GB$. The inverse

$$(x + y)^{-1} = (e + x^{-1}y)^{-1}x^{-1} = \sum_{n=0}^{\infty}(-x^{-1}y)^n x^{-1} = x^{-1} + \sum_{n=1}^{\infty}(-x^{-1}y)^n x^{-1}$$

so

$$||(x + y)^{-1} - x^{-1}|| \leq \sum_{n=1}^{\infty} ||x^{-1}||^{n+1}||y||^n = \frac{||x^{-1}||^2||y||}{1 - ||x^{-1}|| \cdot ||y||}.$$

This converges to zero as $||y|| \to 0$.

2.4 Examples

Example 1

$B = C(K)$, K compact Hausdroff, $f : K \to \mathbb{C}$ continuous.
$GB = \{f \in C(K) \ : \ f(t) \neq 0, \ \forall\, t \in K\}$.

Example 2

$B = C^{n\times n}$.
$GB = \{A \in \mathbb{C}^{n\times n} \,:\, \det A \neq 0\}$.

2.5 Definition:

Let G_0B stand for the connected componet of GB containing e.

Remarks

- the ε-neighborhoods $B_\varepsilon(x) \subseteq B$ are (path-)connected.

$$B_\varepsilon(x) = \{y \in B \,:\, ||x-y|| < \varepsilon\}$$

  For $y_1, y_2 \in B_\varepsilon(x)$, there is a continuous path

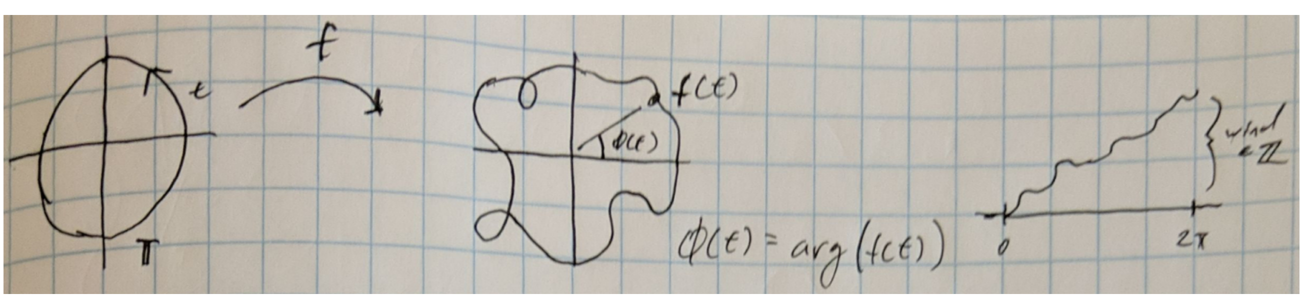
$$\sigma : [0,1] \ni \lambda \mapsto y_1\lambda + y_2(1-\lambda) \in B_\varepsilon(x)$$

- Because GB is open and $B_\varepsilon(x)$ is path-connected, GB is locally (path-)connected (i.e. every $x \in GB$ has a (path-)connected open neighborhood in GB).
- In this context, connectedness and path-connectedness are equivalent. Therefore the components of GB are the path-components of GB.
- GB is the union of disjoint (path-)components where each component is both open and closed in GB.
- $x, y \in GB$ belong to the same path-component if there exists a continuous path $\gamma : [0,1] \to GB$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Here, $x \sim y$ is an equivalence relation.
- $G_0B = \{x \in GB \,:\, \exists$ a path in GB connecting e and $x\}$.

2.6 Examples

Example 1

Take $B = C(\mathbb{T})$ with $\mathbb{T} = \{z \in \mathbb{C} \,:\, |z| = 1\}$ and continuous functions $f : \mathbb{T} \to \mathbb{C}$.
GB is the non-vanishing continuous functions $f : \mathbb{T} \to \mathbb{C}$ ($f(t) \neq 0,\ \forall t \in \mathbb{T}$).
For $f \in GB$ one can define a winding number.



We have $\frac{1}{2\pi}\arg f(e^{ix})$ a continuous function with

$$\text{wind}(t) = \left[\frac{1}{2\pi}\arg f(e^{ix})\right]_{x=0}^{2\pi} = \phi(2\pi) - \phi(0)$$

and $\mathrm{wind}(t) \in \mathbb{Z}$.
The map $GB \ni f \mapsto \mathrm{wind}(t) \in \mathbb{Z}$ is continuous, hence locally constant (i.e. constant on each connected component).
Therefore $G_0C(\mathbb{T}) \subseteq \{f \in GC(\mathbb{T}) : \mathrm{wind}(f) = 0\}$. In fact, we will see that we have equality.
That is, f can be contracted (in GB) to the constant function $e(t) = 1$.

2.7 Proposition

G_0B is a normal subgroup of GB.

Proof

- G_0B is a group.

  For any $x, y \in G_0B$, there exist paths $\gamma_1 : [0,1] \to GB$ and $\gamma_2 : [0,1] \to GB$ with $\gamma_1(0) = \gamma_2(0) = e$, $\gamma_1(1) = x$ and $\gamma_2(1) = y$.
  Define $\gamma(t) = \gamma_1(t)\gamma_2(t)$ a path in GB such that $\gamma(0) = e$ and $\gamma(1) = xy$. Then $xy \in G_0B$.
  Following from Lemma 2.2, $\hat{\gamma} = (\gamma_1(t))^{-1}$ is a continuous path with $\hat{\gamma}_1(0) = e$, $\hat{\gamma}_1(1) = x^{-1}$ and $x^{-1} \in GB$.

- G_0B is a normal subgroup of GB.

  For every $y \in GB$, $yG_0By^{-1} \subseteq G_0B$ if and only if $yG_0B = G_0By$.
  Take $x \in G_0B$ with path γ, then

$$\delta(t) = y\gamma(t)y^{-1}, \quad \delta(0) = yey^{-1} = e, \quad \text{and} \quad \delta(1)yxy^{-1} \in G_0B.$$

2.8 Definition: Abstract Index Group

The quotient group GB/G_0B is called the abstract index group of B.

Remark

GB/G_0B is in 1-to-1 correspondence with the set of connected components of GB.
Indeed, the (path-)connected components of GB are given by $yG_0B = G_0By$ (for $y \in GB$).

$$y_1G_0B = y_2G_0B \iff y_2^{-1}y_1G_0B = G_0B \iff y_2^{-1}y_1 \in G_0B \iff [y_2] = [y_1] \text{ in } GB/G_0B.$$

2.9 Definition: Exponential Map

For $x \in B$, we define the exponential map $B \ni x \mapsto \exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

2.10 Lemma

The exponential map $B \ni x \mapsto \exp(x) \in GB$ is well-defined and continuous.
For $xy = yx$, we have $\exp(x + y) = \exp(x)\exp(y)$.
In particular, $(\exp(x))^{-1} = \exp(-x)$.

Proof

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergent.

$$\sum_{n=0}^{\infty} \frac{||x||^n}{n!} < +\infty.$$

It follows that $s_n = \sum_{n=0}^{k} \frac{x^k}{k!}$ is a Cauchy sequence and therefore converges.
Continuity left as an exercise. Need to show:

$$\left|\left|\sum \frac{x^n}{n!} - \sum \frac{y^n}{n!}\right|\right| \le ||x-y|| \cdot M_{x,y}$$

The fact that $\exp(x+y) = \exp(x)\exp(y)$ follows from multiplying terms and the binomial formula.

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Recall

$GB\ e + x$.
G_0B connected component of GB containing e.
GB/G_0B is the abstract index group.
$B = C(\mathbb{T}) \rightsquigarrow f \in GC(\mathbb{T}) \rightsquigarrow \mathrm{ind}(f)$.

Definition: Exponential Map

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \in GB$$

Lemma:

For $y \in B$, $||y|| < 1$, there exists $x \in B$ such that $\exp(x) = e + y$.

Proof

Define

$$\log(e+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \cdots \in B.$$

This converges absolutely ($||y|| < 1$), therefore it converges in B by completeness.

Identities

$$\exp(\log(e+y)) = \sum_{n=0}^{\infty} \frac{\left(\sum_k \frac{y^k}{k}(-1)^{k-1}\right)^n}{n!} = e + y$$

Proof

G_0B is equal to the set of all finite products of exponentials of elements in B.

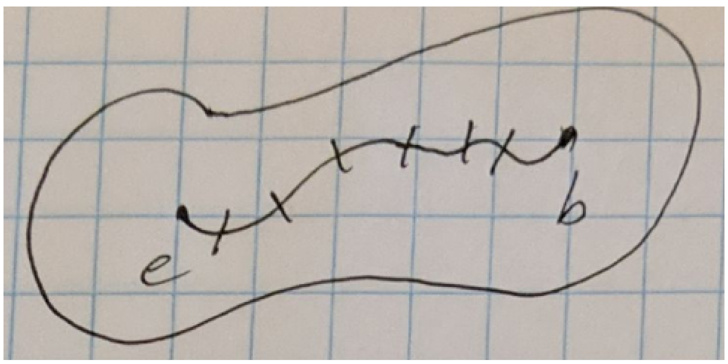
$$G_0B = \bigcup_{n=0}^{\infty} \Gamma_n = \bigcup_{n=0}^{\infty} \{\exp(a_1)\exp(a_2)\cdots \exp_{a_n} \in B\}$$

Proof

Call $\Gamma = \bigcup_{n=0}^{\infty} \Gamma^n$.
Then observe that each Γ_n is path-connected and contains e.
For $b = \exp(a_1) \cdots \exp(a_n) \in \Gamma_n$, define a path

- $\sigma : [0,1] \to \Gamma_n$
- $\sigma(t) = \exp(ta_1) \cdots \exp(ta_n)$ is continuous with $\sigma(0) = e$ and $\sigma(1) = b$.

Therefore, Γ is path-connected and contains e. It follows that $\Gamma \subseteq G_0 B$.
To prove that $G_0 B \subseteq \Gamma$, take $b \in G_0 B$ and show that there exists a path in GB $\gamma : [0,1] \to GB$ continuous with $\gamma(0) = e$ and $\gamma(1) = b$.



We have that $(\gamma(t))^{-1}$ is continuous and bounded in the norm. Then $\gamma(t)$ is uniformly continuous.
$||\gamma^{-1}(t)|| \leq M$.
$(\exists N) : |t - s| \leq \frac{1}{N} \implies ||\gamma(t) - \gamma(b)|| \leq \frac{1}{M} \cdot \frac{1}{2}$. Write

$$b = \gamma(1) \cdot \gamma^{-1}(0) = \gamma(1)\gamma^{-1}\left(\frac{N-1}{N}\right)\gamma\left(\frac{N-1}{N}\right)\gamma^{-1}\left(\frac{N-2}{2}\right) \cdots \gamma\left(\frac{1}{N}\right)\gamma^{-1}\left(\frac{1}{N}\right)\gamma(0) = \prod_{k=1}^{N} \gamma^{-1}\left(\frac{k}{N}\right)\gamma\left(\frac{k-1}{N}\right).$$

Therefore, with $s_k = \gamma^{-1}\left(\frac{k}{N}\right)\gamma\left(\frac{k-1}{N}\right)$, $b = \prod_{k=1}^{N} \exp(\log(s_k))$.

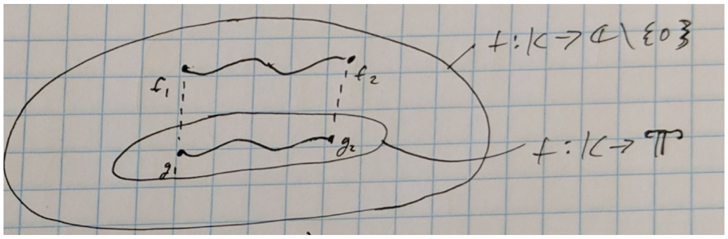
$$||s_k - e|| \leq ||\gamma^{-1}\left(\frac{k}{N}\right)|| \cdot ||\gamma\left(\frac{k-1}{N}\right) - \gamma\left(\frac{k}{N}\right)|| \leq M \cdot \frac{1}{2M} \leq \frac{1}{2}.$$

Corollary

If B is commutative, $G_0 B = \{\exp(a) \, : \, a \in B\}$.

Remark

Special case: $B = C(K)$ (K compact Hausdorff space).
$G_0 B = \{\exp(a) \, : \, a \in C(K)\}$.
$GB / G_0 B$ is an equivalence class of functions $f : K \to \mathbb{C} \setminus \{0\}$ with respect to path-connectedness.
That is, $f_1 \sim f_2$ if and only if there exists continuous $F(t,x) : [0,1] \times K \to \mathbb{C} \setminus \{0\}$ with $F(0,x) = f_1(x)$ and $F(1,x) = f_2(x)$.
These are the homotopy classes of continuous functions $f : K \to \mathbb{C} \setminus \{0\}$.
This corresponds to homotopy classes of continuous functions $f : K \to \mathbb{T}$ (with $\mathbb{T} = \{z \in \mathbb{C} \, : \, |z| = 1\}$) called the 1st co-homotopy group of K $\pi^1(K)$.



$f: K \to \mathbb{C} \setminus \{0\}$ and $\frac{f}{|f|}: K \to \mathbb{C} \setminus \{0\}$ are path-connected by $\sigma(s) = \frac{f}{|f|^s}$, $s \in [0,1]$.
$f_1 \sim f_2$ in $K \to \mathbb{C} \setminus \{0\}$ implies that $\frac{f_1}{||f_1||} \sim \frac{f_2}{||f_2||}$ in $K \to \mathbb{T}$ by $F(s,x)$ and $\frac{F(s,x)}{|F(s,x)|}$.
We conclude that $\pi^1(K) \cong GC(K)/G_0C(K)$.

Example

Let $B = C(\mathbb{T})$.

$$G_0B = \{\exp(a) \,:\, a \in C(\mathbb{T}) = \{f \in GC(\mathbb{T}) \,:\, \text{wind}(f) = 0\}$$

For $f \in GC(\mathbb{T})$, $\text{wind}(f) = 0$ implies that $f = \exp(a)$ has a logarithm.
This implies that $f \in G_0B$ which itself implies that $\text{wind}(f) = 0$, since $\text{wind}(f)$ is continuous on $GC(\mathbb{T})$ and therefore constant on the component.
Therefore, $GB/G_0B \cong \mathbb{Z}$ via the winding number.
For connected components of GB, define $\chi_n(t) = t^n$, $|t| = 1$, where $\text{wind}(\chi_n) = n$.

Remark: Closed Subalgebras and Invertibility

Let A be a closed subalgebra of B (both being unital, $e \in A$, $e \in B$).
Obviously, if $a \in A$ is invertible in A (i.e. $a^{-1} \in A$) then a is invertible in B. Then $GA \subseteq GB \cap A \subseteq GB$.

Example

Take $B = C(\mathbb{T})$ and $A = \{f \in C(\mathbb{T}) \,:\, f_n = 0,\ \forall n < 0\} = C_+(\mathbb{T})$ where $f_n = \frac{1}{2\pi}\int_0^{2\pi} f(e^{ix})e^{-inx}\,dx$ is the nth Fourier coefficient.
Formally: $f(t) \cong \sum_{n=-\infty}^{\infty} f_n t^n$ in $B = C(\mathbb{T})$, $|t| = 1$.
$f \in A: f(t) = \sum_{n=0}^{\infty} f_n t^n$, $|t| = 1$ has an analytic extension into the unit disk $|t| < 1$.
More precisely, $\phi: A(\overline{\mathbb{D}}) \to C_+(\mathbb{T}) \subseteq C(\mathbb{T})$ by $f \mapsto f|_{\mathbb{T}}$.
Where $A(\overline{\mathbb{D}}) = \{f \in \overline{D} \to \mathbb{C}$ continuous, holomorphic on $\mathbb{D}\}$ and $\mathbb{D} = \{t \in \mathbb{C} \,:\, |t| \leq 1\}$.
Then, for $f \in A(\overline{\mathbb{D}})$ with $n \in \{-1, -2, -3, \ldots\}$,

$$f_n = \frac{1}{2\pi}\int_0^{2\pi} f(e^{ix})e^{-inx}\,dx = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(z)}{z^{n+1}}\,dz = \lim_{r\to 1^-} \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(rz)}{z^{n+1}}\,dz = 0$$

- In fact, ϕ is an isometry.

$$||f||_{A(\overline{\mathbb{D}})} = \sup_{|z|\leq 1} |f(z)| = \max_{|z|=1} |f(z)| = ||f|_{\mathbb{T}}||_{C(\mathbb{T})}$$

By maximum modulus principle of holomorphic functions, since ϕ is not constant.

- ϕ is linear and multiplicative.

- $C_+(\mathbb{T})$ is a closed subset of $C(\mathbb{T})$.

$$\Lambda_n : C(\mathbb{T}) \ni f \mapsto f_n \in \mathbb{C}$$

is a continuous linear functional.

$$C_+(\mathbb{T}) = \bigcap_{n=0} \ker \Lambda_n$$

- Less trivaially, ϕ is surjective and $C_+(\mathbb{T})$ is an algebra.

Example

$\chi_1(t) = t$ is invertible in $C(\mathbb{T}) = B$.
$x_1^{-1}(t) = \frac{1}{t} = x_{-1}(t) \notin C_+(\mathbb{T})$ while $\chi_1(t) \in C_+(\mathbb{T})$.
Therefore $GA \subseteq GB \cap A$ may not be equal.

Definition: Boundary

The boundary of a subset U of a topological space X is $\partial U = \overline{U} \setminus \mathrm{int}(U)$.

Remark

For $U \subseteq X$, $X = \mathrm{int}(U) \cup \partial U \cup \mathrm{int}(X \setminus U)$ a union of disjoint sets.

Lemma:

1. if $a \in \partial GA$, then $a \notin GA$ and there exists a sequence $a_n \in GA$ such that $a_n \to a$.
2. if $a \in \partial a$ and $a_n \in GA$ such that $a_n \to a$, then $||a_n^{-1}|| \to +\infty$.

Proof of 1

$a \in GA$ would imply $a \in \mathrm{int}(GA)$ and not a boundary point.

Proof of 2

Otherwise, there would exist a bounded subsequence $||a_{n_i}^{-1}|| \leq M$.

$$||a_{n_i}^{-1} - a_{n_j}^{-1}|| \leq ||a_{n_i}^{-1}|| \cdot ||a_{n_j} - a_{n_i}|| \cdot ||a_{n_j}^{-1}|| \leq M^2 ||a_{n_i} - a_{n_j}||$$

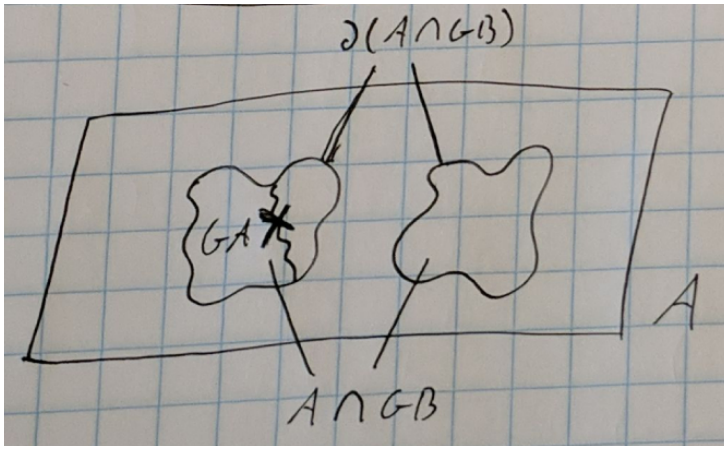
Since a_n converges, $\{a_n\}$ is Cauchy which implies $a_{n_i}^{-1}$ is Cauchy.
Then $a_{n_i}^{-1} \to b \in A$. $e = a_{n_i} a_{n_i}^{-1} \to ab$ implies $a^{-1} = b$ and $a \in GA$. However $a \notin GA$.

Proposition

Let A be a closed subalgebra of B ($e \in A$, $e \in B$). Then $\partial GA \subseteq \partial(A \cap GB)$ (both boundaries are considered in A).

Remark

Both GA and $A \cap GB$ are open subsets of A.



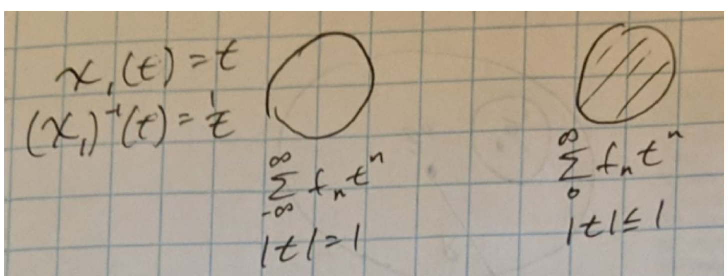
Proof

Take $a \in \partial GA$ and suppose $a \notin \partial(A \cap GB)$.
Take $a \in \partial GA$: $a_n \in GA$, $a \notin GA$, $a_n \to a$, $||a_n^{-1}|| \to +\infty$.

October 9, 2024

Recall

$A \subseteq B$, $GA \subseteq A \cap GB$.
If $A = C_+(\mathbb{T}) \cong A(\overline{\mathbb{D}})$ and $B = C(\mathbb{T})$.



Recall: Theorem

For GA, $A \cap GB$ open sets in A, $U \subseteq X$, $\partial U = \overline{U} \setminus \operatorname{int} U$, we have that $\partial GA \subseteq \partial(A \cap GB)$.

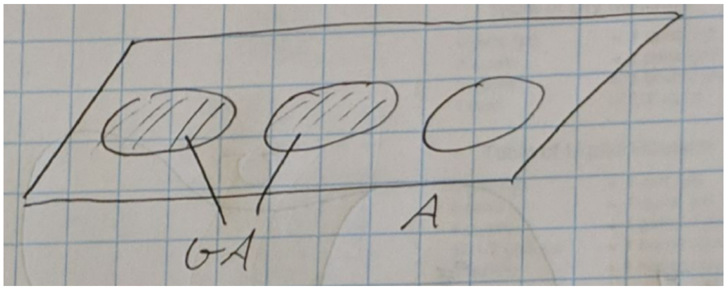
Proof

Take $a \in \partial GA$, $a_n \to a$, $a \notin GA$, $a \in A$.
Since $a_n \in GA$, $||a_n^{-1}|| \to +\infty$.
However, $a \notin GB$ otherwise $a \in GB$, $a_n \to a$ implies $a_n^{-1} \to a^{-1}$ (in GB) and, consequently, $\sup ||a_n^{-1}|| < +\infty$, a contradiction.
Therefore $a \notin A \cap GB$ and, consequently, $a \in \partial(A \cap GB) = \overline{(A \cap GB} \setminus (A \cap GB)$.

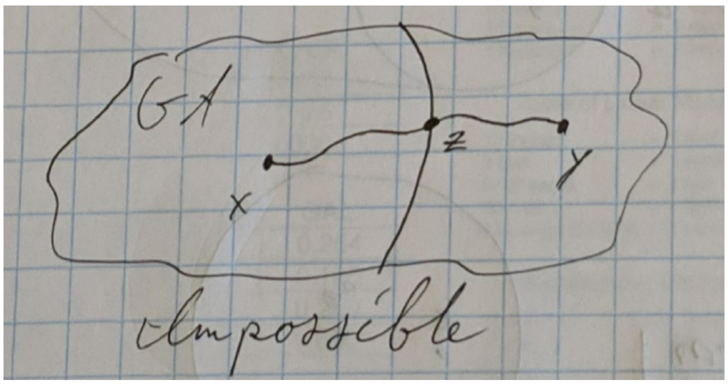
Theorem

Let A be a closed subalgebra of B.
GA is equal to the union of some components of $A \cap GB$.

Proof



Let U be a component of $A \cap GB$.
We want to show that either $U \cap GA \neq \varnothing$ or $U \subseteq GA$.



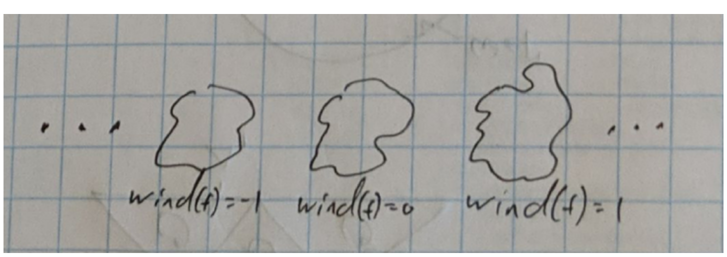
The above cannot occur since, by path-connectedness, for $x, y \in U,\ x \in GA,\ y \neq\in GA$, there would need to be some $z \in \partial GA$ with $z \notin A \cap GB$ a contradiction.
Alternatively, take $A \cap GB$ open in A.
Then $A \cap GB \cap \partial(A \cap GB) = \varnothing$ and $(A \cap GB) \cap \partial GA = \varnothing$ by the previous theorem.
Write $A = GA \cup \partial GA \cup \operatorname{int}(A \setminus GA)$. Then

$$A \cap GB = GA \cup \varnothing \cup \operatorname{int}(A \setminus GA) \cap (A \cap GB)$$

and $U = (GA \cap U) \cup \operatorname{int}(A \setminus GB) \cap U$ where $(GA \cap U) \cap \operatorname{int}(A \setminus GA) = \varnothing$ and open in U.
Therefore either $GA \cap U = \varnothing$ or $GA \cap U = U$ which implies that $U \subseteq GA$.

Example

Take $B(\mathbb{T})$ and $A = C_+(\mathbb{T}) \cong A(\overline{D})$.
Then $GB = \{f : \mathbb{T} \to \mathbb{C} \ : \ f(t) \neq 0$.



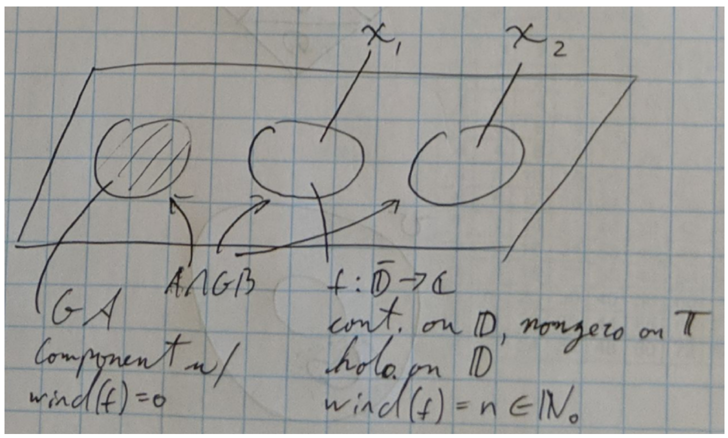
Then take

$$A \cap GB = \{f : \mathbb{T} \to \mathbb{C} \text{ continuous, } f(t) \neq 0,\ |t| = 1 \text{ with analytic continuation into } |t| < 1\}$$

such that $f \in A \cap GB$ which implies $\text{wind}(f) \in \{0, 1, 2, 3, \cdots\}$ gives the number of zeroes of f inside $\mathbb{D}$.

$$\begin{aligned}
\text{wind}(f) &= \frac{1}{2\pi i}\left[\log f(e^{ix})\right]_{x=0}^{2\pi} \\
&= \frac{1}{2\pi i}\lim_{r \to 1^-}\left[\log f(re^{ix})\right]_{x=0}^{2\pi} \\
&= \frac{1}{2\pi i}\lim_{r \to 1}\int_0^{\pi}\frac{f'(re^{ix})}{f(re^{ix})}ire^{ix}\,dx \\
&\underset{z=re^{ix}}{=} \frac{1}{2\pi i}\lim_{r \to 0}\int_{|z|=r}\frac{f'(z)}{f(z)}\,dz
\end{aligned}$$

Which gives the number of zeros of $f(z)$ inside $|z| < 1$



Chapter 3: Holomorphic Vector-Valued Functions

Goal

Define the notion of holomorphic/analytic functions $f : \Omega \to X$ where $\Omega \subset \mathbb{C}$ open and X a (complex) Banach space.

Sumary

- Basically all classical results remain true.
- There is a strong and a weak version of holomorphy, but they are equivalent.

Theorem

For a function $f : \Omega \to X$, $\Omega \subseteq \mathbb{C}$ open and X Banach, the following are equivalent

1. f is differentiable at every $z_0 \in \Omega$, i.e. there exists $f'(z_0) \in X$ such that

$$\lim_{z \to z_0}\left|\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right|\right|_X = 0$$

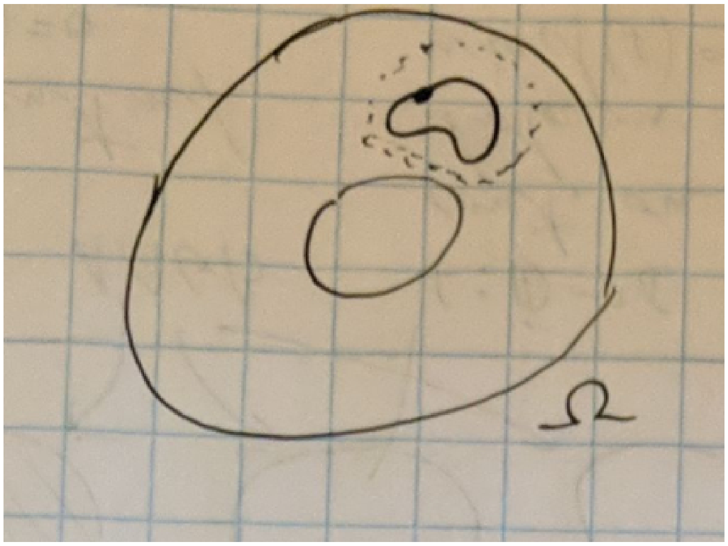
2. f is analytic at each point $z_0 \in \Omega$, i.e. f has a convergent power series at z_0 with radius of convergence $R_{z_0} > 0$.

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n,\ |z - z_0| < R_{z_0},\ a_n \in X$$

which converges in the norm of X.

3. $f : \Omega \to X$ is continuous (in the norm) and for every piecewise smooth closed contour Γ contained in a disk D ($\Gamma \subseteq D \subseteq \Omega$).

$$\int_\Gamma f(z)\, dz = 0$$



Definition: (Strongly) Holomorphic Function

If (1)-(3) hold, then f is (strongly)-holomorphic.

Remarks: Integration of Vector-Valued Functions

A piecewise smooth contour Γ can be parameterized by $\sigma : [0,1] \to \Omega$.

$$\int_\Gamma f(z)\, dz = \int_0^1 \underbrace{f(\sigma(t))\sigma'(t)}_{h(t)\text{ continuous}}\, dt$$

This is independent of the choice of parameterization.
Now $I = \int_0^1 h(t)\, dt$ can be defined via Riemann sums. Given a partition P, $h : [0,1] \to X$ continuous.

$$\lim_{\text{mesh}(P)\to 0} ||S(h, P, \xi) - I||_X = 0$$

where $S(h, P, \xi) = \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1})$, $P = \{x_0, x_1, \ldots, x_n\}$, $\xi_i \in [x_{i-1}, x_i]$.
Note that h is uniformly continuous and $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\text{mesh}(P_1) < \delta$, $\text{mesh}(P_2) < \delta$ implies

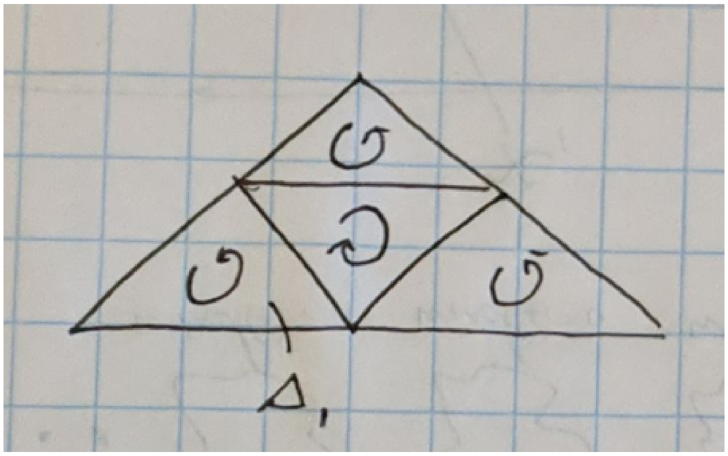
$$\left|\left|S(f, P_1, \xi^{(1)}) - S(f, P_2, \xi^{(2)})\right|\right| < \varepsilon$$

All usual properties of integrals hold.

- linear in integrand
- $||\int_\Gamma f(z)\, dz|| \le \int_\Gamma ||f(z)||\, |dz| \le (\text{length}(\Gamma)) \sup_{z\in\Gamma} ||f(z)||$.

Sketch of Proof (1) to (3)

To show: $\int_\Delta f(z)\, dz = x_0 = 0$ by contradiction that $x_0 \neq 0$.

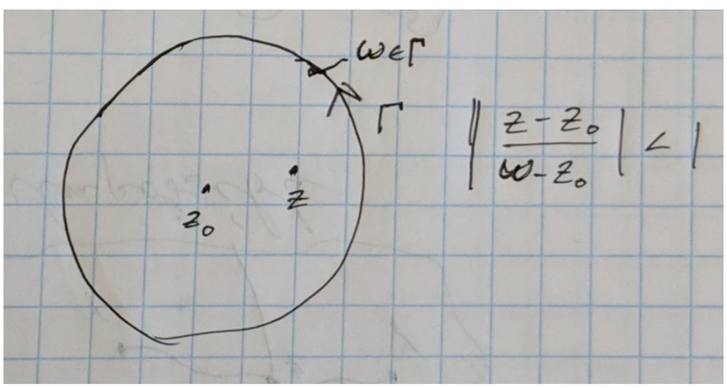


We have $\left|\left|\int_{\Delta_1} f\, dz\right|\right| \geq \frac{||x_0||}{4}$, $\left|\left|\int_{\Delta_n} f\, dz\right|\right| \geq \frac{||x_0||}{4^n}$.

Sketch of Proof (3) to (2)

$\int_\Gamma f\, dz = 0$ implies the Cauchy integral formula. Take

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\omega)}{\omega - z}\, d\omega$$



$$\frac{1}{\omega - z} = \frac{1}{(\omega - z_0) - (z - z_0)} = \frac{1}{w - z_0}\sum_{n=0}^{\infty}\left(\frac{z - z_0}{w - t}\right)^n$$

Therefore

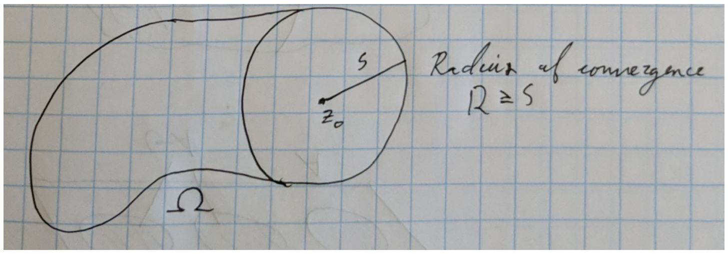
$$f(z) = \frac{1}{2\pi i}\int_\Gamma f(\omega)\sum_{n=0}^{\infty}\frac{(z - z_0)^n}{(\omega - z)^{n\pi}}\, d\omega = \sum_{n=0}^{\infty}(z - z_0)^n \frac{1}{2\pi i}\int_\Gamma \frac{f(\omega)}{(\omega - z)^{n\pi}}\, d\omega = \sum_{n=0}^{\infty}(z - z_0)^n a_n$$

with the sequence converging (in X) on $|z - z_0| < |\omega - z_0|$.

- Radius of Convergence

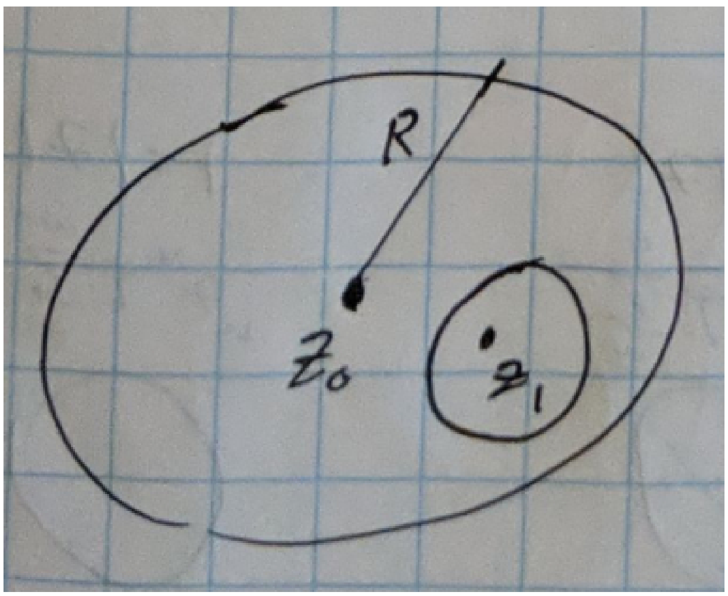
$$R^{-1} = \limsup_{n\to\infty} ||a_n||^{1/n}$$

(Root Test: $|z - z_0| < R$ convergence; $|z - z_0| > R$ divergence)



Sketch of Proof (2) to (1)

One can show that a function defined by convergent power series is differentiable, $f(z) = \sum a_n(z - z_0)^n$, then $f'(z) = \sum a_n \cdot n(z - z_0)^{n-1}$.
The radius of convergence is the same. This also implies that f is infinitely differentiable.



Take $z - z_0 = (z - z_1) + (z_1 - z_0)$ and, by the binomial theorem,

$$f(z) = \sum_{k=0}^{\infty} (z - z_1)^k \left(\sum_{n=k}^{\infty} a_n \binom{n}{k} (z_1 - z_0) \right)$$

which converges for at least $|z - z_1| < R - |z_1 - z_0|$.

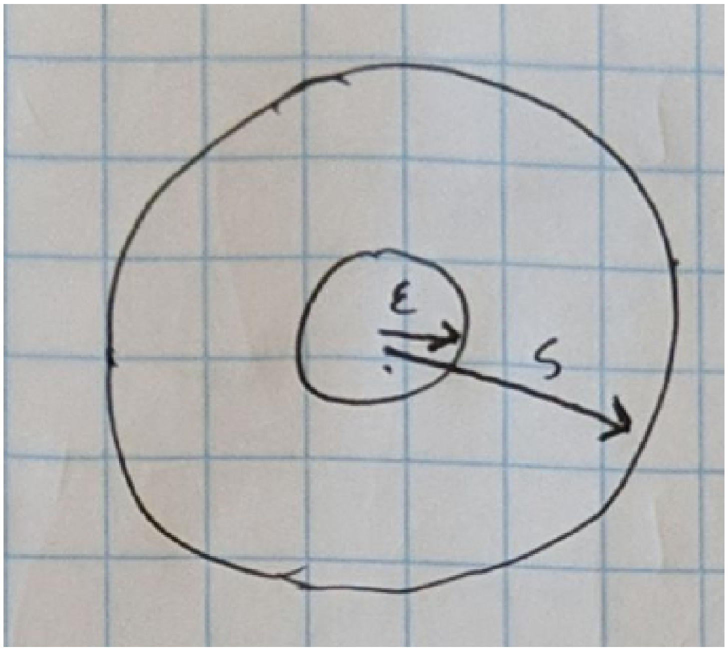
October 14, 2024

Theorem

Let $f : D_\varepsilon(z_0) \to X$ ($D_\varepsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$) be holomorphic.
Then $R = S$ where

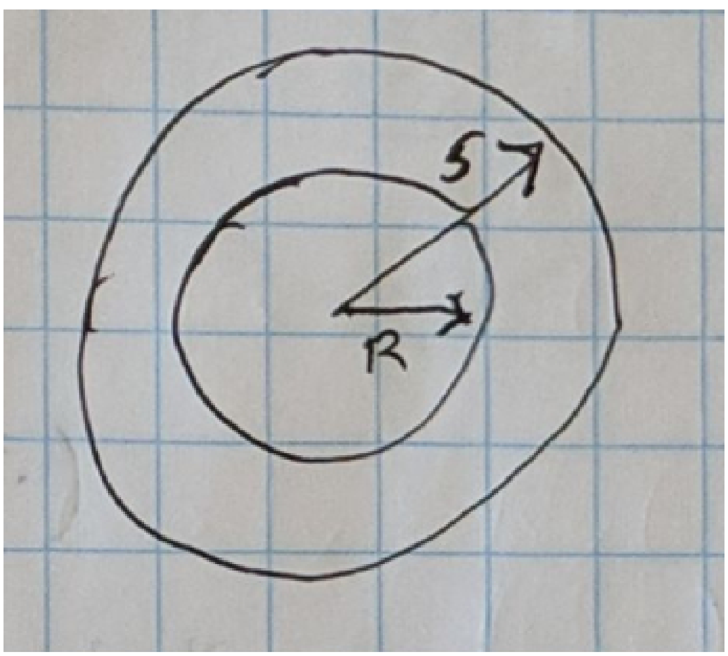
1. R is the radius of convergence of $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ ($R^{-1} = \limsup_{n\to\infty} ||a_n||^{\frac{1}{n}}$).

2. S is the radius of the largest open disk $D_S(z_0)$ such that there exists an analytic extension of f from $D_\varepsilon(z_0)$ to $D_S(z_0)$.



Proof

By definition, $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for $|z-z_0| < R$. Then $|z-z_0| < R$ if and only if $\limsup_{n\to\infty} ||a_n(z-z_0)^n||^{\frac{1}{n}} < 1$ if and only if $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges. Therefore, it converges to a holomorphic function on $R \leq S$.
If $f(z)$ has an analytic extension to $D_S(z_0)$, see step (3) $\implies$ (2) of previous theorem.



Then $f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\omega)}{\omega\cdot z}\, d\omega = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for $|z-z_0| < r < S$ with $a_n = \frac{1}{2\pi i}\int_\Gamma \frac{f(\omega)}{(z-\omega)^{n+1}}\, d\omega$.
From this, we conclude $R \geq S$.

Definition: (Weakly) Holomorphic Function

A function $f : \Omega \to X$ ($\Omega \subseteq \mathbb{C}$ open, X Banach) is called weakly holomorphic if $\phi \circ f : \Omega \to \mathbb{C}$ is holomorphic, $\forall \phi \in X^* = \mathcal{L}(X;\mathbb{C})$ bounded linear functionals.
A function $f : \Omega \to \mathcal{L}(X, Y)$ (X, Y Banach) is weakly-operator holomorphic if $h_{\phi,X} : \Omega \to \mathbb{C}$ is holomorphic for all $\phi \in Y^*$, $x \in X$ where $h_{\phi,X}(z) = \phi(f(z)x)$.

Remarks

Obviously: f strongly holomorphic $\implies$ f weakly holomorphic.

$$\left|\left|\frac{\phi(f(z+h))-\phi(f(z))}{h}-\phi(f'(z))\right|\right| \leq ||\phi||\cdot\left|\left|\frac{f(z+h)-f(z)}{h}-f'(z)\right|\right|$$

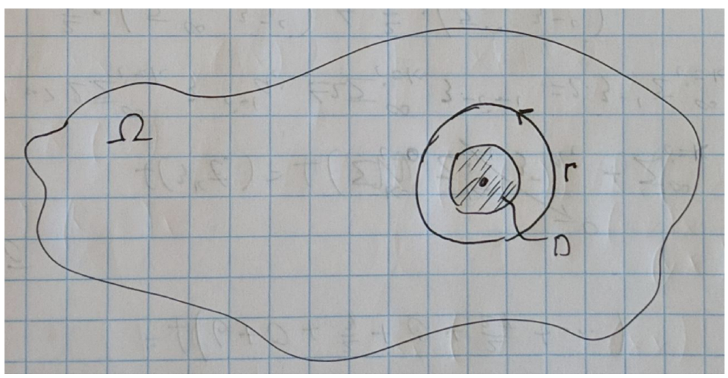
For $f:\Omega\to\mathcal{L}(X,Y)$: f strongly holomorphic $\implies$ f weakly holomorphic $\implies$ f weakly operator holomorphic.
For $x\in X$, $\phi\in Y^*$, $\Lambda_{x,\phi}:\mathcal{L}(X,y)\ni A\mapsto\phi(Ax)\in\mathbb{C}$ and $\Lambda_{x,\phi}\in(\mathcal{L}(X,y))^*$.
All the converses are also true.

Theorem (Dunford)

Take X Banach, $\Omega\subseteq\mathbb{C}$ open.
If $f:\Omega\to X$ is weakly holomorphic, then it is strongly holomorphic.

Proof

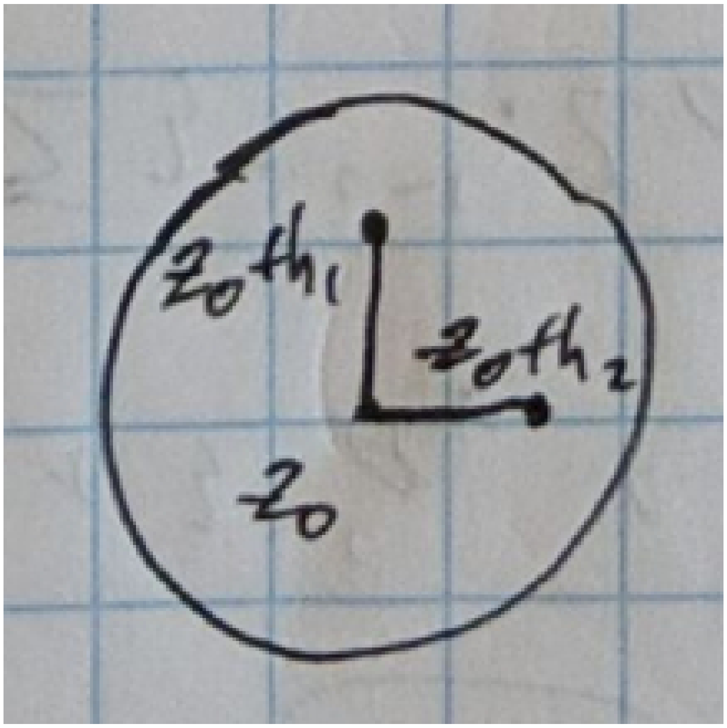
We want to show that for any $z_0\in\Omega$, $\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$ exists in X.
Choose $\varepsilon>0$ such that the disk $D_\varepsilon(z_0)$ and circle $C_{2\varepsilon}(z_0)=\Gamma$ are in Ω.



For $\phi\in X^*$, $\phi(f(z))$ is holomorphic in Ω.

$$\phi(f(z))=\frac{1}{2\pi i}\int_\Gamma\frac{\phi(f(\omega))}{z-\omega}\,d\omega,\ z\in D$$

Apply this to $z=z_0$, $z=z_0+h_1$ and $z=z_0+h_2$ with $0<|h_1|<\varepsilon$, $0<|h_2|<\varepsilon$, $h_1\neq h_2$.



$$
\begin{aligned}
A_{h_1,h_2} &= \frac{1}{h_1-h_2}\left\{\frac{f(z_0+h_1)-f(z_0)}{h_1} - \frac{f(z_{0+h_2)-f(\omega)}}{h_2}\right\}\\
\phi(A_{h_1,h_2}) &= \frac{1}{h_1-h_2}\left\{\frac{\phi(f(z_0+h_1)-\phi(f(z_0))}{h_1} - \frac{\phi(f(z_0+h_2)-\phi(f(\omega))}{h_2}\right\}\\
&= \frac{1}{2\pi i}\int_\Gamma \phi(f(\omega))\frac{1}{h_1-h_2}\left\{\frac{1}{h_1}\left(\frac{1}{z_0+h_1-\omega}-\frac{1}{z_0-\omega}\right)-\frac{1}{h_2}\left(\frac{1}{z_0+h_2-\omega}-\frac{1}{z_0-\omega}\right)\right\}d\omega\\
&= \frac{1}{2\pi i}\int_\Gamma \phi(f(\omega))\frac{1}{h_1-h_2}\left\{\frac{1}{(z+h_1-\omega)(z_0-\omega}-\frac{1}{(z+h_2-\omega)(z_0-\omega)}\right\}d\omega\\
&= \frac{1}{2\pi i}\int_\Gamma \phi(f(\omega))\frac{1}{(z_0+h_1-\omega)(z_0+h_2-\omega)(z_0-\omega)}\,d\omega
\end{aligned}
$$

Observe that the denominator is at least ε^3, therefore $|\phi(A_{h_1,h_2})| \le \frac{\varepsilon^3}{2\pi}\sup_{\omega\in\Gamma}||f(\omega)||\cdot||\phi||$ (so long as f continuous, which will be proven).
Therefore $\forall\phi\in X^*$,

$$\sup_{\substack{0<|h_1|<\varepsilon\\0<|h_2|<\varepsilon\\h_{1\neq h_2}}}|\phi(A_{h_1,h_2})| < +\infty.$$

By the uniform boundedness principle, identify $A_{h_1,h_2}\in X$ with $X^{**}=\mathcal{L}(X^*,\mathbb{C})$.
Then $\sup_{h_1,h_2}||A_{h_1,h_2} < +\infty$ and

$$\left|\left|\frac{f(z_0+h_1)-f(z_0)}{h_1} - \frac{f(z_0-f(z)}{h_2}\right|\right| \le C\cdot|h_1-h_2|.$$

Now, for any sequnce $\{h_n\}_{n=3}^\infty$, $0<|h_n|<\varepsilon$, $h_n\to 0$,

$$\frac{f(z_0+h_n)-f(z_0)}{h_n}$$

is a cauchy sequence. Therefore $\lim_{n\to\infty} \frac{f(z_{0+h_n} - f(z_0)}{h_n}$ exists in X independent of choice of $\{h_n\}$. That is

$$\lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

exists in X.

Chapter 4: Spectrum and Resolvent

Consider a unital Banach algebra B.

Definition: Spectrum

For $b \in B$, the spectrum of b in B $\sigma_B(b) = \{\lambda \in \mathbb{C} : \lambda e - b$ is not invertible in $B\}$.

Definition: Resolvent

The resolvent is a function $R(b;\lambda) = (\lambda e - b)^{-1}$. $R(b,\cdot) : \mathbb{C} \setminus \sigma_B(b) \to B$.
$\mathbb{C} \setminus \sigma_B(b)$ is the resolvent set.

Theorem

1. The spectrum $\sigma_B(b)$ is a non-empty, compact subset of $\mathbb{C}$.

2. The resolvent $R(b,\lambda)$ is an analytic, Banach valued function on $\mathbb{C} \setminus \sigma_B(b)$.

Proof of (a)

$\sigma_B(b)$ is bounded, because $\lambda e - b$ is invertible for $|\lambda| > ||b||$.

$$\lambda e - b = \lambda\left(e - \frac{1}{\lambda} b\right)$$

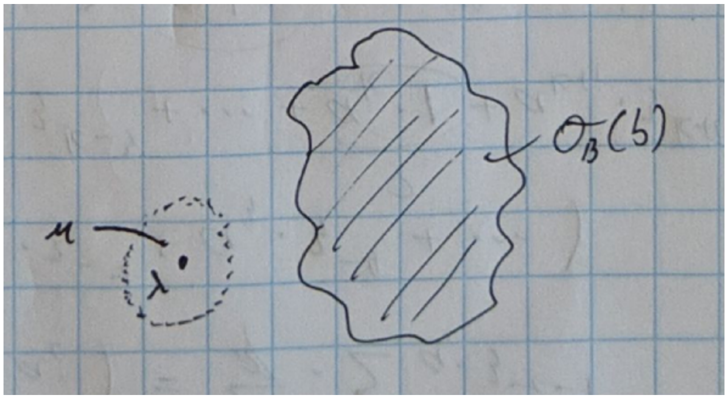
has $\left|\left|\frac{1}{\lambda} b\right|\right| < 1$ for sufficiently large λ. Therefore, $\sigma_B(b) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq ||b||\}$.
To show that $\sigma_B(b)$ is closed, if $\lambda \notin \sigma_B(b)$ then $\forall \mu$ such that $||\lambda - \mu|| < \varepsilon$ we have that $\mu \notin \sigma_B(b)$.

$$\mu e - b = \lambda e - b + (\mu - \lambda)e = (\lambda e - b)\left\{e + \underbrace{(\mu-\lambda)(\lambda e - b)^{-1}}_{||\cdot||<1}\right\}$$

when $|\mu - \lambda| < \frac{1}{||(\lambda e - b)^{-1}||}$.
Therefore $\mathbb{C} \setminus \sigma_B(b)$ is open.

Proof of (b)

Take $\lambda \notin \sigma_B(b)$



$$\begin{aligned}\frac{R(b,\mu)-R(b,\lambda)}{\mu-\lambda} &= \frac{1}{\mu-\lambda}\left((\mu e-b)^{-1}-(\lambda e-b)^{-1}\right)\\ &= \frac{1}{-\mu-\lambda}(\mu e-b)^{-1}\{(\lambda e-b)-(\mu e-b)\}(\lambda e-b)^{-1}\\ &= -(\mu e-b)^{-1})(\lambda e-b)^{-1}\end{aligned}$$

Using continuity with $GB \ni a \mapsto a^{-1} \in GB$ in the norm, $-(\mu e-b)^{-1})(\lambda e-b)^{-1} \to -\left((\lambda e-b)^{-1}\right)^2$ as $\mu \to \lambda$.
Therefore $R^1(b,\lambda) = -(R(b,\lambda))^2$ and $R(b,\lambda)$ is analytic.

Proof of non-empty in (a)

Take $\sigma_B(b) \neq 0$, otherwise $R(b,\lambda)$ is analytic on $\mathbb{C}$ and bounded

$$(\lambda e-b)^{-1} = \frac{1}{\lambda}\left(e-\frac{1}{\lambda}b\right)^{-1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{\lambda^{n+1}}b^n$$

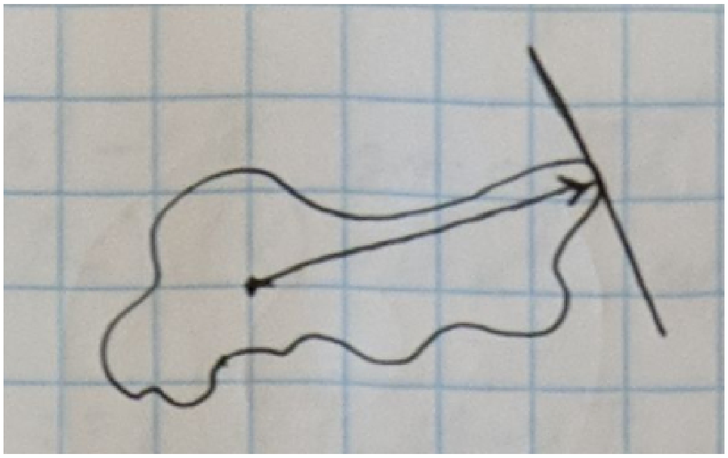
We can estimate

$$||\cdot|| \leq \frac{1}{|\lambda|\left(1-\frac{||b||}{|\lambda|}\right)} = \frac{1}{|\lambda|-||b||}$$

so $\lim_{\lambda\to\infty} ||(\lambda e-b)^{-1}|| = 0$.
By Liouville's theorem, bounded and entire functions are constant. But we may also proceed by weak analyticity.
If $\phi(R(b,\lambda))$ is analytic and bounded on $\mathbb{C}$, $\forall\phi \in B^*$, it follows that $\phi(R(b,\lambda)) \equiv 0$, $\forall\lambda$, $\forall\phi \in B^*$ and that $R(b,\lambda) \equiv 0$ for any λ a contradiction.

Definition: Spectral Radius

For $b \in B$, the spectral radius $r(b) = \max\{|\lambda| : \lambda \in \sigma_B(b)\}$.

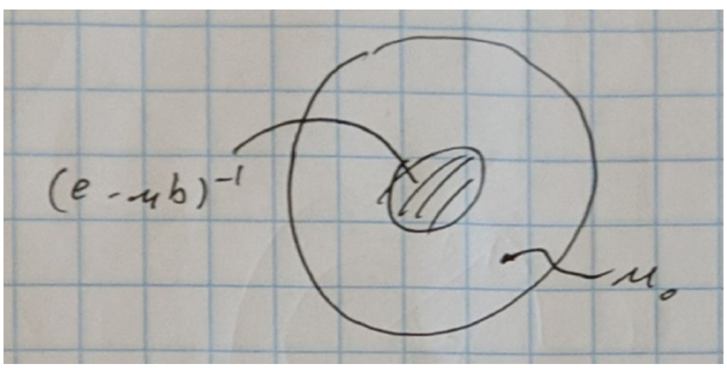


Remark

Write $\frac{1}{r(b)} = \min\{|\lambda|^{-1} : \lambda e - b$ is not invertible$\} = \min\{|\mu| : e - \mu b$ is not invertible$\}$with $\mu = \frac{1}{\lambda}$.

$$\underbrace{(e-\mu b)^{-1}}_{\text{analytic in } |\mu|<\frac{1}{||b||}} = \sum_{n=0} \mu^n b^n$$

converges for $|\mu| < \frac{1}{||b||}$.
Then the radius of convergence $R^{-1} = \limsup_{n\to\infty} ||b^n||^{\frac{1}{n}}$ gives us that R is equal to the largest disk where $(e-\mu b)^{-1}$ has an analytic extension. Therefore $S = \frac{1}{r(b)}$.



Suppose we have an analytic extension $f(\mu)$ beyond S.

$$f(\mu)(e-\mu b) = (e-\mu b)f(\mu) = e$$

implies that and, if $(e-\mu_0 b)$ not invertible, $f(\mu_0)(e-\mu_0 b) = \cdots = e$ a contradiction.

Theorem

$$r(b) = \lim_{n\to\infty} ||b^n||^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} ||b^n||^{\frac{1}{n}}$$

Proof

To demonstrate existence, fix $n_0 \in \mathbb{N}$, $n = q \cdot n_0 + r$, $0 \le r < n_0$.

$$\begin{aligned} ||b^n|| &\le ||b^{n_0}||^q \cdot ||b||^r \\ ||b^n||^{\frac{1}{n}} &\le ||b^{n_0}||^{\frac{q}{n}} \cdot ||b||^{\frac{r}{n}} \\ \limsup_{n\to\infty} ||b^n||^{\frac{1}{n}} &\le ||b^{n_0}||^{\frac{1}{n_0}} \cdot 1 \end{aligned}$$

Since $1 = \frac{q}{n} \cdot n_0 + \frac{r}{n}$. Take $n \to \infty$. Write

$$\limsup_{n\to\infty} ||b^n||^{\frac{1}{n}} \le \inf_{n_0 \in \mathbb{N}} ||b^{n_0}||^{\frac{1}{n_0}} \le \liminf_{n\to\infty} ||b^n||^{\frac{1}{n}}$$