

# Manifolds III

March 31, 2025

## Review

If  $X, Y$  are topological spaces and  $f, g : X \rightarrow Y$  continuous maps, we say  $f$  and  $g$  are homotopic (written  $f \simeq g$ ) if there is a homotopy  $H : X \times I \rightarrow Y$  (where  $I = [0, 1]$ ) such that  $H(x, 0) = f(x)$  and  $H(x, 1) = g(x)$  for all  $x \in X$ . We say that  $f$  is null-homotopic if it is homotopic to a constant map.

## Proposition

Homotopy is an equivalence relation on the collection of continuous maps.

1.  $f \simeq f$  by  $H(x, t) := f(x)$ .
2.  $f \stackrel{\tilde{H}}{\simeq} g \implies g \simeq f$  by defining  $\tilde{H}(x, t) := H(x, 1 - t)$ .
3.  $(f \stackrel{F}{\simeq} g \wedge g \stackrel{G}{\simeq} h) \implies f \simeq h$  by

$$H(x, t) := \begin{cases} F(x, 2t) & 0 \leq t \leq 1/2 \\ G(x, 2t - 1) & 1/2 \leq t \leq 1 \end{cases}.$$

## Proposition

For  $f_0, f_1 : X \rightarrow Y$  and  $g_0, g_1 : Y \rightarrow Z$ , if  $f_0 \stackrel{F}{\simeq} f_1$  and  $g_0 \stackrel{G}{\simeq} g_1$ , then  $g_0 \circ f_0 \simeq g_1 \circ f_1$ .

## Proof

Define  $H(x, t) := G(F(x, t), t)$  such that  $H(x, 0) = G(F(x, 0), 0) = G(f_0(x), 0) = g_0 \circ f_0(x)$ . Similarly,  $H(x, 1) = g_1 \circ f_1(x)$ .

## Definition: Homotopic Spaces

We say that two spaces  $X$  and  $Y$  are homotopic to each other ( $X \simeq Y$ ) if there are continuous maps  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  such that  $f \circ g \simeq \text{id}_Y$  and  $g \circ f \simeq \text{id}_X$ .

## Example

$\mathbb{R}^n$  is homotopic to  $\{0\}$  (or any single point) by  $\iota : 0 \rightarrow \mathbb{R}^n$  and  $r : \mathbb{R}^n \rightarrow 0$ . Then  $r \circ \iota : 0 \rightarrow 0$  is  $\text{id}_0$  and  $\iota \circ r : \mathbb{R}^n \ni x \mapsto 0 \in \mathbb{R}^n$  is homotopic to  $\text{id}_{\mathbb{R}^n}$ . In fact, consider  $H : \mathbb{R}^n \times I \rightarrow \mathbb{R}^n$  where  $H(x, t) = tx$ ,  $H(x, 1) = x = \text{id}_{\mathbb{R}^n}(x)$  and  $H(x, 0) = 0$ .

## Definition: Path

A path in  $X$  from  $p$  to  $q$  is a continuous map  $f : I \rightarrow X$  such that  $f(0) = p$  and  $f(1) = q$ .

## Definition: Path Homotopic

Let  $f, g : I \rightarrow X$  be two paths in  $X$  from  $p$  to  $q$ .

We say that  $f$  and  $g$  are path homotopic (write  $f \sim g$ ) if there is a homotopy  $H : I \times I \rightarrow X$  such that  $H(s, 0) = f(s)$ ,  $H(s, 1) = g(s)$ ,  $H(0, t) = p$  and  $H(1, t) = q$ .

## Proposition

Path homotopy is an equivalence relation on the collection of paths from  $p$  to  $q$ .  
Write  $[f]$ , the equivalence class of  $f$  in the quotient.

## Definition: Loop

In the special case that  $p = q$ , we say that  $f : I \rightarrow X$  is a loop

## Definition: Fundamental Group

Given  $(X, p)$ ,  $\pi_1(X, p)$  (the fundamental group of  $X$  at the point  $p$ ) is the set of all loops at  $p$  modulo the path homotopy.

$$\{\text{loops at } p\} / \sim$$

Equivalently,  $(S^1, 1)$ ,  $\{\text{loops at } p\} = \{\text{continuous maps } f : (S^1, 1) \rightarrow (X, p)\}$  with  $f(1) = p$ . We say this is the homotopy “relative to  $1 \in S^1$ ”. We have  $H : S^1 \times I \rightarrow X$  such that  $H(s, 0) = f(s)$ ,  $H(s, 1) = g(s)$  and  $H(1, t) = p$ .

## Definition: Free Homotopy

For two loops  $f, g : S^1 \rightarrow X$ , we say that  $f$  and  $g$  are free homotopic if  $f \simeq g$ .

## Lemma

When  $f : I \rightarrow X$  is a path from  $p$  to  $q$ , if  $f \circ \varphi$  is a reparameterization of  $f$  then  $(f \circ \varphi) \sim f$  where  $\varphi : I \rightarrow I$  satisfies  $\varphi(0) = 0$  and  $\varphi(1) = 1$ .

## Proof

Note that  $\varphi$  is homotopic to the identity map  $\text{id}_I$  through  $H(s, t) = ts + (1 - t)\varphi(s)$  since  $H(s, 0) = \varphi(s)$  and  $H(s, 1) = s = \text{id}_I(s)$ .

Then consider  $f \circ H : I \times I \rightarrow X$  which is a path homotopy between  $f$  and  $f \circ \varphi$ .

## Fundamental Group

Let  $f, g : I \rightarrow X$  be two paths with  $f(1) = g(0)$ .

Then we can “compose” (concatenate)  $f$  and  $g$  together  $(f \cdot g) : I \rightarrow X$  by

$$(f \cdot g)(s) := \begin{cases} f(2s) & 0 \leq s \leq 1/2 \\ g(2s - 1) & 1/2 \leq s \leq 1 \end{cases}.$$

## Lemma

If  $f_0 \stackrel{F}{\sim} f_1$ ,  $g_0 \stackrel{G}{\sim} g_1$  and  $f_0(1) = f_1(1) = g_0(0) = g_1(0)$ , then  $f_0 \cdot g_0 \sim f_1 \cdot g_1$ .

## Proof

Define

$$H(s, t) := \begin{cases} F(2s, t) & 0 \leq s \leq 1/2 \\ G(2s - 1, t) & 1/2 \leq s \leq 1 \end{cases}.$$

Then

$$H(s, 0) = \begin{cases} F(2s, 0) = f_0(2s) & 0 \leq s \leq 1/2 \\ G(2s - 1, 0) = g_0(2s - 1) & 1/2 \leq s \leq 1 \end{cases}.$$

Similarly  $H(s, 1) = (f_1 \cdot g_1)(s)$ , hence  $f_0 \cdot g_0 \sim f_1 \cdot g_1$ .

With this, we have a well-defined  $[f] \cdot [g] := [f \cdot g]$ .

## Simple Properties

For  $f$  from  $p$  to  $q$  where  $c_p$  is the constant map at  $p$ ,

1.  $[c_p] \cdot [f] = [f] = [f] \cdot [c_q]$  since  $c_p \cdot f$  is a reparameterization of  $f$ .
2. Let  $\bar{f}$  be the inverse path of  $f$  (i.e.  $\bar{f}(s) = f(1 - s)$ ). Then  $[f] \cdot [\bar{f}] = [c_p]$  and  $[\bar{f}] \cdot [f] = [c_q]$ .

$$H(s, t) := \begin{cases} f(2s) & 0 \leq s \leq t/2 \\ f(t) & t/2 \leq s \leq 1 - t/2 \\ f(2 - 2s) & 1 - t/2 \leq s \leq 2 \end{cases}.$$

1.  $([f] \cdot [g]) \cdot [h] = [f] \cdot ([g] \cdot [h])$ , since these are reparameterizations of the same path.

## Group Structure

$\pi_1(X, p) = \{\text{loops at } p\} / \sim$ .

Define  $[f] \cdot [g] := [f \cdot g]$ .

It has an identity element  $[c_p] = e$ .

For any  $f \in \pi_1(X, p)$ , it has an inverse  $[\bar{f}]$  such that  $[f] \cdot [\bar{f}] = [\bar{f}] \cdot [f] = [c_p]$ .

Finally, it is associative by (3) above.

## Proposition

Suppose  $p, q \in X$  with  $X$  path-connected.

Then  $\pi_1(X, p)$  is isomorphic to  $\pi_1(X, q)$ .

Remark: this isomorphism is not canonical.

## Proof

We define a path  $\gamma$  from  $q$  to  $p$  and  $\Phi_\gamma : \pi_1(X, p) \rightarrow \pi_1(X, q)$  by  $[f] \mapsto [\gamma \cdot f \cdot \bar{\gamma}]$ .

$\Phi_\gamma$  is a group homomorphism.

$$\begin{aligned} \Phi_\gamma[f] \cdot \Phi_\gamma[g] &= [\gamma \cdot f \cdot \bar{\gamma}] \cdot [\gamma \cdot g \cdot \bar{\gamma}] \\ &= [\gamma \cdot f \cdot \bar{\gamma} \cdot \gamma \cdot g \cdot \bar{\gamma}] \\ &= [\gamma \cdot f] \cdot \overbrace{[\bar{\gamma} \cdot \gamma]}^{=e} \cdot [g \cdot \bar{\gamma}] \\ &= [\gamma \cdot (f \cdot g) \cdot \bar{\gamma}] \\ &= \Phi_\gamma[f \cdot g]. \end{aligned}$$

$\Phi_\gamma$  has an inverse,  $\Phi_{\bar{\gamma}} : \pi_1(X, q) \rightarrow \pi_1(X, p)$ .

$$\Phi_{\bar{\gamma}} \circ \Phi_\gamma[f] = \Phi_{\bar{\gamma}}[\gamma \cdot f \cdot \bar{\gamma}] = [\bar{\gamma} \cdot \gamma \cdot f \cdot \bar{\gamma} \cdot \gamma] = [f].$$

## Induced Homomorphism

$\varphi : (X, p) \rightarrow (Y, q)$  induces

$$\begin{aligned}\varphi_* : \pi_1(X, p) &\rightarrow \pi_1(Y, q) \\ [f] &\mapsto [\varphi \circ f].\end{aligned}$$

$\varphi_*$  is a homomorphism.

$$(\varphi_*[f]) \cdot (\varphi_*[g]) = [\varphi \circ f] \cdot [\varphi \circ g] = [(\varphi \circ f) \cdot (\varphi \circ g)] = [\varphi \circ (f \cdot g)] = \varphi_*[f \cdot g].$$

## Proposition

If  $\varphi, \psi : (X, p) \rightarrow (Y, q)$  are homotopic, then  $\varphi_* = \psi_* : \pi_1(X, p) \rightarrow \pi_1(Y, q)$ .

## Proof

Let  $[f] \in \pi_1(X, p)$ ,  $\varphi_*[f] = [\varphi \circ f]$  and  $\psi_*[f] = [\psi \circ f]$  and  $H : X \times I \rightarrow Y$  a homotopy between  $\varphi$  and  $\psi$ . Then define  $\tilde{H} : I \times I \rightarrow Y$  by  $\tilde{H}(s, t) = H(f(s), t)$  such that

$$\begin{aligned}\tilde{H}(s, 0) &= H(f(s), 0) = \varphi \circ f(s) \\ \tilde{H}(s, 1) &= H(f(s), 1) = \psi \circ f(s).\end{aligned}$$

## Corollary

If  $X \simeq Y$ , then  $\pi_1(X) \simeq \pi_1(Y)$ .

## Examples (\*)

$\pi_1(S^1) \cong \mathbb{Z}$  and  $\pi_1(S^n) = 0$  for  $n \geq 2$ .

For  $n \geq 2$ , write  $S^n = A_+ \cup A_-$  where  $A_+$  and  $A_-$  are large balls centered at the north and south pole respectively.

Then  $A_+$  and  $A_-$  are both homeomorphic to  $\mathbb{R}^n$  and  $A_+ \cap A_-$  (their intersection about the equator) is homeomorphic to  $S^{n-1} \times \mathbb{R}$ .

We fix a base point  $p \in A_+ \cap A_-$  and let  $f : I \rightarrow S^n$  be a loop based at  $p$ .

There exists a partition of  $I$ ,  $0 = s_0 < s_1 < \dots < s_k = 1$ , such that  $f|_{[s_i, s_{i+1}]}$  is contained in  $A_-$  or  $A_+$ .

Draw a path  $\gamma_i$  from  $p$  to  $f(s_i)$  such that  $\gamma_i \subseteq A_+ \cap A_-$ . Let  $f_i = f|_{[s_i, s_{i+1}]}$  such that  $f = f_0 \cdot f_1 \cdots f_k$ . Then this is path homotopic to

$$(f_0 \cdot \bar{\gamma}_1) \cdot (\gamma_1 \cdot f \cdot \bar{\gamma}_2) \cdots (\gamma_{k-1} \cdot f_{k-1} \cdot \bar{\gamma}_k) \cdot (\gamma_k \cdot f_k).$$

Each  $\gamma_i \cdot f_i \cdot \bar{\gamma}_i$  is contained in  $A_-$  or  $A_+$ , hence  $\gamma_i \cdot f_i \bar{\gamma}_{i+1} \sim c_p$ ,  $f \simeq c_p$  and  $[f] = e$ .

**April 2, 2025**

## Correction

For  $\varphi, \psi : (X, x_0) \rightarrow (Y, y_0)$  where  $\varphi \simeq \psi$ , we say a homotopy  $H$  between  $\varphi$  and  $\psi$  is base point preserving if  $H(x_0, t) = y_0$  for all  $t \in [0, 1]$ .

## Proposition

If  $\varphi \simeq \psi$  through a base point preserving homotopy, then  $\varphi_* = \psi_*$ ,  $\pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ .

For  $X \simeq Y$ ,  $\varphi : X \rightarrow Y$  and  $\psi : Y \rightarrow X$  where  $\psi \circ \varphi = \text{id}_X$  and  $\varphi \circ \psi = \text{id}_Y$ , in general  $\psi \circ \varphi(x_0) \neq x_0$  and  $\varphi \circ \psi(y_0) \neq y_0$ .

Set up:  $\varphi_0, \varphi_1 : X \rightarrow Y$  with  $\varphi_0 \simeq \varphi_1$  through a homotopy  $H$ .

Write  $\varphi_t = H(\cdot, t) : X \rightarrow Y$  and fix a base point  $x_0 \in X$  and set  $\gamma(t) = \varphi_t(x_0)$  for  $t \in [0, 1]$ .

## Proposition 1

$$(\varphi_0)_* = \Phi_\gamma \circ (\varphi_1)_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, \varphi(x_0)).$$

### Proof

Let  $f$  be a loop at  $x_0$ .

IMAGE 1

Let  $\gamma_t$  be  $\gamma|_{[0, t]}$  and then, by rescaling the domain  $[0, t]$  to  $[0, 1]$  i.e.

$$\begin{aligned} \gamma_t : [0, 1] &\rightarrow Y \\ s &\mapsto \gamma(ts). \end{aligned}$$

from  $\varphi_0(x_0)$  to  $\gamma(t) = \varphi_t(x_0)$ . Then  $\gamma_t \cdot (\phi_t \circ f) \cdot \bar{\gamma}_t$  is a homotopy between  $(\varphi_0 \circ f)$  and  $\gamma \cdot (\varphi_1 \circ f) \cdot \bar{\gamma}$ . Hence

$$(\varphi_0)_*[f] = [\varphi_0 \circ f] = [\gamma] \cdot [\varphi_1 \circ f] \cdot [\bar{\gamma}] = \Phi_\gamma \circ (\varphi_1)_*[f].$$

## Proposition 2

If  $X \simeq Y$ , then  $\pi_1(X) \cong \pi_1(Y)$ .

### Proof

Since  $(\psi \circ \varphi) \simeq \text{id}_X$ , by Proposition 1

$$\psi_* \circ \varphi_* = (\psi \circ \varphi)_* = \Phi_\gamma \circ (\text{id}_X)_* = \Phi_\gamma.$$

Hence  $\psi_* \circ \varphi_*$  is an isomorphism (as is  $\varphi_* \circ \psi_*$ ). Therefore  $\varphi_*$  and  $\psi_*$  are isomorphisms.

## Recall: Covering Map

For  $X, \tilde{X}$  connected,  $\pi : \tilde{X} \rightarrow X$  is a covering map if for each  $p \in X$  there exists a neighborhood  $U \subset X$  such that  $\pi^{-1}(U)$  is a disjoint union

$$\pi^{-1}(U) = \bigcup_{\alpha \in A} U_\alpha$$

such that  $\pi|_{U_\alpha} : U_\alpha \rightarrow U$  is a homeomorphism.

## Lifting Properties

A lift is a map  $\tilde{f}$  such that  $f = \pi \circ \tilde{f}$ .

1. Path Lifting: Let  $f : I \rightarrow X$  be a path from  $x_0$ . Then, for any  $\tilde{x}_0 \in \pi^{-1}(x_0)$ , there is a unique lift  $\tilde{f}$  of  $f$  with  $\tilde{f}(0) = \tilde{x}_0$ .
2. Homotopy Lifting: Let  $f_0, f_1 : I \rightarrow X$  be paths in  $X$  with  $f_0(0) = f_1(0) = x_0$  and  $f_0(1) = f_1(1)$ . Suppose  $H$  is a path homotopy between  $f_0$  and  $f_1$ . Then for any  $\tilde{x}_0 \in \pi^{-1}(x_0)$ , there is a unique lift  $\tilde{H} : I \times I \rightarrow \tilde{X}$  of  $H$ . In particular,  $\tilde{H}$  is a path homotopy between  $\tilde{f}_0$  and  $\tilde{f}_1$ . That is if  $H(0, t) = x_0$  then  $\tilde{H}(0, t) \in \pi^{-1}(x_0)$  for all  $t$ . Hence  $\tilde{H}(0, t) = \tilde{x}_0$ ,  $\forall t \in [0, 1]$ . Similarly,  $\tilde{H}(1, t)$  is identically constant. In particular,  $\tilde{f}_0(1) = \tilde{H}(1, 0) = \tilde{H}(1, 1) = \tilde{f}_1(1)$ .

## Fundamental Group of the Circle

$$\pi_1(S^1) = \mathbb{Z}.$$

### Example

$$\pi : \mathbb{R} \rightarrow S^1 \text{ by } s \mapsto e^{2\pi i \cdot s}.$$

### Proof

Take as a base point  $1 = x_0 \in S^1 \subseteq \mathbb{C}$ . For each  $n \in \mathbb{Z}$ , we define a loop  $\omega_n : [0, 1] \rightarrow S^1$  by  $s \mapsto e^{2\pi i \cdot ns}$ . Let  $f$  be a loop at  $x_0 \in S^1$ . We can lift  $f$  to  $\tilde{f} : I \rightarrow \mathbb{R}$  at  $0 \in \mathbb{R}$ . Then  $\tilde{f}(1) \in \pi^{-1}(x_0) = \mathbb{Z} \subseteq \mathbb{R}$ . This defines a map  $\varphi$  that sends a loop  $f$  to  $\tilde{f}(1) \in \mathbb{Z}$ . This  $\varphi$  induces  $\varphi : \pi_1(S^1, x_0) \rightarrow \mathbb{Z}$  well-defined. If  $f_0, f_1 : I \rightarrow S^1$  at  $x_0$  are path homotopic via  $H$ , then we may lift  $H$  to  $\tilde{H} : I \times I \rightarrow \mathbb{R}$  which implies  $\tilde{f}_0(1) = \tilde{f}_1(1)$ .

$\varphi$  is surjective, since for any  $n \in \mathbb{Z}$  we may consider the loop  $\omega_n$  where  $\tilde{\omega}_n(1) = n$ .

$\varphi$  is a group homomorphism since  $\varphi[f \cdot g] = \tilde{f \cdot g}(1) = \tilde{g} + \tilde{f}(1) = \varphi[f] + \varphi[g]$ .

$\varphi$  is injective, since if  $\varphi[f] = 0$  (i.e.  $\tilde{f}(0) = 0$ ) then  $\tilde{f}$  is a loop in  $\mathbb{R}$  and  $\tilde{f}$  is null-homotopic to  $c_0$  by  $H$ . Therefore  $\pi \circ \tilde{H}$  is a path-homotopy between  $f$  and  $c_{x_0}$  (i.e.  $[f] = e$ ).

## Path-Lifting

For  $f : I \rightarrow X$ , we have a special case where  $\text{im } f \subseteq U$  evenly covered. Write  $\pi^{-1}(U) = \dot{\bigcup} \tilde{U}_\alpha$  and pick the  $\tilde{U}_\alpha$  which contains  $\tilde{x}_0$ . Since  $\pi|_{\tilde{U}_\alpha} : \tilde{U}_\alpha \rightarrow U$  is a homeomorphism,  $\tilde{f} := (\pi|_{\tilde{U}_\alpha})^{-1} \circ f$  is the unique lift of  $f$  at  $\tilde{x}_0$ .

In general, pick a partition of  $I = [0, 1]$ ,  $0 = t_0 < t_1 < \dots < t_m = 1$ , such that  $\text{im } f|_{[t_i, t_{i+1}]} \subseteq U_i$  evenly covered. We can lift  $f|_{[0, t_1]}$  at  $\tilde{x}_0$ , giving  $\tilde{f} : [0, t_1] \rightarrow \tilde{X}$ . Next, we lift  $f|_{[t_1, t_2]}$  at  $\tilde{f}(t_1) \in \tilde{X}$ . Since the partition is finite, we may repeat the process until  $f$  is entirely lifted. This lift is unique.

## Homotopy Lifting

For each fixed  $(y_0, t_0) \in I \times I$ , by continuity, there is a neighborhood  $N(y_0) \times (t_0 - \varepsilon, t_0 + \varepsilon)$  such that  $H$  sends  $N(y_0) \times (t_0 - \varepsilon, t_0 + \varepsilon)$  inside an evenly covered neighborhood. By compactness of  $\{y_0\} \times [0, 1]$ , there is a finite collection of  $N_{t_i}(y_0) \times (t_i - \varepsilon_i, t_i + \varepsilon_i)$  such that they cover  $\{y_0\} \times I$  and the image of each under  $H$  is contained in an evenly covered neighborhood. Set  $N = \bigcap_i N_{t_i}(y_0)$ , a neighborhood of  $y_0$ , and construct a partition  $0 = t_0 < t_1 < \dots < t_m = 1$  such that  $H(N \times [t_i, t_{i+1}]) \subseteq U_i$  evenly covered. Then we can start with  $H|_{N \times [0, t_1]}$  and lift it at  $\tilde{x}_0$  by some  $(\pi|_{\tilde{U}_\alpha})^{-1}$ . Then lift each  $H|_{N \times [t_i, t_{i+1}]}$  one by one. Eventually, we have  $\tilde{H} : N \times [0, 1] \rightarrow \tilde{X}$  that lifts  $H : N \times [0, 1] \rightarrow \tilde{X}$  at  $\tilde{x}_0$ . This lift holds for any  $y_0 \in I$  and, if two strips overlap, then the lift must agree there by the uniqueness of path lifting. This assures that  $\tilde{H} : I^2 \rightarrow \tilde{X}$  is continuous.

## Remark

Given a continuous map  $F : Y \times I \rightarrow X$  and a covering  $\pi : \tilde{X} \rightarrow X$ , suppose that we have a map  $\tilde{F} : Y \times \{0\} \rightarrow \tilde{X}$  that lifts  $F|_{Y \times \{0\}} : Y \times \{0\} \rightarrow X$ . Then there is a unique lift  $\tilde{F} : Y \times I \rightarrow \tilde{X}$  of  $F$  which extends  $\tilde{F} : Y \times \{0\} \rightarrow \tilde{X}$ .

## Theorem: Fundamental Theorem of Algebra

A polynomial  $p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$  (with  $a_i \in \mathbb{C}$ ) has a root in  $\mathbb{C}$ .

### Proof

Suppose otherwise. Then  $p(z) \neq 0, \forall z \in \mathbb{C}$ . Consider  $f_r : [0, 1] \rightarrow S^1$  ( $r \geq 0$ ) by

$$f_r(s) = \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}.$$

Then  $f_0(s) \equiv 1$  is a constant loop at  $1 \in \mathbb{C}$ , and  $f_r \simeq f_0$  for each  $r \geq 0$ . Consider  $R \geq 1$  large such that  $R \gg \sum_{i=1}^n |a_i|$ . On  $\{z : |z| = R\}$ , we have

$$|z^n| > \left( \sum_{i=1}^n |a_i| \right) \cdot |z^{n-1}| \geq \sum_{i=1}^n |a_i| \cdot |z^{n-i}| = \left| \sum_{i=1}^n a_i z^{n-i} \right|.$$

This implies that  $p$  does not have any roots on  $\{|z| = R\}$ . Moreover, for  $p_t(z) = z^n + t(a_1 z^{n-1} + \cdots + a_{n-1} z + a_n)$  with  $0 \leq t \leq 1$ ,  $p_t$  does not have any roots on  $\{|z| = R\}$ . Consider

$$f_{R,t}(s) = \frac{p_t(Re^{2\pi i s})/p_t(R)}{|p_t(Re^{2\pi i s})/p_t(R)|}.$$

Then

$$f_{R,0}(s) = \frac{(Re^{2\pi i s})^n / R^n}{|(Re^{2\pi i s})^n / R^n|} = (e^{2\pi i s})^n = \omega_n(s).$$

Therefore  $f_{R,1}(s) \simeq f_R(s)$  and  $f_R \simeq \omega_n$ . But since  $\omega_n \neq \text{constant}$  so this is a contradiction.

**April 7, 2025**

## Definition: Retraction

Let  $X$  be a space and  $A \subseteq X$  be a subset. We say that a continuous map  $r : X \rightarrow A$  is a retraction if  $r|_A = \text{id}_A$ . In particular, because  $r \circ \iota_A = \text{id}_A$ , for  $x_0 \in A$

$$r_* \circ (\iota_A)_* : \pi_1(A, x_0) \rightarrow \pi_1(A, x_0)$$

is an isomorphism. Hence  $r_* : \pi(X, x_0) \rightarrow \pi(A, x_0)$  is surjective.

### Corollary

There is no retraction  $r : D^2 \rightarrow S^1 (= \partial D^2)$ .

## Proof

Suppose there is such a map  $r$ , then

$$r_* : \overbrace{\pi_1(D^2, x_0)}^{=0} \rightarrow \overbrace{\pi_1(S^1, x_0)}^{=\mathbb{Z}}$$

is surjective which is a contradiction.

## Corollary

Every continuous map  $h : D^2 \rightarrow D^2$  has a fixed point.

## Proof

Suppose  $\exists h : D^2 \rightarrow D^2$  without fixed points.

IMAGE 1

Define  $r : D^2 \rightarrow D^2$  as the ray pictured from  $h(x)$  through  $x$  to the boundary. If  $x \in \partial D^2$ , then by construction  $r(x) = x$ . Hence  $r : D^2 \rightarrow S^1$  is a retraction which is a contradiction.

## Corollary (Borsuk-Ulam)

Let  $f : S^2 \rightarrow \mathbb{R}^2$ . Then there exists a pair of antipodal points  $x$  and  $-x$  on  $S^2$  such that  $f(x) = f(-x)$ . This carries analogously to higher dimensions.

## Proof

Suppose that  $f(x) \neq f(-x)$  for all  $x \in S^2$ . We define  $g : S^2 \rightarrow S^1$  by  $g(x) = \frac{f(x)-f(-x)}{\|f(x)-f(-x)\|}$ . On  $S^2 \subseteq \mathbb{R}^3$ , we consider a loop  $\gamma$  at the equator by  $\gamma(s) = (\cos(2\pi s), \sin(2\pi s), 0)$  for  $s \in [0, 1]$ . Because  $S^2$  is simply connected,  $g \circ \gamma : [0, 1] \rightarrow S^1$  is path-homotopic to a constant loop in  $S^1$ . On the other hand, we lift  $h := g \circ \gamma$  to  $\tilde{h} : [0, 1] \rightarrow \mathbb{R}$  with  $\tilde{h}(0) = 0 \in \mathbb{R}$ . Note

$$h(s + 1/2) = g \circ \gamma(s + 1/2) = g(\cos(2\pi s + \pi), \sin(2\pi s + \pi), 0) = g(-\gamma(s)) = -g(\gamma(s)) = -h(s).$$

Hence  $\tilde{h}(s + 1/2) \in \pi^{-1}(-h(s))$  where  $\pi : \mathbb{R} \rightarrow S^1$  is the covering map. Since  $\pi^{-1}(-h(s)) = \frac{1}{2} + \tilde{h}(s) + \mathbb{Z}$ , for each  $s \in [0, 1/2]$  there is an integer  $q_s$  such that  $\tilde{h}(s + 1/2) = \frac{1}{2} + \tilde{h}(s) + q_s$  and

$$\tilde{h}(s + 1/2) - \tilde{h}(s) = \frac{1}{2} + q_s.$$

The left hand side depends continuously on  $s$  and, by continuity,  $q_s$  is a constant (call it  $q$ ). This gives

$$\tilde{h}(1) = \tilde{h}(1/2) + \frac{1}{2} + q = \tilde{h}(0) + 1 + 2q = 1 + 2q \neq 0$$

which contradicts the assertion that  $h$  is homotopic to a constant loop.

## Corollary (Large Fiber Lemma)

If  $f : [0, 1]^{n+1} \rightarrow \mathbb{R}^n$  is a continuous map, then there exist  $a, b \in [0, 1]^{n+1}$  such that  $f(a) = f(b)$  and  $|a - b| \geq 1$ .

Remark: if  $z = f(a) = f(b)$ , then the lemma says that  $\text{diam } f^{-1}(z) \geq 1$ .



## Proof

Take the sphere of radius  $1/2$  in  $[0, 1]^{n+1}$ , then by Borsuk-Ulam there exist a pair of antipodal points  $a, b \in S^1$  such that  $f(a) = f(b)$  and  $|a - b| \geq 1$ .

## Proposition

$$\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$$

## Proof

Write  $F : \pi_1(X \times Y) \rightarrow \pi_1(X) \times \pi_1(Y)$  by  $[f] \mapsto ([g], [h])$ . Then  $f : [0, 1] \rightarrow X \times Y$  is a loop at  $(x_0, y_0)$ ,  $f(s) = (g(s), h(s))$ , and  $g : [0, 1] \rightarrow X$  and  $h : [0, 1] \rightarrow Y$  are loops at  $x_0$  and  $y_0$  respectively.

## Definition: Wedge Sum

Let  $X$  and  $Y$  be path-connected topological spaces. Then  $X \vee Y = (X \amalg Y) / x_0 \sim y_0$

Let  $\{X_\alpha\}$  be a family of such spaces. Then  $\bigvee_\alpha X_\alpha = \bigamalg_\alpha X_\alpha / \sim$ .

## Sketch

$$\pi_1(S_-^1, x_0) \rightarrow \pi_1(X, x_0) \quad \text{gen} \mapsto \alpha$$

$$\pi_1(S_+^1, x_0) \rightarrow \pi_1(X, x_0) \quad \text{gen} \mapsto \beta$$

with  $\alpha \neq \beta$ ,  $\alpha\beta \neq \beta\alpha$ . Then  $\pi_1(X, x_0)$  should be  $\langle \alpha, \beta \rangle$ .

## Definition: Free Product

Let  $\{G_\alpha\}_\alpha$  be a family of groups.  $*_\alpha G_\alpha = \{g_1 g_2 \cdots g_k : \text{each } g_i \text{ is a word in some } A_\alpha\}$ .

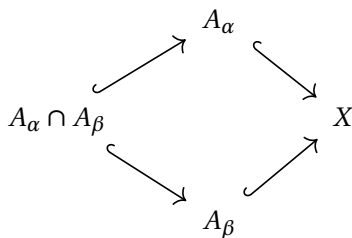
## Proposition

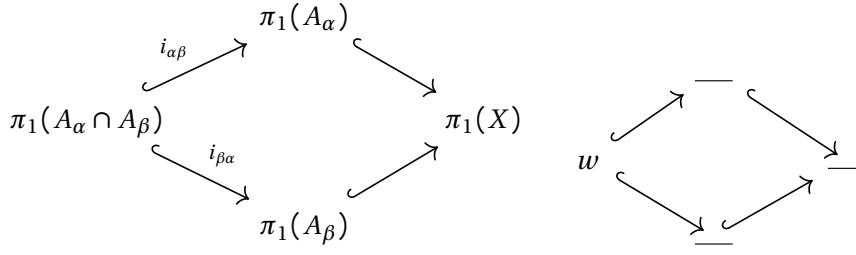
If for each  $\alpha$ , there is a group homomorphism  $\phi_\alpha : G_\alpha \rightarrow H$  then  $\{\phi_\alpha\}$  induces a group homomorphism  $\Phi : *_\alpha G_\alpha \rightarrow H$  by  $g_1 \cdots g_k \mapsto \phi_{\alpha_1}(g_1) \cdots \phi_{\alpha_k}(g_k)$ .

## Van-Kapen Theorem

### Setup

Let  $X = \bigcup_\alpha A_\alpha$ , each  $A_\alpha$  open and connected where  $\{A_\alpha\}$  have a common point  $x_0$ . Assume also that each  $A_\alpha \cap A_\beta$  is path connected. Then  $j_\alpha : A_\alpha \hookrightarrow X$  induces  $j_\alpha : \pi_1(A_\alpha, x_0) \rightarrow \pi_1(X, x_0)$ .  $\{j_\alpha\}_\alpha$  induces  $\Phi : *_\alpha \pi_1(A_\alpha, x_0) \rightarrow \pi_1(X, x_0)$  which is surjective by a similar argument as was used above for Example (\*) ( $S^2 = A_- \cup A_+$ ) applied to  $X = \bigcup_\alpha A_\alpha$ . Now, what is the kernel of  $\Phi$ ?





Then  $i_{\beta\alpha}(w)i_{\alpha\beta}(w)^{-1}$  is NOT id in  $*_{\alpha}\pi_1(A_{\alpha})$ .

But through  $\Phi$ , it should be  $\text{id} \in \pi_1(X, x_0)$ . Hence every element in  $*_{\alpha}\pi_1(A_{\alpha})$  of the form  $i_{\beta\alpha}(w)i_{\alpha\beta}(w)^{-1}$  where  $w \in \pi_1(A_{\alpha} \cap A_{\beta})$  is in the kernel of  $\Phi$ .

### Theorem (Van-Kampen)

If every  $A_{\alpha} \cap A_{\beta} \cap A_{\gamma}$  is path connected,  $\ker \Phi$  is the normal subgroup  $N$  generated by  $\{i_{\beta\alpha}(w)i_{\alpha\beta}(w)^{-1} : \alpha, \beta \in A, w \in \pi_1(A_{\alpha} \cap A_{\beta})\}$ . Hence  $\pi_1(X, x_0) \cong (*_{\alpha}\pi_1(A_{\alpha}, x_0))/N$ .

### Remarks

1. In the case that  $X = A_0 \cup A_1$  with  $A_0 \cap A_1$  path connected, then the intersection condition holds.
2. If  $X = A_0 \cup A_1$  and  $A_0 \cap A_1$  is simply connected, then  $N = \{\text{id}\}$  and  $\pi_1(X) = \pi_1(A_0) * \pi_1(A_1)$ .
3. If  $X = A_0 \cup A_1$  and  $A_1$  is simply connected, then  $\pi_1(X) = \pi_1(A_0)/N$  and  $N$  is the normal subgroup generated by

$$i_{01}(w) \overbrace{i_{10}(w)^{-1}}^{\in \pi_1(A_1, x_0)} = i_{01}(w)$$

i.e.  $N$  is the normal closure of  $i_{01}(\pi_1(A_0 \cap A_1))$ .

### Example

IMAGE 2

For each  $\alpha \in \{1, \dots, 5\}$ , let  $A_{\alpha}$  be a small neighborhood of  $T \cup e_1$ . Every double/triple intersection is a neighborhood of  $T$ . Hence it is path continuous and we have that  $\pi_1(A_{\alpha}) = \mathbb{Z}$ . Thus  $\pi_1(A_{\alpha} \cap A_{\beta}) = \text{id}$ , and  $\pi_1(X) = *_{\alpha}\pi_1(A_{\alpha})/N = *_1^5 \mathbb{Z}$ .

### Example

IMAGE 3

By Van-Kampen,  $\pi_1(X) = \pi_1(A_0)$  modulo the normal closure of  $i(\pi_1(A_0 \cap A_1))$ . That is

$$\langle a, b \mid aba^{-1}b^1 \rangle = \mathbb{Z}^2.$$

### Remark

In general, orientable  $M_g$  is the connected sum of  $g$  many toruses.

April 9, 2025

## Recall: Van-Kampen Theorem

Write  $\pi_1(X) = (\pi_1(A) * \pi_1(B))/N$  where  $N$  is the normal closure of  $i_{\alpha\beta}(w)i_{\beta\alpha}(w)^{-1}$  for  $w \in \pi_1(A \cap B)$ ,  $i_{\alpha\beta} : \pi_1(A \cap B) \rightarrow \pi_1(A)$  and  $i_{\beta\alpha} : \pi_1(A \cap B) \rightarrow \pi_1(B)$ .

### Example

$M_g$  is the connected sum of  $g$  many tori, and  $\pi_1(M_g) = \langle a_1, b_1, \dots, a_g, b_g \mid [a_1 b_1] \cdots [a_g b_g] \rangle$ .

### Example

$N_g$  is the connected sum of  $g$  many  $\mathbb{RP}^2$  (e.g.  $N_2$  is the Klein bottle).  $N_g$  has a polygon-representation by the  $2g$ -gon with boundary identified through  $a_1 a_1 a_2 a_2 \cdots a_g a_g$ . Therefore  $\pi_1(N_g) = \langle a_1 \cdots a_g \mid a_1^2 \cdots a_g^2 \rangle$ .

## Abelianization

1.  $\text{Ab}(\pi_1(M_g))$  is the free abelian group generated by  $\{a_1, b_1, \dots, a_g, b_g\} = \mathbb{Z}^{2g}$ .
2.  $\text{Ab}(\pi_1(N_g)) = \text{Ab}(\langle a_1 \cdots a_{g-1} a_1 a_2 \cdots a_g \mid a_1^2 \cdots a_g^2 \rangle) = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$ .

### Corollary

None of the surfaces in  $\{S^2, M_1, \dots, M_g, \dots, N_1, \dots, N_g, \dots\}$  are homotopic to each other.

## Definition: Cell Complex

0-cells are points; 1-cells,  $e^1$ , are intervals; 2-cells,  $e^2$ , are disks;  $n$ -cells,  $e^n$ , are  $\overline{B}^n$ .

A cell complex for space  $X$  is a decomposition (assuming finite dimensions)  $X = X^0 \cup X^1 \cup \dots \cup X^n$  where  $X^0$  is the discrete set of points (i.e. 0-cells),  $X^1$  is the space obtained by gluing 1-cells to  $X^0$  ( $\varphi_\alpha : \partial e_\alpha^1 \rightarrow X^0$ ),  $X^2$  is the space obtained by gluing 2-cells to  $X^1$  ( $\varphi_\alpha : \partial e_\alpha^2 \rightarrow X^1$ ), and in general  $X^n$  is obtained by gluing  $n$ -cells  $\{e_\alpha^n\}_\alpha$  to  $X^{n-1}$  by  $\varphi_\alpha : \partial e_\alpha^n = S^{n-1} \rightarrow X^{n-1}$ .

### Examples

Cell complexes need not be unique.  $S^2 = X^1 \cup_\alpha e_+^2 \cup_\alpha e_-^2$  and  $S^2 = \{e^0\} \cup_\alpha \{e^2\}$ .

$\mathbb{RP}^2 = \{e^1\} \cup_\alpha \{e^2\}$  where  $\varphi_\alpha$  is given by  $z \mapsto z^2$ .

$\mathbb{T}^2$  is gluing  $e^2$  to  $S^1 \vee S^1$ .

## Theorem (Computing Fundamental Group)

### Set up

Let  $X$  be a path-connected space,  $Y = X \cup_\alpha e_\alpha^2$  (i.e.  $X$  is created by gluing 2-cells  $\{e_\alpha^2\}_\alpha$  to  $X$  via  $\phi_\alpha : \partial e_\alpha^2 \rightarrow X$ ). The inclusion  $\iota : X \rightarrow Y$  induces  $\iota_* : \pi_1(X) \rightarrow \pi_1(Y)$ . Fix a base point  $s_0 \in S^1$ . For each  $\alpha$  we draw a path  $\gamma_\alpha$  from  $x_0$  to  $\varphi_\alpha(s_0)$ . Then  $\gamma_\alpha \cdot \varphi_\alpha \cdot \bar{\gamma}_\alpha$  is a loop based at  $x_0$ . Thus  $\gamma_\alpha \cdot \varphi_\alpha \cdot \bar{\gamma}_\alpha$  is null-homotopic in  $Y$  (because  $\varphi_\alpha$  is null-homotopic in  $e_\alpha^2$ ). That is  $\iota_*[\gamma_\alpha \cdot \varphi_\alpha \cdot \bar{\gamma}_\alpha] = \text{id}$  in  $\pi_1(Y)$  and is therefore in the kernel.

## Theorem

Let  $N$  be the normal subgroup in  $\pi_1(X)$  generated by elements of the form  $[\gamma_\alpha \cdot \varphi_\alpha \cdot \bar{\gamma}_\alpha]$ . Then  $\pi_1(Y) \cong \pi_1(X)/N$ .

IMAGE 1

## Example

$\mathbb{RP}^2$  is  $X^1$  with  $e^2$  glued to it by the map  $\varphi : z \mapsto z^2$ . Then  $\pi_1(\mathbb{RP}^2) = \pi_1(S^1)/N = \langle \gamma \mid \gamma^2 \rangle$  where  $N$  is generated by  $\varphi$ . Similarly, the theorem applies to any  $M_g$  or  $N_g$ .

## Definition: Deformation Retraction

For  $X \subseteq Z$ ,  $r : Z \rightarrow X$  is a retraction if  $r|_X = \text{id}_X$  implies  $r \circ \iota = \text{id}_X$ . If  $\iota \circ r : Z \rightarrow Z$  is homotopic to  $\text{id}_Z$ , then  $r_* : \pi_1(Z) \rightarrow \pi_1(X)$  is an isomorphism.

## Proof

For each  $\alpha$ , we glue a strip  $S_\alpha$  along  $\gamma_\alpha$ . We set the base at  $z_0$  above  $x_0$ ,  $Z = Y \cup_\alpha S_\alpha$ .  $Y$  is a deformation retraction of  $Z$  ( $\pi_1(Y) = \pi_1(Z)$ ).

IMAGE 2

Set  $A = Z - \bigcup_\alpha \{y_\alpha\}$ , where  $y_\alpha$  is a point in  $e_\alpha^2$  not intersecting  $S_\alpha$ .  $B = Z - X$ . A deformation retracts to  $X$   $\pi_1(A) = \pi_1(X)$ .  $B$  is the union of some  $S_\alpha$  (removing  $r_\alpha$ ) and some  $e_\alpha^2$  (removing  $\partial e_\alpha^2$ ).  $B$  is contractible,  $\pi_1(B) = \text{id}$  and  $A \cap B$  is the union of strips  $S_\alpha$  and open disks punctured at  $y_\alpha$ . Therefore

$$\pi_1(Y) = \pi_1(Z) = (\pi_1(A) * \pi_1(B))/N = \pi_1(A)/\iota_*(\pi_1(A \cap B)) \cong \pi_1(X)/\iota_*(\pi_1(A \cap B)).$$

Consider the loop  $\delta_\alpha \cdot \gamma_\alpha \cdot \varphi_\alpha \cdot \bar{\gamma}_\alpha \cdot \bar{\delta}_\alpha$  where  $\delta_\alpha$  runs from  $z_0$  to  $x_0$ , call this  $\lambda_\alpha$ . It suffices to show that these generate  $\pi_1(A \cap B, z_0)$ . Cover  $A \cap B$  by  $A_\alpha = (A \cap B) - \bigcup_{\beta \neq \alpha} e_\beta^2$ . Then  $A_\alpha$  is a union of strips (with trivial fundamental group) and a single punctured, open disk  $e_\alpha^2 - \{y_\alpha\}$  and  $\pi_1(A_\alpha) = \mathbb{Z} = \langle \lambda_\alpha \rangle$ . So  $A_\alpha \cap A_\beta$  is the union of strips, equal to  $A_\alpha \cap A_\beta \cap A_\gamma$  and simply connected. By Van-Kampen,

$$\pi_1(A \cap B) = (*_\alpha \pi_1(A_\alpha))/N = *_\alpha \pi_1(A_\alpha)$$

is the free group generated by  $\{\lambda_\alpha\}_\alpha$ . This completes the proof.

## Generalization (Theorem: Part 2)

If  $Y = X \cup_\alpha e_\alpha^n$  for  $n \geq 3$ , then  $\pi_1(Y) \cong \pi_1(X)$ .

This follows from the same argument where instead  $A_\alpha$  is the union of strips and a single punctured ball  $B^n - \{y_\alpha\} \simeq S^{n-1}$ . So  $\pi_1(A_\alpha) = \text{id}$ ,  $\pi_1(A \cap B) = \text{id}$ , and  $\pi_1(X) \cong \pi_1(Y)$ .

## Theorem: Part 3

Suppose  $X$  has a cell complex  $X = X^0 \cup X^1 \cup \dots \cup X^n$ . Then  $\pi_1(X) \cong \pi_1(X^2)$ .

The proof follows directly from part 2.

## Corollary

Given any group represented by generators and relations  $G = \langle g_\alpha \mid r_\beta \rangle$ , there is a cell complex  $X_G$ , of dimension 2, such that  $\pi_1(X_G) \cong G$ .

## Proof

For each  $g_\alpha$ , we draw a circle  $S_\alpha^1$ . Then  $X^1 = \bigvee_\alpha S_\alpha^1$  has fundamental group  $*_a \pi_1(S_\alpha) = \langle g_\alpha \rangle_\alpha$ . To construct  $X_G$ , for each  $r_\beta$  glue a 2-cell  $e_\alpha^2$  along  $r_\beta$  (think of  $r_\beta$  as a loop in  $X^1$ ). Then in  $X_G := X^1 \cup_\beta e_\beta^2$  we have  $\pi_1(X_G) = \langle g_\alpha \mid r_\beta \rangle$ .

**April 14, 2025**

## Recall: Covering Spaces

Let  $p : \tilde{X} \rightarrow X$ , both  $X$  and  $\tilde{X}$  path-connected.

1. Path-lifting: let  $f : I \rightarrow X$  starting at  $f(0) = x_0$ . There is a unique lifting  $\tilde{f}$  of  $f$  at  $\tilde{x}_0 \in p^{-1}(x_0)$ .
2. Homotopy-lifting: let  $f_0, f_1 : I \rightarrow X$  be two paths with  $f_0(0) = f_1(0) = x_0$  and  $f_0(1) = f_1(1)$ . Suppose  $f_t$  is a path-homotopy between  $f_0$  and  $f_1$ . Then there exists a unique lift  $\tilde{f}_t$  between  $\tilde{f}_0$  and  $\tilde{f}_1$  at  $\tilde{x} \in p^{-1}(x)$ .

These come from the following: let  $f_t : Y \rightarrow X$  be a homotopy between  $f_0$  and  $f_1$ . Given  $\tilde{f}_0 : Y \rightarrow \tilde{X}$  that lifts  $f_0$ , there exists a unique lifting  $\tilde{f}_t$ . For path-lifting, we take  $Y$  a point; for homotopy-lifting,  $Y = [0, 1]$ .

$$\begin{array}{ccc} & \tilde{X} & \\ f \nearrow & & \downarrow p \\ I & \xrightarrow{p \circ f} & X \end{array}$$

## Proposition 1.31 (in Hatcher)

The covering map  $p : \tilde{X} \rightarrow X$  induces  $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x)$ .

1.  $p_*$  is injective.
2.  $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$  are exactly loops at  $x_0$  that lift to loops at  $\tilde{x}_0$ .

## Proof of 1

Suppose  $p_*[f] = \text{id} \in \pi_1(X, x_0)$  where  $[f] \in \pi_1(\tilde{X}, \tilde{x}_0)$ . Then  $[p \circ f] = \text{id}$ , and  $[p \circ f]$  is path-homotopic to the constant loop  $c_{x_0}$ . Hence the lifting  $\tilde{p \circ f} = f$  is path-homotopic to a constant loop  $c_{\tilde{x}_0}$ .

## Proof of 2

Let  $[f] \in \pi_1(\tilde{X}, \tilde{x}_0)$ .  $p_*[f] = [p \circ f]$ ,  $p \circ f$  lifts to  $f$  at  $\tilde{x}_0$  which is a loop at  $\tilde{x}_0$ .

Let  $f$  be a loop at  $\tilde{x}_0$ . Suppose  $f$  lifts to a loop  $\tilde{f}$  at  $\tilde{x}_0$  (i.e.  $p \circ \tilde{f} = f$ ). Hence  $[f] = [p \circ \tilde{f}] = p_*[\tilde{f}] \in p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ .

## Example

If  $p : S^1 \rightarrow S^1$  by  $z \rightarrow z^2$ , then  $p_*(\pi_1(S^1, 1)) = 2\mathbb{Z} \leq \mathbb{Z} = \pi_1(S^1, 1)$ .

## Remark

If  $p : \tilde{X} \rightarrow X$  connected, then  $p^{-1}(x)$  has the same cardinality for all  $x \in X$ .

## Proof

Fix  $x_0 \in X$ . Consider  $\mathcal{A} = \{x \in X : p^{-1}(x) \text{ and } p^{-1}(x_0) \text{ have the same cardinality}\} \neq \emptyset$ . Then  $\mathcal{A}$  is open since for each  $x \in \mathcal{A}$ , there is a neighborhood  $U$  of  $x$  such that  $U$  is evenly covered by  $p$  (i.e.  $p^{-1}(U) = \bigcup_{\alpha \in I} V_\alpha$  where  $V_\alpha \stackrel{p}{\cong} U$ ). Then  $p^{-1}(x')$  has cardinality  $|I|$  for all  $x' \in U$ . It follows, since  $\mathcal{A}^c$  is open, that  $\mathcal{A}$  is also closed.

## Proposition

The number of sheets is given by  $[\pi_1(X, x_0) : p_*(\pi_1(\tilde{X}, \tilde{x}_0))]$ .

## Proof

Write  $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ . Define  $\Phi : \{H\text{-cosets in } \pi_1(X, x_0)\} \rightarrow p^{-1}(x_0)$  by  $H[g] \mapsto \tilde{g}(1)$  where  $\tilde{g}$  is a lift of  $g$  at  $\tilde{x}_0$ . This map is well defined, since for  $[h \cdot g]$  with  $h \in H$ ,  $\overline{h \cdot g}(1) = \tilde{g}(1)$  (because  $\tilde{h}(1) = \tilde{x}_0$ ).  $\Phi$  is surjective. Let  $\tilde{x}_1 \in p^{-1}(x_0)$

IMAGE 1

and let  $\tilde{g}$  be a path from  $\tilde{x}_0$  to  $\tilde{x}_1$ . Define  $g = p \circ \tilde{g}$ , a loop at  $x_0$ . Then  $\Phi(H[g]) = \tilde{g}(1) = \tilde{x}_1$ .  $\Phi$  is injective. Suppose  $\Phi(H[g_1]) = \Phi(H[g_2])$  (i.e.  $\tilde{g}_1(1) = \tilde{g}_2(1)$ ).

IMAGE 2

Consider the loop  $g_1 \bar{g}_2$  in  $X$  at  $x_0$ . It lifts to  $\tilde{g}_1 \bar{\tilde{g}}_2$ , which is a loop at  $\tilde{x}_0$ . This shows that  $[g_1 \bar{g}_2] \in H$  (i.e.  $H[g_1] = H[g_2]$ ).

## Recall (Manifolds 2)

If a smooth manifold  $M$  is non-orientable, then there is a double cover (2 sheets)  $p : \hat{M} \rightarrow M$  ( $\hat{M}$  connected). Consequently,  $\pi_1(M)$  has a subgroup of index 2.

## Definition: Locally Path-Connected

A topological space is called locally path-connected if for each  $x \in X$  and every neighborhood  $U \ni x$ , there is a neighborhood  $V \ni x$  such that  $V \subseteq U$  and  $V$  is path-connected (i.e.  $\forall x \in X$ , there exists a local basis  $\{U_\alpha\}$  at  $x$  such that each  $U_\alpha$  is path-connected). For example, the Topologist's sine curve with endpoints identified is path-connected but not locally path-connected.

## Proposition: Lifting Criterion

Let  $Y$  be path-connected and locally path-connected. Given a covering map  $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$  and a map  $f : (Y, y_0) \rightarrow (X, x_0)$ ,  $f$  has a lift  $\tilde{f}$  at  $\tilde{x}_0$  ( $\tilde{f}(y_0) = \tilde{x}_0$ ) if and only if  $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ .

## Proof

( $\Rightarrow$ )

$$\begin{array}{ccc} & \tilde{X} & \\ \tilde{f} \nearrow & \downarrow p & \\ Y & \xrightarrow{f} & X \end{array} \quad \begin{array}{ccc} & \pi(\tilde{X}) & \\ \tilde{f}_* \nearrow & \downarrow p_* & \\ \pi_1(Y) & \xrightarrow{f_*} & \pi_1(X) \end{array}$$

$$f_*\pi_1(Y) = (p_* \circ \tilde{f}_*)(\pi_1(Y)) \subseteq p_*\pi_1(\tilde{X}).$$

( $\Leftarrow$ ) Let  $y \in Y$ , and draw a path  $\gamma$  from  $y_0$  to  $y$ .

### IMAGE 3

We lift  $f \circ \gamma$  to a path in  $\tilde{X}$  starting at  $\tilde{x}_0$  and define  $\tilde{f}(y)$  as the endpoint (i.e.  $\tilde{f}(y) = \widetilde{f \circ \gamma}(a)$ ).

This is well-defined, since  $(f \circ \gamma) \cdot (f \circ \gamma')$  is a loop at  $x_0$  and  $[(f \circ \gamma) \cdot (f \circ \gamma')] = f_*[\gamma \cdot \gamma'] \in f_*\pi_1(Y, y_0) \leq p_*\pi_1(\tilde{X}, \tilde{x}_0)$ . Hence  $(f \circ \gamma) \cdot (f \circ \gamma')$  lifts to a loop at  $\tilde{x}_0$ .

### IMAGE 4

Therefore  $\widetilde{f \circ \gamma}(1) = \widetilde{f \circ \gamma'}(1)$ .

$\tilde{f}$  is continuous. Fix  $f(y) \in X$  and let  $U$  be a neighborhood of  $f(y)$  that is evenly covered by  $p$ . Choose a path-connected neighborhood  $V$  of  $y$  such that  $f(V) \subseteq U$ . We check  $\tilde{f}|_V$ .

### IMAGE 5

Because  $V$  is path-connected, we may draw a path  $\eta$  in  $V$  from  $y$  to  $y'$ . Then  $\tilde{f}(y') = \widetilde{f \circ \gamma \circ \eta}(1)$ , and  $\widetilde{\gamma \cdot \eta}$  is first lifting  $f \circ \gamma$  at  $\tilde{x}_0$  followed by lifting  $f \circ \eta$  at  $\tilde{\gamma}(1)$ . Let  $\tilde{U} \subseteq \tilde{X}$  such that  $p|_{\tilde{U}} : \tilde{U} \rightarrow U$  is a homeomorphism and  $\widetilde{f \circ \gamma}(1) \in \tilde{U}$ . Then  $\widetilde{f \circ \eta}(1) = (p^{-1})|_U \circ f(y')$ . Hence  $\tilde{f}(y') = f \circ (\gamma \circ \eta)(1) = \widetilde{f \circ \eta}(1) = (p^{-1})|_U \circ f(y')$  (i.e.  $\tilde{f} = (p^{-1})|_U = f$  on  $V$ ). Hence  $\tilde{f}$  is continuous at  $y$ .

$\tilde{f}$  is a lift of  $f$ . In fact,  $(p \circ \tilde{f})(y) = p \circ (\tilde{f} \gamma(1)) = f(y)$ .

### Corollary

$$\begin{array}{ccc} & \tilde{X} & \\ & \downarrow p & \\ Y & \xrightarrow{f} & X \end{array}$$

If  $Y$  is simply connected, then  $f_*\pi_1(Y) \leq p_*\pi_1(\tilde{X})$  always holds (i.e. we can always lift  $f$  to  $\tilde{f} : Y \rightarrow \tilde{X}$  in this case).

### Proposition: Unique Lifting

Given  $p : \tilde{X} \rightarrow X$  and  $f : Y \rightarrow X$ , if two lifts  $\tilde{f}_1$  and  $\tilde{f}_2$  of  $f$  agree at one point, then they agree everywhere on  $Y$ .

### Proof

Take  $\mathcal{A} = \{y \in Y : \tilde{f}_1(y) = \tilde{f}_2(y)\} \neq \emptyset$ . Locally for each  $y \in Y$  there exists a neighborhood  $V$  of  $y$  such that  $\tilde{f} = (p^{-1})|_U \circ f$ . If  $y \in \mathcal{A}$ , then  $\tilde{f}_1(y) = \tilde{f}_2(y)$ . Take a neighborhood  $U$  of  $f(y)$  that is evenly covered and  $\tilde{U}$  of  $\tilde{f}_1(y) = \tilde{f}_2(y)$  such that  $p|_{\tilde{U}} : \tilde{U} \rightarrow U$  is a homeomorphism. Then on  $V$ , a path-connected neighborhood such that  $f(V) \subseteq U$ ,  $\tilde{f}_i = (p^{-1})|_U \circ f$  (i.e.  $\tilde{f}_1 = \tilde{f}_2$  on  $V$ ). If  $y \in \mathcal{A}^c$ ,  $\tilde{f}_1(y) \neq \tilde{f}_2(y)$ . Then  $\tilde{U}_i \ni \tilde{f}_i(y)$  with  $\tilde{U}_1 \cap \tilde{U}_2 = \emptyset$ . Then on  $V$ ,  $\tilde{f}_i = (p^{-1})|_{\tilde{U}_i} \circ f$  (ie  $\tilde{f}_1$  and  $\tilde{f}_2$  never agree on  $V$ ). Hence  $\mathcal{A} = Y$ .

### Remark

If  $p : \tilde{X} \rightarrow X$  is a covering map, recall that a covering transformation is a map  $f : \tilde{X} \rightarrow \tilde{X}$  such that

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{f} & \tilde{X} \\ & \searrow p & \swarrow p \\ & X & \end{array}$$

commutes. This  $f : \tilde{X} \rightarrow \tilde{X}$  is a lift of  $p : \tilde{X} \rightarrow X$ . If we fix  $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$ , the lifting criterion says that  $p_*\pi_1(\tilde{X}, \tilde{x}_1) \leq p_*\pi_1(\tilde{X}, \tilde{x}_2)$ . In particular, if  $\pi_1(\tilde{X})$  is trivial, then this holds. Hence there is a unique lift of  $p$  (i.e. covering transformation)  $f$  such that  $f(\tilde{x}_1) = \tilde{x}_2$ .

April 16, 2025

## Question

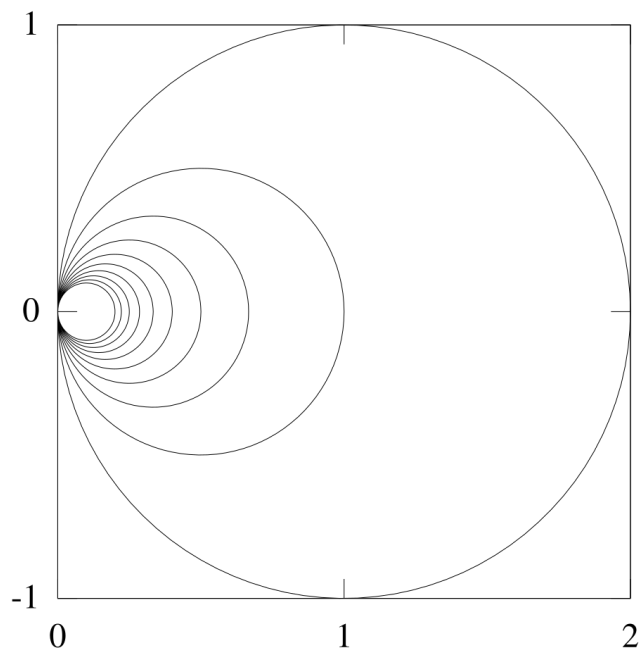
Given  $X$  path-connected and locally path-connected, when does  $X$  admit a simply connected covering space  $p : \tilde{X} \rightarrow X$ ?

## Definition: Semi-locally Simply Connected

We say that  $X$  is semi-locally simply connected if for any  $x \in X$  there exists a neighborhood  $U$  such that every loop in  $U$  is null-homotopic in  $X$ . That is  $\text{Im}(\pi_1(U) \rightarrow \pi_1(X))$  is trivial.

## Non-example

The Hawaiian earring in  $\mathbb{R}^2$ .



## Example

The cones over the Hawaiian earring.

IMAGE 1

In fact, this is simply connected.

## Example

The double Hawaiian earring with cones.

IMAGE 2



## Theorem

$X$  has a simply connected covering space (i.e. a universal covering) if and only if  $X$  is semi-locally simply connected.

## Proof

( $\implies$ ) Let  $x \in X$  and pick a neighborhood  $U$  of  $x$  that is evenly covered by  $p$ . Let  $f$  be a loop at  $x$  in  $U$ .  $f$  lifts to  $\tilde{f}$  at  $\tilde{x}_0$ , which is a loop. Retract  $\tilde{f}$  to  $c_{\tilde{x}_0}$  by a path-homotopy  $H$ . Then  $p \circ H$  shows that  $f$  is null-homotopic in  $X$ .

( $\impliedby$ ) We construct  $\tilde{X}$  as follows: fix  $x_0 \in X$  and set  $\tilde{X} = \{[\gamma] \text{ path homotopies} : \gamma \text{ is a path starting at } x_0\}$ . Let  $\mathcal{U} = \{U : \text{Im}(\pi_1(U) \rightarrow \pi_1(X)) \text{ is trivial}\}$ . By assumption  $\mathcal{U}$  is a basis for  $X$ . For each  $u \in \mathcal{U}$  and each  $\gamma$  from  $x_0$  to a point in  $U$ , we define  $U_{[\gamma]} = \{\gamma \cdot \eta : \eta \text{ starting at } \gamma(1) \text{ stays in } U\}$ . Then  $p : \tilde{X} \rightarrow X$  by  $[\gamma] \rightarrow \gamma(1)$ .

We need to check that  $\{U_{[\gamma]} : U \in \mathcal{U}, \gamma \text{ a path from } x_0 \text{ to a point in } U\}$  generates a topology on  $\tilde{X}$ .

We need also to check that  $p : U_{[\gamma]} \rightarrow U$  is bijective. It is clearly surjective, and if  $p[\gamma \cdot \eta] = p[\gamma \cdot \delta]$  with  $\eta, \delta$  paths starting at  $\gamma(1)$  and staying in  $U$ . Then  $\eta(1) = \delta(1)$  and, since  $\eta, \delta$  share the same endpoints and they stay in  $U_{[\gamma]}$ , then  $[\eta] = [\delta]$ . Hence  $[\gamma \cdot \eta] = [\gamma \cdot \delta]$  and  $p$  is injective.

Further, we need to check that  $p : U_{[\gamma]} \rightarrow U$  is a homeomorphism and that  $p^{-1}(U) = \bigcup_{[\gamma]} U_{[\gamma]}$ . Hence  $p$  is a covering map.

Finally, we need to check that  $\tilde{X}$  is simply connected. Recall that  $p : \tilde{X} \rightarrow X$  induces an injective homomorphism  $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$ . It suffices to show that  $p_*\pi_1(\tilde{X}, \tilde{x}_0) = \text{id}$ . We set  $\tilde{x}_0 = [C_{x_0}] \in \tilde{X}$ . Recall also that elements in  $p_*\pi_1(\tilde{X}, \tilde{x}_0)$  are exactly the loops in  $X$  at  $x_0$  such that they lift to loops at  $\tilde{x}_0$ . Suppose  $[\gamma] \in p_*\pi_1(\tilde{X}, \tilde{x}_0)$ . Then  $\gamma$  lifts to a loop  $\tilde{\gamma}$  at  $\tilde{x}_0 = [C_{x_0}]$ . For  $t \in [0, 1]$ , consider the path  $\gamma_t$  which follows  $\gamma$  on  $[0, t]$  then stays stationary at  $\gamma(t)$  for the remaining time. Then  $t \mapsto [\gamma_t]$  is a path on  $\tilde{X}$ ,  $p([\gamma_t]) = \gamma_t(1) = \gamma(t)$ , and  $t \mapsto [\gamma_t]$  is a lift of  $\gamma$  at  $\tilde{x}_0 = [C_{x_0}]$ . Then  $t \mapsto [\gamma_t]$  is a loop (i.e.  $[\gamma] = [\gamma_1] = \tilde{x}_0 = [C_{x_0}]$ ) and  $\gamma$  is null-homotopic. This shows that  $p_*\pi_1(\tilde{X}, \tilde{x}_0) = \text{id}$  (i.e.  $\tilde{X}$  is simply connected).

## Group Actions on Fibers (Monodromy Action)

Given  $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$  a covering map,  $\pi_1(X, x_0)$  acts on  $p^{-1}$  as follows:  $p^{-1}(x_0) \times \pi_1(X, x_0) \rightarrow p^{-1}(x_0)$  by  $(e, [f]) \mapsto \tilde{f}_e(1)$  where  $\tilde{f}_e$  is the (unique) lift of  $f$  at  $e \in p^{-1}(x_0)$ . This is a right  $\pi_1(X, x)$  action.

We want to check that  $(e \cdot [f]) \cdot [g] = e \cdot [f \cdot g]$ . We have that  $e \cdot [f \cdot g] = (\widetilde{f \cdot g})_e(1)$ , but  $(\widetilde{f \cdot g})_e$  is the lift of  $f$  at  $e$  followed by the lift of  $g$  at the endpoint of  $\tilde{f}_e$ , call it  $\tilde{f}_e(1) = z$ . Then  $(\widetilde{f \cdot g})_e(1) = \tilde{g}_z(1) = z \cdot [g] = (e \cdot [f]) \cdot [g]$ .

This action is transitive. Given  $e$  and  $e'$ , draw a path connecting them  $\tilde{g}$ . Under the map  $p$ , we have that  $p \circ \tilde{g} = g$  which is a loop at  $x_0$ . Then  $e \cdot [g] = \tilde{g}(1) = e'$ .

Recall: Given a right  $G$ -set  $S$ ,  $G_s = \{g \in G : s \cdot g = s\}$  is the isotropy subgroup at  $s \in S$ .

Given  $e \in p^{-1}(x_0)$ , the isotropy subgroup at  $e$  is all the loops such that their lifts at  $e$  are loops (i.e. the isotropy subgroup at  $e$  is precisely  $p_*\pi_1(\tilde{X}, e)$ ).

Recall:  $G \cdot S = G/G_s$ . Here, this tells us that  $p^{-1}(x_0) = \pi_1(X, x_0)/p_*\pi_1(\tilde{X}, e)$ . This recovers the fact that the number of sheets is equal to the index of  $\text{im}(p_*)$ .

In particular, if  $\tilde{X}$  is simply connected, then

- $\pi_1(X, x_0)$  acts freely on  $p^{-1}(x_0)$  and
- the number of sheets equals the cardinality of  $\pi_1(X, x_0)$ .

## Definition: Universal Cover

A covering space  $p : \tilde{X} \rightarrow X$  is called universal if it has the universal property (i.e. for any covering space  $q : Y \rightarrow X$ , there is a covering map  $\tilde{p} : \tilde{X} \rightarrow Y$  such that the associated diagram commutes).

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{p}} & Y \\ p \downarrow & \swarrow q & \\ X & & \end{array}$$

## Definition: Covering Homomorphism

Let  $p_1 : X_1 \rightarrow X$  and  $p_2 : X_2 \rightarrow X$  be two covering spaces. A covering homomorphism is a map  $\varphi : X_1 \rightarrow X_2$  such that the associated diagram commutes

$$\begin{array}{ccc} X_1 & \xrightarrow{\varphi} & X_2 \\ & \searrow p_1 & \swarrow p_2 \\ & X & \end{array}$$

By definition,  $\varphi$  is a lift of  $p_1$ .

## Proposition

1. A covering homomorphism  $\varphi$  is uniquely determined by its value at one point.
2. For each  $x \in X$ ,  $\varphi|_{p_1^{-1}(x)} : p_1^{-1}(x) \rightarrow p_2^{-1}(x)$  is  $\pi_1(X, x_0)$ -equivariant.
3. A covering homomorphism  $\varphi : X_1 \rightarrow X_2$  is a covering map. Assuming this, the universal cover is unique.

Recall: if  $S_1, S_2$  are right  $G$ -sets, a  $G$ -equivariant map  $\varphi : S_1 \rightarrow S_2$  is a map such that the associated diagram commutes

$$\begin{array}{ccc} S_1 & \xrightarrow{\varphi} & S_2 \\ \downarrow \cdot g & & \downarrow \cdot g \\ S_1 & \xrightarrow{\varphi} & S_2 \end{array}$$

## Proof of 2

Let  $e \in p_1^{-1}(x)$ . We need to show that  $\varphi(e) \cdot g = \varphi(e \cdot g)$ . We have that  $g \in \pi_1(X, x_0)$  is represented by a loop  $f$  at  $x_0$ . So  $e \cdot g = e \cdot [f] = \tilde{f}_e(1) \in X_1$ , and  $\varphi(e \cdot g) = \varphi(\tilde{f}_e(1))$ . On the left hand side, we have that  $\varphi(e) \cdot g = f_{\varphi(e)}(1) \in X_2$ . We need to verify that  $\varphi(\tilde{f}_e) = \tilde{f}_{\varphi(e)}$  which are both lifts of  $f$  at  $\varphi(e)$ . But since the diagram commutes,  $p_2(\varphi \circ \tilde{f}_e) = p_1 \circ \tilde{f}_e = f$ .

## Uniqueness in 3

Suppose we have

$$\begin{array}{ccc} X_1 & \xleftarrow{\psi} & X_2 \\ & \searrow p_1 & \swarrow p_2 \\ & X & \end{array}$$

with  $\varphi(e_1) = e_2$  and  $\psi(e_2) = e_1$ . Then  $\psi \circ \varphi(e_1) = e_1$ . Hence  $\psi \circ \varphi = \text{id}$  and, similarly,  $\varphi \circ \psi = \text{id}$ . Hence  $\varphi$  is a bijection and a homomorphism.

## Proof of 3

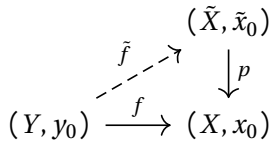
$\varphi$  is surjective. Given any  $e' \in X_2$ , set  $x_0 = p_2(e)$  and let  $e \in p_1^{-1}(x_0)$  so  $\varphi(e) \in p_2^{-1}(x_0)$ . Since  $\pi_1(X, x_0)$  acts transitively on  $p_2^{-1}(x_0)$ , there exists  $g \in \pi_1(X, x_0)$  such that  $e' = \varphi(e) \cdot g = \varphi(e \cdot g)$ .

$\varphi$  is a covering map. Let  $V$  be a neighborhood of  $x_0 \in X$  such that  $V$  is evenly covered by both  $p_1$  and  $p_2$ . Let  $U$  be a component in  $p_2^{-1}(V)$  that contains  $e_2$ . Then  $p_1^{-1}(V) = \bigcup U_\alpha$ .  $U$  as a component in  $p_2^{-1}(V)$  is both open and closed.

Hence  $\varphi^{-1}(U)$  is open and closed in  $p_1^{-1}(V) = \bigcup U_\alpha$ . It follows that  $\varphi^{-1}(U)$  is the disjoint union of several components of  $\{U_\alpha\}_\alpha$ , and each component is homeomorphic to  $V$  and consequently homeomorphic to  $U$ . This shows that  $\varphi$  is a covering map.

**April 21, 2025**

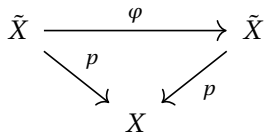
## Recall: Lifting Criterion



There exists a lift  $\tilde{f}$  of  $f$  at  $\tilde{x}_0$  if and only if  $f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)$ .

If  $(\tilde{X}, \tilde{x}_0) \xrightarrow{p} (Y, y_0)$ ,  $\pi_1(X, x_0)$  acts transitively on  $p^{-1}(x_0)$  by path lifting (a right action where  $e \in p^{-1}(x_0)$  by  $e \cdot [\gamma] = \tilde{\gamma}_e(1)$ ). The isotropy subgroup at  $e$  is  $p_*\pi_1(\tilde{X}, e)$ .

## Covering Transformations



Write  $\text{Aut}(\tilde{X} \xrightarrow{p} X)$  for the covering group  $\{\varphi : \tilde{X} \rightarrow \tilde{X} \text{ covering transformations}\}$ .

1.  $\varphi : \tilde{X} \rightarrow \tilde{X}$  is uniquely determined by its value at one point.
2. Given  $e_1, e_2 \in p^{-1}(x)$ , there is  $\varphi \in \text{Aut}(\tilde{X} \xrightarrow{p} X)$  if and only if  $p_*\pi_1(\tilde{X}, e_1) = p_*\pi_1(\tilde{X}, e_2)$ . In fact, for  $\varphi \in \text{Aut}(\tilde{X} \xrightarrow{p} X)$  with  $p_*\pi_1(\tilde{X}, e_1) \subseteq p_*\pi_1(\tilde{X}, e_2)$ .
3.  $\varphi|_{p^{-1}(x)} : p^{-1}(x) \rightarrow p^{-1}(x)$  is  $\pi_1(X, x)$ -equivariant (i.e.  $\varphi(e) \cdot \gamma = \varphi(e \cdot \gamma)$ ).

## Example

Given  $p : \mathbb{R} \rightarrow S^1$ , what is  $\text{Aut}(\mathbb{R} \xrightarrow{p} S^1)$ ?

$1 \in S^1$ ,  $p^{-1}(1) = \mathbb{Z} \subseteq \mathbb{R}$ ,  $\forall \varphi \in \text{Aut}(\mathbb{R} \xrightarrow{p} S^1)$ ,  $\varphi(0) = k \in \mathbb{Z}$ . Then  $\varphi(x) = x + k$ . In fact, the map  $x \mapsto x + k$  is a covering transformation that agrees with  $\varphi$  at  $0 \in \mathbb{R}$ . Hence they agree everywhere (i.e.  $\varphi(x) = x + k$  for all  $x$ ).

## Example

Given  $p : S^2 \rightarrow \mathbb{RP}^2$ , then  $\text{Aut}(S^2 \xrightarrow{p} \mathbb{RP}^2) = \{\text{id}, A\}$  with  $A$  the antipodal map.

## Proposition: Normal Covering

Let  $\tilde{X} \xrightarrow{p} X$  be a covering map. The following are equivalent

1. There exists  $x \in X$  such that  $p_*\pi_1(\tilde{X}, e)$  is normal for one (thus for all)  $e \in p^{-1}(x)$ .
2. For every  $x \in X$  and each  $e \in p^{-1}(x)$ ,  $p_*\pi_1(\tilde{X}, e)$  is normal.

3.  $\text{Aut}(\tilde{X} \xrightarrow{p} X)$  acts transitively on some (thus all) fiber  $p^{-1}(x)$ .

If any of these hold, we say that  $p : \tilde{X} \rightarrow X$  is a normal covering.

### Proof

Suppose  $e, e' \in p^{-1}(x)$  with  $p_*\pi_1(\tilde{X}, e)$  and  $p_*\pi_1(\tilde{X}, e')$ . These are the isotropy subgroups at  $e$  and  $e'$  respectively. We know also  $\pi_1(X, x)$  acts transitively on  $p^{-1}(x)$ .

Fact: If  $S$  is a right  $G$ -set, then  $G_s = \{h \in G : s \cdot h = s\}$  and  $G_{sg} = \{h \in G : s \cdot g \cdot h = s \cdot g\} = \{h \in S : s \cdot g \cdot h \cdot g^{-1} = s\}$ . So  $g \cdot G_{sg} \cdot g^{-1} \in G_s$  which implies that  $G_{sg} = g^{-1} \cdot G_s \cdot g$ . So if  $G_s$  is normal then so is  $G_{sg}$ .

IMAGE 1

$$\begin{array}{ccc} \pi_1(\tilde{X}, e_0) & \xrightarrow{\Phi_{\tilde{h}}} & \pi_1(\tilde{X}, e) \\ \downarrow p_* & & \downarrow p_* \\ \pi_1(X, x_0) & \xrightarrow{\Phi_h} & \pi_1(X, x) \end{array}$$

commutes. Hence  $\Phi_h$  maps  $p_*\pi_1(\tilde{X}, e_0)$  to  $p_*\pi_1(\tilde{X}, e)$ , and  $\Phi_h : \pi_1(X, x_0) \xrightarrow{\sim} \pi_1(X, x)$  preserves normal subgroups.

### (3) implies (1)

Finally, for every  $e_1, e_2 \in p^{-1}(x)$ , there exists  $\varphi \in \text{Aut}(\tilde{X} \xrightarrow{p} X)$  such that  $\varphi(e_1) = e_2$ . This holds if and only if  $p_*\pi_1(\tilde{X}, e_1) = p_*\pi_1(\tilde{X}, e_2)$  for every  $e_1, e_2 \in p^{-1}(x)$ . That is,  $e_2 = e_1 \cdot \gamma$  for some  $\gamma \in \pi_1(X, x)$  and  $H = \gamma^{-1}H\gamma$  for every  $\gamma \in \pi_1(X, x)$ . So  $H$  is normal.

### Remark

The (simply connected) universal cover is always normal because  $\{\text{id}\}$  is normal in  $\pi_1(X, x)$ .

### Theorem

Let  $p : \tilde{X} \rightarrow X$  be a covering map with  $x \in X$  and  $e \in p^{-1}(x)$ . Then  $\text{Aut}(\tilde{X} \xrightarrow{p} X) \cong \frac{N_G(H)}{H}$  where  $G = \pi_1(X, x)$ ,  $H = p_*\pi_1(\tilde{X}, e)$ , and  $N_G(H) = \{g \in G : g^{-1}Hg = H\}$ .

### Special Case 1

If  $p : \tilde{X} \rightarrow X$  is a normal covering, then  $H$  is normal in  $G$ . Then also  $\text{Aut}(\tilde{X} \xrightarrow{p} X) \cong G/H$ .

### Special Case 2

If  $p : \tilde{X} \rightarrow X$  is the (simply connected) universal covering, then  $\text{Aut}(\tilde{X} \xrightarrow{p} X) \cong \pi_1(X, x)$ .

### Proof

Let  $S$  be a right  $G$ -set with transitive action and  $\text{Aut}_G(S) = \{\varphi : S \rightarrow S \text{ } G\text{-equivariant bijections}\}$ . Fix  $s \in S$ . Then  $\text{Aut}_G(S) \cong \frac{N_G(H)}{H}$  where  $h = G_s$ .

Define  $\Phi : N_G(H) \rightarrow \text{Aut}_G(S)$  by  $\gamma \mapsto \Phi(\gamma) = \varphi_\gamma$  with  $\varphi_\gamma : S \rightarrow S$  defined by

$$G_{s \cdot \gamma} = \gamma^{-1}H\gamma = H = G_s.$$

Then there exists a unique  $\varphi_\gamma \in \text{Aut}_G(S)$  such that  $\varphi_\gamma(s) = s \cdot \gamma$ .

• Lemma

For each  $s' \in S$ ,  $s' = s \cdot \gamma'$  for some  $\gamma' \in G$ . Then  $\varphi_\gamma(s') = \varphi_\gamma(s \cdot \gamma') = \varphi_\gamma(s) \cdot \gamma' = s \cdot \gamma \gamma'$ . This is well defined. If  $s' = s \cdot \gamma''$ , then  $s = s(\gamma \cdot \gamma'' \cdot (\gamma')^{-1} \cdot \gamma^{-1})$  which implies that  $\gamma \cdot \gamma''(\gamma')^{-1} \cdot \gamma^{-1} \in G_s$  and  $\gamma'' \cdot (\gamma')^{-1} \in G_s$ .

$\Phi$  is a group homomorphism since

$$\varphi_{\gamma_1} \circ \varphi_{\gamma_2}(s) = \varphi_{\gamma_1}(s \cdot \gamma_2) = \varphi_{\gamma_1}(s) \cdot \gamma_2 = s \cdot \gamma_1 \cdot \gamma_2.$$

$\Phi$  is surjective since letting  $\varphi \in \text{Aut}_G(S)$ , it maps  $s$  to some  $\varphi(s) = s' = s \cdot \gamma$  and hence  $\varphi = \varphi_\gamma$ .

If  $\varphi_\gamma = \text{id}$ , then  $\varphi_\gamma(s) = s$  and  $\gamma \in G_s = H$ . So  $\Phi$  induces  $\frac{N_G(H)}{H} \cong \text{Aut}_G(S)$ .

Take  $G = \pi_1(X, x)$  and  $\text{Aut}(\tilde{X} \xrightarrow{p} X) \rightarrow \text{Aut}_G(p^{-1}(x)) \cong \frac{N_G(H)}{H}$  by  $\varphi \mapsto \varphi|_{p^{-1}(x)}$  where  $H$  is the isotropy subgroup of the  $\pi_1(X, x)$  action at  $e$  ( $p_*\pi_1(\tilde{X}, e)$ ). Then  $\varphi \mapsto \varphi|_{p^{-1}(x)}$  is injective because it is uniquely determined by its value at one point.

$\varphi \mapsto \varphi|_{p^{-1}(x)}$  is surjective. Letting  $\eta \in \text{Aut}_G(p^{-1}(x))$  and  $e_1 \in p^{-1}(x)$ , we set  $e_2 = \eta(e_1)$  and see that  $p_*\pi_1(\tilde{X}, e_1) = G_{e_1} = G_{e_2} = p_*\pi_1(\tilde{X}, e_2)$ . By the lifting criterion, there exists  $\varphi \in \text{Aut}(\tilde{X} \xrightarrow{p} X)$  such that  $\varphi(e_1) = e_2$ . Then  $\varphi|_{p^{-1}(x)} = \eta$  since both are in  $\text{Aut}_G(p^{-1}(x))$  and they agree at one point (hence everywhere). Thus we conclude that the map is a bijection and

$$\text{Aut}(\tilde{X} \xrightarrow{p} X) \cong \text{Aut}_G(p^{-1}(x)) \cong \frac{N_G(H)}{H}.$$

## Definition: Covering Space Action

Let  $X$  be connected and locally path connected with a group action  $\Gamma$  acting by homeomorphism. The quotient map  $p : X \rightarrow X/\Gamma$  will be a covering map if we impose  $(*)$  for all  $x \in X$ , there exists a neighborhood  $U$  of  $x$  such that  $U \cap (g \cdot U) = \emptyset$  for each  $g \in \Gamma - \{\text{id}\}$ . In particular,  $G$  acts freely on  $X$ . We say that a  $\Gamma$ -action on  $X$  is a covering space action if  $(*)$  is fulfilled.

### Counter-example

Consider an  $\mathbb{R}$  action on  $\mathbb{R}^2$  by translation. Then  $U \cap (g \cdot U) \neq \emptyset$ .

IMAGE 2

### Remark

Assuming  $(*)$ ,  $\{g \cdot U : g \in \Gamma\}$  is a disjoint family of open sets.

### Example

Take a  $\mathbb{Z}$ -action by  $\mathbb{R}^2$  given by  $\gamma(x, y) = (x + 1, -y)$ .

IMAGE 3

### Example

$S^2$  with  $\mathbb{Z}_2$ -action  $(\{\text{id}, A\})$ .

## Theorem

If  $\Gamma$  acts on  $X$  as a covering space action, then  $q : X \rightarrow X/\Gamma$  is a normal covering map.

### Proof

Let  $\bar{x} \in X/\Gamma$  and pick  $x \in q^{-1}(\bar{x})$ . By  $(*)$ , we have a neighborhood  $U$  such that  $\{g \cdot U : g \in \Gamma\}$  is a disjoint collection. Let  $V = q(U)$ , an open neighborhood of  $\bar{x}$  in  $X/\Gamma$ . Then  $q^{-1}(V) = \{g \cdot U : g \in \Gamma\}$ . Moreover,  $g \cdot U \rightarrow V$  is a homeomorphism. If there exist  $x', g'x' \in g \cdot U$ , then  $x' = h_1 \cdot u_1$  and  $g' \cdot x' = h_2 \cdot u_2$ . So  $h_1^{-1}x' \in U$  and  $h_2^{-1}g' \cdot x' \in U$  but this holds only for the identity map. So the covering map is injective.

## Classifications of Covering Spaces

Take  $X$  path-connected, locally path-connected and semi-locally simply path connected (guaranteeing a simply connected universal cover). Then there is a 1 – 1 correspondence between

$$\left\{ \begin{array}{l} \text{isomorphism classes of covering} \\ \text{spaces } p: \hat{X} \rightarrow X \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{conjugacy classes of} \\ \text{subgroups in } \pi_1(X, x) \end{array} \right\}$$

$$(p : \hat{X} \rightarrow X) \mapsto [p_* \pi_1(\hat{X}, \hat{x})]$$

April 23, 2025

### Recall: Theorem

For  $X$  path-connected, locally path-connected and semi-locally simply path connected,  $\Gamma$  acts on  $X$  as a covering group action (i.e.  $\forall x \in X$ , there exists a neighborhood  $U$  of  $x$  such that  $U \cap (g \cdot U) = \emptyset$  for all  $g \in \Gamma \setminus \{e\}$ ).

Then  $p : X \rightarrow X/\Gamma$  is a normal covering map. Moreover  $\text{Aut}(X \xrightarrow{p} X/\Gamma) = \Gamma$ .

### Proof

( $\supseteq$ ) this follows from

$$\begin{array}{ccc} X & \xrightarrow{g \cdot} & X \\ & \searrow p & \swarrow p \\ & X/\Gamma & \end{array}$$

( $\subseteq$ ) Let  $\varphi \in \text{Aut}(X \xrightarrow{p} X/\Gamma)$ . That is

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X \\ & \searrow p & \swarrow p \\ & X/\Gamma & \end{array}$$

commutes with  $\varphi$  a homeomorphism. Now let  $x \in p^{-1}(\bar{x})$  where  $\bar{x} \in X/\Gamma$ , and let  $x' = \varphi(x)$ . Then  $p(x) = \bar{x} = p(x')$ , hence  $x, x' \in p^{-1}(\bar{x})$ . Hence there is  $g \in \Gamma$  such that  $gx = x'$ . So we have

$$\begin{aligned} \varphi : X &\rightarrow X \varphi(x) = x' \\ g : X &\rightarrow X g(x) = x' \end{aligned}$$

so  $\varphi$  is equivalent to an action by  $g$ .

## Theorem

Take  $X$  path-connected, locally path-connected and semi-locally simply path connected (guaranteeing a simply connected universal cover). Then there is a 1 – 1 correspondence between

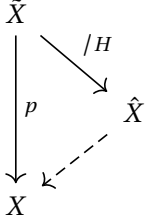
$$\{\text{isomorphism classes of covering maps } p: \tilde{X} \rightarrow X\} \leftrightarrow \{\text{conjugacy classes of subgroups in } \pi_1(X, x_0)\}$$

→ Assign a subgroup  $H = p_*(\hat{X}, \hat{e})$  for  $\hat{e} \in p^{-1}(x_0)$ .

← Given a conjugacy class of subgroups, pick a subgroup  $H$  in the class.

$$H \leq \pi_1(X, x_0) \cong \text{Aut}(\tilde{X} \xrightarrow{p} X)$$

Hence  $H$  acts naturally on  $\tilde{X}$  as covering transformations. Consider  $q: \tilde{X} \rightarrow \tilde{X}/H =: \hat{X}$ , a normal covering map.



Since  $\tilde{X}/\pi_1(X, x_0) = x$ , we have an induced map  $\hat{p}: \hat{X} \rightarrow X$ . We need to show that  $\hat{p}: \hat{X} \rightarrow X$  is a covering map with  $\hat{p}_*\pi_1(\hat{X}, \hat{e}) = H$  for some  $\hat{e} \in \hat{p}^{-1}(x)$ . Let  $U$  be a neighborhood of  $x$  such that  $p^{-1}(U) = \bigcup_{\alpha} \tilde{U}_{\alpha}$ . Then  $\{\tilde{U}_{\alpha}\}$  is a collect iof disjoint open sets and identical to  $\{g \cdot \tilde{U} : g \in \pi_1(X, x)\}$  where  $\tilde{U}$  is a component of  $p^{-1}(U)$ . The  $H$ -action permutes the copies in  $\{g \cdot \tilde{U}\} = \{\tilde{U}_{\alpha}\}$ . Hence  $q|_{\tilde{U}_{\alpha}}: \tilde{U}_{\alpha} \rightarrow \hat{X}$  is a homeomorphism. Let  $\hat{U}$  be a component in  $\hat{p}^{-1}(U)$ . Then  $q^{-1}(\hat{p}^{-1}(U)) = p^{-1}(U) = \bigcup_{\alpha} \tilde{U}_{\alpha}$  where  $q^{-1}(\hat{U})$  is a union of components in  $\bigcup_{\alpha} \tilde{U}_{\alpha}$ . Hence  $\hat{U}$  is homeomorphic to  $U$ , and  $\hat{p}^{-1}(U)$  is a union of components that are homemorphic to  $U$ .

Lastly, we show that  $\hat{p}_*\pi_1(\hat{X}, \hat{e}_0) = H$ . This is the isotropy subgroup of  $\pi_1(X, x_0)$ -actions at  $\hat{e}_0$ .  $q|_{p^{-1}(x_0)}: p^{-1}(x_0) \rightarrow \hat{p}^{-1}(x_0)$  is  $\pi_1(X, x_0)$ -equivariant (i.e.  $q(e \cdot \gamma) = q(e) \cdot \gamma$ ,  $q(e) = \hat{e}$  for  $e \in \tilde{X}$ ). Hence  $\gamma$  fixes  $q(e) = \hat{e}$  if and only if  $q(e \cdot \gamma) = q(e)$ , if and only if  $e \cdot \gamma$  and  $e$  are in the same  $H$ -orbit, if and only if  $\gamma \in H$ .

### Example 1

$X = S^1$  with  $\pi_1(S^1) = \mathbb{Z}$ .

$\mathbb{Z}$  has subgroups  $\mathbb{Z}, 2\mathbb{Z}, 3\mathbb{Z}, \dots, k\mathbb{Z}, \dots$  where  $k\mathbb{Z}$  corresponds to the covering map  $p_k: z \mapsto z^k$ .

### Example 2

$X$  the Mobius strip with  $\pi_1(X) = \mathbb{Z}$  with  $\pi_1(X) = \langle \gamma \rangle$  and  $\gamma(x, y) = (x + 1, -y)$ .

Take  $H = 2\mathbb{Z} = \langle 2\gamma \rangle \leq \mathbb{Z}$ . Then  $2\gamma(x, y) = (x + 2, y)$  and  $\mathbb{R}^2/H$  is the cylinder while the cylinder modulo  $\mathbb{Z}_2$  is the mobius strip.

### Example 3

The Klein bottle,  $K = \mathbb{R}^2/\Gamma$  with  $\Gamma$  generated by  $g(x, y) = (x + 1, -y)$  and  $h(x, y) = (x, y + 1)$ .

So  $\pi_1(K) = \langle g, h \rangle$ .  $g^2(x, y) = (x + 2, y)$  commutes with  $h$ , so  $\mathbb{Z}^2 \cong \langle g^2, h \rangle \leq \pi_1(K)$  and  $\mathbb{R}^2/\langle g^2, h \rangle = \mathbb{T}^2$  covers  $K$ .

## Simplexes

IMAGE 1

The standard  $n$ -simplex is

$$\Delta^n = \left\{ (t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^n t_i = 1, t_i \geq 0, \forall i \right\}$$

$$\Delta^1 = \left\{ (t_0, t_1) \in \mathbb{R}^2 : t_0 + t_1 = 1, t_0, t_1 \geq 0 \right\}$$

IMAGE 2

$$\Delta^2 = \{(t_0, t_1, t_2) \in \mathbb{R}^3 : t_0 + t_1 + t_2 = 1, t_0, t_1, t_2 \geq 0\}$$

IMAGE 3

$\Delta^n$  has  $(n+1)$ -many faces ( $(n+1)$ -simplex) where the  $i$ th face is  $\Delta^{n-1} \rightarrow \Delta^n$  by  $(t_0, \dots, t_{n-1}) \mapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$ . Let  $X$  be a topological space. A  $\Delta$ -complex structure on  $X$  is a family of maps  $\sigma_\alpha : \Delta^n \rightarrow X$  ( $n$  may depend on  $\alpha$ ) such that

1.  $\sigma_\alpha|_{\Delta^n} : \Delta^n \rightarrow X$  is injective and each point is in the image of at most one of  $\sigma_\alpha|_{\Delta^n}$ .
2.  $\sigma_\alpha|_{\text{a face of } \Delta^n}$  is some  $\sigma_\beta : \Delta^{n-1} \rightarrow X$  in the family.
3.  $A \subseteq X$  is open if and only if  $\sigma_\alpha^{-1}(A)$  is open in  $\Delta^n$  for all  $\alpha$ .

$$\sigma_\beta \text{ is } \Delta^{n-1} \xrightarrow{\text{ith face}} \Delta^n \xrightarrow{\sigma} X.$$

### Example

$S^1$  is the following 1-simplex

IMAGE 4

Then the “body” of  $\Delta^1 \xrightarrow{\sigma} X$  is

IMAGE 5

with  $\sigma \circ \delta_0 : \Delta^0 \rightarrow X$  and  $\sigma \circ \delta_1 : \Delta^0 \rightarrow X$ . The boundary  $\partial\sigma = \sum_{i=0}^n (-1)^i \sigma \circ \delta_i$ . They define  $\delta : C_n(X) \rightarrow C_{n-1}(X)$ . For this example, we have  $\partial\sigma = \sigma \circ \delta_0 + (-1)\sigma \circ \delta_1 = 0$ .

The  $i$ th face is  $\delta_i : \Delta^{n-1} \rightarrow \Delta^n$  by  $(t_0, \dots, t_{n-1}) \mapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$ .

In Hatcher’s notation, the boundary is  $\partial\sigma = \sum_{i=0}^n (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$  where we should think of  $[v_0, \dots, \hat{v}_i, \dots, v_n]$  as the  $i$ th face. So  $\sigma : \Delta^n = [v_0, \dots, v_n] \rightarrow X$ . Now we have

$$\cdots \xrightarrow{\partial} C_n(X) \xrightarrow{\partial} C_{n-1}(X) \cdots$$

where  $\partial^2 = 0$ .

### Proof

$$\begin{aligned} \partial(\partial\sigma) &= \partial\left(\sum_{i=0}^n (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}\right) \\ &= \sum_{i=0}^n (-1)^i \partial(\sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}) \\ &= \sum_{j < i} (-1)^i (-1)^j \sigma|_{[v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n]} + \sum_{j > i} (-1)^i (-1)^{j-1} \sigma|_{[v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]} \\ &= 0 \end{aligned}$$



## Homology Associated to the Delta Complex

We have  $\ker \partial \supseteq \operatorname{im} \partial$  where  $\ker \partial$  are the  $n$ -cycles and  $\operatorname{im} \partial$  are the  $n$ -boundaries, and

$$H_n^\delta(X) = \frac{\{n\text{-cycles}\}}{\{n\text{-boundaries}\}} = \frac{\ker \partial}{\operatorname{im} \partial}$$

### Example

For the circle,  $C_1(X) = \mathbb{Z} = \langle \sigma \rangle$  and  $C_0(X) = \mathbb{Z} = \langle \nu \rangle$ . Therefore

$$\overbrace{C_2(X)}^{=0} \rightarrow \overbrace{C_1(X)}^{=\mathbb{Z}} \xrightarrow{0} \overbrace{C_0(X)}^{=\mathbb{Z}} \rightarrow 0$$

Then  $H_1^\Delta(X) = \frac{\ker \partial}{\operatorname{im} \partial} = \mathbb{Z}/\{0\} = \mathbb{Z}$  and  $H_0^\Delta(X) = \frac{\ker \partial}{\operatorname{im} \partial} = \mathbb{Z}$ .

### An Aside

IMAGE 7

$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

### Example

For the torus, draw

IMAGE 6

So  $C_0(X) = \langle \nu \rangle = \mathbb{Z}$ ,  $C_1(X) = \langle a, b, c \rangle = \mathbb{Z}^3$  and  $C_2(X) = \langle U, L \rangle = \mathbb{Z}^2$ . Then also  $\partial U = a + b - c$  and  $\partial L = a + b - c$ , so  $\partial(U - L) = 0$  and  $\ker \partial_2 = \langle U - L \rangle \cong \mathbb{Z}$ . That is  $H_2^\Delta(X) = \frac{\ker \partial}{\operatorname{im} \partial} \cong \mathbb{Z}$ . Now  $\partial a = 0 = \partial b = \partial c$ , so  $\ker \partial_1 = \langle a, b, c \rangle$  and  $H_1^\Delta(X) = \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \frac{\langle a, b, a+b-c \rangle}{\langle a+b-c \rangle} \cong \mathbb{Z}^2$ . Finally we have that  $H_0^\Delta(X) = \frac{\ker \partial_0}{\operatorname{im} \partial_1} = \frac{\langle \nu \rangle}{\{0\}} \cong \mathbb{Z}$ .

### Example

For  $\mathbb{RP}^2$ , draw

IMAGE 8

$AC_0(X) = \langle \nu, w \rangle \cong \mathbb{Z}^2$ ,  $C_1(X) = \langle a, b, c \rangle \cong \mathbb{Z}^3$ , and  $C_2(X) = \langle U, L \rangle \cong \mathbb{Z}^2$ . Then  $\partial U = a + b + c$  while  $\partial L = a + b - c$ , so  $\ker \partial_2 = \{0\}$  and  $H_2^\Delta(X) = \frac{\ker \partial_2}{\operatorname{im} \partial_3} = \{0\}$ .  $\partial_1(a) = w - \nu$ ,  $\partial_1(b) = \nu - w$  and  $\partial_1(c) = 0$ , so  $\ker \partial_1 = \langle c, a - b \rangle$  and

$$H_1^\Delta(X) = \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \frac{\langle c, a + b \rangle}{\langle a + b + c, a + b - c \rangle} = \langle a + b + c, c \rangle / \langle a + b + c, 2c \rangle \cong \langle c \rangle / \langle 2c \rangle \cong \mathbb{Z}^2.$$