Manifolds II

January 6, 2025

Recall: Tangent Bundle

Given a chart (U,ϕ) about a point p, we have coordinates $(x^1,\dots,x^n)$ and a basis for T_qM of $\left(\frac{\partial}{\partial x^1}|_q,\dots,\frac{\partial}{\partial x^n}|_q\right)$ for $q\in U$.
Then given $TM \xrightarrow{\pi} M$, we may write $v_q = v^i\frac{\partial}{\partial x^i}|_q$.

Definition:

For M a topological manifold. A (real) vector bndle of rank k over M is a topological space E with a surjective continuous map $\pi : E \to M$ such that

1. $\forall p\in M$, the fiber $\pi^{-1}(p) =: E_p$ is endowed with the structure of a (real) vector space of dimension k.
2. $\forall p\in M$, there exists a neighborhood U of p in M and a homeomorphism $\Phi : \pi^{-1}(U)\to U\times\mathbb{R}^k$ called a local trivialization.

$$\begin{tikzcd}
\Phi : \pi^{-1}(U) \arrow[rr] \arrow[dr, "\pi"'] & & U\times\mathbb{R}^k \arrow[dl, "\pi_U"] \\
& U &
\end{tikzcd}$$

and $\Phi|_{E_q} : E_q \to \{q\}\times\mathbb{R}^k$ is a linear isometry.

Examples

1. $TM \xrightarrow{\pi} M$
2. $E = M\times\mathbb{R}^k$ with a global trivialization.
3. The Mobius bundle over S^1. $\gamma : \mathbb{R}^2\to\mathbb{R}^2$ by $(x,y)\mapsto(x+1,(-1)\cdot y)$. Then $\langle\gamma\rangle \cong \mathbb{Z}$ a subgroup acting freely and isometrically on $\mathbb{R}^2$. Then $E = \mathbb{R}^2/\langle\gamma\rangle \xrightarrow{\pi} S^1 = \mathbb{R}/\mathbb{Z}$ by $\overline{(x,y)}\mapsto\overline{x}$ is a vector bundle.

IMAGE 1

- We want to show that $\pi^{-1}(U)\cong U\times\mathbb{R}$

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "q"] \arrow[d, "\pi"] & E \arrow[d, "\pi"] & (x,y) \arrow[r, mapsto] \arrow[d, mapsto] & \overline{(x,y)} \arrow[d, mapsto] \\
\mathbb{R} \arrow[r, "\varepsilon"] & S^1 & x \arrow[r, mapsto] & e^{(2\pi i)x}
\end{tikzcd}$$

Then let $p\in S^1$. We choose U a neighborhood of p such that U is evenly covered by ε. This means $\varepsilon^{-1}(U)$ is a disjoint union of open sets difeomorphic to U.

IMAGE 2

Let $\tilde{U}$ be a component in $\pi^{-1}(U)$. Then $\pi_1^{-1}(\tilde{U})\cong\tilde{U}\times\mathbb{R}$ and $\pi^{-1}(U)$ is diffeomorphic to $U\times\mathbb{R}$.

Definition: Transition Function

Take $E \xrightarrow{\pi} M$ with $U, V \subseteq M$ admitting trivializations $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ and $\Psi : \pi^{-1}(V) \to V \times \mathbb{R}^k$. Let $w = U \cap V (\neq \varnothing)$.

$$
\Phi \circ \Psi^{-1} : \quad
\begin{tikzcd}
W \times \mathbb{R}^k \arrow[r] \arrow[dr] & \pi^{-1}(W) \arrow[r] \arrow[d] & W \times \mathbb{R}^k \arrow[dl] \\
& W &
\end{tikzcd}
$$

Then $\Phi \circ \Psi^{-1}|_{\{p\}\times\mathbb{R}^k}$ by $\{p\} \times \mathbb{R}^k \to \{p\} \times \mathbb{R}^k$ is a linear isomorphism.
$\Phi \circ \Psi^{-1}(p, v) = (p, \tau(p)v)$ by $\tau : p \mapsto \tau(p)$ and $\tau(p) \in GL(k, \mathbb{R})$ gives a smooth map $W \to GL(k, \mathbb{R})$.

Definition:

Let $\{E_1, \dots, E_k\}$ be a basis of $\mathbb{R}^k$. Then

$$\tau(p) \cdot E_i = \sum_j \tau(p)_i^j E_j$$

with $\tau(p) = (\tau(p)_i^j)$ and $\tau(p)_j^i \in \mathbb{R}$. It suffices to show each $\tau(*)_i^j$ mapping $W \to \mathbb{R}$ and $p \mapsto (\tau(p)_i^j)$ is smooth. Then if $\sigma(p, v) := \Phi \circ \Psi^{-1}(p, v)$, $\tau(p)_i^j = \pi_j(\sigma(p, E_i))$ and π_j is a projection to the j-th component in $\mathbb{R}^k$.

Lemma 10.6 (Vector Bundle Chart Lemma)

Given M a smooth manifold, suppose that $\forall\, p \in M$ we are given a vector space E_p of dimension k. Let $E = \coprod_{p\in M} E_p$ (as a set) and $\pi : E \to M$ a mapping E_p to p. Suppose also that we have

1. $\{U_\alpha\}_{\alpha\in A}$ an open cover of M with a countable subcover.
2. $\forall\, \alpha \in A$ we hav ea bijection $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ such that $\Phi_\alpha|_{E_p} : E_p \to \{p\} \times \mathbb{R}^k$ is a linear isomorphism.
3. $\forall\, \alpha, \beta \in A$ with $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \varnothing$ we have a smooth map $\tau_{\alpha\beta} : U_{\alpha\beta} \to GL(k, \mathbb{R})$ such that $\Phi_\alpha \circ \phi_\beta^{-1} : U_{\alpha\beta} \times \mathbb{R}^k \to U_{\alpha\beta} \times \mathbb{R}^k$ by $(p, v) \mapsto (p, \tau(p)v)$.

Then $E \xrightarrow{\pi} M$ is a vector bundle.

Example (Whitney Sum):

Suppose we have $E' \xrightarrow{\pi'} M$ and $E'' \xrightarrow{\pi''} M$ two vector bundles over M.
Define $E = E' \oplus E''$ a new vector bundle over M by $E_p = E'_p \oplus E''_p$. Let $\{U_\alpha\}_{\alpha\in A}$ be a countable open cover of M such that each U_α admits trivializations for E' and E''. Then for $\pi : E \to M$, define $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{k'} \times \mathbb{R}^{k''}$ by $(v', v'')_p \mapsto (p, \pi_2 \circ \Phi_\alpha^{-1}(v'), \pi_2 \circ \Phi''_\alpha(v''))$ where

$$\pi'(U_\alpha) \xrightarrow{\Phi'_\alpha} U_\alpha \times \mathbb{R}^{k'} \xrightarrow{\pi_2} \mathbb{R}^{k'}$$

Note that π_2 is the projection into the second component. Then $\tau : U_{\alpha\beta} \to G(k' + k'', \mathbb{R})$ by

$$p \mapsto \begin{pmatrix} \tau'(p) & 0 \\ 0 & \tau''(p) \end{pmatrix}$$

Example

For $\tau_{\alpha\beta} : U_{\alpha\beta} \to GL(k,\mathbb{R})$ by $p \mapsto \tau_{\alpha\beta}(p)$, we can write $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cup U_\gamma (\neq \varnothing)$ and get $\tau_{\alpha\beta} \cdot \tau_{\beta\gamma} = \tau_{\alpha\gamma}$.
Note that this is $\Phi_\alpha \circ (\phi_\beta^{-1} \circ \phi_\beta) \circ \Phi_\gamma^{-1}$.
Without loss of generality, we assume each U_α is a chart for M. Then we want to show that we satisfy Lemma 1.35 from Lee

$$\pi^{-1}(U_\alpha) \overset{\Phi_\alpha}{\to} U_{\alpha\times\mathbb{R}^k} \overset{\phi_\alpha\times\mathrm{id}}{\to} \phi_\alpha(U_\alpha)\times\mathbb{R}^k \subseteq \mathbb{R}^{n+k}$$

$(\pi^{-1}(U_\alpha)\cdot\tilde{\phi}_\alpha = (\phi_\alpha \times \mathrm{id}) \circ \Phi_\alpha)_{\alpha\in A}$ which satisfies (1).
Since

$$\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta) = \pi^{-1}(U_\alpha \cap U_\beta) \to \phi_\alpha(U_{\alpha\beta}) \times \mathbb{R}^K$$

we have that (2) is satisfied.
Finally, for (3),

$$\tilde{\phi_\beta} \circ \tilde{\phi_\alpha}^{-1} = (\Phi_\beta \circ (\phi_\beta \times \mathrm{id})) \circ ((\phi_\alpha \times \mathrm{id})^{-1} \circ \Phi_\alpha^{-1}) = \Phi_\beta \circ ((\phi_\beta \circ \phi_\alpha) \times \mathrm{id}) \circ \Phi_\alpha^{-1}$$

gives $(x,c) \mapsto ((\phi_\beta \circ \phi_\alpha^{-1})x, (\Phi_\beta \circ \Phi_\alpha^{-1})v)$ a diffeomorphism.
(4) and (5) are trivial, and this is indeed a smooth manifold. Now we wish to show that it is a vector bundle. To show that $\pi : E \to M$ is smooth,

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[r, "\pi"] & U_\alpha \arrow[d, "\phi_\alpha"] \\
\phi_\alpha(U_\alpha)\times\mathbb{R}^k \arrow[u, "\tilde{\phi}_\alpha^{-1}"] & \phi_\alpha(U_\alpha)
\end{tikzcd}$$

We have $\tilde{\phi}_\alpha^{-1} = (\phi_\alpha \times \mathrm{id})^{-1} \circ \Phi_\alpha^{-1}$.

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[r, "\Phi_\alpha"] & U_\alpha\times\mathbb{R}^k \arrow[d, "\phi_\alpha\times\mathrm{id}"] \\
\phi_\alpha(U_\alpha)\times\mathbb{R}^k \arrow[u, "\tilde{\phi}_\alpha^{-1}"] & \phi_\alpha(U_\alpha\times\mathbb{R}^k)
\end{tikzcd}$$

Definition: Section of a Bundle

A (smooth) section of $E \overset{\pi}{\to} M$ is a (smooth) map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{id}_M$.
$\Gamma(E) = \{$smooth sections of $E \overset{\pi}{\to} M\}$ and $\Gamma(E)$ is a $C^\infty(M)$-module.
The zero section $Z : M \to E$ is given by $p \mapsto 0_p \in E_p$.
If U has a local trivialization, $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$.

$$\begin{tikzcd}
\Phi: & \pi^{-1}(U) \arrow[rr] & & U\times\mathbb{R}^k & & (p,e_i) \arrow[ll, "\Phi^{-1}"'] \\
& & U \arrow[ul, dashed] \arrow[ur] & & p \arrow[ur, mapsto, "\tilde{e}_i"'] &
\end{tikzcd}$$

Define $\sigma_i : U \to \pi^{-1}(U)$ by $\sigma_i = \Phi^{-1} \circ \tilde{e}_i$ gives a local section that is non-zero on U.
$\{\sigma_1, \ldots, \sigma_n\}$ form a local frame on U (i.e. form a basis in E_p, $\forall\, p \in U$).

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Recall

Last time we had a vector bundle $E \xrightarrow{\pi} M$ of rank k satisfying

1. $\pi^{-1}(p) = E_p$ has a (real) vector space structure of dimension k.

2. We have a local trivialization, $\forall\, p \in M$ there exists a neighborhood U and a diffeomorphism Φ

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^k \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd}$$

   and $\Phi|_{E_p} : E_p \to \{p\} \times \mathbb{R}^k$ is a linear isomorphism.

A section $\sigma : M \to E$ is a smooth map such that $\pi \circ \sigma = \mathrm{id}_M$.
We say that a collection of sections $\{\sigma_1, \ldots, \sigma_k : U \to E\}$ is linearly independent if $\{\sigma_1(x), \ldots, \sigma_k(x)\}$ is linearly independent for each $x \in U$. This is a (local) frame if it is a basis.
If $U \subseteq M$ admits a trivialization

$$\Phi : \quad \begin{tikzcd} \pi^{-1}(U) \arrow[rr] \arrow[dr] & & U \times \mathbb{R}^k \arrow[dl] \\ & U & \end{tikzcd}$$

then there is a local frame $\{\sigma_1, \ldots, \sigma_k\}$ defined on U. Precisely, with $\tilde{e}_i(x) = (x, e_i)$, $\sigma_i = \Phi^{-1} \circ \tilde{e}_i$.

Proposition 10.19

If $U \subseteq M$ admits a local frame, then $\pi^{-1}(U)$ admits a local trivialization.

Remember

If $E \xrightarrow{\pi} M$ admits a global frame, then $E = \pi^{-1}(M)$ has a trivialization. In other words, E is diffeomorphic to a trivial vector bundle $M \times \mathbb{R}^k$.

Examples

Example 1

Mobius bundle over S^1.

IMAGE 1

To check whether it is a trivial bundle of S^1, it suffices to check whether there exists a nowhere zero (global) section. This cannot happen (by itermediate value theorem), hence it is not $S^1 \times \mathbb{R}$.

Example 2

TS^2 becasue there is no non-vanishing vector field over S^2, hence $TS^2 \neq S^2 \times \mathbb{R}^2$.

Example 3

Let G be a Lie group. Every $X \in T_eG(\cong \mathfrak{g})$ uniquely determines a (left-invariant) vector field $\tilde{X} \in \mathfrak{X}(G)$.
Starting with a basis $\{E_i\} \subseteq T_eG$ we get a global frame $\{\tilde{E}_i\}$ for TG. Hence TG is a trivial vector bundle $G \times \mathbb{R}^n$ $(n = \dim G)$. In particular, $TS^1 = S^1 \times \mathbb{R}$, $TS^3 = S^3 \times \mathbb{R}^3$.

Proof of Proposition

Define $\Psi : (x, v^1, \ldots, v^k) \in U \times \mathbb{R}^k \to \pi^{-1}(U) \ni v_x$ where $v_x = v^i \sigma_i(x)$.
Ψ is a bijection. Note that $\Psi|_{E_x} : E_x \to \{x\} \times \mathbb{R}^k$is a linear isomorphism because $\{\sigma_i(x)\}$ is a basis.
Then to show that Ψ is a diffeomorphism, it suffices to show then that Ψ is a local diffeomorphism.
Let $x \in U$ and let V be a neighborhood of x such that $\pi^{-1}(V) \stackrel{\Phi}{\to} V \times \mathbb{R}^k$.

$$V \times \mathbb{R}^k \stackrel{\Psi|_{V\times\mathbb{R}^k}}{\to} \pi^{-1}(V) \stackrel{\Psi}{\to} V \times \mathbb{R}^k$$

We show that this composition is a diffeomorphism. Since $\Phi(\sigma_i(x)) = (x, \sigma_i^1(x), \ldots, \sigma_i^k(x))$

$$\begin{aligned} \Phi \circ \Psi|_{V\times\mathbb{R}^k}(x, v^1, \ldots, v^k) &= \Phi(v^i \sigma_i(x)) \\ &= (x, v^i \sigma_i^1(x), \ldots, v^i \sigma_i^k(x)) \end{aligned}$$

Each $\sigma_i^j(x)$ is smooth. Hence $\Phi \circ \Psi|_{V\times\mathbb{R}^k}$ is smooth.
Let $\vec{v} = (v^1, \ldots, v^k)$ and $\sum(x) = (\sigma_i^j(x))$, then $\Phi \circ \Psi(x, \vec{v}) = (x, \vec{v} \cdot \sum(x))$. Its inverse

$$(\Phi \circ \Psi)^{-1}(x, \vec{w}) = \left(x, \vec{w} \cdot \sum(x)\right)$$

is also smooth. This shows that $\Phi \circ \Psi|_{V\times\mathbb{R}^k}$ is a diffeomorphism. Hence $\Psi|_{V\times\mathbb{R}^k}$ is a diffeomorphism ($V \subseteq U$) and $\Psi : U \times \mathbb{R}^k \to \pi^{-1}(U)$ is also a diffeomorphism.

Definition: Bundle Morphism

A bundle morphism between is a pair of smooth maps (f, F) such that this diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ M & \xrightarrow{f} & M' \end{array}$$

and $F|_{E_p} : E_p \to E'_{f(p)}$ is a linear map ($\forall\, p \in M$).
If it admits an inverse which is itself a bundle morphism, it is a unble isomorphism.
Remember that f is smooth because $f = \pi' \circ F \circ Z$

$$p \stackrel{Z}{\mapsto} 0_p \stackrel{F}{\mapsto} 0_{f(p)} \stackrel{\pi'}{\mapsto} f(p)$$

Remark

$$\begin{tikzcd} E \arrow[rr, "F"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

commutes and $F|_{E_p} : E_p \to E'_p$ is linear ($\forall\, p$).

Remark

$\operatorname{rank}(F|_{E_p})$ may depend on $p \in M$.

$$\begin{array}{ccc} TM & \xrightarrow{Df} & TR \\ \downarrow \pi & & \downarrow \pi \\ M & \xrightarrow{f} & \mathbb{R} \end{array}$$

e.g. $M = \mathbb{R}^2$, $E = E' = TR^2 (= \mathbb{R}^4)$, $F((u,v)_{(x,y)}) = (u, xv)$. For $x \neq 0$, $\operatorname{rank}(F|_{(x,y)}) = 2$ but for $x = 0$ $\operatorname{rank}(F|_{(0,y)}) = 1$.

Proposition 10.26

$$\begin{tikzcd} E \arrow[rr, "F"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

If F is a bijective, smooth bundle homomorphism, then it is a bundle isomorphism.
Proof left as an exercise. We need to show that F^{-1} is smooth.

Definition: Fiber Bundle

$F \to E \overset{\pi}{\to} M$ with fiber F such that $E_x = \pi^{-1}(x)$ is diffeomorphic to F. This diagram commutes.

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd}$$

Fact

If $N \overset{F}{\to} M$ is a submersion from compact manifolds, then F is a fiber bundle.

Chapter 11: Cotangent Bundles

Review: Linear Algebra

Suppose we have a real vector space V of dimension n. Then $V^* = \{f : V \to \mathbb{R} \text{ linear}\}$.

If V has a basis $\{E_1, \ldots, E_n\}$, then we may define the dual basis for V^* $\{\epsilon^1, \ldots, \epsilon^n\}$ by $\epsilon^j(E_i) = \delta_i^j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$.

Remember $V^{**} \cong V$ by $\xi : V \to V^{**}$ by $v \mapsto \xi(v) : V^* \to \mathbb{R}$ and $\omega \mapsto \omega(v)$.
Remember also that if A is a linear map $V \to W$ then we may define $A^* : \omega \in W^* \to V^* \ni A^*\omega$ by $v \in V \to \mathbb{R} \ni \omega(Av)$ (ie. $(A^*\omega)(v) = \omega(Av)$).

Definition: Cotangent Bundle

Let M^n be a smooth manifold, and let (U, ϕ) be a chart. Then T_pM has a basis

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \dots, \frac{\partial}{\partial x^n}\bigg|_p \right\}$$

for every $p \in U$. Take its dual basis

$$\left\{ \lambda^1|_p, \dots, \lambda^n|_p \right\}$$

for T_p^*M. The cotangent bundle $T^*M = \coprod_{p\in M} T_p^*M$.
Similar to the TM case, if $T^*M \xrightarrow{\pi} M$, then $\omega|_p \in \pi^{-1}(U) \xrightarrow{\Phi} U \times \mathbb{R}^n \ni (p, a_1, \dots, a_n)$ where a_i is given by $\omega|_p = a_i \lambda^i|_p$.
In other words, $a_i = \omega|_p \left(\frac{\partial}{\partial x^i}\Big|_p \right)$.

Computing Dual Transition

Suppose $(U, (x^1, \dots, x^n))$ and $(V, (y^1, \dots, y^n))$ are two charts $(W = U \cap V \neq \varnothing)$. Then $\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}$ gives a dual $\{\lambda^i|_p\}$ and $\left\{ \frac{\partial}{\partial y^i}\Big|_p \right\}$ gives $\{\mu^i|_p\}$.
Then, recall, $\frac{\partial}{\partial y^i}\Big|_p = \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j}\Big|_p$ and $x^j(y^1, \dots, y^n)$ is a j-component of $(y^1, \dots, y^n) \to M \to (x^1, \dots, x^n)$.
If $\omega \in T_p^*M$, $\omega = a_i \lambda^i|_p = b_j \mu^j|_p$

$$a_i = \omega|_p \left(\frac{\partial}{\partial x^i}\bigg|_p \right) = \omega_p \left(\frac{\partial y^j}{\partial x_i} \frac{\partial}{\partial y^j} \right) = \frac{\partial y^j}{\partial x^i} \omega \left(\frac{\partial}{\partial y^j} \right) = \frac{\partial y^j}{\partial x^i} b_j$$

In particular, $\mu^j = \omega$, then $a_i = \frac{\partial y^k}{\partial x^i} b_k = \frac{\partial y^j}{\partial x^i}$. Hence $\mu^j = \omega = a_i \lambda^i = \frac{\partial y^i}{\partial x^i} \lambda^i$.

Definition: Smooth Covector Field

A smooth covector field is a smooth section of T^*M, cal it $\Omega^1(M) = \Gamma(T^*M)$.
Given $f \in C^\infty(M)$, we can define a smooth covector field $df \in \Omega^1(M)$ by $df(v|_p) = (v_p)(f)$.
$df(X) = Xf$ is smooth if X and f are smooth.

Differential

Given a local chart $(U, (x^1, \dots, x^n))$ and a smooth function $f : U \to \mathbb{R}$, $df_p = a_i(p)\lambda^i|_p$.

$$\frac{\partial f}{\partial x^j} = df_p \left(\frac{\partial}{\partial x^j}\bigg|_p \right) = a_i(p) \lambda^i|_p \left(\frac{\partial}{\partial x^j}\bigg|_p \right) = a_i(p) \delta^i_j = a_j(p)$$

That is, $df_p = \frac{\partial f}{\partial x^j}(p) \lambda^j|_p$. In particular, if we consider the coordinate function $x^i : U \to \mathbb{R}$, then $dx^i|_p = \frac{\partial x^i}{\partial x^j}(p) \lambda^j|_p = \lambda^i|_p$ for each $p \in U$ (i.e. $dx^i = \lambda^i$ on U).
With this, we can write $df = \frac{\partial f}{\partial x^i} dx^i$ and $dy^j = \frac{\partial y^j}{\partial x^i} \partial x^i$.

Proposition 11.22

For $f \in C^{\infty}(M)$, then $df = 0$ if and only if f is constant on every compnent of M.

Proof

($\Longleftarrow$) is trivial.
($\Longrightarrow$) We assume M is connected. Fix $p \in M$, define $\mathcal{A} = \{q \in M \,:\, f(p) = f(q)\}$ is closed.
Now let $q \in A$ and U a local chart around q. Then $0 = df = \frac{\partial f}{\partial x^i} dx^i$ (i.e. $\frac{\partial f}{\partial x^i} \equiv 0$, $\forall\, i$).
Hence f is constant on U and $f(q) = f(p)$ for $U \in \mathcal{A}$.

Proposition 11.23

Take $\gamma : J \to M$ a smooth curve $f \in C^{\infty}(M)$. Then $(df|_{\gamma(t)})(\gamma'(t)) = (\gamma'(t))f = (f \circ \gamma)'(t)$.

IMAGE 2

Recall that if $v \in T_pM$ and $f \in C^{\infty}(M)$ then $vf = (f \circ \gamma)'(0)$ where $\gamma : (-\varepsilon, \varepsilon) \to M$, $\gamma(0) = p$ and $\gamma'(0) = v$ ($f \circ \gamma : \mathbb{R} \to \mathbb{R}$).

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Recall

T^*M and $\Omega'(M) = \Gamma(T^*M)$. Let $(U, (x^1, \ldots, x^n))$ be a chart. Then inside U, we may write $\omega = \omega_i dx^i$.
$\{dx^i|_p\}$ is a dual basis of $\left\{\frac{\partial}{\partial x^i} \subseteq T_pM\right\}$.
They are also $x^i : U \to \mathbb{R}$ coordinates functions where dx^i is the differential of x^i.
Given $f \in C^{\infty}(M)$ or $C^{\infty}(U)$, $df \in \Omega'(M)$ or $\Omega'(U)$ is defined by $df(X_p) = (Xf)(p)$.
Inside a chart, $df = \frac{\partial f}{\partial x^i} dx^i$.
We have a change of coordinates where $(U, (x^1, \ldots, x^n))$ and $(V, (y^1, \ldots, y^n))$ and $W = U \cap V \neq \varnothing$ gives $dy^j = \frac{\partial y^j}{\partial x^i} dx^i$.

Recall (Linear Algebra)

If $A : V \to W$ is a linear map with $w \in W^*$ and $v \in V$, then $A^* : W^* \to V^*$ is the dual map defined by $(A^* w)(v) := w(Av)$.

Dual of the Tangent Space

Let $F : M \to N$ be a smooth map between manifolds.

$$DF_p : T_pM \to T_{F(p)}N$$
$$(DF_p)^* : T^*_{F(p)}M \to T^*_pN$$

and $(DF^*_p \omega)(v) = \omega(DF_p(v))$ for $\omega \in T^*_{F(p)}N$ and $v \in T_pM$.

Definition: Pullback

Given $\omega \in \Omega'(N)$, we can define $F^*\omega$, a section of T^*M, by $(F^*\omega)_p(v) = \omega(DF_p(v))$ or $(F^*\omega)_p = DF^*_p \omega$.
We call this the pullback of ω by F.

Recall that for $u \in C^\infty(N)$, $M \xrightarrow{F} N \xrightarrow{u} \mathbb{R}$. Then we can define $F^* u \in C^\infty(M)$ by $F^* u = u \circ F$.

Proposition

If $F : M \to N$ is smooth, $u \in C^\infty(N)$ and $\omega \in \Omega^I(N)$, then

1. $F^*(u\omega) = (F^* u)(F^* \omega)$.
2. $F^*(du) = d(F^* u)$.

Proof of 1

$\forall p \in M$, $\forall v \in T_p M$,

$$(F^*(u\omega))_p(v) = DF_p^*(u\omega)(v) = u_{F(p)}\omega_{F(p)}(DF_p(v)) = (u \circ F(p))\omega(DF_p(v)) = (F^* u)(F^* \omega)$$

Proof of 2

$$(F^*(du))(v) = du(DF_p(v)) = (DF_p(v))u = (du)_{F(p)} DF_p(v) = d(u \circ F)(v) = d(F^* u)(v)$$

Change of Coordinates

Locally, $F : M \to N$. Let $(U, (x^1, \dots, x^n))$ be a chart around p and $(V, (y^1, \dots, y^n))$ a chart around $F(p)$. For $\omega \in \Omega^I(N)$, in V $\omega = \omega_i dy^i$ and

$$F^* \omega = F^*(\omega_i dy^i) = (F^* \omega_i)(F^* dy^i) = (F^* \omega_i) d(F^* y^i) = (\omega_i \circ F)(dF^i)$$

where $F^i = y^i \circ F$ is the ith component of F.
When F is smooth and $\omega \in \Omega^I(N)$, then $F^* \omega \in \Omega^I(M)$. In fact, locally, $F^* \omega = (\omega_i \circ F) d(F^i)$. Hence $F^* \omega$ is smooth.

Example 1

Take $F : \mathbb{R}^3 \to \mathbb{R}^2$ by $(x, y, z) \mapsto (u(x, y, z), v(x, y, z)) = (x^2 y, y \sin(z))$.
Then $\omega = u\, dv + v\, du \in \Omega^I(\mathbb{R}^2)$. So

$$\begin{aligned} F^* \omega &= F^*(u\, dv + v\, du) \\ &= (F^* u) d(F^* v) + (F^* v) d(F^* u) \\ &= x^2 y\, d(y \sin(z)) + (y \sin(z))\, d(x^2 y) \\ &= x^2 y(\sin(z)\, dy + y \cos(z)\, dz) + y \sin(z)(2xy\, dx + x^2\, dy) \end{aligned}$$

Example 2

$M = \mathbb{R}^2 - \{0\}$ and $\gamma : [0, 2\pi] \to M$ by $t \mapsto (r \cos(t), r \sin(t))$ for $r > 0$. Take $\omega = \frac{x\, dy - y\, dx}{x^2 + y^2} \in \Omega^I(M)$

$$\begin{aligned} \gamma^* \omega &= \frac{1}{r^2}(r \cos(t)\, d(r \sin(t)) - r \sin(t)\, d(r \cos(t)) \\ &= \cos(t)(\cos(t))\, dt - \sin(t)(\sin(t))\, dt \\ &= dt \end{aligned}$$

Definition: Line Integral

If $\eta \in \Omega^{\prime}(\mathbb{R})$ or $\Omega^{\prime}(I)$ (where $I \subseteq \mathbb{R}$) is an interval), η can be written as $\eta(t) = f(t)\,dt$ and define

$$\int_I \eta = \int_a^b f(t)\,dt$$

Let $\gamma : [a,b] \to M$ be a smooth curve on M. Let $\omega \in \Omega^{\prime}(t)$. Define

$$\int_\gamma \omega = \int_a^b \gamma^* \omega$$

with $\gamma^*(\omega) \in \Omega^{\prime}([a,b])$.

Proposition 11.31

Take $\phi : I \to J$ a diffeomorphism between intervals with $\phi' > 0$. Then

$$\int_J \phi^* \omega = \int_{\phi(J)} \omega$$

Write s for coordinates on J and t for coordinates on I. Then $\omega = f(t)\,dt \in \Omega^1(I)$ and

$$\phi^*\omega = (\phi^* f)\,d(\phi^* t) = (f \circ \phi)\,d(t \circ \phi) = f(\phi(s))\,d(\phi(s)) = f(\phi(s))\phi'(s)\,ds$$

Then

$$\int_J \phi^*\omega = \int_J f(\phi(s))\phi'(s)\,ds \overset{t=\phi(s)}{=} \int_I f(t)\,dt = \int_I \omega$$

Proposition 11.37: Independence of Reparameterization

Suppose $\gamma : I \to M$ is a smooth curve and $\phi : J \to I$ is a diffeomorphism with $\phi' > 0$. Then $\tilde{\gamma} := \gamma \circ \phi : J \to M$ is a reparameterization of γ and

$$\int_\gamma \omega = \int_{\tilde{\gamma}} \omega$$

If $\phi' < 0$, then $\int_\gamma \omega = -\int_{\tilde{\gamma}} \omega$.

Proof

$$\int_\gamma \omega = \int_I \gamma^*\omega \int_J \phi^*\gamma^*\omega = \int_J (\gamma \circ \phi)^*\omega = \int_{\tilde{\gamma}} \omega$$

Example

Take $\gamma : [0, 2\pi] \to M = \mathbb{R}^2 - \{0\}$ by $t \mapsto (r\cos(t), r\sin(t))$ with $r > 0$. If $\omega = \frac{x\,dy - y\,dx}{x^2+y^2}$, then $\gamma^*\omega = dt$ and

$$\int_\gamma \omega = \int_0^{2\pi} \gamma^*\omega = \int_0^{2\pi} dt = 2\pi$$

Proposition 11.38

For $\gamma : I \to M$

$$\int_\gamma \omega = \int_I \omega_{\gamma(t)}(\gamma'(t))\, dt$$

Proof

In a local chart $(U, (x^1, \ldots, x^n))$, we can write $\omega = \omega_i dx^i$. Then $\gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t))$ and

$$\begin{aligned}
\gamma^*\omega &= \gamma^*(\omega_i dx^i) \\
&= (\gamma^*\omega_i) d(\gamma^* x^i) \\
&= (\omega_i \circ \gamma) d\gamma^i \\
&= \omega_i(\gamma(t)) \frac{d\gamma^i}{dt} dt \\
&= \omega_i(\gamma(t)) \dot{\gamma}^i(t) dt
\end{aligned}$$

Since $\omega = \omega_i dx^i$ and $\dot{\gamma}(t) = (\dot{\gamma}^1(t), \ldots, \dot{\gamma}^n(t)) = \dot{\gamma}^i(t)\frac{\partial}{\partial x^i}$, $\omega_{\gamma(t)}(\dot{\gamma}(t)) = \omega_i(\gamma(t))\dot{\gamma}^i(t)$ and

$$\omega_i(\gamma(t))\dot{\gamma}^i(t) dt = \omega_{\gamma(t)}(\dot{\gamma}(t)) dt$$

Hence $\int_\gamma \omega = \int_I \gamma^* \omega = \int_I \omega_{\gamma(t)}(\dot{\gamma}(t))\, dt$.

Corollary

Then, if $f : M \to \mathbb{R}$ is a smooth function,

$$\int_\gamma df = \int_I (df)_{\gamma(t)}(\dot{\gamma}(t))\, dt = \int_I (f \circ \gamma)'(t)\, dt = f(\gamma(b)) - f(\gamma(a))$$

Therefore $\int_\gamma df$ only depends on the value of f at the endpoints of γ.

Definition: Exact and Conservative Forms

Let $\omega \in \Omega^1(M)$. We say that ω is...

1. exact if there exists $f \in C^\infty(M)$ such that $\omega = df$.
2. conservative if $\int_C \omega = 0$ for any closed, piecewise-smooth curve in M

f is called the potential of ω.

Remark

If $\int_C \omega = 0$, we may write C as the concatenation of curves γ then $-\sigma$. Then

$$0 = \int_C \omega = \int_\gamma \omega + \int_{-\sigma} \omega = \int_\gamma \omega - \int_\sigma \omega$$

Remark

Exact implies conservative.

Theorem

If $\omega \in \Omega^1(M)$ is conservative, then it is exact.

Proof

Fix a bse point $p_0 \in M$.
We have that $\int_p^q \omega = \int_\gamma \omega$ is well-defined by the conservative assumption, and we define $f(p) = \int_{p_0}^p \omega$.
Let $q_0 \in M$ and let $(U, (x^1, \ldots, x^n))$ be a chart centered at q_0. Inside U, $\omega = \omega_i dx^i$ and $df = \frac{\partial f}{\partial x^i} dx^i$.
We need to show that $\frac{\partial f}{\partial x^i} = \omega_i$ for each i. Fix an index i and consider a curve $\sigma : (-\varepsilon, \varepsilon) \to U$ by $t \mapsto (0, \ldots, t, \ldots, 0)$.

IMAGE 1

Let $q_- = \sigma(-\varepsilon)$, then

$$f(q_0) = \int_{p_0}^{q} \omega = \int_{p_0}^{q_-} \omega + \int_{q_-}^{q} \omega =: \tilde{f}(q)$$

so $f(q_0) = \text{constant} + \tilde{f}(q)$. Hence $\frac{\partial f}{\partial x^j} = \frac{\partial \tilde{f}}{\partial x^j}$ in U. Therefore

$$\begin{aligned}
\tilde{f}(\sigma(s)) &= \int_{q_-}^{\sigma(s)} \omega \\
&= \int_{\sigma|_{[-\varepsilon,s]}} \omega \\
&= \int_{-\varepsilon}^{s} \omega_{\sigma(t)}(\dot{\sigma}(t))\, dt \\
&= \int_{-\varepsilon}^{s} \omega_{\sigma(t)}\left(\frac{\partial}{\partial x^i}\right) dt \\
&= \int_{-\varepsilon}^{s} \omega_i(\sigma(t))\, dt
\end{aligned}$$

and

$$\frac{\partial f}{\partial x^i}\Big|_{q_0} = \frac{\partial \tilde{f}}{\partial x^i}\Big|_{q_0} = (\tilde{f} \circ \sigma)'(0) = \frac{d}{ds}\Big|_{s=0}\left(\int_{-\varepsilon}^{s} \omega_i(\sigma(t))\, dt\right) = \omega_i(\sigma(0)) = \omega_i(q_0)$$

Remark

Take $\omega = df \in \Omega^1(M)$ which is $\omega_i dx^i$ locally or $\omega_i = \frac{\partial f}{\partial x^i}$ when exact.

$$\frac{\partial \omega_i}{\partial x^j} = \frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial \omega_j}{\partial x^i}$$

Note: $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$ does not, in general, imply $\omega = df$.

January 15, 2025

Recall

If $\omega \in \Omega^1(M)$ and $\gamma : \mathbb{R} \supseteq I \to M$ a (piecewise) smooth curve, then

$$\int_\gamma \omega = \int_I \gamma^* \omega$$

If df is the differential of a smooth function, then

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a))$$

only depends on endpoints. In particular, along a closed (piecewise) smooth curve

$$\int_C df = 0$$

We have that ω is exact if $\omega = df$ and conservative if $\int_C \omega = 0$ for every closed curve.
ω is exact if and only if it is also conservative.

Recall: Checking Exactness

Take $\omega \in \Omega^1(M)$,

$$\omega_i dx^i = \omega = df = \frac{\partial f}{\partial x^i} dx^i$$

Then

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2}{\partial x^j \partial x^i}$$

That is, $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$.

Definition: Closed 1-Form

We say $\omega \in \Omega^1(M)$ is closed if in every chart $(U, (x^i))$, $\omega = \omega_i dx^i$ satisfies $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$.
Exact implies closed, however the converse is not true in general.

Example

$\exists \omega \in \Omega^1(\mathbb{R}^2 - \{0\})$ such that ω is closed but $\int_C \omega = 2\pi$.

Corollary 11.50

If $\omega \in \Omega^1(M)$ is closed, then $\forall p \in M$ there exists a chart U at p such that $\omega_U = df$ for some $f \in C^\infty(U)$

Proposition 11.45

For $\omega \in \Omega^1(M)$, the following are equivalent

1. ω is closed.
2. ω satisfies $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$ in some chart at every point.
3. For every open $U \subseteq M$ and $X, Y \in \mathfrak{X}(U)$, it holds that

$$X(\omega(Y)) - Y(\omega(X)) = \omega([X, Y])$$

The proof that 1 implies 2 is trivial.

Proof 3 Implies 1

Pick U as a chart, $X = \frac{\partial}{\partial x^i}$, and $Y = \frac{\partial}{\partial x^j}$. Then, since $\omega = \omega_i dx^i$,

$$X(\omega(Y)) = X(\omega_j) = \frac{\partial w_j}{\partial x^i}$$

Similarly, $Y(\omega(X)) = \frac{\partial \omega_i}{\partial x^j}$. Then $[X, Y] = \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$ and

$$\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} = 0$$

Proof 2 Implies 3

Fix any $p \in U$. We have a chart $(V, (x^i))$ at p such that $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$. Then

$$X(\omega(y)) = X\left((\omega_i dx^i)\left(Y^j \frac{\partial}{\partial x^j}\right)\right) = X(\omega_i Y^i) = (X\omega_i)Y^i + \omega_i(XY^i) = x^j \frac{\partial w_i}{\partial x^j} Y^i + \omega_i(XY^i) = X^j Y^i \frac{\partial \omega_j}{\partial x^i}$$

Similarly,

$$Y(\omega(X)) = Y(\omega_i x^i) = Y^j \frac{\partial \omega_i}{\partial x^j} x^i - \omega_i(YX^i) = X^i Y^j \frac{\partial \omega_i}{\partial x^j}$$

which is equivalent under a change of indicies. Hence

$$X(\omega(Y)) - Y(\omega(X)) = \omega_i(XY^i) - \omega_i(YX^i) = \omega_i(XY^i - YX^i) = \omega([X, Y])$$

Lemma

Suppose $F : M \to N$ is a local diffeomorphism. Then $F^* : \Omega^1(N) \to \Omega^1(M)$ sends exact (or closed) 1-forms to exact (or closed) ones.

Proof of Exact

If $\omega = df \in \Omega^1(N)$, then $F^*\omega = F^*(df) = d(F^*f)$ is exact on M.

Proof of Closed

If $\omega \in \Omega^1(N)$ is closed, then $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$ in every chart of N.
For any $p \in M$, we consider a chart at p by $(V, \phi \circ F)$

IMAGE 1

Therefore $\phi \circ F \circ (\phi \circ F)^{-1} = \mathrm{id}$ and $F^* = \mathrm{id}$ so $F^*\omega$ is closed.

Poincaré Lemma

Let $\omega \in \Omega^1(M)$ be closed. Fix $p \in M$, and let (U, ϕ) be a chart at p such that $\phi(U) = B_1(0) \subseteq \mathbb{R}^n$.

IMAGE 2

Assuming the above, every closed 1-form on $B_1(0)$ is exact. $(\phi^{-1})^*(\omega|_U) = df$ for some $f \in C^\infty(B_1(0))$ where $\omega|_U = \phi^*(df) = d(\phi^* f) \in C^\infty(U)$

Definition: Star-Shaped Domain

We say that $U \subseteq \mathbb{R}^n$ open is star-shaped with a center $c \in U$ (wlog $c = 0$) if for any $x \in U$, the segment γ_x from c to x is contained in U.

IMAGE 3

If $x = (x^i)$, then $\gamma_x(t) = (tx^i)$.

Theorem 11.49 (Poincaré Lemma)

If $U \subseteq \mathbb{R}^n$ is star-shaped, then every closed 1-form is exact.

Recall

If ω is an exact 1-form, then $f(q) = \int_{p_0}^{p} \omega$ is a potential.
We also have that $\int_\gamma \omega = \int_I \omega_{\gamma(t)}(\dot{\gamma}(t))\, dt$.

Proof

Let $\omega \in \Omega^1(U)$ be a closed 1-form.
We need to construct $f \in C^\infty(U)$ such that $df = \omega$. That is, for all i, $\frac{\partial f}{\partial x^i} = \omega^i$. Define

$$f(x) = \int_{\gamma_x} \omega = \int_0^1 \omega_{\gamma_x(t)}(\dot{\gamma}_x(t))\, dt = \int_0^1 \omega_i|_{\gamma_x(t)} dx^i(x^1, \ldots, x^n)\, dt = \int_0^1 \omega_i|_{tx} x^i\, dt$$

Since everything is smooth,

$$\begin{aligned}\frac{\partial f}{\partial x^j}(x) &= \int_0^1 \frac{\partial}{\partial x^j}(\omega_i(tx)\cdot x^i)\,dt \\ &= \int_0^1 \frac{\partial \omega_i(tx)}{\partial x^j}\cdot x^i + \omega_i(tx)\frac{\partial x^i}{\partial x^j}\,dt \\ &= \int_0^1 \left(\frac{\partial w_i}{\partial x^j}\right)\Big|_{(tx)} tx^i + \omega_j(tx)\,dt \\ &= \int_0^1 \frac{\partial \omega_j}{\partial x^i}\Big|_{tx} tx^i + \omega_j(tx)\,dt \\ &= \int_0^1 \frac{d}{dt}\big(t\omega_j(tx)\big)\,dt \\ &= t\omega_j(tx)\big|_0^1 \\ &= \omega_j(x)\end{aligned}$$

Tensors: Multilinear Maps

All vector spaces will be finite dimensional in our consideration.

$$F : V_1 \times \cdots \times V_k \to W$$

linear in every component. Denote $L(V_1, \ldots, V_k; W)$ to be the set of all such multilinear maps.
Given $\omega \in L(V_1;\mathbb{R}) = V_1^*$ and $\eta \in V_2^*$, we can define $\omega \otimes \eta \in L(V_1, V_2;\mathbb{R})$ by $\omega \otimes \eta(v_1, v_2) = \omega(v_1)\cdot\eta(v_2)$.

- Remark

  $(2\omega)\otimes\eta = \omega\otimes(2\eta)$. We assume $\otimes_{\mathbb{R}}$.

  Similarly, given $\omega_i \in V_i^*$, we can define $\omega_1 \otimes \cdots \otimes \omega_k \in L(V_1, \ldots, V_K;\mathbb{R})$.

Proposition

Let V_j with dimension n_j $(j = 1, \ldots, k)$. Each V_j has a basis $\{E_1^{(j)}, \ldots, E_{n_j}^{(j)}\}$.
Its dual basis $\{\varepsilon^1_{(j)}, \ldots, \varepsilon^{n_j}_{(j)}\} \subseteq V_j^*$. Then $L(V_1, \ldots, V_k;\mathbb{R})$ has a basis

$$\mathcal{B} = \left\{\varepsilon^{i_1}_{(1)} \otimes \cdots \otimes \varepsilon^{i_k}_{(k)} \;:\; 1 \le i_j \le n_j\right\}$$

Proof

For a multi-index $I = (i_1, \ldots, i_k)$ with $i \le i_j \le n_j$, we write $\varepsilon^I = \varepsilon^{i_1}_{(1)} \otimes \cdots \otimes \varepsilon^{i_k}_{(k)}$.
For any $F \in L(V_1, \ldots, V_k;\mathbb{R})$, define $F_I = F(E^{(1)}_{i_1}, \ldots, E^{(k)}_{i_k})$. We claim that $F = F_I\varepsilon^I$.
In fact, for $(v_1, \ldots, v_k) \in V_1 \times \cdots \times V_k$, $v_j = v^i_j E^{(j)}_i$. We may check that $F(v_1, \ldots, v_k) = F_I\varepsilon^I(v_1, \ldots, v_k)$.
Therefore $\mathcal{B}$ spans $L(V_1, \ldots, V_k;\mathbb{R})$.
Then, if $F_I\varepsilon^I = 0$, then applying it to $(E^{(1)}_{i_1}, \ldots E^{(k)}_{i_k})$ gives $F_I = 0$. Therefore $\mathcal{B}$ is linearly independent.
In particular, $\dim\, L(V_1, \ldots, V_k;\mathbb{R}) = \prod_{j=1}^k n_j = \prod_{j=1}^k \dim\, V_j$.

Definition: Formal Linear Combination

Let S be a set. Define

$$\mathcal{F}(S) = \left\{ \sum_{i=1}^{m} a_i s_i \;:\; a_i \in \mathbb{R},\ s_i \in S \right\}$$

This is the free (real) vector space on S containing formal linear combinations of elements of S.
Define $V_1 \otimes \cdots \otimes V_k = \mathcal{F}(V_1 \times \cdots \times V_k)/R$ where R is generated by

$$\begin{aligned}(v_1, \ldots, v_j + v_j', \ldots, v_k) &\sim (v_1, \ldots, v_j, \ldots, v_k) + (v_1, \ldots, v_j', \ldots, v_k) \\ (v_1, \ldots, cv_j, \ldots, v_k) &\sim c(v_1, \ldots, v_k)\end{aligned}$$

In other words, in the quotient $v_1 \otimes \cdots \otimes v_k = \prod(V_1, \ldots, v_k)$.

Proposition

$V_1 \otimes \cdots \otimes V_k$ has a basis $\left\{ E_{i_1}^{(1)} \otimes \cdots \otimes E_{i_k}^{(k)} \;:\; 1 \le i_j \le n_j \right\}$.

Proposition

There exists a canonical isomorphism $(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3)$ by sending $(v_1 \otimes v_2) \otimes v_3 \mapsto v_1 \otimes (v_2 \otimes v_3)$.

Proposition

$L(V_1, \ldots, V_k; \mathbb{R}) \cong V_1^* \otimes \cdots \otimes V_k^*$.

Proof Sketch

Define $\Phi : V_1^* \times \cdots \times V_k^* \to L(V_1, \ldots, V_k; \mathbb{R})$ by $(\omega^1, \ldots, \omega^k) \mapsto \omega^1 \otimes \cdots \otimes \omega^k$. By multilinear, this induces an isomorphism

$$\Phi : V_1^* \otimes \cdots \otimes V_k^* \cong L(V_1, \ldots, V_k; \mathbb{R})$$

Recall

$V^{**} \cong V$ for finite dimensional vector spaces, so $V_1 \otimes \cdots \otimes V_k = L(V_1^*, \ldots V_k^*; \mathbb{R})$.

Definition: Tensor

A tensor of (k, l)-type is an element in $\underbrace{V \otimes \cdots \otimes V}_{k} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{l}$.
The collection of such elements in $T^{(k,l)}V$. Most of the time we consider $T^{(0,l)}V$.

Examples

A vector in V is a $(1, 0)$-tensor.
A covector in V^* is a $(0, 1)$-tensor.
A linear map $A \in L(V)$ is a $(1, 1)$-tensor.
An inner product is a $(0, 2)$-tensor.

Symmetric Tensor

We say that $\alpha \in T^{(0,l)}V$ is symmetric if $\alpha(\dots, v_i, \dots, v_j \dots) = \alpha(\dots, v_j, \dots, v_i, \dots)$.

Alternating Tensor

We say that $\alpha \in T^{(0,l)}V$ is alternating if $\alpha(\dots, v_i, \dots, v_j \dots) = -\alpha(\dots, v_j, \dots, v_i, \dots)$.

January 22, 2024

Alternating/Symmetric Tensors

Let $\sigma \in S_l$ and $\alpha \in T^{(0,l)}V$.
Define σ_α or $(\sigma \cdot \alpha)$ as a new $(0,l)$-tensor by $(\sigma \cdot \alpha)(v_1, \dots, v_l) := \alpha(v_{\sigma(1)}, \dots, v_{\sigma(l)})$.
Then α is symmetric if and only if $\sigma \cdot \alpha = \alpha$.
α is alternating if and only if $\sigma \cdot \alpha = (\operatorname{sign} \sigma) \cdot \alpha$.
Define $\mathrm{Sym} : T^{(0,l)}V \to S^l V$ by

$$\mathrm{Sym}(\alpha) = \frac{1}{l!} \sum_{\sigma \in S^l} (\sigma \cdot \alpha)$$

Then $\mathrm{Sym}(\alpha)$ is symmetric for all $\tau \in S^l$.
Define $\mathrm{Alt} : T^{(0,l)}V \to \Lambda^l V$, the set of alternating (anti)-tensors by

$$\mathrm{Alt}(\alpha) = \frac{1}{l!} \sum_{\sigma \in S^l} (\operatorname{sign} \sigma)(\sigma \cdot \alpha)$$

Definition: Tensor Bundles

Recall that $T_pM \rightsquigarrow TM = \coprod_{p \in M} T_pM$ and $T_p^*M \rightsquigarrow T^*M$.
Then $T^{(k,l)}T_pM \rightsquigarrow T^{(k,l)}TM = \coprod_{p \in M} T^{(k,l)}T_pM$ a tensor bundle.
Mostly, we will consider $T^{(0,l)}TM$.
Inside a chart $(U, (x^1, \dots, x^n))$, $T^{(k,l)}TM$ has a local frame

$$\left\{ \frac{\partial}{\partial x^{i1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{ik}} \otimes dx^{j1} \otimes \cdots \otimes dx^{jl} \right\}$$

Definition: Smooth Tensor Field

A smooth tensor field of type (k,l) is a smooth section of $T^{(k,l)}TM$.
To check that a (o,l)-tensor field A is smooth, we can do either of the following

1. Write A in a local chart, then $A = A_I dx^I$ where A_I are functions in U and $dx^I = dx^{i1} \otimes dx^{il}$ with $I = (i1, \dots, il)$. Then A is smooth if and only if A_I is smooth for all I.

2. Check A testing on any l many smooth vector fields results in a smooth function.

Remark

Every $(0, l)$-tensor field A defines a map

$$\mathcal{A} = \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{l} \to C^{\infty}(M)$$

by $A(x_1, \ldots, X_l)(p) = A_p(X_1(p), \ldots, X_l(p))$. This map $\mathcal{A}$ is $C^{\infty}(M)$-multilinear.

Lemma 12.24

Every $C^{\infty}(M)$-multilinear map $\mathcal{A} : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \mathbb{C}^{\infty}(M)$ defines a smooth $(0, l)$-tensor field

$$A_p(v_1, \ldots, v_l) = (\mathcal{A}(X_1, \ldots, X_l))(p)$$

Example

Given $\omega \in \Omega^1(M)$, define $\mathcal{A} : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \mathbb{C}^{\infty}(M)$ by $(X, Y) \mapsto \omega(L_X Y)$.
If X, Y and X', Y' only agree at a point p, then in general $(L_X Y)(p) \neq (L_{X'} Y')(p)$.

Proof

$\mathcal{A}$ acts locally only depending on the value of $X_1, \ldots, X_l$ in a neighborhood of p, call it U.
It suffices to show that if $X_i = 0$ for some i on U, then $\mathcal{A}(X_1, \ldots, X_l)(p) = 0$.
Let ψ be a bump function with $\operatorname{supp} \psi \subseteq U$ and $\psi(p) = 1$. Let also $V \subseteq U$ such that $\overline{V} \subseteq U$.
Then $\psi X_i \equiv 0$ on M. Then

$$0 = \mathcal{A}(X_1, \ldots, \psi X_i, \ldots, X_l)(p) = \psi(p) A(X_1, \ldots, X_l)(p) = \mathcal{A}(X_1, \ldots, X_l)(p)$$

Now $\mathcal{A}$ acts pointwisely. Write $X_i = a_i^j \frac{\partial}{\partial x^j}$ in U.
Extend each $\frac{\partial}{\partial x^j}\Big|_V$ to $E_j \in \mathfrak{X}(M)$ and each $a_i^j|_V$ to $f_i^j \in C^{\infty}(M)$.
Then inside V,

$$\mathcal{A}(X_1, \ldots, X_l)(p) = \mathcal{A}(X_1, \ldots, f_i^j E_j, \ldots, X_l)(p) = f_i^j(p) \mathcal{A}(X_1, \ldots, X_l)(p)$$

Now let $v_1, \ldots, v_l \in T_p M$. Define A a $(0, l)$-tensor field by $A_p(v_1, \ldots, v_l) = \mathcal{A}(X_1, \ldots, X_l)$ where $X_i \in \mathfrak{X}(M)$ extends v_i.
By assumption, $A(X_1, \ldots, X_l)$ is a smooth function if $X_1, \ldots, X_l \in \mathfrak{X}(M)$ hence A is a smooth $(0, l)$-tensor field.

Definition:

Write $\mathcal{T}^{(0,l)} M = \Gamma(T^{(0,l)} TM)$ where Γ is the section.
Then for $F : M \to N$ a smooth map and $A \in \mathcal{T}^{(0,l)} N$, for $v_i \in T_p M$ define $F^* A \in \mathcal{T}^{(0,l)} M$ by

$$(F^* A)_p(v_1, \ldots, v_l) := A_{F(p)}(DF_p(v_1), \ldots, DF_p(v_l))$$

Lie Derivatives

Recall that if $X, Y \in \mathfrak{X}(M)$, we define $(L_X Y)_p$ where X generates a flow $\phi_t : M \to N$

IMAGE 1

$(\phi_{-t})_* Y_{\phi_t(p)} = ((\phi_{-t})_* Y)_p \in T_pM$ for $Y_p \in T_pM$. Then $L_X Y = \frac{d}{dt}\Big|_{t=0}((\phi_{-t})_* Y)_p$.
If $A \in \mathcal{T}^{(0,l)}M$,

IMAGE 2

$$(\phi_t^* A)_p = (\phi_t)^*(A_{\phi_t(p)} \in T^{(0,l)} T_pM$$

So $L_V A = \frac{d}{dt}\Big|_{t=0}(\phi_t^* A)_p$.

Properties

1. $L_V f = Vf$ (where $f \in C^\infty(M)$ can be thought of as a smooth $(0,0)$-tensor field). Then

$$(L_v f)(p) = \frac{d}{dt}\Big|_{t=0}(\phi_t^* f)_p = \frac{d}{dt}\Big|_{t=0}(f \circ \phi_t(p)) = (Vf)_p$$

1. $L_V(fA) = (Vf)A + fL_V A$.
2. $L_V(A \otimes B) = (L_V A) \otimes B + A \otimes (L_V B)$.
3. $L_V(A(X_1, \ldots, X_l)) = (L_V A)(X_1, \ldots, X_l) + A(L_V X_1, \ldots, X_l) + \ldots + A(X_1, \ldots, L_V X_l)$ for $A \in \mathcal{T}^{(o,l)}M$ and $X_i \in \mathfrak{X}(M)$.

Proof of 2

We have $O := \{p \in M : V_p \neq 0\}$ open in M and $\operatorname{supp} V = \overline{\{p \in M : V_p \neq 0\}}$.

1. (2) holds on O.

   Recall that if $V_p \neq 0$, then there exists a local chart $(U, (x^i))$ centered at p such that on U, $V = \frac{\partial}{\partial x^1}$. In particular, its flow ϕ_t is $(x^1, \ldots, x^n) \mapsto (x^1 + t, x^2, \ldots, x^n)$.
   Then take some chart $U \subseteq O$ centered at p such that $V = \frac{\partial}{\partial x^1}$ in U. Inside U, write $A = A_I dx^I$, and

$$\begin{aligned}
\phi_t^*(fA) &= (\phi_t^* f)(\phi_t^* f)(\phi_t^* A) \\
&= (f \circ \phi_t)\phi_t^*(A_I dx^I) \\
&= f(x^1 + t, x^2, \ldots, x^n) A_I(x^1 + t, \ldots, x^n)\phi_t^* dx^I \\
&= f(x^1 + t, x^2, \ldots, x^n) A_I(x^1 + t, \ldots, x^n) dx^I
\end{aligned}$$

2. (2) holds on $\operatorname{supp} V$ by taking limits.
3. (2) holds outside $\operatorname{supp} V$, since $V \equiv 0$ on open $M \setminus \operatorname{supp} V$ and hence $\phi_t \equiv \mathrm{id}$. So both sides are identically zero.

January 27, 2025

Recall: Prop 12.32(2)

$$L_V(fA) = (Vf)A + fL_V A$$

Proof Step 1:

Show that hte equality holds on $\{p \in M : V(p) \neq 0\}$.

Let $p \in M$ with $V(p) \neq 0$.
Take any chart (U, x^i) centered at p such that $V = \frac{\partial}{\partial x^i}$ on U. Then its flow is

$$\theta_t : (x^1, \ldots, x^n) \mapsto (x^1 + t, x^2, \ldots, x^n)$$

in U. In U, we write $A = A_I dx^I$ (where $dx^I = dx^{i1} \otimes \cdots \otimes dx^{il}$). Recall that

$$\theta_t^*(dx^i) = d(\theta_t^* x^i) = d(x^i \theta_t) = \begin{cases} d(x^1 + t) = dx^1 & i = 1 \\ d(x^i) & i \neq 1 \end{cases}$$

Write the pullback of θ_t

$$\begin{aligned} \theta_t^*(fA) &= (\theta_t^* f)(\theta_t^* A_I dx^I) \\ &= (f \circ \theta_t)(A_I \circ \theta_t)(dx^I) \\ &= f(x^1 + t, x^2, \ldots, x^n) A_I(x^1 + t, \ldots, x^n) dx^I \end{aligned}$$

So for $p = (x^i)$,

$$\begin{aligned} (L_V(fA))_p &= \frac{d}{dt}\bigg|_{t=0} f(x^1 + t, x^2, \ldots, x^n) A_I(x^1 + t, \ldots, x^n) dx^I \\ &= \underbrace{\frac{\partial f}{\partial x^1}(x^1, \ldots, x^n}_{Vf} \underbrace{A_I(x^1, \ldots, x^n) dX^I}_{\theta_t^* A} + f(x^1, \ldots, x^n) \frac{\partial A_I}{\partial x^1 (x^1, \ldots, x^n) dx^I} \end{aligned}$$

inside U. Hence $Vf = \frac{\partial f}{\partial x^1}$.

Corollary

$L_V(df) = d(L_v f)$ for $f \in C^\infty(M)$.

- Proof

For all $X \in \mathfrak{X}(M)$,

$$(L_V(df))(X) = V(df(X)) - df(L_V X) = VXf - [V, X]f = VXf - (VXf - XVf) = XVf$$

and

$$(d(L_V f))(X) = X(L_V f) = XVf.$$

Proof Step 2:

Show that the equality holds on $\overline{\{p \in M : V(p) \neq 0\}}$.

Proof Step 3:

Show that the equality holds elsewhere.

Recall: Invariance

For two vector fields, X and Y, Y is invariant under the flow of X if $L_X Y \equiv 0$.
We say a $(0,l)$-tensor field A is invariant under a map $F: M \to M$ if $F^* A = A$. Equivalently, if under a flow $\theta_t : M \to M$ if $\theta_t^* A = A$ for all t.

Theorem 12.37

A is invariant under θ_t, $\forall\, t$, if and only if $L_V A = 0$.

Note

$$\frac{d}{dt}\bigg|_{t=t_0} (\theta_t^* A)_p = (\theta_{t_0}^* (L_v A))_p = \theta_{t_0})^* (L_V A)_{\theta_{t_0}^*(p)}$$

So

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=t_0} (\theta_t^* A)_p &= \frac{d}{dt}\bigg|_{t=t_0} (\theta_t^*) A_{\theta_t(p)} \\
&\overset{t=s+t_0}{=} \frac{d}{ds}\bigg|_{s=0} \theta_{s+t}^* A_{\theta_{s+t_0}(p)} \\
&= \frac{d}{ds}\bigg|_{s=0} \theta_{t_0}^* \circ \theta_s^* A_{\theta_{t_0}(\theta_s(p))} \\
&= \theta_{t_0}^* (L_V A)_{\theta_{t_0}^*(p)}
\end{aligned}$$

Therefore, if A is invariant under θ_t, then $\theta_t^* = A$ and

$$L_V A = \frac{d}{dt}\bigg|_{t=0} (\theta_t^* A)_p = \frac{d}{dt}\bigg|_{t=0} A_p = 0.$$

In the other direction, if $L_V A \equiv 0$, we show that $(\theta_t^* A)_p = A_p$ for every p and each t. From above,

$$\frac{d}{dt}\bigg|_{t=t_0} (\theta_t^* A)_p = \theta_{t_0}^* \underbrace{(L_V A)_{\theta_{t_0}(p)}}_{=0} = 0$$

Hence $(\theta_t^* A)_p$ is a constant A_p.

Special Tensors (for this course)

Riemannian Metric

g a $(0,2)$-tensor, symmetric and positive definite. That is, at each point p

$$g_p : T_p M \times T_p M \to \mathbb{R}$$

which is bilinear, symmetric and positive definite. This is an inner product.

K (Differential) Form

ω a $(0,k)$-tensor, alternating.

Riemannian Metric

In a chart $(U,(x^i))$, $g = g_{ij}dx^i \otimes dx^j$.
Since it is symmetric, $g(\partial_i,\partial_j) = g(\partial_j,\partial_i)$ (i.e. $g_{ij} = g_{ji}$). We write $dx^i dx^j = \mathrm{Sym}(dx^i \otimes dx^j)$. In this case

$$\mathrm{Sym}(dx^i \otimes dx^j) = \frac{1}{2}\left(dx^i \otimes dx^j + dx^j \otimes dx^i\right)$$

So we may write $g = g_{ij}dx^i dx^j$ and, sometimes, $(dx^1)^2 = dx^1 dx^1$.
We have also that g_{ij} correspinds to a positive definite, symmetric $n \times n$ matrix.

Example

In $\mathbb{R}^n$, $g_E = \delta_{ij}dx^i dx^j$. For $v = v^k\partial_k$ and $w = w^l\partial_l$,

$$g_E(v,w) = \delta_{ij}dx^i dx^j(v^k\partial_k w^l\partial_l) = v^k w^l \delta_{ij}\underbrace{dx^i(\partial_k)}_{\delta^i_k}\underbrace{dx^j(\partial_l)}_{\delta^i_l} = v^1 w^1 + \cdots + v^n w^n$$

Example

Consider $S^2 \subseteq \mathbb{R}^3$ embedded such that $T_pS^2 \hookrightarrow T_p\mathbb{R}^3 \cong \mathbb{R}^3$.
Then $g_p(v,w) = v \cdot w$ defines a Riemannian metric on S^2.

Proposition

Any smooth manifold admits a Riemannian metric.

Proof 1

Embed M into $\mathbb{R}^N$ with N sufficiently large. Then M is an embedded submanifold in $\mathbb{R}^N$ which induces a Riemannian metric on M.

Proof 2

Let $\{U_i\}$ be a countable cover of M (with each U_i a chart) and $\{\psi_i\}$ be a partition of unity with respect to this cover.

IMAGE 1

So $\phi_i^* g_E$ defines a Riemannian metric on U_i and we construct $\sum_i \psi_i(\phi_i^* g_E)$.

Example: Metric Product

Take (M_1, g_1) and (M_2, g_2) and construct $g_1 \oplus g_2$ on $M_1 \times M_2$ by either

$$g_1 \oplus g_2 = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$$

or

$$(g_1 + g_2)((v_1, v_1), (w_1, w_2)) = g_1(v_1, w_1) + g_2(v_2, w_2)$$

e.g. $S^1 \subseteq \mathbb{R}^2$ gives (S^1, g_1), then on the n-torus we construct $(\mathbb{T}^n, g_1 \oplus \cdots \oplus g_1)$.

Example: Warped Product

IMAGE 2

Take $f : M \to \mathbb{R}^+$ smooth, (M, g) and (N, h).
Define a new metric $\tilde{g}$ on $M \times N$ by

$$\tilde{g}_{(x,y)} = g_x + f(x) h_y$$

An example in polar coordinates is

$$(dx)^2 + (dy)^2 = (d(r\cos\theta))^2 + (d(r\sin\theta))^2 = (\cos\theta\, dr - r\sin\theta\, d\theta)^2 + (\sin\theta\, dr + r\cos\theta\, d\theta)^2 = dr^2 + r^2\, d\theta^2$$

Imagine fixing a direction r and at each point attaching a circle of radius r.

IMAGE 3

Recall: Gradient

If $f \in C^\infty(\mathbb{R}^n)$, then

$$\nabla f = \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}$$

Note that this violates our Einstein summation.
If $f \in C^\infty(M)$, its differential df is a 1-form and not a vector field. Why? Because in $\mathbb{R}^n$ we are implicitly using the Euclidean metric.
If we have an inner product on a TVS, say $(V, (\cdot, \cdot))$, then we can construct an isomorphism $V \cong V^*$ by $v \mapsto (v, \cdot)$.
On (M, g) we use g to construct a bundle isomorphism between TM and T^*M by $(p, v) \mapsto g_p(v, \cdot)$.
With this, given $df \in \Omega^1(M)$, we can define a vector field $\nabla f \in \mathfrak{X}(M)$ by

$$g(\nabla f, X) = (df)(X) = Xf$$

In a chart $(U, (x^i))$, set $\nabla f = b^i \frac{\partial}{\partial x^i}$. Then

$$g\left(\nabla f, \frac{\partial}{\partial x^j}\right) = g\left(b^i \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = b^i g_{ij} = (df)\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^j}$$

Let g^{ij} be the inverse of g_{ij}, then

$$b^k = b^i \delta_i^k = b^i g_{ij} g^{jk} = \frac{\partial f}{\partial x^j} g$$

so

$$\nabla f = b^k \frac{\partial}{\partial x^k} = \frac{\partial f}{\partial x^j} g^{jk} \frac{\partial}{\partial^k}$$

Then from above, we actually have

$$\nabla f = \frac{\partial f}{\partial x^i} \delta_{ij} \frac{\partial}{\partial x^j}$$

which satisfies our summation convention.

Example

If $g_E = dr^2 + r^2\, d\theta^2$ in polar coordinates,

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \quad \text{and} \quad (g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

So

$$\nabla f = \frac{\partial f}{\partial x^j} g^{jk} \frac{\partial}{\partial x^k} = \frac{\partial f}{\partial r} \frac{\partial}{\partial r} + \frac{\partial f}{\partial \theta} \frac{1}{r^2} \frac{\partial}{\partial \theta}$$

Isometric Metrics

We say that (M, g) and (N, h) are isometric if there is a diffeomorphism $F : M \to N$ such that $F^* h = g$.
With g, we can define (for $v \in T_p M$), $||v||_g = (g_p(v, v))^{1/2}$ and (for $v, w \in T_p M$)

$$\cos(v, w) = \frac{g_p(v, w)}{||w||_g ||w||_g}$$

Definition: Length

Let $\gamma : I \to M$ be a (piecewise) smooth curve.
Define $\text{length}_g(\gamma) = \int_I ||\gamma'(t)||_g\, dt$.
Remember that $\text{length}_g(\gamma)$ is independent of reparameterization. That is

$J \xrightarrow{\phi} I \xrightarrow{\gamma} M$ with $\tilde{\gamma} = \gamma \circ \phi$ we have

$$\begin{aligned}
\int_J ||\tilde{\gamma}'(t)||\, dt &= \int_J ||(\gamma \circ \phi)'(t)||\, dt \\
&= \int_J ||\gamma'(\phi(t)) \cdot \phi'(t)||\, dt \\
&\overset{\phi' > 0}{=} \int_J ||\gamma'(\phi(t))|| \phi'(t)\, dt \\
&\overset{s = \phi(t)}{=} \int_I ||\gamma'(S)||\, ds
\end{aligned}$$

Definition: Distance

Given (M, g), define

$$d_g(p, q) = \inf\left\{\text{length}_g(\gamma) \,:\, \gamma \text{ is piecewise smooth from } p \text{ to } q\right\}$$

Theorem

(M, d_g) is a metric space.
Moreover, it induces a metric topology that coincides with the manifold topology.

Theorem: Hopf-Rinow

The following are equivalent.

1. (M, d_g) is a complete metric space.
2. $\forall\, p, q \in M$, there exists a length-minimizing curve (a geodesic) from p to q.

Definition: Geodesic

A curve such that the second derivative along $\gamma \equiv 0$.

February 3, 2025

Recall: Wedge Product

$$\begin{aligned} \bigwedge^{k} V^* \times \bigwedge^{l} V^* &\to \bigwedge^{k+l} V^* \\ (\omega, \eta) &\mapsto \omega \wedge \eta \end{aligned}$$

By $\frac{(k+l)!}{k!l!}\operatorname{Alt}(\omega \otimes \eta) = \frac{1}{k!l!}\sum_{\sigma \in S_{k+l}}(\sigma \cdot (\omega \otimes \eta))$.
$\epsilon^I \in \bigwedge^k V^*$, so

$$\epsilon^I(v_1, \dots, v_k) = \det\begin{pmatrix} \epsilon^{i_1}(v_1) & \cdots & \epsilon^{i_1}(v_k) \\ \vdots & & \vdots \\ \epsilon^{i_k}(v_1) & \cdots & \epsilon^{i_k}(v_k) \end{pmatrix}$$

We have a V basis $\{E_I\}$ and a V^* dual basis $\{\epsilon^I\}$ with $I = (i_1, \dots, i_k)$. We also have that $\epsilon^I(E_{j_1}, \dots, E_{j_k}) = \delta^I_J$.
Then $\mathcal{B} = \{E^I \,:\, I \text{ is strictly increasing}\}$ is a basis for $\bigwedge^k V^*$.

Lemma 14.10

$\epsilon^I \wedge \epsilon^J = \epsilon^{IJ}$.

Proof

We show that $\epsilon^I \wedge \epsilon^J(E_{p_k}, \dots, E_{p_{k+l}}) = \epsilon^{IJ}(E_{p_1}, \dots, E_{p_{k+l}})$, $P = (p_1, \dots, p_{k+l})$.
If $I \cup J \neq P$, then both sides are zero.
If IJ or P has repeated index, then both sides are zero.
Then the only nontrivial case is when $P = IJ$ without repeated indecies. Write $IJ = \{i_1, \dots, i_k, j_1, \dots, j_l\}$ such that we can apply a permutation $\gamma \in S_{k+l}$ to generate a strictly increasing $P = \{p_1, \dots, p_{k+l}\}$. Then write $P_1 = \{p_1, \dots, p_k\}$ and $P_2 = \{p_{k+1}, \dots, p_{k+l}\}$, and compute

$$\begin{aligned}
\epsilon^P &= \epsilon^{P_1} \wedge \epsilon^{P_2} \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sign} \sigma) \cdot (\sigma(\epsilon^{P_1} \otimes \epsilon^{P_2})) \\
&= \frac{1}{k!l!} \sum_{\sigma' \in S_{k+l}} (\operatorname{sign} \sigma')(\operatorname{sign} \gamma)((\gamma \cdot \sigma')(\epsilon^{P_1} \otimes \epsilon^{P_2})) \\
&= \operatorname{sign} \gamma(\epsilon^I \wedge \epsilon^J)
\end{aligned}$$

Proposition 14.11

1. If $\omega^i \in V^*$ and $v_j \in V$, then $\omega^1 \wedge \cdots \wedge \omega^k(v_1, \dots, v_k) = \det(w^i(v_j))$.

Proof

It suffices to check (assuming I, J strictly increasing)

$$(\epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k})(E_{j_1}, \dots, E_{j_k}) = \epsilon^I(E_{j_1}, \dots, E_{j_k}) = \delta^I_J = \det(\epsilon^{i_p}(E_{j_q})).$$

Definition: Graded Algebra

Write $\bigwedge V^* = \bigoplus_{k=0}^n \bigwedge^k V^*$ with $\dim \bigwedge V^* = 2^n$.
Remember that $\dim \bigwedge^k V^* = \binom{n}{k}$.
It is graded if $(\bigwedge^k) \wedge (\bigwedge^l) \subseteq \bigwedge^{k+l}$.

Differential Forms on Manifolds

Given a manifold M, a k-form on M $\bigwedge^k(T^*M) = \coprod_{p \in M} \left(\bigwedge^k T_p^* M\right)$ is a section of the bundle $\bigwedge^k(T^*M) \to M$.
$\Omega^k(M)$ is the collection of k-forms on M.
Locally, $\omega \in \Omega^k(M)$ may be written $\omega = \sum \omega_I dx^I$ for a chart $(U, (x^i))$.
Summing over strictly increasing I, $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ and $\omega_I = \omega\left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}}\right)$.

Pullback

For $F : M \to N$ and $\omega \in \Omega^k(N)$, we define $(F^*\omega) \in \Omega^K(M)$ by

$$(F^*\omega)(v_1, \dots, v_k) = \omega(DF(v_1), \dots, DF(v_k)).$$

It follows that

$$F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$$

and

$$
\begin{aligned}
F^*\left(\sum_I{}' \omega_I dx^I\right) &= \sum_I{}' (F^*\omega_I) F^*(dx^{i_1} \wedge \cdots \wedge dx^{ik}) \\
&= \sum_I{}' (\omega_I \circ F)(d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F)) \\
&= \sum_I{}' (\omega_I \circ F) dF^{i_1} \wedge \cdots \wedge dF^{i_k}
\end{aligned}
$$

Example

For $F : \mathbb{R}^2 \to \mathbb{R}^3$ by $F(u,v) = (u, v, u^2 - v^2)$ and $\omega = y\,dx \wedge dz \in \Omega^2(\mathbb{R}^3)$.

$$F^*\omega = F^*(y\,dx \wedge dz) = v\,du \wedge d(u^2 - v^2) = v\,du \wedge (2u\,du - 2v\,dv = -2v^2\,du \wedge dv$$

Proposition 14.20

For $F : M^n \to N^n$ with local coordinates (x^i) and (y^i) respectively, if $u \in C^\infty(N)$ then

$$F^*(u\,dy^1 \wedge \cdots \wedge dy^n) = (u \circ F) \det DF$$

Proof

Write F in components $(F^1, \ldots, F^n)$ where $F^i = y^i \circ F$

$$
\begin{aligned}
F^*(u\,dy^1 \wedge \cdots \wedge dy^n) &= (u \circ F) dF^1 \wedge \cdots \wedge dF^n \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right) \\
&= (u \circ F) \det\left(dF^i\left(\frac{\partial}{\partial x^j}\right)\right) \\
&= (u \circ F) \det(DF)
\end{aligned}
$$

If $(U, (x^i))$ and $(\tilde{U}, (\tilde{x}^i))$ are local charts with $U \cap \tilde{U} \neq \varnothing$, then using $F = \mathrm{id}_{U \cap \tilde{U}}$ we have that $F^* = \mathrm{id}$

$$d\tilde{x}^i \wedge \cdots \wedge d\tilde{x}^n = \det\left(\frac{\partial \tilde{x}^j}{\partial x^i}\right) dx^1 \wedge \cdots \wedge dx^n$$

Definition: Exterior Derivative

For $\omega \in \Omega^k(U)$, $U \subseteq \mathbb{R}^n$ open, $\omega = \sum_I' \omega_I dx^I$ define $d : \omega^k(U) \to \omega^{k+1}(U)$ by $\omega \mapsto d\omega$. Then

$$d\omega = \sum_I{}' \underbrace{d\omega_I}_{\in \Omega^1(U)} \wedge \underbrace{dx^I}_{\in \Omega^k(U)}$$

Example

$\omega \in \Omega^1(U)$, $\omega = \sum_{i=1}^n \omega_i dx^i$.

$$d\omega = \sum_{i=1}^n d\omega_i \wedge dx^i = \sum_{i,j=1}^n \frac{\partial \omega_i}{\partial x^j} dx^j \wedge dx^i = \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} \right) dx^i \wedge dx^j$$

For $\omega = df \in \Omega^1(M)$, $d(df) = \sum_{i<j} \left(\frac{\partial^2 f}{\partial x^j \partial x^i} - \frac{\partial^2 f}{\partial x^i \partial x^j} \right) dx^i \wedge dx^j = 0$. That is, $(d \circ d)(f) = 0$ for any smooth function $f \in C^\infty(M)$.

Proposition

1. d is $\mathbb{R}$-linear.
2. $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge (d\eta)$ with $k = \deg \omega$.
3. $d \circ d = 0$.
4. $F^*(d\omega) = d(F^*\omega)$.

Proof of b

Write $\omega = u\, dx^I$ and$\eta = v\, dx^J$.
Claim: $d(u\, dx^I) = du \wedge dx^I$ for any index (perhaps not strictly increasing) I.
If I has a repeated index, both sides are zero.
If not, let $\sigma \in S_k$ such that $I_\sigma = J$ strictly increasing.

$$d(u\, dx^I) = d((\operatorname{sign} \sigma) u\, dx^J) = \operatorname{sign} \sigma \cdot du \wedge dx^J = du \wedge (\operatorname{sign} \sigma \cdot dx^J) = du \wedge dx^I$$

Then

$$d(\omega \wedge \eta) = d(u\, dx^I \wedge v\, dx^J) = d(uv\, dx^I \wedge dx^J) = d(uv\, dx^{IJ}) = d(uv) \wedge dx^{IJ} = (u\, dv + v\, du) \wedge (dx^I \wedge dx^J)$$

So

$$d\omega \wedge \eta + (-1)^k \omega \wedge d\eta = du \wedge dx^I \wedge (v\, dx^J) + (-1)^k u\, dx^I \wedge (dv \wedge dx^J)$$

and it suffices to show that $dv \wedge dx^I \wedge dx^J = (-1)^k dx^I \wedge dv \wedge dx^J$.

Proof b Implies c

Write

$$d \circ d(\omega_I dx^I) = d(d\omega_I \wedge dx^I) = d(d\omega_I) \wedge dx^I + (-1)^1 d\omega_I \wedge d(dx^I) = 0$$

Proof of d

Write $\omega = u\,dx^I$ such that $d\omega = du \wedge dx^I$.

$$F^*(d\omega) = F^*(du \wedge dx^I) = d(u \circ F) \wedge dF^{i_1} \wedge \cdots \wedge dF^{i_k}$$

and

$$d(F^*\omega) = d((u \circ F)dF^{i_1} \wedge \cdots \wedge dF^{i_k} = d(u \circ F) \wedge dF^{i_1} \wedge \cdots \wedge dF^{i_k}$$

February 5, 2025

Theorem 14.24

There is a unique map $d : \Omega^*(M) \to \Omega^*(M)$ with $d(\Omega^k(M)) \subseteq \Omega^{k+1}(M)$ such that

1. d is $\mathbb{R}$-linear
2. $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$
3. $d \circ d = 0$
4. $df(X) = Xf$ for all $f \in \Omega^0(M) = C^\infty(M)$ and $X \in \mathfrak{X}(M)$.

Proof: Existence

Let $\omega \in \Omega^k(M)$. Then $\omega|_U \in \Omega^k(U)$. We have that $\varphi^{-1*}\omega \in \Omega^k(\varphi(U))$, $d(\varphi^{-1*}\omega) \in \Omega^{k+1}(\varphi(U))$, and $d\omega := \varphi^* d(\varphi^{-1*}\omega) \in \Omega^{k+1}(U)$ on U.

IMAGE 1

Proof: Well-defined

If (V, ψ) is another chart with $U \cap V \neq \varnothing$, we need to show that $\psi^*(d(\psi^{-1*}\omega)) = \varphi^*(d\varphi^{-1*}\omega))$. This is equivalent to

$$\begin{aligned} \iff \quad & d(\psi^{-1*}\omega) = \psi^{-1*}\varphi^*(d(\varphi^{-1*}\omega)) \\ \iff \quad & d(\psi^{-1*}\omega = (\varphi \circ \psi^{-1})^* d(\varphi^{-1*}\omega) \end{aligned}$$

where

$$(\varphi \circ \psi^{-1})^* d(\varphi^{-1*}\omega) = d((\varphi \circ \psi^{-1})^* \varphi^{-1*}\omega) = d(\psi^{-1*} \circ \varphi^* \circ \varphi^{-1*}\omega) = d(\psi^{-1*}\omega)$$

Proof: Uni!

For any $d : \Omega^*(M) \to \Omega^*(M)$ with the property $(d\omega)_p$ only depends on $\omega|_U$ where $p \in U$.
Suppose $\omega_1 = \omega_2$ on U. We need to show that $(d\omega_1)_p = (d\omega_2)_p$.
So set $\eta = \omega_1 - \omega_2$. Then $\omega \equiv 0$ on U, and we need to show that $(d\eta)_p = 0$.
Let ψ be a bump function such that $\operatorname{supp}\psi \subseteq U$ and $\psi(p) = 1$.

IMAGE 2

Then $\psi\eta = 0 \in \Omega^k(M)$.

$$0 = d(\psi\eta) = d\psi \wedge \eta + (-1)^0 \psi \wedge d\eta$$

At point p, it reads

$$0 = 0 \wedge \eta_p + \overbrace{\psi(p)}^{=1} \wedge d\eta_p$$

That is, $0 = d\eta_p$. Let $p \in M$, U a chart around p, say $(U, (x^i))$, and $\omega \in \Omega^k(U)$. We know that $(d\omega)_p$ only depends on $\omega|_U = \sum_I' \omega_I dx^I$. Then for $p \in V \subseteq \overline{V} \subseteq U$, $\omega|_U$ extends functtions $\omega_I, x^I \in C^\infty(V)$ to globally defined functions $\tilde{\omega}_I, \tilde{x}^I \in C^\infty(M)$. Therefore

$$\begin{aligned}
d(\omega|_U) &= \sum_I{}' d(\omega_I dx^I) \\
&= \sum_I{}' d(\tilde{\omega}_I \tilde{x}^I) \\
&= \sum_I{}' (d\tilde{\omega}_I \wedge d\tilde{x}^I + \omega_I \wedge \overbrace{d(d\tilde{x}^{i_1} \wedge \cdots \wedge d\tilde{x}^{i_k})}^{=0} \\
&= \sum_I{}' d\omega_I \wedge dx^I
\end{aligned}$$

which is the same formula for $\mathbb{R}^n$.

Proposition: 14.26

$F^*(d\omega) = d(F^*\omega)$.

Proposition: 14.32

$\mathcal{L}_V(\omega \wedge \eta) = (\mathcal{L}_V\omega) \wedge \eta + \omega \wedge (\mathcal{L}_V\eta)$.

Corollary

$\mathcal{L}_V(d\omega) = d(\mathcal{L}_V\omega)$.

Definition: Interior Multiplication

Given $\omega \in \bigwedge^k V^*$ and $v \in V$, define $\iota_V\omega \in \bigwedge^{k-1} V^*$ (sometimes written $V \lrcorner\, \omega$).

$$(\iota_v\omega)(u_1, \dots, u_{k-1}) = \omega(v, u_1, \dots, u_{k-1})$$

This defines $\iota_V : \bigwedge^k V^* \to \bigwedge^{k-1} V^*$, and we have $\iota_V \circ \iota_V = 0$.

$$\iota_v(\omega \wedge \eta) = (\iota_V\omega) \wedge \eta + (-1)^k \omega \wedge (\iota_V\eta)$$

Proof

It suffices to show that for $\omega^1,\dots,\omega^k \in V^*$

$$\iota_V(\omega^1 \wedge \cdots \wedge \omega^k) = \sum_{i=1}^k (-1)^{i-1}\omega^i(v)\omega^1 \wedge \cdots \wedge \hat{\omega}^i \wedge \cdots \wedge \omega^k$$

Where $\hat{\omega}^i$ is meant to denots “forgetting” a term in the wedge product. That is, the first term has no ω^1, the second no ω^2, etc.
Assuming this, it suffices to consider $\omega = \omega^1 \wedge \cdots \wedge \omega^k$ and $\eta = \eta^1 \wedge \cdots \wedge \eta^l$. Then

$$\begin{aligned}
\iota_V(\omega \wedge \eta) &= \iota_V(\omega^1 \wedge \cdots \omega^k \wedge \eta^1 \wedge \cdots \wedge \eta^l) \\
&= \sum_{i=1}^k (-1)^{i-1}\omega^i(v)\omega^1 \wedge \cdots \wedge \hat{\omega}^i \wedge \cdots \wedge \omega^k \wedge \eta^1 \wedge \cdots \wedge \eta^l + \sum_{i=1}^l (-1)^{k+i-1}\eta^i(v)\omega^1 \wedge \cdots \wedge \omega^k \wedge \eta^1 \wedge \cdots \wedge \hat{\eta}^i \wedge \cdots \wedge \eta^l \\
&= (\iota_V\omega) \wedge \eta + (-1)^k \omega \wedge (\iota_V\eta)
\end{aligned}$$

Write $v_1 = v$, and apply both sides to $(v_2,\dots,v_k)$. The left hand side is

$$\omega^1 \wedge \cdots \omega^k(v_1,\dots,v_k) = \det(\omega^i(v_j)) = \det\begin{pmatrix} \omega^1(v_1) & \cdots & \omega^i(v_1) & \cdots & \omega^k(v_1) \\ \vdots & & & & \vdots \\ \omega^1(v_k) & \cdots & \omega^i(v_1) & \cdots & \omega^k(v_k) \end{pmatrix}$$

The right hand side is given by

$$\sum_{i=1}^k (-1)^{i-1}\omega^i(v_1)(\omega^1 \wedge \cdots \wedge \hat{\omega}^i \wedge \cdots \wedge \omega^k)(v_1,\dots,v_k)$$

which, when expanded, gives $\det(\omega^i(v_j))$ along the first row.

Proposition 14.35 (Cartan)

If $V \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$, then

$$\mathcal{L}_V\omega = V \lrcorner (d\omega) + d(V \lrcorner \omega)$$

Corollary

$$\mathcal{L}_V(d\omega) = d(\mathcal{L}_V\omega)$$

Proof

By assuming Cartan’s formula, the left hand side is

$$V \lrcorner \overbrace{(d \circ d\omega)}^{=0} + d(V \lrcorner d\omega)$$

and the right hand side is

$$d(V \lrcorner d\omega + d(V \lrcorner \omega)) = d(V \lrcorner d\omega) + \overbrace{d \circ d(v \lrcorner \omega)}^{=0}$$

Proof (of Cartan's Formula)

We prove by induction on $\deg(\omega)$. When ω is a function $f \in C^\infty(M) = \Omega^0(M)$, the left hand side is

$$\mathcal{L}_V f = Vf$$

and the right hand side is

$$V \lrcorner (df) + d\underbrace{(V \lrcorner f)}^{=0} = df(V) = Vf$$

since ι_V maps Ω^k to Ω^{k-1}.
Assuming it holds for $k-1$ forms, we consider $\omega \in \Omega^k(M)$ and locally write $\omega = \sum' \omega_I dx^I$.
It suffices to show that the formula holds for $\omega = du \wedge \beta$, $u \in C^\infty(M)$, $\beta \in \Omega^{k-1}(M)$.

$$(\omega_I dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \underbrace{dx^{i_1}}_{du} \wedge \underbrace{(\omega_I dx^{i_1} \wedge \cdots \wedge dx^{i_k})}_{\beta}$$

The left hand side is

$$\begin{aligned}
\mathcal{L}_V(du \wedge \beta) &= \mathcal{L}_V du) \wedge \beta + du \wedge \mathcal{L}_V \beta \\
&= d(\mathcal{L}_V u) \wedge \beta + du \wedge (V \lrcorner d\beta + d(V \lrcorner \beta)) \\
&= d(Vu) \wedge \beta + du \wedge (V \lrcorner d\beta) + du \wedge d(V \lrcorner \beta)
\end{aligned}$$

and the right hand side is

$$\begin{aligned}
V \lrcorner (d(du \wedge \beta)) + d(V \lrcorner (du \wedge \beta)) &= V \lrcorner (\overbrace{(d \circ du)}^{=0} \wedge \beta + (-1) du \wedge d\beta + d(\overbrace{(V \lrcorner du)}^{=Vu} \wedge \beta + du \wedge (V \lrcorner \beta)) \\
&= (-1)(Vu) d\beta + d(Vu) \wedge \beta + (Vu) d\beta
\end{aligned}$$

Proposition 14.32

$$d\omega(X_1, \ldots, X_{k+1}) = \sum_{1 \le i \le k+1} (-1)^{i-1} X_i(\omega(X_1, \ldots, \hat{x}_i, \ldots, X_{k+1})) + \sum_{1 \le i \le j \le k+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1})$$

When $\omega \in \Omega^1$, it reads

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$$

In particular, for ω closed,

$$X(\omega(Y)) - Y(\omega(X)) = \omega([X, Y])$$

Proof

It suffices to prove that for $\omega = udv,\ u, v \in C^{\infty}(M)$ that

$$d(\omega) = d(udv) = du \wedge dv$$

The left hand side

$$(du \wedge dv)(X, Y) = \det\begin{pmatrix} du(X) & du(Y) \\ dv(X) & dv(Y) \end{pmatrix} = \det\begin{pmatrix} X_u & Y_u \\ X_v & Y_v \end{pmatrix}$$

and the right hand side

$$\begin{aligned} X(udv(Y)) - Y(udv(X)) - u(dv([X,Y]) &= X(u(Yv)) - Y(u(Xv)) - u([X,Y]v) \\ &= (Xu)(Yv) + u(XYv) - (Yu)(Xv) - u(YXv) - u([X,Y]v) \\ &= \det\begin{pmatrix} X_u & Y_u \\ X_v & Y_v \end{pmatrix} \end{aligned}$$

Example

For $f \in \Omega^{*}(\mathbb{R}^3)$ and $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \in \Omega^{*+1}(\mathbb{R}^3)$, write $Pdx + Qdy + Rdz$ and

$$\begin{aligned} d(Pdx + Qdy + Rdz) &= \left(\frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz\right) \wedge dx + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial z}dz\right) \wedge dy + \left(\frac{\partial R}{\partial x}dx + \frac{\partial R}{\partial y}dy\right) \wedge dz \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}dx \wedge dy\right) + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}dy \wedge dz\right) + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}dz \wedge dx\right) \end{aligned}$$

Recall that for $X = (P, Q, R) \in \mathfrak{X}(\mathbb{R}^3)$, this is the curl of X.
Let $\omega = udx \wedge dy + vdy \wedge dz + wdz \wedge dx$, then

$$\begin{aligned} d\omega &= \frac{\partial u}{\partial z}dz \wedge dx \wedge dy + \frac{\partial v}{\partial z}dx \wedge dy \wedge dz + \frac{\partial w}{\partial z}dy \wedge dz \wedge dx \\ &= \left(\frac{\partial V}{\partial x} + \frac{\partial W}{\partial y} + \frac{\partial U}{\partial z}\right)dx \wedge dy \wedge dz \end{aligned}$$

Recall that this is divergence. We can also look at the gradient

$$\operatorname{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

we have

$$\operatorname{grad} f \cdot X = Xf = df(X) = \sum \frac{\partial f}{\partial x^i} \cdot x^i$$

Putting this together,

$$\begin{array}{ccccccc} C^{\infty}(\mathbb{R}^3) & \xrightarrow{\text{grad}} & \mathfrak{X}(M) & \xrightarrow{\text{curl}} & \mathfrak{X}(M) & \xrightarrow{\text{div}} & C^{\infty}(\mathbb{R}^3) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Omega^0(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathbb{R}^3) \end{array}$$