Analysis I

October 2, 2023

Lecture Notes

Class will not have dedicated lecture notes. Many are available already.
Undergraduate notes are available on Canvas.
Lecture 1 overview available on Canvas (lecture1.pdf).

Tentative Office Hours

Mondays 2-3pm and Tuesday 1-2pm.

Homework

Nominally due at beginning of class; ask for leeway if needed.
First week homework will be review of undergraduate proofs.
First homework due Wednesday, October 11.

Notation

Natural Numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$
Non Negative Integers: $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
Integers: $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$
Rationals: $\mathbb{Q} = \left\{\frac{p}{q},\ p \in \mathbb{Z},\ q \in \mathbb{Z}\right\} = \mathbb{Z} \times \mathbb{N}/\infty$

- Equivalent representation of rationals: $(p_1, q_1) \sim (p_2, q_2)$ iff $p_1 q_2 = p_2 q_1$

Sequence of Rationals: $\{u_n\}_{n \in \mathbb{N}},\ u_n \in \mathbb{Q},\ \forall n.$

Properties of the Rationals

$(\mathbb{Q}, +, \cdot)$ is a (ii) totally ordered (i) field satisfying the (iii) Archimedean property.

(i) Field

1. $+$ is associative: $(a + b) + c = a + (b + c)$

2. $+$ is commutative: $a + b = b + a$

3. $\cdot$ is associative and commutative.

4. $\exists 0 \in \mathbb{Q}$ such that $\forall a \in \mathbb{Q}$, $0 + a = a + 0$

5. $\exists 1 \in \mathbb{Q} \setminus \{0\}$ such that $\forall a \in \mathbb{Q}$, $1 \cdot a = a \cdot 1 = a$

6. $\forall a \in \mathbb{Q} \setminus \{0\}$ $\exists b \in \mathbb{Q}$, $a \cdot b = b \cdot a = 1$

    - $b = a^{-1} = \frac{1}{a}$

(ii) Totally Ordered

$\exists$ a set $\mathbb{Q}_+ \subseteq Q$ of "Positive Numbers" stable under $+$ and $\cdot$ such that $\forall A \in \mathbb{Q}$ either $a > 0$ ($a \in \mathbb{Q}_+$), $-a > 0$ (also $a < 0$) or $a = 0$.

- Ordering: $\forall a, b \in \mathbb{Q}$, $a < b$ if and only if $b - a > -0$.

- Trichotomy: $\forall a, b \in \mathbb{Q}$ either $a < b$, $a > b$, or $a = b$.

- $\max(a, b) = \begin{cases} a & \text{if } a > b \\ b & \text{otherwise} \end{cases}$.

- $|a| = \max(a, -a)$ (helps measure distance in $\mathbb{Q}$).

- $\operatorname{dist}(a, b) := |b - a|$

- Triangle Inequality: $|u \pm v| \leq |u| + |v|$

- Observe also: $||u| - |v|| \leq |u \pm v|$. The triangle inequality may be used to prove this.

- Proof of Triangle Inequality $-|u| \leq u \leq |u|$ and $-|v| \leq v \leq |v|$, therefore $-|u| - |v| \leq u + v \leq |u| + |v|$. Therefore $u + v \leq |u| + |v|$ and $-(u + v) \leq |u| + |v|$ implies $|u + v| \leq |u| + |v|$.

(iii) Archimedian Property:

$\forall \epsilon > 0,\ \exists N,\ \forall n \geq N,\ \frac{1}{n} < \epsilon$.

Bounded Sequence of Rationals

$\{u_n\}_{n \in \mathbb{N}}$ is bounded if $\exists m \in \mathbb{Q}_+$ such that $|u_n| \leq M,\ \forall n$.
$\{u_n\}_{n \in \mathbb{N}}$ converges to $a \in \mathbb{Q}$ ($\lim_{n \to \infty} u_n = a$) if $\forall \epsilon > 0, \exists N, \forall n \geq N, |u_n - a| < \epsilon$.

Famous Limits

Decaying Rational

1. $\lim_{n\to\infty} \frac{1}{n} = 0$

    - $\forall \epsilon \in \mathbb{Q}_+,\ \exists n \in \mathbb{N},\ 0 < \frac{1}{n} < \epsilon$

    - $\forall n \in \mathbb{N},\ \exists n \in \mathbb{N},\ n \geq N$

        - b. and c. are equivalent.

Decaying Exponential Rational

$r \in \mathbb{Q},\ 0 < r < 1,\ \lim_{n\to\infty} r^n = 0.$

- Proof: Write $r = \frac{1}{1+k}$ for some $k > 0$. Then $r^n = \frac{1}{(1+k)^n} \overset{\text{Bernoulli}}{\leq} \frac{1}{1+nk}$.

Geometric

1. $r \in \mathbb{Q},\ 0 < r < 1,\ u_n = 1 + r + \cdots r^n = \frac{1-r^{n+1}}{1-r} \to \frac{1}{1-r}$

Features of Limits

Limits are Unique

If the limit of a sequence exists, it is unique.

Squeezing Lemma

If $\{a_n\}$, $\{b_n\}$ are such that $0 \leq a_n \leq b_n$, and $b_n \to 0$ as $n \to \infty$, then $a_n \to 0$.

Limits Preserve Order

If $a_n \leq b_n$ $\forall n$ and a_n and b_n converge, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

Limit Algebraic Rules

$\lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = \lim_{n\to\infty}(a_n + b_n)$ when a_n and b_n converge.
If $\lim_{n\to\infty} b_n \neq 0$, then $\frac{a_n}{b_n} \to \frac{\lim a_n}{\lim b_n}$.

Peculiarity of the Rationals

$\mathbb{Q}$ lacks completeness.

Examples

Consider $u_1 = 1$ and $u_{n+1} = \frac{1}{2}(u_n + \frac{2}{u_n})$.
Then $u_n \in \mathbb{Q},\ \forall n \in \mathbb{N}$.
It can further be proven, by induction, that $u_n \geq 1,\ \forall n$. $\left(u_{n+1} - 1 = \frac{1}{2}(u_n + \frac{1}{u_n}) - 1 = \frac{1}{2u_n}((u_n - 1)^2 + 1)\right)$.
$\lim_{n\to\infty} u_n^2 = 2$.

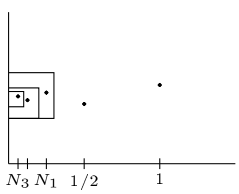
$$\begin{aligned} u_{n+1}^2 - 2 &= \left(\frac{1}{2}(u_n + \frac{2}{u_n})\right)^2 - 2 \\ &= \left(1\frac{1}{2u_n}(u_n^2 + 2)^2 - 4u_n\right) \\ &= 1\frac{4}{u_n^2}\left(u_n^2 - 2)^2\right) \\ &\leq \frac{1}{4}(u_n^2 - 2)^2 \end{aligned}$$

If u_n converged in $\mathbb{Q}$ to L, by algebraic limit rules, $2 = \lim u_n^2 = (\lim u_n)^2 = L^2$, yet $\sqrt{2} \notin \mathbb{Q}$.

Cauchy Criterion

A sequence $\{u_n\}_{n\in\mathbb{N}}$ of rationals is Cauchy if $\forall \epsilon > 0,\ \exists n \in \mathbb{N},\ \forall p, q \geq n,\ |u_p - u_q| < \epsilon$.

Visual Justification



Example 1

The sequence from before is Cauchy.
$|u_p - u_q| = \frac{|u_p^2 - u_q^2|}{|u_p + u_q|} \leq \frac{1}{2}|u_p^2 - u_q^2|$

Example 2

$u_n = \sum_{k=0}^{n} \frac{1}{k!} \in \mathbb{Q}$.

- This is increasing.

- It is bounded above by 3:

$$\begin{aligned} 1 + 1 + \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots + \frac{1}{2\cdots n} &\leq 1 + 1 + \cdots \frac{1}{2^{n-1}} \\ &\leq 1 + \frac{1 - 2^{-n}}{1 - \frac{1}{2}} \\ &\leq 3 \end{aligned}$$

Convergence, Cauchy and Boundedness.

Given a sequence $\{u_n\}_{n\in\mathbb{N}}$,
$\{u_n\}$ converges $\implies$ $\{u_n\}$ is Cauchy $\implies$ $\{u_n\}$ is bounded.
Note that in $\mathbb{Q}$ none of these implications may be reversed.

Construction of the Real Numbers

Short version: If the limit of a sequence isn't in the set, then define the limit as the sequence itself.
Let $C_{\mathbb{Q}} = \{\text{Cauchy sequences of rationals.}\}$.

Two Operations

- Termwise Addition $\{u_n\}_n + \{v_n\}_n := \{u_n + v_n\}_{n\in\mathbb{N}}$

- Termwise Multiplication $\{u_n\}_n \cdot \{v_n\}_n := \{u_n \cdot v_n\}_{n\in\mathbb{N}}$

Closure of Cauchy Sequence

If $\{u_n\}_n,\ \{v_n\}_n \in C_{\mathbb{Q}}$, then $\{u_n\}_n + \{v_n\}_n \in C_n$ and $\{u_n\}_n \cdot \{v_n\}_n \in C_n$.

Example

Infinite decimal expansion.
Fix $N \in \mathbb{Z},\ a_1 \cdots a_n \in \{0, \dots, 9\}$.
Then let $u_n = N + \sum_{k=1}^{n} a_k (10)^{-k}$ (that is the number $N.a_1 a_2 \dots a_n$).
This is always increasing and bounded above by $N + \sum_{k=1}^{n} 9 \cdot (10)^{-k} = N + \frac{9}{10} \cdot \sum_{k=1}^{n} (10)^{-(k+1)} \leq N + 1$.
Hence, it is Cauchy.

Increasing and Bounded Above Implies Cauchy

By contrapositive, increasing and not Cauchy implies not bounded.
By the negation of Cauchy and letting $p \geq q$ without loss of generality, we can force $u_p > u_q + \epsilon$.

Negation of Cauchy

$\exists \epsilon > 0,\ \forall N,\ \exists p, q \geq N,\ |u_p - u_q| > \epsilon$.

Real Numbers as Equivalence Classes of Cauchy Sequences

On $C_{\mathbb{Q}}$ define the relation $\{x_n\}_n \sim \{y_n\}_n$ if and only if $\lim_{n\to\infty} |(x_n - y_n)| = 0$.

Equivalence Relation

Reflexive: $x_n - x_n = 0$
Transitive: Uses algebraic limit rules. $x_n - z_n = x_n - y_n + y_n - z_n$.
Symmetric.

Definition of the Reals

$\mathbb{R} := C_{\mathbb{Q}} / \sim$
Then $x \in \mathbb{R},\ x = [\{x_n\}_n]$.

Addition and Multiplication of Reals

- Addition $x + y := [\{x_n + y_n\}_n]$.

- Multiplication $x \cdot y := [\{x_n \cdot y_n\}_n]$.

Operations Do Not Depend on Choice of Representative

If $\{x_n\}_n \sim \{x_n'\}_n$ and $\{y_n\}_n \sim \{y_n'\}_n$, then $\{x_n\}_n + \{y_n\}_n \sim \{x_n'\}_n + \{y_n'\}_n$.
If $\{x_n\}_n \sim \{x_n'\}_n$ and $\{y_n\}_n \sim \{y_n'\}_n$, then $\{x_n\}_n \cdot \{y_n\}_n \sim \{x_n'\}_n \cdot \{y_n'\}_n$.

The Reals are a Field

There are nine properties to check, eight of which are "obvious":

Commutativity of Addition (and Other "Obvious" Features)

$[\{x_n\}_n] + [\{y_n\}_n] = [\{x_n + y_n\}_n] = [\{y_n + x_n\}] = [\{y_n\}_n] + [\{x_n\}_n]$
That is, the Reals inherit most field features from the Rationals.

- Zero Element $0_{\mathbb{R}} = [\{0_{\mathbb{Q}}\}_n]$

- One Element $1_{\mathbb{R}} = [\{1_{\mathbb{Q}}\}_n]$

Multiplicative Inverses

How to define x^{-1} for $x \in \mathbb{R}$ where $x \neq 0$?

- Idea If $x = [\{x_n\}_n]$ choose $x^{-1} = [\{\frac{1}{x}\}_n]$.
  If $x \in \mathbb{R},\ x \neq 0$ then

  1. $\exists \{x_n\}_n \in C_{\mathbb{Q}}$ representing x with non zero entries.

  2. $\{\frac{1}{x_n}\}_n$ is Cauchy.

  – Proof of 1 Pick any $\{x_n\}_n$ representing x.

    * $x \neq 0$, so NOT ($\lim_{n\to\infty} x_n = 0$: $\exists \epsilon_0 > 0,\ \forall N,\ \exists n \geq N,\ |x_n| > \epsilon_0$.

    * $\{x_n\}$ is Cauchy: $\forall \epsilon > 0, \exists N,\ \forall p, q \geq N,\ |x_p - x_q| < \epsilon$.

Therefore, $\exists N$ such that $\forall p,q \geq N_1,\ |x_p - x_q| < \frac{\epsilon_0}{2}$
And $\exists N_2 \geq N,\ ,|x_{N_2} > \epsilon_0$.
For $q \geq N_2$, the Cauchy Criterion states that $|x_q| = |x_q - x_{N_2} + x_{N_2} \geq |x_{N_2}| - |x_{N_2} - x_q| \geq \epsilon_0 - \frac{\epsilon_0}{2} \geq \frac{\epsilon_0}{2}$.
Therefore, the sought sequence is $\{x_{N_2} + k\}_{k\in\mathbb{N}}$.

- Proof of 2 $\left|\frac{1}{x_p} - \frac{1}{x_q}\right| = \frac{|x_p - x_q|}{|x_p||x_q|} \leq \frac{4}{\epsilon_0^2}|x_p - x_q|$.

Order on the Reals

Let $x \neq 0$, $\exists\{x_n\}_{n\in\mathbb{N}}$ be a representation of x and $\epsilon_0 > 0$.
Then for $|x_n| > \epsilon_0,\ \forall n \in \mathbb{N}$, there is a dichotomy:

- Either $\exists N \in \mathbb{N},\ x_n > \epsilon_0,\ \forall n \geq N$ (in which case we write $x > 0$)

- Or $\exists N \in \mathbb{N},\ x_n < -\epsilon_0,\ \forall n \geq N$ (in which case we write $x < 0$

Thus the Reals are totally ordered.

October 4, 2023

Overview

Completeness of $\mathbb{R}$.
Topology of the Real Line.

Non-zero Reals Are Either Positive or Negative

Given $x \in \mathbb{R} \setminus \{0\},\ \exists \delta \in \mathbb{Q}_+$ such that $\forall\{x_n\}_n$ representing x, $\exists N \in \mathbb{N}$ such that $|x_n| > \delta, \forall n \geq N$.
Moreover, one of the following (but not both) holds:

1. $\forall\{x_n\}_n \in x,\ \exists,\ x_n > \delta,\ \forall n \geq N$ (i.e. $x > 0$)

2. $\forall\{x_n\}_n \in x,\ \exists,\ x_n < -\delta,\ \forall n \geq N$ (i.e. $x < 0$)

Recall that $x \in \mathbb{R} \setminus \{0\}$ is an equivalence class of Cauchy sequences.

Total Ordering of the Reals

$x > 0$ produces a total ordering of $\mathbb{R}$ where $x < y$ if and only if $y - x > 0$.
$\rightsquigarrow \max(x,y) = \begin{cases} x & \text{if } x > y \\ y & \text{otherwise} \end{cases}$
$|x| = \max(x, -x)$ (which satisfies the triangle inequality)

Lemma A

Let $x, y \in \mathbb{R}$. If $\{x_n\}_n, \{y_n\}_n$ represent x, y and satisfy $x_n < y_n,\ \exists N \in \mathbb{N},\ \forall n \geq N$, then $x \leq y$.

- Proof By contradiction, suppose $x > y$ and $\exists \{x_n\}_n, \{y_n\}_n$ representing x, y such that $x_n \leq y_n,\ \forall n \geq N_1$.
  Then, by definition, $x - y > 0 \implies \exists \delta > 0,\ \exists N_2,\ x_n - y_n > \delta$ for $n \geq N_2$.
  But $x_n \leq y_n$ contradicts $x_n - y_n > \delta$.

Sequences of Reals

$\{x_n\}_n,\ x_n \in \mathbb{R}$
The definition of bounded, convergent and Cauchy sequences are the same as in $\mathbb{Q}$.

Injection of Rationals

$\iota : \mathbb{Q} \to \mathbb{R}$ such that $r \mapsto [\{u_n = r\}_n]$
This is isometric in the sense that $|\iota(r) - \iota(s)|_{\mathbb{R}} = |r - s|_{\mathbb{Q}}$

Theorem (Completeness 1)

Let $\{x_n\}_n \in C_{\mathbb{Q}}$ and $x = [\{x_n\}_n]$, then $\{\iota(x_n)\}_n$ converges to x.

Proof

What to show: $\forall \epsilon > 0,\ \exists N,\ \forall n \geq N,\ |\iota(x_n) - x| < \epsilon$.
Let $\epsilon \in \mathbb{Q}_+$. By the Cauchy criterion, $\exists N, \forall q, p \geq N,\ |x_p - x_q| < \epsilon$.
This is equivalent to $x_q - \epsilon \leq x_p \leq x_q + \epsilon$ where p is frozen.
Then by Lemma A, $x - \epsilon \leq \iota(x_p) \leq x + \epsilon$.
It follows that $\forall p \geq N,\ |\iota(x_p) - x \leq \epsilon$.

Corollary

$\mathbb{Q} \cong \iota(\mathbb{Q})$ is dense in $\mathbb{R}$. That is, $\forall \epsilon > 0,\ \forall x \in \mathbb{R},\ \exists r \in \mathbb{Q},\ |\iota(r) - x| < \epsilon$.

The Isometric Copy of Rationals

For brevity, the ι notation will be dropped and the $\mathbb{Q}$ will be understood as $\iota(\mathbb{Q})$.

Completeness of the Real Numbers

A sequence of real numbers converges in $\mathbb{R}$ if and only if it is Cauchy.

Proof

$(\Longrightarrow)$ This is clear.
$(\Longleftarrow)$ Take a Cauchy sequence of reals $\{x_n\}_n$. Then $\forall \epsilon > 0,\ \exists N,\ \forall p, q \geq\ \ |x_p - x_q| < \epsilon$.
Using the density of $\mathbb{Q}$, $\forall n \in \mathbb{N},\ \exists r_n \in \mathbb{Q}$ such that $|x_n - r_n| < \frac{1}{n}$.

Claim: $\{r_n\}_n$ is Cauchy. Indeed,

$$\begin{aligned} |r_p - r_q| &= |r_p - x_p + x_p - x_q + x_q - r_q| \\ &\leq |r_p - x_p| + |x_p - x_q| + |x_q - r_q| \\ &\leq \frac{1}{p} + |x_p - x_q| + \frac{1}{q} \end{aligned}$$

Take $\epsilon > 0$. $\{x_n\}$ cauchy implies $\exists N_1$, $\forall p, q \geq N, |x_p - x_q| \leq \frac{\epsilon}{3}$ and $\exists N_2$, $\forall p, q \geq N_2, \frac{1}{p} \leq \frac{\epsilon}{3}$, $\frac{1}{q} \leq \frac{\epsilon}{3}$ for $p, q \geq \max(N_1, N_2)$ $|r_p - r_q| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$.
Then, for Cauchy $\{r_n\}_n$, call $r = [\{r_n\}_n]$, then $\lim_{n\to\infty} r_n = r$ by the above theorem.
Then my algebraic limit rules, $x_n(x_n - r_n) + r_n$ where $(x_n - r_n) \to 0$ and $r_n \to r$ as $n \to \infty$. So $\{x_n\}$ converges.

Example

Let $x_1 = 1$, $x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n})$.
Then $\{x_n\}_n \in C_{\mathbb{Q}}$, and it converges to $L \in \mathbb{R}$.
By algebraic limit rules, $L^2 (\lim x_n)^2 = \lim x_n^2 = 2$.

Subsets of the Reals, Infimum and Supremum

Notation

Subset: $S \subseteq \mathbb{R}$
Inclusion: $x \in S$
Open Interval: $(a, b) = \{x \in \mathbb{R} | a < x < b\}$
Semiclosed Interval: $(a, b] = \{x \in \mathbb{R} | a < x \leq b\}$
Closed Interval: $[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$
Unbounded Semiclosed Interval: $(-\infty, a] = \{x \in \mathbb{R} | x \leq a\}$
Unbounded Open: $(-\infty, a) = \{x \in \mathbb{R} | x < a\}$

Suprememum

$S \subseteq \mathbb{R}$ is bouned above (respectively below) if $\exists M \in \mathbb{R}$, $\forall x \in S$, $x \leq M$ (respectively $\exists L \in \mathbb{R}$, $\forall x \in S$, $L \leq X$)
S ad mits a least upper bound, LUB, suprememum or sup M if

1. $\forall x \in S, x \leq M$

2. $\forall M' \in \mathbb{R}$, upper bound of S, $M \leq M'$

If $\sup S$ exists, it is unique.
If $x \in S$ and x is an upper bound for S, then $x = \sup S$.

Example 1

$\sup(0, 1) = \sup[0, 1] = 1$

Example 2

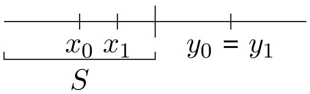
$S = \{x \in \mathbb{Q},\ x^2 < 2\}$ does not have a greatest element in $\mathbb{Q}$, nor a least upper bound in $\mathbb{Q}$.

Theorem (Completness 2)

Every subset $S \subseteq \mathbb{R}$, nonempty and bouned above, has a supremum in $\mathbb{R}$.

Proof

By dichotomy.
$S \neq \varnothing \implies \exists x_0 \in S$ and S bounded above implies $\exists y_0 \in \mathbb{R},\ \forall x \in S,\ x \leq y_0$ (in particular $x_0 \leq y_0$).
If $x_0 = y_0$, done. Otherwise, consider $m_0 = \frac{x_0+y_0}{2}$.



Two options exist: if m_0 is an upper bound for S, set $y_1 = m_0$ and $x_1 = x_0$.
Otherwise, $\exists x_1 \in S$, such that $m_0 < x_1$ so set $y_1 = y_0$.
Repeat this process forever to construct two sequences $x_n,\ y_n$.
$\forall n,\ x_n \in S$, y_n is an upper bound for S.

- $x_n \leq y_n$
- x_n is increasing and bounded above by y_0, so it must be Cauchy and converging to x.
- y_n is decreasing and bounded below by x_0, so it must be Cauchy and converging to y.
- $|x_{n+1} - y_{n+1}| \leq \frac{|x_n - y_n|}{2}$ which implies $|x_n - y_n| \leq \frac{1}{2^n}|x_0 - y_0|$ and $x = y = z$.

Therefore, the process may be understood as $x_0 \leq \cdots \leq x_n \leq x_{n+1} \leq y_{n+1} \leq y_n \leq \cdots \leq y_0$.
There remain two things to check: (1) z is an upper bound for S and (2) z is no larger than any other upper bound for S.

1. Take $x \in S,\ \forall n,\ x \leq y_n \xrightarrow{n\to\infty} x \leq Z$.
2. Take upper bound for S, z'. $x_n \leq z',\ \forall n \xrightarrow{n\to\infty} z \leq z'$.

So $z = \sup S$.

Monotone Convergence Theorem (Completeness 3)

An increasing sequence of reals, $\{x_n\}_n$, that is bounded above, converges to $\sup X = \sup\{x_n | n \in \mathbb{N}\}$.
To prove that this converges, since it is monotone and bounded above it is Cauchy and therefore must be convergent.

Proof

Call x the limit, then $\forall n,\ x_n \leq x$. To see this, suppose $\exists n_0,\ x < x_{n_0}$ then $\forall m \geq m_0,\ x < x_{m_0} \leq x_m \implies |x_m - x| \geq |x_{n_0} - x| > 0,\ \forall m \geq n_0$ is a contradiction.
Let M be an upper bound of X. Then $x_n \leq M,\ \forall n \overset{n\to\infty}{\longrightarrow} x \leq M \implies x = \sup X$.

Theorem (Existence of Roots)

$\forall x \in \mathbb{R}$ where $x > 0,\ p \in \{2, 3, \ldots,\},\ \exists! y > 0$ such that $y^p = x$.

Proof

Left as an exercise.
Either by dichotomy or consider $S = \{y \in \mathbb{R} | y^p < x\}$, show: $S \neq 0$, bounded above and $(\sup S)^p = x$.
For uniqueness, show $y_1^p = y_2^p = x \iff 0 = y_1^p - y_2^p = (y_1 - y_2)(\cdots \neq 0) \implies y_1 = y_2$.

Topological Properties

$S \subseteq \mathbb{R}$ is open if $\forall x \in S,\ \exists a, b \in \mathbb{R},\ x \in (a, b) \subset S$.
x is an accumulation or limit point of S if $\forall \epsilon > 0,\ \exists y \in S,\ 0 < |x - y| < \epsilon$.
$S \subseteq \mathbb{R}$ is closed if it contains all its limit points.
A set may be both open closed, just open, just closed or neither.
Given $S \subseteq \mathbb{R}$, the interior of S is $\bigcup_{S' \text{ open} \subset S} S' = S^{\text{int}} = S^0$.
The closure is $\bigcap_{F \text{ closed} \supseteq S} F = \overline{S} \overset{\text{wts}}{=} S \cup \{\text{limit points of } S\}$.

Example

$\{x\}$ is not open, but, since the limit points of x are $\varnothing$, it is closed.

Propositions

1. Arbitrary unions and finite intersections of open sets are open.
2. S is open if and only the complement $S^c = \mathbb{R} \setminus S$ is closed.
3. Arbitrary intersections and finite unions of closed sets are closed.

Bolzano-Weierstrass Theorem

A bounded sequence in $\mathbb{R}$ ad mits a convergent (Cauchy) subsequence. $\exists M,\ |x_n| \leq M,\ \forall n$

Proof by Dichotomy

Suppose $I_0 = [a, b]$ contains the sequence.
Construct a sequence of intervals by indicators: if $\left[a, \frac{a+b}{2}\right]$ contains infinitely terms of $\{x_n\}_n$, choose n such that $x_{n_1} \in \left[a, \frac{a+b}{2}\right]$ and call $I_1 = \left[a, \frac{a+b}{2}\right]$.

Otherwise, $\left[\frac{a+b}{2}, b\right]$ must contain infinitely many terms. Choose n in a similar fashion as above such that $I_1 = \left[\frac{a+b}{2}, b\right]$.
This process may be repeated to create a sequence of intervals such that $I_k \supseteq I_{k+1} \supseteq I_{k+2}$ and $l(I_k) = \frac{b-a}{2^k}$.
A subsequence $\{u_{n_k}\}_k$ such that $u_{n_k} \in I_l$ for $k \geq l$.

Exercise

Extract a Cauchy criterion out of the above.

October 9, 2023

Overview

- Topology of $\mathbb{R}$ continued.
- Numerical series.

Next Wednesday

- Absolute and Conditional Convergence.
- Rearrangement theorem.

Last Time

Finished with Bolzano-Weierstrass.

Limits

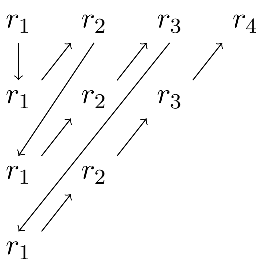
Limit Point

We say $x \in \mathbb{R}$ is a limit point of $\{x_n\}_n$ if a subsequence of $\{x_n\}_n$ converges to x.
Equivalently, $\forall \epsilon > 0,\ \forall n_0 \in \mathbb{N},\ \exists n \geq n_0,\ |x_n - x| < \epsilon$.
That is, the sequence revisits an epsilon neighborhood of x infinitely many times.

Limit Set

The limit set of $\{x_n\}_n : \mathrm{LS}(\{x_n\}_n) =$ the set of limit points of $\{x_n\}_n$.

- Comments
  - if $\lim_{n\to\infty}\{x_n\} = x$, then $\mathrm{LS}(\{x_n\}_n) = \{x\}$.
  - The limit set can be as big as $\mathbb{R}$!



- What Bolzano-Weierstrass says is that if $\{x_n\}$ is bounded, then $\text{LS}(\{x_n\}) \neq \varnothing$.

- Examples $\text{LS}(\{n\}_n) = \varnothing$.
  $\text{LS}(\{x_n\}_n)$ is closed (good exercise).

Limit Superior

If $\{x_n\}_n \in [a, b]$ is bouneded, $\forall k \in \mathbb{N}$, $\sup\{x_j | j \geq k\}$ exists in $\mathbb{R}$.
Because

$$a \leq \sup\{x_j | j \geq k+1\} = y_{k+1} \leq \sup\{x_j | j \geq k\} = y_k$$

by the Monotone Convergence Theorem, $\{y_k\}_k$ converges. Call its limit $\limsup_n x_n = \inf_n \sup\{x_j | j \geq n\}$.

Limit Inferior

Similarly, define $\lim_n \inf x_n = \sup_n \inf\{x_j | j \geq n\}$.

Limit Superior and Limit Inferior Always Exist

What to show: $\limsup x_n, \liminf x_n \in \text{LS}(\{x_n\})$.
Left as an exercise.

Convergence at the Limit

A bounded sequence $\{x_n\}_n$ converges if and only if $\liminf_n x_n = \limsup_n x_n$.

- Proof Technique Often it is useful to structure a proof such that

$$L < \liminf_n x_n \leq \limsup_n x_n < L$$

Topology of the Reals Continued

Compactness

Let $A \subseteq \mathbb{R}$.
A is (sequentially) compact if every sequence in A has a limit point in A.
A is (Heine-Borel) compact if every open cover of A has a finite subcover.

- Open Cover $\{O_\alpha\}_{\alpha \in I}$, with O_α open, is an open cover of A if $A \subseteq \bigcup_{\alpha \in I} O_\alpha$.

- Finite Subcover $O_1, \ldots, O_n, \; n \in \mathbb{N}$.

Heine-Borel Theorem

Let $A \subseteq \mathbb{R}$.
The following are equivalent

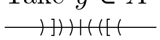
1. A is Heine-Borel compact.

2. A is closed and bounded.

3. A is sequentially compact.

Proof

(1) $\implies$ (2) $\implies$ (3) $\implies$ (1)

- Heine-Borel Compact Implies Closed and Bounded Suppose A satisfies the Heine-Borel property.
  Consider $\{(-n, n)\}_{n \in \mathbb{N}}$. Clearly $\bigcup_n (-n, n) = \mathbb{R} \supseteq A$.
  By Heine-Borel, $\exists n_0, \ldots, n_p$ such that $A \supseteq \bigcup_{j=0}^{p} (-n_j, \; n_j) = (-N, N), \; N = \max(n_0, \ldots, n_p)$. So A is bounded.
  A is closed if $y \notin A \implies y$ is not a limit point of A.
  Take $y \in A^c$, then $A \subseteq \mathbb{R} \setminus \{y\} = \bigcup_{n \in \mathbb{N}} (-\infty, y - \frac{1}{n}) \cup (y + \frac{1}{n}, \infty)$.
  

  By the Heine-Borel property,

$$
\begin{aligned}
A \subseteq & \bigcup_{n_0, \ldots, n_p} (-\infty, y - \frac{1}{n}) \cup (y + \frac{1}{n}, \infty) \\
&= (-\infty, y - \frac{1}{N}) \cup (y + \frac{1}{N}, \infty)
\end{aligned}
$$

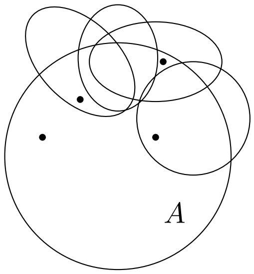
  Which implies $A \cap [y - \frac{1}{N}, y + \frac{1}{N} = \varnothing$ and y is not a limit point of A.
  That is, A contains its limit points.

- Closed and Bounded Implies Sequential Compactness Suppose A is both closed and bounded.
  Let $\{x_n\}_n \in A$. Then $\{x_n\}_n$ is bounded. By Bolzano-Weierstrass, it has a limit point x and a subsequence $\{x_{n_k}\}_k$ converging to x.
  Since A is closed, $\lim_{k \to \infty} x_{n_k} = x \in A$. ∎

- Sequential Compactness Implies Heine-Borel Suppose $A \subseteq \mathbb{R}$ is sequentially compact.
  Consider an open cover of A, $\{O_\alpha | \alpha \in I\}$.
  First, turn it into a countable cover:

  - $\forall \alpha \in I, \; O_\alpha \subseteq \left(r_\alpha^1, r_\alpha^2\right), \; r_\alpha^1, r_\alpha^2 \in \mathbb{Q}$

Assume that $\{O_\alpha\}_\alpha$ can be made countable $(O_1, \dots, O_n)$
By contradiction, suppose $\forall n \in \mathbb{N},\ A \setminus \left(\bigcup_{j=1}^n O_j\right) \neq \varnothing$.
Take $x_n \in A \setminus \left(\bigcup_{j=1}^n O_j\right)$. Since A is sequentially compact, $\exists \{x_{n_k}\}_k$ subsequence of $\{x_n\}_n$ converging to $x \in A$.
Since $A \subset \bigcup_{j \in \mathbb{N}} O_j,\ \exists j_0,\ x \in O_{j_0}$, O_{j_0} is open: $\exists \delta > 0,\ (x - \delta, x + \delta) \subseteq O_{j_0}$.
Then $\exists N,\ k \geq N \implies x_{n_k} \in (x - \delta, x + \delta) \subseteq O_{j_0}$. But if k is such that $n_k > j_0$, we also have $x_{n_k} \notin O_{j_0}$ which is a contradiction!



Structure of Open and Closed Sets

A is open in $\mathbb{R}$ if and only if it can be written as an at most countable, disjoint union of open intervals.

TODO Proof

For $x \in A,\ \exists (a, b)$, such that $x \in (a, b) \subseteq A$.
Let $I_x = \bigcup_{\text{open interval containing} x,\ I \subseteq A} I$. This is the maximal interval containing x in A.
Then, $A \subseteq \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} I_x \subseteq A$.
That is, $A = \bigcup_{x \in A} I_x \quad (*)$.
Next, if $x, y \in A$, then $\begin{cases} \text{either } I_x = I_y \\ \text{or } I_x \cap I_y = \varnothing \end{cases}$
IMAGE HERE
The union $(*)$, as a disjoint union, is at most countable because each distinct one must contain a distinct rational and $\mathbb{Q}$ is countable.

Long Story Short

The topology of the reals avoids exotic sets. Closed sets, on the other hand, can be quite complicated.

TODO Cantor Set

$C := \bigcap_{k \in \mathbb{N}_0} I_k$. I_{k+1} is obtained by removing the middle open third of each interval making I_k.
IMAGE HERE - CANTOR
$I_0 = [0, 1]$. One interval of length 1.
$I_1 = [0, 1/3] \cup [2/3, 1]$. Two intervals of length $2/3$.
$I_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9]$. Four intervals of $(2/3)^2$
I_k is 2^k intervals of length $(2/3)^k$.
$I_{k+1} \subseteq I_k \implies C \subseteq I_k,\ \forall k \implies l(C) \leq l(I_k) = (2/3)^k \implies l(C) = 0$.

TODO Triadic Expansions

Goal:

1. C is perfect (i.e. every point in C is a limit point of C).

2. C contains no open intervals.

Property 2 is easy because $C \subseteq I_k$, which does contain interval of length greater than $(1/3)^k$.

1. C is uncountable.

Every $x \in [0,1]$ can be written in the form $x = \sum_{k=1}^{\infty} \frac{a_k}{3^k},\ a_k \in \{0,1,2\}$.
That is, $x = 0.a_1 a_2 \ldots$ in base 3. This is not always unique (e.g. $1/3 = 0.100\ldots = 0.022\ldots$).
IMAGE HERE - THIRDS OF INTERVAL
Note that the Cantor set is removing all decimal expansions with leading 1s. That is, $x \in C$ if and only if it has a triadic expansion only made of 0s and 2s.

- Proof of 1 If $x \in C,\ x = \sum_{k \geq 1} \frac{a_k}{3^k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{3^k}$, then $x_n \in C,\ \forall n$ and $x_n = 0.a_1 \ldots a_n 0000\ldots$ where $a_1, a_n \in \{0,2\}$.
  Unique representation can be maintained by forcing the behavior of the $n+1$th digit.

- Proof of 3 Every point in $[0,1]$ can also be written as $x = \sum_{n=1}^{\infty} = \frac{b_n}{2^n},\ b_n \in \{0,1\}$ (i.e. a binary expansion).
  Then $C \mapsto [0,1]$ gives $x = \sum \frac{a_k}{3^k} \mapsto \sum \frac{b_k}{2^k},\ b_k = \frac{a_k}{2}$ for $a_k \in \{0,2\}$ is a bijection!

October 11, 2023

Overview: Numeric Series

- Series with non-negative terms.

- Series with general terms.

- Convergence criteria.

- Algebraic rules.

- Rearrangements.

General Notation

Sequence $\{x_n\}_{n \geq n_0}$ (often $n_0 \in \{0,1\}$)

Definition: Partial Sum

$S_n = \sum_{k=n_0}^{n} x_k$ $(x_n = S_n - S_{n-1})$
We say $\sum_n x_n$ converges if $\lim_{n\to\infty} S_n$ exists.
We denote $\sum_{k=n_0}^{\infty} x_k = \lim_{n\to\infty} S_n$

- Example: Geometric Series $\sum_{k=0}^{n} r^k = S_n,\ r \in (0,1)$
  $\frac{1-r^{n-1}}{1-r} \to \frac{1}{1-r}$

- Example: P Series $\sum_{k=1}^{n} \frac{1}{k^p},\ p > 0$

- Example: Exponential $\sum_{k=0}^{n} \frac{1}{k!}$

Series without Non-negative Terms

The series has non-negative terms if $x_n \geq 0,\ \forall n$.

Obvious Algebraic Limit Rules

If $\sum_{n \geq n_0} a_n$ and $\sum_{n \geq n_0} b_n$ converge and $\alpha \in \mathbb{R}$, then $\sum_{n \geq n_0}(a_n + \alpha b_n)$ converges to

$$\sum_{n=n_0}^{\infty} a_n + \alpha \sum_{n=n_0}^{\infty} b_n = \sum_{n=n_0} (a_n + \alpha b_n)$$

- Proof (Sketch) Reason on the partial sums; use algebraic limit rules on sequences.

Proposition

If $\sum_n x_n$ converges in $\mathbb{R}$, then $\lim_{n \to \infty} x_n = 0$.

- Proof $x_n = S_n - S_{n-1} \overset{n\to\infty}{\to} S - S = 0$
  Since $S_n \overset{n\to\infty}{\to} S$ and $S_{n-1} \overset{n\to\infty}{\to} S = \sum_{n=n_0}^{\infty} x_n$.

Series with Non-negative Terms

If $x_n \geq 0,\ \forall n,\ S_n = \sum_{k=n_0}^{n} x_k$ is non-decreasing.
By monotone convergence theorem, S_n is either bouned, and therefore converges, or unbounded from above where

$$\forall m > 0,\ \ \exists n_0 \in \mathbb{N},\ \forall n \geq n_0,\ S_n \geq M$$

This is "diverging to $+\infty$."

Theorem: Convergence Criteria

- Term Test If $0 \leq a_n \leq b_n,\ \forall n \geq n_0$ and $\sum_n b_n$ converges, then $\sum_n a_n$ converges.

  - Proof Suppose $0 \leq a_n \leq b_n$, and $t_n = \sum_{k=n_0}^{n} b_k$ converges and, therefore, is bounded above by $B = \sum_{k=n_0}^{\infty} b_k$.
    Then $\forall n,\ \sum_{k=n_0}^{n} a_k \leq \sum_{k=n_0}^{n} b_k \leq B$.
    Thus, by monotone convergence theorem, $\sum_{k=n_0}^{n} a_k$ converges.

- Ratio Test If $a_n > 0,\ \forall n$ and $\exists n_0 \in \mathbb{R}$ such that $\frac{a_{n+1}}{a_n} \leq r < 1,\ \forall n \geq n_0$, then $\sum_n a_n$ converges.
  - Clarification The harmonic series has ratio $\frac{k}{k+1} < 1$ but since $\frac{k}{k+1} \overset{k\to\infty}{\to} 1$, there is no r which satisfies the ratio test.
  - Proof Suppose $a_{n+1} \leq r a_n$ for $n \geq n_0$.
    Then $a_{m_0+p} \leq a_{m_0+(p-1)} r \leq a_{m_0+(p-2)} r^2 \leq \cdots \leq a_{m_0} r^p$.
    Then for $n \geq n_0$,

$$\sum_{k=n_0}^{n} a_k = \sum_{k=n_0}^{m_0} a_k + \sum_{k=m_0+1}^{n} a_k \leq \sum_{k=m_0}^{m_0+(n-m_0)} a_{m_0} r^{n-m_0} \leq a_{m_0} \sum_{k=m_0}^{n-m_0} r^{n-m_0} \leq \frac{1}{1-r}$$

  - Rate of Convergnce The above proof shows that the ratio test implies a geometric rate of convergence.

- Root Test If $\exists n_0 \in \mathbb{N}$ and $r \in (0,1)$ such that $a_n^{1/n} \leq r$, then $\sum_n a_n$ converges.
  - Proof (Sketch) Same story as the ratio test: $a_n^{1/n} \leq r \implies a_n \leq r^n$.

- Rejection of Ratio/Root If $\exists n_0 \in \mathbb{N}$ such that either $\frac{a_{n+1}}{a_n} \geq 1$ for $n \geq n_0$ or $a_n^{1/n} \geq 1$ for $n \geq n_0$, then $\sum_n a_n$ diverges to $+\infty$.
  - Proof (Sketch) In either case, a_n cannot converge to zero. Therefore the series cannot converge.

Prototype Scales

Geometric Rates

$\sum_{n\geq 1} \frac{1}{n^\alpha}$ converges if and only if $\alpha > 1$ (to $\zeta(\alpha)$)
$a_k = \frac{1}{k^\alpha} \rightsquigarrow 2^k a_{2^k} = \frac{2^k}{2^{k\alpha}} = \left(\frac{1}{2^{\alpha-1}}\right)^k \implies t_n = \sum_{k=1}^n 2^k a_{2^k}$ converges if and only if $\frac{1}{2^{\alpha-1}} < 1$ if and only if $\alpha > 1$.

Log Geometric Case

$\sum_{n\geq 1} \frac{1}{n(\log(n))^\beta}$ converges if and only if $\beta > 1$.
$a_k = \frac{1}{k(\log(k))^\beta} \rightsquigarrow 2^k a_{2^k} = \frac{2^k}{2^k(\log(2^k))^\beta} = \frac{1}{(\log(2)^\beta k^\beta}$ converges if and only if $\beta > 1$.

Lemma:

Suppose a_n decreases to 0.
Then the sequence $S_n = \sum_{k=1}^n a_k$ converges if and only if $t_n = \sum_{k=1}^n 2^k a_{2^k}$ converges.

- Proof

$$S_{2^n} = a_1 + \underbrace{a_2 + a_3}_{a_3 + a_3 \le \quad \le a_2 + a_3} + \underbrace{a_4 + a_5 + a_6 + a_7} + \underbrace{a_8 + \cdots}$$

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \cdots \\ &= a_1 + \sum_{k=1}^{n} \sum_{p=1}^{2^k - 1} a_{2^k + p} \\ &\le a_1 + 2^k a_{2^{k+1}} + \end{aligned}$$

This gives

$$\frac{1}{2}(t_n - a_1) \le S_{2^n} - a_1 \le t_{n-1}$$

Therefore S_{2^n} converges, which implies that t_n converges, and, since S_n is monotone, S_n itself converges.

Series with General Terms

General term is signed.

Trick

Write $a_n = a_n^+ - a_n^-$ and $a_n^\pm = \max(0, \pm a)$. Then

$$S_n = \sum_{k=n_0}^{n} a_k = \left(\sum_{k=n_0}^{n} a_k^+\right) - \left(\sum_{k=n_0}^{n} a_k^-\right)$$

Convergence Outcomes

	$\sum_{k=n_0}^{\infty} a_k^+ < \infty$	$\sum_{k=n_0}^{\infty} a_k^+ = \infty$
$\sum_{k=n_0}^{\infty} a_k^- < \infty$	absolute convergence	$\lim S_n = +\infty$
$\sum_{k=n_0}^{\infty} a_k^+ = \infty$	$\lim S_n = -\infty$	lots of things can happen; divergence, convergence, any limit sequence

If S_n^+ and S_n^- converge, we can return to algebraic limit rules.
S_n converges to $\lim_{n\to\infty} S_n^+ - \lim_{n\to\infty} S_n^-$

Definition: Absolute Convergence

We say $\sum_n a_n$ converges absolutely if and only if $\sum_n |a_n|$ converges.

Note

$|a_n| = a_n^+ + a_n^-$

Proposition: Absolute Convergence Implies Convergence

Proof

Absolute convergence $\implies \sum |a_n|$ converges $\implies \sum a_n^+$ and $\sum a_n^-$ converges $\implies \sum(a_n^+ - a_n^-)$ converges.

Definition: Conditional Convergence

$\sum_n a_n$ converges conditionally if and only if $\sum_n a_n$ converges while $\sum_n |a_n|$ diverges.

Criteria for Convergence

For absolute convergence, run root/ratio/term test on $\sum_n |a_n|$.
Other criteria which might indicate conditional convergence.

Alternating Series Test

If $a_n(-1)^n b_n,\ b_n \geq 0$ decreases to zero, the series is conditionally convergent.

Dirichlet Test

If $a_n = b_n c_n$, where b_n decreases to zero and c_n satisfies $|c_0 + c_1 + \cdots + c_n| \leq C,\ \exists C \in \mathbb{R}, \forall n \in \mathbb{N}$, then $\sum_{n\geq 0} a_n$ converges conditionally.

- Applications $\sum_{n\geq 1} \frac{(-1)^n}{n}$
  $\sum_{n\geq 1} \frac{\cos(n)}{n}$

- Proof Write $C_n = c_0 + c_1 + \cdots + c_n$, such that $|C_n| \leq C,\ \forall n$.
  Then $c_n = C_n - C_{n-1}$, and

$$\sum_{k=0}^{n} b_k c_k = \sum_{k=0}^{n} b_k(C_k - C_{k-1}) = \sum_{k=0}^{n} b_k C_k - \sum_{k=0}^{n} b_k C_{k-1} \overset{l=k-1}{=} \sum_{k=0}^{n} b_k C_k - \sum_{l=0}^{n-1} b_{l+1} C_l = b_n C_n + \sum_{k=0}^{n-1} (b_k - b_{k+1}) C_k$$

  Then, since $b_n C_n \overset{n\to\infty}{\to} 0$, we only need to show that the final term converges absolutely. Consider

$$\sum_{k=0}^{n-1} |b_k - b_{k+1}||C_k| \leq C \sum_{k=0}^{n-1} (b_k - b_{k+1}) = C(b_0 - b_n) \leq C(b_0)$$

  independent of n. Hence, $\sum_{k=0}^{n} b_k c_k$ converges.

Definition: Rearrangement

Take $\sigma : \mathbb{N} \to \mathbb{N}$ a bijection and $\sum_{n\geq 1} a_n$ a series such that $S_n = \sum_{k=1}^{n} a_k$.
Then define a rearranged sum $S_n^{(\sigma)} = \sum_{k=1}^{n} a_{\sigma(k)}$.

Q: When does the rarranged sum converge; to where?

- Theorem: Rearrangement of Absolute Convergence If $\sum a_n$ converges absolutely, then $\forall \sigma,\ \lim_{n\to\infty} S_n^{(\sigma)} = \lim_{n\to\infty} S_n$.

- Theorem: Rearrangement of Conditional Convergence If $\sum a_n$ converges conditionally, then $\forall x \in \mathbb{R},\ \exists \sigma$ such that $\lim_{n\to\infty} S_n^{(\sigma)} = x$.

October 16, 2023

Overview

Sequences and Series of Functions
Things that will be glossed over for time

- Limits
- Continuity
- Differentiability
- Integrabillity

Why care about sequences and series?

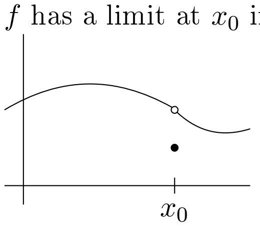
Extending features of functions.
Approximations.

Limits and Continuity

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R},\ x_0 \in I$.

Definition: Limit

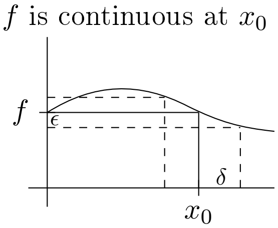
f has a limit at x_0 if $\exists \ell \in \mathbb{R}, \forall \epsilon > 0, \exists \delta > 0, 0 < |x - x_0| < \delta \implies |f(x) - \ell| < \epsilon$



- Equivalently
  For every sequence $\{x_n\}_n$ in I converging to x (but distinct to x), $\lim_{n\to\infty} f(x_n) = \ell$.

Definition: Continuous

f is continuous at x_0 if $\lim_{x\to x_0} f(x) = f(x_0)$.



- Modulus of Continuity $\forall \epsilon > 0, \exists \delta > 0, \forall x \in I, |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$
  Then $\delta(x_0, \epsilon)$ is the modulus of continuity.

Definition: Uniform Continuity on I

f is uniformly continuous on I if $\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in I, |x-y| < \delta \implies |f(x) - f(y)| < \epsilon$.
Where δ is $\delta(\epsilon)$. That is, the modulus of continuity does not depend on the points.

Special Types of Uniform Continuity

Hölder Continuous

f is α-Hölder continuous on I for $\alpha \in (0, i]$, if $\exists c > 0$ such that $\forall x, y \in I, |f(x) - f(y)| \leq c|x-y|^{\alpha}$
$\alpha = 1$ implies that f is "Lipschitz-continuous"

- Example
  If f' exists and is bounded on $[a, b]$ by M, then by the Mean Value Theorem:
  $|f(x) - f(y)| = |f'(\xi)||x-y| \leq M|x-y|$, where $x \leq \xi \leq y$.

Continuity on Compact Sets

Let $K \subseteq \mathbb{R}$ be a compact set and $f : K \to \mathbb{R}$ be continuous.
Then

1. $f(K)$ is compact. In particular, f is bounded on K.

2. f achieves its extrema on K. (e.g. $\exists M \in K$ such that $f(M) = \sup\{f(x) \mid x \in K\}$.

3. f is uniformly continuous on K.

Note: the proofs of these features are good practice. In particular, proofs that exploit the Heine-Borel property.

Proof 1: Compact

Let y_n be a sequence in $f(K)$.
Then, $\forall n, y_n = f(x_n)$ for $x_n \in K$.
It follows that there exists a subsequence $\{x_{n_k}\}_k$ converging to x in K.
By continuity, $y_{n_k} = f(x_{n_k}) \overset{k\to\infty}{\to} f(x) \in f(k)$.

Proof 2: Achieves Its Extrema

Construct M.
By the suprememum property, $S = \sup\{f(x) \mid x \in \mathbb{R}\}$, $\forall n, \exists x_n \in K$ such that $S - \frac{1}{n} \leq f(x_n) < S$.
Since K is compact, there exists a subsequence $\{x_{n_k}\}_k$ converging to $x \in K$.
Since f is continuous at x, $f(x_{n_k}) \overset{k\to\infty}{\to} f(x)$, and also $S - \frac{1}{n_k} \leq f(x_{n_k} \leq S \overset{k\to\infty}{\to} S = f(x)$.

Proof 3: Uniformly Continuous

Suppose, for sake of contradiction, that $\exists \epsilon > 0, \forall \delta > 0, \exists x_\delta, y_\delta \in K, |x_\delta - y_\delta| < \delta$ and $|f(x_\delta) - f(y_\delta)| \geq \epsilon$.
Letting $\delta = \frac{1}{n}$, we may write $x_n, y_n \in K, |x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \epsilon$.
Since K is compact, there exists a subsequence $\{x_{n_k}\}_k$ which converges to $x \in K$.
Since $|x_{n_k} - y_{n_k}| < \frac{1}{n_k}$, then $\{y_{n_k}\}_k$ also converges to x.
By continuity of f at x, $\lim_{k\to\infty} f(x_{n_k}) - f(y_{n_k}) = 0$. However, this contradicts the established fact that $|f(x_n) - f(y_n)| \geq \epsilon$ for $\epsilon > 0$.

Notation

Let $I \subseteq \mathbb{R}$ be an interval.

Sequence of Functions

$f_n(x), x \in I, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$

Series of Functions

$S_n(x) = \sum_{k=0}^{n} f_k(x)$

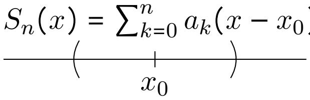
Definition: Pointwise Convergence

A sequence or series of functions converges pointwise on I if and only if $\forall x \in I, \{f_n(x)\}_n$ is convergent.
Call $f(x)$ the limit.

Q: Under what conditions do properties of a sequence (e.g. continuity, differentiability, integrability) propogate to the limit?

Power Series

$\sum_{n\geq 0} a_n(x - x_0)^n$
$S_n(x) = \sum_{k=0}^{n} a_k(x - x_0)^k$



Fourier Series

$S_n = a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx)$ is 2π-periodic.

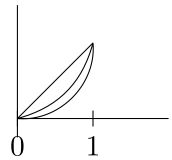
Approximation

For purposes of approximation, it is useful to know if, for example, the integral may be approximated by taking integrals of the partial sums.

Deficiencies of Pointwise Convergence

Example 1

On $[0,1]$, $f_n(x) = x^n \overset{n\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases}$,



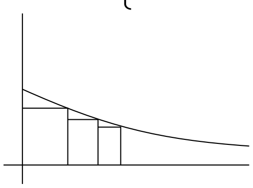
f_n is continuous on $[0,1]$, $\forall n$, but f is not.

- Exercise

  Show that there is no uniform convergence here.
  Hint: negate uniform convergence and prove the negation.

Example 2

$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$ is not Riemann-integrable on $[0,1]$.



If r_n denotes a denumeration of rationals in $[0,1]$, define $f_n(x) = \begin{cases} 1 & \text{if } x \in \{r_0, \dots, r_n \\ 0 & \text{otherwise} \end{cases}$.

So f_n converges pointwise on $\chi_{\mathbb{Q}}$.
Yet, $\forall n$, f_n is Riemann-integrable and $\int_0^1 f_n(x)\,dx = 0$.

Definition: Uniform Convergence

We say $f_n : D \to \mathbb{R}$ (e.g. D an interval) converges uniformly to f on D (notation $f_n \rightrightarrows f$ on D) if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, n \geq N \implies \begin{cases} |f_n(x) - f(x)| < \epsilon, \forall x \in D \\ \sup_D |f_n - f| < \epsilon \end{cases}$$

Compare with Pointwise Convergence

Compare to $f_n \to f$ pointwise on D.
$\forall x \in D, \forall \epsilon > 0, \exists N \in \mathbb{N}, n \geq N \implies |f_n(x) - f(x)| < \epsilon$.
In this case, the behavior is primarily contingent upon the choice of x. That is $N(x,\epsilon)$ is dependent on x.

Theorem: Weierstrass M-Test

Let $f_n : D \to \mathbb{R}$ be bounded by M_n on D.
If $\sum_{n=1}^{\infty} M_n < \infty$, then the series $S_n(x) = \sum_{k=1}^{n} f_k(x)$ converges uniformly to $S(x)$

Proof

$\forall x \in D, |S_n(x) - S(x)| = |\sum_{k=n+1}^{\infty} f_k(x)| \overset{\text{triangle inequality}}{\leq} \sum_{k=n+1}^{\infty} |f_k(x) \leq \sum_{k=n+1}^{\infty} M_k$, where $\sum_{k=n+1}^{\infty} M_k$ is a uniform bound in x.
Let $\epsilon > 0, \exists n, n \geq N \implies \sum_{k=n+1}^{\infty} M_k < \epsilon$.
Then $\forall x \in D, n \geq N, |S_n(x) - S(x)| \leq \sum_{k=n+1}^{\infty} M_k < \epsilon$. ■

Theorem: Continuity and Uniform Limits

Let $f_n D \to \mathbb{R}$ be continuous on D for all n and $f_n f$ on D ($\lim_{n\to\infty} \sup_D |f_n - f| = 0$).
Then f is continuous on D.

Proof

Fix $x \in D$, with x_n converging to x in D.
What To Show: $f(x_n) \overset{n\to\infty}{\longrightarrow} f(x)$.
Scratch: $f(x_n) - f(x) = (f(x_n) - f_p(x_n)) + (f_p(x_n) - f_p(x)) + (f_p(x) - f(x))$.
Let $\epsilon > 0$ be given.
$f_n \rightrightarrows f : \exists N, n \geq N \implies |f_n(y) - f(y)| < \frac{\epsilon}{3}, \forall y \in D$.
For $p \geq N, |f_p(y) - f(y)| < \frac{\epsilon}{3}, \forall y \in D \implies \forall n \in \mathbb{N}, |f(x_n) - f(x)| \overset{\text{triangle inequality}}{\leq} \frac{2\epsilon}{3} + |f_p(x_n - f_p(x)|$.
With $p = N$, since f_p is continuous at x, $\exists N_1, n \geq N_1 \implies |f_p(x_n) - f_p(x)| < \frac{\epsilon}{3}$.
Hence, for $n \geq N_1, |f(x_n) - f(x)| \leq \epsilon$. ■

Riemann-Integrability

Fix $D = [a, b]$ and $g : [a, b] \to \mathbb{R}$ bounded by $|g(x)| \leq M, \forall x$.

Definition: Subdivision

$\sigma = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b$

Definition: Upper and Lower Riemann Sums

$S^+(g, \sigma) = \sum_{k=1}^{n} (x_k - x_{k-1}) M_k$ is the upper sum.
$S^-(g, \sigma) = \sum_{k=1}^{n} (x_k - x_{k-1}) m_k$ is the lower sum.
Where $M_k = \sup_{[x_{k-1}, x_k]} g$ and $m_k = \inf_{[x_{k-1}, x_k]} g$.
This gives $-M(b - a) \leq S^-(g, \sigma) \leq S^+(g, \sigma) \leq (b - a)M$.
If $\mathfrak{S}[a, b] = \{\text{subdivisions of } [a, b]\}$, then
$I^-(g) = \sup_{\sigma \in \mathfrak{S}[a,b]} S^-(g, \sigma)$ and $I^+(g) = \inf_{\sigma \in \mathfrak{S}[a,b]} S^+(g, \sigma)$.

Definition: Riemann Integrable

g is Riemann integrable if $I^+(g) = I^-(g)$ and we denote $\int_a^b g(t)\, dt = I^+(g)$.

Lemma

g is Riemann integrable if and only if $\forall \epsilon > 0, \exists \sigma \in \mathfrak{S}[a, b]$ such that $S^+(g, \sigma) - S^-(g, \sigma) < \epsilon$.

Properties

1. Continous functions and monotone functions are Riemann Integrable.

2. $f \mapsto \int_a^b f(t)\ dt$ is linear.

3. If f, g are Riemann Integrable and $f(x) \le g(x), \forall x \in [a, b]$, then $\int_a^b f(t)\ dt \le \int_a^b g(t)\ dt$.

Theorem:

If $f_n \rightrightarrows f$ on $[a, b]$ and f_n is Riemann Integrable for all n, then f is Riemann Integrable on $[a, b]$ and $\lim_{n\to\infty} \int_a^b f_n(t)\ dt = \int_a^b \lim_{n\to\infty} f_n(t)\ dt = \int_a^b f(t)\ dt$.

Proof

$\forall n, \forall x \in [a, b], f_n(x) - \epsilon \le f(x) \le f_n(x) + \epsilon$ where $\epsilon_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|$ (by hypothesis $e_n \xrightarrow{n\to\infty} 0$)
Then, for any $\sigma \in \mathfrak{S}[a, b]$, $S^-(f_n, \sigma) - \epsilon_n(b - a) \le S^-(f, \sigma) \le S^+(f, \sigma) \le S^+(f_n, \sigma) + \epsilon_n(b - a)$.
It follows that $S^+(f, \sigma) - S^-(f, \sigma) \le S^+(f_n, \sigma) - S^-(f_n, \sigma) + 2\epsilon_n(b - a)$.
Finishing the proof is left as an exercise.

October 18, 2023

Overview

- Sequences/Series

- Power Series

- Exponential and Logarithms

Fundamental Theorems of Calculus

Full proofs in 105A lecture notes.

Differentiation of the Integral

$f : [a, b] \to \mathbb{R}$ continuous.
$\forall x \in [a, b]$, can define $F(X) = \int_a^x f(t)\ dt$.
Then F is continuously differentiable on $[a, b]$
$F'(x) = f(x)$ for $x \in [a, b]$.

Integration of the Derivative

$f \in C^1[a, b]$ with one-sided derivatives at a and b well defined. (e.g. $\frac{f(a+h)-f(a)}{h} \underset{h>0; h\to 0}{\longrightarrow} f'(a)$.
Then $\forall x, y, a \le x \le y \le b$, $f(y) - f(x) = \int_x^y f'(t)\ dt$.

Theorem: Differentiability of Uniform Limits

Let $f_n : (a,b) \to \mathbb{R}$ be a sequence in $C^1[a,b]$, and assume $f_n(x) \to f(x)$ pointwise while $f_n'(x) \rightrightarrows g(x)$ uniformly. Then $f \in C^1(a,b)$ and $f' = g$.

Proof

Fix $a_0 \in (a,b)$.
Then $\forall x \in (a,b)$, by the Fundamental Theorem of Calculus,

$$f_n(x) - f_n(a_0) = \int_{a_0}^{x} f_n'(t)\, dt$$

Observe that $f_n(x) \xrightarrow[n\to\infty]{} f(x)$ and $f_n(a_0) \xrightarrow[n\to\infty]{} f(a_0)$ pointwise, and $\int_{a_0}^{x} f_n'(t)\, dt \to \int_{a_0}^{x} g(t)\, dt$ by the integrability of uniform limits. Then

$$f(x) - f(a_0) = \int_{a_0}^{x} g(t)\, dt,\ \forall x \in (a,b)$$

which implies $f \in C^1$ and $f' = g$. ■

Interesting Applications

$S_n(x) = \sum_{k=0}^{n} f_k(x)$.
Suppose pointwise convergence, that $S_n'(x) = \sum_{k=0}^{n} f_k'(x)$ is continuous, $|f_k'(x)| \le M_k$ and $\sum_{k=0}^{\infty} M_k < \infty$.
Long story short, this implies

$$\left(\sum_{k=0}^{\infty} f_k(x)\right)' = \sum_{k=0}^{\infty} f_k'(x)$$

Example

$f(x) = \sum_{n=0}^{\infty} \frac{\cos(nx)}{n^3}$
Call $u_n(x) = \frac{\cos(nx)}{n^3}$, then $|u_n(x)| \le \frac{1}{n^3}$ summable and $|u_n'(x)| = \left|\frac{-\sin(nx)}{n^2}\right| \le \frac{1}{n^2}$ summable.
This implies $f'(x) = -\sum_{n=0}^{\infty} \frac{\sin(nx)}{n^2}$.
Repetition of this process informs us that $f \in C^2$.

Power Series

$S_n(x) = \sum_{k=1}^{n} a_k(x - x_0)^k$ for, $x_0 \in \mathbb{R}$ fixed, is 'centered at x_0.' Note that each term is $C^\infty(\mathbb{R})$.

Example 1

$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ for $|x| < 1$.

Example 2

$\exp(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges $\forall x \in \mathbb{R}$.

- Why?
  Ratio Test.
  $$\frac{a_{k+1}}{a_k} = \frac{x^{k+1}}{(k+1)!} \cdot \frac{k!}{x^k} = \frac{x}{k+1}$$
  So $\left|\frac{a_{k+1}}{a_k}\right| \xrightarrow[k\to\infty]{} 0$

Lemma: Radius of Convergence

Suppose a power series $\sum_{n\geq 0} a_n x^n$ converges at $b \in \mathbb{R}$.

1. Converges absolutely $\forall x, |x| < |b|$.

2. $\forall a \in (0, b)$ converges uniformly on $[-a, a]$.

- Proof of 1
  Suppose $\sum_{n\geq 0} a_n b^n$ converges.
  Then $a_n b^n \to 0$.
  Let x such that $|x| < b$, then
  $$|a_n x^n| = \left|a_n b^n \left(\frac{x}{b}\right)^n\right| \leq M \left(\frac{|x|}{b}\right)^n$$
  By term test, $\sum_{n=0}^{\infty} |a_n x^n| < \infty \implies \sum a_n x^n$ converges absolutely.

- Proof of 2
  If $|x| \leq a < b$,
  $$|a_n x^n| \leq M \left(\frac{|x|}{b}\right)^n \leq M \left(\frac{a}{b}\right)^n$$
  Thus, by M-test for $x \in [-a, a]$, the series converges uniformly on $[-a, a]$.

- Upshot
  The set where a power series converges is an interval centered at x_0.

Theorem: Radius of Convergence

Given a power series, define R to be such that $\frac{1}{R} = \limsup_n |a_n|^{1/n}$. Then

1. $\forall a \in (0, R)$, the series converges uniformly on $[-a, a]$.

2. If $|x| > R$, the series diverges.

Proof

IMAGE HERE - RADIUS OF CONVERGENCE
Fix x. As an exercise, $\limsup_n |a_n x^n|^{1/n} = |x| \cdot \limsup_n |a_n|^{1/n} = \frac{|x|}{R}$.
Recall that $\limsup_n |a_n x^n|^{1/n} = \lim_{n\to\infty} y_n$ where $y_n = \sup_{k>n}\{|a_k x^k|^{1/k}\}$.
If $\frac{|x|}{R} < 1$, then $\exists N_0, n \geq N_0 \implies y_n < \frac{1+\frac{|x|}{R}}{2} < 1$.
This implies $\forall k \geq N_0, |a_k x^k|^{1/k} \leq \frac{1+\frac{|x|}{R}}{2} < 1$ and, by the root test, the series converges.
If $\frac{|x|}{R} > 1$, $\forall n, \sup_{k>n}\{|a_k x^k|^{1/k}\} \geq \frac{|x|}{R}$.
By the properties of the supremum with $\epsilon = \left(\frac{|x|}{R} - 1\right)/2 > 0$,

$$\forall n, \exists k, 1 \leq \frac{\frac{|x|}{R} + 1}{2} \leq y_n - \epsilon \leq |a_k x^k|^{1/k} \leq y_n$$

Therefore $\forall n, \exists k > n, |a_k x^k|^{1/k} \geq 1$. ■

Observation: Behavior at Endpoints

At the endpoints of $(-R, R)$, a series might

Converge Absolutely

e.g. $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$, $R = 1$, $\frac{1}{R} = \limsup_n \left(\frac{1}{n^2}\right)^{1/n} \xrightarrow{n\to\infty} 1$

Converge Conditionally

e.g. $\sum_{k=1}^{\infty} \frac{x^k}{k}$, $R = 1 \longrightarrow \frac{1}{R} = \limsup_n \left(\frac{1}{n}\right)^{1/n} = 1$
Converges conditionally at $x = -1$.

Diverge

e.g. $\sum_{k=0}^{\infty} x^k$, $R = 1$

Theorem: Power Series Differentiation

Let $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ converge on $(x_0 - R, x_0 + R)$.
Then $\forall k > 0, f \in C^k(x_0 - R, x_0 + R)$ and $f^{(k)}(x) = \sum_{n=k}^{\infty} a_n n(n-1)\cdots(n-k+1)(x-x_0)^{n-k}$, $\forall x \in (x_0 - R, x_0 + R)$

Exercise

Show that if $a_n \to a > 0$, then $\limsup a_n b_n = a \limsup b_n$.

Proof (by Induction)

Consider the series $S_n(x) = \sum_{n=1}^{\infty} a_n n(x - x_0^{n-1} = \sum_{n=0}^{\infty} a_{n+1}(n+1)(x-x_0)^n$.
Then

$$(x - x_0)\frac{1}{R \text{ of series of derivatives}} = \limsup_{n\to\infty}(a_n n)^{1/n} \limsup_{n\to\infty} a_n^{1/n} n^{1/n} = \limsup a_n^{1/n} = \frac{1}{R}$$

This implies $\sum_{k=0}^{\infty} \frac{d}{dx}(a_k(x-x_0)^k)$ converges uniformly on $[x_0 - a, x_0 + a], \forall a \in (0, R)$.
By the Theorem on Differentiability of Uniform Limits, $f'(x)$ exists and $\forall x \in (x_0 - R, x_0 + R)$

$$f'(x) = \sum_{n=1}^{\infty} a_n n(x - x_0)^{n-1}$$

Repeat to get higher derivatives.

Integration

It is similarly possible to integrate term by term.

Famous Power Series

- $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k,\ |x| < 1$
- PSE of $\frac{1}{x}$ centered at $x_0 > 0$

IMAGE HERE - GRAPH

$$\frac{1}{x} = \frac{1}{x - x_0 + x_0} = \frac{1}{x_0} \cdot \frac{1}{1 + \frac{x-x_0}{x_0}} = \frac{1}{x_0}\sum_{k=0}^{\infty}\left(-\frac{x-x_0}{x_0}\right)^k = \sum_{k=0}^{\infty}\frac{(-1)^k}{x_0^{k+1}}(x-x_0)^k \text{ if } |x - x_0| < |x_0|, x \in (0, 2x_0)$$

- $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
- $\exp(0) = 1$
- $\exp'(x) = \sum_{k=1} \frac{kx^{k-1}}{k!} = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} = \exp(x)$

Law of Exponents

$\exp(a)\exp(b) = \exp(a+b), \forall a, b \in \mathbb{R}$

Proof

Special case of the "Cauchy product of convergent series."
If $\sum_{n\geq 0} a_n$ converges absolutely to A and $\sum_{n\geq 0} b_n$ converges to B, then $\sum_{n\geq 0} c_n$ converges to AB, where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$$

- Heuristics

$$\left(\sum_{p=0}^{\infty} a_p x^p\right)\left(\sum_{l=0}^{\infty} b_l x^l\right) = \sum_{p,l \in \mathbb{N}_0^2} a_p b_l x^{p+l}$$

IMAGE HERE - CIRCLES FROM L TO P

$$\{(p,l) : p + l = n, p, l \in \mathbb{N}_0\} = \{(0,n), (1, n-1), \ldots, (n, 0)\}$$

Proof Continued

Aexp$(a) = \sum_{k=0}^{\infty} \frac{a^k}{k!}$ and $\exp(b) = \sum_{l=0}^{\infty} \frac{b^l}{l!}$, thus $\exp(a)\exp(b) = \sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} \frac{(a+b)^n}{n!} = \exp(a+b)$ \\) since

$$c_n = \frac{1}{n!}\sum_{n=0}^{n} \frac{a^k}{k!} \cdot \frac{b^{n-k}}{(n-k)!} \text{ and } n! = \frac{1}{n!}(a+b)^n$$

Power Series Expansion of Exponential

Centered at x_0, we have

$$\exp(x) = \exp(x - x_0)\exp(x_0) = \exp(x_0)\sum_{k=0}^{\infty} \frac{(x-x_0)^k}{k!}$$

Observation:

exp is the only $C^1(\mathbb{R})$ solution of $\begin{cases} \exp'(x) = \exp(x) \\ \exp(0) = 1 \end{cases}$

- Proof If f solves the above, then for some constant c

$$\frac{d}{dx}\left(f(x)\exp(-x)\right) = f'(x)\exp(-x) - f(x)\exp(-x) = 0 \underset{\text{MVT}}{\Longrightarrow} f(x)\exp(-x) = c = f(0)\exp(-0) = 1$$

this implies

$$f(x) = \exp(x) f(x)\exp(-x) = \exp(x)$$

Exponential Features

$$\exp(x) > 0, \forall x \in \mathbb{R} \implies \begin{cases} \text{if } x \geq 0, \exp(x) \geq 1 > 0 \\ \text{if } x < 0, \exp(x) = \frac{1}{\exp(-x)} > 0 \end{cases}$$

Theorem: Exponential and e

$\exp(x) = (\exp(1))^x \forall x \in \mathbb{R}$ and $e = \exp(1)$

Proof

Using law of exponents for

$$x \in \mathbb{N}: \quad \exp(n) = \exp(1 + (n-1) = e \cdot \exp(n-1) = \cdots = e^n \exp(0)$$

$$x = \frac{1}{q}, q \in \mathbb{N}: \quad \left(\exp\left(\frac{1}{q}\right)\right)^q = \exp\left(\frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}\right) = \exp(1) = e$$
$$\therefore \exp\left(\frac{1}{q}\right) = e^{1/q}$$

$$x = \frac{p}{q}, p, q \in \mathbb{N}: \quad \exp\left(\frac{p}{q}\right) = \exp\overbrace{\left(\frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}\right)}^{p} = \left(e^{1/q}\right)^p = e^{p/q}$$

$$x \in -\mathbb{N}, \mathbb{Q} < 0: \quad \text{left as an exercise}$$

Therefore, the functions $x \mapsto \begin{cases} \exp(x) \\ e^x \end{cases}$ are continous on $\mathbb{R}$ and agree on $\mathbb{Q}$. This implies that they must be equal everywhere.

October 23, 2023

Today

Exp and log.
Real-analytic functions. (Newest bit of information.)
Trig functions.

Wednesday, October 25, 2023

Analytic vs C^∞
Approximation by polynomials.

Next Week

Fourier series.

Exponential and Log

Covered Last Lecture

Law of Exponents
$\exp(x) = e^x$ and $e = \exp(1) = \sum_{k=0}^{\infty} \frac{1}{k!}$

Error Estimate

$e = \lim_{n\to\infty} S_n$ where $S_n = \sum_{k=0}^{\infty} \frac{1}{k!}$ (increases).
$e - S_n = \sum_{k=n+1}^{\infty} \frac{1}{k!}$
For $k = n + 1 + p,\ p \geq 0$, $e - S_n = \sum_{p=0}^{\infty} \frac{1}{(n+1+p)!}$.
Then,

$$\begin{aligned}\frac{1}{(n+1+p)!} &= \frac{1}{(n+1)!} \cdot \underbrace{\frac{1}{(n+2)(n+3)\cdots(n+p+1)}}_{p\text{ factors}} \\ &\leq \frac{1}{(n+1)!} \cdot \frac{1}{(n+1)^p}\end{aligned}$$

and

$$\begin{aligned}e - S_n &= \sum_{k=n+1}^{\infty} \frac{1}{k!} \\ &= \sum_{p=0}^{\infty} \frac{1}{(n+1+p)!} \\ &\leq \frac{1}{(n+1)!} \cdot \sum_{p=0}^{\infty} \left(\frac{1}{n+1}\right)^p \\ &= \frac{1}{(n+1)!} \cdot \frac{1}{1 - \frac{1}{n+1}} \\ &= \frac{1}{(n+1)!} \cdot \frac{n+1}{n}\end{aligned}$$

Therefore,

$$0 \leq e - S_n \leq \frac{1}{n!} \cdot \frac{1}{n}$$

Theorem: e is Irrational

Proof

Suppose $e = \frac{p}{q}$, $q > 2$, and p and q coprime. Consider

$$\begin{aligned}0 &< e - S_q \leq \frac{1}{q!} \cdot \frac{1}{q} \\ 0 &< q!(e - S_q) \leq \frac{1}{q} \\ 0 &< q!\left(\frac{p}{q} - \sum_{k=0}^{q} \frac{1}{k!}\right) \leq \frac{1}{q} < \frac{1}{2}\end{aligned}$$

where $q!\left(\frac{p}{q} - \sum_{k=0}^{q} \frac{1}{k!}\right) \in \mathbb{N}$.
This is a contradiction. Thus, e must be irrational. ■

Exponential Decay

$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
$\lim_{x\to+\infty} x^k e^{-k} = 0, \forall k \in \mathbb{N}$
For $x > 0$, $\exp(x) \geq \frac{x^{k+1}}{(k+1)!}$ if and only if $x^k \exp(-x) \leq \frac{(k+1)!}{x} \underset{x\to+\infty}{\longrightarrow} 0$.

Exponential Strictly Positive Over Reals

$\exp(x) > 0, \forall x \in \mathbb{R}$
$x > 0$ is obvious.
$x \leq 0, \exp(x) = \frac{1}{\exp(-x)} > 0$
$\lim_{x\to-\infty} \exp(x) = \lim_{x\to-\infty} \frac{1}{\exp(-x)} = 0$.

Proposition: Exponential is a Bijection

$\exp : \mathbb{R} \to (0, \infty)$ is a C^{∞} ($\exp' = \exp$) bijection (diffeomorphism in the sense that $\exp'(x) > 0, \forall x \in \mathbb{R}$).
By Inverse Function Theorem then, define $\log : (0, \infty) \to \mathbb{R}$ such that $\exp(\log(x)) = x$.
By MATH 105A, $\frac{d}{dx}(\log(x)) = \frac{d}{dx}(\exp^{-1}(x)) = \frac{1}{\exp'(\exp^{-1}(x)}) = \frac{1}{\exp(\log(x))} = \frac{1}{x}$.
$\log(1) = 0$ (since $\exp(0) = 1$) which implies, by the Fundamental Theorem of Calculus, that $\log(x) - \log(1) = \int_1^x \frac{dt}{t}$.

Properties (left as an exercise)

- $\log(xy) = \log(x) + \log(y),\ x, y > 0$
- Power Series Expansion: $\log(1 - x) = -\sum_{k=0}^{\infty} \frac{x^k}{k}$, x near 0, radius of convergence: 1.
- $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = \exp(x)$

Definition: Real-Analytic Functions

A function $f : (a, b) \to \mathbb{R}$ is real-analytic on (a, b) if $\forall x_0 \in (a, b)$, $\exists r > 0$ and a power series $\sum_{n\geq 0}(x - x_0)^n$ converging to f on $(x_0 - r, x_0 + r)$.
When such a power series exists, $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$, then

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

The radius of convergence is related by $\frac{1}{R} = \limsup_n |a_n|^{1/n}$ which provides a contraint on rate of divergence.

Example 1: Polynomial

For every polynomial, $p : \mathbb{R} \to \mathbb{R}$, and $\forall x_0 \in \mathbb{R}$,

$$p(x) = \sum_{k=0}^{\infty} \frac{p^{(k)}(x_0)}{k!}(x - x_0)^t, \forall x \in \mathbb{R}$$

Example 2: Exponential

$$\exp(x) = \exp(x - x_0 + x_0) = \sum_{k=0}^{\infty} \frac{e^{x_0}}{k!}(x - x_0)^t$$

and the radius of convergence, $R = \infty$.

Example 3: 1/x

$\frac{1}{x}$ analytic on $(0, \infty)$
$\frac{1}{x} \sum (x - x_0)^k$ and $R = |x_0|$.

$0 \quad x_0$

Remark: Analyticity Implies Smoothness

f analytic on $(a, b) \implies f$ smooth (C^∞) on (a, b)
The converse is not true. (Example Wednesday)

Proposition:

Suppose $\sum_{n \geq 0} a_n (x - x_0)^n$ converges to $f(x)$ on $(x_0 - R, x_0 + R)$.
Then $f(x)$ is analytic on $(x_0 - R, x_0 + R)$.

$x_0 \quad x_1 \quad x_0 + R$ That is to say, $\forall x_1 \in (x_0 - R, x_0 + r)$, there exists some power series expansion for f, centered at x_1, with positive radius of convergence.

Proof

Let $x_0 = 0$ for simplicity and $x_1 \in (-R, R)$.

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n (x - x_1 + x_1)^n = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} (x - x_1)^k x_1^{n-k}$$

Assuming that rearangement is possible, this is

$$\sum_{n,k,n \geq 0} a_n \binom{n}{k} (x - x_1)^k x_1^{n-k} = \sum_{k=0}^{\infty} \underbrace{\left(\sum_{n=k}^{\infty} a_n \binom{n}{k} x_1^{n-k} \right)}_{b_k} (x - x_1)^k$$

Need to prove two things:

1. b_k is well-defined

2. Interchange of sums valid.

- Proof of 1

  For k fixed, $\binom{n}{k}$ is a d° k (degree k) polynomial in n.
  Letting

$$b_k = \sum_{p=0}^{\infty} a_{p+k} \binom{p+k}{k} x_1^p$$

  where $p = n - k$, we have

$$\limsup_{p\to\infty} \left(|a_{p+k}| \binom{p+k}{k} \right)^{1/p} = \limsup_{p\to\infty} |a_p|^{1/p}$$

  since $x_1 \in (-R, R)$, $b_k < \infty, \forall k$.

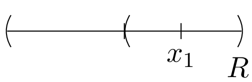
- Proof of 2

  The proof requires invoking Fubini's Theorem to allow rearrangement.
  Need to check that

$$\sum_{n,k,n\geq k} |a_n| \binom{n}{k} \left| (x - x_1)^k x_1^{n-k} \right|$$

  converges.
  Consider

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |a_n| \binom{n}{k} |x - x_1|^k \, |x_1|^{n-k} = \sum_{n=0}^{\infty} |a_n| \left(|x - x_1| + |x_1| \right)^n$$



  Make sure that $|x - x_1| = r < R - |x_1|$, then

$$\sum_{n=0}^{\infty} |a_n| r^n$$

  where $r < R$ which, by absolute convergence of the original power series, is finite. ■

Remark: Analytic Continuation

The process of recentering a power series is also called "analytic continuation."
The radius of convergeence of the new series might actually be larger and allow the orgiginal function.

Example

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

IMAGE HERE - Decaying curve.

Facts: Analytic Functions

- If f, g are analytic on (a, b), then so is $f \cdot g$.

- If f, g are analytic and g does not vanish on (a, b), then $\frac{f}{g}$ is analytic.

- If f is analytic on $(x_0 - R, x_0 + R)$ and g is analytic on $(f(x_0) - \delta, f(x_0) + \delta)$, then $g \circ f$ is analytic on a neighborhood of x_0. (Proof in ; page number in lecture notes).

Remark: No Analytic Bump Functions

IMAGE HERE - BUMP FUNCTION –|-n-|–

Trig Functions

IMAGE HERE – UNIT CIRCLE
We want $(\cos(\theta), \sin(\theta))$ to be the point on the unit circle making an arclength θ from $(1, 0)$.
For x in the right-half plane, $\cos(\theta) \geq 0$.
For x in top right quadrant,

$$\theta = \int_0^{\sin(\theta)} \sqrt{1 + (f'(y))^2}\, dy$$

Then, $y \mapsto (\underbrace{\sqrt{1 - y^2}}_{f(y)}, y)$, $y \in (-1, 1)$. It follows that

$$\theta = \lim_0^{\sin(\theta)} \frac{dy}{\sqrt{1 - y^2}} \underset{\text{FTC}}{\Longrightarrow} \arcsin'(x) = \frac{1}{\sqrt{1 - x^2}} \in C^\infty((-1, 1))$$

and

$$\arcsin(x) = \lim_0^x \frac{dy}{\sqrt{1 - y^2}}$$

Therefore, arcsin is a diffeomorphism from $(-1, 1) \to (\lim_{x \to -1} \arcsin(x), \lim_{x \to 1} \arcsin(x))$.
Since $\frac{1}{\sqrt{1 - x^2}}$ is integrable near ± 1, theese limits are finite.

Definition: Pi

$\pi = 2 \lim_{x \to 1} \arcsin(x)$

Inverse Function Theorem

$\sin : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to (-1, 1)$ exists as a C^1 inverse of arcsin.
On $\left(-\frac{\pi}{2}, \frac{pi}{2}\right)$, define $\cos(\theta) = +\sqrt{1 - \sin^2(\theta)}$. Then

$$\sin'(\theta) = \frac{1}{\arcsin'(\sin(\theta))} = \sqrt{1 - \sin^2(\theta)} = \cos(\theta).$$

Similarly, $\cos'(\theta) = -\sin(\theta) \rightsquigarrow \sin, \cos$ are C^∞ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Extension to the Reals

By graphical or geometric arguments, for $\theta \in \left(0, \frac{\pi}{2}\right)$,

$$\cos(\theta) = -\sin\left(\theta - \frac{\pi}{2}\right)$$
$$\sin(\theta) = -\cos\left(\theta - \frac{\pi}{2}\right)$$

This helps extend to $\mathbb{R}$, with 2π-periodicity such that

$$\begin{cases} \cos' & = -\sin \\ \sin' & = \cos \\ \cos(0) & = 1 \\ \sin(0) & = 0 \end{cases}$$

Therefore, you get all derivatives at $x = 0$ and the corresponding Taylor expansion looks like

$$C(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \qquad S(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

We find that $R = \infty$ for both, and

$$C(0) = 1, \qquad S(0) = 0, \qquad C'(x) = -S(x), \qquad S'(x) = C(x).$$

Take

$$\epsilon(x) = (C(x) - \cos(x))^2 + (S(x) - \sin(x))^2$$

with $\epsilon(0) = 0$. Then, finally,

$$\epsilon'(x) = 0 \implies \epsilon = \text{some constant} = 0.$$

October 25, 2023

Today

Analytic vs C^∞
Approximation by Polynomials

Definition: Real Analytic

f is real analytic on (a, b) if $\forall x_0 \in (a, b), \exists \delta > 0, \{a_n\}_n$ such that $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$, $\forall x \in (x_0 - \delta, x_0 + \delta)$.

Proposition: Analyticity Implies Smoothness

Analytic on $(a, b) \implies C^\infty$ smooth on (a, b).

$$\sum_{n=0}^{\infty} (x - x_0)^n \rightsquigarrow a_n - \frac{f^n(x_0)}{n!}$$

Note: $C^w(a, b) \notin C^\infty(a, b)$
The converse is not true.

Example

Let $x \in \mathbb{R}$ and $f(x) = \begin{cases} 0 & x < 0 \\ \exp\left(\frac{-1}{x^2}\right) & x > 0 \end{cases}$

IMAGE HERE - FUNCTION

$x \neq 0$, $f \in C^\infty(\mathbb{R} \setminus 0)$.

- What about at $x = 0$?

$$\lim_{x \to 0; x<0} f(x) = 0 = \lim_{x \to \varnothing; x>0} e^{-\frac{1}{x^2}}$$

So we can define $f(0) = e$, the resulting function is continuous on $\mathbb{R}$.

- What about higher derivatives?

Claim: $\forall k > 0$, $\lim_{x \to 0; x>0} \frac{d^k}{dx^k}\left(e^{-\frac{1}{x^2}}\right) = 0$

- Proof (Sketch)

$$\frac{d}{dx}\left(e^{-x^2}\right) = 2x^{-3}e^{-x^{-2}}$$

$$\lim_{x \to 0} \frac{e^{-\frac{1}{x^2}}}{x^3} \overset{y=\frac{1}{x}}{=} \lim_{y \to \infty} y^3 e^{-y^{-2}} = 0$$

Claim by induction:

$$\frac{d^k}{dx^k}\left(e^{-\frac{1}{x^2}}\right) = p_k(1/x)e^{-\frac{1}{x^2}}$$

for some polynomial p_k.
If the claim is true, then

$$\lim_{x \to \varnothing} p_k\left(\frac{1}{x}\right)e^{-\frac{1}{x^2}} = \lim_{y \to +\infty} p_k(y)e^{-y^2} = 0 \quad \blacksquare$$

Then we can extend $f^{(k)}$ as a continious function on $\mathbb{R}$ such that $f^{(k)}(0) = 0$.

- Claim

$f(x)$ is not analytic on any neighborhood of $x_0 = 0$.
If it were, it would equal $\sum_{n=0}^\infty a_n x^n$ on $(-\delta, \delta)$ for some a_ks. But,

$$a_k = \frac{f^{(k)}(0)}{k!} = 0 \qquad \text{then} \qquad \sum_{n=0}^\infty a_n x^n = 0, \forall x \in (-\delta, \delta)$$

which is impossible, since $f(x) \neq 0$ whenever $x > 0$. $\blacksquare$

Remark: Contraposition Can Disprove Analyticity

The existance of a non-zero radius of convergence for $\sum a_k (x - x_0)^k$ means

$$\frac{1}{R} = \limsup_n |a_n|^{1/n} = \left(\frac{f^{(n)}(x_0)}{n!}\right)^{1/n} < \infty$$

and
$\left(\frac{f^{(n)}(x_0)}{n!}\right)^{1/n} \rightsquigarrow f^{(n)}(x_0) \leq n! \left(\frac{c}{R}\right)^n$
for some constant c.

Remark: Analyticity is Not Guaranteed

The conditions

$$\begin{cases} h \in C^\infty(R) \\ \limsup_n \left(\frac{h^{(n)}(0)}{n!}\right)^{1/n} < \infty \end{cases}$$

are not sufficient to claim h is analytic on any neighborhood of 0.
Indeed, if h is analytic then $h(x) + f(x)$ will not be for otherwise

$$f(x) = -(h(x) + f(x)) - h(x)$$

would also be analytic, which it isn't.

Definition: Exponential Blip Function

Let $g(x) = \frac{f(x+1)f(1-x)}{f(1)^2}$, where f is the "exponential glue" function.
IMAGE HERE - FUNCTION
Smooth on $\mathbb{R}$; $g(x) \geq 0$.

TODO Theorem: Borel

TODO - Name for theorem?
Given any sequence $\{a_n\}_n$ of reals and any $\begin{cases} x_0 \in \mathbb{R} \\ \lambda > 0 \end{cases}$, $\exists f \in C^\infty(\mathbb{R})$ such that

$$\begin{cases} f^{(k)}(x_0) = a_k & \forall k \\ f(x) = 0 & \text{if } |x - x_0| > \lambda \end{cases}$$

IMAGE HERE - BUMPY MOUNTAIN CLOSE TO X0

Proof

Reductions: $x_0 = 0$ and $\lambda = 1$.
Ansatz: $f(x) = \sum_{k=0}^\infty b_k x^k g\left(\frac{x}{\lambda_k}\right)$ where b_ks and λ_ks need to be tuned.
IMAGE HERE - G(X) and G(X/LAMBDA K)

$g(x) = 0 \iff |x| \geq 1$ and $g\left(\frac{x}{\lambda_k} = 0 \iff \left|\frac{x}{\lambda_k}\right| \geq 1 \iff |x| \geq \lambda_k\right)$
Observations: if $\lambda_k \underset{k\to\infty}{\longrightarrow} 0$, then $\forall x \neq 0$ the series is actuall finite!
Since $g\left(\frac{x}{\lambda_k} = 0\right)$ once $\lambda_k < |x|$.
Therefore, convergent and C^∞ on $\mathbb{R} \setminus \{0\}$.
Constraints:

$$a_0 = f(0) = b_0$$
$$a_1 = f'(0) = \frac{d}{dx}\left(b_0 g\left(\frac{x}{\lambda_0}\right)\right)|_{x=0} + b +_1$$

Generally,

$$a_n = \sum_{k=0}^{n-1} \frac{d^n}{dx^n}\left(b_k x^k g\left(\frac{x}{\lambda_k}\right)\right)|_{x=0} + n! b_n$$

If λ_n are chosen, these constraints uniquely determine the b_ns.

How to Choose Lambdas?

Want to enforce

$$\max_{0\leq k\leq n-1} \sup_{x\in\mathbb{R}} \left|\frac{d^k}{dx^k}\left(b_n x^n g\left(\frac{x}{\lambda_n}\right)\right)\right| \leq 2^{-n}$$

- Example

  Determine λ_2:

$$k = 0: \left|b_n x^n g\left(\frac{x}{\lambda_n}\right)\right| \leq |b_n| \lambda_n^n 2^{-n}$$
$$k = 1: \left|b_n\left(n x^{n-1} g\left(\frac{x}{\lambda_n}\right)\right) + b_n x^n \frac{1}{\lambda_n} g'\left(\frac{x}{\lambda_n}\right)\right| \leq |b_n| \lambda_n^{n-1}(n + ||g'||_\infty) \leq 2^{-n}$$

  In general,

$$a\lambda_n^p < 2^{-n}$$

  for $p > 0$.
  So we construct b_0, then λ_0, then b_1, then $\lambda_1, \ldots$

Claim: Produces Uniform Convergence

When

$$\max_{0\leq k\leq n-1} \sup_{x\in\mathbb{R}} \left|\frac{d^k}{dx^k}\left(b_n x^n g\left(\frac{x}{\lambda_n}\right)\right)\right| \leq 2^{-n}$$

is satisfied, $\forall k \in \mathbb{N}$

$$\sum_{n=0}^{\infty} \frac{d^k}{dx^k}\left(b_n x^n g\left(\frac{x}{\lambda_n}\right)\right)$$

satisfies the Weierstrass M-Test. Therefore it is uniformly convergent. Because

$$\sum_{n=0}^{\infty} \left|\frac{d^k}{dx^k}\left(b_n x^n g\left(\frac{x}{\lambda_n}\right)\right)\right| \le \underbrace{\sum_{n=0}^{k} \left|\frac{d^k}{dx^k}\left(b_n x^n g\left(\frac{x}{\lambda_n}\right)\right)\right|}_{\text{finite sum, uniformly bounded}} + \sum_{n=k+1}^{\infty} 2^{-n}$$

Approximation by Polynomials

Goal (Weierstrass Approximation Theorem):

If $f : [a,b] \to \mathbb{R}$ is continuous on the compact set $[a,b]$, then there exists a sequence of polynomials p_n such that $\lim_{n\to\infty} \sup_{x\in[a,b]} |f(x) - p_n(x)| = 0$.
That is, polynomials are dense in $(C([a,b]), ||\cdot||_\infty)$, where $||f||_\infty := \sup_{x\in[a,b]} |f(x)|$.
How to do this?

Lagrange Interpolation

Give $f \in C([a,b])$.
Idea: subdivide $[a,b]$ with $a = x_0 < x_1 < \cdots < x_n < b$ where $x_k = x_0 + k\left(\frac{b-a}{n}\right)$.
IMAGE HERE - UNIFORM SUBDIVISION
Let $p_n(x) = \sum_{k=0}^{n} f(xk) \prod_{j\neq k} \frac{x-x_j}{x_k - x_j}$.
Problem: the Runge phenomenon.
IMAGE HERE - SMOOTHEST FUNCTION I CAN THINK OF (use the bump again) $1/(1+25x^2)$

Definition: Convolution

Take $f, g : \mathbb{R} \to \mathbb{R}$, define

$$h(x) = f * g(x) = \int_{\mathbb{R}} f(t)g(x-t)\, dt \underset{y=x-t}{=} \int_{\mathbb{R}} f(x-y)g(y)\, dy = g * f(x)$$

Take $f, g \in C(\mathbb{R})$ with compact support ($C_C(\mathbb{R})$). That is, they vanish outside a compact set.
IMAGE HERE - F AND G CONVOLVED

Definition: Approximation of Identity

An approximation of the identity is a sequence $\{g_n\}_n$, all piecewise continuous, defined on $\mathbb{R}$ such that

$$\begin{cases} g_n(x) \ge 0 & \forall x \\ \int_{\mathbb{R}} g_n(x)\, dx = 1 & \\ \forall \delta > 0, & \lim_{n\to\infty} \int_{|x|>\delta} g_n(x)\, dx = 0 \end{cases}$$

IMAGE HERE - CONVOLUTION ACCUMULATING BETWEEN -DELTA AND DELTA

Example

Let $g_n(x) = \frac{n \cdot g(nx)}{\int_{\mathbb{R}} g(x)\,dx}$.

Lemma:

If $\{g_n\}_n$ is an approximation of identity, then $\forall f \in C_C(\mathbb{R})$

$$g_n * f \rightrightarrows f$$

on $\mathbb{R}$.

October 30, 2023

Today

Approximation by polynomials.
Fourier Series.

Recall: Convolution

$f, g \in C_C(\mathbb{R})$, $f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)\,dy$.

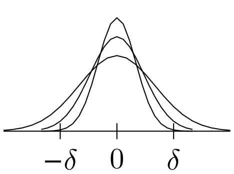
Recall: Approximation of Identity

$\{g_n\}_n$ where $g_n : \mathbb{R} \to \mathbb{R}$ is piecewise continuous (this is overkill but sufficient).

1. $\int g_n\,dx = 1$.

2. $g_n(x) \geq 0$.

3. $\forall \delta > 0, \lim_{n\to\infty} \int_{|x|>\delta} g_n(x)\,dx = 0$.



$f \rightsquigarrow \{g_n * f\}_n$.

Example

Take any $g(x) \geq 0$ (piecewise continuous) with $\int_{\mathbb{R}} g(x)\,dx = 1$.
Define $g_n(x) = n \cdot g(nx)$.
Claim: this defined an approximation of identity.

Lemma: Convolution of Approximation of Identity Converges Uniformly

Suppose $\{g_n\}_n$ is an approximation of identity.
Then, for any $f \in C_C(\mathbb{R})$,

$$g_n * f \quad \text{converges uniformly to } f \text{ on } \mathbb{R}$$

That is to say, $\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |g_n * f(x) - f(x)| = 0$.

Proof

Since $\int_{\mathbb{R}} g_n(y)\ dy = 1$,

$$\begin{aligned} g_n * f(x) - f(x) &= \int_{\mathbb{R}} g_n(y) f(x-y)\ dy - f(x) \cdot \int_{\mathbb{R}} g_n(y)\ dy \\ &= \int_{\mathbb{R}} g_n(y) \left(f(x-y) - f(x) \right)\ dy \\ &= \int_{|y|\geq\delta} g_n(y) \overbrace{\left(f(x-y) - f(x) \right)}^{\leq 2M}\ dy + \int_{|y|>\delta} g_n(y) \overbrace{\left(f(x-y) - f(x) \right)}^{\epsilon}\ dy \end{aligned}$$

By assumption, $f \in C_C(\mathbb{R})$ so f is bounded by M on $\mathbb{R}$.
f is continuous on $\text{supp}(f)$, which is compact, so f is uniformly continuous on $\mathbb{R}$.
Let $\epsilon > 0$ be given.
By uniform continuity, $\exists \delta > 0, \forall x, y \in \mathbb{R}, |x-y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{2}$.
By the Aproximation of Identity property, $\exists N, \forall n \geq N, \int_{|y|\geq\delta} g_n(y)\ dy < \frac{\epsilon}{4M}$.
For $n \geq N$,

$$\begin{aligned} |g_n * f(x) - f(x)| &= \left| \leq \int_{\mathbb{R}} g_n(y) \left(f(x-y) - f(x) \right)\ dy \right| \\ &\leq \int_{|y|\geq\delta} g_n(y) \overbrace{\left| f(x-y) - f(x) \right|}^{\leq 2M}\ dy + \int_{|y|>\delta} g_n(y) \overbrace{\left| f(x-y) - f(x) \right|}^{\leq \frac{\epsilon}{2} \text{ since } |x-y-x|=|y|<\delta}\ dy \\ &\leq 2M \frac{\epsilon}{4M} + \frac{\epsilon}{2} \overbrace{\int_{|y|<\delta} g_n(y)\ dy}^{\leq 1} \\ &\leq \epsilon, \quad \forall x \in \mathbb{R} \quad \blacksquare \end{aligned}$$

Recall: Riemann Integral Properties

If f is Riemann integrable, then

$$\left| \int f\ dx \right| \leq \int |f|\ dx$$

$$\left| \sum_{n=1}^{\infty} S_n \right| \leq \sum_{n=1}^{\infty} |S_n|$$

$$\left| \int f^+\ dx - \int f^-\ dx \right| \leq \int f^+\ dx + \int f^-\ dx = \int (f^+ + f^{-1})\ dx$$

Theorem: Weierstrass Approximation Theorem

If $[a,b]$ is compact, then $\forall f \in C([a,b])$, there exists a sequence of polynomials $p_n(x)$ converging uniformly to f.

Step 1

Extend f into $F \in C_C(\mathbb{R})$.
IMAGE HERE - EXTEND FUNCTION

$$F(x) = \begin{cases} 0 & \text{on } (-\infty, a-1] \cup [b+1, \infty) \\ f(x) & \text{on } [a,b] \\ f(a)(x-(a-1)) & \text{on } [a-1,a] \\ f(b)(b+1-x) & \text{on } [b,b+1] \end{cases}$$

Step 2

Note: $\forall \{g_n\}_n$ Approximation of Identity, $g_n * f \rightrightarrows F(x)$ on $\mathbb{R}$ (by previous lemma),
and $\sup_{x\in[a,b]} |g_n * F(x) - f(x)| \leq \sup_{x\in\mathbb{R}} |g_n * F(x) - F(x)|$.
Trick: Construct g_n such that $g_n * F$ is a polynomial on $[a,b]$.
Answer:

$$g_n(x) = \begin{cases} a_n \left(1 - \frac{x^2}{(b-a+1)^2}\right)^n & \text{if } x \in [-(b-a+1), b-a+1] \\ 0 & \text{otherwise} \end{cases}$$

where a_n is chosen such that $\int_{\mathbb{R}} g_n(x)\, dx = 1$.
IMAGE HERE - NARROWING GAUSSIAN WITH PEAK AT (0,1) BETWEEN -(b-a+1) and b-a+1
If $x \in [a,b]$ and $y \in [a-1, b+1]$, then

$$-b-1 \leq -y \leq -a+1 \implies -(b-a+1) \leq x-y \leq b-a+1$$

Then

$$\begin{aligned} g_n * F(x) &= \int_{a-1}^{b+1} F(y) \underbrace{g_n(x-y)}_{a_n\left(1-\frac{(x-y)^2}{(b-a+1)^2}\right)^n = \sum_{p=0}^{2n} x^p a_{p,n(y)}} dy \\ &= \sum_{p=0}^{2n} x^p \int_{a-1}^{b+1} F(y) a_{p,n(y)}\, dy \quad \blacksquare \end{aligned}$$

Background: Fourier Series

Historical Perspective

In Strichartz.
Associated with solving the wave equation on $[0,L]_x \times [0,T]_t$ (Bernoulli) and the heat equation (Fourier).

Wave Equation

On $[0,L]_x \times [0,T]_t$, $u(x,t)$ displacement field.
IMAGE HERE - WAVE FROM 0 to L PEAK OF FIRST OSCILLATION AT U(X,T)

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c\frac{\partial^2 u}{\partial x^2}(x,t)$$

where c is a fixed coefficient.
Plus Initial Conditions and Boundary Conditions

$$\begin{aligned} \text{Initial Condition :} \quad & u\big|_{t=0}(x) = f(x) \\ & \frac{\partial u}{\partial t}\big|_{t=0}(x) = 0 \\ \text{Boundary Conditions :} \quad & u(0,t) = u(L,t) = 0 \end{aligned}$$

Observation: if $f(x) = \sin\left(\frac{k\pi x}{L}\right)$,
IMAGE HERE - THREE SINUSOIDAL WAVES OVERLAPPING
Ansatz: $u(x,t) = \sin\left(\frac{k\pi x}{L}\right)g(t)$.
Plug into the PDE:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \sin\left(\frac{k\pi x}{L}\right)g''(t) \\ c\frac{\partial^2 u}{\partial x^2} &= -\frac{k^2\pi^2}{L^2}c^2\sin\left(\frac{k\pi x}{L}\right)g(t) \end{aligned}$$

Setting

$$\sin\left(\frac{k\pi x}{L}\right)g''(t) = -\frac{k^2\pi^2}{L^2}c^2\sin\left(\frac{k\pi x}{L}\right)g(t) \underset{g}{\overset{\text{ode for}}{\Longrightarrow}} g'' = -\frac{k^2\pi^2}{L^2}c^2 g$$

Which gives a general solution

$$g(t) = A\cos\left(\frac{k\pi ct}{L}\right) + B\sin\left(\frac{k\pi ct}{L}\right).$$

Initial conditions imply that $g(0) = 1$ and $g'(0) = 0$ which gives

$$g(t) = \cos\left(\frac{k\pi ct}{L}\right).$$

Thus

$$u(x,t) = \sin\left(\frac{k\pi x}{L}\right)\cos\left(\frac{k\pi x}{L}\right)$$

Solves the PDE!

Wave Equation Superposition

Consider instead

$$f(x) = \sum_{k=0}^{n} \sin\left(\frac{k\pi x}{L}\right) a_k$$

Then

$$u(x,t) = \sum_{k=0}^{n} a_k \sin\left(\frac{k\pi x}{L}\right) \cos\left(c\frac{k\pi x}{L}\right)$$

Next Question:

What if f is more general?
$\Longrightarrow$ existence of Fourier series?
In what sense do they converge?

Definition: Fourier Series

Context: $f : [-\pi, \pi) \to \mathbb{R}$ Riemann-Integrable or
$f : \mathbb{R} \to \mathbb{R}$ 2π-periodic. ($f(x + 2\pi) = f(x), \forall x$.
The Fourier series of f:

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k[f] \cos(kx) + b_k[f] \sin(kx)$$

where $a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\, dx$ and $b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\, dx$.
Alternatively,

$$S_n(x) = \sum_{k=-n}^{n} c_k e^{ikx}$$

where $c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx$.
As an exercise: relate c_ks to a_ks and b_ks and prove that these are equivalent.

Question:

In what sense does $S_n(x)$ converge to $f(x)$? That is

- For what topology?
  - Uniform Convergence: $\sup_{x\in[-\pi,\pi)} |S_n(x) - f(x)| \xrightarrow[n\to\infty]{} 0$
  - L^2 Convergence: $\int_{-\pi}^{\pi} |S_n(x) - f(x)|^2\, dx \xrightarrow[n\to\infty]{} 0$
- What are the (smoothness) requirements on f?
  - Observation: if $f(x) = \sum_{k=-N}^{N} f_k e^{ikx}$ is a trigonometric polynomial, then, for $n \geq N$, $S_n(x) = f(x)$.

Lemma: The Kronecker Delta

Fix $N \in \mathbb{N}$
If $\sum_{k=-N}^{N} f_k e^{ikx} = \sum_{k=-N}^{N} c_k e^{ikx}$, then $f_k = c_k, \forall k$.
Note

$$\int_{-\pi}^{\pi} e^{ikx} e^{-imx} \, dx = \int_{-\pi}^{\pi} e^{i(k-m)x} \, dx = \begin{cases} 2\pi & \text{if } k = m \\ \left[\frac{1}{i(k-m)} e^{i(k-m)x} \right]_{-\pi}^{\pi} = 0 & \text{otherwise} \end{cases}$$

Why $-imx$?

$$\langle if, g \rangle = i\langle f, g \rangle$$
$$\langle f, ig \rangle = -i\langle f, g \rangle$$

and

$$\int_{-\pi}^{\pi} f(x)\overline{g(x)} \, dx$$

November 1, 2023

Fourier Series

For f Riemann-integrable on $(-\pi, \pi)$, define

$$S_n(x) = \sum_{k=-n}^{n} c_k e^{ikx}$$

with

$$c_k := \frac{1}{2\pi} \in_{-\pi}^{\pi} f(x) e^{-ikx} \, dx$$

Then $f : [-\pi, \pi) \to \mathbb{R}$.

Recall

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} e^{-ilx} \, dx = \begin{cases} 0 & \text{if } k \neq l \\ 1 & \text{if } k = l \end{cases} = \delta_{kl} \text{ (the Kronecker delta)}$$

Definition: Norm

$|| \cdot || : E \to \mathbb{R}_{\geq 0}$ is a "norm" on E if

1. $||f|| = 0 \iff f \equiv 0$

2. $||\lambda f|| = |\lambda| \cdot ||f||, \forall \lambda \in \mathbb{R}, f \in E$

3. $||f + g|| \leq ||f|| + ||g||$

Definition: Normed Space

$(E, ||\cdot||)$ is a normed space.
e.g. $(\mathbb{R}, |\cdot|)$ or $(\mathbb{Q}, |\cdot|)$

Definition: Complete Space

$(E, ||\cdot||)$ is complete if every cauchy sequence in E converges in E.

In what sense does a Fourier series converge?

Depends on regularity of f and the topology used.

Note

On $C([-\pi,\pi])$, can put 2 norms.

- $||f||_\infty = \sup_{x\in[-\pi,\pi]} |f(x)|$

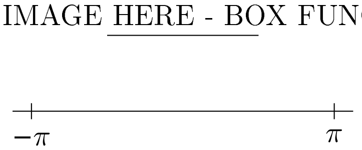
  $d(f,g) = ||f-g||_\infty$: "f_n converges uniformly to f" $\leftrightarrow \lim_{n\to\infty} ||f_n - f||_\infty = 0$.
  $(C([-\pi,\pi]), ||\cdot||_\infty)$ is complete.

- $||f||_2 := \left(\int_{-\pi}^{\pi} f^2(x)\, dx\right)^{1/2}$

  "f_n converges to f in L^2" $\leftrightarrow \lim_{n\to\infty} ||f_n - f||_2 = 0$.
  $(C([-\pi,\pi]), ||\cdot||_2)$ is not complete.

Example

Take $f(x) = \begin{cases} 1 & \text{if } |x| \le \pi/2 \\ 0 & \text{if } |x| > \pi/2 \end{cases}$
IMAGE HERE - BOX FUNCTION FROM -pi/2 to pi/2



Then

$$c_k = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{-ikx}\, dx = \frac{1}{2\pi}\left[\frac{1}{-ik}e^{-kx}\right]_{-\pi/2}^{\pi/2} \frac{1}{2\pi}\frac{1}{-ik}\left[e^{-ik(\pi/2)} - e^{ik(\pi/2)}\right] = \frac{1}{k\pi}\sin(k(\pi/2))$$

So $c_k = 0$ and for $k = 2p+1$: $c_{2p+1} = \frac{(-1)^p}{\pi(2p+1)}$.
IMAGE HERE - BOX FUNCTION WITH SUNUSOIDALS APPROXIMATING
However, the approximation will over and undershoot at the boundaries. This is the "Gibbs Phenomenon", and the discrepency is roughly 12%.
For $k < 0$:

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{ikx}\, dx = \overline{c_{-k}} = c_k$$

In the end,

$$S_{2p+1}(x) = \sum_{l=1}^{p} \frac{(-1)^p}{\pi(2p+1)} \underbrace{\left(e^{i(2p+1)x} + e^{-i(2p+1)x}\right)}_{2\cos((2p+1)x)}$$

Theorem: Uniform Convergence of Continuously Differentiable Continuous Functions

1. If f is C^2, 2π-periodic, then $S_n \rightrightarrows f$ on $[-\pi, \pi)$.

   Moreover, $||S_n - f||_\infty \le \frac{c}{n}$ for some $c > 0$.

1. If $f \in C^1$, 2π-periodic, same conclusion with $||S_n - f||_\infty \le \frac{c}{\sqrt{n}}$ for some $c > 0$.

Proof of Part 1

Write

$$S_n(x) = \sum_{k=-n}^{n} c_k e^{ikx} = \sum_{k=-n}^{n} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iky}\, dy\, e^{ikx} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \underbrace{\left[\sum_{k=-n}^{n} e^{ik(x-y)}\right]}_{D_n(x-y)} dy$$

Where $D_n(t) = \sum_{k=-n}^{n} e^{ikt}$ is the "Dirichlet kernel."
That is S_n is a convolution of $f(y)$ with some kernel.

$$e^{it} \cdot D_n(t) = \sum_{k=-n}^{n} e^{i(k+1)t} \underset{l=k+1}{=} \sum_{l=-n+1}^{n+1} e^{ilt} = D_n(t) + e^{i(n+1)t} - e^{-int}$$

Therefore

$$D_n(t) = \frac{e^{i(n+1)t} - e^{-int}}{e^{it} - 1} = \frac{e^{(it)/2}\left(e^{i(n+(1/2))t} - e^{-i(n+(1/2))t}\right)}{e^{(it)/2}\left(e^{(it)/2} - e^{-(it)/2}\right)} = \frac{\sin((n+1/2)t)}{\sin(t/2)}$$

IMAGE HERE - DN(T) OSCILLATING WITH MANY ZEROS THEN PEAKING TO 2N+1 at X=0
So

$$\int_{-\pi}^{\pi} D_n(t)\, dt = 2\pi$$

Then

$$\begin{aligned} S_n(x) - f(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) D_n(x-y)\, dy - f(x) \\ &\underset{z=x-y}{=} \frac{1}{2\pi} \int_{-\pi}^{pi} f(x-z) D_n(z)\, dz - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) D_n(z)\, dz \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x-z) - f(x)) D_n(z)\, dz \end{aligned}$$

and

$$S_n(x) \cdot f(x) = \frac{1}{2\pi} \underbrace{\frac{(f(x-y) - f(x))}{\sin(y/2)}}_{\text{call } g_x(y) = \frac{f(x-y)-f(x)}{\sin(y/2)}} \sin((n + (1/2)y)\; dy$$

If $g_x(y)$ was differentiable (in fact C^1), then integrating by parts

$$S_n(x) - f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_x(y) \sin((n + (1/2)y)\; dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_x'(y) \frac{\cos((n + (1/2)y)}{n + (1/2)}\; dy$$

Then

$$|S_n(x) - f(x)| \le \sup_{y \in [-\pi, \pi]} \left| g_x'(y) \right| \frac{1}{n + (1/2)}$$

- Claim
  If $f \in C^2$, 2π-periodic, then $\sup_{x \in [-\pi,\pi]} |g_x'(y)| < \infty$. Then the first part of the theorem is proved.

  - Proof of Claim
    $f \in C^2 \implies g_x \in C^2$ away from $y = 0$. ($g_x''(y)$ = differentiation rules).
    At $y = 0$, write

$$f(x-y) - f(x) = \int_x^{x-y} f'(t)\; dt$$

Changing variables such that $t = x + u(x - y - x) = x - uy$ for $u \in [0,1]$ gives $dt = -y\; du$

$$= -y \int_0^1 f'(x - uy)\; du$$

Therefore

$$g_x(y) = \underbrace{\left(\frac{-y}{\sin(y/2)} \right)}_{\text{smooth near } y=0} \int_0^1 f'(x - uy)\; du$$

Calling the smooth piece $h(y)$,

$$g_x(y) = h(y) \int_0^1 f'(x - yu)\; du$$

is differentiable at 0 if and only if $\frac{d}{dy} \left(\int_0^1 f'(x - yu)\; du \right) = \int_0^1 f''(x - yu)(-u)\; du$ exists. ■

Proof of Part 2 (Sketch)

$$S_n(x) - f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_x(y) \sin(n + (1/2)y)\; dy$$

If f is only C^1, then g is C^1 away from 0, so it is unclear near $y=0$. So, for some δ to be chosen later

$$S_n(x) - f(X) = \frac{1}{2\pi} \underbrace{\int_{[-\delta,\delta]} g_x(y)\sin((n+(1/2))y)\,dy}_{\leq \frac{2\delta}{2\pi}(||f'||_\infty + ||f||_\infty)} + \frac{1}{2\pi} \underbrace{\int_{[-\pi,\pi]\setminus[-\delta,\delta]} g_x(y)\sin((n+(1/2))y)\,dy}_{\text{integrate by parts}}$$

Study $\int_\delta^\pi$ (study of $\int_{-\pi}^{-\delta}$ is similar)

$$\begin{aligned}
\int_\delta^\pi g_x(y)\sin((n+(1/2)y)\,dy &= \int_\delta^\pi g_x(y)\frac{d}{dy}\left(\frac{-\cos((n+(1/2))y)}{n+(1/2)}\right)dy \\
&= \int_\delta^\pi \frac{d}{dy}\left(g_x(y)\frac{-\cos((n+(1/2))y)}{n+(1/2)}\right) - \int_\delta^\pi g_x'(y)\frac{\cos((n+(1/2))y)}{n+(1/2)}\,dy \\
&= -g_x(\pi)\frac{\cos((n+(1/2))\pi)}{n+(1/2)} + g_x(\delta)\frac{\cos((n+(1/2))\delta)}{n+(1/2)} - \int_\delta^\pi g_x'(y)\frac{\cos((n+)1/2))y)}{n+(1/2)}\,dy
\end{aligned}$$

Problem:

$$g_x'(y) = \frac{-f'(x-y)\sin(y/2) - (1/2)\cos(y/2)(f(x-y)-f(x))}{(\sin(y/2))^2} \approx \frac{c}{y} \text{ near } y=0$$

So

$$\left|\int_\delta^\pi g_x(y)\sin((n+(1/2))y)\,dy\right| \leq \frac{1}{n+(1/2)}\cdot\frac{1}{\delta}$$

Combining all estimates, for $\delta > 0$

$$|S_n(x) - f(x)| \leq C_1\delta + C_2\frac{1}{n\delta}$$

Since we are free to choose δ, we may optimize over δ.
Balancing out the terms is done by choosing $\delta = \delta(n)$ such that

$$\delta \overset{n\to\infty}{\sim} \frac{1}{n\delta} \iff n\delta^2 \sim 1 \iff \delta \sim \frac{1}{\sqrt{n}}$$

which gives

$$|S_n(x) - f(x)| \leq C_1\delta + C_2\frac{1}{n\delta} = \frac{C_1}{\sqrt{n}} + C_2\frac{1}{n\frac{1}{\sqrt{n}}} \leq \frac{C_1 + c_2}{\sqrt{n}}$$

- Comment on the Sketch

  Morally, we want $|g_x'(y)| \leq \frac{c}{y}$ for some constant c.
  Numerator:

$$\left|-f'(x-y)\sin(y/2) - (1/2)\cos(y/2)(f(x-y)-f(x))\right| \leq ||f'||_\infty(y/2) + (\cdots)y \leq Cy$$

  Since $|\sin(y/2)| \leq (y/2)$,

$$\begin{aligned} |\sin(x) - \sin(0)| &= |\cos(\xi)||x - 0| \\ &= 1|x| \end{aligned}$$

Denominator

$$(\sin(y/2))^2 \geq \left(\frac{2y}{2\pi}\right)^2 = \frac{y}{\pi}^2$$

So,

$$\left|g_x'(y)\right| \leq \frac{Cy}{\left(\frac{y}{\pi}\right)^2} \leq \frac{C^1}{y}$$

Theorem: Continuous, Periodic Functions Converge in L2

If f is continuous, 2π-periodic, then $\lim_{n\to\infty} ||S_n - f||_2 = 0$.
That is, $\lim_{n\to\infty} \int_{-\pi}^{\pi} |S_n - f(x)|^2 \, dx = 0$.
IMAGE HERE - PERIODIZE f(x)=x THEN APPROXIMATE WITH FOURIER

November 6, 2023

Recall: Fourier Series

$$f : [-\pi, \pi] \to \mathbb{R} \text{ or } \mathbb{C}$$

Fourier Coefficient:

$$c_k(f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{ikx} \, dx, \ k \in \mathbb{Z}$$

$$s_n(x) = \sum_{k=-n}^{n} c_k e^{ikx} = \frac{1}{2\pi} \in_{-\pi}^{\pi} D_n(x-y)f(y) \, dy$$

Dirichlet Kernel:

$$D_n(y) := \frac{\sin((n+1/2)y)}{\sin((1/2)y)}$$

Theorem: L2 Convergence of Sn to N

If f is C^0, 2π-periodic, then

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |s_n(x) - f(x)|^2 \, dx = 0$$

Recall: Kronecker Delta

For $m, n \in \mathbb{Z}$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{imx}e^{-imx}\,dx = \delta_{m,n} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

That is $\{e^{inx}\}_{n\in\mathbb{Z}}$ is an orthnormal system for the inner product

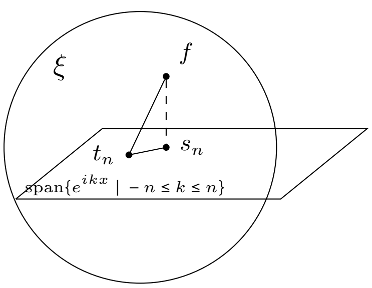
$$\begin{aligned} \xi \times \xi &\to \mathbb{C} \\ (f,g) &\mapsto \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx \end{aligned}$$

where $\xi = \{f : \mathbb{R} \to \mathbb{C},\ 2\pi\text{-periodic},\ \text{continuous}\}$.

Example

For $f \in \xi$, fixing $n \in \mathbb{N}_0$, consider the map

$$\begin{aligned} \mathbb{C}^{2n+1} &\to \mathbb{R} \\ (d_{-n}, \ldots, d_n) &\mapsto \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - \sum_{k=-n}^{n} d_k e^{ikx}|^2\,dx \end{aligned}$$



- Claim:
  F_n is minimal if and only if $\lambda_k = c_k(f),\ \forall -n \leq k \leq n$.

  - Proof:
    Take any $\lambda_n, \lambda_{n+1}, \ldots, \lambda_n$ and set $t_n(x) = \sum_{k=-n}^{n} \lambda_k e^{ikx}$. Then

    $$\int_{-\pi}^{\pi} |f(x) - t_n(x)|^2\,dx = \int_{-\pi}^{\pi} |f(x) - s_n(x) + s_n(x) - t_n(x)|^2\,dx$$

    Then, since

    $$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) = |z_1|^2 + |z_2|^2 + 2 \cdot \Re(z_1\overline{z_2})$$

$$\begin{aligned} &\int_{-\pi}^{\pi} |f(x) - t_n(x)|^2\,dx \\ = &\int_{-\pi}^{\pi} |f(x) - s_n(x)|^2\,dx + \int_{-\pi}^{\pi} |s_n(x) - t_n(x)|^2\,dx + 2 \cdot \Re \int_{-\pi}^{\pi} (f(x) - s_n(x))(\overline{s_n(x) - t_n(x)})\,dx \end{aligned}$$

What to Show: Integral on real part is zero.

$$
\begin{aligned}
A &= \int_{-\pi}^{\pi} (f(x) - s_n(x)) \overline{\sum_{k=-n}^{n} (c_k - \lambda_k) e^{ikx}} \, dx \\
&= \sum_{k=-n}^{n} \overline{(c_k - \lambda_k} \underbrace{\int_{-\pi}^{\pi} (f(x) - s_n(x)) e^{-ikx} \, dx}_{2\pi(c_k - c_k) = 0}
\end{aligned}
$$

Since

$$
\int_{-\pi}^{\pi} s_n(x) e^{-ikx} \, dx = \int_{-\pi}^{\pi} \sum_{p=-n}^{n} c_p e^{ipx} e^{-ikx} \, dx = 2\pi c_k
$$

It follows that

$$
\begin{aligned}
\underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - t_n(x)|^2 \, dx}_{F_n(\lambda_{-n}, \ldots, \lambda_n)} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 \, dx + \frac{1}{2\pi} \int_{-\pi}^{\pi} |t_n(x) - s_n(x)|^2 \, dx \\
&\geq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_n(x)| \, dx \\
&\geq F_n(c_{-n}, \ldots, c_n)
\end{aligned}
$$

Moreover:

$$
\begin{aligned}
\frac{1}{2\pi} \int_{-\pi}^{\pi} | \underbrace{t_n(x) - s_n(x)|^2 |}_{\underbrace{(t_n - s_n)}_{\sum_{p=-n}^{n} (\lambda_p - c_p) e^{ipx}} \quad \overline{\cdot (t_n - s_n)}} |^2 \, dx &= \frac{1}{2\pi} \sum_{p,l=-n}^{n} (\lambda_p - c_p) \overline{(\lambda_l - c_l)} \underbrace{\int_{-\pi}^{\pi} e^{ipx} e^{-ilx} \, dx}_{\delta_{p,l}} \\
&= \frac{1}{2\pi} \sum_{p=-n}^{n} |\lambda_p - c_p|^2
\end{aligned}
$$

Conclusion:

* $\forall (\lambda_{-n}, \ldots, \lambda_n \neq (c_{-n}, \ldots, c_n), F_n(\lambda_{-n}, \ldots, \lambda_n) > F_n(c_{-n}, \ldots, c_n)$
* $F_n(c_{-n}, \ldots, c_n) = F_n(c_{-n}, \ldots, c_n)$
* Lemma

For all trigonometric polynomials of degree at most n, of the form $\sum_{k=-n}^{n} \lambda_k e^{ikx} = t_n(x)$,

$$
\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 \, dx \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - t_n(x)|^2 \, dx
$$

and

$$
\frac{1}{2\pi} \int_{-\pi}^{\pi} |t_n(x)|^2 \, dx = \sum_{k=-n}^{n} |\lambda_k|^2
$$

Apply this to $(\lambda_{-n}, \dots, \lambda_n) = (0, \dots, 0)$:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 \, dx + \sum_{k=-n}^{n} |c_k|^2$$

As a consequence, for all n,

$$\sum_{k=-n}^{n} |c_k|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx$$

which implies that $\sum_{k=-n}^{n} |c_k|^2$ converges absolutely and, in particular, $c_k \to 0$ as $k \to \infty$.

Riemann-Lebesgue Lemma

The above proves that if $f \in \xi$ (more generally, if f is Riemann-integrable), then

$$\lim_{k \to \infty} \int_{-\pi}^{\pi} f(x) e^{\pm ikx} \, dx = 0$$

Moreover, sending $n \to \infty$, we get

$$\lim_{n \to \infty} \sum_{k=-n}^{n} |c_k|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx$$

Importantly, there is equality whenever $\lim_{n \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 \, dx = 0$.
When does that happen?

Theorem:

If $f \in \xi$, then

$$\lim_{n \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 \, dx = 0$$

Proof

For $n \geq 0$, define $\sigma_n = \frac{1}{n+1} \sum_{k=0}^{n} s_k$ (the "Cesano sum").
Then

$$\sigma_n \in \operatorname{span}\langle e^{-inx}, \dots, e^{inx} \rangle.$$

In particular,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 \, dx \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - \sigma_n(x)|^2 \, dx \leq \left(\sup_{[-\pi,\pi]} |f - \sigma_n| \right)^2$$

What to show: $\sigma_n \rightrightarrows f$ on $[-\pi, \pi]$.
Recall that

$$s_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x-y)f(y)\,dy$$

$$\begin{aligned}\sigma_n(x) &= \frac{1}{n+1}\sum_{k=0}^{n} s_k(x)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x-y)f(y)\,dy\end{aligned}$$

Where

$$\begin{aligned}K_n(y) &= \frac{1}{n+1}\sum_{k=0}^{n} D_k(y)\\ &= \frac{1}{n+1}\frac{1}{\sin(y/2)}\sum_{k=0}^{n}\sin((k+1/2)y)\end{aligned}$$

Using $2\sin((k+1/2)y)\sin(y/2) = \cos(ky) - \cos((k+1)y)$.

$$= \frac{1}{n+1}\frac{1}{(\sin(y/2))^2}\frac{1}{2}\underbrace{\sum_{k=0}\cos(ky) - \cos((k+1)y)}_{\underbrace{\frac{1-\cos((n+1)y)}{2}}_{\sin^2((\frac{n+1}{2})y)}}$$

$$= \frac{1}{n+1}\left(\frac{\sin\left(\left(\frac{n+1}{2}\right)y\right)}{\sin(y/2)}\right)^2$$

This is the Féjer kernel.
IMAGE HERE - FÉJER KERNEL
Claims:

1. $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)\,dy = 1$

2. $K_n(y) \geq 0$ on $[-\pi, \pi]$ (obvious)

3. $\forall\delta > 0$, $K_n \rightrightarrows 0$

- Proof of 1

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)\,dy = \frac{1}{2\pi}\frac{1}{n+1}\sum_{k=0}^{n}\underbrace{\int_{-\pi}^{\pi} D_k(y)\,dy}_{2\pi} = 1$$

- Proof of 3 If $|y| \geq \delta$,

$$|K_n(y)| = \frac{1}{n+1}\frac{\overbrace{|\sin((n+1)y/2)|^2}^{\leq 1}}{|\sin(y/2)|^2}$$

Recall $|sin(x)| \geq \frac{2|x|}{\pi}$

$$\leq \frac{1}{n+1}\frac{1}{(|y|/\pi)^2}$$
$$\leq \frac{1}{n+1}\frac{1}{(\delta/\pi)^2}$$

Which goes unfiormly to 0 on $[-\pi,\pi]\setminus[-\delta,\delta]$ as $n \to \infty$.
What to show: $K_n * f \rightrightarrows f$ on $[-\pi,\pi]$.
The proof scheme is dentical to: if $f \in C_c(\mathbb{R})$ and K_n is an approximation of identity, then $K_n * f \rightrightarrows f$ on $\mathbb{R}$.
Left as an exercise.

Corollary: Parseval's Equality

$\forall \delta \in \xi$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx = \lim_{n\to\infty}\sum_{k=-n}^{n} |c_k|^2$$

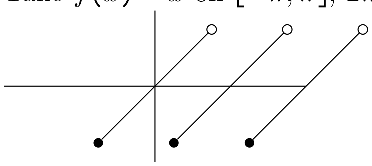
Remark:

This should hold for a larger class of function.

- Piecewise Continuous
- L^2 functions

Example

Take $f(x) = x$ on $[-\pi,\pi]$, 2π-periodized



Then $\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$.

Application to Solving the Heat Equation

On $[0,L]_x \times \mathbb{R}_+$, $u(x,t)$ is the "heat distribution"
IMAGE HERE - ONE DIMENSIONAL ROD HEAT EQUATION YADA YADA

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial x}\right) = 0$$

Problem

PDE	$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$	on $[0,L]\times[0,T]$
Boundary Conditions	$u(0,t) = u(L,t) = 0$	
Initial Conditions	$u(x,0) = f(x)$	f continuous, $f(0) = f(L) = 0$

IMAGE HERE - POSITION TIME PLANE

- Step 1: Separation of Variables
  Seek an ansatz of the form

$$u(x,t) = g(x)h(t)$$

Where

$$\begin{aligned} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} &\iff g(x)h'(t) = g''(t)h(t) \\ &\iff \frac{h'(t)}{h(t)} = \frac{g''(x)}{g(x)} = c \end{aligned}$$

Left Solving:

$$g''(x) = cg(x) \quad g(0) = 0 = g(L)$$

$$h'(t) = ch(t) \rightsquigarrow h(t) = h(0)e^{ct}$$

Then

$$\begin{aligned} g''(x) - cg(x) = 0 \rightsquigarrow c = 0. &\quad g(x) = a + bx \\ c > 0. &\quad g(x) = ae^{\sqrt{c}x} + be^{-\sqrt{c}x} \\ c < 0. &\quad g(x) = a\cos(\sqrt{-c}x) + b\sin(\sqrt{-c}x) \end{aligned}$$

and

$$g(0) = 0 = g(L) \implies \begin{cases} c = 0: \quad g \equiv 0 & \text{(no solution)} \\ c > 0: \quad g \equiv 0 & \text{(no solution)} \\ c < 0: \quad a = 0. \quad g(x) = b\sin(\sqrt{-c}x) \end{cases}$$

$$\begin{aligned} g(L) = 0 &\implies \sin(\sqrt{-c}k) = 0 \\ &\implies L\sqrt{-c} = k\pi \\ &\implies c = -\left(\frac{k\pi}{L}^2\right), k \in \mathbb{N}_0 \end{aligned}$$

For $c = -\left(\frac{k\pi}{L}\right)^2 = \lambda_k$,

$$g_k(x) = \sin\left(\frac{k\pi x}{L}\right)$$
$$h_k(x) = h_k(0)\exp\left(-\left(\frac{k\pi}{L}\right)^2 t\right)$$

For all $k \in \mathbb{N}_0$,

$$u_k(x,t) = g_k(x)h_k(t)$$

solves the heat equation with boundary conditions.
Initial conditions $g_k(x)$, fix $h_k(0) = 1$.
Ansatz for a solution:

$$u(x,t) = \sum_{k=0}^{\infty} a_k g_k(x) h_k(t) \implies u(x,0) = \sum_{k=0}^{\infty} a_k \sin\left(\frac{k\pi x}{L}\right) = f(x)$$

Thus, the left hand side is the solution.

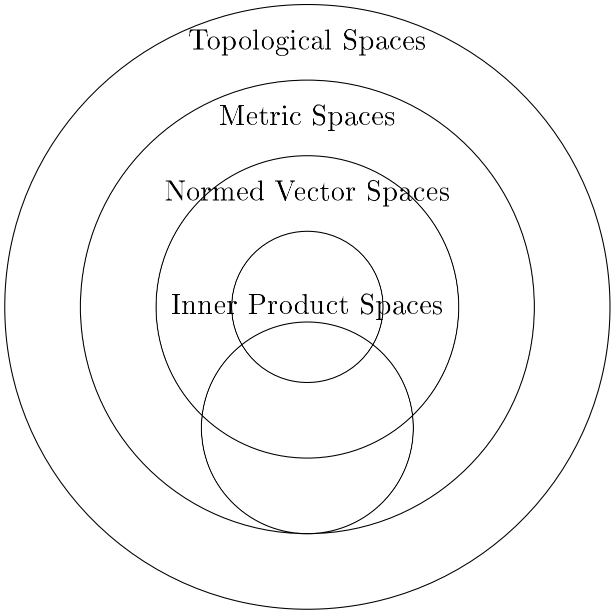
November 8, 2023

Topology of Metric Spaces

Definition: Topological Space

A pair (X,τ) is a topological space if

- X is a set.
- $\tau \subseteq \mathcal{P}(x)$ and satisfies
  1. $\varnothing, X \in \tau$
  2. τ is stable under arbitrary unions and finite intersections.
- Elements of τ are called "open sets".
  IMAGE HERE - ADD COMPLETE BANACH HILBERT



Definition: Vector Space

$(E, +, \cdot)$ is a vector space (over $\mathbb{R}$) if
There are two operations $+ : E \times E \to E$ and $\cdot : \mathbb{R} \times E \to E$ ($(\lambda, x) \mapsto \lambda x$) such that

$$(E,+) \text{ is a a commutative group} \begin{cases} x + y = y + x & \forall x, y \in E \\ (x+y)+z = x+(y+z) & \forall x, y, z \in E \\ \exists 0 \in E \text{ such that } x + 0 = 0 + x = x & \forall x \in E \\ \forall x \in E, \ \exists -x \in E, \text{ such that } x + (-x) = (-x) + x = 0 \end{cases}$$

$$\text{and} \begin{cases} \lambda(x+y) = \lambda \cdot x + \lambda \cdot y & \forall \lambda \in \mathbb{R}, \ x, y \in E \\ (\lambda \cdot \mu) \cdot x = \lambda \cdot (\mu \cdot x) & \forall \lambda, \mu \in \mathbb{R}, \ x \in E \\ (\lambda + \mu) \cdot x = \lambda \cdot x + \mu \cdot x & \forall \lambda, \mu \in \mathbb{R}, \ x \in E \end{cases}$$

Example 1

$(\mathbb{R}, +, \cdot)$

Example 2

$$\begin{aligned} \mathbb{R}^n : & x = (x_1, \ldots, x_n) \\ & x + y = (x_1 + y_1, \ldots, x_n + y_n) \\ & \lambda \cdot x = (\lambda x_1, \ldots, \lambda x_n) \end{aligned}$$

Example 3

Functions from $\mathbb{R} \to \mathbb{R}$

$$\begin{aligned} \text{“}f + g\text{''}(x) &= f(x) + g(x) \\ \lambda \cdot f(x) &= \lambda f(x) \end{aligned}$$

Sequences $\mathbb{N}_0 \to \mathbb{R}$
$C(\mathbb{R})$; $C^k(\mathbb{R}), \forall k$; $C^\infty(\mathbb{R})$; $C^w(\mathbb{R})$ (real-analytic functions.

Definition: Normed Vector Space

A norm on a vector space E is a map $||\cdot|| : E \to \mathbb{R}$ such that

1. $||x|| \geq 0,\ \forall x \in E$, with equality if and only if $x = 0$.

2. $||\lambda x|| = |\lambda| ||x||,\ \forall \lambda \in \mathbb{R},\ \forall x \in E$.

3. $||x + y|| \leq ||x|| + ||y||,\ \forall x, y \in E$ (triangle inequality).

$(E, ||\cdot||)$ is a normed vector space.

Example 1

$$\text{On } \mathbb{R}^n : ||x||_\infty = \max_{1 \leq i \leq n} |x_i|$$
$$1 \leq p \leq \infty ||x||_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/}$$

Example 2

$$\text{On } \mathbb{C}([a,b]) : ||f||_\infty = \sup_{x \in [a,b]} |f(x)|$$
$$[a,b] \text{ compact } ||f||_p = \left(\int_a^b |f(x)|^p \, dx \right)^{1/p}$$

Definition: Inner Product Space

An inner product on $\mathbb{R}$-vector space E is a map $\langle \cdot, \cdot \rangle : E \times E \to \mathbb{R}$ such that

1. It is bilinear: $\langle \lambda f + \mu g, h \rangle = \lambda \langle f, h \rangle + \mu \langle g, h \rangle,\ forall \lambda, \mu \in \mathbb{R},\ f, g, h \in E$

2. It is symmetric: $\langle f, g \rangle = \langle g, f \rangle,\ \forall f, g \in E$

3. It is positive definite: $\langle f, f \rangle \geq 0$, with equality if and only if $f = 0$.

The pair $(E, \langle \cdot, \cdot \rangle)$ is called an inner product space (or a pre-hilber space).

Example 1

$$\text{On } \mathbb{R}^n : \langle x, y\rangle = x_1y_1 + \cdots + x_ny_n$$
$$x = (x_1, \ldots, x_n)$$

- Proof

  1. Satisfied by properties of $\mathbb{R}$.

  2. Satisfied by mutliplicative commutativity.

  3. $\langle x, x\rangle = \sum_{i=1}^n x_i^2 = 0$ if and only if $x_i = 0, \forall i$.

Example 2

On $\mathbb{R}^n$: $A(a_{ij})_{i,j=1}^n$ symmetric positive definite matrix.
Then $\langle x, y\rangle_A := \langle x, Ay\rangle$ is an inner product.
Notice $\langle x, x\rangle = ||x||_2^2$.

Example 3

On $C([a, b])$, $\langle f, g\rangle = \int_a^b f(t)g(t)\, dt$.

Fact: Every Inner Product Gives Rise to a Norm

If the inner product is $\langle\cdot,\cdot\rangle$, then the norm is $||x|| = \langle x, x\rangle^{1/2}$.
But not every norm comes from an inner product.

Proposition:

Let $(E, \langle\cdot,\cdot\rangle$ an inner product space.
Denote $||x|| := \langle x, x\rangle^{1/2}$ for $x \in E$.
Then

1. $\forall x, y \in E,\ |\langle x, y\rangle| \le ||x||||y||$ (Cauchy-Schwarz)

2. $\forall x, y \in E,\ ||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2)$ (Parallelogram Identity)

3. $\forall x, y \in E,\ ||x + y|| \le ||x|| + ||y|| \implies ||\cdot||$ is a norm.

- Proof of 1
  $\forall t \in \mathbb{R}$, $\langle x + ty, x + ty\rangle \ge 0$

  $$0 \le \langle x + ty, x + ty\rangle = \langle x, x\rangle + 2t\langle x, y\rangle + t^2\langle y, y\rangle$$

  Therefore the discriminant is less than 0 and

$$(2\langle x,y\rangle)^2 - 4\langle x,x\rangle\langle y,y\rangle \leq 0$$

implies that $\langle x,y\rangle^2 \leq ||x||^2||y||^2$.

- Proof of 2

$$\begin{aligned} ||x+y||^2 + ||x-y||^2 &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\ &= ||x||^2 + 2\langle x,y\rangle + ||y||^2 + ||x||^2 - 2\langle x,y\rangle + ||y||^2 \\ &= 2(||x||^2 + ||y||^2) \end{aligned}$$

- Proof of 3

$$\begin{aligned} || &= ||x||^2 + 2\langle x,y\rangle + ||y||^2 \\ &\overset{\text{CS}}{\leq} ||x||^2 + 2||x||||y|| + ||y||^2 \\ &\leq (||x|| + ||y||)^2 \\ &\xrightarrow{\sqrt{}} ||x+y|| \leq ||x|| + ||y|| \end{aligned}$$

Proposition:

Let $(E, ||\cdot||)$ be a normed space such that $||\cdot||$ satisfies the parallelogram law, then

$$\langle x,y\rangle := \underbrace{\frac{1}{4}(||x+y||^2 - ||x-y||^2)}_{\text{"polarization identity"}}$$

is an inner product on E.

Definition: Metric Space

A pair (X,d) is a metric space if

- X is a set.

- $d : X \times X \to \mathbb{R}$ such that

  1. $d(x,y) \geq 0,\ \forall x,y \in X$ with equality if $x = y$.

  2. $d(x,y) = d(y,x),\ \forall x,y \in X$.

  3. $d(x,y) \leq d(x,z) + d(z,y),\ \forall x,y,z \in X$.

d is a "distance function."

Example 1

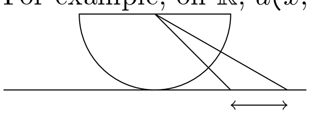
On $\mathbb{R}$, $d(x,y) = |x-y|$.

Example 2

On $(E, ||\cdot||)$, $d(x,y) = ||x-y||$ is a disatnce function.
Note that $d(x+z, y+z) = ||x+z-y-z|| = ||x-y||$ (translation-invaraniance).
$||x-y|| = ||x-z+z-y|| \le ||x-z|| + ||z-y||$.
Therefore, every normed space gives rise to a metric space.

Proposition:

Not every metric space is a normed space.
If E is a vector space, might be interested in non-transatio-invariant distances.
For example, on $\mathbb{R}$, $d(x,y) = |\tan^{-1}(x) - \tan^{-1}(y)|$.



Proposition:

Also, E might not be a vector space.
For example S^1, manifolds, graphs, etc.

Definition: Open Ball

Let (x,d) be a metric space.
$x \in X, \delta > 0$ define $B_\delta(x) := \{y \in x, d(x,y) \le \delta\}$ (and "open ball").
We say $A \subseteq X$ is open if and only if $\forall x \in A,\ \exists \delta > 0,\ B_\delta(x) \subseteq A$.

Definition: Open Neighborhood

An open neighborhood of $x \in X$ is any open set A containing x.

Proposition:

Let (X,d) be a metric space.

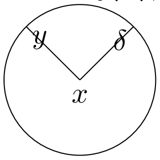
1. Open balls in X are open sets.

2. Arbitrary unions and finite intersections of open sets are open.

3. $\varnothing, X$ are open.

Proof of 1

Take $B_\delta(x),\ x \in X,\ \delta > 0$.



Take $y \in B_\delta(x)$, then $d(c, y) < \delta$. Set $\epsilon = \frac{\delta - d(x,y)}{2}$
Consider $B_\epsilon(y)$: if $z \in B_\epsilon(y)$, $d(z, y) < \frac{\delta - d(x,y)}{2}$, then

$$d(x, z) \le d(x, y) + \underbrace{d(y, z)}_{\frac{\delta - d(x,y)}{2}} < \frac{d(x, y) + \delta}{2} < \delta$$

Hence $B_\epsilon(y) \subseteq B_\delta(x)$.

Proof of 2

Arbitrary Union:
Suppose $A_\alpha,\ \alpha \in I$ are all open.
$x \in \bigcup_\alpha A_\alpha,\ \exists \alpha_0,\ x \in A_{\alpha_0}$.
$\exists \delta > 0,\ B_\delta(x) \subseteq A_0 \subseteq \bigcup_\alpha A_\alpha$
Finite Intersection:
Suppose $A_1 \cdot A_n$ are open.

$$x \in \bigcap_{j=1}^{n} A_j,\ \forall j,\ \exists \delta_j > 0$$

$B_{\delta_j}(x) \subseteq A_j$.
Take $\delta \min(\delta_1, \dots, \delta_n)$, then

$$B_\delta \subseteq \bigcap_{j=1}^{n} A_j$$

Definitions that Generalize on a Metric Space

Limits of sequences: x_n converges to x if $\lim_{n\to\infty} d(x_n, x) = 0$.
Cauchy sequences.
Boundedness

Definition: Limit Points of Sets

$a \in x$ is a limit point of $A \subseteq X$ if $\forall \epsilon > 0$, $\exists b \in A \setminus a$, $d(a, b) < \epsilon$.
Equivalenetly, every open neighborhood of a has a point in $A \setminus a$.

Definition: Closed Set in Metric Space

A is closed if and only if it contains all its limit points.

Proposition:

On a metric space (x, d),

1. $A \subseteq X$ is closed if and only if A^C is open ($A^C = X \setminus A$).

2. Finite unions and countable intersections of closed sets are closed.

Proof of 1

Suppose A is open. Want to show that A^C is closed.
If $x \notin A^C$, then $x \in A$, then $\exists \delta > 0$ such that $B_\delta(x) \subseteq A$ (i.e. $B_\delta(x) \cap A^C = \varnothing$.
Therefore x is not a a limit point of A^C.

Suppose A not open.
Then $\exists x \in A$ such that $\forall \delta > 0$, $B_\delta(x) \cap A^C \neq \varnothing$.
Therefore $x \neq A^C$ and x is a limit point of A^C, so A^C is not closed.

Proof of 2 (Finite Union)

Take $F_1, \ldots, F_n$ closed sets and consider

$$\bigcup_{j=1}^{n} F_j = \left(\bigcup_{j=1}^{n} F_j\right)^{CC} = \left(\bigcap_{j=1}^{n} \underbrace{F_j^C}_{\text{open by 1}}\right)^C$$

is closed.

Definition: Completeness

A metric space (X, d) is complete if and only if every cauchy sequence converges to a point in X.
If (X, d) comes from a normed vector space $(X, ||\cdot||)$, it is called Banach.
If (X, d) comes from an inner product space, it is called Hilber.

Examples

These are complete.
$(\mathbb{R}, |\cdot|)$
$(\mathbb{R}^n, ||\cdot||_2)$
$(C([a, b]), ||f - g||_\infty)$

Counter Examples

This is not complete.
$(C([a, b]), ||f - g||_2)$ where $||f - g||_2 = \left(\int_a^b (f(t) - g(t))^2 \, dt\right)^{1/2}$.
Consider x^n on $[0, 1]$.
$\left(\int_0^1 (x^n - 0)^2 \, dx\right)^{1/2} = \frac{1}{\sqrt{2n+1}}$.

Theorem:

$\forall n \in \mathbb{N}$, $(\mathbb{R}^n, ||\cdot||_2)$ is complete.
Let x_p be a cauchy sequence in $\mathbb{R}^n$, $x_p = (x_{p,1}, \ldots, x_{p,n})$.
Note for $1 \leq j \leq n$, $|x_{p,j} - x_{q,j}| \leq ||x_p - x_q||_2$.
Therefore $\forall 1 \leq j \leq n$, $\{x_{p,j}\}_p$ is Cauchy in $\mathbb{R}$.

November 13, 2023

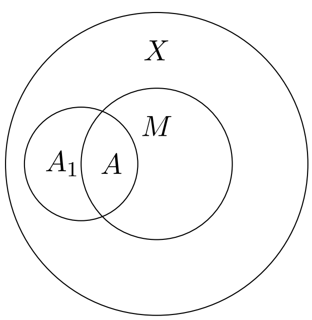
Topology of metric spaces.

- Induced Topology
- Continuous Maps
- Connectedness
- Compact Sets / Separability

Induced Topology

Let (X, d) be a metric space, and $M \subseteq X$.
Can restrict the distance function to $M \times M$, then $(M, d|_{M\times M})$ is a metric space.
$U \subseteq M$ is open if $\forall a \in U$, $\exists \delta > 0$, $\{x \in M \mid d(a,x) < \delta\} = B_\delta(a) \subset U$.
Note: if $a \in M \subseteq X$,

$$B_\delta^M(a) = \{x \in M \mid d(a,x) < \delta\} = B_\delta^X \cap M$$
$$B_\delta^X(a) = \{x \in X \mid d(a,x) < \delta\}$$



Open sets for both topologies may differ!
Note: M is always open for $(M, d|_{M\times M})$.
e.g. $X = \mathbb{R}$ and $M = [0,1]$.

Proposition:

In the setting above, $A \subset M$ is open in (M, d) if and only if $\exists A_1 \subseteq X$ open in (X, d) such that $A = A_1 \cap M$.

- Proof

  ($\Longrightarrow$) Suppose A open in (M,d), $\forall a \in A$, $\exists \delta_a > 0$, $B^M_{\delta_a}(A) \subset A$.

$$A = \bigcup_{a \in A} B^M_{\delta_a}(a) = \bigcup_{a \in A} \left(B^X_{\delta_a}(a) \cap M \right) = \underbrace{\left(\bigcup_{a \in A} B^X_{\delta_a}(A) \right)}_{A_1} \cap M$$

  ($\Longleftarrow$) Suppose $A = A_1 \cap M$, A_1 open in (X,d). Let $a \in A_1$.
  $a \in A_1 : \exists \delta > 0$, $B^X_\delta(a) \subseteq A_1$.
  Then

$$B^M_\delta(a) = \underbrace{B^X_\delta(a)}_{\subseteq A_1} \cap M \subseteq A_1 \cap M = A \quad \blacksquare$$

Proposition:

A closed subspace M of a complete metric space (X,d) is also complete.

- Proof

  Take a Cauchy sequence in M, $\{x_k\}_k$, then it is also Cauchy in X.
  Therefore it converges to $x \in X$.
  Since M contains its limit points, $x \in M$.

Theorem:

Let $f : (M, d_M) \to (N, d_N)$, where (M, d_m) and (N, d_n) are metric spaces. The following are equivalent.

1. $\forall x \in M$, $\forall \epsilon > 0$, $\exists \delta > 0$

$$d_m(x,y) < \delta \implies d_N(f(x), f(y)) < \epsilon$$

   (in short: $\forall \epsilon > 0, \exists \delta > 0, f\left(B^M_\delta(x)\right) \subset B^N_\epsilon(f(x))$)

2. $\forall x \in M$ and $\{x_n\}_n$ convergin to $x \in M$, $f(x_n)$ converges to $f(x)$ in N.

3. $\forall O$ open in N, $f^{-1}(O)$ is open in M.

Definition: Continuity

When (1), (2) or (3) is satisfied, we say "f is continuous on M".

Proof that 1 Implies 2

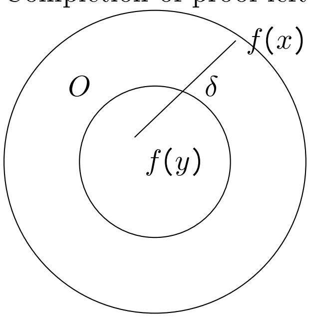
Let $x \in M$, $\{x_n\}_n$ converges to x.
What to show: $\lim_{n\to\infty} d_N(f(x_n), f(x)) = 0$.
Let $\epsilon > 0$, by (1), $\exists \delta > 0$, $f\left(B^m_\delta(x)\right) \subseteq B^N_\epsilon(f(x)) \quad (*)$.
Since $x_n \to x$, $\exists n_0, n \geq n_0 \implies d_M(x_n, x) < \delta$ (i.e. $x_n \in B^M_\delta(x)$).
Then, by $(*)$, $f(x_n) \subseteq B^N_\epsilon(f(x))$ (i.e. $d_N(f(x_n), f(x)) < \epsilon$, $\forall n \geq n_0$)

Proof that 2 Implies 3

Assume (2) and let $\exists O$ open in N such that $f^{-1}(O)$ is not open:

$$\exists y \in f^{-1}(O), \forall \delta = \frac{1}{n} > 0, \exists x_n, d_M(x_n, y) < \frac{1}{n} \text{ and } f(x_n) \notin O$$

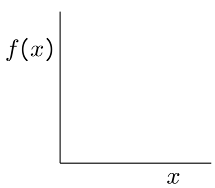
Then x_n converges to y, but $f(x_n)$ cannot converge to $f(y)$.
Completion of proof left as an exercise.



Proof that 3 Implies 1

Let $x \in M$ and $\epsilon > 0$. $B_\epsilon^N(f(x))$ is open.
By (3), $f^{-1}\left(B_\epsilon^N(f(x))\right)$ is open and contains x.
Then $\exists \delta > 0$, $B_\delta^M(x) \subseteq f^{-1}\left(B_\epsilon^N(f(x))\right)$, which implies

$$f\left(B_\delta^M(x)\right) \subseteq B_\epsilon^N(f(x))$$

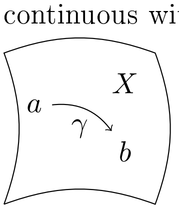


Definition: Path Connected

Let (X, d) be a metric space.
We say X is path-connected if $\forall a, b \in X$,

$$\exists \gamma : [0,1] \to X$$

continuous with $\gamma(0) = a$ and $\gamma(1) = b$.



Definition: Connected

We say X is connected if the only subsets of X that are both open and closed are $\varnothing$ and X.
Alternatively, $X = A_1 \cup A_2$, $A_1 \neq \varnothing$, $A_2 \neq \varnothing$, $A_1 \cap A_2 \neq \varnothing$, A_1, A_2 open.

Proposition: Path-connectedness Implies Connectedness

Let (X, d) be a metric space. If (X, d) is path-connected, then it is connected.

Remarks

If $(X, d) = (\mathbb{R}, |\cdot|)$, $A \subseteq \mathbb{R}$ is connected if and only if its path-connected if and only if it is an interval.
In general, connectedness does not imply path connectedness.

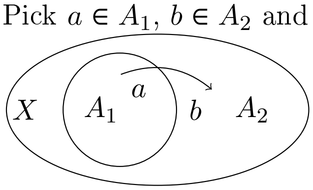
- Counterexample
  In $\mathbb{R}^2$, the topologist's sine wave

$$A = \left\{(0,0)\} \cup \{\left(x, \sin\frac{1}{x}\right) \mid x \in \mathbb{R} \setminus \{0\}\right\}$$

IMAGE HERE - TOPOLOGISTS SINE WAVE

Proof

Suppose X is path-connected but not connected. That is
$X = A_1 \cup A_2$, A_1, A_2 open and nonempty, $A_1 \cap A_2 = \varnothing$.
Pick $a \in A_1$, $b \in A_2$ and consider $\gamma : [0,1] \to X$ continuous with $\gamma(0) = a$ and $\gamma(1) = b$.



$a \in A_1$, which is open, so $\exists \delta > 0$, $B_\delta(a) \subseteq A_1$.
Then $\gamma^{-1}(B_\delta(a))$ is open and contains 0, therefore it contains $[0, \epsilon)$ for some $\epsilon > 0$.
Let $l = \sup\{\epsilon > 0 \mid \gamma([0, \epsilon)) \subset A_1\}$.
Then $\gamma([0, l)] \subset A_1$.
If $l < 1$, $\gamma(l) \in A_1$ open. Then $\exists \delta' > 0$, $B_{\delta'}(\gamma(l)) \subseteq A_1$, and $\gamma^{-1}(B_{\delta'}(\gamma(l)))$ is open in $[0,1]$, of the form $(l - \epsilon', l + \epsilon')$.
This contradicts the supremum property of l, so $l = 1$.
But then $\gamma(l) \in A_1$ and $\gamma(l) = \gamma(1) = b \in A_2$, which is a contradiction.

Definition: Compact

Let (X, d) be a metric space.

1. We say that $A \subseteq X$ is compact if every sequence in A has a limit point in A.

2. We say $A \subseteq X$ satisfies Heine-Borel property if every open cover of A has a finite subcover, still covering A.

Definition: Dense

We say A is dense in X if $\forall x \in X, \forall \epsilon > 0, \exists a \in A, d(x, a) < \epsilon$.

Definition: Separable

We say B is separable if B has a countable dense subset.
i.e. $\exists \{x_n\}_n \in B$ such that every point in B is a limit point of $\{x_n\}_n$.

Example 1

$(\mathbb{R}, |\cdot|)$, with dense subset $\mathbb{Q}$.

Example 2

$(C([a,b]), ||\cdot||_\infty)$, with dense subset polynomials with rational coefficients.
Proof left as an exercise.

Theorem:

Suppose (X,d) is a compact metric space. Then it is separable.

Distance Between Sets.

Given a finite collection $\{x_1,\ldots,x_n\}$, write $d(x,\{x_1,\ldots,x_n\}) = \min_{1\le j\le n} d(x,x_j)$.

Proof

Pick $x_1 \in X$.
Look at $R_1 := \sup\{d(x,x_1) \mid x \in X\}$.
Claim: $R_1 < \infty$. Otherwise, construct a sequence with no convergent subsequence.
Then, pick x_2 such that $d(x_1,x_2) > \frac{1}{2}R_1$.
Look at $R_2 := \sup\{d(x,\{x_1,x_2\}) \mid x \in X\} < \infty$.
Pick x_3 such that $d(x_3,\{x_1,x_2\}) > \frac{1}{2}R_2$.
Repeat: if $x_1,\ldots,x_k$ are constructed, look at $R_k := \sup\{d(x,\{x_1,\ldots,x_k\} \mid x \in X\} < \infty$ and set x_{k+1} such that $d(x,\{x_1,\ldots,x_k\}) > \frac{1}{2}R_k$.
Claim: $R_k \underset{k\to\infty}{\longrightarrow} 0$, otherwise $\{x_n\}_n$ has no convergent subsequences.
But then, $\forall x \in X$, $d(x,\{x_1,\ldots,x_k\}) \le R_k$. Hence $\lim_{k\to\infty} d(x,\{x_1,\ldots,x_k\} = 0$.
Then for $\epsilon > 0, \exists k$ such that $d(x,\{x_1,\ldots,x_k\}) < \epsilon$.
i.e. $\exists k_0 \in \{1,\ldots,k\}, d(x,x_{k_0}) < \epsilon$. ■

Theorem:

A subsets A of a metric space (X,d) is compact if and only if it satisfies the Heine-Borel property.

Proof

($\Longleftarrow$) (Note: true even if A is not separable)
Take a seqeuence $\{x_n\}_n$ in A and argue by contradiction.
Case 1: $\{x_n\}_n$ has a limit point $b \notin A$.
Then $U_k = \left\{x \in X \mid d(x,b) > \frac{1}{k}\right\}$.
IMAGE HERE - COVERS AROUND B WITH X AND A
$\bigcup_k U_k = X \setminus \{b\}$ covers A, but no finite subcover covers A since b is a limit point of A.

Case 2: $\{x_n\}_n$ has no imit points at all.
$V_k = X \setminus \{x_k, x_{k+1}, \dots\}$ is open and $\bigcup_k V_k$ covers A, but no finite subcover covers A. ■
($\implies$) Sketch
To do: assume compactness show that it leads to Heine-Borel.
Take an arbitrary cover.
Separability implies one can extract a countable subcover of O covering A.

November 15, 2023

Contraction Mapping Theorem
Application 1: Local Existence Theorem for ODEs
Application 2: Implicit Function Theorem

Definition: Contraction

$f : M \to N$, (M, d_M), (N, d_N) two metric spaces.
f is a contraction if $\exists C \in [0, 1)$ such that

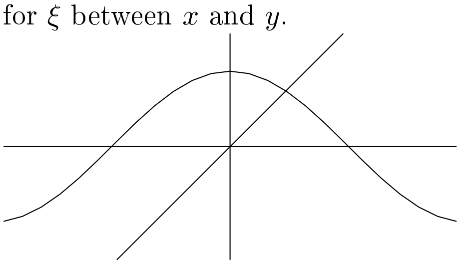
$$\forall x, y \in M \quad d_N(f(x), f(y)) \leq C d_M(x, y)$$

Example:

$f(x) = \frac{1}{2}\cos(x)$, $f : \mathbb{R} \to \mathbb{R}$.

$$|f(x) - f(y) \underbrace{=}_{MeanValueTheorem} | \underbrace{f'(\xi)}_{\left|\frac{1}{2}\sin(\xi)\right|} ||x - y| \leq \frac{1}{2}|x - y|$$

for ξ between x and y.



Theorem: Contraction Mapping Theorem

Suppose (M, d) is a complete metric space and $f : M \to M$ a contraction. Then

$$\exists! x \in M, f(x) = x$$

(i.e. f has a unique fixed point).

Proof

- Existence.

  Pick any $x_0 \in M$. Consider the sequence $x_{k+1} = f(x_k)$.

Claim 1: $\{x_n\}_n$ is Cauchy (then, by completeness, it converges to some $x \in M$)
Claim 2: If Caim 1 is true, by continuity of f at x,

$$f(x) = f\left(\lim_{k\to\infty} x_k\right) \underbrace{=}\text{continuity @ } x \lim_{k\to\infty} \underbrace{f(x_k)}_{x_{k+1}} = \lim_{k\to\infty} x_{k+1} = x$$

What to show: $\{x_n\}_n$ is Cauchy.
$\forall \epsilon > 0, \exists N, \forall p \geq N, k \geq 0, d(x_{p+k}, x_p) < \epsilon$

– Scratch Work

$$\begin{aligned} d(x_{p+k}, x_p) &\leq d(x_{p+k}, x_{p+k-1}) + d(x_{p+k-1}, x_{p+k-2}) + \cdots + d(x_{p+1}, x_p) \\ &\leq \sum_{q=0}^{k-1} \underbrace{d(x_{p+q+1}, x_{p+q}))}_{C^q d(x_{p+1}, x+p} \\ &\leq \underbrace{d(x_{p+1}, x_p)}_{c^p d(x_1, x_0)} \cdot \underbrace{\sum_{q=0}^{k+1} C^q}_{\leq \frac{1}{1-c}} \end{aligned}$$

$$d(x_{p+2}, x_{p+1}) = d(f(x_{p+1}), f(x_p) \leq Cd(x_{p+1}, x_p)$$

$$\begin{aligned} d(x_2, x_1) &= d(f(x_1), f(x_0)) \leq Cd(x_1, x_0) \\ d(x_3, x_2) &\leq Cd(x_2, x_1) \leq CCd(x_1, x_0) \end{aligned}$$

Ultimately,

$$(*) \quad d(x_{p+k}, x_p) \leq \frac{d(x_1, x_0)}{1-C} C^p$$

- Proof of Cauchy

Let $\epsilon > 0$, since $\lim_{p\to\infty} \frac{d(x_1,x_0)}{1-C} C^p = 0$,

$$\exists N, \forall p \geq N, \frac{d(x_1, x_0)}{1-C} C^p < \epsilon$$

Then, for $p \geq N$ and $k \geq 0$,

$$d(x_{p+k}, x_p \overset{(*)}{\leq} \frac{d(x_1, x_0)}{1-C} C^p < \epsilon \quad \blacksquare$$

- Uniqueness

If x, y satisfy $f(x) = x$ and $f(y) = y$

$$d(x, y) = d(f(x), f(y)) \leq Cd(x, y)$$

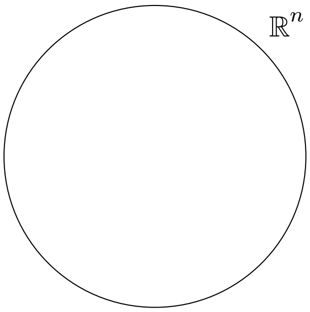
If $d(x, y) \neq 0$, then $1 \leq C$ is a contradiction.

Application to ODEs

A system of 1st order Ordinary Differential Equations takes the form

$$\begin{cases} \frac{dx}{dt} = f(t, x(t)) \quad (*) \\ x(0) = x_0 \end{cases}$$

$$x : [0, b]_t \to \mathbb{R}^n$$
$$f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$



Under what conditions on f does $(*)$ have unique solution $x(t)$ on some interval $[0, \epsilon]$?

Example

$\frac{d^2x}{dt^2} = \sin(x(t))$
IMAGE HERE - PENDULUM WITH GRAVITY VECTOR, LENGTH AND SIN(T) PERIOD

Meta-Principle:

An ODE of order k in $\mathbb{R}^n$: $\frac{d^k}{dt^k}x(t) = f(t, x(t), x^1(t), \ldots, x^{k-1}(t))$ can be rephrased as a 1st-order system in $\mathbb{R}^{nk}$ upon introducing variables

$$x_1(t) = \frac{dx}{dt}(t)$$
$$x_2(t) = \frac{dx_1}{dt}(t)$$
$$\vdots$$
$$x_{k-1}(t) = \frac{dx_{k-2}}{dt}(t)$$

The ODE becomes

$$\frac{d}{dt}\underbrace{\begin{bmatrix} x(t) \\ x_1(t) \\ \vdots \\ x_{k-1}(t) \end{bmatrix}}_{U(t)} = \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{k-1}(t) \\ f(t, x(t), \ldots, x_{k-1}(t)) \end{bmatrix}}_{F(U(t))}$$

Theorem: Local Existence Theorem of ODEs

Take $x_0 \in \mathbb{R}^n$. Suppose $\exists \delta > 0$ such that f is continuous on $\mathbb{R} \times \overline{B_{\delta(x_0)}}$.
Where $\overline{B_\delta(x_0)} = \{x \in \mathbb{R}^n \mid ||x - x_0||_2 < \delta\}$

And f is Lipschitz with respect to $x: |f(t,x) - f(t,y)| \leq C|x-y|$, $\forall t \in \mathbb{R}$, $\forall x, y \in \overline{B_\delta(x_0)}$.
Notation: $|x| = ||x||_2$.
Then $\exists > 0$ and $\exists! x \in C^1([0,\epsilon]; \mathbb{R}^n)$ solution to $(*)$.

Proof

If x solves $(*)$,

$$\int_0^t \rightsquigarrow x(t) - x_0 = \int_0^t f(u, x(u))\, dw \iff \underbrace{x(t) = x_0 + \int_0^t f(u, x(u))\, dw}_{Tx(t)}$$

This is a fixed point problem.
Goal:

1. find (E,d) a complete metric space

2. such that $T(E) \subseteq E$ and T is a contraction.

3. Then, by the Contraction Mapping Theorem, $\exists! x \in E$.

4. Finally, show that x is actually C^1.

- Part A

  $\overline{B_\delta(x_0)}$ is closed in $(\mathbb{R}^n, ||\cdot||_2)$ which is complete $\implies (\overline{B_\delta(x_0)}, ||\cdot||_2)$ is complete.
  Then $\forall \epsilon > 0$, $C([0,\epsilon], \overline{B_\delta(x_0)})$, with norm $||x(t)||_\infty = \sup_{t\in[0,\epsilon]} ||x(t)||_2$, is complete.
  Set

  $$E = C([0,\epsilon], \overline{B_\delta(x_0)})$$

- Part B

  When is $T(E) \subseteq E$?
  Suppose $x \in E$, i.e. $||x(t) - x_0||_2 \leq \delta$, $\forall t \in [0,\epsilon]$.

  $$||x(t) - x_0||_2 = \left|\left| \int_0^t f(u, x(u))\, du \right|\right|_2 \leq \int_0^t \underbrace{||f(u, x(u))||_2}_{\text{Note}}\, du \leq tM \leq \epsilon M \quad \forall t \in [0,\epsilon]$$

  Note: since f is continuous on $\mathbb{R} \times \overline{B_\delta(x_0} \implies f$ bounded by M.
  Upon making $\epsilon M \leq \delta$, then $\sup_{t\in[0,\epsilon]} ||Tx(t) - x_0||_2 \leq \delta$
  $\implies T(E) \subseteq E$ if $\epsilon \leq \frac{\delta}{M}$.

  Is T continuous on E?

$$
\begin{aligned}
||Tx(t) - Ty(t)||_2 &= \left|\left|\int_0^t (f(u, x(u)) - f(u, y(u)))\, du\right|\right|_2 \\
&\underbrace{\leq \int_0^t ||f(u, x(u)) - f(u, y(u))||_2 \, du}_{\underbrace{\underbrace{\leq L||x(u) - y(u)||_2}_{L \sup_{u \in [0,\epsilon]} ||x(u)-y(u)||_2}}_{||x-y||_E}} \\
&\leq tL||x - y||_E \leq \epsilon L||x - y||_E
\end{aligned}
$$

Therefore $||Tx - Ty||_E \leq \epsilon L||x - y||_E$ is a contraction if $\epsilon L < 1$, i.e. $\epsilon < \frac{1}{L}$.

- Part C

  By CMT, $\exists! x \in E(\epsilon)$ as long as $\epsilon \leq \frac{\delta}{M}$ and $\epsilon < \frac{1}{L}$.

- Part D

  $x(t) \in C([0, t], \overline{B_\delta(x_0)})$, but also notice

$$
x(t) = \underbrace{x_0 + \int_0^t f(u, x(u))\, du}_{\text{differentiable with derivative } f(t, x(t))}
$$

  which is continuous.
  Therefore x is C^1, $x'(t) = f(t, x(t))$, and $x(0) = x_0 + \int_0^0 = x_0$.

Remark: Lipschitz Requirement for Uniquness

f being locally Lipschitz is necessary for uniquness purposes.

Example

$$
\begin{cases}
\frac{dx}{dt} = 3x^{2/3} \\
x(0) = 0
\end{cases}
$$

This ODE has two distinct solutions:

$$
\begin{cases}
x(t) = 0 \\
x(t) = t^3 \rightsquigarrow \frac{dx}{dt} = 3t^2 = 3x^{2/3}
\end{cases}
$$

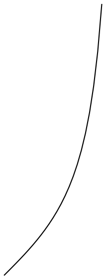
Remark:

The time of existence is not sharply controlled (could be infinite, finite, very small)

Example

$$\begin{cases} \frac{dx}{dt} = \overbrace{x^2}^{\text{Lipschitz}} \\ x(0) = x_0 \end{cases} \rightsquigarrow \frac{1}{x^2}\frac{dx}{dt} = 1 \implies -\frac{d}{dt}\left(\frac{1}{x}\right) \implies \frac{-1}{x(t)} + \frac{1}{x_0} = t \implies x(t) = \frac{x_0}{1-x_0 t}$$

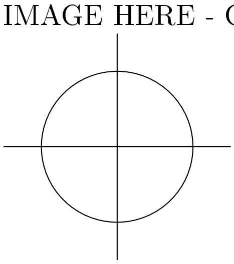
Lifetime: $t^\star = \frac{1}{x_0}$.
IMAGE HERE - FROM X0 EXPLODING AT 1/X0; FASTER FOR 2X0



Theorem: Implicit Function Theorem / Inverse Function Theorem

Example

$F : \mathbb{R}^2 \to \mathbb{R}$, $F(x, y) = x^2 + y^2$.
Question: Can the set $F(x, y) = 1$ be described by equations "$x(y)$" or "$y(x)$" ?
IMAGE HERE - CIRCLE AS TWO EQUATIONS WITH VERTICAL LINE TEST



Implicit Function Theorem gives situations where it is possible.

Setting

$F : \mathbb{R}^p_x \times \mathbb{R}^q_y \to \mathbb{R}^q$ continuously differentiable.
Fix $(a, b) \in \mathbb{R}^p \times \mathbb{R}^q$ and set $F(a, b) = c$.
If the $q \times q$ matrix

$$\frac{\partial F}{\partial y}(a,b) = \begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} & \cdots & \frac{\partial F_1}{\partial y_q} \\ & \ddots & & \\ \vdots & & & \vdots \\ \frac{\partial F_q}{\partial y_1} & \cdots & \cdots & \frac{\partial F_q}{\partial y_q} \end{bmatrix}$$

is invertible, then $\exists \Omega$ neighborhood of a and a function $y : \Omega \to \mathbb{R}^q$ such that $\forall x \in \Omega$,

$$F(x, y(x)) = c$$

Moreover:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial x} = 0 \implies \frac{\partial y}{\partial x} = \left[\frac{\partial F}{\partial y}(x, y(x))\right]^{-1} \frac{\partial F}{\partial x}(x, y(x))$$

Parallel: solve $f(x) = 0$ by Newton's iteration.
x_{n+1} = the zero of the line $(f(x_n) + (x - x_n)f'(x_n) = 0) = x_n - (f'(x_n))^{-1}f(x_n)$.
Left iterating
$\phi(x) = x - (f'(x))^{-1}f(x)$
Claim: ϕ is a contraction near fixed points $x^\star$ whenever $f'(x^\star) \neq 0$.

- Proof (Sketch)

  Uses CMT.
  Freeze x; approximately solve for y the equation $F(x,y) - c = 0$.
  Iterate $\phi_x(y) = y - \frac{\partial F}{\partial Y}^{-1}(a,b)(F(x,y) - c)$.

November 20, 2023

Continuity and compactness.
Uniform equicontinuity and Arzéla-Ascoli.
Bolzano's example of a continuious, nowhere differentiable function.

Continuity on Compact Domains

Context:
(M, d_m), (N, d_n) two metric spaces.
$f : M \to N$ continuous.
M compact.

Proposition:

In the context above, we can deduce:

1. f isuniformly continuous in the sense that

$$\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in M, d_M(x,y) < \delta \implies d_n(f(x), f(y)) < \epsilon$$

2. $f(M)$ is compact.

- Corollary:

  If (M, d_M) is a compact metric space and $f(M) \to (\mathbb{R}, |\cdot|)$ continuous, then f is bounded and achieves its extrema.

$$\exists x_\pm \in M \text{ such that } f(x_+) = \sup\{|f(x)| \mid x \in M\} \text{ and } f(x_-) = \inf\{|f(x)| \mid x \in M\}$$

- Proof of 1

  Alternative proof in lecture notes.
  By contradiction, suppose

$$\exists \epsilon > 0, \forall \underbrace{\delta}_{1/n} > 0, \exists \underbrace{x, y}_{x_n, y_n} \in \underbrace{M, d_M(x,y) < \delta}_{d_M(x_n, y_n) < 1/n} \text{ and } d_N(f(x), f(y)) \geq \epsilon$$

x_n lives in M compact, therefore a subsequence x_{n_k} converges to x in M.

$$d_M(x_{n_k}, y_{n_k}) < \frac{1}{n_k} \implies d_M(y_{n_k}, x) \leq d_M(y_{n_k}, x_{n_k}) + d_M(x_{n_k}, x)$$
$$\leq d_M(x_{n_k}, x) + \frac{1}{n_k} \xrightarrow[k\to\infty]{} 0$$

So y_{n_k} converges to x.
Since f is continuous at x, $f(x_{n_k}) \to f(x)$ and $f(y_{n_k}) \to f(x)$ as $k \to \infty$.
Hence $d_N(f(x_{n_k}, y_{n_k}) \xrightarrow[k\to\infty]{} d_N(f(x), f(y)) = 0$ (cf. homework problem).
This contradicts the assumption.

- Proof of 2

  What to show: $\forall \mathcal{O}$ open covers of $f(M)$, there exists a finite subcover $\mathcal{O}' \subset \mathcal{O}$ still covering $f(M)$.
  Take $\mathcal{O} = \{O_\alpha \mid \alpha \in I\}$ covering $f(M)$.
  Set $V_\alpha = f^{-1}(\mathcal{O}_\alpha)$ open by the continuity assumption.
  Claim: $\mathcal{V} = \{V_\alpha \mid \alpha \in I\}$ covers M: let $x \in M, f(x) \in f(M)$, then $f(x) \in \mathcal{O}_\alpha$ for some $\alpha \in I$.
  By Heine-Borel Compactness,

$$\exists \{ \underbrace{V_1}_{f'^1(O_1)}, \dots, \underbrace{V_n}_{f^{-1}(O_n)} \}$$

  subcollection of $\mathcal{V}$ covering M.
  Claim: $(O_1, \dots, O_n)$ covers $f(M)$: Pick $y \in f(M), y = f(x)$ for $x \in M$
  M covered by

$$V_1, \dots, V_n \implies \exists j, x \in V_j \implies \underbrace{f(x)}_{y} \in O_j$$

Arzéla-Ascoli

Q: What are the compacts sets of $(C([a,b]), ||\cdot||_\infty)$?
Context:
$[a,b] \subset \mathbb{R}$ a compact interval.
$C([a,b])$ continuous functions on $[a,b]$.
$||f||_\infty := \sup_{x\in[a,b]} |f(x)|$ is a norm.
$(C([a,b], ||\cdot||_\infty)$ is complete.

Definition: Uniform Boundedness / Uniform Equicontinuity

For $\mathcal{F} \subset C([a,b])$, we say

1. $\mathcal{F}$ is uniformly bounded if $\exists M \geq 0, \forall x \in [a,b], \forall f \in \mathcal{F}, |f(x)| \leq M$

2. $\mathcal{F}$ is uniformly equicontinuous (UEC) if $\forall \epsilon > 0, \exists \delta > 0,$

$$\forall x, y \in [a,b], |x-y| < \delta \implies |f(x) - f(y)| < \epsilon, \quad \forall f \in \mathcal{F}$$

- Example 1

  Any finite $\mathcal{F} = \{f_1, \ldots, f_n\}$ is uniformly bounded and uniformly equicontinuous.
  $M = \max\{M_1, \ldots, M_n\}$
  $\delta(\epsilon) = \min\{\delta_1(\epsilon), \ldots, \delta_n(\epsilon)\}$

- Example 2

  Sets satisfying a uniform Lipschitz/Hölder criterion, i.e. $\exists L > 0, \alpha \in (0, 1]$,

$$|f(x) - f(y)| \leq L|x - y|^{\alpha}, x, y \in [a, b], f \in \mathcal{F}, \left(\delta(\epsilon) = \left(\frac{\epsilon}{L}\right)^{1/\alpha}\right)$$

- Example 3

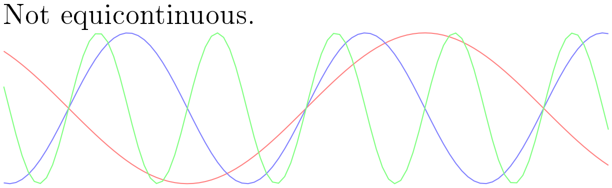
  Sets satisfying a uniform bound on first derivatives (assuming they exist).

$$\mathcal{F} = \{f \in C^1([a, b]) \mid |f'(x)| \leq M, x \in [a, b]\}$$

- Non-example

$$\mathcal{F} = \{\sin(nx)\ x \in [0, \pi], n \in \mathbb{N}\}$$

  Uniformly bounded by 1.
  Not equicontinuous.



Proposition:

If $f_n \in C([a, b])$ converges uniformly to $f \in C([a, b])$, then $\{f_n \mid n \in \mathbb{N}\}$ is uniformly bounded and UEC.

Proof

- Unfiorm Boundedness

  Since $||f_n - f||_{\infty} \underset{n\to\infty}{\longrightarrow} 0$ and f is continuous on $[a, b]$ compact, then f bounded implies $||f||_{\infty} < \infty$.
  For

$$\implies ||f_{\mathrm{n}} - f||_{\infty} < 1$$
$$\implies ||f_{\mathrm{n}}||_{\infty} = ||f_{\mathrm{n}} - f + f||_{\infty}$$

Then $M = \max\{||f_1||_\infty, \dots, ||f_n||_\infty, ||f||_\infty + 1\}$ is a uniform bound.

- Uniform Equicontinuity

$$\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in [a,b], |x-y| < \delta \implies |f_n(x) - f_n(y)| < \epsilon \quad \forall n \in \mathbb{N}$$

  f is continuous, hence uniform continuous on $[a,b]$ with modulus of continuity $d_f(\epsilon)$.
  Let $\epsilon > 0$, by uniform convergence, (a) $\exists N, n \geq N \implies |f_n(x) - f(x)| \leq \epsilon/3\ \forall x \in [a,b]$.
  For $x, y, |x-y| < \delta_f$, and $n \geq N$,

$$|f_n(x) - f_n(y)| = \underbrace{|f_n(x) - f(x)|}_{<\frac{\epsilon}{3} \text{ by (a)}} + \underbrace{|f(x) - f(y)|}_{<\frac{\epsilon}{3} \text{ by uniform convergence of } f} + \underbrace{|f(y) - f_n(y)|}_{<\frac{\epsilon}{3} \text{ by (a)}}$$

Therefore $\mathcal{F} = \{f_n\}n \geq N$ is UEC.
$\delta(\epsilon) = \min\{d_f(\epsilon/3), \delta_1(\epsilon), \dots, \delta_N(\epsilon)\}$.

Theorem: Arzéla-Ascoli

Let $[a,b] \subset \mathbb{R}$ be compact and $f_k \in C([a,b])$ be uniformly bounded and uniformly equicontinuous.
Then a subsequence of f_k converges uniformly.

Proof

Step A: construct a subsequence converging at all rationals in $[a,b]$.
(uniform boundedness is enough)
Step B: show that the subsequence is uniformly Cauchy.
$\forall \epsilon > 0, \exists N, \forall p, q \geq N, ||f_p - f_q||_\infty < \epsilon$
(uses UEC)

- Step A
  Let r_k be a denumeration of all rationals in $[a,b]$.
  $\{f_n(r_1)\}_n$ has a convergent subsequence (Bolzano-Weierstrass), $\{f_{1,n}(r_1)\}_n$.
  $\{f_{1,n}(x)\}_n$ converges at r_1; a subsequence $f_{2,n}$ converges at $r_2 \implies f_{2,n}$ converges at r_1, r_2.
  Repeat $\{f_{2,n}(r_3)\}_n$ is bounded $\implies$ a subsequence $\{f_{3,n}(r_3)\}_n$ converges $\implies f_{3,n}$ converges at r_1, r_2, r_3.
  $\forall p, \{f_{p+1,n}\}_n$ subsequence of $\{f_{p,n}\}_n$ converges at $r_1, \dots, r_p, r_{p+1}$.
  Consider $\{f_{k,k}\}_{k \in \mathbb{N}}$ subsequence of $\{f_n\}_n$.
  $\forall p$, $f_{k,k}(r_p)$ converges because $\{f_{k,k}(r_p)\}_{k \geq p}$ is a subsequence of $\{f_{p,n}\}_n$.

- Step B
  Simplified statement: if f_n is uniformly bounded, uniformly equicontinuous and converges at all rationals in $[a,b]$, then f_n is uniformly Cauchy.
  Let $\epsilon > 0$. By UEC, $\exists \delta 0, \forall x, y \in [a,b], |x-y| < \delta \implies |f_n(x) - f_n(y)| < \delta\ \forall n \in \mathbb{N}$.
  $\exists r_{i_1}, \dots, r_{i_l}$ rationals such that $\forall x \in [a,b], \min_{1 \leq j \leq l} |x - r_{i_j}| < \delta$.
  Since $f_n(r_{i_j})$ converges $\forall j$,

$$\exists N, p, q \geq N, |f_p(r_{i_j}) - f_q(r_{i_j})| < \frac{\epsilon}{3}$$

Now for any $x \in [a, b]$ and $p, q \geq N$,

$$|f_p(x) - f_q(x)| \leq \underbrace{|f_p(x) - f_p(r_{i_j})|}_{<\frac{\epsilon}{3} \text{ by UEC}} + \underbrace{|f_p(r_{i_j}) - f_q(r_{i_j})|}_{<\frac{\epsilon}{3}} + \underbrace{|f_q(r_{i_j}) - f_q(x)|}_{<\frac{\epsilon}{3} \text{ by UEC}}$$

We showed $\forall \epsilon > 0, \exists N, p, q \geq N \implies ||f_p - f_q||_\infty < \epsilon$. ■

Consequence

$\mathcal{F} \subseteq C([a, b])$ is compact if and only if it is closed, uniformly bounded and uniformly equicontinuous.

Interesting Functional Analytic Consequence

Consider $\left(C^1([a, b]), f \mapsto ||f||_\infty + ||f'||_\infty\right)$
One can show that it is complete.
Arzéla-Ascoli $\implies$ bounded sets in $C^1([a, b])$ are precompact (i.e. have compact closures) in $(C([a, b]), || \cdot ||_\infty)$.
We say the injection

$$C^1([a, b]) \hookrightarrow C([a, b])$$

is compact.

November 22, 2023

Topological spaces.

- Definitions
- Hausdorff
- Pathological Cases
- Base for a Topology

Definition:

Given X a set and $\tau \subset \mathcal{P}(X)$, we say that τ is a topology if

1. $\varnothing, X \in \tau$
2. τ is stable under finite intersection and arbitrary union.

- $O_1, \dots, O_n \in \tau \implies \bigcap_{j=1}^{n} O_j \in \tau$
- $\{O_\alpha\}_\alpha \in \tau \implies \bigcup_\alpha O_\alpha \in \tau$

We say (X, τ) is a topological space and elements of τ are called open sets.

Examples

1. (X, d) a metric space, then $\tau = \{$open sets defined by $\forall x \in O, \exists \delta > 0, B_{\delta(x)} \subset O$
2. $(X, \{\varnothing, X\})$
3. $(X, \mathcal{P}(X))$

Topological Definitions

Fix (X, τ) a topological space.

Definition: Open Neighborhood

An open neighborhood of $x \in X$ is an open set $U \ni x$.

Definition: Interior Point

$A \subseteq X$, then $a \in A$ is an interior point of A if $\exists U$ an open neighborhood of x such that $x \in U \subseteq A$.

Definition: Interior

The interior of $A \subseteq X$, denoted A°, is the set of all interior points.

Definition: Convergence

x_n converges to x if and only if for every neighborhood U of x, $\exists N, n \geq N \implies x_n \in U$.

Definition: Limit Point

x is a limit point of A if for every neighborhood U of x, $A \cap (U \setminus \{x\}) \neq \varnothing$.

Definition: Closed Set

A is closed if it contains all its limit points.

Definition: Closure

The closure of A, called $\overline{A}$, is $A \cup \{$limit poiints of $A\}$.

Proposition: Induced Topology

Given (X, τ) a topological space and $A \subseteq X$, then $\tau_A = \{U \cap A \mid U \in \tau\}$ is a topology on A called the induced topology.

Topological Propositions

Take (X, τ) a topological space (fixed) and $A \subseteq X$.

1. $A \in \tau$ if and only if every point in A is an interior point of A. (Then A deserves to be called open)

2. A is closed if and only if A^c is open

3. Arbitrary intersection and finite union of closed sets are closed.

4. A° is open and $A^\circ = \bigcup_{\substack{O \subset A \\ O \in \tau}} O$ (which implies A open if and only if $A = A^\circ$.

5. $\overline{A}$ is closed and $\overline{A} = \bigcap_{\substack{F \supset A \\ F^c \in \tau}} F$ (which implies A is closed if and only if $\overline{A} = A$)

Proof of 1

($\Longleftarrow$) Take A.

$$\begin{aligned} \forall x \in A, \exists U_x \in \tau, x \in U_x \subseteq A &\implies A = \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} U_x \subseteq A \\ &\implies A = \bigcup_{x \in A} U_x \in \tau \end{aligned}$$

($\implies$) $A \in \tau$, $x \in A$, take $U \in \tau$, containing x, then $A \cap U \in \tau, x \in A \cap U \subseteq A$.

Proof of 2

A open $\implies$ A^c closed.
If $x \notin A^c$, then $x \in A$.
Therefore $\exists U \in \tau, x \in U \subseteq A \implies U \cap A^c \varnothing \implies x$ not a limit point of A^c.
So A^c contains its limit points.
Converse left as an exercise.

Proof of 3 (Technique)

De Morgan's Laws.

Proof of 4

Take $x \in A^\circ : \exists U \in \tau, x \in U \subseteq A$.
Claim $U \subseteq A^\circ$. Why?
If $y \in V, V \in \tau$ so $\exists V \in \tau, y \in V \subseteq U \subseteq A \implies y \in A^\circ$.
Therefore A° is open.

$A^\circ = \bigcup_{\substack{O\subset A\\ O\in\tau}} O$:
($\subset$) $A^\circ \subseteq A$, so $A^\circ \subseteq \bigcup_{\substack{O\subset A\\ O\in\tau}} O$.
($\supset$) Claim: if $O \in \tau, O \subseteq A$, then $O \subseteq A^\circ$.
Pick $x \in O$, $\exists U \in \tau, x \in U \subseteq A \implies x \in A^\circ$.
Therefore $\bigcup_{\substack{O\subset A\\ O\in\tau}} O \subseteq A^\circ$.

Proof of 5 (Exercise)

$(\overline{A})^c = (A^c)^\circ$? Prove or disprove.

Definition: Continuous Maps

Fix (X,τ), (Y,σ) topological spaces.
$f : X \to Y$ is continuous if $\forall O \in \sigma$, $f^{-1}(O) \in \tau$.
Can also define continuity at $x \in X$:
$\forall V$ open neighborhood of $f(x)$, $\exists U$ open neighborhood of x such that $f(U) \subseteq V$.
IMAGE HERE - EPSILON DELTA NEIGHRBORHOODS ON R2

Proposition:

f is continuous on X if and only if it is continuous at every $x \in X$.

Proposition:

f continuous at x if and only if $\forall x_n$ such that $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} f(x_n) = f(x)$.

Definition: Homeomorphism

$f : (X,\tau) \to (Y,\sigma)$ is a homeomorphism if and only if f is continuous, bijective and $f^{-1} : (Y,\sigma) \to (X,\tau)$ is continuous.
A homeomorphism induces a one to one correspondence between τ and σ.

$$O \longrightarrow f(O) = (f^{-1})^{-1}(O)$$

i.e. $(X,\tau) \sim (Y,\sigma)$ if and only if $\exists f : X \to Y$ homemorphic is an equivalence relation.

Examples

1. $\tan : \left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right), |\cdot|\right) \to (\mathbb{R}, |\cdot|)$ is a homemorphism.

2. X, d_1, d_2 two metrics such that $\exists C_1, C_2 > 0$, $\forall x, y \in X, C_1 d_2(x,y) \le d_1(x,y) \le C_2 d_2(x,y)$, then

$$\mathrm{id} : (X, \tau(d_1)) \to (X, \tau(d_2))$$

   is a homemorphism.

Topological Connectedness

Definition: Connected

(X, τ) is connected if $X = A \cup B$, $A \cap B = \varnothing$, and $A, B \in \tau$ implies $A = \varnothing$ or $B = \varnothing$.
Equivalently, the only two subsets of X that are open and closed are $\varnothing$ and X.

Definition: Path Connected

(X, τ) is path-connected under the same definition as before.

Definition: Connected Subspace

A is connected if and only if (A, τ_A) is connected where τ_A is the induced topology).

Proposition:

Let $f : (X, \tau) \to (Y, \sigma)$ be continuous.

1. If X is connected, so is $f(x)$.

2. If X is path-connected, so is $f(x)$.

Topological Compactness

Keep the Heine-Borel definition only.

Definition: Compact

A set $A \subseteq X$ is (HB)-compact if $\forall$ open covers $\mathcal{O}$ of A, a finite subcollection of $\mathcal{O}$ still covers A.

Proposition:

If $K \subset X$ is compact and $F \subseteq K$ is closed, then F is compact.

- Proof
  Take $\{O_\alpha\}_\alpha$ open cover of F. Then $\{O_\alpha\}_\alpha$, F^c covers K.
  By HBP, a finite subcover (e.g. $\{O_\alpha, \ldots, O_{\alpha_n}, F^c\}$) covers K, in particular F.
  And since $F \cap F^c = \varnothing$, $O_\alpha, \ldots, O_{\alpha_n}$ covers F. $\blacksquare$

Proposition:

If $f : X \to Y$ continuous and K compact in X, then $f(K)$ comapct.

Definition: Hausdorff Property

We say that a topological space (X, τ) is Hausdorff if and only if "it separates points":

$$\forall x, y \in X, x \neq y, \exists U_x \ni X, \exists U_Y \ni Y, U_X, U_Y \in \tau, U_X \cap U_Y = \varnothing$$

IMAGE HERE - HAUSDORFF SEPARABILITY DRAWING

Example 1

Any $(X, \tau(d))$ induced by a metric d is Hausdorff.

- Proof
  IF $x \neq y$, let $\delta = d(x, y) > 0$, take $U_X = B_{\delta/3}(x)$ and $U_Y = B_{\delta/3}(y)$.
  IMAGE HERE – POINTS X Y WITH DELTA/3 BALLS

Non-example 2

$(X = \{0, 1\}, \tau = \{\varnothing, X\})$ not Hausdorff.

Example 3

$(X, \mathcal{P}(X))$ is Hausdorff.

Example 4

If X is not Hausdorff, the singleton $\{x\}$ need not be closed.
In Non-example 2, $\{0\}$ is neither open nor closed.

Theorem:

If (X, τ) is Hausdorff, then for $K \subseteq X$ compact and $x \notin K$, $\exists U, V$ open sets such that $U \cap V = \varnothing$, $x \in U$, $K \subset V$.

Proof

$\forall y \in K$, $\exists U_y, V_y \in \tau, x \in U_y, y \in V_y, U_y \cap V_y = \varnothing$.
Then $\{V_y \mid y \in K\}$ is an open cover of K, so $\exists y_1, \ldots y_n$ such that K is covered by $V_{y_1}, \ldots V_{y_n}$.
$\tau \ni \bigcup_{j=1}^n V_{y_j} \supseteq K$ and $U = \bigcap_{j=1}^n V_{y_j} \in \tau$, contains x.

$$\forall j, U \cap V_{y_j} = \bigcap_{p=1}^{n} U_{y_p} \cap V_{y_j} \subseteq U_{y_j} \cap V_{y_j} = \varnothing$$

Therefore $U \cap \bigcup_{j=1}^n V_{y_j} = \varnothing$,

Corollary

Let (X, τ) be Hausdorff.

1. $K \subseteq X$ compact implies K closed.

2. K compact and F closed implies $F \cap K$ compact (i.e. K^c contains no limit points of K).

Theorem:

Let (X, τ) be Hausdroff.
Suppose that $\{K_\alpha\}_\alpha$ is a collection of compact sets in X such that $\bigcap_\alpha K_\alpha = \varnothing$.
Then $\exists \alpha_1, \ldots, \alpha_n$, $\bigcap_{j=1}^n K_{\alpha_j} = \varnothing$.

Proof

Single out K_{α_1}.
Then $K_{\alpha_1} \cap \bigcap_{\alpha \neq \alpha_1} K_\alpha = \varnothing$ i.e. $K_{\alpha_1} \subset \left(\bigcap_{\alpha \neq \alpha_1} K_\alpha\right)^c = \bigcup_{\alpha \neq \alpha_1} K_\alpha^c$.
Therefore $\{K_\alpha^c\}_{\alpha \neq \alpha_1}$ is an open cover of K_α.
Take a finite subcover $K_{\alpha_2}, \ldots, K_{\alpha_n}$, then $K_{\alpha_1} \subseteq \bigcup_{j=2}^n K_{\alpha_j}^c = \left(\bigcap_{j=1}^n K_{\alpha_j}\right)^c$.
Therefore $\bigcap_{j=1}^n K_{\alpha_j} K_{\alpha_j} = \varnothing$. ■

Some Heuristic Statements

One usually seeks a topology that is not to small (coarse) and not too large (fine).
On X, given τ_1 and τ_2, we say τ_1 is finer (contains more open sets) than t_2 if $\tau_2 \subseteq \tau_1$.
The finest of all is $\mathcal{P}(X)$.

- Hausdorff.

- Completely disconnected (every set is open and closed).

- Very few compact sets (only the finite sets).

The coarsets of all is $\{\varnothing, X\}$.

- Very few open sets.

- Not Hausdorff (as soon as X has more than one element)

- Compact

- Connected

In general, due to Heine-Borel, more open sets means fewer compact sets and vice versa.
IMAGE HERE - PRODUCT AND QUOTIENT TOPOLOGIES

November 27, 2023

Quotient Topology
Base for a topology.
Topology on finite products.

Definition: Equivalence Relation on X

Let X be a set and $\sim$ an equivalence relation on X:

- $\sim \in \mathcal{P}(X \times X)$.
- Reflexive: $\forall x \in X, x \sim x$.
- Symmetric: $\forall x \in X, y \in X, x \sim y \implies y \sim x$
- Transitive $\forall x, y, z \in X, x \sim y \wedge y \sim z \implies x \sim z$

Definition: Equivalence Class on X

$[x] := \{y \in x \mid y \sim x\}$

Lemma

Two equivalence classes are either disjoint or equal.

Definition: Quotient Space

The equivalence relation induces a partitoning of X into equivalence classes.
Define $X/\sim = \{[x] \mid x \in X\}$ (the quotient space), then there exists a natural projection map $\pi : \underset{x \mapsto [x]}{X \to X/\sim}$.

Question:

If X carries a topology, τ, can we induce one on $X/\sim$?
Answer: Yes. We say U is open (i.e. $U \in \tau_\sim$) in $X/\sim$ if and only if $\pi^{-1}(U)$ is open in X.

- Claim:
  $\tau_\sim$ is a topology.
  - Proof
    $\pi^{-1}(\varnothing) = \varnothing \in \tau$ so $\varnothing \in \tau_\sim$.
    $\pi^{-1}(X/\sim) = X \in \tau$ so $X/\sim \in \tau_\sim$.
    Stability under finite intersection and arbitrary union:
    $\pi^{-1}\left(\bigcup_\alpha U\alpha\right) = \bigcup_\alpha \pi^{-1}(U_\alpha)$.
    $\pi^{-1}\left(\bigcap_{n=1}^k V_n\right) = \bigcap_{n=1}^k \pi^{-1}(V_n)$.

- Claims:
  $\tau_\sim$ makes π continuous.
  $\tau_\sim$ is the finest topology making π continuous.

Obvious Corollaries

If X is compact/connected/path-connected, so is $X/\sim$.

- Proof:

  \ (X/∼=π(x) \), π continuous.

IMAGE HERE - COMMUTATIVE DIAGRAM

$$\begin{tikzcd}
X \arrow[d, "\pi"'] \arrow[dr, "f \circ \pi"] \arrow[dr, bend left, "h"] & \\
X/\sim \arrow[r, "f"'] & Y
\end{tikzcd}$$

Proposition

Let $X, \tau, \sim$ be as above.
Let $f : X/\sim \to (Y, \tau_Y)$.
Then f is continuous if and only if $f \circ \pi$ is continuous.

Proof

($\Longrightarrow$) Obvious (composition of continuous maps).
($\Longleftarrow$) Suppose $f \circ \pi$ is continuous.
Take $U \in \tau_Y$, then $(f \circ \pi)^{-1}(U) = \pi^{-1}(f^{-1}(U))$ is open.
Then $f^{-1}(U)$ is open. Therefore f is continuous.

Proposition:

Let $X, \sim \tau, (Y, \tau_Y)$ as above and $h : X \to Y$.
Then $\exists f : X/\sim \to Y$ such that $h = f \circ \pi$ if and only if $\forall [a] \in X/\sim, \forall x \in [a], h(x) = h(a)$.
Moreover (by previous propositions), f continuous if and only if h is continuous.

Proof:

($\Longrightarrow$) If $h = f \circ \pi, [a] \in X/\sim, x \in [a]$

$$h(x) = f(\pi(x)) = f([x]) = f([a]) = f(\pi(a)) = h(a)$$

($\Longleftarrow$) If $\forall [a] \in X/\sim, \forall x \in [a], h(x) = h(a) (*)$, define $f([a]) := h(a)$ ro h(any representative of $[a]$).
f is well defined thanks to $(*)$.

Note:

Hausdorff property can be lost in a quotient construction.
IMAGE HERE - ON TWO UNIT INTERVALS SEND ALL POSITIVE VALUES BUT NOT ZERO

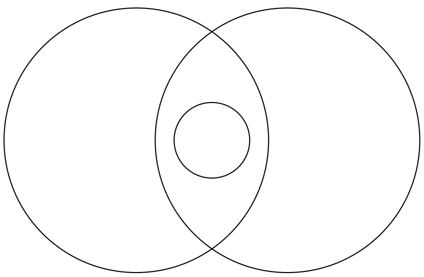
Definition: Base for a Topology

Take X a set.
Suppose $\sigma \subset \mathcal{P}(X)$ satisfies

1. σ covers X (i.e. $X \subseteq \bigcup_{A \in \sigma} A$)

2. $\forall A, B \in \sigma, x \in A \cap B, \exists C \in \sigma$ such that $x \in C \subset A \cap B$.

IMAGE HERE - VENN DIAGRAM WITH A, B, INTERSECT SIGMA AND C IN INTERSECT



Then $\tau := \{textarbitraryunionsofelementin\sigma\} \cup \{\varnothing\} = \mathcal{T}(\sigma)$ is a topology.
$\mathcal{T} : \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(\mathcal{P}(X))$ if τ is a topology, $\mathcal{T}(\tau) = \tau$.
σ is called a base for τ.

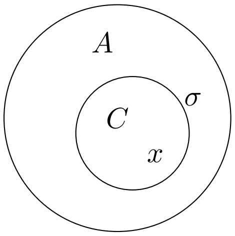
Proof

τ is stable under arbitrary unions.
τ is stable under intersection: let $A, B \in \sigma$.
Then $A \cap B = \bigcup_{x \in A \cap B} C_x$ where C_x is given by (2), so $A \cap B \in \tau$.
If $A, B \in \tau$, $A = \bigcup_\alpha A_\alpha \in \sigma$ and $B = \bigcup_\beta B_\beta \in \sigma$, so

$$A \cap B = \left(\bigcup_\alpha A_\alpha\right) \cap \left(\bigcup_\beta B_\beta\right) = \underbrace{\bigcup_{\alpha,\beta} \underbrace{(A_\alpha \cap B_\beta)}_{\in \tau}}_{\in \tau}$$

Proposition: Criterion for a Basis

(X, τ) topological space and $\sigma \subset \tau$.
Then σ is a base for τ if and only if $\forall A \in \tau, \forall x \in A, \exists C \in \sigma, x \in C \subset A$.



Proof

($\Longrightarrow$) Let $A \in \tau$, then $A = \bigcup_\alpha C_\alpha, C_\alpha \in \sigma$. If $x \in A$, then $x \in C_{\alpha_0}$ for some α_0.
($\Longleftarrow$) Let $A \in \tau$,

$$A = \bigcup_{x \in A} C_x$$

Where $C_x \in \sigma$ comes from the hypothesis.

Proposition:

Let $\sigma_1, \sigma_2 \in \mathcal{P}(\mathcal{P}(X))$, then $\mathcal{T}(\sigma_1) \subset \mathcal{T}(\sigma_2)$ if and only if $\forall A \in \sigma_1, \forall x \in A, \exists B \in \sigma_2, x \in B \subset A$.
This helps give a criterion for when $\mathcal{T}(\sigma_1) = \mathcal{T}(\sigma_2)$.

Example

A base for the standard topology on $\mathbb{R}$ is

$$\sigma = \left\{ \left(r - \frac{1}{n}, r + \frac{1}{n} \right) \mid r \in \mathbb{Q}, n \in \mathbb{N} \right\}$$

Note: it's countable.

Definition: Dense Space

Take (X, τ) a topological space. We say $A \subset X$ is dense in X if $\forall x \in X, \forall U$ open neighborhood of x, $U \cap A \neq \varnothing$.

Definition: Separable Space

Take (X, τ) a topological space. We say X is separable if it has a countable dense subset.

Proposition:

Given (X, τ) a topological space, if τ has a countable base then

1. X is separable.

2. Any cover of X has a countable subcover.

Proof of 1

Call $\sigma = \{C_k \mid k \in \mathbb{N}\}$ a countable base.
$\forall k$, let $x_k \in C_k$.
Claim: $A := \{x_k\}_k$ is dense in X.
Indeed, if $x \in X$, U open neighborhood of x, $U \in \tau$, $U = \bigcup_{j \in \mathbb{N}} C_{k_j}$ then $x_{k_j} \in U$ for all j.

Proof of 2

Write $\mathcal{O} = \{O_\alpha\}_\alpha$ a cover of X.
Construct $\mathcal{O}'$ as follows:
$\forall k \in \mathbb{N}$, if $C_k \subset O_\alpha$ for some α_k, adjoin $O\alpha_k$ to $\mathcal{O}'$.
That is at most countably many.
Let $x \in X$.
Since $\mathcal{O}$ covers X, $\exists O_\alpha$ such that $x \in O_\alpha$.

$$O_\alpha \in \tau \implies O_\alpha = \bigcup_{j \in \mathbb{N}} C^\alpha_{k_j} \implies \exists j, x \in C^\alpha_{k_j}$$

Then either $O_\alpha \in \mathcal{O}'$ or another $x \in O_{\alpha_k} \in \mathcal{O}'$.
Either way, $\exists V \in \mathcal{O}', x \in V$.

Topology of Finite Products

Setup: (X, τ_X) and (Y, τ_Y), $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$.
Two natural projections π_X and π_Y.

$$\begin{tikzcd}
 & & X \times Y \arrow[dddll] \arrow[dddrr] & & \\
 & & (x,y) \arrow[ddl, mapsto, "\pi_X"'] \arrow[ddr, mapsto, "\pi_Y"] & & \\
 & & & & \\
X & x & & y & Y
\end{tikzcd}$$

How to put a topology on $X \times Y$?
It should make π_X, π_Y continuous.
That is, $\forall U \in \tau_X$, $\pi_X^{-1}(U) = U \times Y \in \tau$ and $\forall V \in \tau_Y$, $\pi_Y^{-1}(V) = X \times V \in \tau$.
By stability under intersection, τ should contain

$$(U \times Y) \cap (X \times V) = (U \cap X) \times (Y \cap V) = U \times V$$

Now define $\sigma := \{U \times V \mid U \in \tau_X, V \in \tau_Y\}$. σ satisfies

1. It covers $X \times Y$, since $X \times Y \in \sigma$.

2. $A, B \in \sigma \implies A \cap B \in \sigma$.Indeed,

$$U_1 \times V_1 \cap U_2 \times V_2 = \underbrace{(U_1 \cap U_2)}_{\in \tau_X} \times \underbrace{(V_1 \cap V_2)}_{\in \tau_Y}$$

Definition: Product Topology

The product topology on $X \times Y$ is $\mathcal{T}_{X \times Y}$.
It is the coarsest topology making π_X, π_Y continuous (i.e. if τ is a topology on $X \times Y$ such that π_X, π_Y continuous, then $\tau \supset \mathcal{T}(\sigma)$.

Proposition:

Given (Z, τ_Z) and $f : Z \to (X \times Y, \mathcal{T}_{x \times Y})$, f is continuous if and only if $\pi_X \circ f$ and $\pi_Y \circ f$ are continuous.

Proposition

($\Longrightarrow$) Clear: composition of continuous functions.
($\Longleftarrow$) Take $U \in \mathcal{T}_{X \times Y}$, write $U = \bigcup_\alpha U_\alpha \times V_\alpha$, then

$$\begin{aligned} f^{-1}(U) &= f^{-1}\left(\bigcup_\alpha U_\alpha \times V_\alpha\right) \\ &= \bigcup_\alpha f^{-1}(U_\alpha \times V_\alpha) \\ &= \bigcup_\alpha \underbrace{(\pi_X \circ f)(U_\alpha)}_{\in \tau_Z} \cap \underbrace{(\pi_Y \circ f)(V_\alpha)}_{\in \tau_Z} \end{aligned}$$

where

$$(U_\alpha \times V_\alpha) = \{z \in Z \mid f(z) \in U_\alpha \times V_\alpha\} = \{z \in Z \mid \pi_X \circ f(z) \in U_\alpha, \pi_Y \circ f(z) \in V_\alpha\}$$

Proposition:

Let X, Y be two topological spaces and $X \times Y$ with the product topology.

1. If X, Y are Hausdorff, so is $X \times Y$.
2. If X, Y are connected, so is $X \times Y$.
3. If X, Y are compact, so is $X \times Y$.

Proof of 1

Pick $(x_1, y_1), (x_2, y_2) \in X \times Y$.
If $x_1 \neq x_2$, then $\exists U_1, U_2 \in \tau_X, U_1 \cap U_2 = \varnothing, x_1 \in U_1, x_2 \in U_2$.
Then $U_1 \times Y$ and $U_2 \times Y$ separate (x_1, y_1) and (x_2, y_2).
IMAGE HERE - X Y PLANE WITH VERTICAL LINES DEMONSTRATING
If $x_1 = x_2$, then $y_1 \neq y_2$ by a similar construction.

Proof of 2

Suppose $A \neq \varnothing$ both open and closed in $X \times Y$.
What to show: $A = X \times Y$.
Will follow from: $(x, y) \in A \implies \{x\} \times Y \subset A$ and $X \times \{y\} \in A$.
Claim: $\{x\} \times Y \cap A$ is both open and closed in $\{x\} \times Y$.
Since $\{x\} \times Y$ is connected, then $\{x\} \times Y \cap A = \{x\} \times Y$.
Therefore $\{x\} \times Y \subset A$, $\forall x \in A$.
Similarly, $X \times \{y\} \subset A$, $\forall y \in A$.

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Compcatness of products.

- Finite products
- Zorn's lemma
- Arbitrary products (Tychonoff's Theorem)
- Mapping spaces

Theorem: Compactness of Product

Let (X, τ_X), (Y, τ_Y) be topological spaces, $\sigma = \{U \times V \mid U \in \tau_X, V \in \tau_Y\}$ and on $X \times Y$ equip the topology $\mathcal{T}(\sigma) = \tau_{X \times Y}$.
If X, Y are compact, then so is $X \times Y$.

Proof

Take $(C_\alpha)_{\alpha \in A} = \mathcal{C}$ a cover of $X \times Y$.
Since $\tau_{X \times Y}$ is generated by σ, each $C_\alpha = \bigcup_\beta U_{\alpha,\beta} \times V_{\alpha,\beta}$ where $U_{\alpha,\beta} \in \tau_X$ and $V_{\alpha,\beta} \in \tau_Y$.
Then $X \times Y$ is covered by $\mathcal{C}' = \{U_{\alpha,\beta} \times V_{\alpha,\beta} \mid \alpha, \beta\}$.
Extract a finite subset of $\mathcal{C}' \underset{rewrite}{=} \{U_\alpha \times V_\alpha \mid \alpha \in A\}$.
Then, $\forall y \in Y$, $X \times \{y\}$ is compact. So $X \times \{y\}$ is coverd by $\{U^y_{\alpha_j} \times V^y_{\alpha_j} \mid i \le j \le k_y\}$.
Then $V_y = \bigcap_{j=1}^{k_y} V^y_{\alpha_j}$ is an open neighborhood of y. Then $\{V_y\}_{y \in Y}$ is an open cover of Y which is compact.
Therefore, Y is coverd by $V_{y_1}, \dots, V_{y_p}$ for some $p \in \mathbb{N}$, $y_1, \dots, y_p \in Y$.
Check that: $X \times Y$ is coverd by $\{U^{y_j}_{\alpha_i} \times V^{y_j}_{\alpha_i} \mid 1 \le j \le p, 1 \le i \le k_{y_j}\}$.
Take $(x, y) \in X \times Y$, $y \in V_{\alpha_{j_0}}$ for some j_0 (i.e. $y \in V_{\alpha_i}, \forall i \in 1, 2, \dots, k_{y_{j0}}$).
Since $X \subseteq \bigcup_{i=1}^{k_{y_{j0}}} U^{y_{j0}}_{\alpha_i}$, $\exists i_0$ such that $(x, y) \in U^{y_{j0}}_{\alpha_{i_0}} \times V^{y_{j0}}_{\alpha_{i_0}}$.

Recall: Axiom of Choice

Given an arbitrary collection of nonempty sets $\{S_\alpha\}_{\alpha \in A}$, there exists a function defined on A, f, such that $\forall \alpha \in A, f(\alpha) \in S_\alpha$.

Definition: Partially Ordered Set

Take $X \neq \emptyset$ a set. A relation "$\le$" on X is a "partial order" if

1. $x \le x, \forall x \in X$.
2. $x \le y$ and $y \le z$ implies $x \le z, \forall x, y, z \in X$.
3. $x \le y$ and $y \le x$ implies $x = y, \forall x, y \in X$.

It is a "total order" if additionally

1. $\forall x, y \in X$ either $x \le y$ or $y \le x$.

$(X, \leq)$ is a partially ordered set or "poset".

Example

$(\mathbb{R}, \leq)$ is totally ordered.
$X \neq \varnothing$, consider $\mathcal{P}(X), \subseteq)$.
e.g. $X = \{0,1\}$ and $\mathcal{P}(X) = \{\varnothing, \{0\}, \{1\}, \{0,1\}\}$.

Definition: Totally Ordered Subset

If $(X, \leq)$ is a poset, we say $A \subset X$ is a totally ordered subset if $\leq|_{A\times A}$ is a total order.
IMAGE HERE - ILLUSTRATION OF INCLUSION AND A= \0, 1, 0,1 A TOTALY ORDERED SUBSET
Given $(X, \leq)$ a poset, $A \subset X$, we say

- A has an upper bound if $\exists m \in X$, $a \leq m$, $\forall a \in A$.
- X has a maximum element if $\exists m \in X$, $\forall x \in X$, $m \leq x \implies m = x$.

a maximum element need not be unique.

Lemma: Zorn's Lemma

Let $(X, \leq)$ be a poset.
If every totally ordered subset of X has an upperbound, then X has a maximum element.

Example

Zorn's lemma true for $(X, \mathcal{P}(X))$: a totally ordered subset of X might look like $A_1 \subseteq A_2 \subseteq \cdots \subseteq X$.
Write $\mathcal{A} = \{A_\alpha \mid \alpha \in J\}$.
Guess for upperbound: $m_{\mathcal{A}} = \bigcup_{\alpha\in J} A_\alpha$.

Infinite Product

Take A an index set. For $\alpha \in A$, (X_α, τ_α) is a topological space.
Define $X = \prod_{\alpha\in A} X_\alpha$ the set of maps $x : A \to \bigcup_\alpha X_\alpha$ such that $\forall \alpha \in A, x(\alpha) \in X_\alpha$.
Topology on $\prod_\alpha X_\alpha$?
$\forall \alpha \in A$, there is a natural projection map $\pi_\alpha : X \to X_\alpha$ ($\pi_\alpha : x \mapsto x(\alpha)$).
On X, we want τ making π_α continuous for every α.
Therefore τ should contain $\pi_\alpha^{-1}(U_\alpha)$ ("slabs"), $\forall U_\alpha \in \tau_\alpha$.
By stability under finite intersection, it should contain $\bigcap_{i=1}^n \pi_\alpha^{-1}(U_{\alpha_i})$, $\forall n \in \mathbb{N}$, $\alpha_1, \dots, \alpha_n \in A$, $U_{\alpha_j} \in \tau_{\alpha_j}$, $\forall j = 1, \dots, n$.
Then $\sigma = \{\bigcap_{i=1}^n \pi_{\alpha_i}^{-1}(U_{\alpha_i}) \mid n \in N, \alpha_1, \dots, \alpha_n \in A, U_{\alpha_j} \in \tau_{\alpha_j}, j = 1, \dots, n\}$ satisfies the axioms of a base.
Define $\tau_X : \mathcal{T}(\sigma)$, the coarsest topology on X making π_α continuous for every α.

Notation:

C is a subset of X (i.e. $\in \mathcal{P}(X)$)
$\mathcal{C}$ is a cover of X (i.e. $\in \mathcal{P}(\mathcal{P}(X))$)

$\mathbf{C}$ is a set of covers (i.e. $\in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$)

Theorem: Tychonoff's Theorem

Following from above: if X_α is compact $\forall \alpha \in A$, then X is compact.
An arbitrary product of compact spaces (equipped with the product topology) is compact.

Proof

- Step 1
  If $\mathcal{C}$ is an open cover of X made of slabs ($\pi_\alpha^{-1}(U_\alpha), \alpha \in A, U\alpha \in \tau_\alpha$), then it admits a finite subcover.
  Suppose not and write $\mathcal{C} = \coprod_{\alpha\in A} \mathcal{C}_\alpha$ where $\mathcal{C}_\alpha = \{O \in \mathcal{C} \mid O = \pi_\alpha^{-1}(U_\alpha)$ for some $U_\alpha \in \tau_\alpha\}$ and $\mathcal{U}_\alpha = \{U_\alpha \mid \pi_\alpha^{-1}(U_\alpha) \in \mathcal{C}_\alpha\}$.
  Note that $\mathcal{U}_\alpha$ covers X_α if and only if $\mathcal{C}_\alpha$ covers X.
  If $\mathcal{U}_\alpha$ covers X_α, since X_α is compact, then $\exists U_1^\alpha, \dots, U_n^\alpha$ covering X_α.
  Then $\{\pi_\alpha^{-1}(U_j^\alpha) \mid j = 1, \dots, n\}$ covers X, a contradiction.
  Then $\mathcal{U}_\alpha$ does not cover $X_\alpha, \forall \alpha \implies \exists x_\alpha$ not covered by $\mathcal{U}_\alpha$.
  Define $x \in X$ by $x(\alpha) = x_\alpha, \forall \alpha \in A$. Then X is not covered by $\mathcal{C}_\alpha, \forall \alpha \implies$ not covered by $\mathcal{C}$.

- Step 2
  Take $\mathcal{B}$ an arbitrary open cover of X. Suppose $\mathcal{B}$ has no finite subcover. We will construct a subcover made of slabs and appeal to step 1, making a contradiction.
  Let $\mathbf{P} = \{$open covers $\mathcal{A} \supset \mathcal{B}$ with no finite subcover$\}$, poset for set-inclusion ($\mathcal{B} \in \mathbf{P} \neq \varnothing$).
  Claim: by Zorn, $\mathbf{P}$ has a maximum element $\mathcal{O}$.
  To show this, prove: if $\mathbf{W}$ is totally ordered subset of $\mathbf{P}$, it has an upper bound.

  - Proof
    Let $\mathbf{W}$ as above, let $\mathcal{A}_\mathbf{W} := \bigcup_{\mathcal{A}\in\mathbf{W}} \mathcal{A}$.
    Crux to prove: $\mathcal{A}_\mathbf{W} \in \mathbf{P}$.
    By contradiction: if $\mathcal{A}_\mathbf{W}$ has a finite subcover, call it $\{A_1, \dots, A_n\}$, $\forall j \in \{1, \dots, n\}$, $\exists \mathcal{A}_j \in \mathbf{W}, A_j \in \mathcal{A}_j$.
    Since $\mathbf{W}$ is totally ordered, $\exists j_0, \forall j, A_j \in \mathcal{A}_j \subset \mathcal{A}_{j_0}$.
    But then $\{A_1, \dots, A_n\}$ is a finite subcover of $\mathcal{A}_{j_0}$, a contradiction.
    Note: "$\mathcal{O}$ max element in $\mathbf{P}$" means $\forall U \notin \mathcal{O}$, $\mathcal{O} \cup \{u\}$ has a finite subcover.

- Final Step
  Let $\mathcal{O}' = \{$slabs in $\mathcal{O}\}$.
  Claim: $\mathcal{O}'$ covers X ($\implies$ Step 1 $\implies$ Contradiction)
  Let $x \in X$, since $\mathcal{O}$ covers X, $\exists O \in \mathcal{O}$ containing x.
  Since the product topology is genreated by σ, $\exists n, \alpha_1, \dots, \alpha_n, U_{\alpha_j} \in \tau_{\alpha_j}$ such that $x \in \bigcap_{j=1}^n \pi_{\alpha_j}^{-1}(U_{\alpha_j})$.
  Suppose by contradiction that $\forall j = 1, \dots, n$, $\pi_{\alpha_j}^{-1}(U_{\alpha_j}) \notin \mathcal{O}$.
  Then, by maximality of $\mathcal{O}$, $\exists O_{j,1}, \dots, O_{j,k} \in \mathcal{O}$ such that X is covered by $O_{j,1}, \dots, O_{j,k} \cup \pi_{\alpha_j}^{-1}(U_{\alpha_j})$.
  $\forall j$ $X \setminus (O_{j,1}, \dots, O_{j,k}) \subset \pi_{\alpha_j}^{-1}(U_{\alpha_j})$, so $X \setminus (\bigcup_{j=1}^n \bigcup_{i=1}^{k_j} O_{j,i}) \subset \bigcap_{j=1}^n \pi_{\alpha_j}^{-1}(U_{\alpha_j})$.
  Therefore X is covered by $U = \bigcap_{j=1}^n \pi_{\alpha_j}^{-1}(U_{\alpha_j})$ and $\{O_{i,j} \mid j = 1, \dots, n, i = 1, \dots, k_j\}$, hence $O \supseteq U$.