# Analysis I

## October 2, 2023

### Lecture Notes

Class will not have dedicated lecture notes. Many are available already.
Undergraduate notes are available on Canvas.
Lecture 1 overview available on Canvas (lecture1.pdf).

### Tentative Office Hours

Mondays 2-3pm and Tuesday 1-2pm.

### Homework

Nominally due at beginning of class; ask for leeway if needed.
First week homework will be review of undergraduate proofs.
First homework due Wednesday, October 11.

### Notation

Natural Numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$
Non Negative Integers: $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
Integers: $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$
Rationals: $\mathbb{Q} = \left\{\frac{p}{q},\ p \in \mathbb{Z},\ q \in \mathbb{Z}\right\} = \mathbb{Z} \times \mathbb{N} / \infty$

- Equivalent representation of rationals: $(p_1, q_1) \sim (p_2, q_2)$ iff $p_1 q_2 = p_2 q_1$

Sequence of Rationals: $\{u_n\}_{n \in \mathbb{N}},\ u_n \in \mathbb{Q},\ \forall n.$

### Properties of the Rationals

$(\mathbb{Q}, +, \cdot)$ is a (ii) totally ordered (i) field satisfying the (iii) Archimedean property.

#### (i) Field

1. $+$ is associative: $(a + b) + c = a + (b + c)$

2. $+$ is commutative: $a + b = b + a$

3. $\cdot$ is associative and commutative.

4. $\exists 0 \in \mathbb{Q}$ such that $\forall a \in \mathbb{Q}$, $0 + a = a + 0$

5. $\exists 1 \in \mathbb{Q} \setminus \{0\}$ such that $\forall a \in \mathbb{Q}$, $1 \cdot a = a \cdot 1 = a$

6. $\forall a \in \mathbb{Q} \setminus \{0\}$ $\exists b \in \mathbb{Q}$, $a \cdot b = b \cdot a = 1$

    - $b = a^{-1} = \frac{1}{a}$

**(ii) Totally Ordered**

$\exists$ a set $\mathbb{Q}_+ \subseteq Q$ of "Positive Numbers" stable under $+$ and $\cdot$ such that $\forall A \in \mathbb{Q}$ either $a > 0$ ($a \in \mathbb{Q}_+$), $-a > 0$ (also $a < 0$) or $a = 0$.

- Ordering: $\forall a, b \in \mathbb{Q}$, $a < b$ if and only if $b - a > -0$.

- Trichotomy: $\forall a, b \in \mathbb{Q}$ either $a < b$, $a > b$, or $a = b$.

- $\max(a, b) = \begin{cases} a & \text{if } a > b \\ b & \text{otherwise} \end{cases}$.

- $|a| = \max(a, -a)$ (helps measure distance in $\mathbb{Q}$).

- $\text{dist}(a, b) := |b - a|$

- Triangle Inequality: $|u \pm v| \leq |u| + |v|$

- Observe also: $||u| - |v|| \leq |u \pm v|$. The triangle inequality may be used to prove this.

- Proof of Triangle Inequality $-|u| \leq u \leq |u|$ and $-|v| \leq v \leq |v|$, therefore $-|u| - |v| \leq u + v \leq |u| + |v|$. Therefore $u + v \leq |u| + |v|$ and $-(u + v) \leq |u| + |v|$ implies $|u + v| \leq |u| + |v|$.

**(iii) Archimedian Property:**

$\forall \epsilon > 0,\ \exists N,\ \forall n \geq N,\ \frac{1}{n} < \epsilon$.

## Bounded Sequence of Rationals

$\{u_n\}_{n \in \mathbb{N}}$ is bounded if $\exists m \in \mathbb{Q}_+$ such that $|u_n| \leq M,\ \forall n$.
$\{u_n\}_{n \in \mathbb{N}}$ converges to $a \in \mathbb{Q}$ ($\lim_{n \to \infty} u_n = a$) if $\forall \epsilon > 0, \exists N, \forall n \geq N, |u_n - a| < \epsilon$.

## Famous Limits

### Decaying Rational

1. $\lim_{n\to\infty} \frac{1}{n} = 0$

    - $\forall \epsilon \in \mathbb{Q}_+,\ \exists n \in \mathbb{N},\ 0 < \frac{1}{n} < \epsilon$

    - $\forall n \in \mathbb{N},\ \exists n \in \mathbb{N},\ n \geq N$

        - b. and c. are equivalent.

### Decaying Exponential Rational

$r \in \mathbb{Q},\ 0 < r < 1,\ \lim_{n\to\infty} r^n = 0.$

- Proof: Write $r = \frac{1}{1+k}$ for some $k > 0$. Then $r^n = \frac{1}{(1+k)^n} \overset{\text{Bernoulli}}{\leq} \frac{1}{1+nk}$.

### Geometric

1. $r \in \mathbb{Q},\ 0 < r < 1,\ u_n = 1 + r + \cdots r^n = \frac{1-r^{n+1}}{1-r} \to \frac{1}{1-r}$

## Features of Limits

### Limits are Unique

If the limit of a sequence exists, it is unique.

### Squeezing Lemma

If $\{a_n\}$, $\{b_n\}$ are such that $0 \leq a_n \leq b_n$, and $b_n \to 0$ as $n \to \infty$, then $a_n \to 0$.

### Limits Preserve Order

If $a_n \leq b_n$ $\forall n$ and $a_n$ and $b_n$ converge, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

### Limit Algebraic Rules

$\lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = \lim_{n\to\infty}(a_n + b_n)$ when $a_n$ and $b_n$ converge.
If $\lim_{n\to\infty} b_n \neq 0$, then $\frac{a_n}{b_n} \to \frac{\lim a_n}{\lim b_n}$.

## Peculiarity of the Rationals

$\mathbb{Q}$ lacks completeness.

### Examples

Consider $u_1 = 1$ and $u_{n+1} = \frac{1}{2}(u_n + \frac{2}{u_n})$.
Then $u_n \in \mathbb{Q},\ \forall n \in \mathbb{N}$.
It can further be proven, by induction, that $u_n \geq 1,\ \forall n$. $\left(u_{n+1} - 1 = \frac{1}{2}(u_n + \frac{1}{u_n}) - 1 = \frac{1}{2u_n}((u_n - 1)^2 + 1)\right)$.
$\lim_{n\to\infty} u_n^2 = 2$.

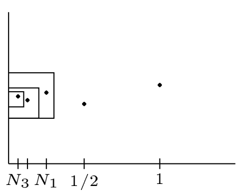
$$\begin{aligned} u_{n+1}^2 - 2 &= \left(\frac{1}{2}(u_n + \frac{2}{u_n})\right)^2 - 2 \\ &= \left(1\frac{1}{2u_n}(u_n^2 + 2)^2 - 4u_n\right) \\ &= 1\frac{4}{u_n^2}\left(u_n^2 - 2)^2\right) \\ &\leq \frac{1}{4}(u_n^2 - 2)^2 \end{aligned}$$

If $u_n$ converged in $\mathbb{Q}$ to $L$, by algebraic limit rules, $2 = \lim u_n^2 = (\lim u_n)^2 = L^2$, yet $\sqrt{2} \notin \mathbb{Q}$.

## Cauchy Criterion

A sequence $\{u_n\}_{n\in\mathbb{N}}$ of rationals is Cauchy if $\forall \epsilon > 0,\ \exists n \in \mathbb{N},\ \forall p, q \geq n,\ |u_p - u_q| < \epsilon$.

### Visual Justification




### Example 1

The sequence from before is Cauchy.
$|u_p - u_q| = \frac{|u_p^2 - u_q^2|}{|u_p + u_q|} \leq \frac{1}{2}|u_p^2 - u_q^2|$

### Example 2

$u_n = \sum_{k=0}^{n} \frac{1}{k!} \in \mathbb{Q}$.

- This is increasing.

- It is bounded above by 3:

$$\begin{aligned} 1 + 1 + \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots + \frac{1}{2\cdots n} &\leq 1 + 1 + \cdots \frac{1}{2^{n-1}} \\ &\leq 1 + \frac{1 - 2^{-n}}{1 - \frac{1}{2}} \\ &\leq 3 \end{aligned}$$

## Convergence, Cauchy and Boundedness.

Given a sequence $\{u_n\}_{n\in\mathbb{N}}$,
$\{u_n\}$ converges $\implies$ $\{u_n\}$ is Cauchy $\implies$ $\{u_n\}$ is bounded.
Note that in $\mathbb{Q}$ none of these implications may be reversed.

## Construction of the Real Numbers

Short version: If the limit of a sequence isn't in the set, then define the limit as the sequence itself.
Let $C_{\mathbb{Q}} = \{\text{Cauchy sequences of rationals.}\}$.

### Two Operations

- Termwise Addition $\{u_n\}_n + \{v_n\}_n := \{u_n + v_n\}_{n\in\mathbb{N}}$

- Termwise Multiplication $\{u_n\}_n \cdot \{v_n\}_n := \{u_n \cdot v_n\}_{n\in\mathbb{N}}$

### Closure of Cauchy Sequence

If $\{u_n\}_n,\ \{v_n\}_n \in C_{\mathbb{Q}}$, then $\{u_n\}_n + \{v_n\}_n \in C_n$ and $\{u_n\}_n \cdot \{v_n\}_n \in C_n$.

### Example

Infinite decimal expansion.
Fix $N \in \mathbb{Z},\ a_1 \cdots a_n \in \{0, \dots, 9\}$.
Then let $u_n = N + \sum_{k=1}^{n} a_k (10)^{-k}$ (that is the number $N.a_1 a_2 \dots a_n$).
This is always increasing and bounded above by $N + \sum_{k=1}^{n} 9 \cdot (10)^{-k} = N + \frac{9}{10} \cdot \sum_{k=1}^{n} (10)^{-(k+1)} \leq N + 1$.
Hence, it is Cauchy.

## Increasing and Bounded Above Implies Cauchy

By contrapositive, increasing and not Cauchy implies not bounded.
By the negation of Cauchy and letting $p \geq q$ without loss of generality, we can force $u_p > u_q + \epsilon$.

### Negation of Cauchy

$\exists \epsilon > 0,\ \forall N,\ \exists p, q \geq N,\ |u_p - u_q| > \epsilon$.

## Real Numbers as Equivalence Classes of Cauchy Sequences

On $C_{\mathbb{Q}}$ define the relation $\{x_n\}_n \sim \{y_n\}_n$ if and only if $\lim_{n\to\infty} |(x_n - y_n)| = 0$.

### Equivalence Relation

Reflexive: $x_n - x_n = 0$
Transitive: Uses algebraic limit rules. $x_n - z_n = x_n - y_n + y_n - z_n$.
Symmetric.

### Definition of the Reals

$\mathbb{R} := C_{\mathbb{Q}}/\sim$
Then $x \in \mathbb{R},\ x = [\{x_n\}_n]$.

### Addition and Multiplication of Reals

- Addition $x + y := [\{x_n + y_n\}_n]$.

- Multiplication $x \cdot y := [\{x_n \cdot y_n\}_n]$.

### Operations Do Not Depend on Choice of Representative

If $\{x_n\}_n \sim \{x_n'\}_n$ and $\{y_n\}_n \sim \{y_n'\}_n$, then $\{x_n\}_n + \{y_n\}_n \sim \{x_n'\}_n + \{y_n'\}_n$.
If $\{x_n\}_n \sim \{x_n'\}_n$ and $\{y_n\}_n \sim \{y_n'\}_n$, then $\{x_n\}_n \cdot \{y_n\}_n \sim \{x_n'\}_n \cdot \{y_n'\}_n$.

## The Reals are a Field

There are nine properties to check, eight of which are "obvious":

### Commutativity of Addition (and Other "Obvious" Features)

$[\{x_n\}_n] + [\{y_n\}_n] = [\{x_n + y_n\}_n] = [\{y_n + x_n\}] = [\{y_n\}_n] + [\{x_n\}_n]$
That is, the Reals inherit most field features from the Rationals.

- Zero Element $0_{\mathbb{R}} = [\{0_{\mathbb{Q}}\}_n]$

- One Element $1_{\mathbb{R}} = [\{1_{\mathbb{Q}}\}_n]$

### Multiplicative Inverses

How to define $x^{-1}$ for $x \in \mathbb{R}$ where $x \neq 0$?

- Idea If $x = [\{x_n\}_n]$ choose $x^{-1} = [\{\frac{1}{x}\}_n]$.
  If $x \in \mathbb{R},\ x \neq 0$ then

  1. $\exists \{x_n\}_n \in C_{\mathbb{Q}}$ representing $x$ with non zero entries.

  2. $\{\frac{1}{x_n}\}_n$ is Cauchy.

  - Proof of 1 Pick any $\{x_n\}_n$ representing $x$.

    * $x \neq 0$, so NOT ($\lim_{n\to\infty} x_n = 0$: $\exists \epsilon_0 > 0,\ \forall N,\ \exists n \geq N,\ |x_n| > \epsilon_0$.

    * $\{x_n\}$ is Cauchy: $\forall \epsilon > 0, \exists N,\ \forall p, q \geq N,\ |x_p - x_q| < \epsilon$.

Therefore, $\exists N$ such that $\forall p, q \geq N_1,\ |x_p - x_q| < \frac{\epsilon_0}{2}$
And $\exists N_2 \geq N,\ , |x_{N_2} > \epsilon_0$.
For $q \geq N_2$, the Cauchy Criterion states that $|x_q| = |x_q - x_{N_2} + x_{N_2} \geq |x_{N_2}| - |x_{N_2} - x_q| \geq \epsilon_0 - \frac{\epsilon_0}{2} \geq \frac{\epsilon_0}{2}$.
Therefore, the sought sequence is $\{x_{N_2} + k\}_{k \in \mathbb{N}}$.

– Proof of 2 $\left|\frac{1}{x_p} - \frac{1}{x_q}\right| = \frac{|x_p - x_q|}{|x_p||x_q|} \leq \frac{4}{\epsilon_0^2}|x_p - x_q|$.

### Order on the Reals

Let $x \neq 0$, $\exists \{x_n\}_{n \in \mathbb{N}}$ be a representation of $x$ and $\epsilon_0 > 0$.
Then for $|x_n| > \epsilon_0,\ \forall n \in \mathbb{N}$, there is a dichotomy:

- Either $\exists N \in \mathbb{N},\ x_n > \epsilon_0,\ \forall n \geq N$ (in which case we write $x > 0$)
- Or $\exists N \in \mathbb{N},\ x_n < -\epsilon_0,\ \forall n \geq N$ (in which case we write $x < 0$

Thus the Reals are totally ordered.

## October 4, 2023

### Overview

Completeness of $\mathbb{R}$.
Topology of the Real Line.

### Non-zero Reals Are Either Positive or Negative

Given $x \in \mathbb{R} \setminus \{0\},\ \exists \delta \in \mathbb{Q}_+$ such that $\forall \{x_n\}_n$ representing $x$, $\exists N \in \mathbb{N}$ such that $|x_n| > \delta, \forall n \geq N$.
Moreover, one of the following (but not both) holds:

1. $\forall \{x_n\}_n \in x,\ \exists,\ x_n > \delta,\ \forall n \geq N$ (i.e. $x > 0$)
2. $\forall \{x_n\}_n \in x,\ \exists,\ x_n < -\delta,\ \forall n \geq N$ (i.e. $x < 0$)

Recall that $x \in \mathbb{R} \setminus \{0\}$ is an equivalence class of Cauchy sequences.

### Total Ordering of the Reals

$x > 0$ produces a total ordering of $\mathbb{R}$ where $x < y$ if and only if $y - x > 0$.
$\rightsquigarrow \max(x, y) = \begin{cases} x & \text{if } x > y \\ y & \text{otherwise} \end{cases}$
$|x| = \max(x, -x)$ (which satisfies the triangle inequality)

### Lemma A

Let $x, y \in \mathbb{R}$. If $\{x_n\}_n, \{y_n\}_n$ represent $x, y$ and satisfy $x_n < y_n,\ \exists N \in \mathbb{N},\ \forall n \geq N$, then $x \leq y$.

- Proof By contradiction, suppose $x > y$ and $\exists \{x_n\}_n, \{y_n\}_n$ representing $x, y$ such that $x_n \leq y_n,\ \forall n \geq N_1$.
  Then, by definition, $x - y > 0 \implies \exists \delta > 0,\ \exists N_2,\ x_n - y_n > \delta$ for $n \geq N_2$.
  But $x_n \leq y_n$ contradicts $x_n - y_n > \delta$.

## Sequences of Reals

$\{x_n\}_n,\ x_n \in \mathbb{R}$
The definition of bounded, convergent and Cauchy sequences are the same as in $\mathbb{Q}$.

## Injection of Rationals

$\iota : \mathbb{Q} \to \mathbb{R}$ such that $r \mapsto [\{u_n = r\}_n]$
This is isometric in the sense that $|\iota(r) - \iota(s)|_{\mathbb{R}} = |r - s|_{\mathbb{Q}}$

## Theorem (Completeness 1)

Let $\{x_n\}_n \in C_{\mathbb{Q}}$ and $x = [\{x_n\}_n]$, then $\{\iota(x_n)\}_n$ converges to $x$.

### Proof

What to show: $\forall \epsilon > 0,\ \exists N,\ \forall n \geq N,\ |\iota(x_n) - x| < \epsilon$.
Let $\epsilon \in \mathbb{Q}_+$. By the Cauchy criterion, $\exists N, \forall q, p \geq N,\ |x_p - x_q| < \epsilon$.
This is equivalent to $x_q - \epsilon \leq x_p \leq x_q + \epsilon$ where $p$ is frozen.
Then by Lemma A, $x - \epsilon \leq \iota(x_p) \leq x + \epsilon$.
It follows that $\forall p \geq N,\ |\iota(x_p) - x \leq \epsilon$.

### Corollary

$\mathbb{Q} \cong \iota(\mathbb{Q})$ is dense in $\mathbb{R}$. That is, $\forall \epsilon > 0,\ \forall x \in \mathbb{R},\ \exists r \in \mathbb{Q},\ |\iota(r) - x| < \epsilon$.

## The Isometric Copy of Rationals

For brevity, the $\iota$ notation will be dropped and the $\mathbb{Q}$ will be understood as $\iota(\mathbb{Q})$.

## Completeness of the Real Numbers

A sequence of real numbers converges in $\mathbb{R}$ if and only if it is Cauchy.

### Proof

($\Longrightarrow$) This is clear.
($\Longleftarrow$) Take a Cauchy sequence of reals $\{x_n\}_n$. Then $\forall \epsilon > 0,\ \exists N,\ \forall p, q \geq\ \ |x_p - x_q| < \epsilon$.
Using the density of $\mathbb{Q}$, $\forall n \in \mathbb{N},\ \exists r_n \in \mathbb{Q}$ such that $|x_n - r_n| < \frac{1}{n}$.

Claim: $\{r_n\}_n$ is Cauchy. Indeed,

$$\begin{aligned}|r_p - r_q| &= |r_p - x_p + x_p - x_q + x_q - r_q| \\ &\le |r_p - x_p| + |x_p - x_q| + |x_q - r_q| \\ &\le \frac{1}{p} + |x_p - x_q| + \frac{1}{q}\end{aligned}$$

Take $\epsilon > 0$. $\{x_n\}$ cauchy implies $\exists N_1,\ \forall p,q \ge N, |x_p - x_q| \le \frac{\epsilon}{3}$ and $\exists N_2,\ \forall p,q \ge N_2, \frac{1}{p} \le \frac{\epsilon}{3},\ \frac{1}{q} \le \frac{\epsilon}{3}$ for $p,q \ge \max(N_1,N_2)$ $|r_p - r_q| \le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$.
Then, for Cauchy $\{r_n\}_n$, call $r = [\{r_n\}_n]$, then $\lim_{n\to\infty} r_n = r$ by the above theorem.
Then my algebraic limit rules, $x_n(x_n - r_n) + r_n$ where $(x_n - r_n) \to 0$ and $r_n \to r$ as $n \to \infty$. So $\{x_n\}$ converges.

### Example

Let $x_1 = 1$, $x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n})$.
Then $\{x_n\}_n \in C_{\mathbb{Q}}$, and it converges to $L \in \mathbb{R}$.
By algebraic limit rules, $L^2(\lim x_n)^2 = \lim x_n^2 = 2$.

## Subsets of the Reals, Infimum and Supremum

### Notation

Subset: $S \subseteq \mathbb{R}$
Inclusion: $x \in S$
Open Interval: $(a,b) = \{x \in \mathbb{R} | a < x < b\}$
Semiclosed Interval: $(a,b] = \{x \in \mathbb{R} | a < x \le b\}$
Closed Interval: $[a,b] = \{x \in \mathbb{R} | a \le x \le b\}$
Unbounded Semiclosed Interval: $(-\infty, a] = \{x \in \mathbb{R} | x \le a\}$
Unbounded Open: $(-\infty, a) = \{x \in \mathbb{R} | x < a\}$

## Suprememum

$S \subseteq \mathbb{R}$ is bouned above (respectively below) if $\exists M \in \mathbb{R},\ \forall x \in S,\ x \le M$ (respectively $\exists L \in \mathbb{R},\ \forall x \in S,\ L \le X$)
$S$ ad mits a least upper bound, LUB, suprememum or sup $M$ if

1. $\forall x \in S, x \le M$

2. $\forall M' \in \mathbb{R}$, upper bound of $S$, $M \le M'$

If $\sup S$ exists, it is unique.
If $x \in S$ and $x$ is an upper bound for $S$, then $x = \sup S$.

### Example 1

$\sup(0,1) = \sup[0,1] = 1$

**Example 2**

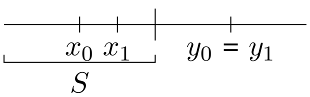
$S = \{x \in \mathbb{Q},\ x^2 < 2\}$ does not have a greatest element in $\mathbb{Q}$, nor a least upper bound in $\mathbb{Q}$.

## Theorem (Completness 2)

Every subset $S \subseteq \mathbb{R}$, nonempty and bouned above, has a supremum in $\mathbb{R}$.

## Proof

By dichotomy.
$S \neq \varnothing \implies \exists x_0 \in S$ and $S$ bounded above implies $\exists y_0 \in \mathbb{R},\ \forall x \in S,\ x \leq y_0$ (in particular $x_0 \leq y_0$).
If $x_0 = y_0$, done. Otherwise, consider $m_0 = \frac{x_0+y_0}{2}$.




Two options exist: if $m_0$ is an upper bound for $S$, set $y_1 = m_0$ and $x_1 = x_0$.
Otherwise, $\exists x_1 \in S$, such that $m_0 < x_1$ so set $y_1 = y_0$.
Repeat this process forever to construct two sequences $x_n,\ y_n$.
$\forall n,\ x_n \in S,\ y_n$ is an upper bound for $S$.

- $x_n \leq y_n$
- $x_n$ is increasing and bounded above by $y_0$, so it must be Cauchy and converging to $x$.
- $y_n$ is decreasing and bounded below by $x_0$, so it must be Cauchy and converging to $y$.
- $|x_{n+1} - y_{n+1}| \leq \frac{|x_n - y_n|}{2}$ which implies $|x_n - y_n| \leq \frac{1}{2^n}|x_0 - y_0|$ and $x = y = z$.

Therefore, the process may be understood as $x_0 \leq \cdots \leq x_n \leq x_{n+1} \leq y_{n+1} \leq y_n \leq \cdots \leq y_0$.
There remain two things to check: (1) $z$ is an upper bound for $S$ and (2) $z$ is no larger than any other upper bound for $S$.

1. Take $x \in S,\ \forall n,\ x \leq y_n \overset{n\to\infty}{\longrightarrow} x \leq Z$.
2. Take upper bound for $S$, $z'$. $x_n \leq z',\ \forall n \overset{n\to\infty}{\longrightarrow} z \leq z'$.

So $z = \sup S$.

## Monotone Convergence Theorem (Completeness 3)

An increasing sequence of reals, $\{x_n\}_n$, that is bounded above, converges to $\sup X = \sup\{x_n | n \in \mathbb{N}\}$.
To prove that this converges, since it is monotone and bounded above it is Cauchy and therefore must be convergent.

### Proof

Call $x$ the limit, then $\forall n,\ x_n \leq x$. To see this, suppose $\exists n_0,\ x < x_{n_0}$ then $\forall m \geq m_0,\ x < x_{m_0} \leq x_m \implies |x_m - x| \geq |x_{n_0} - x| > 0,\ \forall m \geq n_0$ is a contradiction.
Let $M$ be an upper bound of $X$. Then $x_n \leq M,\ \forall n \xrightarrow{n\to\infty} x \leq M \implies x = \sup X$.

### Theorem (Existence of Roots)

$\forall x \in \mathbb{R}$ where $x > 0,\ p \in \{2, 3, \ldots ,\},\ \exists! y > 0$ such that $y^p = x$.

### Proof

Left as an exercise.
Either by dichotomy or consider $S = \{y \in \mathbb{R} | y^p < x\}$, show: $S \neq 0$, bounded above and $(\sup S)^p = x$.
For uniqueness, show $y_1^p = y_2^p = x \iff 0 = y_1^p - y_2^p = (y_1 - y_2)(\cdots \neq 0) \implies y_1 = y_2$.

### Topological Properties

$S \subseteq \mathbb{R}$ is open if $\forall x \in S,\ \exists a, b \in \mathbb{R},\ x \in (a, b) \subset S$.
$x$ is an accumulation or limit point of $S$ if $\forall \epsilon > 0,\ \exists y \in S,\ 0 < |x - y| < \epsilon$.
$S \subseteq \mathbb{R}$ is closed if it contains all its limit points.
A set may be both open closed, just open, just closed or neither.
Given $S \subseteq \mathbb{R}$, the interior of $S$ is $\bigcup_{S' \text{ open} \subset S} S' = S^{\text{int}} = S^0$.
The closure is $\bigcap_{F \text{ closed} \supseteq S} F = \overline{S} \overset{\text{wts}}{=} S \cup \{\text{limit points of } S\}$.

### Example

$\{x\}$ is not open, but, since the limit points of $x$ are $\varnothing$, it is closed.

### Propositions

1. Arbitrary unions and finite intersections of open sets are open.

2. $S$ is open if and only the complement $S^c = \mathbb{R} \setminus S$ is closed.

3. Arbitrary intersections and finite unions of closed sets are closed.

### Bolzano-Weierstrass Theorem

A bounded sequence in $\mathbb{R}$ ad mits a convergent (Cauchy) subsequence. $\exists M,\ |x_n| \leq M,\ \forall n$

### Proof by Dichotomy

Suppose $I_0 = [a, b]$ contains the sequence.
Construct a sequence of intervals by indicators: if $\left[a, \frac{a+b}{2}\right]$ contains infinitely terms of $\{x_n\}_n$, choose $n$ such that $x_{n_1} \in \left[a, \frac{a+b}{2}\right]$ and call $I_1 = \left[a, \frac{a+b}{2}\right]$.

Otherwise, $\left[\frac{a+b}{2}, b\right]$ must contain infinitely many terms. Choose $n$ in a similar fashion as above such that $I_1 = \left[\frac{a+b}{2}, b\right]$.
This process may be repeated to create a sequence of intervals such that $I_k \supseteq I_{k+1} \supseteq I_{k+2}$ and $l(I_k) = \frac{b-a}{2^k}$.
A subsequence $\{u_{n_k}\}_k$ such that $u_{n_k} \in I_l$ for $k \geq l$.

### Exercise

Extract a Cauchy criterion out of the above.

## October 9, 2023

### Overview

- Topology of $\mathbb{R}$ continued.

- Numerical series.

Next Wednesday

- Absolute and Conditional Convergence.

- Rearrangement theorem.

### Last Time

Finished with Bolzano-Weierstrass.

### Limits

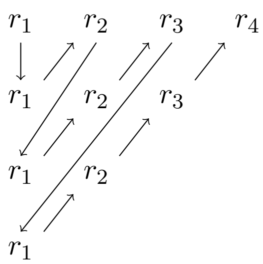
#### Limit Point

We say $x \in \mathbb{R}$ is a limit point of $\{x_n\}_n$ if a subsequence of $\{x_n\}_n$ converges to $x$.
Equivalently, $\forall \epsilon > 0,\ \forall n_0 \in \mathbb{N},\ \exists n \geq n_0,\ |x_n - x| < \epsilon$.
That is, the sequence revisits an epsilon neighborhood of $x$ infinitely many times.

#### Limit Set

The limit set of $\{x_n\}_n : \mathrm{LS}(\{x_n\}_n) =$ the set of limit points of $\{x_n\}_n$.

- Comments
  - if $\lim_{n\to\infty}\{x_n\} = x$, then $\mathrm{LS}(\{x_n\}_n) = \{x\}$.
  - The limit set can be as big as $\mathbb{R}$!



  - What Bolzano-Weierstrass says is that if $\{x_n\}$ is bounded, then $\mathrm{LS}(\{x_n\}) \neq \varnothing$.

- Examples $\mathrm{LS}(\{n\}_n) = \varnothing$.
  $\mathrm{LS}(\{x_n\}_n)$ is closed (good exercise).

### Limit Superior

If $\{x_n\}_n \in [a,b]$ is bouneded, $\forall k \in \mathbb{N}$, $\sup\{x_j | j \geq k\}$ exists in $\mathbb{R}$.
Because

$$a \leq \sup\{x_j | j \geq k+1\} = y_{k+1} \leq \sup\{x_j | j \geq k\} = y_k$$

by the Monotone Convergence Theorem, $\{y_k\}_k$ converges. Call its limit $\limsup_n x_n = \inf_n \sup\{x_j | j \geq n\}$.

### Limit Inferior

Similarly, define $\lim_n \inf x_n = \sup_n \inf\{x_j | j \geq n\}$.

### Limit Superior and Limit Inferior Always Exist

What to show: $\limsup x_n, \liminf x_n \in \mathrm{LS}(\{x_n\})$.
Left as an exercise.

### Convergence at the Limit

A bounded sequence $\{x_n\}_n$ converges if and only if $\liminf_n x_n = \limsup_n x_n$.

- Proof Technique Often it is useful to structure a proof such that

$$L < \liminf_n x_n \leq \limsup_n x_n < L$$

## Topology of the Reals Continued

### Compactness

Let $A \subseteq \mathbb{R}$.
$A$ is (sequentially) compact if every sequence in $A$ has a limit point in $A$.
$A$ is (Heine-Borel) compact if every open cover of $A$ has a finite subcover.

- Open Cover $\{O_\alpha\}_{\alpha\in I}$, with $O_\alpha$ open, is an open cover of $A$ if $A \subseteq \bigcup_{\alpha\in I} O_\alpha$.

- Finite Subcover $O_1, \ldots, O_n, n \in \mathbb{N}$.

## Heine-Borel Theorem

Let $A \subseteq \mathbb{R}$.
The following are equivalent

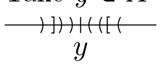
1. $A$ is Heine-Borel compact.

2. $A$ is closed and bounded.

3. $A$ is sequentially compact.

## Proof

(1) $\implies$ (2) $\implies$ (3) $\implies$ (1)

- Heine-Borel Compact Implies Closed and Bounded Suppose $A$ satisfies the Heine-Borel property.
  Consider $\{(-n, n)\}_{n\in\mathbb{N}}$. Clearly $\bigcup_n (-n, n) = \mathbb{R} \supseteq A$.
  By Heine-Borel, $\exists n_0, \ldots, n_p$ such that $A \supseteq \bigcup_{j=0}^{p} (-n_j, n_j) = (-N, N)$, $N = \max(n_0, \ldots, n_p)$. So $A$ is bounded.
  $A$ is closed if $y \notin A \implies y$ is not a limit point of $A$.
  Take $y \in A^c$, then $A \subseteq \mathbb{R} \setminus \{y\} = \bigcup_{n\in\mathbb{N}} (-\infty, y - \frac{1}{n}) \cup (y + \frac{1}{n}, \infty)$.
  
  
  By the Heine-Borel property,

$$A \subseteq \bigcup_{n_0, \ldots, n_p} (-\infty, y - \frac{1}{n}) \cup (y + \frac{1}{n}, \infty)$$
$$= (-\infty, y - \frac{1}{N}) \cup (y + \frac{1}{N}, \infty)$$

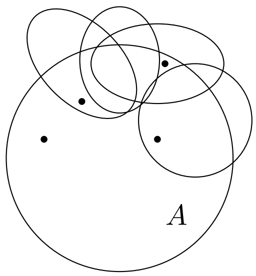
  Which implies $A \cap [y - \frac{1}{N}, y + \frac{1}{N} = \varnothing$ and $y$ is not a limit point of $A$.
  That is, $A$ contains its limit points.

- Closed and Bounded Implies Sequential Compactness Suppose $A$ is both closed and bounded.
  Let $\{x_n\}_n \in A$. Then $\{x_n\}_n$ is bounded. By Bolzano-Weierstrass, it has a limit point $x$ and a subsequence $\{x_{n_k}\}_k$ converging to $x$.
  Since $A$ is closed, $\lim_{k\to\infty} x_{n_k} = x \in A$. ∎

- Sequential Compactness Implies Heine-Borel Suppose $A \subseteq \mathbb{R}$ is sequentially compact.
  Consider an open cover of $A$, $\{O_\alpha | \alpha \in I\}$.
  First, turn it into a countable cover:

  - $\forall \alpha \in I,\ O_\alpha \subseteq \left(r_\alpha^1, r_\alpha^2\right),\ r_\alpha^1, r_\alpha^2 \in \mathbb{Q}$

Assume that $\{O_\alpha\}_\alpha$ can be made countable $(O_1, \dots, O_n)$
By contradiction, suppose $\forall n \in \mathbb{N},\ A \setminus \left(\bigcup_{j=1}^n O_j\right) \neq \varnothing$.
Take $x_n \in A \setminus \left(\bigcup_{j=1}^n O_j\right)$. Since $A$ is sequentially compact, $\exists \{x_{n_k}\}_k$ subsequence of $\{x_n\}_n$ converging to $x \in A$.
Since $A \subset \bigcup_{j\in\mathbb{N}} O_j$, $\exists j_0,\ x \in O_{j_0}$, $O_{j_0}$ is open: $\exists \delta > 0,\ (x-\delta, x+\delta) \subseteq O_{j_0}$.
Then $\exists N,\ k \geq N \implies x_{n_k} \in (x-\delta, x+\delta) \subseteq O_{j_0}$. But if $k$ is such that $n_k > j_0$, we also have $x_{n_k} \notin O_{j_0}$ which is a contradiction!




## Structure of Open and Closed Sets

$A$ is open in $\mathbb{R}$ if and only if it can be written as an at most countable, disjoint union of open intervals.

### TODO Proof

For $x \in A,\ \exists (a,b)$, such that $x \in (a,b) \subseteq A$.
Let $I_x = \bigcup_{\text{open interval containing} x,\ I \subseteq A} I$. This is the maximal interval containing $x$ in $A$.
Then, $A \subseteq \bigcup_{x\in A}\{x\} \subseteq \bigcup_{x \in A} I_x \subseteq A$.
That is, $A = \bigcup_{x\in A} I_x \quad (*)$.
Next, if $x, y \in A$, then $\begin{cases} \text{either } I_x = I_y \\ \text{or } I_x \cap I_y = \varnothing \end{cases}$
IMAGE HERE
The union $(*)$, as a disjoint union, is at most countable because each distinct one must contain a distinct rational and $\mathbb{Q}$ is countable.

### Long Story Short

The topology of the reals avoids exotic sets. Closed sets, on the other hand, can be quite complicated.

## TODO Cantor Set

$C := \bigcap_{k\in\mathbb{N}_0} I_k$. $I_{k+1}$ is obtained by removing the middle open third of each interval making $I_k$.
IMAGE HERE - CANTOR
$I_0 = [0,1]$. One interval of length 1.
$I_1 = [0,1/3] \cup [2/3,1]$. Two intervals of length $2/3$.
$I_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9]$. Four intervals of $(2/3)^2$
$I_k$ is $2^k$ intervals of length $(2/3)^k$.
$I_{k+1} \subseteq I_k \implies C \subseteq I_k,\ \forall k \implies l(C) \leq l(I_k) = (2/3)^k \implies l(C) = 0$.

### TODO Triadic Expansions

Goal:

1. $C$ is perfect (i.e. every point in $C$ is a limit point of $C$).

2. $C$ contains no open intervals.

Property 2 is easy because $C \subseteq I_k$, which does contain interval of length greater than $(1/3)^k$.

1. $C$ is uncountable.

Every $x \in [0,1]$ can be written in the form $x = \sum_{k=1}^{\infty} \frac{a_k}{3^k},\ a_k \in \{0,1,2\}$.
That is, $x = 0.a_1a_2\ldots$ in base 3. This is not always unique (e.g. $1/3 = 0.100\ldots = 0.022\ldots$).
IMAGE HERE - THIRDS OF INTERVAL
Note that the Cantor set is removing all decimal expansions with leading 1s. That is, $x \in C$ if and only if it has a triadic expansion only made of 0s and 2s.

- Proof of 1 If $x \in C,\ x = \sum_{k\geq 1} \frac{a_k}{3^k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{3^k}$, then $x_n \in C,\ \forall n$ and $x_n = 0.a_1\ldots a_n0000\ldots$ where $a_1, a_n \in \{0,2\}$.
  Unique representation can be maintained by forcing the behavior of the $n+1$th digit.

- Proof of 3 Every point in $[0,1]$ can also be written as $x = \sum_{n=1}^{\infty} = \frac{b_n}{2^n},\ b_n \in \{0,1\}$ (i.e. a binary expansion). Then $C \mapsto [0,1]$ gives $x = \sum \frac{a_k}{3^k} \mapsto \sum \frac{b_k}{2^k},\ b_k = \frac{a_k}{2}$ for $a_k \in \{0,2\}$ is a bijection!

## October 11, 2023

### Overview: Numeric Series

- Series with non-negative terms.

- Series with general terms.

- Convergence criteria.

- Algebraic rules.

- Rearrangements.

### General Notation

Sequence $\{x_n\}_{n\geq n_0}$ (often $n_0 \in \{0,1\}$)

#### Definition: Partial Sum

$S_n = \sum_{k=n_0}^{n} x_k$ $(x_n = S_n - S_{n-1})$
We say $\sum_n x_n$ converges if $\lim_{n\to\infty} S_n$ exists.
We denote $\sum_{k=n_0}^{\infty} x_k = \lim_{n\to\infty} S_n$

- Example: Geometric Series $\sum_{k=0}^{n} r^k = S_n,\ r \in (0,1)$
  $\frac{1-r^{n-1}}{1-r} \to \frac{1}{1-r}$

- Example: P Series $\sum_{k=1}^{n} \frac{1}{k^p},\ p > 0$

- Example: Exponential $\sum_{k=0}^{n} \frac{1}{k!}$

### Series without Non-negative Terms

The series has non-negative terms if $x_n \geq 0,\ \forall n$.

### Obvious Algebraic Limit Rules

If $\sum_{n\geq n_0} a_n$ and $\sum_{n\geq n_0} b_n$ converge and $\alpha \in \mathbb{R}$, then $\sum_{n\geq n_0}(a_n + \alpha b_n)$ converges to

$$\sum_{n=n_0}^{\infty} a_n + \alpha \sum_{n=n_0}^{\infty} b_n = \sum_{n=n_0} (a_n + \alpha b_n)$$

- Proof (Sketch) Reason on the partial sums; use algebraic limit rules on sequences.

### Proposition

If $\sum_n x_n$ converges in $\mathbb{R}$, then $\lim_{n\to\infty} x_n = 0$.

- Proof $x_n = S_n - S_{n-1} \overset{n\to\infty}{\to} S - S = 0$
  Since $S_n \overset{n\to\infty}{\to} S$ and $S_{n-1} \overset{n\to\infty}{\to} S = \sum_{n=n_0}^{\infty} x_n$.

### Series with Non-negative Terms

If $x_n \geq 0,\ \forall n,\ S_n = \sum_{k=n_0}^{n} x_k$ is non-decreasing.
By monotone convergence theorem, $S_n$ is either bouned, and therefore converges, or unbounded from above where

$$\forall m > 0,\ \ \exists n_0 \in \mathbb{N},\ \forall n \geq n_0,\ S_n \geq M$$

This is "diverging to $+\infty$."

### Theorem: Convergence Criteria

- Term Test If $0 \leq a_n \leq b_n,\ \forall n \geq n_0$ and $\sum_n b_n$ converges, then $\sum_n a_n$ converges.

  - Proof Suppose $0 \leq a_n \leq b_n$, and $t_n = \sum_{k=n_0}^{n} b_k$ converges and, therefore, is bounded above by $B = \sum_{k=n_0}^{\infty} b_k$.
    Then $\forall n,\ \sum_{k=n_0}^{n} a_k \leq \sum_{k=n_0}^{n} b_k \leq B$.
    Thus, by monotone convergence theorem, $\sum_{k=n_0}^{n} a_k$ converges.

- Ratio Test If $a_n > 0,\ \forall n$ and $\exists n_0 \in \mathbb{R}$ such that $\frac{a_{n+1}}{a_n} \le r < 1,\ \forall n \ge n_0$, then $\sum_n a_n$ converges.

  - Clarification The harmonic series has ratio $\frac{k}{k+1} < 1$ but since $\frac{k}{k+1} \overset{k\to\infty}{\to} 1$, there is no $r$ which satisfies the ratio test.

  - Proof Suppose $a_{n+1} \le r a_n$ for $n \ge n_0$.
    Then $a_{m_0+p} \le a_{m_0+(p-1)} r \le a_{m_0+(p-2)} r^2 \le \cdots \le a_{m_0} r^p$.
    Then for $n \ge n_0$,

$$\sum_{k=n_0}^{n} a_k = \sum_{k=n_0}^{m_0} a_k + \sum_{k=m_0+1}^{n} a_k \le \sum_{k=m_0}^{m_0+(n-m_0)} a_{m_0} r^{n-m_0} \le a_{m_0} \sum_{k=m_0}^{n-m_0} r^{n-m_0} \le \frac{1}{1-r}$$

  - Rate of Convergnce The above proof shows that the ratio test implies a geometric rate of convergence.

- Root Test If $\exists n_0 \in \mathbb{N}$ and $r \in (0,1)$ such that $a_n^{1/n} \le r$, then $\sum_n a_n$ converges.

  - Proof (Sketch) Same story as the ratio test: $a_n^{1/n} \le r \implies a_n \le r^n$.

- Rejection of Ratio/Root If $\exists n_0 \in \mathbb{N}$ such that either $\frac{a_{n+1}}{a_n} \ge 1$ for $n \ge n_0$ or $a_n^{1/n} \ge 1$ for $n \ge n_0$, then $\sum_n a_n$ diverges to $+\infty$.

  - Proof (Sketch) In either case, $a_n$ cannot converge to zero. Therefore the series cannot converge.

## Prototype Scales

### Geometric Rates

$\sum_{n\ge 1} \frac{1}{n^\alpha}$ converges if and only if $\alpha > 1$ (to $\zeta(\alpha)$)
$a_k = \frac{1}{k^\alpha} \rightsquigarrow 2^k a_{2^k} = \frac{2^k}{2^{k\alpha}} = \left(\frac{1}{2^{\alpha-1}}\right)^k \implies t_n = \sum_{k=1}^{n} 2^k a_{2^k}$ converges if and only if $\frac{1}{2^{\alpha-1}} < 1$ if and only if $\alpha > 1$.

### Log Geometric Case

$\sum_{n\ge 1} \frac{1}{n(\log(n))^\beta}$ converges if and only if $\beta > 1$.
$a_k = \frac{1}{k(\log(k))^\beta} \rightsquigarrow 2^k a_{2^k} = \frac{2^k}{2^k(\log(2^k))^\beta} = \frac{1}{(\log(2)^\beta k^\beta}$ converges if and only if $\beta > 1$.

### Lemma:

Suppose $a_n$ decreases to 0.
Then the sequence $S_n = \sum_{k=1}^{n} a_k$ converges if and only if $t_n = \sum_{k=1}^{n} 2^k a_{2^k}$ converges.

$$S_{2^n} = a_1 + \underbrace{a_2 + a_3} + \underbrace{a_4 + a_5 + a_6 + a_7} + \underbrace{a_8 + \cdots}$$

- Proof $\quad a_3 + a_3 \leq \qquad \leq a_2 + a_3$

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \cdots \\ &= a_1 + \sum_{k=1}^{n} \sum_{p=1}^{2^k-1} a_{2^k+p} \\ &\leq a_1 + 2^k a_{2^{k+1}} + \end{aligned}$$

This gives

$$\frac{1}{2}(t_n - a_1) \leq S_{2^n} - a_1 \leq t_{n-1}$$

Therefore $S_{2^n}$ converges, which implies that $t_n$ converges, and, since $S_n$ is monotone, $S_n$ itself converges.

### Series with General Terms

General term is signed.

#### Trick

Write $a_n = a_n^+ - a_n^-$ and $a_n^\pm = \max(0, \pm a)$. Then

$$S_n = \sum_{k=n_0}^{n} a_k = \left(\sum_{k=n_0}^{n} a_k^+\right) - \left(\sum_{k=n_0}^{n} a_k^-\right)$$

#### Convergence Outcomes

| | $\sum_{k=n_0}^{\infty} a_k^+ < \infty$ | $\sum_{k=n_0}^{\infty} a_k^+ = \infty$ |
|---|---|---|
| $\sum_{k=n_0}^{\infty} a_k^- < \infty$ | absolute convergence | $\lim S_n = +\infty$ |
| $\sum_{k=n_0}^{\infty} a_k^+ = \infty$ | $\lim S_n = -\infty$ | lots of things can happen; divergence, convergence, any limit sequence |

If $S_n^+$ and $S_n^-$ converge, we can return to algebraic limit rules.
$S_n$ converges to $\lim_{n\to\infty} S_n^+ - \lim_{n\to\infty} S_n^-$

### Definition: Absolute Convergence

We say $\sum_n a_n$ converges absolutely if and only if $\sum_n |a_n|$ converges.

#### Note

$|a_n| = a_n^+ + a_n^-$

### Proposition: Absolute Convergence Implies Convergence

#### Proof

Absolute convergence $\implies \sum |a_n|$ converges $\implies \sum a_n^+$ and $\sum a_n^-$ converges $\implies \sum(a_n^+ - a_n^-)$ converges.

### Definition: Conditional Convergence

$\sum_n a_n$ converges conditionally if and only if $\sum_n a_n$ converges while $\sum_n |a_n|$ diverges.

### Criteria for Convergence

For absolute convergence, run root/ratio/term test on $\sum_n |a_n|$.
Other criteria which might indicate conditional convergence.

#### Alternating Series Test

If $a_n(-1)^n b_n$, $b_n \geq 0$ decreases to zero, the series is conditionally convergent.

#### Dirichlet Test

If $a_n = b_n c_n$, where $b_n$ decreases to zero and $c_n$ satisfies $|c_0 + c_1 + \cdots + c_n| \leq C$, $\exists C \in \mathbb{R}, \forall n \in \mathbb{N}$, then $\sum_{n\geq 0} a_n$ converges conditionally.

- Applications $\sum_{n\geq 1} \frac{(-1)^n}{n}$
  $\sum_{n\geq 1} \frac{\cos(n)}{n}$

- Proof Write $C_n = c_0 + c_1 + \cdots + c_n$, such that $|C_n| \leq C$, $\forall n$.
  Then $c_n = C_n - C_{n-1}$, and

  $$\sum_{k=0}^{n} b_k c_k = \sum_{k=0}^{n} b_k (C_k - C_{k-1}) = \sum_{k=0}^{n} b_k C_k - \sum_{k=0}^{n} b_k C_{k-1} \overset{l=k-1}{=} \sum_{k=0}^{n} b_k C_k - \sum_{l=0}^{n-1} b_{l+1} C_l = b_n C_n + \sum_{k=0}^{n-1} (b_k - b_{k+1}) C_k$$

  Then, since $b_n C_n \overset{n\to\infty}{\to} 0$, we only need to show that the final term converges absolutely. Consider

  $$\sum_{k=0}^{n-1} |b_k - b_{k+1}||C_k| \leq C \sum_{k=0}^{n-1} (b_k - b_{k+1}) = C(b_0 - b_n) \leq C(b_0)$$

  independent of $n$. Hence, $\sum_{k=0}^{n} b_k c_k$ converges.

### Definition: Rearrangement

Take $\sigma : \mathbb{N} \to \mathbb{N}$ a bijection and $\sum_{n\geq 1} a_n$ a series such that $S_n = \sum_{k=1}^{n} a_k$.
Then define a rearranged sum $S_n^{(\sigma)} = \sum_{k=1}^{n} a_{\sigma(k)}$.

#### Q: When does the rarranged sum converge; to where?

- Theorem: Rearrangement of Absolute Convergence If $\sum a_n$ converges absolutely, then $\forall \sigma$, $\lim_{n\to\infty} S_n^{(\sigma)} = \lim_{n\to\infty} S_n$.

- Theorem: Rearrangement of Conditional Convergence If $\sum a_n$ converges conditionally, then $\forall x \in \mathbb{R}$, $\exists \sigma$ such that $\lim_{n\to\infty} S_n^{(\sigma)} = x$.

# October 16, 2023

## Overview

Sequences and Series of Functions
Things that will be glossed over for time

- Limits
- Continuity
- Differentiability
- Integrabillity

## Why care about sequences and series?

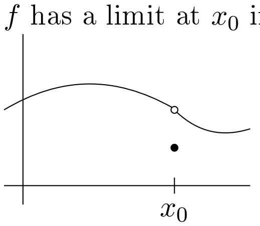
Extending features of functions.
Approximations.

## Limits and Continuity

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R},\ x_0 \in I$.

### Definition: Limit

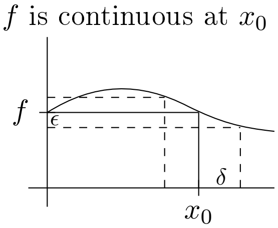
$f$ has a limit at $x_0$ if $\exists \ell \in \mathbb{R}, \forall \epsilon > 0, \exists \delta > 0, 0 < |x - x_0| < \delta \implies |f(x) - \ell| < \epsilon$




- Equivalently
  For every sequence $\{x_n\}_n$ in $I$ converging to $x$ (but distinct to $x$), $\lim_{n\to\infty} f(x_n) = \ell$.

### Definition: Continuous

$f$ is continuous at $x_0$ if $\lim_{x\to x_0} f(x) = f(x_0)$.




- Modulus of Continuity $\forall \epsilon > 0, \exists \delta > 0, \forall x \in I, |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$
  Then $\delta(x_0, \epsilon)$ is the modulus of continuity.

**Definition: Uniform Continuity on I**

$f$ is uniformly continuous on $I$ if $\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in I, |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$.
Where $\delta$ is $\delta(\epsilon)$. That is, the modulus of continuity does not depend on the points.

## Special Types of Uniform Continuity

### Hölder Continuous

$f$ is $\alpha$-Hölder continuous on $I$ for $\alpha \in (0, i]$, if $\exists c > 0$ such that $\forall x, y \in I, |f(x) - f(y)| \leq c|x - y|^\alpha$
$\alpha = 1$ implies that $f$ is "Lipschitz-continuous"

- Example
  If $f'$ exists and is bounded on $[a, b]$ by $M$, then by the Mean Value Theorem:
  $|f(x) - f(y)| = |f'(\xi)||x - y| \leq M|x - y|$, where $x \leq \xi \leq y$.

## Continuity on Compact Sets

Let $K \subseteq \mathbb{R}$ be a compact set and $f : K \to \mathbb{R}$ be continuous.
Then

1. $f(K)$ is compact. In particular, $f$ is bounded on $K$.

2. $f$ achieves its extrema on $K$. (e.g. $\exists M \in K$ such that $f(M) = \sup\{f(x) \mid x \in K\}$.

3. $f$ is uniformly continuous on $K$.

Note: the proofs of these features are good practice. In particular, proofs that exploit the Heine-Borel property.

**Proof 1: Compact**

Let $y_n$ be a sequence in $f(K)$.
Then, $\forall n, y_n = f(x_n)$ for $x_n \in K$.
It follows that there exists a subsequence $\{x_{n_k}\}_k$ converging to $x$ in $K$.
By continuity, $y_{n_k} = f(x_{n_k}) \overset{k\to\infty}{\to} f(x) \in f(k)$.

**Proof 2: Achieves Its Extrema**

Construct $M$.
By the suprememum property, $S = \sup\{f(x) \mid x \in \mathbb{R}\}$, $\forall n, \exists x_n \in K$ such that $S - \frac{1}{n} \leq f(x_n) < S$.
Since $K$ is compact, there exists a subsequence $\{x_{n_k}\}_k$ converging to $x \in K$.
Since $f$ is continuous at $x$, $f(x_{n_k}) \overset{k\to\infty}{\to} f(x)$, and also $S - \frac{1}{n_k} \leq f(x_{n_k} \leq S \overset{k\to\infty}{\to} S = f(x)$.

### Proof 3: Uniformly Continuous

Suppose, for sake of contradiction, that $\exists \epsilon > 0, \forall \delta > 0, \exists x_\delta, y_\delta \in K, |x_\delta - y_\delta| < \delta$ and $|f(x_\delta) - f(y_\delta)| \geq \epsilon$.
Letting $\delta = \frac{1}{n}$, we may write $x_n, y_n \in K, |x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \epsilon$.
Since $K$ is compact, there exists a subsequence $\{x_{n_k}\}_k$ which converges to $x \in K$.
Since $|x_{n_k} - y_{n_k}| < \frac{1}{n_k}$, then $\{y_{n_k}\}_k$ also converges to $x$.
By continuity of $f$ at $x$, $\lim_{k\to\infty} f(x_{n_k}) - f(y_{n_k}) = 0$. However, this contradicts the established fact that $|f(x_n) - f(y_n)| \geq \epsilon$ for $\epsilon > 0$.

## Notation

Let $I \subseteq \mathbb{R}$ be an interval.

### Sequence of Functions

$f_n(x), x \in I, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$

### Series of Functions

$S_n(x) = \sum_{k=0}^{n} f_k(x)$

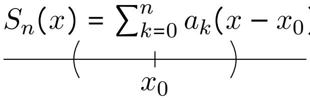
## Definition: Pointwise Convergence

A sequence or series of functions converges pointwise on $I$ if and only if $\forall x \in I, \{f_n(x)\}_n$ is convergent.
Call $f(x)$ the limit.

## Q: Under what conditions do properties of a sequence (e.g. continuity, differentiability, integrability) propogate to the limit?

### Power Series

$\sum_{n\geq 0} a_n (x - x_0)^n$
$S_n(x) = \sum_{k=0}^{n} a_k (x - x_0)^k$




### Fourier Series

$S_n = a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx)$ is $2\pi$-periodic.

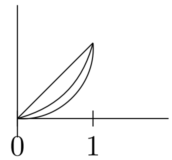
### Approximation

For purposes of approximation, it is useful to know if, for example, the integral may be approximated by taking integrals of the partial sums.

## Deficiencies of Pointwise Convergence

### Example 1

On $[0,1]$, $f_n(x) = x^n \xrightarrow{n\to\infty} \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases}$,



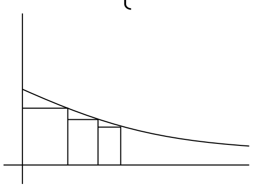

$f_n$ is continuous on $[0,1]$, $\forall n$, but $f$ is not.

- Exercise

  Show that there is no uniform convergence here.
  Hint: negate uniform convergence and prove the negation.

### Example 2

$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$ is not Riemann-integrable on $[0,1]$.



If $r_n$ denotes a denumeration of rationals in $[0,1]$, define $f_n(x) = \begin{cases} 1 & \text{if } x \in \{r_0, \dots, r_n \\ 0 & \text{otherwise} \end{cases}$.

So $f_n$ converges pointwise on $\chi_{\mathbb{Q}}$.
Yet, $\forall n, f_n$ is Riemann-integrable and $\int_0^1 f_n(x)\, dx = 0$.

## Definition: Uniform Convergence

We say $f_n : D \to \mathbb{R}$ (e.g. $D$ an interval) converges uniformly to $f$ on $D$ (notation $f_n \rightrightarrows f$ on $D$) if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, n \geq N \implies \begin{cases} |f_n(x) - f(x)| < \epsilon, \forall x \in D \\ \sup_D |f_n - f| < \epsilon \end{cases}$$

### Compare with Pointwise Convergence

Compare to $f_n \to f$ pointwise on $D$.
$\forall x \in D, \forall \epsilon > 0, \exists N \in \mathbb{N}, n \geq N \implies |f_n(x) - f(x)| < \epsilon$.
In this case, the behavior is primarily contingent upon the choice of $x$. That is $N(x,\epsilon)$ is dependent on $x$.

## Theorem: Weierstrass M-Test

Let $f_n : D \to \mathbb{R}$ be bounded by $M_n$ on $D$.
If $\sum_{n=1}^{\infty} M_n < \infty$, then the series $S_n(x) = \sum_{k=1}^{n} f_k(x)$ converges uniformly to $S(x)$

**Proof**

$\forall x \in D, |S_n(x) - S(x)| = |\sum_{k=n+1}^{\infty} f_k(x)| \overset{\text{triangle inequality}}{\leq} \sum_{k=n+1}^{\infty} |f_k(x) \leq \sum_{k=n+1}^{\infty} M_k$, where $\sum_{k=n+1}^{\infty} M_k$ is a uniform bound in $x$.
Let $\epsilon > 0, \exists n, n \geq N \implies \sum_{k=n+1}^{\infty} M_k < \epsilon$.
Then $\forall x \in D, n \geq N, |S_n(x) - S(x)| \leq \sum_{k=n+1}^{\infty} M_k < \epsilon$. ■

## Theorem: Continuity and Uniform Limits

Let $f_n D \to \mathbb{R}$ be continuous on $D$ for all $n$ and $f_n f$ on $D$ ($\lim_{n\to\infty} \sup_D |f_n - f| = 0$).
Then $f$ is continuous on $D$.

**Proof**

Fix $x \in D$, with $x_n$ converging to $x$ in $D$.
What To Show: $f(x_n) \xrightarrow{n\to\infty} f(x)$.
Scratch: $f(x_n) - f(x) = (f(x_n) - f_p(x_n)) + (f_p(x_n) - f_p(x)) + (f_p(x) - f(x))$.
Let $\epsilon > 0$ be given.
$f_n \rightrightarrows f : \exists N, n \geq N \implies |f_n(y) - f(y)| < \frac{\epsilon}{3}, \forall y \in D$.
For $p \geq N, |f_p(y) - f(y)| < \frac{\epsilon}{3}, \forall y \in D \implies \forall n \in \mathbb{N}, |f(x_n) - f(x)| \overset{\text{triangle inequality}}{\leq} \frac{2\epsilon}{3} + |f_p(x_n - f_p(x)|$.
With $p = N$, since $f_p$ is continuous at $x$, $\exists N_1, n \geq N_1 \implies |f_p(x_n) - f_p(x)| < \frac{\epsilon}{3}$.
Hence, for $n \geq N_1, |f(x_n) - f(x)| \leq \epsilon$. ■

## Riemann-Integrability

Fix $D = [a, b]$ and $g : [a, b] \to \mathbb{R}$ bounded by $|g(x)| \leq M, \forall x$.

### Definition: Subdivision

$\sigma = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b$

### Definition: Upper and Lower Riemann Sums

$S^+(g, \sigma) = \sum_{k=1}^{n} (x_k - x_{k-1}) M_k$ is the upper sum.
$S^-(g, \sigma) = \sum_{k=1}^{n} (x_k - x_{k-1}) m_k$ is the lower sum.
Where $M_k = \sup_{[x_{k-1}, x_k]} g$ and $m_k = \inf_{[x_{k-1}, x_k]} g$.
This gives $-M(b - a) \leq S^-(g, \sigma) \leq S^+(g, \sigma) \leq (b - a)M$.
If $\mathfrak{S}[a, b] = \{\text{subdivisions of } [a, b]\}$, then
$I^-(g) = \sup_{\sigma \in \mathfrak{S}[a,b]} S^-(g, \sigma)$ and $I^+(g) = \inf_{\sigma \in \mathfrak{S}[a,b]} S^+(g, \sigma)$.

## Definition: Riemann Integrable

$g$ is Riemann integrable if $I^+(g) = I^-(g)$ and we denote $\int_a^b g(t)\, dt = I^+(g)$.

### Lemma

$g$ is Riemann integrable if and only if $\forall \epsilon > 0, \exists \sigma \in \mathfrak{S}[a, b]$ such that $S^+(g, \sigma) - S^-(g, \sigma) < \epsilon$.

### Properties

1. Continous functions and monotone functions are Riemann Integrable.

2. $f \mapsto \int_a^b f(t)\, dt$ is linear.

3. If $f, g$ are Riemann Integrable and $f(x) \leq g(x), \forall x \in [a,b]$, then $\int_a^b f(t)\, dt \leq \int_a^b g(t)\, dt$.

### Theorem:

If $f_n \rightrightarrows f$ on $[a,b]$ and $f_n$ is Riemann Integrable for all $n$, then $f$ is Riemann Integrable on $[a,b]$ and $\lim_{n\to\infty} \int_a^b f_n(t)\, dt = \int_a^b \lim_{n\to\infty} f_n(t)\, dt = \int_a^b f(t)\, dt$.

### Proof

$\forall n, \forall x \in [a,b], f_n(x) - \epsilon \leq f(x) \leq f_n(x) + \epsilon$ where $\epsilon_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|$ (by hypothesis $e_n \xrightarrow{n\to\infty} 0$)
Then, for any $\sigma \in \mathfrak{S}[a,b]$, $S^-(f_n, \sigma) - \epsilon_n(b-a) \leq S^-(f,\sigma) \leq S^+(f,\sigma) \leq S^+(f_n,\sigma) + \epsilon_n(b-a)$.
It follows that $S^+(f,\sigma) - S^-(f,\sigma) \leq S^+(f_n,\sigma) - S^-(f_n,\sigma) + 2\epsilon_n(b-a)$.
Finishing the proof is left as an exercise.

## October 18, 2023

### Overview

- Sequences/Series

- Power Series

- Exponential and Logarithms

### Fundamental Theorems of Calculus

Full proofs in 105A lecture notes.

### Differentiation of the Integral

$f : [a,b] \to \mathbb{R}$ continuous.
$\forall x \in [a,b]$, can define $F(X) = \int_a^x f(t)\, dt$.
Then $F$ is continuously differentiable on $[a,b]$
$F'(x) = f(x)$ for $x \in [a,b]$.

### Integration of the Derivative

$f \in C^1[a,b]$ with one-sided derivatives at $a$ and $b$ well defined. (e.g. $\frac{f(a+h)-f(a)}{h} \underset{h>0;h\to 0}{\longrightarrow} f'(a)$.
Then $\forall x, y, a \leq x \leq y \leq b$, $f(y) - f(x) = \int_x^y f'(t)\, dt$.

### Theorem: Differentiability of Uniform Limits

Let $f_n : (a,b) \to \mathbb{R}$ be a sequence in $C^1[a,b]$, and assume $f_n(x) \to f(x)$ pointwise while $f_n'(x) \rightrightarrows g(x)$ uniformly. Then $f \in C^1(a,b)$ and $f' = g$.

### Proof

Fix $a_0 \in (a,b)$.
Then $\forall x \in (a,b)$, by the Fundamental Theorem of Calculus,

$$f_n(x) - f_n(a_0) = \int_{a_0}^{x} f_n'(t)\, dt$$

Observe that $f_n(x) \underset{n\to\infty}{\longrightarrow} f(x)$ and $f_n(a_0) \underset{n\to\infty}{\longrightarrow} f(a_0)$ pointwise, and $\int_{a_0}^{x} f_n'(t)\, dt \to \int_{a_0}^{x} g(t)\, dt$ by the integrability of uniform limits. Then

$$f(x) - f(a_0) = \int_{a_0}^{x} g(t)\, dt,\ \forall x \in (a,b)$$

which implies $f \in C^1$ and $f' = g$. ■

### Interesting Applications

$S_n(x) = \sum_{k=0}^{n} f_k(x)$.
Suppose pointwise convergence, that $S_n'(x) = \sum_{k=0}^{n} f_k'(x)$ is continuous, $|f_k'(x)| \le M_k$ and $\sum_{k=0}^{\infty} M_k < \infty$.
Long story short, this implies

$$\left(\sum_{k=0}^{\infty} f_k(x)\right)' = \sum_{k=0}^{\infty} f_k'(x)$$

### Example

$f(x) = \sum_{n=0}^{\infty} \frac{\cos(nx)}{n^3}$
Call $u_n(x) = \frac{\cos(nx)}{n^3}$, then $|u_n(x)| \le \frac{1}{n^3}$ summable and $|u_n'(x)| = \left|\frac{-\sin(nx)}{n^2}\right| \le \frac{1}{n^2}$ summable.
This implies $f'(x) = -\sum_{n=0}^{\infty} \frac{\sin(nx)}{n^2}$.
Repetition of this process informs us that $f \in C^2$.

### Power Series

$S_n(x) = \sum_{k=1}^{n} a_k(x - x_0)^k$ for, $x_0 \in \mathbb{R}$ fixed, is 'centered at $x_0$.' Note that each term is $C^\infty(\mathbb{R})$.

### Example 1

$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ for $|x| < 1$.

**Example 2**

$\exp(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges $\forall x \in \mathbb{R}$.

- Why?
  Ratio Test.

$$\frac{a_{k+1}}{a_k} = \frac{x^{k+1}}{(k+1)!} \cdot \frac{k!}{x^k} = \frac{x}{k+1}$$

  So $\left|\frac{a_{k+1}}{a_k}\right| \xrightarrow[k\to\infty]{} 0$

**Lemma: Radius of Convergence**

Suppose a power series $\sum_{n\geq 0} a_n x^n$ converges at $b \in \mathbb{R}$.

1. Converges absolutely $\forall x, |x| < |b|$.

2. $\forall a \in (0, b)$ converges uniformly on $[-a, a]$.

- Proof of 1
  Suppose $\sum_{n\geq 0} a_n b^n$ converges.
  Then $a_n b^n \to 0$.
  Let $x$ such that $|x| < b$, then

$$|a_n x^n| = \left|a_n b^n \left(\frac{x}{b}\right)^n\right| \leq M\left(\frac{|x|}{b}\right)^n$$

  By term test, $\sum_{n=0}^{\infty} |a_n x^n| < \infty \implies \sum a_n x^n$ converges absolutely.

- Proof of 2
  If $|x| \leq a < b$,

$$|a_n x^n| \leq M\left(\frac{|x|}{b}\right)^n \leq M\left(\frac{a}{b}\right)^n$$

  Thus, by $M$-test for $x \in [-a, a]$, the series converges uniformly on $[-a, a]$.

- Upshot
  The set where a power series converges is an interval centered at $x_0$.

## Theorem: Radius of Convergence

Given a power series, define $R$ to be such that $\frac{1}{R} = \limsup_n |a_n|^{1/n}$. Then

1. $\forall a \in (0, R)$, the series converges uniformly on $[-a, a]$.

2. If $|x| > R$, the series diverges.

**Proof**

IMAGE HERE - RADIUS OF CONVERGENCE
Fix $x$. As an exercise, $\limsup_n |a_n x^n|^{1/n} = |x| \cdot \limsup_n |a_n|^{1/n} = \frac{|x|}{R}$.
Recall that $\limsup_n |a_n x^n|^{1/n} = \lim_{n\to\infty} y_n$ where $y_n = \sup_{k>n}\{|a_k x^k|^{1/k}\}$.
If $\frac{|x|}{R} < 1$, then $\exists N_0, n \geq N_0 \implies y_n < \frac{1+\frac{|x|}{R}}{2} < 1$.
This implies $\forall k \geq N_0, |a_k x^k|^{1/k} \leq \frac{1+\frac{|x|}{R}}{2} < 1$ and, by the root test, the series converges.
If $\frac{|x|}{R} > 1$, $\forall n, \sup_{k>n}\{|a_k x^k|^{1/k}\} \geq \frac{|x|}{R}$.
By the properties of the supremum with $\epsilon = \left(\frac{|x|}{R} - 1\right)/2 > 0$,

$$\forall n, \exists k, 1 \leq \frac{\frac{|x|}{R}+1}{2} \leq y_n - \epsilon \leq |a_k x^k|^{1/k} \leq y_n$$

Therefore $\forall n, \exists k > n, |a_k x^k|^{1/k} \geq 1$. ■

## Observation: Behavior at Endpoints

At the endpoints of $(-R, R)$, a series might

### Converge Absolutely

e.g. $\sum_{k=1}^{\infty} \frac{x^k}{k^2},\ R = 1,\ \frac{1}{R} = \limsup_n \left(\frac{1}{n^2}\right)^{1/n} \xrightarrow{n\to\infty} 1$

### Converge Conditionally

e.g. $\sum_{k=1}^{\infty} \frac{x^k}{k},\ R = 1 \longrightarrow \frac{1}{R} = \limsup_n \left(\frac{1}{n}\right)^{1/n} = 1$
Converges conditionally at $x = -1$.

### Diverge

e.g. $\sum_{k=0}^{\infty} x^k,\ R = 1$

## Theorem: Power Series Differentiation

Let $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ converge on $(x_0 - R, x_0 + R)$.
Then $\forall k > 0, f \in C^k(x_0 - R, x_0 + R)$ and $f^{(k)}(x) = \sum_{n=k}^{\infty} a_n n(n-1)\cdots(n-k+1)(x-x_0)^{n-k}$, $\forall x \in (x_0 - R, x_0 + R)$

### Exercise

Show that if $a_n \to a > 0$, then $\limsup a_n b_n = a \limsup b_n$.

**Proof (by Induction)**

Consider the series $S_n(x) = \sum_{n=1}^{\infty} a_n n(x - x_0^{n-1} = \sum_{n=0}^{\infty} a_{n+1}(n+1)(x-x_0)^n$.
Then

$$(x - x_0)\frac{1}{R \text{ of series of derivatives}} = \limsup_{n\to\infty}(a_n n)^{1/n} \limsup_{n\to\infty} a_n^{1/n} n^{1/n} = \limsup a_n^{1/n} = \frac{1}{R}$$

This implies $\sum_{k=0}^{\infty} \frac{d}{dx}(a_k(x-x_0)^k)$ converges uniformly on $[x_0 - a, x_0 + a]$, $\forall a \in (0, R)$.
By the Theorem on Differentiability of Uniform Limits, $f'(x)$ exists and $\forall x \in (x_0 - R, x_0 + R)$

$$f'(x) = \sum_{n=1}^{\infty} a_n n(x - x_0)^{n-1}$$

Repeat to get higher derivatives.

**Integration**

It is similarly possible to integrate term by term.

## Famous Power Series

- $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \; |x| < 1$
- PSE of $\frac{1}{x}$ centered at $x_0 > 0$

IMAGE HERE - GRAPH

$$\frac{1}{x} = \frac{1}{x - x_0 + x_0} = \frac{1}{x_0} \cdot \frac{1}{1 + \frac{x-x_0}{x_0}} = \frac{1}{x_0}\sum_{k=0}^{\infty}\left(-\frac{x-x_0}{x_0}\right)^k = \sum_{k=0}^{\infty}\frac{(-1)^k}{x_0^{k+1}}(x-x_0)^k \text{ if } |x - x_0| < |x_0|, x \in (0, 2x_0)$$

- $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
- $\exp(0) = 1$
- $\exp'(x) = \sum_{k=1} \frac{kx^{k-1}}{k!} = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} = \exp(x)$

**Law of Exponents**

$\exp(a)\exp(b) = \exp(a+b), \forall a, b \in \mathbb{R}$

### Proof

Special case of the "Cauchy product of convergent series."
If $\sum_{n\geq 0} a_n$ converges absolutely to $A$ and $\sum_{n\geq 0} b_n$ converges to $B$, then $\sum_{n\geq 0} c_n$ converges to $AB$, where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$$

- Heuristics

$$\left(\sum_{p=0}^{\infty} a_p x^p\right)\left(\sum_{l=0}^{\infty} b_l x^l\right) = \sum_{p,l\in\mathbb{N}_0^2} a_p b_l x^{p+l}$$

  IMAGE HERE - CIRCLES FROM L TO P

$$\{(p,l) : p+l = n, p, l \in \mathbb{N}_0\} = \{(0,n),(1,n-1),\ldots,(n,0)\}$$

### Proof Continued

Aexp$(a) = \sum_{k=0}^{\infty} \frac{a^k}{k!}$ and $\exp(b) = \sum_{l=0}^{\infty} \frac{b^l}{l!}$, thus $\exp(a)\exp(b) = \sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} \frac{(a+b)^n}{n!} = \exp(a+b)$ \\) since

$$c_n = \frac{1}{n!}\sum_{n=0}^{n} \frac{a^k}{k!} \cdot \frac{b^{n-k}}{(n-k)!} \text{ and } n! = \frac{1}{n!}(a+b)^n$$

## Power Series Expansion of Exponential

Centered at $x_0$, we have

$$\exp(x) = \exp(x-x_0)\exp(x_0) = \exp(x_0)\sum_{k=0}^{\infty}\frac{(x-x_0)^k}{k!}$$

### Observation:

exp is the only $C^1(\mathbb{R})$ solution of $\begin{cases} \exp'(x) = \exp(x) \\ \exp(0) = 1 \end{cases}$

- Proof If $f$ solves the above, then for some constant $c$

$$\frac{d}{dx}\left(f(x)\exp(-x)\right) = f'(x)\exp(-x) - f(x)\exp(-x) = 0 \underset{\text{MVT}}{\Longrightarrow} f(x)\exp(-x) = c = f(0)\exp(-0) = 1$$

  this implies

$$f(x) = \exp(x)f(x)\exp(-x) = \exp(x)$$

## Exponential Features

$$\exp(x) > 0, \forall x \in \mathbb{R} \implies \begin{cases} \text{if } x \geq 0, \exp(x) \geq 1 > 0 \\ \text{if } x < 0, \exp(x) = \frac{1}{\exp(-x)} > 0 \end{cases}$$

### Theorem: Exponential and e

$\exp(x) = (\exp(1))^x \forall x \in \mathbb{R}$ and $e = \exp(1)$

### Proof

Using law of exponents for

$$x \in \mathbb{N}: \quad \exp(n) = \exp(1 + (n-1) = e \cdot \exp(n-1) = \cdots = e^n \exp(0)$$

$$x = \frac{1}{q}, q \in \mathbb{N}: \quad \left(\exp\left(\frac{1}{q}\right)\right)^q = \exp\left(\frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}\right) = \exp(1) = e$$
$$\therefore \exp\left(\frac{1}{q}\right) = e^{1/q}$$

$$x = \frac{p}{q}, p, q \in \mathbb{N}: \quad \exp\left(\frac{p}{q}\right) = \exp\overbrace{\left(\frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}\right)}^{p} = \left(e^{1/q}\right)^p = e^{p/q}$$

$$x \in -\mathbb{N}, \mathbb{Q} < 0: \quad \text{left as an exercise}$$

Therefore, the functions $x \mapsto \begin{cases} \exp(x) \\ e^x \end{cases}$ are continous on $\mathbb{R}$ and agree on $\mathbb{Q}$. This implies that they must be equal everywhere.

## October 23, 2023

### Today

Exp and log.
Real-analytic functions. (Newest bit of information.)
Trig functions.

### Wednesday, October 25, 2023

Analytic vs $C^\infty$
Approximation by polynomials.

### Next Week

Fourier series.

### Exponential and Log

#### Covered Last Lecture

Law of Exponents
$\exp(x) = e^x$ and $e = \exp(1) = \sum_{k=0}^{\infty} \frac{1}{k!}$

### Error Estimate

$e = \lim_{n\to\infty} S_n$ where $S_n = \sum_{k=0}^{\infty} \frac{1}{k!}$ (increases).
$e - S_n = \sum_{k=n+1}^{\infty} \frac{1}{k!}$
For $k = n+1+p,\ p \geq 0$, $e - S_n = \sum_{p=0}^{\infty} \frac{1}{(n+1+p)!}$.
Then,

$$\begin{aligned} \frac{1}{(n+1+p)!} &= \frac{1}{(n+1)!} \cdot \underbrace{\frac{1}{(n+2)(n+3)\cdots(n+p+1)}}_{p \text{ factors}} \\ &\leq \frac{1}{(n+1)!} \cdot \frac{1}{(n+1)^p} \end{aligned}$$

and

$$\begin{aligned} e - S_n &= \sum_{k=n+1}^{\infty} \frac{1}{k!} \\ &= \sum_{p=0}^{\infty} \frac{1}{(n+1+p)!} \\ &\leq \frac{1}{(n+1)!} \cdot \sum_{p=0}^{\infty} \left(\frac{1}{n+1}\right)^p \\ &= \frac{1}{(n+1)!} \cdot \frac{1}{1 - \frac{1}{n+1}} \\ &= \frac{1}{(n+1)!} \cdot \frac{n+1}{n} \end{aligned}$$

Therefore,

$$0 \leq e - S_n \leq \frac{1}{n!} \cdot \frac{1}{n}$$

## Theorem: e is Irrational

### Proof

Suppose $e = \frac{p}{q}$, $q > 2$, and $p$ and $q$ coprime. Consider

$$\begin{aligned} 0 < e - S_q &\leq \frac{1}{q!} \cdot \frac{1}{q} \\ 0 < q!(e - S_q) &\leq \frac{1}{q} \\ 0 < q!\left(\frac{p}{q} - \sum_{k=0}^{q} \frac{1}{k!}\right) &\leq \frac{1}{q} < \frac{1}{2} \end{aligned}$$

where $q!\left(\frac{p}{q} - \sum_{k=0}^{q} \frac{1}{k!}\right) \in \mathbb{N}$.
This is a contradiction. Thus, $e$ must be irrational. ∎

## Exponential Decay

$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
$\lim_{x\to+\infty} x^k e^{-k} = 0, \forall k \in \mathbb{N}$
For $x > 0$, $\exp(x) \geq \frac{x^{k+1}}{(k+1)!}$ if and only if $x^k \exp(-x) \leq \frac{(k+1)!}{x} \underset{x\to+\infty}{\longrightarrow} 0$.

## Exponential Strictly Positive Over Reals

$\exp(x) > 0, \forall x \in \mathbb{R}$
$x > 0$ is obvious.
$x \leq 0, \exp(x) = \frac{1}{\exp(-x)} > 0$
$\lim_{x\to-\infty} \exp(x) = \lim_{x\to-\infty} \frac{1}{\exp(-x)} = 0$.

## Proposition: Exponential is a Bijection

$\exp : \mathbb{R} \to (0,\infty)$ is a $C^\infty$ ($\exp' = \exp$) bijection (diffeomorphism in the sense that $\exp'(x) > 0, \forall x \in \mathbb{R}$).
By Inverse Function Theorem then, define $\log : (0,\infty) \to \mathbb{R}$ such that $\exp(\log(x)) = x$.
By MATH 105A, $\frac{d}{dx}(\log(x)) = \frac{d}{dx}(\exp^{-1}(x)) = \frac{1}{\exp'(\exp^{-1}(x))} = \frac{1}{\exp(\log(x))} = \frac{1}{x}$.
$\log(1) = 0$ (since $\exp(0) = 1$) which implies, by the Fundamental Theorem of Calculus, that $\log(x) - \log(1) = \int_1^x \frac{dt}{t}$.

## Properties (left as an exercise)

- $\log(xy) = \log(x) + \log(y),\ x, y > 0$

- Power Series Expansion: $\log(1-x) = -\sum_{k=0}^{\infty} \frac{x^k}{k}$, $x$ near 0, radius of convergence: 1.

- $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = \exp(x)$

## Definition: Real-Analytic Functions

A function $f : (a,b) \to \mathbb{R}$ is real-analytic on $(a,b)$ if $\forall x_0 \in (a,b)$, $\exists r > 0$ and a power series $\sum_{n\geq 0}(x-x_0)^n$ converging to $f$ on $(x_0 - r, x_0 + r)$.
When such a power series exists, $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$, then

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

The radius of convergence is related by $\frac{1}{R} = \limsup_n |a_n|^{1/n}$ which provides a contraint on rate of divergence.

### Example 1: Polynomial

For every polynomial, $p : \mathbb{R} \to \mathbb{R}$, and $\forall x_0 \in \mathbb{R}$,

$$p(x) = \sum_{k=0}^{\infty} \frac{p^{(k)}(x_0)}{k!}(x-x_0)^t, \forall x \in \mathbb{R}$$

### Example 2: Exponential

$$\exp(x) = \exp(x - x_0 + x_0) = \sum_{k=0}^{\infty} \frac{e^{x_0}}{k!}(x - x_0)^t$$

and the radius of convergence, $R = \infty$.

### Example 3: 1/x

$\frac{1}{x}$ analytic on $(0, \infty)$
$\frac{1}{x} \sum (x - x_0)^k$ and $R = |x_0|$.

$0 \quad x_0$

## Remark: Analyticity Implies Smoothness

$f$ analytic on $(a, b) \implies f$ smooth $(C^\infty)$ on $(a, b)$
The converse is not true. (Example Wednesday)

## Proposition:

Suppose $\sum_{n \geq 0} a_n (x - x_0)^n$ converges to $f(x)$ on $(x_0 - R, x_0 + R)$.
Then $f(x)$ is analytic on $(x_0 - R, x_0 + R)$.

$x_0 \quad x_1 \quad x_0 + R$ That is to say, $\forall x_1 \in (x_0 - R, x_0 + r)$, there exists some power series expansion for $f$, centered at $x_1$, with positive radius of convergence.

### Proof

Let $x_0 = 0$ for simplicity and $x_1 \in (-R, R)$.

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n (x - x_1 + x_1)^n = \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} (x - x_1)^k x_1^{n-k}$$

Assuming that rearangement is possible, this is

$$\sum_{n,k,n \geq 0} a_n \binom{n}{k} (x - x_1)^k x_1^{n-k} = \sum_{k=0}^{\infty} \underbrace{\left( \sum_{n=k}^{\infty} a_n \binom{n}{k} x_1^{n-k} \right)}_{b_k} (x - x_1)^k$$

Need to prove two things:

1. $b_k$ is well-defined

2. Interchange of sums valid.

- Proof of 1

  For $k$ fixed, $\binom{n}{k}$ is a d° k (degree k) polynomial in $n$.
  Letting

$$b_k = \sum_{p=0}^{\infty} a_{p+k} \binom{p+k}{k} x_1^p$$

  where $p = n - k$, we have

$$\limsup_{p\to\infty} \left( |a_{p+k}| \binom{p+k}{k} \right)^{1/p} = \limsup_{p\to\infty} |a_p|^{1/p}$$

  since $x_1 \in (-R, R)$, $b_k < \infty, \forall k$.

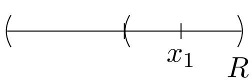
- Proof of 2

  The proof requires invoking Fubini's Theorem to allow rearrangement.
  Need to check that

$$\sum_{n,k,n\geq k} |a_n| \binom{n}{k} \left| (x - x_1)^k x_1^{n-k} \right|$$

  converges.
  Consider

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |a_n| \binom{n}{k} |x - x_1|^k \, |x_1|^{n-k} = \sum_{n=0}^{\infty} |a_n| \left( |x - x_1| + |x_1| \right)^n$$




  Make sure that $|x - x_1| = r < R - |x_1|$, then

$$\sum_{n=0}^{\infty} |a_n| r^n$$

  where $r < R$ which, by absolute convergence of the original power series, is finite. ■

## Remark: Analytic Continuation

The process of recentering a power series is also called "analytic continuation."
The radius of convergeence of the new series might actually be larger and allow the orgiginal function.

### Example

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

IMAGE HERE - Decaying curve.

## Facts: Analytic Functions

- If $f, g$ are analytic on $(a, b)$, then so is $f \cdot g$.

- If $f, g$ are analytic and $g$ does not vanish on $(a, b)$, then $\frac{f}{g}$ is analytic.

- If $f$ is analytic on $(x_0 - R, x_0 + R)$ and $g$ is analytic on $(f(x_0) - \delta, f(x_0) + \delta)$, then $g \circ f$ is analytic on a neighborhood of $x_0$. (Proof in ; page number in lecture notes).

## Remark: No Analytic Bump Functions

IMAGE HERE - BUMP FUNCTION –|-n-|–

## Trig Functions

IMAGE HERE – UNIT CIRCLE
We want $(\cos(\theta), \sin(\theta))$ to be the point on the unit circle making an arclength $\theta$ from $(1, 0)$.
For $x$ in the right-half plane, $\cos(\theta) \geq 0$.
For $x$ in top right quadrant,

$$\theta = \int_0^{\sin(\theta)} \sqrt{1 + (f'(y))^2}\, dy$$

Then, $y \mapsto (\underbrace{\sqrt{1 - y^2}}_{f(y)}, y)$, $y \in (-1, 1)$. It follows that

$$\theta = \lim_0^{\sin(\theta)} \frac{dy}{\sqrt{1 - y^2}} \underset{\text{FTC}}{\Longrightarrow} \arcsin'(x) = \frac{1}{\sqrt{1 - x^2}} \in C^\infty((-1, 1))$$

and

$$\arcsin(x) = \lim_0^x \frac{dy}{\sqrt{1 - y^2}}$$

Therefore, arcsin is a diffeomorphism from $(-1, 1) \to (\lim_{x \to -1} \arcsin(x), \lim_{x \to 1} \arcsin(x))$.
Since $\frac{1}{\sqrt{1 - x^2}}$ is integrable near $\pm 1$, theese limits are finite.

## Definition: Pi

$\pi = 2 \lim_{x \to 1} \arcsin(x)$

### Inverse Function Theorem

$\sin : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to (-1, 1)$ exists as a $C^1$ inverse of arcsin.
On $\left(-\frac{\pi}{2}, \frac{pi}{2}\right)$, define $\cos(\theta) = +\sqrt{1 - \sin^2(\theta)}$. Then

$$\sin'(\theta) = \frac{1}{\arcsin'(\sin(\theta))} = \sqrt{1 - \sin^2(\theta)} = \cos(\theta).$$

Similarly, $\cos'(\theta) = -\sin(\theta) \rightsquigarrow \sin, \cos$ are $C^\infty$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

### Extension to the Reals

By graphical or geometric arguments, for $\theta \in \left(0, \frac{\pi}{2}\right)$,

$$\cos(\theta) = -\sin\left(\theta - \frac{\pi}{2}\right)$$
$$\sin(\theta) = -\cos\left(\theta - \frac{\pi}{2}\right)$$

This helps extend to $\mathbb{R}$, with $2\pi$-periodicity such that

$$\begin{cases} \cos' & = -\sin \\ \sin' & = \cos \\ \cos(0) & = 1 \\ \sin(0) & = 0 \end{cases}$$

Therefore, you get all derivatives at $x = 0$ and the corresponding Taylor expansion looks like

$$C(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \qquad S(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

We find that $R = \infty$ for both, and

$$C(0) = 1, \qquad S(0) = 0, \qquad C'(x) = -S(x), \qquad S'(x) = C(x).$$

Take

$$\epsilon(x) = (C(x) - \cos(x))^2 + (S(x) - \sin(x))^2$$

with $\epsilon(0) = 0$. Then, finally,

$$\epsilon'(x) = 0 \implies \epsilon = \text{some constant} = 0.$$

## October 25, 2023

### Today

Analytic vs $C^\infty$
Approximation by Polynomials

### Definition: Real Analytic

$f$ is real analytic on $(a, b)$ if $\forall x_0 \in (a, b), \exists \delta > 0, \{a_n\}_n$ such that $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$, $\forall x \in (x_0 - \delta, x_0 + \delta)$.

### Proposition: Analyticity Implies Smoothness

Analytic on $(a, b) \implies C^\infty$ smooth on $(a, b)$.

$$\sum_{n=0}^{\infty} (x - x_0)^n \rightsquigarrow a_n - \frac{f^n(x_0)}{n!}$$

Note: $C^w(a, b) \notin C^\infty(a, b)$
The converse is not true.

**Example**

Let $x \in \mathbb{R}$ and $f(x) = \begin{cases} 0 & x < 0 \\ \exp\left(\frac{-1}{x^2}\right) & x > 0 \end{cases}$
IMAGE HERE - FUNCTION
$x \neq 0$, $f \in C^{\infty}(\mathbb{R} \setminus 0)$.

- What about at $x = 0$?

$$\lim_{x \to 0; x<0} f(x) = 0 = \lim_{x \to \varnothing; x>0} e^{-\frac{1}{x^2}}$$

So we can define $f(0) = e$, the resulting function is continuous on $\mathbb{R}$.

- What about higher derivatives?

Claim: $\forall k > 0$, $\lim_{x \to 0; x>0} \frac{d^k}{dx^k}\left(e^{-\frac{1}{x^2}}\right) = 0$

- Proof (Sketch)

$$\frac{d}{dx}\left(e^{-x^2}\right) = 2x^{-3}e^{-x^{-2}}$$

$$\lim_{x \to 0} \frac{e^{-\frac{1}{x^2}}}{x^3} \overset{y=\frac{1}{x}}{=} \lim_{y \to \infty} y^3 e^{-y^{-2}} = 0$$

Claim by induction:

$$\frac{d^k}{dx^k}\left(e^{-\frac{1}{x^2}}\right) = p_k(1/x)e^{-\frac{1}{x^2}}$$

for some polynomial $p_k$.
If the claim is true, then

$$\lim_{x \to \varnothing} p_k\left(\frac{1}{x}\right)e^{-\frac{1}{x^2}} = \lim_{y \to +\infty} p_k(y)e^{-y^2} = 0 \quad \blacksquare$$

Then we can extend $f^{(k)}$ as a continious function on $\mathbb{R}$ such that $f^{(k)}(0) = 0$.

- Claim

$f(x)$ is not analytic on any neighborhood of $x_0 = 0$.
If it were, it would equal $\sum_{n=0}^{\infty} a_n x^n$ on $(-\delta, \delta)$ for some $a_k$s. But,

$$a_k = \frac{f^{(k)}(0)}{k!} = 0 \qquad \text{then} \qquad \sum_{n=0}^{\infty} a_n x^n = 0, \forall x \in (-\delta, \delta)$$

which is impossible, since $f(x) \neq 0$ whenever $x > 0$. $\blacksquare$

**Remark: Contraposition Can Disprove Analyticity**

The existance of a non-zero radius of convergence for $\sum a_k(x-x_0)^k$ means

$$\frac{1}{R} = \limsup_n |a_n|^{1/n} = \left(\frac{f^{(n)}(x_0)}{n!}\right)^{1/n} < \infty$$

and
$\left(\frac{f^{(n)}(x_0)}{n!}\right)^{1/n} \rightsquigarrow f^{(n)}(x_0) \leq n! \left(\frac{c}{R}\right)^n$
for some constant $c$.

**Remark: Analyticity is Not Guaranteed**

The conditions

$$\begin{cases} h \in C^{\infty}(R) \\ \limsup_n \left(\frac{h^{(n)}(0)}{n!}\right)^{1/n} < \infty \end{cases}$$

are not sufficient to claim $h$ is analytic on any neighborhood of 0.
Indeed, if $h$ is analytic then $h(x) + f(x)$ will not be for otherwise

$$f(x) = -(h(x) + f(x)) - h(x)$$

would also be analytic, which it isn't.

## Definition: Exponential Blip Function

Let $g(x) = \frac{f(x+1)f(1-x)}{f(1)^2}$, where $f$ is the "exponential glue" function.
IMAGE HERE - FUNCTION
Smooth on $\mathbb{R}$; $g(x) \geq 0$.

## TODO Theorem: Borel

TODO - Name for theorem?
Given any sequence $\{a_n\}_n$ of reals and any $\begin{cases} x_0 \in \mathbb{R} \\ \lambda > 0 \end{cases}$, $\exists f \in C^{\infty}(\mathbb{R})$ such that

$$\begin{cases} f^{(k)}(x_0) = a_k & \forall k \\ f(x) = 0 & \text{if } |x - x_0| > \lambda \end{cases}$$

IMAGE HERE - BUMPY MOUNTAIN CLOSE TO X0

**Proof**

Reductions: $x_0 = 0$ and $\lambda = 1$.
Ansatz: $f(x) = \sum_{k=0}^{\infty} b_k x^k g\left(\frac{x}{\lambda_k}\right)$ where $b_k$s and $\lambda_k$s need to be tuned.
IMAGE HERE - G(X) and G(X/LAMBDA K)

$g(x) = 0 \iff |x| \geq 1$ and $g\left(\frac{x}{\lambda_k} = 0 \iff \left|\frac{x}{\lambda_k}\right| \geq 1 \iff |x| \geq \lambda_k\right)$
Observations: if $\lambda_k \xrightarrow[k\to\infty]{} 0$, then $\forall x \neq 0$ the series is actuall finite!
Since $g\left(\frac{x}{\lambda_k} = 0\right)$ once $\lambda_k < |x|$.
Therefore, convergent and $C^\infty$ on $\mathbb{R} \setminus \{0\}$.
Constraints:

$$\begin{aligned} a_0 &= f(0) = b_0 \\ a_1 &= f'(0) = \frac{d}{dx}\left(b_0 g\left(\frac{x}{\lambda_0}\right)\right)|_{x=0} + b+_1 \end{aligned}$$

Generally,

$$a_n = \sum_{k=0}^{n-1} \frac{d^n}{dx^n}\left(b_k x^k g\left(\frac{x}{\lambda_k}\right)\right)|_{x=0} + n! b_n$$

If $\lambda_n$ are chosen, these constraints uniquely determine the $b_n$s.

### How to Choose Lambdas?

Want to enforce

$$\max_{0\leq k\leq n-1} \sup_{x\in\mathbb{R}} \left|\frac{d^k}{dx^k}\left(b_n x^n g\left(\frac{x}{\lambda_n}\right)\right)\right| \leq 2^{-n}$$

- Example

  Determine $\lambda_2$:

$$\begin{aligned} k = 0 &: \left|b_n x^n g\left(\frac{x}{\lambda_n}\right)\right| \leq |b_n| \lambda_n^n 2^{-n} \\ k = 1 &: \left|b_n \left(n x^{n-1} g\left(\frac{x}{\lambda_n}\right)\right) + b_n x^n \frac{1}{\lambda_n} g'\left(\frac{x}{\lambda_n}\right)\right| \leq |b_n| \lambda_n^{n-1}(n + ||g'||_\infty) \leq 2^{-n} \end{aligned}$$

  In general,

$$a\lambda_n^p < 2^{-n}$$

  for $p > 0$.
  So we construct $b_0$, then $\lambda_0$, then $b_1$, then $\lambda_1$,...

### Claim: Produces Uniform Convergence

When

$$\max_{0\leq k\leq n-1} \sup_{x\in\mathbb{R}} \left|\frac{d^k}{dx^k}\left(b_n x^n g\left(\frac{x}{\lambda_n}\right)\right)\right| \leq 2^{-n}$$

is satisfied, $\forall k \in \mathbb{N}$

$$\sum_{n=0}^{\infty} \frac{d^k}{dx^k}\left(b_n x^n g\left(\frac{x}{\lambda_n}\right)\right)$$

satisfies the Weierstrass M-Test. Therefore it is uniformly convergent. Because

$$\sum_{n=0}^{\infty} \left|\frac{d^k}{dx^k}\left(b_n x^n g\left(\frac{x}{\lambda_n}\right)\right)\right| \le \underbrace{\sum_{n=0}^{k} \left|\frac{d^k}{dx^k}\left(b_n x^n g\left(\frac{x}{\lambda_n}\right)\right)\right|}_{\text{finite sum, uniformly bounded}} + \sum_{n=k+1}^{\infty} 2^{-n}$$

### Approximation by Polynomials

**Goal (Weierstrass Approximation Theorem):**

If $f : [a,b] \to \mathbb{R}$ is continuous on the compact set $[a,b]$, then there exists a sequence of polynomials $p_n$ such that $\lim_{n\to\infty} \sup_{x\in[a,b]} |f(x) - p_n(x)| = 0$.
That is, polynomials are dense in $(C([a,b]), ||\cdot||_\infty)$, where $||f||_\infty := \sup_{x\in[a,b]} |f(x)|$.
How to do this?

**Lagrange Interpolation**

Give $f \in C([a,b])$.
Idea: subdivide $[a,b]$ with $a = x_0 < x_1 < \cdots < x_n < b$ where $x_k = x_0 + k\left(\frac{b-a}{n}\right)$.
IMAGE HERE - UNIFORM SUBDIVISION
Let $p_n(x) = \sum_{k=0}^{n} f(xk) \prod_{j\neq k} \frac{x-x_j}{x_k-x_j}$.
Problem: the Runge phenomenon.
IMAGE HERE - SMOOTHEST FUNCTION I CAN THINK OF (use the bump again) 1/(1+25x$^2$)

### Definition: Convolution

Take $f, g : \mathbb{R} \to \mathbb{R}$, define

$$h(x) = f * g(x) = \int_{\mathbb{R}} f(t)g(x-t)\,dt \underset{y=x-t}{=} \int_{\mathbb{R}} f(x-y)g(y)\,dy = g * f(x)$$

Take $f, g \in C(\mathbb{R})$ with compact support ($C_C(\mathbb{R})$). That is, they vanish outside a compact set.
IMAGE HERE - F AND G CONVOLVED

### Definition: Approximation of Identity

An approximation of the identity is a sequence $\{g_n\}_n$, all piecewise continuous, defined on $\mathbb{R}$ such that

$$\begin{cases} g_n(x) \ge 0 & \forall x \\ \int_{\mathbb{R}} g_n(x)\,dx = 1 & \\ \forall \delta > 0, & \lim_{n\to\infty} \int_{|x|>\delta} g_n(x)\,dx = 0 \end{cases}$$

IMAGE HERE - CONVOLUTION ACCUMULATING BETWEEN -DELTA AND DELTA

### Example

Let $g_n(x) = \frac{n \cdot g(nx)}{\int_{\mathbb{R}} g(x)\, dx}$.

### Lemma:

If $\{g_n\}_n$ is an approximation of identity, then $\forall f \in C_C(\mathbb{R})$

$$g_n * f \rightrightarrows f$$

on $\mathbb{R}$.

## October 30, 2023

### Today

Approximation by polynomials.
Fourier Series.

### Recall: Convolution

$f, g \in C_C(\mathbb{R})$, $f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)\, dy$.

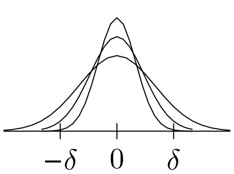
### Recall: Approximation of Identity

$\{g_n\}_n$ where $g_n : \mathbb{R} \to \mathbb{R}$ is piecewise continuous (this is overkill but sufficient).

1. $\int g_n\, dx = 1$.

2. $g_n(x) \geq 0$.

3. $\forall \delta > 0, \lim_{n\to\infty} \int_{|x|>\delta} g_n(x)\, dx = 0$.




$f \rightsquigarrow \{g_n * f\}_n$.

### Example

Take any $g(x) \geq 0$ (piecewise continuous) with $\int_{\mathbb{R}} g(x)\, dx = 1$.
Define $g_n(x) = n \cdot g(nx)$.
Claim: this defined an approximation of identity.

### Lemma: Convolution of Approximation of Identity Converges Uniformly

Suppose $\{g_n\}_n$ is an approximation of identity.
Then, for any $f \in C_C(\mathbb{R})$,

$$g_n * f \quad \text{converges uniformly to } f \text{ on } \mathbb{R}$$

That is to say, $\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |g_n * f(x) - f(x)| = 0$.

### Proof

Since $\int_{\mathbb{R}} g_n(y)\, dy = 1$,

$$\begin{aligned}
g_n * f(x) - f(x) &= \int_{\mathbb{R}} g_n(y) f(x-y)\, dy - f(x) \cdot \int_{\mathbb{R}} g_n(y)\, dy \\
&= \int_{\mathbb{R}} g_n(y) \left( f(x-y) - f(x) \right)\, dy \\
&= \int_{|y|\geq\delta} g_n(y) \overbrace{\left( f(x-y) - f(x) \right)}^{\leq 2M}\, dy + \int_{|y|>\delta} g_n(y) \overbrace{\left( f(x-y) - f(x) \right)}^{\epsilon}\, dy
\end{aligned}$$

By assumption, $f \in C_C(\mathbb{R})$ so $f$ is bounded by $M$ on $\mathbb{R}$.
$f$ is continuous on $\operatorname{supp}(f)$, which is compact, so $f$ is uniformly continuous on $\mathbb{R}$.
Let $\epsilon > 0$ be given.
By uniform continuity, $\exists \delta > 0, \forall x, y \in \mathbb{R}, |x-y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{2}$.
By the Aproximation of Identity property, $\exists N, \forall n \geq N, \int_{|y|\geq\delta} g_n(y)\, dy < \frac{\epsilon}{4M}$.
For $n \geq N$,

$$\begin{aligned}
|g_n * f(x) - f(x)| &= \left| \leq \int_{\mathbb{R}} g_n(y) \left( f(x-y) - f(x) \right)\, dy \right| \\
&\leq \int_{|y|\geq\delta} g_n(y) \overbrace{\left| f(x-y) - f(x) \right|}^{\leq 2M}\, dy + \int_{|y|>\delta} g_n(y) \overbrace{\left| f(x-y) - f(x) \right|}^{\leq \frac{\epsilon}{2} \text{ since } |x-y-x| = |y| < \delta}\, dy \\
&\leq 2M \frac{\epsilon}{4M} + \frac{\epsilon}{2} \overbrace{\int_{|y|<\delta} g_n(y)\, dy}^{\leq 1} \\
&\leq \epsilon, \quad \forall x \in \mathbb{R} \quad \blacksquare
\end{aligned}$$

### Recall: Riemann Integral Properties

If $f$ is Riemann integrable, then

$$\left| \int f\, dx \right| \leq \int |f|\, dx$$

$$\left| \sum_{n=1}^{\infty} S_n \right| \leq \sum_{n=1}^{\infty} |S_n|$$

$$\left| \int f^+\, dx - \int f^-\, dx \right| \leq \int f^+\, dx + \int f^-\, dx = \int (f^+ + f^{-1})\, dx$$

### Theorem: Weierstrass Approximation Theorem

If $[a,b]$ is compact, then $\forall f \in C([a,b])$, there exists a sequence of polynomials $p_n(x)$ converging uniformly to $f$.

#### Step 1

Extend $f$ into $F \in C_C(\mathbb{R})$.
IMAGE HERE - EXTEND FUNCTION

$$F(x) = \begin{cases} 0 & \text{on } (-\infty, a-1] \cup [b+1, \infty) \\ f(x) & \text{on } [a,b] \\ f(a)(x-(a-1)) & \text{on } [a-1,a] \\ f(b)(b+1-x) & \text{on } [b,b+1] \end{cases}$$

#### Step 2

Note: $\forall \{g_n\}_n$ Approximation of Identity, $g_n * f \rightrightarrows F(x)$ on $\mathbb{R}$ (by previous lemma),
and $\sup_{x\in[a,b]} |g_n * F(x) - f(x)| \leq \sup_{x\in\mathbb{R}} |g_n * F(x) - F(x)|$.
Trick: Construct $g_n$ such that $g_n * F$ is a polynomial on $[a,b]$.
Answer:

$$g_n(x) = \begin{cases} a_n \left(1 - \frac{x^2}{(b-a+1)^2}\right)^n & \text{if } x \in [-(b-a+1), b-a+1] \\ 0 & \text{otherwise} \end{cases}$$

where $a_n$ is chosen such that $\int_{\mathbb{R}} g_n(x)\, dx = 1$.
IMAGE HERE - NARROWING GAUSSIAN WITH PEAK AT (0,1) BETWEEN -(b-a+1) and b-a+1
If $x \in [a,b]$ and $y \in [a-1,b+1]$, then

$$-b-1 \leq -y \leq -a+1 \implies -(b-a+1) \leq x-y \leq b-a+1$$

Then

$$\begin{aligned} g_n * F(x) &= \int_{a-1}^{b+1} F(y) \underbrace{g_n(x-y)}_{a_n\left(1-\frac{(x-y)^2}{(b-a+1)^2}\right)^n = \sum_{p=0}^{2n} x^p a_{p,n(y)}} dy \\ &= \sum_{p=0}^{2n} x^p \int_{a-1}^{b+1} F(y) a_{p,n(y)}\, dy \quad \blacksquare \end{aligned}$$

## Background: Fourier Series

### Historical Perspective

In Strichartz.
Associated with solving the wave equation on $[0,L]_x \times [0,T]_t$ (Bernoulli) and the heat equation (Fourier).

**Wave Equation**

On $[0,L]_x \times [0,T]_t$, $u(x,t)$ displacement field.
IMAGE HERE - WAVE FROM 0 to L PEAK OF FIRST OSCILLATION AT U(X,T)

---

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c\frac{\partial^2 u}{\partial x^2}(x,t)$$

where $c$ is a fixed coefficient.
Plus Initial Conditions and Boundary Conditions

$$\begin{aligned} \text{Initial Condition :} \quad & u\big|_{t=0}(x) = f(x) \\ & \frac{\partial u}{\partial t}\big|_{t=0}(x) = 0 \\ \text{Boundary Conditions :} \quad & u(0,t) = u(L,t) = 0 \end{aligned}$$

Observation: if $f(x) = \sin\left(\frac{k\pi x}{L}\right)$,
IMAGE HERE - THREE SINUSOIDAL WAVES OVERLAPPING
Ansatz: $u(x,t) = \sin\left(\frac{k\pi x}{L}\right)g(t)$.
Plug into the PDE:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \sin\left(\frac{k\pi x}{L}\right)g''(t) \\ c\frac{\partial^2 u}{\partial x^2} &= -\frac{k^2\pi^2}{L^2}c^2\sin\left(\frac{k\pi x}{L}\right)g(t) \end{aligned}$$

Setting

$$\sin\left(\frac{k\pi x}{L}\right)g''(t) = -\frac{k^2\pi^2}{L^2}c^2\sin\left(\frac{k\pi x}{L}\right)g(t) \underset{g}{\overset{\text{ode for}}{\Longrightarrow}} g'' = -\frac{k^2\pi^2}{L^2}c^2 g$$

Which gives a general solution

$$g(t) = A\cos\left(\frac{k\pi ct}{L}\right) + B\sin\left(\frac{k\pi ct}{L}\right).$$

Initial conditions imply that $g(0) = 1$ and $g'(0) = 0$ which gives

$$g(t) = \cos\left(\frac{k\pi ct}{L}\right).$$

Thus

$$u(x,t) = \sin\left(\frac{k\pi x}{L}\right)\cos\left(\frac{k\pi x}{L}\right)$$

Solves the PDE!

### Wave Equation Superposition

Consider instead

$$f(x) = \sum_{k=0}^{n} \sin\left(\frac{k\pi x}{L}\right) a_k$$

Then

$$u(x,t) = \sum_{k=0}^{n} a_k \sin\left(\frac{k\pi x}{L}\right) \cos\left(c\frac{k\pi x}{L}\right)$$

### Next Question:

What if $f$ is more general?
$\implies$ existence of Fourier series?
In what sense do they converge?

## Definition: Fourier Series

Context: $f : [-\pi, \pi) \to \mathbb{R}$ Riemann-Integrable or
$f : \mathbb{R} \to \mathbb{R}$ $2\pi$-periodic. ($f(x + 2\pi) = f(x), \forall x$.
The Fourier series of $f$:

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k[f] \cos(kx) + b_k[f] \sin(kx)$$

where $a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\, dx$ and $b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\, dx$.
Alternatively,

$$S_n(x) = \sum_{k=-n}^{n} c_k e^{ikx}$$

where $c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx$.
As an exercise: relate $c_k$s to $a_k$s and $b_k$s and prove that these are equivalent.

### Question:

In what sense does $S_n(x)$ converge to $f(x)$? That is

- For what topology?
  - Uniform Convergence: $\sup_{x \in [-\pi,\pi)} |S_n(x) - f(x)| \xrightarrow[n\to\infty]{} 0$
  - $L^2$ Convergence: $\int_{-\pi}^{\pi} |S_n(x) - f(x)|^2\, dx \xrightarrow[n\to\infty]{} 0$
- What are the (smoothness) requirements on $f$?
  - Observation: if $f(x) = \sum_{k=-N}^{N} f_k e^{ikx}$ is a trigonometric polynomial, then, for $n \geq N$, $S_n(x) = f(x)$.

### Lemma: The Kronecker Delta

Fix $N \in \mathbb{N}$
If $\sum_{k=-N}^{N} f_k e^{ikx} = \sum_{k=-N}^{N} c_k e^{ikx}$, then $f_k = c_k, \forall k$.
Note

$$\int_{-\pi}^{\pi} e^{ikx} e^{-imx}\, dx = \int_{-\pi}^{\pi} e^{i(k-m)x}\, dx = \begin{cases} 2\pi & \text{if } k = m \\ \left[\frac{1}{i(k-m)} e^{i(k-m)x}\right]_{-\pi}^{\pi} = 0 & \text{otherwise} \end{cases}$$

Why $-imx$?

$$\langle if, g\rangle = i\langle f, g\rangle$$
$$\langle f, ig\rangle = -i\langle f, g\rangle$$

and

$$\int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx$$

## November 1, 2023

### Fourier Series

For $f$ Riemann-integrable on $(-\pi, \pi)$ , define

$$S_n(x) = \sum_{k=-n}^{n} c_k e^{ikx}$$

with

$$c_k := \frac{1}{2\pi} \in_{-\pi}^{\pi} f(x) e^{-ikx}\, dx$$

Then $f : [-\pi, \pi) \to \mathbb{R}$.

### Recall

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} e^{-ilx}\, dx = \begin{cases} 0 & \text{if } k \neq l \\ 1 & \text{if } k = l \end{cases} = \delta_{kl} \text{ (the Kronecker delta)}$$

### Definition: Norm

$|| \cdot || : E \to \mathbb{R}_{\geq 0}$ is a "norm" on $E$ if

1. $||f|| = 0 \iff f \equiv 0$

2. $||\lambda f|| = |\lambda| \cdot ||f||, \forall \lambda \in \mathbb{R}, f \in E$

3. $||f + g|| \leq ||f|| + ||g||$

**Definition: Normed Space**

$(E, ||\cdot||)$ is a normed space.
e.g. $(\mathbb{R}, |\cdot|)$ or $(\mathbb{Q}, |\cdot|)$

**Definition: Complete Space**

$(E, ||\cdot||)$ is complete if every cauchy sequence in $E$ converges in $E$.

## In what sense does a Fourier series converge?

Depends on regularity of $f$ and the topology used.

**Note**

On $C([-\pi, \pi])$, can put 2 norms.

- $||f||_\infty = \sup_{x \in [-\pi,\pi]} |f(x)|$

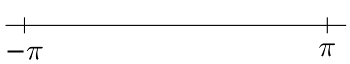
  $d(f, g) = ||f - g||_\infty$: "$f_n$ converges uniformly to $f$" $\leftrightarrow \lim_{n\to\infty} ||f_n - f||_\infty = 0$.
  $(C([-\pi, \pi]), ||\cdot||_\infty)$ is complete.

- $||f||_2 := \left(\int_{-\pi}^{\pi} f^2(x)\, dx\right)^{1/2}$

  "$f_n$ converges to $f$ in $L^2$" $\leftrightarrow \lim_{n\to\infty} ||f_n - f||_2 = 0$.
  $(C([-\pi, \pi]), ||\cdot||_2)$ is not complete.

**Example**

Take $f(x) = \begin{cases} 1 & \text{if } |x| \le \pi/2 \\ 0 & \text{if } |x| > \pi/2 \end{cases}$
IMAGE HERE - BOX FUNCTION FROM -pi/2 to pi/2




Then

$$c_k = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{-ikx}\, dx = \frac{1}{2\pi}\left[\frac{1}{-ik}e^{-kx}\right]_{-\pi/2}^{\pi/2} \frac{1}{2\pi}\frac{1}{-ik}\left[e^{-ik(\pi/2)} - e^{ik(\pi/2)}\right] = \frac{1}{k\pi}\sin(k(\pi/2))$$

So $c_k = 0$ and for $k = 2p+1$: $c_{2p+1} = \frac{(-1)^p}{\pi(2p+1)}$.
IMAGE HERE - BOX FUNCTION WITH SUNUSOIDALS APPROXIMATING
However, the approximation will over and undershoot at the boundaries. This is the "Gibbs Phenomenon", and the discrepency is roughly 12%.
For $k < 0$:

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{ikx}\, dx = \overline{c_{-k}} = c_k$$

In the end,

$$S_{2p+1}(x) = \sum_{l=1}^{p} \frac{(-1)^p}{\pi(2p+1)} \underbrace{\left(e^{i(2p+1)x} + e^{-i(2p+1)x}\right)}_{2\cos((2p+1)x)}$$

**Theorem: Uniform Convergence of Continuously Differentiable Continuous Functions**

1. If $f$ is $C^2$, $2\pi$-periodic, then $S_n \rightrightarrows f$ on $[-\pi, \pi)$.

   Moreover, $||S_n - f||_\infty \leq \frac{c}{n}$ for some $c > 0$.

1. If $f \in C^1$, $2\pi$-periodic, same conclusion with $||S_n - f||_\infty \leq \frac{c}{\sqrt{n}}$ for some $c > 0$.

**Proof of Part 1**

Write

$$S_n(x) = \sum_{k=-n}^{n} c_k e^{ikx} = \sum_{k=-n}^{n} \frac{1}{2\pi} \in_{-\pi}^{\pi} f(y) e^{-iky}\, dy\, e^{ikx} = \frac{1}{2\pi} \in_{-\pi}^{\pi} f(y) \underbrace{\left[\sum_{k=-n}^{n} e^{ik(x-y)}\right]}_{D_n(x-y)} dy$$

Where $D_n(t) = \sum_{k=-n}^{n} e^{ikt}$ is the "Dirichlet kernel."
That is $S_n$ is a convolution of $f(y)$ with some kernel.

$$e^{it} \cdot D_n(t) = \sum_{k=-n}^{n} e^{i(k+1)t} \underset{l=k+1}{=} \sum_{l=-n+1}^{n+1} e^{ilt} = D_n(t) + e^{i(n+1)t} - e^{-int}$$

Therefore

$$D_n(t) = \frac{e^{i(n+1)t} - e^{-int}}{e^{it} - 1} = \frac{e^{(it)/2}\left(e^{i(n+(1/2))t} - e^{-i(n+(1/2))t}\right)}{e^{(it)/2}\left(e^{(it)/2} - e^{-(it)/2}\right)} = \frac{\sin((n+1/2)t)}{\sin(t/2)}$$

IMAGE HERE - DN(T) OSCILLATING WITH MANY ZEROS THEN PEAKING TO 2N+1 at X=0

So

$$\int_{-\pi}^{\pi} D_n(t)\, dt = 2\pi$$

Then

$$\begin{aligned} S_n(x) - f(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) D_n(x-y)\, dy - f(x) \\ &\underset{z=x-y}{=} \frac{1}{2\pi} \int_{-\pi}^{pi} f(x-z) D_n(z)\, dz - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) D_n(z)\, dz \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x-z) - f(x)) D_n(z)\, dz \end{aligned}$$

and

$$S_n(x) \cdot f(x) = \frac{1}{2\pi} \underbrace{\frac{(f(x-y) - f(x))}{\sin(y/2)}}_{\text{call } g_x(y) = \frac{f(x-y)-f(x)}{\sin(y/2)}} \sin((n + (1/2)y)\, dy$$

If $g_x(y)$ was differentiable (in fact $C^1$), then integrating by parts

$$S_n(x) - f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_x(y) \sin((n + (1/2)y)\, dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_x'(y) \frac{\cos((n + (1/2)y)}{n + (1/2)}\, dy$$

Then

$$|S_n(x) - f(x)| \leq \sup_{y \in [-\pi,\pi]} \left| g_x'(y) \right| \frac{1}{n + (1/2)}$$

- Claim
  If $f \in C^2$, $2\pi$-periodic, then $\sup_{x \in [-\pi,\pi]} |g_x'(y)| < \infty$. Then the first part of the theorem is proved.

  - Proof of Claim
    $f \in C^2 \implies g_x \in C^2$ away from $y = 0$. ($g_x''(y)$ = differentiation rules).
    At $y = 0$ , write

    $$f(x-y) - f(x) = \int_x^{x-y} f'(t)\, dt$$

    Changing variables such that $t = x + u(x - y - x) = x - uy$ for $u \in [0, 1]$ gives $dt = -y\, du$

    $$= -y \int_0^1 f'(x - uy)\, du$$

    Therefore

    $$g_x(y) = \underbrace{\left( \frac{-y}{\sin(y/2)} \right)}_{\text{smooth near } y=0} \int_0^1 f'(x - uy)\, du$$

    Calling the smooth piece $h(y)$,

    $$g_x(y) = h(y) \int_0^1 f'(x - yu)\, du$$

    is differentiable at 0 if and only if $\frac{d}{dy} \left( \int_0^1 f'(x - yu)\, du \right) = \int_0^1 f''(x - yu)(-u)\, du$ exists. ■

**Proof of Part 2 (Sketch)**

$$S_n(x) - f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_x(y) \sin(n + (1/2)y)\, dy$$

If $f$ is only $C^1$, then $g$ is $C^1$ away from 0, so it is unclear near $y = 0$. So, for some $\delta$ to be chosen later

$$S_n(x) - f(X) = \frac{1}{2\pi}\underbrace{\int_{[-\delta,\delta]} g_x(y)\sin((n+(1/2))y)\ dy}_{\leq \frac{2\delta}{2\pi}(||f'||_\infty + ||f||_\infty)} + \frac{1}{2\pi}\underbrace{\int_{[-\pi,\pi]\setminus[-\delta,\delta]} g_x(y)\sin((n+(1/2))y)\ dy}_{\text{integrate by parts}}$$

Study $\int_\delta^\pi$ (study of $\int_{-\pi}^{-\delta}$ is similar)

$$\begin{aligned}
\int_\delta^\pi g_x(y)\sin((n+(1/2)y)\ dy &= \int_\delta^\pi g_x(y)\frac{d}{dy}\left(\frac{-\cos((n+(1/2))y)}{n+(1/2)}\right)\ dy \\
&= \int_\delta^\pi \frac{d}{dy}\left(g_x(y)\frac{-\cos((n+(1/2))y)}{n+(1/2)}\right) - \int_\delta^\pi g_x'(y)\frac{\cos((n+(1/2))y)}{n+(1/2)}\ dy \\
&= -g_x(\pi)\frac{\cos((n+(1/2))\pi)}{n+(1/2)} + g_x(\delta)\frac{\cos((n+(1/2))\delta)}{n+(1/2)} - \int_\delta^\pi g_x'(y)\frac{\cos((n+)1/2))y)}{n+(1/2)}\ dy
\end{aligned}$$

Problem:

$$g_x'(y) = \frac{-f'(x-y)\sin(y/2) - (1/2)\cos(y/2)(f(x-y)-f(x))}{(\sin(y/2))^2} \approx \frac{c}{y} \text{ near } y = 0$$

So

$$\left|\int_\delta^\pi g_x(y)\sin((n+(1/2))y)\ dy\right| \leq \frac{1}{n+(1/2)}\cdot\frac{1}{\delta}$$

Combining all estimates, for $\delta > 0$

$$|S_n(x) - f(x)| \leq C_1\delta + C_2\frac{1}{n\delta}$$

Since we are free to choose $\delta$, we may optimize over $\delta$.
Balancing out the terms is done by choosing $\delta = \delta(n)$ such that

$$\delta \overset{n\to\infty}{\sim} \frac{1}{n\delta} \iff n\delta^2 \sim 1 \iff \delta \sim \frac{1}{\sqrt{n}}$$

which gives

$$|S_n(x) - f(x)| \leq C_1\delta + C_2\frac{1}{n\delta} = \frac{C_1}{\sqrt{n}} + C_2\frac{1}{n\frac{1}{\sqrt{n}}} \leq \frac{C_1 + c_2}{\sqrt{n}}$$

- Comment on the Sketch

  Morally, we want $|g_x'(y)| \leq \frac{c}{y}$ for some constant $c$.
  Numerator:

  $$\left|-f'(x-y)\sin(y/2) - (1/2)\cos(y/2)(f(x-y)-f(x))\right| \leq ||f'||_\infty(y/2) + (\cdots)y \leq Cy$$

  Since $|\sin(y/2)| \leq (y/2)$,

$$\begin{aligned}|\sin(x) - \sin(0)| &= |\cos(\xi)||x - 0| \\ &= 1|x|\end{aligned}$$

Denominator

$$(\sin(y/2))^2 \geq \left(\frac{2y}{2\pi}\right)^2 = \frac{y}{\pi}^2$$

So,

$$\left|g_x'(y)\right| \leq \frac{Cy}{\left(\frac{y}{\pi}\right)^2} \leq \frac{C^1}{y}$$

### Theorem: Continuous, Periodic Functions Converge in L2

If $f$ is continuous, $2\pi$-periodic, then $\lim_{n\to\infty} ||S_n - f||_2 = 0$.
That is, $\lim_{n\to\infty} \int_{-\pi}^{\pi} |S_n - f(x)|^2 \, dx = 0$.
IMAGE HERE - PERIODIZE f(x)=x THEN APPROXIMATE WITH FOURIER

## November 6, 2023

### Recall: Fourier Series

$$f : [-\pi, \pi] \to \mathbb{R} \text{ or } \mathbb{C}$$

Fourier Coefficient:

$$c_k(f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{ikx} \, dx, \ k \in \mathbb{Z}$$

$$s_n(x) = \sum_{k=-n}^{n} c_k e^{ikx} = \frac{1}{2\pi} \in_{-\pi}^{\pi} D_n(x-y) f(y) \, dy$$

Dirichlet Kernel:

$$D_n(y) := \frac{\sin((n+1/2)y)}{\sin((1/2)y)}$$

### Theorem: L2 Convergence of Sn to N

If $f$ is $C^0$, $2\pi$-periodic, then

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} |s_n(x) - f(x)|^2 \, dx = 0$$

## Recall: Kronecker Delta

For $m, n \in \mathbb{Z}$,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{imx} e^{-imx} \, dx = \delta_{m,n} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

That is $\{e^{inx}\}_{n \in \mathbb{Z}}$ is an orthnormal system for the inner product

$$\begin{aligned} &\xi \times \xi \to \mathbb{C} \\ &(f, g) \mapsto \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} \, dx \end{aligned}$$

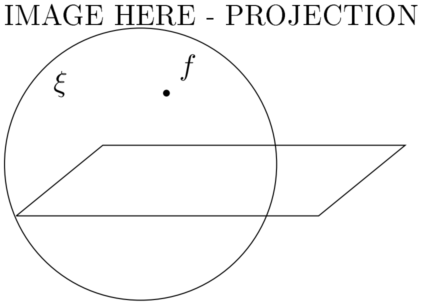
where $\xi = \{f : \mathbb{R} \to \mathbb{C}, \ 2\pi\text{-periodic}, \ \text{continuous}\}$.

## Example

For $f \in \xi$, fixing $n \in \mathbb{N}_0$, consider the map

$$\begin{aligned} &\mathbb{C}^{2n+1} \to \mathbb{R} \\ &(d_{-n}, \dots, d_n) \mapsto \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - \sum_{k=-n}^{n} d_k e^{ikx}|^2 \, dx \end{aligned}$$

IMAGE HERE - PROJECTION




- Claim:
  $F_n$ is minimal if and only if $\lambda_k = c_k(f), \ \forall -n \leq k \leq n$.

  - Proof:
    Take any $\lambda_n, \lambda_{n+1}, \dots, \lambda_n$ and set $t_n(x) = \sum_{k=-n}^{n} \lambda_k e^{ikx}$. Then

    $$\int_{-\pi}^{\pi} |f(x) - t_n(x)|^2 \, dx = \int_{-\pi}^{\pi} |f(x) - s_n(x) + s_n(x) - t_n(x)|^2 \, dx$$

    Then, since

    $$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) = |z_1|^2 + |z_2|^2 + 2 \cdot \Re(z_1 \overline{z_2})$$

    $$\int_{-\pi}^{\pi} |f(x) - t_n(x)|^2 \, dx = \int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 \, dx + \int_{-\pi}^{\pi} |s_n(x) - t_n(x)|^2 \, dx + 2 \cdot \Re \int_{-\pi}^{\pi} (f(x) - s_n(x))(\overline{s_n(x) -$$

What to Show: Integral on real part is zero.

$$
\begin{aligned}
A &= \int_{-\pi}^{\pi} (f(x) - s_n(x)) \overline{\sum_{k=-n}^{n} (c_k - \lambda_k) e^{ikx}} \, dx \\
&= \sum_{k=-n}^{n} \overline{(c_k - \lambda_k} \underbrace{\int_{-\pi}^{\pi} (f(x) - s_n(x)) e^{-ikx} \, dx}_{2\pi(c_k - c_k) = 0}
\end{aligned}
$$

Since

$$
\int_{-\pi}^{\pi} s_n(x) e^{-ikx} \, dx = \int_{-\pi}^{\pi} \sum_{p=-n}^{n} c_p e^{ipx} e^{-ikx} \, dx = 2\pi c_k
$$

It follows that

$$
\begin{aligned}
\underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - t_n(x)|^2 \, dx}_{F_n(\lambda_{-n}, \ldots, \lambda_n)} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 \, dx + \frac{1}{2\pi} \int_{-\pi}^{\pi} |t_n(x) - s_n(x)|^2 \, dx \\
&\geq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_n(x)| \, dx \\
&\geq F_n(c_{-n}, \ldots, c_n)
\end{aligned}
$$

Moreover:

$$
\begin{aligned}
\frac{1}{2\pi} \int_{-\pi}^{\pi} | \underbrace{t_n(x) - s_n(x)|^2|}_{\underbrace{(t_n - s_n)}_{\sum_{p=-n}^{n} (\lambda_p - c_p) e^{ipx}} \cdot \overline{(t_n - s_n)}} |^2 \, dx &= \frac{1}{2\pi} \sum_{p,l=-n}^{n} (\lambda_p - c_p) \overline{(\lambda_l - c_l)} \underbrace{\int_{-\pi}^{\pi} e^{ipx} e^{-ilx} \, dx}_{\delta_{p,l}} \\
&= \frac{1}{2\pi} \sum_{p=-n}^{n} |\lambda_p - c_p|^2
\end{aligned}
$$

Conclusion:

* $\forall (\lambda_{-n}, \ldots, \lambda_n \neq (c_{-n}, \ldots, c_n), F_n(\lambda_{-n}, \ldots, \lambda_n) > F_n(c_{-n}, \ldots, c_n)$
* $F_n(c_{-n}, \ldots, c_n) = F_n(c_{-n}, \ldots, c_n)$
* Lemma

For all trigonometric polynomials of degree at most $n$, of the form $\sum_{k=-n}^{n} \lambda_k e^{ikx} = t_n(x)$,

$$
\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 \, dx \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - t_n(x)|^2 \, dx
$$

and

$$
\frac{1}{2\pi} \int_{-\pi}^{\pi} |t_n(x)|^2 \, dx = \sum_{k=-n}^{n} |\lambda_k|^2
$$

Apply this to $(\lambda_{-n}, \dots, \lambda_n) = (0, \dots, 0)$:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 \, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 \, dx + \sum_{k=-n}^{n} |c_k|^2$$

As a consequence, for all $n$,

$$\sum_{k=-n}^{n} |c_k|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 \, dx$$

which implies that $\sum_{k=-n}^{n} |c_k|^2$ converges absolutely and, in particular, $c_k \to 0$ as $k \to \infty$.

### Riemann-Lebesgue Lemma

The above proves that if $f \in \xi$ (more generally, if $f$ is Riemann-integrable), then

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} f(x) e^{\pm ikx} \, dx = 0$$

Moreover, sending $n \to \infty$, we get

$$\lim_{n\to\infty} \sum_{k=-n}^{n} |c_k|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 \, dx$$

Importantly, there is equality whenever $\lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 \, dx = 0$.
When does that happen?

### Theorem:

If $f \in \xi$, then

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 \, dx = 0$$

### Proof

For $n \geq 0$, define $\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} s_k$ (the "Cesano sum").
Then

$$\sigma_n \in \operatorname{span}\langle e^{-inx}, \dots, e^{inx} \rangle.$$

In particular,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 \, dx \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - \sigma_n(x)|^2 \, dx \leq \left( \sup_{[-\pi,\pi]} |f - \sigma_n| \right)^2$$

What to show: $\sigma_n \rightrightarrows f$ on $[-\pi, \pi]$.
Recall that

$$s_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x-y)f(y)\,dy$$

$$\begin{aligned}\sigma_n(x) &= \frac{1}{n+1}\sum_{k=0}^{n} s_k(x)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x-y)f(y)\,dy\end{aligned}$$

Where

$$\begin{aligned}K_n(y) &= \frac{1}{n+1}\sum_{k=0}^{n} D_k(y)\\ &= \frac{1}{n+1}\frac{1}{\sin(y/2)}\sum_{k=0}^{n}\sin((k+1/2)y)\end{aligned}$$

Using $2\sin((k+1/2)y)\sin(y/2) = \cos(ky) - \cos((k+1)y)$.

$$\begin{aligned}&= \frac{1}{n+1}\frac{1}{(\sin(y/2))^2}\frac{1}{2}\underbrace{\sum_{k=0}\cos(ky)-\cos((k+1)y)}_{\underbrace{\frac{1-\cos((n+1)y)}{2}}_{\sin^2((\frac{n+1}{2})y)}}\\ &= \frac{1}{n+1}\left(\frac{\sin\left(\left(\frac{n+1}{2}\right)y\right)}{\sin(y/2)}\right)^2\end{aligned}$$

This is the Féjer kernel.
IMAGE HERE - FÉJER KERNEL
Claims:

1. $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)\,dy = 1$

2. $K_n(y) \geq 0$ on $[-\pi,\pi]$ (obvious)

3. $\forall\delta > 0$, $K_n \rightrightarrows 0$

- Proof of 1

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)\,dy = \frac{1}{2\pi}\frac{1}{n+1}\sum_{k=0}^{n}\underbrace{\int_{-\pi}^{\pi} D_k(y)\,dy}_{2\pi} = 1$$

- Proof of 3 If $|y| \geq \delta$,

$$|K_n(y)| = \frac{1}{n+1}\frac{\overbrace{|\sin((n+1)y/2)|^2}^{\leq 1}}{|\sin(y/2)|^2}$$

Recall $|sin(x)| \geq \frac{2|x|}{\pi}$

$$\leq \frac{1}{n+1}\frac{1}{(|y|/\pi)^2}$$
$$\leq \frac{1}{n+1}\frac{1}{(\delta/\pi)^2}$$

Which goes unfiormly to 0 on $[-\pi,\pi]\setminus[-\delta,\delta]$ as $n\to\infty$.
What to show: $K_n * f \rightrightarrows f$ on $[-\pi,\pi]$.
The proof scheme is dentical to: if $f\in C_c(\mathbb{R})$ and $K_n$ is an approximation of identity, then $K_n * f \rightrightarrows f$ on $\mathbb{R}$.
Left as an exercise.

### Corollary: Parseval's Equality

$\forall \delta \in \xi$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx = \lim_{n\to\infty}\sum_{k=-n}^{n}|c_k|^2$$

### Remark:

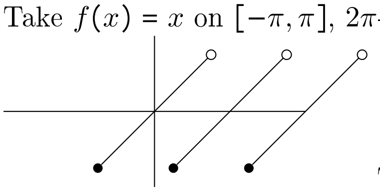
This should hold for a larger class of function.

- Piecewise Continuous

- $L^2$ functions

#### Example

Take $f(x) = x$ on $[-\pi,\pi]$, $2\pi$-periodized



Then $\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$.

### Application to Solving the Heat Equation

On $[0,L]_x\times\mathbb{R}_+$, $u(x,t)$ is the "heat distribution"
IMAGE HERE - ONE DIMENSIONAL ROD HEAT EQUATION YADA YADA

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial x}\right) = 0$$

**Problem**

| | | |
|---|---|---|
| PDE | $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ | on $[0,L] \times [0,T]$ |
| Boundary Conditions | $u(0,t) = u(L,t) = 0$ | |
| Initial Conditions | $u(x,0) = f(x)$ | $f$ continuous, $f(0) = f(L) = 0$ |

IMAGE HERE - POSITION TIME PLANE

- Step 1: Separation of Variables

  Seek an ansatz of the form

$$u(x,t) = g(x)h(t)$$

Where

$$\begin{aligned} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} &\iff g(x)h'(t) = g''(t)h(t) \\ &\iff \frac{h'(t)}{h(t)} = \frac{g''(x)}{g(x)} = c \end{aligned}$$

Left Solving:

$$g''(x) = cg(x) \quad g(0) = 0 = g(L)$$

$$h'(t) = ch(t) \rightsquigarrow h(t) = h(0)e^{ct}$$

Then

$$\begin{aligned} g''(x) - cg(x) = 0 \rightsquigarrow c = 0. &\quad g(x) = a + bx \\ c > 0. &\quad g(x) = ae^{\sqrt{c}x} + be^{-\sqrt{c}x} \\ c < 0. &\quad g(x) = a\cos(\sqrt{-c}x) + b\sin(\sqrt{-c}x) \end{aligned}$$

and

$$g(0) = 0 = g(L) \implies \begin{cases} c = 0: & g \equiv 0 & \text{(no solution)} \\ c > 0: & g \equiv 0 & \text{(no solution)} \\ c < 0: & a = 0. \quad g(x) = b\sin(\sqrt{-c}x) \end{cases}$$

$$\begin{aligned} g(L) = 0 &\implies \sin(\sqrt{-c}k) = 0 \\ &\implies L\sqrt{-c} = k\pi \\ &\implies c = -\left(\frac{k\pi}{L}^2\right), k \in \mathbb{N}_0 \end{aligned}$$

For $c = -\left(\frac{k\pi}{L}\right)^2 = \lambda_k$,

$$g_k(x) = \sin\left(\frac{k\pi x}{L}\right)$$
$$h_k(x) = h_k(0)\exp\left(-\left(\frac{k\pi}{L}\right)^2 t\right)$$

For all $k \in \mathbb{N}_0$,

$$u_k(x,t) = g_k(x)h_k(t)$$

solves the heat equation with boundary conditions.
Initial conditions $g_k(x)$, fix $h_k(0) = 1$.
Ansatz for a solution:

$$u(x,t) = \sum_{k=0}^{\infty} a_k g_k(x) h_k(t) \implies u(x,0) = \sum_{k=0}^{\infty} a_k \sin\left(\frac{k\pi x}{L}\right) = f(x)$$

Thus, the left hand side is the solution.