Partial Differential Equations I

January 8, 2024

Homework

Assigned exercises and concept maps. Graded by completion.

Presentations

Assigned topics; responsible for giving a class.

Definition: Partial Differential Equation(s) (PDE)

An identity relating an uknown function, its partial derivatives and its variables.

$$F(D^k u, \ldots, D^2 u, Du, u, x) = 0, \quad x \in U \subseteq \mathbb{R}^n$$

where U is an open subset of $\mathbb{R}^n$, $u : U \subset \mathbb{R}^n \to \mathbb{R}$, $Du = (\partial_{x_1} u_1, \ldots, \partial_{x_n} u)$.

Then $F : \mathbb{R}^{n^k} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, where F is given.

$x = (x_1 \ldots, x_n)$ is (are) the independent variable(s).

u is the unknown function or dependent variable.

k is the order of the PDE.

Goal

Given a PDE, we consider

- Existence

- Uniqueness

- Stability

Recall: Multiindex Notation

$\alpha = (\alpha_1, \ldots, \alpha_n)$ a vector such that $\alpha_i \in \mathbb{Z}_{\geq 0}$.

Then we say that α is a multiindex with order $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Notation

$u : U \subseteq \mathbb{R}^n \to \mathbb{R}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$.

$u^\alpha := D^\alpha u = \partial_{x_n}^{\alpha_n} \cdots \partial_{x_1}^{\alpha_1} u$, where $\partial^0 u = u$.

Definition: Linear Partial Differential Equation

A linear PDE of order k is of the form

$$(*) \sum_{|\alpha|=k} a_\alpha(x) D^\alpha u = f(x)$$

Remark

This means that F is multilinear in the first $n^k + n^{k-1} + \cdots$ variables.

Definition: Homogeneous Linear Partial Differential Equation

A linear given by $(*)$ is homogeneous if $f(x) \equiv 0$.
Otherwise, it is non-homogeneous.

Example 1: Linear Transport Equation

$$\nabla u \cdot (1, b) = u_t + b \cdot Du = f(t, x)$$

This is a linear PDE of order 1 on $\mathbb{R} \times \mathbb{R}^n \equiv \mathbb{R}^{n+1}$ where (t, x) are independent variables and u is dependent.
Here, x is the spatial variable while t is time and Du represents the gradient.
$\nabla u = (\partial_t u, \nabla u)$, $b \cdot Du = \sum_{i=1}^{n} b_i \partial_{x_i} u$, $(b_1, \ldots, b_n) \in \mathbb{R}^n$ is fixed.

Example 2: Laplace Equation

$$\Delta u := \sum_{i=1}^{n} \partial_{x_i} u = 0$$

This is a linear, homogeneous PDE of order 2.

Example 3: Poisson Equation

$-\Delta u := f(u)$
This is a nonlinear PDE of order 2.
Consider $f(u) = u^2$.

Example 4: Heat Equation (Diffusion Equation)

$$u_t - \Delta u = 0$$

This is a linear, homogeneous PDE of order 2.

Example 5: Wave Equation

$$u_{tt} - \Delta u = 0$$

This is a linear, homogeneous PDE of order 2.

Transport Equation

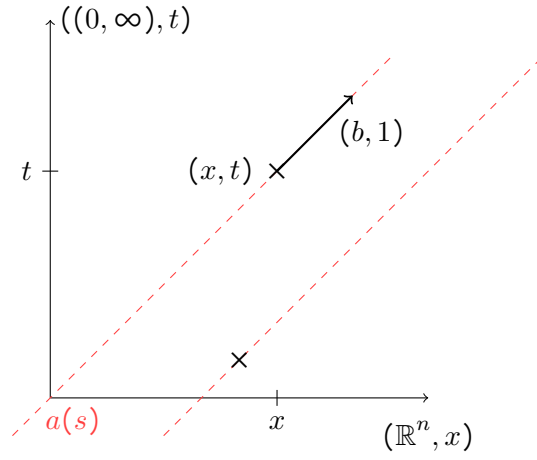$u : \mathbb{R}^n (0, \infty) \to \mathbb{R}$ given by

$$u_t + b \cdot Du = 0, \quad b \in \mathbb{R}^n$$

In order to get a solution, first assume that ther exists a "nice" (e.g. smooth, C^1, differentiable, etc.) solution.

Step 1

Notice that the PDE is equivalent to

$$\nabla u \cdot (b, 1) = 0$$

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Step 2

Consider a curve on $\mathbb{R}^{n+1}$ with velocity $(1, b)$ which passes through (x, t). i.e.

$$\alpha(s) = (x + sb, t + s)$$

Notice $\alpha'(s) = (b, 1)$.

Then, let us study u along the curve $\alpha(s)$.

$$z(s) := u(\alpha(s))$$

Taking the derivative with respect to s,

$$z'(s) := \frac{d}{ds}(u \circ \alpha(s)) = \nabla u|_{\alpha(s)} \cdot \alpha'(s) = \nabla u|_{\alpha(s)} \cdot (b, 1) = 0$$

That is $z'(s) = 0$, $z(s)$ is constant, and u along $\alpha(s)$ is constant.
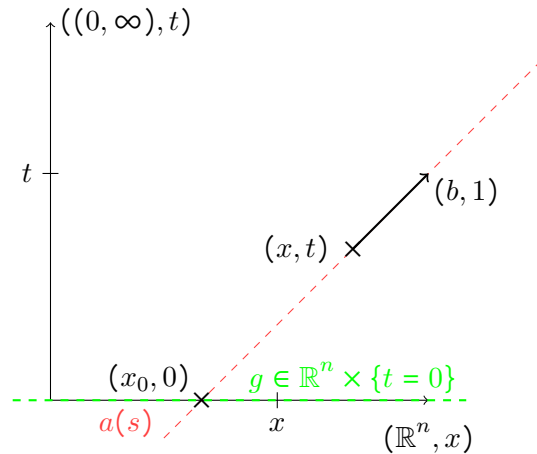
Conclusion

If we know some value of u along $\alpha(s)$, then we know all values along $\alpha(s)$.
If we have some value of u along every $\alpha(s)$, then we know u on $\mathbb{R}^n \times (0, \infty)$.

Transport Equation - Homogeneous Initial Value Problem

$$(*) \begin{cases} \nabla u \cdot (b, 1) = 0, & \mathbb{R}^n \times (0, \infty) \\ u = g, & \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Here, $g : \mathbb{R}^n \to \mathbb{R}$ is given.

Consider (x, t); we want to find $(x_0, 0)$.

We know $\alpha(s) = (x + sb, t + s) = (x_0, 0)$, therefore

$$\begin{cases} x + sb = x_0 & (1) \\ t + s = 0 \implies s = -t & (2) \end{cases}$$

Then, by replacing (2) in (1),

$$x_0 = x - tb$$

Then from the conclusion

$$u(x, t) = u(x_0, 0) = g(x_0) = g(x - tb)$$

Therfore, $u(x, t) := g(x - tb)$ (♥).

Remark

1. If there exists a regular (differentiable or C^1) solution u for ∗, then u should look like ♥.

2. If g is (differentiable or C^1), then u defined by ♥ is a (differentiable or C^1) solution for my problem.

Homework

Show that ♥ satisfies ∗.

Transport Equation - Non-homogeneous Initial Value Problem

$$(*) \begin{cases} \nabla u \cdot (b, 1) = f(x, t), & \mathbb{R}^n \times (0, \infty) \\ u = g, & \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Where $g : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ are given.

Solution

Notice that the PDE is equivalent to

$$\nabla u \cdot (b, 1) = f(x, t)$$

Define the "characteristic curve"

$$\alpha(s) = (x + sb, t + s)$$

and

$$z(s) := u(\alpha(s))$$

Taking $\frac{d}{ds}$,

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$$z'(s) = \nabla u|_{\alpha(s)} \cdot (b, 1) = f(\alpha(s)) \implies z'(s) = f(x + sb, t + s) \;\; (c)$$

Notice that c is an ordinary differential equation. Integrating from $-t$ to 0.

$$\int_{-t}^{0} z'(s) \, ds = \int_{-t}^{0} f(x + sb, t + s) \, ds$$

$$z(0) - z(-t) = \int_{-t}^{0} f(x + sb, t + s) \, ds$$

Notice that $z(0) = u(x, t)$ and $z(-t) = u(\alpha(-t)) = u(x - tb, 0)$.

$$u(x, t) = u(x - tb, 0) + \int_{-t}^{0} f(x + sb, t + s) \, ds$$

Then

$$u(x, t) = g(x - tb) + \int_{-t}^{0} f(x + sb, t + s) \, ds$$

$$\underset{\bar{s} = s + t}{=} g(x - tb) + \int_{0}^{t} f(x + (\bar{s} - t)b, \bar{s}) \, d\bar{s}$$

$$= g(x - tb) + \int_{0}^{t} f(x + (s - t)b, s) \, ds$$

Remark: Method of Characteristics

Try to vert the PDE into an ODE and solve using characteristic curves.

January 10, 2024

Definition: Harmonic Function

If $u \in C^2$ such that $\Delta u = 0$, then u is a harmonic function.

Laplace Equation

Consider $u : U \subseteq \mathbb{R}^n \to \mathbb{R}$ with U open, then the homogeneous (Laplace) form is given by

$$\Delta u = 0$$

and the non-homogeneous (Poisson) form is

$$-\Delta u = f$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is given.

Remark / Exercise

The Laplace equation is invariant under translation and rotation.
That is, if $\Delta u(x) = 0$ and $v(x) = u(x - y)$, then $\Delta v = 0$.
Similarly, if $w(x) = u(O(x))$ then $\Delta w = 0$ where O is an orthogonal matrix.

Fundamental Solution of the Laplace Equation

Remark: since the Laplace equation is invariant under rotation, we can consider a function in terms of the radius $v(x) = v(|x|)$.

Recall $r = |x| = (x_1^2 + \cdots + x_n^2)^{1/2}$.

Because of this remark, assume that $u(x) = v(|x|) = v(r(x))$ $\quad (*)$ where $v : (0, \infty) \to \mathbb{R}$.

Therefore, we need

$$\frac{\partial r}{\partial x_i} = \frac{1}{2} \cdot \frac{2x_i}{(x_1^2 + \cdots + x_n^2)^{1/2}} = \frac{x_i}{r}$$

Replace $(*)$ in the PDE

$$\frac{\partial u}{\partial x_i} = \frac{\partial}{\partial x_i} v(r(x)) = v'(r(x)) \cdot \frac{\partial r}{\partial x_i} = v'(r(x)) \cdot \frac{x_i}{r}$$

and

$$\frac{\partial^2 u}{\partial x_i^2} = \frac{\partial}{\partial x_i} \left(v'(r(x)) \cdot \frac{x_i}{r} \right)$$

$$= \frac{\partial}{\partial x_i} (v'(r(x))) \cdot \frac{x_i}{r} + v'(r(x)) \cdot \frac{\partial}{\partial x_i} \left(\frac{x_i}{r} \right)$$

$$= v''(r(x)) \cdot \frac{x_i^2}{r^2} + v'(r(x)) \left[\frac{1}{r} + x_i \frac{\partial}{\partial x_i} (r) \right]$$

$$= v'' \frac{x_i^2}{r^2} + v' \left[\frac{1}{r} - \frac{x_i^2}{r^3} \right]$$

Thenm, summing across i,

$$\Delta u = v'' + v' \left[\frac{n}{r} + \frac{1}{r} \right] = 0$$

Then the PDE is equivalent to

$$v''(r) + \frac{v'}{r}(n - 1) = 0 \; (\square)$$

We need to find a solution for $\square$.

$$v''(r) = \frac{(1 - n)v'}{r}$$

Assume, without loss of generality, that $v' \neq 0$ such that

$$\frac{v''(r)}{v'(r)} = \frac{1 - n}{r} \implies (\log(|v'|))' = \frac{1 - n}{r}$$

Then, integrating,

$$\log(|v'|) = (1 - n) \log(r) + C = \log \left(r^{1-n} \right) + C$$

such that

$$|v'| = Cr^{1-n} \implies v' = Cr^{1-n} \implies v(r) = Cr^{1-n+1} + D = Cr^{2-n} + D$$

Definition: Fundamental Solution of the Laplace Equation

The function Φ given by

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x|, & n = 2 \\ \frac{1}{n(n-2)\alpha(n)} \cdot \frac{1}{|x|^{n-2}}, & n \geq 3 \end{cases}$$

where $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$ is called the fundamental solution.

Remark

Φ solves the Laplace equation away from 0.

Lemma: Estimates for the Fundamental Solution

- First Estimate
  $|D\Phi(x)| \leq \frac{C}{|x|^{n-1}}$, for $x \neq 0$.

$$\frac{\partial\Phi}{\partial x_i} = C\frac{\partial}{\partial x_i}\left(|x|^{2-n}\right) = \frac{C(2-n)}{1-n}|x|^{2-n-1}\frac{\partial|x|}{\partial x_i} = |x|^{1-n}\cdot\frac{x_i}{|x|} = Cx_i|x|^{-n}$$

  Therefore

$$|D\Phi(x)| \leq C|x||x|^{-n} \implies |D\Phi(x)| \leq C|X|^{1-n}$$

  - Exercise
    Compute for $n = 2$.

- Second Estimate
  $|D^2\Phi(x)| \leq \frac{C}{|x|^n}$, for $x \neq 0$.

$$\begin{aligned}\frac{\partial^2}{\partial x_J\partial x_i}\Phi &= C\frac{\partial}{\partial x_J}\left(x_i|x|^{-n}\right) \\ &= C\left[\delta_{iJ}|x|^{-n} + x_i\frac{\partial}{\partial x_J}|x|^{-n}\right] \\ &= C\left[\delta_{iJ}|x|^{-n} + (-n)\cdot\frac{x_i|x|^{-n-1}x_J}{|x}\right] \\ &= C\left[\frac{\delta_{iJ}|x|}{|x|^n} + \frac{Cx_ix_J}{|x|^{n+1}}\right]\end{aligned}$$

  Then

$$\left|\frac{\partial\Phi}{\partial x_i\partial x_J}\right| \leq \frac{C}{|x|^n} + \frac{C|x_i||x_J|}{|x|^{n+2}} \leq \frac{2C}{|x|^n} = \frac{C}{|x|^n}$$

  Then, we are done since

$$|D^2\Phi(x)| = \sqrt{\sum_J\left(\frac{\partial\Phi}{\partial x_i\partial x_J}\right)}$$

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Poisson Equation

Motivation

Suppose we have $\Phi(x)$, the fundamental solution.
Then for an arbitrary, fixed element $y \in \mathbb{R}^n$, then we have $x \to \Phi(x - y)$ harmonic for $x \neq y$.
Consider $f : \mathbb{R}^n \to \mathbb{R}$ such that $y \to f(y)$ then $x \to f(y)\Phi(x - y)$ is similarly harmonic for $x \neq y$.
Now, if given $\{y_1, \ldots, y_m\}$ where $y_i \in \mathbb{R}^n$, then $x \to \sum_{i=1}^m f(y_i)\Phi(x - y_i)$ is harmonic $\forall x \neq \{y_1, \ldots, y_m\}$.
Then, what happens if we consider

$$u(x) := \int_{\mathbb{R}^n} f(y)\Phi(x - y) \, dy \quad (\square_3)$$

Is u harmonic? No, since $\Delta\Phi(x - y)$ is not summable in $\mathbb{R}^n$ we may not pass the limit into the integral.
(to be covered later) However, since $\Delta\Phi(x - y)$ acts as δ_{xy} in distribution, this may solve the Poisson equation.

Remark / Exercise

Assume that $f \in C_C^2(\mathbb{R}^n)$ (i.e f is twice continuously differentiable with compact support on $\mathbb{R}^n$).
The function Φ is integrable near the singularity on compact sets.
Prove using spherical coordinates.
Therefore, u defined by $\square_3$ is well defined

$$|u| = \left| \int_{\mathbb{R}^n} f(y)\Phi(x - y) \, dy \right| = \left| \int_K \Phi(x - y) \, dy \right| < \infty$$

Theorem: Solving the Poisson Equation

If $f \in C_C^2(\mathbb{R}^n)$ and u is defined by $\square_3$, then

1. $u \in C^2(\mathbb{R}^n)$

2. $-\Delta u = f$, in $\mathbb{R}^n$

- Proof of 1
  Since Φ presents a problem at $x = y$ but f is well behaved, we will change variables such that $\overline{y} = x - y$,
  $y = x - \overline{y}$, and $\frac{dy}{d\overline{y}}(-1)I_{m \times m}$ and then redefine $\overline{y} = y$.

  $$u(x) = \int_{\mathbb{R}^n} f(y)\Phi(x - y) \, dy = \int_{\mathbb{R}^n} f(x - \overline{y})\Phi(\overline{y}) \, d\overline{y} = \int_{\mathbb{R}^n} f(x - y)\Phi(y) \, dy$$

  In short, we have sent the problem from Φ to f.
  Now, let us consider $e_i = (0, \ldots, 1, \ldots, 0)$.
  Then for $h > 0$,

  $$\frac{u(x + he_i) - u(x)}{h} = \frac{1}{h} \int_{\mathbb{R}^n} \Phi(y) \left[f(x + he_i - y) - f(x - y) \right] \, dy$$

  Now, the limit as $h \to 0$

  $$\lim_{h \to 0} \frac{u(x + he_i) - u(x)}{h} = \lim_{h \to 0} \int_{\mathbb{R}^n} \Phi(y) \overbrace{\left[\frac{f(x + he_i - y) - f(x - y)}{h} \right]}^{H(h,y)} \, dy$$

  $$= \int_{\mathbb{R}^n} \Phi(y) \cdot \frac{\partial f(x - y)}{\partial x_i} \, dy$$

To justify passing the limit into the integral, take an arbitrary sequence $h_m \xrightarrow{\to 0} 0$ and consider

$$f_m(y) := H(h_m, y)$$

We want to consider

$$|H(h_m, y)| \leq \Phi(y) \left[\frac{f(x + h_m e_i - y) - f(x - y)}{h} \right]$$
$$\leq \Phi(y) f'(c)$$

Where c is along the curve between $f(x + h_m e_i - y)$ and $f(x - y)$ and chosen by mean value theorem.

– Exercise

$$|H(h_m, y)| \leq \Phi(y) ||f'||_{L^\infty} \chi_{B(x,R)}(y)$$

Note that

$$C \int_{\mathbb{R}^n} |\Phi(y)| \chi_{B(x,R)}(y) \, dy = \int_{B(x,R)} |\Phi(y)| \, dy < \infty$$

– Exercise

Using the fact that a continuous function is uniformly continuous on a compact set, show that $u \in C^2(\mathbb{R}^n)$.

Dominated Convergence Theorem

If $f_m(x)$ such that $f_m(x) \xrightarrow[\text{pointwise}]{m \to \infty} f(x)$, and $|f_m(x)| \leq g(x)$ for $g \in L^1$, then f is integrable and

$$\lim_{m \to \infty} \int f_m(x) \, dx = \int f(x) \, dx$$

January 17, 2024

Recall: Averages

$f : \{1, \ldots, n\} \to \mathbb{R}$
$i \to a(i)$
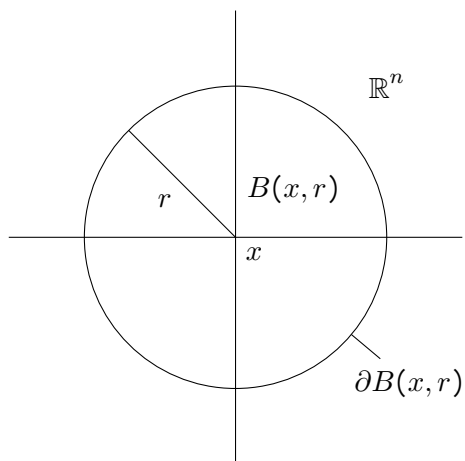The average is given as $\frac{a(i) + \cdots + a(n)}{n}$.
Then for $f : \Omega \to \mathbb{R}$, the average is given as

$$\frac{1}{|\Omega|} \int f(y) \, dy := \fint_\Omega f \, d\mu$$

In our case, $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$

$$\fint_{B(x,n)} f \, d\mu \equiv \frac{1}{|B(x,n)|} \fint_{B(x,n)} f \, d\mu$$

$$\fint_{\partial B(x,n)} f \, d\mu = \frac{1}{|\partial B(x,n)|} \fint_{\partial B(x,b)} f \, d\mu$$
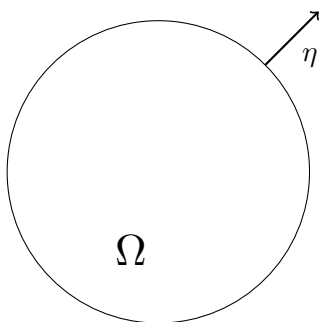
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Theorem: Lebesgue Differentiation

$$u|x| = \lim_{r \to 0} \fint_{B(x,n)} u \; d\mu = \lim_{r \to 0} \fint_{\partial B(x,n)} u \; d\mu$$

Integration by Parts

$$\int_{\Omega} u \Delta v = - \int_{\Omega} \nabla u \cdot \nabla v + \int_{\partial \Omega} u \cdot \frac{\partial v}{\partial \eta}$$

Recall: Poisson's PDE

$f \in C_c^2 |\mathbb{R}^n|$, $u(x) = \int_{\mathbb{R}^n} \Phi(x - y) f(y) \; dy$.

$$\Phi(x) = \left\{ \frac{1}{n(n-2)|\alpha(n)|} \frac{1}{|X|^{(n-2)}} \right.$$

$$u(x) \; " = " \int_{\mathbb{R}^n} f(x - y) \Phi(y) \; dy$$

Part A

$u \in C^2(\mathbb{R}^n)$
Then

$$\frac{\partial u}{\partial x_1} = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_1}(x - y) \Phi(y) \; dy$$

$$\frac{\partial^2 u}{\partial x_1 x_T} = \int_{\mathbb{R}^n} \frac{\partial^2 f}{\partial x_1 x_T}(x - y) \Phi(y) \; dy$$

Notice that if we prove that

$$g(x) = \int_{\mathbb{R}^n} h(x-y)\Phi(y) \, dy$$

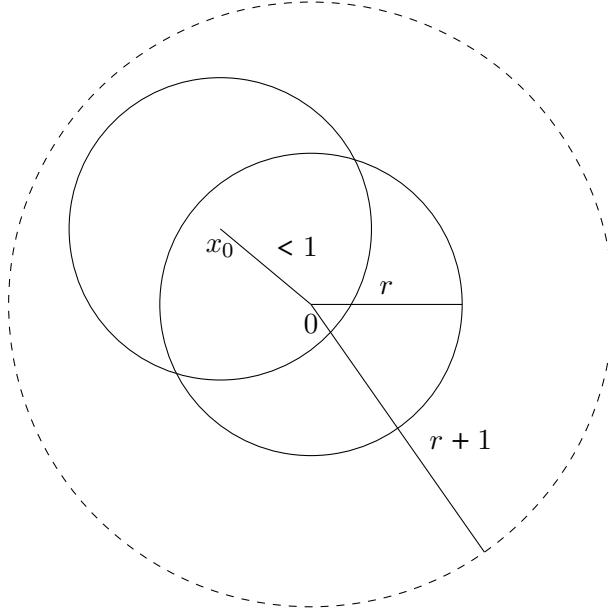– where h is continuous with compact support – is continuous then we are done.

Let us prove that g is continuous.

Let $\varepsilon > 0$,

$$|g(x) - g(x_0)| \leq \int_{\mathbb{R}^n} \Phi(y)|h(x-y) - h(x_0 - y)| \, dy$$

Without loss of generality, h has compact support on $B(0, r)$ for some radius r.

Therefore $h(x, y)$ has compact support on $B(x, r)$ and $h(x_0, y)$ has compact support on $B(x_0, r)$.

Consider $|x - x_0| < 1$, then $|h(x-y) - h(x_0 - y)|$ has compact support on $B(x_0, r+1)$. Then

$$|g(x) - g(x_0)| \leq \int_{B(x_0, r+1)} \Phi(y)|h(x-y) - h(x_0 - y)| \, dy$$

Since h is continuous on a compact domain, it is uniformly continuous.

Therefore $\exists \delta > 0$ such that $|w - z| < \delta \implies |h(w) - h(z)| < \epsilon$.

Set $w = x - y$ and $z = x_0 - y$ such that $|w - z| = |x - x_0| < \delta$, then $|h(x-y) - h(x_0 - y)| < \epsilon$. Thus,
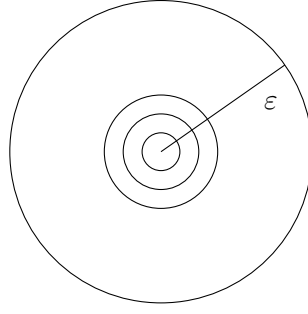
$$|g(x) - g(x_0)| \leq \varepsilon \int_{B(x_0, r+1)} \Phi(y) \, dy$$

Part B

$-\Delta u = f$.
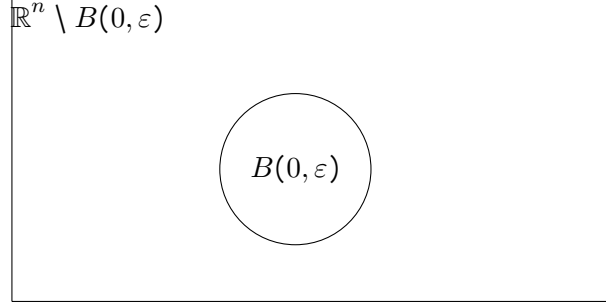
Letting $\varepsilon > 0$ and taking the Laplacian of both sides,

$$\Delta_x u(x) = \int_{\mathbb{R}^n} \Delta_x f(x-y)\Phi(y) \, dy$$

$$= \overbrace{\int_{B(0,\varepsilon)} \Delta_x f(x-y)\Phi(y) \, dy}^{I_\varepsilon} + \overbrace{\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Delta_x f(x-y)\Phi(y) \, dy}^{J_\varepsilon}$$

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Then

$$|I_\varepsilon| \le \int_{B(0,\varepsilon)} |\Delta_x f(x-y)| \Phi(y)\, dy$$

$$\le ||\overset{2}{\nabla} f||_{L^\infty} \int_{B(0,\varepsilon)} \Phi(y)\, dy$$

$$\le c \int_0^\varepsilon \int_{\partial B(0,r)} \Phi(y)\, dS(y)\, dr$$

$$\le c \int_0^\varepsilon \int_{\partial B(0,r)} \frac{1}{|y|^{n-2}}\, dS(y)\, dr$$

$$= c \int_0^\varepsilon \int_{\partial B(0,r)} \frac{1}{r^{n-2}}\, dS(y)\, dr$$

$$= c \int_0^\varepsilon \frac{1}{r^{n-2}} \int_{\partial B(0,r)} dS(y)\, dr$$

$$\le c \int_0^\varepsilon \frac{r^{n-1}}{r^{n-2}}\, dr$$

$$c \int_0^\varepsilon r\, dr = c\varepsilon^2$$

As an exercise, attempt the same argument with $n = 2$.
Therefore,

$$\Delta_x u = I_\varepsilon + J_\varepsilon$$

and $\lim_{\varepsilon \to 0} I_\varepsilon = 0$.
Now, we need to control J_ε.

$$J_\varepsilon = \int_{\mathbb{R}^n} \Delta_x f(x-y) \Phi(y)\, dy$$

$$\Delta_x f(x-y) = \sum \frac{\partial^2 f}{\partial x^2} f(x-y)$$

$$\frac{\partial f}{\partial x}(x-y) = \nabla f|_{z=(x-y)} \cdot e_i = \frac{\partial f}{\partial z_i}|z = (x-y)$$

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{\partial^2 f}{\partial z_i^2}|_{z=(x-y)}$$

$$\Delta_y f(x-y) = \sum \frac{\partial f^2}{\partial y_i}(x-y)$$

$$\frac{\partial f}{\partial y_i}(x-y) = \nabla f|_{z=(x-y)} \cdot -e_i = -\frac{\partial f}{\partial z_i}|_{z=(x-y)}$$

$$\frac{\partial^2 f}{\partial y_i^2} = \frac{\partial^2}{\partial y_i^2}|_{z=x-y}$$

So

$$J_\varepsilon = \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Delta_y f(x-y) \Phi(y)\, dy$$

$$= \overbrace{-\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \nabla_y f(x-y) \nabla \Phi(y)\, dy}^{K_\varepsilon} + \overbrace{\int_{\partial(\mathbb{R}^n \setminus B(0,\varepsilon))} \frac{\partial f}{\partial \eta} \Phi(y)\, dS(y)}^{L_\varepsilon}$$

and

$$\Delta_x u = I_\varepsilon + K_\varepsilon + L_\varepsilon$$

To control L_ε, since

$$|L_\varepsilon| \le \int_{\partial B(0,\varepsilon)} \frac{|\partial f|}{\partial \eta} \Phi(y)\, dy$$

$$\le \int_{\partial B(0,\varepsilon)} |\nabla f| \Phi(y)\, dy$$

$$\le ||\nabla f||_{L^\infty} \int_{\partial B(0,\varepsilon)} \Phi(y)\, dy$$

$$\le c \int_{\partial B(0,\varepsilon)} \frac{1}{|y|^{n-2}}\, dy$$

$$= \frac{c}{\varepsilon^{n-2}} \int_{\partial B(0,\varepsilon)} dy$$

$$\le \frac{c\varepsilon^{n-1}}{\varepsilon^{n-2}}$$

$$= c\varepsilon$$

and K_ε, since $\frac{\partial \Phi}{\partial \eta} = \nabla \phi \cdot \eta = \frac{-y}{n\alpha(n)|y|^n} \cdot \eta = \frac{|y|^2}{n\alpha(n)|y|^n|y|} = \frac{1}{n\alpha(n)|y|^{n-1}}$
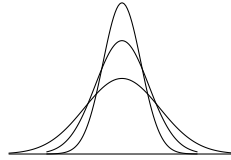
$$|K_\varepsilon| = -\int_{\mathbb{R}^n \backslash B(0,\varepsilon)} \nabla_y f(x-y) \nabla_y \Phi(y) \, dy$$

$$= \overbrace{\int_{\mathbb{R}^n \backslash B(0,\varepsilon)} f(x-y) \Delta_y \Phi(y) \, dy}^{0} - \int_{\partial(\mathbb{R}^n \backslash B(0,\varepsilon))} f(x-y) \frac{\partial \Phi}{\partial \eta}$$

$$= -\int_{\partial B(0,\varepsilon)} f(x-y) \frac{\partial \Phi}{\partial \eta}$$

$$= -\int_{\partial B(0,\varepsilon)} f(x-y) \frac{1}{n\alpha(n)|y|^{n-1}} \, dS(y)$$

$$= -\frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\delta B(0,\varepsilon)} f(x-y) \, dS(y)$$

$$= -\underbrace{\frac{1}{n\alpha(n)\varepsilon^{n-1}}}_{} \overset{\text{volume}}{} \int_{\partial B(0,\varepsilon)} f(z) \, dS(z)$$

$$= \frac{1}{|\partial B(0,\varepsilon)|} \int_{\partial B(0,\varepsilon)} f(z) \, dz$$

$$= -\fint_{\partial B(x,\varepsilon)} f(z) \, dz$$

Laplacian as a Distribution

$-\Delta\Phi(y) = \delta(y)$
Define the Dirac delta "function" as

$$\delta(y) = \begin{cases} \infty & y = 0 \\ 0 & y \neq 0 \end{cases}$$

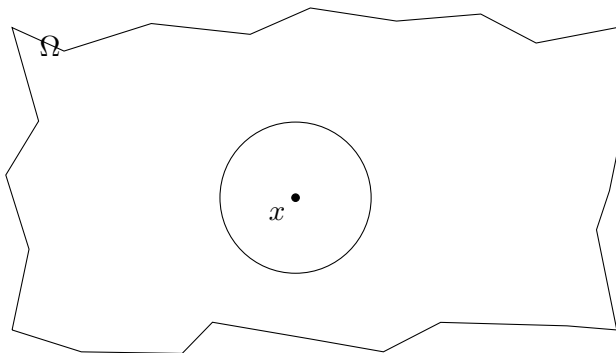such that $\int_{\mathbb{R}^n} \delta = 1$.
Translate the Dirac delta as

$$\delta_x = \begin{cases} \infty & y = x \\ 0 & y \neq x \end{cases}$$

and recall that

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$$\Delta u(x) = \Delta \left(\int_{\mathbb{R}^n} \Phi(x-y) f(y) \, dy \right)$$

$$= \int_{\mathbb{R}^n} \overbrace{\Delta \Phi(x-y)}^{-\delta_x(y)} f(y) \, dy$$

$$= - \int_{\mathbb{R}^n} \delta_x(y) f(y) \, dy$$

$$= - \int_{\mathbb{R}^n} \delta_x(y) f(x) \, dy$$

$$= -f(x) \overbrace{\int_{\mathbb{R}^n} \delta_x(y) \, dy}^{1}$$

$$= -f(x)$$

Harmonic Functions

Suppose u is harmonic

$u : \Omega \to \mathbb{R}^n$ harmonic.

Mean-value Formulas

Let U be an open set in $\mathbb{R}^n$, $u : U \to \mathbb{R}$ such that $\Delta u = 0$ in U. Then

$$u(x) = \fint_{\partial B(0,r)} -u(y) \, dS(y)$$

$$= \fint_{B(x,r)} u(y) \, dy$$

where $B(x,r) \subseteq U$.
IMAGE HERE

Proof

Consider $\phi(r) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y) \, dS(y)$.
If $\phi'(r) = 0$, when we are done since that would make ϕ constant and $\phi(r) = \lim_{s \to 0} \phi(s) = u(x)$. Then

$$\phi(r) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y)\, dS(y)$$

$$\underset{y=x+rz}{=} \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(0,1)} u(x+rz)r^{n-1}\, dS(z)$$

$$= \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} u(x+rz)\, dS(z)$$

Therefore

$$\phi'(r) = \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} \nabla u(x+rz) \cdot z\, dS(z)$$

$$\underset{y=x+rz}{=} \frac{1}{|\partial B(0,r)|} \int_{\partial B(x,r)} \nabla u(y) \cdot \frac{y-x}{r}\, dS(y)$$

$$= \frac{1}{|\partial B(0,r)|} \int_{\partial B(x,r)} \nabla u(y) \cdot \eta\, dS(y)$$

$$= \frac{1}{|\partial B(0,r)|} \int_{\partial B(x,r)} \frac{\partial y}{\partial \eta}\, dS(y)$$

$$= \frac{1}{|\partial B(0,r)|} \int_{B(x,r)} \Delta u$$

$$= 0$$

January 22, 2024

Mean Value Formula

For $U \subseteq \mathbb{R}^n$, U open with $u : U \to \mathbb{R}$ such that $u \in C^2(U)$, $\Delta u = 0$, we have

$$u(x) \underset{(a)}{=} \fint_{\partial B(x,r)} u \underset{(b)}{=} \fint_{B(x,r)} u$$

for all $B(x,r) \subseteq U$.

Recall that (a) was proven above by setting $\phi(r) = \fint_{\partial B(r)} u(y)\, dS(y)$ and showing $\phi'(r) = 0$.

For (b), we again apply spherical coordinates such that

$$\int_{B(x,r)} u(y)\, dy = \int_0^r \int_{\partial B(x,s)} u(y)\, dS(y)ds$$

$$= \int_0^r |\partial B(x,s)| \overbrace{\fint_{\partial B(x,s)} u(y)\, dS(y)}^{u(x)} ds$$

$$= u(x) \int_0^r |\partial B(x,s)|\, ds$$

$$= u(x) \int_0^r n\alpha(n)S^{n-1}\, ds$$

$$= \frac{u(x)n\alpha(n)S^n}{n}\Big|_0^r$$

$$= u(x) \overbrace{\alpha(n)r^n}^{|B(x,r)|}$$

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Converse

Recall that

$$\phi'(r) = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u(y) \, dy$$

Suppose then that we do not know that $\Delta u = 0$ but we have

$$u(x) = \fint_{\partial B(x,r)} u, \quad \forall B(x,r) \subseteq U$$

Then, necessarily, $\Delta u = 0$ in U.

- Proof
  Suppose, for sake of contradiction, that $\Delta u \neq 0$. Then, without loss of generality, there exists $y \in U$ such that $\Delta u(x) > 0$ for $x \in B(y,n) \subseteq U$.
  IMAGE HERE

$$\phi(r) = \fint_{\partial B(x,r)} u(x) \, dS(x)$$

and

$$\phi'(r) = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(y,r)} \Delta u(x) \, dS(x) > 0$$

which contradicts $\phi'(x) = 0$.

Strong Maximum Principle

Let $U \subseteq \mathbb{R}^n$ be a bounded open set, $u \in C^2(U) \cap C(\overline{U})$, $\Delta u = 0$ on U. Then

1. $\max_{\overline{U}}(u) = \max_{\partial U}(u)$.

2. If U is connected and u has its maximum in an interior point, then u is constant on $\overline{U}$.

IMAGE HERE - 2

Proof of A

Since $\partial U \subseteq \overline{U}$, $\max_{\partial U}(u) \leq \max_{\overline{U}}(u)$.
Let $x_0 \in \overline{U}$ such that $u(x_0) = \max_{\overline{U}}(u)$.
IMAGE HERE - 4
So either $x_0 \in \partial U$ or $x_0 \in U$.
Let U' be the connected component which contains x_0. Then $x_0 \in U'$, so by part (b) u is constant on $\overline{U'}$. So

$$\max_{\overline{U}}(u) = u(x_0) = \max_{\partial U'}(u) \leq \max_{\partial U}(u)$$

Proof of B

Then there exists $x_0 \in U$ such that $\max_{\overline{U}}(u) = u(x_0) = M$.
Let us define $\Omega = \{y \in U \mid u(y) = M\}$. Then

1. $\Omega \neq \emptyset$, $B \setminus x_0 \in \Omega$.

2. Ω open set.

IMAGE HERE - 3

1. Ω is closed, since $\Omega = u^{-1}(\{M\})$.

It follows that $\Omega = U$ and, therefore, $u(y) = M$, $\forall y \in U$.

- Proof of 2
  Let $y \in \Omega$, $y \in U$, $u(y) = M$. Then there exists $B(y, r) \subseteq U$, and

$$M = u(y) = \fint_{B(y,r)} u(x)\, dS(x) \leq M$$

Then

$$\fint_{B(y,r)} u(x)\, dx = M$$

so $u(x) = M$, $\forall x \in B(y, r)$ and, therefore $B(y, r) \subseteq \Omega$ and Ω is open.

Remark: Boundedness Is Important

1. Consider $f(x) = x$ on $\mathbb{R}_{\geq 0}$.

2. IMAGE HERE - 5

Remark: Strong Minimum Principle Is Equivalent

Consequences

1. Positivity of harmonic functions.

2. Uniqueness of the Poisson problem.

Corollary: Positivity of Harmonic Functions

Suppose that U is connected and $u : U \to \mathbb{R}$, $u \in C^2(U) \cap C(\overline{U})$ solves the following problem

$$\begin{cases} \Delta u = 0 & \text{on } U \\ u = g & \text{on } \partial U \end{cases}$$

If $g \geq 0$ on ∂U, then u is positive on U as long as g is positive in some point.

Proof

Assume $\exists x_0 \in \partial U$ where x_0 is the minimum. Then $u(x_0) = \min_{\overline{U}}(u)$ and, $\forall x \in U$,

$$0 \le u(x_0) = \min_{\overline{U}}(u) \le u(x)$$

so u is non-negative. If $u(x) = 0$, then $u(x_0) = 0$ and the minimum is achieved in the interior. That would mean $u(x) = 0$, $\forall x \in \overline{U} \supseteq \partial U$ and $g(x) = 0$, $\forall x \in \partial U$ which would be a contradiction.

Theorem: Uniqueness of the Poisson Problem

Suppose $U \subseteq \mathbb{R}^n$ is open, connected and bounded. Then, there exists at most one solution to

$$(*) \begin{cases} -\Delta u = f & \text{in } U \\ \quad u = g & \text{on } \partial U \end{cases}$$

where $u \in C^2(U) \cap C(\overline{U})$.

Proof

Let u_1 and u_2 be two solutions of $*$.
Consider $w = u_1 - u_2$ and observe that

$$\Delta w = \Delta u_1 - \Delta u_2 = -f + f = 0, \quad \text{in } U$$

and $w|_{\partial U} = g - g = 0$ on ∂U. Therefore

$$\begin{cases} \Delta w = 0 & \text{in } U \\ \quad w = 0 & \text{on } \partial U \end{cases}$$

By applying the minimum and maximum principles,

$$w(x_0) = \min_{\overline{U}}(w) \le w(x) \le \max_{U}(w) = w(x)$$

so $w(x) = 0$, $\forall x \in \overline{U}$ and therefore $u_1 = u_2$.

Example

Let's consider $f : \mathbb{C} \to \mathbb{C}$ analytic (i.e. $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $a_n, z \in \mathbb{C}$). Then

$$f(z) = u(z) + v(z)$$

If $\mathbb{C} \cong \mathbb{R}^2$,

$$f(x + y) = u(x, y) + v(x, y)$$

for $u : \mathbb{R}^2 \to \mathbb{R}$ and $v : \mathbb{R}^2 \to \mathbb{R}$.
Claim: u and v are Harmonic.

$$u(x, y) + v(x, y) = \sum_{n=0}^{\infty} a_n (x + iy)^n$$

and

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \sum_{n=1}^{\infty} a_n n |x + iy|^{n-1}$$

$$\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} = \sum_{n=1}^{\infty} a_n n |x + iy|^{n-1} i$$

So

$$i \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) = \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

Recall: Convolution and Smoothing

Let $U \subseteq \mathbb{R}^n$ be an open set.
For $\varepsilon > 0$, define $U_\varepsilon = \{x \in U \mid d(x, \partial U) > \varepsilon\}$.
IMAGE HERE - 6
Define

$$\eta(x) \begin{cases} ce\left(\frac{1}{|x|^2 - 1}\right) & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

with c such that $\int_{\mathbb{R}^n} \eta(x) \, dx = 1$, $\eta \in C^\infty(\mathbb{R}^n)$
IMAGE HERE - 7
Note that $\text{supp}(\eta) = B(0, 1)$ and take

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right), \quad \eta_\varepsilon \in C^\infty(\mathbb{R}^n)$$

Then

$$\int_{\mathbb{R}^n} \eta_\varepsilon(x) \, dx = 1$$

and $\text{supp}(\eta_\varepsilon) \subseteq B(0, \varepsilon)$.
If f is locally integrable on U, define its mollification

$$f^\varepsilon = (x) = \int_U \eta_\varepsilon(x - y) f(y) \, dy \quad \forall x \in U_\varepsilon$$

January 24, 2024

Recall: Mollifiers

Define

$$\eta(x) = \begin{cases} ce\left(\frac{1}{|x|^2-1}\right) & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

where $\eta \in C^\infty(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \eta(x) = 1$ and $\text{supp}(\eta) \subseteq B(0,1)$.
Then for $\varepsilon > 0$, $\eta_\varepsilon(x) = \frac{1}{\varepsilon}\left(\frac{x}{\varepsilon}\right)$ where $\eta_\varepsilon \in C^\infty(\mathbb{R}^n)$.
So $\int_{\mathbb{R}^n} \eta_\varepsilon(x) = 1$ and $\text{supp}(\eta_\varepsilon) \subseteq B(0,\varepsilon)$
Given f locally summable; $f : U \to \mathbb{R}$,

$$f^\varepsilon(x) := \int_U \eta_\varepsilon(x-y)f(y)\, dy \quad x \in U_\varepsilon$$
$$= \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)f(y)\, dy \quad x \in U_\varepsilon$$

Properties

1. $f^\varepsilon \in C^\infty(U_\varepsilon)$.

2. $f^\varepsilon \xrightarrow[\varepsilon \to 0]{} f$ a.e.

3. If f continuous, then $f^\varepsilon \xrightarrow[\varepsilon \to 0]{} f$ uniformly on compact sets of U.

Theorem 6:

Let $u \in C(U)$ with $U \in \mathbb{R}^n$ open and such that u satisfies the mean-value property (i.e. $u(x) = \fint_{\partial B(x,r)} u(y)\, dS(y)$, $\forall B(x,r) \subseteq U$), then $u \in C^\infty$.

Corollary

If $u \in C^2(U)$ is harmonic, then $u \in C^\infty(U)$.

Proof

Let us take $x_0 \in U$
IMAGE HERE - 1
Notice, that if we prove that $u = u_\varepsilon$ on U_ε then we are done.
Let $x \in U_\varepsilon$, and noticing that $\eta(x) = \eta(|x|)$,

$$u_\varepsilon(x) = \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)u(y)\,dy$$

$$= \frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} \eta\frac{|x-y|}{\varepsilon}u(y)\,dy$$

$$\underset{\text{spherical}}{=} \frac{1}{\varepsilon^n} \int_0^\varepsilon \int_{\partial B(x,r)} \eta\frac{\overbrace{|x-y|}^{r}}{\varepsilon}u(y)\,dS(y)dr$$

$$= \frac{1}{\varepsilon^n} \int_0^\varepsilon \eta\frac{r}{\varepsilon} \int_{\partial B(x,r)} u(y)\,dS(y)dr$$

$$= \frac{1}{\varepsilon n} \int_0^\varepsilon \eta\frac{r}{\varepsilon} \underbrace{|\partial B(x,r)|}_{|\partial B(0,r)|}\, u(x)\,dr$$

$$= \frac{u(x)}{\varepsilon^n} \int_0^r \eta\frac{r}{\varepsilon} \int_{\partial B(0,r)\,dS(y)dr}$$

$$= u(x) \int_0^\varepsilon \frac{1}{\varepsilon^n}\eta\frac{r}{\varepsilon}\,dS(y)dr$$

$$= u(x) \overbrace{\int_{B(0,\varepsilon)} \eta_\varepsilon(y)\,dy}^{1} = u(x)$$

Theorem 7: Local Estimates of Harmonic Functions

Suppose $u \in C^2(U)$ a harmonic function.
Then $|D^\alpha u(x_0)| \le \frac{C_k}{r^{n+k}}||u||_{L^1(B(x_0,r))}$, $B(x_0,r) \subseteq U$, where α is a multiindex of order k, $C_0 = \frac{1}{\alpha(n)}$ and
$C_k = \frac{(2^{n+1}nk)^k}{\alpha(n)}$ for $k = 1, 2, \ldots$.
We may take α since, by previous theorem, $u \in C^\infty(U)$.

Proof

By induction.
Consider $k = 0$.

$$u(x_0) = \fint_{B(x_0,r)} u(y)\,dy$$

$$= \frac{1}{\alpha(n)r^n} \int_{B(x_0,r)} u(y)\,dy$$

$$|u(x_0)| \le \frac{1}{\alpha(n)r^n} \int_{B(x_0,r)} |u(y)|\,dy$$

$$= \frac{C_0}{r^n}||u||_{L^1(B(x_0,r))}$$

For $k = 1$, if $|\alpha| = k = 1$ then $D^\alpha u(X_0) = \frac{\partial u}{\partial x_i}(x)$ for $i = 1, 2, \ldots$.
Notice that $\frac{\partial u}{\partial x_i}$ is also harmonic.

$$\Delta \frac{\partial u}{\partial x_i} = \sum_{t=1}^n \frac{\partial^2}{\partial x_t^2}\frac{\partial u}{\partial x_i}$$

$$= \frac{\partial}{\partial x_i} \underbrace{\sum_{t=1}^\infty \frac{\partial^2 u}{\partial x_t^2}}_{0}$$

Applying the mean-value formula to $\frac{\partial u}{\partial x_i}(x_0)$ in $B(x, r/2)$.

IMAGE HERE - 2

$$\frac{\partial u}{\partial x_i}(x_0) = \fint_{B(x_0, r/2)} \frac{\partial u}{\partial x_i}(y)\ dy$$

$$= \frac{2^n}{\alpha(n)r^n} \int_{B(x_0, r/2)} \frac{\partial u}{\partial x_i}(y)\ dy$$

Recall $\int_\Omega w\Delta v = -\int_\Omega \nabla w \cdot \nabla v + \int_\Omega w\frac{\partial v}{\partial \eta}$.

$$\underset{e_i = \nabla y_i}{=} \frac{2^n}{\alpha(n)r^n} \int_{B(x_0, r/2)} \nabla \underbrace{u(y)}_{w} \cdot \nabla \underbrace{y_i}_{v}\ dy$$

$$\underset{IBP}{=} \frac{2^n}{\alpha(n)r^n} \left[-\int_{B(x_0, r/2)} u(y)\Delta y_i\ dy + \int_{\partial B(x_0, r/2)} u(y)\frac{\partial y_i}{\partial \eta} \right]$$

Note that

$$\frac{\partial y_i}{\partial \eta} = \nabla y_i \cdot \eta = e_i \cdot \eta = \eta_i$$

and

$$\left| \frac{\partial y_i}{\partial \eta} \right| = |\eta_i| \leq |\eta| = 1$$

So,

$$\left| \frac{\partial u}{\partial x_i} x_0 \right| \leq \frac{2^n}{\alpha(n)r^n} \int_{\partial B(x_0, r/2)} |u(y)|\ dS(y)$$

$$= \frac{2^n n\alpha(n)\left(\frac{r}{2}\right)^{n-1}}{\alpha(n)r^n} ||u||_{L^\infty(\partial B(x_0, r/2))}$$

$$= \frac{2n}{r} \underbrace{||u||_{L^\infty(\partial B(x_0, r/2))}}_{*}$$

Let's analyze $*$.

Let $x \in \partial B(x_0, r/2)$, then $B(x, r/2) \subseteq B(x_0, r)$.

IMAGE HERE - 3

Then we may apply $k = 0$.

$$|u(x)| \leq \frac{C_0}{\left(\frac{r}{2}\right)^n} ||u||_{L^1(B(x, r/2))}$$

$$\leq \frac{C_0}{\left(\frac{r}{2}\right)^n} ||u||_{L^1(B(x_0, r))}$$

Then

$$\left| \frac{\partial u}{\partial x_i}(x_0) \right| \leq \frac{2n}{r} \frac{C_0}{\left(\frac{r}{2}\right)^n} ||u||_{L^1(B(x_0, r))}$$

$$= \frac{2^{n+1}n}{r^{n+1}\alpha(n)} ||u||_{L^1(B(x_0, r))}$$

HOMEWORK: Induct for arbitrary k.

Theorem 8: Liouville's Theorem

Suppose $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic and bounded. Then u is constant.

Proof

$$|D^\alpha u(x)| = \sqrt{\sum_{i=1}^n \left[\frac{\partial u}{\partial x_i}\right]^2} \leq \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}\right|$$

Let $r > 0$, $B(x,r) \subseteq \mathbb{R}^n$. Then, using estimates

$$\left|\frac{\partial u}{\partial x_i}(x)\right| \leq \frac{C_1}{r^{n+1}}||u||_{L^1(B(x,r))}$$

Therefore,

$$
\begin{aligned}
|D^\alpha u(x)| &\leq \frac{nC_1}{r^{n+1}}||u||_{L^1(B(x,r))} \\
&= \frac{nC_1}{r^{n+1}}\int_{B(x,r)} |u(y)|\, dy \\
&\leq \frac{nC_1}{r^{n+1}}||u||_{L^\infty(B(x,r))}\alpha(n)r^n \\
&= \frac{C||u||_{L^\infty(B(x,r))}}{r}
\end{aligned}
$$

and

$$\left|\frac{\partial u}{\partial x_i}(x)\right| \leq \frac{C||u||_{L^\infty(B(x,r))}}{r}$$

$$\left|\frac{\partial u}{\partial x_i}(x)\right| \leq C||u||_{L^\infty(B(x,r))} \lim_{r\to\infty}\frac{1}{r} \implies \frac{\partial u}{\partial x_i}(x) = 0 \implies u = Ck$$

Theorem: Representation Formula

Recall: $f \in C_c^2(\mathbb{R}^n)$, $(*) - \Delta u = f$ in $\mathbb{R}^n$, we saw that

$$\tilde{u}(x) : \int_{\mathbb{R}^n} \Phi(x-y)f(y)\, dy$$

solves $*$.

Let us consider $u \in C^2(\mathbb{R}^n)$ solving $-\Delta u = f$ for $n \geq 3$ where $f \in C_c^2(\mathbb{R}^n)$ and u is bounded.
Then $u(x) = \tilde{u}(x) + C = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\, dy + C$.

Proof

Notice that if $\tilde{u}$ is bounded, then we are done. Because we may consider $w = u - \tilde{u}$ on $\mathbb{R}^n$ where

$$\Delta w = \Delta u - \Delta \tilde{u} = -f - (-f) = 0$$

Therefore w is bounded and, by Liouville's Theorem, $w = C$ and $u - \tilde{u} = c \iff u = \tilde{u} + C$.

$$|\tilde{u}(x)| \leq \int_{B(0,k)} \Phi(x-y)f(y) \, dy$$

$$\leq ||f||_{L^\infty(\mathbb{R}^n)} \int_{B(0,k)} \Phi(x-y) \, dy$$

If this is less than some C which does not depend on x, we are done.

Since $\Phi(x) \to 0$ for $|x| \to \infty$, for any $C_1 \; \exists \alpha$ such that if $|x| > \alpha$ then $|\Phi(x)| < C_1$.

IMAGE HERE - 4

$\text{dist}(x, B(0,k)) = \text{dist}(x, y_0)$ where $y_0 \in \overline{B(0,k)}$.

IMAGE HERE - 5

There are two cases.

- Case 1
  $\text{dist}(x, B(0,k)) \leq \alpha$
  $B(x,k) \subseteq B(0, \alpha + Ck)$
  Let $y \in B(x,k)$, then $|y - x| < k$ so $|x - y_0| < \alpha$.
  Therefore $|y - y_0| \leq k + \alpha \implies |y| \leq k + \alpha + |y_0| = 2k + \alpha \implies y \in B(0, 2k + \alpha)$. Then

$$||f||_{L^\infty(\mathbb{R}^n)} \int_{B(x,k)} \Phi(y) \, dy \leq ||f||_{L^\infty(\mathbb{R}^n)} \int_{B(0, \alpha+2k)} \Phi(y) \, dy$$

  HOMEWORK - Consider the other case.

January 29, 2024

Recall: Representation Formula

For $n \geq 3$.

$$\tilde{u}(x) : \int_{\mathbb{R}^n} \Phi(x-y)f(y) \, dy$$

It is sufficient to show that $\tilde{u}$ is bounded. Then

$$|\tilde{u}| \leq C \int_{B(0,k)} \Phi(x-y) \, dy$$

$\forall C_1, \; \exists \alpha$ such that $|z| \geq \alpha \implies |\Phi(z)| \leq C_1$.

Case 2

For $\text{dist}(x, B(0,k)) \geq \alpha$, $\text{dist}(x, y) \geq \alpha$, $\forall y \in B(0,k)$. Then

$$|x - y| \geq \alpha$$
$$\frac{1}{|x-y|} \leq \frac{1}{\alpha}$$
$$\frac{1}{|x-y|^{n-2}} \leq \frac{1}{\alpha^{n-1}}$$

and

$$|\tilde{u}(x)| \leq C \int_{B(0,k)} \frac{1}{|x-y|^{n-2}} \, dy \leq \frac{C}{\alpha^{n-2}} \int_{B(0,k)} dy$$

Theorem 10: Harmonic Implies Analytic

Let $U \subseteq \mathbb{R}^n$ open, $u \in C^2(U)$ harmonic. Then u is analytic in U.

Proof

Let $x_0 \in U$. We want to prove that the power series converges to $u(x)$ for x in a neighborhood around x_0.
Let $r = \text{dist}\left(x_0, \frac{\partial U}{4}\right)$, $M = \frac{1}{\alpha(n)r^n}||u||_{L^1(B(x_0,r))} \subseteq U$.
IMAGE HERE - 1
We want to analyze $x \in B(x_0, r)$.
Notice that $B(x, r) \leq B(x_0, 2r)$, and $z \in B(x, r)$ gives $|z - x| < r$ so

$$|z - x_0| \leq \underbrace{|z - x|}_{\leq r} + \underbrace{|x - x_0|}_{\leq r} \leq 2r$$

Applying estimates on $B(x, r)$, $|\alpha| = k$,

$$|D^\alpha u(x)| \leq \frac{C_k}{r^{n+k}}||u||_{L^1(B(x,r))}$$

$$\leq \frac{C_k}{r^{n+k}}||u||_{L^1(B(x_0,2r))}$$

and

$$\sup_{x \in B(x_0,r)} |D^\alpha u(x)| \leq \frac{\left(2^{n+1}nk\right)^k}{\alpha(n)r^{n+k}}||u||_{L^1(B(x_0,2r))}$$

Notice, by Stirling's approximation or Taylor expansion, $\frac{k^k}{k!} < e^k$, $\forall k \geq 1$. So

$$|\alpha|^{|\alpha|} < e^{|\alpha|}|\alpha|!$$

and

$$n^k = \underbrace{(1 + \cdots + 1)}_{n-\text{times}}^k = \sum_{|\beta|=k} \frac{|\beta|!}{\beta!} \geq \frac{|\alpha|!}{\alpha!}$$

where $|\alpha|! \leq \alpha! n^k$, $\beta = (\beta_1, \ldots, \beta_2)$ and $\beta! := \beta_1!\beta_2!\cdots\beta_n!$. Therefore

$$|\alpha|^{|\alpha|} \leq e^{|\alpha|}|\alpha|! \leq e^{|\alpha|}\alpha! n^k$$

and finally

$$(*) \quad k^k \leq e^k \alpha! n^k$$

Applying $*$ to the above inequality,

$$\sup_{X \in B(x_0, r)} |D^\alpha u(x)| \le \frac{\left(2^{n+1} n\right)^k e^k \alpha! n^k}{\alpha(n) r^n r^k} ||u||_{L^1(B(x_0, 2r))}$$

$$\le \left(\frac{2^{n+1} n^2 e}{r}\right)^k \cdot \alpha! M$$

Let us analyze the Taylor expansion

$$\sum_{k=0}^{N} \sum_{|\alpha|=k} \frac{D^\alpha u(x_0)}{\alpha!} (x - x^0)^\alpha$$

Where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $y \in \mathbb{R}^n$ and $y^\alpha = y_1^{\alpha_1} \cdots y_n^{\alpha_n}$.
Pick $|x - x_0| \le \frac{r}{2^{n+2} n^3 e}$. We want to prove that the remainder $R_N(x) \xrightarrow[N \to \infty]{} 0$.

$$R_N(x) = u(x) - \sum_{k=0}^{N-1} \sum_{|\alpha|=k} \frac{D^\alpha u(x_0)}{\alpha!} (x - x_0)^\alpha$$

$$= \sum_{|\alpha|=N} \frac{D^\alpha u(x_0 + t(x - x_0))(x - x_0)^\alpha}{\alpha!}, \quad \text{for some } |t| \le 1$$

Using the remainder of the Taylor expansion with $g(t) = u(x_0 + t(x - x_0))$ for $g : I \to \mathbb{R}$.
Homework: show this around $t = 0$ at $t = 1$.
Note that $u(x_0 + t(x - x_0))$ describes a straight long with $t = 0 \implies u(x_0)$ and $t = 1 \implies u(x)$.
Notice also that $x_0 + t(x - x_0) \in B(x_0, r)$. Then, considering the superemum of the remainder,

$$|R_n(x)| \le \sum_{|\alpha|=N} \left(\frac{2^{n+1} n^2 e}{r}\right)^N \cdot M\alpha! \cdot \frac{|(x - x_0)^\alpha|}{\alpha!}$$

Remark: for $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $y = (y_1, \ldots, y_n)$,

$$\begin{aligned}
|y^\alpha| = |y_1^{\alpha_1} \cdots y_n^{\alpha_n}| &\le |y_1|^{\alpha_1} \cdots |y_n|^{\alpha_n} \\
&\le |y|^{\alpha_1} \cdots |y|^{\alpha_n} \\
&= |y|^{\alpha_1 + \alpha_2 + \cdots + \alpha_n} \\
&= |y|^\alpha
\end{aligned}$$

Therefore

$$|R_n(x)| \le \sum_{|\alpha|=N} \left(\frac{2^{n+1}n^2 e}{r}\right)^N \cdot M|x - x_0|^N$$

$$\le M \cdot \sum_{|\alpha|=N} \left(\frac{2^{n+1}n^2 e}{r}\right)^N \left(\frac{r}{2^{n+2}n^3 e}\right)^N$$

$$= M \cdot \sum_{|\alpha|=N} \left(\frac{1}{2n}\right)^N$$

$$\le M\left(\frac{1}{2n}\right)^N \sum_{|\alpha|=N}$$

$$\le M\left(\frac{1}{2n}\right)^N n^N$$

$$= M\left(\frac{1}{2}\right)^N$$

Note that $\sum_{|\alpha|=N}(1) \le n^N$ since

$$\alpha = (\alpha_1, \ldots, \alpha_n) = (\alpha_{1_N}, \ldots, \alpha_{i_N}) = n^N$$

Theorem 11: Harnack's Inequality

Define $V \subset\subset U$ as "V totally contained in U" meaning $\overline{V}$ compact and $V \subseteq \overline{V} \subseteq U$.
IAMGE HERE - 2
Let U open and $u \in C^2(U)$ harmonic and non-negative.
Then for each connected open set $V \subset\subset U$

$$\sup_V u \le C \inf_V u$$

for some C that depends on V.

Remark

Then

$$\frac{1}{C}u(y) \le u(x) \le Cu(y), \quad \forall x, y \in V$$

Since

$$u(x) \le \sup_V u \le C \inf_V u \le Cu(y)$$

and

$$\frac{1}{C}u(y) \le \frac{1}{C}\sup_V u \le \inf_V u \le u(x).$$

Proof

Take $r = \frac{\text{dist}(v, \partial U)}{4} > 0$.

- Case 1
  Let us suppose that $x, y \in V$ such that $|x - y| < r$.
  IMAGE HERE - 3
  Notice $B(x, 2r) \subseteq U$. Applying mean-value formulas,

$$u(x) = \fint_{B(x,2r)} u = \frac{1}{\alpha(n)(2r)^n} \int_{B(x,2r)} u$$

But notice that $B(y, r) \subseteq B(x, 2r)$, so

$$u(x) \geq \frac{1}{\alpha(n)2^n r^n} \int_{B(y,r)} u = \frac{1}{2^n} \fint_{B(y,r)} u = \frac{1}{2^n} u(y)$$

That is, if $x, y \in V$ such that $|x - y| < r$, then $u(x) \geq \frac{1}{2^n} u(y)$ and, mutatis mutandis, $u(y) \geq \frac{1}{2^n} u(x)$.

- Case 2
  Let us cover $\overline{V}$ by an open covering of balls $\{B_i\}_{i=1}^N$ such that the radius of each ball is $\frac{r}{2}$ and $B_i \cap B_{i-1} \neq \varnothing$.
  IMAGE HERE - 4
  Then $u(x) \geq \frac{1}{2^n} u(z) \frac{1}{2^n 2^n} u(y)$, so $u(x) \geq \frac{1}{2^{2n}} u(y)$.
  In the same way, $u(y) \geq \frac{1}{2^{2n}} u(x)$.
  IMAGE HERE - 5
  For three balls, $u(x) \geq \frac{1}{2^{3n}} u(y)$ and $u(y) \geq \frac{1}{2^{3n}} u(x)$.
  Since we have a finite covering of N balls, the same strategy gives

$$u(x) \geq \frac{1}{2^{Nn}} u(y) \qquad\qquad u(y) \geq \frac{1}{2^{Nn}} u(x)$$

and

$$\frac{1}{2^{Nn}} \leq u(x)$$

Taking the supremum $y \in V$;

$$\sup_{y \in V} u(y) \leq 2^{Nn} u(x)$$

taking the infemum $x \in V$

$$\inf_{x \in V} u(y)$$

Recap: Laplace Equation

- Fundemental Solution

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    – Poisson Equation in $\mathbb{R}^n$

- Mean-value Formulas

- Properties

    – Strong Maximum / Minimum Principles

        * Uniqueness of the Poisson Equation on Bounded Domains

    – Regularity

    – Derivative Estimates

    – Liouville's Theorem

        * Representation Formula

            · Uniqueness of the Poisson Equation up to a Constant on $\mathbb{R}^n$ for Bounded Functions

    – Analyticity

    – Harnack's Inequality

Green's Functions

For U open and bounded, $\partial U \in C^1$.
Goal: We want to solve $-\Delta u = f$ on U and $u = g$ on ∂U.

Recall: Green's Formula

$$\int_\Omega u\Delta v - v\Delta u = \int_{\partial\Omega} u\frac{\partial v}{\partial \eta} - v\frac{\partial u}{\partial \eta}$$

Obtaining Green's Formula

Let $x \in U$ and consider $u(y)$, $\Phi(y-x)$ as functions of y.
Let $\varepsilon > 0$ and consider $V_\varepsilon = U \setminus B_\varepsilon(x)$. Applying Green's formula; $\Omega = V_\varepsilon$,

$$\int_{V_\varepsilon} u(y)\underbrace{\Delta_y\Phi(y-x)}_{=0} - \Phi(y-x)\Delta_y u = \int_{\partial V_\varepsilon} u(y)\frac{\partial\Phi(y-x)}{\partial \eta} - \Phi(y-x)\frac{\partial u(y)}{\partial \eta}$$

IMAGE HERE - 6

January 31, 2024

Green's Functions

Goal is to solve for $U \subseteq \mathbb{R}^n$ open and bounded,

$$\begin{cases} -\Delta u = f, & \text{in } U \\ \quad u = g, & \text{on } \partial U \end{cases}$$

by obtaining Green's function.

Let $x \in U$ and assume $u \in C^2(U)$, and consider $u(y)$ and $\Phi(y - x)$.

Recall Green's formula $\int_\Omega u\Delta v - v\Delta u = \int_{\partial\Omega} u\frac{\partial u}{\partial\eta} - v\frac{\partial v}{\partial\eta}$.

Then, let $\varepsilon > 0$ and define $V_\varepsilon U \setminus B(x, \varepsilon)$.

IMAGE HERE - 1

By applying Green's Formula,

$$\int_{V_\varepsilon} u(y)\underbrace{\Delta\Phi(y - x)}_{0} - \Phi(y - x)\Delta u(y) = \int_{\partial V_\varepsilon} \overbrace{\underbrace{u\frac{\partial\Phi(y - x)}{\partial\eta}}_{\square_1} - \underbrace{\Phi(y - x)\frac{\partial u}{\partial\eta}}_{\square_2}}^{\square}$$

Notice that $\partial V_\varepsilon = \partial U \cup \partial B(x, \varepsilon)$.

Let us analyze $\square$ along $\partial B(x, \varepsilon)$

For $\square_2$ along $\partial B(x, \varepsilon)$,

$$\left| \int_{\partial B(x,\varepsilon)} \Phi(y - x)\frac{\partial u(y)}{\partial\eta} \right| \le \sup_{\overline{U}} |\nabla U| \int_{\partial B(x,\varepsilon)} \Phi(y - x)\, dS(y)$$

$$= \frac{C}{\varepsilon^{n-2}} \int_{\partial B(x,\varepsilon)} dS(y)$$

$$= \frac{C\varepsilon^{n-1}}{\varepsilon^{-2}}$$

$$= c\varepsilon$$

Then $\lim_{\varepsilon\to 0} \int_{\partial B(x,\varepsilon)} \square_2 = 0$.

Now, for $\square_1$ along $\partial B(x, \varepsilon)$ and recalling $\nabla\Phi(z) = \frac{-z}{n\alpha(n)|z|^n}$ while $\eta(z) = \frac{-z}{|z|}$ such that

$$\frac{\partial\Phi}{\partial\eta}(z) = \nabla\Phi\cdot\eta = \frac{|z|^2}{n\alpha(n)|z|^{n+1}} = \frac{1}{n\alpha(n)|z|^{n-1}}$$

we have

$$\int_{\partial B(x,\varepsilon)} u(y)\frac{\partial\Phi(y - x)}{\partial\eta}\, dS(y) \underset{z=y-x}{=} \int_{\partial U(0,\varepsilon)} u(z + x)\frac{\partial\Phi(z)}{\partial\eta}|z|\, ds(z)$$

$$= \frac{1}{n\alpha(n)} \int_{\partial B(0,\varepsilon)} \frac{u(z + x)}{|z|^{n-1}}\, dS(z)$$

$$= \frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B(0,\varepsilon)} u(z + x)\, dS(z)$$

$$\underset{y=z+x}{=} \frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B(x,\varepsilon)} u(y)\, dS(y)$$

$$= \fint_{\partial B(x,\varepsilon)} u(y)\, dS(y)$$

Then $\lim_{\varepsilon\to 0} \int_{\partial B(x,\varepsilon)} \square_1 = u(x)$. It follows, then, that

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$$\int_U -\Phi(y-x)\Delta u(y) = \int_{\partial U} \overbrace{\underbrace{u\frac{\partial \Phi(y-x)}{\partial \eta}}_{\square_1} - \underbrace{\Phi(y-x)\frac{\partial u}{\partial \eta}}_{\square_2}}^{\square} + u(x)$$

That is

$$u(x) \underset{\square_4}{=} -\int_U \Phi(y-x)\Delta u + \int_{\partial u} \Phi(y-x)\frac{\partial u}{\partial \eta} - u\frac{\partial \Phi(y-x)}{\partial \eta}$$

Notice that we have $-\Delta u = f$ in U and $u = g$ on ∂U, but we will need $\frac{\partial u}{\partial \eta}\big|_{\partial U}$.

Definition: Corrector Function

Given a domain $U \subseteq \mathbb{R}^n$ open and bounded with $x \in U$, define the function $\phi^x(y)$ that satisfies the following

$$\begin{cases} \Delta \phi^x(y) = 0, & \text{in } U \\ \phi^x(y) = \Phi(y-x), & \text{on } y \in \partial U \end{cases}$$

Note that we do not know that such a function exists.

Green's Function Continued

Suppose that we have $\phi^x(y)$. Then, applying green's formula for $u(y)$ and $\phi^x(y)$,

$$\int_U u\Delta \underbrace{\phi^x(y)}_{0} - \phi^x(y)\Delta u = \int_{\partial U} u\frac{\partial \phi^x(y)}{\partial \eta} - \underbrace{\phi^x(y)\frac{\partial u}{\partial \eta}}_{\Phi(y-x)\frac{\partial u}{\partial \eta}}$$

Then

$$\int_{\partial U} \Phi(y-x)\frac{\partial u}{\partial \eta} \underset{\square_3}{=} \int_{\partial U} u\frac{\partial \phi^x(y)}{\partial \eta} + \int_U \phi^x(y)\Delta u$$

Replacing $\square_3$ in $\square_4$,

$$u(x) = -\int_U \Phi(y-x)\Delta u + \int_{\partial U} u\frac{\partial \phi^x(y)}{\partial \eta} + \int_U \phi^x(y)\Delta u - \int_{\partial U} u\frac{\partial \Phi(y-x)}{\partial \eta}$$

and, therefore,

$$u(x) = -\int_U \Delta u\left[\Phi(y-x) - \phi^x(y)\right] - \int_{\partial U} u\frac{\partial}{\partial \eta}\left[\Phi(y-x) - \phi^x(y)\right]$$

Definition: Green's Function

Given a domain $U \subseteq \mathbb{R}^n$, the Green's function for $x \in U$ is defined by

$$G(x,y) := \Phi(y-x) - \phi^x(y)$$

Theorem: Representation Formula

Suppose $U \subseteq \mathbb{R}^n$, and $u \in C^2(U)$ that solves

$$\begin{cases} -\Delta u = f, & \text{in } U \\ u = g, & \text{on } \partial U \end{cases}$$

Then,

$$u(x) = \int_U f G(x,y) - \int_{\partial U} g \frac{\partial G(x,y)}{\partial \eta}$$

Interpretation of the Green's Functions

$$\Delta_y G(x,y) = \Delta_y \Phi(y-x) - \underbrace{\Delta_y \phi^x(y)}_{0} = \delta^x(y)$$

and

$$G(x,y) = \Phi(y-x) - \phi^x(y) = 0, \quad y \in \partial U$$

That is, it is the Dirac delta on the interior which vanishes at the boundary.

Theorem: Symmetry of the Green's Function

For all $x, y \in U$, $x \neq y$, we want to show that $G(x,y) = G(y,x)$.

Proof

Let $x, y \in U$, $x \neq y$.
Define $V(z) := G(x,z)$ and $W(z) := G(y,z)$.
Notice that $\Delta_z V = 0$ for $z \neq x$ and $\Delta_z W = 0$ for $w \neq y$ and $V(z) = W(z) = 0$ for $z \in \partial U$.
IMAGE HERE - 2
Then, let us consider $\varepsilon > 0$ and

$$\Omega_\varepsilon := U \setminus \left[B(x,\varepsilon) \bigsqcup B(y,\varepsilon) \right]$$

Then

$$0 = \int_{\Omega_\varepsilon} W \underbrace{\Delta V}_{0} - V \underbrace{\Delta W}_{0} = \int_{\partial \Omega_\varepsilon} W \frac{\partial V}{\partial \eta} - V \frac{\partial W}{\partial \eta}$$

$$= \int_{\partial U} W \frac{\partial V}{\partial \eta} - V \frac{\partial W}{\partial \eta} + \int_{\partial B(x,\varepsilon)} W \frac{\partial V}{\partial \eta} - V \frac{\partial W}{\partial \eta} + \int_{\partial B(y,\varepsilon)} W \frac{\partial V}{\partial \eta} - V \frac{\partial W}{\partial \eta}$$

It follows that

$$\underbrace{\int_{\partial B(x,\varepsilon)} \overset{(a)}{\overbrace{W \frac{\partial V}{\partial \eta}}} - \overset{(b)}{\overbrace{V \frac{\partial W}{\partial \eta}}}}_{\heartsuit_1} = \underbrace{\int_{\partial B(y,\varepsilon)} V \frac{\partial W}{\partial \eta} - W \frac{\partial V}{\partial \eta}}_{\heartsuit_2}$$

Let us analyze (b), fixing $\varepsilon_0 > 0$ such that $\varepsilon < \varepsilon_0$

$$\left| \int_{B(x,\varepsilon)} V \frac{\partial W}{\partial \eta} \right| \leq \sup_{z \in \partial B(x,\varepsilon)} |V(z)| \int_{B(x,\varepsilon)} \left| \frac{\partial W}{\partial \eta}(z) \right| dS(z)$$

$$\leq \sup_{z \in \partial B(x,\varepsilon_0)} |\nabla W(z)| \int_{\partial B(x,\varepsilon)} dS(z)$$

$$\leq C\varepsilon^{n-1} \sup_{z \in \partial B(x,\varepsilon)} |V(z)|$$

$$\leq C\varepsilon^{n-1} \left(\frac{C}{\varepsilon^{n-2} + C} \right)$$

$$= C\varepsilon + C\varepsilon^{n-1}$$

Since, given $z \in \partial B(x,\varepsilon)$,

$$V(z) = G(x,z) = \Phi(z-x) - \phi^x(z)$$

we have

$$|V(z)| \leq |\Phi(z-x)| + |\phi^x(z)|$$

$$\leq \frac{C}{\varepsilon^{n-2}} + \sup_{z \in B(x,\varepsilon_0)} |\phi^x(z)|$$

Thus, we have $\lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} V \frac{\partial W}{\partial \eta} = 0$.
Let us analyze (a),

$$\int_{\partial B(x,\varepsilon)} W(z) \frac{\partial V}{\partial \eta}(z)\, dS(z) = \int_{\partial B(x,\varepsilon)} W(z) \left[\frac{\Phi(z-x)}{\partial \eta} - \frac{\partial \phi^x(z)}{\partial \eta} \right] dS(z)$$

$$= \int_{\partial B(x,\varepsilon)} \overbrace{W(z) \frac{\partial \Phi(z-x)}{\partial \eta}}^{(e)} - \overbrace{W(z) \frac{\partial \phi^x(z)}{\partial \eta}}^{(h)}\, dS(z)$$

Analyzing (h),

$$\left| \int_{\partial B(x,\varepsilon)} W(z) \frac{\partial \phi^x(z)}{\partial \eta} \right| \leq \sup_{\partial B(x,\varepsilon_0)} |\nabla \phi^x(z)||W(z)| \int_{\partial B(x,\varepsilon)} dS(z)$$

$$= C\varepsilon^{n-1}$$

Then $\lim_{\varepsilon \to 0} h = 0$ and

$$\lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} W(z) \frac{\partial \Phi(z-x)}{\partial \eta} = W(x)$$

So $\lim_{\varepsilon \to 0}(a) = W(x)$. Then

$$\lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} W \frac{\partial V}{\partial \eta} - V \frac{\partial W}{\partial \eta} = W(x)$$

Applying the same process,

$$\lim_{\varepsilon \to 0} \int_{\partial B(y,\varepsilon)} V \frac{\partial W}{\partial \eta} - W \frac{\partial V}{\partial \eta} = V(y)$$

Therefore $W(x) = V(y)$ and $G(y,x) = G(x,y)$.

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Definition: Half Space

Define the half space $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \mid x_n > 0.$
IMAGE HERE - 3

Definition: Reflection

For a $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$, define its reflection $\tilde{x} = (x_1, \ldots, -x_n)$.

Green's Function in the Half Space

We want to find $\phi^x(y)$ that solves

$$(*)\begin{cases} \Delta \phi^x(y) = 0, & \text{in } \mathbb{R}^n_+ \\ \phi^x(y) = \Phi(y - x), & y \in \partial \mathbb{R}^n_+ \end{cases}$$

Let us consider $\phi^x(y) := \Phi(y - \tilde{x})$, $x, y \in \mathbb{R}^n_+$. Then $\phi^x(y)$ satisfies $*$.
Then we can see that $\Delta \phi^x(y) = 0$.
Let $y \in \partial \mathbb{R}^n_+$ such that $y = (y_1, \ldots, y_{n-1}, 0)$. So

$$\begin{aligned} \phi^x(y) &= \Phi(y - \tilde{x}) \\ &= \Phi(|y - \tilde{x}|) \\ &= \Phi\left(\sqrt{(y_1 - x_1)^2 + \cdots + (y_{n-1} - x_{n-1})^2 + (0 + x^n)^2}\right) \\ &= \Phi\left(|y - x|^2\right) \\ &= \Phi(y - x) \end{aligned}$$

February 5, 2024

Recall: Green's Function

$G(x, y) = \Phi(y - x) - \phi^x(y)$.
For $U \subset \mathbb{R}^n_+$, when

$$G(x, y) = \Phi(y - x) - \Phi(y - \tilde{x})$$

we proved that if $u \in C^2(\overline{U})$,

$$\begin{cases} -\Delta u = f, & U \\ u = g, & \partial U \end{cases}$$

then

$$u(x) = \int_U f G(x, y) \, dy - \int_{\partial U} g \frac{\partial G(x, y)}{\partial \eta} \, dS(y)$$

Let us consider

$$\begin{cases} \Delta u = 0, & \mathbb{R}^n_+ \\ u = g, & \partial \mathbb{R}^n_+ \end{cases}$$

such that

$$u(x) = -\int_{\partial \mathbb{R}^n_+} g \frac{\partial G(x,y)}{\partial \eta} \, dS(y)$$

Let us compute $\frac{\partial G}{\partial \eta}$.

IMAGE HERE - 1 UPPER HALF SPACE WITH NORMAL VECOTR ETA

Recall

$$\nabla \Phi(z) = \frac{-z}{n\alpha(n)|z|^n}$$

$$\frac{\partial \Phi(z)}{\partial z_n} = \frac{-z_n}{n\alpha(n)|z|^n}$$

so, since $y - \tilde{x}_n = y_n + x_n$,

$$\begin{aligned}
\frac{\partial G}{\partial \eta} &= \nabla G(x,y) \cdot \eta \\
&= -\frac{\partial G(x,y)}{\partial y_{n+1}} \\
&= -\frac{\partial}{\partial y_{n+1}} \left(\Phi(y-x) - \Phi(y-\tilde{x}) \right) \\
&= -\left[\frac{-(y_n - x_n)}{n\alpha(n)|y-x|^n} - \frac{-(y_n + x_n)}{n\alpha(n)|x - \tilde{x}|^n} \right]
\end{aligned}$$

But recall that if $y \in \partial\mathbb{R}^n_+$, $|y - x| = |y - \tilde{x}|$. Then $y \in \partial\mathbb{R}^n_+$,

$$\frac{\partial G(x,y)}{\partial \eta} = -\frac{1}{n\alpha(n)|y-x|^n} \left[-y_n + x_n + y_n + x_n \right] = -\frac{2x_n}{n\alpha(n)|y-x|^n}$$

Then

$$u(x) = \int_{\partial\mathbb{R}^n_+} \frac{g(y)2x_n}{n\alpha(n)|y-x|^n} \, dS(y)$$

Definition: Poisson Kernel

$$K(x,y) = \frac{2x_n}{n\alpha(n)|y-x|^n} = \frac{\partial G}{\partial y_n}$$

is called the Poisson Kernel and

$$u(x) \int_{\partial\mathbb{R}^n_+} g(x)K(x,y) \, dS(y)$$

is called the Poisson Formula.

Notice (HW): $\int_{\partial\mathbb{R}^n_+} K(x,y) \, dy = 1$, $\forall x \in \mathbb{R}^n_+$ (hint: apply spherical coordinates).

Theorem 14:

Define

$$(*) \quad u(x) = \int_{\partial \mathbb{R}_+^n} K(x,y)g(y)\, dS(y)$$

Suppose that $g \in C^\infty(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$.
Then

1. $u \in C^\infty(\mathbb{R}_+^n) \cap L^\infty(\mathbb{R}_+^n)$.

2. $\Delta u = 0$, $\mathbb{R}_+^n$.

3. $\lim_{x \to x^0} u(x) = g(x^0)$, $x \in \mathbb{R}_+^n$, $x^0 \in \partial\mathbb{R}_+^n$.

Proof

We know $G(x,y)$ satisfies

$$\Delta_y G(x,y) = \delta^x(y).$$

Notice that $y \to G(x,y)$ is harmonic for $x \neq y$.
Recall that $G(x,y) = G(y,x)$, so $x \to G(x,y)$ is harmonic for $x \neq y$.
Then $x \to \frac{\partial G(x,y)}{\partial y_n}$ is harmonic $(*_2)$ for $x \neq y$ and for $y \in \partial\mathbb{R}_+^n$.
Homework: compute this directly.
Noticing that K is smooth when $x \neq y$, then

$$\frac{\partial u}{\partial x_i} = \int_{\partial \mathbb{R}_+^n} \frac{\partial}{\partial x_i} K(x,y)g(y)\, dS(y)$$

Homework: justify puting the limit inside the integral.
Homework: prove that $\frac{\partial u}{\partial x_i}$ is continuous.
By repeatedly taking derivaties, we see $u \in C^\infty(\mathbb{R}_+^n)$.
Moreover,

$$\Delta_x u = \int_{\partial \mathbb{R}^n} \underbrace{\Delta_x K(x,y)}_{=0} g(y)\, dS(y) = 0$$

by $*_2$. Then

$$|u(x)| \leq \int_{\partial \mathbb{R}_+^n} |K(x,y)||g(y)|\, dS(y) \leq ||g||_{L^\infty(\mathbb{R}^{n-1})} \underbrace{\int_{\partial \mathbb{R}_+^n} K(x,y)\, dS(y)}_{=1} < \infty$$

For part c, consider $x^0 \in \partial\mathbb{R}_+^n$ and $\varepsilon > 0$.
Since $g \in C^\infty(\mathbb{R}^{n-1})$, let $\delta > 0$ such that $|y - x^0| < \delta \implies |g(y) - g(x^0)| < \varepsilon$ for $y \in \partial\mathbb{R}_+^n$.
IMAGE HERE - 2 DELTA BALL AROUND X0 HALF DELTA BALL WITH X INSIDE
Now, let us consider $|x - x_0| < \frac{\delta}{2}$.

$$|u(x) - g(x^0)| = \left| \int_{\partial \mathbb{R}^n_+} K(x,y)g(y) - K(x,y)g(x^0) \, dS(y) \right|$$

$$\leq \int_{\partial \mathbb{R}^n_+} K(x,y) \left| g(y) - g(x^0) \right| \, dS(y)$$

$$= \underbrace{\int_{\partial \mathbb{R}^n_+ \cap B(x^0,\delta)} K(x,y)|g(y) - g(x^0)| \, dS(y)}_{I} + \underbrace{\int_{\partial \mathbb{R}^n_+ \cap B^c(x^0,\delta)} K(x,y)|g(y) - g(x^0)| \, dS(y)}_{II}$$

Then

$$I \leq \varepsilon \int_{\partial \mathbb{R}^n_+ \cap B(x^0,\delta)} K(x,y) \leq \varepsilon$$

Now, we want to control II

$$\int_{\partial \mathbb{R}^n_+ \cap B^c(x^0,\delta)} K(x,y)|g(y) - g(x^0)| \, dS(y) \leq C||g||_{L^\infty} \int_{\partial \mathbb{R}^n_+ \cap B^c(x^0,\delta)}$$

$$= \frac{2C}{n\alpha(n)} \int_{\partial \mathbb{R}^n_+ \cap B^c(x^0,\delta)} \frac{x_n}{|x-y|^n} \, dS(y)$$

We want to control $|x^0 - y|$ with something related to $|x - y|$.
We know $|y - x^0| > \delta$ and we will consider $|x - x^0| < \frac{\delta}{2}$. So

$$|y - x^0| \leq |y - x| + |x - x^0| \leq |y - x| + \frac{\delta}{2} \leq |y - x| + \frac{|y - x^0|}{2}$$

So $\frac{|y - x_0|}{2} \leq |y - x|$ implies that $\frac{1}{|y-x|^n} \leq \frac{2^n}{|y-x_0|^n}$. Therefore

$$\int_{\partial \mathbb{R}^n_+ \cap B^c(x^0,\delta)} K(x,y)|g(y) - g(x^0)| \, dS(y) \leq C x_n \int_{\partial \mathbb{R}^n_+ \cap B^c(x^0,\delta)} \frac{1}{|y - x^0|^n} \, dS(y)$$

$$= \int_\delta^\infty \int_{\partial B^{n-1}(x^0,r)} \frac{1}{r^n} \, dS(y) dr$$

$$= C \int_\delta^\infty \frac{1}{r^n} r^{n-2} \, dr$$

$$= C \int_\delta^\infty \frac{1}{r^2} \, dr$$

$$= C(\frac{1}{r})|_\delta^\infty$$

$$= \frac{C}{\delta}$$

Then $II \leq \frac{C x_n}{\delta}$. Now let us consider $|x - x^0| < \frac{\delta}{J}$ where $\frac{1}{J} < \varepsilon$. Then

$$II \leq \frac{C|x - x^0|}{\delta} \leq C \frac{\delta}{\delta J} \leq C\varepsilon$$

Energy Methods: Uniqueness

Consider the boundary value problem

$$(*) \quad \begin{cases} -\Delta u = f, & U, \ f \in C(U) \\ \quad u = g, & \partial U, \ g \in C(\partial U) \end{cases}$$

with U open and bounded in $\mathbb{R}^n$, $u \in C^2(\overline{U})$ and $\partial U \in C^1$.

Theorem 16: Uniquness

There exists at most one solution $u \in C^2(\overline{U})$ for $*$.

Proof

Let us suppose that $\tilde{u}$ is another solution.
Then $w := u - \tilde{u}$ solves

$$\begin{cases} \Delta w = 0, & U, \ w \in C^2(\overline{U}) \\ w = 0, & \partial U \end{cases}$$

where

$$0 = \int_U w \Delta w = - \int_U |\nabla W|^2 + \int_{\partial U} w \frac{\partial w}{\partial \eta}$$

so

$$0 = \int_U |\nabla w|^2 \implies \nabla w = 0 \implies w = 0 \implies u = \tilde{u}$$

Definition: Energy Functional

Let us consider

$$A = \left\{ w \in C^2(\overline{U}) \mid W|_{\partial U} = g \right\}$$

for $g \in C(\partial U)$ and $f \in C(U)$.
Define the energy functional $I : A \to \mathbb{R}$ given by $I(w) := \int_U \frac{|\nabla w|^2}{2} - fw$.

Energy Methods: Dirichlet Principle

Calculus of variations applied to the Laplace equation.

Theorem:

Suppose $u \in C^2(\overline{U})$ is a solution to the problem

$$\square \quad \begin{cases} -\Delta u = f, & U \\ \quad u = g, & \partial U \end{cases}$$

Then,

$$(*) \quad I(u) = \min_{w \in A}\{I(w)\}$$

Moreover, if $u \in A$ such that $*$ happens, then u satisfies $\square$.

Proof

($\Longrightarrow$) For $w \in A$,

$$0 = \int_U \underbrace{(-\Delta u - f)}_{=0}(u - w)$$

$$= \int_U -\Delta u(u - w) - \int_U f(u - w)$$

$$= \int_U \nabla(u - w) \cdot \nabla u - \underbrace{\int_{\partial U}(u - w) \cdot \frac{\partial u}{\partial \eta}}_{=0} - \int_U f(u - w)$$

$$= \int_U |\nabla u|^2 - \int_U \nabla w \cdot \nabla u - \int_U fu + \int_U fw$$

Notice that, since $|a - b|^2 \geq 0$ implies $\frac{a^2 + b^2}{2} \geq ab$,

$$\int_U \nabla w \cdot \nabla u \leq \int_U |\nabla w||\nabla u| \leq \frac{1}{2}\int_U |\nabla w|^2 + \frac{1}{2}\int_U |\nabla u|^2$$

Therefore

$$\int_U |\nabla u|^2 - \int_U fu = \int_U \nabla w \cdot \nabla u - \int_U fw$$

$$\leq \int_U \frac{|\nabla w|^2}{2} + \int_U \frac{|\nabla u|^2}{2} - \int_U fw$$

$$\int_U \frac{|\nabla u|^2}{2} - fu \leq \int_U \frac{|\nabla w|^2}{2} - fw$$

Then

$$I(u) \leq I(w), \quad \forall w \in A$$

$B/u \in A$.

February 7, 2024

Recall: Energy Functional

For $U \subseteq \mathbb{R}^n$ bounded, $g \in C(\partial U)$, $f \in C(\overline{U})$

$$A = \left\{w \in C^2(\overline{U}) \mid w|_{\partial U} = g\right\}$$

we have

$$I(w) := \int_U \frac{1}{2}|\nabla w|^2 - fw$$

Theorem:

Suppose $u \in A$ such that $I(u) = \min\{I(w) \mid w \in A\}$. Then u satisfies

$$\begin{cases} -\Delta u = f, & \text{in } U \\ \quad u = g, & \text{on } \partial U \end{cases}$$

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Proof

Consider $v \in C_c^\infty(U)$.
Define $i : \mathbb{R} \to \mathbb{R}$ such that $\tau \mapsto I(\tau) := I(u + \tau v)$.
Notice that $u + \tau v$ is a perturbation of u and, since $u + \tau v \in C^2(\overline{U})$ while $u + \tau v|_{\partial U} = u|_{\partial U} = g$, $u + \tau v \in A$. Then

$$i(0) = I(u) \le I(u + \tau v) = i(\tau)$$

so i has a minimum point at $\tau = 0$. Compute

$$
\begin{aligned}
i(\tau) &= I(u + \tau v) \\
&= \int_U \frac{|\nabla(u + \tau v)|^2}{2} - f(u + \tau v) \\
&= \int_U \frac{|\nabla u|^2}{2} + \tau \langle \nabla u, \nabla v \rangle + \frac{\tau^2 |\nabla v|^2}{2} - \int_U fu - \tau \int_U fv \\
&= \int_U \frac{|\nabla u|^2}{2} + \tau \int_U \langle \nabla u, \nabla v \rangle + \frac{\tau^2}{2} \int_U |\nabla v|^2 - \int_U fu - \tau \int_U fv
\end{aligned}
$$

So i is a polynomial in τ, and

$$i'(0) = i'(\tau)_{\tau=0} = \left(\int_U \langle \nabla u, \nabla v \rangle + \tau \int_U |\nabla v|^2 - \int_U fv \right)_{\tau=0}$$

So

$$
\begin{aligned}
0 = i'(0) \\
&= \int_U \langle \nabla u, \nabla v \rangle - \int_U fv \\
&= \int_U -\Delta u \cdot v + \underbrace{\int_{\partial U} \frac{\partial u}{\partial \eta} \cdot v}_{=0} - \int_U fv \\
&= \int_U \underbrace{(-\Delta u - f)}_{=0} v
\end{aligned}
$$

Since $0 = \int gv$, $\forall v \in C_c^\infty(U)$ requires $g \equiv 0$.
Then $-\Delta u - f = 0$.

Heat Equation (Diffusion Equation)

The equations

$$(*) \begin{cases} u_t - \Delta u = 0, & \text{homogeneous case} \\ u_t - \Delta u = f, & \text{non-homogeneous case} \end{cases}$$

(note that $\Delta u = \Delta_x u$)
subject to some boundary and initial conditions $t \ge 0$ time and $x \in \mathbb{R}^n$, space variable, $x \in U$ and opsen set of $\mathbb{R}^n$.
$u : U \times (0, \infty) \to \mathbb{R}$ defined as $(x, t) \mapsto u(x, t)$ with u unknown.
IMAGE HERE - 1

Motivation: Fundamental Solution of the Heat Equation

We would like to have the following:
If u solves

$$\begin{cases} u_t - \Delta u = 0, & \mathbb{R}^n \times (0, \infty) \\ u(x, 0) = g(x), & \mathbb{R}^n \times \{0\} \end{cases}$$

then

$$u(x, t) = \int_{\mathbb{R}^n} G(x - y, t) g(y) \, dy$$

How do we get G?
Let us suppose that $u(\tilde{x}, \tilde{t})$ solves

$$\begin{cases} u_{\tilde{t}} - \Delta_{\tilde{x}} u = 0 \\ u(\tilde{x}, 0) = g(\tilde{x}) \end{cases}$$

We would like to have invariance under dilation.

$$v(x, t) := u(\lambda x, \lambda^2 t)$$

Such that

$$v_t = \nabla U|_{(\lambda x, \lambda^2 t)} - \frac{\partial}{\partial t} \begin{bmatrix} \lambda x \\ \lambda^2 t \end{bmatrix}$$

$$= \lambda^2 u_{\tilde{t}}(\lambda x, \lambda^2 t)$$

$$v_{x_i} = \lambda u_{\tilde{x}_i}(\lambda x, \lambda^2 t)$$

$$v_{x_i x_i} = \lambda^2 u_{\tilde{x}_i \tilde{x}_i}(\lambda x, \lambda^2 t)$$

Therefore

$$v_t - \Delta_x v = \lambda^2 u_{\tilde{t}} - \lambda^2 \Delta_{\tilde{x}} u = \lambda^2 (\underbrace{u_{\tilde{t}} - \Delta_{\tilde{x}} u}_{=0}) = 0$$

with

$$v(x, 0) = u(\lambda x, 0) = g(\lambda x)$$

Then, applying the motivation,

$$v(x, t) = \int_{\mathbb{R}^n} G(x - y, t) g(\lambda y) \, dy \underset{z = \lambda y}{=} \int_{\mathbb{R}^n} G\left(x - \frac{z}{\lambda}, t\right) g(z) \frac{dz}{\lambda^n}$$

On the other hand,

$$v(x, t) = u(\lambda x, \lambda^2 t) = \int_{\mathbb{R}^n} G(\lambda x - z, \lambda^2 t) g(z) \, dz$$

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It follows that

$$\frac{1}{\lambda^n} G\left(\overbrace{x - \frac{z}{\lambda}}^{w}, t \right) = G(\lambda x - z, \lambda^2 t)$$

$$\frac{1}{\lambda^n} G(w, t) = G(\lambda w, \lambda^2 t)$$

If $\lambda^2 t = 1$, then

$$G(w, t) = \frac{1}{t^{n/2}} G\left(\frac{1}{\sqrt{t}} w, 1 \right)$$

If we call $G\left(\frac{w}{\sqrt{t}}, 1 \right) = v\left(\frac{w}{\sqrt{t}} \right)$, then we are looking at $G(w, t) = \frac{1}{t^{n/2}} v\left(\frac{w}{t^{1/2}} \right)$.
So, we have motivation to define

$$u(x, t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta} \right)$$

for α, β appropriate and $v(y) : \mathbb{R}^n \to \mathbb{R}$.

Obtaining a Fundamental Solution to the Heat Equation

Let us compute u_t and $\Delta_x u$.

$$u_t = \frac{\partial}{\partial t} \left(\frac{1}{t^\alpha} v\left(\frac{x}{t^\beta} \right) \right)$$

$$= \frac{(-\alpha)}{t^{\alpha+1}} v\left(\frac{x}{t^\beta} \right) + \frac{1}{t^\alpha} \frac{\partial}{\partial t} \left(v\left(\frac{x}{t^\beta} \right) \right)$$

$$= \frac{(-\alpha)}{t^{\alpha+1}} v\left(\frac{x}{t^\beta} \right) + \frac{1}{t^\alpha} \cdot \nabla v|_{\frac{x}{t^\beta}} \cdot \frac{\partial}{\partial t} \left(\frac{x}{t^\beta} \right)$$

$$u_t = \frac{(-\alpha)}{t^{\alpha+1}} v\left(\frac{x}{t^\beta} \right) + \frac{(-\beta)}{t^\alpha t^{\beta+1}} \nabla v|_{\frac{x}{t^\beta}} \cdot x \quad \square_1$$

and

$$\frac{\partial u}{\partial x_i} = \frac{1}{t^\alpha} \frac{\partial}{\partial x_i} \left(v\left(\frac{x}{t^\beta} \right) \right)$$

$$= \frac{1}{t^\alpha} \nabla v|_{\frac{x}{t^\beta}} \cdot \frac{\partial}{\partial x_i} \left(\frac{x}{t^\beta} \right)$$

$$= \frac{1}{t^{\alpha+\beta}} \frac{\partial v}{\partial x_i}|_{\frac{x}{t^\beta}}$$

while

$$\frac{\partial^2 u}{\partial x_i x_i} = \frac{1}{t^{\alpha+2\beta}} \frac{\partial^2 v}{\partial x_i x_i}|_{\frac{x}{t^\beta}} \quad \square_2$$

Then, replacing $\square_1$ and $\square_2$ in $*$,

$$-\frac{\alpha}{t^{\alpha+1}} v\left(\frac{x}{t^\beta} \right) - \frac{\beta}{t^{\alpha+\beta+1}} \nabla v|_{\frac{x}{t^\beta}} \cdot x - \frac{1}{t^{\alpha+2\beta}} \Delta v|_{\frac{x}{t^\beta}} \stackrel{?}{=} 0$$

Set $y := \frac{x}{t^\beta}$

$$-\frac{\alpha}{t^{\alpha+1}}v(y) - \frac{\beta}{t^{\alpha+1}}\nabla v(y) \cdot y - \frac{1}{t^{\alpha+2\beta}}\Delta v(y) = 0$$

Multiplying through by $-t^{\alpha+1}$,

$$\alpha v(y) + \beta \nabla v(y) \cdot y + \frac{1}{t^{2\beta-1}}\Delta v(y) = 0$$

Let us assume that $2\beta - 1 = 0$ such that $\beta = \frac{1}{2}$, giving

$$\alpha v(y) + \frac{1}{2}\nabla v(y) \cdot y + \Delta v(y) = 0$$

Since the Laplacian is rotationally invariant, assume $v(y) = w(|y|)$ for $w : \mathbb{R}^+ \to \mathbb{R}$.
Recall that $\frac{\partial}{\partial y_i}|y| = \frac{\partial}{\partial y_i}\left(\sqrt{y_1^2 + \cdots + y_n^2}\right) = \frac{y_i}{|y|}$. Now

$$\frac{\partial}{\partial y_i}v(y) = \frac{\partial}{\partial y_i}(w(|y|)) = w'(|y|) \cdot \frac{\partial}{\partial y_i}(|y|) = w'(|y|) \cdot \frac{y_i}{|y|}$$

$$\begin{aligned}\frac{\partial^2 v(y)}{\partial y_i y_i} &= \frac{\partial}{\partial y_i}\left(w'(|y|)\right)\frac{y_i}{|y|} + w'(|y|) \cdot \frac{\partial}{\partial y_i}\left(\frac{y_i}{|y|}\right)\\ &= w''(|y|) \cdot \frac{y_i^2}{|y|^2} + w'(|y|)\left[\frac{1}{|y|} + y_i\frac{\partial}{\partial y_i}\left(\frac{1}{|y|}\right)\right]\\ &= w''(|y|)\frac{y_i^2}{|y|^2} + w'(|y|)\left[\frac{1}{|y|} - \frac{y_i^2}{|y|^3}\right]\end{aligned}$$

Replacing in the PDE of v,

$$\begin{aligned}0 &= \alpha w(|y|) + \frac{1}{2}\frac{w'(|y|)y}{|y|} \cdot y + \sum_{i=1}^n w''(|y|)\frac{y_i^2}{|y|^2} + w'(|y|)\left[\frac{1}{|y|} - \frac{y_i^2}{|y|^3}\right]\\ &= \alpha w(|y|) + \frac{1}{2}w'(|y|)|y| + w''(|y|) + w'(|y|)\left[\frac{n}{|y|} - \frac{1}{|y|}\right]\end{aligned}$$

If $|y| = r$

$$0 = \alpha w(r) + \frac{1}{2}w'(r)r + w''(r) + w'(r)\frac{n-1}{r}$$

Take $\alpha = \frac{n}{2}$ and multiply through by r^{n-1},

$$\begin{aligned}0 &= \frac{nr^{n-1}}{2}w(r) + \frac{r^n}{2}w'(r) + w''(r)r^{n-1} + w'(r)(n-1)r^{n-2}\\ &= \frac{1}{2}\left[w(r)r^n\right]' + \left[w'(r)r^{n-1}\right]'\end{aligned}$$

Then by the fundamental theorem of calculus, $w'(r)r^{n-1} + \frac{w(r)r^n}{2} = C$.

We would like $w, w' \xrightarrow[r \to \infty]{} 0$. Then $C = 0$, so

$$w'(r)r^{n-1} = -\frac{w(r)r^n}{2}$$

Which gives

$$w' = \frac{-wr}{2} \iff \frac{w'}{w} = -\frac{r}{2} \iff (\ln(w))' = \frac{-r}{2} \iff \ln(w) = -\frac{r^2}{4} + d$$

and, finally,

$$w(r) = be^{-\frac{r^2}{4}}$$

Then define

$$\begin{aligned}
u(x,t) &:= \frac{1}{t^{n/2}}v\left(\frac{x}{t^{1/2}}\right) \\
&= \frac{1}{t^{n/2}}w\left(\left|\frac{x}{t^{1/2}}\right|\right) \\
&= \frac{b}{t^{n/2}}e^{-\frac{1}{4}\left|\frac{x}{t^{1/2}}\right|^2} \\
&= \frac{b}{t^{n/2}}e^{-\frac{1}{4t}|x|^2}
\end{aligned}$$

Where b is chosen such that the expression integrates to 1.

Definition: Fundamental Solution of the Heat Equation

The fundamental solution for the heat equation is given by

$$\begin{cases} \Phi(x,t) = \frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x|^2}{4t}}, & x \in \mathbb{R}^n,\ t > 0 \\ \qquad\qquad 0, & x \in \mathbb{R}^n,\ t < 0 \end{cases}$$

where we have chosen $b = \frac{1}{(4\pi)^{n/2}}$.

IMAGE HERE - 2

Notice that these match in the limit away from the origin $(\lim_{(x,t) \to (x_0,0)} \Phi(x,t) = 0)$.

Remark: $\Phi(x,t)$ has a unique singularity at $(0,0)$.