

Advanced Analysis

September 25, 2025

Suppose we have some function of the form $-\Delta + q \in \mathbb{L}(H)$ satisfying $R_A(\lambda)(A - \lambda I)^{-1}$ bounded on $\text{Im}(\lambda) > 0$ and not surjective for $\text{Im}(\lambda) = 0$.

IMAGE 1

Waves: solutions to $\partial_{tt}u + Au = 0$ on \mathbb{R}^n .

Mathematical Tools

- Spectral theory of unbounded operators
- Complex analysis
- Functional analysis
- Microlocal analysis
- Semiclassical analysis

Classical Resonances in ODEs

IMAGE 2

A harmonic oscillator assuming no friction.

We have an acceleration force, $m\ddot{x}(t) = -kx(t)$ which gives $\ddot{x} + \omega_0^2 x = 0$ with $\omega_0 = \sqrt{\frac{k}{m}}$ and has solution $x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$.

With forcing, i.e. $m\ddot{x}(t) = -kx(t) + A\sin(\omega t)$, we have $\ddot{x} + \omega_0^2 x' = A\sin(\omega t)$.

If $|\omega| \neq |\omega_0|$, then $x(t) \sim \text{trig}\left(\left(\frac{\omega-\omega_0}{2}\right)t\right)\left(\left(\frac{\omega+\omega_0}{2}\right)t\right)$ the low and high frequencies respectively.

IMAGE 3

Beats (non-amplified)

If instead $|\omega| = |\omega_0|$, then $x(t) \propto \text{trig}(\omega t)t$.

IMAGE 4

In general, $\dot{x} + Ax = 0$ for $x \in \mathbb{R}^n$, $x(t) = \exp(-tA)x(0)$.

In the case where A is skew-adjoint, i.e. $\text{sp}(A) \subseteq i\mathbb{R}$, $(x, Ax) = 0 \forall x \in \mathbb{R}^n$, then

$$\frac{d}{dt}(x, x) = (\dot{x}, x) + (x, \dot{x}) = (-Ax, x) - (x, Ax) = 0$$

Which implies that $\|x(t)\|$ is constant and the dynamics are norm preserving.

To generate resonant solutions, if (iw, v) is an eigenpair of A ($\omega \in \mathbb{R}$), consider $\dot{x} + Ax = e^{-i\omega t}v$. As an ansatz, we look for a solution of the form $x(t) = a(t)v$ and the equation becomes $(\dot{a}(t) + i\omega a)v = e^{-i\omega t}v$. Then

$$\begin{aligned} e^{-i\omega t} \frac{d}{dt}(e^{i\omega t}a) &= e^{-i\omega t} \\ \frac{d}{dt}(e^{i\omega t}a) &= 1 \\ a(t) &= te^{-i\omega t}. \end{aligned}$$

Resonances in PDEs

Consider one-dimensional waves on $[0, L]$, $L > 0$.

$$\begin{cases} \partial_{tt}u + \partial_{xx}u = 0 \\ u|_{t=0} = f & x \in [0, L] \\ \partial_t u|_{t=0} = g & x \in [0, L] \\ u(0, t) = u(L, t) = 0 & \forall t \geq 0 \end{cases}$$

We want to think about this as $\partial_{tt}u = Au = 0$ where A is the Dirichlet Laplacian $Au = -\partial_{xx}u$ with Dirichlet boundary conditions. We then want to find the spectral decomposition of A , $Au - \lambda u = 0 = -\partial_x^2 u - \lambda u$.

$$\begin{aligned} \lambda = 0. \quad u(x) &= A + Bx \implies A = B = 0 \\ \lambda = -p^2. \quad u(x) &= Ae^{px} + be^{-px} \implies A = B = 0 \\ \lambda = p^2. \quad u(x) &= A\cos(px) + B\sin(px) \implies 0 = u(0) = A \quad 0 = u(L) = B\sin(pl) \implies p = k\pi, k \in \mathbb{N} \end{aligned}$$

Therefore there are infinitely many eigenpairs $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $\phi_n(x) = \sin\left(\frac{k\pi x}{L}\right)$.

IMAGE 5

The family $\{\phi_n, n \in \mathbb{N}\}$ is dense in $L^2([0, L])$ where the unbounded operator $(-\partial_x^2)$ with Dirichlet boundary conditions is self-adjoint.

Other Prototypes

(of unbounded self-adjoint operators with discrete spectrum)

- Laplace-Beltrami operators on compact manifolds without boundary.

IMAGE 6

- On compact domains with boundary there is the Laplacian with Dirichlet boundary conditions.

The (Quantum) Harmonic Oscillator

$H = -\frac{d^2}{dx^2} + x^2$ on \mathbb{R} , on $L^2(\mathbb{R})$ with $(f, g) = \int_{\mathbb{R}} f(x)\overline{g(x)} dx$.

H acts on the Schwarz space $\mathcal{S}(\mathbb{R}) := \left\{ f \in C^\infty(\mathbb{R}), \forall k, \ell \geq 0, \sup_{x \in \mathbb{R}} \left| x^k \left(\frac{d}{dx}\right)^\ell f(x) \right| < \infty \right\}$.

- The action of $H : \mathcal{S}(\mathbb{R})$ is continuous.
- H is L^2 -symmetric: $\int_{\mathbb{R}} -f''\bar{g} + x^2 f\bar{g} dx = (Hf, g) = (f, Hg) = \int_{\mathbb{R}} -\bar{g}''f + x^2 f\bar{g} dx$ (integrating by parts).

We seek eigenvalues $Hu = \lambda u$. If (u, λ) and (v, μ) are eigenpairs, then

$$0 = (Hu, v) - (u, Hv) = (\lambda u, v) - (u, \mu v) = (\lambda - \mu)(u, v)$$

Where if the difference is nonzero then $(u, v) = 0$.

We can write $H = L^+ L^- + I$ where $L^+ = -\frac{d}{dx} + x$ and $L^- = \frac{d}{dx} + x$ and also $[H, L^+] = 2L^+$ and $[H, L^-] = -2L^-$.

Note that H is a non-negative operators

$$(Hf, f) = \int_{\mathbb{R}} ((f')^2 + x^2 f^2) dx > 0$$

for $f \neq 0$ and $f \in \mathcal{S}(\mathbb{R})$. Thus $\text{sp}(H) \subseteq (0, \infty)$. If $Hv = \lambda v$, then $H(L^+ v) = [H, L^+]v + L^+(Hv) = (\lambda + 2)L^+ v$. Similarly $H(L^- v) = (\lambda - 2)L^- v$.

Now we want to solve $L^- \phi_0 = 0$. $\frac{d}{dx} \phi_0 + x \phi_0 = 0$ tells us that $\phi_0(x) = \frac{1}{\sqrt{\pi}} e^{-x^2/2}$ (L^2 -normalized). Therefore $H\phi_0 = \phi_0$ and we have an eigenvalues of one. So we may construct $\phi_n = \frac{(L^+)^n \phi_0}{\|(L^+)^n \phi_0\|}$ which gives an eigenvector of H with eigenvalues $2n + 1$. Note that $\|(L^+)^n \phi_0\| = \sqrt{2^n n!}$.

Fact: $\phi_n = p_n(x) e^{-x^2/2}$ where p_n is the Hermite polynomial of degree n .

$$\delta_{nq} = (\phi_n, \phi_q) = \int_{\mathbb{R}} p_n(x) p_q(x) e^{-x^2} dx$$

Theorem

$\{\phi_n\}_{n \geq 0}$ is dense in $L^2(\mathbb{R})$ (if $\int_{\mathbb{R}} g \phi_n dx = 0$ for all n , then $g = 0$).

Proof (Sketch)

For $g \in L^2$, $\xi \in \mathbb{R}$, $F_g(\xi) = \int_{\mathbb{R}} e^{ix\xi} g(x) \phi_0(x) dx = \widehat{g\phi_0}(\xi)$. We observe that

- F_g is real-analytic in ξ .
- $F_g^{(k)}(0) = \int_{\mathbb{R}} (-ix)^k g(x) \phi_0(x) dx = 0$ by assumption.

So we have a real-analytic function where all derivatives vanish at a point. So $F_g \equiv 0$, $g\phi_0 = 0$, and $g = 0$.

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One of the overarching goals is to obtain large time asymptotics of the solution $v(x, t)$ ($x \in \mathbb{R}$, $t > 0$) to

$$\begin{cases} -\partial_{tt} v - P_V v = F(x, t) & \text{on } \mathbb{R}_x \times (0, \infty)_t \\ v(x, 0) = \partial_t v(x, 0) = 0, & F \in C_C^\infty(\mathbb{R}_x \times (0, \infty)_t) \end{cases}$$

where $P_V = D_x^2 + V(x) = -\left(\frac{\partial}{\partial}\right)^2 + V(x)$ and $D_x = \frac{1}{i} \frac{\partial}{\partial x}$. The operator D_x is symmetric and self-adjoint on appropriately chosen domains. For $f(x)$ and $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx$, $\widehat{D_x f} = \xi \hat{f}(\xi)$. $V \in L_{\text{comp.}}^\infty(\mathbb{R})$ (i.e. compactly supported L^∞) is the potential. If $f, g \in \mathcal{S}(\mathbb{R})$, then $(P_V f, g)_{L^2(\mathbb{R})} = (f, P_V g)_{L^2(\mathbb{R})}$.

IMAGE 1

Another way to look at this assuming v exists, we can consider $u(x, \lambda) := \int_0^\infty e^{it\lambda} v(x, t) dt$ (the Fourier-Laplace transform of v) with $\lambda \in \mathbb{C}$, $\text{Im}(\lambda) > 0$. Write $\lambda = \xi + ic$, $c > 0$, such that $u(x, \xi + ic) = \int_0^\infty e^{it\xi} e^{-ct} v(x, t) dt = \mathcal{F}_{t \mapsto \xi}(t \mapsto$

$e^{-ct}v(x, t))(x, -\xi)$. Then $u(x, \lambda)$ solves

$$\begin{aligned} \int_0^\infty e^{it\lambda}(-\partial_{tt}v - P_V v) dt &= \int_0^\infty e^{it\lambda}F(x, t) = \hat{F}(x, \lambda) \\ (\lambda^2 - P_V) \underbrace{\int_0^\infty e^{it\lambda}v(x, t) dt}_{u(x, \lambda)} &= \hat{F}(x, \lambda) \end{aligned}$$

which is an entire function in λ .

To Do:

- Study solvability of $(\lambda^2 - P_V)u = \hat{F}(x, \lambda)$.
- Return to v .

For frozen c , we can get $v(x, t)$ back by Fourier inversion.

$$\begin{aligned} e^{-ct}v(x, t) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it\xi} u(x, \xi + ic) d\xi \\ v(x, t) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it(\xi+ic)} u(x, \xi + ic) d\xi \\ v(x, t) &= \frac{1}{2\pi} \int_{\text{Im}(\lambda)=c} e^{-it\lambda} u(x, \lambda) d\lambda \end{aligned}$$

IMAGE 2

where the spectral problem is invertible.

1D Waves in the Time Domain

Suppose $R > 0$ is such that $\text{supp } V \subset [-R, R]$ and $\text{supp } F \subset [-R, R] \times (0, \infty)$. If $|x| > R$, the PDE looks like $\partial_{tt}v - \partial_{xx}v = 0 = (\partial_t + \partial_x)(\partial_t - \partial_x)v$. Setting $\xi = x + t$ and $\mu = x - t$, then it follows that

$$\partial_\xi \partial_\mu v = 0 \implies v = F(\xi) + G(\mu) = F(x+t) + G(x-t)$$

IMAGE 3

On $x > R$, we can expect $v(x, t) = F_+(x+t) + G_+(x-t)$; on $x < R$, we expect $v(x, t) = F_-(x+t) + G_-(x-t)$. The terms G_+ and F_- are outgoing whereas the terms F_+ and G_- are incoming and, given that we assumed a source, we expect to be zero.

What does incoming/outgoing look like on the spectral side? $(\lambda^2 - P_V)u = \hat{F}(x, \lambda)$ supported in $|x| \leq R$. For $|x| > R$, $(\lambda^2 + \partial_x^2)u = 0$ leads to $u = Ae^{ix\lambda} + Be^{-ix\lambda}$. For $x > R$, $u(x) = a_+e^{i\lambda|x|} + b_+e^{-i\lambda|x|}$ for $x < -R$, $u(x) = a_-e^{i\lambda|x|} + b_-e^{-i\lambda|x|}$. u is outgoing if and only if $b_\pm = 0$ and incoming if and only if $a_\pm = 0$.

P_V is an unbounded, symmetric operator on a Hilbert space. For $z \in \mathbb{C}$, $\text{sp}(P_V)$ is the set on the complement of which $(P_V - z)$ is boundedly invertible. That is, $\forall f, \exists! u$ such that $(P_V - z)u = f$ and $\|u\| \lesssim \|f\|$.

Waves in the Time Domain [Evans, §2.4]

Goal: if v solves

$$\begin{aligned}\partial_{tt}v - \partial_{xx}v &= f(x, t) \quad x \in \mathbb{R}, t > 0, f \in C_C^\infty(\mathbb{R} \times (0, \infty)) \\ v(x, 0) &= \partial_t v(x, 0) = 0 \quad x \in \mathbb{R}\end{aligned}$$

then $v(x, t) = \frac{1}{2} \int_0^t \int_{x-s}^{x+s} f(y, t-s) dy ds$. We look at

$$\begin{cases} \partial_{tt}v - \partial_{xx}v = 0 \rightsquigarrow v(x, t) = F(x+t) + G(x-t) \\ v(x, 0) = g(x), \partial_t v(x, 0) = h(x) \end{cases}$$

Initial conditions gives us

$$\begin{cases} F(x) + G(x) = g(x) \\ F'(x) - G'(x) = h(x) \end{cases} \quad \begin{cases} G'(x) = \frac{1}{2}(g'(x) - h(x)) \\ F'(x) = \frac{1}{2}(g'(x) + h(x)) \end{cases}$$

So

$$\begin{aligned}F(x) &= \frac{1}{2} \left(g(x) + \int_0^x h(s) ds \right) + C_1 \\G(x) &= \frac{1}{2} \left(g(x) - \int_0^x h(s) ds \right) + C_2 \\v(x, t) &= \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} h(s) ds + C\end{aligned}$$

IMAGE 4

This has a finite speed of propagation in the sense that if we suppose $\text{supp}(g, h) \subset [-R, R]$ then $v(x, t) = 0$ whenever $x > R + t$ or $x < -R - t$.

Now we want to go from the homogeneous problem to the inhomogeneous problem. The idea is to think about $v(x, t) = \int_0^t v(x, t; s) ds$ where $v(x, t; s)$ solves the homogeneous problem

$$\begin{cases} \partial_{tt}v(\cdot, \cdot; s) - \partial_{xx}v(\cdot, \cdot; s) = 0 \\ v(\cdot, s; s) = 0, \partial_t v(\cdot, s; s) = f(x, s) \end{cases}$$

Then

$$\partial_{tt}v - \partial_{xx}v = 0 \iff \partial_t \begin{pmatrix} v \\ \partial_t v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \partial_{xx} & 0 \end{pmatrix} \begin{pmatrix} v \\ \partial_t v \end{pmatrix} \quad \text{and} \quad \begin{bmatrix} v \\ \partial_t v \end{bmatrix}_{t=s} = \begin{bmatrix} * \\ * \end{bmatrix}$$

So $v(x, t; s) = \frac{1}{2} \int_{x-(t-s)}^{x+(t-s)} f(y, s) dy$ and $v(x, t) = \frac{1}{2} \int_0^t \int_{x-s}^{x+s} f(y, t-s) dy ds$ follows.

Going back to the original PDE, $(-\partial_{tt} - P_V)v = F$ is equivalent to $(\partial_{tt} - \partial_{xx})v = -(Vv + F)$ which leads to the conclusion that $v(x, t) = -\frac{1}{2} \int_0^t \int_{x-s}^{x+s} (Vv + F)(y, t-s) dy ds$. For $|x| > R$, v is outgoing.

IMAGE 5

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Take come complex vector space and consider the Hilbert space $(\mathcal{H}, (\cdot, \cdot))$ with (\cdot, \cdot) satisfying

$$\begin{cases} (\lambda f, g) = \lambda(f, g) \\ (f, \lambda g) = \overline{\lambda}(f, g) \\ (f, g) = (g, f) \\ f \mapsto (f, g) =: ||f||^2 \text{ a norm} \\ (\mathcal{H}, ||\cdot||) \text{ complete with respect to the norm} \end{cases}$$

- Examples

- $(\mathbb{C}^n, (a, b) = \sum_{j=1}^n a_j \bar{b}_j), a = (a_1, \dots, a_n), b = (b_1, \dots, b_n)$
- $L^2(x, \mu)$ (e.g. $[0, 1]$ and the Lebesgue measure), $(f, g) = \int_X f \bar{g} d\mu$.

Bounded Operators: $T : \mathcal{H} \rightarrow \mathcal{H}$ bounded if and only if $\sup_{||x||=1} ||Tx|| < \infty$ satisfying

$$\begin{cases} \mathcal{B}(\mathcal{H}) \text{ the space of bunched operators on } \mathcal{H} \text{ (a complex vector space)} \\ ||\cdot|| \text{ is a norm on } \mathcal{B}(\mathcal{H}), \text{ making it complete} \\ \text{There is a multiplication, } \mathcal{B}(\mathcal{H}) \ni A, B \mapsto AB \text{ and } ||AB|| \leq ||A|| ||B|| \end{cases}$$

Adjoint: if $A \in \mathcal{B}(\mathcal{H})$, $\exists! A^* \in \mathcal{B}(\mathcal{H})$ such that $\forall f, g \in \mathcal{H}, (Af, g) = (f, A^*g)$ where A is symmetric/self-adjoint if $A = A^*$.
The notions are different in the world of unbounded operators.

- Example

- $\mathcal{H} = \mathbb{C}^n$: $T \in M_n(\mathbb{C})$ symmetric if and only if T is Hermitian. $t_{ij} = \overline{t_{ji}}$.
- $\mathcal{H} = L^2([0, 1])$, $Tf(t) = tf(t)$. $(Tf, g) = \int_0^1 tf(t) \overline{g(t)} dt = \int_0^1 f(t) \overline{tg(t)} dt = (f, Tg)$.
- $\mathcal{H} = L^2(\mathbb{R})$ with the Fourier transform. $||f(x)||^2 = c ||\hat{f}(\xi)||^2$ (Parseval's Equality).

Finite Dimensional Spectral Theorem

If $A \in M_n(\mathbb{C})$ is Hermitian, there exists an orthonormal basis (ϕ_1, \dots, ϕ_n) of \mathbb{C}^n and real eigenvalues $\lambda_1, \dots, \lambda_n$ such that $A\phi_j = \lambda_j \phi_j$.

Important observation: if A is Hermitian, then λ_j is real for each j , and $(\overline{A\phi_j}, \phi_j) = (\phi_j, A\phi_j) = (A\phi_j, \phi_j) = \lambda_j ||\phi_j||^2$. So $\lambda_j = \frac{(A\phi_j, \phi_j)}{||\phi_j||^2}$ is real. If $\lambda_j \neq \lambda_k$, then $(\phi_j, \phi_k) = 0$ since $(A\phi_j, \phi_k) - (\phi_j, A\phi_k) = (\lambda_j - \lambda_k)(\phi_j, \phi_k)$.

Notation: Let $u, v \in \mathbb{C}^n$, denote $u \otimes \bar{v}$ the operator $(u \otimes \bar{v})w = (w, v)u$.

With A as in the theorem, we can write $A = \sum_{j=1}^n \lambda_j \phi_j \otimes \bar{\phi}_j$ ($I = \sum_{j=1}^n \phi_j \otimes \bar{\phi}_j$). A second way of writing this is

IMAGE 1

Where $U^* = U^{-1}$ and $A = U \Lambda U^*$. This allows us to construct a functional calculus for A where

$$\begin{cases} A^2 = U \Lambda U^* U \Lambda U^* = U \Lambda^2 U^* \\ A^n = U \Lambda^n U^* \\ p(A) = U \cdot p(\Lambda) \cdot U^*, p \text{ a polynomial} \end{cases}$$

Defining $f(A) := U \cdot f(\Lambda) \cdot U^*$, we obtain a Banach algebra homomorphism. Then $f \in C([-||A||, ||A||])$ is also a Banach algebra with sup norm and pointwise multiplication.

IMAGE 2

Then we can map $C([-||A||, ||A||]) \ni f \mapsto f(A) \in \mathcal{B}(\mathcal{H})$. This is useful for solving ODEs.

- Prototypes

- Heat equation: $\partial_t u + Au = 0$, $u|_{t=0} = u_0$, $u(t) = e^{-tA}u_0$.
- Schrödinger equation: $i\partial_t u + Au = 0$, $u|_{t=0} = u_0$, $u(t) = e^{-itA}u_0$.
- Wave equation: $\partial_{tt} u + Au = 0$, $u|_{t=0} = u_0$, $\partial_t u|_{t=0} = u_1$.

Write $u(t) := \sum_{j=1}^n u_j(t)\phi_j$ with the PDE $\sum_{j=1}^n (u_j'' + \lambda_j u_j)\phi_j = 0$. Then $u_j'' + \lambda_j u_j = 0$, $u_j(0) = u_{j,0}$, and $u_j'(0) = u_{j,1}$. Suppose $\lambda_j > 0$ for all j . Then $u_j(t) = u_{j,0} \cos(\sqrt{\lambda_j}t) + \frac{u_{j,1}}{\sqrt{\lambda_j}} \sin(\sqrt{\lambda_j}t)$. So

$$u(t) = \sum_{j=1}^n \cos(\sqrt{\lambda_j}t)u_{j,0}\phi_j + \frac{1}{\sqrt{\lambda_j}} \sin(\sqrt{\lambda_j}t)u_{j,1}\phi_j$$

Therefore $u = \cos(t\sqrt{A})u_0 + A^{-1/2} \sin(t\sqrt{A})u_1$.

Spectrum of a Bounded Operator

Take $T \in \mathcal{B}(\mathcal{H})$. We say that T is invertible (within $\mathcal{B}(\mathcal{H})$) if and only if $\exists S \in \mathcal{B}(\mathcal{H})$ such that $TS = ST = I$.

Counterexample: take $\mathcal{H} = \ell^2(\mathbb{N}_0) = \{u = (u_n)_{n \geq 0} : \sum |u_n|^2 < \infty\}$ and $Au = \left(\frac{1}{n}u_n\right)_{n \geq 0}$. Then the proxy for $A^{-1}u = (nu_n)_{n \geq 0}$ is not bounded.

Given $T \in \mathcal{B}(\mathcal{H})$, the resolvent set of T is $\rho(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \text{ is invertible}\}$. Invertibility is equivalent to $\forall y \in \mathcal{H}$, $\exists!x$ such that $Tx - \lambda x = y$ with an estimate $||x|| \lesssim ||y||$.

For $\lambda \in \rho(T)$, denote $R(\lambda)$ or $R_T(\lambda) = (T - \lambda I)^{-1} \in \mathcal{B}(\mathcal{H})$ the resolvent of T . Properties of the resolvent set:

1. $\rho(T) \neq \emptyset$ (in fact, if $|\lambda| > ||T||$ then $\lambda \in \rho(T)$).

2. $\rho(T)$ is open.

3. the map $\rho(T) \ni \lambda \mapsto R_T(\lambda) \in \mathcal{B}(\mathcal{H})$ is holomorphic in the sense that $\forall \lambda_0 \in \rho(T)$, $\exists R'_T(\lambda_0) \in \mathcal{B}(\mathcal{H})$ such that $\lim_{\lambda \rightarrow \lambda_0} \left| \left| \frac{R_T(\lambda) - R_T(\lambda_0)}{\lambda - \lambda_0} - R'_T(\lambda_0) \right| \right| = 0$.

For a., if $|\lambda| > ||T||$, $Tx - \lambda x = y \iff \left(I - \frac{T}{\lambda}\right)x = -\frac{y}{\lambda} \iff x = \sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}}y$. Then $R_T(\lambda) = \frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{T}{\lambda}\right)^k$ and

$$||R_T(\lambda)|| \leq \frac{1}{||\lambda||} \frac{1}{1 - ||T/\lambda||} \leq \frac{1}{|\lambda| - ||T||}$$

For b., pick $\lambda_0 \in \rho(T)$ and find $r > 0$ such that $|\lambda - \lambda_0| < r \implies \lambda \in \rho(T)$. Then $Tx - \lambda x = y \iff (T - \lambda_0)x - (\lambda - \lambda_0)x = y \iff x - (\lambda - \lambda_0)R_T(\lambda_0)x = R_T(\lambda_0)y$ where if $||(\lambda - \lambda_0)R_T(\lambda_0)|| < 1$ it is boundedly solvable by Neumann series.

For c.,

$$\begin{aligned}
R_T(\lambda) - R_T(\lambda_0) &= (T - \lambda I)^{-1} - (T - \lambda_0 I)^{-1} \\
(T - \lambda I)(R_T(\lambda) - R_T(\lambda_0)) &= I - (T - \lambda_0 I + (\lambda_0 - \lambda)I)(T - \lambda_0 I)^{-1} \\
(T - \lambda I)(R_T(\lambda) - R_T(\lambda_0)) &= I - I + (\lambda - \lambda_0)R_T(\lambda_0) \\
R_T(\lambda) - R_T(\lambda_0) &= (\lambda - \lambda_0)R_T(\lambda)R_T(\lambda_0)
\end{aligned}$$

$$\text{So } \frac{R_T(\lambda) - R_T(\lambda_0)}{\lambda - \lambda_0} - R_T(\lambda_0)^2 = o(\lambda - \lambda_0).$$

Then we define the spectrum $\sigma(T) := \mathbb{C} \setminus \rho(T)$ which is closed since $\rho(T)$ is open.

Lemma

If $T \in \mathcal{B}(\mathcal{H})$ is self-adjoint, then $\sigma(T) \subseteq [-||T||, ||T||]$.

- Proof

First we know $\sigma(T) \subseteq \{|\lambda| \leq ||T||\}$. We want to show that it is real, and that if $\lambda = a+ib$ and $b \neq 0$ then $T - (a+ib)I$ is invertible.

$T - (a+bi)I$ is injective.

$$\begin{aligned}
||(T - (a+ib))x||^2 &= (Tx - (a+ib)x, Tx - (a+ib)x) \\
&= ||Tx||^2 + (a^2 + b^2)||x||^2 - (Tx, (a+ib)x) - ((a+ib)x, Tx) \\
&= ||Tx||^2 + (a^2 + b^2)||x||^2 - (a-ib)(Tx, x) - (a+ib)(x, Tx) \\
&= ||Tx||^2 + a^2||x||^2 - 2a(x, Tx) + b^2||x||^2 \geq b^2||x||^2
\end{aligned}$$

since $||Tx||^2 + a^2||x||^2 - 2a(x, Tx) \geq 0$ by Cauchy-Schwarz. Therefore $T - (a+ib)$ is injective and, by the open mapping theorem, $(T - (a+ib))^* = T - (a-ib)$ is surjective. Similarly for $T - (a-ib)$, and the norm estimate is $||(T - (a+ib))^{-1}|| \leq \frac{1}{b}$. Note that $\frac{1}{b} = \frac{1}{\text{dist}(a+ib, \mathbb{R})}$.

Note that the spectrum of T may no longer be made of eigenvalues in the non-finite case. There may exist λ such that $T - \lambda I$ is not injective, $\exists v \neq 0$ $Tv = \lambda v$. Recall the example $Tf(t) = tf(t)$ with $f \in L^2((\cdot, \cdot), dt)$. T is self-adjoint, $||T|| \leq 1$, and $(Tf, f) = \int_0^1 t|f(t)|^2 dt \geq 0$. So $\sigma(T) \subseteq [0, 1]$. For $\lambda \in [0, 1]$ is $T - \lambda I$ injective? $Tf = \lambda f \iff tf(t) = \lambda f(t) \iff (t-\lambda)f(t) = 0$ which implies $f \equiv 0$ in $L^2([0, 1])$. Is $T - \lambda I$ surjective? $(t-\lambda)f(t) = g(t) \iff f(t) = \frac{g(t)}{t-\lambda}$, so $g(t) \equiv 1 \in L^2([0, 1])$ which implies $f(t) = \frac{1}{t-\lambda}$ is not $L^2([0, 1])$ and $\sigma(T) = [0, 1]$.

October 9, 2025

Spectral Resolution

Take \mathcal{H} a Hilbert space, and say that $P \in \mathcal{B}(\mathcal{H})$ is an orthogonal projection if $P^2 = P$ and $P^* = P$. Then let $\mathcal{P}(\mathcal{H}) = \{P \in \mathcal{B}(\mathcal{H}), P \text{ is an orthogonal projection}\}$.

- Examples

– $\phi \in \mathcal{H}$, $||\phi|| = 1$, $P := \phi \otimes \bar{\phi}$. Then $P\psi = (\psi, \phi)\phi$ and $P^2\psi = P(\psi, \phi)\phi = (\psi, \phi)(\phi, \phi)\phi = P\psi$.

- ϕ_1, \dots, ϕ_n an orthonormal family with $P = \sum_{k=1}^n \phi_k \otimes \bar{\phi}_k$.
- $\mathcal{H} = \ell^2(\mathbb{N})$, $e_j = (0, 0, \dots, 0, 1, 0, \dots)$ and $I = \sum_{j=1}^{\infty} e_j \otimes \bar{e}_j$.
- $\mathcal{H} = L^2(\mathbb{R})$. Fix I an interval with χ_I the characteristic function for I . Then take $Pf := \chi_I f$.
 - * $PPf = \chi_I \chi_I f = \chi_I f = Pf$.
 - * $\int_{\mathbb{R}} \chi_I f \bar{g} dx = \int_{\mathbb{R}} f \bar{\chi_I g} dx$.
 - * If I has a nonempty interior, then $\text{Range}(P) = \{f \in L^2(\mathbb{R}), \text{supp } f \subset I\} \simeq L^2(I)$.

Definition: Spectral Resolution

A spectral resolution is a map $\mathbb{R} \ni \lambda \mapsto E(\lambda) \in \mathcal{P}(\mathcal{H})$ satisfying

1. $\forall f \in \mathcal{H}$, $\|E(\lambda)f\|$ is increasing.
2. $\exists [a, b]$ such that $E(\lambda) = 0$ if $\lambda < a$ and $E(\lambda) = \text{Id}$ if $\lambda \geq b$.
3. $E(\lambda)$ is right continuous. That is, $\forall f \in \mathcal{H}, \lambda \in \mathbb{R}$,

$$\lim_{\substack{\mu \rightarrow \lambda \\ \mu > \lambda}} \|E(\mu)f - E(\lambda)f\| = 0$$

Alternatively, we can require $E(\lambda)E(\mu) = E(\min\{\mu, \lambda\})$.

Long story short: the collection of self-adjoint bounded operators is in one-to-one correspondence with the collection of spectral resolutions.

- Examples

- $A \in M_n(\mathbb{C})$, $A^* = A$, with eigencouples $(\lambda_1, \phi_1), \dots, (\lambda_n, \phi_n)$ and simple spectrum $\lambda_1 < \lambda_2 < \dots < \lambda_n$. Define $E(\lambda) := \sum_{j: \lambda_j \leq \lambda} \phi_j \otimes \bar{\phi}_j$.
 - * $A = I$ gives $E(\lambda) = 0$ for $\lambda < 1$ and $E(\lambda) = \text{Id}$ for $\lambda \geq 1$.
 - * $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ gives $E(\lambda) = \text{Id}_{\lambda \geq 1} e_1 \otimes e_1 + \text{Id}_{\lambda \geq 2} e_2 \otimes e_2$.
 - If $f = f_1 e_1 + f_2 e_2$, then $\|E(\lambda)f\|^2 = \text{Id}_{\lambda \geq 1}(\lambda) |f_1|^2 + \text{Id}_{\lambda \geq 2}(\lambda) |f_2|^2$.

Spectral Measures

A spectral resolution gives rise to spectral measures

$$f, g \in \mathcal{H} \quad \lambda \mapsto (E(\lambda)f, g) = F(\lambda) \in \mathbb{C}$$

This defines a Lebesgue-Stieltjes measure

$$\mu_F : \mu_F((a, b]) = F(b) - F(a), \quad \forall a, b \in \mathbb{R}$$

We can construct this as follows:

- When $f = g$, $\lambda \mapsto (E(\lambda)f, f) = (E(\lambda)^2 f, f) = ||E(\lambda)f||^2$ (increasing).
- When $f \neq g$,

$$(E(\lambda)f, g) = (E\lambda(f), E\lambda(g)) = \frac{1}{4} \left(||E(\lambda)(f+g)||^2 - ||E(\lambda)(f-g)||^2 - i||E(\lambda)(f+ig)||^2 + i||E(\lambda)(f-ig)||^2 \right)$$

$\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ defines a projection-valued measure E_Ω , $\Omega \subset \mathbb{R}$ a Borel set. Start with $E_{(a,b]} = E(b) - E(a)$. We would like for E_Ω to satisfy $E_{\Omega_1}E_{\Omega_2} = E_{\Omega_1 \cap \Omega_2}$, $E_\emptyset = 0$, $E_{\mathbb{R}} = \text{Id}$.

Theorem: Spectral Theorem

For $A \in \mathcal{B}(\mathcal{H})$ self-adjoint, there exist $a, b \in \mathbb{R}$ and an A -dependent spectral resolution $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ such that

$$A = \int_{a_-}^b \lambda dE(\lambda)$$

in the sense that $(Af, g) = \int_{a_-}^b \lambda d(E(\lambda)f, g)$ for all $f, g \in \mathcal{H}$. This is amenable to creating a functional calculus

$C([-||A||, ||A||]) \rightarrow$ bounded self-adjoint operators that commute with A

$$h \mapsto h(A) := \int_{a_-}^b h(\lambda) dE(\lambda)$$

The idea is that $E(\lambda) = \chi_{(-\infty, \lambda]}(A)$. Once E is constructed, this leads to E_Ω for Ω Borel. We say that a measure μ is supported in G (Borel) if for every Ω Borel, $\mu(\Omega) = \mu(\Omega \cap G)$. Then $\text{supp } E \subset \sigma(A)$.

Functional Calculus

We want to make sense of $h(A)$ for h in a large enough class. If p is a polynomial, we can make sense of $p(A) = \sum_{k=0}^n a_k A^k$ which is self-adjoint and bounded.

For $h \in C(-||A||, ||A||)$, h is uniformly approximated by polynomials. We want to show that p_n is uniformly Cauchy which implies that $p_n(A)$ converges to some $h(A)$.

For $h = \chi_{(-\infty, \lambda]}$, we proceed by approximation by tent functions.

IMAGE 1

Definition: Positive Operator

If S is a self-adjoint, bounded operator on \mathcal{H} , we say that S is positive ($S \geq 0$) if $(Sf, f) \geq 0$, $\forall f \in \mathcal{H}$. For S_1, S_2 self-adjoint and bounded, we say that $S_1 \geq S_2$ if and only if $S_1 - S_2 \geq 0$.

For $T \in \mathcal{B}(\mathcal{H})$, self-adjoint, set $a := \inf_{||f||=1} (Tf, f)$ and $b := \sup_{||f||=1} (Tf, f)$. Then $a \text{Id} \leq T \leq b \text{Id}$.

$$((T - a \text{Id})f, f) = (Tf, f) - a(f, f) = (f, f) \left(\left(T \frac{f}{||f||}, \frac{f}{||f||} \right) - a \right) \geq 0$$

We want to show that if p is a polynomial on $[-||A||, ||A||]$, then $(\inf_{[-||A||, ||A||]} p) \text{Id} \leq p(A) \leq (\sup_{[-||A||, ||A||]} p) \text{Id}$.

Lemma

If T_1 and T_2 are positive and commute, then $T_1 T_2 \geq 0$.

Square Root Lemma

If $A \geq 0$ (i.e. bounded, self-adjoint, and positive), then $\exists! B \geq 0$ such that $B^2 = A$ and B commutes with any operator that commutes with A .

- Proof

Use the power series of $z \mapsto \sqrt{1-z}$ at $z=0$.

$$1 + \sum_{k=1}^{\infty} c_k z^k$$

We can find that $c_k < 0$ for all $k \geq 1$ and that the series converges uniformly on $\{|z| \leq 1\}$.

Now let $A \geq 0$ which implies that $0 \text{Id} \leq I - A \leq 1 \text{Id}$. Without loss of generality, suppose $\text{supp } ||A|| \leq 1$. The idea is to write

$$B = \sqrt{A} = \sqrt{I - (I - A)} = I + \sum_{k=1}^{\infty} c_k (I - A)^k$$

which converges strongly because the series converges uniformly. Then $B^2 = A$. We see that $B \geq 0$ using the fact that $\text{sign}(c_k) < 0$ which implies $\sum_{k \geq 1} c_k \geq -1$. The proof of uniqueness can be found in the text.

Proof of Lemma

Assuming the square root lemma, write $T_2 = B^2$. Then since $[T_1, T_2] = 0$, $[B, T_1] = 0$. Then

$$(T_1 T_2 f, f) = (T_1 B^2 f, f) = (B T_1 B f, f) = (T_1 (B f), B f) \geq 0$$

Weaker Version

Instead of $(\inf_{[-||A||, ||A||]} p) \text{Id} \leq p(A) \leq (\sup_{[-||A||, ||A||]} p) \text{Id}$, we have that if $\min_{[-||A||, ||A||]} p \geq 0$, then $p(A) \geq 0$.

Proof

If $p \geq 0$ on $[-||A||, ||A||]$, we can factor it as a product of positive pieces

$$\begin{aligned} p(x) &= \prod_{j=1}^{r_j < -||A||} \underbrace{(x - r_j)}_{\geq 0} \underbrace{(s_j - x)}_{\geq 0} ((x - a_j)^2 + b_j^2) \\ p(A) &= \prod_{j=1}^{r_j < -||A||} \underbrace{(A - r_j)}_{\geq 0} \underbrace{(s_j - A)}_{\geq 0} \underbrace{((A - a_j)^2 + b_j^2)}_{\geq 0} \end{aligned}$$

Using the previous lemma, we have that $P(A) \geq 0$.

Proof

Finally, to show that $(\inf_{[-||A||, ||A||]} p) \text{Id} \leq p(A) \leq (\sup_{[-||A||, ||A||]} p) \text{Id}$, we see that $p - \inf p$ and $\text{supp } f - p$ are positive polynomials and apply the weaker version to them.

Definition

We can define $h(A)$ for $h \in C(-||A||, ||A||)$ by Weierstrass approximation. There exist p_n polynomials such that $\sup_{[-||A||, ||A||]} |p_n - h| \xrightarrow{n \rightarrow \infty} 0$. Then p_n is uniformly Cauchy, so

$$\inf(p_n - p_m) \text{Id} \leq p_n(A) - p_m(A) \leq \sup(p_n - p_m) \text{Id}$$

which implies that

$$||p_n(A) - p_{n-1}(A)|| \leq \max(\sup(p_n - p_m), -\inf(p_n - p_m)) \xrightarrow{n, m \rightarrow \infty} 0.$$

So $p_n(A)$ is Cauchy in $(\mathcal{B}(\mathcal{H}), ||\cdot||)$ which means it converges. We call $h(A) = \lim_{n \rightarrow \infty} p_n(A)$. We still want to show that $h(A)$ is bounded and self-adjoint.

October 14, 2025

Spectral Theorem for Bounded Self-Adjoint Operators

If $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint such that $a||f||^2 \leq (Af, f) \leq b||f||^2 \quad \forall f \in \mathcal{H}$, then there exists a spectral resolution $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ such that $A = \int_{a^-}^b \lambda dE(\lambda)$.

Proof (Continued)

We would like that $E(\lambda) = \phi^\lambda(A)$ where $\phi^\lambda := \chi_{(-\infty, \lambda]}$, however ϕ^λ is not continuous so instead we can approximate it as

$$\phi_n^\lambda := \begin{cases} 1, & x \leq \lambda \\ \text{linear on } [\lambda, \lambda + \frac{1}{n}] \\ 0, & x \geq \lambda + \frac{1}{n} \end{cases}$$

IMAGE 1

To demonstrate this, we need the following proposition.

Proposition

If T_n is a sequence of positive operators and $T_n \geq T_{n+1} \geq 0$, then there exists some $T \geq 0$ such that $T_n f \rightarrow Tf, \forall f \in \mathcal{H}$.

Proof

Fix $f \in \mathcal{H}$, and consider $(T_n f, f)$ which is decreasing, bounded from below, and therefore converges and is Cauchy. Now as an estimate, we can say that if $S \in \mathcal{B}(\mathcal{H})$ is self-adjoint where $0 \leq S \leq MI$, then $\forall f \in \mathcal{H}$ we know that

$$||Sf||^2 \leq (Sf, f)^{1/2} M^{3/2} ||f||.$$

To see this, we look at $t \mapsto (S(S+tI)f, (S+tI)f) \geq 0$ since $S \geq 0$. Then

$$(S(S+tI)f, (S+tI)f) = (S^2 f, Sf) + 2t ||Sf||^2 + t^2 (Sf, f)$$

So $\Delta < 0$ if and only if

$$\begin{aligned} ||Sf||^4 - (S^2 f, Sf)(Sf, f) &< 0 \\ ||Sf||^4 &\leq (Sf, f)(S^2 f, Sf) \\ &\leq (Sf, f) \underbrace{||S^2 f|| ||Sf||}_{M^3 ||f||^2} \end{aligned}$$

We apply this to $T_n - T_m = S$ where $T_n - T_m \leq T_0$ for all $n \leq m$. Then

$$||(T_n - T_m)f||^2 \leq ((T_n - T_m)f, f)^{1/2} ||f|| ||T_0||^{3/2}$$

Since $(T_n f, f)$ is Cauchy, $T_n f$ is Cauchy and therefore converges. It remains to check that T is linear, positive, satisfies, $T \leq T_0$, etc.

Spectral Theorem Proof Continued

For each n , $\phi_n^\lambda(A)$ makes sense and is positive since $\phi_n^\lambda(t) \geq 0$. Since $\phi_n^\lambda \geq \phi_{n+1}^\lambda$, $\phi_n^\lambda(A) \geq \phi_{n+1}^\lambda(A)$. Thus, by the preceding proposition, $\phi^\lambda(A) := \lim_{n \rightarrow \infty} \phi_n^\lambda(A)$ exists as a bounded, self-adjoint, positive operator.

It remains to check the spectral resolution properties.

The final property to check is that $\forall f \in \mathcal{H}, (Af, f) = \int_a^b \lambda d(E(\lambda)f, f)$. The idea is to approximate multiplication by t with piecewise constant functions. We fix a partition $a = \lambda_0 \leq \dots \leq \lambda_k = b$ such that $\sup(\lambda_{j+1} - \lambda_j) < \delta$. Then

$$t\phi^{\lambda_k}(t) = t = \phi^{\lambda_0}(t) + \sum_{j=1}^k t(\phi^{\lambda_j}(t) - \phi^{\lambda_{j-1}}(t))$$

for all $a \leq t \leq b$.

IMAGE 2

Then

$$\begin{aligned} \lambda^{j-1} &\leq t \leq \lambda^j \\ \lambda^{j-1}(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) &\leq t(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \leq \lambda^j(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \\ \sum_{j=1}^k \lambda^{j-1}(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) &\leq t \sum_{j=1}^k (\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \leq \sum_{j=1}^k \lambda^j(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \\ &\leq \sum_{j=1}^k (\lambda^j - \delta)(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \end{aligned}$$

Adding $\lambda_0 \phi^{\lambda_0}$, we see that

$$t \leq \lambda_0 \phi^{\lambda_0} + \sum_{j=1}^k \lambda^j (\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \leq t + \delta$$

IMAGE 3

When we apply this to A ,

$$A \leq \lambda_0 E(\lambda_0) + \sum_{j=1}^k \lambda^j (E(\lambda_j) - E(\lambda_{j-1})) \leq A + \delta I$$

Finally, we see that

As one refines the partition, $\delta \rightarrow 0$ and $(A, f, f) = \int_{a^-}^b \lambda d(E(\lambda)f, f)$.

Then if $\phi \in C([-||A||, ||A||])$, $\phi(A) = \int_{a^-}^b \phi(\lambda) dE(\lambda)$.

$$\left| (Af, f) - \lambda_0(E(\lambda_0)f, f) - \sum_{j=1}^k \lambda^j(E(\lambda_j)f, f) - (E(\lambda_{j-1}f, f)) \right| \leq \delta ||I||^2$$

Functional Calculus

We observe that for $g \in C_C^\infty(\mathbb{C})$, $g(w) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{w-z} \partial_{\bar{z}} g(z) d^2 z$. Then $f(A) := \frac{i}{\pi} \int_{\mathbb{C}} (A-z)^{-1} \partial_{\bar{z}} \tilde{f}(z) d^2 z$ where $\partial_{\bar{z}} \tilde{f} = O((\text{Im}(z))^n)$ for $n \geq 3$ as $\text{Im}(z) \rightarrow 0$.

Proposition

For every $f \in \mathcal{H}$, the Lebesgue-Stieltjes measure corresponding to $F(\lambda) = (E(\lambda f, f))$ is supported on $\sigma(A)$.

Since $\sigma(A) \subseteq [a, b]$ is closed, $[a, b] \setminus \sigma(A)$ is open (i.e. $\bigcup_{k \in \mathbb{N}} J_k$ for open intervals J_k). We want to show that $F(\lambda)$ is constant on each J_k (equivalently: $\int_{J_k} dF(\lambda) = 0$).

IMAGE 4

Fix $J_k \ni x_0$. Then $R_A(x_0) = (A - x_0 I)^{-1}$ exists, and we can pick $\varepsilon > 0$ such that $\forall z \in \overline{B_\varepsilon(x_0)}$, $||R_A(z)|| \leq M$. Then for $z \in B_\varepsilon(x_0) + \text{Im}(z) \neq 0$,

$$R_A(z) = (A - zI)^{-1} = \phi_z(A)$$

and $\phi_z(t) = \frac{1}{t-z} \in C([a, b])$. Consider

$$R_A(z)R_A(\bar{z}) = \psi_z(A) = \int \frac{1}{|\lambda - z|^2} dE(\lambda)$$

with $\psi_z(t) = \frac{1}{|t-z|^2}$. It follows that for all $z \in \overline{B_\varepsilon(x_0)} \setminus \mathbb{R}$,

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{1}{|\lambda - z|^2} dF(\lambda) \leq \int \frac{1}{|\lambda - z|^2} dF(\lambda) = (R_A(z)R_A(\bar{z})f, f) \leq M^2 ||f||^2$$

which stays true for all $z \in \overline{B_\varepsilon(x_0)}$. In particular for $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$,

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} dx \int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{1}{|\lambda - z|^2} dF(\lambda) \leq 2\varepsilon M^2 ||f||^2$$

Since Fubini holds, we observe that $\int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{1}{|\lambda-x|^2} dx = \infty$, so it must be the case that $\int_{x_0-\varepsilon}^{x_0+\varepsilon} dF(\lambda) = 0$.

Discrete Spectrum vs Essential Spectrum

Recall the spectral measure of A , $E_\Omega(A)$ for $\Omega \subset \mathbb{R}$ Borel where $E_{(a,b]} = E(b) - E(a)$. We say that $\lambda \in \sigma_d(A)$ (the discrete spectrum of A) if there exists $\varepsilon > 0$ such that $\dim(\text{range}(E_{(\lambda-\varepsilon, \lambda+\varepsilon)})) < \infty$. Likewise, $\lambda \in \sigma_{\text{ess}}(A)$ (the essential spectrum) if $\forall \varepsilon > 0$, $\dim(\text{range}(E_{(\lambda-\varepsilon, \lambda+\varepsilon)})) = \infty$.

As an example, take $\mathcal{H} = \ell^2(\mathbb{N})$ with $(Au)_n = \frac{u_n}{n}$ and $A = \sum_{j=1}^{\infty} \frac{1}{j} e_j \otimes e_j$.

IMAGE 5

Discrete spectra include eigenvalues of finite multiplicity.

Essential spectra include accumulation points of eigenvalues, eigenvalues of infinite multiplicity, absolutely continuous spectrum, s.c. spectrum.

Another example if $Af(t) = tf(t)$ on $L^2([0, 1])$. Then $E(\lambda)f(t) = \chi_{(-\infty, \lambda]}f(t)$, and $E_{(a, b]}f(t) = \chi_{(a, b]}f(t)$. $\forall x_0 \in [0, 1]$, we have that $\text{range}(E_{(x_0-\varepsilon, x_0+\varepsilon)}) = L^2((x_0 - \varepsilon, x_0 + \varepsilon))$.

IMAGE 6

October 16, 2025

Compact Operators and Analytic Fredholm Theorem

Definition: Spectral Radius

For $A \in \mathcal{B}(\mathcal{H})$, we say that the spectral radius of A is $r(A) = \sup_{\lambda \in \sigma(A)} |\lambda| < \infty$

Theorem

1. if $A \in \mathcal{B}(\mathcal{H})$, then $r(A) = \limsup_{n \rightarrow \infty} \|A^n\|^{1/n}$.
2. If A is, in addition, self-adjoint, then $r(A) = \|A\|$.

As a non-example, consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $\sigma(A) = \{0\}$, but $r(A) = 0 \neq \|A\|$.

Proof

Recall Hadamard's Formula: $\sum_{k=0}^{\infty} a_k z^k$ has radius of convergence R computed by $\frac{1}{R} = \limsup_{n \rightarrow \infty} |a_n|^{1/n}$. This holds even when a_k are members of a Banach algebra (e.g. $\mathcal{B}(\mathcal{H})$).

Set $z = \frac{1}{\lambda}$, such that $0 < |z| < \frac{1}{r(A)}$ and implies the reexistence of

$$R_A\left(\frac{1}{z}\right) = \left(A - \frac{1}{z}I\right)^{-1} = -(I - zA)^{-1} = -z \sum_{k=0}^{\infty} A^k z^k.$$

So we have that $r(A) = \frac{1}{R} = \limsup_{k \rightarrow \infty} \|A^k\|^{1/k}$.

Now if A is self-adjoint, $\|A^2\| = \|A\|^2$ since $\|A^2\| \leq \|A\|^2$ by submultiplicativity and $\|A^2\| \geq \sup_{\|x\|=1} (x, A^2 x) = \sup_{\|x\|=1} \|Ax\|^2 = \|A\|^2$ using Cauchy-Schwarz.

By induction, $\|A^{2^k}\|^{1/2^k} = \|A\|$ which implies that $r(A) = \|A\|$.

Another Spectral Decomposition

For $T : \mathcal{H} \rightarrow \mathcal{H}$ bounded, we say that λ is in the point spectrum of T if $T - \lambda I$ is not-injective. We say that λ is in the residual spectrum of T if $T - \lambda I$ is injective but does not have dense range.

IMAGE 1

Self-adjoint operators have no residual spectrum (RS, Thm VI.8).

Definition: Compact Operators

$K \in \mathcal{B}(\mathcal{H})$ is compact if K maps bounded sequences to sequences with a limit point. Equivalently, if $B_{\mathcal{H}} = \{x \in H : ||x|| \leq 1\}$ then K is compact if $K(B_{\mathcal{H}})$ has compact closure (i.e. is precompact).

$\mathcal{K}(\mathcal{H})$, the collection of compact operators on \mathcal{H} , is a closed linear subspace of $\mathcal{B}(\mathcal{H})$ since if $T, S \in \mathcal{K}(\mathcal{H})$, then $T + S \in \mathcal{K}(\mathcal{H})$. We have also that $\mathcal{K}(\mathcal{H})$ is a 2-sided ideal of $\mathcal{B}(\mathcal{H})$ since when T is compact and S is bounded, ST and TS are compact.

Examples

- For a finite-rank operator $A : \mathcal{H} \rightarrow \mathcal{H}$, $\text{range}(A) < \infty$. The general form of A is $A = \sum_{j=1}^n \phi_j \otimes \bar{\psi}_j$ for $\phi_j, \psi_j \in \mathcal{H}$.
- Strong limits of finite-rank operators.
- $(Au)_n = \frac{1}{n} u_n$ on $\ell^2(\mathbb{N})$. Note that $A_n = \sum_{j=1}^n \frac{1}{j} e_j \otimes e_j$ shows that $||A - A_n|| \leq \frac{1}{n+1}$.
- The inclusion $h^1 \hookrightarrow \ell^2$ where $h^1 = \{u \in \mathbb{C}^{\mathbb{N}} : \sum_{j=1}^{\infty} j^2 |u_j|^2 < \infty\}$.

Proposition

If \mathcal{H} is separable, all compact operators arise as limits of finite-rank operators.

Proof

We want to show that for $A \in \mathcal{K}(\mathcal{H})$, $\forall \varepsilon > 0$, $\exists A_{\varepsilon}$ finite-rank such that $||A - A_{\varepsilon}|| < \varepsilon$.

Let $\varepsilon > 0$ be given. Since $A(B_{\mathcal{H}})$ is precompact, it is totally bounded. That is $\exists y_1, \dots, y_n \in A(B_{\mathcal{H}})$ such that $\forall x \in A(B_{\mathcal{H}})$, $\min_{1 \leq j \leq n} ||x - y_j|| < \varepsilon$.

Let P_{ε} be the orthogonal projection onto $\text{span}\{y_1, \dots, y_n\}$, and set $A_{\varepsilon} = P_{\varepsilon} A$. Then for $f \in B_{\mathcal{H}}$ and $x = Af \in A(B_{\mathcal{H}})$,

$$||Af - A_{\varepsilon}f|| = ||x - P_{\varepsilon}x|| \leq \min_{1 \leq j \leq n} ||x - y_j|| < \varepsilon$$

So $||A - A_{\varepsilon}|| < \varepsilon$.

Exercise: confirm whether this argument needs separability.

Theorem: Analytic Fredholm Theorem

Let $D \subset \mathbb{C}$ be open and connected. Let $f : D \rightarrow \mathcal{B}(\mathcal{H})$ be an analytic, operator-valued function such that $f(z) \in \mathcal{K}(\mathcal{H})$ for all $z \in D$. Then

1. either $(I - f(z))^{-1}$ exists for no $z \in D$
2. or $(I - f(z))^{-1}$ exists for all $z \in D \setminus S$, where S is a discrete set (finite-rank, no accumulation points) in D .

Then $(I - f(z))^{-1}$ is meromorphic in D , analytic in $D \setminus S$, residues at poles are finite-rank, and if $z \in S$, $f(z)\psi = \psi$ has a nonzero solution for $\psi \in \mathcal{H}$.

Application

If K is compact, consider $f(z) = zK$. At $z = 0$, $I - f(z) = I$ so this is invertible which implies that the theorem holds for $\frac{1}{z} \in \mathbb{C}$ (taking $D = \mathbb{C} \setminus \{0\}$). We have $R_K(\lambda) = -\frac{1}{\lambda} (I - f(\frac{1}{\lambda}))^{-1}$. Note that K is not necessarily self-adjoint. Note that K is not necessarily self-adjoint.

Proof

We want to prove that either (a) or (b) hold locally for any $z_0 \in D$.

Fix $z_0 \in D$, $r > 0$ such that $\|f(z) - f(z_0)\| < \frac{1}{2}$ for $z \in D_r(z_0)$. Choose $F = \sum_{j=1}^n \phi_j \otimes \bar{\psi}_j$ a finite-rank operator such that $\|f(z_0) - F\| < \frac{1}{2}$. Then $\forall z \in D_r(z_0)$,

$$\|f(z) - F\| \leq \|f(z) - f(z_0)\| + \|f(z_0) - F\| < 1$$

which implies that $(I - (f(z) - F))^{-1}$ exists and is holomorphic on $D_r(z_0)$.

Define $g(z) = F(I - f(z) + F)^{-1}$, and observe that $(I - f(z)) = (I - g(z))(I - f(z) + F) = I - f(z) + F - \overbrace{g(z)(I - f(z) + F)}^F$. Write

$$g(z) = \sum_{j=1}^n \phi_j \otimes \bar{\psi}_j \cdot (I - f(z) + F)^{-1} = \sum_{j=1}^n \phi_j \otimes \overline{\psi_j(z)}$$

where $\psi_j(z) = ((I - f(z) + F)^{-1})^* \psi_j$ is holomorphic in z . Then $I - f(z)$ is invertible if and only if $I - g(z)$ is invertible. We claim that this holds if and only if $d(z) \neq 0$ for some holomorphic function d .

When is $I - g(z)$ invertible?

Injectivity: if $g(z)\phi = \phi$, we expect $\phi = \sum_{j=1}^n \beta_j \phi_j$. So $g(z)\phi = \phi$ if and only if

$$\sum_{j=1}^n \phi_j \left(\sum_{k=1}^n \beta_k \phi_k, \psi_j \right) = \sum_{j=1}^n \beta_j \phi_j$$

where $\beta_j = \sum_{k=1}^n \beta_k (\phi_k, \psi_j(z))$. If $A_{jk}(z) := (\phi_k, \psi_j(z))$, then this has a solution if and only if $\det(I - A(z)) = 0$. Call $d := \det(I - A(z))$. Moreover, if $d(z) \neq 0$ then $I - g(z)$ is invertible. We can solve $(I - g(z))\phi = \psi$ for ϕ given $\psi \in \mathcal{H}$. So $\phi = \psi + g(z)\phi$ which motivates an ansatz $\phi = \psi + \sum_{j=1}^n \beta_j \phi_j$. Then

$$(I - g(z))\phi = (I - g(z))\psi + \sum_{j=1}^n (\beta_j - A_{jk}\beta_k) \phi_j = \psi$$

If and only if

$$\sum_{j=1}^n (\beta_j - A_{jk}\beta_k) \phi_j = \sum_{j=1}^n (\psi, \psi_j(z)) \phi_j$$

which is boundedly invertible as long as $d(z) \neq 0$.

October 21, 2025

Definition: Unbounded Operator

Let $(\mathcal{H}, (\cdot, \cdot))$ a Hilbert space. An unbounded operator is a linear map $A : \mathcal{D}(A) \rightarrow \mathcal{H}$, $\mathcal{D}(A) \subsetneq \mathcal{H}$ ($\mathcal{D}(A)$ the domain of A).

Hypothesis: $\overline{\mathcal{D}(A)} = \mathcal{H}$.

Why “unbounded?” BEcause, generally, $\sup_{f \in \mathcal{D}(A) \setminus \{0\}} \frac{\|Af\|}{\|f\|} = \infty$.

Examples

1. $\mathcal{H} = L^2(\mathbb{R})$, $Af(t) = tf(t)$. Then $A\left(\frac{1}{|t|^2}\right) \notin L^2(\mathbb{R})$, and $\mathcal{D}(A) = C_C(\mathbb{R})$. Consider $f_n(t)$ like

IMAGE 1

1. $\mathcal{H} = L^2(\mathbb{R})$, $Af(x) = D_x f(x) = \frac{1}{i} \frac{\partial}{\partial_x} f(x)$. Then $\mathcal{D}(A) = \mathcal{S}(\mathbb{R})$, and we can see unboundedness from $f(x) = e^{-\frac{x^2}{\sigma}}$ for $\sigma > 0$.

$$\int f^2(x) dx = \int e^{-\frac{x^2}{\sigma}} dx \stackrel{y=\frac{x}{\sqrt{\sigma}}}{=} \sqrt{\sigma} \int e^{-2y^2} dy$$

$$\int (f')^2(x) dx = \int \frac{4x^2}{\sigma^2} e^{-\frac{x^2}{\sigma}} dx \stackrel{y=\frac{x}{\sqrt{\sigma}}}{=} \frac{1}{\sqrt{\sigma}} \int 4y^2 e^{-2y^2} dy$$

So $\frac{\|Af\|^2}{\|f\|^2} = \frac{C}{\sigma} \xrightarrow[\sigma \rightarrow 0]{} \infty$.

Definition: Extension of an Unbounded Operator

We say that B extends A (and write $A \subseteq B$) if $\mathcal{D}(A) \subset \mathcal{D}(B)$ and $B|_{\mathcal{D}(A)} = A$. We have that $A = B$ when $A \subset B$ and $B \subseteq A$.

Definition: Closed Operator

To recover a notion of continuity, we look at closed operators.

We say that $(A, \mathcal{D}(A))$ is closed if and only if $\forall u_n \in \mathcal{D}(A)$ such that $u_n \rightarrow u \in \mathcal{H}$ and $Au_n \rightarrow v \in \mathcal{H}$, $u \in \mathcal{D}(A)$ and $v = Au$.

Equivalently, A is closed if the graph of A , $\Gamma(A) = \{(u, Au) : u \in \mathcal{D}(A)\}$, is closed in $\mathcal{H} \times \mathcal{H}$.

We say that A is closable if there exists some B such that $A \subseteq B$.

Example

- $(P, \mathcal{D}(P), \mathcal{H}) = (D_x, C_C^\infty(\mathbb{R}), L^2(\mathbb{R}))$.
 - P is not closed, because there exists $u_n \in C_C^\infty(\mathbb{R})$ such that $u_n \rightarrow u \in \mathcal{H}$, $D_x u_n \rightarrow v \in \mathcal{H}$, yet $u \notin C_C^\infty(\mathbb{R})$.
 - Take $u \in H^1 \setminus C_C^\infty(\mathbb{R})$, $u_n \in C_C^\infty(\mathbb{R})$ converging to u in H^1 .
- Recall that
 - $H^1(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : f'(x) \in L^2(\mathbb{R})\}$, $\|f\|_{H^1}^2 = \|f\|_{L^2(\mathbb{R})}^2 + \|f'\|_{L^2(\mathbb{R})}^2$.
 - $C_C^\infty(\mathbb{R})$ is dense in $H^1(\mathbb{R})$.
 - $H^1(\mathbb{R})$ is complete.

Exercise: prove that if $\mathcal{D}(B) := H^1(\mathbb{R})$ and $Bf = D_x f$, then B is closed and $B|_{C_C^\infty(\mathbb{R})} = P$.

Adjoints of Unbounded Operators

In the bounded case, we have $A \in \mathcal{B}(\mathcal{H})$, $u \in \mathcal{H}$ and define $\ell_u(v) = (u, Av) \in \mathcal{H}'$ (\mathcal{H}' the dual space). By Riesz Representation Theorem, $\exists! w := A^* u$ such that $\forall v \in \mathcal{H}$, $(A^* u, v) = (u, A^* v)$.

In the unbounded case, we need to define A^* and $\mathcal{D}(A^*)$.

For $u \in \mathcal{H}$, set $\ell_u = (u, Av)$, $v \in \mathcal{D}(A)$. If ℓ_u extends to an element of \mathcal{H}' (i.e. if we can prove an estimate $|\ell_u(v)| \leq$

$c||v||_{\mathcal{H}}$, then by Riesz Representation Theorem $\exists! w \in \mathcal{H}$, called $A^* u$, such that $\forall v \in \mathcal{D}(A)$, $(A^* u, v) = (u, A^* v)$. Then

$$\mathcal{D}(A^*) = \{u \in \mathcal{H} : \underset{v \mapsto (u, Av)}{\mathcal{D}(A) \rightarrow \mathbb{C}} \text{ extends to an element of } H'\}.$$

Example

- $(P, \mathcal{D}(P), \mathcal{H}) = (D_x, C_C^\infty(\mathbb{R}), L^2(\mathbb{R}))$.

- Observe that $\forall f, g \in C_C^\infty(\mathbb{R})$, $(D_x f, g) = (f, D_x g)$.
- Take $f \in \mathcal{H}$ and $g \in C_C^\infty(\mathbb{R})$. Then if $f \in C_C^\infty$ or $f \in H^1(\mathbb{R})$, $g \mapsto (f, D_x g) = (D_x f, g)$ and

$$|(D_x f, g)| \leq \|D_x f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})} \leq \|f\|_{H^1(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}$$

- Then $B(f, g) = D_x f, g - (f, D_x g)$.
- So $\mathcal{D}(P^*) \supseteq C_C^\infty(\mathbb{R})$ and, in fact, $\mathcal{D}(P^*) = H^1(\mathbb{R})$.
- P^* is closed (this is always true of any adjoint).
- $P \subset P^*$, we say that P is symmetric.

Generally, if $(A, \mathcal{D}(A))$ is an unbounded operator, then $\Gamma(A^*) = J((\Gamma(A))^\perp)$ where $J: \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H} \times \mathcal{H}$ by $J(u, v) = (-v, u)$. Recall that $(\Gamma(A))^\perp$ is always closed, so A^* is always closed.

Definitions:

We say that P is symmetric if $P \subseteq P^*$.

We say that P is self-adjoint if $P = P^*$.

We say that P is essentially self-adjoint if $P^* = \overline{P}$.

Important Questions

- Given a symmetric operator $(P \subseteq P^*)$, to find a self-adjoint operator A such that $P \subseteq A = A^* \subseteq P^*$. We call A a self-adjoint extension of P . If $P^* = \overline{P}$ (essentially self-adjoint), then it must be the unique extension.
- How to find or parameterize all self-adjoint extensions? (This problem is hard.)

Example

- $P = (D_x, H_0^1([0, 1]), L^2(0, 1))$ (where elements of $H_0^1([0, 1])$ vanish on the boundary).

- $P^* = (D_x, H^1([0, 1])) \supsetneq P$.
- There exists a “circle” of self-adjoint extensions, parameterized by any point on the unit circle.

General Principle

If $\Omega \subset \mathbb{R}^n$ is open, bounded and withs mooth boundary $\partial\Omega$ ($\mathcal{H} = L^2(\Omega)$), and P is some differential operator (e.g. $P = \Delta$) such that $\forall f, g \in C_C^\infty(\Omega)$. $(Pf, g) = (f, Pg)$. By Green's Identity,,

$$\begin{aligned}\int_{\Omega} (\Delta f g - f \Delta g) dx &= \int_{\Omega} \nabla((\nabla f)g(-f(\nabla g))) dx \\ &= \int_{\partial\Omega} (g \nabla f - f \nabla g) \cdot \nu ds \\ &= \int_{\partial\Omega} \left(g \frac{\partial f}{\partial \nu} - f \frac{\partial g}{\partial \nu} \right) ds\end{aligned}$$

Examples

- $P = (P = \Delta, C_C^\infty(\Omega), L^2(\Omega))$.
 - P symmetric.
 - $\mathcal{D}(P^*) = \{u \in L^2(\Omega) : \Delta u \in L^2\}$.
 - Let $P_{\min} = \overline{(P, C_C^\infty(\Omega))}$ and define $P_{\max} = P_{\min}^*$. Then $\mathcal{D}(P_{\max}) = \{u \in L^2(\Omega) : Pu \in L^2(\Omega)\}$.
 - The self-adjoint extensions lie inbetween.
- $(D_x, H_0^1([0, 1])) = \{u \in H^1([0, 1]) : u(0) = u(1) = 0\}, L^2([0, 1])$.
 - Observe that the evaluation map $C^\infty([0, 1]) \ni f \mapsto f(0)$ extends boundedly to $\tau : H^1([0, 1]) \rightarrow \mathbb{C}$.
 - For $f, g \in C^\infty([0, 1])$,

$$(D_x f, g) - (f, D_x g) = \int_0^1 \frac{1}{i} f' \bar{g} - f \left(\overline{\frac{1}{i} g'} \right) = \frac{1}{i} \int_0^1 (f \bar{g})' = \frac{1}{i} (f(1) \bar{g}(1) - f(0) \bar{g}(0))$$

- Then if $g \in \mathcal{D}(P_{\min})$,

$$\begin{aligned}|(f, D_x g)| &= |(D_x f, g) + \frac{1}{i} (\overbrace{f(1) \bar{g}(1)}^{=0} - \overbrace{f(0) \bar{g}(0)}^{=0})| \\ &\leq \|f\|_{H^1} \|g\|_{L^2}\end{aligned}$$

- Therefore $\mathcal{D}(P_{\min}^*) = \mathcal{D}(P_{\max}) = H^1([0, 1])$, and we note that $P_{\max}^* = P_{\min}$.

Now suppose that T is a self-adjoint extension of P ($P_{\min} \subseteq T \subseteq P_{\max}$). If T is self-adjoint, then $\forall f, g \in \mathcal{D}(T) \subseteq H^1([0, 1])$, $f(0) \bar{g}(0) - f(1) \bar{g}(1) = 0$.
For $f = g$, $|f(0)|^2 = |f(1)|^2$ for some fixed f , then $\exists \alpha \in S^1$ such that $f(1) = \alpha f(0)$. Then for any other g ,

$$0 = f(0)(\bar{g}(0) - \alpha \bar{g}(1)) = \alpha f(0)(\overline{\alpha g(0) - g(1)})$$

Therefore $g(1) = \alpha g(0)$ for the same fixed α .

October 23, 2025

Definition: Resolvent Set and Spectrum of and Unbounded Operator

For $(A, \mathcal{D}(A), \mathcal{H})$ unbounded and closed, the resolvent set of A , $\rho(A)$, is the set of $\lambda \in \mathbb{C}$ such that $(A - \lambda) : \mathcal{D}(A) \rightarrow \mathcal{H}$ is bijective with bounded inverse $R_A(\lambda) := (A - \lambda)^{-1}$.

The spectrum of A , $\sigma(A)$, is $\sigma(A) = \mathbb{C} \setminus \rho(A)$. If $\rho(A)$ is open, then $\sigma(A)$ is closed.

Example

- Take $P = (D_x, H_0^1([0, 1]), L^2(0, 1))$ and define two closed extensions P_0 and P_1

$$- \mathcal{D}(P_0) = \{u \in H^1([0, 1]) : u(0) = 0\}$$

$$\begin{aligned} * \text{ For } \lambda \in \mathbb{C}, f \in L^1([0, 1]), \text{ we solve } & \begin{cases} D_x u + \lambda u = f \\ u(0) = 0 \end{cases} \\ * \text{ So } u(x) = i e^{-\lambda x} \int_0^x e^{i \lambda t} f(t) dt \text{ and } \sigma(P_0) = \emptyset. \end{aligned}$$

$$- \mathcal{D}(P_1) = H^1([0, 1])$$

$$* \text{ For } \lambda \in \mathbb{C}, \ker(D_x - \lambda) \cap L^2([0, 1]) \neq \emptyset. \text{ So } \sigma(P_1) = \mathbb{C}.$$

Proposition

If $(A, \mathcal{D}(A))$ is self-adjoint, then $\sigma(A) \subseteq \mathbb{R}$, $\sigma(A) \neq \emptyset$, and we have a resolvent estimate $\|R_A(z)\| \leq \frac{1}{|\operatorname{Im}(z)|}$ for $\operatorname{Im}(z) \neq 0$.

The proof of real value and the estimate are the same as in the bounded case.

To show that $\sigma(A) \neq \emptyset$, by contradiction if $\sigma(A) = \emptyset$ then $A^{-1} = R_A(0)$ exists and is bounded. We claim that this requires $\sigma(A^{-1}) = \{0\}$ and pick $\lambda \neq 0$, such that

$$\begin{aligned} (A^{-1} - \lambda)u &= f \\ \left(\frac{1}{\lambda} - A\right)u &= \frac{1}{\lambda}Af \end{aligned}$$

But $\frac{1}{\lambda} \notin \sigma(A)$ implies $u = -R_A\left(\frac{1}{\lambda}\right)\frac{1}{\lambda}Af$. Therefore $R_{A^{-1}}(\lambda) = -\frac{1}{\lambda}AR_A\left(\frac{1}{\lambda}\right)$ and $\lambda \notin \sigma(A^{-1})$. Therefore $A^{-1} = 0$ which contradicts the assumption that $A^{-1}A = \operatorname{Id}_{\mathcal{D}(A)}$.

Generalizing Spectral Resolution

We can extend $\{E_\lambda\}_{a \leq \lambda \leq b}$ to $a = -\infty$ and $b = \infty$ by setting strong limits $E_{-\infty} = 0$ and $E_\infty = I$.

Example

- On $L^2(\mathbb{R})$, $E_\lambda f(t) = \chi_{(-\infty, \lambda)}(t)f(t)$.

Theorem: Spectral Theorem for Unbounded Self-Adjoint Operators

Theorem 8.15 Helffer.

Any self-adjoint operator A on a Hilbert space \mathcal{H} admits a spectral decomposition $\{E_\lambda\}_\lambda$ such that $\forall x, y \in \mathcal{H}$,

$$(Ax, y) = \int_{\mathbb{R}} \lambda d(E_\lambda x, y)$$

$$Ax = \int_{\mathbb{R}} \lambda d(E_\lambda x)$$

Proof

Largely, we will use the proof for the bounded case.

Suppose A is semibounded from below (i.e. $\exists \mu \in \mathbb{R}, \forall x \in \mathcal{D}(A), (Ax, x) \geq \mu ||x||^2$) then $R_A(\lambda)$ exists for $\lambda_0 < \mu$ ($((A - \lambda)x, x) \geq (\mu - \lambda_0)||x||^2$ coercive in the sense that $(\mu - \lambda_0) > 0$). Then it is also a self-adjoint bounded operator which implies that $R_A(\lambda_0)$ has a spectral representation. Then $A = f(R_A(\lambda_0))$, $f(x) = \lambda_0 + \frac{1}{x}$. We can work with a general case $(A - i)^{-1}$.

Example

- Take $(D_x, H^1(\mathbb{R}), L^2(\mathbb{R}))$, and define the Fourier transform $f(x) \mapsto \mathcal{F}f = \hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx$. Then $\widehat{D_x f}(\xi) = \xi \hat{f}(\xi)$, and we have the following commutative diagram

$$\begin{array}{ccc} H^1(\mathbb{R}) & \xrightarrow{D_x} & L^2(\mathbb{R}) \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\ H^1(\mathbb{R}) & \xrightarrow[f(t) \mapsto t\hat{f}(t)]{} & L^2(\mathbb{R}) \end{array}$$

- So $E_\lambda f = \mathcal{F}^{-1} \chi_{(-\infty, \lambda)} \mathcal{F}f$.
- $(D_x^2, H^2(\mathbb{R}), L^2(\mathbb{R}))$. Then $\widehat{D_x^2 f}(\xi) = \xi^2 \hat{f}(\xi)$, $\sigma(D_x^2) = [0, \infty)$ and $D_x^2 f = \mathcal{F}^{-1} \xi^2 \mathcal{F}f$.
 - So $E_\lambda f = 0$ for $\lambda < 0$; $E_\lambda f = \mathcal{F}^{-1} \chi_{\xi^2 < \lambda}(\xi) \mathcal{F}f$.
- $(D_x^2 + V(x), C_C^\infty(\mathbb{R}), L^2(\mathbb{R}))$, $V \in C(\mathbb{R}; \mathbb{R})$ bounded from below. Then $D_x^2 + V$ is semibounded from below and the spectrum depends drastically on V .
 - $V = 0$ gives $\sigma = [0, \infty)$.
 - $V = x^2$ gives $\sigma = \{2n+1 : n \geq 0\}$
 - $V \in C_C(\mathbb{R})$ gives $\sigma = [0, \infty)$

Proposition

If for some $z_0 \in \rho(A) \cap \mathbb{R}$ $R_A(z_0)$ is compact, then A has purely discrete spectrum $(z_n)_{n \geq 1}$, $|z_n| \rightarrow \infty$ as $n \rightarrow \infty$.

Proof

$R_A(z_0)$ is compact and self-adjoint, so there exists a basis for \mathcal{H} made of normalized eigenvectors $(\phi_n)_{n \in \mathbb{N}}$ with eigenvalues $|\lambda_n| \searrow 0$.

$$(A - z_0)^{-1} \phi_n = \lambda_n \phi_n$$

$$A\phi_n = \left(z_0 + \frac{1}{\lambda_n} \right) \phi_n$$

$$\mathcal{H} = \left\{ u = \sum u_n \phi_n : \sum |u_n|^2 < \infty \right\}$$

$$\mathcal{D}(A) = \left\{ u = \sum u_n \phi_n : \sum |u_n|^2 \left(z_0 + \frac{1}{\lambda_n} \right)^2 < \infty \right\}$$

Applications

- Laplacian (Leplace-Beltrami Operator) on closed Riemann manifolds or Dirichlet Laplace-Beltrami if $\partial M \neq 0$.
- Dirichlet Laplacian ($\Delta = -\sum_{k=1}^n \delta_{x^k}^2$) on bounded domains in \mathbb{R}^n (Ω open in \mathbb{R}^n , regular, $\partial\Omega$ smooth).
 - $\Delta_D = (\Delta, H_0^1(\Omega) \cap H^2(\Omega), L^2(\Omega))$ where $H_0^1 = \{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\}$ ($u|_{\partial\Omega}$ makes sense due to a trace theorem from Evans 5.5) and $H^2(\Omega) = \{u \in L^2(\Omega) : \partial_i \partial_j u \in L^2, \partial_i u \in L^2, \forall i, j\}$.
 - $\int_{\Omega} |\nabla f|^2 dx = (\Delta f, f) \geq 0$ (Green's Theorem)
 - Claim: $\Delta_D + 1$ has compact inverse.
- Solvability of $\Delta_D + 1$

Our goal is, given $f \in L^2$, to find $u \in ?$ such that $\begin{cases} \Delta u + u = f \\ u|_{\partial\Omega} = 0 \end{cases}$. We have a weak formulation, if $v \in H_0^1$, we multiply by v and get an IBP

$$\underbrace{\int \nabla u \cdot \nabla v + \int uv}_{B(u,v)} = \int fv$$

where $|\int fv| \leq \|f\|_{L^2} \|v\|_{L^2} \leq \|f\|_{L^2} (B(v, v))^{1/2}$. By Riesz Representation Theorem, there exists some unique $u \in H_0^1$ such that $\forall v \in H_0^1$, $B(v, v) = \int fv$. Moreover, we can recover H^2 regularity on u . Also, $|B(u, u)| = |\int fu| \leq \|f\|_{L^2} (B(u, u))^{1/2}$ if and only if $\|(\Delta_D + 1)^{-1} f\|_{H_0^1(\Omega)} = \|u\|_{H_0^1(\Omega)} \leq \|f\|_{L^2}$. Therefore $(\Delta_D + 1)^{-1} : L^2(\Omega) \rightarrow H_0^1(\Omega)$ is bounded. By Rellich Compactness, $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ compactly. Hence $(\Delta_D + 1)^{-1} : L^2 \rightarrow L^2$ is compact.

Theorem

On \mathbb{R} , $H^1(\mathbb{R})$ does not embed compactly into $L^2(\mathbb{R})$. For example, $f \in H^1$ with $\text{supp } f \subset [0, 1]$ defined by $f_n(x) = f(x - n)$.

So if $H \subset L^2(\mathbb{R})$ where $\|u\|_H^2 = \|u'\|_L^2 + \|a(x)^{-1} u\|_{L^2}^2$ for $a(x) > 0$ and $\lim_{|x| \rightarrow \infty} a(x) = 0$, then H embeds compactly in L^2 .

It follows that $D_x^2 + x^2$ has compact resolvent and discrete spectrum. $\|u\|_H^2 = \|d_x u\|_{L^2}^2 + \|xu\|_{L^2}^2$.

October 30, 2025

Friedrichs Extension

Let $(A, \mathcal{D}(A), \mathcal{H})$ be symmetric, bounded from below in the sense that there exists $\alpha \in \mathbb{R}$ such that $\forall x \in \mathcal{D}(A)$ we have $(Ax, x)_{\mathcal{H}} \geq \alpha \|x\|_{\mathcal{H}}^2$.

Theorem:

If $(A, \mathcal{D}(A), \mathcal{H})$ is densely defined, symmetric, and bounded below, then A admits a self-adjoint extension $(S, \mathcal{D}(S))$. The procedure for producing this extension is the Friedrichs Extensions.

Proof

Assume that $\alpha = 1$, otherwise consider $A - \lambda I$ for some well-chosen λ . Define the symmetric quadratic form

$$q_A(u, v) = (Au, v), \quad (u, v) \in \mathcal{D}(A) \times \mathcal{D}(A).$$

Note that $q_A(u, u) = (Au, u) \geq \|u\|_{\mathcal{H}}^2$, so this form is coercive. Next, we construct $V \subset \mathcal{H}$ by taking $p(u) = (q_A(u, u))^{1/2}$ and letting $u \in \mathcal{D}(A)$ belong to V if $\exists u_n \in \mathcal{H}$ such that $\|u_n - u\|_{\mathcal{H}} \xrightarrow{n \rightarrow \infty} 0$ and $\{u_n\}$ is p -cauchy (i.e. $\forall \varepsilon > 0, \exists N, m, n \geq N \implies p(u_n - u_m) < \varepsilon$).

Then we can define a norm on V : $\|u\|_V := \lim_{n \rightarrow \infty} p(u_n)$ where $u_n \xrightarrow{\mathcal{H}} u$ and u_n is p -Cauchy.

Exercise: show that this definition does not depend on $\{u_n\}$.

We also define $(u, v)_V := \lim_{n \rightarrow \infty} q_A(u_n, v_n)$ for u_n and v_n similarly converging and p -cauchy.

So $(V, \|\cdot\|_V)$ is complete and $p(u_n) \geq \|u_n\|_{\mathcal{H}}$ becomes $\|u\|_V \geq \|u\|_{\mathcal{H}}$ as $n \rightarrow \infty$. Hence $V \hookrightarrow \mathcal{H}$ is continuous.

IMAGE 1

We now construct S starting with a domain $D(S) = \{u \in V : V \ni v \mapsto q_A(u, v) \text{ satisfies an continuity estimate of the form } \leq C\|v\|_{\mathcal{H}}\}$. For $u \in D(S)$, the map $v \mapsto q_A(u, v)$ extends to an element of \mathcal{H}' (bilinear functionals on \mathcal{H}). By Riesz-Representation Theorem, there exists some $w := Su$ such that $\forall v \in V, q_A(u, v) = (Su, v)_{\mathcal{H}}$.

It remains to show that S is self-adjoint, $D(S) \supset D(A)$, $S|_{D(A)} = A$. Self-adjointness is left as an exercise. If $u \in \mathcal{D}(A)$, then

$$|q_A(u, v)| = |(Au, v)| \leq \|Au\|_{\mathcal{H}}\|v\|_{\mathcal{H}}$$

so $u \in D(S)$ and $Su = Au$.

Remark

We could start with a quadratic form $q(u, v)$ which is sesquilinear, coercive and $\mathcal{D}(q) \subset \mathcal{H}$ densely. This construction produces a self-adjoint operator as well.

Example (Dirichlet Laplacian)

$\Delta = \sum_{j=1}^n \partial_{x_j}^2$ with $\Omega = \mathbb{R}^n$ open, bounded, and regular.

$(A = -\Delta, \mathcal{D}(A) = C_C^\infty(\Omega), \mathcal{H} = L^2(\Omega))$. For $u \in \mathcal{D}(A)$,

$$\begin{aligned} (Au, u) &= \int_{\Omega} -\Delta u \cdot \bar{u} \, dx \\ &= \int_{\Omega} -\nabla(\nabla u \cdot \bar{u}) + \nabla u \cdot \nabla \bar{u} \, dx \\ &= \underbrace{\int_{\partial\Omega} -\partial_{\nu} u \, ds}_{=0} + \int_{\Omega} \nabla u \cdot \nabla \bar{u} \, dx \geq 0 \end{aligned}$$

So Friedrich's procedure produces a self-adjoint extension

$$q_A(u, v) = \int_{\Omega} \nabla u \cdot \nabla \bar{u} \, dx, \quad u, v \in C_C^\infty(\Omega)$$

which leads to $V = \overline{C_C^\infty(\Omega)}^{H^1(\Omega)} = H_0^1(\Omega)$ (that is the H^1 closure of $C_C^\infty(\Omega)$). What about $\mathcal{D}(S)$?

IMAGE 2

For $v \in C_C^\infty(\Omega)$, $\int_\Omega \nabla u \cdot \nabla \bar{v} dx = \int_\Omega -\Delta u \cdot v \leq C ||v||_{L^2}$. Then we have

$$\mathcal{D}(S) = \{u \in H_0^1(\Omega) : \Delta u \in L^2(\Omega)\} = H_0^1(\Omega) \cap H^2(\Omega)$$

Example (Neumann Laplacian)

Start from a quadratic form $q(u, v) = \int_\Omega \nabla u \cdot \nabla \bar{v} + u \bar{v} dx$ with $u, v \in C^\infty(\overline{\Omega})$. Then $V = H^1(\Omega)$. We compute $\mathcal{D}(S)$ by first recalling that for $u, v \in C^\infty(\overline{\Omega})$,

$$\int_\Omega -\Delta u \bar{v} dx = \int_\Omega \nabla u \cdot \nabla \bar{v} dx - \int_{\partial\Omega} \partial_\nu u \bar{v} dx$$

We sense that $\Delta u \in L^2(\Omega)$, so we at least need $u \in W(\Omega) = \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\}$. How do we make sense of the Neumann trace on $W(\Omega)$?

- Lemma

- The restriction map $C(\overline{\Omega}) \rightarrow C(\partial\Omega)$ extends to a bounded surjective “Dirichlet trace” $\tau_D : H^1(\Omega) \rightarrow H^{1/2}(\partial\Omega)$ with bounded right inverse R .

- Example

- if $\Omega = \mathbb{D}$, $\partial\Omega = S^1$, $f \in C^\infty(S^1) \leftrightarrow f(\theta) \sum_{k \in \mathbb{Z}} f_k e^{ik\theta}$ where $f_k = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-ik\theta} d\theta$, we can for each $s \in \mathbb{R}$ define

$$H^s(S^1) = \{f \in \mathcal{D}'(S^1) : \sum_{k \in \mathbb{Z}} (1+k^2)^s |f_k|^2 < \infty\} = \mathcal{D}((-i\partial_\theta)^{s/2})$$

- Fact

- For $s \geq 0$, $H^{-s}(\partial\Omega) \cong (H^s(\partial\Omega))'$.

- If $f \in H^{-s}$, $g \in C^\infty(S^1)$,

$$\langle f, g \rangle_{\mathcal{D}', \mathcal{D}} = \int (f, g) = c \sum_{k \in \mathbb{Z}} f_k \overline{g_k} \frac{(1+k^2)^{s/2}}{(1+k^2)^{s/2}} \leq c \left(\sum_{k \in \mathbb{Z}} (1+k^2)^{-s} |f_k|^2 \right) \left(\sum_{k \in \mathbb{Z}} (1+k^2)^s |g_k|^2 \right)$$

Returning to the Neumann Laplacian, the Neumann trace $C^1(\overline{\Omega}) \rightarrow C(\partial\Omega)$ by $f \mapsto \partial_\nu f$ extends to a bounded operator $\tau_N : W(\Omega) \rightarrow H^{-1/2}(\partial\Omega)$.

- Proof

If $u \in W(\Omega)$, set

$$\begin{aligned} \Phi_u(v) &:= \int_\Omega -\Delta u \bar{v} dx - \int_\Omega \nabla u \cdot \nabla \bar{v} dx, \quad v \in H^1(\Omega) \\ &\leq ||\Delta u||_{L^2} ||v||_{L^2} + ||\nabla u||_{L^2} ||\nabla v||_{L^2} \end{aligned}$$

Using $R : H^{1/2}(\partial\Omega) \rightarrow H^1(\Omega)$, $w \in H^{1/2}(\partial\Omega)$,

$$|\Phi_u(Rw)| \leq (\|\Delta u\|_{L^2} + \|\nabla u\|_{L^2}) \|Rw\|_{H^1} \leq (\|\Delta u\|_{L^2} + \|\nabla u\|_{L^2}) \|w\|_{H^{1/2}(\Omega)}$$

which implies that there exists some $h \in H^{-1/2}(\partial\Omega)$ such that $\Phi_u(Rw) = \langle h, w \rangle_{H^{-1/2}, H^{1/2}}$. We want to extend

$$\int_{\Omega} -\Delta u \bar{v} \, dx = \int_{\Omega} \nabla u \cdot \nabla \bar{v} \, dx - \int_{\partial\Omega} \partial_{\nu} u \bar{v} \, dx$$

to $u \in W(\Omega)$ and $v \in H^1(\Omega)$. First, if $v \in H_0^1(\Omega)$ then

$$\int_{\Omega} -\Delta u \bar{v} \, dx = \int_{\Omega} \nabla u \cdot \nabla \bar{v} \, dx$$

(by passing to the limit with a sequence $v_n \in C_C^\infty(\Omega)$ converging to v in H^1).

Next, if $v \in H^1$, we write $v = v_0 + R\tau_D v$ with $v_0 \in H_0^1$. Then

$$\int_{\Omega} -\Delta u \bar{v} - \nabla u \cdot \nabla v \, dx = \overbrace{\int_{\Omega} -\Delta u \bar{v}_0 - \nabla u \cdot \nabla \bar{v}_0 \, dx}^{=0} + \Phi_u(R\tau_D v) = \langle \tau_N u, \tau_D v \rangle_{H^{-1/2}, H^{1/2}}$$

We conclude that for each $u \in W(\Omega)$, $v \in H^1(\Omega)$

$$\int_{\Omega} -\Delta u \cdot \bar{v} \, dx = \int_{\Omega} \nabla u \cdot \nabla \bar{v} \, dx + \langle \tau_N u, \tau_D v \rangle_{H^{-1/2}, H^{1/2}}$$

Therefore we have

$$q(u, v) = \int_{\Omega} (-\Delta u + u) \bar{v} \, dx + \langle \tau_N u, \tau_D v \rangle_{H^{-1/2}, H^{1/2}}$$

and $v \mapsto q(u, v)$ is L^2 -bounded if and only if $v \mapsto \langle \tau_N u, \tau_D v \rangle_{H^{-1/2}, H^{1/2}}$ is L^2 -bounded which holds if and only if $\tau_N u = 0$. So

$$\mathcal{D}(S) = \{u \in H^1(\Omega) : -\Delta u \in L^2, \tau_N u = 0\} = \{u \in H^2(\Omega) : \tau_N u = 0\}$$

Observation

The form domains satisfy $\mathcal{D}(q_D^{-\Delta}) \subseteq \mathcal{D}(q_N^{-\Delta})$.

November 04 and 06, 2025

See François' notes on variational principles for eigenvalues and Weyl's criterion for essential spectrum.

November 13, 2025

Consider $(D_x^2 - \lambda^2)^{-1} = R_0(\lambda)$, the “outgoing” resolvent and let V be in $C_C^\infty(\mathbb{R}, \mathbb{R})$ or $L_{\text{comp}}^\infty(\mathbb{R}, \mathbb{R})$. Then we may define $R_V(\lambda) = (D_x^2 + V - \lambda^2)^{-1}$. We want to show that $-R_V(\lambda)$ makes sense for $\text{Im}(\lambda) \gg 0$ as an $L^2 \rightarrow L^2$ operator and extends meromorphically to either $\{\text{Im}(\lambda) > 0\}$ as an $L^2 \rightarrow L^2$ bounded operator or to \mathbb{C} as a $L_{\text{comp}}^2 \rightarrow L_{\text{loc}}^2$ operator. Recall that $L_{\text{comp}}^2 = \{f \in L^2 : \text{supp } f \text{ compact}\}$ and $L_{\text{loc}}^2 = \{f : \mathbb{R} \rightarrow \mathbb{R} : \forall K \text{ compact}, \int_K |f|^2 < \infty\}$.

IMAGE 1

We will further see that there are no poles (embedded eigenvalues) on $\mathbb{R} \setminus \{0\}$.

Outgoing/Incoming Solution

We have that $(D_x^2 - \lambda^2)u = 0$ if and only if $u(x) = Ae^{i\lambda x} + Be^{-i\lambda x}$. A solution to $(D_x^2 - \lambda^2)u = f \in C_C^\infty(\mathbb{R})$ is said to be outgoing if $x \gg 0$ implies that $u(x) = A_+ e^{i\lambda x}$ and $x \ll 0$ implies that $u(x) = B_+ e^{-i\lambda x}$.

IMAGE 2

Likewise, the solution is incoming if $x \gg 0$ implies $u(x) = B_- e^{-i\lambda x}$ and $x \ll 0$ implies $u(x) = A_- e^{i\lambda x}$. We have seen previously that the D'Alembert Solution to

$$\begin{cases} (\partial_t^2 - \partial_x^2 + V)u = f \\ u|_{t=0} = \partial_t u|_{t=0} = 0 \end{cases}$$

is outgoing.

We will find that an outgoing solution to $(D_x^2 - \lambda^2)u = f$ is $R_0(\lambda)f := u_+ = \frac{i}{2\lambda} \int_{\mathbb{R}} f(y) e^{i\lambda|x-y|} dy$. The incoming solution, then, is $u_-(x, \lambda) = -\frac{i}{2\lambda} \int_{\mathbb{R}} f(y) e^{-i\lambda|x-y|} dy$.

Note that $u_+(x, \lambda) = u_-(x, -\lambda) = \overline{u_-(x, \lambda)}$. Note also that the outgoing solution is in $L^2(\mathbb{R})$ if and only if $\text{Im}(\lambda) > 0$, and the incoming solution if and only if $\text{Im}(\lambda) < 0$.

The above solution may be derived via Fourier transform. Namely

$$(\xi^2 - \lambda^2)\hat{u}(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx$$

which implies that $\hat{u}(\xi) = \frac{\hat{f}(\xi)}{\xi^2 - \lambda^2}$ if and only if $\text{dist}(\lambda^2, [0, \infty)) > 0$. Considering the outgoing solution, this occurs only when $\text{Im}(\lambda) > 0$.

Note, that with respect to Plancherel Theorem giving $\|u\|_{L^2} = c\|\hat{u}(\xi)\|_{L^2}$,

$$\frac{1}{c}\|u\|_{L^2} = \|\hat{u}(\xi)\|_{L^2} \leq \frac{1}{\text{dist}(\lambda^2, [0, \infty))} \|\hat{f}(\xi)\|_{L^2} = \frac{1}{c}\|f\|_{L^2}$$

This gives continuity estimates for $R_0(\lambda)$. Then $u(x) = h_\lambda(x) * f(x)$ where $h_\lambda(x) = \int_{\mathbb{R}} e^{ix\xi} \frac{d\xi}{\xi^2 - \lambda^2}$. We claim, computing by residue theorem, that $h_\lambda(x) = \frac{i}{2\lambda} e^{i\lambda|x|}$.

Exercise: do the residue computation.

So we have $R_0(\lambda) = \frac{i}{2\lambda} \int_{\mathbb{R}} f(y) e^{i\lambda|x-y|} dy$ with Schwarz kernel $R_0(x, y, \lambda) = \frac{i}{2\lambda} e^{i\lambda|x-y|}$. When x and y are frozen, this looks meromorphic in λ with a simple pole at $\lambda = 0$ and residue $\frac{i}{2}$. This still makes sense as a map $C_c^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})$ or $L^2_{\text{comp}}(\mathbb{R}) \rightarrow L^2_{\text{loc}}(\mathbb{R})$.

Alternatively, we can formulate this as $\forall \rho \in C_C^\infty(\mathbb{R})$, $\rho R_0 \rho : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ extends meromorphically to \mathbb{C} . Then $\rho R_0(\lambda) \rho$ has Schwarz kernel $\frac{i}{2\lambda} \rho(x) \rho(y) e^{i\lambda|x-y|}$.

Exercise: if $\lambda \neq 0$, this is $L^2 \rightarrow L^2$ bounded.

Then $R_0(\lambda) = \frac{P}{\lambda} + Q(\lambda)$ where $Pf(x) = \frac{i}{2} \int_{\mathbb{R}} f(y) dy$, $P = \phi \otimes \bar{\phi}$ and $\phi(x) = \frac{e^{i\pi/4}}{\sqrt{2}}$. Here $Q : L^2_{\text{comp}}(\mathbb{R}) \rightarrow L^2_{\text{loc}}(\mathbb{R})$ is entire.

Compactly Supported Potential

Take $V \in L_{\text{comp}}^\infty(\mathbb{R})$, where $\sigma_{\text{ess}}(D_x^2 + V) = [0, \infty) + (\text{possibly negative eigenvalues}) + (\text{embedded eigenvalues})$.

Theorem:

The outgoing resolvent $(D_x^2 + V - \lambda^2)^{-1}$ extends meromorphically to \mathbb{C} with no poles at $\lambda \in \mathbb{R} \setminus \{0\}$.

Proof

Goal: when can we solve $(D_x^2 + V - \lambda^2)u = f$?

We start with the identity $(D_x^2 + V - \lambda^2)R_0(\lambda) = I + VR_0(\lambda)$. For $\text{Im}(\lambda) > 0$, $\|VR_0(\lambda)\|_{L^2 \rightarrow L^2} \leq \frac{\|V\|_\infty}{\text{dist}(\lambda^2, [0, \infty))}$. So if $\text{Im}(\lambda) \gg 1$, $\|VR_0(\lambda)\|_{L^2 \rightarrow L^2} < 1$. Then

$$\overbrace{(D_x^2 + V - \lambda^2)R_0(\lambda) \sum_{p=0}^{\infty} (-VR_0(\lambda))^p}^{R_V(\lambda)} = I.$$

So $R_V(\lambda)$ is an inverse for $D_x^2 + V - \lambda^2$ on $\text{Im} \gg 1$, bounded with operator norm below $\|R_0(\lambda)\| \frac{1}{1 - \|VR_0(\lambda)\|}$. It produced outgoing solutions due to its factored form.

Now, to extend $R_V(\lambda)$ meromorphically as $L_{\text{comp}}^2 \rightarrow L_{\text{loc}}^2$, we look at $\rho R_V(\lambda) \rho$ for $\rho \in C_C^\infty(\mathbb{R})$ such that $\rho V = V$. Then for $\text{Im}(\lambda) \gg 0$,

$$R_V(\lambda)\rho = R_0(\lambda)(I + VR_0(\lambda))^{-1}\rho = R_0(\lambda)\rho \sum_{p=0}^{\infty} (VR_0(\lambda)\rho)^p = R_0(\lambda)\rho(I + VR_0(\lambda)\rho)^{-1}$$

Therefore

$$\rho R_V(\lambda)\rho = \rho R_0(\lambda)\rho(I + \rho VR_0(\lambda)\rho)^{-1}$$

We claim that $\lambda \mapsto \rho VR_0(\lambda)\rho$ is a meromorphic on $\mathbb{C} \setminus \{0\}$ family of compact operators (at least on $V \in C_C^\infty(\mathbb{R})$). To see this, we use $R_0(\lambda) : L^2(\mathbb{R}) \rightarrow H^2(\mathbb{R})$, then $\rho VR_0(\lambda)\rho : L^2(\mathbb{R}) \rightarrow \{u \in H^2(\mathbb{R}) : \text{supp } u \subset \text{supp } V\}$ embeds compactly into $L^2(\mathbb{R})$.

By analytic Fredholm theory, since $I + \rho VR_0(\lambda)\rho$ is invertible for $\text{Im}(\lambda) \gg 1$, it is invertible on $\mathbb{C} \setminus \{0\}$ outside of a discrete set with finite-dimensional obstructions at that discrete set.

To upgrade this to a meromorphic extension of $R_V(\lambda) : L_{\text{comp}}^2 \rightarrow L_{\text{loc}}^2$, pick an increasing sequence of cutoff functions. If ρ_1 is such that $\rho_1\rho = \rho$, then $\rho_1 R_V(\lambda) \rho_1$ extends $\rho R_V(\lambda) \rho$ in the sense that $R_V(\lambda) \rho_1 f = R_V(\lambda) \rho f$ when $\text{supp } f \subset \{\rho = 1\}$.

Claim: if $\lambda \in \mathbb{R} \setminus \{0\}$, then λ is not a pole of $R_V(\lambda)$.

- Step 1: if λ is a pole, then there exists an outgoing solution to $(D_x^2 + V - \lambda^2)u = 0$.
- Step 2: if $\lambda \in \mathbb{R} \setminus \{0\}$, and u is an outgoing solution to $(D_x^2 + V - \lambda^2)u = 0$, then u has compact support.
- Step 3: if u is a compactly supported solution of $(D_x^2 + V - \lambda^2)u = 0$, then $u = 0$.

Step 1. If $\lambda = \lambda_0$ is a pole, we may write $R_V(\lambda) = \frac{P_N}{(\lambda - \lambda_0)^N} + \dots + \frac{P_1}{\lambda - \lambda_0} + Q(\lambda)$ near $\lambda = \lambda_0$ where each term is $L_{\text{comp}}^2 \rightarrow L_{\text{loc}}^2$. If $\lambda \neq \lambda_0$, then

$$(\lambda - \lambda_0)^N \overbrace{(D_x^2 + V - \lambda^2)R_V(\lambda)}^I = (D_x^2 + V - \lambda^2)P_N + (\lambda - \lambda_0)(\dots)$$

Then as $\lambda \rightarrow \lambda_0$, $(D_x^2 + V - \lambda^2)P_N = 0$. So P_N produces outgoing solutions in L_{loc}^2 .

Step 2. If $\lambda \in \mathbb{R}$ and u solves $(D_x^2 + V - \lambda^2)u = 0$, then \bar{u} also solves $(D_x^2 + V - \lambda^2)\bar{u} = 0$.

Since u has compact support, write $u(x) = A_+ e^{i\lambda x} + B_- e^{-i\lambda x}$ when $x \gg 0$ and $u(x) = A_- e^{i\lambda x} + B_+ e^{-i\lambda x}$ when $x \ll 0$. Compute the Wronskian, $W(u, \bar{u}) = u\bar{u}' - u'\bar{u}$ and $\frac{d}{dx}W(u, \bar{u}) = u\bar{u}'' - u''\bar{u} = 0$. We find that this is $|A_+|^2 - |B_-|^2$ for

$x \gg 0$ and $|A_-|^2 - |B_+|^2$ for $x \ll 0$.

Therefore $|A_+|^2 + |B_+|^2 = |A_-|^2 + |B_-|^2$. That is, if u is outgoing then $A_- = B_- = 0$ and $A_+ = B_+ = 0$.

Step 3 is insane.

- Lemma

If we take $u \in L^\infty(\mathbb{R})$, $(D_x^2 + W)u = 0$ where $W \in L^\infty(\mathbb{R})$, then if $u = 0$ on $(-\infty, 0)$ then $u \equiv 0$.

Fix $h > 0$ and let $v = e^{-x/h} u$.

$$\begin{aligned} \|e^{-x/h}(hD_x)^2 e^{x/h} v\|_{L^2} &= \|(h^2 D_x^2 - 2ihD_x - 1)v\|_{L^2} \\ &= \|(h\xi - i)^2 \hat{v}\|_{L^2} \\ &\geq \|\hat{v}\|_{L^2} \\ &= \|v\|_L^2 \end{aligned}$$

Equivalently

$$\|e^{-x/h} u\|_{L^2} \leq h^2 \|e^{-x/h} D_x^2 u\| \leq h^2 \|W\|_{L^\infty} \|e^{-x/h} u\|_{L^2}$$