

September 30, 2024

## Chapter 1: Banach Algebras

### 1.1: Definitions and Basic Properties

#### Definition: Banach Space

A Banach space  $X$  (over  $\mathbb{C}$ ) is a normed vector space with algebraic operations

$$\begin{aligned}(x, y) &\mapsto x + y && \text{addition} \\ (\lambda, y) &\mapsto \lambda y && \text{scalar multiplication}\end{aligned}$$

and a norm

$$x \mapsto \|x\|$$

which is complete (i.e. every Cauchy sequence converges).

#### Definition: (Complex) Banach Algebra

A (complex) Banach algebra  $B$  is a Banach space in which there is multiplication

$$B \times B \ni (x, y) \mapsto xy \in B$$

such that

1.  $x(yz) = (xy)z$
2.  $(x + y)z = xz + yz$  and  $x(y + z) = xy + xz$
3.  $\lambda(xy) = (\lambda x)y = x(\lambda y)$
4.  $\|xy\| \leq \|x\| \cdot \|y\|$

#### Definition: Unital Banach Algebra

$B$  is called a unital Banach algebra if  $\exists e \in B$  such that

$$xe = ex = x \quad \text{and} \quad \|e\| = 1.$$

If  $e$  exists, it is unique.

### 1.2: Examples

#### Example 1

If  $X$  is a Banach space, then  $B = \mathcal{L}(X)$  (the set of all bounded linear operators  $A : X \rightarrow X$ ) equipped with algebraic operations

$$(A+B)x = Ax + Bx$$

$$(\lambda A)x = \lambda(Ax)$$

$$(AB)x = A(Bx)$$

and the operator norm

$$\|A\|_{\mathcal{L}(X)} = \sup_{x \neq 0} \frac{\|Ax\|_X}{\|x\|_X}.$$

$B = \mathcal{L}(X)$  is complete because  $X$  is complete.

The unit element is given by  $I_X x = x$ .

### Example 2

If  $X = \mathbb{C}^n$ , then  $B = \mathcal{L}(\mathbb{C}^n) \cong \mathbb{C}^{n \times n}$ .

$$A = (a_{ij})_{i,j=1}^n$$

$$Ax = y$$

$$\sum_{j=1}^n a_{ij} x_j = y_i.$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

The norm in  $\mathbb{C}^n$  leads to a norm in  $\mathbb{C}^{n \times n}$

$$\|(x_i)\| = \left( \sum |x_i|^2 \right)^{1/2}$$

$$\|(x_i)\| = \sum |x_i|$$

$$\|(x_i)\| = \max |x_i|$$

$$\|A\| =$$

$$\|A\| = \max_j \sum_i |a_{ij}|$$

$$\|A\| = \max_i \sum_j |a_{ij}|$$

All norms are equivalent.

### Example 3

Take  $B = C(K)$  with  $K$  a compact Hausdorff space,  $f : K \rightarrow \mathbb{C}$  continuous and  $\|f\| = \max_{t \in K} |f(t)|$ .

### Example 4

Take  $B = A(K)$ ,  $K \subseteq \mathbb{C}$  compact with  $\text{int}(K) \neq \emptyset$ ,  $f : K \rightarrow \mathbb{C}$  continuous where  $f$  is holomorphic on  $\text{int}(K)$  and

$$\|f\| = \max_{t \in K} |f(t)| = \max_{t \in K \setminus \text{int}(K)} |f(t)|$$

e.g.  $K = \overline{\mathbb{D}} = \{t \in \mathbb{C} : |t| \leq 1\}$ . Then  $A(K) \subseteq C(K)$ .

### Example 5

Take  $B = \ell^\infty(\mathbb{N})$  or  $B = L^\infty(S, \sigma, \mu)$  with  $(S, \sigma, \mu)$  a measure space,  $f : S \rightarrow \mathbb{C}$  essentially bounded functions and

$$||f|| = \text{ess sup}_{t \in S} |f(t)| = \inf_{\substack{N \subseteq S \\ \mu(N) = 0}} \left( \sup_{t \in S \setminus N} |f(t)| \right)$$

### Example 6

Take  $B = \ell^1(\mathbb{Z})$  or  $B = L^1(\mathbb{R}^d)$  with  $||\{x_n\}|| = \sum |x_n|$  and  $||f|| = \int_{\mathbb{R}^d} |f(t)| dt$  respectively. Multiplication is given by the convolution. e.g.

$$fg = (f * g)(x) = \int_{\mathbb{R}^d} f(x-t)g(t) dt$$

$\ell^1(\mathbb{Z})$  is unital, but  $L^1(\mathbb{R}^d)$  is non-unital (since the unit of convolution is the Dirac delta; see Example 7).

### Example 7

Take  $B = M(\mathbb{R}^d)$  the complex measures on  $\mathbb{R}^d$  with bounded variation. Then multiplications is given as

$$(\mu * \nu)(A) = \int_{\mathbb{R}^d} \mu(A-x) d\nu(x)$$

and norm

$$||\mu|| = \sup_{\substack{\mathbb{R}^d = \bigcup_{i=1}^n A_i \\ \text{disjoint}}} \sum_{i=1}^n |\mu(A_i)| < +\infty.$$

Then,  $f dm = d\mu$  gives  $L^1(\mathbb{R}^d) \rightarrow M(\mathbb{R}^d)$ .

### Example 8

Take  $B = C^{n \times n}[K]$  with  $K$  compcat and Hausdorff, continuous functions  $f : K \rightarrow \mathbb{C}^{n \times n}$  and norm

$$||f||_B = \max_{t \in K} ||f(t)||_{C^{n \times n}}.$$

Then  $B \cong (C(K))^{n \times n}$  the  $n \times n$  matrices with entries from  $C(K)$ .

### 1.3: Remarks

- If  $B$  does not have a unit element, consider  $B_1 = B \times \mathbb{C}$  with operations

$$\begin{aligned} (b_1, \lambda_1) + (b_2, \lambda_2) &= (b_1 + b_2, \lambda_1 + \lambda_2) \\ \alpha(b, \lambda) &= (\alpha b, \alpha \lambda) \\ (b_1, \lambda_1)(b_2, \lambda_2) &= b_1 b_2 + \lambda_1 b_2 + \lambda_2 b_1, \lambda_1 \lambda_2 \end{aligned}$$

and norm

$$||(b, \lambda)|| = ||b|| + |\lambda|.$$

Then  $B_1$  is a unital Banach algebra with  $e = (0, 1)$ . One writes  $(b, \lambda) = (b, 0) + \lambda(0, 1) = b + \lambda \cdot e$ . In some sense,  $B \subseteq B_1$  where  $b \in B \mapsto (b, 0) \in B_1$ .

## 1.4: Definitions

### Definition: Commutative Banach Algebra

$B$  is called commutative if  $xy = yx$ .

### Definition: Banach Subalgebra

A subset  $B_0$  of a  $B$ -algebra is called a subalgebra if it is closed with respect to the algebraic operations

$$x, y \in B_0, \lambda \in \mathbb{C} \leadsto x + y, xy, \lambda x \in B$$

### Definition: Closed Subalgebra

$B_0$  is a closed subalgebra or Banach subalgebra if it is norm-closed.

- Proposition:  $B_0$  is a Banach algebra.

### Definition: Generated Subalgebra

Let  $M \neq \emptyset$  be a subset of a Banach algebra  $B$ .

The Banach subalgebra generated by  $M$  is the smallest closed subalgebra containing  $M$ .

$$\text{alg } M = (\text{clos alg}_B M)$$

- Remark

$\text{alg } M$  is the intersection of all closed subalgebras containing  $M$ .

$\text{alg } M = \text{clos} \left\{ \sum_{i=1}^N \lambda_i a_1^{(i)} a_2^{(i)} \cdots a_{n_i}^{(i)} \right\}$  is the norm-closure of finite linear combinations of finite products of  $a_j^{(i)} \in M$ .

## 1.5: Examples

### Example 1

Take  $B$  unital,  $b \in B$ . Then

$$\text{alg}\{e, b\} = \text{clos}_B \left\{ \sum_{i=0}^N \lambda_i b^i : \lambda_i \in \mathbb{C}, N \in \mathbb{N} \right\}$$

where  $b^0 = e$ .

## 1.6 Definitions

### Definition: Banach Algebra Homomorphism

A Banach algebra homomorphism is a map  $\phi : B_1 \rightarrow B_2$  between Banach algebras  $B_1$  and  $B_2$  such that

- $\phi$  is linear
- $\phi$  is bounded (continuous)
- $\phi$  is multiplicative

$$\phi(b_1 b_2) = \phi(b_1) \cdot \phi(b_2)$$

- $\phi$  is unital if both  $B_1, B_2$  have units and  $\phi(e_{B_1}) = e_{B_2}$ .

### Definition: Banach Algebra Isomorphism

A Banach algebra homomorphism which is bijective is called a Banach algebra isomorphism.

Then  $\phi^{-1} : B_2 \rightarrow B_1$  is an isomorphism as well.

### Definition: Banach Algebra Isometry

$\phi$  is an isometry if  $||\phi(x)|| = ||x||$ .

**October 2, 2024**

### Recall

Given  $M \subseteq \mathcal{L}(X)$  with  $X$  a Banach space (and  $\mathcal{L}(X)$  itself a Banach algebra), we may construct  $B = \text{alg}_{\mathcal{L}(X)} M$ .

### 1.7 Proposition

Let  $B$  be a unital Banach algebra. Then the map

$$\phi : B \ni x \rightarrow L_x \in \mathcal{L}(B)$$

is an isometric isomorphism onto a closed subalgebra of  $\mathcal{L}(B)$  where

$$L_x : B \ni z \mapsto xz \in B$$

is the left-representation of  $x$ .

### Proof

$L_x$  is in  $\mathcal{L}(B)$  since  $L_x z = xz$

- is linear in  $z$  and
- $||L_x z|| = ||xz|| \leq ||x|| \cdot ||z||$  implies  $||L_x|| \leq ||x||$  (i.e.  $L_x$  is a bounded).

The map  $\phi : x \mapsto L_x$  is linear

$$L_{x_1+x_2}z = (x_1+x_2)z = x_1z + x_2z = L_{x_1}z + L_{x_2}z = (L_{x_1} + L_{x_2})z$$

$\phi$  is multiplicative

$$L_{x_1x_2}z = (x_1x_2)z = x_1(x_2z) = L_{x_1}(L_{x_2}z)$$

From the above, we conclude that  $\phi$  is a homomorphism.

To show that  $\phi$  is an isometry,

$$\|L_x\| = \sup_{z \neq 0} \frac{\|L_x z\|}{\|z\|} \geq \frac{\|L_x e\|}{\|e\|} = \frac{\|x\|}{1} = \|x\|.$$

Then also  $\phi$  is injective and  $\text{im } \phi$  is closed. Since  $\text{im } \phi$  is a Banach algebra, it is therefore a closed subalgebra.

### 1.7 Remark: Right-Regular Representation

Every unital Banach algebra is isometrically isomorphic to a Banach algebra of operators.

Right-regular representation:

$$R_x = z \mapsto zx$$

## Section 1.2: Group of Invertible Elements in a Banach Algebra

### 2.1 Definition: Invertible Element

Let  $B$  be a unital Banach algebra. An element  $x \in B$  (in  $B$ ) if there exists  $y \in B$  such that  $xy = yx = e$ .

Note that  $y = x^{-1}$  is uniquely determined.

Write  $GB$  for the set of all invertible elements of  $B$ .

#### Remark

$GB$  is a (multiplicative group).

- $x, y \in GB \implies xy \in GB$  and  $(xy)^{-1} = y^{-1}x^{-1}$ ,
- $x \in GB \implies x^{-1} \in GB$  and  $(x^{-1})^{-1} = x$ , and
- $e \in GB$ .

### 2.2 Lemma

If  $x \in B$  and  $\|x\| < 1$ , then  $e - x \in GB$ .

#### Proof

Take the Neumann series

$$e + x + x^2 + x^3 + \dots$$

which converges to some  $s \in B$

$$s_n = e + x + \cdots + x^n$$

where  $s_n$  are Cauchy:

$$||s_{n+k} - s_n|| = ||x^{n+1} + \cdots + x^{n+k}|| \leq ||x||^{n+1} + ||x||^{n+2} + \cdots = \frac{||x||^{n+1}}{1 - ||x||}.$$

So  $s_n \rightarrow S$ ,

$$(e - x)s_n = s_n(e - x)e - x^{n+1}.$$

Taking  $n \rightarrow \infty$

$$(e - x)s = s(e - x) = e.$$

## 2.3 Proposition

The group  $GB$  is open in  $B$  and the map  $\Lambda : GB \ni x \mapsto x^{-1} \in GB$  is continuous (in the norm).

### Proof

Take  $x \in GB$  and consider  $y \in B$  with  $||y|| < \frac{1}{||x^{-1}||} = \varepsilon$ .

Then  $x + y \in B_\varepsilon(x)$  is invertible,

$$x + y = x(e + x^{-1}y),$$

and

$$||x^{-1}y|| \leq ||x^{-1}|| \cdot ||y|| < 1.$$

Therefore  $GB$  is open, since  $B_\varepsilon(X) \subseteq GB$ . The inverse

$$(x + y)^{-1} = (e + x^{-1}y)^{-1}x^{-1} = \sum_{n=0}^{\infty} (-x^{-1}y)^n x^{-1} = x^{-1} + \sum_{n=1}^{\infty} (-x^{-1}y)^n x^{-1}$$

so

$$||(x + y)^{-1} - x^{-1}|| \leq \sum_{n=1}^{\infty} ||x^{-1}||^{n+1} ||y||^n = \frac{||x^{-1}||^2 ||y||}{1 - ||x^{-1}|| \cdot ||y||}.$$

This converges to zero as  $||y|| \rightarrow 0$ .

## 2.4 Examples

### Example 1

$B = C(K)$ ,  $K$  compact Hausdorff,  $f : K \rightarrow \mathbb{C}$  continuous.

$GB = \{f \in C(K) : f(t) \neq 0, \forall t \in K\}$ .

## Example 2

$$B = \mathbb{C}^{n \times n}.$$

$$GB = \{A \in \mathbb{C}^{n \times n} : \det A \neq 0\}.$$

### 2.5 Definition:

Let  $G_0 B$  stand for the connected component of  $GB$  containing  $e$ .

### Remarks

- the  $\varepsilon$ -neighborhoods  $B_\varepsilon(x) \subseteq B$  are (path-)connected.

$$B_\varepsilon(x) = \{y \in B : \|x - y\| < \varepsilon\}$$

For  $y_1, y_2 \in B_\varepsilon(x)$ , there is a continuous path

$$\sigma : [0, 1] \ni \lambda \mapsto y_1 \lambda + y_2(1 - \lambda) \in B_\varepsilon(x)$$

- Because  $GB$  is open and  $B_\varepsilon(x)$  is path-connected,  $GB$  is locally (path-)connected (i.e. every  $x \in GB$  has a (path-)connected open neighborhood in  $GB$ ).
- In this context, connectedness and path-connectedness are equivalent. Therefore the components of  $GB$  are the path-components of  $GB$ .
- $GB$  is the union of disjoint (path-)components where each component is both open and closed in  $GB$ .
- $x, y \in GB$  belong to the same path-component if there exists a continuous path  $\gamma : [0, 1] \rightarrow GB$  such that  $\gamma(0) = x$  and  $\gamma(1) = y$ . Here,  $x \sim y$  is an equivalence relation.
- $G_0 B = \{x \in GB : \exists \text{ a path in } GB \text{ connecting } e \text{ and } x\}.$

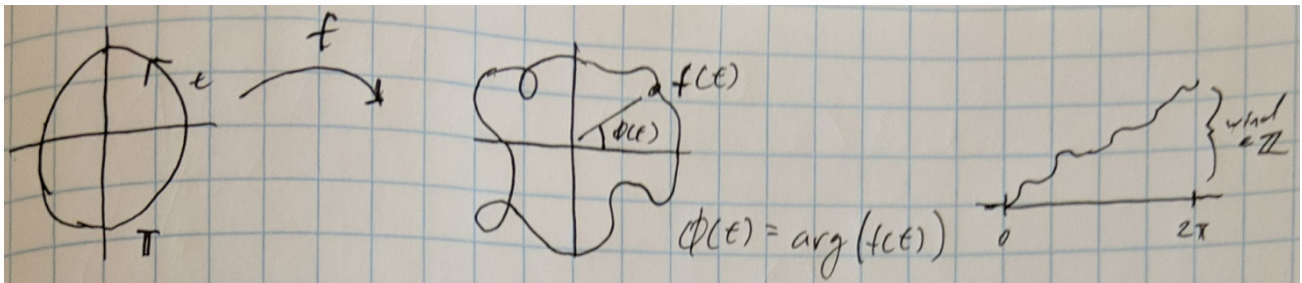
## 2.6 Examples

### Example 1

Take  $B = C(\mathbb{T})$  with  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$  and continuous functions  $f : \mathbb{T} \rightarrow \mathbb{C}$ .

$GB$  is the non-vanishing continuous functions  $f : \mathbb{T} \rightarrow \mathbb{C}$  ( $f(t) \neq 0, \forall t \in \mathbb{T}$ ).

For  $f \in GB$  one can define a winding number.



We have  $\frac{1}{2\pi} \arg f(e^{ix})$  a continuous function with

$$\text{wind}(t) = \left[ \frac{1}{2\pi} \arg f(e^{ix}) \right]_{x=0}^{2\pi} = \phi(2\pi) - \phi(0)$$



and  $\text{wind}(t) \in \mathbb{Z}$ .

The map  $GB \ni f \mapsto \text{wind}(t) \in \mathbb{Z}$  is continuous, hence locally constant (i.e. constant on each connected component).

Therefore  $G_0C(\mathbb{T}) \subseteq \{f \in GC(\mathbb{T}) : \text{wind}(f) = 0\}$ . In fact, we will see that we have equality.

That is,  $f$  can be contracted (in  $GB$ ) to the constant function  $e(t) = 1$ .

## 2.7 Proposition

$G_0B$  is a normal subgroup of  $GB$ .

### Proof

- $G_0B$  is a group.

For any  $x, y \in G_0B$ , there exist paths  $\gamma_1 : [0, 1] \rightarrow GB$  and  $\gamma_2 : [0, 1] \rightarrow GB$  with  $\gamma_1(0) = \gamma_2(0) = e$ ,  $\gamma_1(1) = x$  and  $\gamma_2(1) = y$ .

Define  $\gamma(t) = \gamma_1(t)\gamma_2(t)$  a path in  $GB$  such that  $\gamma(0) = e$  and  $\gamma(1) = xy$ . Then  $xy \in G_0B$ .

Following from Lemma 2.2,  $\hat{\gamma} = (\gamma_1(t))^{-1}$  is a continuous path with  $\hat{\gamma}_1(0) = e$ ,  $\hat{\gamma}_1(1) = x^{-1}$  and  $x^{-1} \in G_0B$ .

- $G_0B$  is a normal subgroup of  $GB$ .

For every  $y \in GB$ ,  $yG_0By^{-1} \subseteq G_0B$  if and only if  $yG_0B = G_0By$ .

Take  $x \in G_0B$  with path  $\gamma$ , then

$$\delta(t) = y\gamma(t)y^{-1}, \quad \delta(0) = yey^{-1} = e, \quad \text{and} \quad \delta(1) = yxy^{-1} \in G_0B.$$

## 2.8 Definition: Abstract Index Group

The quotient group  $GB/G_0B$  is called the abstract index group of  $B$ .

### Remark

$GB/G_0B$  is in 1-to-1 correspondence with the set of connected components of  $GB$ .

Indeed, the (path-)connected components of  $GB$  are given by  $yG_0B = G_0By$  (for  $y \in GB$ ).

$$y_1G_0B = y_2G_0B \iff y_2^{-1}y_1G_0B = G_0B \iff y_2^{-1}y_1 \in G_0B \iff [y_2] = [y_1] \text{ in } GB/G_0B.$$

## 2.9 Definition: Exponential Map

For  $x \in B$ , we define the exponential map  $B \ni x \mapsto \exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$ .

### 2.10 Lemma

The exponential map  $B \ni x \mapsto \exp(x) \in GB$  is well-defined and continuous.

For  $xy = yx$ , we have  $\exp(x+y) = \exp(x)\exp(y)$ .

In particular,  $(\exp(x))^{-1} = \exp(-x)$ .

### Proof

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$  is absolutely convergent.

$$\sum_{n=0}^{\infty} \frac{||x||^n}{n!} < +\infty.$$

It follows that  $s_n = \sum_{k=0}^n \frac{x^k}{k!}$  is a Cauchy sequence and therefore converges. Continuity left as an exercise. Need to show:

$$\left\| \sum \frac{x^n}{n!} - \sum \frac{y^n}{n!} \right\| \leq \|x - y\| \cdot M_{x,y}$$

The fact that  $\exp(x + y) = \exp(x) \exp(y)$  follows from multiplying terms and the binomial formula.

## October 7, 2024

### Recall

$GB$   $e + x$ .

$G_0B$  connected component of  $GB$  containing  $e$ .

$GB/G_0B$  is the abstract index group.

$B = C(\mathbb{T}) \leadsto f \in GC(\mathbb{T}) \leadsto \text{ind}(f)$ .

### Definition: Exponential Map

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \in GB$$

### Lemma:

For  $y \in B$ ,  $\|y\| < 1$ , there exists  $x \in B$  such that  $\exp(x) = e + y$ .

### Proof

Define

$$\log(e + y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \dots \in B.$$

This converges absolutely ( $\|y\| < 1$ ), therefore it converges in  $B$  by completeness.

### Identities

$$\exp(\log(e + y)) = \sum_{n=0}^{\infty} \frac{\left( \sum_k \frac{y^k}{k} (-1)^{k-1} \right)^n}{n!} = e + y$$

### Proof

$G_0B$  is equal to the set of all finite products of exponentials of elements in  $B$ .

$$G_0B = \bigcup_{n=0}^{\infty} \Gamma_n = \bigcup_{n=0}^{\infty} \{ \exp(a_1) \exp(a_2) \cdots \exp(a_n) \in B \}$$

## Proof

Call  $\Gamma = \bigcup_{n=0}^{\infty} \Gamma^n$ .

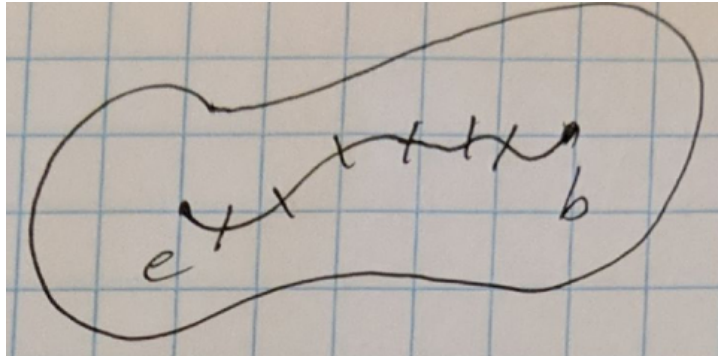
Then observe that each  $\Gamma_n$  is path-connected and contains  $e$ .

For  $b = \exp(a_1) \cdots \exp(a_n) \in \Gamma_n$ , define a path

- $\sigma : [0, 1] \rightarrow \Gamma_n$
- $\sigma(t) = \exp(ta_1) \cdots \exp(ta_n)$  is continuous with  $\sigma(0) = e$  and  $\sigma(1) = b$ .

Therefore,  $\Gamma$  is path-connected and contains  $e$ . It follows that  $\Gamma \subseteq G_0 B$ .

To prove that  $G_0 B \subseteq \Gamma$ , take  $b \in G_0 B$  and show that there exists a path in  $GB$   $\gamma : [0, 1] \rightarrow GB$  continuous with  $\gamma(0) = e$  and  $\gamma(1) = b$ .



We have that  $(\gamma(t))^{-1}$  is continuous and bounded in the norm. Then  $\gamma(t)$  is uniformly continuous.

$$\|\gamma^{-1}(t)\| \leq M.$$

$(\exists N) : |t - s| \leq \frac{1}{N} \implies \|\gamma(t) - \gamma(s)\| \leq \frac{1}{M} \cdot \frac{1}{2}$ . Write

$$b = \gamma(1) \cdot \gamma^{-1}(0) = \gamma(1) \gamma^{-1}\left(\frac{N-1}{N}\right) \gamma\left(\frac{N-1}{N}\right) \gamma^{-1}\left(\frac{N-2}{N}\right) \cdots \gamma\left(\frac{1}{N}\right) \gamma^{-1}\left(\frac{1}{N}\right) \gamma(0) = \prod_{k=1}^N \gamma^{-1}\left(\frac{k}{N}\right) \gamma\left(\frac{k-1}{N}\right).$$

Therefore, with  $s_k = \gamma^{-1}\left(\frac{k}{N}\right) \gamma\left(\frac{k-1}{N}\right)$ ,  $b = \prod_{k=1}^N \exp(\log(s_k))$ .

$$\|s_k - e\| \leq \|\gamma^{-1}\left(\frac{k}{N}\right)\| \cdot \|\gamma\left(\frac{k-1}{N}\right) - \gamma\left(\frac{k}{N}\right)\| \leq M \cdot \frac{1}{2M} \leq \frac{1}{2}.$$

## Corollary

If  $B$  is commutative,  $G_0 B = \{\exp(a) : a \in B\}$ .

## Remark

Special case:  $B = C(K)$  ( $K$  compact Hausdorff space).

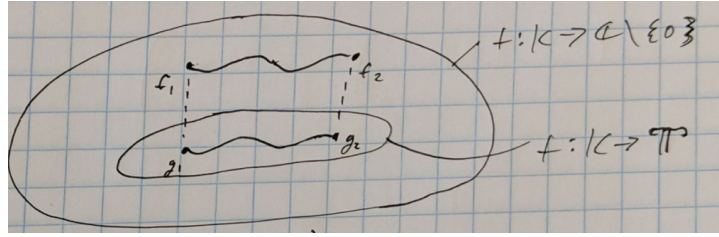
$$G_0 B = \{\exp(a) : a \in C(K)\}.$$

$GB/G_0 B$  is an equivalence class of functions  $f : K \rightarrow \mathbb{C} \setminus \{0\}$  with respect to path-connectedness.

That is,  $f_1 \sim f_2$  if and only if there exists continuous  $F(t, x) : [0, 1] \times K \rightarrow \mathbb{C} \setminus \{0\}$  with  $F(0, x) = f_1(x)$  and  $F(1, x) = f_2(x)$ .

These are the homotopy classes of continuous functions  $f : K \rightarrow \mathbb{C} \setminus \{0\}$ .

This corresponds to homotopy classes of continuous functions  $f : K \rightarrow \mathbb{T}$  (with  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ ) called the 1st co-homotopy group of  $K$   $\pi^1(K)$ .



$f : K \rightarrow \mathbb{C} \setminus \{0\}$  and  $\frac{f}{|f|} : K \rightarrow \mathbb{C} \setminus \{0\}$  are path-connected by  $\sigma(s) = \frac{f}{|f|^s}$ ,  $s \in [0, 1]$ .

$f_1 \sim f_2$  in  $K \rightarrow \mathbb{C} \setminus \{0\}$  implies that  $\frac{f_1}{\|f_1\|} \sim \frac{f_2}{\|f_2\|}$  in  $K \rightarrow \mathbb{T}$  by  $F(s, x)$  and  $\frac{F(s, x)}{|F(s, x)|}$ .

We conclude that  $\pi^1(K) \cong GC(K)/G_0C(K)$ .

### Example

Let  $B = C(\mathbb{T})$ .

$$G_0B = \{\exp(a) : a \in C(\mathbb{T}) = \{f \in GC(\mathbb{T}) : \text{wind}(f) = 0\}\}$$

For  $f \in GC(\mathbb{T})$ ,  $\text{wind}(f) = 0$  implies that  $f = \exp(a)$  has a logarithm.

This implies that  $f \in G_0B$  which itself implies that  $\text{wind}(f) = 0$ , since  $\text{wind}(f)$  is continuous on  $GC(\mathbb{T})$  and therefore constant on the component.

Therefore,  $GB/G_0B \cong \mathbb{Z}$  via the winding number.

For connected components of  $GB$ , define  $\chi_n(t) = t^n$ ,  $|t| = 1$ , where  $\text{wind}(\chi_n) = n$ .

### Remark: Closed Subalgebras and Invertibility

Let  $A$  be a closed subalgebra of  $B$  (both being unital,  $e \in A$ ,  $e \in B$ ).

Obviously, if  $a \in A$  is invertible in  $A$  (i.e.  $a^{-1} \in A$ ) then  $a$  is invertible in  $B$ . Then  $GA \subseteq GB \cap A \subseteq GB$ .

### Example

Take  $B = C(\mathbb{T})$  and  $A = \{f \in C(\mathbb{T}) : f_n = 0, \forall n < 0\} = C_+(\mathbb{T})$  where  $f_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{ix}) e^{-inx} dx$  is the  $n$ th Fourier coefficient.

Formally:  $f(t) \cong \sum_{n=-\infty}^{\infty} f_n t^n$  in  $B = C(\mathbb{T})$ ,  $|t| = 1$ .

$f \in A : f(t) = \sum_{n=0}^{\infty} f_n t^n$ ,  $|t| = 1$  has an analytic extension into the unit disk  $|t| < 1$ .

More precisely,  $\phi : A(\overline{\mathbb{D}}) \rightarrow C_+(\mathbb{T}) \subseteq C(\mathbb{T})$  by  $f \mapsto f|_{\mathbb{T}}$ .

Where  $A(\overline{\mathbb{D}}) = \{f \in \overline{D} \rightarrow \mathbb{C} \text{ continuous, holomorphic on } \mathbb{D}\}$  and  $\mathbb{D} = \{t \in \mathbb{C} : |t| \leq 1\}$ .

Then, for  $f \in A(\overline{\mathbb{D}})$  with  $n \in \{-1, -2, -3, \dots\}$ ,

$$f_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{ix}) e^{-inx} dx = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(z)}{z^{n+1}} dz = \lim_{r \rightarrow 1^-} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(rz)}{z^{n+1}} dz = 0$$

- In fact,  $\phi$  is an isometry.

$$\|f\|_{A(\overline{\mathbb{D}})} = \sup_{|z| \leq 1} |f(z)| = \max_{|z|=1} |f(z)| = \|f|_{\mathbb{T}}\|_{C(\mathbb{T})}$$

By maximum modulus principle of holomorphic functions, since  $\phi$  is not constant.

- $\phi$  is linear and multiplicative.

- $C_+(\mathbb{T})$  is a closed subset of  $C(\mathbb{T})$ .

$$\Lambda_n : C(\mathbb{T}) \ni f \mapsto f_n \in \mathbb{C}$$

is a continuous linear functional.

$$C_+(\mathbb{T}) = \bigcap_{n=0} \ker \Lambda_n$$

- Less trivially,  $\phi$  is surjective and  $C_+(\mathbb{T})$  is an algebra.

### Example

$\chi_1(t) = t$  is invertible in  $C(\mathbb{T}) = B$ .  
 $\chi_1^{-1}(t) = \frac{1}{t} = x_{-1}(t) \notin C_+(\mathbb{T})$  while  $\chi_1(t) \in C_+(\mathbb{T})$ .  
 Therefore  $GA \subseteq GB \cap A$  may not be equal.

### Definition: Boundary

The boundary of a subset  $U$  of a topological space  $X$  is  $\partial U = \overline{U} \setminus \text{int}(U)$ .

### Remark

For  $U \subseteq X$ ,  $X = \text{int}(U) \cup \partial U \cup \text{int}(X \setminus U)$  a union of disjoint sets.

### Lemma:

1. if  $a \in \partial GA$ , then  $a \notin GA$  and there exists a sequence  $a_n \in GA$  such that  $a_n \rightarrow a$ .
2. if  $a \in \partial a$  and  $a_n \in GA$  such that  $a_n \rightarrow a$ , then  $\|a_n^{-1}\| \rightarrow +\infty$ .

### Proof of 1

$a \in GA$  would imply  $a \in \text{int}(GA)$  and not a boundary point.

### Proof of 2

Otherwise, there would exist a bounded subsequence  $\|a_{n_i}^{-1}\| \leq M$ .

$$\|a_{n_i}^{-1} - a_{n_j}^{-1}\| \leq \|a_{n_i}^{-1}\| \cdot \|a_{n_j} - a_{n_i}\| \cdot \|a_{n_j}^{-1}\| \leq M^2 \|a_{n_i} - a_{n_j}\|$$

Since  $a_n$  converges,  $\{a_n\}$  is Cauchy which implies  $a_{n_i}^{-1}$  is Cauchy.

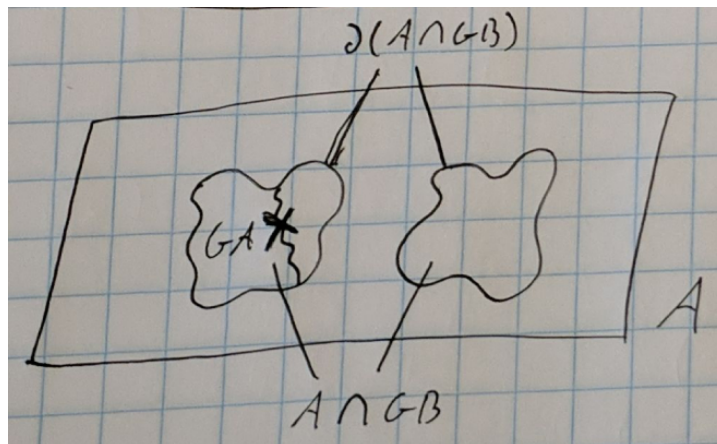
Then  $a_{n_i}^{-1} \rightarrow b \in A$ .  $e = a_{n_i} a_{n_i}^{-1} \rightarrow ab$  implies  $a^{-1} = b$  and  $a \in GA$ . However  $a \notin GA$ .

### Proposition

Let  $A$  be a closed subalgebra of  $B$  ( $e \in A$ ,  $e \in B$ ). Then  $\partial GA \subseteq \partial(A \cap GB)$  (both boundaries are considered in  $A$ ).

## Remark

Both  $GA$  and  $A \cap GB$  are open subsets of  $A$ .



## Proof

Take  $a \in \partial GA$  and suppose  $a \notin \partial(A \cap GB)$ .

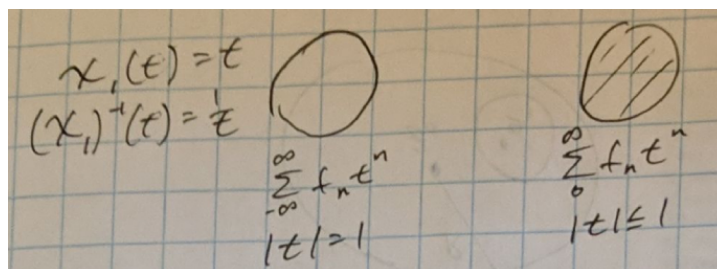
Take  $a \in \partial GA$ :  $a_n \in GA$ ,  $a \notin GA$ ,  $a_n \rightarrow a$ ,  $\|a_n^{-1}\| \rightarrow +\infty$ .

**October 9, 2024**

## Recall

$A \subseteq B$ ,  $GA \subseteq A \cap GB$ .

If  $A = C_+(\mathbb{T}) \cong A(\overline{\mathbb{D}})$  and  $B = C(\mathbb{T})$ .



## Recall: Theorem

For  $GA$ ,  $A \cap GB$  open sets in  $A$ ,  $U \subseteq X$ ,  $\partial U = \overline{U} \setminus \text{int } U$ , we have that  $\partial GA \subseteq \partial(A \cap GB)$ .

## Proof

Take  $a \in \partial GA$ ,  $a_n \rightarrow a$ ,  $a \notin GA$ ,  $a \in A$ .

Since  $a_n \in GA$ ,  $\|a_n^{-1}\| \rightarrow +\infty$ .

However,  $a \notin GB$  otherwise  $a \in GB$ ,  $a_n \rightarrow a$  implies  $a_n^{-1} \rightarrow a^{-1}$  (in  $GB$ ) and, consequently,  $\sup \|a_n^{-1}\| < +\infty$ , a contradiction.

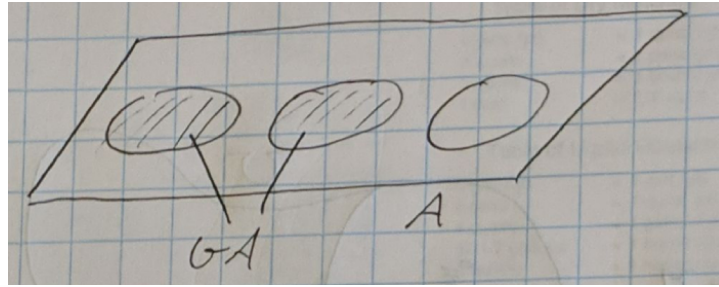
Therefore  $a \notin A \cap GB$  and, consequently,  $a \in \partial(A \cap GB) = \overline{(A \cap GB)} \setminus (A \cap GB)$ .

## Theorem

Let  $A$  be a closed subalgebra of  $B$ .

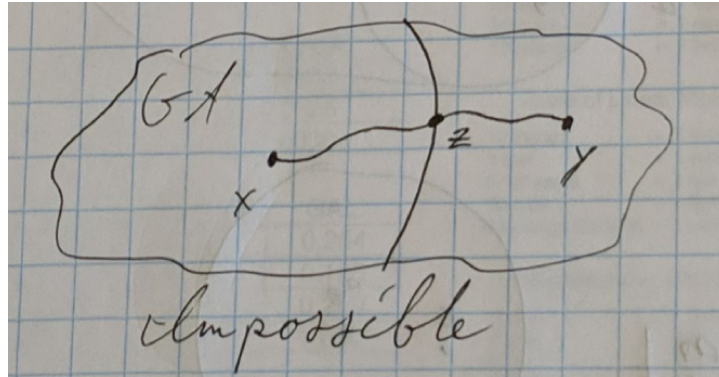
$GA$  is equal to the union of some components of  $A \cap GB$ .

## Proof



Let  $U$  be a component of  $A \cap GB$ .

We want to show that either  $U \cap GA \neq \emptyset$  or  $U \subseteq GA$ .



The above cannot occur since, by path-connectedness, for  $x, y \in U$ ,  $x \in GA$ ,  $y \notin GA$ , there would need to be some  $z \in \partial GA$  with  $z \notin A \cap GB$  a contradiction.

Alternatively, take  $A \cap GB$  open in  $A$ .

Then  $A \cap GB \cap \partial(A \cap GB) = \emptyset$  and  $(A \cap GB) \cap \partial GA = \emptyset$  by the previous theorem.

Write  $A = GA \cup \partial GA \cup \text{int}(A \setminus GA)$ . Then

$$A \cap GB = GA \cup \emptyset \cup \text{int}(A \setminus GA) \cap (A \cap GB)$$

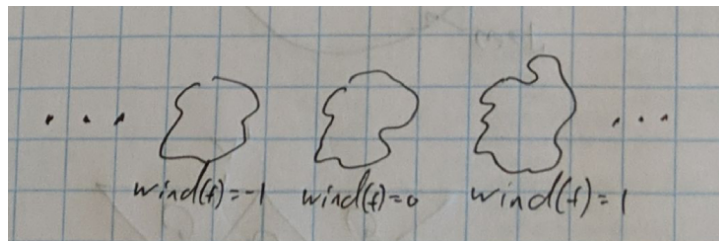
and  $U = (GA \cap U) \cup \text{int}(A \setminus GB) \cap U$  where  $(GA \cap U) \cap \text{int}(A \setminus GA) = \emptyset$  and open in  $U$ .

Therefore either  $GA \cap U = \emptyset$  or  $GA \cap U = U$  which implies that  $U \subseteq GA$ .

## Example

Take  $B(\mathbb{T})$  and  $A = C_+(\mathbb{T}) \cong A(\overline{D})$ .

Then  $GB = \{f : \mathbb{T} \rightarrow \mathbb{C} : f(t) \neq 0\}$ .



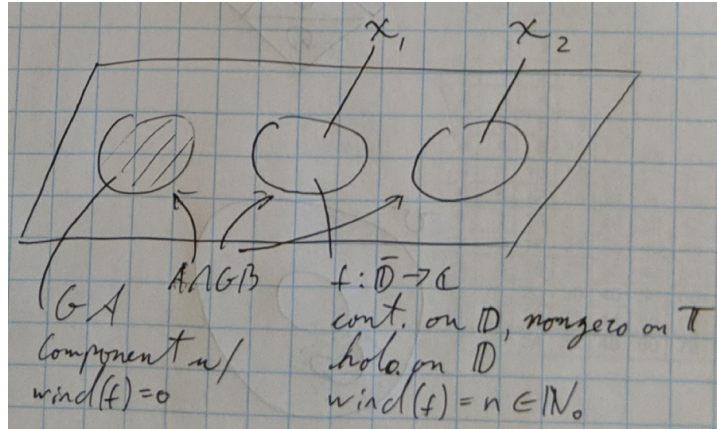
Then take

$$A \cap GB = \{f : \mathbb{T} \rightarrow \mathbb{C} \text{ continuous, } f(t) \neq 0, |t| = 1 \text{ with analytic continuation into } |t| < 1\}$$

such that  $f \in A \cap GB$  which implies  $\text{wind}(f) \in \{0, 1, 2, 3, \dots\}$  gives the number of zeroes of  $f$  inside  $\mathbb{D}$ .

$$\begin{aligned}\text{wind}(f) &= \frac{1}{2\pi i} \left[ \log f(e^{ix}) \right]_{x=0}^{2\pi} \\ &= \frac{1}{2\pi i} \lim_{r \rightarrow 1^-} \left[ \log f(re^{ix}) \right]_{x=0}^{2\pi} \\ &= \frac{1}{2\pi i} \lim_{r \rightarrow 1^-} \int_0^{2\pi} \frac{f'(re^{ix})}{f(re^{ix})} ire^{ix} dx \\ &= \frac{1}{2\pi i} \lim_{r \rightarrow 0} \int_{|z|=r} \frac{f'(z)}{f(z)} dz\end{aligned}$$

Which gives the number of zeroes of  $f(z)$  inside  $|z| < 1$



## Section 1.3: Holomorphic Vector-Valued Functions

### Goal

Define the notion of holomorphic/analytic functions  $f : \Omega \rightarrow X$  where  $\Omega \subset \mathbb{C}$  open and  $X$  a (complex) Banach space.

### Summary

- Basically all classical results remain true.
- There is a strong and a weak version of holomorphy, but they are equivalent.

### Theorem

For a function  $f : \Omega \rightarrow X$ ,  $\Omega \subseteq \mathbb{C}$  open and  $X$  Banach, the following are equivalent

1.  $f$  is differentiable at every  $z_0 \in \Omega$ , i.e. there exists  $f'(z_0) \in X$  such that

$$\lim_{z \rightarrow z_0} \left\| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right\|_X = 0$$

2.  $f$  is analytic at each point  $z_0 \in \Omega$ , i.e.  $f$  has a convergent power series at  $z_0$  with radius of convergence  $R_{z_0} > 0$ .

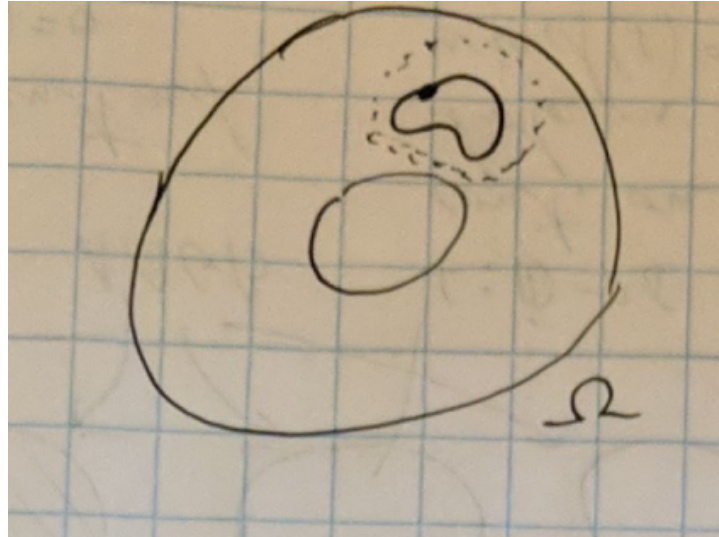
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad |z - z_0| < R_{z_0}, \quad a_n \in X$$



which converges in the norm of  $X$ .

3.  $f : \Omega \rightarrow X$  is continuous (in the norm) and for every piecewise smooth closed contour  $\Gamma$  contained in a disk  $D$  ( $\Gamma \subseteq D \subseteq \Omega$ ).

$$\int_{\Gamma} f(z) dz = 0$$



### Definition: (Strongly) Holomorphic Function

If (1)-(3) hold, then  $f$  is (strongly)-holomorphic.

### Remarks: Integration of Vector-Valued Functions

A piecewise smooth contour  $\Gamma$  can be parameterized by  $\sigma : [0, 1] \rightarrow \Omega$ .

$$\int_{\Gamma} f(z) dz = \int_0^1 \underbrace{f(\sigma(t))\sigma'(t)}_{h(t) \text{ continuous}} dt$$

This is independent of the choice of parameterization.

Now  $I = \int_0^1 h(t) dt$  can be defined via Riemann sums. Given a partition  $P$ ,  $h : [0, 1] \rightarrow X$  continuous.

$$\lim_{\text{mesh}(P) \rightarrow 0} \|S(h, P, \xi) - I\|_X = 0$$

where  $S(h, P, \xi) = \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1})$ ,  $P = \{x_0, x_1, \dots, x_n\}$ ,  $\xi_i \in [x_{i-1}, x_i]$ .

Note that  $h$  is uniformly continuous and  $\forall \varepsilon > 0$ ,  $\exists \delta > 0$  such that  $\text{mesh}(P_1) < \delta$ ,  $\text{mesh}(P_2) < \delta$  implies

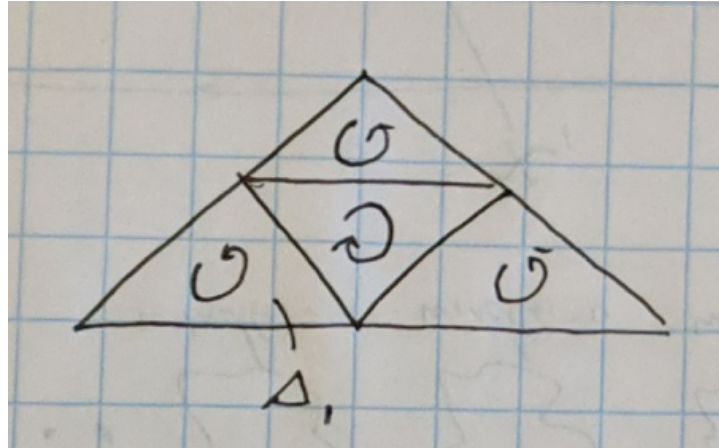
$$\|S(f, P_1, \xi^{(1)}) - S(f, P_2, \xi^{(2)})\| < \varepsilon$$

All usual properties of integrals hold.

- linear in integrand
- $\left\| \int_{\Gamma} f(z) dz \right\| \leq \int_{\Gamma} \|f(z)\| |dz| \leq (\text{length}(\Gamma)) \sup_{z \in \Gamma} \|f(z)\|.$

### Sketch of Proof (1) to (3)

To show:  $\int_{\Delta} f(z) dz = x_0 = 0$  by contradiction that  $x_0 \neq 0$ .

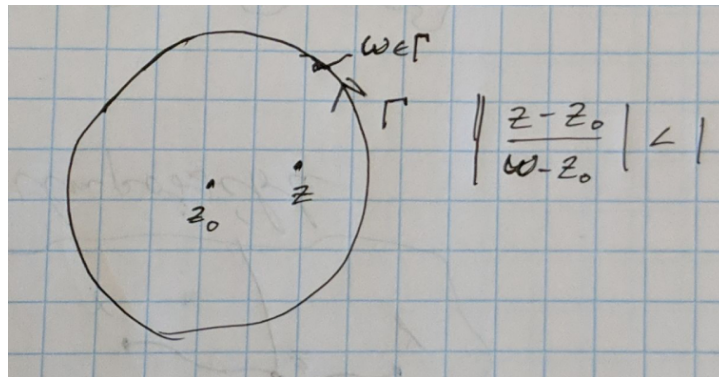


We have  $\left| \int_{\Delta_1} f dz \right| \geq \frac{\|x_0\|}{4}, \left| \int_{\Delta_n} f dz \right| \geq \frac{\|x_0\|}{4^n}.$

### Sketch of Proof (3) to (2)

$\int_{\Gamma} f dz = 0$  implies the Cauchy integral formula. Take

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\omega)}{\omega - z} d\omega$$



$$\frac{1}{\omega - z} = \frac{1}{(\omega - z_0) - (z - z_0)} = \frac{1}{\omega - z_0} \sum_{n=0}^{\infty} \left( \frac{z - z_0}{\omega - z_0} \right)^n$$

Therefore

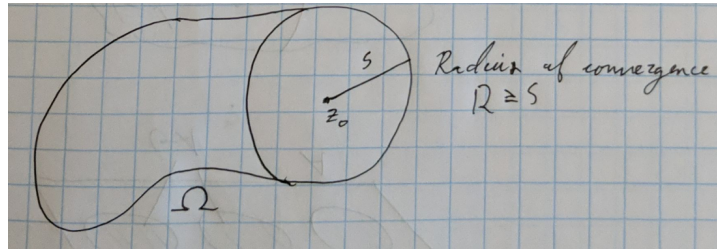
$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} f(\omega) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\omega - z)^{n+1}} d\omega = \sum_{n=0}^{\infty} (z - z_0)^n \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\omega)}{(\omega - z)^{n+1}} d\omega = \sum_{n=0}^{\infty} (z - z_0)^n a_n$$

with the sequence converging (in  $X$ ) on  $|z - z_0| < |\omega - z_0|$ .

- Radius of Convergence

$$R^{-1} = \limsup_{n \rightarrow \infty} \|a_n\|^{1/n}$$

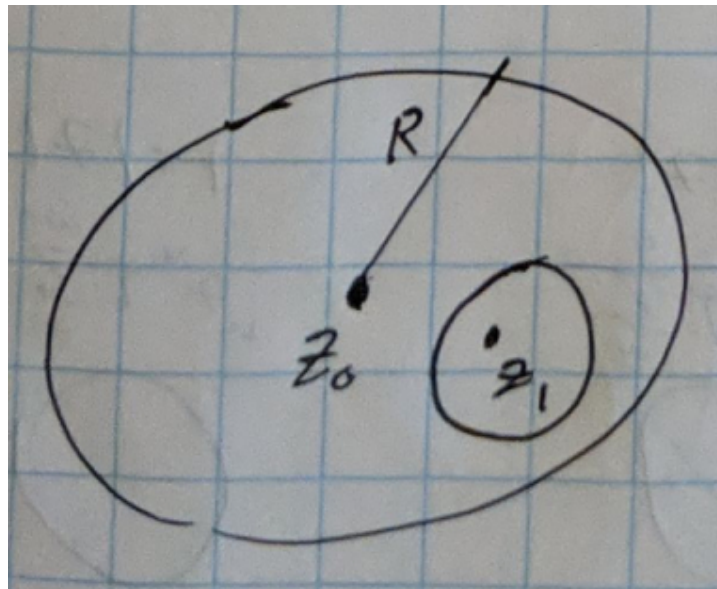
(Root Test:  $|z - z_0| < R$  convergence;  $|z - z_0| > R$  divergence)



### Sketch of Proof (2) to (1)

One can show that a function defined by convergent power series is differentiable,  $f(z) = \sum a_n(z - z_0)^n$ , then  $f'(z) = \sum a_n \cdot n(z - z_0)^{n-1}$ .

The radius of convergence is the same. This also implies that  $f$  is infinitely differentiable.



Take  $z - z_0 = (z - z_1) + (z_1 - z_0)$  and, by the binomial theorem,

$$f(z) = \sum_{k=0}^{\infty} (z - z_1)^k \left( \sum_{n=k}^{\infty} a_n \binom{n}{k} (z_1 - z_0)^{n-k} \right)$$

which converges for at least  $|z - z_1| < R - |z_1 - z_0|$ .

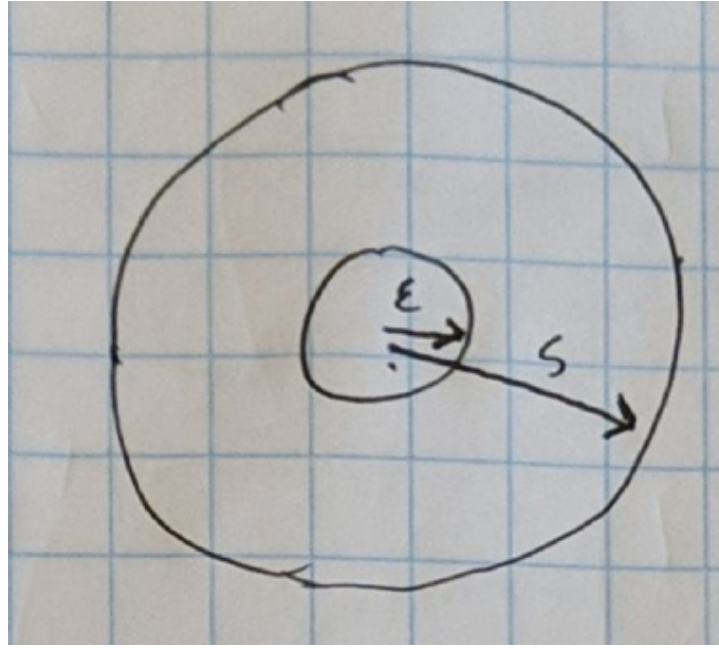
**October 14, 2024**

### Theorem

Let  $f : D_\varepsilon(z_0) \rightarrow X$  ( $D_\varepsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$ ) be holomorphic.

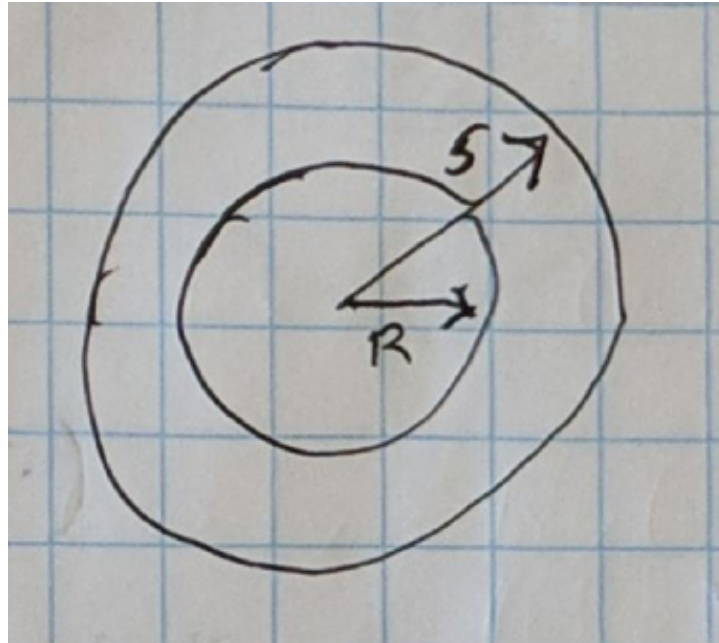
Then  $R = S$  where

1.  $R$  is the radius of convergence of  $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$  ( $R^{-1} = \limsup_{n \rightarrow \infty} \|a_n\|^{\frac{1}{n}}$ ).
2.  $S$  is the radius of the largest open disk  $D_S(z_0)$  such that there exists an analytic extension of  $f$  from  $D_\varepsilon(z_0)$  to  $D_S(z_0)$ .



### Proof

By definition,  $\sum_{n=0}^{\infty} a_n(z-z_0)^n$  converges for  $|z-z_0| < R$ . Then  $|z-z_0| < R$  if and only if  $\limsup_{n \rightarrow \infty} ||a_n(z-z_0)^n||^{\frac{1}{n}} < 1$  if and only if  $\sum_{n=0}^{\infty} a_n(z-z_0)^n$  converges. Therefore, it converges to a holomorphic function on  $R \leq S$ .  
If  $f(z)$  has an analytic extension to  $D_S(z_0)$ , see step (3)  $\implies$  (2) of previous theorem.



Then  $f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\omega)}{\omega \cdot z} d\omega = \sum_{n=0}^{\infty} a_n(z-z_0)^n$  converges for  $|z-z_0| < r < S$  with  $a_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\omega)}{(z-\omega)^{n+1}} d\omega$ .  
From this, we conclude  $R \geq S$ .

### Definition: (Weakly) Holomorphic Function

A function  $f : \Omega \rightarrow X$  ( $\Omega \subseteq \mathbb{C}$  open,  $X$  Banach) is called weakly holomorphic if  $\phi \circ f : \Omega \rightarrow \mathbb{C}$  is holomorphic,  $\forall \phi \in X^* = \mathcal{L}(X; \mathbb{C})$  bounded linear functionals.

A function  $f : \Omega \rightarrow \mathcal{L}(X, Y)$  ( $X, Y$  Banach) is weakly-operator holomorphic if  $h_{\phi, X} : \Omega \rightarrow \mathbb{C}$  is holomorphic for all  $\phi \in Y^*$ ,  $x \in X$  where  $h_{\phi, X}(z) = \phi(f(z)x)$ .

## Remarks

Obviously:  $f$  strongly holomorphic  $\implies f$  weakly holomorphic.

$$\left\| \frac{\phi(f(z+h)) - \phi(f(z))}{h} - \phi(f'(z)) \right\| \leq \|\phi\| \cdot \left\| \frac{f(z+h) - f(z)}{h} - f'(z) \right\|$$

For  $f : \Omega \rightarrow \mathcal{L}(X, Y)$ :  $f$  strongly holomorphic  $\implies f$  weakly holomorphic  $\implies f$  weakly operator holomorphic.

For  $x \in X$ ,  $\phi \in Y^*$ ,  $\Lambda_{x,\phi} : \mathcal{L}(X, Y) \ni A \mapsto \phi(Ax) \in \mathbb{C}$  and  $\Lambda_{x,\phi} \in (\mathcal{L}(X, Y))^*$ .

All the converses are also true.

## Theorem (Dunford)

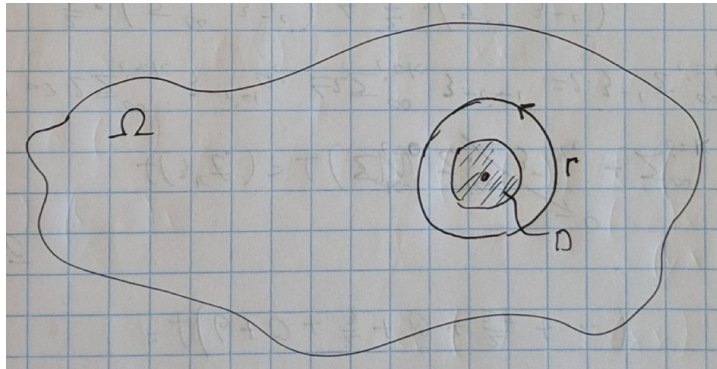
Take  $X$  Banach,  $\Omega \subseteq \mathbb{C}$  open.

If  $f : \Omega \rightarrow X$  is weakly holomorphic, then it is strongly holomorphic.

## Proof

We want to show that for any  $z_0 \in \Omega$ ,  $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$  exists in  $X$ .

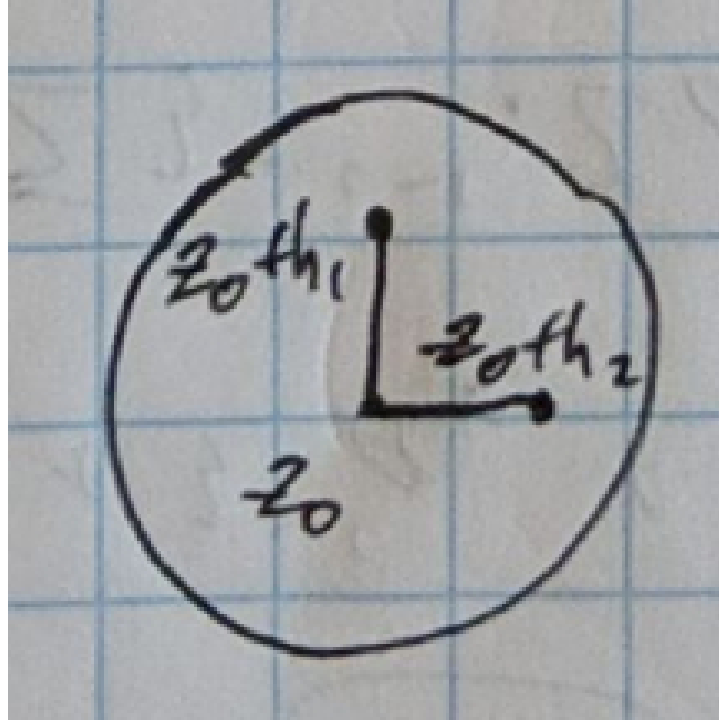
Choose  $\varepsilon > 0$  such that the disk  $D_\varepsilon(z_0)$  and circle  $C_{2\varepsilon}(z_0) = \Gamma$  are in  $\Omega$ .



For  $\phi \in X^*$ ,  $\phi(f(z))$  is holomorphic in  $\Omega$ .

$$\phi(f(z)) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\phi(f(\omega))}{z - \omega} d\omega, \quad z \in D$$

Apply this to  $z = z_0$ ,  $z = z_0 + h_1$  and  $z = z_0 + h_2$  with  $0 < |h_1| < \varepsilon$ ,  $0 < |h_2| < \varepsilon$ ,  $h_1 \neq h_2$ .



$$\begin{aligned}
A_{h_1, h_2} &= \frac{1}{h_1 - h_2} \left\{ \frac{f(z_0 + h_1) - f(z_0)}{h_1} - \frac{f(z_0 + h_2) - f(z_0)}{h_2} \right\} \\
\phi(A_{h_1, h_2}) &= \frac{1}{h_1 - h_2} \left\{ \frac{\phi(f(z_0 + h_1)) - \phi(f(z_0))}{h_1} - \frac{\phi(f(z_0 + h_2)) - \phi(f(z_0))}{h_2} \right\} \\
&= \frac{1}{2\pi i} \int_{\Gamma} \phi(f(\omega)) \frac{1}{h_1 - h_2} \left\{ \frac{1}{h_1} \left( \frac{1}{z_0 + h_1 - \omega} - \frac{1}{z_0 - \omega} \right) - \frac{1}{h_2} \left( \frac{1}{z_0 + h_2 - \omega} - \frac{1}{z_0 - \omega} \right) \right\} d\omega \\
&= \frac{1}{2\pi i} \int_{\Gamma} \phi(f(\omega)) \frac{1}{h_1 - h_2} \left\{ \frac{1}{(z_0 + h_1 - \omega)(z_0 - \omega)} - \frac{1}{(z_0 + h_2 - \omega)(z_0 - \omega)} \right\} d\omega \\
&= \frac{1}{2\pi i} \int_{\Gamma} \phi(f(\omega)) \frac{1}{(z_0 + h_1 - \omega)(z_0 + h_2 - \omega)(z_0 - \omega)} d\omega
\end{aligned}$$

Observe that the denominator is at least  $\varepsilon^3$ , therefore  $|\phi(A_{h_1, h_2})| \leq \frac{\varepsilon^3}{2\pi} \sup_{\omega \in \Gamma} \|f(\omega)\| \cdot \|\phi\|$  (so long as  $f$  continuous, which will be proven).

Therefore  $\forall \phi \in X^*$ ,

$$\sup_{\substack{0 < |h_1| < \varepsilon \\ 0 < |h_2| < \varepsilon \\ h_1 \neq h_2}} |\phi(A_{h_1, h_2})| < +\infty.$$

By the uniform boundedness principle, identify  $A_{h_1, h_2} \in X$  with  $X^{**} = \mathcal{L}(X^*, \mathbb{C})$ .

Then  $\sup_{h_1, h_2} \|A_{h_1, h_2}\| < +\infty$  and

$$\left\| \frac{f(z_0 + h_1) - f(z_0)}{h_1} - \frac{f(z_0 + h_2) - f(z_0)}{h_2} \right\| \leq C \cdot |h_1 - h_2|.$$

Now, for any sequence  $\{h_n\}_{n=3}^{\infty}$ ,  $0 < |h_n| < \varepsilon$ ,  $h_n \rightarrow 0$ ,

$$\frac{f(z_0 + h_n) - f(z_0)}{h_n}$$

is a cauchy sequence. Therefore  $\lim_{n \rightarrow \infty} \frac{f(z_0+h_n)-f(z_0)}{h_n}$  exists in  $X$  independent of choice of  $\{h_n\}$ . That is

$$\lim_{h \rightarrow 0} \frac{f(z_0+h)-f(z_0)}{h}$$

exists in  $X$ .

## Section 1.4: Spectrum and Resolvent

Consider a unital Banach algebra  $B$ .

### Definition: Spectrum

For  $b \in B$ , the spectrum of  $b$  in  $B$   $\sigma_B(b) = \{\lambda \in \mathbb{C} : \lambda e - b \text{ is not invertible in } B\}$ .

### Definition: Resolvent

The resolvent is a function  $R(b; \lambda) = (\lambda e - b)^{-1}$ .  $R(b, \cdot) : \mathbb{C} \setminus \sigma_B(b) \rightarrow B$ .  
 $\mathbb{C} \setminus \sigma_B(b)$  is the resolvent set.

### Theorem

1. The spectrum  $\sigma_B(b)$  is a non-empty, compact subset of  $\mathbb{C}$ .
2. The resolvent  $R(b, \lambda)$  is an analytic, Banach valued function on  $\mathbb{C} \setminus \sigma_B(b)$ .

### Proof of (a)

$\sigma_B(b)$  is bounded, because  $\lambda e - b$  is invertible for  $|\lambda| > \|b\|$ .

$$\lambda e - b = \lambda \left( e - \frac{1}{\lambda} b \right)$$

has  $\left\| \frac{1}{\lambda} b \right\| < 1$  for sufficiently large  $\lambda$ . Therefore,  $\sigma_B(b) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|b\|\}$ .

To show that  $\sigma_B(b)$  is closed, if  $\lambda \notin \sigma_B(b)$  then  $\forall \mu$  such that  $\|\lambda - \mu\| < \varepsilon$  we have that  $\mu \notin \sigma_B(b)$ .

$$\mu e - b = \lambda e - b + (\mu - \lambda)e = (\lambda e - b) \left\{ e + \underbrace{(\mu - \lambda)(\lambda e - b)^{-1}}_{\|\cdot\| < 1} \right\}$$

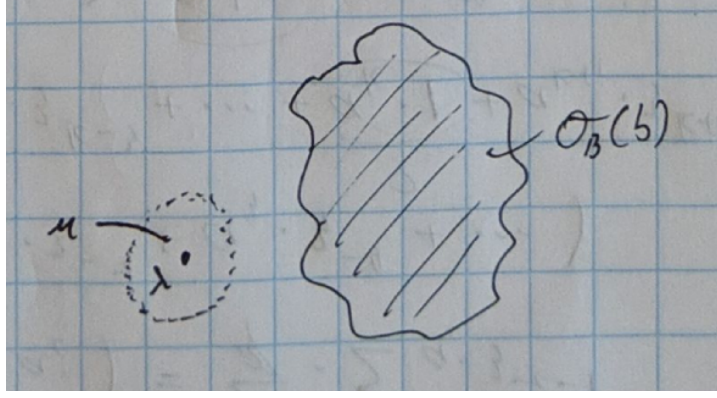
when  $|\mu - \lambda| < \frac{1}{\|(\lambda e - b)^{-1}\|}$ .

Therefore  $\mathbb{C} \setminus \sigma_B(b)$  is open.

### Proof of (b)

Take  $\lambda \notin \sigma_B(b)$





$$\begin{aligned}
 \frac{R(b, \mu) - R(b, \lambda)}{\mu - \lambda} &= \frac{1}{\mu - \lambda} ((\mu e - b)^{-1} - (\lambda e - b)^{-1}) \\
 &= \frac{1}{-\mu - \lambda} (\mu e - b)^{-1} \{(\lambda e - b) - (\mu e - b)\} (\lambda e - b)^{-1} \\
 &= -(\mu e - b)^{-1} (\lambda e - b)^{-1}
 \end{aligned}$$

Using continuity with  $GB \ni a \mapsto a^{-1} \in GB$  in the norm,  $-(\mu e - b)^{-1} (\lambda e - b)^{-1} \rightarrow -((\lambda e - b)^{-1})^2$  as  $\mu \rightarrow \lambda$ . Therefore  $R^1(b, \lambda) = -(R(b, \lambda))^2$  and  $R(b, \lambda)$  is analytic.

### Proof of non-empty in (a)

Take  $\sigma_B(b) \neq 0$ , otherwise  $R(b, \lambda)$  is analytic on  $\mathbb{C}$  and bounded

$$(\lambda e - b)^{-1} = \frac{1}{\lambda} \left( e - \frac{1}{\lambda} b \right)^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\lambda^{n+1}} b^n$$

We can estimate

$$\| \cdot \| \leq \frac{1}{|\lambda| \left( 1 - \frac{\|b\|}{|\lambda|} \right)} = \frac{1}{|\lambda| - \|b\|}$$

so  $\lim_{\lambda \rightarrow \infty} \|(\lambda e - b)^{-1}\| = 0$ .

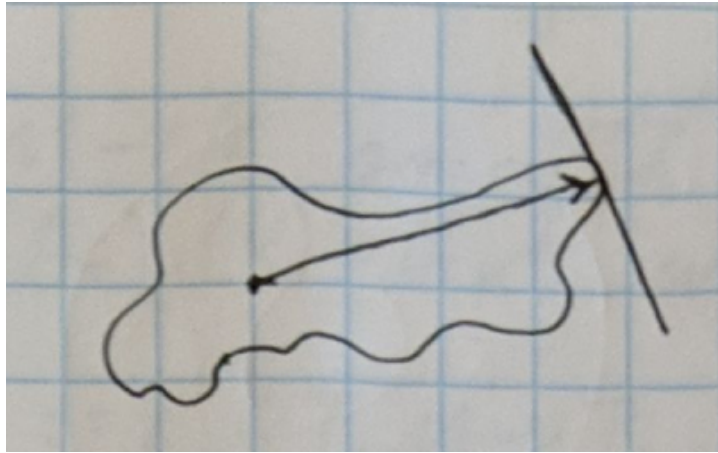
By Liouville's theorem, bounded and entire functions are constant. But we may also proceed by weak analyticity.

If  $\phi(R(b, \lambda))$  is analytic and bounded on  $\mathbb{C}$ ,  $\forall \phi \in B^*$ , it follows that  $\phi(R(b, \lambda)) \equiv 0$ ,  $\forall \lambda$ ,  $\forall \phi \in B^*$  and that  $R(b, \lambda) \equiv 0$  for any  $\lambda$  a contradiction.

### Definition: Spectral Radius

For  $b \in B$ , the spectral radius  $r(b) = \max\{|\lambda| : \lambda \in \sigma_B(b)\}$ .





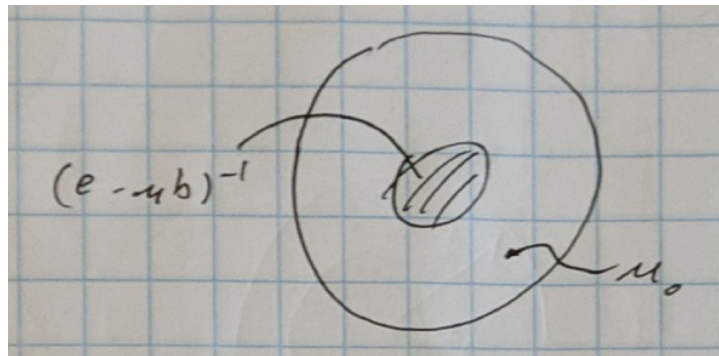
### Remark

Write  $\frac{1}{r(b)} = \min\{|\lambda|^{-1} : \lambda e - b \text{ is not invertible}\} = \min\{|\mu| : e - \mu b \text{ is not invertible}\}$  with  $\mu = \frac{1}{\lambda}$ .

$$\underbrace{(e - \mu b)^{-1}}_{\text{analytic in } |\mu| < \frac{1}{\|b\|}} = \sum_{n=0}^{\infty} \mu^n b^n$$

converges for  $|\mu| < \frac{1}{\|b\|}$ .

Then the radius of convergence  $R^{-1} = \limsup_{n \rightarrow \infty} \|b^n\|^{\frac{1}{n}}$  gives us that  $R$  is equal to the largest disk where  $(e - \mu b)^{-1}$  has an analytic extension. Therefore  $S = \frac{1}{r(b)}$ .



Suppose we have an analytic extension  $f(\mu)$  beyond  $S$ .

$$f(\mu)(e - \mu b) = (e - \mu b)f(\mu) = e$$

implies that and, if  $(e - \mu_0 b)$  not invertible,  $f(\mu_0)(e - \mu_0 b) = \dots = e$  a contradiction.

### Theorem

$$r(b) = \lim_{n \rightarrow \infty} \|b^n\|^{\frac{1}{n}} = \inf_{n \in \mathbb{N}} \|b^n\|^{\frac{1}{n}}$$

## Proof

To demonstrate existence, fix  $n_0 \in \mathbb{N}$ ,  $n = q \cdot n_0 + r$ ,  $0 \leq r < n_0$ .

$$\begin{aligned} ||b^n|| &\leq ||b^{n_0}||^q \cdot ||b||^r \\ ||b^n||^{\frac{1}{n}} &\leq ||b^{n_0}||^{\frac{q}{n}} \cdot ||b||^{\frac{r}{n}} \\ \limsup_{n \rightarrow \infty} ||b^n||^{\frac{1}{n}} &\leq ||b^{n_0}||^{\frac{1}{n_0}} \cdot 1 \end{aligned}$$

Since  $1 = \frac{q}{n} \cdot n_0 + \frac{r}{n}$ . Take  $n \rightarrow \infty$ . Write

$$\limsup_{n \rightarrow \infty} ||b^n||^{\frac{1}{n}} \leq \inf_{n_0 \in \mathbb{N}} ||b^{n_0}||^{\frac{1}{n_0}} \leq \liminf_{n \rightarrow \infty} ||b^n||^{\frac{1}{n}}$$

**October 16, 2024**

## Note: Closed Subalgebras

Assume  $A$  is a closed subalgebra of  $B$  ( $e \in A \subseteq B$ ).

Take  $b \in A \subseteq B$ .

Obviously,  $b - \lambda e$  being invertible in  $A$  implies  $b - \lambda e$  is invertible in  $B$ . We also have

$$\mathbb{C} \setminus \text{sp}_A(b) \subseteq \mathbb{C} \setminus \text{sp}_B(b)$$

(confer.  $GA \subseteq GB$  with  $\partial GA = \partial(A \cap GB)$ ) and, equivalently,

$$\text{sp}_B(b) \subseteq \text{sp}_A(b).$$

One can show similarly that

$$\begin{aligned} \partial(\mathbb{C} \setminus \text{sp}_A(b)) &\subseteq \partial(\mathbb{C} \setminus \text{sp}_B(b)) \\ &= \\ \partial \text{sp}_A(b) &\subseteq \partial \text{sp}_B(b) \end{aligned}$$

## Proposition

1.  $\mathbb{C} \setminus \text{sp}_A(b)$  is the union of some components of  $\mathbb{C} \setminus \text{sp}_B(b)$ .
2.  $\text{sp}_A(b) = \text{sp}_B(b) \cup \bigcup_{\omega} H_{\omega}$  where  $H_{\omega}$  are some components of  $\mathbb{C} \setminus \text{sp}_B(b)$ .

## Example 1

Suppose  $\text{sp}_B(b)$  looks like

IMAGE 1

Now  $\text{sp}_A(b)$  can only be one of the 4 possibilities.

IMAGE 2

## Example 2

$B = C(\mathbb{T})$ ,  $A = C_+(\mathbb{T}) \simeq A(\mathbb{D})$ ,  $\chi_1(t) = t$ ,  $\text{sp}_B \chi_1 = \mathbb{T}$ .

IMAGE 3

## Theorem: Spectral Mapping Theorem (Simple Version)

For a polynomial  $p(z) = \sum_{n=0}^N p_n t^n$  we define  $p(b) = \sum_{n=0}^N p_n b^n$  for  $b \in B$  where  $b^0 = e$ .

Let  $p$  be a polynomial and  $b \in B$  with  $B$  a unital Banach algebra, then  $\text{sp}(p(b)) = p(\text{sp}(b)) := \{p(z) : z \in \text{sp}(b)\}$ .

### Proof

For  $\lambda \in \mathbb{C}$ , consider  $q(z) = p(z) - \lambda = c \prod_{i=1}^N (z - \gamma_i)$ .

Now,  $q(b) = p(b) - \lambda e = c \prod_{i=1}^N (b - \gamma_i e)$ . It follows that

$$\begin{aligned}
 \lambda \notin \text{sp}(p(b)) &\iff p(b) - \lambda e \text{ is invertible.} \\
 &\quad \text{a commuting product} \\
 &\iff \underbrace{\prod_{i=1}^N (b - \gamma_i e)}_{\text{a commuting product}} \text{ is invertible.} \\
 &\iff \forall i, b - \gamma_i e \text{ is invertible.} \\
 &\iff \forall i, \gamma_i \notin \text{sp}(b) \\
 &\iff \forall z \in \text{sp}(b), q(z) = c \prod_{i=1}^N (z - \gamma_i) \neq 0 \\
 &\iff \forall z \in \text{sp}(b), p(z) \neq \lambda \\
 &\iff \lambda \notin p(\text{sp}(b))
 \end{aligned}$$

### Applications

If  $p(b) = 0$ , then  $\text{sp}(b) \subseteq \{z \in \mathbb{C} : p(z) = 0\}$ , because

$$\{0\} = \text{sp } 0 = \text{sp } p(b) \stackrel{\text{SMT}}{=} p(\text{sp } b).$$

It follows that if  $b$  is nilpotent, such that  $b^n = 0$  for some  $n$  ( $p(z) = z^n$ ), then  $\text{sp}(b) = \{0\}$ .

If  $b$  is idempotent, such that  $b^2 = b$  ( $p(z) = z^2 - z$ ), then  $\text{sp}(b) \subseteq \{0, 1\}$ .

If  $b$  is unipotent (or flip), such that  $b^2 = e$ , then  $\text{sp}(b) = \{\pm 1\}$ .

## Section 1.5: Riesz Functional Calculus

### Question:

Can one define  $f(b)$  for  $b \in B$  a unital Banach algebra for more general functions  $f$ ?

### Definition: Set of Functions Holomorphic on the Spectrum

For a unital Banach algebra  $B$  and  $b \in B$ , let  $A[\text{sp}(b)]$  stand for the set of all functions  $f$  which are holomorphic on some open neighborhood  $U$  of  $\text{sp}(b)$ .

IMAGE 4

## Lemma

Let  $f \in A[\text{sp}(b)]$ , i.e.  $f : U \rightarrow \mathbb{C}$  holomorphic. Then there exists an open set  $W$  with (piece-)smooth boundary such that

$$\text{sp}(f) \subseteq W \subseteq \overline{W} \subseteq U$$

(i.e.  $\partial \overline{W} \subseteq U \setminus \text{sp}(b)$ ) and

$$\frac{1}{2\pi} \int_{\partial W} \frac{d\omega}{\omega - z} = \begin{cases} 1 & z \in \text{sp}(b) \\ 0 & z \notin U \end{cases}.$$

## Example

IMAGE 5

IMAGE 6

- Proof

IMAGE 7

SQUARES

## Definition:

Using the lemma, we define for  $f \in A[\text{sp}(b)]$

$$f(b) := \frac{1}{2\pi i} \int_{\partial W} f(\lambda)(\lambda e - b)^{-1} d\lambda$$

(where  $\text{sp}(b) \subseteq W \subseteq \overline{W} \subseteq U$ ).

One can show that this is independent of choice of  $W$  (and also of  $U$ ).

Note  $f(\lambda)(\lambda e - b)^{-1}$  is holomorphic on  $U \setminus \text{sp}(b)$ .

IMAGE 8

## Remark

$f_1, f_2 \in A[\text{sp}(b)]$  implies  $f_1 + f_2 \in A[\text{sp}(b)]$  and  $(f_1 + f_2)(b) = f_1(b) + f_2(b)$ .

## Proposition

For a polynomial  $f(z) = p(z) = \sum p_i z^i$ , we get  $f(b) = p(b) = \sum p_i b^i$ .

## Proof

$$\frac{1}{2\pi i} \int_{\partial W} (\lambda e - b)^{-1} d\lambda = \frac{1}{2\pi i} \int_{|\lambda|=R} (\lambda e - b)^{-1} d\lambda = \frac{1}{2\pi i} \int_{|\lambda|=R} \sum_{n=0}^{\infty} \frac{b^n}{\lambda^{n+1}} d\lambda = e$$

IMAGE 9

Therefore,  $p(b) = \frac{1}{2\pi i} \int_{\partial W} p(b)(\lambda e - b)^{-1} d\lambda$ , and

$$\begin{aligned} f(b) - p(b) &= \frac{1}{2\pi i} \int_{\partial W} (f(\lambda)e - p(b))(\lambda e - b)^{-1} d\lambda \\ &= \frac{1}{2\pi i} \int_{\partial W} \sum_{n=0}^N \underbrace{(\lambda^n e - b^n)}_{(\lambda^{n-1}e + \dots + b^{n-1})(\lambda e - b)} (\lambda e - b)^{-1} d\lambda \\ &= \frac{1}{2\pi i} \int_{\partial W} \text{"polynomial in } \lambda, b \text{" } d\lambda = 0 \end{aligned}$$

### Proposition

If  $f_1, f_2 \in A[\text{sp}(b)]$ , then  $f_1 f_2 \in A[\text{sp}(b)]$ .

$$(f_1 f_2)(b) = f_1(b) \cdot f_2(b)$$

### Proof

IMAGE 10

We assume  $\partial V$  is inside  $\partial W$ .

$$\begin{aligned} f_1(b) &= \frac{1}{2\pi i} \int_{\partial W} f_1(\lambda)(\lambda e - b)^{-1} d\lambda \\ f_2(b) &= \frac{1}{2\pi i} \int_{\partial V} f_2(\xi)(\xi e - b)^{-1} d\xi \end{aligned}$$

Then

$$f_1(b)f_2(b) = \frac{1}{(2\pi i)^2} \int_{\partial W} \int_{\partial V} f_1(\lambda)f_2(\xi)(\lambda e - b)^{-1}(\xi e - b)^{-1} d\xi d\lambda$$

Recall that

$$(\lambda e - b)^{-1}(\xi e - b)^{-1} = (\lambda e - b)^{-1} \left[ \frac{(\lambda e - b) - (\xi e - b)}{\lambda - \xi} \right] (\xi e - b)^{-1} = \frac{(\xi e - b)^{-1} - (\lambda e - b)^{-1}}{\lambda - \xi}$$

Therefore

$$\begin{aligned} f_1(b)f_2(b) &= \frac{1}{(2\pi i)^2} \int_{\partial V} \int_{\partial W} f_1(\lambda)f_2(\xi)(\xi e - b)^{-1} \frac{1}{\lambda - \xi} d\lambda d\xi - \frac{1}{(2\pi i)^2} \int_{\partial W} \int_{\partial V} f_1(\lambda)f_2(\xi)(\lambda e - b)^{-1} \frac{1}{\lambda - \xi} d\xi d\lambda \\ &= \frac{1}{(2\pi i)^2} \int_{\partial V} f_2(\xi)(\xi e - b)^{-1} \underbrace{\int_{\partial W} \frac{f_1(\lambda)}{\lambda - \xi} d\lambda}_{\equiv f_2(\xi)} d\xi - \frac{1}{(2\pi i)^2} \int_{\partial W} f_1(\lambda)(\lambda e - b)^{-1} \underbrace{\int_{\partial V} \frac{f_2(\xi)}{\lambda - \xi} d\xi}_{\equiv 0} d\lambda \\ &= \frac{1}{2\pi i} \int_{\partial V} f_2(\xi)f_1(\xi)(\xi e - b)^{-1} d\xi \\ &= (f_1 f_2)(b) \end{aligned}$$

October 21, 2024

## Recall

$f \in A[\text{sp } b]$ ,  $f : U \rightarrow \mathbb{C}$ ,  $\text{sp}(b) \subseteq U$  open. Define

$$(1) \quad f(b) = \frac{1}{2\pi i} \int_{\partial W} \underbrace{f(\lambda)(\lambda e - b)^{-1}}_{\text{analytic in } U \setminus \text{sp } b} dz$$

IMAGE 1

with  $\text{sp } b \subseteq W \subseteq \overline{W} \subseteq U$  and  $\partial W$  piecewise smooth.

From the above lemma, applied to  $W$ , we get  $W_0$  such that  $\text{sp } b \subseteq W_0 \subseteq \overline{W}_0 \subseteq W \subseteq \overline{W} \subseteq U$ . Then

$$(2) \quad \frac{1}{2\pi i} \int_{\partial W_0} f(\lambda)(\lambda e - b)^{-1} dz$$

IMAGE 2

with  $V = W \setminus W_0$ ,  $\partial V = \partial W \cup \partial W_0$ .

$$(1) - (2) = \frac{1}{2\pi i} \int_{\partial V} \underbrace{f(\lambda)(\lambda e - b)^{-1}}_{\text{holomorphic on } V} dz = 0$$

and  $V \subseteq \overline{V} \subseteq U \setminus \text{sp}(b)$ .

## Results

$$f_1, f_2 \in A[\text{sp } b] \implies f_1 + f_2 \in A[\text{sp } b]$$

$$f_1(b) + f_2(b) = (f_1 + f_2)(b).$$

For  $f$  polynomial,  $\sum_{n=0}^N f_n t^n$ ,  $f(b) = \sum_{n=0}^N f_n b^n$ .

## Proposition

$$f_1(b)f_2(b) = (f_1 f_2)(b).$$

## Theorem: Spectral Mapping Theorem

Let  $b \in B$  and  $f \in A[\text{sp } b]$ . Then  $\text{sp}(f(b)) = f(\text{sp } b) := \{f(z) : z \in \text{sp } b\}$ .

## Proof

1. take  $\mu \notin f(\text{sp } b)$ .

Then  $\mu \notin f(z)$ ,  $\forall z \in \text{sp } b$  and  $\mu - f(z) \neq 0$ .

IMAGE 3

Therefore, there exist an open  $U_1 \ni \text{sp}(b)$ ,  $U_1 \subseteq U$ , such that  $\mu - f(z) \neq 0$ ,  $\forall z \in U_1$ .

Define  $g(z) = \frac{1}{\mu - f(z)}$  holomorphic on  $U_1$ , and

$$g(z) \cdot (\mu - f(z)) = 1 \implies g(b) \cdot (\mu e - f(b)) = e$$

by the previous proposition and the polynomial result. So  $\mu e - f(b)$  is invertible, and  $\mu \notin \text{sp}(f(b))$ .

- Remark

$$(\mu e - f(b))^{-1} = \frac{1}{2\pi i} \int_{\partial W_1} \frac{1}{\mu - f(z)} (ze - b)^{-1} dz$$

for  $\text{sp } b \subseteq W_1 \subseteq \overline{W_1} \subseteq U_1$ .

- take  $\mu \notin \text{sp}(f(b))$  and, for contradiction, assume  $\mu \in f(\text{sp } b)$ .

Then  $\mu e - f(b)$  is invertible,  $\mu = f(\lambda)$  for some  $\lambda \in \text{sp } b$ .

- Idea

$$\mu e - f(b) = f(\lambda)e - f(b) = (\lambda e - b) \cdot g_\lambda(b)$$

We define

$$g_\lambda(z) = \begin{cases} \frac{f(\lambda)e - f(z)}{\lambda - z} & z \in U \supseteq \text{sp}(b) \\ f'(\lambda) & z = \lambda \end{cases}$$

such that  $g_\lambda(z)$  is holomorphic on  $U$ . Therefore  $g_\lambda(b) \in B$ ,

$$(\lambda - z)g_\lambda(z) = f(\lambda) - f(z), \quad \forall z \in U$$

and  $(\lambda e - b)g_\lambda(b) = f(\lambda)e - f(b) = g_\lambda(b)(\lambda e - b)$ . Since this is invertible,  $(\lambda e - b)$  is left and right invertible.

## Remark

$$g_\lambda(b) = \frac{1}{2\pi i} \int_{\partial W} \frac{f(\lambda) - f(z)}{\lambda - z} (ze - b)^{-1} dz$$

## Theorem: Composition of Functions

Let  $b \in B$  unital,  $f \in A[\text{sp } b]$ , and  $g \in A[\text{sp}(f(b))] = A[f(\text{sp } b)]$ .

Then  $h = g \circ f \in A[\text{sp } b]$  and  $h(b) = g(f(b))$ .

## Remark

IMAGE 4

$f$  is an open mapping and maps  $U$  to the open set  $V \supseteq \text{sp}(f(b))$ .

## Applications

- Exponentials

$$\exp(b) = \sum_{n=0}^{\infty} \frac{b^n}{n!} = \frac{1}{2\pi i} \int_{|z|=R} e^z (ze - b)^{-1} dz$$

- Logarithms

$\log b$ ,  $b \in B$  under the assumption that

- $0 \notin \text{sp } b$
- There exists a path connecting 0 to  $\infty$  in  $\mathbb{C} \setminus \text{sp } b$ .

This gives us that  $\log z$  is analytic on  $U \supseteq \text{sp } b$ .

IMAGE 5

$\mathbb{C} \setminus \text{path}$  is simply connected, so there exists an analytic  $\log z$  on  $\mathbb{C} \setminus \text{path}$ .

- if  $\log b$  is well-defined, then  $\exp(\log b) = b$  (via composition)
- likewise, one can define powers  $f(z) = z^\alpha$  ( $\alpha \in \mathbb{C}$ )

### Application: Spectral Idempotents (Riesz Idempotents)

$p$  is idempotent if  $p^2 = p$ .

Assume that  $b \in B$  and that  $\text{sp } b$  is not connected.

$$\text{sp } b = \sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_N$$

with  $\sigma_i$  closed and disjoint subsets of  $\text{sp } b$ .

IMAGE 6

Now let  $U_1, \dots, U_n$  be open neighborhoods of  $\sigma_1, \dots, \sigma_n$  which are themselves disjoint.

IMAGE 7

Write  $U = U_1 \cup \cdots \cup U_n \supseteq \text{sp } b$ , and consider

$$\chi_i(x) = \begin{cases} 1 & x \in U_i \\ 0 & x \in U_j, j \neq i \end{cases}$$

Then  $\chi_i$  is analytic on  $U \supseteq \text{sp}(b)$ .

Put  $p_i = \chi_i(b)$  the spectral or Riesz idempotents.

### Properties / Remarks

- $p_i^2 = p_i$  because  $(\chi_i)^2 = \chi_i$ .
- $e = p_1 + \cdots + p_N$ , mutually orthogonal such that  $p_i p_j = 0$ ,  $\forall i \neq j$ , because  $\chi_1 + \cdots + \chi_N = 1$  and  $\chi_i \chi_j = 0$ .
- $p_i b = b p_i$ , because  $\chi_i f = f \chi_i$  for  $f(z) = z$ .



- $p_i = \frac{1}{2\pi i} \int_{\partial W} \chi_i(z)(ze - b)^{-1} dz$  where  $\text{sp } b \subseteq W \subseteq \overline{W} \subseteq U$ .

IMAGE 8

$W_i = W \cap U_i$ ,  $W_1 \cup \dots \cup W_N = W$ . Therefore

$$p_i = \frac{1}{2\pi i} \sum_{j \neq i}^N \int_{\partial W_j} \chi_i(z)(ze - b)^{-1} dz = 0, \quad i \neq j$$

Then  $p_i = \frac{1}{2\pi i} \int_{\partial W_i} (ze - b)^{-1} dz$ .

IMAGE 9

- Write

$$b = (p_1 + p_2 + \dots + p_N)b(p_1 + p_2 + \dots + p_N) = p_1bp_1 + p_2bp_2 + \dots + p_Nbp_N$$

since  $p_i bp_j = bp_i p_j = 0$ .

- For an idempotent  $p \neq 0$ ,

$$B_p = \{pap : a \in B\}$$

and, therefore,  $B_p$  has a unit element  $p$ .

### Lemma

Assume  $b \in B$  with Riesz idempotents  $p_1, \dots, p_N \neq 0$ .

Then  $b$  is invertible if and only if  $p_i bp_i$  is invertible in  $B_{p_i}$  for all  $i$ .

### Proof

$b^{-1} = c$ ,  $bc = e$ , then

$$\begin{aligned} (p_1 + \dots + p_N)b(p_1 + \dots + p_N)c &= e \\ \sum p_i b(p_i p_i)c &= e \\ (p_i b_i)(p_i c p_i) &= p_i \end{aligned}$$

Suppose  $p_i bp_i$  invertible in  $B_{p_i}$ . Then  $p_i bp_i c = p_i$ ,  $c_i = p_i c p_i$  and  $b^{-1} = c = \sum_{i=1}^N p_i c_i p_i$ .

### Remark

$$B = \begin{pmatrix} B_1 & & 0 \\ & B_2 & \\ 0 & & \ddots & \\ & & & B_N \end{pmatrix} \quad P_1 = \begin{pmatrix} I & & \\ & 0 & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \quad P_2 = \begin{pmatrix} 0 & & \\ & I & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

Therefore  $B$  invertible if and only if  $B_i$  are invertible.  $B_i \cong P_i B P_i$ .

$$\begin{pmatrix} 0 & & \\ & B_i & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

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### Lemma

Let  $b \in B$  and  $p_1, \dots, p_n \in B$  satisfying  $p_i^2 = p_i$ ,  $p_i p_j = 0$  ( $i \neq j$ ),  $p_1 + \dots + p_n = e$ ,  $b p_i = p_i b$ . Then  $b$  is invertible in  $B$  if and only if for each  $i$ ,  $p_i b p_i$  are invertible in  $B_{p_i}$  and

$$\text{sp}_B(b) = \bigcup_{i=1}^N \text{sp}_{B_{p_i}}(p_i b p_i)$$

where  $B_{p_i} = \{p_i a p_i : a \in B \text{ is a unital Banach algebra with unit } p_i\}$ .

### Theorem

Let  $p_1, \dots, p_N$  be spectral idempotents of  $b$  with respect to  $\text{sp}(b) = \sigma_1 \cup \sigma_2 \cup \dots \cup \sigma_N$  (closed and disjoint).

If  $\sigma_1, \dots, \sigma_N \neq \emptyset$ , then  $p_1, \dots, p_N \neq 0$  and  $\text{sp}_{B_{p_i}}(p_i b p_i) = \sigma_i$ .

Note: if  $\sigma_i = \emptyset$  then

$$p_i = \chi_{U_i}(b) = \frac{1}{2\pi i} \int_{\partial W_i} \underbrace{|ze - b|}_{\text{analytic}}^{-1} dz.$$

### Proof

Without loss of generality, we may assume  $p_1, \dots, p_M \neq 0$  ( $M \geq 1$ ) and  $p_{M+1} = \dots = p_N = 0$ .

Then by the above lemma  $p_1 + \dots + p_M = e$  and

$$\text{sp}_B(b) = \bigcup_{i=1}^M \text{sp}_{B_{p_i}}(p_i b p_i)$$

Assuming  $\text{sp}_{B_{p_i}}(p_i b p_i) = \sigma_i$  is proven for  $j = 1, \dots, M$ , then

$$\text{sp}_B(b) = \bigcup_{i=1}^M \sigma_i = \bigcup_{i=1}^N \sigma_i$$

and therefore that  $M = N$ .

To prove that  $\text{sp}_{B_{p_i}}(p_i b p_i) = \sigma_i$  for each  $p_i \neq 0$ ,

$$\text{sp}_{B_{p_i}}(p_i b p_i) = \{\lambda \in \mathbb{C} : p_i(b - \lambda)p_i + e - p_i \text{ not invertible in } B\}$$

For fixed  $\lambda$ ,  $f_\lambda(z) = \chi_i(z)(z - \lambda)\chi_i(z) + (1 - \chi_i(z))$  is analytic in a neighborhood of  $\text{sp}(b)$ .

$$f_\lambda(b) = p_i(b - \lambda e)p_i + (1 - p_i)$$

Then  $\lambda \in \text{sp}_{B_{p_i}}(p_i b p_i)$  if and only if  $f_\lambda(b)$  is not invertible in  $B$ .

Equivalently that  $0 \in \text{sp}(f_\lambda(b))$  or, by spectral mapping theorem,  $0 \in f_\lambda(\text{sp } b)$ .

This is further equivalent to there existing some  $\xi \in \text{sp } b : 0 = f_\lambda(\xi)$

$$f_\lambda(z) = \begin{cases} 1 & z \in \sigma_j \subseteq U_j, i \neq j \\ z - \lambda & z \in \sigma_i \end{cases}$$

That is, if  $\xi \in \text{sp } b : \xi \in \sigma_i$  and  $\xi = \lambda$  or, simply,  $\lambda \in \sigma_i$ .

## Chapter 2: Commutative Banach Algebras

### Section 2.1: Homomorphisms, Ideals and Quotient Algebras.

$B$  need not be commutative.

#### Definition: Banach Algebra Homomorphisms

$\phi : A \rightarrow B$  is a Banach algebra homomorphism if it is linear, multiplicative and bounded.

#### Definition: Banach Algebra Ideal

A (two-sided) ideal  $J$  of a Banach algebra is a linear subspace  $J \subseteq A$  such that  $\forall a \in A, \forall j \in J, aj, ja \in J$ .

#### Remark

If  $\phi : A \rightarrow B$  is a Banach algebra homomorphism then  $\ker \phi$  is a closed two-sided ideal of  $A$ .

#### Proof

Put  $J \in \ker \phi, a \in A, j \in J$ . Then  $\phi(j) = 0, \phi(aj) = \phi(a)\phi(j) = 0 = \phi(j)\phi(a) = \phi(ja)$  and  $aj, ja \in J$ .

#### Definition: Quotient Algebra

If  $J$  is a closed, two-sided ideal of  $A$  ( $J \neq A$ ), then  $A/J$  is a Banach algebra.  $A/J$  is a Banach algebra  $[a] = a + J$ .  $A/J$  is a vector space, a normed space ( $J$  closed) with  $||[a]_J|| = \inf_{j \in J} ||a + j||$ , and a Banach space because  $A$  is complete.

$$[a_1] + [a_2] = [a_1 + a_2] \text{ and } [a_1] \cdot [a_2] = [a_1 \cdot a_2]$$

$$(a_1 + j_1)(a_2 + j_2) = a_1 a_2 + \underbrace{a_1 j_2 + a_2 j_1 + j_1 j_2}_{\in J}$$

#### Definition: Quotient Map

Take  $\pi : A \rightarrow A/J$  by  $a \mapsto [a]$ .

This is a Banach algebra homomorphism which is surjective with  $\ker \pi = J$ .

#### Proposition

Let  $\phi : A \rightarrow B$  be a Banach algebra homomorphism and  $J \subseteq \ker \phi$  a closed, two-sided ideal of  $A$ .

Then there exists a Banach algebra homomorphism  $\phi^J : A/J \rightarrow B$  such that  $\phi = \phi^J \circ \pi$

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ & \searrow \pi & \nearrow \phi^J \\ & A/J & \end{array}$$

Write  $\phi^J([a]_J) = \phi(a)$ , and  $[a_1] = [a_2]$  implies  $a_1 - a_2 \in J \subseteq \ker \phi$  and subsequently that

$$\phi(a_1) = \phi(a_2).$$

#### Remark

$J = \{0\}$  and  $J = A$  are always closed, two-sided ideals of  $A$ .

## Examples

- $A = \mathbb{C}^{n \times n}$ . Only ideals are  $\{0\}$  and  $A$ .
- $A = L(X)$  (continuous operators) for  $X$  a Banach space. Then at least  $\{0\}$ ,  $K(X)$  (compact operators), and  $A$  are ideals.
- $X$  a separable hilbert space. Only  $\{0\}$ ,  $K(X)$  and  $A$ .
- $A = \mathbb{C}_{\text{upper}}^{n \times n}$  upper triangular matrices. Then there are many (one sided) ideals for  $n = 2$ .
- $A = C(X)$  for  $X$  compact Hausdorf spaces. Then every closed set  $E \subseteq X$  generates a closed ideal

$$J_E = \{f \in C(X) : f|_E \equiv 0\}$$

In particular,  $E = \{x_0\}$ ,  $J_{x_0} = \{f \in C(X) : f(x_0) = 0\}$ ,  $\dim(A/J_{x_0}) = 1$  implies  $A/J_{x_0} \cong \mathbb{C}$ .

## Remark

Every closed (2-sided) ideal is a closed subalgebra of  $A$  but not vice versa.

For a set  $S \subseteq A$ , let  $J = \text{clos id}_A(S)$  be the smallest closed 2-sided ideal containing  $S$  (i.e. the ideal generated by  $S$  or the intersection of all ideals containing  $S$ ). One can show that

$$J = \text{clos}_A \left\{ \sum_{i=1}^N a_i j_i b_i : a_i, b_i \in A, j_i \in S \right\}$$

## Section 2.2: Maximal Ideals and Multiplicative Linear Functionals

From now on,  $B$  is a unital, commutative Banach algebra.

### Definition: Multiplicative Linear Functional

A multiplicative linear functional on  $B$  is a linear map  $\phi : B \rightarrow \mathbb{C}$  such that  $\phi(ab) = \phi(a)\phi(b)$  ( $\phi \neq 0$ ).

### Proposition

A multiplicative linear functional on  $B$  is bounded. In fact  $\phi \in B^*$ ,  $\|\phi\| = 1$ ,  $\phi(e) = 1$ .

### Proof

$\phi \neq 0$  means that there exists  $a \in B$  such that  $\phi(a) \neq 0$ .

Then  $\phi(e)\phi(a) = \phi(ea) = \phi(a)$  so  $\phi(e) = 1$  and consequently that  $\|\phi\| \geq 1$ .

If  $|\phi(a)| \leq \|a\|$ , then  $\|\phi\| \leq 1$ . If this were not the case,

$$|\phi(a)| > \|a\| \iff \left\| \frac{a}{\phi(a)} \right\| < 1$$

and  $e - \frac{a}{\phi(a)}$  is invertible. Call the inverse  $b$ . Then

$$b \left( e - \frac{a}{\phi(a)} \right) = e \implies \underbrace{\phi(b) \left( e - \frac{a}{\phi(a)} \right)}_{=\phi(e) - \frac{1}{\phi(a)}\phi(a)=0} = \phi(e) = 1$$

which is a clear contradiction.

### **Definition: Maximal Ideal**

A (two-sided) ideal  $I$  of  $B$  is called maximal if

- $I \neq B$  ( $I$  is a proper ideal)
- if  $J$  is another ideal of  $B$  such that  $I \subseteq J \subseteq B$ , then either  $I = J$  or  $J = B$ .

### **Proposition**

A maximal ideal  $I$  is closed and  $B/I$  is a field.

### **Proof**

We have  $I \subseteq \bar{I} \subseteq B$  with  $\bar{I}$  an ideal. Since  $I$  is maximal, either  $I = \bar{I}$  or  $\bar{I} = B$ . But then  $e \in \bar{I}$ , and there exists  $a \in I$  such that  $\|a - e\| < 1$ . Then  $a = e + (a - e)$  is invertible and for each  $b \in B$ ,  $b = ba^{-1}a \in I$  and  $I = B$  a contradiction.