Random Matrix Theory

April 1, 2025

Preliminaries

Let ξ_{ij}, η_{ij} be normal random variables (i.e. Gaussian, mean 0, variance 1).
e.g. $\mathbb{P}(\xi_{11} < s) = \int_{-\infty}^{s} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$.
$\int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$ is the variance.
$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is the Probability Density Function (PDF).
$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$ is the probability measure on our probability space (i.e. totally finite measure space).
We build matrices

$$\begin{bmatrix} \xi_{11} & \frac{\xi_{12}+i\eta_{12}}{\sqrt{2}} & \frac{\xi_{13}+i\eta_{13}}{\sqrt{2}} & \cdots \\ \frac{\xi_{21}+i\eta_{21}}{\sqrt{2}} & \xi_{22} & \frac{\xi_{22}+i\eta_{22}}{\sqrt{2}} & \\ \frac{\xi_{31}+i\eta_{31}}{\sqrt{2}} & \frac{\xi_{32}+i\eta_{32}}{\sqrt{2}} & \xi_{33} & \\ \vdots & & & \ddots \end{bmatrix}$$

Computing Random Matrices in Matlab

Gassuain, real valued 1x1 matrix.

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1.472038790162054
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Gaussian, real valued 2x2 matrix.

-0.7151347737777703	2.554608872047015
1.061153666892094	-0.4894993627612058

Gaussian, complex valued 2x2 matrix.

0.9289825830129438-0.4813485333930206i	0.9662134522578358-0.4294592473872408i
0.5408810406545974+0.03634893478821161i	-1.712789776568709-0.6297486061153271i

Gaussian, complex valued, self-adjoint 2x2 matrix.
Note that appending ’ to a matrix takes the conjugate transpose, and matlab reserves i for the imaginary unit.

-0.01880611454872252+0i	-0.2653045793341844+2.032048899446623i
-0.2653045793341844-2.032048899446623i	0.4961821165653084+0i

Producing eigenvalues.

```
ans =

  -1.8312
   0.2405
```

Running tests to see how many hits we get within the interval $[0, 2]$. Note that matlab uses histcounts rather than histc.

Homework

Is the PDF of $\frac{a+b}{2}$ the same as $\frac{\xi_{12}}{\sqrt{2}}$ for normal RVs a, b, ξ_{12}?
i.e. $\mathbb{P}\left(\frac{a+b}{2} < s)\right) \stackrel{?}{=} \left(\mathbb{P}\frac{\xi_{12}}{\sqrt{2}}\right)$

2x2 Random Matrix

Our matrix L corresponds to eigvenvalues λ_1, λ_2 which are random variables determined by $\{\xi_{ij}, \eta_{ij}\}$.
Then the number of evaulations in the interval B is given by $\sum_{j=1}^{2} \chi_B(\lambda_j)$. We may take the average by

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{j=1}^{2} \chi_B(\lambda_j) \frac{1}{\sqrt{2\pi}} e^{-\xi_{11}^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\xi_{22}^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\xi_{12}^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\eta_{12}^2} \, d\xi_{11} d\xi_{22} d\xi_{12} d\eta_{12}.$$

Expected Evaluations

We have that the expecation of the number of evaluations in the interval (a, b) is given by $\int_a^b G(s)\, ds$ where

$$G(s) = e^{-\frac{s^2}{2}} \sum_{\ell=0}^{2} P_\ell(s)^2$$

and $P_\ell(s)$ is the Hermite polynomial of degree d.