Manifolds III

March 31, 2025

Review

If X, Y are topological spaces and $f, g : X \to Y$ continuous maps, we say f and g are homotopic (written $f \simeq g$) if there is a homotopy $H : X \times I \to Y$ (where $I = [0,1]$) such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$.
We say that f is null-homotopic if it is homotopic to a constant map.

Proposition

Homotopy is an equivalence relation on the collection of continuous maps.

1. $f \simeq f$ by $H(x,t) := f(x)$.

2. $f \overset{\tilde{H}}{\simeq} g \implies g \simeq f$ by defining $\tilde{H}(x,t) := H(x, 1-t)$.

3. $(f \overset{F}{\simeq} g \wedge g \overset{G}{\simeq} h) \implies f \simeq h$ by

$$H(x,t) := \begin{cases} F(x,2t) & 0 \le t \le 1/2 \\ G(x, 2t-1) & 1/2 \le t \le 1 \end{cases}.$$

Proposition

For $f_0, f_1 : X \to Y$ and $g_0, g_1 : Y \to Z$, if $f_0 \overset{F}{\simeq} f_1$ and $g_0 \overset{G}{\simeq} g_1$, then $g_0 \circ f_0 \simeq g_1 \circ f_1$.

Proof

Define $H(x,t) := G(F(x,t),t)$ such that $H(x,0) = G(F(x,0),0) = G(f_0(x),0) = g_0 \circ f_0(x)$.
Similarly, $H(x,1) = g_1 \circ f_1(x)$.

Definition: Homotopic Spaces

We say that two spaces X and Y are homotopic to each other ($X \simeq Y$) if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.

Example

$\mathbb{R}^n$ is homotopic to $\{0\}$ (or any single point) by $\iota : 0 \to \mathbb{R}^n$ and $r : \mathbb{R}^n \to 0$. Then $r \circ \iota : 0 \to 0$ is id_0 and $\iota \circ r : \mathbb{R}^n \ni x \mapsto 0 \in \mathbb{R}^n$ is homotopic to $\mathrm{id}_{\mathbb{R}^n}$. In fact, consider $H : \mathbb{R}^n \times I \to \mathbb{R}^n$ where $H(x,t) = tx$, $H(x,1) = x = \mathrm{id}_{\mathbb{R}^n}(x)$ and $H(x,0) = 0$.

Definition: Path

A path in X from p to q is a continuous map $f : I \to X$ such that $f(0) = p$ and $f(1) = q$.

Definition: Path Homotopic

Let $f, g : I \to X$ be two paths in X from p to q.
We say that f and g are path homotopic (write $f \sim g$) if there is a homotopy $H : I \times I \to X$ such that $H(s,0) = f(s)$, $G(s,1) = g(s)$, $H(0,t) = p$ and $H(1,t) = q$.

Proposition

Path homotopy is an equivalence relation on the collection of paths from p to q.
Write $[f]$, the equivalence class of f in the quotient.

Definition: Loop

In the special case that $p = q$, we say that $f : I \to X$ is a loop

Definition: Fundamental Group

Given (X, p), $\pi_1(X, p)$ (the fundamental group of X at the point p) is the set of all loops at p modulo the path homotopy.

$$\{\text{loops at } p\}/ \sim$$

Equivalently, $(S^1, 1)$, $\{\text{loops at } p\} = \{\text{continuous maps } f : (S^1, 1) \to (X, p)\}$ with $f(1) = p$. We say this is the homotopy "relative to $1 \in S^1$". We have $H : S^1 \times I \to X$ such that $H(s, 0) = f(s)$, $H(s, 1) = g(s)$ and $H(1, t) = p$.

Definition: Free Homotopy

For two loops $f, g : S^1 \to X$, we say that f and g are free homotopic if $f \simeq g$.

Lemma

When $f : I \to X$ is a path from p to q, if $f \circ \varphi$ is a reparameterization of f then $(f \circ \varphi) \sim f$ where $\varphi : I \to I$ satisfies $\varphi(0) = 0$ and $\varphi(1) = 1$.

Proof

Note that φ is homotopic to the identity map id_I through $H(s, t) = ts + (1 - t)\varphi(s)$ since $H(s, 0) = \varphi(s)$ and $H(s, 1) = s = \mathrm{id}_I(s)$.
Then consider $f \circ H : I \times I \to X$ which is a path homotopy between f and $f \circ \varphi$.

Fundamental Group

Let $f, g : I \to X$ be two paths with $f(1) = g(0)$.
Then we can "compose" (concatenate) f and g together $(f \cdot g) : I \to X$ by

$$(f \cdot g)(s) := \begin{cases} f(2s) & 0 \le s \le 1/2 \\ g(2s - 1) & 1/2 \le s \le 1 \end{cases}.$$

Lemma

If $f_0 \overset{F}{\sim} f_1$, $g_0 \overset{G}{\sim} g_1$ and $f_0(1) = f_1(1) = g_0(0) = g_1(0)$, then $f_0 \cdot g_0 \sim f_1 \cdot g_1$.

Proof

Define

$$H(s, t) := \begin{cases} F(2s, t) & 0 \le s \le 1/2 \\ G(2s - 1, t) & 1/2 \le s \le 1 \end{cases}.$$

Then

$$H(s,0) = \begin{cases} F(2s,0) = f_0(2s) & 0 \le s \le 1/2 \\ G(2s-1,0 = g_0(2s-1) & 1/2 \le s \le 1 \end{cases}.$$

Similarly $H(s,1) = (f_1 \cdot g_1)(s)$, hence $f_0 \cdot g_0 \sim f_1 \cdot g_1$.
With this, we have a well-defined $[f] \cdot [g] := [f \cdot g]$.

Simple Properties

For f from p to q where c_p is the constant map at p,

1. $[c_p] \cdot [f] = [f] = [f] \cdot [c_q]$ since $c_p \cdot f$ is a reparameterization of f.

2. Let $\overline{f}$ be the inverse path of f (i.e. $\overline{f}(s) = f(1-s)$). Then $[f] \cdot [\overline{f}] = [c_p]$ and $[\overline{f}] \cdot [f] = [c_q]$.

$$H(s,t) := \begin{cases} f(2s) & 0 \le s \le t/2 \\ f(t) & t/2 \le s \le 1 - t/2 \\ f(2-2s) & 1 - t/2 \le s \le 2 \end{cases}.$$

1. $([f] \cdot [g]) \cdot [h] = [f] \cdot ([g] \cdot [h])$, since these are reparameterizations of the same path.

Group Structure

$\pi_1(X,p) = \{\text{loops at } p\}/\sim$.
Define $[f] \cdot [g] := [f \cdot g]$.
It has an identity element $[c_p] = e$.
For any $f \in \pi_1(X,p)$, it has an inverse $[\overline{f}]$ such that $[f] \cdot [\overline{f}] = [\overline{f}] \cdot [f] = [c_p]$.
Finally, it is associative by (3) above.

Proposition

Suppose $p, q \in X$ with X path-connected.
Then $\pi_1(X,p)$ is isomorphic to $\pi_1(X,q)$.
Remark: this isomorphism is not canonical.

Proof

We define a path γ from q to p and $\Phi_\gamma : \pi_1(X,p) \to \pi_1(X,q)$ by $[f] \mapsto [\gamma \cdot f \cdot \overline{\gamma}]$.
Φ_γ is a group homomorphism.

$$\begin{aligned} \Phi_\gamma[f] \cdot \Phi_\gamma[g] &= [\gamma \cdot f \cdot \overline{\gamma}] \cdot [\gamma \cdot g \cdot \overline{\gamma}] \\ &= [\gamma \cdot f \cdot \overline{\gamma} \cdot \gamma \cdot g \cdot \overline{\gamma}] \\ &= [\gamma \cdot f] \cdot \overbrace{[\overline{\gamma} \cdot \gamma]}^{=e} \cdot [g \cdot \overline{\gamma}] \\ &= [\gamma \cdot (f \cdot g) \cdot \overline{\gamma}] \\ &= \Phi_\gamma[f \cdot g]. \end{aligned}$$

Φ_γ has an inverse, $\Phi_{\overline{\gamma}} : \pi_1(X,q) \to \pi_1(X,p)$.

$$\Phi_{\overline{\gamma}} \circ \Phi_\gamma[f] = \Phi_{\overline{\gamma}}[\gamma \cdot f \cdot \overline{\gamma}] = [\overline{\gamma} \cdot \gamma \cdot f \cdot \overline{\gamma} \cdot \gamma] = [f].$$

Induced Homomorphism

$\varphi : (X, p) \to (Y, q)$ induces

$$\varphi_* : \pi_1(X, p) \to \pi_1(Y, q)$$
$$[f] \mapsto [\varphi \circ f].$$

φ_* is a homomorphism.

$$(\varphi_*[f]) \cdot (\varphi_*[g]) = [\varphi \circ f] \cdot [\varphi \circ g] = [(\varphi \circ f) \cdot (\varphi \circ g)] = [\varphi \circ (f \cdot g)] = \varphi_*[f \cdot g].$$

Proposition

If $\varphi, \psi : (X, p) \to (Y, q)$ are homotopic, then $\varphi_* = \psi_* : \pi_1(X, p) \to \pi_1(Y, q)$.

Proof

Let $[f] \in \pi_1(X, p)$, $\varphi_*[f] = [\varphi \circ f]$ and $\psi_*[f] = [\psi \circ f]$ and $H : X \times I \to Y$ a homotopy between φ and ψ.
Then define $\tilde{H} := I \times I \to Y$ by $\tilde{H}(s, t) = H(f(s), t)$ such that

$$\tilde{H}(s, 0) = H(f(s), 0) = \varphi \circ f(s)$$
$$\tilde{H}(s, 1) = H(f(s), 1) = \psi \circ f(s).$$

Corollary

If $X \simeq Y$, then $\pi_1(X) \simeq \pi_1(Y)$.

Examples (*)

$\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(S^n) = 0$ for $n \geq 2$.
For $n \geq 2$, write $S^n = A_+ \cup A_-$ where A_+ and A_- are large balls centered at the north and south pole respectively.
Then A_+ and A_- are both homeomorphic to $\mathbb{R}^n$ and $A_+ \cap A_-$ (their intersection about the equator) is homemorphic to $S^{n-1} \times \mathbb{R}$.
We fix a base point $p \in A_+ \cap A_-$ and let $f : I \to S^n$ be a loop based at p.
There exists a partition of I, $0 = s_0 < s_1 < \cdots < s_k = 1$, such that $f|_{[s_i, s_{i+1}]}$ is contained in A_- or A_+.
Draw a path γ_i from p to $f(s_i)$ such that $\gamma_i \subseteq A_+ \cap A_-$. Let $f_i = f|_{[s_i, s_{i+1}]}$ such that $f = f_0 \cdot f_1 \cdots f_k$. Then this is path homotopic to

$$(f_0 \cdot \overline{\gamma}_1) \cdot (\gamma_1 \cdot f \cdot \overline{\gamma}_2) \cdots (\gamma_{k-1} \cdot f_{k-1} \cdot \overline{\gamma}_k) \cdot (\gamma_k \cdot f_k).$$

Each $\gamma_i \cdot f_i \cdot \overline{\gamma}_i$ is contained in A_- or A_+, hence $\gamma_i \cdot f_i \overline{\gamma}_{i+1} \sim c_p$, $f \simeq c_p$ and $[f] = e$.

April 2, 2025

Correction

For $\varphi, \psi : (X, x_0) \to (Y, y_0)$ where $\varphi \simeq \psi$, we say a homotopy H between φ and ψ is base point preserving if $H(x_0, t) = y_0$ for all $t \in [0, 1]$.

Proposition

If $\varphi \simeq \psi$ through a base point preserving homotopy, then $\varphi_* = \psi_*$, $\pi_1(X, x_0) \to \pi_1(Y, y_0)$.
For $X \simeq Y$, $\varphi : X \to Y$ and $\psi : Y \to X$ where $\psi \circ \varphi = \text{id}_X$ and $\varphi \circ \psi = \text{id}_Y$, in general $\psi \circ \varphi(x_0) \neq x_0$ and $\varphi \circ \psi(y_0) \neq y_0$.
Set up: $\varphi_0, \varphi_1 : X \to Y$ with $\varphi_0 \simeq \varphi_1$ through a homotopy H.
Write $\varphi_t = H(\cdot, t) : X \to Y$ and fix a base point $x_0 \in X$ and set $\gamma(t) = \varphi_t(x_0)$ for $t \in [0, 1]$.

Proposition 1

$(\varphi_0)_* = \Phi_\gamma \circ (\varphi_1)_* : \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$.

Proof

Let f be a loop at x_0.

<div align="center">IMAGE 1</div>

Let γ_t be $\gamma|_{[0,t]}$ and then, by rescaling the domain $[0, t]$ to $[0, 1]$ i.e.

$$\gamma_t : [0, 1] \to Y$$
$$s \mapsto \gamma(ts).$$

from $\varphi_0(x_0)$ to $\gamma(t) = \varphi_t(x_0)$. Then $\gamma_t \cdot (\phi_t \circ f) \cdot \overline{\gamma}_t$ is a homotopy between $(\varphi_0 \circ f)$ and $\gamma \cdot (\varphi_1 \circ f) \cdot \overline{\gamma}$. Hence

$$(\varphi_0)_*[f] = [\varphi_0 \circ f] = [\gamma] \cdot [\varphi_1 \circ f] \cdot [\overline{\gamma}] = \Phi_\gamma \circ (\varphi_1)_*[f].$$

Proposition 2

If $X \simeq Y$, then $\pi_1(X) \cong \pi_1(Y)$.

Proof

Since $(\psi \circ \varphi) \simeq \text{id}_X$, by Proposition 1

$$\psi_* \circ \varphi_* = (\psi \circ \varphi)_* = \Phi_\gamma \circ (\text{id}_x)_* = \Phi_\gamma.$$

Hence $\psi_* \circ \varphi_*$ is an isomorphism (as is $\varphi_* \circ \psi_*$). Therefre φ_* and ψ_* are isomorphisms.

Recall: Covering Map

For $X, \tilde{X}$ connected, $\pi : \tilde{X} \to X$ is a covering map if for each $p \in X$ there exists a neighborhood $U \subset X$ such that $\pi^{-1}(U)$ is a disjoint union

$$\pi^{-1}(U) = \dot{\bigcup}_{\alpha \in A} U_\alpha$$

such that $\pi|_{U_\alpha} : U_\alpha \to U$ Is a homeomorphism.

<div align="center">5</div>

Lifting Properties

A lift is a map $\tilde{f}$ such that $f = \pi \circ \tilde{f}$.

1. Path Lifting: Let $f : I \to X$ be a path from x_0. Then, for any $\tilde{x}_0 \in \pi^{-1}(x_0)$, there is a unique lift $\tilde{f}$ of f with $\tilde{f}(0) = \tilde{x}_0$.

2. Homotopy Lifting: Let $f_0, f_1 : I \to X$ be paths in X with $f_0(0) = f_1(0) = x_0$ and $f_0(1) = f_1(1)$. Suppose H is a path homotopy between f_0 and f_1. Then for any $\tilde{x}_0 \in \pi^{-1}(x_0)$, there is a unique lift $\tilde{H} : I \times I \to \tilde{X}$ of H. In particular, $\tilde{H}$ is a path homotopy between $\tilde{f}_0$ and $\tilde{f}_1$. That is if $H(0,t) = x_0$ then $\tilde{H}(0,t) \in \pi^{-1}(x_0)$ for all t. Hence $\tilde{H}(0,t) = \tilde{x}_0$, $\forall t \in [0,1]$. Similarly, $\tilde{H}(1,t)$ is identically constant. In particular, $\tilde{f}_0(1) = \tilde{H}(1,0) = \tilde{H}(1,1) = \tilde{f}_1(1)$.

Fundamental Group of the Circle

$\pi_1(S^1) = \mathbb{Z}$.

Example

$\pi : \mathbb{R} \to S^1$ by $s \mapsto e^{2\pi i \cdot s}$.

Proof

Take as a base point $1 = x_0 \in S^1 \subseteq \mathbb{C}$. For each $n \in \mathbb{Z}$, we define a loop $\omega_n : [0,1] \to S^1$ by $s \mapsto e^{2\pi i \cdot ns}$. Let f be a loop at $x_0 \in S^1$. We can lift f to $\tilde{f} : I \to \mathbb{R}$ at $0 \in \mathbb{R}$. Then $\tilde{f}(1) \in \pi^{-1}(x_0) = \mathbb{Z} \subseteq \mathbb{R}$. This defines a map φ that sends a loop f to $\tilde{f}(1) \in \mathbb{Z}$. This φ induces $\varphi : \pi_1(S^1, x_0) \to \mathbb{Z}$ well-defined. If $f_0, f_1 : I \to S^1$ at x_0 are path homotopic via H, then we may lift H to $\tilde{H} : I \times I \to \mathbb{R}$ which implies $\tilde{f}_0(1) = \tilde{f}_1(1)$.
φ is surjective, since for any $n \in Z$ we may consider the loop ω_n where $\tilde{\omega}_n(1) = n$.
φ is a group homorphism since $\varphi[f \cdot g] = \widetilde{f \cdot g}(1) = \tilde{g} + \tilde{f}(1) = \varphi[f] + \varphi[g]$.
φ is injective, since if $\varphi[f] = 0$ (i.e. $\tilde{f}(0) = 0$) then $\tilde{f}$ is a loop in $\mathbb{R}$ and $\tilde{f}$ is null-homotopic to c_0 by H. Therefore $\pi \circ \tilde{H}$ is a path-homotopy between f and c_{x_0} (i.e. $[f] = e$).

Path-Lifting

For $f : I \to X$, we have a special case where $\operatorname{im} f \subseteq U$ evenly covered. Write $\pi^{-1}(U) = \bigsqcup \tilde{U}_\alpha$ and pick the $\tilde{U}_\alpha$ which contains $\tilde{x}_0$. Since $\pi|_{\tilde{U}_\alpha} : \tilde{U}_\alpha \to U$ is a homemorphism, $\tilde{f} := (\pi|_{\tilde{U}_\alpha})^{-1} \circ f$ is the unique lift of f at $\tilde{x}_0$.
In general, pick a partititon of $I = [0,1]$, $0 = t_0 < t_1 < \cdots < t_m = 1$, such that $\operatorname{im} f|_{[t_i, t_{i+1}]} \subseteq U_i$ evenly covered. We can lift $f|_{[0,t_1]}$ at $\tilde{x}_0$, giving $\tilde{f} : [0,t] \to \tilde{X}$. Next, we lift $f|_{t_1, t_2}$ at $\tilde{f}(t_1) \in \tilde{X}$. Since the partition is finite, we may repeat the process until f is entirely lifted. This lift is unique.

Homotopy Lifting

For each fixed $(y_0, t_0) \in I \times I$, by continuity, there is a neighborhood $N(y_0) \times (t_0 - \varepsilon, t_0 + \varepsilon)$ such that H sends $N(y_0) \times (t_0 - \varepsilon, t_0 + \varepsilon)$ inside an evenly covered neighborhood. By compactness of $\{y_0\} \times [0,1]$, there is a finite collection of $N_{t_i}(y_0) \times (t_i - \varepsilon_i, t_i + \varepsilon_i)$ such that they cover $\{y_0\} \times I$ and the image of each under H is contained in an evenly covered neighborhood. Set $N = \bigcap_i N_{t_i}(y_0)$, a neighborhood of y_0, and construct a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ such that $H(N \times [t_i, t_{i+1}]) \subseteq U_i$ evenly covered. Then we can start with $H|_{N \times [0,t_1]}$ and lift it at $\tilde{x}_0$ by some $(\pi|_{\tilde{U}_\alpha})^{-1}$. Then lift each $H|_{N \times [t_i, t_{i+1}]}$ one by one. Eventually, we have $\tilde{H} : N \times [0,1] \to \tilde{X}$ that lifts $H : N \times [0,1] \to \tilde{X}$ at $\tilde{x}_0$. This lift holds for any $y_0 \in I$ and, if two strips overlap, then the lift must agree there by the uniqueness of path lifting. This assures that $\tilde{H} : I^2 \to \tilde{X}$ is continuous.

Remark

Given a continuous map $F : Y \times I \to X$ and a covering $\pi : \tilde{X} \to X$, suppose that we have a map $\tilde{F} : Y \times \{0\} \to \tilde{X}$ that lifts $F|_{Y \times \{0\}} : Y \times \{0\} \to X$. Then there is a unique lift $\tilde{F} : Y \times I \to \tilde{X}$ of F which extends $\tilde{F} : Y \times \{0\} \to \tilde{X}$.

Theorem: Fundamental Theorem of Algebra

A polynomial $p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ (with $a_i \in \mathbb{C}$) has a root in $\mathbb{C}$.

Proof

Suppose otherwise. Then $p(z) \neq 0$, $\forall z \in \mathbb{C}$. Consider $f_r : [0,1] \to S^1$ ($r \geq 0$) by

$$f_r(s) = \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}.$$

Then $f_0(s) \equiv 1$ is a constant loop at $1 \in \mathbb{C}$, and $f_r \simeq f_0$ for each $r \geq 0$. Consider $R \geq 1$ large such that $R \gg \sum_{i=1}^n |a_i|$. On $\{z : |z| = R\}$, we have

$$|z^n| > \left(\sum_{i=1}^n |a_i| \right) \cdot |z^{n-1}| \geq \sum_{i=1}^n |a_i| \cdot |z^{n-i}| = \left| \sum_{i=1}^n a_i z^{n-i} \right|.$$

This implies that p does not have any roots on $\{|z| = R\}$. Moreover, for $p_t(z) = z^n + t(a_1 z^{n-1} + \cdots a_{n-1} z + a_n)$ with $0 \leq t \leq 1$, p_t does not have any roots on $\{|z| = R\}$. Consider

$$f_{R,t}(s) = \frac{p_t(Re^{2\pi i s})/p_t(R)}{|p_t(Re^{2\pi i s})/p_t(R)|}.$$

Then

$$f_{R,0}(s) = \frac{(Re^{2\pi i s})^n / R^n}{|(Re^{2\pi i s})^n / R^n|} = (e^{2\pi i s})^n = \omega_n(s).$$

Therefore $f_{R,1}(s) \simeq f_R(s)$ and $f_R \simeq \omega_n$. But since $\omega_n \not\simeq$ constant so this is a contradiction.

April 7, 2025

Definition: Retraction

Let X be a space and $A \subseteq X$ be a subset. We say that a continuous map $r : X \to A$ is a retraction if $r|_A = \mathrm{id}_A$. In particular, becasue $r \circ \iota_A = \mathrm{id}_A$, for $x_0 \in A$

$$r_* \circ (\iota_A)_* : \pi_1(A, x_0) \to \pi_1(A, x_0)$$

is an isomorphism. Hence $r_* : \pi(X, x_0) \to \pi(A, x_0)$ is surjective.

Corollary

There is no retraction $r : D^2 \to S^1 (= \partial D^2)$.

Proof

Suppose there is such a map r, then

$$r_* : \overbrace{\pi_1(D^2, x_0)}^{=0} \to \overbrace{\pi_1(S^1, x_0)}^{=\mathbb{Z}}$$

is surjective which is a contradiction.

Corollary

Every continuous map $h : D^2 \to D^2$ has a fixed point.

Proof

Suppose $\exists h : D^2 \to D^2$ without fixed points.

<div align="center">IMAGE 1</div>

Define $r : D^2 \to D^2$ as the ray pictured from $h(x)$ through x to the boundary. If $x \in \partial D^2$, then by construction $r(x) = x$. Hence $r : D^2 \to S^1$ is a retraction which is a contradiction.

Corollary (Borsuk-Ulam)

Let $f : S^2 \to \mathbb{R}^2$. Then there exists a pair of antipodal points x and $-x$ on S^2 such that $f(x) = f(-x)$. This carries analogously to higher dimensions.

Proof

Suppose that $f(x) \neq f(-x)$ for all $x \in S^2$. We define $g : S^2 \to S^1$ by $g(x) = \frac{f(x)-f(-x)}{||f(x)-f(-x)||}$. On $S^2 \subseteq \mathbb{R}^3$, we consider a loop γ at the equator by $\gamma(s) = (\cos(2\pi s), \sin(2\pi s), 0)$ for $s \in [0,1]$. Because S^2 is simply connected, $g \circ \gamma : [0,1] \to S^1$ is path-homotopic to a constant loop in S^1. On the other hand, we lift $h := g \circ \gamma$ to $\tilde{h} : [0,1] \to \mathbb{R}$ with $\tilde{h}(0) = 0 \in \mathbb{R}$. Note

$$h(s + 1/2) = g \circ \gamma(s + 1/2) = g(\cos(2\pi s + \pi), \sin(2\pi s + \pi), 0) = g(-\gamma(s)) = -g(\gamma(s)) = -h(s).$$

Hence $\tilde{h}(s + 1/2) \in \pi^{-1}(-h(s))$ where $\pi : \mathbb{R} \to S^1$ is the covering map. Since $\pi^{-1}(-h(s)) = \frac{1}{2} + \tilde{h}(s) + \mathbb{Z}$, for each $s \in [0, 1/2]$ there is an integer q_s such that $\tilde{h}(s + 1/2) = \frac{1}{2} + \tilde{h}(s) + q_s$ and

$$\tilde{h}(s + 1/2) - \tilde{h}(s) = \frac{1}{2} + q_s.$$

The left hand side depends continuously on s and, by continuity, q_s is a constant (call it q). This gives

$$\tilde{h}(1) = \tilde{h}(1/2) + \frac{1}{2} + q = \tilde{h}(0) + 1 + 2q = 1 + 2q \neq 0$$

which contradicts the assertion that h is homotopic to a constant loop.

Corollary (Large Fiber Lemma)

If $f : [0,1]^{n+1} \to \mathbb{R}^n$ is a continuous map, then there exist $a, b \in [0,1]^{n+1}$ such that $f(a) = f(b)$ and $|a - b| \geq 1$.
Remark: if $z = f(a) = f(b)$, then the lemma says that diam $f^{-1}(z) \geq 1$.

Proof

Take the sphere of radius $1/2$ in $[0,1]^{n+1}$, then by Borsuk-Ulam there exist a pair of antipodal points $a, b \in S^1$ such that $f(a) = f(b)$ and $|a - b| \geq 1$.

Proposition

$\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$

Proof

Write $F : \pi_1(X \times Y) \to \pi_1(X) \times \pi_1(Y)$ by $[f] \mapsto ([g],[h])$. Then $f : [0,1] \to X \times Y$ is a loop at (x_0, y_0), $f(s) = (g(s), h(s))$, and $g : [0,1] \to X$ and $h : [0,1] \to Y$ are loops at x_0 and y_0 respectively.

Definition: Wedge Sum

Let X and Y be path-connected topological spaces. Then $X \vee Y = (X \coprod Y)/x_0 \sim y_0$
Let $\{X_\alpha\}$ be a family of such spaces. Then $\bigvee_\alpha X_\alpha = \coprod_\alpha X_\alpha / \sim$.

Sketch

$$\pi_1(S^1_-, x_0) \to \pi_1(X, x_0) \quad \text{gen} \mapsto \alpha$$
$$\pi_1(S^1_+, x_0) \to \pi_1(X, x_0) \quad \text{gen} \mapsto \beta$$

with $\alpha \neq \beta$, $\alpha\beta \neq \beta\alpha$. Then $\pi_1(X, x_0)$ should be $\langle \alpha, \beta \rangle$.

Definition: Free Product

Let $\{G_\alpha\}_\alpha$ be a family of groups. $*_\alpha G_\alpha = \{g_1 g_2 \cdots g_k : \text{each } g_i \text{ is a word in some } A_\alpha\}$.
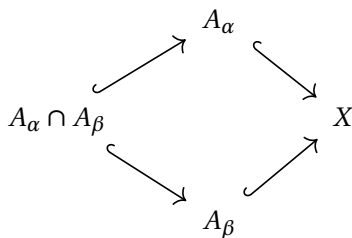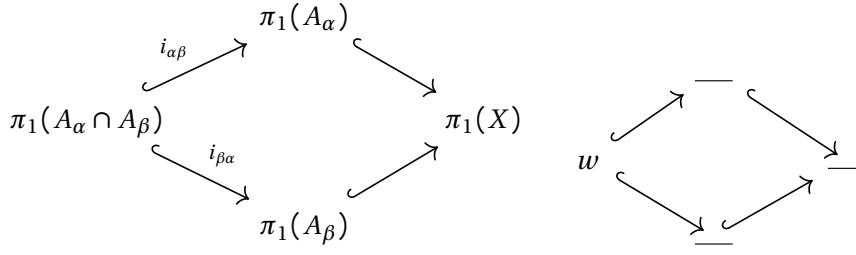
Proposition

If for each α, there is a group homomorphism $\phi_\alpha : G_\alpha \to H$ then $\{\phi_\alpha\}$ induces a group homomorphism $\Phi : *_\alpha G_\alpha \to H$ by $g_1 \cdots g_k \mapsto \phi_{\alpha_1}(g_1) \cdots \phi_{\alpha_k}(g_k)$.

Van-Kapen Theorem

Setup

Let $X = \bigcup_\alpha A_\alpha$, each A_α open and connected where $\{A_\alpha\}$ have a common point x_0. Assume also that each $A_\alpha \cap A_\beta$ is path connected. Then $j_\alpha : A_\alpha \hookrightarrow X$ induces $j_\alpha : \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$. $\{j_\alpha\}_\alpha$ induces $\Phi : *_\alpha \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$ which is surjective by a similar argument as was used above for Example (*) ($S^2 = A_- \cup A_+$) applied to $X = \bigcup_\alpha A_\alpha$. Now, what is the kernel of Φ?

$$\begin{array}{ccc}
 & \pi_1(A_\alpha) & \\
\pi_1(A_\alpha \cap A_\beta) & & \pi_1(X) \\
 & \pi_1(A_\beta) &
\end{array}$$

with maps $i_{\alpha\beta}$, $i_{\beta\alpha}$.

Then $i_{\beta\alpha}(w)i_{\alpha\beta}(w)^{-1}$ is NOT id in $*_\alpha \pi_1(A_\alpha)$. But through Φ, it hsould be id $\in \pi_1(X, x_0)$. Hence every element in $*_\alpha \pi_1(A_\alpha)$ of the form $i_{\beta\alpha}(w)i_{\alpha\beta}(w)^{-1}$ where $w \in \pi_1(A_\alpha \cap A_\beta)$ is in the kernel of Φ.

Theorem (Van-Kampen)

If every $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected, ker Φ is the normal subgroup N generated by $\{i_{\beta\alpha}(w)i_{\alpha\beta}(w)^{-1} : \alpha, \beta \in A, \ w \in \pi_1(A_\alpha \cap A_\beta)\}$. Hence $\pi_1(X, x_0) \cong (*_\alpha \pi_1(A_\alpha, x_0))/N$.

Remarks

1. In the case that $X = A_0 \cup A_1$ with $A_0 \cap A_1$ path connected, then the intersection condition holds.

2. If $X = A_0 \cup A_1$ and $A_0 \cap A_1$ is simply connected, then $N = \{\mathrm{id}\}$ and $\pi_1(X) = \pi_1(A_0) * \pi_1(A_1)$.

3. If $X = A_0 \cup A_1$ and A_1 is simply connected, then $\pi_1(X) = \pi_1(A_0)/N$ and N is the normal subgroup generated by

$$i_{01}(w) \overbrace{i_{10}(w)^{-1}}^{\in \pi_1(A_1, x_0)} = i_{01}(w)$$

i.e. N is the normal closure of $i_{01}(\pi_1(A_0 \cap A_1))$.

Example

IMAGE 2

For each $\alpha \in \{1, \ldots, 5\}$, let A_α be a small neighborhood of $T \cup e_1$. Every double/triple intersection is a neighborhood of T. Hence it is path continuous and we have that $\pi_1(A_\alpha) = \mathbb{Z}$. Thus $\pi_1(A_\alpha \cap A_\beta) = \mathrm{id}$, and $\pi_1(X) = *_\alpha \pi_1(A_\alpha)/N = *_1^5 \mathbb{Z}$.

Example

IMAGE 3

By Van-Kampen, $\pi_1(X) = \pi_1(A_0)$ modulo the normal closure of $i(\pi_1(A_0 \cap A_1))$. That is

$$\langle a, b \mid aba^{-1}b^1 \rangle = \mathbb{Z}^2.$$

Remark

In general, orientable M_g is the connected sum of g many toruses.

April 9, 2025

Recall: Van-Kampen Theorem

Write $\pi_1(X) = (\pi_1(A) * \pi_1(B))/N$ where N is the normal closure of $i_{\alpha\beta}(w)i_{\beta\alpha}(w)^{-1}$ for $w \in \pi_1(A \cap B)$, $i_{\alpha\beta} : \pi_1(A \cap B) \to \pi_1(A)$ and $i_{\beta\alpha} : \pi_1(A \cap B) \to \pi_1(B)$.

Example

M_g is the connected sum of g many tori, and $\pi_1(M_g) = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1 b_1] \cdots [a_g b_g] \rangle$.

Example

N_g is the connected sum of g many $\mathbb{RP}^2$ (e.g. N_2 is the Klein bottle). N_g has a polygon-representation by the $2g$-gon with boundary identified through $a_1 a_1 a_2 a_2 \cdots a_g a_g$. Therefore $\pi_1(N_g) = \langle a_1 \cdots a_g \mid a_1^2 \cdots a_g^2 \rangle$.

Abelianiztion

1. $\text{Ab}(\pi_1(M_g))$ is the free abelian group generated by $\{a_1, b_1, \ldots, a_g, b_g\} = \mathbb{Z}^{2g}$.

2. $\text{Ab}(\pi_1(N_g)) = \text{Ab}(\langle a_1 \cdots a_{g-1} a_1 a_2 \cdots a_g \mid a_1^2 \cdots a_g^2 \rangle = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$.

Corollary

None of the surfaces in $\{S^2, M_1, \ldots, M_g, \ldots, N_1, \ldots, N_g, \ldots\}$ are homotopic to each other.

Definition: Cell Complex

0-cells are points; 1-cells, e^1, are intervals; 2-cells, e^2, are disks; n-cells, e^n, are $\overline{B}^n$.
A cell complex for space X is a decomposition (assuming finite dimensions) $X = X^0 \cup X^1 \cup \cdots \cup X^n$ where X^0 is the discrete set of points (i.e. 0-cells), X^1 is the space obtained by gluing 1-cells to X^0 ($\varphi_\alpha : \partial e_\alpha^1 \to X^0$), X^2 is the space obtained by gluing 2-cells to X^1 ($\varphi_\alpha : \partial e_\alpha^2 \to X^1$), and in general X^n is obtained by gluing n-cells $\{e_\alpha^n\}_\alpha$ to X^{n-1} by $\varphi_\alpha : \partial e_\alpha^n = S^{n-1} \to X^n$.

Examples

Cell complexes need not be unique. $S^2 = X^1 \cup_\alpha e_+^2 \cup_\alpha e_-^2$ and $S^2 = \{e^0\} \cup_\alpha \{e^2\}$.
$\mathbb{RP}^2 = \{e^1\} \cup_\alpha \{e^2\}$ where φ_α is given by $z \mapsto z^2$.
$\mathbb{T}^2$ is gluing e^2 to $S^1 \vee S^1$.

Theorem (Computing Fundamental Group)

Set up

Let X be a path-connected space, $Y = X \cup_\alpha e_\alpha^2$ (i.e. X is created by gluing 2-cells $\{e_\alpha^2\}_\alpha$ to X via $\phi_\alpha : \partial e_\alpha^2 \to X$). The inclusion $\iota : X \to Y$ induces $\iota_* : \pi_1(X) \to \pi_1(Y)$. Fix a base point $s_0 \in S^1$. For each α we draw a path γ_α from x_0 to $\varphi_\alpha(s_0)$. Then $\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha$ is a loop based at x_0. Thus $\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha$ is null-homotopic in Y (because φ_α is null-homotopic in e_α^2). That is $\iota_*[\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha] = \text{id}$ in $\pi_1(Y)$ and is therefore in the kernel.

Theorem

Let N be the normal subgroup in $\pi_1(X)$ generated by elements of the form $[\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha]$. Then $\pi_1(Y) \cong \pi_1(X)/N$.

IMAGE 1

Example

$\mathbb{RP}^2$ is X^1 with e^2 glued to it by the map $\varphi : z \mapsto z^2$. Then $\pi_1(\mathbb{RP}^2) = \pi_1(S^1)/N = \langle \gamma \mid \gamma^2 \rangle$ where N is generated by φ. Similarly, the theorem applies to any M_g or N_g.

Definition: Deformation Retraction

For $X \subseteq Z$, $r : Z \to X$ is a retraction if $r|_X = \mathrm{id}_X$ implies $r \circ \iota = \mathrm{id}_X$. If $\iota \circ r : Z \to Z$ is homotopic to id_X, then $r_* : \pi_1(Z) \to \pi_1(X)$ is an isomorphism.

Proof

For each α, we glue a strip S_α along γ_α. We set the base at z_0 above x_0, $Z = Y \cup_\alpha S_\alpha$. Y is a deformation retraction of Z ($\pi_1(Y) = \pi_1(Z)$).

IMAGE 2

Set $A = Z - \bigcup_\alpha \{y_\alpha\}$, where y_α is a point in e_α^2 not intersecting S_α. $B = Z - X$. A deformation retracts to X $\pi_1(A) = \pi_1(X)$. B is the union of some S_α (removing r_α) and some e_α^2 (removing ∂e_α^2). B is contractible, $\pi_1(B) = \mathrm{id}$ and $A \cap B$ is the union of strips S_α and open disks punctured at y_α. Therefore

$$\pi_1(Y) = \pi_1(Z) = (\pi_1(A) * \pi_1(B))/N = \pi_1(A)/\iota_*(\pi_1(A \cap B)) \cong \pi_1(X)/\iota_*(\pi_1(A \cap B)).$$

Consider the loop $\delta_\alpha \cdot \gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha \cdot \overline{\delta}_\alpha$ where δ_α runs from z_0 to x_0, call this λ_α. It suffices to show that these generate $\pi_1(A \cap B, z_0)$. Cover $A \cap B$ by $A_\alpha = (A \cap B) - \bigcup_{\beta \neq \alpha} e_\beta^2$. Then A_α is a union of strips (with trivia fundamental group) and a single punctured, open disk $e_\alpha^2 - \{y_\alpha\}$ and $\pi_1(A_\alpha) = \mathbb{Z} = \langle \lambda_\alpha \rangle$. So $A_\alpha \cap A_\beta$ is the union of strips, equal to $A_\alpha \cap A_\beta \cap A_\gamma$ and simply connected. By Van-Kampen,

$$\pi_1(A \cap B) = (*_\alpha \pi_1(A_\alpha))/N = *_\alpha \pi_1(A_\alpha)$$

is the free group generated by $\{\lambda_\alpha\}_\alpha$. This completes the proof.

Generalization (Theorem: Part 2)

If $Y = X \cup_\alpha e_\alpha^n$ for $n \geq 3$, then $\pi_1(Y) \cong \pi_1(X)$.
This follows from the same argument where instead A_α is the union of strips and a single punctured ball $B^n - \{y_\alpha\} \simeq S^{n-1}$. So $\pi_1(A_\alpha) = \mathrm{id}$, $\pi_1(A \cap B) = \mathrm{id}$, and $\pi_1(X) \cong \pi_1(Y)$.

Theorem: Part 3

Suppose X has a cell complex $X = X^0 \cup X^1 \cup \cdots \cup X^n$. Then $\pi_1(X) \cong \pi_1(X^2)$.
The proof follows directly from part 2.

Corollary

Given any group represented by generators and relations $G = \langle g_\alpha \mid r_\beta \rangle$, there is a cell complex X_G, of dimension 2, such that $\pi_1(X_G) \cong G$.

Proof

For each g_α, we draw a circle S^1_α. Then $X^1 = \bigvee_\alpha S^1_\alpha$ has fundamental group $*_\alpha \pi_1(S_\alpha) = \langle g_\alpha \rangle_\alpha$. To construct X_G, for each r_β glue a 2-cell e^2_α along r_β (think of r_β as a loop in X^1). Then in $X_G := X^1 \cup_\beta e^2_\beta$ we have $\pi_1(X_G) = \langle g_\alpha \mid r_\beta \rangle$.

April 14, 2025

Recall: Covering Spaces

Let $p : \tilde{X} \to X$, both X and $\tilde{X}$ path-connected.

1. Path-lifting: let $f : I \to X$ starting at $f(0) = x_0$. There is a unique lifting $\tilde{f}$ of f at $\tilde{x}_0 \in p^{-1}(x_0)$.

2. Homotopy-lifting: let $f_0, f_1 : I \to X$ be two paths with $f_0(0) = f_1(0) = x_0$ and $f_0(1) = f_1(1)$. Suppose f_t is a path-homotopy between f_0 and f_1. Then there exists a unique lift $\tilde{f}_t$ between $\tilde{f}_0$ and $\tilde{f}_1$ at $\tilde{x} \in p^{-1}(x)$.

These come from the following: let $f_t : Y \to X$ be a homotopy between f_0 and f_1. Given $\tilde{f}_0 : Y \to \tilde{X}$ that lifts f_0, there exists a unique lifting $\tilde{f}_t$. For path-lifting, we take Y a point; for homotopy-lifting, $Y = [0,1]$.

$$
\begin{array}{ccc}
 & & \tilde{X} \\
 & \overset{f}{\nearrow} & \downarrow p \\
I & \underset{p \circ f}{\longrightarrow} & X
\end{array}
$$

Proposition 1.31 (in Hatcher)

The covering map $p : \tilde{X} \to X$ induces $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x)$.

1. p_* is injective.

2. $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ are exactly loops at x_0 that lift to loops at $\tilde{x}_0$.

Proof of 1

Suppose $p_*[f] = \mathrm{id} \in \pi_1(X, x_0)$ where $[f] \in \pi_1(\tilde{X}, \tilde{x}_0)$. Then $[p \circ f] = \mathrm{id}$, and $[p \circ f]$ is path-homotopic to the constant loop c_{x_0}. Hence the lifting $\widetilde{p \circ f} = f$ is path-homotopic to a constant loop $c_{\tilde{x}_0}$.

Proof of 2

Let $[f] \in \pi_1(\tilde{X}, \tilde{x}_0)$. $p_*[f] = [p \circ f]$, $p \circ f$ lifts to f at $\tilde{x}_0$ which is a loop at $\tilde{x}_0$.
Let f be a loop at x_0. Suppose f lifts to a loop $\tilde{f}$ at $\tilde{x}_0$ (i.e. $p \circ \tilde{f} = f$). Hence $[f] = [p \circ \tilde{f}] = p_*[\tilde{f}] \in p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

Example

If $p : S^1 \to S^1$ by $z \to z^2$, then $p_*(\pi_1(S^1, 1)) = 2\mathbb{Z} \leq \mathbb{Z} = \pi_1(S^1, 1)$.

Remark

If $p : \tilde{X} \to X$ connected, then $p^{-1}(x)$ has the same cardinality for all $x \in X$.

Proof

Fix $x_0 \in X$. Consider $\mathcal{A} = \{x \in X \; : \; p^{-1}(x)$ and $p^{-1}(x_0)$ have the same cardinality$\} \neq \varnothing$. Then $\mathcal{A}$ is open since for each $x \in \mathcal{A}$, there is a neighborhood U of x such that U is evenly covered by p (i.e. $p^{-1}(U) = \dot{\bigcup}_{\alpha \in I} V_\alpha$ where $V_\alpha \overset{p}{\cong} U$). Then $p^{-1}(x')$ has cardinality $|I|$ for all $x' \in U$. It follows, since $\mathcal{A}^c$ is open, that $\mathcal{A}$ is also closed.

Proposition

The number of sheets is given by $[\pi_1(X, x_0) : p_*(\pi_1(\tilde{X}, \tilde{x}_0))]$.

Proof

Write $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Define $\Phi : \{H\text{-cosets in } \pi_1(X, x_0)\} \to p^{-1}(x_0)$ by $H[g] \mapsto \tilde{g}(1)$ where $\tilde{g}$ is a lift of g at $\tilde{x}_0$. This map is well defined, since for $[h \cdot g]$ with $h \in H$, $\widetilde{h \cdot g}(1) = \tilde{g}(1)$ (because $\tilde{h}(1) = \tilde{x}_0$).
Φ is surjective. Let $\tilde{x}_1 \in p^{-1}(x_0)$

IMAGE 1

and let $\tilde{g}$ be a path from $\tilde{x}_0$ to $\tilde{x}_1$. Define $g = p \circ \tilde{g}$, a loop at x_0. Then $\Phi(H[g]) = \tilde{g}(1) = \tilde{x}_1$.
Φ is injective. Suppose $\Phi(H[g_1]) = \Phi(H[g_2])$ (i.e. $\tilde{g}_1(1) = \tilde{g}_2(1)$).

IMAGE 2

Consider the loop $g_1 \overline{g}_2$ in X at x_0. It lifts to $\tilde{g}_1 \overline{\tilde{g}}_2$, which is a loop at $\tilde{x}_0$. This shows that $[g_1 \overline{g}_2] \in H$ (i.e. $H[g_1] = H[g_2]$).

Recall (Manifolds 2)

If a smooth manifold M is non-orientable, then there is a double cover (2 sheets) $p : \hat{M} \to M$ ($\hat{M}$ connected). Consequently, $\pi_1(M)$ has a subgroup of index 2.

Definition: Locally Path-Connected

A topological space is called locally path-connected if for each $x \in X$ and every neighborhood $U \ni X$, there is a neighborhood $V \ni x$ such that $V \subseteq U$ and V is path-connected (i.e. $\forall x \in X$, there exists a local basis $\{U_\alpha\}$ at x such that each U_α is path-connected). For example, the Topologist's sine curve with endpoints identified is path-connected but not locally path-connected.

Proposition: Lifting Criterion

Let Y be path-connected and locally path-connected. Given a covering map $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and a map $f : (Y, y_0) \to (X, x_0)$, f has a lift $\tilde{f}$ at $\tilde{x}_0$ ($\tilde{f}(y_0) = \tilde{x}_0$) if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

Proof

($\Longrightarrow$)

$$\begin{array}{ccccc} & \tilde{X} & & & \pi(\tilde{X}) \\ {\scriptstyle\tilde{f}}\nearrow & \downarrow{\scriptstyle p} & & {\scriptstyle\tilde{f}_*}\nearrow & \downarrow{\scriptstyle p_*} \\ Y \xrightarrow{\;f\;} & X & & \pi_1(Y) \xrightarrow{\;f_*\;} & \pi_1(X) \end{array}$$

$$f_* \pi_1(Y) = (p_* \circ \tilde{f}_*)(\pi_1(Y)) \leq p_* \pi_1(\tilde{X}).$$

($\Longleftarrow$) Let $y \in Y$, and draw a path γ from y_0 to y.

We lift $f \circ \gamma$ to a path in $\tilde{X}$ starting at $\tilde{x}_0$ and define $\tilde{f}(y)$ as the endpoint (i.e. $\tilde{f}(y) = \widetilde{f \circ \gamma}(a)$).
This is well-defined, since $(f \circ \gamma) \cdot (f \circ \overline{\gamma}')$ is a loop at x_0 and $[(f \circ \gamma) \cdot (f \circ \overline{\gamma}')] = f_*[\gamma \cdot \overline{\gamma}'] \in f_* \pi_1(Y, y_0) \leq p_* \pi_1(\tilde{X}, \tilde{x}_0)$.
Hence $(f \circ \gamma) \cdot (f \circ \overline{\gamma}')$ lifts to a loop at $\tilde{x}_0$.

Therefore $\widetilde{f \circ \gamma}(1) = \widetilde{f \circ \gamma'}(1)$.
$\tilde{f}$ is continuous. Fix $f(y) \in X$ and let U be a neighborhood of $f(y)$ that is evenly covered by p. Choose a path-connected neighborhood V of y such that $f(V) \subseteq U$. We check $\tilde{f}|_V$.

Because V is path-connected, we may draw a path η in V from y to y'. Then $\tilde{f}(y') = \widetilde{f \circ \gamma \circ \eta}(1)$, and $\widetilde{\gamma \cdot \eta}$ is first lifting $f \circ \gamma$ at $\tilde{x}_0$ followed by lifting $f \circ \eta$ at $\tilde{\gamma}(1)$. Let $\tilde{U} \subseteq \tilde{X}$ such that $p|_{\tilde{U}} : \tilde{U} \to U$ is a homeomorphism and $\widetilde{f \circ \gamma}(1) \in \tilde{U}$.
Then $\widetilde{f \circ \eta}(1) = (p^{-1})|_U \circ f(y')$. Hence $\tilde{f}(y') = f \circ \widetilde{(\gamma \circ \eta)}(1) = \widetilde{f \circ \eta}(1) = (p^{-1})|_U \circ f(y')$ (i.e. $\tilde{f} = (p^{-1})|_U = f$ on V). Hence $\tilde{f}$ is continuous at y.
$\tilde{f}$ is a lift of f. In fact, $(p \circ \tilde{f})(y) = p \circ (\widetilde{f\gamma}(1)) = f(y)$.

Corollary

$$
\begin{array}{ccc}
 & & \tilde{X} \\
 & & \downarrow p \\
Y & \xrightarrow{\ f\ } & X
\end{array}
$$

If Y is simply connected, then $f_* \pi_1(Y) \leq p_* \pi_1(\tilde{X})$ always holds (i.e. we can always lift f to $\tilde{f} : Y \to \tilde{X}$ in this case).

Proposition: Unique Lifting

Given $p : \tilde{X} \to X$ and $f : Y \to X$, if two lifts $\tilde{f}_1$ and $\tilde{f}_2$ of f agree at one point, then the agree everywhere on Y.

Proof

Take $\mathcal{A} = \{y \in Y : \tilde{f}_1(y) = \tilde{f}_2(y)\} \neq \varnothing$. Locally for each $y \in Y$ there exists a neighborhood V of y such that $\tilde{f} = (p^{-1})|_U \circ f$. If $y \in \mathcal{A}$, then $\tilde{f}_1(y) = \tilde{f}_2(y)$. Take a neighborhood U of $f(y)$ that is evenly covered and $\tilde{U}$ of $\tilde{f}_1(y) = \tilde{f}_2(y)$ such that $p|_{\tilde{U}} : \tilde{U} \to U$ is a homeomorphism. Then on V, a path-connected neighborhood such that $f(V) \subseteq U$, $\tilde{f}_i = (p^{-1})|_U \circ f$ (i.e. $\tilde{f}_1 = \tilde{f}_2$ on V). If $y \in \mathcal{A}^c$, $\tilde{f}_1(y) \neq \tilde{f}_2(y)$. Then $\tilde{U}_i \ni \tilde{f}_i(y)$ with $\tilde{U}_1 \cap \tilde{U}_2 = \varnothing$. Then on V, $\tilde{f}_i = (p^{-1})|_{\tilde{U}_i} \circ f$ (ie $\tilde{f}_1$ and $\tilde{f}_2$ never agree on V). Hence $\mathcal{A} = Y$.

Remark

If $p : \tilde{X} \to X$ is a covering map, recall that a covering transformation is a map $f : \tilde{X} \to \tilde{X}$ such that

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\ f\ } & \tilde{X} \\
 & p \searrow \quad \swarrow p & \\
 & X &
\end{array}
$$

commutes. This $f : \tilde{X} \to \tilde{X}$ is a lift of $p : \tilde{X} \to X$. If we fix $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$, the lifting criterion says that $p_* \pi_1(\tilde{X}, \tilde{x}_1) \leq p_* \pi_1(\tilde{X}, \tilde{x}_2)$. In particular, if $\pi_1(\tilde{X})$ is, then this holds. Hence there is a unique lift of p (i.e. covering transformation) f such that $f(\tilde{x}_1) = \tilde{x}_2$.
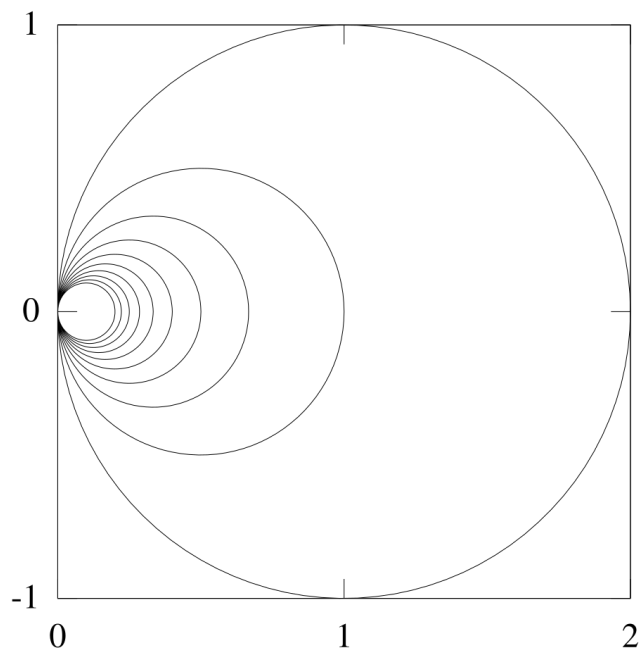
Question

Given X path-connected and locally path-connected, when does X admit a simply connected covering space $p : \tilde{X} \to X$?

Definition: Semi-locally Simply Connected

We say that X is semi-locally simply connected if for any $x \in X$ there exists a neighborhood U such that every loop in U is null-homotopic in X. That is $\mathrm{Im}(\pi_1(U) \to \pi_1(X))$ is trivial.

Non-example

The Hawaiian earing in $\mathbb{R}^2$.

Example

The cones over the Hawaiian earing.

IMAGE 1

In fact, this is simply connected.

Example

The double Hawaiian earing with cones.

IMAGE 2

Theorem

X has a simply connected covering space (i.e. a universal covering) if and only if X is semi-locally simply connected.

Proof

($\implies$) Let $x \in X$ and pick a neighborhood U of x that is evenly covered by p. Let f be a loop at x in U. f lifts to $\tilde{f}$ at $\tilde{x}_0$, which is a loop. Retract $\tilde{f}$ to $c_{\tilde{x}_0}$ by a path-homotopy H. Then $p \circ H$ shows that f is null-homotopic in X.
($\impliedby$) We construct $\tilde{X}$ as follows: fix $x_0 \in X$ and set $\tilde{X} = \{[\gamma] \text{ path homotopies} : \gamma \text{ is a path starting at } x_0\}$. Let $\mathcal{U} = \{U : \mathrm{Im}(\pi_1(U) \to \pi_1(X)) \text{ is trivial}\}$. By assumption $\mathcal{U}$ is a basis for X. For each $u \in \mathcal{U}$ and each γ from x_0 to a point in U, we define $U_{[\gamma]} = \{\gamma \cdot \eta\} : \eta \text{ starting at } \gamma(1) \text{ stays in } U\}$. Then $p : \tilde{X} \to X$ by $[\gamma] \to \gamma(1)$.
We need to check that $\{U_{[\gamma]} : U \in \mathcal{U}, \gamma \text{ a path from } x_0 \text{ to a point in } U\}$ generates a topology on $\tilde{X}$.
We need also to check that $p : U_{[\gamma]} \to U$ is bijective. It is clearly surjective, and if $p[\gamma \cdot \eta] = p[\gamma \cdot \delta]$ with η, δ paths starting at $\gamma(1)$ and staying in U. Then $\eta(1) = \delta(1)$ and, since η, δ share the same endpoints and they stay in $U_{[\gamma]}$, then $[\eta] = [\delta]$. Hence $[\gamma \cdot \eta] = [\gamma \cdot \delta]$ and p is injective.
Further, we need to check that $p : U_{[\gamma]} \to U$ is a homemorphism and that $p^{-1}(U) = \dot{\bigcup}_{[\gamma]} U_{[\gamma]}$. Hence p is a covering map.
Finally, we need to check that $\tilde{X}$ is simply connected. Recall that $p : \tilde{X} \to X$ induces an injective homomorphism $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$. It suffices to show that $p_* \pi_1(\tilde{X}, \tilde{x}_0) = \mathrm{id}$. We se t$\tilde{x}_0 = [C_{x_0}] \in \tilde{X}$. Recall also that elements in $p_* \pi_1(\tilde{X}, \tilde{x}_0)$ are exactly the loops in X at x_0 such that they lift to loops at $\tilde{x}_0$. Suppose $[\gamma] \in p_* \pi_1(\tilde{X}, \tilde{x}_0)$. Then γ lifts to a loop $\tilde{\gamma}$ at $\tilde{x}_0 = [C_{x_0}]$. For $t \in [0,1]$, consider the path γ_t which follows γ on $[0,t]$ then stays stationary at $\gamma(t)$ for the remaining time. Then $t \mapsto [\gamma_t]$ is a path on $\tilde{X}$, $p([\gamma_t]) = \gamma_t(1) = \gamma(t)$, and $t \mapsto [\gamma_t]$ is a lift of γ at $\tilde{x}_0 = [C_{x_0}]$. Then $t \mapsto [\gamma_t]$ is a loop (i.e. $[\gamma] = [\gamma_1] = \tilde{x}_0 = [C_{x_0}]$) and γ is null-homotopic. This shows that $p_* \pi_1(\tilde{X}, \tilde{x}_0) = \mathrm{id}$ (i.e. $\tilde{X}$ is simply connected).

Group Actions on Fibers (Monodromy Action)

Given $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ a covering map, $\pi_1(X, x_0)$ acts on p^{-1} as follows: $p^{-1}(x_0) \times \pi_1(X, x_0) \to p^{-1}(x_0)$ by $(e, [f]) \mapsto \tilde{f}_e(1)$ where $\tilde{f}_e$ is the (unique) lift of f at $e \in p^{-1}(x_0)$. This is a right $\pi_1(X, x)$ action.
We want to check that $(e \cdot [f]) \cdot [g] = e \cdot [f \cdot g]$. We have that $e \cdot [f \cdot g] = \widetilde{(f \cdot g)}_e(1)$, but $\widetilde{(f \cdot g)}_e$ is the lift of f at e followed by the lift of g at the endpoint of $\tilde{f}_e$, call it $\tilde{f}_e(1) = z$. Then $\widetilde{(f \cdot g)}_e(1) = \tilde{g}_z(1) = z \cdot [g] = (e \cdot [f]) \cdot [g]$.
This action is transitive. Given e and e', draw a path connecting them $\tilde{g}$. Under the map p, we have that $p \circ \tilde{g} = g$ which is a loop at x_0. Then $e \cdot [g] = \tilde{g}(1) = e'$.
Recall: Given a right G-set S, $G_S = \{g \in G : s \cdot g = s\}$ is the isotropy subgroup at $s \in S$.
Given $e \in p^{-1}(x_0)$, the isotropy subgroup at e is all the loops such that their lfts at e are loops (i.e. the isotropy subgroup at e is precisely $p_* \pi_1(\tilde{X}, e)$).
Recall: $G \cdot S = G / G_s$. Here, this tells us that $p^{-1}(x_0) = \pi_1(X, x_0) / p_* \pi_1(\tilde{X}, e)$. This recovers the fact that the number of sheets is equal to the index of $\mathrm{im}(p_*)$.
In particular, if $\tilde{X}$ is simply connected, then

- $\pi_1(X, x_0)$ acts freely on $p^{-1}(x_0)$ and

- the number of sheets equals the cardinality of $\pi_1(X, x_0)$.

Definition: Universal Cover

A covering space $p : \tilde{X} \to X$ is called universal if it has the universal property (i.e. for any covering space $q : Y \to X$, there is a covering map $\tilde{p} : \tilde{X} \to Y$ such that the associated diagram commutes).

$$\tilde{X} \xrightarrow{\ \tilde{p}\ } Y$$
$$p \downarrow \quad \swarrow q$$
$$X$$

Definition: Covering Homomorphism

Let $p_1 : X_1 \to X$ and $p_2 : X_2 \to X$ be two covering spaces. A covering homomorphism is a map $\varphi : X_1 \to X_2$ such that the associated diagram commutes

$$X_1 \xrightarrow{\ \varphi\ } X_2$$
$$\searrow^{p_1} \quad \swarrow_{p_2}$$
$$X$$

By definition, φ is a lift of p_1.

Proposition

1. A covering homomorphism φ is uniquely determined by its value at one point.

2. For each $x \in X$, $\varphi|_{p_1^{-1}(x)} : p_1^{-1}(x) \to p_2^{-1}(x)$ is $\pi_1(X, x_0)$-equivariant.

3. A covering homomorphism $\varphi : X_1 \to X_2$ is a covering map. Assuming this, the universal cover is unique.

Recall: if S_1, S_2 are right G-sets, a G-equivariant map $\varphi : S_1 \to S_2$ is a map such that the associated diagram commutes

$$S_1 \xrightarrow{\ \varphi\ } S_2$$
$$\downarrow^{\cdot g} \qquad \downarrow^{\cdot g}$$
$$S_1 \xrightarrow{\ \varphi\ } S_2$$

Proof of 2

Let $e \in p_1^{-1}(x)$. We need to show taht $\varphi(e) \cdot g = \varphi(e \cdot g)$. We have that $g \in \pi_1(X, x_0)$ is represented by a loop f at x_0. So $e \cdot g = e \cdot [f] = \tilde{f}_e(1) \in X_1$, and $\varphi(e \cdot g) = \varphi(\tilde{f}_e(1))$. On the left hand side, we have that $\varphi(e) \cdot g = f_{\varphi(e)}(1) \in X_2$. We need to verify that $\varphi(\tilde{f}_e) = \tilde{f}_{\varphi(e)}$ which are both lifts of f at $\varphi(e)$. But since the diagram commutes, $p_2(\varphi \circ \tilde{f}_e) = p_1 \circ \tilde{f}_e = f$.

Uniqueness in 3

Suppose we have

$$X_1 \overset{\psi}{\underset{\varphi}{\rightleftarrows}} X_2$$
$$\searrow^{p_1} \quad \swarrow_{p_2}$$
$$X$$

with $\varphi(e_1) = e_2$ and $\psi(e_2) = e_1$. Then $\psi \circ \varphi(e_1) = e_1$. Hence $\psi \circ \varphi = \mathrm{id}$ and, similarly, $\psi \circ \varphi = \mathrm{id}$. Hence φ is a bijection and a homemorphism.
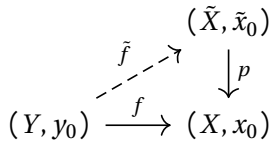
Proof of 3

φ is surjective. Given any $e' \in X_2$, set $x_0 = p_2(e)$ and let $e \in p_1^{-1}(x_0)$ so $\varphi(e) \in p_2^{-1}(x_0)$. Since $\pi_1(X, x_0)$ acts transitively on $p_2^{-1}(x_0)$, there exists $g \in \pi_1(X, x_0)$ such that $e' = \varphi(e) \cdot g = \varphi(e \cdot g)$
φ is a covering map. Let V be a neighborhood of $x_0 \in X$ such that V is evenly covered by both p_1 and p_2. Let U be a component in $p_2^{-1}(V)$ that contains e_2. Then $p_1^{-1}(X) = \bigcup U_\alpha$. U as a component in $p_2^{-1}(V)$ is both open and closed.

Hence $\varphi^{-1}(U)$ is open and closed in $p_1^{-1}(V) = \dot{\bigcup} U_\alpha$. It follows that $\varphi^{-1}(U)$ is the disjoint union of several components of $\{U_\alpha\}_\alpha$, and each component is homemorphic to V and consequently homeomorphic to U. This shows that φ is a covering map.
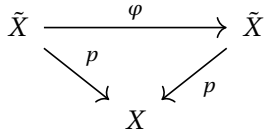
April 21, 2025

Recall: Lifting Criterion

$$
\begin{array}{ccc}
 & & (\tilde{X}, \tilde{x}_0) \\
 & \overset{\tilde{f}}{\nearrow} & \downarrow p \\
(Y, y_0) & \overset{f}{\longrightarrow} & (X, x_0)
\end{array}
$$
There exists a lift $\tilde{f}$ of f at $\tilde{x}_0$ if and only if $f_* \pi_1(Y, y_0) \subseteq p_* \pi_1(\tilde{X}, \tilde{x}_0)$.

If $(\tilde{X}, \tilde{x}_0) \overset{p}{\to} (Y, y_0)$, $\pi_1(X, x_0)$ acts transitively on $p^{-1}(x_0)$ by path lifting (a right action where $e \in p^{-1}(x_0)$ by $e \cdot [\gamma] = \tilde{\gamma}_e(1)$). The isotropy subgroup at e is $p_* \pi_1(\tilde{X}, e)$.

Covering Transformations

$$
\begin{array}{ccc}
\tilde{X} & \overset{\varphi}{\longrightarrow} & \tilde{X} \\
 & \searrow p \quad p \swarrow & \\
 & X &
\end{array}
$$
Write $\mathrm{Aut}(\tilde{X} \overset{p}{\to} X)$ for the covering group $\{\varphi : \tilde{X} \to \tilde{X} \text{ covering transformations}\}$.

1. $\varphi : \tilde{X} \to \tilde{X}$ is uniquely determined by its value at one point.

2. Given $e_1, e_2 \in p^{-1}(x)$, there is $\varphi \in \mathrm{Aut}(\tilde{X} \overset{p}{\to} X)$ if and only if $p_* \pi_1(\tilde{X}, e_1) = p_* \pi_1(\tilde{X}, e_2)$. In fact, for $\varphi \in \mathrm{Aut}(\tilde{X} \overset{p}{\to} X)$ with $p_* \pi_1(\tilde{X}, e_1) \subseteq p_* \pi_1(\tilde{X}, e_2)$.

3. $\varphi|_{p^{-1}(x)} : p^{-1}(x) \to p^{-1}(x)$ is $\pi_1(X, x)$-equivariant (i.e. $\varphi(e) \cdot \gamma = \varphi(e \cdot \gamma)$.

Example

Given $p : \mathbb{R} \to S^1$, what is $\mathrm{Aut}(\mathbb{R} \overset{p}{\to} S^1)$?
$1 \in S^1$, $p^{-1}(1) = \mathbb{Z} \subseteq \mathbb{R}$, $\forall \varphi \in \mathrm{Aut}(\mathbb{R} \overset{p}{\to} S^1)$, $\varphi(0) = k \in \mathbb{Z}$. Then $\varphi(x) = x + k$. In fact, the map $x \mapsto x + k$ is a covering transofmratio that agrees with φ at $0 \in \mathbb{R}$. Hence they agree everywhere (i.e. $\varphi(x) = x + k$ for all x).

Example

Given $p : S^2 \to \mathbb{RP}^2$, then $\mathrm{Aut}(S^2 \overset{p}{\to} \mathbb{RP}^2) = \{\mathrm{id}, A\}$ with A the antipodal map.

Proposition: Normal Covering

Let $\tilde{X} \overset{p}{\to} X$ be a covering map. The following are equivalent

1. There exists $x \in X$ such that $p_* \pi_1(\tilde{X}, e)$ is normal for one (thus for all) $e \in p^{-1}(x)$.

2. For every $x \in X$ and each $e \in p^{-1}(x)$, $p_* \pi_1(\tilde{X}, e)$ is normal.

3. $\mathrm{Aut}(\tilde{X} \xrightarrow{p} X)$ acts transitively on some (thus all) fiber $p^{-1}(x)$.

If any of these hold, we say that $p : \tilde{X} \to X$ is a normal covering.

Proof

Suppose $e, e' \in p^{-1}(x)$ with $p_* \pi_1(\tilde{X}, e)$ and $p_* \pi_1(\tilde{X}, e')$. These are the isotropy subgroups at e and e' respectively.
We know also $\pi_1(X, x)$ acts transitively on $p^{-1}(x)$.
Fact: If S is a right G-set, then $G_s = \{h \in G : s \cdot h = s\}$ and $G_{sg} = \{h \in G : s \cdot g \cdot h = s \cdot g\} = \{h \in S : s \cdot g \cdot h \cdot g^{-1} = s\}$.
So $g \cdot G_{sg} \cdot g^{-1} \in G_s$ which implies that $G_{sg} = g^{-1} \cdot G_s \cdot g$. So if G_s is normal then so is G_{sg}.

IMAGE 1

$$
\begin{array}{ccc}
\pi_1(\tilde{X}, e_0) & \xrightarrow{\Phi_{\tilde{h}}} & \pi_1(\tilde{X}, e) \\
\downarrow{p_*} & & \downarrow{p_*} \\
\pi_1(X, x_0) & \xrightarrow{\Phi_h} & \pi_1(X, x)
\end{array}
$$

commutes. Hence Φ_h maps $p_* \pi_1(\tilde{X}, e_0)$ to $p_* \pi_1(\tilde{X}, e)$, and $\Phi_h : \pi_1(X, x_0) \xrightarrow{\sim} \pi_1(X, x)$ preserves normal subgroups.

(3) implies (1)

Finally, for every $e_1, e_2 \in p^{-1}(x)$, there exists $\varphi \in \mathrm{Aut}(\tilde{X} \xrightarrow{p} X)$ such that $\varphi(e_1) = e_2$. This holds if and only if $p_* \pi_1(\tilde{X}, e_1) = p_* \pi_1(\tilde{X}, e_2)$ for every $e_1, e_2 \in p^{-1}(x)$. That is, $e_2 = e_1 \cdot \gamma$ for some $\gamma \in \pi_1(X, x)$ and $H = \gamma^{-1} H \gamma$ for every $\gamma \in \pi_1(X, x)$. So H is normal.

Remark

The (simply connected) universal cover is always normal because $\{\mathrm{id}\}$ is normal in $\pi_1(X, x)$.

Theorem

Let $p : \tilde{X} \to X$ be a covering map with $x \in X$ and $e \in p^{-1}(x)$. Then $\mathrm{Aut}(\tilde{X} \xrightarrow{p} X) \cong \frac{N_G(H)}{H}$ where $G = \pi_1(X, x)$, $H = p_* \pi_1(\tilde{X}, e)$, and $N_G(H) = \{g \in G : g^{-1} H g = H\}$.

Special Case 1

If $p : \tilde{X} \to X$ is a normal covering, then H is normal in G. Then also $\mathrm{Aut}(\tilde{X} \xrightarrow{p} X) \cong G/H$.

Special Case 2

If $p : \tilde{X} \to X$ is the (simply connected) universal covering, then $\mathrm{Aut}(\tilde{X} \xrightarrow{p} X) \cong \pi_1(X, x)$.

Proof

Let S be a right G-set with transitive action and $\mathrm{Aut}_G(S)\{\varphi : S \to S \ G\text{-equivariant bijections}\}$. Fix $s \in S$. Then $\mathrm{Aut}_G(S) \cong \frac{N_G(H)}{H}$ where $h = G_s$.
Define $\Phi : N_G(H) \to \mathrm{Aut}_G(S)$ by $\gamma \mapsto \Phi(\gamma) = \varphi_\gamma$ with $\varphi_\gamma : S \to S$ defined by

$$
G_{s \cdot \gamma} = \gamma^{-1} H \gamma = H = G_s.
$$

Then there exists a unique $\varphi_\gamma \in \operatorname{Aut}_G(S)$ such that $\varphi_\gamma(s) = s \cdot \gamma$.

- Lemma

  For each $s' \in S$, $s' = s \cdot \gamma'$ for some $\gamma' \in G$. Then $\varphi_\gamma(s') = \varphi_\gamma(s \cdot \gamma') = \varphi_\gamma(s) \cdot \gamma' = s \cdot \gamma\gamma'$. This is well defined. If $s' = s \cdot \gamma''$, then $s = s(\gamma \cdot \gamma'' \cdot (\gamma')^{-1} \cdot \gamma^{-1})$ which implies that $\gamma \cdot \gamma''(\gamma')^{-1} \cdot \gamma^{-1} \in G_s$ and $\gamma'' \cdot (\gamma')^{-1} \in G_s$.

Φ is a group homomorphism since

$$\varphi_{\gamma_1} \circ \varphi_{\gamma_2}(s) = \varphi_{\gamma_1}(s \cdot \gamma_2) = \varphi_{\gamma_1}(s) \cdot \gamma_2 = s \cdot \gamma_1 \cdot \gamma_2.$$

Φ is surjective since letting $\varphi \in \operatorname{Aut}_G(S)$, it maps s to some $\varphi(s) = s' = s \cdot \gamma$ and hence $\varphi = \varphi_\gamma$.

If $\varphi_\gamma = \operatorname{id}$, then $\varphi_\gamma(s) = s$ and $\gamma \in G_s = H$. So Φ induces $\frac{N_G(H)}{H} \cong \operatorname{Aut}_G(S)$.

Take $G = \pi_1(X, x)$ and $\operatorname{Aut}(\tilde{X} \xrightarrow{p} X) \to \operatorname{Aut}_G(p^{-1}(x)) \cong \frac{N_G(H)}{H}$ by $\varphi \mapsto \varphi|_{p^{-1}(x)}$ where H is the isotropy subgroup of the $\pi_1(X, x)$ action at e ($p_* \pi_1(\tilde{X}, e)$). Then $\varphi \mapsto \varphi|_{p^{-1}(x)}$ is injective because it is uniquely determined by its value at one point.

$\varphi \mapsto \varphi|_{p^{-1}(x)}$ is surjective. Letting $\eta \in \operatorname{Aut}_g(p^{-1}(x))$ and $e_1 \in p^{-1}(x)$, we set $e_2 = \eta(e_1)$ and see that $p_* \pi_1(\tilde{X}, e_1) = G_{e_1} = G_{e_2} = p_* \pi_1(\tilde{X}, e_2)$. By the lifting criterion, there exists $\varphi \in \operatorname{Aut}(\tilde{X} \xrightarrow{p} X)$ such that $\varphi(e_1) = e_2$. Then $\varphi|_{p^{-1}(x)} = \eta$ since both are in $\operatorname{Aut}_G(p^{-1}(x))$ and they agree at one point (hence everywhere). Thus we conclude that the map is a bijection and

$$\operatorname{Aut}(\tilde{X} \xrightarrow{p} X) \cong \operatorname{Aut}_G(p^{-1}(x)) \cong \frac{N_G(H)}{H}.$$

Definition: Covering Space Action

Let X be connected and locally path connected with a group action Γ acting by homeomorphism. The quotient map $p : X \to X/\Gamma$ will be a covering map if we impose $(*)$ for all $x \in X$, there exists a neighborhood U of x such that $U \cap (g \cdot U) = \varnothing$ for each $g \in \Gamma - \{\operatorname{id}\}$. In particular, G acts freely on X. We say that a Γ-action on X is a covering space action if $(*)$ if fulfilled.

Counter-example

Consider an $\mathbb{R}$ action on $\mathbb{R}^2$ by translation. Then $U \cap (g \cdot U) \neq \varnothing$.

IMAGE 2

Remark

Assuming $(*)$, $\{g \cdot U : g \in \Gamma\}$ is a disjoint family of open sets.

Example

Take a $\mathbb{Z}$-action by $\mathbb{R}^2$ given by $\gamma(x, y) = (x + 1, -y)$.

IMAGE 3

Example

S^2 with $\mathbb{Z}_2$-action ($\{\operatorname{id}, A\}$).

Theorem

If Γ acts on X as a covering space action, then $q : X \to X/\Gamma$ is a normal covering map.

Proof

Let $\overline{x} \in X/\Gamma$ and pick $x \in q^{-1}(\overline{x})$. By $(*)$, we have a neighborhood U such that $\{g \cdot U : g \in \Gamma\}$ is a disjoint collection. Let $V = q(U)$, an open neighborhood of $\overline{x}$ in X/Γ. Then $q^{-1}(V) = \{g \cdot U : g \in \Gamma\}$. Moreover, $g \cdot U \to V$ is a homeomorphism. If there exist $x', g'x' \in g \cdot U$, then $x' = h_1 \cdot u_1$ and $g' \cdot x' = h_2 \cdot u_2$. So $h_1^{-1} x' \in U$ and $h_2^{-1} g' \cdot x' \in U$ but this holds only for the identity map. So the covering map is injective.

Classifications of Covering Spaces

Take X path-connected, locally path-connected and semi-locally simply path connected (guaranteeing a simply connected universal cover). Then there is a $1-1$ correspondence between

$$\left\{ \begin{array}{c} \text{isomorphism classes of covering} \\ \text{spaces } p:\hat{X} \to X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{conjugacy classes of} \\ \text{subgroups in } \pi_1(X,x) \end{array} \right\}$$

$$(p : \hat{X} \to X) \leftrightarrows [p_* \pi_1(\hat{X}, \hat{e})]$$

April 23, 2025

Recall: Theorem

For X path-connected, locally path-connected and semi-locally simply path connected, Γ acts on X as a covering group action (i.e. $\forall x \in X$, there exists a neighborhood U of x such that $U \cap (g \cdot U) = \varnothing$ for all $g \in \Gamma \setminus \{e\}$).

Then $p : X \to X/\Gamma$ is a normal covering map. Moreover $\mathrm{Aut}(X \overset{p}{\to} X/\Gamma) = \Gamma$.

Proof

($\supseteq$) this follows from

$$X \xrightarrow{g\cdot} X$$
$$\searrow^{p} \quad \swarrow_{p}$$
$$X/\Gamma$$

($\subseteq$) Let $\varphi \in \mathrm{Aut}(X \overset{p}{\to} X/\Gamma)$. That is

$$X \xrightarrow{\varphi} X$$
$$\searrow^{p} \quad \swarrow_{p}$$
$$X/\Gamma$$

commutes with φ a homeomorphism. Now let $x \in p^{-1}(\overline{x})$ where $\overline{x} \in X/\Gamma$, and let $x' = \varphi(x)$. Then $p(x) = \overline{x} = p(x')$, hence $x, x' \in p^{-1}(\overline{x})$. Hence there is $g \in \Gamma$ such that $gx = x'$. So we have

$$\varphi : X \to X \varphi(x) = x'$$
$$g : X \to X g(x) = x'$$

so φ is equivalent to an action by g.

Theorem

Take X path-connected, locally path-connected and semi-locally simply path connected (guaranteeing a simply connected universal cover). Then there is a $1-1$ correspondence between
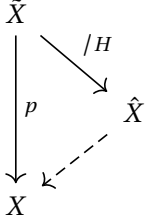
$$\left\{\begin{array}{l}\text{isomorphism classes of}\\\text{covering maps } p{:}\hat{X}{\to}X\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{conjugacy classes of}\\\text{subgroups in } \pi_1(X,x_0)\end{array}\right\}$$

$\rightarrow$ Assign a subgroup $H = p_*(\hat{X}, \hat{e})$ for $\hat{e} \in p^{-1}(x_0)$.

$\leftarrow$ Given a conjugacy class of subgroups, pick a subgroup H in the class.

$$H \leq \pi_1(X, x_0) \cong \mathrm{Aut}(\tilde{X} \xrightarrow{p} X)$$

Hence H acts naturally on $\tilde{X}$ as covering transformations. Consider $q : \tilde{X} \to \tilde{X}/H =: \hat{X}$, a normal covering map.

$$\tilde{X}$$
$$/H$$
$$p \qquad \hat{X}$$
$$X$$

Since $\tilde{X}/\pi_1(X, x_0) = x$, we have an induced map $\hat{p} : \hat{X} \to X$. We need to show that $\hat{p} : \hat{X} \to X$ is a covering map with $\hat{p}_* \pi_1(\hat{X}, \hat{e}) = H$ for some $\hat{e} \in \hat{p}^{-1}(x)$. Let U be a neighborhood of x such that $p^{-1}(U) = \bigsqcup_\alpha \tilde{U}_\alpha$. Then $\{\tilde{U}_\alpha\}$ is a collect iof disjoint open sets and identical to $\{g \cdot \tilde{U} : g \in \pi_1(X, x)\}$ where $\tilde{U}$ is a component of $p^{-1}(U)$. The H-action permutes the copies in $\{g \cdot \tilde{U}\} = \{\tilde{U}_\alpha\}$. Hence $q|_{\tilde{U}_\alpha} : \tilde{U}_\alpha \to \hat{X}$ is a homeomorphism. Let $\hat{U}$ be a component in $\hat{p}^{-1}(U)$. Then $q^{-1}(\hat{p}^{-1}(U)) = p^{-1}(U) = \bigsqcup_\alpha \tilde{U}_\alpha$ where $q^{-1}(\hat{U})$ is a union of components in $\bigsqcup_\alpha \tilde{U}_\alpha$. Hence $\hat{U}$ is homeomorphic to U, and $\hat{p}^{-1}(U)$ is a union of components that are homemorphic to U.

Lastly, we show that $\hat{p}_* \pi_1(\hat{X}, \hat{e}_0) = H$. This is the isotropy subgroup of $\pi_1(X, x_0)$-actions at $\hat{e}_0$. $q|_{p^{-1}(x_0)} : p^{-1}(x_0) \to \hat{p}^{-1}(x_0)$ is $\pi_1(X, x_0)$-equivariant (i.e. $q(e \cdot \gamma) = q(e) \cdot \gamma$, $q(e) = \hat{e}$ for $e \in \tilde{X}$). Hence γ fixes $q(e) = \hat{e}$ if and only if $q(e \cdot \gamma) = q(e)$, if and only if $e \cdot \gamma$ and e are in the same H-orbit, if and only if $\gamma \in H$.

Example 1

$X = S^1$ with $\pi_1(S^1) = \mathbb{Z}$.

$\mathbb{Z}$ has subgroups $\mathbb{Z}, 2\mathbb{Z}, 3\mathbb{Z}, \ldots, k\mathbb{Z}, \ldots$ where $k\mathbb{Z}$ corresponds to the covering map $p_k : z \mapsto z^k$.

Example 2

X the Mobius strip with $\pi_1(X) = \mathbb{Z}$ with $\pi_1(X) = \langle \gamma \rangle$ and $\gamma(x, y) = (x + 1, -y)$.

Take $H = 2\mathbb{Z} = \langle 2\gamma \rangle \leq \mathbb{Z}$. Then $2\gamma(x, y) = (x + 2, y)$ and $\mathbb{R}^2/H$ is the cylinder while the cylinder modulo $\mathbb{Z}_2$ is the mobius strip.

Example 3

The Klein bottle, $K = \mathbb{R}^2/\Gamma$ with Γ genereted by $g(x, y) = (x + 1, -y)$ and $h(x, y) = (x, y + 1)$.

So $\pi_1(K) = \langle g, h \rangle$. $g^2(x, y) = (x + 2, y)$ commutes with h, so $\mathbb{Z}^2 \cong \langle g^2, h \rangle \leq \pi_1(K)$ and $\mathbb{R}^2/\langle g^2, h \rangle = \mathbb{T}^2$ covers K.

Simplexes

IMAGE 1

The standard n-simplex is

$$\Delta^n = \left\{ (t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} t_i = 1, \; t_i \geq 0, \forall i \right\}$$

$$\Delta^1 = \left\{ (t_0, t_1) \in \mathbb{R}^2 : t_0 + t_1 = 1, \; t_0, t_1 \geq 0 \right\}$$

$$\Delta^2 = \left\{ (t_0, t_1, t_2) \in \mathbb{R}^3 \ : \ t_0 + t_1 + t_2 = 1, \ t_0, t_1, t_2 \geq 0 \right\}$$

IMAGE 3

Δ^n has $(n+1)$-many faces $((n+1)$-simplex) where the ith face is $\Delta^{n-1} \to \Delta^n$ by $(t_0, \ldots, t_{n-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$. Let X be a topological space. A Δ-complex structure on X is a family of maps $\sigma_\alpha : \Delta^n \to X$ (n may depend on α) such that

1. $\sigma_\alpha|_{\overset{\circ}{\Delta}{}^n} : \overset{\circ}{\Delta}{}^n \to X$ is injective and each point is in the image of at most one of $\sigma_\alpha|_{\circ\Delta^n}$.

2. $\sigma_\alpha|_{\text{a face of } \Delta^N}$ is some $\sigma_\beta : \Delta^{n-1} \to X$ in the family.

3. $A \subseteq X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for all α.

σ_β is $\Delta^{n-1} \overset{i\text{th face}}{\to} \Delta^n \overset{\sigma}{\to} X$.

Example

S^1 is the following iwht 1-simplex

IMAGE 4

Then the "body" of $\Delta^1 \overset{\sigma}{\to} X$ is

IMAGE 5

with $\sigma \circ \delta_0 : \Delta^0 \to X$ and $\sigma \circ \delta_1 : \Delta^0 \to X$. The boundary $\partial\sigma = \sum_{i=0}^n (-1)^i \sigma \circ \delta_i$. They define $\delta : C_n(X) \to C_{n-1}(X)$. For this example, we have $\partial\sigma = \sigma \circ \delta_0 + (-1)\sigma \circ \delta_1 = 0$.
The ith face is $\delta_i : \Delta^{n-1} \to \Delta^n$ by $(t_0, \ldots, t_{n-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$.
In Hatcher's notation, the boundary is $\partial\sigma = \sum_{i=0}^n (-1)^i \sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}$ where we shoild think of $[v_0, \ldots, \hat{v}_i, \ldots, v_n]$ as the ith face. So $\sigma : \Delta^n = [v_0, \ldots, v_n] \to X$. Now we have

$$\cdots \overset{\partial}{\to} C_n(X) \overset{\partial}{\to} C_{n-1}(X) \cdots$$

where $\partial^2 = 0$.

Proof

$$\begin{aligned}
\partial(\partial\sigma) &= \partial\left(\sum_{i=0}^n (-1)^i \sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]} \right) \\
&= \sum_{i=0}^n (-1)^i \partial(\sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}) \\
&= \sum_{j<i} (-1)^i (-1)^j \sigma_{[v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_n]} + \sum_{j>i} (-1)^i (-1)^{j-1} \sigma_{[v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_n]} \\
&= 0
\end{aligned}$$

Homoology Associated to the Delta Complex

We have $\ker\partial \supseteq \operatorname{im}\partial$ where $\ker\partial$ are the n-cycles and $\operatorname{im}\partial$ are the n-bodies, and

$$H_n^\delta(X) = \frac{\{n\text{-cycles}\}}{\{n\text{-bodies}\}} = \frac{\ker\partial}{\operatorname{im}\partial}$$

Example

For the circle, $C_1(X) = \mathbb{Z} = \langle \sigma \rangle$ and $C_0(X) = \mathbb{Z} = \langle v \rangle$. Therefore

$$\overbrace{C_2(X)}^{=0} \to \overbrace{C_1(X)}^{=\mathbb{Z}} \xrightarrow{0} \overbrace{C_0(X)}^{=\mathbb{Z}} \to 0$$

Then $H_1^\Delta(X) = \frac{\ker\partial}{\operatorname{im}\partial} = \mathbb{Z}/\{0\} = \mathbb{Z}$ and $H_0^\Delta(X) = \frac{\ker\partial}{\operatorname{im}\partial} = \mathbb{Z}$.

An Aside

IMAGE 7

$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

Example

For the torus, draw

IMAGE 6

So $C_0(X) = \langle v \rangle = \mathbb{Z}$, $C_1(X) = \langle a, b, c \rangle = \mathbb{Z}^3$ and $C_2(X) = \langle U, L \rangle = \mathbb{Z}^2$. Then also $\partial U = a + b - c$ and $\partial L = a + b - c$, so $\partial(U - L) = 0$ and $\ker\partial_2 = \langle U - L \rangle \cong \mathbb{Z}$. That is $H_2^\Delta(X) = \frac{\ker\partial}{\operatorname{im}\partial} \cong \mathbb{Z}$. Now $\partial a = 0 = \partial b = \partial c$, so $\ker\partial_1 = \langle a, b, c \rangle$ and $H_1^\Delta(X) = \frac{\ker\partial_1}{\operatorname{im}\partial_2} = \frac{\langle a, b, a+b-c \rangle}{\langle a+b-c \rangle} \cong \mathbb{Z}^2$. Finally we have that $H_0^\Delta = \frac{\ker\partial_0}{\operatorname{im}\partial_1} = \frac{\langle v \rangle}{\{0\}} \cong \mathbb{Z}$.

Example

For $\mathbb{RP}^2$, draw

IMAGE 8

$AC_0(X) = \langle v, w \rangle \cong \mathbb{Z}^2$, $C_1(X) = \langle a, b, c \rangle \cong \mathbb{Z}^3$, and $C_2(X) = \langle U, L \rangle \cong \mathbb{Z}^2$. Then $\partial U = a + b + c$ while $\partial L = a + b - c$, so $\ker\partial_2 = \{0\}$ and $H_2^\Delta \frac{\ker\partial_2}{\operatorname{im}\partial_3} = \{0\}$. $\partial_1(a) = w - v$, $\partial_1(b) = v - w$ and $\partial_1(c) = 0$, so $\ker\partial_1 = \langle c, a - b \rangle$ and

$$H_1^\Delta(X) = \frac{\ker\partial_1}{\operatorname{im}\partial_2} = \frac{\langle c, a + b \rangle}{\langle a + b + c, a + b - c \rangle} = \langle a + b + c, c \rangle / \langle a + b + c, 2c \rangle \cong \langle c \rangle / \langle 2c \rangle \cong \mathbb{Z}^2.$$

April 28th, 2025

Recall:

For X with a Δ-complex structure, we have $H_n^\Delta(X)$.

Definition: Singular Simplex

A singular n-simplex is a continuous map $\sigma : \Delta^n \to X$.
The singular chain $C_n(X)$ is the free Abelian group generated by singular n-simplecies. Write

$$C_n(X) = \left\{ \sum n_i \sigma_i \ : \ \left| \sum n_i \sigma_i \right| < \infty, \ n_i \in \mathbb{Z}, \ \sigma_i : \Delta^n \to X \right\}$$

$$\cdots \xrightarrow{\partial} C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial} \cdots$$

$$\partial(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma |_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}$$

While $\partial^2 = 0$ and $H_n(X) = \ker \partial_n / \operatorname{Im} \partial_{n+1}$ is the singular homology.

Proposition

If $X = \coprod_\alpha X_\alpha$ with X_α connected components of X, then $H_n(X) \cong \oplus_\alpha H_n(X_\alpha)$.

Proof

$\sigma : \Delta^n \to x$, $\operatorname{im} \sigma \subseteq X_\alpha$ for some α. So $C_n(X) = \oplus_\alpha C_n(X_\alpha)$ and $\partial : C_n(X) \to C_{n-1}(X)$ maps $C_n(X_\alpha)$ to $C_{n-1}(X_\alpha)$.
Therefore $\ker \partial_n = \oplus_\alpha \ker(\partial|_{C_n(X_\alpha)})$ and $\operatorname{im} \partial_{n+1} = \oplus_\alpha \operatorname{im}(\partial|_{C_{n+1}(X_\alpha)})$. Then $H_n(X) \cong \oplus_\alpha \ker(\partial|_{C_n(X_\alpha)})/\oplus_\alpha \operatorname{im}(\partial|_{C_{n+1}(X_\alpha)}) \cong$
$\oplus_\alpha H_n(X_\alpha)$.

Proposition

Let X be a point. Then

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \geq 1 \end{cases}$$

Proof

For each n, $C_n(X)$ is generated by a single element $\sigma_n : \Delta^n \to p$ so $C_n(X) \cong \mathbb{Z}$. Then

$$\partial \sigma_n = \sum_{i=0}^{n} (-1)^i \sigma_n|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]} = \sum_{i=0}^{n} (-1)^i \sigma_{n-1} = \begin{cases} 0 & n \text{ odd} \\ \sigma_{n-1} & n \text{ even} \end{cases}$$

$$\cdots \longrightarrow C_{n+1}(X) \longrightarrow C_n(X) \longrightarrow C_{n-1}(X) \longrightarrow \cdots$$

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \cdots$$

We see that

$$\partial_n = \begin{cases} \cong & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Therefore $\ker / \operatorname{im} = 0$ or $\ker / im = \mathbb{Z}/\mathbb{Z} = 0$. Because

$$C_1(X) \xrightarrow{0} C_0(X) \xrightarrow{0} 0$$

we have thata $H_0(X) = \ker / \operatorname{im} = \mathbb{Z}/\{0\} = \mathbb{Z}$.

Proposition

If X is path connected, then $H_0(X) \cong \mathbb{Z}$.

Proof

Define a map $\epsilon : C_0(X) \to Z$ by $\sum n_i \sigma_i \mapsto \sum n_i$ given that $\sigma_i : \{v\} \to X$. Then ϵ is surjective. Also,

$$H_0(X) = \ker \partial_0 / \operatorname{im} \partial_1 = C_0(X) / \operatorname{im} \partial_1 = C_0(X) / \ker \epsilon \cong \operatorname{im} \epsilon \cong \mathbb{Z}$$

We claim that $\ker \epsilon = \operatorname{im} \delta$.
($\supseteq$) Let $\sigma : \Delta^1 \to X$, $\epsilon(\delta_1(\sigma)) = \epsilon(v_1 - v_0) = 1 - 1 = 0$.
($\subseteq$) Let $\sum n_i \sigma_i \in C_0(X)$ such that $0 = \epsilon(\sum n_i \sigma_i) = \sum n_i$. We fix a point $x_0 \in X$. Because X is path-connected, we can draw paths τ_i from x_0 to σ_i. Consider $\sum n_i \tau_i \in C_1(X)$, then

$$\partial(\sum n_i \tau_i) = \sum n_i(\partial \tau_i) = n_i(\sigma_i - x_0) = \sum n_i \sigma_i - \overbrace{\sum n_i x_0}^{=0} = \sum n_i \sigma_i$$

Reduced Homology

$$\cdots \longrightarrow C_n(X) \longrightarrow \cdots \longrightarrow C_1(X) \longrightarrow C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

Usually written as $\tilde{H}_n(X)$, and $\tilde{H}_n(X) = H_n(X)$ if $n \geq 1$. We have that $\tilde{H}_0(X) = \ker \epsilon / \operatorname{im} \partial_1$ and $\epsilon|_{\operatorname{im} \partial_1} = 0$ so ϵ induces a map $\tilde{\epsilon} H_0(X) \hookrightarrow \mathbb{Z}$. Then $\ker \tilde{\epsilon} = \tilde{H}_0(X)$. It follows that

$$0 \longrightarrow \tilde{H}_0(X) \longrightarrow H_0(X) \longrightarrow \mathbb{Z} \longrightarrow 0$$

is a split exact sequence since $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$. In particualr, $\tilde{H}(\text{pt}) = \{0\}$.

Remark

$$\pi_1 / [\pi_1, \pi_1] \cong H_1$$

Homotopy Invariance

Suppose we have $f : X \to Y$ continuous. It induces $f_\sharp : C_n(X) \to C_n(Y)$ by $\sigma \mapsto f \circ \sigma$. $f_\sharp$ is called a chain map and the following diagram commutes

$$\cdots \longrightarrow C_n(X) \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots$$
$$\downarrow{f_\sharp} \qquad \downarrow{f_\sharp}$$
$$\cdots \longrightarrow C_n(X) \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots$$

Let $\sigma \in C_n(X)$ and

$$f_\sharp(\partial \sigma) = f_\sharp\left(\sum_{i=0}^{n}(-1)^i \sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}\right) = \sum_{i=0}^{n}(-1)^i(f \circ \sigma)|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]} = \partial(f_\sharp \sigma)$$

Then $f_\sharp$ maps cycles to cycles ($\partial c = 0$, $\partial(f_\sharp c) = f_\sharp(\sigma c) = 0$) and boundaries to boundaries ($f_\sharp(\partial c) = \partial(f_\sharp c)$). So $f_\sharp$ induces $f_* : H_n(X) \to H_n(Y)$.

Theorem

If $f, g : X \to Y$ are homotopic, then $f_* = g_* : H_n(X) \to H_n(Y)$ for all n.

Corollary

If $X \simeq Y$ are homotopic, then $H_n(X) \cong H_n(Y)$. $g \circ f \simeq \mathrm{id}_X$, $f \circ g \simeq \mathrm{id}_Y$,

$$g_* \circ f_* = (g \circ f)_* = (\mathrm{id}_X)_* = \mathrm{id}$$

and similarly $g_* \circ f_* = \mathrm{id}$. So f_* and g_* are isomorphisms.

Definition

Let $f, g : C.(X) \to C.(Y)$ be two chain maps. We say that f and g are chain homotopic if there is a map $p : C_n(X) \to C_{n+1}(Y)$ such that $\partial P + P\partial = g - f$.

$$\cdots \longrightarrow C_n(X) \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots$$

with vertical maps f,g and diagonal maps P

$$\cdots \longrightarrow C_n(X) \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots$$

Theorem

If $f \simeq g$ are homotopic, then

1. $f_\sharp$ and $g_\sharp$ are chain homotopic,

2. $f_* = g_*$ on homoology

3. For any n-cycle, $c \in C_n(X)$, $g(c) - f(c) = \partial P(c) + \overbrace{P(\partial c)}^{=0}$. Hence $g_*[c] = f_*[c]$.

Proof

Consider $\Delta^n \times I$, and set $\Delta^n \times \{0\} = [v_0, \ldots, v_n]$ and $\Delta^n \times \{1\} = [w_0, \ldots, w_n]$. Then the following are all n-simplicies

$$[v_0, v_1, \ldots, v_{n-1}, v_n]$$
$$[v_0, v_1, \ldots, v_{n-1}, w_n]$$
$$[v_0, v_1, \ldots, w_{n-1}, w_n]$$
$$\vdots$$
$$[v_0, w_1, \ldots, w_{n-1}, w_n]$$
$$[w_0, w_1, \ldots, w_{n-1}, w_n]$$

They divide $\Delta^n \times I$ into $(n+1)$-simplicies, $\{[v_0, \ldots, v_i, w_i, \ldots, w_n] : i = 0, \ldots, n\}$. Now let $F : X \times I \to Y$ be a homotopy between f and g. Consider

$$\Delta^n \times I \xrightarrow{\sigma \times \mathrm{id}} X \times I \xrightarrow{F} Y$$

and define $P : C_N(X) \to C_{n+1}(Y)$ by $\sigma \mapsto \sum_{i=0}^n (-1)^i F \circ (\sigma \times \mathrm{id})|_{[v_0, \ldots, v_i, w_i, \ldots, w_n]}$. We need to check that $\partial P + P\partial = g_\sharp - f_\sharp$.

Short Exact Sequences of Chain Complexes Induce Long Exact Sequences of Homology Groups

Applications

1. Rleative homology group.

2. Meyer-Vietoris sequence.

Short Exact Sequences Induce Long Exact Sequences

Suppose we have sequences

$$
\begin{array}{ccccccccc}
0 & & 0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
A_* \longrightarrow \cdots \longrightarrow & A_{n+1} & \xrightarrow{\partial} & A_n & \xrightarrow{\partial} & A_{n-1} & \longrightarrow & \cdots \\
\downarrow i & & \downarrow & & \downarrow & & \downarrow & & \\
B_* \longrightarrow \cdots \longrightarrow & B_{n+1} & \xrightarrow{\partial} & B_n & \xrightarrow{\partial} & B_{n-1} & \longrightarrow & \cdots \\
\downarrow j & & \downarrow & & \downarrow & & \downarrow & & \\
C_* \longrightarrow \cdots \longrightarrow & C_{n+1} & \xrightarrow{\partial} & C_n & \xrightarrow{\partial} & C_{n-1} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & &
\end{array}
$$

So H induces a long exact sequence

$$
\xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C)
$$

$$
\xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} \cdots
$$

where $\partial : H_n(C) \to H_{n-1}(A)$ by $[c] \mapsto [a]$, our connecting homomorphism, for $c \in C_n$. Then we have that the following commutes

$$
\begin{array}{ccc}
a & \xmapsto{\partial} & \\
\uparrow i & & \downarrow i \\
b & \longmapsto \partial b \longmapsto & 0 \\
\downarrow j & \downarrow j & \\
c & \longmapsto & 0
\end{array}
$$

So a is a cycle. We need to show that $\partial a = 0$. Note that $i(\partial a) = \partial(ia) = \partial(\partial b) = 0$. Because i is injective, $\partial a = 0$. ∂ is well defined since

- choice of a: i is injective

- choice of b: suppose $b' \in B_n$ such that $j(b') = j(b) = c$. Then $b - b'$ satisfies $j(b - b') = 0$ and $b - b' \in \ker j = \operatorname{im} i$ (i.e. there exists $a' \in A_n$ such that $i(a') = b - b'$, so $b' = b + i(a')$. Then

$$a' \longmapsto \partial a'$$
$$\downarrow$$
$$b - b'$$
$$\downarrow$$
$$0$$

So $a + \partial a'$ satisfies

$$i(a + \partial a') = i(a) + i(\partial a') = \partial b + \partial(i a') = \partial b'$$

and

$$a + \partial a'$$
$$\downarrow$$
$$b' \longmapsto \partial b'$$
$$\downarrow$$
$$c$$

since $[a + \partial a'] = [a]$.

- We need to check choice of c, but we will skip this.

- We need to check that δ is a homomorphism, which follows from the definitions.

- Finally, check that the induced long sequence is exact. We will check only exactness about $H_n(C)$ (i.e. $\operatorname{im} j_* = \ker \delta$).

$\operatorname{im} j_* \subseteq \ker \delta$: $\delta(j_*[b]) = 0$ because

$$0$$
$$\downarrow$$
$$b \xmapsto{\ \partial\ } 0$$
$$\downarrow j$$
$$j(b)$$

$\ker \delta \subseteq \operatorname{im} j_*$: Suppose $[c] \in H_n(C)$ such that $\partial[c] = 0$, then

$$a' \xmapsto{\ \partial\ } a = \partial a'$$
$$\downarrow$$
$$b \longmapsto \partial b$$
$$\downarrow j$$
$$c$$

Consider $b - i(a')$, then $j(b - i(a')) = j(b) - \overbrace{j \circ i(a')}^{=0} = j(b) = c$. So $[c] = j_*[b - i(a')] \in \operatorname{im} j_*$. This is a cycle, since $\partial(b) - \partial(i(a')) = \partial b - i(\partial a') = \partial b - \partial b = 0$.

April 30, 2025

Recall

1. if $f, g : X \to Y$ are homotopic, then $f_* = g_* : H_n(X) \to H_n(Y)$.

$$\cdots \xrightarrow{\partial} C_{n+1}(X) \xrightarrow{\partial} C_n(X) \xrightarrow{\partial} \cdots$$

$$\downarrow f_\# = g_\# \qquad \downarrow f_\# = g_\#$$

$$\cdots \xrightarrow{\partial} C_{n+1}(Y) \xrightarrow{\partial} C_n(Y) \xrightarrow{\partial} \cdots$$

$$\partial P + P \partial = f_\# - g_\#.$$

1. Short exact sequence of chain complexes

$$0 \longrightarrow A_* \xrightarrow{i} B_* \xrightarrow{j} C_* \longrightarrow 0$$

induces a long exact sequence of homology groups

$$\cdots \longrightarrow H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(C) \longrightarrow \cdots$$

$$\xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots$$

Relative Homoology Group

Setup: $A \subseteq X$, A closed and non-empty. Then

$$C_n(A) = \{ c \in C_n(X) : c = \sum n_i \sigma_i, \operatorname{im} \sigma_i \subseteq A \}.$$

Define $C_n(X, A) = C_n(X)/C_n(A)$ such that

$$0 \longrightarrow C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X, A) \longrightarrow 0$$

is a short exact sequence. Then $C_*(X, A)$ is a chain complex

$$\cdots \xrightarrow{\partial} C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \xrightarrow{\partial} \cdots$$

with $\partial^2 = 0$. Note that $\partial : C_n(X) \to C_{n-1}(X)$ maps $C_n(A)$ to $C_{n-1}(A)$. Hence it induces $\partial : C_n(X)/C_n(A) \to C_{n-1}(X)/C_{n-1}(A)$. It gives homoology groups $H_n(X, A) = \ker \partial_n / \operatorname{im} \partial_{n+1}$ and induces a long exact sequence

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \longrightarrow \cdots$$

$$\xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots$$

Remarks

1. the elements in $H_n(X, A)$ are represented by relative cycles (i.e. $\alpha \in C_n(X)$ such that $\partial \alpha \in C_{n-1}(A)$).

2. A relative cycle α is trivial in $H_n(X, A)$ means α is a "relative boundary" (i.e. $\alpha = \partial \beta + \gamma$ for $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$).

$\partial : H_n(X, A) \to H_{n-1}(A)$ is defined by $[\alpha] \mapsto [\partial \alpha]$

$$\partial \alpha$$
$$\downarrow i$$
$$\alpha \in C_n(X) \xmapsto{\partial} \partial \alpha \in C_{n-1}(A)$$
$$\downarrow j$$
$$\alpha \in C_n(X, A)$$

We can also define the relative version.

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\cdots \xrightarrow{\partial} C_0(A) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0 \\
\downarrow & & \downarrow{\scriptstyle\cong} \\
\cdots \xrightarrow{\partial} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0 \\
\downarrow & & \downarrow{\scriptstyle 0} \\
\cdots \xrightarrow{\partial} C_0(X,A) \xrightarrow{0} 0 \longrightarrow 0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

$$\begin{array}{ccc}
\sum n_i \sigma_i & \longrightarrow & \sum n_i \\
\downarrow & & \downarrow{\scriptstyle =} \\
\sum n_i \sigma_i & \longmapsto & \sum n_i
\end{array}$$

$$\tilde{H}_n(A) \longrightarrow \tilde{H}_n(X) \longrightarrow H_n(X,A)$$
$$\xrightarrow{\ \partial\ }$$
$$\longrightarrow \tilde{H}_0(A) \longrightarrow \tilde{H}_0(X) \longrightarrow H_0(X,A)$$
$$\longrightarrow 0$$

Example

$H_n(X,X) = 0$ for all n, because $C_n(X,X) = C_n(X)/C_n(X) = \{0\}$. So $H_n(X,X_0) \cong \tilde{H}_n(X)$

$$\overbrace{\tilde{H}_n(X_0)}^{=0} \longrightarrow \tilde{H}_n(X) \xrightarrow{\ \cong\ } H_n(X,X_0)$$

$$\xrightarrow{\ \partial\ } \overbrace{\tilde{H}_{n-1}(X_0)}^{=0} \longrightarrow \cdots$$

Fact

$H_n(X,A) \cong \tilde{H}_n(X/A)$ if (X,A) is a "good" pair (i.e. there exists a neighborhood V of A which deformation retracts to A).

Example

$(X,A) = (D^n, \partial D^n)$ is a good pair, so $H_i(X,A) \cong \tilde{H}_i(D^n/\partial D^n) = \tilde{H}_i(S^n)$. This give a long exact sequence

$$\tilde{H}_i(S^{n-1}) \longrightarrow \overbrace{\tilde{H}_i(D^n)}^{=0} \longrightarrow H_i(X,A)$$

$$\xrightarrow{\ \partial\ } \tilde{H}_{i-1}(S^{n-1}) \longrightarrow \overbrace{\tilde{H}_{i-1}(D^n)}^{=0} \longrightarrow \cdots$$
and $\tilde{H}_{i-1}(S^{n-1}) \cong H_i(D^n, \partial D^n) \cong \tilde{H}_i(S^n)$. We conclude
that $\tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1})$.

For $n = 0$, S^0 is two points, $\tilde{H}_0(S^0) = \mathbb{Z}$, and $\tilde{H}_i(S^0) = \tilde{H}_i(\text{pt}) \oplus \tilde{H}_i(\text{pt}) = 0$ for each $i \geq 1$.
For $n = 1$, $\tilde{H}_1(S^1) \cong \tilde{H}_0(S^0) \cong \mathbb{Z}$ and $\tilde{H}_0(S^1) = 0$.
For $n = 2$, $\tilde{H}_2(S^2) \cong \tilde{H}_1(S^1) \cong \mathbb{Z}$, $\tilde{H}_1(S^2) \cong \tilde{H}_0(S^1) = 0$ and $\tilde{H}_0(S^2) = 0$.
So $\tilde{H}_i(S^n$ is $\mathbb{Z}$ when $i = n$ and 0 otherwise.

Induced Maps on Pairs

Write $f : (X,A) \to (Y,B)$ for a continuous map $f : X \to Y$ such that $f(A) \subseteq B$. Then $f_\sharp : C_n(X) \to C_n(Y)$ ($f_\sharp : C_n(A) \to C_n(B)$) induces $f_\sharp : C_n(X,A) \to C_n(Y,B)$ a chain map $\partial f_\sharp = f\sharp \partial$. This induces $f_* : H_n(X,A) \to H_n(Y,B)$.

Proposition

Given $f, g : (X,A) \to (Y,B)$ which are homotopic through maps between pairs $(X,A) \to (Y,B)$, then $f_* = g_* : H_n(X,A) \to H_n(Y,B)$.

$$\cdots \longrightarrow C_{n+1}(X,A) \longrightarrow C_n(X,A) \longrightarrow \cdots$$

(diagram with maps p)

$$\cdots \longrightarrow C_{n+1}(Y,B) \longrightarrow C_n(Y,B) \longrightarrow \cdots$$

such that $\partial P + P\partial = g_\sharp - f_\sharp$ (i.e. $f_* = g_*$). $P : C_n(X) \to C_{n+1}(Y)$ maps $C_n(A)$ to $C_{n+1}(B)$. P defined by $P(\sigma) \sum (-1)^i F \circ (0 \times \mathrm{id})|_{[v_0,\dots,v_i,w_i,\dots,w_j]}$

$$\Delta^n \times I \xrightarrow{\ 0\times\mathrm{id}\ } X \xrightarrow{\ F\ } Y$$

If $\sigma : \Delta^n \to A$, then $P(\sigma) : \Delta^{n+1} \to B$.

Excision

Given a good pair (X,A), $H_n(X,A) \cong \tilde{H}_n(X/A)$.
Suppose we have $Z \subseteq A \subseteq X$ such that $\overline{Z} \subseteq A^\circ$ (i.e. the closure of Z is in the interior of A). Then $H_n(X,A) \cong H_n(X-Z, A-Z)$. Equivalently, if $B = X - Z$ then $A \cap B = A - Z$ and $\overline{Z} \subseteq A^\circ \implies A^\circ \cup B^\circ = X$. If A and B satisfy $A^\circ \cup B^\circ = X$, then by excision $H_n(X,A) \cong H_n(B, A \cap B)$.

Remark

If X has a Δ-complex structure such that A, $X-Z$ and $A-Z$ are subcomplexes, then we claim that $C_n^\Delta(X,A) = C_n^\Delta(X-Z, A-Z)$ (and $H_n^\Delta(X,A) = H_n^\Delta(X-Z, A-Z)$). In fact, consider $\varphi : C_n^\Delta(X-Z) \to C_n^\Delta(X)/C_n^\Delta(A)$ which factors through

$$C_n^\Delta(X-Z) \xhookrightarrow{\ \iota\ } C_n^\Delta(X) \longrightarrow C_n^\Delta(X,A) = C_n^\Delta(X)/C_n^\Delta(A)$$

Then φ is surjective, $\ker \varphi = C_n^\Delta(A-Z)$ and

$$C_n^\Delta(X,A) = C_n^\Delta(X)/C_n^\Delta(A) = C_n^\Delta(X-Z)/\ker\varphi = C_n^\Delta(X-Z, A-Z)$$

Proof

Let $\{U_\alpha\}_\alpha = \mathcal{U}$ be a collection of subsets such that $\{U_\alpha^\circ\}_\alpha$ is an open cover of X (it will suffices to consider $\mathcal{U} = \{A,B\}$). Write

$$C_n^{\mathcal{U}}(X) = \left\{ \sum n_i \sigma_i \in C_n(X) \ : \ \mathrm{im}\,\sigma_i \subseteq U_j^\circ \text{ for some } j \right\}.$$

Then $\partial : C_n(X) \to C_{n-1}(X)$ maps $C_n^{\mathcal{U}}(X)$ to $C_{n-1}^{\mathcal{U}}(X)$. The chain complex $C_*^{\mathcal{U}}(X)$ gives homoology groups $H_*^{\mathcal{U}}(X)$.

Proposition

$\iota : C_n^{\mathcal{U}} \to C_n(X)$ induces an isomorphism $H_n^{\mathcal{U}}(X) \cong H_n(X)$.
The sketch of this proof is to construct a map $\rho : C_n(X) \to C_n^{\mathcal{U}}(X)$ by subdivision. That is, if the simplex $\sigma : \Delta^n \to X$ does not sit inside any U_α we may subdivide into further simplices that do. Then $\rho \circ \iota = \mathrm{id}$ and $\iota \circ \rho$ is chain homotopic to the identity.

$$\cdots \longrightarrow C_n(X) \longrightarrow C_{n-1}(X) \longrightarrow \cdots$$

(diagram with maps D and $\iota \circ \rho$)

$$\cdots \longrightarrow C_n(X) \longrightarrow C_{n-1}(X) \longrightarrow \cdots$$

where $D : C_{n-1}(X) \to C_n(X)$ such that $\partial D + D\partial = \mathrm{id} - \iota \circ \rho$ which implies $(\iota \circ \rho)_* : H_n(X) \to H_n(X)$ is the identity map. There also exists a relative version. For simplicty, say $\mathcal{U} = \{A,B\}$ and denote $C_n^{\mathcal{U}}(X) \overset{\Delta}{=} C_n(A+B)$ so we have $H_n(A+B, A) \cong H_n(X,A)$.

Proof Continued

We have that $H_n(A + B, A) \cong H_n(X, A)$ (proof in Hatcher).
The left hand side comes from the chain complex of

$$C_n(A + B, A) = C_n(A + B)/C_n(A) = C_n(B)/C_n(A \cap B) = C_n(B, A \cap B)$$

so $H_n(A + B, A) = H_n(B, A \cap B)$.

Proposition

Let (X, A) be a good pair. Then the quotient map $q : (X, A) \to (X/A, A/A)$ induces an isomorphism $q_* : H_n(X, A) \xrightarrow{\sim} H_n(X/A, \text{pt}) \cong \tilde{H}_n(X/A)$.

Proof

Let V be a neighborhood of A which deformation retracts to A.

$$
\begin{array}{ccccc}
H_n(X, A) & \xrightarrow{(1)} & H_n(X, V) & \xrightarrow{\sim} & H_n(X - A, V - A) \\
\downarrow{q_*} & & & & \downarrow{\sim} \\
H_n(X/A, A/A) & \xrightarrow{(2)} & H_n(X/A, V/A) & \xleftarrow[\text{excision}]{} & \underset{=}{H_n((X-A)/A,(V-A)/A)} \\
& & & & H_n(X/A - A/A, V/A - A/A)
\end{array}
$$

It remains to show that (1) and (2) are isomorphisms. For (2), V/A deformation retracts to A/A in X/A. So consider the triple $A \subseteq V \subseteq X$. It induces a short exact sequence

$$
0 \longrightarrow \underset{=}{C_n(V,A)} \xrightarrow{i} \underset{=}{C_n(X,A)} \xrightarrow{j} \underset{=}{C_n(X,V)} \longrightarrow 0
$$
$$
\quad\quad C_n(V)/C_n(A) \quad\quad C_n(X)/C_n(A) \quad\quad C_n(X)/X_n(V)
$$

So $\ker j = \operatorname{im} i$, and this induces a long exact sequence

$$
\longrightarrow \overset{=0}{\overbrace{H_n(V, A)}} \longrightarrow H_n(X, A) \xrightarrow{\sim} H_n(X, V)
$$

$$
\xrightarrow{\partial} \overset{=0}{\overbrace{H_{n-1}(V, A)}} \longrightarrow
$$

where the terms zero since V deformation retracts to A.

May 5, 2025

Recall

For $A \subseteq X$, we have

$$
0 \longrightarrow C_*(A) \longrightarrow C_*(X) \longrightarrow C_*(X, A) = C_*(X)/C_*(A) \longrightarrow 0
$$

which induces

$$
\cdots \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \longrightarrow \cdots
$$

$$
\xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots
$$

Also, we have excision where

1. if $Z \subseteq A \subseteq X$ such that $\overline{Z} \subseteq A^\circ$, then $H_n(X - Z, A - Z) = H_n(X, A)$.

2. if (X, A) is a good pair, i.e. A has a neighborhood V such that V deformation retracts to A, then $H_n(X, A) = \tilde{H}_n(X/A)$.

Simplicial and Singular Homology

Goal: given X with Δ-complex structure, $H_n^\Delta(X) \cong H_n(X)$.

Example

$H_n(D^n, \partial D^n) \cong \tilde{H}_n(D^n/\partial D^n) = \tilde{H}_n(S^n) \cong \mathbb{Z}$. We can construct a generator for this $\mathbb{Z}$. We consider $H_n(\Delta^n, \partial\Delta^n)$ and claim that it is generated by $i_n : \Delta^n \to \Delta^n$ as the identity map. We prove by induction, first observing that $n = 0$ is good. Then suppose $n - 1$ and let $\Lambda \subseteq \Delta^n$ be the space obtained by removing a face from the boundary $\partial\Delta^n$. Then take

$$H_n(\Delta^n, \partial\Delta^n) \xrightarrow{\partial} H_n(\partial\Delta^n, \Lambda) \xleftarrow[(2)]{} H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$$

Consider the triple $\Lambda \subseteq \partial\Delta^n \subseteq \Delta^n$ and the short exact sequence on the chain level

$$0 \longrightarrow C_\bullet(\partial\Delta^n, \Lambda) \xrightarrow{i} C_\bullet(\Delta^n, \Lambda) \xrightarrow{j} C_\bullet(\Delta^n, \partial\Delta^n) \longrightarrow 0$$

which induces the long exact sequence

$$\cdots \longrightarrow H_n(\partial\Delta^n, \Lambda) \longrightarrow \overbrace{H_n(\Delta^n, \Lambda)}^{=0} \longrightarrow H_n(\Delta^n, \partial\Delta^n) \longrightarrow \cdots$$

$$\xrightarrow{\partial} H_{n-1}(\partial\Delta^n, \Lambda) \longrightarrow \overbrace{H_{n-1}(\Delta^n, \lambda)}^{=0} \longrightarrow \cdots$$

since Δ^n deformation retracts to Λ, $H_*(\Delta^n, \Lambda) = 0$. Hence $H_n(\Delta^n, \partial\Delta^n) \underset{\partial}{\cong} H_{n-1}(\partial\Delta^n, \Lambda)$.

For (2), let Δ^{n-1} be the face that is not in Λ. Then $\Delta^{n-1} \hookrightarrow \partial\Delta^n$ induces a homeomorphism $\Delta^{n-1}/\partial\Delta^{n-1} \cong \partial\Delta^n/\Lambda$. Hence $(\partial\Delta^n, \Lambda)$ Is a good pair, and

$$H_{n-1}(\partial\Delta^n, \Lambda) \cong \tilde{H}_{n-1}(\partial\Delta^n/\Lambda) \cong \tilde{H}_{n-1}(\Delta^{n-1}/\partial\Delta^{n-1}) \cong H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$$

We have

$$\partial i_n \in C_{n-1}(\partial\Delta^n, \Lambda)$$
$$\downarrow$$
$$i_n \in C_n(\Delta^n, \Lambda) \xrightarrow{\partial} \partial i_n \in C_{n-1}(\Delta^n, \partial\Delta^n)$$
$$\downarrow$$
$$i_n \in C_n(\Delta^n, \partial\Delta^n)$$

so $\delta^{-1} : [\partial i_n] \mapsto [i_n]$. Through the isomorphism $H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) \cong H_n(\Delta^n, \partial\Delta^n)$, $[i_n]$ is identified with $[\partial i_n]$ for $i_n : \Delta^n \to \Delta^n$. Hence $[\partial i_n]$ is $[\pm i_{n-1}]$ in $H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$.

Corollary

Let $\bigvee_\alpha X_\alpha$ by identifying $x_\alpha \in X_\alpha$ for each α. Suppose (X_α, x_α) is a good pair for each α. Then $\bigoplus_\alpha \tilde{H}_n(X_\alpha) \cong \tilde{H}_n(\bigvee_\alpha X_\alpha)$.

Proof

Consider the good pair $(X, A) := \left(\coprod_\alpha X_\alpha, \coprod_\alpha \{x_\alpha\} \right)$ where $X / A = \bigvee_\alpha X_\alpha$ such that

$$\tilde{H}_n \left(\bigvee_\alpha X_\alpha \right) \cong H_n(X, A) \cong \bigoplus_\alpha H_n(X_\alpha, x_\alpha) = \bigoplus_\alpha \tilde{H}_n(X_\alpha).$$

Theorem

Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open sets. If U and V are homeomorphic, then $m = n$.

Proof

Let $x \in U$. By excision,

$$H_i(U, U - \{x\}) \cong H_i(\mathbb{R}^m, \mathbb{R}^m - \{x\}$$

where we note that $(\mathbb{R}^m, \mathbb{R}^m - \{x\})$ is not a good pair. However, it still induces a long exact sequence

$$\longrightarrow \tilde{H}_i(\mathbb{R}^m - \{x\}) \longrightarrow \overbrace{\tilde{H}_i(\mathbb{R}^m)}^{=0} \longrightarrow H_i(\mathbb{R}^m, \mathbb{R}^m - \{x\})$$

$$\longrightarrow \tilde{H}_{i+1}(\mathbb{R}^m - \{x\}) \longrightarrow \overbrace{\tilde{H}_{i-1}(\mathbb{R}^m)}^{=0} \longrightarrow \cdots$$

Hence

$$H_i(\mathbb{R}^m, \mathbb{R}^m - \{x\}) \cong \tilde{H}_{i-1}(\mathbb{R}^m - \{x\}) \cong \tilde{H}_{i-1}(S^{m-1}) = \begin{cases} \mathbb{Z} & i = m \\ 0 & i \neq m \end{cases}.$$

If U and V are homemorphisms, then $H_i(U, U - \{x\}) \cong H_i(V, V - \{\varphi(x)\})$ and $m = n$.

Naturality of Long Exact Sequences of Pairs

$f : (X, A) \to (Y, B)$ with $f(A) \subseteq B$,

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C_\bullet(A) & \longrightarrow & C_\bullet(X) & \longrightarrow & C_\bullet(X, A) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle f_\sharp} & & \downarrow{\scriptstyle f_\sharp} & & \downarrow{\scriptstyle f_\sharp} & & \\
0 & \longrightarrow & C_\bullet(B) & \longrightarrow & C_\bullet(Y) & \longrightarrow & C_\bullet(Y, B) & \longrightarrow & 0
\end{array}
$$

commutes. Then the long exact sequence

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A) & \xrightarrow{\delta} & H_{n-1}(A) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & \\
\cdots \longrightarrow & H_n(B) & \longrightarrow & H_n(Y) & \longrightarrow & H_n(Y, B) & \xrightarrow{\delta} & H_{n-1}(B) & \longrightarrow \cdots
\end{array}
$$

$$\partial\alpha \in C_{n-1}(A)$$
$$\downarrow$$
$$\alpha \in C_n(X) \xrightarrow{\;\partial\;} \partial\alpha \in C_{n-1}(X))$$
$$\downarrow$$
$$\alpha \in C_n(X,A)$$

So $\delta : [\alpha] \to [\partial\alpha]$ and

$$f_*(\delta[\alpha]) = f_*[\partial\alpha] = [f_*(\partial\alpha)] = [\partial f_*(\alpha)] = \delta(f_*[\alpha]).$$

Recall: the Five Lemma

$$\cdots \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow E \longrightarrow \cdots$$
$$\downarrow{\cong} \qquad \downarrow{\cong} \qquad \downarrow \qquad \downarrow{\cong} \qquad \downarrow{\cong}$$
$$\cdots \longrightarrow A' \longrightarrow B' \longrightarrow C' \longrightarrow D' \longrightarrow E' \longrightarrow \cdots$$

implies that $C \cong C'$.

Equivalence Between Simplicial and Singular Homology

Given X with a finite dimensional Δ-complex structure, then $C_n^\Delta(X) \hookrightarrow C_n(X)$ induces an isomorphism $H_n^\Delta(X) \cong H_n(X)$.

Proof

Suppose it holds for all (X,Δ) with dimension less than $k-1$. We condier the k-dimensional case.
Let X^i be the i-skeleton of X. Note that $X^k = X$, so the pair (X^k, X^{k-1}) induces a long exact sequence

$$H_{n+1}^\Delta(X^k, X^{k-1}) \longrightarrow H_n^\Delta(X^{k-1}) \longrightarrow H_n^\Delta(X^k) \longrightarrow H_n^\Delta(X^k, X^{k-1}) \longrightarrow H_{n-1}^\Delta(X^{k-1})$$
$$\downarrow{(1)} \qquad\qquad \downarrow{(2)} \qquad\qquad \downarrow{(3)} \qquad\qquad \downarrow{(4)} \qquad\qquad \downarrow{(5)}$$
$$H_{n+1}(X^k, X^{k-1}) \longrightarrow H_n(X^{k-1}) \longrightarrow H_n(X^k) \longrightarrow H_n(X^k, X^{k-1}) \longrightarrow H_{n-1}(X^{k-1})$$

We have that (2) and
(5) are isomorphisms per our inductive assumption. Note also that $C_n^\Delta(X^k) = 0$ for $n \ge k$, so

$$C_n^\Delta(X^k, X^{k-1}) = C_n^\Delta(X^k)/C_n^\Delta(X^{k-1}) = \begin{cases} C_n^\Delta(X^k) & k = n \\ 0 & n < k \end{cases}.$$

So the chain complex $C_\bullet^\Delta(X^k, X^{k-1})$ is

$$0 \longrightarrow 0 \longrightarrow C_n^\Delta(X^k, X^{k-1}) = C_n^\Delta(X^k) \longrightarrow 0 \longrightarrow 0$$

and $H_n^\Delta(X^k, X^{k-1}) \cong \begin{cases} C_k^\Delta(X^k) & k = n \\ 0 & k \ne n \end{cases}$. Now consider

$\Phi : \left(\coprod_\alpha \Delta_\alpha^k, \coprod_\alpha \partial\Delta_\alpha^k \right) \to (X^k, X^{k-1})$. It induces a homemorphism $X^k/X^{k-1} \cong \left(\coprod_\alpha \Delta_\alpha^k \right)/\left(\coprod_\alpha \partial\Delta_\alpha^k \right)$. So \(

$$H_n(X^k, X^{k-1}) \cong \tilde{H}_n(X^k, X^{k-1}) \cong \tilde{H}_n\left(\left(\coprod_\alpha \Delta_\alpha^n \right)/\left(\coprod_\alpha \partial\Delta_\alpha^k \right) \right) \cong H_n\left(\coprod_\alpha \Delta_\alpha^k, \coprod_\alpha \partial\Delta_\alpha^k \right) \cong \bigoplus_\alpha H_n(\Delta_\alpha^k, \partial\Delta_\alpha^k)$$

where each $H_n(\Delta_\alpha^k, \partial\Delta_\alpha^k)$ is generated by $i_\alpha^k : \Delta_\alpha^k \to \Delta_\alpha^k$ (the identity map) if $n = k$ or $H_n(\Delta_\alpha^k, \partial\Delta_\alpha^k)$ when $n \ne k$. Finally, we observe that

$$C_k^\Delta(X^k) \cong \bigoplus_\alpha \langle i_\alpha^k \rangle \cong \bigoplus_\alpha H_n(\Delta_\alpha^k, \partial\Delta_\alpha^k).$$

So (1) and (4) are isomorphisms and, by the five lemma, (3) is an isomorphy as well.

Remark

$H_n^\Delta(X, A) \cong H_n(X, A)$ if X has a Δ-complex structure and $A \subseteq X$ is a sub-complex.

$$
\begin{array}{ccccccccc}
H_n^\Delta(A) & \longrightarrow & H_n^\Delta(X) & \longrightarrow & H_n^\Delta(X, A) & \longrightarrow & H_{n-1}^\Delta(A) & \longrightarrow & H_{n-1}^\Delta(X) \\
\downarrow{\scriptstyle(1)} & & \downarrow{\scriptstyle(2)} & & \downarrow{\scriptstyle(3)} & & \downarrow{\scriptstyle(4)} & & \downarrow{\scriptstyle(5)} \\
H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A) & \longrightarrow & H_{n-1}(A) & \longrightarrow & H_{n-1}(X)
\end{array}
$$

where $(1), (2), (4), (5)$ are isomorphisms,

so we have the conclusion by the five lemma.

May 7, 2025

Definition: Degree

Let $f : S^n \to S^n$ which induces $f_* : H_n(S^n) \to H_n(S^n)$ (i.e. $\mathbb{Z} \to \mathbb{Z}$). Hence f_* is multiplication by some integer $d \in \mathbb{Z}$. Define $\deg(f) = d$.

Properties

1. $\deg(\mathrm{id}) = 1$.

2. If $f, g : S^n \to S^n$ are homotopic, then $f_* = g_*$ thus $\deg(f) = \deg(g)$.

3. $\deg(f \circ g) = \deg(f) \cdot \deg(g)$, because $(f \circ g)_* = f_* \circ g_*$. In particular, if $f \circ g \simeq \mathrm{id}_{S^n}$ then $\deg(f) \cdot \deg(g) = \deg(f \circ g) = 1$ and $\deg(f) = \pm 1$.

4. Suppose $f : S^n \to S^n$ is not surjective, say $x_0 \in S^n \backslash \mathrm{im}\, f$. Then $f : S^n \to S^n \backslash \{x_0\} \cong \mathbb{R}^n$. So f is $S^n \xrightarrow{f} S^n \backslash \{x_0\} \xrightarrow{\iota} S^n$ and

$$
H_n(S^n) \longrightarrow \overbrace{H_n(S^n \backslash \{x_0\})}^{=0} \longrightarrow H_n(S^n)
$$

So $f_* : H_n(S^n) \to H_n(S^n)$ is the zero map (i.e. $\deg(f) = 0$).

1. $f : S^n \to S^n$ a reflection has degree -1. In general, if we take two copies of Δ^n glued along corresponding edges by the identity map then we get S^n. Then $H_n^\Delta(S^n)$ has a generator $U - L$, and reflection of f maps $U - L$ to $L - U$ (i.e. $f_* : \mathbb{Z} \to \mathbb{Z}$ is $1 \mapsto -1$).

2. $f : S^n \to S^n$ an antipodal map $(-\mathrm{id})$ which sends $(x^1, \ldots, x^{n+1}) \mapsto (-x^1, \ldots, -x^{n+1})$ has $\deg(-1\,\mathrm{id}) = (-1)^{n+1}$.

3. Theorem (Hopf) if $f, g : S^n \to S^n$ have the same degree, then $f \simeq g$.

4. If $f : S^n \to S^n$ has no fixed points, then $f \simeq -\mathrm{id}$ and $\deg(f) = (-1)^{n+1}$. Proof: if $x \neq f(x)$, then the segment $(1-t)f(x) + t(-x)$ does not pass through $0 \in \mathbb{R}^{n+1}$. Consider $f_t(x) = \frac{(1-t)f(x)+t(-x)}{||(1-t)f(x)+t(-x)||}$ where $f_0(x) = f(x)$ and $f_1(x) = -x$ show that $f_t(x)$ gives a homotopy between f and $-\mathrm{id}$.

5. S^n has a continuous, non-vanishing vector field if and only if n is odd. Proof: ($\Longleftarrow$) say $n = 2k - 1$ such that $S^n \subseteq \mathbb{R}^{2k}$. Define $V(x_1, \ldots, x_{2k}) = (-x_2, x_1, -x_4, x_3, \ldots)$. Then $V(\vec{x}) \perp \vec{x}$. ($\Longrightarrow$) Think of $V(\vec{x})$ starting at $\vec{x}$

and without loss of generality that $||V(\vec{x})|| = 1$. Consider $f_t(x) = (\cos t)\vec{x} + (\sin t)V(\vec{x})$ where $f_\pi(x) = -x$ and $f_0(x) = x$ such that $\{f_t\}$ is a homotopy between id and $-$id. Hence $1 = \deg(\mathrm{id}) = \deg(-\mathrm{id}) = (-1)^{n+1}$ and n is odd.

6. If n is even, then $\mathbb{Z}_2$ is the only non-trivial group that can act freely on S^n. For example, S^1 acts on S^3 freely if we consider $(z_1, z_2) \in S^3 \subseteq C^2$ and $\theta(z_1, z_2) = (e^{i\theta}z_1, e^{i\theta}z_2)$. Proof: suppose $G \neq \mathrm{id}$ acts freely on S^n. Consider $\deg : G \to \mathbb{Z}$ where $\mathrm{im}(\deg) \subseteq \{\pm 1\} \subseteq \mathbb{Z}$ and for $g \neq e$ then $\deg(g) = (-1)^{n+1} = -1$. Then $G/\ker \cong \mathrm{im} = \{-1, 1\}$ since $\ker = \{e\}$. Hence $G \cong \mathrm{im} = (\{\pm 1\}, \cdot) = \mathbb{Z}_2$.

Theorem

Below, we assume that S^n has a point y such that $f^{-1}(y) = \{x_1, \ldots, x_m\}$ is a finite set. If f is smooth, then by Sard's theorem we may pick a regular point y. Then $f^{-1}(y)$ is an embedded submanifold of dimension zero (i.e. $f^{-1}(y)$ is a collection of finitely many points). That is, when f is smooth this assumption holds automatically.
For each $i = 1, \ldots, m$, we choose a small ball U_i about x_i and a ball V about y such that $f(U_i) \subseteq V$. The pair $(S^n, S^n \setminus \{x\})$ induces

$$\cdots \longrightarrow \overbrace{H_n(S^n \setminus \{x\})}^{=0} \longrightarrow H_n(S^n) \xrightarrow{j} H_n(S^n, S^n \setminus \{x\}) \longrightarrow \overbrace{H_{n-1}(S^n \setminus \{x\})}^{=0} \longrightarrow \cdots$$

The pair $(U, U \setminus \{x\})$ gives

$$\cdots \longrightarrow H_n(U \setminus \{x\}) \longrightarrow \overbrace{H_n(U)}^{=0} \longrightarrow H_n(U, U \setminus \{x\}) \xrightarrow{\delta} H_{n-1}(U \setminus \{x\}) \longrightarrow \overbrace{H_{n-1}(U)}^{=0} \longrightarrow \cdots$$

and we observe that $H_n(S^n, S^n \setminus \{x\}) \cong H_n(U, U \setminus \{x\})$ by excision.

We have that $f_* : H_n(U_i, U_i \setminus \{x_i\}) \to H_n(V, V \setminus \{y\})$ is $\mathbb{Z} \to \mathbb{Z}$ and hence it gives an integer. We call this the local degree $\deg(f|_{x_i})$.
Theorem: $\deg(f) = \sum_{i=1}^{m} \deg(f|_{x_i})$.
Write

$$H_n(S^n, S^n - f^{-1}(y)) \underset{\text{excision}}{\cong} H_n\left(\bigsqcup_i U_i, \bigsqcup_i (U_i \setminus \{x_i\})\right) \cong \bigoplus_i H_n(U_i, U_i \setminus \{x_i\}) \cong \mathbb{Z}^m.$$

then $k_i : H_n(U_i, U_i \setminus \{x_i\}) \to \bigoplus_i H_n(U_i, U_i \setminus \{x_i\})$ by $1 \mapsto (0, \ldots, 0, 1, 0, \ldots, 0) =: e_i$. Consider the triple $S^n - f^{-1}(y) \subseteq S^n \setminus \{x_i\} \subseteq S^n$ which induces

$$0 \longrightarrow C_\bullet(S^n \setminus \{x_i\}, S^n \setminus f^{-1}(y)) \longrightarrow C_\bullet(S^n, S^n \setminus f^{-1}(y)) \longrightarrow C_\bullet(S^n, S^n \setminus \{x_i\}) \longrightarrow 0$$

So we have $p_i : H_n(S^n, S^n \setminus f^{-1}(y)) \to H_n(S^n, S^n \setminus \{x_i\})$. Then

$$\begin{array}{ccc} & & \mathbb{Z} \\ & \nearrow & \downarrow {\scriptstyle k_i} \\ & {\scriptstyle \mathrm{id}} & \\ \mathbb{Z} & \xleftarrow{\;p_i\;} & \mathbb{Z}^m \end{array}$$

commutes and $1 = p_i(k_i(1)) = p_i(e_i)$, hence p_i is the projection to the i-th component. Similarly

$$\begin{array}{ccc} \mathbb{Z} & \xleftarrow{\;p_i\;} & \mathbb{Z}^m \\ & {\scriptstyle \mathrm{id}} \nwarrow & \uparrow {\scriptstyle j} \\ & & \mathbb{Z} \end{array}$$

commutes so $1 = p_i(j(1))$ and the i-th component of $j(1)$ is 1 (i.e. $j(1) = (1,1,\dots,1) \in \mathbb{Z}^m$. Then $\deg(f|_{x_i}) = f_*(k_i(1)) = f_*(e_i)$. Finally,

$$\deg f = f_*(1) = f_*(j(1)) = f_*\left(\sum e_i\right) = \sum f_*(e_i) = \sum \deg(f_*|_{x_i})$$

Remark

If f is smooth and y is a regular value, then we can pick U_i and V such that each $f|_{U_i} : U_i \to V$ is a diffeomorphism. Hence $\deg(f|_{x_i}) = \pm 1$.

Example

If $f : S^1 \to S^1$ by $z \mapsto z^k$, $f^{-1}(1)$ has k many points (viz. the roots of unity). $f|_{U_i} : U_i \to V$ is diffeomorphic (by rotation and scaling) and $\deg(f|_{x_i}) = 1$. $\deg(f) = \sum \deg(f|_{x_i}) = k$.

IMAGE 1

Definition: Suspension of a Space

Recall that the cone of X is $C(X) = X \times I / X \times \{1\}$.

IMAGE 2

The suspension of X is $S(X) = C(X)/X \times \{0\}$.

IMAGE 3

Examples

$S(S^1) = S^2$. In general $S(S^n) = S^{n+1}$.

Definition: Suspension of a Map

$f : X \to Y$ induces $f : X \times I \to Y \times I$ by $(x,t) \mapsto (f(x), t)$. This induces $Cf : C(X) \to C(Y)$ and $Sf : S(X) \to S(Y)$.

Examples

$f : S^n \to S^n$ induces a map $Sf : S^{n+1} \to S^{n+1}$. $f : S^1 \to S^1$ by $z \mapsto z^2$ induces $Sf : S^2 \to S^2$

IMAGE 4

Proposition

$\deg(Sf) = \deg(f)$.

Proof

Consider the pair $(C(S^n), S^n \times \{0\})$ which induces

$$\overbrace{\tilde{H}_{n+1}(S^N)}^{=0} \longrightarrow \overbrace{\tilde{H}_{n+1}(C(S^n))}^{=0} \longrightarrow H_{n+1}(C(S^n), S^n \times \{0\}) \xrightarrow{\delta} \overbrace{\tilde{H}_n(S^n)}^{=\mathbb{Z}} \longrightarrow \overbrace{\tilde{H}_n(C(S^n))}^{=0} \longrightarrow$$

Hence $\mathbb{Z} \cong H_{n+1}(C(S^n), S^n \times \{0\}) \cong \tilde{H}_{n+1}(S(S^n)) = \tilde{H}_{n+1}(S^{n+1})$. Therefore

$$
\begin{array}{ccccc}
\tilde{H}_{n+1}(S^{n+1}) & \xrightarrow{\sim} & H_{n+1}(C(S^n), S^n \times \{0\}) & \xrightarrow{\delta} & \tilde{H}_n(S^N) \\
\downarrow {(Sf)_*} & & \downarrow {(Cf)_*} & & \downarrow {f_*} \\
H_{n+1}(S^{n+1}) & \xrightarrow{\sim} & H_{n+1}(C(S^n), S^n \times \{0\}) & \longrightarrow & \tilde{H}_n(S^n)
\end{array}
$$

So $\deg(Sf) = \deg(f)$.

Remark

For any $k, n \in \mathbb{Z}_+$, by iterated suspension of the map $z \mapsto z^k$, we can construct $f : S^n \to S^n$ of degree k.

Remark

$Sf : S^{n+1} \to S^{n+1}$, pick $p \in S^{n+1}$ a pole, then $(Sf)^{-1}(p) = \{p\}$.

IMAGE 5

Hence $\deg(Sf|_p) = \deg(Sf) = k$.

May 12, 2025

Recall

Let X be a CW-Complex of finite dimension $X = X^0 \cup X^1 \cup \cdots \cup X^{\dim X}$.
X^0 is a discrete set of points.
X^1 is a gluing of $\{e_\alpha^1\}_{\alpha \in A}$ to X^0, where $e^1 = [-1, 1]$, by the attaching map $\varphi_\alpha : \partial e_\alpha^1 \to X^0$.
X^{k+1} is the gluing of $\{e_\alpha^{k+1}\}_{\alpha \in A}$, where $e^{k+1} \cong D^{k+1}$, by $\varphi_\alpha : \partial e_\alpha^{k+1} \cong S^k \to X^k$.

Lemma

(a)

Let X be a CW-Complex of $\dim X$. Then

$$H_k(X^n, X^{n-1}) = \begin{cases} 0 & k \neq n \\ \text{free abelian with a basis in 1-1 correspondence to} \{n\text{-cells}\} \end{cases}$$

Proof

(X^n, X^{n-1}) is a good pair. So

$$H_k(X^n, X^{n-1}) \cong \tilde{H}_k(X^n/X^{n-1}) = \tilde{H}_k\left(\bigvee_\alpha S_\alpha^n\right) = \bigoplus_\alpha \tilde{H}_k(S_\alpha^n).$$

If $k \neq n$, then $\tilde{H}_k(S_\alpha^n) = 0$.
If $k = n$, then $\tilde{H}_k(S_2^n) = \mathbb{Z}$ and $H_n(X^n, X^{n-1}) \cong \bigoplus_\alpha \mathbb{Z}$.

(b)

$H_k(X^n) = 0$ if $k > n$.

Proof

The pair (X^n, X^{n-1}) gives a long exact sequence.

$$\cdots \longrightarrow H_{k+1}(X^n, X^{n-1}) \xrightarrow{\delta} H_k(X^{n-1}) \longrightarrow H_k(X^n)$$

$$\longrightarrow H_k(X^n, X^{n-1}) \xrightarrow{\delta} \cdots$$

Supposing both $k \neq n$ and $k + 1 \neq n$, the first and last terms are zero and $H_k(X^{n-1}) \cong H_k(X^n)$. Then

$$H_k(X^n) \cong H_k(X^{n-1}) \cong H_k(X^{n-2}) \cong \cdots \cong H_k(X^0) = 0$$
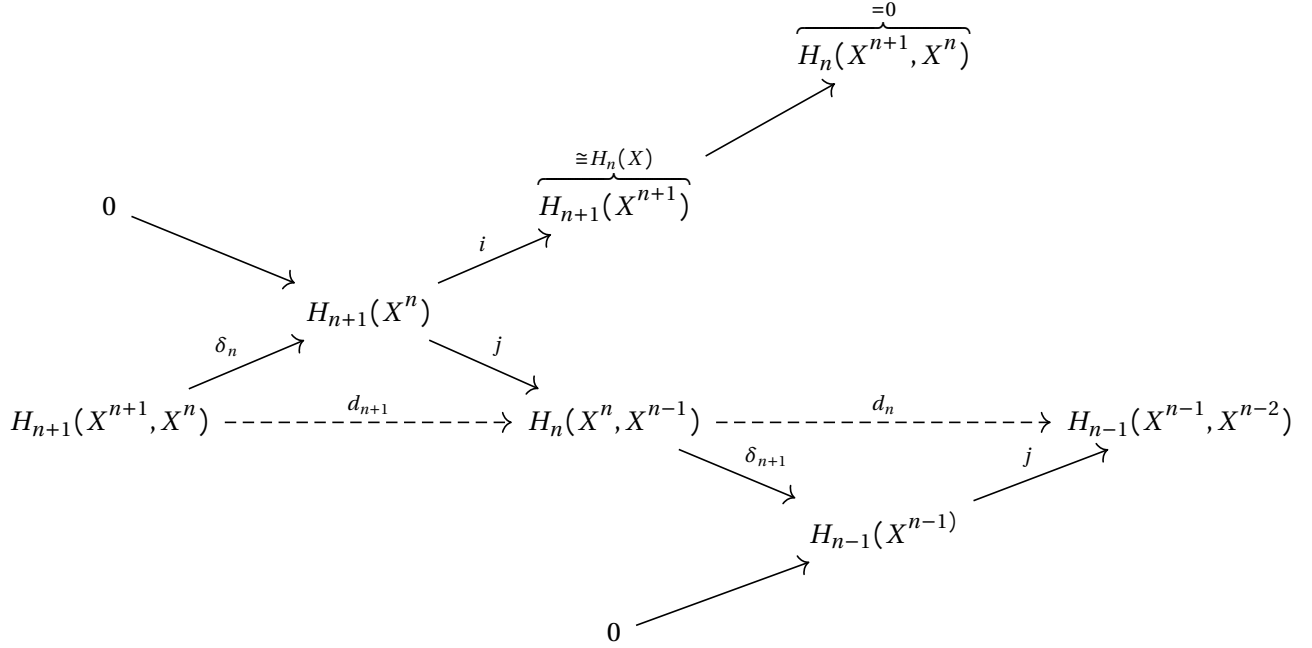
(c)

$i : X^n \hookrightarrow X$ induces an isomorphism $i_* : H_k(X^n) \to H_k(X)$ if $k < n$.

Proof

If $k < n$, then

$$H_k(X^n) \cong H_k(X^{n+1}) \cong \cdots \cong H_k(X^{\dim X}) = H_k(X)$$

Chain Complexes

$$\overbrace{H_n(X^{n+1}, X^n)}^{=0}$$

$$\overbrace{H_{n+1}(X^{n+1})}^{\cong H_n(X)}$$

```
0                                                                    H_n(X^{n+1}, X^n)
   \                                                             ↗
    ↘                        ≅H_n(X)
                        H_{n+1}(X^{n+1})
              i    ↗              ↘  j
    H_{n+1}(X^n)
  δ_n ↗        ↘ j
H_{n+1}(X^{n+1},X^n) ---d_{n+1}---> H_n(X^n,X^{n-1}) ---d_n---> H_{n-1}(X^{n-1},X^{n-2})
                         δ_{n+1} ↘           ↗ j
                            H_{n-1}(X^{n-1})
                       ↗
                  0
```

This give a cellular chain complex $\{H_n(X^n, X^{n-1}), d_n\}$ with $d_n \circ d_{n+1} = 0$ because $\overset{j}{\to} \cdot \overset{\delta}{\to} = 0$. This defines a cellular homology $H_k^{CW}(X)$. We claim that $H_n^{CW}(X) \cong H_n(X)$.

Proof

$$
\begin{aligned}
H_n(X) &\cong H_n(X^{n+1} \\
&\cong H_n(X^n)/\ker i && \text{because } i \text{ is surjective} \\
&= H_n(X^n)/\operatorname{im}\delta_{n+1} && \text{because } \overset{\delta_{n+1}}{\to} \cdot \overset{i}{\to} \text{ is exact} \\
&\cong j(H_n(X^n))/j(\operatorname{im}\delta_{n+1}) && \text{because } j \text{ is injective} \\
&= \ker(\delta_n)/\operatorname{im}(d_{n+1}) && j(\operatorname{im}\delta_{n+1}) = \operatorname{im}(j \circ \delta_{n+1}) = \operatorname{im}(d_{n+1}) \\
&= \ker(d_n)/\operatorname{im}(d_{n+1}) && \ker(\delta_n) = \ker(j \circ \delta_n) = \ker d_n \\
&= H_n^{CW}(X)
\end{aligned}
$$

Applications

For

$$\cdots \longrightarrow H_n(X^n, X^{n-1}) \overset{d_n}{\longrightarrow} H_{n-1}(X^{n-1}, X^{n-2}) \overset{d_{n-1}}{\longrightarrow} \cdots$$

where $H_n(X^n, X^{n-1}) = \bigoplus_\alpha \mathbb{Z}$

(1)

If a CW-Complex does not have any n-cells, then $H_n(X^n, X^{n-1}) = 0$ and $H_n(X) \cong H_n^{CW}(X) = 0$.

(2)

If a CW-Complex X has k-many n-cells, then $H_n(X^n, X^{n-1}) \cong \mathbb{Z}^k$. Then $H_n(X) \cong H_n^{CW}(X) = ker\, d_n/\operatorname{im} d_{n-1}$. $\ker d_n \leq H_n(X^n, X^{n-1}) = \mathbb{Z}^k$. Hence $\ker d_n$ and $H_n(X)$ can be generated by at most k many elements.

(3)

If X and Y are CW-complexes with $\{\varphi_\alpha : e_\alpha^n \to X^{n-1}\}$ and $\{\psi_\beta : e_\beta^n \to Y^{n-1}\}$ respectively, then $X \times Y$ has $\{\varphi_\alpha \times \psi_\beta : e_\alpha^m \times e_\beta^n \to (X \times Y)^{m+n-1}\}$ where $e_\alpha^m \times e_\beta^n \cong e^{m+n}$.

Consider $S^n \times S^n$ (for $n \geq 2$) where S^n is constructed by one 0-cell and one n-cell. Then $S^n \times S^n$ has one 0-cell ($\mathbb{Z}^1$), two n-cells ($\mathbb{Z}^2$) and one $2n$-cell ($\mathbb{Z}^1$).

$$0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots \longrightarrow \mathbb{Z}^2 \longrightarrow 0 \longrightarrow \cdots \longrightarrow \mathbb{Z} \longrightarrow 0$$

so

$$H_k(S^n \times S^n) = \begin{cases} \mathbb{Z} & k = 0, 2n \\ \mathbb{Z}^2 & k = n \\ 0 & \text{otherwise} \end{cases}.$$

(4)

Take $\mathbb{CP}^n$ as $\mathbb{C}^{n+1}/\sim$ or as $S^{2n+1}/\sim$ where $v \sim \lambda v$ and $\lambda v = (e^{i\theta} z_1, \ldots, e^{i\theta} z_{n+1})$. Consider the set of vectors in S^{2n+1} whose last component is real and nonnegative. $D_+^{2n} = \{(w, \sqrt{1-|w|^2}) \in \mathbb{C}^{n+1} : w \in \mathbb{C}^n, |w| \leq 1\}$ is the graph of the function $w \mapsto \sqrt{1-|w|^2}$ defined on $\{w : |w| \leq 1\} \subseteq \mathbb{C}^n$. So D_+^{2n} is homeomorphic to a disk $\{|w| \leq 1\} = D^{2n} \subseteq \mathbb{C}^n$. For any vector $v \in S^{2n+1}$, $v = (z_1, \ldots, z_{n+1})$ if $z_{n+1} \neq 0$, then v is equivalent to a unique vector in D_+^{2n}. If $z_{n+1} = 0$, $\{(z_1, \ldots, z_n, 0) \in S^{2n-1} \times \{0\}\} = S^{2n-1}$. So $q : S^{2n+1} \to \mathbb{CP}^n$ has that $q|_{D_+^{2n}}$ is a homeomorphism. Then $S^{2n-1}/\sim$ is exactly $\mathbb{CP}^{n-1}$. Therefore, we may view $\mathbb{CP}^n$ as gluing e^{2n} to $\mathbb{CP}^{n-1}$ by the attaching map $\partial e^{2n} = S^{2n-1} \to \mathbb{CP}^{n-1}$. So $\mathbb{CP}^n$ has cells $e^0, e^2, \ldots, e^{2n}$ and the cellular chain complex is

$$\mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

$$H_k(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & k = 0, 2, \ldots, 2n \\ 0 & \text{otherwise} \end{cases}.$$

Recall that $\mathbb{RP}^n$ by $\S^n/\sim$ with $S^n \subseteq \mathbb{R}^{n+1}$ and $v \sim -v$, we may take the upper hemisphere D_+^n. For every $v \in S^n = (x_1, \ldots, x_n)$, if $x_{n+1} \neq 0$ then v is equivaent to a unique vector in D_+^n where $q|_{D_+^n} : D_+^n \to \mathbb{RP}^n$ homemorphic to its image. If $x_{n+1} = 0$, then $\{(x_1, \ldots, x_n, 0) \in S^n\}/\sim$ and $\mathbb{RP}^n$ is gluing e^n to $\mathbb{RP}^{n-1}$ via the attaching map $\varphi : \partial e^n = S^{n-1} \to \mathbb{RP}^{n-1}$ as the quotient map.

Computation

We want $d_n : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$. For $n = 1$ we have

where $d_1 = \delta : H_1(X^1, X^0) \to H_0(X^0)$. If X is connected, and $X^0 = \{v\}$, then $H_0(X^0) = \mathbb{Z}$ and $H_0(X) = H_0(X^0)/\operatorname{im} d_1$ implies that $\operatorname{im} d_1 = 0$.

For $n \geq 2$, $H_n(X^n, X^{n-1})$ is $\bigoplus_\alpha \mathbb{Z}$ and the generators are in one-to-one correspondence with $\{e_\alpha^n\}_\alpha$. We have a cellular boundary formula

$$d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1}$$

where $d_{\alpha\beta} = \deg(\Delta_{\alpha\beta})$ and $\Delta_{\alpha\beta} : S^{n-1} = \partial e_\alpha^n \overset{\varphi_\alpha}{\to} X^{n-1} \overset{q_\beta}{\to} S_\beta^{n-1}$. $q_\beta : X^{n-1} \to S_\beta^{n-1}$ is obtained by collapsing everything in X^{n-1} except $(e_\beta^{n-1})^\circ$. For every n-cel e_α^n and every $(n-1)$-cell e_β^{n-1}, we obtain $d_{\alpha\beta} = \deg(\Delta_{\alpha\beta})$.

Example

Suppose we have M_g, an orientable surface of genus g. M_g has one 0-cell, $2g$ 1-cells $(a_1, b_1, \ldots, a_g, b_{g\_}$ and one 2-cell. Then $d_1 = 0$, and $d_2(e_2)$ comes from $\Delta_{\alpha\beta} : S^2 = \partial e^2 \overset{\alpha}{\to} X^1 = \bigvee S^1 \overset{q_\beta}{\to} S_\beta^1$ which glues S^1 to S^1 by $a \cdot a^{-1}$. So $\deg(\Delta_{\alpha\beta}) = 0$ and $d_2(e_2) = 0$.

$$0 \longrightarrow \mathbb{Z} \overset{0}{\longrightarrow} \mathbb{Z}^{2g} \overset{0}{\longrightarrow} \mathbb{Z} \longrightarrow 0$$

so $H_2 = \mathbb{Z}$, $H_1 = \mathbb{Z}^{2g}$ and $H_0 = \mathbb{Z}$.

Example

N_g is a non-orientable surface of genus g. N_g has one 0-cell, g 1-cells $(a_1^2 a_2^2 \cdots a_g^2)$, and one 2-cell. We know that $d_1 = 0$. Consider $\Delta_{\alpha\beta} : S_\alpha^1 \to X^1 \to S_\beta^1$ which glues S^1 to S^1 by a^2 (i.e. $z \mapsto z^2$) and $\deg(\Delta_{\alpha\beta}) = 2$. So $d_2(e_2) = \sum_\beta 2 e_\beta^1 = (2, 2, \ldots, 2) \in \mathbb{Z}^g$ and

$$0 \longrightarrow \mathbb{Z} \overset{d_2}{\longrightarrow} \mathbb{Z}^g \overset{0}{\longrightarrow} \mathbb{Z} \longrightarrow 0$$

So $H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z}^g / \operatorname{im} d_2 = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$ and $H_2 = \ker d_2 / 0 = 0$.

May 14, 2025

Remaining Homology Topics

1. More examples of H_*^{CW}.

2. Euler characteristic.

3. Homology with coefficients.

4. Mayer-Vietoris Sequence.

5. $\pi_1/[\pi_1, \pi_1] = H_1$.

Recall: Cellular Chain Complex

$(H_n(X^n, X^{n-1}), d_n)$ where $H_n(X^n, X^{n-1})$ has a basis in one-to-one correspondence with the n-cells in X and $d_n(e_n^\alpha) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1}$ with $d_{\alpha\beta} = \deg(\Delta_{\alpha\beta})$ for $\Delta_{\alpha\beta} : \partial e_n^\alpha = S_\alpha^{n-1} \overset{\varphi_\alpha}{\to} X^{n-1} \overset{q_\beta}{\to} S_\beta^{n-1}$ where q_β collapses everything in X^{n-1} except

$\text{int}(e_\beta^{n-1})$ to a point.

Example

$\mathbb{RP}^n = e^0 \cup e^1 \cup \cdots \cup e^n$ with the attaching map $S^{k-1} = \partial e^k \to \mathbb{RP}^{k-1} = X^{n-1}$ as the quotient map (2-sheet covering).

IMAGE 1

So the cellular chain complex is

$$0 \longrightarrow \mathbb{Z} \longrightarrow \cdots \longrightarrow \mathbb{Z} \longrightarrow 0$$

with $d_n(e^n) = \deg(\Delta)e^{n-1}$ for $\Delta : S^{n-1} \xrightarrow{\varphi} X^{n-1} = \mathbb{RP}^{n-1} \xrightarrow{q} \mathbb{RP}^{n-1}/\mathbb{RP}^{n-2} = S^{n-1}$. Then Δ restricted to the upper (or lower) hemisphere is a homeomorphism to $S^{n-1} - \{pt\}$. So antipodal open sets differe only be the antipodal map

IMAGE 2

So $\deg(\Delta) = 1 + (-1)^n = \begin{cases} 0 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}$. If n is even, the chain complex is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \cdots \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

which means $H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}_2 & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$. Similarly for odd n,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

so $H_n(\mathbb{RP}^n) = \mathbb{Z}$ and we conclude

$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & k = n \text{ odd} \\ \mathbb{Z}_2 & 0 < k < n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

Example

Recall for the torus $\mathbb{T}^2$, $0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z} \to 0$, the degree is zero (see above).
For $\mathbb{T}^3$,

IMAGE 3

we have one 0-cell, three 1-cells, three 2-cells and one 3-cell and a chain complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_3} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

So $d_3(e^3) = \sum_\beta \deg(\Delta_{\alpha\beta})e_\beta^2$ and $\Delta_{\alpha\beta} : \partial e^3 = S^2 \xrightarrow{\varphi} X^2 \xrightarrow{q_\beta} S_\beta^2$.

IMAGE 4

$$H_k(T^3) = \begin{cases} \mathbb{Z}^3 & k = 1,2 \\ \mathbb{Z} & k = 0,3 \\ 0 & \text{otherwise} \end{cases}.$$

46

Example

Consider $K \times S^1$ where K is the Klein bottle.

So $d_2(A) = 2b$, $d_2(B) = 0$ and $d_2(C) = 0$. However, for d_3 the front face changes by reflection in the back

So $d_3(e^3) = 2C$ and the degree is $1 + (-1) \cdot (-1) = 2$ and

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_3} \mathbb{Z}^3 \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

We have that $\ker d_3 = 0$, $H_3 = 0$; $\ker d_2 = \langle B, C \rangle$, $\operatorname{im} d_3 = \langle 2C \rangle$, $H_2 = \ker d_2 / \operatorname{im} d_3 = \mathbb{Z} \oplus \mathbb{Z}_2$; $\ker d_1 = \langle a, b, c \rangle$, $\operatorname{im} d_2 = \langle 2b \rangle$, $H_1 = \mathbb{Z}^2 \oplus \mathbb{Z}_2$.

Homology with Coefficients (Moore Space)

Let G be a finitely generated abelian group and $n \in \mathbb{N}$ with $n \geq 1$. We can construct a CW-complex X such that

$$\tilde{H}_k(X) = \begin{cases} G & k = n \\ 0 & \text{otherwise} \end{cases}.$$

First, let us consider $G = 2m$. We start with $X^n = S^n$ (1 0-cell, 1 n-cell). Then construct $X = X^{n+1}$ by gluing in a $(n+1)$-cell e^{n+1} via the attaching map $\varphi : S^n = \partial e^{n+1} \to X^{n+1} = S^n$. Then $d_{n+1}(e^{n+1}) = \deg(\Delta)e^n$ where $\Delta : S^n = \partial e^{n+1} \to X^n = S^n$ is φ. If $\varphi : S^n \to S^n$ has degree m, then d_{n+1} is multiplication by m.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_{n+1}} \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots$$

So $H_n(X) = \ker / \operatorname{im} = \mathbb{Z} / m\mathbb{Z} = \mathbb{Z}_m = G$. In general, $G = \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_\ell} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ a k-term sum. Use $X^n = \bigvee_{i=1}^k S^n$. For each $\mathbb{Z}_{m_i}$-factor, glue in e_α^{n+1} by $\varphi : \partial e_\alpha^{n+1} \to S_i^k$. Consider X with ℓ-many $(n+1)$-cells, k-many n-cells and a single 0-cell for $\ell \leq k$. Then

$$0 \longrightarrow \mathbb{Z}^\ell \longrightarrow \mathbb{Z}^k \longrightarrow 0$$

$$(1, 0, \cdots, 0) \longmapsto (m_i, 0, \ldots)$$

So $\operatorname{im} d_{n+1} = m_1 \mathbb{Z} \oplus \cdots \oplus m_\ell \mathbb{Z} \oplus 0 \subseteq \mathbb{Z}^k$.

Let G be an abelian group ($G = \mathbb{Z}$ or $G = \mathbb{Z}_m$). Then $C_n(X; G) = \{\sum_i n_i \sigma_i : \text{finite formal sums of } \sigma_i : \Delta^n \to X \text{ with } n_i \in G\}$. We can similarly define $H_n(X, G)$, $\tilde{H}_n(X; G)$, $H_n(X, A; G)$, $H_n^{CW}(X; G)$, etc.

If we use $\mathbb{Z}_m = G$ as the coefficient with $\tilde{H}_k(X) = \begin{cases} \mathbb{Z}_m & k = m \\ 0 & k \neq m \end{cases}$,

$$0 \longrightarrow \mathbb{Z}_m \xrightarrow{\cdot m} \mathbb{Z}_m \longrightarrow 0$$

Consider the map $f : X \to X/S^n = S^{n+1}$. This induces $f_* : H_k(X) \to H_k(S^{n+1})$ which is $f_* = 0$ on all $H_k(X)$. If we use coefficients $\mathbb{Z}_m$ instead, we still induce $f_* : H_k(X; \mathbb{Z}_m) \to H_k(S^{n+1}; \mathbb{Z}_m)$, $H^{n+1}(X; \mathbb{Z}_m) = \mathbb{Z}_m$ and $H^n(X; \mathbb{Z}_m) = \mathbb{Z}_m$. For $k = n+1$, $f_* : \mathbb{Z}_m \to \mathbb{Z}_m$ is the identity map which implies that f is not null-homotopic.

Lemma

If $f : S^k \to S^k$ is of degree m, then $f_* : H_k(S^k; G) \to H_k(S^k; G)$ by $g \mapsto mg$. A homomorphism $\varphi : G_1 \to G_2$ induces $\varphi_\sharp : C_n(X; G_1) \to C_n(X, G_2)$ by $\sum n_i \sigma_i \mapsto \sum \varphi(n_i)\sigma_i$. Then $\partial \varphi_\sharp = \varphi_\sharp \partial$ and $\varphi_* : H_n(X; G_1) \to H_n(X; G_2)$. If $f : S^k \to S^k$

is of degree k, fix any $g \in G$ and set $\varphi : \mathbb{Z} \to G$ by $1 \mapsto g$.

$$
\begin{array}{ccc}
H_k(S^k; \mathbb{Z}) & \xrightarrow{\ f_*\ } & H_k(S^k; \mathbb{Z}) \\
\downarrow{\varphi_*} & & \downarrow{\varphi_*} \\
H_k(S^k; G) & \xrightarrow{\ f_*\ } & H_k(S^k; G)
\end{array}
$$

Euler Characteristic

Let X be a finite CW-complex with c_i the number of i-cells. Then $\chi(X) = \sum_{i=0}^{n}(-1)^i c_i$ with $n = \dim X$. For example, $\chi(S^2) = 1 + 1 = 2$, $\chi(\mathbb{T}^2) = 1 - 2 + 1 = 0$, $\chi(\mathbb{RP}^2) = 1 - 1 + 1 = 1$, and $\chi(\mathbb{CP}^n) = 1 + 1 + \cdots + 1 = n$.

Theorem

$\chi(X) = \sum_{i=0}^{n}(-1)^i \operatorname{rank}(H_i(X))$.

Proof

$$
0 \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow 0
$$

So $Z_i = \ker d_i$, $B_i = \operatorname{im} d_{i+1}$ and $H_i = Z_i / B_i$. Then

$$
0 \longrightarrow B_i \longrightarrow Z_i \longrightarrow H_i \longrightarrow 0
$$

$$
0 \longrightarrow Z_i = \ker d_i \longrightarrow C_i \longrightarrow B_{i-1} = \operatorname{im} d_i \longrightarrow 0
$$

So $\operatorname{rank} Z_i = \operatorname{rank} B_i + \operatorname{rank} H_i$ and $\operatorname{rank} C_i = \operatorname{rank} Z_i + \operatorname{rank} B_{i-1} = \operatorname{rank} B_i + \operatorname{rank} H_i + \operatorname{rank} B_{i-1}$. Therefore

$$
\sum (-1)^i \overbrace{\operatorname{rank} C_i}^{= c_i} = \sum (-1)^i \operatorname{rank} H_i.
$$

So $\chi(M_g) = 2 - 2g$ and $\chi(N_g) = 2 - g$.

May 19, 2025

Mayer-Vietoris Sequences

Given a space X and open sets $A, B \subseteq X$ such that $A \cup B = X$ (i.e. $\mathcal{U} = \{A, B\}$ is an open cover of X), we have the chain

$$
C_n^{\mathcal{U}}(X) = \left\{ \sum_i n_i \sigma_i \ : \ \sigma_i : \Delta^n \to \text{ some } U \in \mathcal{U} \right\}
$$

Fact: $H_n^{\mathcal{U}}(X) \cong H_n(X)$.
For convenience, we write $C_n^{\mathcal{U}}(X) = C_n(A + B)$ which induces a short exact sequence

$$
0 \longrightarrow C_N(A \cap B) \xrightarrow{\ \varphi\ } C_n(A) \oplus C_n(B) \xrightarrow{\ \psi\ } C_n(A + B) \longrightarrow 0
$$

$$
\alpha \longmapsto (\alpha, -\alpha)
$$

$$
(\alpha, \beta) \longmapsto \alpha + \beta
$$

Verifying Exactness

φ is injective, since if $0 = \varphi(\alpha) = (\alpha, -\alpha)$, then $\alpha = 0$.
ψ is surjective, since any element in $C_n(A+B)$ and for $\sigma_i : \Delta^n \to A$ or B with $\alpha_i : \Delta^n \to A$ and $\beta_j : \Delta^n \to B$, we have

$$\sum_i n_i \sigma_i = \sum_i n_i \alpha_i + \sum_i n_i \beta_i = \psi \left(\sum_i n_i \alpha_i, \sum_i n_i \beta_i \right)$$

$\operatorname{im}\varphi \subseteq \ker\psi$ since $\psi(\varphi(\alpha)) = \psi(\alpha, -\alpha) = 0$
$\ker\psi \subseteq \operatorname{im}\varphi$, since if we suppose $(\alpha, \beta) \in C_n(A) \oplus C_n(B)$ such that $\psi(\alpha, \beta) = 0$. Then $\alpha \in C_n(A \cap B)$ and $\varphi(\alpha) = (\alpha, -\alpha) = (\alpha, \beta)$ (i.e. $(\alpha, \beta) \in \operatorname{im}\varphi$).

Long Exact Sequence

The short exact sequence induces

$$\cdots \longrightarrow H_n(A \cap B) \xrightarrow{\varphi_*} H_n(A) \oplus H_n(B) \xrightarrow{\psi_*} H_n(X)$$

$$\xrightarrow{\partial} H_{n-1}(A \cap B) \longrightarrow \cdots$$

Example: Klein Bottle

The Klein bottle is created by gluing two Mobius strips along their boundary

IMAGE 1

Let A, B be two Mobius strips as subsets of the Klein bottle. Let U, V be neighborhoods of A and B respectively that deformation retract to A and B. Then $A \cap B$ is homotopic to the circle. Then

$$\overbrace{H_2(A \cap B)}^{=0} \longrightarrow \overbrace{H_2(A) \oplus H_2(B)}^{=0} \longrightarrow H_2(X)$$

$$\longrightarrow H_1(A \cap B) \longrightarrow H_1(A) \oplus H_1(B) \longrightarrow H_1(X)$$

$$\longrightarrow \overbrace{\tilde{H}_0(A \cap B)}^{=0}$$

Which leads to

$$0 \longrightarrow H_2(X) \longrightarrow H_1(A \cap B) \xrightarrow{\varphi_*} H_1(A) \oplus H_1(B) \xrightarrow{\psi_*} H_1(X) \longrightarrow 0$$

$$\alpha \longmapsto (\alpha, -\alpha)$$

$$0 \longrightarrow H_2(X) \longrightarrow \mathbb{Z} \xrightarrow{\varphi_*} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\psi_*} H_1(X) \longrightarrow 0$$

$$1 \longmapsto (2, -2)$$

In particular, ψ_* is injective. Hence $H_2(X) \to \mathbb{Z}$ is the zero map and $H_2(X) = 0$. Then also

$$H_1(X) = (\mathbb{Z} \oplus \mathbb{Z})/\ker\psi_* = (\mathbb{Z} \oplus \mathbb{Z})/\operatorname{im}\varphi_* = \langle a, b \rangle / \langle 2a - 2b \rangle = \langle a, a - b \rangle / \langle 2a - 2b \rangle \cong \mathbb{Z} \oplus \mathbb{Z}_2.$$

Definition: Commutator

Let G be a group. $[G, G]$ is the normal subgroup generated by elements of the form $[g, h] = g^{-1}h^{-1}gh$. Then $G/[G, G]$ is Abelian.

Commutator of the Fundamental Group

$H_1 = \pi_1/[\pi_1, \pi_1]$.
We define a map $h : \pi_1(X, x_0) \to H_1(X)$. Let $f : \Delta^1 = [0, 1] \to X$ be a loop at x_0, such that f is also a singular 1-simplex (i.e. a cycle). So $\partial f = f(1) - f(0) = x_0 - x_0 = 0$. Hence we may assign $[f] \in H_1(X)$ to the loop f (i.e. $h(f) = [f]$). Write $f \simeq g$ for path homotopy and $f \sim g$ when $f - g$ is a boundary.
h is well defined, since

1. if f is homotopic to a constant loop, then $f \sim 0$ (i.e. f is a boundary). We treat $H : D^2 \to X$ as a singular 2-simplex, call it σ. Then $\partial\sigma = f$.

2. if $f \simeq g$, then $f \sim g$. Let H be a path homotopy between f and g.

<div align="center">IMAGE 2</div>

Then $\partial L = d - g - Cx_0$, $\partial R = d - Cx_0 - f$ and $\partial(L - R) = f - g$. This shows that $h : \pi_1(X, x_0) \to H_1(X)$ is well-defined. h is a group homomorphism. We need to show that $h(f \cdot g) = h(f) + h(g)$ (i.e. $f \cdot g \sim f + g$).

<div align="center">IMAGE 3</div>

So $\sigma : \Delta^2 \to X$ defined by the filling in of the 2-simplex has $\partial\sigma = f \cdot g - f - g$.
h is surjective. For $\sigma_i : \Delta^i \to X$, let $\sum_i n_i \sigma_i \in C_1(X)$ be a 1-cycle. Then

$$0 = \partial\left(\sum_i n_i\sigma_i\right) = \sum_i n_i\left(\sigma_i(1) - \sigma_i(0)\right).$$

Let S be the set of distinct points in the list $\{\sigma_i(0), \sigma_i(1) : |i = 1, \ldots, k\}$ for $m_p \in \mathbb{Z}$. Then

$$\sum_i n_i(\sigma_i(1) - \sigma_i(0)) = \sum_{p \in S} m_p \cdot p.$$

and $m_p = 0$ for all $p \in S$. For each σ_i, we consider a loop η_i at x_0 by

<div align="center">IMAGE 4</div>

For any $p \in S$, β_p is a path from x_0 to p. Then $h(\eta_i) = \beta_{\sigma_i(0)} + \sigma_i - \beta_{\sigma_i(1)}$. Now consider a loop $\eta_1^{n_1}\cdots\eta_k^{n_k}$ at x_0.

$$h(\eta_1^{n_1}\cdots\eta_k^{n_k}) = \sum_i n_i h(\eta_i) = \sum_i n_i\sigma_i + \sum_i n_i(\beta_{\sigma_1(0)} - \beta_{\sigma_1(1)}) = \sum_i n_i\sigma_i + 0$$

$\ker(h) = [\pi_1, \pi_1]$. Since $[\pi_1, \pi_1]$ is generated by $f^{-1}g^{-1}fg$, $h(f^{-1}g^{-1}fg) = -h(f) - h(g) + h(f) + h(g) = 0$. So $[\pi_1, \pi_1] \subseteq \ker h$.

Lemma:

Let G be a group, and let w be a word in G. Suppose that for each $g \in G$ its exponent in w adds up to zero, then $w \in [G, G]$.

Proof

Let $\pi : G \to G/[G, G]$ be the quotient map and $\pi(w) = 0$ (i.e. $w \in \ker \pi = [G, G]$ because if $w = g^{k_1} \cdots g^{k_2} \cdots g^{k_3} \cdots$ then $\pi(w) = \pi(g)^{k_1 + k_2 + k_3 + \cdots} = 0$.

Commutator of the Fundamental Group Continued

Let f be a loop at x_0 such that $h(f) = 0$ (i.e. as a singular 1-simplex). f is a boundary. Hence there is $\sum_i n_i \sigma_i \in C_2(X)$ such that $(\sigma_i : \Delta^2 \to X)$

IMAGE 5

$$f = \partial \left(\sum_i n_i \sigma_i \right) = \sum n_i (\alpha_i + \beta_i + \gamma_i)$$

Let S be the set of disjoint edges in the list $\{\alpha_i, \beta_i, \gamma_i : i = 1, \ldots, k\}$. Then

$$f = \sum n_i (\alpha_i + \beta_i + \gamma_i) = \sum_{e \in S} m_e \cdot e$$

in $C_1(X)$. Hence $m_e = 1$ when $e = f$ and $m_e = 0$ otherwise. For each σ_i, we draw a loop η_i at x_0 by joining

IMAGE 6

Then each η_i is homotopic to the boundary of σ_i (i.e. null-homotopic). Let us consider a loop $\eta_1^{n_1} \cdots \eta_k^{n_k} f^{-1} \simeq f^{-1}$. Each η_i is a product of 3 loops. Hence $\eta_1^{n_1} \cdots \eta_k^{n_k} f^{-1}$ is a word in π_1 in $S' \cup \{f\}$ where S' is the collection of all the generated loops with basepoint at x_0. The exponent of such a loop in this word is either m_e when $e \neq f$ or $m_e - 1$ when $e = f$. So it is always zero by the precedeing. This shows that

$$[f^{-1}] = [\eta_1^{n_1} \cdots \eta_k^{n_k} f^{-1}] \in [\pi_1, \pi_1].$$

Mayer-Vietoris for Reduced Homology

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
C_0(A \cap B) & \xrightarrow{\epsilon} & \mathbb{Z} \longrightarrow 0 \\
\downarrow{\varphi_*} & & \downarrow{\varphi} \\
C_0(A) \oplus C_0(B) & \xrightarrow{\epsilon \oplus \epsilon} & \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0 \\
\downarrow{\psi_*} & & \downarrow{\psi} \\
C_0(A + B) & \xrightarrow{\epsilon} & \mathbb{Z} \longrightarrow 0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

Where $\epsilon : C_0(X) \to \mathbb{Z}$ by $\sum_i n_i \sigma_i \mapsto \sum n_i$ is surjective. Then $\alpha \in \sum_i n_i \sigma_i$, and we have $\varphi\left(\sum_i n_i \right)$ and $(\epsilon \oplus \epsilon)\left(\sum_i n_i \sigma_i, -\sum_i n_i \sigma_i \right) = \left(\sum_i n_i, -\sum_i n_i \right)$. So assign $\varphi : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ by $1 \mapsto (1, -1)$ and let $\left(\sum_i n_i \alpha_i, \sum_i m_i \beta_i \right) \in C_0(A) \oplus C_0(B)$. Then we have that $\epsilon\left(\sum_i n_i \alpha_i, \sum_i m_i \beta_i \right) = \sum_i n_i + \sum_i m_i$ and $\psi\left(\sum_i n_i, \sum_i m_i \right)$ is assigned to $\psi_i : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ by $(m, n) \mapsto m + n$. Then the above diagram commutes.

May 21, 2025

Recall: de Rahm Cohomology

If M^n is a continuous manifold, $\Omega^p(M)$ is the collection of k-forms on M and $d : \Omega^p(M) \to \Omega^{p+1}(M)$ with $d \circ d = 0$. This gives a cochain

$$\cdots \xrightarrow{d} \Omega^p(M) \xrightarrow{d} \Omega^{p+1}(M) \xrightarrow{d} \cdots$$

It defines the de Rahm cohomology $H^p_{\mathsf{dR}}(M) = \ker d_p / \operatorname{im} d_{p-1}$.
Our goal is the de Rahm Theorem:

$$H^p_{\mathsf{dR}}(M) \cong H^p(M;\mathbb{R})(= \operatorname{Hom}(H_p(M),\mathbb{R})).$$

For $p = 1$ we can construct a map $I : H^1_{\mathsf{dR}}(M) \to \operatorname{Hom}(\pi_1(M,x),\mathbb{R})$ by $[\omega] \mapsto I[\omega] : \pi_1(M,x) \to \mathbb{R}$ where $(I[\omega])[\gamma] := \int_{\tilde{\gamma}} \omega$. TO be precise, we pick a piecewise smooth $\tilde{\gamma}$ at x_0 such that $[\tilde{\gamma}] = [\gamma]$ (well-defined because for α, β piecewise smooth and "smoothly homotopic", $\int_\alpha \omega = \int_\beta \omega$).
Then I is well defined because it is independent of the choice of $\tilde{\gamma}$ and ω. In the latter case, if $\omega = \omega' + df$ for $f \in \Omega^0(M)$, then

$$\int_{\tilde{\gamma}} \omega - \omega' = \int_{\tilde{\gamma}} df = f(\tilde{\gamma}(1)) - f(\tilde{\gamma}(0)) = 0.$$

I is injective, because if we suppose $I[\omega] = 0 \in \operatorname{Hom}(\pi_1(M,x),\mathbb{R})$, then $\int_{\tilde{\gamma}} \omega = I[\omega][\gamma] = 0$ for all $\gamma \in \pi_1(M,x)$. Then we claim that ω is conservative, because for any piecewise smooth loop α we can let y be a point on α and draw a composed curve with β from x to y. Then $\beta\alpha\beta^{-1}$ is based at x and

$$0 = \int_{\beta\alpha\beta^{-1}} \omega = \int_\beta \omega + \int_\alpha \omega - \int_\beta \omega = \int_\alpha \omega$$

Then, recall that conservative implies exactness since $f(y) = \int_{\gamma_{xy}} \omega$ holds for γ_{xy} from x to y. Therefore $[\omega] = 0$.

I is surjective (to be proved later) since if $\gamma \in \pi_1(M,x)$ such that $\gamma^k = \mathrm{id}$, then for every $\varphi \in \operatorname{Hom}(\pi_1(M,x),\mathbb{R})$ we have that $\varphi(\gamma) = 0$.

Corollary

If $\pi_1(M)$ is torsion (in particular if $\pi_1(M)$ is finite), then $H^1_{\mathsf{dR}}(M) = \{0\}$.

(A Little Bit of) Cohomology

In homology, we start with a chain complex

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \longrightarrow \cdots$$

Let G be an abelian group ($G = \mathbb{R}$ or $\mathbb{Z}$ or $\mathbb{Z}_m$), and define $C_n^* = \operatorname{Hom}(C_n, G)$ and $\partial^* : C_n^* \to C_{n+1}^*$ by $\varphi \mapsto \partial^* \varphi$ where $\partial^* \varphi$ is defined by $\varphi \circ \partial$. This gives a cochain

$$\cdots \longleftarrow C_{n+1}^* \xleftarrow{\partial^*} C_n^* \xleftarrow{\partial^*} C_{n-1}^* \longleftarrow \cdots$$

with $\partial^* \circ \partial^* = 0$.

Definition: Cohomology Group

The cohomology group $H^n(X, G)$ is $\ker \partial^* / \operatorname{im} \partial^*$.

Example

Given

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ 0\ } \mathbb{Z} \xrightarrow{\ \cdot 2\ } \mathbb{Z} \xrightarrow{\ 0\ } \mathbb{Z} \longrightarrow 0$$

we have a dual

$$0 \longleftarrow \mathbb{Z} \xleftarrow{\ 0\ } \mathbb{Z} \xleftarrow{\ \cdot 2\ } \mathbb{Z} \xleftarrow{\ 0\ } \mathbb{Z} \longleftarrow 0$$

Then we have $H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z}_2$, $H_2 = 0$ and $H_3 = \mathbb{Z}$ while $H^0 = \mathbb{Z}$, $H^1 = 0$, $H^2 = \mathbb{Z}_2$ and $H^3 = \mathbb{Z}$. In general, $H^n(X, G) \not\cong \operatorname{Hom}(H_n(X), G)$.

Fact (1)

Let T_n be the torsion subgroup of H_n. Then $H^n(X, \mathbb{Z}) \cong T_{n-1} \oplus (H_n / T_n)$.

Fact (2)

$H^n(X, \mathbb{R}) \cong \operatorname{Hom}(H_n(X), \mathbb{R})$. For example, we can dual by $G = \mathbb{R}$

$$0 \longleftarrow \mathbb{R} \xleftarrow{\ 0\ } \mathbb{R} \xleftarrow{\ \sim\ } \mathbb{R} \xleftarrow{\ 0\ } \mathbb{R} \longleftarrow 0$$

with $\mathbb{R}$ coefficients. Then $H^0 = \mathbb{R}$, $H^1 = 0$, $H^2 = 0$ and $H^3 = \mathbb{R}$.

Integration

Take $H_p(M)$ and a singular n-simplex $\sigma : \Delta^p \to M$. We want $I : H^p_{\mathrm{dR}}(M) \to \operatorname{Hom}(H_p(M), \mathbb{R})$ by $[\omega] \mapsto I[\omega]$ where $(I[\omega])[\sigma] = \int_\sigma \omega$. To make this work, we consider smooth σ and the collection $C_p^\infty(M) = \{\sigma : \Delta^p \to M \text{ smooth}\}$. This gives a chain complex

$$\cdots \longrightarrow C_{p+1}^\infty \longrightarrow C_p^\infty \longrightarrow C_{p-1}^\infty \longrightarrow \cdots$$

Then it has homology group $H_p^\infty(M)$.

Fact

The inclusion map $\iota : C_p^\infty(M) \to C_p(M)$ induces an isomorphism $\iota_* : H_p^\infty(M) \to H_p(M)$. Then we can consider instead $I : H_{\mathrm{dR}^p(M)} \to \operatorname{Hom}(H_p^\infty(M), \mathbb{R})$. Then if $\sigma : \Delta^p \to M$ is smooth, we can write $\int_\sigma \omega := \int_{\Delta^p} \sigma^* \omega$ where $\Delta^p \subseteq \mathbb{R}^p$ is formed by vertices $[v_0, \ldots, v_p]$ of the form $v_0 = (0, \ldots, 0)$ and $v_i = (0, \ldots, 1, \ldots, 0)$.

Stoke's Theorem (for Integration over Smooth Chains)

$$\int_\sigma d\omega = \int_{\partial \sigma} \omega$$

where $\partial \sigma = \sum_i (-1)^i \sigma|_{i\text{-th face}}$. More precisely, $F_i : \Delta^{p-1} \to i$-th face of Δ^p by $[v_0, \ldots, v_{p-1}] \mapsto [v_0, \ldots, \hat{v}_i, \ldots, v_p]$. So $\partial \sigma = \sum_i (-1)^i \sigma \circ F_i$. We need to check orientation, so write

$$\int_\sigma d\omega = \int_{\Delta^p} \sigma^*(d\omega) = \int_{\Delta^p} d(\sigma^*\omega) = \int_{\partial\Delta^p} \sigma^*\omega = \sum_i \int_{\partial_i\Delta^p} \sigma^*\omega$$

where $\partial\Delta^p$ has outward orientation and $\partial_i\Delta^p$ is the i-th face with outward orientation. On the right-hand side

$$\int_{\partial\sigma} \omega = \sum_i (-1)^i \int_{\sigma\circ F_i} \omega = \sum_i (-1)^i \int_{F_i(\Delta^{p-1})} \sigma^*\omega$$

where $\Delta^{p-1} \subseteq \mathbb{R}^p$ has a standard orientation given by $[e_1^{(p-1)}, \ldots, e_{p-1}^{(p-1)}]$. So $F_i : \Delta^{p-1} \to \partial_i\Delta^p$, where the domain has standard orientation and the image has outward orientation. Then F_i maps $e_1^{(p-1)} \mapsto e_1^{(p)}, \ldots, e_{i-1}^{(p-1)} \mapsto e_{i-1}^{(p)}$, $e_i^{(p-1)} \mapsto e_{i+1}^{(p)}, \ldots, e_{p-1}^{(p-1)} \mapsto e_p^{(p)}$. Then

$$((-e_i) \lrcorner d\mathrm{vol}_p)[F_i(e_1^{(p-1)}), \ldots, F_i(e_{p-1}^{(p-1)})] = d\mathrm{vol}_p[-e_i, e_1, \ldots, \hat{e}_i, \ldots, e_p] = (-1)^i \sum_i (-1)^i \int_{F_i(\Delta^{p-1})} \sigma^*\omega = \sum_i (-1)^{2i} \int_{\partial_i\Delta^p} \sigma^*\omega$$

Continuing...

For $I : H_{\mathrm{dR}}^p(M) \to \mathrm{Hom}(H_p(M), \mathbb{R})$ by $[\omega] \mapsto I[\omega]$ with $I[\omega][c] = \int_c \omega$, recall that $c = \sum_i n_i\sigma_i$ and $\sigma_i : \Delta^p \to M$ smooth. Then $\int_c \omega := \sum_i n_i \int_{\sigma_i} \omega$.

I is well defined, since if $\sigma' = \sigma + \partial\eta$ for $\eta : \Delta^{p-1} \to M$ smooth, $\int_{\sigma'} \omega = \int_\sigma \omega + \int_{\partial\eta} \omega$. But by Stokes', $\int_{\partial\eta} \omega = \int_\eta d\omega = 0$.

Secondly, if $\omega' = \omega + d\eta$ for $\eta \in \Omega^{p-1}(M)$, then $\int_\sigma \omega' = \int_\sigma \omega + \int_\sigma d\eta$ and again $\int_\sigma d\eta = \int_{\partial\sigma} \eta = 0$.

Naturality in de Rahm Cohomology

Given $F : M \to N$ smooth,

$$
\begin{array}{ccc}
H_{\mathrm{dR}}^p(N) & \xrightarrow{\ F^*\ } & H_{\mathrm{dR}}^p(M) \\
\downarrow{\scriptstyle I} & & \downarrow{\scriptstyle I} \\
\mathrm{Hom}(H_p(N), \mathbb{R}) & \xrightarrow{\ F^*\ } & \mathrm{Hom}(H_p(M), \mathbb{R})
\end{array}
$$

Let $[\omega] \in H_{\mathrm{dR}}^p(N)$ and $[0] \in H^p(M)$, then

$$(I \circ F^*[\omega])([0]) = (I \circ [F^*\omega])([0]) = \int_\sigma F^*\omega = \int_{F\circ\sigma} \omega = I[\omega](F \circ \sigma) = (F^* \circ I[\omega])(\sigma).$$

so the diagram commutes.

Mayer-Vietoris for de Rahm Cohomology

If $M = U \cup V$ for U, V open, then

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & H_{\mathrm{dR}}^p(M) & \longrightarrow & H_{\mathrm{dR}}^p(U) \oplus H_{\mathrm{dR}}^p(V) & \longrightarrow & H_{\mathrm{dR}}^p(U\cap V) & \xrightarrow{\ \delta\ } & H_{\mathrm{dR}}^{p+1}(M) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle I\oplus I} & & \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle I} & \\
\cdots \longrightarrow & H_{\mathrm{dR}}^p(M, \mathbb{R}) & \longrightarrow & H_{\mathrm{dR}}^p(U, \mathbb{R}) \oplus H_{\mathrm{dR}}^p(V, \mathbb{R}) & \longrightarrow & H_{\mathrm{dR}}^p(U\cap V, \mathbb{R}) & \xrightarrow{\ \delta\ } & H^{p+1}(M, \mathbb{R}) & \longrightarrow \cdots
\end{array}
$$

Recall that

$$0 \longrightarrow \Omega^k(M) \xrightarrow{k^* \oplus l^*} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{i^* - j^*} \Omega^k(U \cap V) \longrightarrow 0$$

$$\omega \longmapsto (\omega|_U, \omega|_V) \longrightarrow 0$$

$$(\omega, \eta) \longmapsto (\omega|_{U \cap V} - \eta|_{U \cap V})$$

So we have a short exact sequence

$$0 \longrightarrow C_p(U \cap V) \xrightarrow{\alpha} C_p(U) \oplus C_p(V) \xrightarrow{\beta} C_p(M) \longrightarrow 0$$

$$\sigma \longmapsto (\sigma, -\sigma)$$

$$(\omega, \eta) \longmapsto \sigma + \eta$$

which we dualize to

$$0 \longrightarrow C_p^*(M) \xrightarrow{\beta^*} C_p^*(U) \oplus C_p^*(V) \xrightarrow{\alpha^*} C_p^*(U \cap V) \longrightarrow 0$$

$$\varphi \longmapsto \beta^* \varphi$$

$$(\varphi, \psi) \longmapsto \alpha^*(\varphi, \psi)$$

So ultimately we have

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^p(M) & \longrightarrow & \Omega^p(U) \oplus \Omega^p(V) & \longrightarrow & \Omega^p(U \cap V) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \\
0 & \longrightarrow & \Omega^{p+1}(M) & \longrightarrow & \Omega^{p+1}(U) \oplus \Omega^{p+1}(V) & \longrightarrow & \Omega^{p+1}(U \cap V) & \longrightarrow & 0
\end{array}
$$

and

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C_p^*(M) & \longrightarrow & C_p^*(U) \oplus C_p^*(V) & \longrightarrow & C_p^*(U \cap V) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \\
0 & \longrightarrow & C_{p+1}^*(M) & \longrightarrow & C_{p+1}^*(U) \oplus C_{p+1}^*(V) & \longrightarrow & C_{p+1}^*(U \cap V) & \longrightarrow & 0
\end{array}
$$

Where I maps from $\Omega^p(M)$ to $C_p^*(M)$ and the entire three dimensional diagram commutes.

May 28, 2025

Recall

If $M = U \cap V$, then we may dualize by $C^p = \mathrm{Hom}(C_p; \mathbb{R})$ and

$$
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
C_p(U \cap V) \quad c & C^p(M) \xleftarrow{\;\;I\;\;} \Omega^p(M) & \omega \\
\downarrow{\scriptstyle\alpha} \quad \downarrow & \downarrow{\scriptstyle\beta^*} \quad\quad \downarrow & \downarrow \\
C_p(U) \oplus C_p(V) \quad {\scriptstyle(c,-c) \atop (c,c')} & C^p(U) \oplus C^p(V) \xleftarrow{\;\;I \oplus I\;\;} \Omega^p(U) \oplus \Omega^p(V) & {\scriptstyle(\omega|_U, \omega|_V) \atop (\omega, \omega')} \\
\downarrow{\scriptstyle\beta} \quad \downarrow & \downarrow{\scriptstyle\alpha^*} \quad\quad \downarrow & \downarrow \\
C_p(M) \quad c + c' & C^p(U \cap V) \xleftarrow{\;\;I\;\;} \Omega^p(U \cap V) & \omega|_{U \cap V} - \omega'|_{U \cap V} \\
\downarrow & \downarrow \quad\quad \downarrow & \\
0 & 0 \quad\quad 0 &
\end{array}
$$

where $(\beta^* \varphi)(c, c') = (\varphi \circ \beta)(c, c') = \varphi(c + c')$ and $(\alpha(\varphi, \varphi'))(c) = (\varphi, \varphi')(\alpha(c)) = (\varphi, \varphi')(c, -c) = \varphi(c) - \varphi'(c)$. $C^p(U) \oplus C^p(V)$ acts on $C_p(U) \oplus C_p(V)$ by $(\varphi, \varphi')(c, c') := \varphi(c) + \varphi'(c')$. $I : \Omega^p(*) \to C^p(*)$ by $I(\omega)(c) := \int_c \omega$. We have that the following commutes

$$
\begin{array}{ccc}
& C^p(M) \xrightarrow{\;\partial^*\;} C^{p+1}(M) & \\
{\scriptstyle I}\nearrow & {\scriptstyle I}\nearrow & \\
\Omega^p(M) \xrightarrow{\;d\;} \Omega^{p+1}(M) &
\end{array}
$$

since for $\omega \in \Omega^p(M)$ we have $(I(d\omega))(c) = \int_c d\omega = \int_{\partial c} \omega = I(\omega)(\partial c) = \partial^*(I(\omega))(c)$. Similarly, for $(\omega, \omega') \in \Omega^p(U) \oplus \Omega^p(V)$,

$$
\begin{aligned}
(\alpha^*(I \oplus I)(\omega, \omega'))(c) &= (I \oplus I)(\omega, \omega')(c, -c) \\
&= (I(\omega), I(\omega'))(c, -c) \\
&= I(\omega)(c) + I(\omega')(-c) \\
&= \int_c \omega - \omega' \\
&= I(\omega|_{U \cap V} - \omega'|_{U \cap V})(c).
\end{aligned}
$$

Then we may apply the five lemma to see that

$$
\begin{array}{ccccccccc}
H_{\mathrm{dR}}^{p-1}(U) \oplus H_{\mathrm{dR}}^{p-1}(V) & \longrightarrow & H_{\mathrm{dR}}^{p-1}(U \cap V) & \longrightarrow & H_{\mathrm{dR}}^p(M) & \longrightarrow & H_{\mathrm{dR}}^p(U) \oplus H_{\mathrm{dR}}^p(V) & \longrightarrow & H_{\mathrm{dR}}^p(U \cap V) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{p-1}(U) \oplus H^{p-1}(V) & \longrightarrow & H^{p-1}(U \cap V) & \longrightarrow & H^p(M) & \longrightarrow & H^p(U) \oplus H^p(V) & \longrightarrow & H^p(U \cap V)
\end{array}
$$

Theorem (de Rahm Theorem)

$I : H_{\mathrm{dR}}^p(M) \to H^p(M; \mathbb{R})$ is an isomorphism.

Proof

For convenience, we say a manifold M is de Rahm is $I : H_{dR}^p(M) \to H^p(M; \mathbb{R})$ is an isomorphism.

Step 1

A disjoint union of de Rahm manifolds is de Rahm. In fact, $M = \coprod_{\alpha \in A} M_\alpha$. We have shown that $H_{dR}^p(M) = \prod_{\alpha \in A} H_{dR}^p(M_\alpha)$ and can show the same for $H^p(M) = \prod_{\alpha \in A} H^p(M_\alpha)$. If $I : H_{dR}^p(M_\alpha) \to H^p(M_\alpha; \mathbb{R})$ is an isomorphism for each α, then $I : H_{dR}^p(M) \to H^p(M; \mathbb{R})$.

Step 2

Every convex open set in $\mathbb{R}^n$ is de Rahm. Let $U \subseteq \mathbb{R}^n$ convex. We know that

$$H_{dR}^p(U) = \begin{cases} \mathbb{R}, & p = 0 \\ 0, & p \geq 1 \end{cases} \qquad\qquad H^p(U; \mathbb{R}) = \begin{cases} \mathbb{R}, & p = 0 \\ 0, & p \geq 1 \end{cases}$$

and only need to check $p = 0$. We see that $H_{dR}^0(U) = \{f : U \to \mathbb{R} \,:\, df = 0\}$ are the constant functions and $C_0(U)$ is generated by $\sigma : \{p\} \to U$. For $I : H_{dR}^0(M) \to H^0(M; \mathbb{R})$, we know tha t$I$ is injective (and therefore an isomorphism) since if $f \in \ker I$ (i.e. $f(p) = I(f)(p) = 0, \, \forall p \in U$), then $f \equiv 0$ on U and $\ker I = \{0\}$.

Definition: de Rahm Cover / de Rahm Basis

We say that an open cover $\mathcal{U} = \{U_\alpha\}$ of M is de Rahm if every non-empty, finite intersection $\bigcap_{i=1}^k U_{\alpha_i}$ is de Rahm. If, in addition, $\mathcal{U}$ is a basis for M, then we say that $\mathcal{U}$ is a de Rahm basis.

Step 3

If M has a finite de Rahm cover, then M is de Rahm. Let $\{U_i\}_{i=1}^k$ be a de Rahm cover. We prove by induction on k. For $k = 2$, $M = U \cap V$, we conclude by Mayer-Vietoris sequences and the five lemma (above) that $I : H_{dR}^p(M) \to H^p(M; \mathbb{R})$. Now, supposing this holds for arbitrary k, we set $U = \bigcup_{j=1}^k U_j$ and $V = U_{k+1}$. Then U is de Rahm by inductive hypothesis, and again by Mayer-Vietoris and five lemma.

Remark

As a fact, if M^n is a closed manifold, then M has a finite good cover $\{U_i\}_{i=1}^k$ (i.e. every non-empty intersection is diffeomorphic to $\mathbb{R}^n$).

Step 4

If M has a de Rahm basis $\mathcal{U}$, then M is de Rahm.
Fact: M admits an exhaustion function $f : M \to \mathbb{R}$ such that $f^{-1}([0, a])$ is compact (i.e. f is proper) for all $a > 0$. For each $m \in \mathbb{Z}_+$ consider $A_m \in \{x \in M \,:\, m \leq f(x) \leq m+1\}$ and $A'_m = \{x \in M \,:\, m - 1/2 \leq f(x) \leq m + 3/2\}$. Then A_m and A'_m are compact and $A_m \subseteq A'_m$. For each $x \in A_m$, there is $U_x \in \mathcal{U}$ such that $x \in U_x \subseteq A'_m$ so $\{U_x \,:\, x \in A_m\}$ is an open cover of A_m and admits a finite subcover. Let B_m be the union of such a finite subcover. Then $A_m \subseteq B_m \subseteq A'_m$. Moreover, B_m fulfills the assumptions of Step 3 and is therefore de Rahm.
Note: if $m \neq \tilde{m}$, then B_m cannot intersect $B_{\tilde{m}}$ when $\tilde{m} \neq m - 1, m + 1$.
Set $U = \bigcup_{m \text{ odd}} B_m$ and $V = \bigcup_{m \text{ even}} B_m$ which are both de Rahm by step 1. We observe that $(U \cap V) = \bigcup (B_m \cap B_{m+1})$ which are each de Rahm by step 3. This further implies that $U \cap V$ is de Rahm by step 1. Now, $M = U \cup V$ with $U, V, U \cap V$ de Rahm. By step 3 again, M is de Rahm.

Step 5

Every open set in $\mathbb{R}^n$ is de Rahm. This is because for $U \subseteq \mathbb{R}^n$ open, U has a basis $\mathcal{U}$ whose elements are Euclidean balls. Therefore $\mathcal{U}$ is de Rahm by step 2 and U is de Rahm by step 4.

Step 5

Every manifold is de Rahm, because they have a basis whose elements are diffeomorphic to open sets in $\mathbb{R}^n$.

Poincaré Duality (for de Rahm Cohomology)

Let M^n be a closed, orientable manifold. Define $P : \Omega^k(M) \to \Omega^{n-k}(M)^*$ by $(P(\omega))(\eta) = \int_M \omega \wedge \eta$ for $\omega \in \Omega^k$ and $\eta \in \Omega^{n-k}$. It induces $P : H_{\mathrm{dR}}^k(M) \to H_{\mathrm{dR}}^{n-k}(M)^*$ by $(P[\omega])[\eta] = \int_M \omega \wedge \eta$.

Then P is well defined, since we observe that for η closed $d(\alpha \wedge \eta) = d\alpha \wedge \eta \pm \overset{=0}{\overbrace{\alpha \wedge d\eta}}$ and

$$\int_M (\omega + d\alpha) \wedge \eta = \int_M \omega \wedge \eta + \int_M d\alpha \wedge \eta = \int_M \omega \wedge \eta + \int_M d(\alpha \wedge \eta) = \int_M \omega \wedge \eta + \overset{=0}{\overbrace{\int_{\partial M} \alpha \wedge \eta}}$$

Similarly, $\int_M \omega \wedge \eta = \int_M \omega \wedge (\eta + d\rho)$.

If, instead, M is an orientable manifold without boundary, then $P : H_{\mathrm{dR}}^k(M) \to H_C^{n-k}(M)^*$ (i.e. compactly supported) by $(P[\omega])[\eta] = \int_M \omega \wedge \eta$.

Theorem (Poincaré)

If M^n is closed and orientable, then $H_{\mathrm{dR}}^k(M) \cong H_{\mathrm{dR}}^{n-k}(M)^* \cong H_{\mathrm{dR}}^k(M)^{**}$ which implies that $\dim H_{\mathrm{dR}}^k(M) < +\infty$ which implies that $\dim H_{\mathrm{dR}}^k(M) = \dim H_{\mathrm{dR}}^{n-k}(M)$. Also

$$\dim H_{\mathrm{dR}}^k(M) = \dim H^k(M; \mathbb{R}) = \dim H_k(M; \mathbb{R}) = \operatorname{rank} H_k(M).$$

When n is odd, the Euler characteristic

$$\chi(M) = \sum_k (-1)^k \operatorname{rank} H_k(M) = \sum_k (-1)^k \dim H_{\mathrm{dR}}^k(M) = 0.$$

Ingredients to Prove Poincaré Theorem

We say that a manifold M is Poincaré if $P : H_{\mathrm{dR}}^k(M) \to H_C^{n-k}(M)^*$ is an isomorphism. Similarly to the proof of the de Rahm theorem, we can define a Poincaré cover and a Poincaré basis.

- Step 1: If $M = \coprod M_\alpha$ with each M_α Poincaré, then M is Poincaré.

- Step 2: If M is a convex open subset in $\mathbb{R}^n$, then M is Poincaré. If $M = U$ is a convex open subset of $\mathbb{R}^n$, we know that

$$H_C^{n-k}(U) = \begin{cases} \mathbb{R}, & k = 0 \\ 0, & k \neq 0 \end{cases} \qquad\qquad H_{\mathrm{dR}}^k(U) = \begin{cases} \mathbb{R}, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

so we only need check $k = 0$. Then $P : H_{\mathrm{dR}}^0(U) \to H_C^n(U)^*$ is given by $P(c)(\omega) = \int_U c\omega$ for $c : U \to \mathbb{R}$ constant and $\omega \in \Omega_C^n(U)$. If $P(c) = 0$, then $\int_U c\omega = 0$ for all $\omega \in \Omega_C^n(U)$. In particular, we can use a bump function to construct $\omega \in \Omega_C^n(U)$ such that $\int_U \omega = 1$ ($\omega = f\, dx^1 \wedge \cdots \wedge dx^n$). Then $c \int_U \omega = 0$ implies $c = 0$ and P is injective.