

Partial Differential Equations I

January 8, 2024

Homework

Assigned exercises and concept maps. Graded by completion.

Presentations

Assigned topics; responsible for giving a class.

Definition: Partial Differential Equation(s) (PDE)

An identity relating an unknown function, its partial derivatives and its variables.

$$F(D^k u, \dots, D^2 u, Du, u, x) = 0, \quad x \in U \subseteq \mathbb{R}^n$$

where U is an open subset of \mathbb{R}^n , $u : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$, $Du = (\partial_{x_1} u, \dots, \partial_{x_n} u)$.

Then $F : \mathbb{R}^{n^k} \times \dots \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$, where F is given.

$x = (x_1, \dots, x_n)$ is (are) the independent variable(s).

u is the unknown function or dependent variable.

k is the order of the PDE.

Goal

Given a PDE, we consider

- Existence
- Uniqueness
- Stability

Recall: Multiindex Notation

$\alpha = (\alpha_1, \dots, \alpha_n)$ a vector such that $\alpha_i \in \mathbb{Z}_{\geq 0}$.

Then we say that α is a multiindex with order $|\alpha| = \alpha_1 + \dots + \alpha_n$.

Notation

$u : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$, $\alpha = (\alpha_1, \dots, \alpha_n)$.

$u^\alpha := D^\alpha u = \partial_{x_n}^{\alpha_n} \dots \partial_{x_1}^{\alpha_1} u$, where $\partial^0 u = u$.

Definition: Linear Partial Differential Equation

A linear PDE of order k is of the form

$$(*) \quad \sum_{|\alpha|=k} a_\alpha(x) D^\alpha u = f(x)$$

Remark

This means that F is multilinear in the first $n^k + n^{k-1} + \dots$ variables.

Definition: Homogeneous Linear Partial Differential Equation

A linear given by $(*)$ is homogeneous if $f(x) \equiv 0$.

Otherwise, it is non-homogeneous.

Example 1: Linear Transport Equation

$$\nabla u \cdot (1, b) = u_t + b \cdot Du = f(t, x)$$

This is a linear PDE of order 1 on $\mathbb{R} \times \mathbb{R}^n \equiv \mathbb{R}^{n+1}$ where (t, x) are independent variables and u is dependent. Here, x is the spatial variable while t is time and Du represents the gradient.

$\nabla u = (\partial_t u, \nabla u)$, $b \cdot Du = \sum_{i=1}^n b_i \partial_{x_i} u$, $(b_1, \dots, b_n) \in \mathbb{R}^n$ is fixed.

Example 2: Laplace Equation

$$\Delta u := \sum_{i=1}^n \partial_{x_i}^2 u = 0$$

This is a linear, homogeneous PDE of order 2.

Example 3: Poisson Equation

$$-\Delta u := f(u)$$

This is a nonlinear PDE of order 2.

Consider $f(u) = u^2$.

Example 4: Heat Equation (Diffusion Equation)

$$u_t - \Delta u = 0$$

This is a linear, homogeneous PDE of order 2.

Example 5: Wave Equation

$$u_{tt} - \Delta u = 0$$

This is a linear, homogeneous PDE of order 2.

Transport Equation

$u : \mathbb{R}^n(0, \infty) \rightarrow \mathbb{R}$ given by

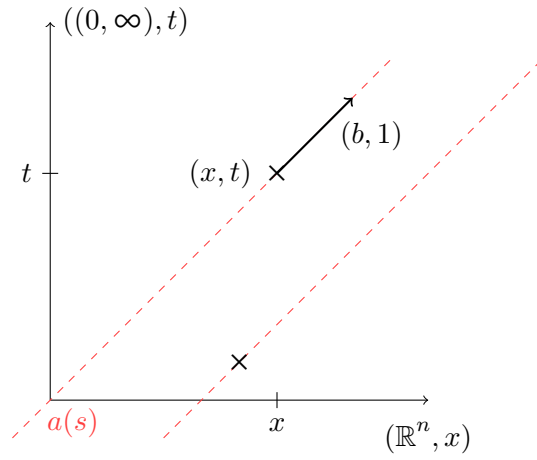
$$u_t + b \cdot Du = 0, \quad b \in \mathbb{R}^n$$

In order to get a solution, first assume that there exists a “nice” (e.g. smooth, C^1 , differentiable, etc.) solution.

Step 1

Notice that the PDE is equivalent to

$$\nabla u \cdot (b, 1) = 0$$



Step 2

Consider a curve on \mathbb{R}^{n+1} with velocity $(1, b)$ which passes through (x, t) . i.e.

$$\alpha(s) = (x + sb, t + s)$$

Notice $\alpha'(s) = (b, 1)$.

Then, let us study u along the curve $\alpha(s)$.

$$z(s) := u(\alpha(s))$$

Taking the derivative with respect to s ,

$$z'(s) := \frac{d}{ds}(u \circ \alpha(s)) = \nabla u|_{\alpha(s)} \cdot \alpha'(s) = \nabla u|_{\alpha(s)} \cdot (b, 1) = 0$$

That is $z'(s) = 0$, $z(s)$ is constant, and u along $\alpha(s)$ is constant.

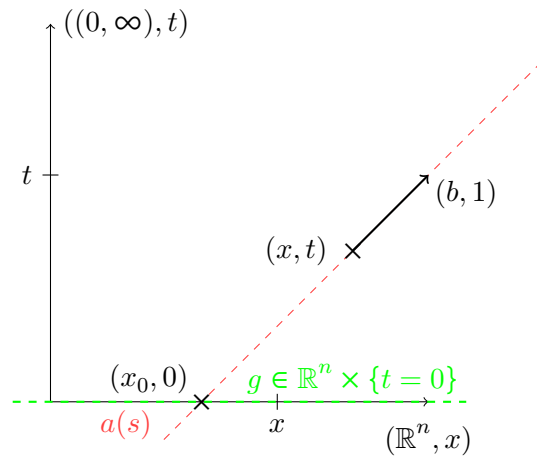
Conclusion

If we know some value of u along $\alpha(s)$, then we know all values along $\alpha(s)$.

If we have some value of u along every $\alpha(s)$, then we know u on $\mathbb{R}^n \times (0, \infty)$.

Transport Equation - Homogeneous Initial Value Problem

$$(*) \begin{cases} \nabla u \cdot (b, 1) = 0, & \mathbb{R}^n \times (0, \infty) \\ u = g, & \mathbb{R}^n \times \{t = 0\} \end{cases}$$



Here, $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is given.

Consider (x, t) ; we want to find $(x_0, 0)$.

We know $\alpha(s) = (x + sb, t + s) = (x_0, 0)$, therefore

$$\begin{cases} x + sb = x_0 \\ t + s = 0 \end{cases} \implies s = -t \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

Then, by replacing (2) in (1),

$$x_0 = x - tb$$

Then from the conclusion

$$u(x, t) = u(x_0, 0) = g(x_0) = g(x - tb)$$

Therefore, $u(x, t) := g(x - tb)$ (♥).

Remark

1. If there exists a regular (differentiable or C^1) solution u for $*$, then u should look like ♥.
2. If g is (differentiable or C^1), then u defined by ♥ is a (differentiable or C^1) solution for my problem.

Homework

Show that ♥ satisfies $*$.

Transport Equation - Non-homogeneous Initial Value Problem

$$(*) \begin{cases} \nabla u \cdot (b, 1) = f(x, t), & \mathbb{R}^n \times (0, \infty) \\ u = g, & \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Where $g : \mathbb{R}^n \rightarrow \mathbb{R}$ and $f : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}$ are given.

Solution

Notice that the PDE is equivalent to

$$\nabla u \cdot (b, 1) = f(x, t)$$

Define the “characteristic curve”

$$\alpha(s) = (x + sb, t + s)$$

and

$$z(s) := u(\alpha(s))$$

Taking $\frac{d}{ds}$,

$$z'(s) = \nabla u|_{\alpha(s)} \cdot (b, 1) = f(\alpha(s)) \implies z'(s) = f(x + sb, t + s) \quad (c)$$

Notice that c is an ordinary differential equation. Integrating from $-t$ to 0 .

$$\begin{aligned} \int_{-t}^0 z'(s) ds &= \int_{-t}^0 f(x + sb, t + s) ds \\ z(0) - z(-t) &= \int_{-t}^0 f(x + sb, t + s) ds \end{aligned}$$

Notice that $z(0) = u(x, t)$ and $z(-t) = u(\alpha(-t)) = u(x - tb, 0)$.

$$u(x, t) = u(x - tb, 0) + \int_{-t}^0 f(x + sb, t + s) ds$$

Then

$$\begin{aligned} u(x, t) &= g(x - tb) + \int_{-t}^0 f(x + sb, t + s) ds \\ &\stackrel{\bar{s}=s+t}{=} g(x - tb) + \int_0^t f(x + (\bar{s} - t)b, \bar{s}) d\bar{s} \\ &= g(x - tb) + \int_0^t f(x + (s - t)b, s) ds \end{aligned}$$

Remark: Method of Characteristics

Try to vert the PDE into an ODE and solve using characteristic curves.

January 10, 2024

Definition: Harmonic Function

If $u \in C^2$ such that $\Delta u = 0$, then u is a harmonic function.

Laplace Equation

Consider $u : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ with U open, then the homogeneous (Laplace) form is given by

$$\Delta u = 0$$

and the non-homogeneous (Poisson) form is

$$-\Delta u = f$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is given.

Remark / Exercise

The Laplace equation is invariant under translation and rotation.

That is, if $\Delta u(x) = 0$ and $v(x) = u(x - y)$, then $\Delta v = 0$.

Similarly, if $w(x) = u(O(x))$ then $\Delta w = 0$ where O is an orthogonal matrix.

Fundamental Solution of the Laplace Equation

Remark: since the Laplace equation is invariant under rotation, we can consider a function in terms of the radius $v(x) = v(|x|)$.

Recall $r = |x| = (x_1^2 + \dots + x_n^2)^{1/2}$.

Because of this remark, assume that $u(x) = v(|x|) = v(r(x))$ (*) where $v : (0, \infty) \rightarrow \mathbb{R}$.

Therefore, we need

$$\frac{\partial r}{\partial x_i} = \frac{1}{2} \cdot \frac{2x_i}{(x_1^2 + \dots + x_n^2)^{1/2}} = \frac{x_i}{r}$$

Replace (*) in the PDE

$$\frac{\partial u}{\partial x_i} = \frac{\partial}{\partial x_i} v(r(x)) = v'(r(x)) \cdot \frac{\partial r}{\partial x_i} = v'(r(x)) \cdot \frac{x_i}{r}$$

and

$$\begin{aligned} \frac{\partial^2 u}{\partial x_i^2} &= \frac{\partial}{\partial x_i} \left(v'(r(x)) \cdot \frac{x_i}{r} \right) \\ &= \frac{\partial}{\partial x_i} (v'(r(x))) \cdot \frac{x_i}{r} + v'(r(x)) \cdot \frac{\partial}{\partial x_i} \left(\frac{x_i}{r} \right) \\ &= v''(r(x)) \cdot \frac{x_i^2}{r^2} + v'(r(x)) \left(\frac{1}{r} - \frac{x_i}{r^2} \frac{\partial r}{\partial x_i} \right) \\ &= v''(r(x)) \frac{x_i^2}{r^2} + v'(r(x)) \left(\frac{1}{r} - \frac{x_i^2}{r^3} \right) \end{aligned}$$

Then, summing across i ,

$$\Delta u = v'' + v' \left(\frac{n}{r} - \frac{1}{r} \right) = 0$$

Then the PDE is equivalent to

$$v''(r) + \frac{v'}{r}(n-1) = 0 \quad (\square)$$

We need to find a solution for \square .

$$v''(r) = \frac{(1-n)v'}{r}$$

Assume, without loss of generality, that $v' \neq 0$ such that

$$\frac{v''(r)}{v'(r)} = \frac{1-n}{r} \implies ((|v'|))' = \frac{1-n}{r}$$

Then, integrating,

$$(|v'|) = (1-n)(r) + C = \frac{1-n}{2} r^2 + C$$

such that

$$|v'| = Cr^{1-n} \implies v' = Cr^{1-n} \implies v(r) = Cr^{1-n+1} + D = Cr^{2-n} + D$$

Definition: Fundamental Solution of the Laplace Equation

The function Φ given by

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} |x|, & n = 2 \\ \frac{1}{n(n-2)\alpha(n)} \cdot \frac{1}{|x|^{n-2}}, & n \geq 3 \end{cases}$$

where $\alpha(n)$ is the volume of the unit ball in \mathbb{R}^n is called the fundamental solution.

Remark

Φ solves the Laplace equation away from 0.

Lemma: Estimates for the Fundamental Solution

- First Estimate

$$|D\Phi(x)| \leq \frac{C}{|x|^{n-1}}, \text{ for } x \neq 0.$$

$$\frac{\partial \Phi}{\partial x_i} = C \frac{\partial}{\partial x_i} |x|^{2-n} = \frac{C(2-n)}{1-n} |x|^{2-n-1} \frac{\partial |x|}{\partial x_i} = |x|^{1-n} \cdot \frac{x_i}{|x|} = C x_i |x|^{-n}$$

Therefore

$$|D\Phi(x)| \leq C |x| |x|^{-n} \implies |D\Phi(x)| \leq C |X|^{1-n}$$

– Exercise

Compute for $n = 2$.

- Second Estimate

$$|D^2\Phi(x)| \leq \frac{C}{|x|^n}, \text{ for } x \neq 0.$$

$$\begin{aligned} \frac{\partial^2}{\partial x_J \partial x_i} \Phi &= C \frac{\partial}{\partial x_J} |x|^{-n} \\ &= C \left(\delta_{iJ} |x|^{-n} + x_i \frac{\partial}{\partial x_J} |x|^{-n} \right) \\ &= C \left(\delta_{iJ} |x|^{-n} + (-n) \cdot \frac{x_i |x|^{-n-1} x_J}{|x|} \right) \\ &= C \left(\frac{\delta_{iJ} |x|}{|x|^n} + \frac{C x_i x_J}{|x|^{n+1}} \right) \end{aligned}$$

Then

$$\left| \frac{\partial \Phi}{\partial x_i \partial x_J} \right| \leq \frac{C}{|x|^n} + \frac{C |x_i| |x_J|}{|x|^{n+2}} \leq \frac{2C}{|x|^n} = \frac{C}{|x|^n}$$

Then, we are done since

$$|D^2\Phi(x)| = \sqrt{\sum_J \left| \frac{\partial \Phi}{\partial x_i \partial x_J} \right|^2}$$

Poisson Equation

Motivation

Suppose we have $\Phi(x)$, the fundamental solution.

Then for an arbitrary, fixed element $y \in \mathbb{R}^n$, then we have $x \rightarrow \Phi(x - y)$ harmonic for $x \neq y$.

Consider $f : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $y \rightarrow f(y)$ then $x \rightarrow f(y)\Phi(x - y)$ is similarly harmonic for $x \neq y$.

Now, if given $\{y_1, \dots, y_m\}$ where $y_i \in \mathbb{R}^n$, then $x \rightarrow \sum_{i=1}^m f(y_i)\Phi(x - y_i)$ is harmonic $\forall x \neq \{y_1, \dots, y_m\}$.

Then, what happens if we consider

$$u(x) := \int_{\mathbb{R}^n} f(y)\Phi(x - y) dy \quad (\square_3)$$

Is u harmonic? No, since $\Delta\Phi(x - y)$ is not summable in \mathbb{R}^n we may not pass the limit into the integral.

(to be covered later) However, since $\Delta\Phi(x - y)$ acts as δ_{xy} in distribution, this may solve the Poisson equation.

Remark / Exercise

Assume that $f \in C_C^2(\mathbb{R}^n)$ (i.e f is twice continuously differentiable with compact support on \mathbb{R}^n).

The function Φ is integrable near the singularity on compact sets.

Prove using spherical coordinates.

Therefore, u defined by \square_3 is well defined

$$|u| = \left| \int_{\mathbb{R}^n} f(y)\Phi(x - y) dy \right| = \left| \int_K \Phi(x - y) dy \right| < \infty$$

Theorem: Solving the Poisson Equation

If $f \in C_C^2(\mathbb{R}^n)$ and u is defined by \square_3 , then

1. $u \in C^2(\mathbb{R}^n)$
2. $-\Delta u = f$, in \mathbb{R}^n

• Proof of 1

Since Φ presents a problem at $x = y$ but f is well behaved, we will change variables such that $\bar{y} = x - y$, $y = x - \bar{y}$, and $\frac{dy}{d\bar{y}}(-1)I_{m \times m}$ and then redefine $\bar{y} = y$.

$$u(x) = \int_{\mathbb{R}^n} f(y)\Phi(x - y) dy = \int_{\mathbb{R}^n} f(x - \bar{y})\Phi(\bar{y}) d\bar{y} = \int_{\mathbb{R}^n} f(x - y)\Phi(y) dy$$

In short, we have sent the problem from Φ to f .

Now, let us consider $e_i = (0, \dots, 1, \dots, 0)$.

Then for $h > 0$,

$$\frac{u(x + he_i) - u(x)}{h} = \frac{1}{h} \int_{\mathbb{R}^n} \Phi(y) [f(x + he_i - y) - f(x - y)] dy$$

Now, the limit as $h \rightarrow 0$

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{u(x + he_i) - u(x)}{h} &= \lim_{h \rightarrow 0} \int_{\mathbb{R}^n} \Phi(y) \overbrace{\frac{f(x + he_i - y) - f(x - y)}{h}}^{H(h,y)} dy \\ &= \int_{\mathbb{R}^n} \Phi(y) \cdot \frac{\partial f(x - y)}{\partial x_i} dy \end{aligned}$$

To justify passing the limit into the integral, take an arbitrary sequence $h_m \xrightarrow{0} 0$ and consider

$$f_m(y) := H(h_m, y)$$

We want to consider

$$\begin{aligned} |H(h_m, y)| &\leq \Phi(y) \times \frac{f(x + h_m e_i - y) - f(x - y)}{h} \times \\ &\leq \Phi(y) f'(c) \end{aligned}$$

Where c is along the curve between $f(x + h_m e_i - y)$ and $f(x - y)$ and chosen by mean value theorem.

– Exercise

$$|H(h_m, y)| \leq \Phi(y) \|f'\|_{L^\infty} \chi_{B(x, R)}(y)$$

Note that

$$C \int_{\mathbb{R}^n} |\Phi(y)| \chi_{B(x, R)}(y) dy = \int_{B(x, R)} |\Phi(y)| dy < \infty$$

– Exercise

Using the fact that a continuous function is uniformly continuous on a compact set, show that $u \in C^2(\mathbb{R}^n)$.

Dominated Convergence Theorem

If $f_m(x)$ such that $f_m(x) \xrightarrow[\text{pointwise}]{m \rightarrow \infty} f(x)$, and $|f_m(x)| \leq g(x)$ for $g \in L^1$, then f is integrable and

$$\lim_{m \rightarrow \infty} \int f_m(x) dx = \int f(x) dx$$

January 17, 2024

Recall: Averages

$$\begin{aligned} f : \{1, \dots, n\} &\rightarrow \mathbb{R} \\ i &\rightarrow a(i) \end{aligned}$$

The average is given as $\frac{a(1) + \dots + a(n)}{n}$.

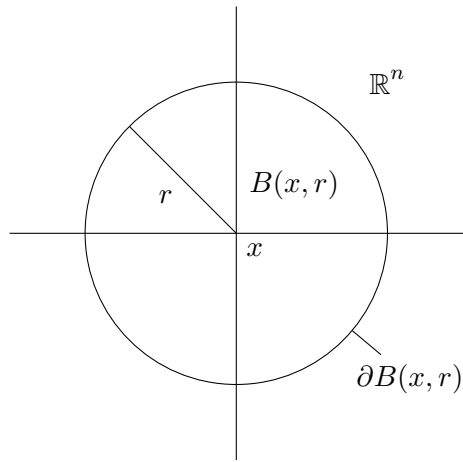
Then for $f : \Omega \rightarrow \mathbb{R}$, the average is given as

$$\frac{1}{|\Omega|} \int f(y) dy := \oint_{\Omega} f d\mu$$

In our case, $f : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$

$$\oint_{B(x, n)} f d\mu \equiv \frac{1}{|B(x, n)|} \oint_{B(x, n)} f d\mu$$

$$\oint_{\partial B(x, n)} f d\mu = \frac{1}{|\partial B(x, n)|} \oint_{\partial B(x, n)} f d\mu$$

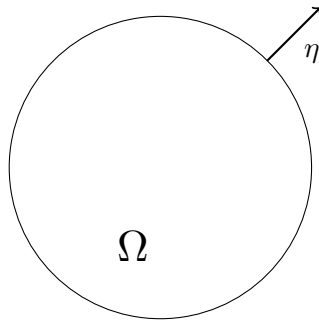


Theorem: Lebesgue Differentiation

$$u|_x = \lim_{r \rightarrow 0} \int_{B(x,r)} u \, d\mu = \lim_{r \rightarrow 0} \int_{\partial B(x,r)} u \, d\mu$$

Integration by Parts

$$\int_{\Omega} u \Delta v = - \int_{\Omega} \nabla u \cdot \nabla v + \int_{\partial \Omega} u \cdot \frac{\partial v}{\partial \eta}$$



Recall: Poisson's PDE

$$f \in C_c^2(\mathbb{R}^n), \quad u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) \, dy.$$

$$\Phi(x) = \left\{ \frac{1}{n(n-2)|\alpha(n)|} \frac{1}{|X|^{(n-2)}}$$

$$u(x) = \int_{\mathbb{R}^n} f(x-y) \Phi(y) \, dy$$

Part A

$$u \in C^2(\mathbb{R}^n)$$

Then

$$\frac{\partial u}{\partial x_1} = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_1}(x-y) \Phi(y) \, dy$$

$$\frac{\partial^2 u}{\partial x_1 \partial x_T} = \int_{\mathbb{R}^n} \frac{\partial^2 f}{\partial x_1 \partial x_T}(x-y) \Phi(y) \, dy$$

Notice that if we prove that

$$g(x) = \int_{\mathbb{R}^n} h(x-y)\Phi(y) dy$$

– where h is continuous with compact support – is continuous then we are done.

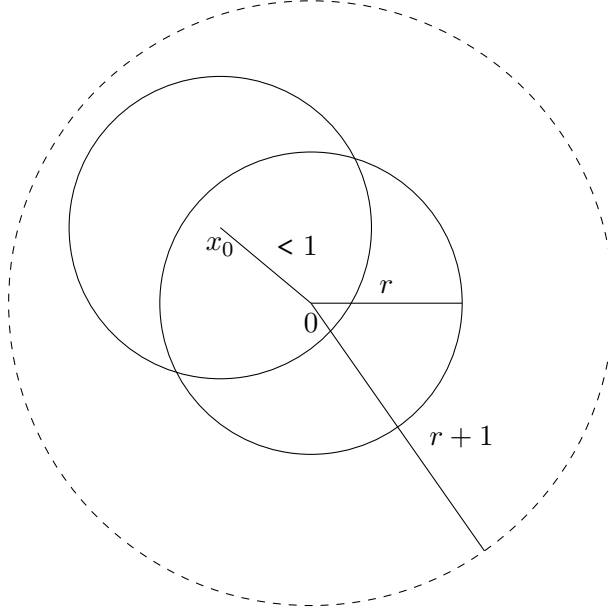
Let us prove that g is continuous.

Let $\varepsilon > 0$,

$$|g(x) - g(x_0)| \leq \int_{\mathbb{R}^n} \Phi(y) |h(x-y) - h(x_0-y)| dy$$

Without loss of generality, h has compact support on $B(0, r)$ for some radius r .

Therefore $h(x, y)$ has compact support on $B(x, r)$ and $h(x_0, y)$ has compact support on $B(x_0, r)$.



Consider $|x - x_0| < 1$, then $|h(x-y) - h(x_0-y)|$ has compact support on $B(x_0, r+1)$. Then

$$|g(x) - g(x_0)| \leq \int_{B(x_0, r+1)} \Phi(y) |h(x-y) - h(x_0-y)| dy$$

Since h is continuous on a compact domain, it is uniformly continuous.

Therefore $\exists \delta > 0$ such that $|w - z| < \delta \implies |h(w) - h(z)| < \varepsilon$.

Set $w = x - y$ and $z = x_0 - y$ such that $|w - z| = |x - x_0| < \delta$, then $|h(x-y) - h(x_0-y)| < \varepsilon$. Thus,

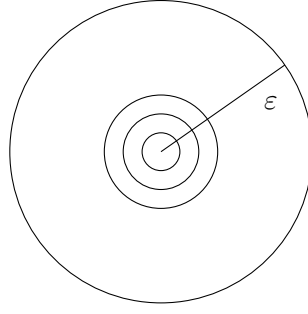
$$|g(x) - g(x_0)| \leq \varepsilon \int_{B(x_0, r+1)} \Phi(y) dy$$

Part B

$$-\Delta u = f$$

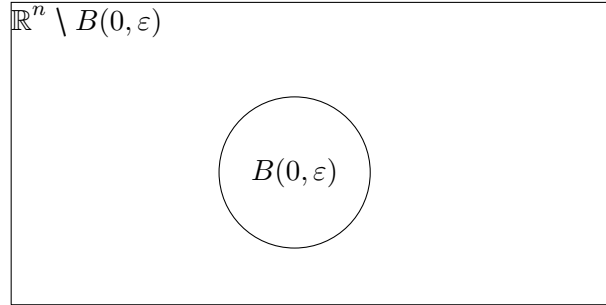
Letting $\varepsilon > 0$ and taking the Laplacian of both sides,

$$\begin{aligned} \Delta_x u(x) &= \int_{\mathbb{R}^n} \Delta_x f(x-y)\Phi(y) dy \\ &= \overbrace{\int_{B(0, \varepsilon)} \Delta_x f(x-y)\Phi(y) dy}^{I_\varepsilon} + \overbrace{\int_{\mathbb{R}^n \setminus B(0, \varepsilon)} \Delta_x f(x-y)\Phi(y) dy}^{J_\varepsilon} \end{aligned}$$



Then

$$\begin{aligned}
|I_\varepsilon| &\leq \int_{B(0,\varepsilon)} |\Delta_x f(x-y)| \Phi(y) \, dy \\
&\leq \|\nabla^2 f\|_{L^\infty} \int_{B(0,\varepsilon)} \Phi(y) \, dy \\
&\leq c \int_0^\varepsilon \int_{\partial B(0,r)} \Phi(y) \, dS(y) \, dr \\
&\leq c \int_0^\varepsilon \int_{\partial B(0,r)} \frac{1}{|y|^{n-2}} \, dS(y) \, dr \\
&= c \int_0^\varepsilon \int_{\partial B(0,r)} \frac{1}{r^{n-2}} \, dS(y) \, dr \\
&= c \int_0^\varepsilon \frac{1}{r^{n-2}} \int_{\partial B(0,r)} dS(y) \, dr \\
&\leq c \int_0^\varepsilon \frac{r^{n-1}}{r^{n-2}} \, dr \\
&= c \int_0^\varepsilon r \, dr = c\varepsilon^2
\end{aligned}$$



As an exercise, attempt the same argument with $n = 2$.

Therefore,

$$\Delta_x u = I_\varepsilon + J_\varepsilon$$

and $\lim_{\varepsilon \rightarrow 0} I_\varepsilon = 0$.

Now, we need to control J_ε .

$$J_\varepsilon = \int_{\mathbb{R}^n} \Delta_x f(x-y) \Phi(y) \, dy$$

$$\Delta_x f(x-y) = \sum \frac{\partial^2 f}{\partial x^2} f(x-y)$$

$$\begin{aligned}\frac{\partial f}{\partial x}(x-y) &= \nabla f|_{z=(x-y)} \cdot e_i = \frac{\partial f}{\partial z_i}|_{z=(x-y)} \\ \frac{\partial^2 f}{\partial x_i^2} &= \frac{\partial^2 f}{\partial z_i^2}|_{z=(x-y)}\end{aligned}$$

$$\begin{aligned}\Delta_y f(x-y) &= \sum \frac{\partial^2 f}{\partial y_i^2}(x-y) \\ \frac{\partial f}{\partial y_i}(x-y) &= \nabla f|_{z=(x-y)} \cdot -e_i = -\frac{\partial f}{\partial z_i}|_{z=(x-y)} \\ \frac{\partial^2 f}{\partial y_i^2} &= \frac{\partial^2}{\partial y_i^2}|_{z=x-y}\end{aligned}$$

So

$$\begin{aligned}J_\varepsilon &= \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Delta_y f(x-y) \Phi(y) dy \\ &= \overbrace{- \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \nabla_y f(x-y) \nabla \Phi(y) dy}^{K_\varepsilon} + \overbrace{\int_{\partial(\mathbb{R}^n \setminus B(0,\varepsilon))} \frac{\partial f}{\partial \eta} \Phi(y) dS(y)}^{L_\varepsilon}\end{aligned}$$

and

$$\Delta_x u = I_\varepsilon + K_\varepsilon + L_\varepsilon$$

To control L_ε , since

$$\begin{aligned}|L_\varepsilon| &\leq \int_{\partial B(0,\varepsilon)} \frac{|\partial f|}{\partial \eta} \Phi(y) dy \\ &\leq \int_{\partial B(0,\varepsilon)} |\nabla f| \Phi(y) dy \\ &\leq \|\nabla f\|_{L^\infty} \int_{\partial B(0,\varepsilon)} \Phi(y) dy \\ &\leq c \int_{\partial B(0,\varepsilon)} \frac{1}{|y|^{n-2}} dy \\ &= \frac{c}{\varepsilon^{n-2}} \int_{\partial B(0,\varepsilon)} dy \\ &\leq \frac{c\varepsilon^{n-1}}{\varepsilon^{n-2}} \\ &= c\varepsilon\end{aligned}$$

and K_ε , since $\frac{\partial \Phi}{\partial \eta} = \nabla \phi \cdot \eta = \frac{-y}{n\alpha(n)|y|^n} \cdot \eta = \frac{|y|^2}{n\alpha(n)|y|^n|y|} = \frac{1}{n\alpha(n)|y|^{n-1}}$

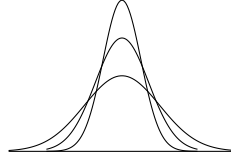
$$\begin{aligned}
|K_\varepsilon| &= - \int_{\mathbb{R}^n \setminus B(0, \varepsilon)} \overbrace{\nabla_y f(x-y) \nabla_y \Phi(y)}^0 dy \\
&= \int_{\mathbb{R}^n \setminus B(0, \varepsilon)} f(x-y) \Delta_y \Phi(y) dy - \int_{\partial(\mathbb{R}^n \setminus B(0, \varepsilon))} f(x-y) \frac{\partial \Phi}{\partial \eta} \\
&= - \int_{\partial B(0, \varepsilon)} f(x-y) \frac{\partial \Phi}{\partial \eta} \\
&= - \int_{\partial B(0, \varepsilon)} f(x-y) \frac{1}{n\alpha(n)|y|^{n-1}} dS(y) \\
&= - \frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B(0, \varepsilon)} f(x-y) dS(y) \\
&= - \underbrace{\frac{1}{n\alpha(n)\varepsilon^{n-1}}}_{\text{volume}} \int_{\partial B(0, \varepsilon)} f(z) dS(z) \\
&= \frac{1}{|\partial B(0, \varepsilon)|} \int_{\partial B(0, \varepsilon)} f(z) dz \\
&= - \oint_{\partial B(x, \varepsilon)} f(z) dz
\end{aligned}$$

Laplacian as a Distribution

$$-\Delta \Phi(y) = \delta(y)$$

Define the Dirac delta “function” as

$$\delta(y) = \begin{cases} \infty & y = 0 \\ 0 & y \neq 0 \end{cases}$$



such that $\int_{\mathbb{R}^n} \delta = 1$.

Translate the Dirac delta as

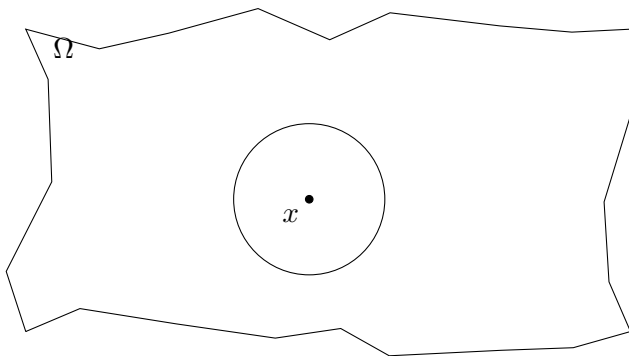
$$\delta_x = \begin{cases} \infty & y = x \\ 0 & y \neq x \end{cases}$$

and recall that

$$\begin{aligned}
 \Delta u(x) &= \Delta \left(\int_{\mathbb{R}^n} \Phi(x-y) f(y) \, dy \right) \\
 &= \int_{\mathbb{R}^n} \overbrace{\Delta \Phi(x-y)}^{-\delta_x(y)} f(y) \, dy \\
 &= - \int_{\mathbb{R}^n} \delta_x(y) f(y) \, dy \\
 &= - \int_{\mathbb{R}^n} \delta_x(y) f(x) \, dy \\
 &= -f(x) \int_{\mathbb{R}^n} \overbrace{\delta_x(y)}^1 \, dy \\
 &= -f(x)
 \end{aligned}$$

Harmonic Functions

Suppose u is harmonic



$u : \Omega \rightarrow \mathbb{R}^n$ harmonic.

Mean-value Formulas

Let U be an open set in \mathbb{R}^n , $u : U \rightarrow \mathbb{R}$ such that $\Delta u = 0$ in U . Then

$$\begin{aligned}
 u(x) &= \oint_{\partial B(0,r)} -u(y) \, dS(y) \\
 &= \oint_{B(x,r)} u(y) \, dy
 \end{aligned}$$

where $B(x,r) \subseteq U$.

IMAGE HERE

Proof

Consider $\phi(r) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y) \, dS(y)$.

If $\phi'(r) = 0$, when we are done since that would make ϕ constant and $\phi(r) =_{s \rightarrow 0} \phi(s) = u(x)$. Then

$$\begin{aligned}
\phi(r) &= \frac{1}{|\partial B(x, r)|} \int_{\partial B(x, r)} u(y) \, dS(y) \\
&\stackrel{y=x+rz}{=} \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(0, 1)} u(x + rz)r^{n-1} \, dS(z) \\
&= \frac{1}{n\alpha(n)} \int_{\partial B(0, 1)} u(x + rz) \, dS(z)
\end{aligned}$$

Therefore

$$\begin{aligned}
\phi'(r) &= \frac{1}{n\alpha(n)} \int_{\partial B(0, 1)} \nabla u(x + rz) \cdot z \, dS(z) \\
&\stackrel{y=x+rz}{=} \frac{1}{|\partial B(0, r)|} \int_{\partial B(x, r)} \nabla u(y) \cdot \frac{y - x}{r} \, dS(y) \\
&= \frac{1}{|\partial B(0, r)|} \int_{\partial B(x, r)} \nabla u(y) \cdot \eta \, dS(y) \\
&= \frac{1}{|\partial B(0, r)|} \int_{\partial B(x, r)} \frac{\partial y}{\partial \eta} \, dS(y) \\
&= \frac{1}{|\partial B(0, r)|} \int_{B(x, r)} \Delta u \\
&= 0
\end{aligned}$$

January 22, 2024

Mean Value Formula

For $U \subseteq \mathbb{R}^n$, U open with $u : U \rightarrow \mathbb{R}$ such that $u \in C^2(U)$, $\Delta u = 0$, we have

$$u(x) \stackrel{(a)}{=} \oint_{\partial B(x, r)} u \stackrel{(b)}{=} \oint_{B(x, r)} u$$

for all $B(x, r) \subseteq U$.

Recall that (a) was proven above by setting $\phi(r) = \oint_{\partial B(x, r)} u(y) \, dS(y)$ and showing $\phi'(r) = 0$.

For (b), we again apply spherical coordinates such that

$$\begin{aligned}
\int_{B(x, r)} u(y) \, dy &= \int_0^r \int_{\partial B(x, s)} u(y) \, dS(y) \, ds \\
&= \int_0^r |\partial B(x, s)| \overbrace{\oint_{\partial B(x, s)} u(y) \, dS(y)}^{u(x)} \, ds \\
&= u(x) \int_0^r |\partial B(x, s)| \, ds \\
&= u(x) \int_0^r n\alpha(n)S^{n-1} \, ds \\
&= \frac{u(x)n\alpha(n)S^n}{n} \Big|_0^r \\
&= u(x) \frac{|\overbrace{B(x, r)}|}{\alpha(n)r^n} \\
&= u(x) \frac{|B(x, r)|}{\alpha(n)r^n}
\end{aligned}$$

Converse

Recall that

$$\phi'(r) = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u(y) dy$$

Suppose then that we do not know that $\Delta u = 0$ but we have

$$u(x) = \oint_{\partial B(x,r)} u, \quad \forall B(x,r) \subseteq U$$

Then, necessarily, $\Delta u = 0$ in U .

- Proof

Suppose, for sake of contradiction, that $\Delta u \neq 0$. Then, without loss of generality, there exists $y \in U$ such that $\Delta u(x) > 0$ for $x \in B(y,n) \subseteq U$.

IMAGE HERE

$$\phi(r) = \oint_{\partial B(x,r)} u(x) dS(x)$$

and

$$\phi'(r) = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(y,r)} \Delta u(x) dS(x) > 0$$

which contradicts $\phi'(x) = 0$.

Strong Maximum Principle

Let $U \subseteq \mathbb{R}^n$ be a bounded open set, $u \in C^2(U) \cap C(\overline{U})$, $\Delta u = 0$ on U . Then

1. $\overline{U}(u) = \partial U(u)$.
2. If U is connected and u has its maximum in an interior point, then u is constant on \overline{U} .

IMAGE HERE - 2

Proof of A

Since $\partial U \subseteq \overline{U}$, $\partial U(u) \leq \overline{U}(u)$.

Let $x_0 \in \overline{U}$ such that $u(x_0) = \overline{U}(u)$.

IMAGE HERE - 4

So either $x_0 \in \partial U$ or $x_0 \in U$.

Let U^I be the connected component which contains x_0 . Then $x_0 \in U^I$, so by part (b) u is constant on $\overline{U^I}$. So

$$\overline{U}(u) = u(x_0) = \partial U^I(u) \leq \partial U(u)$$

Proof of B

Then there exists $x_0 \in U$ such that $\overline{u}(u) = u(x_0) = M$.

Let us define $\Omega = \{y \in U \mid u(y) = M\}$. Then

1. $\Omega \neq \emptyset, B \setminus x_0 \in \Omega$.

2. Ω open set.

IMAGE HERE - 3

1. Ω is closed, since $\Omega = u^{-1}(\{M\})$.

It follows that $\Omega = U$ and, therefore, $u(y) = M, \forall y \in U$.

• Proof of 2

Let $y \in \Omega, y \in U, u(y) = M$. Then there exists $B(y, r) \subseteq U$, and

$$M = u(y) = \oint_{B(y,r)} u(x) dS(x) \leq M$$

Then

$$\oint_{B(y,r)} u(x) dx = M$$

so $u(x) = M, \forall x \in B(y, r)$ and, therefore $B(y, r) \subseteq \Omega$ and Ω is open.

Remark: Boundedness Is Important

1. Consider $f(x) = x$ on $\mathbb{R}_{\geq 0}$.

2. IMAGE HERE - 5

Remark: Strong Minimum Principle Is Equivalent

Consequences

1. Positivity of harmonic functions.

2. Uniqueness of the Poisson problem.

Corollary: Positivity of Harmonic Functions

Suppose that U is connected and $u : U \rightarrow \mathbb{R}, u \in C^2(U) \cap C(\overline{U})$ solves the following problem

$$\begin{cases} \Delta u = 0 & \text{on } U \\ u = g & \text{on } \partial U \end{cases}$$

If $g \geq 0$ on ∂U , then u is positive on U as long as g is positive in some point.

Proof

Assume $\exists x_0 \in \partial U$ where x_0 is the minimum. Then $u(x_0) = \overline{U}(u)$ and, $\forall x \in U$,

$$0 \leq u(x_0) = \overline{U}(u) \leq u(x)$$

so u is non-negative. If $u(x) = 0$, then $u(x_0) = 0$ and the minimum is achieved in the interior. That would mean $u(x) = 0, \forall x \in \overline{U} \supseteq \partial U$ and $g(x) = 0, \forall x \in \partial U$ which would be a contradiction.

Theorem: Uniqueness of the Poisson Problem

Suppose $U \subseteq \mathbb{R}^n$ is open, connected and bounded. Then, there exists at most one solution to

$$(*) \begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

where $u \in C^2(U) \cap C(\overline{U})$.

Proof

Let u_1 and u_2 be two solutions of $*$.

Consider $w = u_1 - u_2$ and observe that

$$\Delta w = \Delta u_1 - \Delta u_2 = -f + f = 0, \quad \text{in } U$$

and $w|_{\partial U} = g - g = 0$ on ∂U . Therefore

$$\begin{cases} \Delta w = 0 & \text{in } U \\ w = 0 & \text{on } \partial U \end{cases}$$

By applying the minimum and maximum principles,

$$w(x_0) = \overline{U}(w) \leq w(x) \leq \underline{U}(w) = w(x)$$

so $w(x) = 0, \forall x \in \overline{U}$ and therefore $u_1 = u_2$.

Example

Let's consider $f : \mathbb{C} \rightarrow \mathbb{C}$ analytic (i.e. $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $a_n, z \in \mathbb{C}$). Then

$$f(z) = u(z) + v(z)$$

If $\mathbb{C} \cong \mathbb{R}^2$,

$$f(x + iy) = u(x, y) + v(x, y)$$

for $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $v : \mathbb{R}^2 \rightarrow \mathbb{R}$.

Claim: u and v are Harmonic.

$$u(x, y) + v(x, y) = \sum_{n=0}^{\infty} a_n (x + iy)^n$$

and

$$\begin{aligned}\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} &= \sum_{n=1}^{\infty} a_n n |x + iy|^{n-1} \\ \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} &= \sum_{n=1}^{\infty} a_n n |x + iy|^{n-1} i\end{aligned}$$

So

$$i \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

Recall: Convolution and Smoothing

Let $U \subseteq \mathbb{R}^n$ be an open set.

For $\varepsilon > 0$, define $U_\varepsilon = \{x \in U \mid d(x, \partial U) > \varepsilon\}$.

IMAGE HERE - 6

Define

$$\eta(x) = \begin{cases} ce^{-\frac{1}{|x|^2-1}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

with c such that $\int_{\mathbb{R}^n} \eta(x) dx = 1$, $\eta \in C^\infty(\mathbb{R}^n)$

IMAGE HERE - 7

Note that $\text{supp}(\eta) = B(0, 1)$ and take

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right), \quad \eta_\varepsilon \in C^\infty(\mathbb{R}^n)$$

Then

$$\int_{\mathbb{R}^n} \eta_\varepsilon(x) dx = 1$$

and $\text{supp}(\eta_\varepsilon) \subseteq B(0, \varepsilon)$.

If f is locally integrable on U , define its mollification

$$f^\varepsilon(x) = \int_U \eta_\varepsilon(x-y) f(y) dy \quad \forall x \in U_\varepsilon$$

January 24, 2024

Recall: Mollifiers

Define

$$\eta(x) = \begin{cases} ce^{-\frac{1}{|x|^2-1}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

where $\eta \in C^\infty(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \eta(x) = 1$ and $\text{supp}(\eta) \subseteq B(0, 1)$.
Then for $\varepsilon > 0$, $\eta_\varepsilon(x) = \frac{1}{\varepsilon} \eta\left(\frac{x}{\varepsilon}\right)$ where $\eta_\varepsilon \in C^\infty(\mathbb{R}^n)$.
So $\int_{\mathbb{R}^n} \eta_\varepsilon(x) = 1$ and $\text{supp}(\eta_\varepsilon) \subseteq B(0, \varepsilon)$
Given f locally summable; $f : U \rightarrow \mathbb{R}$,

$$\begin{aligned} f^\varepsilon(x) &:= \int_U \eta_\varepsilon(x - y) f(y) \, dy \quad x \in U_\varepsilon \\ &= \int_{B(x, \varepsilon)} \eta_\varepsilon(x - y) f(y) \, dy \quad x \in U_\varepsilon \end{aligned}$$

Properties

1. $f^\varepsilon \in C^\infty(U_\varepsilon)$.
2. $f^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} f$ a.e.
3. If f continuous, then $f^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} f$ uniformly on compact sets of U .

Theorem 6:

Let $u \in C(U)$ with $U \subseteq \mathbb{R}^n$ open and such that u satisfies the mean-value property (i.e. $u(x) = \int_{\partial B(x, r)} u(y) \, dS(y)$, $\forall B(x, r) \subseteq U$), then $u \in C^\infty$.

Corollary

If $u \in C^2(U)$ is harmonic, then $u \in C^\infty(U)$.

Proof

Let us take $x_0 \in U$

IMAGE HERE - 1

Notice, that if we prove that $u = u_\varepsilon$ on U_ε then we are done.

Let $x \in U_\varepsilon$, and noticing that $\eta(x) = \eta(|x|)$,

$$\begin{aligned}
u_\varepsilon(x) &= \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y) u(y) dy \\
&= \frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} \eta \frac{|x-y|}{\varepsilon} u(y) dy \\
&\stackrel{\text{spherical}}{=} \frac{1}{\varepsilon^n} \int_0^\varepsilon \int_{\partial B(x,r)} \eta \frac{\overbrace{|x-y|}^r}{\varepsilon} u(y) dS(y) dr \\
&= \frac{1}{\varepsilon^n} \int_0^\varepsilon \eta \frac{r}{\varepsilon} \int_{\partial B(x,r)} u(y) dS(y) dr \\
&= \frac{1}{\varepsilon^n} \int_0^\varepsilon \eta \frac{r}{\varepsilon} \underbrace{|\partial B(x,r)|}_{|\partial B(0,r)|} u(x) dr \\
&= \frac{u(x)}{\varepsilon^n} \int_0^r \eta \frac{r}{\varepsilon} \int_{\partial B(0,r)} dS(y) dr \\
&= u(x) \int_0^\varepsilon \frac{1}{\varepsilon^n} \eta \frac{r}{\varepsilon} dS(y) dr \\
&= u(x) \overbrace{\int_{B(0,\varepsilon)} \eta_\varepsilon(y) dy}^1 = u(x)
\end{aligned}$$

Theorem 7: Local Estimates of Harmonic Functions

Suppose $u \in C^2(U)$ a harmonic function.

Then $|D^\alpha u(x_0)| \leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0,r))}$, $B(x_0,r) \subseteq U$, where α is a multiindex of order k , $C_0 = \frac{1}{\alpha(n)}$ and $C_k = \frac{(2^{n+1}nk)^k}{\alpha(n)}$ for $k = 1, 2, \dots$

We may take α since, by previous theorem, $u \in C^\infty(U)$.

Proof

By induction.

Consider $k = 0$.

$$\begin{aligned}
u(x_0) &= \int_{B(x_0,r)} u(y) dy \\
&= \frac{1}{\alpha(n)r^n} \int_{B(x_0,r)} u(y) dy \\
|u(x_0)| &\leq \frac{1}{\alpha(n)r^n} \int_{B(x_0,r)} |u(y)| dy \\
&= \frac{C_0}{r^n} \|u\|_{L^1(B(x_0,r))}
\end{aligned}$$

For $k = 1$, if $|\alpha| = k = 1$ then $D^\alpha u(X_0) = \frac{\partial u}{\partial x_i}(x)$ for $i = 1, 2, \dots$

Notice that $\frac{\partial u}{\partial x_i}$ is also harmonic.

$$\begin{aligned}
\Delta \frac{\partial u}{\partial x_i} &= \sum_{t=1}^n \frac{\partial^2}{\partial x_t^2} \frac{\partial u}{\partial x_i} \\
&= \frac{\partial}{\partial x_i} \underbrace{\sum_{t=1}^\infty \frac{\partial^2 u}{\partial x_t^2}}_0
\end{aligned}$$

Applying the mean-value formula to $\frac{\partial u}{\partial x_i}(x_0)$ in $B(x, r/2)$.

IMAGE HERE - 2

$$\begin{aligned}\frac{\partial u}{\partial x_i}(x_0) &= \oint_{B(x_0, r/2)} \frac{\partial u}{\partial x_i}(y) dy \\ &= \frac{2^n}{\alpha(n)r^n} \oint_{B(x_0, r/2)} \frac{\partial u}{\partial x_i}(y) dy\end{aligned}$$

Recall $\int_{\Omega} w \Delta v = - \int_{\Omega} \nabla w \cdot \nabla v + \int_{\Omega} w \frac{\partial v}{\partial \eta}$.

$$\begin{aligned}e_i &= \nabla_{y_i} \frac{2^n}{\alpha(n)r^n} \int_{B(x_0, r/2)} \underbrace{\nabla u(y)}_w \cdot \underbrace{\nabla y_i}_v dy \\ &\stackrel{IBP}{=} \frac{2^n}{\alpha(n)r^n} \times - \int_{B(x_0, r/2)} u(y) \Delta y_i dy + \int_{\partial B(x_0, r/2)} u(y) \frac{\partial y_i}{\partial \eta} \times\end{aligned}$$

Note that

$$\frac{\partial y_i}{\partial \eta} = \nabla y_i \cdot \eta = e_i \cdot \eta = \eta_i$$

and

$$\left| \frac{\partial y_i}{\partial \eta} \right| = |\eta_i| \leq |\eta| = 1$$

So,

$$\begin{aligned}\left| \frac{\partial u}{\partial x_i} x_0 \right| &\leq \frac{2^n}{\alpha(n)r^n} \int_{\partial B(x_0, r/2)} |u(y)| dS(y) \\ &= \frac{2^n n \alpha(n) \left| \frac{r}{2} \right|^{n-1}}{\alpha(n)r^n} \|u\|_{L^\infty(\partial B(x_0, r/2))} \\ &= \frac{2n}{r} \underbrace{\|u\|_{L^\infty(\partial B(x_0, r/2))}}_*\end{aligned}$$

Let's analyze $*$.

Let $x \in \partial B(x_0, r/2)$, then $B(x, r/2) \subseteq B(x_0, r)$.

IMAGE HERE - 3

Then we may apply $k = 0$.

$$\begin{aligned}|u(x)| &\leq \frac{C_0}{\left| \frac{r}{2} \right|^n} \|u\|_{L^1(B(x, r/2))} \\ &\leq \frac{C_0}{\left| \frac{r}{2} \right|^n} \|u\|_{L^1(B(x_0, r))}\end{aligned}$$

Then

$$\begin{aligned}\left| \frac{\partial u}{\partial x_i}(x_0) \right| &\leq \frac{2n}{r} \frac{C_0}{\left| \frac{r}{2} \right|^n} \|u\|_{L^1(B(x_0, r))} \\ &= \frac{2^{n+1} n}{r^{n+1} \alpha(n)} \|u\|_{L^1(B(x_0, r))}\end{aligned}$$

HOMEWORK: Induct for arbitrary k .

Theorem 8: Liouville's Theorem

Suppose $u : \mathbb{R}^n \rightarrow \mathbb{R}$ is harmonic and bounded. Then u is constant.

Proof

$$|D^\alpha u(x)| = \sqrt{\sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^2} \leq \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|$$

Let $r > 0$, $B(x, r) \subseteq \mathbb{R}^n$. Then, using estimates

$$\left| \frac{\partial u}{\partial x_i}(x) \right| \leq \frac{C_1}{r^{n+1}} \|u\|_{L^1(B(x, r))}$$

Therefore,

$$\begin{aligned} |D^\alpha u(x)| &\leq \frac{nC_1}{r^{n+1}} \|u\|_{L^1(B(x, r))} \\ &= \frac{nC_1}{r^{n+1}} \int_{B(x, r)} |u(y)| \, dy \\ &\leq \frac{nC_1}{r^{n+1}} \|u\|_{L^\infty(B(x, r))} \alpha(n) r^n \\ &= \frac{C \|u\|_{L^\infty(B(x, r))}}{r} \end{aligned}$$

and

$$\begin{aligned} \left| \frac{\partial u}{\partial x_i}(x) \right| &\leq \frac{C \|u\|_{L^\infty(B(x, r))}}{r} \\ \left| \frac{\partial u}{\partial x_i}(x) \right| &\leq C \|u\|_{L^\infty(B(x, r))} \frac{1}{r} \implies \frac{\partial u}{\partial x_i}(x) = 0 \implies u = Ck \end{aligned}$$

Theorem: Representation Formula

Recall: $f \in C_c^2(\mathbb{R}^n)$, $(*) - \Delta u = f$ in \mathbb{R}^n , we saw that

$$\tilde{u}(x) : \int_{\mathbb{R}^n} \Phi(x - y) f(y) \, dy$$

solves $*$.

Let us consider $u \in C^2(\mathbb{R}^n)$ solving $-\Delta u = f$ for $n \geq 3$ where $f \in C_c^2(\mathbb{R}^n)$ and u is bounded. Then $u(x) = \tilde{u}(x) + C = \int_{\mathbb{R}^n} \Phi(x - y) f(y) \, dy + C$.

Proof

Notice that if \tilde{u} is bounded, then we are done. Because we may consider $w = u - \tilde{u}$ on \mathbb{R}^n where

$$\Delta w = \Delta u - \Delta \tilde{u} = -f - (-f) = 0$$

Therefore w is bounded and, by Liouville's Theorem, $w = C$ and $u - \tilde{u} = C \iff u = \tilde{u} + C$.

$$\begin{aligned} |\tilde{u}(x)| &\leq \int_{B(0, k)} \Phi(x - y) f(y) \, dy \\ &\leq \|f\|_{L^\infty(\mathbb{R}^n)} \int_{B(0, k)} \Phi(x - y) \, dy \end{aligned}$$

If this is less than some C which does not depend on x , we are done.

Since $\Phi(x) \rightarrow 0$ for $|x| \rightarrow \infty$, for any $C_1 \exists \alpha$ such that if $|x| > \alpha$ then $|\Phi(x)| < C_1$.

IMAGE HERE - 4

$\text{dist}(x, B(0, k)) = \text{dist}(x, y_0)$ where $y_0 \in \overline{B(0, k)}$.

IMAGE HERE - 5

There are two cases.

- Case 1

$$\text{dist}(x, B(0, k)) \leq \alpha$$

$$B(x, k) \subseteq B(0, \alpha + Ck)$$

Let $y \in B(x, k)$, then $|y - x| < k$ so $|x - y_0| < \alpha$.

Therefore $|y - y_0| \leq k + \alpha \implies |y| \leq k + \alpha + |y_0| = 2k + \alpha \implies y \in B(0, 2k + \alpha)$. Then

$$\|f\|_{L^\infty(\mathbb{R}^n)} \int_{B(x, k)} \Phi(y) dy \leq \|f\|_{L^\infty(\mathbb{R}^n)} \int_{B(0, \alpha + 2k)} \Phi(y) dy$$

HOMEWORK - Consider the other case.

January 29, 2024

Recall: Representation Formula

For $n \geq 3$.

$$\tilde{u}(x) : \int_{\mathbb{R}^n} \Phi(x - y) f(y) dy$$

It is sufficient to show that \tilde{u} is bounded. Then

$$|\tilde{u}| \leq C \int_{B(0, k)} \Phi(x - y) dy$$

$\forall C_1, \exists \alpha$ such that $|z| \geq \alpha \implies |\Phi(z)| \leq C_1$.

Case 2

For $\text{dist}(x, B(0, k)) \geq \alpha$, $\text{dist}(x, y) \geq \alpha$, $\forall y \in B(0, k)$. Then

$$\begin{aligned} |x - y| &\geq \alpha \\ \frac{1}{|x - y|} &\leq \frac{1}{\alpha} \\ \frac{1}{|x - y|^{n-2}} &\leq \frac{1}{\alpha^{n-1}} \end{aligned}$$

and

$$|\tilde{u}(x)| \leq C \int_{B(0, k)} \frac{1}{|x - y|^{n-2}} dy \leq \frac{C}{\alpha^{n-2}} \int_{B(0, k)} dy$$

Theorem 10: Harmonic Implies Analytic

Let $U \subseteq \mathbb{R}^n$ open, $u \in C^2(U)$ harmonic. Then u is analytic in U .

Proof

Let $x_0 \in U$. We want to prove that the power series converges to $u(x)$ for x in a neighborhood around x_0 .

Let $r = \text{dist} \left(x_0, \frac{\partial U}{4} \right)$, $M = \frac{1}{\alpha(n)r^n} \|u\|_{L^1(B(x_0, r))} \subset U$.

IMAGE HERE - 1

We want to analyze $x \in B(x_0, r)$.

Notice that $B(x, r) \subset B(x_0, 2r)$, and $z \in B(x, r)$ gives $|z - x| < r$ so

$$|z - x_0| \leq \underbrace{|z - x|}_{\leq r} + \underbrace{|x - x_0|}_{\leq r} \leq 2r$$

Applying estimates on $B(x, r)$, $|\alpha| = k$,

$$\begin{aligned} |D^\alpha u(x)| &\leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x, r))} \\ &\leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0, 2r))} \end{aligned}$$

and

$$|D^\alpha u(x)| \leq \frac{2^{n+1} n^k}{\alpha(n) r^{n+k}} \|u\|_{L^1(B(x_0, 2r))}$$

Notice, by Stirling's approximation or Taylor expansion, $\frac{k^k}{k!} < e^k$, $\forall k \geq 1$. So

$$|\alpha|^{|\alpha|} < e^{|\alpha|} |\alpha|!$$

and

$$n^k = \underbrace{(1 + \dots + 1)}_{n\text{-times}} = \sum_{|\beta|=k} \frac{|\beta|!}{\beta!} \geq \frac{|\alpha|!}{\alpha!}$$

where $|\alpha|! \leq \alpha! n^k$, $\beta = (\beta_1, \dots, \beta_n)$ and $\beta! := \beta_1! \beta_2! \dots \beta_n!$. Therefore

$$|\alpha|^{|\alpha|} \leq e^{|\alpha|} |\alpha|! \leq e^{|\alpha|} \alpha! n^k$$

and finally

$$(*) \quad k^k \leq e^k \alpha! n^k$$

Applying $*$ to the above inequality,

$$\begin{aligned} |D^\alpha u(x)| &\leq \frac{2^{n+1} n^k e^k \alpha! n^k}{\alpha(n) r^n r^k} \|u\|_{L^1(B(x_0, 2r))} \\ &\leq \frac{2^{n+1} n^2 e^k}{r} \cdot \alpha! M \end{aligned}$$

Let us analyze the Taylor expansion

$$\sum_{k=0}^N \sum_{|\alpha|=k} \frac{D^\alpha u(x_0)}{\alpha!} (x - x_0)^\alpha$$

Where $\alpha = (\alpha_1, \dots, \alpha_n)$, $y \in \mathbb{R}^n$ and $y^\alpha = y_1^{\alpha_1} \cdots y_n^{\alpha_n}$.

Pick $|x - x_0| \leq \frac{r}{2^{n+2}n^3e}$. We want to prove that the remainder $R_N(x) \xrightarrow{N \rightarrow \infty} 0$.

$$\begin{aligned} R_N(x) &= u(x) - \sum_{k=0}^{N-1} \sum_{|\alpha|=k} \frac{D^\alpha u(x_0)}{\alpha!} (x - x_0)^\alpha \\ &= \sum_{|\alpha|=N} \frac{D^\alpha u(x_0 + t(x - x_0))(x - x_0)^\alpha}{\alpha!}, \quad \text{for some } |t| \leq 1 \end{aligned}$$

Using the remainder of the Taylor expansion with $g(t) = u(x_0 + t(x - x_0))$ for $g : I \rightarrow \mathbb{R}$.

Homework: show this around $t = 0$ at $t = 1$.

Note that $u(x_0 + t(x - x_0))$ describes a straight line with $t = 0 \implies u(x_0)$ and $t = 1 \implies u(x)$.

Notice also that $x_0 + t(x - x_0) \in B(x_0, r)$. Then, considering the supremum of the remainder,

$$|R_N(x)| \leq \sum_{|\alpha|=N} \left| \frac{2^{n+1}n^2e}{r} \right|^N \cdot M \alpha! \cdot \frac{|(x - x_0)^\alpha|}{\alpha!}$$

Remark: for $\alpha = (\alpha_1, \dots, \alpha_n)$ and $y = (y_1, \dots, y_n)$,

$$\begin{aligned} |y^\alpha| &= |y_1^{\alpha_1} \cdots y_n^{\alpha_n}| \leq |y_1|^{\alpha_1} \cdots |y_n|^{\alpha_n} \\ &\leq |y|^{\alpha_1} \cdots |y|^{\alpha_n} \\ &= |y|^{\alpha_1 + \alpha_2 + \cdots + \alpha_n} \\ &= |y|^\alpha \end{aligned}$$

Therefore

$$\begin{aligned} |R_N(x)| &\leq \sum_{|\alpha|=N} \left| \frac{2^{n+1}n^2e}{r} \right|^N \cdot M |x - x_0|^N \\ &\leq M \cdot \sum_{|\alpha|=N} \left| \frac{2^{n+1}n^2e}{r} \right|^N \left| \frac{r}{2^{n+2}n^3e} \right|^N \\ &= M \cdot \sum_{|\alpha|=N} \left| \frac{1}{2n} \right|^N \\ &\leq M \left| \frac{1}{2n} \right|^N \sum_{|\alpha|=N} 1 \\ &\leq M \left| \frac{1}{2n} \right|^N n^N \\ &= M \left| \frac{1}{2} \right|^N \end{aligned}$$

Note that $\sum_{|\alpha|=N} 1 \leq n^N$ since

$$\alpha = (\alpha_1, \dots, \alpha_n) = (\alpha_{1_N}, \dots, \alpha_{i_N}) = n^N$$

Theorem 11: Harnack's Inequality

Define $V \subset\subset U$ as “ V totally contained in U ” meaning \overline{V} compact and $V \subseteq \overline{V} \subseteq U$.
 IMAGE HERE - 2

Let U open and $u \in C^2(U)$ harmonic and non-negative.

Then for each connected open set $V \subset\subset U$

$$\sup_V u \leq C \inf_V u$$

for some C that depends on V .

Remark

Then

$$\frac{1}{C}u(y) \leq u(x) \leq Cu(y), \quad \forall x, y \in V$$

Since

$$u(x) \leq \sup_V u \leq C \inf_V u \leq Cu(y)$$

and

$$\frac{1}{C}u(y) \leq \frac{1}{C} \sup_V u \leq \inf_V u \leq u(x).$$

Proof

Take $r = \frac{\text{dist}(v, \partial U)}{4} > 0$.

• Case 1

Let us suppose that $x, y \in V$ such that $|x - y| < r$.

IMAGE HERE - 3

Notice $B(x, 2r) \subseteq U$. Applying mean-value formulas,

$$u(x) = \int_{B(x, 2r)} u = \frac{1}{\alpha(n)(2r)^n} \int_{B(x, 2r)} u$$

But notice that $B(y, r) \subseteq B(x, 2r)$, so

$$u(x) \geq \frac{1}{\alpha(n)2^n r^n} \int_{B(y, r)} u = \frac{1}{2^n} \int_{B(y, r)} u = \frac{1}{2^n} u(y)$$

That is, if $x, y \in V$ such that $|x - y| < r$, then $u(x) \geq \frac{1}{2^n} u(y)$ and, mutatis mutandis, $u(y) \geq \frac{1}{2^n} u(x)$.

• Case 2

Let us cover \overline{V} by an open covering of balls $\{B_i\}_{i=1}^N$ such that the radius of each ball is $\frac{r}{2}$ and $B_i \cap B_{i-1} \neq \emptyset$.

IMAGE HERE - 4

Then $u(x) \geq \frac{1}{2^n} u(z) \frac{1}{2^{n-1}} u(y)$, so $u(x) \geq \frac{1}{2^{2n}} u(y)$.

In the same way, $u(y) \geq \frac{1}{2^{2n}} u(x)$.

IMAGE HERE - 5

For three balls, $u(x) \geq \frac{1}{2^{3n}}u(y)$ and $u(y) \geq \frac{1}{2^{3n}}u(x)$.

Since we have a finite covering of N balls, the same strategy gives

$$u(x) \geq \frac{1}{2^{Nn}}u(y) \qquad u(y) \geq \frac{1}{2^{Nn}}u(x)$$

and

$$\frac{1}{2^{Nn}} \leq u(x)$$

Taking the supremum $y \in V$;

$$\sup_{y \in V} u(y) \leq 2^{Nn}u(x)$$

taking the infimum $x \in V$

$$\inf_{x \in V} u(x)$$

Recap: Laplace Equation

- Fundamental Solution
 - Poisson Equation in \mathbb{R}^n
- Mean-value Formulas
- Properties
 - Strong Maximum / Minimum Principles
 - * Uniqueness of the Poisson Equation on Bounded Domains
 - Regularity
 - Derivative Estimates
 - Liouville's Theorem
 - * Representation Formula
 - Uniqueness of the Poisson Equation up to a Constant on \mathbb{R}^n for Bounded Functions
 - Analyticity
 - Harnack's Inequality

Green's Functions

For U open and bounded, $\partial U \in C^1$.

Goal: We want to solve $-\Delta u = f$ on U and $u = g$ on ∂U .

Recall: Green's Formula

$$\int_{\Omega} u \Delta v - v \Delta u = \int_{\partial \Omega} u \frac{\partial v}{\partial \eta} - v \frac{\partial u}{\partial \eta}$$

Obtaining Green's Formula

Let $x \in U$ and consider $u(y)$, $\Phi(y-x)$ as functions of y .

Let $\varepsilon > 0$ and consider $V_{\varepsilon} = U \setminus B_{\varepsilon}(x)$. Applying Green's formula; $\Omega = V_{\varepsilon}$,

$$\int_{V_{\varepsilon}} \underbrace{u(y) \Delta_y \Phi(y-x) - \Phi(y-x) \Delta_y u}_{=0} = \int_{\partial V_{\varepsilon}} u(y) \frac{\partial \Phi(y-x)}{\partial \eta} - \Phi(y-x) \frac{\partial u(y)}{\partial \eta}$$

IMAGE HERE - 6

January 31, 2024

Green's Functions

Goal is to solve for $U \subseteq \mathbb{R}^n$ open and bounded,

$$\begin{cases} -\Delta u = f, & \text{in } U \\ u = g, & \text{on } \partial U \end{cases}$$

by obtaining Green's function.

Let $x \in U$ and assume $u \in C^2(U)$, and consider $u(y)$ and $\Phi(y-x)$.

Recall Green's formula $\int_{\Omega} u \Delta v - v \Delta u = \int_{\partial \Omega} u \frac{\partial v}{\partial \eta} - v \frac{\partial u}{\partial \eta}$.

Then, let $\varepsilon > 0$ and define $V_{\varepsilon} U \setminus B(x, \varepsilon)$.

IMAGE HERE - 1

By applying Green's Formula,

$$\int_{V_{\varepsilon}} u(y) \underbrace{\Delta \Phi(y-x) - \Phi(y-x) \Delta u(y)}_0 = \int_{\partial V_{\varepsilon}} \underbrace{u \frac{\partial \Phi(y-x)}{\partial \eta}}_{\square_1} - \underbrace{\Phi(y-x) \frac{\partial u}{\partial \eta}}_{\square_2}$$

Notice that $\partial V_{\varepsilon} = \partial U \cup \partial B(x, \varepsilon)$.

Let us analyze \square along $\partial B(x, \varepsilon)$

For \square_2 along $\partial B(x, \varepsilon)$,

$$\begin{aligned} \left| \int_{\partial B(x, \varepsilon)} \Phi(y-x) \frac{\partial u(y)}{\partial \eta} \right| &\leq \bar{U} |\nabla U| \int_{\partial B(x, \varepsilon)} \Phi(y-x) dS(y) \\ &= \frac{C}{\varepsilon^{n-2}} \int_{\partial B(x, \varepsilon)} dS(y) \\ &= \frac{C \varepsilon^{n-1}}{\varepsilon^{-2}} \\ &= c\varepsilon \end{aligned}$$

Then $\varepsilon \rightarrow 0 \int_{\partial B(x, \varepsilon)} \square_2 = 0$.

Now, for \square_1 along $\partial B(x, \varepsilon)$ and recalling $\nabla \Phi(z) = \frac{-z}{n\alpha(n)|z|^n}$ while $\eta(z) = \frac{-z}{|z|}$ such that

$$\frac{\partial \Phi}{\partial \eta}(z) = \nabla \Phi \cdot \eta = \frac{|z|^2}{n\alpha(n)|z|^{n+1}} = \frac{1}{n\alpha(n)|z|^{n-1}}$$

we have

$$\begin{aligned} \int_{\partial B(x, \varepsilon)} u(y) \frac{\partial \Phi(y-x)}{\partial \eta} dS(y) &\stackrel{z=y-x}{=} \int_{\partial U(0, \varepsilon)} u(z+x) \frac{\partial \Phi(z)}{\partial \eta} |z| ds(z) \\ &= \frac{1}{n\alpha(n)} \int_{\partial B(0, \varepsilon)} \frac{u(z+x)}{|z|^{n-1}} dS(z) \\ &= \frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B(0, \varepsilon)} u(z+x) dS(z) \\ &\stackrel{y=z+x}{=} \frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B(x, \varepsilon)} u(y) dS(y) \\ &= \oint_{\partial B(x, \varepsilon)} u(y) dS(y) \end{aligned}$$

Then $\varepsilon \rightarrow 0 \int_{\partial B(x, \varepsilon)} \square_1 = u(x)$. It follows, then, that

$$\int_U -\Phi(y-x) \Delta u(y) = \int_{\partial U} \overbrace{u \frac{\partial \Phi(y-x)}{\partial \eta} - \Phi(y-x) \frac{\partial u}{\partial \eta}}^{\square} + u(x)$$

$\square_1 \qquad \qquad \square_2$

That is

$$u(x) \stackrel{\square_4}{=} - \int_U \Phi(y-x) \Delta u + \int_{\partial U} \Phi(y-x) \frac{\partial u}{\partial \eta} - u \frac{\partial \Phi(y-x)}{\partial \eta}$$

Notice that we have $-\Delta u = f$ in U and $u = g$ on ∂U , but we will need $\frac{\partial u}{\partial \eta} |_{\partial U}$.

Definition: Corrector Function

Given a domain $U \subseteq \mathbb{R}^n$ open and bounded with $x \in U$, define the function $\phi^x(y)$ that satisfies the following

$$\begin{cases} \Delta \phi^x(y) = 0, & \text{in } U \\ \phi^x(y) = \Phi(y-x), & \text{on } y \in \partial U \end{cases}$$

Note that we do not know that such a function exists.

Green's Function Continued

Suppose that we have $\phi^x(y)$. Then, applying green's formula for $u(y)$ and $\phi^x(y)$,

$$\int_U u \Delta \underbrace{\phi^x(y)}_0 - \phi^x(y) \Delta u = \int_{\partial U} u \frac{\partial \phi^x(y)}{\partial \eta} - \underbrace{\phi^x(y) \frac{\partial u}{\partial \eta}}_{\Phi(y-x) \frac{\partial u}{\partial \eta}}$$

Then

$$\int_{\partial U} \Phi(y-x) \frac{\partial u}{\partial \eta} \stackrel{\square_3}{=} \int_{\partial U} u \frac{\partial \phi^x(y)}{\partial \eta} + \int_U \phi^x(y) \Delta u$$

Replacing \square_3 in \square_4 ,

$$u(x) = - \int_U \Phi(y-x) \Delta u + \int_{\partial U} u \frac{\partial \phi^x(y)}{\partial \eta} + \int_U \phi^x(y) \Delta u - \int_{\partial U} u \frac{\partial \Phi(y-x)}{\partial \eta}$$

and, therefore,

$$u(x) = - \int_U \Delta u \times \Phi(y-x) - \phi^x(y) \times - \int_{\partial U} u \frac{\partial}{\partial \eta} \times \Phi(y-x) - \phi^x(y) \times$$

Definition: Green's Function

Given a domain $U \subseteq \mathbb{R}^n$, the Green's function for $x \in U$ is defined by

$$G(x, y) := \Phi(y-x) - \phi^x(y)$$

Theorem: Representation Formula

Suppose $U \subseteq \mathbb{R}^n$, and $u \in C^2(U)$ that solves

$$\begin{cases} -\Delta u = f, & \text{in } U \\ u = g, & \text{on } \partial U \end{cases}$$

Then,

$$u(x) = \int_U f G(x, y) - \int_{\partial U} g \frac{\partial G(x, y)}{\partial \eta}$$

Interpretation of the Green's Functions

$$\Delta_y G(x, y) = \Delta_y \Phi(y-x) - \underbrace{\Delta_y \phi^x(y)}_0 = \delta^x(y)$$

and

$$G(x, y) = \Phi(y-x) - \phi^x(y) = 0, \quad y \in \partial U$$

That is, it is the Dirac delta on the interior which vanishes at the boundary.

Theorem: Symmetry of the Green's Function

For all $x, y \in U$, $x \neq y$, we want to show that $G(x, y) = G(y, x)$.

Proof

Let $x, y \in U$, $x \neq y$.

Define $V(z) := G(x, z)$ and $W(z) := G(y, z)$.

Notice that $\Delta_z V = 0$ for $z \neq x$ and $\Delta_z W = 0$ for $z \neq y$ and $V(z) = W(z) = 0$ for $z \in \partial U$.

IMAGE HERE - 2

Then, let us consider $\varepsilon > 0$ and

$$\Omega_\varepsilon := U \setminus \bigtimes B(x, \varepsilon) \bigsqcup B(y, \varepsilon) \bigtimes$$

Then

$$\begin{aligned} 0 &= \int_{\Omega_\varepsilon} W \underbrace{\Delta V}_0 - V \underbrace{\Delta W}_0 = \int_{\partial \Omega_\varepsilon} W \frac{\partial V}{\partial \eta} - V \frac{\partial W}{\partial \eta} \\ &= \int_{\partial U} W \frac{\partial V}{\partial \eta} - V \frac{\partial W}{\partial \eta} + \int_{\partial B(x, \varepsilon)} W \frac{\partial V}{\partial \eta} - V \frac{\partial W}{\partial \eta} + \int_{\partial B(y, \varepsilon)} W \frac{\partial V}{\partial \eta} - V \frac{\partial W}{\partial \eta} \end{aligned}$$

It follows that

$$\underbrace{\int_{\partial B(x, \varepsilon)} \overbrace{W \frac{\partial V}{\partial \eta} - V \frac{\partial W}{\partial \eta}}^{(a)}}_{\heartsuit_1} = \underbrace{\int_{\partial B(y, \varepsilon)} \overbrace{V \frac{\partial W}{\partial \eta} - W \frac{\partial V}{\partial \eta}}^{(b)}}_{\heartsuit_2}$$

Let us analyze (b), fixing $\varepsilon_0 > 0$ such that $\varepsilon < \varepsilon_0$

$$\begin{aligned} \left| \int_{\partial B(x, \varepsilon)} V \frac{\partial W}{\partial \eta} \right| &\leq \sup_{z \in \partial B(x, \varepsilon)} |V(z)| \int_{\partial B(x, \varepsilon)} \left| \frac{\partial W}{\partial \eta}(z) \right| dS(z) \\ &\leq \sup_{z \in \partial B(x, \varepsilon_0)} |\nabla W(z)| \int_{\partial B(x, \varepsilon)} dS(z) \\ &\leq C \varepsilon^{n-1} \sup_{z \in \partial B(x, \varepsilon)} |V(z)| \\ &\leq C \varepsilon^{n-1} \left(\frac{C}{\varepsilon^{n-2} + C} \right) \\ &= C \varepsilon + C \varepsilon^{n-1} \end{aligned}$$

Since, given $z \in \partial B(x, \varepsilon)$,

$$V(z) = G(x, z) = \Phi(z - x) - \phi^x(z)$$

we have

$$\begin{aligned} |V(z)| &\leq |\Phi(z - x)| + |\phi^x(z)| \\ &\leq \frac{C}{\varepsilon^{n-2}} + \sup_{z \in B(x, \varepsilon_0)} |\phi^x(z)| \end{aligned}$$

Thus, we have $\lim_{\varepsilon \rightarrow 0} \int_{\partial B(x, \varepsilon)} V \frac{\partial W}{\partial \eta} = 0$.

Let us analyze (a),

$$\begin{aligned}
\int_{\partial B(x,\varepsilon)} W(z) \frac{\partial V}{\partial \eta}(z) dS(z) &= \int_{\partial B(x,\varepsilon)} W(z) \times \frac{\Phi(z-x)}{\partial \eta} - \frac{\partial \phi^x(z)}{\partial \eta} \times dS(z) \\
&= \int_{\partial B(x,\varepsilon)} \overbrace{W(z) \frac{\partial \Phi(z-x)}{\partial \eta}}^{(e)} - \overbrace{W(z) \frac{\partial \phi^x(z)}{\partial \eta}}^{(h)} dS(z)
\end{aligned}$$

Analyzing (h),

$$\begin{aligned}
\left| \int_{\partial B(x,\varepsilon)} W(z) \frac{\partial \phi^x(z)}{\partial \eta} \right| &\leq_{\partial B(x,\varepsilon_0)} |\nabla \phi^x(z)| |W(z)| \int_{\partial B(x,\varepsilon)} dS(z) \\
&= C\varepsilon^{n-1}
\end{aligned}$$

Then $\varepsilon \rightarrow 0$ $h = 0$ and

$$\varepsilon \rightarrow 0 \int_{\partial B(x,\varepsilon)} W(z) \frac{\partial \Phi(z-x)}{\partial \eta} = W(x)$$

So $\varepsilon \rightarrow 0(a) = W(x)$. Then

$$\varepsilon \rightarrow 0 \int_{\partial B(x,\varepsilon)} W \frac{\partial V}{\partial \eta} - V \frac{\partial W}{\partial \eta} = W(x)$$

Applying the same process,

$$\varepsilon \rightarrow 0 \int_{\partial B(y,\varepsilon)} V \frac{\partial W}{\partial \eta} - W \frac{\partial V}{\partial \eta} = V(y)$$

Therefore $W(x) = V(y)$ and $G(y, x) = G(x, y)$.

Definition: Half Space

Define the half space $\mathbb{R}_+^n = \{(x_1, \dots, x_n) \mid x_n > 0\}$.

IMAGE HERE - 3

Definition: Reflection

For a $x = (x_1, \dots, x_n) \in \mathbb{R}_+^n$, define its reflection $\tilde{x} = (x_1, \dots, -x_n)$.

Green's Function in the Half Space

We want to find $\phi^x(y)$ that solves

$$(*) \left\{ \begin{array}{l} \Delta \phi^x(y) = 0, \quad \text{in } \mathbb{R}_+^n \\ \phi^x(y) = \Phi(y-x), \quad y \in \partial \mathbb{R}_+^n \end{array} \right.$$

Let us consider $\phi^x(y) := \Phi(y - \tilde{x})$, $x, y \in \mathbb{R}_+^n$. Then $\phi^x(y)$ satisfies $*$.
Then we can see that $\Delta \phi^x(y) = 0$.

Let $y \in \partial \mathbb{R}_+^n$ such that $y = (y_1, \dots, y_{n-1}, 0)$. So

$$\begin{aligned}\phi^x(y) &= \Phi(y - \tilde{x}) \\ &= \Phi(|y - \tilde{x}|) \\ &= \Phi \left(\sqrt{(y_1 - x_1)^2 + \dots + (y_{n-1} - x_{n-1})^2 + (0 - x_n)^2} \right) \\ &= \Phi(|y - x|) \\ &= \Phi(y - x)\end{aligned}$$

February 5, 2024

Recall: Green's Function

$$G(x, y) = \Phi(y - x) - \phi^x(y).$$

For $U \subset \mathbb{R}_+^n$, when

$$G(x, y) = \Phi(y - x) - \Phi(y - \tilde{x})$$

we proved that if $u \in C^2(\overline{U})$,

$$\begin{cases} -\Delta u = f, & U \\ u = g, & \partial U \end{cases}$$

then

$$u(x) = \int_U f G(x, y) \, dy - \int_{\partial U} g \frac{\partial G(x, y)}{\partial \eta} \, dS(y)$$

Let us consider

$$\begin{cases} \Delta u = 0, & \mathbb{R}_+^n \\ u = g, & \partial \mathbb{R}_+^n \end{cases}$$

such that

$$u(x) = - \int_{\partial \mathbb{R}_+^n} g \frac{\partial G(x, y)}{\partial \eta} \, dS(y)$$

Let us compute $\frac{\partial G}{\partial \eta}$.

IMAGE HERE - 1 UPPER HALF SPACE WITH NORMAL VECTOR η

Recall

$$\begin{aligned}\nabla \Phi(z) &= \frac{-z}{n\alpha(n)|z|^n} \\ \frac{\partial \Phi(z)}{\partial z_n} &= \frac{-z_n}{n\alpha(n)|z|^n}\end{aligned}$$

so, since $y - \tilde{x}_n = y_n + x_n$,

$$\begin{aligned}
\frac{\partial G}{\partial \eta} &= \nabla G(x, y) \cdot \eta \\
&= -\frac{\partial G(x, y)}{\partial y_{n+1}} \\
&= -\frac{\partial}{\partial y_{n+1}} (\Phi(y - x) - \Phi(y - \tilde{x})) \\
&= -\left(\frac{-(y_n - x_n)}{n\alpha(n)|y - x|^n} - \frac{-(y_n + x_n)}{n\alpha(n)|x - \tilde{x}|^n} \right)
\end{aligned}$$

But recall that if $y \in \partial\mathbb{R}_+^n$, $|y - x| = |y - \tilde{x}|$. Then $y \in \partial\mathbb{R}_+^n$,

$$\frac{\partial G(x, y)}{\partial \eta} = -\frac{1}{n\alpha(n)|y - x|^n} [-y_n + x_n + y_n + x_n] = -\frac{2x_n}{n\alpha(n)|y - x|^n}$$

Then

$$u(x) = \int_{\partial\mathbb{R}_+^n} \frac{g(y)2x_n}{n\alpha(n)|y - x|^n} dS(y)$$

Definition: Poisson Kernel

$$K(x, y) = \frac{2x_n}{n\alpha(n)|y - x|^n} = \frac{\partial G}{\partial y_n}$$

is called the Poisson Kernel and

$$u(x) = \int_{\partial\mathbb{R}_+^n} g(y)K(x, y) dS(y)$$

is called the Poisson Formula.

Notice (HW): $\int_{\partial\mathbb{R}_+^n} K(x, y) dy = 1$, $\forall x \in \mathbb{R}_+^n$ (hint: apply spherical coordinates).

Theorem 14:

Define

$$(*) \quad u(x) = \int_{\partial\mathbb{R}_+^n} K(x, y)g(y) dS(y)$$

Suppose that $g \in C^\infty(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$.

Then

1. $u \in C^\infty(\mathbb{R}_+^n) \cap L^\infty(\mathbb{R}_+^n)$.
2. $\Delta u = 0$, \mathbb{R}_+^n .
3. $\lim_{x \rightarrow x^0} u(x) = g(x^0)$, $x \in \mathbb{R}_+^n$, $x^0 \in \partial\mathbb{R}_+^n$.

Proof

We know $G(x, y)$ satisfies

$$\Delta_y G(x, y) = \delta^x(y).$$

Notice that $y \rightarrow G(x, y)$ is harmonic for $x \neq y$.

Recall that $G(x, y) = G(y, x)$, so $x \rightarrow G(x, y)$ is harmonic for $x \neq y$.

Then $x \rightarrow \frac{\partial G(x, y)}{\partial y_n}$ is harmonic $(*_2)$ for $x \neq y$ and for $y \in \partial \mathbb{R}_+^n$.

Homework: compute this directly.

Noticing that K is smooth when $x \neq y$, then

$$\frac{\partial u}{\partial x_i} = \int_{\partial \mathbb{R}_+^n} \frac{\partial}{\partial x_i} K(x, y) g(y) dS(y)$$

Homework: justify putting the limit inside the integral.

Homework: prove that $\frac{\partial u}{\partial x_i}$ is continuous.

By repeatedly taking derivatives, we see $u \in C^\infty(\mathbb{R}_+^n)$.

Moreover,

$$\Delta_x u = \int_{\partial \mathbb{R}^n} \underbrace{\Delta_x K(x, y)}_{=0} g(y) dS(y) = 0$$

by $*_2$. Then

$$|u(x)| \leq \int_{\partial \mathbb{R}_+^n} |K(x, y)| |g(y)| dS(y) \leq \|g\|_{L^\infty(\mathbb{R}^{n-1})} \underbrace{\int_{\partial \mathbb{R}_+^n} K(x, y) dS(y)}_{=1} < \infty$$

For part c, consider $x^0 \in \partial \mathbb{R}_+^n$ and $\varepsilon > 0$.

Since $g \in C^\infty(\mathbb{R}^{n-1})$, let $\delta > 0$ such that $|y - x^0| < \delta \implies |g(y) - g(x^0)| < \varepsilon$ for $y \in \partial \mathbb{R}_+^n$.

IMAGE HERE - 2 DELTA BALL AROUND x^0 HALF DELTA BALL WITH x INSIDE

Now, let us consider $|x - x_0| < \frac{\delta}{2}$.

$$\begin{aligned} |u(x) - g(x^0)| &= \left| \int_{\partial \mathbb{R}_+^n} K(x, y) g(y) - K(x, y) g(x^0) dS(y) \right| \\ &\leq \int_{\partial \mathbb{R}_+^n} K(x, y) |g(y) - g(x^0)| dS(y) \\ &= \underbrace{\int_{\partial \mathbb{R}_+^n \cap B(x^0, \delta)} K(x, y) |g(y) - g(x^0)| dS(y)}_I + \underbrace{\int_{\partial \mathbb{R}_+^n \cap B^c(x^0, \delta)} K(x, y) |g(y) - g(x^0)| dS(y)}_{II} \end{aligned}$$

Then

$$I \leq \varepsilon \int_{\partial \mathbb{R}_+^n \cap B(x^0, \delta)} K(x, y) \leq \varepsilon$$

Now, we want to control II

$$\begin{aligned} \int_{\partial \mathbb{R}_+^n \cap B^c(x^0, \delta)} K(x, y) |g(y) - g(x^0)| dS(y) &\leq C \|g\|_{L^\infty} \int_{\partial \mathbb{R}_+^n \cap B^c(x^0, \delta)} \\ &= \frac{2C}{n\alpha(n)} \int_{\partial \mathbb{R}_+^n \cap B^c(x^0, \delta)} \frac{x_n}{|x - y|^n} dS(y) \end{aligned}$$

We want to control $|x^0 - y|$ with something related to $|x - y|$.
We know $|y - x^0| > \delta$ and we will consider $|x - x^0| < \frac{\delta}{2}$. So

$$|y - x^0| \leq |y - x| + |x - x^0| \leq |y - x| + \frac{\delta}{2} \leq |y - x| + \frac{|y - x^0|}{2}$$

So $\frac{|y - x^0|}{2} \leq |y - x|$ implies that $\frac{1}{|y - x|^n} \leq \frac{2^n}{|y - x^0|^n}$. Therefore

$$\begin{aligned} \int_{\partial \mathbb{R}_+^n \cap B^c(x^0, \delta)} K(x, y) |g(y) - g(x^0)| dS(y) &\leq C x_n \int_{\partial \mathbb{R}_+^n \cap B^c(x^0, \delta)} \frac{1}{|y - x^0|^n} dS(y) \\ &= \int_{\delta}^{\infty} \int_{\partial B^{n-1}(x^0, r)} \frac{1}{r^n} dS(y) dr \\ &= C \int_{\delta}^{\infty} \frac{1}{r^n} r^{n-2} dr \\ &= C \int_{\delta}^{\infty} \frac{1}{r^2} dr \\ &= C \left(\frac{1}{r} \right) \Big|_{\delta}^{\infty} \\ &= \frac{C}{\delta} \end{aligned}$$

Then $II \leq \frac{C x_n}{\delta}$. Now let us consider $|x - x^0| < \frac{\delta}{J}$ where $\frac{1}{J} < \varepsilon$. Then

$$II \leq \frac{C |x - x^0|}{\delta} \leq C \frac{\delta}{\delta J} \leq C \varepsilon$$

Energy Methods: Uniqueness

Consider the boundary value problem

$$(*) \quad \begin{cases} -\Delta u = f, & U, f \in C(U) \\ u = g, & \partial U, g \in C(\partial U) \end{cases}$$

with U open and bounded in \mathbb{R}^n , $u \in C^2(\overline{U})$ and $\partial U \in C^1$.

Theorem 16: Uniqueness

There exists at most one solution $u \in C^2(\overline{U})$ for $*$.

Proof

Let us suppose that \tilde{u} is another solution.
Then $w := u - \tilde{u}$ solves

$$\begin{cases} \Delta w = 0, & U, w \in C^2(\overline{U}) \\ w = 0, & \partial U \end{cases}$$

where

$$0 = \int_U w \Delta w = - \int_U |\nabla w|^2 + \int_{\partial U} w \frac{\partial w}{\partial \eta}$$

so

$$0 = \int_U |\nabla w|^2 \implies \nabla w = 0 \implies w = 0 \implies u = \tilde{u}$$

Definition: Energy Functional

Let us consider

$$A = \{w \in C^2(\overline{U}) \mid W|_{\partial U} = g\}$$

for $g \in C(\partial U)$ and $f \in C(U)$.

Define the energy functional $I : A \rightarrow \mathbb{R}$ given by $I(w) := \int_U \frac{|\nabla w|^2}{2} - fw$.

Energy Methods: Dirichlet Principle

Calculus of variations applied to the Laplace equation.

Theorem:

Suppose $u \in C^2(\overline{U})$ is a solution to the problem

$$\square \quad \begin{cases} -\Delta u = f, & U \\ u = g, & \partial U \end{cases}$$

Then,

$$(*) \quad I(u) = \min_{w \in A} \{I(w)\}$$

Moreover, if $u \in A$ such that $*$ happens, then u satisfies \square .

Proof

(\implies) For $w \in A$,

$$\begin{aligned} 0 &= \int_U \underbrace{(-\Delta u - f)}_{=0} (u - w) \\ &= \int_U -\Delta u (u - w) - \int_U f(u - w) \\ &= \int_U \nabla(u - w) \cdot \nabla u - \underbrace{\int_{\partial U} (u - w) \cdot \frac{\partial u}{\partial \eta}}_{=0} - \int_U f(u - w) \\ &= \int_U |\nabla u|^2 - \int_U \nabla w \cdot \nabla u - \int_U fu + \int_U fw \end{aligned}$$

Notice that, since $|a - b|^2 \geq 0$ implies $\frac{a^2 + b^2}{2} \geq ab$,

$$\int_U \nabla w \cdot \nabla u \leq \int_U |\nabla w| |\nabla u| \leq \frac{1}{2} \int_U |\nabla w|^2 + \frac{1}{2} \int_U |\nabla u|^2$$

Therefore

$$\begin{aligned} \int_U |\nabla u|^2 - \int_U fu &= \int_U \nabla w \cdot \nabla u - \int_U fw \\ &\leq \int_U \frac{|\nabla w|^2}{2} + \int_U \frac{|\nabla u|^2}{2} - \int_U fw \\ \int_U \frac{|\nabla u|^2}{2} - fu &\leq \int_U \frac{|\nabla w|^2}{2} - fw \end{aligned}$$

Then

$$I(u) \leq I(w), \quad \forall w \in A$$

$$B/u \in A.$$

February 7, 2024

Recall: Energy Functional

For $U \subseteq \mathbb{R}^n$ bounded, $g \in C(\partial U)$, $f \in C(\overline{U})$

$$A = \{w \in C^2(\overline{U}) \mid w|_{\partial U} = g\}$$

we have

$$I(w) := \int_U \frac{1}{2} |\nabla w|^2 - fw$$

Theorem:

Suppose $u \in A$ such that $I(u) = \{I(w) \mid w \in A\}$. Then u satisfies

$$\begin{cases} -\Delta u = f, & \text{in } U \\ u = g, & \text{on } \partial U \end{cases}$$

Proof

Consider $v \in C_c^\infty(U)$.

Define $i : \mathbb{R} \rightarrow \mathbb{R}$ such that $\tau \mapsto I(\tau) := I(u + \tau v)$.

Notice that $u + \tau v$ is a perturbation of u and, since $u + \tau v \in C^2(\overline{U})$ while $u + \tau v|_{\partial U} = u|_{\partial U} = g$, $u + \tau v \in A$. Then

$$i(0) = I(u) \leq I(u + \tau v) = i(\tau)$$

so i has a minimum point at $\tau = 0$. Compute

$$\begin{aligned} i(\tau) &= I(u + \tau v) \\ &= \int_U \frac{|\nabla(u + \tau v)|^2}{2} - f(u + \tau v) \\ &= \int_U \frac{|\nabla u|^2}{2} + \tau \langle \nabla u, \nabla v \rangle + \frac{\tau^2 |\nabla v|^2}{2} - \int_U fu - \tau \int_U fv \\ &= \int_U \frac{|\nabla u|^2}{2} + \tau \int_U \langle \nabla u, \nabla v \rangle + \frac{\tau^2}{2} \int_U |\nabla v|^2 - \int_U fu - \tau \int_U fv \end{aligned}$$

So i is a polynomial in τ , and

$$i'(0) = i'(\tau)_{\tau=0} = \left[\int_U \langle \nabla u, \nabla v \rangle + \tau \int_U |\nabla v|^2 - \int_U f v \right]_{\tau=0}$$

So

$$\begin{aligned} 0 &= i'(0) \\ &= \int_U \langle \nabla u, \nabla v \rangle - \int_U f v \\ &= \int_U -\Delta u \cdot v + \underbrace{\int_{\partial U} \frac{\partial u}{\partial \eta} \cdot v}_{=0} - \int_U f v \\ &= \int_U \underbrace{(-\Delta u - f) v}_{=0} \end{aligned}$$

Since $0 = \int g v$, $\forall v \in C_c^\infty(U)$ requires $g \equiv 0$.

Then $-\Delta u - f = 0$.

Heat Equation (Diffusion Equation)

The equations

$$(*) \begin{cases} u_t - \Delta u = 0, & \text{homogeneous case} \\ u_t - \Delta u = f, & \text{non-homogeneous case} \end{cases}$$

(note that $\Delta u = \Delta_x u$)

subject to some boundary and initial conditions $t \geq 0$ time and $x \in \mathbb{R}^n$, space variable, $x \in U$ and open set of \mathbb{R}^n .

$u : U \times (0, \infty) \rightarrow \mathbb{R}$ defined as $(x, t) \mapsto u(x, t)$ with u unknown.

IMAGE HERE - 1

Motivation: Fundamental Solution of the Heat Equation

We would like to have the following:

If u solves

$$\begin{cases} u_t - \Delta u = 0, & \mathbb{R}^n \times (0, \infty) \\ u(x, 0) = g(x), & \mathbb{R}^n \times \{0\} \end{cases}$$

then

$$u(x, t) = \int_{\mathbb{R}^n} G(x - y, t) g(y) dy$$

How do we get G ?

Let us suppose that $u(\tilde{x}, \tilde{t})$ solves

$$\begin{cases} u_{\tilde{t}} - \Delta_{\tilde{x}} u = 0 \\ u(\tilde{x}, 0) = g(\tilde{x}) \end{cases}$$

We would like to have invariance under dilation.

$$v(x, t) := u(\lambda x, \lambda^2 t)$$

Such that

$$\begin{aligned} v_t &= \nabla U|_{(\lambda x, \lambda^2 t)} - \frac{\partial}{\partial t} \times_{\lambda^2 t}^{\lambda x} \times \\ &= \lambda^2 u_{\tilde{t}}(\lambda x, \lambda^2 t) \\ v_{x_i} &= \lambda u_{\tilde{x}_i}(\lambda x, \lambda^2 t) \\ v_{x_i x_i} &= \lambda^2 u_{\tilde{x}_i \tilde{x}_i}(\lambda x, \lambda^2 t) \end{aligned}$$

Therefore

$$v_t - \Delta_x v = \lambda^2 u_{\tilde{t}} - \lambda^2 \Delta_{\tilde{x}} u = \lambda^2 \underbrace{(u_{\tilde{t}} - \Delta_{\tilde{x}} u)}_{=0} = 0$$

with

$$v(x, 0) = u(\lambda x, 0) = g(\lambda x)$$

Then, applying the motivation,

$$v(x, t) = \int_{\mathbb{R}^n} G(x - y, t) g(\lambda y) dy \stackrel{z=\lambda y}{=} \int_{\mathbb{R}^n} G\left(x - \frac{z}{\lambda}, t\right) g(z) \frac{dz}{\lambda^n}$$

On the other hand,

$$v(x, t) = u(\lambda x, \lambda^2 t) = \int_{\mathbb{R}^n} G(\lambda x - z, \lambda^2 t) g(z) dz$$

It follows that

$$\begin{aligned} \frac{1}{\lambda^n} G\left(x - \overbrace{\frac{z}{\lambda}}^w, t\right) &= G(\lambda x - z, \lambda^2 t) \\ \frac{1}{\lambda^n} G(w, t) &= G(\lambda w, \lambda^2 t) \end{aligned}$$

If $\lambda^2 t = 1$, then

$$G(w, t) = \frac{1}{t^{n/2}} G\left(\frac{w}{\sqrt{t}}, 1\right)$$

If we call $G\left(\frac{w}{\sqrt{t}}, 1\right) = v\left(\frac{w}{\sqrt{t}}\right)$, then we are looking at $G(w, t) = \frac{1}{t^{n/2}} v\left(\frac{w}{t^{1/2}}\right)$. So, we have motivation to define

$$u(x, t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right)$$

for α, β appropriate and $v(y) : \mathbb{R}^n \rightarrow \mathbb{R}$.

Obtaining a Fundamental Solution to the Heat Equation

Let us compute u_t and $\Delta_x u$.

$$\begin{aligned}
 u_t &= \frac{\partial}{\partial t} \left(\frac{1}{t^\alpha} v \left(\frac{x}{t^\beta} \right) \right) \\
 &= \frac{(-\alpha)}{t^{\alpha+1}} v \left(\frac{x}{t^\beta} \right) + \frac{1}{t^\alpha} \frac{\partial}{\partial t} v \left(\frac{x}{t^\beta} \right) \\
 &= \frac{(-\alpha)}{t^{\alpha+1}} v \left(\frac{x}{t^\beta} \right) + \frac{1}{t^\alpha} \cdot \nabla v \left(\frac{x}{t^\beta} \right) \cdot \frac{\partial}{\partial t} \left(\frac{x}{t^\beta} \right) \\
 u_t &= \frac{(-\alpha)}{t^{\alpha+1}} v \left(\frac{x}{t^\beta} \right) + \frac{(-\beta)}{t^\alpha t^{\beta+1}} \nabla v \left(\frac{x}{t^\beta} \right) \cdot x \quad \square_1
 \end{aligned}$$

and

$$\begin{aligned}
 \frac{\partial u}{\partial x_i} &= \frac{1}{t^\alpha} \frac{\partial}{\partial x_i} v \left(\frac{x}{t^\beta} \right) \\
 &= \frac{1}{t^\alpha} \nabla v \left(\frac{x}{t^\beta} \right) \cdot \frac{\partial}{\partial x_i} \left(\frac{x}{t^\beta} \right) \\
 &= \frac{1}{t^{\alpha+\beta}} \frac{\partial v}{\partial x_i} \left(\frac{x}{t^\beta} \right)
 \end{aligned}$$

while

$$\frac{\partial^2 u}{\partial x_i \partial x_i} = \frac{1}{t^{\alpha+2\beta}} \frac{\partial^2 v}{\partial x_i \partial x_i} \left(\frac{x}{t^\beta} \right) \quad \square_2$$

Then, replacing \square_1 and \square_2 in $*$,

$$-\frac{\alpha}{t^{\alpha+1}} v \left(\frac{x}{t^\beta} \right) - \frac{\beta}{t^{\alpha+\beta+1}} \nabla v \left(\frac{x}{t^\beta} \right) \cdot x - \frac{1}{t^{\alpha+2\beta}} \Delta v \left(\frac{x}{t^\beta} \right) \stackrel{?}{=} 0$$

Set $y := \frac{x}{t^\beta}$

$$-\frac{\alpha}{t^{\alpha+1}} v(y) - \frac{\beta}{t^{\alpha+1}} \nabla v(y) \cdot y - \frac{1}{t^{\alpha+2\beta}} \Delta v(y) = 0$$

Multiplying through by $-t^{\alpha+1}$,

$$\alpha v(y) + \beta \nabla v(y) \cdot y + \frac{1}{t^{2\beta-1}} \Delta v(y) = 0$$

Let us assume that $2\beta - 1 = 0$ such that $\beta = \frac{1}{2}$, giving

$$\alpha v(y) + \frac{1}{2} \nabla v(y) \cdot y + \Delta v(y) = 0$$

Since the Laplacian is rotationally invariant, assume $v(y) = w(|y|)$ for $w : \mathbb{R}^+ \rightarrow \mathbb{R}$.

Recall that $\frac{\partial}{\partial y_i} |y| = \frac{\partial}{\partial y_i} \sqrt{y_1^2 + \cdots + y_n^2} = \frac{y_i}{|y|}$. Now

$$\frac{\partial}{\partial y_i} v(y) = \frac{\partial}{\partial y_i} (w(|y|)) = w'(|y|) \cdot \frac{\partial}{\partial y_i} (|y|) = w'(|y|) \cdot \frac{y_i}{|y|}$$

$$\begin{aligned}
\frac{\partial^2 v(y)}{\partial y_i \partial y_i} &= \frac{\partial}{\partial y_i} \left(w'(|y|) \frac{y_i}{|y|} + w'(|y|) \cdot \frac{\partial}{\partial y_i} \left(\frac{y_i}{|y|} \right) \right) \\
&= w''(|y|) \cdot \frac{y_i^2}{|y|^2} + w'(|y|) \times \frac{1}{|y|} + y_i \frac{\partial}{\partial y_i} \left(\frac{1}{|y|} \right) \\
&= w''(|y|) \frac{y_i^2}{|y|^2} + w'(|y|) \times \frac{1}{|y|} - \frac{y_i^2}{|y|^3}
\end{aligned}$$

Replacing in the PDE of v ,

$$\begin{aligned}
0 &= \alpha w(|y|) + \frac{1}{2} \frac{w'(|y|)y}{|y|} \cdot y + \sum_{i=1}^n w''(|y|) \frac{y_i^2}{|y|^2} + w'(|y|) \times \frac{1}{|y|} - \frac{y_i^2}{|y|^3} \\
&= \alpha w(|y|) + \frac{1}{2} w'(|y|)|y| + w''(|y|) + w'(|y|) \times \frac{n}{|y|} - \frac{1}{|y|}
\end{aligned}$$

If $|y| = r$

$$0 = \alpha w(r) + \frac{1}{2} w'(r)r + w''(r) + w'(r) \frac{n-1}{r}$$

Take $\alpha = \frac{n}{2}$ and multiply through by r^{n-1} ,

$$\begin{aligned}
0 &= \frac{nr^{n-1}}{2} w(r) + \frac{r^n}{2} w'(r) + w''(r)r^{n-1} + w'(r)(n-1)r^{n-2} \\
&= \frac{1}{2} \times w(r)r^n \times' + \times w'(r)r^{n-1} \times'
\end{aligned}$$

Then by the fundamental theorem of calculus, $w'(r)r^{n-1} + \frac{w(r)r^n}{2} = C$.

We would like $w, w' \xrightarrow{r \rightarrow \infty} 0$. Then $C = 0$, so

$$w'(r)r^{n-1} = -\frac{w(r)r^n}{2}$$

Which gives

$$w' = \frac{-wr}{2} \iff \frac{w'}{w} = -\frac{r}{2} \iff ((w))' = \frac{-r}{2} \iff (w) = -\frac{r^2}{4} + d$$

and, finally,

$$w(r) = be^{-\frac{r^2}{4}}$$

Then define

$$\begin{aligned}
u(x, t) &:= \frac{1}{t^{n/2}} v \left(\frac{x}{t^{1/2}} \right) \\
&= \frac{1}{t^{n/2}} w \left(\left| \frac{x}{t^{1/2}} \right| \right) \\
&= \frac{b}{t^{n/2}} e^{-\frac{1}{4} \left| \frac{x}{t^{1/2}} \right|^2} \\
&= \frac{b}{t^{n/2}} e^{-\frac{1}{4t} |x|^2}
\end{aligned}$$

Where b is chosen such that the expression integrates to 1.

Definition: Fundamental Solution of the Heat Equation

The fundamental solution for the heat equation is given by

$$\begin{cases} \Phi(x, t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, & x \in \mathbb{R}^n, t > 0 \\ 0, & x \in \mathbb{R}^n, t < 0 \end{cases}$$

where we have chosen $b = \frac{1}{(4\pi)^{n/2}}$.

IMAGE HERE - 2

Notice that these match in the limit away from the origin $((x, t) \rightarrow (x_0, 0) \Phi(x, t) = 0)$.

Remark: $\Phi(x, t)$ has a unique singularity at $(0, 0)$.

February 12, 2024

Recall: Heat Equation

$$\Phi(x, t) = \begin{cases} \frac{b}{(t)^{n/2}} e^{-\frac{|x|^2}{4t}}; & t > 0, x \in \mathbb{R}^n \\ 0; & t < 0 \end{cases}$$

Remark: Φ is radial such that $\Phi(x, t) = \Phi(|x|, t)$.

Lemma:

For each $t > 0$,

$$\int_{\mathbb{R}^n} \Phi(x, t) dx = 1$$

Proof

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi(x, t) dx &= \frac{b}{t^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}} dx \\ &= \frac{b}{t^{n/2}} \int_{\mathbb{R}^n} e^{-\left|\frac{x}{2\sqrt{t}}\right|^2} \\ &\stackrel{z=\frac{x}{2\sqrt{t}}}{=} \frac{b}{t^{n/2}} \int_{\mathbb{R}^n} e^{-|z|^2} (2\sqrt{t})^n dz \\ &= b 2^n \int_{\mathbb{R}^n} e^{-|z|^2} dz \\ &= 2^n b \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{-z_1^2 - \dots - z_n^2} dz_1 \dots dz_n \\ &= 2^n b \times \int_{-\infty}^{\infty} e^{-x} dx \times^n \end{aligned}$$

We need

$$\begin{aligned}
A &= \int_{-\infty}^{\infty} e^{-x^2} dx \\
A^2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} dx dy \\
&= \int_{\mathbb{R}^n} e^{-|z|^2} dz \\
&= \int_0^{\infty} \int_{\partial B_r^2} e^{-r^2} dS(z) dr \\
&= 2\pi \int_0^{\infty} e^{-r^2} r dr \\
&= \pi \int_0^{\infty} e^{-s} ds \\
&= -\pi(e^{-s})|_0^{\infty} = -\pi(0-1) = \pi
\end{aligned}$$

Therefore $A^2 = \pi$ and $A = \sqrt{\pi}$. So, picking $b = \frac{1}{(4\pi)^{n/2}}$,

$$\int_{\mathbb{R}^n} \Phi(x, t) dx = b 2^n A^n = b 2^n \pi^{n/2} = 1$$

Remark:

Φ solves the Heat Equation, except at the point $(x, t) = (0, 0)$.

Remark:

Φ is infinitely differentiable on $\mathbb{R}^n \times (\delta, \infty)$, $\forall \delta > 0$.

Cauchy Problem (Initial Value Problem)

$$\begin{cases} u_t - \Delta_x u = 0, & \mathbb{R}^n \times (0, \infty) \\ u(x, 0) = g(x), & x \in \mathbb{R}^n \end{cases}$$

Recall $y \in \mathbb{R}^n$,

$$(x, t) \rightarrow \Phi(x - y)$$

solves the heat equation except at $(y, 0)$.

Define, $x \in \mathbb{R}^n$, $t > 0$,

$$(*) \quad u(x, t) = \int_{\mathbb{R}^n} \Phi(x - y, t) g(y) dy = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y) dt$$

Theorem (#?): Solution to the Cauchy Problem

Assume $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then u defined by $*$ satisfies

1. $u \in C^\infty(\mathbb{R}^n \times (0, \infty))$.
2. $u_t(x, t) - \Delta_x u(x, t) = 0$, $(x, t) \in \mathbb{R}^n \times (0, \infty)$.
3. $\lim_{\substack{(x,t) \rightarrow (x_0,0) \\ x \in \mathbb{R}^n, t > 0}} u(x, t) = g(x_0)$, $x_0 \in \mathbb{R}^n$.

Proof

Homework: justify putting the limit inside to prove (1).

For (2), observe that

$$u_t - \Delta_x u(x, t) = \int_{\mathbb{R}^n} \underbrace{[\Phi_t(x - y, t) - \Delta_x \Phi(x - y, t)]}_{=0} g(y) dy$$

For (3), let $\varepsilon > 0$. Let $\delta > 0$ such that $|x - x_0| < \delta \implies |g(x) - g(x_0)| < \varepsilon$ (since g continuous). Then

$$\begin{aligned} |u(x, t) - g(x_0)| &= \left| \int_{\mathbb{R}^n} \Phi(x - y, t) g(y) dy - g(x_0) \underbrace{\int_{\mathbb{R}^n} \Phi(x - y, t) dy}_{=1} \right| \\ &\leq \int_{\mathbb{R}^n} \Phi(x - y, t) |g(y) - g(x_0)| dy \\ &= \underbrace{\int_{B(x_0, \delta)} \Phi(x - y, t) |g(y) - g(x_0)| dy}_I + \underbrace{\int_{\mathbb{R}^n \setminus B(x_0, \delta)} \Phi(x - y, t) |g(y) - g(x_0)| dy}_J \end{aligned}$$

Bounding I , $|y - x_0| < \delta \implies |g(y) - g(x_0)| < \varepsilon$ gives

$$I \leq \varepsilon \underbrace{\int_{B(x_0, \delta)} \Phi(x - y, t) dy}_{\leq 1} \leq \varepsilon$$

Bounding J , assume $|x - x_0| < \frac{\delta}{2}$. Then

$$|J| \leq \|g\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n \setminus B(x_0, \delta)} \Phi(x - y, t) dy$$

Now we want to compare $|x - y|$ with $|x_0 - y|$. Then, for $|x - x_0| < \frac{\delta}{2}$ and $|y - x_0| > \delta$,

$$|y - x_0| < |y - x| + |x - x_0| < |y - x| + \frac{\delta}{2} < |y - x| + \frac{|y - x_0|}{2}$$

so $\frac{|y - x_0|}{2} < |y - x|$. It follows that

$$\begin{aligned} \frac{|y - x_0|^2}{4} &\leq |y - x|^2 \\ -\frac{|y - x|^2}{4t} &\leq -\frac{|y - x_0|^2}{16t} \\ e^{-\frac{|y - x|^2}{4t}} &\leq e^{-\frac{|y - x_0|^2}{16t}} \end{aligned}$$

Then

$$\begin{aligned} |J| &\leq 2\|g\|_{L^\infty} \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n \setminus B(x_0, \delta)} e^{-\frac{|y - x_0|^2}{16t}} dy \\ &= \frac{C}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n \setminus B(x_0, \delta)} e^{-\frac{1}{16} \left| \frac{y - x_0}{\sqrt{t}} \right|^2} dy \end{aligned}$$

Letting $z = \frac{y-x_0}{\sqrt{t}}$ such that $\sqrt{t} dz = dy$,

$$\begin{aligned} |J| &\leq \frac{C}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n \setminus B(0, \delta/\sqrt{t})} e^{-\frac{|z|^2}{16}} \underbrace{(\sqrt{t})^n}_{dy} dz \\ &= \frac{C}{(4\pi)^{n/2}} \int_{\mathbb{R}^n \setminus B(0, \delta/\sqrt{t})} e^{-\frac{|z|^2}{16}} dz \end{aligned}$$

Let $\delta_2 > 0$ such that $\delta_2 = \left\{ \frac{\delta}{2}, \delta^3 \right\}$.
If $|(x, t) - (x_0, 0)| < \delta_2$,

$$\begin{aligned} t &< \delta_2 < \delta^3 \\ \sqrt{t} &< \delta^{3/2} \\ \frac{1}{\delta^{3/2}} &< \frac{1}{\sqrt{t}} \\ \frac{1}{\delta^{1/2}} &< \frac{\delta}{\sqrt{t}} \end{aligned}$$

so

$$B(0, 1/\delta^{1/2}) \subseteq B(0, \delta/\sqrt{t}) \quad \text{and} \quad \mathbb{R}^n \setminus B(0, \delta/\sqrt{t}) \subseteq \mathbb{R}^n \setminus B(0, 1/\delta^{1/2})$$

Therefore,

$$|u| \leq C \int_{\mathbb{R}^n \setminus B(0, 1/\sqrt{\delta})} e^{-\frac{|z|^2}{16}} dz \rightarrow 0$$

Interpretation of Fundamental Solution for the Heat Equation

$$\begin{cases} \Phi_t - \Delta_x \Phi(x, t) = 0, & x \in \mathbb{R}^n, t > 0 \\ \Phi(x, 0) = \delta_0(x), & x \in \mathbb{R}^n \end{cases}$$

Then

$$u(x, t) = \int_{\mathbb{R}^n} \Phi(x - y, t) g(y) dy$$

if $t = 0$,

$$\begin{aligned} u(x, 0) &= \int_{\mathbb{R}^n} \Phi(x - y, 0) g(y) dy \\ &= \int_{\mathbb{R}^n} \underbrace{\delta^x(y) g(y)}_{y=x} dy \\ &= \int_{\mathbb{R}^n} \delta^x(y) g(x) dy \\ &= g(x) \underbrace{\int_{\mathbb{R}^n} \delta^x(y) dy}_{=1} = g(x) \end{aligned}$$

Remark: Infinite Propagation Speed

Let $g \in C(\mathbb{R}^n \cap L^\infty(\mathbb{R}^n))$, $g \geq 0$, $g \neq 0$. Then

$$u(x, t) \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y) dy > 0, \quad \forall x \in \mathbb{R}^n, \quad \forall t > 0$$

IMAGE HERE - 1

That is, the heat equation forces infinite propagation speed for disturbances.

Non-Homogeneous Heat Problem

$$(*_2) \quad \begin{cases} u_t - \Delta_x u = f, & f : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R} \\ u(x, 0) = 0, & x \in \mathbb{R}^n \end{cases}$$

Motivation

Let $y \in \mathbb{R}^n$, $s > 0$. Then $(x, t) \rightarrow \Phi(x - y, t - s)$ solves the heat equation except at $x = y$ and $t = s$. That is, it satisfies the equation on $\mathbb{R}^n \times (s, \infty)$.

Then for s fixed, define

$$(\square) \quad u(x, t; s) := \int_{\mathbb{R}^n} \Phi(x - y, t - s) f(y; s) dy$$

which solves

$$\begin{cases} u_t(x, t; s) - \Delta_x u(x, t; s) = 0, & \mathbb{R}^n \times (s, \infty) \\ u(x, s; s) = f(x; s), & \mathbb{R}^n \times \{s\} \end{cases}$$

which is the IVP with $t = 0 \iff t = s$ and $g(y) \iff f(y; s)$.

Definition: Duhamel's Principle

If we integrate \square from 0 to t ,

$$u(x, t) := \int_0^t u(x, t; s) ds$$

Let us consider,

$$(\square_2) \quad u(x, t) := \int_0^t \int_{\mathbb{R}^n} \Phi(x - y, t - s) f(y; s) dy ds$$

as a candidate solution for $*_2$.

Theorem: Solution to the Non-Homogeneous Heat Equation

Suppose $f \in C_c^2(\mathbb{R}^n \times (0, \infty))$ with compact support.

If we define u by \square_2 , then

1. $u \in C_c^2(\mathbb{R}^n \times (0, \infty))$.
2. $u_t(x, t) - \Delta_x u(x, t) = f(x, t); x \in \mathbb{R}^n, t > 0$.
3. $_{\substack{(x,t) \rightarrow (x_0,0) \\ x \in \mathbb{R}^n, t > 0}} u(x, t) = 0, \forall x_0 \in \mathbb{R}^n$.