

Analysis III

Homework

Exercises (homework) can be found in the script at the end of each Chapter.

Chapter I: # 3 (only for convex sets), # 4 due Th 4-25

Chapter II: # 2,4,5,6 due Th 5-2

Chapter III: # 3c, 4 due Th 5-9

Chapter IV: # 2b, 3, 4, 6 due Th 5-16

Chapter V: # 2,4,6 due Th 5-25

Chapter VI: # 2,3,4 due Th 6-1

Key Dates

Instruction begins: Mo, April 1

Instruction ends: Fr, June 7

Final's week: June 10-13 (Mo-Th)

Holiday: Mo, May 27

April 2, 2024

No class Thursday, April 04.

Makeup class (tentatively) on Friday, April 12 at 10:30.

Discussion sections on Fridays (tentatively) at 11:40.

Topological Vector Spaces

Definition: Vector Spaces

V over a field \mathbb{F} (\mathbb{R} or \mathbb{C}).

Definition: Topological Spaces

(X, τ) where $\tau \subseteq \mathcal{P}(X)$ satisfying

1. $\emptyset, X \in \tau$
2. $A, B \in \tau \implies A \cap B \in \tau$
3. $A_\omega \in \tau \implies \bigcup_\omega A_\omega \in \tau$

Recall: $A \in \tau \iff A$ open $\iff X \setminus A$ closed.

$A^\circ = \bigcup_{\substack{U \subseteq A \\ U \text{ open}}} U$ the set of interior points of A .

$\overline{A} = \bigcap_{\substack{F \supseteq A \\ F \text{ closed}}} F$ the closure of A .

A' limit points of A .

Compact sets.

Locally compact sets.

Recall: X is Hausdorff iff $\forall x, y \in X, \exists U, V \in \tau$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

Definition: Bases for Topological Spaces

Definition: Let (X, τ) be a topological space. $\sigma \subseteq \tau$ is called a base for topology τ if $\forall x \in X, \forall U \in \tau, x \in U, \exists W \in \sigma$ such that $x \in W \subseteq U$.

Proposition

$\sigma \subseteq \tau$ is a base for τ if and only if every $U \in \tau$ is the union of certain sets taken from σ .

$$(*) \quad \tau = \left\{ \bigcup_{\omega \in \Omega} W_\omega : \{W_\omega\}_{\omega \in \Omega} \subseteq \sigma, \Omega \right\}$$

Proof

(\Leftarrow) \checkmark

(\Rightarrow) Take $U \in \tau$ and let $x \in U$, \leadsto find $W_x \in \sigma, x \in W_x \subseteq U$.

$$U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} W_x \subseteq U$$

Therefore $\bigcup_{x \in U} W_x = U$.

Proposition

If σ is a base for some topology τ on X , then

1. $\forall x \in X, \exists W \in \sigma$ such that $x \in W$.
2. $\forall U, V \in \sigma, \forall x \in U \cap V, \exists W \in \sigma$ such that $x \in W \subseteq U \cap V$.

Conversely, if $\sigma \in \mathcal{P}(X)$ ($\emptyset \notin \sigma$) satisfies (1) and (2), then σ is the base for a topology τ (and τ is given by $(*)$).

Note that $U, V \in \tau \implies U \cap V \in \tau$ (requires (2)).

If $U = \bigcup U_\alpha$ and $V = \bigcup V_\beta$, then $U \cap V = \bigcup_{\alpha, \beta} (U_\alpha \cap V_\beta) = \bigcup_{\alpha, \beta} \bigcup_{x \in U_\alpha \cap V_\beta} W_{\alpha, \beta, x}$.

Example: Metric Spaces

(X, d) is a metric space if $d : X \times X \rightarrow [0, +\infty)$ satisfies

1. $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

Definition: Epsilon Neighborhoods

$$B_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$$

$A \subseteq X$ is open if and only if $\forall x \in A, \exists \varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A$. $x \in B_\varepsilon(x)$.

τ = set of all open sets.

$$\sigma_1 = \{B_\varepsilon(x) : x \in X, \varepsilon > 0\}$$

is a base for the topology τ .

$$\sigma_2 = \{B_{1/n}(x) : x \in X, n \in \mathbb{N}\}$$

is also a base for τ .

Definition: Direct Product - Product Topology

Let (X_1, τ_1) and (X_2, τ_2) be topological spaces.

Consider $X = X_1 \times X_2$. The product topology τ on X is given by the base

$$\sigma = \{U_1 \times U_2 : U_1 \in \tau_1, U_2 \in \tau_2\}$$

More explicitly,

$$\tau = \left\{ \bigcup_{\omega} U_{1,\omega} \times U_{2,\omega} : U_{i,\omega} \in \tau_i \right\}$$

(X_ω, τ_ω) topological spaces $(\omega \in \Omega)$

$$X = \prod_{\omega \in \Omega} X_\omega = \{(x_\omega)_{\omega \in \Omega} : x_\omega \in X_\omega\}$$

Formally, $f \cong (x_\omega)_{\omega \in \Omega}$, $x_\omega = f(\omega)$, $f : \Omega \rightarrow \bigcup_{\omega \in \Omega} X_\omega$ such that $f(\omega) \in X_\omega$.

[$x \neq \emptyset \iff X_\omega \neq \emptyset$ axiom of choice]

$$\sigma = \left\{ \prod_{\omega \in \Omega} U_\omega : U_\omega \in \tau_\omega \text{ and all but finitely many } U_\omega = X_\omega \right\}$$

Definition: Subspace Topology

Given (X, τ) and $Y \subseteq X$, then (Y, τ_Y) is also a topological space where

$$\tau_Y \{U \cap Y : U \in \tau\}$$

Definition: Local Bases for Topological Spaces

A collection $\gamma \subseteq \tau$ is called a local base at $x \in X$ if

1. $\forall U \in \tau, x \in U, \exists W \in \gamma$ such that $x \in W \subseteq U$.
2. $\forall W \in \gamma, x \in W$

Example

Let (X, d) be a metric space.

$$\gamma_x = \{B_\varepsilon(x) : \varepsilon > 0\}$$

is a local base at x . Similarly,

$$\tilde{\gamma}_x = \{B_{1/n} : n \in \mathbb{N}\}$$

is a countable local base.

Proposition

If γ_x ($x \in X$) are local bases for τ at X , then

$$\sigma = \bigcup_{x \in X} \gamma_x$$

is a bse for τ .

Proposition

$\{\gamma_x\}_{x \in X}$ are local bases at x for some topology τ if and only if

1. $\forall x \in X$, γ_x is a non-empty collection of subsets containing x .
2. If $U \in \gamma_x$, $V \in \gamma_y$, and $z \in U \cap V$, then $\exists W \in \gamma_z$ such that $z \in W \subseteq U \cap V$.

Definition: Topological Vector Spaces

Suppose V is a vector space over $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and let τ be a topology on V . Then V is a topological vector space (TVS) if

1. $\forall x \in V$, $\{x\}$ is closed.
2. The functions f, g (i.e. algebraic operations) are continuous.

$$\begin{aligned} f : V \times V &\rightarrow V, f(x, y) = x + y \\ g : \mathbb{F} \times V &\rightarrow V, g(\lambda, x) = \lambda \cdot x \end{aligned}$$

Notation

For $A_1, A_2 \subseteq V$ and $B \subseteq \mathbb{F}$,

$$\begin{aligned} A_1 + A_2 &= \{a_1 + a_2 : a_1 \in A_1, a_2 \in A_2\} \\ a + A_1 &= \{a + \alpha : \alpha \in A_1\} \\ B \cdot A &= \{\beta \cdot a : \beta \in B, a \in A\} \\ \alpha \cdot A &= \{\alpha \cdot a : a \in A\} \end{aligned}$$

Lemma

Let V be a TVS. Then

1. $\forall x, y \in V, \forall \text{ open } U_{x+y} \ni x + y, \exists \text{ open } U_x \ni x, \text{ open } U_y \ni y \text{ such that } U_x + U_y \subseteq U_{x+y}.$
2. $\forall x \in V, \alpha \in \mathbb{F}, \forall \text{ open } U_{\alpha x} \ni \alpha x, \exists \text{ open } U_x \ni x, U_\alpha \ni \alpha \text{ such that } U_\alpha \cdot U_x \subseteq U_{\alpha x}.$

Proof of 1

Given $x, y \in X, x + y \in U_{x+y}$ open.

$$f(x, y) = x + y \in U_{x+y}$$

and $(x, y) \in f^{-1}(U_{x+y})$ open. In the product topology

$$(x, y) \in U_x \times U_y \subseteq f^{-1}(U_{x+y})$$

which implies $x \in U_x$ and $y \in U_y$, both open, and $U_x + U_y \subseteq U_{x+y}$.

April 9, 2024

Office Hours: Th 11:30 AM - 1:00 PM

Makeup Class: Friday, April 12 at 11:00 AM.

Lemma 1

Let V be a TVS

1. $\forall x, y \in V, \forall U_{x+y} \ni x + y \text{ open}, \exists U_x \ni x, U_y \ni y \text{ such that } U_x + U_y \subseteq U_{x+y}.$
2. $\forall \alpha \in F, \forall U_{\alpha x} \ni \alpha x \text{ open}, \exists U_\alpha \ni \alpha \text{ open in } F, U_x \ni x \text{ such that } U_\alpha \cdot U_x \subseteq U_{\alpha x}.$

For 2. with $\alpha = 0, \forall x \in X, \forall U \ni 0 \text{ open}, \exists \delta > 0, U_\delta \ni x \text{ open such that } B_\delta(0) \cdot U_x \subseteq U. \text{ That is, } \beta U_x \subseteq U, \forall |\beta| < \delta.$

Proposition

In a TVS, the maps

1. Translation: $T_a : x \in V \mapsto x + a \in V \ (a \in V)$
2. Multiplication: $M_\lambda : x \in V \mapsto \lambda \cdot x \in V \ (\lambda \in \mathbb{F}, \lambda \neq 0)$

are continuous (in fact, homeomorphic).

Proof

We know $(x, y) \mapsto x + y$ and $(\lambda, x) \mapsto \lambda \cdot x$ are continuous.

Inversions

$T_a \circ T_{-a} = \text{id}$, $T_{-a} \circ T_a = \text{id}$, $M_\lambda \circ M_{1/\lambda} = \text{id}$, and $M_{1/\lambda} \circ M_\lambda = \text{id}$.
Therefore they are bijective and the inverses are continuous.

Remark

If U is open, then $a + U$ is also open.

If γ_0 is a local base at 0, then $\gamma_x = \{x + U : U \in \gamma_0\} = x + \gamma_0$ is a local base at x .

Recall that γ_x is a local base at x if $\forall W \ni x$ open, $\exists U \in \gamma_x$ such that $x \in U \subseteq W$.

That is, in a TVS only local bses at 0 are needed. We may interpret “local base” as “local base at 0”.

$$\sigma = \bigcup_{x \in V} (x + \gamma_0)$$

is a base for the TVS.

Types of Topological Vector Spaces

Normed Spaces / Banach Spaces

A normed space is a vector space over \mathbb{F} together with a norm $|| \cdot ||$, i.e. a map $|| \cdot || : x \in V \mapsto ||x|| \in [0, \infty)$ such that

1. $||x|| = 0 \iff x = 0$.
2. $||x + y|| \leq ||x|| + ||y||$.
3. $||\lambda x|| = |\lambda| \cdot ||x||$.

Remarks

A normed space is a metric space with $d(x, y) = ||x - y||$.

A local base (at 0) is given by ε -neighborhoods:

$$\gamma_0 = \{B_\varepsilon(0) : \varepsilon > 0\}$$

where

$$B_\varepsilon(0) = \{x \in V : ||x|| < \varepsilon\}$$

(open ball with radius $\varepsilon > 0$).

Convergence in Normed Space

A sequence $\{x_n\}$ ($x_n \in V$) converges to $\lambda \in V$ if $\lim_{n \rightarrow \infty} ||x_n - \lambda|| = 0$.

A sequence $\{x_n\}$ is Cauchy if $\forall \varepsilon > 0$, $\exists N$ such that $\forall j, k \geq N$, $||x_j - x_k|| < \varepsilon$.

A normed space is complete if $\{x_n\}$ Cauchy implies $\exists x \in V$ such that $x_n \rightarrow x$.

Complete normed spaces are called Banach spaces.

Example 1

$\ell^p(\mathbb{N})$, $1 \leq p < \infty$, the set of all sequences $\{x_n\}_{n=1}^\infty = x$ such that

$$||x|| = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} < +\infty$$

Recall $\{x_n\} + \{y_n\} = \{x_n + y_n\}$ and $\lambda\{x_n\} = \{\lambda x_n\}$.

ℓ^p spaces are complete and therefore Banach.

If $\{x_n\} \in \ell^p$ and $\{y_n\} \in \ell^q$, then $\{x_n y_n\} \in \ell^r$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \in [0, 1]$ (e.g. $\ell^2 \cdot \ell^2 \leq \ell^1$)

Example 2

$\ell^\infty(\mathbb{N})$, the set of all bounded sequences

$$||x|| = \sup_{n \in \mathbb{N}} |x_n| < +\infty$$

Example 3

$C_0(\mathbb{N}) \subseteq \ell^p(\mathbb{N})$, the set of all sequences $\{x_n\}$

$$\lim_{n \rightarrow \infty} x_n = 0$$

C_0 is a closed subspace, and both are Banach.

Example 4

$L^p(\Omega)$, $1 \leq p < \infty$, $\Omega \subseteq \mathbb{R}^d$ a Lebesgue measurable set with $m(\Omega) > 0$, the space of all equivalence classes of Lebesgue measurable functions $f : \Omega \rightarrow \mathbb{F}$ such that

$$||f|| = \left(\int_{\Omega} |f(x)|^p dx \right)^{1/p} < +\infty$$

Example 5

$L^\infty(\Omega)$, the measurable and essentially bounded functions

$$\begin{aligned} ||f|| &= \inf_{\substack{N \subseteq \Omega \\ m(N)=0}} \sup_{x \in \Omega \setminus N} |f(x)| < +\infty \\ &= \text{ess sup}_{x \in \Omega} |f(x)| \end{aligned}$$

$L^p(\Omega)$ spaces, $1 \leq p \leq \infty$, are Banach.

Example 6

For $\Omega \neq \emptyset$, let $B(\Omega)$ the set of all bounded functions $f : \Omega \rightarrow \mathbb{F}$ with

$$||f|| = \sup_{x \in \Omega} |f(x)|$$

is a Banach space.

$f_n \rightarrow f$ in $B(\Omega)$ if and only if f_n converges uniformly on Ω to f .

Example 7

Let Ω be a topological space and $BC(\Omega)$ the set of all bounded, continuous functions $f : \Omega \rightarrow \mathbb{F}$. Then $BC(\Omega) \subseteq B(\Omega)$ is a closed Banach subspace under the same norm. That is, the uniform limit of continuous functions is a continuous function.

$$\lim_{f_n \in BC(\Omega)} f_n \rightarrow f \implies f \in BC(\Omega)$$

Example 8

Let K be a compact, Hausdorff space.

Then $C(K)$ is the set of all continuous functions $f : K \rightarrow \mathbb{F}$ and $C(K) = BC(K)$.

F Spaces / pre-F Spaces

A pre- F -space is a TVS where the topology is given by some invariant metric $d(x+z, y+z) = d(x, y)$ or $d(x, y) = d(x-y, 0)$.

An F -space is a complete pre- F -space.

A local base (at 0) is given by

$$\gamma_0 = \{B_\varepsilon(0) : \varepsilon > 0\}, \quad B_\varepsilon(x) = \{y \in V : d(x, y) < \varepsilon\}$$

Example 1

$\ell^p(\mathbb{N})$, $0 < p < 1$, the set of all $\{x_n\}_{n=1}^\infty$ such that

$$\sum_{n=1}^{\infty} |x_n|^p < +\infty$$

with

$$d(x, y) = \sum_{n=1}^{\infty} |x_n - y_n|^p$$

Note that this obeys the triangle inequality specifically because it has not been raised to $1/p$.

$$d(\lambda x, \lambda y) = |\lambda|^p \cdot d(x, y)$$

means that $d(z, 0)$ is not a norm.

Here, $B_\varepsilon(x)$ are not convex sets.

Side Remark

Given \mathbb{R}^2 , the ℓ^p norm for $1 \leq p \leq \infty$ is given by

$$|| (x_1, x_2) || = (|x_1|^p + |x_2|^p)^{1/p}$$

and the distance for $0 < p < 1$ by

$$d((x_1, x_2)) = |x_1|^p + |x_2|^p$$

The ε neighborhoods for $p = 1$ are diamonds, $p = 2$ circles, $p = \infty$ squares with smooth transition between them. However, for $0 < p < 1$, we have concave diamond shapes. These norms and metrics are all equivalent on \mathbb{R}^2 in the sense that they give the same topology.

Locally Convex TVS

A TVS which has a local base γ at 0 consisting of open neighborhoods of 0 which are all convex.

Definition: Convex Set

A set $A \subseteq V$ is convex if $\forall x, y \in A, \lambda \in [0, 1]$, then $\lambda x + (1 - \lambda)y \in A$.
Alternatively, the line segment between x and y is contained in A ($[x, y] \subseteq A$).

Fréchet Spaces and pre-Fréchet Spaces

A pre-Fréchet space is locally convex.
A Fréchet space is a locally convex F -space.

April 11, 2024

Fréchet Spaces

Example

$\mathcal{S} = \{\{x_n\}_{n=1}^{\infty} \mid \text{the space of all sequences } x_n \in \mathbb{F}\}$.

$$d(\{x_n\}, \{y_n\}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|x_n - y_n|}{1 + |x_n - y_n|} < +\infty$$

Triangle inequality:

$$\frac{a+b}{1+a+b} \leq \frac{a}{1+a} + \frac{b}{1+b}, \quad a, b \geq 0$$

invariant metric, complete.

$\gamma_0 = \{B_\varepsilon(0) : \varepsilon > 0\}$ is a local base.

$\hat{\gamma}_0 = \{U_{\varepsilon, N} : \varepsilon > 0, N \in \mathbb{N}\}$.

$U_{\varepsilon, N} = \{\{x_n\}_{n=1}^{\infty} : |x_n| < \varepsilon, \forall n = 1, \dots, N\}$.

$\forall \varepsilon > 0, \exists \hat{\varepsilon} > 0, N$ such that $U_{\hat{\varepsilon}, N} \subseteq B_\varepsilon(0)$.

$\forall \hat{\varepsilon} > 0, N, \exists \varepsilon > 0$ such that $B_\varepsilon(0) \subseteq U_{\hat{\varepsilon}, N}$.

$x^{(m)} \rightarrow x$ in metric of \mathcal{S} as $m \rightarrow \infty$.

$x^{(m)} = \{x_n^{(m)}\}_{n=1}^{\infty}, x = \{x_n\}_{n=1}^{\infty}$ if and only if $\forall n \in \mathbb{N}, x_n^{(m)} \rightarrow x_n$ as $m \rightarrow \infty$ (pointwise, componentwise convergence).

Example

$C(\mathbb{R}^d)$, the set of continuous functions $f : \mathbb{R}^d \rightarrow \mathbb{F}$.

$$|||f|||_N = \sup_{\substack{x \in \mathbb{R}^d \\ ||x|| \leq N}} |f(x)|$$

$$d(f, g) = \sum_{N=1}^{\infty} \frac{1}{2^N} \cdot \frac{|||f - g|||_N}{1 + |||f - g|||_N}$$

Complete Fréchet space.

“Locally uniform convergence” such that $f_n \rightarrow f$ in metric of $C(\mathbb{R}^d)$ if and only if \forall compact set $K \subseteq \mathbb{R}^d$, f_n converges to f uniformly on K .

Example

$C^\infty[0,1]$ the set of infinitely differentiable functions $f : [0,1] \rightarrow \mathbb{F}$.

$$|||f|||_n = \sup_{x \in [0,1]} |f^{(n)}(x)|$$

$$d(f, g) = \sum_{n=0}^{\infty} \frac{1}{2^n} \cdot \frac{|||f-g|||_n}{1 + |||f-g|||_n}$$

Fréchet space.

$f_m \rightarrow f$ in $C^\infty[0,1]$ as $m \rightarrow \infty$ if and only if for every $m \in \{0,1,\dots\}$, $f_m^{(n)} \rightarrow f^{(n)}$ uniformly on $[0,1]$ as $m \rightarrow \infty$.

Proposition

Every TVS is Hausdorff.

Proof

Let $x, y \in V$, $x \neq y$.

For $U = V \setminus \{0\}$, and open set, $x - y \in U$.

Using the continuity of $(x^2, y^2) \mapsto x^2 - y^2$ and Lemma 1, there exist $U_x \ni x$ and $U_y \ni y$ open such that $U_x - U_y \subseteq U$.

Note that $U_x \cap U_y = \emptyset$, otherwise there would exist $z \in U_x \cap U_y$ such that $0 = z - z \in U_x - U_y \subseteq U$ a contradiction.

Definition: Balancedness

A subset U of a vector space V is called balanced if $\forall \lambda \in \mathbb{F}$, $|\lambda| \leq 1$, $\lambda U \subseteq U$.

Example

For $V = \mathbb{R}^2$, $\mathbb{F} = \mathbb{R}$, an ellipse is convex and balanced.

Note that since $\lambda = 0$ is a valid choice, 0 is always in a balanced set.

A rectangle, offset from the origin, is convex but not balanced.

A concave diamond centered at 0 may be balanced.

An annulus is neither.

Exercise

Show that for $V = \mathbb{C}$, $\mathbb{F} = \mathbb{C}$, the balanced, convex sets are the open and closed disks along with the entire plane.

Proposition

1. Every TVS has a balanced, local base.
2. Every locally convex TVS has a balanced and convex local base.

Proof of A

e.g. $\gamma = \{U : U \text{ open}, 0 \in U\}$.

For every $U \in \gamma$, construct another \hat{U} open, $0 \in \hat{U} \subseteq U$ balanced.

Then $\hat{\gamma} = \{\hat{U} : U \text{ taken from } \gamma\}$ is a local base.

Use Lemma 1 again and the continuity of $(\lambda, x') \mapsto \lambda \cdot x'$ at $\lambda = 0, x' = 0$.

Given open $U \ni 0$, find $\delta > 0$ and open $U_0 \ni 0$ such that $B_{2\delta}(0) \cdot U_0 \subseteq U$.

Then for $\alpha \in \mathbb{F}, |\alpha| \leq \delta, \alpha \cdot U_0 \subseteq U$. Take

$$\hat{U} = \bigcup_{\substack{\alpha \in \mathbb{F} \\ |\alpha| \leq \delta}} \alpha \cdot U_0$$

Therefore \hat{U} is a union of open sets and $0 \in \hat{U} \subseteq U$. Finally, for $|\lambda| \leq 1$,

$$\lambda \cdot \hat{U} = \bigcup_{\substack{\alpha \in \mathbb{F} \\ |\alpha| < \delta}} \lambda \cdot \alpha \cdot U_0 = \bigcup_{\substack{\beta \\ |\beta| \leq |\lambda| \cdot \delta \leq \delta}} \beta U_0 = \hat{U}$$

Proof of B

We have a local base $\gamma = \{U_\omega\}$, $U_\omega \ni 0$ open and convex.

We want to construct $\hat{\gamma} = \{\hat{U}_\omega\}$, $\hat{U}_\omega \ni 0$ open, convex and balanced.

Given U convex, define

$$\hat{U} = \bigcap_{|\alpha| \leq \delta} \alpha U$$

convex and balanced.

Need to show that $\hat{U} \ni 0$ is an open neighborhood.

Rest of the owl left to the reader.

Recall

The intrinsic characterization of a base for a topological space or a local base for a topological space X , $\{\gamma_x\}_{x \in X}$.

- $\forall x \in X, U \in \gamma_x, x \in U$
- $\forall x, y \in X, U \in \gamma_x, V \in \gamma_y, z \in U \cap V, \exists W \in \gamma_z, W \subseteq U \cap V$.

Proposition

A balanced, local base γ (at 0) of a TVS V has the following properties:

1. γ is a nonempty collection of subsets of V containing 0.
2. $\forall U_1, U_2 \in \gamma, \exists U \in \gamma$ such that $U \subseteq U_1 \cap U_2$.
3. $\forall U \in \gamma, x \in U, \exists W \in \gamma$ such that $x + W \subseteq U$.

4. $\forall U \in \gamma, \exists W \in \gamma$ such that $W + W \subseteq U$ (continuity of $(x, y) \mapsto x + y$ at $(x = y = 0)$).
 5. $\forall U \in \gamma, \forall x \in V, \exists t > 0, x \in t \cdot U$ (continuity of scalar multiplication $(\lambda, x') \mapsto \lambda x'$ at $\lambda = 0, x' = x$).
- $$\frac{1}{t} \cdot x \in U, \frac{\delta}{2} \cdot x \subset B_\delta(0) \cdot \hat{U} \subseteq U.$$
6. $\forall x \in V, x \neq 0, \exists U \in \gamma, x \notin U$ ($\{x\}$ closed; $0 \in V \setminus \{x\}$ open; $0 \in U \subseteq V \setminus \{x\}$). (Hausdorff)

Converse

Conversely, if γ satisfies properties 1-6, then there exists a unique topology on V such that γ is a balanced, local base for V and V with this topology is a TVS.

Theorem:

Any two TVS of finite dimension d (over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) are homeomorphic to each other.

Proof

Let V be a TVS with $\dim(V) = d$.

We want to show that $V \cong \mathbb{F}^d$. We have

$$V = \text{lin}\{v_1, \dots, v_d\}$$

a basis and

$$f : (\lambda_1, \dots, \lambda_n) \in \mathbb{F}^d \mapsto \sum_{i=1}^d \lambda_i v_i \in V$$

an isomorphism between \mathbb{F}^d and V as vector spaces. Further, f is continuous. Consider \mathbb{F}^d equipped with the product topology and the continuity of addition and scalar multiplication.

We need only that f^{-1} is continuous at 0 which is equivalent to $\forall U \ni 0$ open in $\mathbb{F}^d, \exists W \ni 0$ open in V such that $W \subseteq f(U) ((f^{-1})^{-1}(U))$.

April 12, 2024

Lemma

$\forall U \ni 0$ open in $\mathbb{F}^d, \exists W \ni 0$ open such that $f(U) \supseteq W$.

That is, 0 is an interior point of $f(U)$.

Proof

$f : \mathbb{F}^d \rightarrow V$, continuous.

We may assume without loss of generality that $U = B_1(0)$.

Let $S = \{\lambda \in \mathbb{F}^d : \|\lambda\| = 1\}$, a compact set.

Since f continuous, $f(S)$ is compact in V . Since V is Hausdorff, $f(S)$ is closed.

Take $\hat{U} = V \setminus f(S) \ni 0$ open (because $0 \notin f(S)$ else $f(\lambda) = 0$ would imply $\|\lambda\| = 1$)

Now, there exists a balanced, open set $0 \in W \subseteq \hat{U}$. Therefore, $W \subseteq f(U)$.

Otherwise, $x \in W, x \notin f(U), x = f(\lambda), \lambda \notin U, \|\lambda\| \geq 1$ would give $\frac{x}{\|\lambda\|} = \frac{1}{\|\lambda\|} \cdot f(\lambda) = f\left(\frac{\lambda}{\|\lambda\|}\right) \in f(S)$.

But, $\frac{x}{\|\lambda\|} \in W \subseteq \hat{U}$ because $x \in W, \frac{1}{\|\lambda\|} \in [0, 1]$ and W is balanced shows a contradiction.

Theorem

Any finite-dimensional subspace in a TVS is closed.

Theorem

Every locally compact TVS is finite-dimensional.

Definition: Locally Compact

V is locally compact if $\forall x \in V, \exists U \ni x$ open and $K \subseteq V$ such that $U \subseteq K$.
For Hausdorff spaces, $\forall x \in V, \exists U \ni x$ open such that \overline{U} compact.

Example

Let V be a normed space, $\dim(V) = +\infty$.
Then $\overline{B_1(0)} = \{x \in V : \|x\| \leq 1\}$ is not compact.

Definition: Semi-norm

A semi-norm on a metric space V (over $\mathbb{F} = \mathbb{R}, \mathbb{C}$) is a map

$$p : V \rightarrow [0, +\infty)$$

such that

1. $p(x + y) \leq p(x) + p(y)$
2. $p(\lambda x) = |\lambda| \cdot p(x)$.

Note that $p(0) = 0$ and $(p(x - y) \geq |p(x) - p(y)|$.

$$N_p = \{x \in V : p(x) = 0\}$$

is a linear subspace of V : $x, y \in N$ such that $p(x + y) \leq p(x) + p(y) = 0$, $p(\lambda x) = 0$.
A semi-norm on V induces a norm on the quotient space V/N_p .

$$\|[x]_{N_p}\| = p(x) \quad [x]_N = \{x + z : z \in N\} = x + N$$

Definition: Absorbing

A set $A \subseteq V$ is called absorbing if $\forall x \in V, \exists \lambda > 0$ such that $\lambda x \in A$.

Equivalently, $\bigcup_{\lambda > 0} \frac{1}{\lambda} A = V$.

There is a relationship between semi-norms on V and balanced, convex and absorbing subsets of V .

Proposition

If p is a semi-norm on a vector space V , then

$$A = \{x \in V : p(x) < 1\}$$

is balanced, convex and absorbing.

Proof

Convex: $x, y \in A$, $p(x) < 1$, $p(y) < 1$,

$$p(\lambda x + (1 - \lambda)y) \leq \lambda \cdot p(x) + (1 - \lambda)p(y) < 1$$

Balanced: $x \in A$, $|\lambda| \leq 1$, $p(x) < 1$,

$$p(\lambda x) = |\lambda| \cdot p(x) < 1$$

Absorbing: $x \in V$. If $p(x) = 0$, then $x \in A$ ($\lambda = 1$).

If $p(x) > 0$, $\lambda = \frac{1}{2p(x)}$ gives $p(\lambda x) = |\lambda| \cdot p(x) = \frac{1}{2} < 1$.

Example

Let $V = \mathbb{R}^2$ and $\mathbb{F} = \mathbb{R}$.

An ellipse is balanced, convex and absorbing.

A band is also balanced, convex and absorbing.

An open interval along the axis is balanced and convex, but it is not absorbing.

Proposition

Each open neighborhood of 0 in a TVS is absorbing.

Proof

Continuity of the map $(\lambda, x) \mapsto \lambda x'$ at $\lambda = 0$ and $x' = x$.

Given $x \in V$, $U \ni 0$ open, $\exists \delta > 0$, $W \ni x$ such that $B_r(0) \cdot W \subseteq U$ and $\frac{\delta}{2} \cdot x \in U$.

Definition: Minkowski Functional

Let A be a subset in a vector space V .

If A is absorbing, one can define the Minkowski functional

$$\mu_A(x) = \inf \left\{ \lambda > 0 : \frac{x}{\lambda} \in A \right\} = \inf \{ \lambda > 0 : x \in \lambda \cdot A \}$$

Proposition

If A is convex, balanced and absorbing, then μ_A is a semi-norm.

Proof

Absorbing $\leadsto \mu_A$ is well defined, $\mu_A(x) \in [0, +\infty)$. For $\alpha \neq 0$,

$$\begin{aligned} \mu_A(\lambda x) &= \inf \left\{ \lambda > 0 : \frac{\alpha \cdot x}{\lambda} \in A \right\} \\ &= \inf \left\{ |\alpha| \cdot \lambda > 0 : \frac{\alpha \cdot x}{|\alpha| \cdot \lambda} \in A \right\} \quad (\lambda \mapsto |\alpha| \cdot \lambda) \\ &= |\alpha| \cdot \inf \left\{ \lambda > 0 : \frac{x}{\lambda} \in \frac{|\alpha|}{\alpha} A \right\} \\ &= |\alpha| \cdot \inf \left\{ \lambda > 0 : \frac{x}{\lambda} \in A \right\} \\ &= |\alpha| \cdot \mu_A(x) \end{aligned}$$

since A is balanced, $\frac{\alpha}{|\alpha|}A = A$.

Note that $\mu_A(0) = 0$ since $0 \in A$ balanced.

Given $x, y \in V$ and $\varepsilon > 0$, let $s = \mu_A(x) + \varepsilon$ and $t = \mu_A(y) + \varepsilon$. Then, since A is balanced, $\frac{x}{s}, \frac{y}{t} \in A$. By convexity,

$$\frac{x+y}{t+s} = \underbrace{\frac{s}{t+s}}_{(1-\sigma)} \cdot \underbrace{\frac{x}{s}}_{\in A} + \underbrace{\frac{t}{t+s}}_{\sigma} \cdot \underbrace{\frac{y}{t}}_{\in A} \in A$$

Therefore, $\mu_A(x+y) \leq t+s$ which implies $\mu_A(x+y) \leq \mu_A(x) + \mu_A(y) + 2\varepsilon$ for all $\varepsilon > 0$.

Equivalence between Semi-norm and ABC Sets

$p \rightsquigarrow A = \{x : p(x) < 1\} \rightsquigarrow \mu_a = p$.

A bounded, convex, absorbing $\rightsquigarrow \mu_A \rightsquigarrow \tilde{A} = \{x : \mu_A(x) < 1\}$ where $\tilde{A} \subseteq A$ differing possibly by the boundary.

Question: which TVS are normable?

That is a norm such that the topology is given by this norm.

Definition: Bounded Sets

A subset A in a TVS is bounded if $\forall U \ni 0$ open, $\exists \delta > 0$ such that $A \subseteq t \cdot U$, $\forall t > \delta$.

Theorem:

A TVS is normable if and only if there exists a bounded, convex, balanced, open neighborhood of 0.

Proof (Sketch)

Suppose V is a normed space with norm $\|\cdot\|$.

$$B = \{x \in V : \|x\| < 1\} = B_1(0)$$

is convex, balanced, and an open neighborhood of 0.

B is bounded, since given $U \ni 0$ open, $B_\varepsilon(0) \subseteq U$, so $B = \frac{1}{\varepsilon} \cdot B_\varepsilon(0) \subseteq \lambda B_\varepsilon(0) \subseteq \lambda \cdot U$ for $\lambda \geq \frac{1}{\varepsilon}$.

Now, let B be a bounded, convex, balanced, open neighborhood of 0 in a TVS.

Therefore B is absorbing (as an open neighborhood of 0).

It follows that the semi-norm $\mu_B(x)$ may be defined.

Then $\mu_B(x) = 0 \implies x = 0$ since B is bounded, otherwise $0 \in U = V \setminus \{x\}$ open gives $B \subseteq t \cdot U$, $\forall t > \delta$ and $\frac{1}{t}B \subseteq U$, $\forall t > \delta$.

Thus, $\|x\| = \mu_B(x)$ is a norm on V .

One need only demonstrate that the norm topology is the same as the original topology on V .

That is, $\forall U \ni 0$ open, $\exists \varepsilon > 0$ such that $\varepsilon \cdot B \subseteq U$.

$\forall \varepsilon > 0$, $\exists \hat{U} \ni 0$ open such that $\hat{U} \subseteq \varepsilon B$.

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Recall

Given p a semi-norm,

$$A = \{x \in V : p(x) < 1\}$$

a convex, balanced absorbing set gives the Minkowski formula for a seminorm μ_a .

The TVS V is normable if and only if there exist bounded, convex, balanced, open $U \ni 0$.

Definition: Separating Family of Semi-norms

Let V be a vector space.

A family of semi-norms $\{p_\omega\}_{\omega \in \Omega}$ is called separating if $\forall x \in V, x \neq 0, \exists \omega \in \Omega$ such that $p_\omega(x) \neq 0$.

Equivalently,

$$\{x \in V : \forall \omega \in \Omega, p_\omega(x) = 0\} = \{0\}$$

or

$$\bigcap_{\omega \in \Omega} N_{p_\omega} = \bigcap_{\omega \in \Omega} \{x \in V : p_\omega(x) = 0\} = \{0\}.$$

For a given separating family of semi-norms, define

$$U_{n,\omega} = \left\{x \in V : p_\omega(x) < \frac{1}{n}\right\}$$

$$U_{n,\omega_1,\dots,\omega_N} = \left\{x \in V : p_{\omega_i}(x) < \frac{1}{n}, i = 1, \dots, N\right\} = \bigcap_{i=1}^N U_{n,\omega_i}$$

and put

$$\gamma = \{U_{n,\omega_1,\dots,\omega_N} : n \in \mathbb{N}, \omega_1, \dots, \omega_N \in \Omega, N \in \mathbb{N}\}.$$

One can show by earlier propositions that γ is a local base at 0 for some topology τ .

Perhaps unsurprisingly, if $\{p_\omega\}$ is separating, then this locally convex TVS is Hausdorff.

Theorem:

Let $\{p_\omega\}$ be a separating family of semi-norms on a vector space V . Then with local base γ defined above, V becomes a locally convex TVS, and all $p_\omega : V \rightarrow [0, +\infty)$ continuous.

Example

$$\mathcal{S} = \{\{x_n\}_{n=1}^\infty \text{ all sequences}\}$$

$$\text{with } p_n(x) = |x_n|, x = \{x_n\}_{n=1}^\infty, d(x, y) = \sum_{n=1}^\infty \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$

Remark

Local base at x

$$\gamma_x = x + \gamma = \{U_{n,\omega_1,\dots,\omega_N}[x] : n, N \in \mathbb{N}, \omega_1, \dots, \omega_N \in \Omega\}$$

$$U_{n,\omega_1,\dots,\omega_N}[x] = \left\{y \in V : p_{\omega_i}(x - y) < \frac{1}{n}, i = 1, \dots, N\right\}$$

Theorem:

Let V be a locally convex TVS. Then there exists a separating family of semi-norms $\{p_\omega\}_{\omega \in \Omega}$ on V such that the topology defined by $\{p_\omega\}$ coincides with the original topology.

Proof (Sketch)

V is locally convex, so it has a convex, balanced, local base at 0

$$\hat{\gamma} = \{U_\omega\}_{\omega \in \Omega}$$

where $U_\omega \ni 0$ are open, convex, balanced, and absorbing.

Put $p_\omega = \mu_{U_\omega}$ (i.e. set the Minkowski functional to be the semi-norm).

Then we need to show that these are separating.

Then define $U_{n,\omega_1,\dots,\omega_N}$, $\gamma = \{U_{n,\omega_1,\dots,\omega_N}\}$, $U_\omega = U_{1,\omega}$, $\hat{\gamma} \subseteq \gamma$ and show that γ and $\hat{\gamma}$ induce the same topology.

Theorem:

A TVS V is a pre-Fréchet space if and only if V has a countable, convex, balanced local base.

Proof

(\implies) Assume that V is a pre-Fréchet space.

Then we have an invariant metric d and

$$B_\varepsilon(x) = \{y \in V : d(x, y) < \varepsilon\}.$$

It follows that $\gamma_1 = \{B_{1/n}(0) : n \in \mathbb{N}\}$ is a local base.

The fact that V is locally convex means that $\gamma_2 = \{U_\omega : \omega \in \Omega\}$ with $U_\omega \ni 0$ open, convex and balanced is a convex, balanced local base.

To every $n \in \mathbb{N}$, $B_{1/n}(0)$ is an open neighborhood of 0, and there exists $\omega_n \in \Omega$, $0 \in U_{\omega_n} \subseteq B_{1/n}(0)$. Put

$$\gamma_3 = \{U_{\omega_n} : n \in \mathbb{N}\}$$

a countable, convex, balanced collection.

Then, for any $U \ni 0$ open, $\exists n$ such that $U_{\omega_n} \subseteq B_{1/n}(0) \subseteq U$. So γ_3 is a local base.

(\impliedby) Assume a TVS V has a countable, convex, balanced local base:

$$\gamma = \{U_n : n \in \mathbb{N}\}$$

Without loss of generality, we may assume that $U_{n+1} \subseteq U_n$. Otherwise, we may take $\hat{U}_n = U_1 \cap \dots \cap U_n \subseteq U_n$ such that $\{\hat{U}_n : n \in \mathbb{N}\}$ is also a local base where $\hat{U}_{n+1} \subseteq \hat{U}_n$.

Then, since U_n are open, they are absorbing and $p_n = \mu_{U_n}$ gives a separating semi-norm from the Minkowski functional. Define a metric

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x-y)}{1 + p_n(x-y)}$$

where $d(x, y) = 0 \implies x = y$ since $\{p_n\}$ are separating.

Claim: the metric topology (local base $\tilde{\gamma}$) is the same as the original topology (local base γ). Write

$$\tilde{\gamma} = \{B_{1/2^m}(0) : m \in \mathbb{N}\}$$

Then for all $m \in \mathbb{N}$,

$$\frac{1}{2^{m+1}}U_{m+1} \subseteq B_{1/2^m}(0)$$

there exists $n \in \mathbb{N}$ such that $U_n \subseteq \frac{1}{2^{m+1}}U_{m+1}$.

Also, $\forall n, B_{1/2^{n+1}}(0) \subseteq U_n$. Then V is locally convex (γ) and has an invariant metric ($\tilde{\gamma}$). That is, V is pre-Fréchet space.

Corollary

In a pre-Fréchet space, the metric can always be defined by

$$\forall n, B_{1/2^{n+1}}(0) \subseteq U_n$$

where $\{p_n\}$ are separating family of semi-norms.

A separating family of semi-norms leads to a locally convex TVS.

A countable, separating family of semi-norms leads to a pre-Fréchet space.

A finite, separating family of semi-norms leads to a normed space.

Quotient Spaces

For a vector space X and a linear subspace $N \subseteq X$, $X/N = \{[x]_N : x \in X\}$, $[x]_N = x + N$.

$\pi : X \rightarrow X/N$ is the quotient map to the vector space X/N .

For a TVS X , $N \subseteq X$ a subspace, $\pi : X \rightarrow X/N$ where τ is the topology of X and $\hat{\tau}$ is the topology of X/N given by

$$\hat{\tau} = \{\pi(U) : U \in \tau\}.$$

N is closed if and only if X/N is Hausdorff.

Thoeerem:

For X a TVS and $N \subseteq X$ a linear subspace, X/N is a TVS and $\pi : X \rightarrow X/N$ is open and continuous.

Normed / Banach

For X a normed (Banach) space, X/N is a normed (Banach) space where $\|[x]\|_{X/N} = \inf_{z \in N} \|x + z\|$.

Pre-Fréchet / Fréchet

For X a (pre-)Fréchet space, X/N is a (pre-)Fréchet space where $d_{X/N}(x, y) = \inf_{z \in N} d(x + z, y) = \inf_{z_1, z_2} d(x + z_1, y + z_2)$.

Definition: Linear Operator

A map $T : V \rightarrow W$ between vector spaces V, W is linear (or a linear operator) if

$$T(x + y) = Tx + Ty \quad \text{and} \quad T(\alpha x) = \alpha(Tx)$$

Notation

$M(V, W)$ is the set of all linear operators.

$M(V, V) = M(V)$.

$V' = M(V, \mathbb{F})$ (linear functionals) is the algebraic dual of V .

Note that $M(V, W)$ is a vector space.

$$(T_1 + T_2)(x) := T_1x + T_2x \quad \text{and} \quad (\lambda T)(x) := \lambda(Tx)$$

If T_1, T_2 are linear, then $T_1 + T_2$ is linear; likewise, λT is linear precisely when T is linear.

Definition: Continuous Linear Operator

For V, W TVS, T is a continuous linear operator if $T \in M(V, W)$ and T is continuous with respect to the topologies.

Notation

$L(V, W)$ is the set of all continuous linear operators.

$L(V, V) = L(V)$.

$V^* = L(V, \mathbb{F})$, the set of continuous linear functionals on V , is the dual space of V .

Example

Let $V = \mathbb{R}^n, W = \mathbb{R}^m$.

$M(V, W) = L(V, W)$.

To an $m \times n$ matrix $A = (a_{ij})_{i=1, j=1}^{m, n}$, one associates the linear operator T_A

$$T_A : (x_j)_{j=1}^n \mapsto (y_i)_{i=1}^m, \quad y_i = \sum_{j=1}^n a_{ij} x_j$$

$V' = V^*$. Given $\phi = (\phi_1, \dots, \phi_n) \in \mathbb{R}^n$ (a row vector),

$$\phi(x) = \phi \cdot x = \sum_{j=1}^n \phi_j x_j$$

In this case, $V^* \cong \mathbb{R}^n$.

Definition: Image or Range

For $T \in M(V, W)$, $T : V \rightarrow W$,

$$\text{im } T = R(T) = \{Tx : x \in V\}$$

Definition: Kernel or Nullspace

$$\ker T = N(T) = \{x \in V : Tx = 0\}$$

Remarks

$R(T)$ is a linear subspace of W while $N(T)$ is a linear subspace of V .

T is injective if and only if $N(T) = \{0\}$.

If T is injective, then one has an inverse map $T^{-1} : R(T) \rightarrow V$. T^{-1} is linear.

T is invertible if and only if T is injective and surjective if and only if $N(T) = \{0\}$ and $R(T) = W$.

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Proposition

Let V, W be TVS.

1. a linear operator $T : V \rightarrow W$ is continuous if and only if T is continuous at some $x_0 \in V$.
2. if T is a continuous linear operator, then $N(T) = \ker(T)$ is a closed, linear subspace of V .

Proof of A

(\implies) continuous at all points imply continuous at x_0 .

(\impliedby) Write $f(x) = T(x + x_0 - x_1) - T(x_0 - x_1)$ and assume T is continuous at $x = x_0$.

Then $T(x + x_0 - x_1)$ is continuous at $x = x_1$.

Proof of B

We have that $\ker(T) = \{x \in V : Tx = 0\} = T^{-1}(\{0\})$ where $\{0\}$ is closed and so must be its preimage.

Definition: Bounded Linear Operator

Let V, W be normed spaces with norms $\|\cdot\|_V, \|\cdot\|_W$.

A linear operator $T : V \rightarrow W$ is called bounded if there exists some $c \geq 0$ such that

$$\|Tx\|_W \leq c \cdot \|x\|_V, \quad \forall x \in V$$

Proposition:

A linear operator $T : V \rightarrow W$ (V, W normed spaces) is continuous if and only if it is bounded.

Proof

(\impliedby) We know that $\|Tx\|_W \leq c \cdot \|x\|_V, \forall x$.

Consider $\{x_n\}, x_n \rightarrow a$ in V . Then

$$\lim_{n \rightarrow \infty} \|x_n - a\|_V = 0$$

so $\|Tx_n - Ta\|_W \leq c \cdot \|x_n - a\|_V, \|Tx_n - Ta\|_W = 0$, and $Tx_n \rightarrow Ta$ in W .

(\implies) For every $n \in \mathbb{N}$, find $x_n \in V$ such that

$$\|Tx_n\|_W > n \cdot \|x_n\|_V$$

Then $y_n = \frac{x_n}{\|Tx_n\|}$, since $\|y_n\| = \frac{\|x_n\|}{\|Tx_n\|} < \frac{1}{n}$ it must be $y_n \rightarrow 0$.

Hence, $Ty_n \rightarrow T0 = 0$ (T continuous) $\implies Ty_n = \frac{Tx_n}{\|Tx_n\|}$.

But $\|Ty_n\| = 1$, so $Ty_n \not\rightarrow 0$ a contradiction.

Remark

The following statements are equivalent

- T is continuous.
- T is bounded.
- $Tx_n \rightarrow 0$ whenever $x_n \rightarrow 0$.
- $\{Tx_n\}$ is bounded whenever $\{x_n\}$ is bounded.

Definition: Operator Norm

For V, W normed spaces.

For $T : V \rightarrow W$ a bounded linear operator, we define

$$\|T\| = \sup_{\substack{x \in V \\ x \neq 0}} \frac{\|Tx\|_W}{\|x\|_V}$$

the operator norm of T .

Remark

$\|T\| \in [0, +\infty)$ and it is equal to the smallest $c \geq 0$ such that $\|Tx\|_W \leq c \cdot \|x\|_V, \forall x \in V$.

Indeed, if this holds for some $c \geq 0$, then $\|T\| \leq c$.

Conversely, from the definition $\|Tx\|_W \leq \|T\| \cdot \|x\|_V$.

That is, $\|T\| = \min\{c \geq 0 : \|Tx\|_W \leq c \cdot \|x\|_V, \forall x\}$.

Remark

$$\|T\| = \sup_{\substack{x \in V \\ \|x\|=1}} \|Tx\| = \sup_{\substack{x \in V \\ \|x\| \leq 1}} \|Tx\|$$

Note that

$$\sup_{x \neq 0} \frac{\|Tx\|_W}{\|x\|_V} = \sup_{x \neq 0} \left\| T \left(\frac{x}{\|x\|_V} \right) \right\|_W = \sup_{\|z\|_V=1} \|Tz\|_W$$

Remark

$M(V, W)$ and $L(V, W)$ are linear spaces,

$$(T + S)(x) = Tx + TS$$

$$(\lambda T)(x) = \lambda(Tx)$$

If T, S are continuous, linear operators, then $T + S$ and λT are continuous linear operators.

Further Properties

- $||T|| = 0$ if and only if $T = 0$ (i.e. $Tx = 0, \forall x \in V$).
- $||T + S|| \leq ||T|| + ||S||$, because

$$||(T + S)x||_W = ||Tx + Sx||_W \leq ||Tx||_W + ||Sx||_W \leq ||T|| \cdot ||x||_V + ||S|| \cdot ||x||_V \leq \underbrace{(||T|| + ||S||)}_c \cdot ||x||_V$$

Since $T + S$ is bounded. $\frac{||(T+S)x||_W}{||x||_V} \leq ||T|| + ||S||$, etc.

- $||\alpha T|| = |\alpha| \cdot ||T||$.
- if $T \in L(U, V)$ and $S \in L(V, W)$, then $ST \in L(U, W)$ and

$$||ST|| \leq ||S|| \cdot ||T||$$

Proposition

Let V, W be normed spaces.

Then $L(V, W)$ is a normed space with the operator norm.

If, in addition, W is Banach, then $L(V, W)$ is also Banach.

Proof

Part A

$|| \cdot ||$ is a norm.

Part B

Let W be a Banach space, and let $T_n \in L(V, W)$ be such that $\{T_n\}$ is a Cauchy sequence in the operator norm.

Then, $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $\forall j, k \geq N, ||T_j - T_k|| < \varepsilon$.

So $\forall x \in V, \{T_n x\}$ is Cauchy in W .

$$||T_j x - T_k x|| = ||(T_j - T_k)x|| \leq ||T_j - T_k|| \cdot ||x|| \leq \varepsilon \cdot ||x||$$

By completeness, for every $x \in V, T_n x$ converges in W . Define

$$Tx = \lim_{n \rightarrow \infty} T_n x$$

such that $||Tx - T_n x|| \rightarrow 0$ as $n \rightarrow \infty$.

We need to show that T is a linear operator:

$$T(x + y) = \lim_{n \rightarrow \infty} T_n(x + y) = \lim_{n \rightarrow \infty} T_n x + \lim_{n \rightarrow \infty} T_n y = Tx + Ty.$$

$$T(\lambda x) = \lambda \cdot Tx.$$

We need also show that T is bounded:

$$\frac{||Tx||_W}{||x||_V} = \lim_{n \rightarrow \infty} \frac{||T_n x||_W}{||x||_V} = \liminf_{n \rightarrow \infty} ||T_n||$$

Since $\{T_n\}$ is Cauchy, it is bounded and $\liminf_{n \rightarrow \infty} \|T_n\| \leq c$ for some c .

We have that $\lim_{n \rightarrow \infty} \|Tx - T_n x\| = 0$ such that T_n converges pointwise.

We need that $\lim_{n \rightarrow \infty} \|T - T_n\| = 0$.

For given $\varepsilon > 0$, we find N such that $\forall j, k \geq N, x \in V$:

$$\|T_j x - T_k x\| \leq \varepsilon \cdot \|x\|$$

Then

$$\|T_j x - Tx\| = \|T_j x - T_k x + T_k x - Tx\| \leq \varepsilon \cdot \|x\| + \|T_k x - Tx\|$$

and sending $k \rightarrow \infty$ sends $T_k x - Tx$ to 0.

Therefore, $\|T_j x - Tx\| \leq \varepsilon \cdot \|x\|, \forall j \geq N, \forall x \in V$. It follows that

$$\frac{\|T_j x - Tx\|}{\|x\|} \leq \varepsilon$$

and, taking the supremum over x , that $\|T_j - T\| \leq \varepsilon, \forall j \geq N, \forall x \in V$.

Hence, $\lim_{n \rightarrow \infty} \|T_n - T\| = 0$.

That is, $L(V, W)$ is complete.

Corollary

The dual space of a normed space is a Banach space. Recall $V^* = L(V, \mathbb{F})$, and both \mathbb{R} and \mathbb{C} are complete.

Notation

Read $\dot{+}$ as a direct sum implied to be between components of a larger space.

Read $\text{lin}\{v_1, \dots, v_n\}$ as the linear combinations of v_1, \dots, v_n .

Definition: Codimension

If V is a vector space and W is a subspace, we say that W has codimension n in V if there exists a subspace $\hat{W} \subseteq V$ such that

$$V = W \dot{+} \hat{W}$$

and $\dim(\hat{W}) = n$.

Equivalently, $\dim(V/W) = n, V/W = \text{lin}\{[e_1], \dots, [e_n]\}$ basis and $\hat{W} = \text{lin}\{e_1, \dots, e_n\}$ implies $V = W \dot{+} \hat{W}$.

Proposition:

Let V be a vector space and $\phi \in V^I, \phi \neq 0$. Then $\ker(\phi)$ is a subspace of V of codimension 1.

Proof

$\phi \neq 0$. Find $x_0 \in V$ such that $\phi(x_0) = 1$.

Claim: $V = \ker(\phi) \dot{+} \text{lin}\{x_0\}$.

Indeed, for $x \in V$ write

$$x = \underbrace{x - \phi(x) \cdot x_0}_{\in \ker(\phi)} + \underbrace{\phi(x)}_{\in \text{lin}\{x_0\}} \cdot x_0$$

so

$$\phi(x - \phi(x) \cdot x_0) = \phi(x) - \phi(\phi(x) \cdot x_0) = \phi(x) - \phi(x) \cdot \phi(x_0) = 0$$

and

$\ker(\phi) \cap \text{lin}\{x_0\} = \{0\}$ which means $z = \lambda \cdot x_0 \in \ker(\phi)$. Therefore

$$0 = \phi(\lambda x_0) = \lambda \cdot 1$$

so $\lambda = 0$ and $z = 0$.

Proposition:

Let V be a normed space and $\phi \in V'$.

Then ϕ is bounded if and only if $\ker(\phi)$ is closed in V .

Proof

(\implies) ϕ continuous, as a linear operator, implies $\ker(\phi) = \phi^{-1}(\{0\})$ is closed.

(\impliedby) assume that $\ker(\phi)$ is closed. Then

$$V = \ker(\phi) \dot{+} \text{lin}\{x_0\}$$

for some $x_0 \in V$ and $x_0 \notin \ker(\phi)$.

Without loss of generality, we may assume $\phi(x_0) = 1$.

Claim: $\inf_{x \in \ker(\phi)} \|x_0 - x\| = \text{dist}(\ker(\phi), x_0) > 0$.

Otherwise, there would exist some sequence $\{x_n\} \subseteq \ker(\phi)$ such that $\|x_0 - x_n\| \rightarrow 0$.

From the assumption of closure, this would mean $x_0 \in \ker(\phi)$ a contradiction.

Therefore, $\exists c > 0$ such that $\|x_0 - x\| \geq c, \forall x \in \ker(\phi)$. So

$$\begin{aligned} \|\lambda x_0 - \lambda x\| &\geq c \cdot |\lambda| \\ \|\lambda x_0 - u\| &\geq c \cdot |\lambda|, \quad \forall u \in \ker(\phi) \end{aligned}$$

Write $y \in V$ as $y = \underbrace{-u}_{\in \ker(\phi)} + \underbrace{\lambda x_0}_{\in \text{lin}\{x_0\}}$. So $\phi(y) = 0 + \lambda \cdot \phi(x_0) = \lambda$.

Thus, $\forall x \in V, \|x\| \geq c \cdot |\phi(x)|$ and $|\phi(x)| \leq \frac{1}{c} \cdot \|x\|$ and ϕ is bounded.