Analysis II

January 9, 2024

(Real) Analysis

- Calculus
  - Differential
  - Integral (Riemann)
- Functions and Maps
  - Measure Theory
  - (Lebesgue) Integration
- Topology
  - Completeness (as a metric space)
  - Compactness (Bolzano-Weierstrass theorem [real]) (Arzela-Ascoli)
  - Paracompactness / Metrizable / Baire Category Theorem
  - Algebraic / Combinatoric (continuous maps or functions)

Definition: Cardinality

For sets A, B, Card(A) = Card(B) if there exists a one-to-one correspondence $q : A \leftrightarrow B$.
Counting, labelling, indexing, etc.
Card$(A) \leq$ Card(B) if $A \subset B$ or there exists a one-to-one mapping $A \to B$.

Definition: Countable

If $A \hookrightarrow \mathbb{N}$, then A is countable.

Theorem

The countable union of countable sets is countable.

Proof

Let $A_i = \{a_j\}_{j=1}^{\infty}$, $i = 1, 2, \ldots$.

$$\begin{matrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ \vdots & & & \\ a_{k1} & a_{k2} & a_{k3} & \cdots \end{matrix}$$

Index by diagonalization.

Theorem

The cartesian product of countable sets is countable.

Proof

$X \times Y = \{(x_i, y_j \,:\, x_i \in X, y_j \in Y\}$

$$\begin{array}{cccc} (x_1, y_1) & (x_1, y_2) & (x_1, y_3) & \cdots \\ (x_2, y_1) & (x_2, y_2) & (x_2, y_3) & \cdots \\ \vdots & & & \\ (x_k, y_1) & (x_k, y_2) & (x_k, y_3) & \cdots \end{array}$$

Theorem

Card$\left(2^X\right) >$ Card(X), where $2^X = \{A \subset X\}$ is the power set of X.

Proof

For all $x \in X$, $\{x\} \subset 2^X$, so Card$(X) \leq$ Card$\left(2^X\right)$.
Assume, for sake of contradiction, that Card$(X) =$ Card$\left(2^X\right)$.
Then, by definition, there exists a one-to-one correspondence $\phi : X \leftrightarrow 2^X$.
Set $A = \{x \subset X \,:\, x \notin \phi(x)\}$, and let $a = \phi^{-1}(A)$ (i.e. $A = \phi(a)$).
If $a \in A$, then $a \notin A \subset \phi(a)$; but if $a \notin A$, then $a \in A$, a contradiction.

Theorem

Card$(\mathbb{R}) =$ Card$\left(2^{\mathbb{N}}\right)$.

Topology of the Real Line

Completeness (as a metric space)

$d(a, b) = |a - b|, \quad \forall a, b \in \mathbb{R}$.

1. $x_i \to x$ if $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ such that $|x_i - x| < \varepsilon$, $\forall i \geq n$.

2. $\{x_i\}$ is Cauchy if $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ such that $|x_i - x_j| < \varepsilon$, $\forall i, j \geq n$.

Definition: Open Inteval

(a, b) is an open set on the real line.
There exist interior points for any subset A of real numbers.
$\forall x \in A$, x is interior if $\exists (a, b)$ such that (1) $x \in (a, b)$ and (2) $(a, b) \subset A$.

- Theorem
  The union of open sets is open.
  The intersection of finitely many open sets is open.
  $\varnothing$ and $\mathbb{R}$ are open.

Definition: Limit Point

A limit point $x \in \mathbb{R}$ of a subset A is a limit point in A if for every open neighborhood U of X, $(U \setminus \{x\}) \cap A \neq \varnothing$.

Definition: Closed

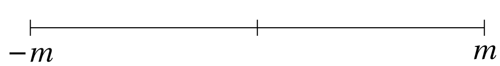
A is closed if A contains all of its limit points.

- Theorem
  A is closed if and only if $A^c = \mathbb{R} \setminus A$ is open.
  - Proof
    A closed $\implies$ A^c open.
    Otherwise, $\exists x \in A^c$ such that for every neighborhood U of X, $(U \setminus \{x\}) \cap A = \varnothing$ which would make it a limit point of A not in A. By assumption, A contains all its limit points so this is a contradiction.
    A^c open $\implies$ A closed.
    For any x a limit point of A, assume otherwise that $x \in A^c$.
    Then there exists some neighborhood U of x such that $U \subset A^c$ (since A^c is open).
    It follows that $(U \setminus \{x\}) \cap A = \varnothing$ and x is not a limit point of A, which is a contradiction.

Definition: Sequential Compactness

A is compact if $\forall \{x_i\}$, $x_i \in A$ there exists a convergent subsequence $\{x_{i_k}\}$ and $x_{i_k} \to x \in A$.

- Theorem: Bolzano-Weierstrass
  For $A \subseteq \mathbb{R}$, A is compact if and only if A is closed and bounded.
  - Proof
    A compact $\implies$ A closed and bounded.
    Assume that A is not bounded from abvove.
    Then there exists a sequence $\{x_i\}$, $x_i \in A$ where $x_{i+1} > x_i + 1$ and $\{x_i\}$ has no convergent subsequences.
    Then compactness implies closedness.
    A closed and bounded $\implies$ A (sequentially) compact.
    Let any $\{x_i\}$, $x_i \in A$.
    Claim: $\forall \{x_i\}$ of reals, if there exists $m \in \mathbb{R}$ such that $|x_i| \leq m$, $\forall m$ then there is some convergent subsequence.

    Divide and conquer: dividing the interval in half necessitates that at least one half contains infinitely many points. Repeat indefinitely.

- Theorem: Heine-Borel)
  $A \subseteq \mathbb{R}$ is (sequentially) compact if and only if any open cover has a finite subcover.
  - Proof
    Heine-Borel Property $\implies$ closed and bounded.
    Assume that A is unbounded, $U_n = (-n, n)$ and $\{U_n\}_{n=1}^{\infty}$ an open cover for $A \subseteq \mathbb{R}$ has no finite subcover.
    Assume A is not closed, then $x \in \dot{A}$ (where $\dot{A}$ is the limit set of A) and $x \notin A$, $U_n \left\{\left(-\infty, x - \frac{1}{n}\right) \cup \left(x + \frac{1}{n}, +\infty\right)\right\}$.
    Then $\{U_n\}$ covers $\mathbb{R} \setminus \{x\} \supset A$ has no finite subcover of A.

A is bounded and closed $\implies$ A is Heine-Borel
Divide and conquer: using open sets with respect to open covers.

Definition: Cantor Set

$C = \{x \in [0,1] :$ the ternary expansion of x has only the digits $\{0,2\}\}$.
Equivalenetly, let $C_0 = [0,1]$, $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$.
Then $C_n = \bigcup_{k=1}^{2^n} C_n^k$ and $C = \bigcap_{n=1}^{\infty} C_n$.
$|C_n| = 2^n \left(\frac{1}{3}\right)^n \to 0$.

Definition: Perfectly Symmetric Sets

Let $\{\xi_n\}$ where $\xi_n \in \left(0, \frac{1}{2}\right)$.
$E_0 = [0,1]$, $E_1 = [0, \xi_1] \cup [1 - \xi_1, 1]$, $E_2 = [0, \xi_1\xi_2] \cup [\xi_1 - \xi_1\xi_2, \xi_1] \cup [1 - \xi_1, 1 - \xi_1 + \xi_1\xi_2] \cup [1 - \xi_1\xi_2, 1]$.
Then the cantor set is given by $\xi_n = \frac{1}{3}$.
$E_n = \bigcup_{k=1}^{2^n} E_n^k$, $|E_n^k| = \xi_1\xi_2\cdots\xi_n$, and $|E_n| = \sum |E_n^k| = 2^n \xi_1\xi_2\cdots\xi_n$.
Therefore, $E = \bigcap_{n=1}^{\infty} E_n$ and we define $|E| = \lim_{n\to\infty} |E_n| = \lim_{n\to\infty} \left(2^n \xi_1\xi_2\cdots\xi_n\right) = \lambda$ where $\lambda \in [0,1)$.
Let

$$2\xi_n = \frac{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n-1}\right)^{n-1}}{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n}\right)^{n}} < 1$$

, then

$$2^n \xi_1 \cdots \xi_n = \frac{1}{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n}\right)^{n}} \to \lambda.$$

Proof

$\lim_{n\to\infty} \left(\left(1 + \frac{x}{n}\right)^{n/x}\right)^x = e^x$, then $\lim_{y\to 0} (1+y)^{1/y} = e$, $\log(1+y)^{1/y} = \frac{\log(1+y)}{y} \underset{y\to 0}{\to} 1$.
Observe that

$$\left(\frac{\log(1+y)}{y}\right)' = \frac{\frac{y}{1+y} - \log(1+y)}{y^2} = \left(1 + \frac{1}{1+y} - \log(1+y)\right)' = \frac{1}{(1+y)^2} - \frac{1}{1+y} = -\frac{y}{(1+y)^2} < 0$$

Theorem

Cantor sets and perfect symmetric sets are closed, perfect, uncountable, and nowhere dense.

January 11, 2024

Last Week

Cardinality.
Topology of the reals.

- Cantor (perfect symmetric sets)

  $C_0 = [0,1]$
  $C_1 = [0,1/3] \cup [2/3,1]$
  $C_2 = [0,1/9] \cup [2/9,3/9] \cup [6/9,7/9] \cup [8/9,1]$
  $C_n = \bigcup_{n=1}^{2^n} C_n^k$
  $|C_n^k| = \left(\frac{1}{3}\right)^n$
  $C = \bigcap_{n=1}^{\infty} C_n$
  $|C_n| = 2^n \frac{1}{3^n} = \left(\frac{2}{3}\right)^n \implies |C| = \lim_{n\to\infty} |C_n| = 0$
  Closed, no interior points and uncountable.

- Perfect Symmetric Sets

  $\{\xi_k\} \in \left(0, \frac{1}{2}\right)$
  $E_0 = [0,1]$
  $E_1 = [0,\xi_1] \cup [1-\xi_1, 1]$
  $E_2 = [0,\xi_1\xi_2] \cup [\xi_1 - \xi_1\xi_2, \xi_1] \cup [1-\xi_1, 1-\xi_1+\xi_1\xi_2] \cup [1-\xi_1\xi_2, 1]$
  $E_n = \bigcup_{n=1}^{2^n} E_n^k$
  $|E_n^k| \xi_1\xi_2 \cdots \xi_n$
  $|E_n| = 2^n \xi_1\xi_2\cdots\xi_n$
  $2\xi_n = \dfrac{\left(1+\frac{\log\left(\frac{1}{n}\right)}{n-1}\right)^{n-1}}{\left(1+\frac{\log\left(\frac{1}{n}\right)}{n}\right)^{n}} < 1$
  $|E_n| = \dfrac{1}{\left(1+\frac{\log\left(\frac{1}{n}\right)}{n}\right)^{n}}$
  $|E| = \lim_{n\to\infty} |E_n| = \frac{1}{e^{\log\left(\frac{1}{n}\right)}} = \lambda, \quad \lambda \in (0,1)$

Volterra's Function

$$\phi(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

IMAGE HERE - graph of phi(x)

$$\phi'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$f(x) = \begin{cases} 0 & x \in E \\ \phi(x-a) & x \in (a, a+y) \\ -\phi(b-x) & x \in (b-y, b) \\ \phi(y) & x \in (a+y, b-y) \end{cases}, \quad (a,b) \in E^c$$

IMAGE HERE - f interval (a,b)

Propositions

1. $f'(x) = 0$ for $x \in E$.
2. $f'(x)$ discontinuous on E.
3. f' exists on $[0,1]$ and is bounded.

Since $|E| > 0$, $f'(x)$ is not Riemann integrable and, therefore, the fundamental theorem of calculus does not apply.

Lebesgue Outer Measure

$|(a,b)| = b - a$.
Let $A \subseteq \mathbb{R}$, then $m^*(A) = \inf\{\sum_{n=1}^{\infty} I_n \;:\; A \subseteq \bigcup_{n=1}^{\infty}\}$
Question: $m^*(A \cup B) \stackrel{?}{=} m^*(A) + m^*(B)$ for $A \cap B \neq \varnothing$?

Properties

1. $A \subseteq B \implies m^*(A) \leq m^*(B)$.
2. $m^*(\varnothing) = 0$.
3. If I is an interval, then $m^*(I) = |I|$.
4. If $\{A_i\}$ is countable, $m^*\left(\bigcup A_i\right) \leq \sum m^*(A_i)$.

- Proof of 4
  $\forall A_i$, $\exists\{I_n\}$ open intervals such that $\sum_n |I_n| < m^*(A_i) + \frac{\varepsilon}{2^i}$.
  Then $\bigcup_i \bigcup_n I_n^i \supset \bigcup_i A_i$, and $\sum_{n,i} |I_n^i| = \sum_i \left(\sum_n |I_n^i|\right) \leq \sum_i \left(m^*(A_i) + \frac{\varepsilon}{2^i}\right)$.
  - Corollary

    If A is countable, then $m^*(A) = 0$.
    Thus, by contraposition, every interval is uncountable.

Proposition

For $A \subseteq \mathbb{R}$, $\forall \varepsilon > 0$, $\exists U$ open such that $A \subseteq U$ and $m^*(U) \leq m^*(A) + \varepsilon$.

Corollary

There exists G in the intersection of countable open sets such that $m^*(G) = m^*(A)$ and $G \supseteq A$.

Caratheodory Criteria

If $\forall E$, $m^*(E \cap A) + m^*(E \cap A^c) = m^*(E)$, then A is Lebesgue measurable.

- Remark: $m^*(E \cap A) + m^*(E \cap A^c) \leq m^*(E) \leq +\infty$

Propositions

1. If A is measurable, then A^c is measurable.
2. $m^*(A) = 0$, then A is measurable.
3. If A, B are measurable, then $A \cup B,\ A \cap B,\ A \setminus B$ are measurable.
4. If $\{A_i\}_{i=1}^k$ are disjoint and measurable, then $m^*\left(\bigcup_{i=1}^k A_i\right) = \sum_{i=1}^k m^*(A_i)$.

- Proof of 3

$$\begin{aligned} m^*(E \cap (A \cup B)) + m^*(E \cap (A \cup B)^c) &= m^*((E \cap A) \cup (E \cap B)) + m^*(E \cap A^c \cap B^c) \\ &= m^*(E \cap A) + m^*((E \cap A^c) \cap B) + m^*((E \cap A^c) + B^c) \\ &\le m^*(E) \end{aligned}$$

Since o$(A \cap B)^C = A^c \cup B^c$, this holds from before; similarly, $A \setminus B = A \cap B^c = A^c \cup B$.
If A, B disjoint, then

$$\begin{aligned} m^*(A \cup B) &= m^*(E \cap A) + m^*(E \cap A^c) \\ &= m^*(A) + m^*(B) \end{aligned}$$

Theorem

If $\{A_i\}$ is a countable collection of disjoint and measurable sets, then

1. $\bigcup_i A_i$ is measurable.
2. $m^*\left(\bigcup_i A_i\right) = \sum_i m^*(A_i)$.

Proof of 1

Want to show:

$$m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) + m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right) \le m^*(E)$$

By assumption, since the measure of E is finite, $m^*\left(E \cap \bigcup_{i=1}^{\infty} A_i\right) < +\infty$.
Claim: $\forall \varepsilon > 0$, $\exists k$ such that
Therefore $m^*\left(E \cap \bigcup_{i=1}^{k} A_i\right) \ge m^*\left(E \cap \bigcup_{i=1}^{\infty} A_i\right) - \varepsilon$.

$$m^*(E) \le m^*\left(E \cap \bigcup_{i=1}^{k} A_i\right) + \varepsilon + m^*\left(E \cap \left(\bigcup_{i=1}^{k} A_i\right)^c\right) \le m^*(E) + \varepsilon$$

Proof of 2

We have shown $m^*\left(\bigcup_i A_i\right) \le \sum_{i=1}^{\infty} m^*(A_i)$.
Assume $m^*\left(\bigcup_i A_i\right) < +\infty$, then

$$\sum_{i=1}^{k} m^*(A_i) = m^*\left(\bigcup_{i=1}^{k} A_i\right) \le m^*\left(\bigcup_{i}^{\infty} A_i\right) \implies \sum_{i=1}^{\infty} m^*(A_i) \le m^*\left(\bigcup_{i}^{\infty} A_i\right)$$

January 16, 2024

Office Hours Tuesday / Thursday 10 AM - 11:30 AM
A note on notation: Latin characters are to be understood as countable indecies; greek as possible uncountable.

Lebesgue Outer Measure

$A \subset \mathbb{R}$
$m^*(A) = \inf\{\sum_{i=1}^{\infty} |I_i| \,:\, \bigcup_{i=1}^{\infty} I_i \supset A,\ I_i \text{ open intervals}\}$

Properties

1. $A \subset B \implies m^*(A) \leq m^*(B)$.

2. $m^*(\varnothing) = 0$.

3. $m^*(I) = |I|$ for I an interval.

4. Countable Subadditivity: $\{A_i\}_{i=1}^{\infty} \implies m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} m(A_i)$.

5. $\forall A \subset \mathbb{R}$, $\forall \varepsilon > 0$, $\exists$ open neighborhood $U \supseteq A$ such that $m^*(U) \leq m^*(A) + \varepsilon$.

6. $\exists G \in \bigcap_{n=1}^{\infty} U_n$, U_n open, $U_n \supseteq A \implies G \supseteq A$, such that $m^*(G) = m^*(A)$.

Measurable (Caratheodory Criterion)

$\forall A \subseteq \mathbb{R}$ is Lebesgue measurable if

$$m^*(A) = m^*(E \cap A) + m^*(E \cap A^c)$$

Essentially, $m^*(E \cap A) + m^*(E \cap A^c) \leq m^*(E) \leq +\infty$.

- Propositions
  1. A measurable $\implies A^c$ measurable.

  2. $m^*(A) = 0 \implies A$ measurable.

  3. $\{A_i\}_{i=1}^{\infty}$ countable with A_i measurable, then

     (a) $\bigcap_{i=1}^{\infty} A_i$ are measurable.

     (b) Moreover, $A_i \cap A_j = \varnothing \implies m^*\left(\bigcup_{i=1}^{\infty}(A_i\right) = \sum_{i=1}^{\infty} m^* A_i)$.

     (c) A, B measurable $\implies A \cup B$, $A \cap B$, $A \setminus B$ measurable.

     (d) $A \cap B = \varnothing \implies m^*(A \cup B) = m^*(A) + m^*(B)$.

     (e) $\{A_i\}_i^{\infty}$ with A_i measurable, then $\bigcup_{i=1}^{\infty} A_i$ is measurable and $A_i \cap A_j \varnothing \implies m^*\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m^*(A_i)$.

  – Proof of e $\forall E \subset \mathbb{R}$, $m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) + m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right)$.
    Claim: $m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) = \sum_{i=1}^{\infty} m^*(E \cap A_I)$ for $A_i \cap A_j = \varnothing$.

Then, $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$,

$$m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) = \sum_{i=1}^{\infty} m^*(E \cap A_i) \leq \sum_{i=1}^{n} m^*(E \cap A_i) + \varepsilon$$
$$\implies m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) + m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right) \leq m^*\left(E \cap \left(\bigcup_{i=1}^{n} A_i\right)\right) + m^*\left(E \cap \left(\bigcup_{i=1}^{n} A_i\right)^c\right) + \varepsilon \leq m^*(E) + \varepsilon$$
$$\implies \bigcup_{i=1}^{\infty} A_i \text{ measurable}$$

Proof of Claim:
Step 1: A, B measurable and $A \cap B = \varnothing$. Since A is measurable,

$$\begin{aligned} m^*(E \cap (A \cup B)) &= m^*((E \cap (A \cup B)) \cap A) + m^*((E \cap (A \cup B)) \cap A^c) \\ &= m^*(E \cap A) + m^*(E \cap A^c) \end{aligned}$$

For $\{A_i\}_{i=1}^{\infty}$, $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} A_i'$ with $A_1 = A_1'$ and $A_i' = A_i \setminus \bigcup_{k=1}^{i-1} A_k$, $\forall i \geq 2$.
Therefore $A_i' \cap A_j' = \varnothing$ and A_i' is measurable.

$$m^*\left(\bigcup_{i=1}^{n} A_i\right) \leq m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} m^*(A_i)$$
$$m^*\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} m^*(A_i) \leq m^*\left(\bigcup_{k=1}^{\infty} A_k\right) < +\infty \implies \sum_{i=1}^{\infty} m^*(A_i) \leq m^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{i=1}^{\infty} m^*(A_i)$$

Sigma Algebra and Borel Sets

Definition: Sigma Algebra

Let $S \subset 2^X$ for some set X. Then S is said to be a σ-algebra if

1. $\varnothing \in S$.
2. $A^c \in S$ if A^c.
3. $\bigcup_{i=1}^{\infty} A_i \in S$ if $A_i \in S$.
    - Equivalently, $\bigcap_{i=1}^{\infty} A_i \in S$ if $A_i \in S$.

Theorem:

The collection $\mathcal{L}$ of all Lebesgue measurable sets is a σ-algebra.

Definition: Borel Set

Let B be the σ-algebra generated by open sets of reals (i.e. the smallet σ-algebra containing all open sets of reals).
Then $b \in B$ is called a Borel set.

Remark

B is generated by $\{(a,+\infty) : a \in \mathbb{R}\}$.

1. $(a,+\infty)^c = (-\infty, a]$.
2. $\bigcap_{n=1}^{\infty}\left(a-\frac{1}{n},+\infty\right) = [a,+\infty)$.
3. $[a,+\infty)^c = (-\infty, a)$.
4. $(-\infty, b) \cap (a,+\infty) = (a,b)$.
5. $(-\infty, b] \cap [a,+\infty) = [a,b]$.

Theorem:

Any Borel set is Lebesgue measurable.

Proof

It suffices to demonstrate that $(a,+\infty)$ is measurable $\forall a \in \mathbb{R}$.
$\forall E \subset \mathbb{R}$, we want to show that $m^*(E \cap (a,+\infty)) + m^*(-\infty, a]) \leq m^*(E)$.
Then, $\forall \varepsilon > 0$, $\exists \mathcal{C} = \{I_i\}$ with I_i open intervals such that $\sum_{I_i \in \mathcal{C}} |I_i| \leq m^*(E) + \varepsilon/2$. Set

$$\begin{aligned}
\mathcal{C}^{\ell} &= \{I \in \mathcal{C} : x < a, \forall x \in I\} \\
\mathcal{C}^{r} &= \{I \in \mathcal{C} : x > a, \forall x \in I\} \\
\mathcal{C}^{m} &= \{I \in \mathcal{C} : a \in I\} = \{I_k\}
\end{aligned}$$

Then A$\mathcal{C} = \mathcal{C}^{\ell} \cup \mathcal{C}^{r} \cup \mathcal{C}^{m}$.
$\forall I_k \in \mathcal{C}^m = \{I_k\}$, $I_k = (c_k, d_k)$ for some $c_k, d_k \in \mathbb{R}$, define

$$\begin{aligned}
I_k^{\ell} &= \left(c_k, a + \frac{\varepsilon}{2^{k+1}}\right) \\
I_k^{r} &= (a, d_k)
\end{aligned}$$

Let $\mathcal{C}^m = \{I_k^{\ell}\} \cup \{I_k^{r}\} = \overline{\mathcal{C}}^{m\ell} \cup \overline{\mathcal{C}}^{mr}$. Then

$$\begin{aligned}
&\mathcal{C}^{\ell} \cup \overline{\mathcal{C}}^{m\ell} \text{ covers } E \cap (-\infty, k] \\
&\mathcal{C}^{r} \cup \overline{\mathcal{C}}^{mr} \text{ covers } E \cap (k, +\infty) \\
&\mathcal{C}^{\ell} \cup \mathcal{C}^{r} \cup \mathcal{C}^{m} \text{ covers } E
\end{aligned}$$

Observe that

$$|I_k^{\ell}| + |I_k^{r}| \leq |I_k| + \frac{\varepsilon}{2^{k+1}}$$

Therefore

$$
\begin{aligned}
m^*(E \cap (a, +\infty)) &\leq \sum_{I \in \mathcal{C}^R + \overline{\mathcal{C}}^{mr}} |I| \\
m^*(E \cap [-\infty, a]) &\leq \sum_{I \in \mathcal{C}^\ell + \overline{\mathcal{C}}^{m\ell}} |I|
\end{aligned}
$$

Therefore

$$
\begin{aligned}
m^*(E \cap (a, +\infty)) + m^*(E \cap (-\infty, a]) &\leq \sum_{I \in \mathcal{C}^r \cup \overline{\mathcal{C}}^{mr}} |I| + \sum_{I \in \mathcal{C}^\ell |I| \cup \overline{\mathcal{C}}^{m\ell}} |I| \\
&= \sum_{I \in \mathcal{C}^r} |I| + \sum_{I \in \mathcal{C}^\ell} |I| + \sum_k \left(|I_k^\ell| + |I_k^r| \right) \\
&\leq \sum_{I \in \mathcal{C}} |I| + \sum_{k=1}^{+\infty} \frac{\varepsilon}{2^{k+1}} \\
&\leq m^*(E) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&\leq m^*(E) + \varepsilon
\end{aligned}
$$

Lebesgue Measurable vs Borel

Theorem

The following statements are equivalent

1. A is measurable.
2. $\forall \varepsilon > 0$, $\exists U$ open, $U \supset A$ such that $m(U \setminus A) < \varepsilon$.
3. $\forall \varepsilon > 0$, $\exists C$ closed, $C \subset A$ such that $m(A \setminus C) < \varepsilon$.
4. $\forall A \in \mathbb{R}$, $\exists \bigcap_{n=1}^{\infty} U_n = F \in B$, U_n open, $U_n \supset A$ such that $F \supset A$ and $m(F \setminus A) = 0$.
5. $\exists \{C_n\}$, C_n closed and $C_n \subset A$ such that $G = \bigcup_{n=1}^{\infty} C_n \subset A$ and $m(A \setminus G) = 0$.

Corollary

Every measurable set is the union of a Borel set and a measure zero set.

Proof 1 Implies 2

Step 1: if $m(A) < \infty$, then for $\varepsilon > 0$, $\exists U$ open and $U \supset A$, then

$$
m(U) \leq m(A) + \varepsilon \iff m(U \setminus A) = m(U) - m(A) \leq \varepsilon
$$

Step 2: let $A_n = A \cap (-n, n)$, $n \in \mathbb{N}$.
Then $m(A_n) \leq 2n < +\infty$.
For ech A_n, $\exists U_n$ open with $U_n \supset A_n$ and $m(U_n \setminus A_n) < \frac{\varepsilon}{2^n}$

Let $U = \bigcup_{n=1}^{\infty} U_n$ and $A = \bigcup_{n=1}^{\infty} A_n$.
Now verify that

$$m(U \setminus A) = m\left(\bigcup_{n=1}^{\infty} U_n \setminus \bigcup_{n=1}^{\infty} A_n\right) = m\left(\bigcup_{n=1}^{\infty} U_n \setminus A_n\right) \le \sum_{n=1}^{\infty} m(U_n \setminus A_n) \le \varepsilon$$

Proof 2 Implies 3

Write

$$A \setminus C = A \cap C^c = C^c \cap A = C^c \setminus A^c$$

Apply (2).

Proof 3 Implies 4

U_n comes from 2.

Proof 4 Implies 5

Follows from 4.

Proof 5 Implies 1

$A = G \cup (A \setminus G) \implies A$ is measurable.

Example: Non-measurable Set

Define $x \sim y$ if $x - y \in \mathbb{Q}$, $\forall x, y \in \mathbb{R}$.
Let $A = \{x \in (0,1) : x$ is a representative of each class $\mathbb{R}/\sim\} \subset (0,1) \subset \mathbb{R}$.
Claim: A is not Lebesgue measurable.
Let $(-1,1) \supset S = \bigcup_{r \in \mathbb{Q} \cap (0,1)} (A + r) \supset (0,1)$, and observe that $\mathbb{Q} \cap (0,1)$ is countable.
So $(A + r) \cap (A + s) = \varnothing$ for $s \neq r$.
Then $1 < m(S) < 2$, so $m(A) = 0$ and $m(A) > 0$ are both contradictions.

January 18, 2024

Abstract measure theory.

Definition: Topological Space

A set X equipped with a collection of subsets $\tau \subset 2^X$ where τ is a topology if

1. $\varnothing, X \in \tau$
2. Union of subsets in τ remains in τ.
3. Intersection of finitely many subsets in τ remains in τ.

Any subset of τ is called an open set of X.

Definition: Measure Space

For a set X with $\Lambda \subset 2^X$ a σ-algebra such that

1. $\varnothing \in \Lambda$
2. $A^c \in \Lambda$ if $A \in \Lambda$.
3. $\bigcup_{i=1}^{\infty} A_i \in \Lambda$ if $A_i \in \Lambda$.
4. Remark: Borel Sigma Algebra

The σ-algebra generated by τ for a topological space (X, τ).
The measure space (X, Λ, μ), $\Lambda \subset 2^X$ a σ-algebra equipped with set function $\mu : \Lambda \to [0, +\infty]$ such that

1. $\mu(\varnothing) = 0$
2. $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m(A_i)$ for $A_i \in \Lambda$ and $A_i \cap A_j = \varnothing$ for all $i \neq j$ (countable additivity).

Proposition: Monotonicity

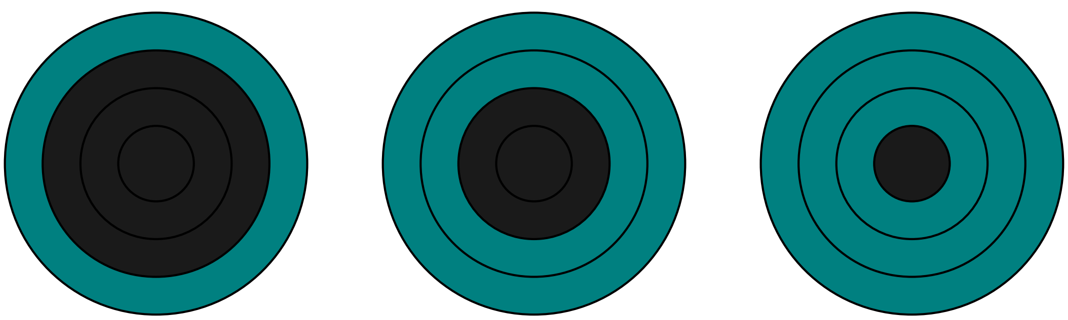
$A, B \in \Lambda,\ A \subseteq B \implies \mu(A) \leq \mu(B).$

Proposition: Countable Subadditivity

$\mu(\bigcup A_i) \leq \sum \mu(A_i)$ if $A_i \in \Lambda$

Proposition: Monotone Convergence

Given $A_i \subset \Lambda$ such that $A_i \subset A_{i+1}$ where $A = \bigcup_{i=1}^{\infty} A_i$, then $\mu(A_i) \to \mu(A)$.
Similarly, if $A_i \supset A_{i+1}$ such that $A = \bigcap_{i=1}^{\infty} A_i$, then $\mu(A_i) \to \mu(A)$ if $\mu(A_k) < +\infty$ for some $k = 1, 2, 3, \ldots$.



Given $A_i' = \begin{cases} A_1 & i = 1 \\ A_i \setminus \bigcup_{j=1}^{i-1} A_j & i > 1 \end{cases}$, $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} A_i'$ and

$$\mu(A) \sum_{i=1}^{\infty} A_i' = \lim_{n\to\infty} \sum_{i=1}^{\infty} \mu(A_i')$$

and

$$\sum_{i=1}^{n} \mu(A_i') = \mu(A_1) + (\mu(A_2) - \mu(A_1)) + (\mu(A_3) - \mu(A_2)) + \cdots + (\mu(A_n) - \mu(A_{n-1})) = \mu(A_n)$$

Similarly, $A_1 \setminus A = \bigcup_{i=2}^{\infty}(A_i \setminus A_{i-1})$ where $\mu(A_1) < +\infty$ gives

$$\mu(A_1) - \mu(A) = \mu(A_1) + \sum_{i=2}^{\infty}(\mu(A_i) - \mu(A_{i-1})) = \lim_{n\to\infty}\mu(A_n)$$

Definition: Complete Measure Space

A measure space (X, Λ, μ) is complete if $\forall A \in \Lambda$ with $\mu(A) = 0$, then $\forall B \subset A$ and $B \in \Lambda$.

Example

The Lebesgue measure space on the reals $(\mathbb{R}, \mathcal{L}, m)$ is complete.

Theorem: Completion of a Measure Space

Given a measure space (X, Λ, μ), then there exists $(X, \overline{\Lambda}, \overline{\mu})$ such that

1. $\Lambda \subset \overline{\Lambda}$.
2. If $A \in \Lambda$, then $\overline{\mu}(A) = \mu(A)$.
3. $(X, \overline{\Lambda}, \overline{\mu})$ is complete.

Proof (Construction)

Let $\overline{\Lambda} = \{A \cup Z \,:\, A \in \Lambda, \exists D \in \Lambda, m(D) = 0, Z \subset D\}$ and $\overline{\mu}(A \cup Z) := \mu(A)$.
Verify:

1. $\overline{\Lambda}$ is a σ-Algebra.
   (a) If $A \cup Z \in \overline{\Lambda}$, then $(A \cup Z)^c \in \overline{\Lambda}$.
   (b) If $A_i \cup Z_i \in \overline{\Lambda}$, then $\bigcup(A_i \cup Z_i) \in \overline{\Lambda}$.
2. $\overline{\mu}$ is a well-defined measure on $\overline{\Lambda}$.
3. $(X, \overline{\Lambda}, \overline{\mu})$ is complete.

- Proof of 1
  Given $A \in \Lambda$ and $Z \subset D$ where $\mu(D) = 0$ and $D \in \Lambda$, we know $D^c \subset Z^c$ and $Z^c = D^c \cup (Z^c \cap D)$. Therefore

$$(A \cup Z)^C = A^c \cap Z^c = A^c \cap (D^c \cup (Z^c \cap D)) = (A^c \cap D^c) \cup (A^c \cap Z^c \cap D) \in \overline{\Lambda}$$

  Since $A^c \cap D^c \in \Lambda$ and $A^c \cap Z^c \cap D \in D$
  Since $\bigcup A_i \in \Lambda$ and $\bigcup Z_i \subset \bigcup D_i$,

$$\bigcup_{i=1}^{\infty}(A_i \cup Z_i) = \left(\bigcup_{i=1}^{\infty} A_i\right) \cup \left(\bigcup_{i=1}^{\infty} Z_i\right) \in \overline{\Lambda}$$

- Proof of 2
  Given $A_1 \cup Z_1 = A_2 \cup Z_2$, $A_1 \subset A_2 \cup Z_2 \subset A_2 \cup D_2$ implies $\mu(A_1) \leq \mu(A_2)$.
  Then, $\mu(A_2) \leq \mu(A_1) \implies \mu(A_1) = \mu(A_2)$. So $\overline{\mu}$ is well defined.
  Given $\{A_i \cup Z_i\}$ with $(A_i \cup Z_i) \cap (A_j \cup Z_j) = \varnothing$ for all $i \neq j$,

$$\overline{\mu}\left(\bigcup_{i=1}^{\infty}(A_i \cup Z_i)\right) = \overline{\mu}\left(\left(\bigcup_{i=1}^{\infty} A_i\right) \cup \bigcup_{i=1}^{\infty} Z_i\right) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty} \overline{\mu}(A_i \cup Z_i)$$

  So $\overline{\mu}$ is countably additive and therefore a measure.

Borel Measure and Radon Measure

Given a measure space (X, Λ, μ) and an underlying topology (X, τ),

Definition: Borel Measure

μ is a Borel measure if all borel sets $\tau \subset \Lambda$.

Definition: Locally Finite Measure

μ is locally finite if $\forall x \in X$, $\exists U \subset X$ a neighborhood such that $\mu(U) < +\infty$.

Definition: Borel Regularity

μ is Borel regular if $\forall A \in \Lambda$, $\exists B$ a Borel set such that $B \supseteq A$ and $\mu(B) = \mu(A)$.

Definition: Radon Measure

μ is a Radon measure if

1. it is a Borel measure.
2. $\mu(K) \leq +\infty$ for K compact.
3. $\mu(V) = \sup\{\mu(K) : K \subset V, K \text{ compact}\}$, V open.
4. $\mu(A) = \inf\{\mu(V) : A \subset V, V \text{ open}\}$, $\forall A \in \Lambda$.

- Example 1
  Lebesgue measure.

- Example 2
  Point charge: $\mu(\{x\}) = 1$ and $\mu(A) = 0$ if $x \notin A$.

Theorem:

Let (X, Λ, μ) be a Borel regular measure space where the underlying topology (X, τ) is a metric space. Then

1. For $A \in \Lambda$ with $\mu(A) < +\infty$, $\forall \varepsilon > 0$, $\exists C \subseteq A$ closed such that $\mu(A \setminus C) < \varepsilon$.

2. For $A \in \Lambda$, $\exists\{V_i\}$ open sets such that $A \subset \bigcup_{i=1}^{\infty} V_i$ and $\mu(V_i) < +\infty$. Then $\forall \varepsilon > 0$, $\exists U$ open with $A \subset U$ and $\mu(U \setminus A) < \varepsilon$.

Proof

Given $\mu(A) < +\infty$, $\nu(B) = \mu(B \cap A) < +\infty$, $\forall B \in \Lambda$ and (X, Λ, ν).
Let $F = \{B \in \Lambda : \forall \varepsilon > 0, \exists C \subset B \text{ closed, with } \nu(B \setminus C) < \varepsilon\}$.
Note that closed sets are in F.
Claim 1: the Borel σ-algebra is in F.
Claim 2: if $A_i \in F$, $\bigcup A_i, \bigcap A_i \in F$.
Given claim 2, $\forall U$ open, U^c is closed. Then $U_\varepsilon = \{x \in U : \text{dist}(x, U^c) \leq \varepsilon\}$ is closed and, therefore, $U = \bigcup_{i=1}^{\infty} U_{1/i}$.
So, given $A_i \in F$, $\exists C_i \subset A_i$ closed where $\nu(A_i \setminus C_i) < \varepsilon/2^{i+1}$. We want to show that $\nu\left(\bigcap A_i \setminus \bigcap C_i\right) < \varepsilon$.
Then, for $x \in \bigcap A_i \setminus \bigcap C_i$, $x \in A_i$ for all i and $x \notin C_{i_0}$ for some i_0.
Therefore $x \in A_{i_0}$, $x \notin C_{i_0}$, and $x \in A_{i_0} \setminus C_{i_0}$. It follows that

$$\bigcap_{i=1}^{\infty} A_i \setminus \bigcap_{i=1}^{\infty} C_i \subset \bigcup_{i=1}^{\infty} (A_i \setminus C_i)$$
$$\nu\left(\bigcap_{i=1}^{\infty} A_i \setminus \bigcap_{i=1}^{\infty} C_i\right) \leq \sum_{i=1}^{\infty} \nu(A_i \setminus C_i) < \varepsilon$$

Therefore

$$\nu\left(\bigcup_{i=1}^{\infty} A_i \setminus \bigcup_{i=1}^{n} C_i\right) \to \nu\left(\bigcup_{i=1}^{\infty} A_i \setminus \bigcup_{i=1}^{\infty} C_i\right) \leq \nu\left(\bigcup_{i=1}^{\infty} (A_i \setminus C_i\right) < \frac{\varepsilon}{2}$$

so $\exists N >> 1$ such that $\nu\left(\bigcup_{i=1}^{\infty} \setminus \bigcup_{i=1}^{N} C_i < \varepsilon\right)$ with $\bigcup_{i=1}^{N} C_i$ closed.

Restatement

For A Borel,

$$\varepsilon > \nu(A \setminus C) = \mu((A \setminus C) \cap A) = \mu(A \setminus C)$$

January 23, 2024

Review - Abstract Measure

Given (X, Λ, μ) where $\Lambda \subseteq 2^X$ is a σ-algebra, $\mu : \Lambda \to [0, +\infty]$

1. $\mu(\varnothing) = 0$.

2. $m\left(\bigcup A_i\right) = \sum \mu(A_i)$, $A_i \cap A_j = \varnothing$.

Properties of a Measure

Monotonicity

$\mu(A) \subseteq \mu(B)$, $A, B \in \Lambda$, $A \subseteq B$

Countable Subadditivity

$\mu(\bigcup A_i) \leq \sum \mu(A_i)$

Monotone Convergence

$A_i \subset A_{i+1},\ A_i \to \bigcup A_i \implies \mu(A) = \mu(\bigcup A_i)$.
$A_i \supset A_{i+1},\ A_i \to \bigcap A_i \implies \mu(A_i) \to \mu(\bigcap A_i)$ if $\mu(A_1) < \infty$

- Example
  $A_n = (n, +\infty)$ gives $\bigcap A_n = \varnothing$

Completeness of a Measure

(X, Λ, μ) is complete if $\forall A \in \Lambda$ with $\mu(A) = 0$, then $\forall B \in \Lambda$ if $B \subseteq A$.

Theorem:

Given (X, Λ, μ), there exists $(X, \overline{\Lambda}, \overline{\mu})$ such that $\Lambda \subset \overline{\Lambda}$ and $\overline{\mu}(A) = \mu(A)$ if $A \in \Lambda$.

$$\overline{\Lambda} = \{A \cup Z \,:\, A \in \Lambda, Z \subset D, D \in \Lambda \text{ with } \mu(D) = 0\}$$

$$\overline{\mu}(A \cup Z) = \mu(A)$$

$(X, \overline{\Lambda}, \overline{\mu})$ is complete.

Measure Space with Topology

Given a topological space (X, τ), a measure space (X, Λ, μ)

Definition: Locally Finite

The measure μ is locally finite if $\forall x \in X$, there exists an open neighborhood U of x such that $U \in \Lambda$ and $\mu(U) < +\infty$.

Definition: Borel Measure

μ is a Borel measure if the Borel σ-algebra generated by τ, $\mathcal{B}$, is a subset of Λ.

Definition: Borel Regular

$\forall A \in \Lambda$, $\exists B \in \mathcal{B}$ such that $B \supset A$ and $\mu(B) = \mu(A)$.

Definition: Radon Measure

1. Borel.
2. $\mu(K) < +\infty$ for K compact.
3. $\mu(V) = \sup\{\mu(K) \,:\, K \text{ compact}, K \subset V\}$, $\forall V$ open.
4. $\mu(A) = \inf\{\mu(V) \,:\, V \text{ open}, A \subset V\}$, $\forall A \in \Lambda$.

Theorem:

If X is a metric space equipped with a Borel regular (X,Λ,μ), then

1. $\forall A \in \Lambda$, $\mu(A) < +\infty$, $\forall \varepsilon > 0$, $\exists C$ closed where $C \subset A$ and $\mu(C \setminus A) < \varepsilon$.

2. If $\exists \{V_i\}$, V_i open and $\mu(V_i) < +\infty$, and $A \in \Lambda$ with $A \subset \bigcup V_i$, then $\exists U$ open such that $A \subset U$ and $\mu(U \setminus A) < \varepsilon$.

Proof of 1

Define $\nu(B) = \mu(B \cap A)$ such that (X,Λ,ν) is a new measure space.
Define $F = \{B \in \Lambda : \forall \varepsilon > 0, \exists C \text{ closed}, C \subset B, \nu(B \setminus C) < \varepsilon\}$, all closed sets in F.
Claim 1: $\bigcap A_i, \bigcap A_i \in F$ if $A_i \in F$.
Claim 2: U is open.
$U = \bigcup U_i$, $U_i = \left\{x \in U : \text{dist}(x, U^c) \le \frac{1}{i}\right\}$, therefore $\mathcal{B} \subset F$.
IMAGE HERE - 1
If A is Borel, then $\forall \varepsilon > 0$, $\exists C$ closed with $C \subset A$ and $\mu(A \setminus C) < \varepsilon$.
To finish, $\forall A \subset \Lambda$ by Borel Regularity of μ, $\exists B \in \mathcal{B}$ such that $B \supset A$ and $\mu(B) = \mu(A)$.
Note also that this requires $\mu(B \setminus A) = 0$ since $\mu(A) < +\infty$.
IMAGE HERE - 2
Then $B \setminus A \in \Lambda$, $\exists D \in \mathcal{B}$ such that $D \supset B \setminus A$ and $\mu(D) = \mu(B \setminus A) = 0$. Then

$$\begin{aligned} B \cap A^c = B \setminus A &\subset D \\ (B \cap A^c)^c &\supset D^c \\ B \cap (B^c \cup A) &\supset D^c \cap B \\ A &\supset B \setminus D \end{aligned}$$

$$A \setminus (B \setminus D) = A \cap (B \cap D^c)^c = A \cap (B^c \cup D = \overbrace{(A \cap B^c)}^{\varnothing} \cup A \cap D = A \cap D \subset D$$

Therefore $B \setminus D \subset A$, and $\mu(A \setminus (B \setminus D)) = 0$.
$B \setminus D \in \mathcal{B}$, $\forall \varepsilon > 0$, $\exists C$ closed such that $C \subset B \setminus D \subset A$, $\mu((B \setminus D) \setminus C) < \varepsilon$.
This implies that $\mu(A \setminus C) = \mu(A \setminus (B \setminus D)) + \mu((B \setminus D) \setminus C) < \varepsilon$.

Proof of 2

Consider $V_i \setminus A$ where $\mu(V_i \setminus A) \le \mu(V_i) < +\infty$.
By (1), $\exists C_i$ closed with $C_i \subset V_i \setminus A$ and $\mu((V_i \setminus A) \setminus C_i) < \varepsilon/2^{i+1}$. Write

$$(V_i \setminus A) \setminus C_i = (V_i \setminus A) \cap C_i^c = V_i \cap A^c \cap C_i^c = (V_i \cap C_i^c) \cap A^c = (V_i \setminus C_i) \setminus A$$

Note that $V_i \setminus C_i$ is open, since C_i is closed.
Define $U = \bigcup (V_i \setminus C_i) \supset A$. Then,

$$U \setminus A = \left(\bigcup (V_i \setminus C_i)\right) \setminus A = \bigcup ((V_i \setminus C_i) \setminus A)$$

Therefore $\mu(U \setminus A) \le \varepsilon \frac{\varepsilon}{2^{1+1}} = \varepsilon$.

Remark

$X = \bigcup V_i$, V_i open and $\mu(V_i) < +\infty$.
Then $\forall A \in \Lambda$, $\forall \varepsilon > 0$, $\exists U$ open such that $U \supset A$ and $\mu(U \setminus A) < \varepsilon$.
For A^c, $\exists U \supset A^c$ ($\implies U^c \subset A$), $\mu(U \setminus A^c) < \varepsilon$. So

$$U \cap A = U \setminus A^c = A \setminus U^c = A \cap U$$

and $\mu(A \setminus U^c) < \varepsilon$, $U^c \subset A$ with U^c closed.

Corollary

For $\mathbb{R}^n$, a measure is Radon if and only if it is locally finite and Borel regular.

- Proof
  ($\implies$)
  Let $B(r, x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$ and $\overline{B(r, x_0)} = \{x \in \mathbb{R}^n : |x - x_0| \leq r, \text{ compact}\}$.
  Then $\mu(B(r, x_0)) \leq \mu(\overline{B(r, x_0)}) < +\infty$. So μ is locally finite.
  For $A \in \Lambda$, we may assume without loss of generality that $\mu(A) < +\infty$.
  Then $\forall i$, $\exists U_i$ open where $U_i \supset A$ and $\mu(A) \leq \mu(U_i) \leq \mu(A) + \frac{1}{i} < +\infty$.
  Set $G = \bigcap U_i \in \mathcal{B}$, then $\mu(G) = \mu(A)$.
  ($\impliedby$)

  1. Borel regular implies Borel.

  2. For K compact, $\forall x \in K \ni U_x$ open where $\mu(U_x) < +\infty$.

     $\{U_\lambda\}_{\lambda \in k}$ is an open cover. Therefore there is a finite subcover $\{U_{\lambda_i}\}_{i=1}^{\lambda}$ where

     $$\mu(K) \leq \mu\left(\bigcup_{i=1}^{k} U_{\lambda_i}\right) \leq \sum_{i=1}^{k} \mu\left(U_{x_i}\right) < +\infty$$

  3. $\forall V$ open, $B(i) = B(i, 0)$, $V \cap B(i)$, $\mu(V \cap B(i)) < +\infty$, $\exists C_i$ closed where $C_i \subset V_{\cap B(i)}$ so C_i is bounded and therefore compact.

     So $\mu(C_i) \leq \mu((V \cap B(i)) \setminus C_i) < \frac{1}{i}$ and $\mu(V \cap B(i)) \leq \mu(C_i) + \frac{1}{i}$.
     Then $\mu(V) = \lim_{i \to \infty} \mu(V \cap B(i)) = \lim_{i \to \infty} \mu(C_i)$, and $C_i \subset V \cap B(i) \subset V$ compact.
     Therefore $\mu(V) = \sup\{\mu(K) : K \text{ compact}, K \subset V\}$.

  4. $\forall A \in \Lambda$, $\forall i$, $\exists U_i$ open where $U_i \supset A$ and $\mu(U_i \setminus A) < \frac{1}{i}$

     This implies that $\mu(A) \leq \mu(U_i) \leq \mu(A) + \frac{1}{i}$ and therefore $\mu(A) = \inf\{\mu(U) : U \supset A, U \text{ open}\}$.

Caratheodory Construction

Definition: Outer Measure

$\mu^*(A)$, $\forall A \in 2^X$

1. $\mu^*(\varnothing) = 0$.

2. $\mu^*(A) \le \mu^*(B)$ if $A \subseteq B$.

3. $\mu^*(\bigcup A_i) \le \sum \mu^*(A_i)$, $\forall A_i \in 2^X$ (countable subadditivity)

Define $\Lambda = \{A \in 2^x : \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c), \forall E \in 2^X\}$.
Then $\mu(A) = \mu^*(A)$ if $A \in \Lambda$.
(X, Λ, μ) is complete.

January 25, 2024

Theorem: Caratheodory Construction

Outer Measure

$u^* : 2^X \to [0, +\infty]$.

1. $\mu^*(\varnothing) = 0$

2. Monotonicity: $\mu^*(A) \le \mu^*(B)$, $A \subseteq B$

3. Countable Subadditivity: $\mu^*\left(\bigcup_i A_i\right) \le \sum_i \mu^*(A_i)$.

Caratheodory Criterion

$A \subset X$ is measurable if $\forall E \in X$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

Theorem

The collection Λ of all measurable sets is a σ-algebra.
(X, Λ, μ) is a complete measure space (cf. proof of Lebesgue completeness).

Hausdorff Measure

$\forall A \subseteq \mathbb{R}^n$, $\forall s \ge 0$, $H_s^\delta(A) = \inf\left\{\sum_i (d(E_i))^s : \bigcup_i E_i \supset A,\ d(E_i) \le \delta\right\}$ where $d(E_i)$ is the diameter of E_i.
Notice that $H_s^{\delta_1}(A) \le H_s^{\delta_2}(A)$ if $\delta_2 \le \delta_1$.
Let $H_s^*(A) = \lim_{\delta \to 0} H_s^\delta(A)$, $\forall A \in 2^{\mathbb{R}^n}$.
Claim: H_s^* is an outer measure.

- Verify

  1. $H_s^*(\varnothing) = 0$.

  2. $H_s^*(A) \le H_s^*(B)$, $\forall A \subseteq B \subseteq \mathbb{R}^n$.

  3. Given $A_i \subset \mathbb{R}^N$,

$\exists\delta_0 > 0$ such that $\forall \delta < \delta_0$, $H_s^*\left(\bigcup_i A_i\right) \leq H_s^\delta\left(\bigcup_i A_i\right) + \frac{\varepsilon}{2}$.
Then $\forall\delta < \delta_0$ fixed, $\forall A_i$, $\exists\{E_i^j\}$ such that $\bigcup_j E_i^j \supset A_i$, $\sum_j (d(E_i^j))^s \leq H_s^\delta(A_i) + \frac{\varepsilon}{2^{i+1}}$, and $d(E_j^j) \leq \delta$. So

$$\begin{aligned} H_s^\delta\left(\bigcup_i A_i\right) &\leq \sum_{i,j}(d(E_i^j))^s \\ &= \sum_i\left(\sum_j (d(E_i^j)^s\right) \\ &= \sum_i\left(H_s^\delta(A_i) + \frac{\varepsilon}{2^{i+1}}\right) \\ &= \sum_i H_s^\delta(A_i) + \frac{\varepsilon}{2} \end{aligned}$$

and

$$H_s^*\left(\bigcup_i A_i\right) \leq \sum_i H_s^\delta(A_i) + \varepsilon \leq \sum_i H_s^*(A_i) + \varepsilon, \quad \forall\varepsilon > 0.$$

Then, since H_s^* is an outer measure, it is a measure by the Caratheodory construction.

Definition: Hausdorff Measure

The Hausdroff Measure $H_s : \Lambda \to [0, +\infty)$ on a σ-algebra $\Lambda \subset 2^{\mathbb{R}^n}$.

Not Locally Finite

Consider $B(0,1) = \{x : |x| < 1\}$.
Then $H_s(B(0,1)) = \infty$ for $s < n$.
That is, the Hausdorff measure is not locally finite for $s < n$.

Complete

The Hausdorff measure, by the Caratheodory construction, is complete.

Symmetry

1. Translation Invariance: $H_s(A + x) = H_s(A)$.
2. Rotation Invariance: $H_s(RA) = H_s(A)$.
3. Scaling: $H_s(\lambda A) = \lambda^s H_s(A)$.

Open Balls Measurable

What about $B(0,1) \subset \mathbb{R}^n$. For $\delta > 0$,

$$H_s^*(E \cap B(0,1)) + H_s^*(E \cap B(0,1)^c) \leq H_s^*(E \cap B(0,1-\delta)) + H_s^*(E \cap (B(0,1) \setminus B(0,1-\delta))) + H_s^*(E \cap B(0,1)^c)$$

Want to show that for all $\varepsilon > 0$, this is $\leq H_s^*(E) + \varepsilon$.

- Lemma 1

$$H_s^*(E\cap B(0,1-\delta))+H_s^*(E\cap B(0,1)^c) = H_s^*(E\cap(B(0,1-\delta)\cup B(0,1)^c))$$
$$\leq H_s^*(E)$$

- Lemma 2

$$H_s^*(E\cap(B(0,1)\setminus B(0,1-\delta))<\varepsilon.$$

- Lemma 1'

  If $A,B\subset\mathbb{R}^n$, dist$(A,B)>0$, then $H_s^*(A\cup B)=H_s^*(A)+H_s^*(B)$.
  Since $\{E_i\}$ covering $A\cup B$, $d(E_i)<\frac{1}{4}$dist(A,B) gives

$$\delta<\frac{1}{4}\text{dist}(A,B)\iff \{E_j^A\}\cup\{E_k^B\}$$

  if and only if $\{E_j^A\}$ covers A and $\{E_k^B\}$ covers B. Therefore,

$$\sum_i (d(E_i))^s=\sum_j (d(E_j^A))^s+\sum_k (d(E_k^B))^s$$
$$\inf\left\{\sum_i (d(E_i))^s\right\}=\inf\left\{\sum_j (d(E_j^A))^s\right\}+\inf\left\{\sum_k (d(E_k^B))^s\right\}$$

  and $H_s^\delta(A\cup B)=H_s^\delta(A)+H_s^\delta(B)$.
  Thus $H_s^*(A\cup B)=H_s^*(A)+H_s^*(B)$.

Let $T_i=E\cap\left(B\left(0,1-\frac{1}{i+1}\right)\right)\setminus B\left(0,1-\frac{1}{i}\right)$.
IMAGE HERE - 1 CONCENTRIC RINGS
We want to show that $H_s^*\left(E\cap\left(B(0,1)\setminus B\left(0,\frac{1}{i}\right)\right)\right)<\varepsilon$ for $i>>1$. Then

$$\bigcup_{k=1} T_k=(B(0,1)\setminus\{0\})\cap E$$
$$\bigcup_{k=i} T_k=\left(B(0,1)\setminus B\left(0,1-\frac{1}{i}\right)\right)\cap E$$

Claim: $\sum_i H_s^*(T_i)<+\infty$. It suffices to prove this claim.

$$\sum_{i\text{ even}}^{2k} H_s^*(T_i)=H_s^*\left(\bigcup_{i\text{ even}}^{2k}\right)\leq H_s^*(E)<+\infty$$

$$\sum_{i\text{ odd}}^{2k+1} H_s^*(T_i)=H_s^*\left(\bigcup_{i\text{ odd}}^{2k+1}\right)\leq H_s^*(E)<+\infty$$

Then $\sum_i^k H_s^*(T_i)<=\infty$.

Borel

Take a countable, dense set $\{q_i\} \subset \mathbb{R}^n$ and $\left\{B\left(q_i, \frac{1}{k}\right)\right\}_{i,k}$.
Claim: $\forall V \subseteq \mathbb{R}^n$ open, then $V = \bigcup_l B\left(q_{i_l}, \frac{1}{k_l}\right)$.
Then $\mathcal{B} \subseteq \Lambda$ and the Hausdorff measure is Borel.

Borel Regular

$\forall A \subset \Lambda$, $\exists B \in \mathcal{B}$ such that $B \supset A$ and $H_s(B) = H_s(A)$.
$\forall \delta = \frac{1}{j}$, $\{E_i^j\}$ E_i^j closed balls with $d(E_i^j) < \frac{1}{j}$,

$$\sum_i (d(E_i))^s \leq H_s^{\frac{1}{j}}(A) + \frac{1}{j}$$

Take $B = \bigcap_j \left(\bigcup_i E_i^j\right) \in \mathcal{B}$ since $B = \bigcap_j \bigcup_i E_i^j \supset A$. Then

$$\begin{aligned} H_s^{\frac{i}{j}}(B) &\leq H_s^{\frac{1}{j}}\left(\bigcup_i E_i^j\right) \\ &\leq \sum_i H_s^{\frac{1}{j}}\left(E_i^j\right) \\ &\leq \sum_i \left(d(E_i^j)\right)^s \\ &\leq H_s^{\frac{1}{j}}(A) + \frac{1}{j} \end{aligned}$$

and in the limit as $j \to \infty$

$$H_s^*(A) \leq H_s^*(B) \leq H_s^*(A)$$

Fractional or Hausdorff Dimension

Theorem:

1. $H_s^*(A) < +\infty \implies H_t^*(A) = 0$, $\forall t > s \geq 0$.

2. $H_t^s > 0 \implies H_s(A) = \infty$, $\forall 0 \leq s < t$

Proof

$$\begin{aligned} H_s^{\delta}(A) &\sim \sum_i (d(E_i))^s \\ &= \sum_i (d(E_i))^t (d(E_i))^{s-t} \end{aligned}$$

So $s < t$ gives $\geq \delta^{s-t}$.
In the other direction, when $s < t$

$$\begin{aligned} \sum_i (d(E_i))^t &= \sum (d(E_i))^s (d(E_i))^{t-s} \\ &\leq \delta^{t-s} \sum_i (d(E_i))^s \end{aligned}$$

Definition: Hausdorff Dimension

Given $A \subset \mathbb{R}^n$,

$$\begin{aligned}\dim_H(A) &= \sup\{s : H_s^*(A) = \infty\} \\ &= \sup\{s : H_s^*(A) > 0\} \\ &= \inf\{s : H_s^*(A) = 0\} \\ &= \inf\{s : H_s^*(A) < +\infty\}\end{aligned}$$

Example 1

$\mathbb{R}^n$ has n Hausdorff dimension.
Consider the n-cube with sides d, $C(d)$. Then

$$H_s(C(d)) = C(n,s)d^s$$

So $C(n,s) = C(n,s)2^{nk}\frac{1}{(2^k)^s} = C(n,s)2^{(n-1)k}$.
If $s < n$, this tends to infinity as $k \to \infty$.
Is $s > n$ it tends to 0.

Example 2

Cantor set has Hausdorff dimension $\frac{\log(2)}{\log(3)}$.

$$\bigcup_{k=1}^{2^n} C_n^k = \frac{\log(2)}{\log(3)}$$

where $|C_n^k| = \frac{1}{3^n}$, so $H_s^\delta(C^n) \sim \frac{2^n}{(3^n)^s} = \left(\frac{2}{3^s}\right)^n$.

Example 3

The Koch snowflake has dimension $\frac{\log(4)}{\log(3)}$.

January 30, 2024

Lemma:

Given a measure space (X, Λ, μ) and an extended real-valued function $f : X \to [-\infty, +\infty]$, the following are equivalent

1. $\forall \alpha \in \mathbb{R}$, $\{x \in X : f(x) > \alpha\} \in \Lambda$.

2. $\forall \alpha \in \mathbb{R}$, $\{x \in X : f(x) \geq \alpha\} \in \Lambda$.

3. $\forall \alpha \in \mathbb{R}$, $\{x \in X : f(x) < \alpha\} \in \Lambda$.

4. $\forall \alpha \in \mathbb{R}$, $\{x \in X : f(x) \leq \alpha\} \in \Lambda$.

5. $\forall U \subset \mathbb{R}$ open, $f^{-1}(U) \in \Lambda$ and $f^{-1}(\pm\infty) \in \Lambda$.

Proof 1 Implies 2

$\{x \in X : f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \left\{x \in X : f(x) > \alpha - \frac{1}{n}\right\}$.

Proof 2 Implies 3

$\{x \in X : f(x) < \alpha\} = \{x \in X : f(x) \geq \alpha\}^c$

Proof 3 Implies 4

$\{x \in X : f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty} \left\{x \in X : f(x) < \alpha + \frac{1}{n}\right\}$

Proof 4 Implies 1

$\{x \in X : f(x) > \alpha\} = \{x \in X : f(x) \leq \alpha\}^c$

Proof of 5

$\forall U \subset \mathbb{R}$ open, $V = \bigcup_i I_i$ disjoint open intervals.
Therefore $f^{-1}((a,b)) = \{x \in X : f(x) > a\} \cap \{x \in X : f(x) < b\}$.
Similarly, $f^{-1}(-\infty) = \bigcap_n \{x \in X : f(x) < -n\}$ and $f^{-1}(\infty) = \bigcap_n \{x \in X : f(x) > n\}$.

Proof 5 Implies 1

$\{x \in X : f(x) > \alpha\} = f^{-1}((\alpha, +\infty)) \cup f^{-1}(+\infty) \in \Lambda$.

Definition: Measurable Function

For a measure space (X, Λ, μ), an extended real-valued function $f : X \to [-\infty, +\infty]$ is said to be measurable if one or all of (1)-(5) hold.

Remark:

If (X, Λ, μ) is Borel, then continuous functions are always measurable.

Remark:

The characteristic function

$$\chi_A = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

is measurable if $A \in \Lambda$.

Definition: Simple Functions

The function ϕ is simple if

$$\phi(x) = \sum_{i=1}^{k} \lambda_i \chi_{A_i}, \quad \lambda_I \in \mathbb{R},\ A_i \in \Lambda$$

Proposition:

Given a measure space (X, Λ, μ) and measurable, real-valued f, g,

- $f \pm g$ is measruable.

  $\{x \in X : f(x) + g(x) < \alpha\} = \bigcup_{r \in \mathbb{Q}} (\{x \in X : f(x) < r\} \cup \{x \in X : g(x) < \alpha - r\})$.

- f^2 is measurable

  $\forall \alpha \geq 0$, $\{x \in X : f^2(x) < \alpha\} = \{x \in x : f(x) < \sqrt{\alpha}\} \cap \{x \in X : f(x) > -\sqrt{\alpha}\}$.

- $f \cdot g$ is measurable

  $f(x) \cdot g(x) = \frac{1}{2}\left((f+g)^2 - f^2 - g^2\right)$.

Definition: Almost Everywhere Equality

Measurable functions f and g on the space (X, Λ, μ) are the same almost everywhere with respect to μ (written μ-a.e.) if

$$\mu(\{x \in X : f(x) \neq g(x)\}) = 0$$

Propositon:

For a complete measure space (X, Λ, μ), if f and g are equal μ-a.e., then f is measurable if and only if g is measurable.

Proof

$$\begin{aligned}\{x \in X : f(x) > \alpha\} &= (\{x \in X : f(x) > \alpha\} \cap \{x \in X : f(x) = g(x)\}) \cup \{x \in X : f(x) > \alpha\} \cap \{x \in X : f(x) \neq g(x)\} \\ &= (\{x \in X : g(x) > \alpha\} \cap \{x \in X : f(x) = g(x)\}) \cup \{x \in X : f(x) > \alpha\} \cap \underbrace{\{x \in X : f(x) \neq g(x)\}}_{\mu=0}\end{aligned}$$

Proppsotion:

Given $\{f_k(x)\}$ measurable.

1. $g_n(x) = \sup\{f_1(x), f_2(x), \ldots, f_n(x)\}$ and $h_n(x) = \inf\{f_1(x), f_2(x), \ldots, f_n(x)\}$ measurable.

2. $g(x) = \sup\{f_n(x)\}$ and $h(x) = \inf\{f_n(x)\}$ measurable.

3. $\limsup_{n \to +\infty} f_n(x) = \inf_n \sup\{f_n(x), f_{n+1}(x), \ldots\}$ and $\liminf_{n \to +\infty} f_n(x) = \sup_n \inf\{f_n(x), f_{n+1}(x), \ldots\}$ measurable.

4. $f_n(x) \to f(x)$ pointwise $\implies$ f measurable.

Proof of A

$\{x \in X : g_n(x) > \alpha\} = \bigcup_{k=1}^{n} \{x \in X : f_k(x) > \alpha\}$
$\{x \in X : h_n(x) < \alpha\} = \bigcup_{k=1}^{n} \{x \in X : f_k(x) < \alpha\}$

Proof of B

$\{x \in X : g(x) > \alpha\} = \bigcup_n \{x \in X : f_n(x) > \alpha\}$
$\{x \in X : h(x) < \alpha\} = \bigcup_n \{x \in X : f_n(x) < \alpha\}$

Definition: Almost Everywhere Convergence

For $f_n(x)$ measurable, $f_n(x) \to f(x)$ μ-a.e. in X if $f_n(x) \to f(x)$ in $A \subset X$ pointwise where $\mu(X \setminus A) = 0$.

Proposition:

On a complete measure space (X, Λ, μ) with f_n measurable and $f_n(x) \to f(x)$ μ-a.e. in X, $f(x)$ is measurable.

Proof

$f_n(x) \to f(x)$ pointwise in A and $\mu(A^c) = 0$.
$\{x \in X : f(x) > \alpha\} = (\{x \in X : f(x) > \alpha\} \cap A) \cup (\{x \in X : f(x) > \alpha\} \cap A^c)$.

Theorem:

With (X, Λ, μ) a measure space and f measurable, there exist simple functions ϕ_n such that

1. $|\phi_n(x)| \leq |\phi_{n+1}(x)|$.
2. $\phi_n(x) \to f(x)$ pointwise in X.
3. If f is bounded, then $\phi_n(x) \rightrightarrows f(x)$ in X.

Proof

Consider $(-\infty, -n] \cup (-n, n) \cup [n, +\infty)$, and define $N_n = \{x \in X : f(x) \leq -n\}$ and $P_n = \{x \in X : f(x) \geq n\}$.
Then $\bigcap_n (N_n \cup P_n) = \varnothing$.
Define

$$\begin{aligned}
A_{n,k} &= \left\{x \in X : \frac{k-1}{2^n} < f(x) \leq \frac{k}{2^n}\right\}_{k=-1,-2,\ldots,-n2^n+1} \\
A_{n,0} &= \left\{x \in X : \frac{-1}{2^n} < f(x) < 0\right\} \\
A_{n,1} &= \left\{x \in X : 0 < f(x) < \frac{1}{2^n}\right\} \\
A_{n,k} &= \left\{x \in X : \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n}\right\}_{k=2,3,\ldots,n2^n}
\end{aligned}$$

and set

$$\phi_n(x) = -n\chi_{N_n} + \sum_{k=0}^{-n2^n+1} \frac{k}{2^n}\chi_{A_{n,k}} + \sum_{k=1}^{n2^n} \frac{k-1}{2^n}\chi_{A_{n,k}} + n\chi_{P_n}$$

Claim:

1. $\forall x \in X$, $\phi_n(x) \to f(x)$.
2. if $\exists N \in \mathbb{N}$ such that $|f(x)| < N \implies \phi_n(x) \rightrightarrows f(x)$ in X.

Proof

$|\phi_n(x) - f(x)| \le \frac{1}{2^n}$, $\forall x \in X \setminus (U_n \cup P_n)$
Note $\forall x \in X$, $\exists m \in \mathbb{N}$ such that $x \notin N_m \cup P_m$. So $|f(x)| < m$.
Then boundedness implies $\exists N$ such that $N_N \cup P_N = \varnothing$.
Therefore $\forall x \in X$, $|\phi_n(x) - f(x)| < \frac{1}{2^n}$, $\forall n \ge N$.

Theorem: Egoroff

Given a measure space (X, Λ, μ), $\mu(x) < +\infty$ and $f_n \to f$ μ-a.e. in X, then $\forall \delta > 0$, $\exists A \in \Lambda$ such that $\mu(X \setminus A) < \delta$ and $f_n(x) \rightrightarrows f(x)$ in A.

Recall: Pointwise Convergence

$\forall x \in X$, $f_n(x) \to f(x)$ if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$, $\forall n \ge N$.
$Bjj_{N,\varepsilon} = \{x \in X : \exists N \in \mathbb{N}, |f_n(x) - f(x)| < \varepsilon, \forall n \ge N\}$
In negation, $\exists \varepsilon > 0$ such that $\forall N \in \mathbb{N}$, $\exists m \ge N$ such that $|f_n(x) - f(x)| \ge \varepsilon$.
$A_{N,\varepsilon} = B^c_{N,\varepsilon} = \{x \in X : \exists m \ge N, |f_n(x) - f(x)| \ge \varepsilon\}$
Then $\{x \in X : f_n(x) \to f(x)\} = \bigcap_{\varepsilon > 0} \bigcup_N B_{N,\varepsilon} = \bigcap_{\varepsilon_i \to 0} \bigcup_i B_{N_i,\varepsilon_i}$
and $\{x \in X : f_n(x) \not\to f(x)\} = \bigcup_{\varepsilon > 0} \bigcap_N A_{N,\varepsilon} = \bigcup_{\varepsilon_i \to 0} \bigcap_i A_{N_i,\varepsilon_i}$ where $\varepsilon_i = \frac{1}{i}$.

February 2, 2024

Review: Measurable Function

An extended, real-valued function $f : X \to [-\infty, +\infty]$ is measurable if one or all of the following hold

1. $\forall \alpha \in \mathbb{R}$, $\{x : f(x) > \alpha\} \in \Lambda$.
2. $\forall \alpha \in \mathbb{R}$, $\{x : f(x) \ge \alpha\} \in \Lambda$.
3. $\forall \alpha \in \mathbb{R}$, $\{x : f(x) < \alpha\} \in \Lambda$.
4. $\forall \alpha \in \mathbb{R}$, $\{x : f(x) \le \alpha\} \in \Lambda$.
5. $\forall V \subseteq \mathbb{R}$ open, $f^{-1}(U) = \{x : f(x) \in V\}$ and $f^{-1}(-\infty), f^{-1}(+\infty) \in \Lambda$.

Properties

1. For $f = g$ μ-a.e., f is measurable if and only if g is measurable.
2. For f, g measurable, $f + g$ and $f \cdot g$ are measurable.
3. For $\{f_n\}$ measurable,
   (a) $\sup_{n \le k}\{f_n\}$ and $\inf_{n \le k}\{f_n\}$ are measurable.
   (b) $\sup_n\{f_n\}$ and $\inf_n\{f_n\}$ are measurable.
   (c) $\limsup_{n\to\infty} f_n$ and $\liminf_{n\to\infty} f_n$ are measurable.

(d) if $f_n \to f$ μ-a.e. in X, then f is measurable.

Examples

Characteristic Functions

$$\chi_A = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}, \quad A \in \Lambda$$

Simple Functions

$$\sum_{i=1}^{k} \alpha_i \chi_{A_i}, \quad \alpha_i \in \mathbb{R},\ A_i \in \Lambda,\ A_j \cap A_k = \varnothing$$

Step Functions

$$\sum_{i=1}^{k} \alpha_i \chi_{I_i}, \quad I_i \text{ interval}$$

Theorem:

On a measure space (X, Λ, μ), suppose f is measurable.
There exists a sequence of simple functions $\{\phi_n\}$ such that

1. $\phi_n \to f$ pointwise.
2. $\phi_n \rightrightarrows f$ for f bounded.

Proof

Let $N_n = \{x : f(x) \leq -n\}$ and $A_{n,k} = \left\{x : \frac{k-1}{2^n} < f(x) \leq \frac{k}{2^n}\right\}$. Then

$$A_{n,0} = \left\{x : -\frac{1}{2^n} < f(x) < 0\right\}$$
$$A_{n,1} = \left\{x : 0 < f(x) < \frac{1}{2n}\right\}$$
$$A_{n,k} = \left\{x : \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n}\right\}$$
$$P_n = \{x : f(x) \geq n\}$$

and

$$\phi_n = -n\chi_{N_n} + \sum_{k=-n2^n+1}^{D} \frac{k}{2^n} \chi_{A_{n,k}} + \sum_{1}^{n2^n} \chi_{A_{n,k}} + n\chi_{\phi_n}$$

So

$$|\phi_n(x) - f(x)| \leq \frac{1}{2^n}, \quad x \in X \setminus (N_n \cup P_n), \quad \bigcap_n (N_n \cap P_p) = \varnothing$$

Egoroff Theorem

Given (X, Λ, μ) where $\mu(X) < +\infty$, if

1. $f_n(x) \to f(x)$ μ-a.e. in X and
2. f_n, f μ-a.e. finite.

Then, $\forall \delta > 0$, $\exists A \in \Lambda$ with $\mu(A) < \delta$ such that $f_n(x) \rightrightarrows f(x)$ on A^c.

Proof

Define $D = \{x : f_n(x) \to f(x)\} = X$.
Then $\forall \varepsilon > 0$, $\exists m \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$, $\forall n \geq m$.
Say that the universal quantifier $\forall$ is equivalent to grand intersection and the existential quantifier $\exists$ is equivalent to grand union. Then

$$D_{m,\varepsilon} = \{x : f_n(x) - f(x) < \varepsilon, \ \forall n \geq m\}$$

and

$$\bigcap_{\varepsilon > 0} \bigcup_m D_{m,\varepsilon} = X.$$

The negation is

$$D^c_{n,\varepsilon} = \{x : \exists n \geq m, \ |f_n(x) - f(x)| \geq \varepsilon\}$$

Then injection is equivalent to the complement.
Set $\varepsilon_i = \frac{1}{i}$ such that

$$D = \bigcap_i \bigcup_{m_i} D_{m_i,1/i}$$
$$\varnothing = D^c = \bigcup_i \bigcap_m D^c_{m,1/i}$$

So $\bigcap_m D^c_{m,1/i} = \varnothing$,

$$D^c_{m,1/i} = A_{m,1/i} = \left\{x : \exists n \geq m, |f_n(x) - f(x)| \geq \frac{1}{i}\right\}$$

and $A_{n,1/i} \supset A_{n+1,1/i} \supset \cdots$. Therefore

$$\mu(A_{n,1/i}) \to \mu\left(\bigcap_m A_{m,1/i}\right) = 0$$

for $\mu(X) < +\infty$.
Thus, $\forall i$, $\exists m_i$ such that $\mu(A_{m_i,1/i}) < \frac{\delta}{2^{i+1}}$. It follows that $A = \bigcup_i (A_{m_i,1/i})$,

$$\mu(A) \leq \sum \mu(A_{m_i,1/i}) < \delta$$

and

$$x \in A^c = \bigcap_i A^c_{m_i,1/i} = \bigcap_i D_{m_i,1/i} = \bigcap_i \left\{x \,:\, |f_n(x) - f(x)| < \frac{1}{i},\ \forall n \geq m_i\right\}$$

Finally, this implies $f_n(x) \rightrightarrows f(x)$ in A^c.

Example

Take $f_n = \chi_{[n,n+1]}$ on $\mathbb{R}$, then $f_n(x) \to 0$ in $\mathbb{R}$ but $A \subset \mathbb{R}$, $\mu(A) < \frac{1}{2}$, $A^c \cap [n, n+1] \neq \varnothing$, $\forall n$.
That is, $\forall n$, $\exists x \in A^c$ such that $f_n(x) = 1$ but $f(x) = 0$.
Therefore $f_n(x) \not\rightrightarrows f(x)$ on $\mathbb{R}$.

Definition: Essential Bounds

On a measure space (X, Λ, μ) with f measurable, define $||f||_\infty = \inf\{M \,:\, \mu(\{x \,:\, |f(x)| > M\}) = 0\}$.
This is the L^∞-norm.

Proposition:

$f_n \rightrightarrows f$ on A where $\mu(A^c) = 0$ if and only if $||f_n - f||_\infty \to 0$.

Proof

$(\Longrightarrow)$
$\forall \varepsilon > 0$, $\exists m \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$, $\forall x \in A$.
Claim: $||f_n(x) - f(x)||)\infty < \varepsilon$, $\forall n \geq m$.

$$||f_n(x) - f(x)||_\infty = \inf\{M \,:\, \mu(\{x \,:\, |f_n(x) - f(x)| > M\}) = 0\}$$

Where $\{x \,:\, |f_n(x) - f(x)| > n\} \subset A^c$ and $n \geq m$ and $M \geq \varepsilon/2$.
$(\Longleftarrow)$

Recall: Urysohn's Lemma

For X locally compact and Hausdorff, $K \subset U$ for K compact and U open, $\exists \phi$ continuous such that $\phi = \begin{cases} 1 & K \\ 0 & U^c \end{cases}$.

Theorem: Vitali-Lusin

On measure space (X, Λ, μ) with X locally compact and Hausdorff and μ a Radon measure.
For f measurable, μ-a.e. finite and vanishing outside A where $\mu(A) < +\infty$,
$\forall \varepsilon > 0$, $\exists g$ continuous with compact support such that $\mu(\{x \,:\, f(x) \neq g(x)\}) < \varepsilon$.

Proof

1. $\exists C \subset A$ compact with $\mu(A \setminus C) < \varepsilon$.

2. For A compact with $\mu(A) < +\infty$, $\exists U \supset A$ open neighborhood with compact closure and $\mu(U \setminus A) < \varepsilon$.

3. $\phi_n = -n\chi_{N_n} + \sum_{-n2^n+1}^{0} \frac{k}{2^n}\chi_{A_{n,k}} + \sum_1^{n2^n} \frac{k-1}{2^n}\chi_{A_{n,k}} + n\chi_{P_n}$

Since we may minimize $\mu(N_n \cup P_n) < \varepsilon$,

$$\phi_n = \sum_{-n2^n+1}^{0} \frac{k}{2^n} \chi_{A_{n,k}} + \sum_{1}^{n2^n} \frac{k-1}{2^n} \chi_{A_{n,k}}$$

Take $C_{1,k} \subset A_{1,k}$ compact with $\mu(C_{1,k}) \geq \mu(A_{1,k}) - 2^{-1}2^{-|k|+1}\varepsilon$. Write

$$C_1 = \bigcup_k C_{1,k}$$

and inductively define $C_{n-1,k}$ and $C_{n-1} = \bigcup_k C_{n-1,k}$ such that $C_{n,k} \subset A_{n,k} \cap C_{n-1}$ compact and

$$\mu(C_{n,k}) \geq \mu(A_{n,k} \cap C_{n-1}) - 2^{-1}2^{-|k|+1}\varepsilon$$

Define, by Urysohn's Lemma,

$$\tilde{\chi}_{A_{n,k}} := \begin{cases} 1 & C_{n,k} \\ 0 & U^c \cup \bigcup_{l \neq k} C_{n,l} \end{cases}$$

where $C_n \subset C_{n-1}$, $C = \bigcap C_n$, $C_n = \bigcup_k C_{n,k}$.
Then define

$$g_n := \sum_{-n2^n+1}^{0} \frac{k}{2^n} \tilde{\chi}_{A_{n,k}} + \sum_{1}^{n2^n} \frac{k-1}{2^n} \tilde{\chi}_{A_{n,k}}$$

Then $g_n = \phi_n$ on C for all n.
Therefore $g_n = \phi_n \rightrightarrows \hat{g} = f$ on C.
By uniform convergence, $\hat{g}$ is continuous on C.
So, again by Urysohn's Lemma, $g = \phi\hat{g}$ and $\{x : g \neq f\} = U \setminus C$.

February 8, 2024

Midterm Review

Problem 2

Given a finite measure space (X, Λ, μ), $\mu(X) < +\infty$ and a function f which is μ-a.e. finite.
Monotone Convergence Theorem:

1. $A_1 \subset A_2 \subset \cdots$, then $\mu\left(\bigcup_i A_i\right) = \lim_{i\to\infty} \mu(A_i)$.
2. $A_1 \supset A_2 \supset \cdots$, then $\mu\left(\bigcap_i A_i\right) \lim_{i\to\infty} \mu(A_i)$ for $\mu(A_1) < +\infty$.

If $A_k = \{x : |f(x)| > k\}$ and

$$F = \bigcap_{k=1}^{\infty} A_k$$

then $\mu(F) = \lim_{k\to\infty} \mu(A_k) = 0$ since $\mu(X) < +\infty$.
If instead we consider A_k^c, then

$$\bigcup_k A_k^c = X \setminus F$$

Problem 3

1. Borel

   Given $(\alpha, +\infty)$, we want $\forall E \subset \mathbb{R}$

   $$m^*(E \cap (\alpha, +\infty)) + m^*(E \cap (-\infty, \alpha]) \leq m^*(E)$$

   $\forall \varepsilon > 0$, $\exists \{I_i\}$ pen intervals

   $$\bigcup_i I_i \supset E \quad \sum_i |I_i| \leq m^*(E) + \varepsilon/2$$

   Divide $\{I_i\}$ into 3 groups,

   $$\begin{aligned} C^\ell &= \{I \in \{I_i\} \,:\, I \text{ is to the left of } \alpha\} \\ C^r &= \{I \in \{I_i\} \,:\, I \text{ is to the right of } \alpha\} \\ C^m &= \{I \in \{I_i\} \,:\, \alpha \in I\} \end{aligned}$$

   Then, $\forall I_k^m \in C^m = \{I_k^m\}$, and

   $$\begin{aligned} {}^\ell I_k^n &= \left(a_k, \alpha + \frac{2}{2^{k+2}}\right) \\ {}^r I_k^n &= \left(\alpha - \frac{2}{2^{k+2}}, b_k\right) \\ {}^m I_k^n &= (a_k, b_k) \end{aligned}$$

   where also

   $$\begin{aligned} A_n \supset (\alpha, +\infty)^c \quad A_n &= \left(-\infty, \alpha + \frac{1}{2^n}\right) \\ B_n \supset (\alpha, +\infty) \quad B_n &= \left(\alpha + \frac{1}{2^n}, +\infty\right) \\ A_n \cap B_n &= \left(\alpha - \frac{1}{2^n}, \alpha + \frac{1}{2^n}\right) \end{aligned}$$

   So ${}^\ell I_k^n \cup {}^r I_k^n = I_k^n$, and $|{}^\ell I_k^n| + |{}^r I_k^n| = |I_k^n| + \frac{\varepsilon}{2^{k+1}}$.
   Finally

   $$\begin{aligned} m^*(E \cap (\alpha, +\infty)) + m^*(E \cap (-\infty, \alpha]) &\leq \sum_{I \in C^r} |I| + \sum_k |{}^r I_k^n| + \sum_{I \in C^\ell} |I| + \sum_k |{}^\ell I_k^n| \\ &\leq \sum_{I \in C^r} |I| + \sum_{I \in C^\ell} |I| + \sum_k |I_k^n| + \frac{\varepsilon}{2} \\ &\leq m^*(E) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \end{aligned}$$

2. $\mu(K) < +\infty$ for $K \subset \mathbb{R}$ compact.

   K is bounded, $k \subset (-M, M)$ for large M.
   Therefore $\mu(K) \leq 2M < +\infty$.

3. $\forall U \subset \mathbb{R}$ open, we want to show $\exists K_n$ compact such that $K_n \subset U$ and $\mu(K_n) \to \mu(U)$.

   Let $U = \bigcup_i I_i$ a union of countably many disjoint open intervals (e.g. $I_i = (a_i, b_i)$).
   Then $m(U) = \sum_i m(I_i)$.
   Set $I_i^n = \left[a_i + \frac{1}{n2^{i+1}}, b_i - \frac{1}{n2^{i+1}}\right]$. Then

$$\sum_i^k |I_i^n \geq \sum_{i=1}^k |I_i| - \frac{1}{n}, \quad \forall k$$

   It follows that

$$\sum_{i=1}^k |I_i| \to \sum_{i=1}^\infty |I_i|, \quad \text{as } k \to +\infty$$

   and

$$K_k^n = \bigcup_{i=1}^k U_i^n \subset U \quad \text{compact}$$

$$m(U) \geq m(K_k^n) = \sum_{i=1}^n |I_i^n| \geq \underbrace{\sum_{i=1}^\infty |I_i|}_{m(U)} - \frac{1}{n}$$

   Alternatively, we have the theorem that if X is a metric space and μ is Borel regular on (X, Λ), then

   (a) $A \in \Lambda$, $\mu(A) < +\infty$, $\forall \varepsilon > 0$, $\exists C$ closed with $C \subset A$ such that $\mu(A \setminus C) < \varepsilon$.

   (b) $\exists\{U_i\}$, $\mu(U_i) < +\infty$, U_i open where $A \subset \bigcup_i U_i$, $\forall \varepsilon > 0$ there exists V open such that $V \supset A$ and $\mu(V \setminus A)\varepsilon$.

   With the corollary that for μ on $\mathbb{R}^n$, μ is Radon if and only if it is locally finite and Borel regular.

4. For $A \in \Lambda$, $m(A) = \inf\{m(V) \,:\, V \supset A,\ V \text{ open}\}$

   Recall Borel regularity: $\forall A \in \Lambda$, there is some Borel set $B \supset A$ with $m(B) = m(A)$.
   We may assume $m(A) < +\infty$. Then, $\forall \varepsilon > 0$, there is some collection of open intervals $\{I_i^n\}$ containing A where

$$\sum_i |I_i^n| \leq m(A) + \varepsilon$$

   Set $\varepsilon = \frac{1}{n}$ and let $U^n = \bigcup_i I_i^n \supset A$ open. Then

$$m(A) \leq m(U^n) \leq \sum_i |I_i^n| \leq m(A) + \frac{1}{n}$$

   If $B = \bigcap_n U_n$, then $\lim_{m\to\infty} m(U^n) = m(A)$ and $m(B) = m(A)$.

Problem 4

Given $f : \mathbb{R} \to \mathbb{R}$, continuous outisde a measure zero set D.
That is, $\overline{f} : \mathbb{R} \setminus D \to \mathbb{R}$ is continuous.
$\forall V \subset \mathbb{R}$, $f^{-1}(V) = (f^{-1})V \cap (\mathbb{R} \setminus D)) \cup (f^{-1}(V) \cap D)$.
By measure completeness, we are automatically safe on $f^{-1}(V) \cap D$.
Claim: $f^{-1}(V) \cap (\mathbb{R} \setminus D) = \overline{f}^{-1}(V)$.
Claim: $\overline{f}^{-1}$ is measurable.
Claim: $\overline{f}^{-1}(V) = U \cap (\mathbb{R} \setminus D)$ where $U \subset \mathbb{R}$ open.
Since $U \cap (\mathbb{R} \setminus D)$ is open in the subspace topology, we are done.
Alternatively (similary to Probelm 8 below), for D such that $m(D) = 0$, $\forall n$, $\exists U^n$ such that $m(U^n) \leq 2^{-n}$, $U^n \supset D$ and $U^n = \bigcup_i (a_i, b_i)$ where $(a_j, b_j) \cap (a_k, b_k) = \varnothing$ and $a_i, b_i \in \mathbb{R} \setminus D$. So

$$f_n = \begin{cases} f(x), & x \in (U^n)^c \\ f(a_i) + \frac{f(b_i) - f(a_i)}{b_i - a_i}(x - a_i), & x \in (a_i, b_i) \subset U^n \end{cases}$$

Then $\{x : f_n(x) \neq f(x)\} \subset U^n$ and $m(\{x : f_n(x) \neq f(x)\}) \leq 2^{-n}$.

Homework 4 Problem 8

Assume $f(x)$ is decreasing.

1. Discontinuities are limited to jump discontinuities.

2. Discontinuities are countable.

3. $D = \{x_i\}_i$, $\forall n$ there exists an open cover $\{I_i^n = (a_i, b_i)\}$ where $\bigcup_i I_i^n = C^n \supset \{x_i\}_i$ and $m(C^n) \leq 2^{-n}$.

Then $\{x : f_n(x) \neq f(x)\} \subset C^n\}$ and $\mu(\{x : f_n(x) \neq f(x)\}) \leq 2^{-n}$.
Claim: $f_n(x) \to f(x)$ on $\mathbb{R} \setminus G$ where $G = \bigcap_n^\infty \bigcup_{k=n}^\infty \{x : f_k(x) \neq f(x)\}$.
By monotone convergence, $\mu(g) = \lim_{n \to +\infty} \mu\left(\bigcup_{k=n}^\infty \{x : f_n(x) \neq f(x)\}\right) = \lim_{n \to +\infty} \left(\sum_{k=n}^{+\infty} 2^{ik}\right) = 0$.
Consider the complement, $G^c = \bigcap_{n=1}^\infty \bigcap_{k=n}^{+\infty} \{x : f_k(x) \neq f(x)\}$.
Then $\forall x \in G^c$, $x \in \bigcap_{k=n_0}^{+\infty} \{x : f_k(x) = f(x)$, so $f_n(x) = f(x)$ $\forall n \geq n_0$.

Riemann Integration

Given a function $f : [a, b] \to \mathbb{R}$ bounded and P a partiation of $[a, b]$ where

$$a = x_0 < x_1 < \cdots < x_n = b$$

The Cauchy sum

$$C(P, [a, b]) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i+1}), \quad \xi_i \in [x_i, x_{i+1})$$

alternatively

$$\phi(P, [a, b]) = \sum_i f(\xi_i) \chi_{[x_i, x_i+1)}$$

Consider the upper Riemann sum

$$S(P,[a,b]) = \sum_i M_i(x_i, x_{i+1}), \quad M_i = \sup_{[x_i,x_{i+1}]} f(x)$$

and the lower Riemann sum

$$s(P,[a,b]) = \sum_i m_i(x_i, x_{i+1}), \quad m_i = \inf_{[x_i,x_{i+1}]} f(x)$$

then define

$$S = \inf_P S(P,[a,b]) = s = \sup_P s(P,[a,b]) \implies \int_a^b f(x)\,dx = \lim_{l(P)\to 0} C(P,[a,b])$$

Theorem:

f is Riemann integrable on $[a,b]$ if and only if f is continuous m-a.e. (w.r.t Lebesgue measure) on $[a,b]$.

Proof

($\Longrightarrow$) Let f be Riemann integrable on $[a,b]$.
Define the oscillation

$$\text{Osc}_I(f) = \sup_I f(x) - \inf_I f(x)$$
$$\text{Osc}_x(f) = \lim_{\delta\to 0} \text{Osc}_{(x-\delta,x+\delta)}(f)$$

and observe that f is continuous at x if and only if $\text{Osc}_x(f) = 0$.
Let $D = \{x : \text{Osc}_x(f) > 0\}$ and $D_k = \left\{x : \text{Osc}_x(f) > \frac{1}{k}\right\}$ such that $D_k \subset D_{k+1}$ and $D = \bigcup_k D_k$.
Therefore $m(D_k) \to m(D)$.
To show that $m(D) = 0$, assume otherwise that $m(D) > 0$.
Therefore, $\exists k$ such that $m(D_k) > d_{k_0}$ for any $k \geq k_0$.
Then, for any partition P we may examine

$$S(P,[a,b]) - s(P,[a,b]) = \sum_{I_i} (M_i - m_i)|I_i|$$

We want to show that this is $\geq \delta > 0$ for any P.

February 13, 2024

Recall: Riemann Integration

$f(x) \geq 0$ on $[a,b]$ bounded.
Partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, $[x_{i-1}, x_i]$.
IMAGE HERE - Riemann Integration
Upper Riemann Sum: $S_P = \sum_{i=1}^n M_i(x_i - x_{i-1})$ where $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$.
Lower Riemann Sum: $s_P = \sum_{i=1}^n m_i(x_i - x_{i-1})$ where $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$.
Step Functions: $\phi_{P,\alpha} = \sum_i \alpha_i \chi_{I_i}$ where $I_i = [x_{i-1}, x_i]$.
Set $S = \inf_P S_P = \inf\left\{\sum_i \alpha_i |I_i| : \phi_{P,\alpha}(x) \geq f(x)\right\}$
and $s = \sup_P s_P = \sup\left\{\sum_i \alpha_i |I_i| : \phi_{P,\alpha}(x) \leq f(x)\right\}$.

Definition: Riemann Integrable

The function f is Riemann integrable if $S = s$.

Remark:

$$S_P - s_P = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \to 0 \text{ as } \ell(P) \to 0$$

Remark:

If f is continuous, then it is Riemann integrable.

Theorem:

Given $f : [a, b] \to \mathbb{R}$ bounded, then f is Riemann integrable if and only if f is continuous m-a.e.
$m(D) = 0$ if and only if f is Riemann integrable.

Proof

Recall that $\text{Osc}_I(f) = \sup_I f(x) - \inf_I f(x)$ and $\text{Osc}_{x_0}(f) = \lim_{\delta\to 0} \text{Osc}_{(x_0-\delta, x_0+\delta)}(f)$.
IMAGE HERE - 2 Oscillation
Write $D = \{x \in [a, b] : f \text{ is not continuous at } x\}$, and $D_k\{x \in [a, b] : \text{Osc}_x(f) \geq 1/k\}$ closed (since D_k^C open). Then

$$D = \bigcup_k D_k = \{x \in [a, b] : \text{Osc}_x(f) > 0\}$$

We have $m(D_k) \underset{k\to\infty}{\longrightarrow} m(D)$.
Then there exists an open cover of D_k, $\{I_i\}$ such that $m(D_k) + \varepsilon \geq \sum_i |I_i| \geq m(D_k) - \varepsilon$.
Since D_k is closed and bounded, it is compact and there exists finite subcover $\{I_{i_k}\}_{k=1}^{\ell} \subset \{I_i\}$.
($\Longleftarrow$) Assume that f is Riemann integrable and, for sake of contradiction, that $m(D) > 0$.
Then $m(D_k) \geq m > 0$, $\forall k \geq k_0$.
Now for any partition $P = \{x_0, x_1, \ldots, x_n\}$,

$$\begin{aligned}
S_P - s_P &= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \\
&\geq \sum_{(x_{i-1}, x_i)\cap D_k \neq \varnothing} (M_i - m_i)(x_i - x_{i-1}) \\
&\geq \frac{1}{k} \sum_{(x_{i-1}, x_i)\cap D_k \neq \varnothing} (x_i - x_{i-1})
\end{aligned}$$

Since $\bigcup_{(x_{i-1}, x_i)\cap D_k \neq \varnothing} [x_{i-1}, x_i] \supset D_k$,

$$\sum_{(x_i, x_{i-1})\cap D_k \neq \varnothing} (x_i - x_{i-1}) = m\left(\bigcup_{(x_{i-1}, x_i)\cap D_k \neq 0} [x_{i-1}, x_i] \right) \geq m(D_k)$$

we conclude that

$$S_P - s_P \geq \frac{m}{k_0} \geq 0$$

($\Longrightarrow$) Assume $m(D) = 0$.
Then, for any k satisfying $\frac{1}{k} < \frac{\varepsilon}{2(b-a)}$, $m(D_k) = 0$ and $\{I_{i_k}\}_{k=1}^{\ell} \subset \{I_i\}$ for open intervals I_i.
We have, also, $\bigcup_{k=1}^{\ell} I_{i_k} \supset D_k$ so

$$\sum_{k=1}^{\ell} |I_{i_k}| \le \sum_i |I_i| \le \frac{\varepsilon}{2M}$$

and

$$[a,b] \setminus \bigcup_{k=1}^{\ell} I_{i_k} \subset D_k^c$$

compact.
Claim: there exists some partition $P = \{x_i\}_{i=0}^{n}$ such that $S_P - s_P < \varepsilon = \frac{1}{k}$.
Given $\text{Osc}_x(f) \le 2M$,

$$\begin{aligned} S_P - s_P &= \sum_i (M_i - m_i)(x_i - x_{i-1}) \\ &= \sum_{[x_{i-1},x_i]\cap D_k = \varnothing} + \sum_{[x_{i-1},x_i]\cap D_k \neq \varnothing} \\ &\le \frac{\varepsilon}{2(b-a)}(b-a) + 2M \cdot \frac{\varepsilon}{4M} \end{aligned}$$

Definition: Lebesgue Integration

Given a measure space (X, Λ, μ) and simple function $s = \sum_i \alpha_i \chi_{A_i}$ for $\alpha_i \in \mathbb{R}$ and $A_i \in \Lambda$,

$$\int_E s \, d\mu = \sum_i \alpha_i \mu(A_i \cap E)$$

Then, for extended real-valued $f \ge 0$,

$$\int_E f \, d\mu = \sup\left\{ \sum_i \alpha_i \mu(A \cap E) \,:\, 0 \le s(x) \le f(x) \right\}$$

Properties

1. For $0 \le f \le g$ on E, $\int_E f \, d\mu \le \int_E g \, d\mu$.

2. For $A \subset B$ where $A, B \in \Lambda$, $\int_A f \, d\mu \le \int_B f \, d\mu$.

3. Since $f \ge 0$, $\forall c \in \mathbb{R}_{\ge 0}$ $\int_E cf \, d\mu = c \int_E f \, d\mu$.

4. $f = 0$ μ-a.e. if and only if $\int_X f \, d\mu = 0$.

5. $\int_E f \, d\mu = \int_X f \chi_E \, d\mu$.

6. For $f, g \ge 0$, $\int_E f + g \, d\mu = \int_E f \, d\mu + \int_E g \, d\mu$.

7. For $A, B \in \Lambda$ where $A \cap B = \varnothing$, $\int_{A\cup B} f\, d\mu = \int_A f\, d\mu + \int_B f\, d\mu$.

- Proof of 4
  $(\Longrightarrow)$ $\sum_i \alpha_i \chi_{A_i} = s(x) = f(x) \implies \alpha_i > 0 \implies \mu(A_i) = 0$.
  $(\Longleftarrow)$ $f \geq \alpha > 0$ and $\mu(A) > 0 \implies f(x) \geq \alpha\chi_A \implies \int_X f\, d\mu \geq \alpha_{\mu(A)} > 0$ a contradiction.

- Proof of 5
  $s\chi_E = \sum_i \alpha_i \chi_{A_i \cap E}$.

- Proof of 6
  If $0 \leq s_1 \leq f$ and $0 \leq s_2 \leq g$, then $0 \leq s_1 + s_2 \leq f + g$.

Monotone Convergence of Lebesgue Integration

On a measure space (X, Λ, μ), let $f_n \geq 0$ be a sequence of measurable functions which is monotone $f_i(x) \leq f_{i+1}(x)$ and converging $f_n(x) \to f(x)$ for any $x \in X$. Then

$$\lim_{n\to+\infty} \int_X f_n\, d\mu = \int_X f\, d\mu = \int_X \left(\lim_{n\to+\infty} f_n\right) d\mu$$

Proof

Observe that $f_n(x) \leq f(x)$, $\forall x \in X$, so

$$\int_X f_n\, d\mu \leq \int_X f_{n+1}\, d\mu \leq \int_X f\, d\mu$$

so

$$\lim_{n\to+\infty} \int_X f_n\, d\mu \leq \int_X f\, d\mu$$

We want to show that

$$\lim_{n\to+\infty} \int_X f_n\, d\mu \geq \int_X f\, d\mu$$

Let s be a simple function satisfying $0 \leq s(x) \leq f(x)$, and define

$$E_n = \{x \in X \,:\, f_n(x) \geq cs(x)\}$$

for some $c \in (0, 1)$.
Then $E_n \subset E_{n+1}$ and $\bigcup_n E_n = X$. Consider

$$\int_X f_n\, d\mu \geq \int_{E_n} f_n\, d\mu \geq c\int_{E_n} s(x)\, d\mu = c\sum_i \alpha_i \mu(A_i \cap E_n)$$

For any i, $A_i \cap E_n \to A_i$. Therefore $\mu(A_i \cap E_n) \underset{n\to+\infty}{\longrightarrow} \mu(A_i)$. So

$$\lim_{n\to+\infty} \int_X f_n\, d\mu \geq c\sum_i \alpha_i \mu(A_i)$$

for $0 \leq s = \sum \alpha_i \chi_{A_i} \leq f(x)$. Since this hold for any c,

$$\lim_{n\to+\infty} \int_X f_n\, d\mu \geq \int_X f\, d\mu$$

Corollary

Given a measurable sequence $f_n \geq 0$ with $f(x) = \sum_n f_n(x)$,

$$\int_X f \, d\mu = \sum_n \int_X f_n \, d\mu$$

and

$$\phi_n(x) = \sum_{k=1}^{n} f_k(x) \to f(x)$$

Definition: Fatou's Lemma

Given a sequence of measurable functions $f_n \geq 0$,

$$\int_X \left(\liminf_{n\to+\infty} f_n \right) \, d\mu \leq \liminf_{n\to+\infty} \int_X f_n \, d\mu$$

Proof

Observe that

$$\liminf_{n\to+\infty} f_n = \lim_{n\to+\infty} \overbrace{(\inf\{f_n(x), f_{n+1}(x), \ldots\})}^{g_n(x)}$$

so, by monotone convergence,

$$\int_X \left(\lim_{n\to+\infty} g_n(x) \right) \, d\mu = \lim_{n\to+\infty} \int_X g_n(x) \, d\mu$$

and $g_n(x) \leq f_n(x)$ gives

$$\int_X g_n(x) \, d\mu \leq \int_X f_n(x) \, d\mu$$

and implies

$$\lim_{n\to+\infty} \int_X g_n(x) \, d\mu \leq \liminf_{n\to+\infty} \int_X f_n(x) \, d\mu$$

Space of Integrable Functions

Write

$$f(x) = f^+(x) - f^-(x)$$

where

$$f^+(x) = \max\{f(x), 0\} \geq 0$$
$$f^+(x) = \min\{-f(x), 0\} \geq 0$$

Then for $\int_X f^+ \, d\mu$ and $\int_X f^- \, d\mu$, $\int_X f \, d\mu$ is defined when at least one is finite.
If both are finite, then

$$L^1_\mu(x) = \int_X |f| \, d\mu = \int_X f^+ \, d\mu + \int_X f^- \, d\mu \leq +\infty$$

Properties

1. For any $\alpha, \beta \in \mathbb{R}$,

$$\int_X (\alpha f + \beta g)\, d\mu = \alpha \int_X f\, d\mu + \beta \int_X g\, d\mu$$

   if $f, g \in L^1_\mu(x)$.

2. For $f \in L^1_\mu(x)$,

$$\left| \int_X f\, d\mu \right| \leq \int_X |f|\, d\mu$$
$$\left| \int_X f^+\, d\mu - \int_x f^{-1}\, d\mu \right| \leq \int_X f^+\, d\mu + \int_X f^-\, d\mu$$

3. For $f \leq g$, $f, g \in L^1_\mu(x)$, $\int_X f\, d\mu \leq \int_X g\, d\mu$.

4. $\int_{A \cup B} f\, d\mu = \int_A f\, d\mu + \int_B f\, d\mu$.

5. $f = 0$ μ-a.e. if and only if $\int_X |f|\, d\mu = 0$.

February 15, 2024

Recall

Given (X, Λ, μ) a measure space and X topological.
$M_\mu(x) = \{f : X \to \mathbb{R} : \text{ measurable}\}$.
$L^1_\mu(x) = \left\{f \in M_\mu(x) : \int_X |f|\, d\mu < +\infty\right\}$.
$||f||_1 = ||f||_{L^1_\mu(x)} = \int_X |f|\, d\mu$.
$L^\infty_\mu(x) = \left\{f \in M_\mu(x) : ||f||_{L^\infty_\mu(x)} < +\infty\right\}$.
$||f||_\infty = ||f||_{L^\infty_\mu(x)} = \inf\{M = \mu(\{x \in X : |f(x)| > M\} = 0\}$.
$C_c(x)$ the space of continuous functions with compact support.

Remark

In $L^1_\mu(x)$ and $L^\infty_{\mu(x)}$, $[f] = [g]$ if and only if $f = g$ μ-a.e.

Topologies

1. $f_n, f \in M_\mu(x)$, $f_n \to f$ μ-a.e. in X.

2. $f_n \to f$ in $L^\infty_\mu(x)$ if and only if $\exists A \in \Lambda$, $\mu(A) = 0$, $f_n \rightrightarrows +\infty$ in $X \setminus A$.

3. $f_n \to f$ in $L^1_\mu(x)$, $\lim_{n \to +\infty} ||f_n - f|| = \lim_{n \to +\infty} \int_X |f_n - f|\, d\mu$.

4. $f_n \to f$ in measure if $\forall \varepsilon > 0$, $\lim_{n \to +\infty} \mu(\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}) = 0$.

Theorem:

For (X, Λ, μ) with $\mu(x) < +\infty$, asssume

1. $f_n \to f$ μ-a.e. in X.

2. $||f_n||_\infty \leq M \leq +\infty$, $\forall n$

Then, $f_n \to f$ in $L^1_\mu(x)$. Therefore

$$\lim_{n\to+\infty} \int_X f_n \, d\mu = \int_X \left(\lim_{n\to+\infty} f_n \right) \, d\mu$$

Proof

Step 1: $f \in L^\infty_\mu(x)$ and $||f||_\infty \leq M$.
Given $\varepsilon > 0$, $\{x \in X : |f(x)| > M + \varepsilon\} \subset \{x : |f_n(x)| > M + \varepsilon\}$, $\forall n \geq n_0$.
Then, $\mu(\{x : |f(x)| > M + \varepsilon\}) = 0$. Therefore $||f||_\infty \leq M$.
Step 2: consider $\int_X |f_n - f| \, d\mu$.
Since $\mu(X) < +\infty$, by Egoroff's theorem $\exists A \subset X$ with $\mu(X \setminus A) < \frac{\varepsilon}{4M}$ where $f_n(x) \rightrightarrows f(x)$ in A.
Then, $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \frac{\varepsilon}{2\mu(x)}$, $\forall x \in A$, $\forall n \geq n_0$.

$$\begin{aligned} \int_X |f_n - f| \, d\mu &= \int_A |f_n - f| \, d\mu + \int_{X \setminus A} |f_n - f| \, d\mu \\ &= \frac{\varepsilon}{2\mu(x)} \mu(A) + 2M\mu(X \setminus A) \frac{\varepsilon}{4M} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

So $f_n \to f$ in $L^1_\mu(x)$.
Step 3: observe

$$\left| \int_X f_n \, d\mu - \int_X f \, d\mu \right| = \left| \int_X (f_n - f) \, d\mu \right| \leq \int_X |f_n - f| \, d\mu \overset{n\to+\infty}{\longrightarrow} 0$$

Remark

For $\mu(X) < +\infty$,

1. $L^\infty_\mu(x) \subset L^1_\mu(x)$.

2. $f_n \to f$ in $L^\infty_\mu(x) \implies f_n \to f$ in $L^1_\mu(x)$.

Theorem: Dominated Convergence

Let (X, Λ, μ) and $f_n \in M_\mu(x)$. If $\exists g \in L^1_\mu(x)$ such that $|f_n(x)| \leq g(x)$, $\forall n$ and $f_n \to f$ μ-a.e. in X, then $f_n \to f$ in $L^1_\mu(x)$.
In particular,

$$\lim_{n\to+\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$$

Proof

Note that $|f_n(x)| \le g(x)$, $\forall\, n$ means $|f(x)| \le g(x)$ and, conseuqently, that $f_n, f \in L^1_\mu(x)$.
Define $\phi_n(x) := 2g(x) - |f_n(x) - f(x)|$. Since

$$|f_n(x) - f(x)| \le |f_n(x)| + |f(x)| \le 2g(x)$$

$\phi_n \ge 0$.
By Fatou's lemma,

$$\begin{aligned}\int_X \left(\liminf_{n\to+\infty} \phi_n\right) d\mu &\le \liminf_{n\to+\infty} \int_X \phi_n \, d\mu \\ &\le \liminf_{n\to+\infty}\left(2\int_X g\, d\mu - \int_X |f_n - f| \, d\mu\right) \\ &= 2\int_X g\, d\mu - \limsup_{n\to+\infty} \int_X |f_n - f| \, d\mu\end{aligned}$$

Therefore

$$\limsup_{n\to+\infty} \int_X |f_n - f| \, d\mu \le 0 \implies \lim_{n\to+\infty} \int_X |f_n - f| \, d\mu = 0$$

and $f_n \to f$ in $L^1_\mu(x)$.

Definition: Vitality Continuity

On a measure space (X, Λ, μ), $\nu : \Lambda \to \mathbb{R}$ is said to be Vitali continuous if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\nu(A) < \varepsilon, \ \forall\, A \in \Lambda, \ \mu(A) < \delta$$

Write $\forall\, f \in L^1_\mu(x)$, $\nu_f(A) = \int_A |f| \, d\mu$.

Lemma

If $f \in L^1_\mu$, then ν_f is Vitali continuous.

- Proof
  Set $f_n(x) = \begin{cases} f(x) & |f(x)| \le n \\ n & |f(x)| > n \end{cases}$.
  Then $f_n \to f$ in X and $|f_n(x)| \le |f(x)|$. Therefore,

$$\int_A |f| \, d\mu \le \int_A \big||f| - |f_n|\big| \, d\mu + \int_A |f_n| \, d\mu$$

  By dominated convergence, for $\varepsilon > 0$, $\exists n_0$ such that $\int_X |f_n - f| \, d\mu < \frac{\varepsilon}{2}$ for all $n \ge n_0$. Then

$$\int_A \big||f| - |f_n|\big| \, d\mu \le \int_X \big||f| - |f_n|\big| \, d\mu \le \frac{\varepsilon}{2}, \quad \forall\, n \ge n_0$$

In particular

$$\int_A |f_{n_0}| \, d\mu \le n_0 \mu(A)$$

Letting $\delta = \frac{\varepsilon}{2n_0}$ gives

$$\int_A |f| \, d\mu \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

if $\mu(A) < \delta$.

Lemma

If (X, Λ, μ), $\mu(X) < +\infty$, and $f_n \to f$ μ-a.e. in X, then $f_n \to f$ in measure μ.

Remark

Proof can be done through Egoroff's Theorem.

Proof

Set $A_{n,\varepsilon} = \{x : |f_n(x) - f(x)| \ge \varepsilon\}$ and $A_\varepsilon = \bigcap_{n=1}^{\infty} \bigcup_{j \ge n} A_{j,\varepsilon}$, and $N = \bigcup_{\varepsilon > 0} A_\varepsilon$.
Then $N^c = \bigcap_{\varepsilon > 0} A_\varepsilon^c$, $A_\varepsilon^c = \bigcup_{n=1}^{j \ge n} A_{j,\varepsilon}^c$, and $A_{j,\varepsilon}^c = \{x : |f_j(x) - f(x)| < \varepsilon\}$.
Therefore, $\forall x \in N^c$, $f_n(x) \to f(x)$ and $\forall x \in N$, $f_n \not\to f(x)$.
So $\mu(N) = 0$ implies $\mu(A_\varepsilon) = 0$ for any $\varepsilon > 0$. Therefore

$$\mu\left(\bigcup_{j \ge n} A_{j,\varepsilon}\right) \to \mu(A_\varepsilon) = 0$$

since $\mu(X) < +\infty$. Then

$$\bigcup_{j \ge n}^{\infty} A_{j,\varepsilon} \supset \bigcup_{j \ge n+1}^{\infty} A_{j,\varepsilon}$$

and

$$A_{n,\varepsilon} \subset \bigcup_{j \ge n}^{\infty} A_{j,\varepsilon}$$

which implies $\mu(A_{n,\varepsilon}) \to 0$ as $n \to +\infty$.

Lemma (Chebyshev's Inequality)

~ Very Trivial ♥ ~
If $f \in L^1_\mu(x)$ and $f \ge 0$, then $\mu(\{x : f > \alpha\}) \le \frac{1}{\alpha} \int_X f \, d\mu$.

Proof

$$\int_X f \, d\mu \ge \int_{\{x : f(x) > \alpha\}} f \, d\mu \ge \int_{\{x : f(x) \ge \alpha\}} f \, d\mu = \alpha\mu(\{x : f(x) > \alpha\})$$

Corollary

$f_n \to f$ in $L^1_\mu(x)$ implies $f_n \to f$ in measure.
Since $\forall \varepsilon > 0$,

$$\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) \leq \frac{1}{\varepsilon}\int_X |f_n - f|\, d\mu \to 0$$

Definition: Vitali Equicontinuity

S sequence $\{\nu_n\}$ of Vitali continuous functions is Vitali equicontinuous if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\nu_n(A) < \varepsilon$, $\forall n$, $\forall A \in \Lambda$, $\mu(A) < \delta$.

Theorem

On (X, Λ, μ) with $\mu(X) < +\infty$, $f_n \to f$ in $L^1_\mu(x)$ if and only if ν_{f_n} is Vitali equicontinuous and $f_n \to f$ in measure μ.

Proof

($\Longrightarrow$) By assumption, $\int_X |f_n - f|\, d\mu \to 0$ as $n \to +\infty$.
Therefore, $\exists n_0 \in \mathbb{N}$ such that $\int_X |f_n - f|\, d\mu < \frac{\varepsilon}{2}$, $\forall n \geq n_0$. See that for all $n \geq n_0$,

$$\begin{aligned}\left|\int_A |f_n|\, d\mu - \int_A |f|\, d\mu\right| &= \int_A \left||f_n| - |f| big\right|\, d\mu \\ &\leq \int_X |f_n - f|\, d\mu \\ &< \frac{\varepsilon}{2}\end{aligned}$$

and therefore $\int_A |f_n|\, d\mu \leq \int_A |f|\, d\mu + \frac{\varepsilon}{2}$.
So there exists $\delta_0 > 0$ such that $\int_A |f_n|\, d\mu \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for any $n \geq n_0$ and $\mu(A) < \delta_0$.
Then $\exists \delta_n > 0$ such that $\int_A |f_n|\, d\mu < \varepsilon$, $\forall A \in \Lambda$ and $\mu(A) < \delta_n$.
Set $\delta = \min\{\delta_0, \ldots, \delta_{n_0 - 1}\} > 0$. Then $\int_A |f_n|\, d\mu < \varepsilon$, $\forall n$, $\forall A \in \Lambda$, $\mu(A) < \delta$.
($\Longleftarrow$)
By Vitali equicontinuity, $\exists \delta > 0$ giving $\int_A (|f_n| + |f|)\, d\mu < \frac{\varepsilon}{2}$, $\forall A \in \Lambda$, $\mu(A) < \delta$. Then

$$\begin{aligned}\int_X |f_n - f|\, d\mu &= \int_{\left\{x : |f_n(x) - f(x)| < \frac{\varepsilon}{2\mu(x)}\right\}} |f_n - f|\, d\mu + \int_{\left\{x : |f_n(x) - f(x)| \geq \frac{\varepsilon}{2\mu(x)}\right\}} |f_n - f|\, d\mu \\ &\leq \frac{\varepsilon}{2\mu(x)}\mu(x) + \int_{A_{n,\varepsilon}} (|f_n| + |f|)\, d\mu\end{aligned}$$

for $\varepsilon > 0$, $\mu(A_{n,\varepsilon}) \to 0$ as $n \to +\infty$.
So $\exists n_0 \in \mathbb{N}$ where $\mu(A_{n,\varepsilon}) < \delta$ for $n \geq n_0$ such that

$$\int_X |f_n - f|\, d\mu < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Theorem: Riesz Theorem

On (X, Λ, μ), $\mu(X) < +\infty$, if $f_n, f \in M_\mu(x)$ and $f_n \to f$ in measure then there exists a subsequence $\{f_{n_k}\} \subset \{f_n\}$ such that $f_{n_k} \to f$ μ-a.e.

Proof

Take

$$A_{n,\varepsilon}\{x : |f_n(x) - f(x)| \geq \varepsilon\}$$

and $f_n \to f$ in measure.
Then $\forall \varepsilon > 0$, $\mu(A_{n,\varepsilon}) \to 0$ as $n \to +\infty$.
Let $\varepsilon = \frac{1}{i}$. There exists n_i such that $\mu\left(A_{n,\frac{1}{i}}\right) < 2^{-i}$. Set

$$A = \bigcap_n \bigcup_{j \geq n} A_{n_j,\frac{1}{i}}$$

Claim

1. $\mu(A) = 0$.

2. $f_{n_k} \to f$ in $X \setminus A$.

   Since $\mu(X) < +\infty$,

$$\mu(A) = \lim_{n \to +\infty} \mu\left(\bigcup_{j \geq n} A_{n_j,\frac{1}{i}}\right)$$

where

$$\begin{aligned}
\mu\left(\bigcup_{j \geq n} A_{n_j,\frac{1}{i}}\right) &\leq \sum_{j \geq n} \mu\left(A_{n,\frac{1}{i}}\right) \\
&\leq \sum_{j \geq n} 2^{-i} \\
&\xrightarrow{n \to +\infty} 0
\end{aligned}$$

Then

$$X \setminus A = \bigcup_{n=1}^{+\infty} \bigcap_{j \geq n} A^c_{n_j,\frac{1}{i}}$$

where $A^c_{n_j,\frac{1}{i}} = \left\{x : |f_{n_j}(x) - f(x)| < \frac{1}{j}\right\}$, $\forall \varepsilon > \frac{1}{j_0}$.
So for some n_0, $x \in X \setminus A$ implies that $x \in \bigcap_{j \geq n_0} A^c_{n_j,\frac{1}{j}}$ where $j = \max\{n_0, j_0\}$.

February 20, 2024

Riesz Representation Theorem

Linear Functionals

On a vector space V, a map $T : V \to \mathbb{R}$ such that $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$, $\forall \alpha, \beta \in \mathbb{R}$, $\forall x, y \in V$ is called a linear functional.
A linear functional is positive if $Tf \geq 0$ when $f \geq 0$.

Example

On (X, Λ, μ), $L^1_\mu(X) = V$, take $Tf = \int_X f\, d\mu$. Then

$$T(\alpha x + \beta g) = \int_X \alpha x + \beta g\, d\mu = \alpha \int_X x\, d\mu + \beta \int_X g\, d\mu = \alpha Tf + \beta Tg$$

Example

On (X, Λ, μ), X locally compact Hausdorff, μ Radon.
$C_c(X)$, the space of continuous functions with compact support.
Recall: supp$(f) = \{x : f(x) \neq 0\}$ and supp$(f)^c = \{x : \exists$ open neighborhood U of $X, f = 0$ in $U\}$.
Then, $Tf = \int_X f\, d\mu$ on $C_c(x) \subset L^1_\mu(X)$ is a linear functional.

Theorem: Riesz Representation

Let X be a locally compact Hausdorff space and T be a positive linear functional on $C_c(X)$.
Then there exists a unique, complete Radon measure μ such that $Tf = \int_X f\, d\mu$.

Lemma 0

If X is locally compact Hausdorff, if $K \subset U \subset X$ with K comapct, U open, then there exists some V open with $\overline{V}$ compact such that $K \subset V \subset \overline{V} \subset U$.

Lemma 1 (Urysohn's)

If X is locally compact Hausdorff, if $K \subset U \subset X$ with K compact, U open, then there exists some continous function f with compact support such that

1. supp$(f) \subset U$
2. $0 \leq f \leq 1$
3. $f \equiv 1$ in K

Write $K \prec f \prec U$.

Radon Measure

For (X, Λ, μ), μ is a Radon measure if

1. μ is Borel
2. $\mu(K) < +\infty$ for K compact
3. $\mu(V) = \sup\{\mu(K) : K \subset V,\ K$ compact$\}$ for every V open.
4. $\mu(A) = \inf\{\mu(V) : A \subset V,\ V$ open$\}$ for every V open.

Proof: Step 1 (Uniqueness)

Suppose μ_1 and μ_2 such that $Tf = \int_X f\, d\mu_1 = \int_X f\, d\mu_2$, $\forall f \in C_c(X)$.
We want to show that $\mu_1(K) = \mu_2(K)$, $\forall K$ compact so that $\mu_1 = \mu_2$.
So, for any K compact, there is some V open with $V \supset K$ such that $\mu_2(V) < \mu_2(K) + \varepsilon$.
By Urysohn's lemma, $K \prec f \prec V$. So

$$\mu_1(K) = \int_K d\mu_1 = \int_X \chi_K\, d\mu \leq \int_X f\, d\mu_1 = \int_X f\, d\mu_2 \leq \mu_2(V) < \mu_2(K) + \varepsilon$$

Assuming $\mu_1(V) < \mu_1(V) + \varepsilon$ and repeating the proof mutatis mutandis shows $\mu_1 = \mu_2$.

Proof: Step 2 (Construction)

Let T be a positive linear function on $C_c(X)$.
We want to construct a complete Radon measure μ such that $Tf = \int_X f\, d\mu$, $\forall f \in C_c(X)$.

- Outer Measure
  For any U open, let $\mu^*(U) = \sup\{Tf : f \prec U\}$.
  Then for any $A \subset X$, $\mu^*(A) = \inf\{\mu^*(U) : A \subset U,\ U \text{ open}\}$.

  1. $\mu^*(\varnothing) = 0$.

  2. $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$.

  3. $\mu^*\left(\bigcup_i A_i\right) \leq \sum_i \mu^*(A_i)$, $\forall A_i \subset X$.

  – Lemma: Partition of Unity
    For X LCH, $U_1, U_2, \ldots, U_n$ open, K compact and $K \subset \bigcup_{i=1}^n U_i$.
    Then there exists a partition of unity $h_i \prec U_i$ and $\sum_{i=1}^n h_i = 1$ on K.
    Since, $\forall x \in K$, $\exists V_x$ open, $\overline{V}_x \subset U_i$ for some i.
    Then there exists a subcover $\{U_{x_i}\}_{i=1}^m$ and $H_i = \bigcup_i V_{x_i}$ while $\overline{V}_{x_i} \subset U_i$.
    Thus $\overline{H}_i$ is compact and $H_i \subset \overline{H}_i \subset U_i$.
    By Urysohn's lemma, $\exists \overline{A}_i \prec g_i \prec U_i$.
    Write $h_1 = g_1$, $h_2(1-g_1)g_2$, $h_k = (1-g_1)(1-g_2)\cdots g_k$, $h_n = (1-g_1)(1-g_2)\cdots(1-g_m)g_n$. Then

    1. $h_i \prec U_i$, since we have not modified the support.

    2. $K \prec \sum_i h_i$, since $\forall x \in K \subset \bigcup_i A_i \subset \bigcup_i \overline{A}_i \subset \bigcup_i U$.

       Then $x \in \overline{H}_{i_0}$ for some i_0 implies that $g_{i_0}(x) = 1$.

    $$\sum_i h_i(x) = \sum_{i \leq i_0} h_i(x) = g_1(x) + (1-g_1(x))g_2(x) + \cdots + (1-g_1(x))\cdots(1-g_{i_0-1}) = g_1(x) + (1-g_1(x)) = 1$$

    Therefore, $K \subset \bigcup_i \overline{A}_i \prec \sum_{i=1}^n h_i$.

  – Proof of 3
    Take $\bigcup_i U_i$, U_i open and consider $\mu^*\left(\bigcup_i U_i\right)$.
    Then $\forall f \prec \bigcup_i U_i$, there exists a finite subcover $f \prec \bigcup_{j=1}^n U_{i_j}$, $\{U_{i_j}\} \subset \{U_i\}$.

By the partition of unity, $\exists h_j \prec U_{i_j}$ where $\sum h_j = 1$ on supp(f). So

$$f = \left(\sum_j h_j\right) f = \sum_j (h_j f)$$

and

$$Tf = \sum_j T(h_j f) \le \sum_j \mu^*(U_{i_j} \quad \text{and} \quad h_j f \prec U_{i_j}$$

It follows that $\mu^*\left(\bigcup_i U_i\right) \le \sum_i \mu^*(U_i)$.
For $\bigcup_i A_i$, $A_i \subset X$, by definition there exists U_i open with $U_i \supset A_i$ and $\mu^*(U_i) \le \mu^*(A_i) + \frac{\varepsilon}{2i}$. Thus

$$\mu^*\left(\bigcup_i A_i\right) \le \mu^*\left(\bigcup_i U_i\right) \le \sum_i \mu^*(U_i) \le \sum_i \left(\mu^*(A_i) + \frac{\varepsilon}{2i}\right) \le \sum_i \mu^*(A_i) + \varepsilon$$

Therefore μ^* is an outer measure and, by the Caratheodory construction, (X, Λ, μ) complete.

- Radon Measure
  1. Borel.
  2. $\mu(K) < +\infty$ for K compact.
  3. $\mu(V) = \sup\{\mu(K) : K \subset V,\ K$ compact$\}$.
  4. $\mu(A) = \inf\{\mu(V) : A \subset V,\ V$ open$\}$.

  – Proof of 2
  By definition of μ^*, for any K compact there is some V open such that $K \subset V$ and $\mu(K) \le \mu(V)$.
  By Urysohn's lemma, $K \subset \bigcup_i H_i \subset \bigcup \overline{H}_i \prec f \prec V$ and

$$\mu(K) \le \mu\left(\bigcup_i H_i\right) \le Tf < +\infty, \quad f \in C_c(X)$$

  since $\mu^*\left(\bigcup_i H_i\right) = \sup\{Tg : g \prec \bigcup_i H_i\}$ for $g \le f$.

  – Proof of 3
  $\forall K \subset V$, K compact, V open, $\mu(K) \le \mu(V)$, by the definition of the outer measure $\exists f \prec V$ such that

$$\mu^*(V) \le Tf + \frac{\varepsilon}{2}$$

  We have supp$(f) = K \subset V$, so there exists U open $U \supset K$ such that $\mu^*(U) \le \mu^*(K) + \frac{\varepsilon}{2}$.
  By Urysohn's lemma, $\exists K \prec g \prec U$ and

$$\mu^*(V) < Tf + \frac{\varepsilon}{2} \le Tg + \frac{\varepsilon}{2} \le \mu^*(U) + \frac{\varepsilon}{2} \le \mu^*(K) + \varepsilon$$

  Therefore, $\mu^*(V) = \sup\{\mu^*(K) : K \subset V,\ K$ compact$\}$.

– Lemma
If $A, B \subset X$, $\exists U \supset A$ U open, $\exists V \supset B$ V open, such that $U \cap V = \varnothing$.
Then $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.

* Proof

For $\forall W$ open, $W \supset A \cup B$, take

$$\begin{cases} W_1 = W \cap A \\ W_2 = W \cap B \end{cases}$$

such that $W_1 \cap W_2 = \varnothing$.
Fact: $f \prec W$ if and only if $f = f_1 + f_2$ where $f_1 \prec W_1$ and $f_2 \prec W_2$.
Since $Tf = Tf_1 + Tf_2$ gives $\mu^*(W) = \mu^*(W_1) + \mu^*(W_2) \geq \mu^*(A) + \mu^*(B)$, we have

$$\mu^*(A) + \mu^*(B) \geq \mu^*(A \cup B) \geq \mu^*(A) + \mu^*(B)$$

– Lemma (Proof of 1)
If for any A open, $\mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \mu^*(E)$, then μ is Borel.

* Proof

For any open set $V \supset E$, $\mu^*(V) \leq \mu^*(E) + \frac{\varepsilon}{2}$.
By 3, $V \cap A$ is open and $\exists K$ comapct with $K \subset V \cap A$ such that $\mu^*(V \cap A) \leq \mu^*(K) + \frac{\varepsilon}{2}$. So

$$\mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \mu^*(V \cap A) + \mu^*(E \cap A^c) \leq \frac{\varepsilon}{2} + \mu^*(K) + \mu^*(E \cap A^c)$$

Since $K \subset V \cap A \subset A$ and A open, we may find $K \subset W \subset \overline{W} \subset A$ where $K \subset W$ and $A^c \subset \overline{W}^c$. Therefore

$$\frac{\varepsilon}{2} + \mu^*(K \cup (E \cap A^c)) \leq \frac{\varepsilon}{2} + \mu^*((V \cap A) \cup (V \cap A^c)) \leq \frac{\varepsilon}{2} + \mu^*(V) \leq \varepsilon + \mu^*(E)$$

Therefore $A \in \Lambda$, and $\mathcal{B} \subset \Lambda$.

Proof: Step 3 (Verify)

For any $f \in C_c(X)$, write $f(x) \in [a, b]$.
Take $P = \{a = y_0 < y_1 < \cdots < y_{n-1} < y_n = b\}$ with $\ell(P) = \max\{y_i - y_{i-1} : i = 1, \ldots, n\}$.
Then, take $A_i = \{x \in X : y_{i-1} < f(x) \leq y_i\} \cap \text{supp}(f)$.
We have $\bigcup_i A_i = \text{supp}(f)$.
So for each A_i there is some V_i open where $V_i \supset A_i$, $f(x) < y_i + \varepsilon$, $\forall x \in V_i$, and

$$\text{supp}(f) = \bigcup_i A_i \subset \bigcup_i V_i$$

By partition of unity, $\exists h_i \prec V_i$ such that $\sum_i h_i = 1$ in $\text{supp}(f)$.
Therefore $f = \sum_i (h_i f)$ and $Tf = \sum_i T(h_i f)$.
We want to show that $Tf \leq \int_X f \, d\mu$ since linarity will make the reverse true by taking $-f$.

Since $fh_i \le (y_i + \varepsilon)h_i$,

$$\begin{aligned}
T(h_i f) &\le (y_i + \varepsilon)Th_i \\
&\le (|a| + y_i + \varepsilon)Th_i - |a|Th_i \\
&\le (|a| + y_i + \varepsilon)\mu(V_i) - |a|Th_i \\
&\le y_{i-1}\mu(A_i) \\
&\le \int_{A_i} f\, d\mu + c\varepsilon
\end{aligned}$$

By summing each term, we get

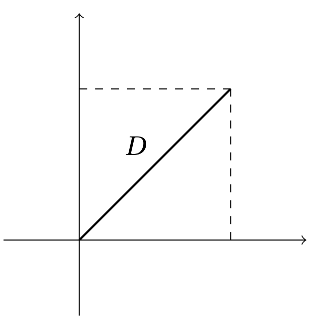
$$\sum_i T(h_i f) \le \int_X f\, d\mu + c\varepsilon$$

February 22, 2024

Fubini's Theorem

Product of measure spaces.

Example 1



Given m a Lebesgue measure, m_c a counting measure, $\chi_D(x,y)$, $\forall\, x \in [0,1]$,

$$\int \chi_D(x,y)\, dm_c(y) = \int_{[0,1]} \chi_{\{x=y\}}(y)\, dm_c(y) = \chi_{[0,1]}(x)$$

$$\int_{[0,1]}\int_{[0,1]} \chi_D(x,y)\, dm(y)dm(x) = \int_{[0,1]} \chi_{[0,1]}\, dm(x) = 1$$

And $\forall\, y \in [0,1]$,

$$\int_{[0,1]} \chi_D(x,y)\, dm(x) = \int \chi_{\{x=y\}}\, dm(x) = 0$$

$$\int_{[0,1]}\int_{[0,1]} \chi_D(x,y)\, dm(x)dm(y) = 0$$

Example 2

For

$$0 = \alpha_1 < \alpha_2 < \cdots \to 1$$

and $g_n(x) = \frac{1}{\alpha_{n+1}-\alpha_n}\chi_{[\alpha_n,\alpha_{n+1}]}$, $x \in [0,1]$.

1. $\int_{[0,1]} g_n(x)\, dm(x) = 1$

2. $f(x,y) = \sum_{n=1}^{+\infty}(g_n(x) - g_{n+1}(x))g_n(y)$

3.

$$\forall x \in [0,1], \quad \int_{[0,1]} f(x,y)\, dm(y)$$

$$\forall x \in [\alpha_n, \alpha_{n+1}], n > 1, \quad \int_{[0,1]} -g_n(x)g_{n-1}(y) + g_n(x)g_n(y)\, dm(y) = 0$$

$$\forall x \in [\alpha_1, \alpha_2], n = 1, \quad \int_{[0,1]} g_1(x)g_1(y)\, dm(y)$$

$$\int_{x,y} f(x,y)\, dm(y) = g_1(x)$$

$$\int_{[0,1]}\left(\int_{[0,1]} f(x,y)\, dm(y)\right) dm(x) = \int_{[0,1]} g(x)\, dm(x) = 1$$

For $\forall n \in [0,1]$, $y \in [\alpha_n, \alpha_m]$

$$\int_{[0,1]} f(x,y)\, dm(x) = \left(\int (g_n(x) - g_{n+1}(x))\, dm(x)\right) g_n(y) = 0$$

$$\int_{[0,1]}\left(\int_{[0,1]} f(x,y)\, dm(x)\right) dm(y) = 0$$

Therefore, with (X, Λ, μ) and (Y, Γ, ν), $(x \times y, \Lambda \times \Gamma, \mu \times \nu)$?
We want

$$\int_X \int_Y f(x,y)\, d\nu(y) d\mu(x) = \int_{X\times Y} f(x,y)\, dm(\mu \times \nu) = \int_Y \int_X f(x,y)\, d\mu(x) d\nu(y)$$

Definition: Elementary Set

Take $A \in \Lambda$, $B \in \Gamma$ and construct $R = A \times B \subset X \times Y$ a measurable rectangle.
Define $Q = \bigcup_{i=1}^{k} R_i$ where $\{R_i\}$ are finitely many disjoint, measurable rectangles.
Then $(\mu \times \nu)(R) = \mu(A)\nu(B)$.
Take $\Lambda \times \Gamma$ the σ-algebra generated by all measurable rectangles.

Definition: Monotone Class

A collection M of subsets is a monotone class if

1. $A_i \in M$, $A_i \subset A_{i+1} \implies \bigcup_i A_i) \in M$.

2. $A_i \in M$, $A_i \supset A_{i+1} \implies \bigcap_i A_i \in M$.

Proposition:

Let M be the monotone class generated by the set E of all elementary sets, then $M = \Lambda \times \Gamma$.

Proof

$M \subset \Lambda \times \Gamma$.
Then, $\forall P \subset X \times Y$, define $\Omega(P) = \{Q : P \setminus Q, Q \setminus P, P \cup Q \in M\}$ with

1. $Q \in \Omega(P)$ if and only i $P \in \Omega(Q)$.
2. $\Omega(P)$ is a monotone class.
3. If $P \in E$, then $E \subset \Omega(P)$. Therefore $M \subset \Omega(P)$.
4. So $\forall P \in M$, $M \subset \Omega(P)$ and $\forall P, Q \in M$, $P \setminus Q$, $Q \setminus P$, $P \cup Q \in M$.
5. $X \times Y \in E \in M$, so $\forall P \in M$, $P^c = X \times Y \setminus P \in M$.

Proposition:

If $E \in \Lambda \times \Gamma$, then $E_X = \{y : (x, y) \in E\} \in \Gamma$ and $E^Y = \{x : (x, y) \in E\} \in \Lambda$.

Proof

1. For any measurable rectangle $R = A \times B$, $R_X = B \in \Gamma$ and $R^Y = A \in \Lambda$.
2. For $(A_i)_X \in \Gamma$ and $(A_i)^Y \in \Lambda$, $\left(\bigcup_i A_i\right)_X \in \Gamma$ and $\left(\bigcup_i a_i\right)^Y \in \Lambda$.
3. For A with $A_X \in \Gamma$ and $A^Y \in \Lambda$, $(A^c)_X \in \Gamma$ and $(A^c)^Y \in \Lambda$.

Product Measure on Elementary Sets

Given $\mu \times \nu$, $(\mu \times \nu)(R) = \mu \times \nu)(A \times B) = \mu(A)\nu(B)$.

$$\int_{X \times Y} \chi_{A \times B}(x, y)\, d(\mu \times \nu) = (\mu \times \nu)(A \times B) = \mu(A)\nu(B)$$

Define

$$\begin{aligned}
\phi(x) &= \int_Y \chi_{A \times B}(x, y)\, d\nu(y) = \nu(B)\chi_A \\
\psi(y) &= \int_X \chi_{A \times B}(x, y)\, d\mu(x) = \mu(A)\chi_B
\end{aligned}$$

so

$$\int_X \phi\, d\mu = \int_X \int_Y \chi_{A \times B}\, d\nu d\mu = \mu(A)\nu(B) = \int_Y \int_X \chi_{A \times B}\, d\mu d\nu = \int_Y \psi(y)\, d\nu$$

Now $\forall P \in \Lambda \times \Gamma$,

$$\begin{aligned}
\phi(x) &= \int_Y \chi_P(x, y)\, d\nu(y) = \int_Y \chi_{P_x}\, d\nu \\
\psi(y) &= \int_X \chi_P(x, y)\, d\mu(x) = \int_X \chi_{P^y}\, d\mu
\end{aligned}$$

so

$$(*) \quad (\mu\times\nu)(P) = \int_X\int_Y \chi_P \, d\nu d\mu = \int_X \phi \, d\mu = \int_Y\int_X \chi_P \, d\mu d\nu = \int_Y \psi \, d\nu$$

Theorem:

On (X,Λ,μ) and (Y,Γ,ν) σ-finite, the equality $*$ holds.
Recall that a space is σ-finite if $X = \bigcup_i X_i$, $X_i \in \Lambda$, $\mu(X_i) < +\infty$.
One may assume $X_i \subset X_{i+1}$.

Proof

1. E ok!

2. $P_i \in \Lambda\times\Gamma$, $P_i \subset P_{i+1}$, and the equality of the product measure holds for any i.

   If $P_i \subset P_{i+1}$, $\chi_{P_i} \le \chi_{P_{i+1}}$, $\phi_i \le \phi_{i+1}$, $\psi_i \le \psi_{i+1}$, $\phi_i \to \phi$ and $\psi_i \to \psi$.
   Apply monotone convergence theorem for integration.

3. $P_i \in \Lambda\times\Gamma = M$, $P_i \supset P_{i+1}$, $\int \phi_1 \, d\mu < +\infty$, and $\int \psi_1 \, d\nu < +\infty$.

   If 1, 2 and 3 hold, then $M = \Lambda\times\Gamma$.

4. $X = \bigcup_k X_k$, $Y = \bigcup_k Y_k$, $\Lambda_k = \{A\cap X_k \,:\, A\in\Lambda\}$, $\Gamma_k = \{B\cap Y_k \,:\, B\in\Gamma\}$.

   Then take $\Lambda_k\times\Gamma_k = M_k$. By 2, $M_k \to M$ and 4 implies 3 holds.

Definition: Product Measure

Define

$$(\mu\times\nu)(P) = \int_X \phi \, d\mu + \int_Y \psi \, d\nu = \int_X\int_Y \chi_P \, d\nu d\mu = \int_Y\int_X \chi_P \, d\mu d\nu$$

Then

$$\int_{X\times Y} \chi_P \, d(\mu\times\nu)$$

On $(X\times Y, \Lambda\times\Gamma, \mu\times\nu)$.

Proposition:

If $f(x,y)$ is measurable, then $\forall\, y\in Y$, $f_y(x)$ is measurable and $\forall\, x\in X$, $f_x(y)$ is measurable.

Proof

1. χ_P measurable gives $P\in\Lambda\times\Gamma$ which implies $P_x\in\Gamma$ for all $x\in X$ and $P^y\in\Lambda$ for any $y\in Y$.

2. $\phi_n(x,y) \to f(x,y)$ pointwise on $X\times Y$, then $(\phi_n)_x(y) \to f_x(y)$ in Y and $(\phi_n)_y(x) \to f_y(x)$ in X for fixed $x\in X$, $y\in Y$ respectively.

Therefore,

$$\phi_n = \sum_{j=1}^{k} \alpha_j \chi_{P_j} \quad \text{and} \quad \forall\, x \in X,\ (\phi_n)_x(y) = \sum_{j=1}^{k} \alpha_j \chi_{(P_j)_x}$$

$$\forall\, y \in Y,\ (\phi_n)_y(x) = \sum_{j=1}^{k} \alpha_j \chi_{(P_j)^y}$$

Theorem: Fubini Theorem

Let (X, Λ, μ) and (Y, Γ, ν) be σ-finite measure spaces, and take $f(x, y)$ measurable on $(X \times Y, \Lambda \times \Gamma, \mu \times \nu)$. Assume also that $f \geq 0$.

$$\int_X \left(\int_Y f(x, y)\, d\nu(y) \right) d\mu(x) = \int_{X \times Y} f(x, y)\, d(\mu \times \nu) = \int_Y \left(\int_X f(xy)\, d\mu(x) \right) d\nu(y)$$

Proof

There exist ϕ_n simple such that $\phi_n \to f$ monotonically.

Corollary

When f assumes negative values, if

$$\int_X \int_Y |f(x, y)|\, d\nu(y) d\mu(x) < +\infty$$

then Fubini holds for f. Likewise when

$$\int_{X \times Y} |f(x, y)|\, d(\mu \times \nu) < +\infty$$

February 27, 2024

Definition: Lp Space

For (X, Λ, μ) a complete measure space,

$$L^p_\mu(x) = \left\{ f \,:\, \int_X |f|^p\, d\mu < +\infty \right\}$$

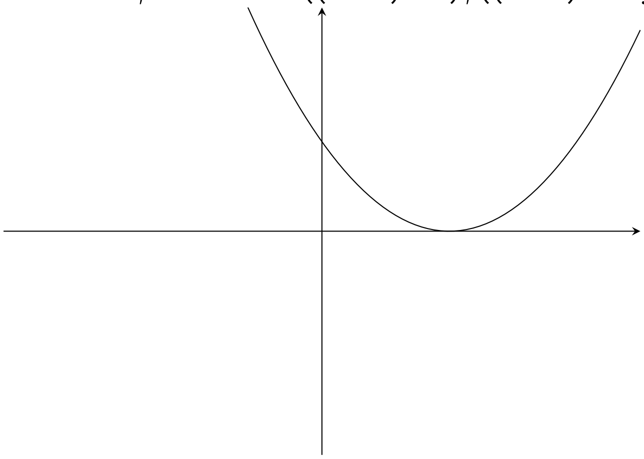
where $1 \leq p \leq +\infty$ and we identify $[f] = [g]$ if $f = g$ μ-a.e.

Definition: Banach Space

A normed, complete vector space.

Definition: Convec Funxtions

A function ϕ is convex if $((1-\lambda)+\lambda)\phi((1-\lambda)x+\lambda y) \leq (1-\lambda)\phi(x)+\lambda\phi(y)$,



Equivalently,

$$\frac{[\phi((1-\lambda)x+\lambda y)-\phi(x)]}{\lambda(y-x)} \leq \frac{[\phi(y)-\phi((1-\lambda)x+\lambda y)]}{(1-\lambda)(y-x)}$$
$$\frac{\phi(z)-\phi(x)}{z-x} \leq \frac{\phi(y)-\phi(z)}{y-z}$$
$$\phi'(a) \leq \phi'(b)$$

Theorem:

If ϕ is differentiable, then ϕ is convex if and only if ϕ' is non decreasing.
And if ϕ is twice differentiable, ϕ is convex if and only if $\phi'' \geq 0$.

Corollary

e^x is convex, since

$$e^{(1-\lambda)x+\lambda y} \leq (1-\lambda)e^x + e^y$$

Then if $e^x = a$ and $e^y = b$

$$a^{1-\lambda}b^{\lambda} \leq (1-\lambda)a + \lambda b$$

for $\lambda \in (0,1)$.
If $\lambda = \frac{1}{2}$, then $\sqrt{ab} \leq \frac{a+b}{2}$.

Theorem: Jensen's Inequality

For ϕ convex and (X, Λ, μ) with $\mu(X) = 1$,

$$\phi\left(\int_X f\, d\mu\right) \leq \int_X \phi \circ f\, d\mu$$

where the range of f is in the domain of ϕ.
Compare: $\phi\left(\frac{a+b}{2}\right) \leq \frac{1}{2}(\phi(a)+\phi(b))$.

Proof

Write $t = \int_X f\, d\mu$. Then $\forall\, a < t < b$,

$$\frac{\phi(t) - \phi(a)}{t-a} \leq \frac{\phi(b) - \phi(t)}{b-t}$$

Set $\beta = \sup_a \frac{\phi(t)-\phi(a)}{t-a}$, then

$$\begin{aligned} \frac{\phi(t) - \phi(a)}{t-a} &\leq \beta \\ \phi(t) &\leq \beta(t-a) + \phi(a) \end{aligned}$$

$$\begin{aligned} \frac{\phi(b) - \phi(t)}{b-t} &\geq \beta \\ \phi(b) - \phi(t) &\geq \beta(b-t) \\ \phi(t) &\leq \phi(b) + \beta(t-b) \end{aligned}$$

Therefore

$$\begin{aligned} \phi(t) &\leq \phi(s) + \beta(t-s), \quad \forall\, s \\ \phi(t) &\leq \phi \circ f + \beta(t-s), \quad \forall\, x \in X \\ \phi(t) &\leq \int_X \phi \circ f\, d\mu + \beta \overbrace{\left(t - \int_x f\, d\mu\right)}^{=0} \\ \phi\left(\int_X f\, d\mu\right) &\leq \int_X \phi \circ f\, d\mu \end{aligned}$$

Compare: $e^{\int_X f\, d\mu} \leq \int_X e^{f(x)}\, d\mu$.

Theorem: Holder Inequality

On (X, Λ, μ) with $1 \leq p \leq +\infty$,

$$\left|\int_X fg\, d\mu\right| \leq \left(\int_X |f|^p\right)^{\frac{1}{p}} \left(\int_X |g|^q\right)^{\frac{1}{q}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p = 1 \implies q = \infty$ and $p = \infty \implies q = 1$.

Proof

Take $||f||_p = \left(\int_X |f|^p\, d\mu\right)^{\frac{1}{p}}$.
For $p = 1$, $q = \infty$ or $p = \infty$, $q = 1$,

$$\left|\int_X fg\, d\mu\right| \leq |f||g|\, d\mu \leq ||g||_\infty \int_X |f|\, d\mu = ||f||_1 ||g||_\infty$$

We have $\frac{1}{p}+\frac{1}{q}=1$ and $1-\lambda=\frac{1}{p}$ while $\lambda=\frac{1}{q}$, so

$$\begin{aligned}\frac{|f|}{||f||_p}\cdot\frac{|g|}{||g||_q} &= \left(\frac{|f|^p}{||f||_p}\right)^{\frac{1}{p}}\left(\frac{|g|^q}{||g||_q}\right)^{\frac{1}{q}}\\ &= \left(\frac{|f|^p}{||f||_p}\right)^{\frac{1}{p}}\left(\frac{|g|^q}{||g||_q}\right)^{\frac{1}{q}}\end{aligned}$$

For

$$\begin{aligned}\left|\int_X fg\,d\mu\right| &\leq \int_X (|f||g|)\,d\mu\\ \int_X \frac{|f|}{||f||_p}\cdot\frac{|g|}{||g||_q} &\leq \int_X \frac{1}{p}\frac{|f|^p}{||f||_p^p}+\frac{1}{q}\frac{|g|^q}{||g||_q^q}\\ \frac{\int_X |fg|\,d\mu}{||f||_p||g||_q} &\leq \frac{1}{p}\frac{\int_X |f|^p\,d\mu}{\int_X |f|^p\,d\mu}+\frac{1}{q}\frac{\int_X |g|^q\,d\mu}{\int_X |g|^q\,d\mu}\\ &\leq \frac{1}{p}+\frac{1}{q}\end{aligned}$$

Theorem: Minkowsky Inequality

On (X,Λ,μ) with $1\leq p\leq +\infty$,

$$\left(\int_X |f+g|^p\,d\mu\right)^{\frac{1}{p}} \leq \left(\int_X |f|^p\,d\mu\right)^{\frac{1}{p}} + \leq \left(\int_X |g|^p\,d\mu\right)^{\frac{1}{p}}$$

Proof

If $p=1$,

$$\begin{aligned}\int_X |f+g|\,d\mu &\leq \int_X |f|\,d\mu+\int_X |g|\,d\mu\\ ||f+g||_{L^\infty} &\leq ||f||_\infty+||g||_\infty\end{aligned}$$

For $1<p<+\infty$, $1<q<+\infty$, $\frac{1}{p}+\frac{1}{q}=1$,

$$\frac{1}{q}=1-\frac{1}{p}=\frac{p-1}{p}\quad\text{and}\quad\frac{p}{p+1}=q$$

therefore

$$\begin{aligned}||f+g||_p^p=\int_X |f+g|^p\,d\mu &= \int_X |f+g|^{p-1}|f+g|\,d\mu\\ &\leq \int_X |f+g|^{p-1}|f|\,d\mu+\int_X |f+g|^{p-1}|g|\,d\mu\\ &\leq \left(\int_X |f+g|^{p-1\frac{p}{p-1}}\,d\mu\right)^{\frac{p-1}{p}}\left(\int_X |f|^p\,d\mu\right)^{\frac{1}{p}}+\left(\int_X |f+g|^{p-1\frac{p}{p-1}}\,d\mu\right)^{\frac{p-1}{p}}\left(\int_X |g|^p\,d\mu\right)^{\frac{1}{p}}\\ &= ||f+g||_p^{p-1}(||f||_p+||g||_p)\end{aligned}$$

Theorem:

$L_\mu^p(x)$ is a Banach space with $1 \le p \le +\infty$.

Proof

It suffices to verify $L_\mu^p(x)$ is complete, but the $p = +\infty$ case must be considered separately.
For $1 \le p < +\infty$, let $\{f_n\}$ with $f_n \in L_\mu^p(x)$ be Cauchy.
We want to show that $\exists f \in L_\mu^p(x)$ such that $||f_n - f||_p \to 0$ as $n \to +\infty$.
Recall: a sequence is cauchy if $\forall \varepsilon > 0$, $\exists k \in \mathbb{N}$ such that $||f_n - f_m||_p < \varepsilon$, $\forall n, m \ge k$.
Pick f_{n_k} such that $||f_{n_{i+1}} - f_{n_i}||_p \le 2^{-i}$.
Take $g_k = \sum_{i=1}^{k} |f_{n_{i+1}}(x) - f_{n_i}(x)|$ and define $g(x) = \sum_{i=1}^{\infty} |f_{n_{i+1}}(x) - f_n(x)|$.
By the Minkowski inequality,

$$||g_k||_p \le \sum_{i=1}^{k} ||f_{n_{i+1}} f_{n_i}||_p \le 1$$

Therefore $\int_X |g_k|^p \, d\mu \le 1$, $\forall k$.
Then, by Fatou's Lemma

$$\int_X |g|^p \, d\mu \le 1$$

so g is μ-a.e. finite. So

$$s_k(x) = \sum_{i=1}^{k} \left(f_{n_{i+1}}(x) - f_{n_i}(x)\right) \to s(x) = \sum_{i=1}^{\infty} \left(f_{n_{i+1}}(x) - f_{n_i}(x)\right)$$

Therefore, by dominated convergence,

$$s_k \to s \text{ in } L_\mu^p(x) \quad \text{and} \quad f_{n_k} \to s + f_{n_i}(x) = f(x) \text{ in } L_\mu^p(x)$$

For $p = +\infty$, let

$$\begin{aligned} B_k &= \{x : |f_k(x)| > ||f_k||_\infty\} \\ B_{m,n} &= \{x : |f_m(x) - f_n(x)| > ||f_m - f_n||_\infty\} \end{aligned}$$

Then $B = \left(\bigcup_k B_k\right) \cup \left(\bigcup_{m,n} B_{m,n}\right)$ and $\mu(B) = 0$. Examining the convergence on $X \setminus B$ completes the proof.

Theorem:

Let (X, Λ, μ) be a complete measure space with X Locally Compact Hausdorff and μ Radon.
Then $C_c(X) \subset L_\mu^p(x)$, $1 \le p < +\infty$.

Remark

Write $||f||_C = \sup_X |f(x)|$, and take $C_0(X)$ the collection of continuous functions vanishing at infinity to be the completion.

Proof

Step 1: $s_n(x) \to f$, where $s_n = \sum_{i=1}^{k} \alpha_i \chi_{A_i} \in L_\mu^p(x)$.
Step 2: If f is bounded, and $\mu(\text{supp}(f)) < +\infty$, we may use Vitali-Lusin.

February 29, 2024

Recall: Lp Space is Banach

Given (X, Λ, μ), $L_\mu^p(x)$ is a Banach space given $||f||_p = \left(\int_X |f|^p \, d\mu\right)^{1/p}$,
$1 \leq p \leq +\infty$ and $||f||_\infty = \inf\{\mu : \mu(\{x \, : \, |f| > \mu\}) = 0\}$.

Definition: Linear Operator

Given vector spaces $V \to W$, $\alpha, \beta \in \mathbb{R}$, and $u, v \in V$, the map (or operator) $T : V \to W$ is linear if

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$$

Definition: Linear Functional

If $L : V \to \mathbb{R}$ for linear operator L, then L is called a linear functional.

Definition: Operator Norm

For normed vector spaces, we have $||T|| = \sup\{||Tx|| \, : \, ||x|| \leq 1\}$.

Definition: Bounded Linear Functional

A linear functional $L : V \to \mathbb{R}$ which sastisfies $|L(v)| \leq ||L||||v||$.

Definition: Dual Space

If V is a normed vector space, then the dual space V^* is the collection of all bounded linear functionals $L : V \to \mathbb{R}$.

Theorem:

$(L^p)^* = L^q$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p, \, q < +\infty$.

Proof

The general proof will require Radon-Nikodym.
In this case, $\forall g \in L^q \implies L_g : L^p \to \mathbb{R}$.
Take $\phi(g) = L_g : L^q \to (L^p)^*$ so $L_g = \int_X f \cdot g \, d\mu$, $\forall f \in L^p$. Then

$$|L_g(f)| = \left|\int_X f \cdot g \, d\mu\right| \leq \int_X |f||g| \, d\mu \leq ||g||_q||f||_p$$

So $||L_g|| \leq ||g||_q$. We claim that $||L_g|| = ||g||_q$. Take

$$f = \frac{\text{sign}(g)|g|^{q-1}}{||g||_q^{q-1}}$$

and, since, $||g||_q^q = \int_X |g|^q \, d\mu$ and $q = p(q-1)$,

$$\int_X |f|^p \, d\mu = \int_X \frac{|g|^{p(q-1)}}{||g||_q^{p(q-1)}} \, d\mu = \frac{\int_X |g|^q \, d\mu}{\int_X |g|^q \, d\mu} = 1$$

Therefore,

$$L_g(f) = \int_X f \cdot g \, d\mu = \frac{\int_X |g|^q \, d\mu}{||g||_q^{q-1}} = ||g||_q$$

Since L_g is a linear operator, $L_g f_1 - L_g f_2 = L_g(f_1 - f_2)$ and $L_{g_1}(f) + L_{g_2}(f) = L_{g_1+g_2}(f)$.
That is, $||L_g|| = ||g||_q$ and L_g is injective. We claim that $L_G : L^q \to (L^p)^*$ is an isometric isomorphism.
Step 1 of proving isometry is that $\forall L \in (L^p)^*$, $\exists \nu$ such that $L(f) = \int_X f \, d\nu$, $\forall f \in L^p$.
Step 2, Radon-Nikodym, $\exists g \in L^q$ where $d\nu = g d\mu$. That is $\frac{d\nu}{d\mu} = g$.

Useful Inequalities

Chebyshev's Inequality

Suppose $f \in L^p$, then

$$\mu(\{x : |f| > \alpha\}) \leq \frac{||f||_p^p}{\alpha^p}$$

- Proof

$$||f||_p^p = \int_X |f|^p \, d\mu \geq \int_{\{x : |f| > \alpha\}} |f|^p \, d\mu \geq \int_{\{x : |f| > \alpha\}} \alpha^p \, d\mu$$

Minkowski's Inequality

$$\left|\left| \int_Y f(x,y) \, d\nu(y) \right|\right|_p \leq \int_Y ||f(x,y)||_p \, d\nu(y)$$

Equivalently

$$\left(\int_X \left| \int_Y f(x,y) \, d\nu(y) \right|^p \, d\mu(x) \right)^{\frac{1}{p}} \leq \int_Y \left(\int_X |f(x,y)|^p \, d\mu(x) \right)^{\frac{1}{p}} \, d\nu(y)$$

Recall

$$\int_X |fg| \, d\mu \leq ||f||_p ||g||_q$$

for $\frac{1}{p} + \frac{1}{q} = 1$.
Then

$$||f||_p \leq ||f||_r^{\theta} ||f||_s^{1-\theta}$$

if $\frac{1}{p} = \frac{\theta}{r} + \frac{1-\theta}{s}$ for $r < p < s$. Since $p\left(\frac{\theta}{r} + \frac{1-\theta}{s}\right) = 1$,

$$\frac{1}{\frac{r}{p\theta}} + \frac{1}{\frac{s}{p(1-\theta)}} = 1$$

and

$$\int_X |f|^p \, d\mu = \left(\int_X |f|^{p\theta} |f|^{p(1-\theta)}\right)^{\frac{1}{p}} \le \left(\int_X |f|^r\right)^{\frac{\theta}{r}} \left(\int_X |f|^s\right)^{\frac{1-\theta}{s}} = ||f||_r^{\theta} ||f||_s^{1-\theta}$$

For $r < p < \infty$,

$$\left(\int_X |f|^p\right)^{\frac{1}{p}} = \left(\int_X |f|^r |f|^{p-r}\right)^{\frac{1}{p}} \le ||f||_{\infty}^{1-\frac{r}{p}} \left(\left(\int_X |f|^r\right)^{\frac{1}{r}}\right)^{\frac{r}{p}} = ||f||_{\infty}^{\frac{r}{p}} ||f||_{\infty}^{1-\frac{r}{p}}$$

Homework 6 Problem 5

$$\int_X f \, d\mu = \sup\left\{\int_X s \, d\mu \, : \, 0 \le s \le f\right\}$$

so

$$\int_X f \, d\mu - \frac{1}{n} \le \int_X s_n \, d\mu \le \int_X f \, d\mu$$

Alternatively, $\forall f \ge 0$, $\exists s_n$ simple $0 \le s_n \le f$, $0 \le s_n \le s_{n+1}$. So

$$s_n = \sum \frac{k}{2^i} \chi_{A_{n,k}}$$

gives

$$\int_X s_n \, d\mu \to \int_X f \, d\mu$$

by monotone convergence theorem.

Homework 6 Problem 6

$$F(x) = \int_{-\infty}^{x} f(t) \, dt$$

was shown to be Vitali continuous, so

$$F(x) - F(y) = \left|\int_{(x,y)} f(t) \, dt\right| < \varepsilon$$

when $\mu((x,y)) = y - x < \delta$.

Homework 6 Problem 7

Given

$$\int_{\mathbb{R}} f_n \, dm \to \int_{\mathbb{R}} f \, dm$$

and $A \subset \mathbb{R}$, Fatou's Lemma gives

$$\int_A f \, dm \leq \liminf_{n \to +\infty} \int_A f_n \, dm$$

$$\int_{A^c} f \, dm \leq \liminf_{n \to +\infty} \left(\int_{\mathbb{R}} f_n \, dm - \int_A f_n \, dm \right)$$

Therefore

$$\int_{\mathbb{R}} f \, dm - \int_A f \, dm \leq \int_R f \, dm - \limsup_{n \to +\infty} \int_A f_n \, dm$$

Homework 6 Problem 8

Given

$$\int_{\mathbb{R}} g(x)(f(x+t) - f(x)) \, dx \to 0$$

with f, g integrable and $|g| \leq M$.
Part 1
If $f(x)$ is continuous with compact support, we would have
$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|f(x+t) - f(x)| < \frac{\varepsilon}{2kM}$, $\forall |f| < \delta$ where $\operatorname{supp}(f)[-k, k]$.
Then, $\forall \varepsilon > 0$, $\exists \delta > 0$,

$$\begin{aligned}
\left| \int_{\mathbb{R}} g(x)(f(x+t) - f(x)) \, dx \right| &\leq \int_{\mathbb{R}} |g(x)||f(x+t) - f(x)| \, dx \\
&\leq M \int_{-k}^{k} |f(x+t) - f(x)| \, dx \\
&\leq M(2k) \frac{\varepsilon}{2kM} \\
&= \varepsilon
\end{aligned}$$

when $|f| < \delta$.
Part 2
$||f - g||_{L^1} \leq \frac{\varepsilon}{2M}$, we have

$$\begin{aligned}
\left| \int_{\mathbb{R}} g(x)(f(x+t) - f(x)) \, dx - \int_{\mathbb{R}} g(x)(f(x+t) - f(x)) \, dx \right| &= \left| \int_{\mathbb{R}} g(x) \left((f(x+t) - f(x)) - (f(x) - g(x)) \right) \, dx \right| \\
&\leq M \int_{\mathbb{R}} (|f(x+t) - g(x+t)| + |f(x) - g(x)|) \\
&\leq 2M ||f - g||_{L^1(\mathbb{R})} \\
&< \frac{\varepsilon}{2}
\end{aligned}$$

Part 3
We need $C_c(\mathbb{R}) \subset L^1(\mathbb{R})$ to be dense.
We may patch our functions with Urysohn's Lemma or, more explicitly,
Since $f_n = f\chi_{[-n,n]} \xrightarrow{n\to\infty} f$, $f_n \to f$ in L^1 by dominated convergence theorem. Then

$$\phi_n = \begin{cases} f & |f| \le n \\ n & f \ge n \\ -n & f \le -n \end{cases} \to f$$

Homework 7

1: Calculate.
2: Fatou's Lemma to $g \pm f_n$.
3: Part 3 of Homework 6 Problem 8.
5: Use monotone class and monotone convergence.
7: Do the rectangles.

Problem 4

Part 1
With Riemann integration, take

$$\begin{aligned} \int_a^b f(x)\sin(nx)\,dx &= \int_a^b f(x)\frac{1}{n}d(-\cos(nx)) \\ &= \overbrace{\frac{1}{n}f(x)(-\cos(nx))\big|_a^b}^{0} + \frac{1}{n}\int_a^b f'(x)cos(nx)\,dx \end{aligned}$$

and $\int_a^b |f'(x)|\,dx < +\infty$.
Part 2

$$\left|\int f(x)\sin(nx)\,dx - \int g(x)\sin(nx)\right| \le \int |f-g|\,dx$$

Part 3
Density. We need smooth

$$h(x) = \int g_n(x-y)f(x)\,dy$$

Problem 6

Write

$$\int_0^\infty \frac{\sin(x)}{x}\,dx = \int_0^\infty\int_0^\infty e^{-tx}\sin(x)\,dtdx = \int_0^\infty\int_0^\infty e^{-tx}\sin(x)\,dxdt$$

By integration by parts,

$$\int_0^\infty \left(\int_0^\infty e^{-tx}\sin(x)\,dx\right)dt = \int_0^\infty \frac{1}{1+t^2}\,dt$$

March 5, 2024

Definition: Signed Measure

A function $\nu : \Lambda \to \mathbb{R}$, $\forall A \in \Lambda$, $\nu(A) \in \mathbb{R}$ which is countably additive (i.e. if $A_i \cap A_j = \varnothing$ then $\nu(\bigcup A_i) = \sum \nu(A_i)$).

Remarks

1. $\nu : \Lambda \to \mathbb{R}_+ = \{r \in \mathbb{R} : r \geq 0\}$ is a signed measure and a finite measure.
2. $f \in L^1_\mu(x)$, (X, Λ, μ), $\nu(A) = \int_A f \, d\mu$.

Lemma: Signed Measure is Bounded from Above

On (X, Λ) with ν a signed measure, $\exists M > 0$ such that $|\nu(A)| \leq M$, $\forall A \in \Lambda$.

Proof

Assume, for sake of contradiction, that there is no such M.
Claim: Then $\exists E \in \Lambda$ such that $\nu(E) > 1$ and $\nu(A) \leq \nu(E) + 1$, $\forall A \subset E$.

- Proof of Claim

  Assume, again for sake of contradiction, that $\forall E \in \Lambda$ such that $\nu(E) > 1$, $\exists A \subset E$ such that $\nu(A) > \nu(E) + 1 > 1$.
  Then there exists $E_{i+1} \subset E_i \subset \cdots \subset E$ with $\nu(E_{i+1}) > \nu(E_i) + 1$. This gives

  $$E \setminus \bigcap_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} E_{i-1} \setminus E_i$$

  but since $\nu(E_{i-1} \setminus E_i) = \nu(E_{i-1}) - \nu(E_i) < -1$,

  $$\nu\left(E \setminus \bigcap_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \nu(E_{i-1} \setminus E_i) = -\infty$$

  a contradiction.

- By the Claim

  $\exists E_n \in A$ with $\nu(E_n) > n + \sum_{i=1}^{n-1} \nu(E_i)$ and $\nu(A) \leq \nu(E_n) + 1$, $\forall A \subset E_n$.
  For $A_i \subset E_i \cap E_n \subset E_n$ with $A_i \cap A_j = \varnothing$, we have $\bigcup_{i=1}^{n-1} A_i = \bigcup_{i=1}^{n-1} (E_i \cap E_n)$, so

$$
\begin{aligned}
\left(\bigcup_{n=1}^{\infty} E_n\right) &= \nu\left(\bigcup_{n=1}^{\infty}\left(E_n \setminus \bigcup_{i=1}^{n-1} E_i\right)\right) \\
&= \sum_{n=1}^{\infty} \nu\left(E_n \setminus \bigcup_{i=1}^{n-1} E_i\right) \\
&= \sum_{n=1}^{\infty}\left[\nu(E_n) - \nu\left(E_n \cap \left(\bigcup_{i=1}^{n-1} E_i\right)\right)\right] \\
&\geq \sum_{n=1}^{\infty}\left[\nu(E_n) - \sum_{i=1}^{n-1}(\nu(E_n)+1)\right] \\
&\geq \sum_{n=1}^{\infty} 1 \\
&\geq \infty
\end{aligned}
$$

a contradiction.

Definition: Variation

$$
|\nu|(A) = \sup\left\{\sum_i |\nu(E_i)| \; : \; \{E_i\} \text{ is a partition of } A\right\}
$$

Definition: Total Variation

$$
||\nu|| = |\nu|(X)
$$

Lemma: Variation is a Finite Measure

Given (X, Λ) and ν a signed measure, $(X, \Lambda, |\nu|)$ is a finite measure space.

Proof

Monotonicity is given by the definition.
For finite, we claim $|\nu|(A) \leq 2M$, $\forall A \in \Lambda$.
By the definition, $\exists\{E_i\}$ a partition of A such that

$$
\begin{aligned}
|\nu|(A) &\leq \sum_i |\nu(E_i)| + \varepsilon \\
&= \sum_{\nu(E_i)>0} \nu(E_i) - \sum_{\nu(E_i)<0} \nu(E_i) + \varepsilon \\
&= \nu\left(\bigcup_{\nu(E_i)>0} E_i\right) - \nu\left(\bigcup_{\nu(E_i)<0} E_i\right) + \varepsilon \\
&\leq 2M + \varepsilon
\end{aligned}
$$

For countable additivity, take $\{A_i\} \subset \Lambda$ a countably disjoint collection.
Then for all i, $\exists\left\{E_j^i\right\}_j$ a partition of A_i such that

$$
|\nu|(A_i) \leq \sum_j \left|\nu\left(E_j^i\right)\right| + 2^{-i+1}\varepsilon
$$

and where $\left\{E_j^i\right\}_{\substack{j=1,\dots,\infty \\ i=1,\dots,k}}$ is a partition for $\bigcup_{i=1}^k A_i$,

$$\begin{aligned}
\sum_{i=1}^k |\nu|(A_i) &\le \left(\sum_{i=1}^k \sum_j \left|\nu\left(E_j^i\right)\right|\right) + \varepsilon \\
&\le |\nu|\left(\bigcup_{i=1}^k A_i\right) + \varepsilon \\
&\le |\nu|\left(\bigcup_{i=1}^\infty A_i\right) + \varepsilon
\end{aligned}$$

So $\sum_{i=1}^\infty |\nu|(A_i) \le |\nu|\left(\bigcup_{i=1}^\infty A_i\right)$.
Then, given $\{E_i\}$ a partition of $\bigcup_{i=1}^\infty A_i$ such that

$$|\nu\left(\bigcup_{i=1}^\infty A_i\right) \le \sum_k |\nu(E_k)| + \varepsilon$$

we have that $\{A_i \cap E_k\}_k$ partitions A_i. So

$$\begin{aligned}
|\nu|(A_i) &\ge \sum_i \sum_k |\nu(A_i \cap E_k)| \\
&= \sum_k \sum_i |\nu(A_i \cap E_k)| \\
&\ge \sum_k \left|\sum_i \nu(A_i \cap E_k)\right| \\
&= \sum_k |\nu(E_k)| \\
&\ge |\nu|\left(\bigcup_{i=1}^\infty A_i\right) - \varepsilon
\end{aligned}$$

Therefore $\sum_{i=1}^\infty |\nu|(A_i) = |\nu|\left(\bigcup_{i=1}^\infty A_i\right)$.

Theorem: Jordan Decomposition

For any (X, Λ) with ν a signed measure, we have two finite measures ν^+ and ν^- such that $\nu = \nu^+ - \nu^-$.

Proof

Set $\nu \le \nu^+ = \frac{1}{2}(|\nu| + \nu) \le |\nu|$ and $\nu^- = \frac{1}{2}(|\nu| - \nu) \le |\nu|$.

Lemma:

$\nu^+(A) = \sup\{\nu(F) : F \subset A\}$ and $\nu^- = -\inf\{\nu(F) : F \subset A\}$.

Proof

$$\nu(F) \le \nu^+(F) \le \nu^+(A) \quad \text{and} \quad \sup\{\nu(F) : F \subset A\} \le \nu^+(A)$$

Then, if $\{B, C\}$ is a partition of A for positive and negative values,

$$|\nu|(A) \le \nu(B) - \nu(C) + \varepsilon \quad \text{and} \quad \nu(A) = \nu(B) - \nu(C)$$

therefore $\nu^+(A) \le \nu(B) + \frac{\varepsilon}{2} \le \sup\{\nu(F) : F \subset A\} + \frac{\varepsilon}{2}$ and $\nu^+(A) \le \sup\{\nu(F) : F \subset A\}$.

Theorem: Hahn Decomposition

For any (X, Λ) with ν a signed measure, we have $X = E \cup F$, $E \cap F = \varnothing$, and $\nu(A) \ge 0$ for $A \subset E$ while $\nu(A) \le 0$ for $A \subset F$.

Proof

We have $\nu^+(X) = \sup\{\nu(A) : A \subset X\}$, so $\exists A_n$ such that $\nu^+(x) - 2^{-n} \le \nu(A_n) \le \nu^+(X)$.
For $i \ge n+1$, since $\nu\left(A_i \cap \bigcup_{k=n}^{i-1} A_k\right) \le \nu^+\left(A_i \cap \bigcup_{k=n}^{i-1} A_k\right) \le \nu^+(X)$,

$$\begin{aligned}\nu\left(A_i \setminus \bigcup_{k=n}^{i-1} A_k\right) &= \nu(A_i) - \nu\left(A_i \cap \bigcup_{k=n}^{i-1} A_k\right) \\ &\ge \nu^+(X) - 2^{-i} - \nu^+(X) \\ &\ge -2^{-i}\end{aligned}$$

so $\nu\left(\bigcup_{i=n}^{\infty} A_i\right) \ge \nu(A_n) + \nu\left(\bigcup_{i=n+1}^{\infty}\left(A_i \setminus \bigcup_{k=n}^{i-1} A_k\right)\right) \ge \nu^+(X) - 2^{-n}$.
Take $E = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i$, and we claim that $\nu(E) = \nu^+(X)$.

- Proof of Claim

$$\nu^+(X) \ge \nu(E) = \nu\left(\bigcup_{i=n}^{\infty} A_i\right) - \nu\left(\bigcup_{i=n}^{\infty} A_i \setminus E\right) \ge \nu^+(X) - 2^{-n}$$

- Verify

$$\nu^+(X) = \nu(E) = \nu(A) + \nu(E \setminus A) \le \nu(A) + \nu^+(E \setminus A) \le \nu(A) + \nu^+(X)$$

  such that $\nu(A) \ge 0$.
  Then take $F = E^c$. For all $A \subset F$,

$$\nu^+(X) \ge \nu^+(E \cup A) \ge \nu(E \cup A) = \nu(E) + \nu(A) = \nu^+(X) + \nu(A)$$

  such that $\nu(A) \le 0$.

Remark

On (X, Λ, μ) with $f \in L^1_\mu(X)$

$$\begin{aligned}\nu(A) &= \int_A f \, d\mu \\ |\nu|(A) &= \int_A |f| \, d\mu \\ \nu^+(A) &= \int_A f^+ \, d\mu \\ \nu^-(A) &= \int_A f^- \, d\mu\end{aligned}$$

so $\nu = \nu^+ - \nu^-$ and $X = \{x : f(x) \ge 0\} \cup \{x : f(x) < 0\}$.

Example: Point Charge

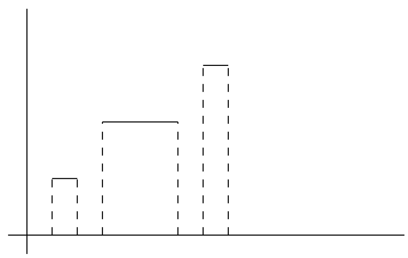
For $x_0 \in X$,

$$\nu(A) = \begin{cases} 1 & \text{if } x_0 \in A \\ 0 & \text{otherwise} \end{cases}$$

Then $\nu(A) \neq \int_A f \, d\mu$ for any $f \in L^1_\mu(X)$.

Example: Cantor Function

Also called the double stairs.
A function ϕ with the graph



For $\phi \in C$, we have $\phi(r) = \lim_{\substack{x \to r \\ x \in C^c}} \phi(x)$ and $\mu_\phi((a,b)) = \phi(b) - \phi(a)$.
Furthermore, $\mu_\phi(C) = 1$ and $\mu(C^c) = 0$.
The conclusion is that one necessary condition is $\nu(A) = 0$ if $\mu(A) = 0$.

March 7, 2024

Recall: Signed Measure

On (X, Λ) with Λ a σ-algebra, a function $\nu : \Lambda \to \mathbb{R}$ such that

$$\nu\left(\bigcup_i A_i\right) = \sum_i \nu(A_i)$$

for $A_i \cap A_j = \varnothing$.

Example

(X, Λ, μ), $f \in L^1_\mu(X)$,

$$\nu_f(A) = \int_A f \, d\mu$$

$\forall\, A \in \Lambda$.

Question

Given (X, Λ, μ) and $\nu : \Lambda \to \mathbb{R}$, is $\int_A |f| \, d\mu = 0$ when $\mu(A) = 0$ sufficient to make $\nu = \nu_f$ when $f \in L^1_\mu(X)$?

Recall: Signed Measure Bounded from Above

Given (X, Λ) and $\nu : \Lambda \to \mathbb{R}$, $\exists M > 0$ such that $|\nu(A)| \leq M$, $\forall A \in \Lambda$.

Recall: Variation of Signed Measure

$$|\nu|(A) = \sup\left\{\sum_i |\nu(E_i)\ :\ \{E_i\} \text{ is a partiation of } A\right\}$$

Recall: Norm from Variation

$$||\nu|| + |\nu|(X)$$

Recall: Variation is a Finite Measure

$(X, \Lambda, |\nu|)$ is a finite measure space.

Recall: Jordan Decomposition

Given (X, Λ) and $\nu : \Lambda \to \mathbb{R}$ a signed measure, then

$$\nu^+ = \frac{1}{2}(|\nu| + \nu), \quad \nu^- = \frac{1}{2}(|\nu| - \nu), \quad \text{and} \quad \nu = \nu^+ - \nu^-$$

where ν^+ and ν^- are finite measures.

Recall: Lemma

Given (X, Λ) and $\nu : \Lambda \to \mathbb{R}$ a signed measure, we have

$$\nu^+ = \sup\{\nu(F)\ :\ F \subseteq A\} \quad \text{and} \quad \nu^- = -\inf\{\nu(F)\ :\ F \subseteq A\}$$

Recall: Hahn Decomposition

Given (X, Λ) and $\nu : \Lambda \to \mathbb{R}$ a signed measure, we have $X = E \cup F$ with $E \cap F = \varnothing$ such that $\nu(A) \geq 0$ for $A \subseteq E$ and $\nu(A) \leq 0$ for $A \subseteq F$.

Proof

By the preceding lemma, $\forall n$, $\exists A_n \in \Lambda$ such that

$$\nu^+(X) - 2^{-n} \leq \nu(A_n) \leq \nu^+(A_n) \leq \nu^+(X)$$

where $E = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i$.
Claim: $\nu(E) = \nu^+(X)$.
Part 1

$$\nu^+(X) \geq \nu\left(\bigcup_{i=n}^{\infty} A_i\right) = \nu\left(A_n \cup (A_{n+1} \setminus A_n) \cup \cdots \cup \left(A_k \setminus \bigcup_{i=n}^{k-1} A_i\right) \cup \cdots\right) \geq \nu^+(X) - 2^{-n+1}$$

since

$$\nu\left(A_k \setminus \bigcup_{i=n}^{k-1} A_i\right) = \nu(A_k) - \nu\left(A_k \cap \bigcup_{i=n}^{k-1} A_i\right) \geq \nu^+(X) - 2^{-k} - \nu^+(X) \geq -2^{-k}$$

Part 2
For all n,

$$\nu(E) = \nu\left(\bigcup_{i=n}^{\infty} A_i\right) - \nu\left(\bigcup_{i=n}^{\infty} \setminus E\right) \geq \nu^+(X) - 2^{-n+1}$$

and

$$\nu\left(\bigcup_{i=n}^{\infty} A_i \setminus E\right)$$

where

$$\bigcup_{i=n}^{\infty} A_i = \left(\bigcup_{i=n}^{\infty} A_i \setminus \bigcup_{i=n+1}^{\infty} A_i\right) \cup \left(\bigcup_{i=n+1}^{\infty} A_i \setminus \bigcup_{i=n+2}^{\infty} A_i\right) \cup \cdots$$

so

$$\nu\left(\bigcup_{i=k}^{\infty} A_i \setminus \bigcup_{i=k+1}^{\infty} A_i\right) = \nu\left(\bigcup_{i=k}^{\infty} A_i\right) - \nu\left(\bigcup_{i=k+1}^{\infty} A_i\right) \leq \nu^+(X) - (\nu^+(X) - 2^{-k-2}) \leq 2^{-k+2}$$

Therefore, $\forall A \subset E$, we have

$$\nu^+(X) = \nu(E) = \nu(A) + \nu(E \setminus A) \leq \nu(A) + \nu^+(X)$$

and $\nu(A) \geq 0$ while $\forall A \subset F$

$$\nu^+(X) \geq \nu(A \cup E) = \nu(A) + \nu(E) = \nu(A) + \nu^+(X)$$

so $\nu(A) \leq 0$.

Example: Jordan

Given (X, Λ, μ), $f \in L^1_\mu(X)$ and $\nu_f(A) = \int_A f \, d\mu$,

$$|\nu_f|(A) = \int_A |f| \, d\mu, \quad \nu_f^+(A) = \int_A f^+ \, d\mu, \quad \nu_f^-(A) = \int_A f^- \, d\mu \quad \text{and} \quad \nu_f = \nu_f^+ - \nu_f^-$$

Example: Hahn

Given $E = \{x : f(x) \geq 0\}$ and $F = \{x : f(x) < 0\}$, $X = E \cup F$.

Definition: Absolute Continuity

Given (X, Λ, μ) and $\nu : \Lambda \to \mathbb{R}$ a signed measure, we say $\nu << \mu$ (ν is absolutely continuous with respect to μ) if

$$\mu(A) = 0 \implies |\nu|(A) = 0$$

Lemma:

Given (X, Λ, μ) and $\nu : \Lambda \to \mathbb{R}$ a signed measure, $\nu << \mu$ if and only if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|\nu|(A) < \varepsilon$, $\forall A \in \Lambda$, $\mu(A) < \delta$.

Proof

$(\impliedby)$ Trivial.
$(\implies)$ Assume, for sake of contradiction, that there exists $\varepsilon_0 > 0$ such that $\forall n$, $\exists A_n$ where $|\nu|(A_n) \geq \varepsilon_n$ while $\mu(A_n) \leq 2^{-n}$.
Write $A = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i$ such that $\mu\left(\bigcup_{i=n}^{\infty} A_i \leq 2^{-n+1}\right)$ and

$$\mu(A) = \lim_{n \to +\infty} \mu\left(\bigcup_{i=n}^{\infty} A_i\right) = 0$$

but since

$$|\nu|\left(\bigcup_{i=n}^{\infty} A_i\right) \geq |\nu|(A_n) \geq \varepsilon_0$$

we have

$$|\nu|(A) = \lim_{n \to +\infty} |\nu|\left(\bigcup_{i=n}^{\infty} A_i\right) \geq \varepsilon_0$$

a contradiction.

Theorem:

Let (X, Λ, μ) be a complete σ-finite measure space, $\nu : \Lambda \to \mathbb{R}$ a signed measure and $\nu << \mu$.
Then, $\exists! f \in L^1_\mu(X)$ such that $\nu(A) = \nu_f(A) = \int_A f \, d\mu$.

Proof: Uniqueness

;

$$\implies$$

Proof: Step 1

Assume ν and μ are finite measures and define

$$G = \left\{ g \,:\, g \geq 0, \text{ measurable, and } \int_A g\, d\mu \leq \nu(A) \right\}$$

then set

$$M = \sup\left\{ \int_X g\, d\mu \,:\, g \in G \right\} \leq \nu(X)$$

For any n, $\exists g_n \in G$ such that $M - \frac{1}{n} < \int_X g_n\, d\mu \leq M$. Then for $f_n = \max\{g_1, \ldots, g_n\}$,

$$M - \frac{1}{n} \leq \int_X f_n\, d\mu \leq M$$

Since $f_n \to f$ with $f_n, f \in G$, by monotone convergence $\int_X f\, d\mu = M$.
Claim: $\nu(A) = \int_A f\, d\mu$, $\forall A \in \Lambda$.
Otherwise, $\exists A_0 \in \Lambda$ such that $\int_{A_0} f\, d\mu < \nu(A_0)$ $(\nu(A_0) > 0)$
Therefore $\exists \varepsilon > 0$ such that $\int_A (f + \varepsilon)\, d\mu < \nu(A_0)$.
Then take $\xi(A) = \nu(A) - \int_A (f+\varepsilon)\, d\mu$.
We have the Hahn decompositon $A_0 = E_0 \cup F_0$. Therefore $\xi(A) \geq 0$, $\forall A \subseteq E_0$ and $\xi(A) \leq 0$, $\forall A \subseteq F_0$. Then

$$g = \begin{cases} f & E_0^c \\ f + \varepsilon & E_0 \end{cases} \in G$$

since $\int_A g\, d\mu = \int_{A \cap E_0} g\, d\mu + \int_{A \cap E_0^c} f\, d\mu \leq \nu(A \cap E_0^c(A) \leq \nu(A)$.
So

$$\int_X g\, d\mu = \int_{E_0} g\, d\mu + \int_{E_0^c} g\, d\mu = \int_{E_0} (f+\varepsilon)\, d\mu + \int_{E_0^c} f\, d\mu = \varepsilon\mu(E_0) + M$$

Then $\nu << \mu$ implies $\mu(E_0) > 0$.

Corollary

For (X, Λ, μ) a σ-finite measure space, ν a finite measure and $\nu << \mu$, then $\forall g \in L^1_\nu(X)$, $\exists f \in L^1_\mu(X)$

$$\int_A g\, d\nu = \int_A fg\, d\mu$$

since $\nu(A) = \int_A f\, d\mu$. Therefore $f = \frac{d\nu}{d\mu}$.

Definition: Mutual Singularity

Signed measures ν_1 and ν_2 are said to be mutually singular if $\exists X = E \cup F$ $(E \cap F = \varnothing)$ such that

$$\begin{cases} |\nu_1|(E) = 0 \\ |\nu_2|(F) = 0 \end{cases}$$

Write $\nu_1 \perp \nu_2$.

Remark

If ν_1 is a signed measure and μ is a measure where $\nu_1 \perp \mu$ and $\nu_1 << \mu$, then $\nu = 0$.

Recall: Cantor Set

Given μ_ϕ a measure from the cantor set and Lebesgue measure m, we have $\mu_\phi \perp m$.

Theorem:

Given (X, Λ, μ) a σ-finite measure space and ν a signed measure, there are unique $\nu = \nu_s + \nu_a$ where $\nu_s \perp \mu$ and $\nu_a << \mu$.

Proof: Uniqueness

$$\nu_s + \nu_a = \nu_s^* + \mu_a^* \implies \nu_s - \nu_s^* = \mu_a^* - \mu_a \implies \text{uniqueness}$$

Proof: Step 1

For $\nu << \nu + \mu$, $\exists f$ where

$$\nu(A) = \int_A f\, d(\nu + \mu) = \int_A f\, d\nu + \int_A f\, d\mu$$

so take $E = \{x : f \geq 1\}$ and $F = \{x : f < 1\}$.

1. $\nu(E) \geq \nu(E) + \mu(E) \implies \mu(E) = 0$.
2. $\forall A \subseteq F$, $\nu(A) \leq \int_A f\, d\nu + \mu(A)$ if $\mu(A) = 0 \implies \nu(A) = 0$.

Then $\nu_a(A) = \nu(A \cap F)$ and $\nu_s(A) = \nu(A \cap E)$.

Duality of Lp and Lq

On (X, Λ, μ) a σ-finite measure space, given $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p, q < +\infty$, we have $L^q = (L^p)^*$ and $\phi : L^q \to (L^p)^*$.
For $L \in (L^p)^*$, we want $\exists g \in L^q$ such that $L(X) = \int_X fg\, d\mu$.
$\forall L \in (L^p)^*$, $\mu(X) < +\infty$, $\nu_L(A) = L(\chi_A)$, $\nu\left(\bigcup_i A_i\right) = L\left(\sum \chi_{A_i}\right)$, $\sum_{i=1}^k \chi_{A_i} \to \sum_{i=1}^\infty \chi_{A_i}$ in L^p.
Then $\mu\left(\bigcup_{i=1}^k A_i\right) \to \mu\left(\bigcup_{i=1}^\infty A_i\right)$.
So $L(\chi_A) = \nu_L(A) = 0$ if $\mu(A) = 0$; $\chi_A = 0$ μ-a.e. if $\mu(A) = 0$.
Therefore for g, $\nu_L(A) = \int_A g\, d\mu$, therefore $\forall s$ simple functions

$$\int_X s\, d\nu = \int_X sg\, d\mu$$

and $\forall f \in L_\mu^\infty(X)$, $s_n \to f$, $L(s_n) \to L(f)$.

$$\int_X s_n\, d\mu = \int_X s_n g\, d\mu \to \int_X fg\, d\mu$$

Then $f_n = \text{sign}(g)|g|^{q-1}\chi_{\{x : |q| \leq n\}} \in L_\mu^\infty \subset L_\mu^p$.

$$L(f_n) = \int_x f_n g\, d\mu = \int_X |g\chi_{\{x : |g| \leq n\}}|^q\, d\mu$$

$$||f_n||^p = ||g\chi_{\{x\,:\,|g|\le n\}}||_q^{q-1}$$

$$||L|| \ge \frac{|Lf_n|}{||f_n||_p} = ||g\chi_{\{x\,:\,|g|\le n\}}||_q$$

Therefore $g \in L^\infty_\mu(X)$, $g\chi_{\{x\,:\,|g|\le n\}} \to g$.
Then for any $f \in L^p_\mu(X)$, $f\chi_{\{x\,:\,|f|\le n\}} \to f$.

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Recall: Fundemental Theorem of Calculus

$$f(a) - f(b) = \int_a^b f'(t)\,dt$$

for $f \in C^1$, continuously differentiable.

Definition: Vitali Covering

Given $E \subset \mathbb{R}$, a collection of intervals $F = \{I\}$ such that $\forall \epsilon > 0$, $\forall x \in E$, there exists $I \in F$ such that

1. $x \in I$
2. $|I| < \varepsilon$

Lemma: Vitali Covering Lemma

Given $E \subset \mathbb{R}$, $m^*(E) < +\infty$ and F a Vitali covering of E, then $\forall \varepsilon > 0$ there exists $\{I_i\} \subset F$ finite and disjoint such that $m^*\left(E \setminus \bigcup_i I_i\right) < \varepsilon$.

Proof

For $\{I\} = F$, we may assume I closed and assume that there exists some open set $U \supset E$ such that $m^*(U) \le m^*(E) + 1 < +\infty$.
Note that we may limit the size of our intervals such that $\forall I \in F$, $I \subset U$.

- Step 1
  1. I_1 is arbitrary.
  2. Inductively, assume $\{I_i\}_{i=1}^k$ with $I_i \cap I_j = \varnothing$ and select I_{k+1}.
     (a) Select $l_k = \sup\{|I| \,:\, I \in F, I \cap I_i = \varnothing, i = 1, \dots, k\}$.
     (b) Pick I_{k+1} such that $|I_{k+1}| \ge \frac{1}{2} l_k$ and $I_{k+1} \cap I_i = \varnothing$ for $i = 1, \dots, k$.
- Step 2

  For $\bigcup_{i=1}^\infty I_i \subset U$, $\sum_{i=1}^\infty |I_i| \le m^*(U) < +\infty \implies l_k \to 0$ as $k \to +\infty$.
  Further, $\exists N$ such that $\sum_{i=N+1}^\infty |I_i| < \frac{\varepsilon}{5}$.

- Step 3

  $\forall\, x \in E \setminus \bigcup_{i=1}^N I_i$, $x \in \bigcup_{i=N+1}^\infty 5I_i$. Where $5I$ is the interval I with the same center and five times the length.
  Then $\exists I$, $x \in I$ where $I \cap I_i = \varnothing$, $\forall\, i = 1, \ldots, N$.
  Define

  $$n_0 = \max\{n \,:\, I \cap I_i = 0, i = 1, \ldots, n\}$$

  such that $I \cap I_{n_0+1} = \neq$. Then $|I| \leq l_{n_0}$ and $|I_{n_0}| \geq \frac{1}{2} l_{n_0}$. Therefore $I \subset 5I_{n_0+1}$.
  Note that we know n_0 exists by the fact that $l_k \to 0$.
  Therefore $E \setminus \bigcup_{i=1}^N I_i \subset \bigcup_{i=N+1}^\infty 5I_i$ and, subsequently, $m^*\left(E \setminus \bigcup_{i=1}^N I_i\right) < \varepsilon$.

Definition: Dini Derivative

Define

$$D^r f(x) = \limsup_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$
$$D_r f(x) = \liminf_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$
$$D^l f(x) = \limsup_{h \to 0^+} \frac{f(x) - f(x-h)}{h}$$
$$D_l f(x) = \liminf_{h \to 0^+} \frac{f(x) - f(x-h)}{h}$$

Fact

$f'(x)$ exists if and only if $D^r f(x) = D_r f(x) = D^l f(x) = D_l f(x)$.

Theorem: Differentiability of Monotone Functions

If $f : [a, b] \to \mathbb{R}$ is nondecreasing, then it is differentiable m-a.e.

Proof

Let

$$B = \{x \,:\, D^r f(x) > D_l f(x)\} \cup \{x \,:\, D^l f(x) > D_r f(x)\}$$

and

$$B^c = \{x \,:\, D^r f(x) \leq D_l f(x)\} \cap \{x \,:\, D^l f(x) \leq D_r f(x)\}$$

That is, B is the collection of points at which f is not differentiable and

$$D_r f(x) \leq D^r f(x) \leq D_l f(x) \leq D^l f(x) \leq D_r f(x)$$

gives B^c the feature of including all differentiable points. Write

$$\{x \,:\, D^r f(x) > D_l f(x)\} = \bigcup_{\substack{p,q \in \mathbb{Q} \\ p > q}} \underbrace{\{x \,:\, D^r f(x) > p > q > D_l f(x)\}}_{E_{p,q}}$$

Claim: $m^*(E_{p,q}) = 0$.
Assume, for sake of contradiction, that $m^*(E_{p,q} = \delta > 0$.

- Step 1

  Then there exists U open such that $U \supset E_{p,q}$ and $m^*(U) < \delta + \varepsilon$.
  Then $\forall x \in E_{p,q}$, $D_l f(x) < q \implies \exists [x-h, x]$ such that $f(x) - f(x-h) < hq$ for arbitarily small h.
  Therefore $\{[x-h, x]\} = F_l$ is a Vitali covering of $E_{p,q}$.
  By the Vitali Covering Lemma, $\exists \{I_i\}_{i=1}^k \subset F_l$ finite and disjoint and $m^*\left(E_{p,q} \setminus \bigcup_{i=1}^k I_i\right) < \varepsilon$.
  It follows that $m^*\left(E_{p,q} \cap \bigcup_{i=1}^k I_i\right) > \delta - \varepsilon$. Letting $I_i = [x_i - h_i, x_i]$,

$$\sum_{i=1}^k (f(x_i) - f(x_i - h_i)) < q \sum_{i=1}^k h_i < q(\delta + \varepsilon)$$

- Step 2

  Consider $A_{p,q} = E_{p,q} \cap \bigcup_{i=1}^k \mathring{I}_i$.

$$m^*\left(E_{p,q} \cap \bigcup_{i=1}^k I_i\right) > \delta - \varepsilon$$

  Then $\forall x \in A_{p,q}$, since $D^r f(x) > p$, $[x, x+h] \subset \bigcup_{i=1}^k \mathring{I}_i$, $f(x+h) - f(x) > p \cdot h$ for $h_i \to 0$.
  So $\{[x, x+h]\} = F_r$ is a Vitali covering for $A_{p,q}$.
  Therefore there exists $\{I_i'\}_{i=1}^k \subset F_r$ finite and disjoint where $I_i' = [x_i', x_i' + h_i']$ and

$$\sum_{i=1}^k \left(f(x_i' + h_i') - f(x_i')\right) > p \sum_{i=1}^k h_i' \geq p(\delta - 2\varepsilon)$$

- Conclusion

  It follows that

$$p(\delta - 2\varepsilon) \leq p \sum_{i=1}^k h_i' < \sum_{i=1}^k \left(f(x_i' + h_i') - f(x_i')\right) \leq \sum_{i=1}^k (f(x_i) - f(x_i - h_i)) < q \sum_{i=1}^k h_i \leq q(\delta + \varepsilon)$$

  That is, $p > q$ but $p(\delta - 2\varepsilon) \leq q(\delta + \varepsilon)$ which means it must be the case that $\delta = 0$ which contradicts our assumption.

Definition: Function of Bounded Variation

A function $f : [a, b] \to \mathbb{R}$ is said to be of bounded variation if for any partition of $[a, b]$

$$P = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

we define the variation of f over $[a, b]$ as

$$V(f, [a, b]) \sup \left\{ \sum_{i=1}^n |f(x_i) - f(x_{i-1})| = P \right\}$$

Observe that

$$f(x) = \int_a^b g(t)\,dt \implies V(f,[a,b]) = \int_a^b |g(t)|\,dt$$

Positive and Negative Variation

Define

$$PV(f,[a,b]) = \sup\left\{\sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))^+ = P\right\}$$
$$NV(f,[a,b]) = \sup\left\{\sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))^- = P\right\}$$

Lemma

$$V(f,[a,b]) = PV(f,[a,b]) + NV(f,[a,b])$$

and

$$f(b) - f(a) = PV(f,[a,b]) - NV(f,[a,b])$$

- Proof

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))^+ + \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))^-$$
$$f(b) - f(a) = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}) = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))^+ - \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))^-$$

Corollary

$$f(x) - f(a) = PV(f,[a,x]) - NV(f,[a,x])$$

So if f is of bounded variation, then $f = g_1 - g_2$ where g_1, g_2 nondecreasing.

Corollary

If f is of bounded variation, it is m-a.e. differentiable.

Example: The Cantor Function

We have $\phi' = 0$ m-a.e., but $\phi(1) - \phi(0) \neq \int_0^1 \phi'\,d\mu$.