Partial Differential Equations I

January 8, 2024

Homework

Assigned exercises and concept maps. Graded by completion.

Presentations

Assigned topics; responsible for giving a class.

Definition: Partial Differential Equation(s) (PDE)

An identity relating an uknown function, its partial derivatives and its variables.

$$F(D^k u, \ldots, D^2 u, Du, u, x) = 0, \quad x \in U \subseteq \mathbb{R}^n$$

where U is an open subset of $\mathbb{R}^n$, $u : U \subset \mathbb{R}^n \to \mathbb{R}$, $Du = (\partial_{x_1} u_1, \ldots, \partial_{x_n} u)$.
Then $F : \mathbb{R}^{n^k} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, where F is given.
$x = (x_1 \ldots, x_n)$ is (are) the independent variable(s).
u is the unknown function or dependent variable.
k is the order of the PDE.

Goal

Given a PDE, we consider

- Existence
- Uniqueness
- Stability

Recall: Multiindex Notation

$\alpha = (\alpha_1, \ldots, \alpha_n)$ a vector such that $\alpha_i \in \mathbb{Z}_{\geq 0}$.
Then we say that α is a multiindex with order $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Notation

$u : U \subseteq \mathbb{R}^n \to \mathbb{R}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$.
$u^\alpha := D^\alpha u = \partial_{x_n}^{\alpha_n} \cdots \partial_{x_1}^{\alpha_1} u$, where $\partial^0 u = u$.

Definition: Linear Partial Differential Equation

A linear PDE of order k is of the form

$$(*) \sum_{|\alpha|=k} a_\alpha(x) D^\alpha u = f(x)$$

Remark

This means that F is multilinear in the first $n^k + n^{k-1} + \cdots$ variables.

Definition: Homogeneous Linear Partial Differential Equation

A linear given by $(*)$ is homogeneous if $f(x) \equiv 0$.
Otherwise, it is non-homogeneous.

Example 1: Linear Transport Equation

$$\nabla u \cdot (1, b) = u_t + b \cdot Du = f(t, x)$$

This is a linear PDE of order 1 on $\mathbb{R} \times \mathbb{R}^n \equiv \mathbb{R}^{n+1}$ where (t, x) are independent variables and u is dependent.
Here, x is the spatial variable while t is time and Du represents the gradient.
$\nabla u = (\partial_t u, \nabla u)$, $b \cdot Du = \sum_{i=1}^n b_i \partial_{x_i} u$, $(b_1, \ldots, b_n) \in \mathbb{R}^n$ is fixed.

Example 2: Laplace Equation

$$\Delta u := \sum_{i=1}^n \partial_{x_i} u = 0$$

This is a linear, homogeneous PDE of order 2.

Example 3: Poisson Equation

$-\Delta u := f(u)$
This is a nonlinear PDE of order 2.
Consider $f(u) = u^2$.

Example 4: Heat Equation (Diffusion Equation)

$$u_t - \Delta u = 0$$

This is a linear, homogeneous PDE of order 2.

Example 5: Wave Equation

$$u_{tt} - \Delta u = 0$$

This is a linear, homogeneous PDE of order 2.

Transport Equation

$u : \mathbb{R}^n (0, \infty) \to \mathbb{R}$ given by

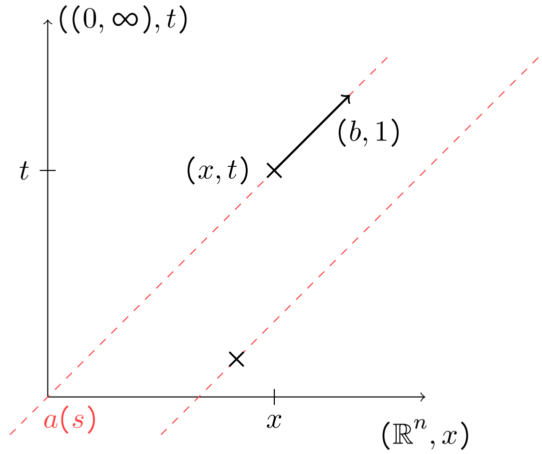
$$u_t + b \cdot Du = 0, \quad b \in \mathbb{R}^n$$

In order to get a solution, first assume that ther exists a “nice” (e.g. smooth, C^1, differentiable, etc.) solution.

Step 1

Notice that the PDE is equivalent to

$$\nabla u \cdot (b, 1) = 0$$



Step 2

Consider a curve on $\mathbb{R}^{n+1}$ with velocity $(1,b)$ which passes through (x,t). i.e.

$$\alpha(s) = (x + sb, t + s)$$

Notice $\alpha'(s) = (b,1)$.
Then, let us study u along the curve $\alpha(s)$.

$$z(s) := u(\alpha(s))$$

Taking the derivative with respect to s,

$$z'(s) := \frac{d}{ds}(u \circ \alpha(s)) = \nabla u|_{\alpha(s)} \cdot \alpha'(s) = \nabla u|_{\alpha(s)} \cdot (b,1) = 0$$

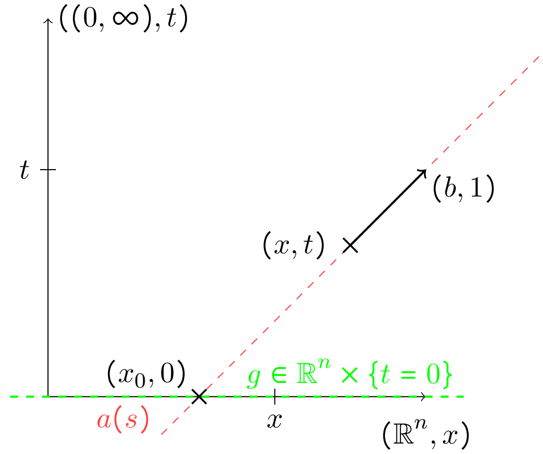
That is $z'(s) = 0$, $z(s)$ is constant, and u along $\alpha(s)$ is constant.

Conclusion

If we know some value of u along $\alpha(s)$, then we know all values along $\alpha(s)$.
If we have some value of u along every $\alpha(s)$, then we know u on $\mathbb{R}^n \times (0,\infty)$.

Transport Equation - Homogeneous Initial Value Problem

$$(*) \begin{cases} \nabla u \cdot (b,1) = 0, & \mathbb{R}^n \times (0,\infty) \\ u = g, & \mathbb{R}^n \times \{t = 0\} \end{cases}$$



Here, $g : \mathbb{R}^n \to \mathbb{R}$ is given.
Consider (x,t); we want to find $(x_0, 0)$.
We know $\alpha(s) = (x + sb, t + s) = (x_0, 0)$, therefore

$$\begin{cases} x + sb = x_0 & (1) \\ t + s = 0 \implies s = -t & (2) \end{cases}$$

Then, by replacing (2) in (1),

$$x_0 = x - tb$$

Then from the conclusion

$$u(x,t) = u(x_0, 0) = g(x_0) = g(x - tb)$$

Therfore, $u(x,t) := g(x - tb)$ (♥).

Remark

1. If there exists a regular (differentiable or C^1) solution u for $*$, then u should look like ♥.
2. If g is (differentiable or C^1), then u defined by ♥ is a (differentiable or C^1) solution for my problem.

Homework

Show that ♥ satisfies $*$.

Transport Equation - Non-homogeneous Initial Value Problem

$$(*) \begin{cases} \nabla u \cdot (b, 1) = f(x,t), \quad \mathbb{R}^n \times (0, \infty) \\ u = g, \quad \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Where $g : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ are given.

Solution

Notice that the PDE is equivalent to

$$\nabla u \cdot (b, 1) = f(x,t)$$

Define the "characteristic curve"

$$\alpha(s) = (x + sb, t + s)$$

and

$$z(s) := u(\alpha(s))$$

Taking $\frac{d}{ds}$,

$$z'(s) = \nabla u|_{\alpha(s)} \cdot (b, 1) = f(\alpha(s)) \implies z'(s) = f(x + sb, t + s) \ (c)$$

Notice that c is an ordinary differential equation. Integrating from $-t$ to 0.

$$\int_{-t}^{0} z'(s) \, ds = \int_{-t}^{0} f(x + sb, t + s) \, ds$$
$$z(0) - z(-t) = \int_{-t}^{0} f(x + sb, t + s) \, ds$$

Notice that $z(0) = u(x, t)$ and $z(-t) = u(\alpha(-t)) = u(x - tb, 0)$.

$$u(x, t) = u(x - tb, 0) + \int_{-t}^{0} f(x + sb, t + s) \, ds$$

Then

$$\begin{aligned} u(x, t) &= g(x - tb) + \int_{-t}^{0} f(x + sb, t + s) \, ds \\ &\underset{\bar{s}=s+t}{=} g(x - tb) + \int_{0}^{t} f(x + (\bar{s} - t)b, \bar{s}) \, d\bar{s} \\ &= g(x - tb) + \int_{0}^{t} f(x + (s - t)b, s) \, ds \end{aligned}$$

Remark: Method of Characteristics

Try to vert the PDE into an ODE and solve using characteristic curves.

January 10, 2024

Definition: Harmonic Function

If $u \in C^2$ such that $\Delta u = 0$, then u is a harmonic function.

Laplace Equation

Consider $u : U \subseteq \mathbb{R}^n \to \mathbb{R}$ with U open, then the homogeneous (Laplace) form is given by

$$\Delta u = 0$$

and the non-homogeneous (Poisson) form is

$$-\Delta u = f$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is given.

Remark / Exercise

The Laplace equation is invariant under translation and rotation.
That is, if $\Delta u(x) = 0$ and $v(x) = u(x - y)$, then $\Delta v = 0$.
Similarly, if $w(x) = u(O(x))$ then $\Delta w = 0$ where O is an orthogonal matrix.

Fundamental Solution of the Laplace Equation

Remark: since the Laplace equation is invariant under rotation, we can consider a function in terms of the radius $v(x) = v(|x|)$.
Recall $r = |x| = (x_1^2 + \cdots + x_n^2)^{1/2}$.
Because of this remark, assume that $u(x) = v(|x|) = v(r(x)) \quad (*)$ where $v : (0, \infty) \to \mathbb{R}$.
Therefore, we need

$$\frac{\partial r}{\partial x_i} = \frac{1}{2} \cdot \frac{2x_i}{(x_1^2 + \cdots + x_n^2)^{1/2}} = \frac{x_i}{r}$$

Replace $(*)$ in the PDE

$$\frac{\partial u}{\partial x_i} = \frac{\partial}{\partial x_i} v(r(x)) = v'(r(x)) \cdot \frac{\partial r}{\partial x_i} = v'(r(x)) \cdot \frac{x_i}{r}$$

and

$$\begin{aligned} \frac{\partial^2 u}{\partial x_i^2} = \frac{\partial}{\partial x_i} \left(v'(r(x)) \cdot \frac{x_i}{r} \right) \\ &= \frac{\partial}{\partial x_i}(v'(r(x))) \cdot \frac{x_i}{r} + v'(r(x)) \cdot \frac{\partial}{\partial x_i} \left(\frac{x_i}{r} \right) \\ &= v''(r(x)) \cdot \frac{x_i^2}{r^2} + v'(r(x)) \left[\frac{1}{r} + x_i \frac{\partial}{\partial x_i}(r) \right] \\ &= v'' \frac{x_i^2}{r^2} + v' \left[\frac{1}{r} - \frac{x_i^2}{r^3} \right] \end{aligned}$$

Thenm, summing across i,

$$\Delta u = v'' + v' \left[\frac{n}{r} + \frac{1}{r} \right] = 0$$

Then the PDE is equivalent to

$$v''(r) + \frac{v'}{r}(n-1) = 0 \ (\square)$$

We need to find a solution for $\square$.

$$v''(r) = \frac{(1-n)v'}{r}$$

Assume, without loss of generality, that $v' \neq 0$ such that

$$\frac{v''(r)}{v'(r)} = \frac{1-n}{r} \implies (\log(|v'|))' = \frac{1-n}{r}$$

Then, integrating,

$$\log(|v'|) = (1-n)\log(r) + C = \log\left(r^{1-n}\right) + C$$

such that

$$|v'| = Cr^{1-n} \implies v' = Cr^{1-n} \implies v(r) = Cr^{1-n+1} + D = Cr^{2-n} + D$$

Definition: Fundamental Solution of the Laplace Equation

The function Φ given by

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x|, & n = 2 \\ \frac{1}{n(n-2)\alpha(n)} \cdot \frac{1}{|x|^{n-2}}, & n \geq 3 \end{cases}$$

where $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$ is called the fundamental solution.

Remark

Φ solves the Laplace equation away from 0.

Lemma: Estimates for the Fundamental Solution

- First Estimate
  $|D\Phi(x)| \leq \frac{C}{|x|^{n-1}}$, for $x \neq 0$.

  $$\frac{\partial \Phi}{\partial x_i} = C\frac{\partial}{\partial x_i}\left(|x|^{2-n}\right) = \frac{C(2-n)}{1-n}|x|^{2-n-1}\frac{\partial |x|}{\partial x_i} = |x|^{1-n} \cdot \frac{x_i}{|x|} = Cx_i|x|^{-n}$$

  Therefore

  $$|D\Phi(x)| \leq C|x||x|^{-n} \implies |D\Phi(x)| \leq C|X|^{1-n}$$

  - Exercise
    Compute for $n = 2$.

- Second Estimate
  $|D^2\Phi(x)| \leq \frac{C}{|x|^n}$, for $x \neq 0$.

  $$\begin{aligned} \frac{\partial^2}{\partial x_J \partial x_i}\Phi &= C\frac{\partial}{\partial x_J}\left(x_i|x|^{-n}\right) \\ &= C\left[\delta_{iJ}|x|^{-n} + x_i\frac{\partial}{\partial x_J}|x|^{-n}\right] \\ &= C\left[\delta_{iJ}|x|^{-n} + (-n)\cdot\frac{x_i|x|^{-n-1}x_J}{|x}\right] \\ &= C\left[\frac{\delta_{iJ}|x|}{|x|^n} + \frac{Cx_ix_J}{|x|^{n+1}}\right] \end{aligned}$$

  Then

  $$\left|\frac{\partial \Phi}{\partial x_i \partial x_J}\right| \leq \frac{C}{|x|^n} + \frac{C|x_i||x_J|}{|x|^{n+2}} \leq \frac{2C}{|x|^n} = \frac{C}{|x|^n}$$

  Then, we are done since

  $$|D^2\Phi(x)| = \sqrt{\sum_J\left(\frac{\partial \Phi}{\partial x_i \partial x_J}\right)}$$

Poisson Equation

Motivation

Suppose we have $\Phi(x)$, the fundamental solution.
Then for an arbitrary, fixed element $y \in \mathbb{R}^n$, then we have $x \to \Phi(x-y)$ harmonic for $x \neq y$.
Consider $f : \mathbb{R}^n \to \mathbb{R}$ such that $y \to f(y)$ then $x \to f(y)\Phi(x-y)$ is similarly harmonic for $x \neq y$.
Now, if given $\{y_1, \dots, y_m\}$ where $y_i \in \mathbb{R}^n$, then $x \to \sum_{i=1}^m f(y_i)\Phi(x-y_i)$ is harmonic $\forall x \neq \{y_1, \dots, y_m\}$.
Then, what happens if we consider

$$u(x) := \int_{\mathbb{R}^n} f(y)\Phi(x-y)\, dy \quad (\square_3)$$

Is u harmonic? No, since $\Delta\Phi(x-y)$ is not summable in $\mathbb{R}^n$ we may not pass the limit into the integral.
(to be covered later) However, since $\Delta\Phi(x-y)$ acts as δ_{xy} in distribution, this may solve the Poisson equation.

Remark / Exercise

Assume that $f \in C_C^2(\mathbb{R}^n)$ (i.e f is twice continuously differentiable with compact support on $\mathbb{R}^n$).
The function Φ is integrable near the singularity on compact sets.
Prove using spherical coordinates.
Therefore, u defined by $\square_3$ is well defined

$$|u| = \left| \int_{\mathbb{R}^n} f(y)\Phi(x-y)\, dy \right| = \left| \int_K \Phi(x-y)\, dy \right| < \infty$$

Theorem: Solving the Poisson Equation

If $f \in C_C^2(\mathbb{R}^n)$ and u is defined by $\square_3$, then

1. $u \in C^2(\mathbb{R}^n)$

2. $-\Delta u = f$, in $\mathbb{R}^n$

- Proof of 1
  Since Φ presents a problem at $x = y$ but f is well behaved, we will change variables such that $\overline{y} = x - y$, $y = x - \overline{y}$, and $\frac{dy}{d\overline{y}}(-1)I_{m \times m}$ and then redefine $\overline{y} = y$.

$$u(x) = \int_{\mathbb{R}^n} f(y)\Phi(x-y)\, dy = \int_{\mathbb{R}^n} f(x-\overline{y})\Phi(\overline{y})\, d\overline{y} = \int_{\mathbb{R}^n} f(x-y)\Phi(y)\, dy$$

  In short, we have sent the problem from Φ to f.
  Now, let us consider $e_i = (0, \dots, 1, \dots, 0)$.
  Then for $h > 0$,

$$\frac{u(x+he_i) - u(x)}{h} = \frac{1}{h} \int_{\mathbb{R}^n} \Phi(y) \left[f(x+he_i-y) - f(x-y) \right] dy$$

  Now, the limit as $h \to 0$

$$\begin{aligned} \lim_{h \to 0} \frac{u(x+he_i) - u(x)}{h} &= \lim_{h \to 0} \int_{\mathbb{R}^n} \overbrace{\Phi(y) \left[\frac{f(x+he_i-y) - f(x-y)}{h} \right]}^{H(h,y)} dy \\ &= \int_{\mathbb{R}^n} \Phi(y) \cdot \frac{\partial f(x-y)}{\partial x_i}\, dy \end{aligned}$$

To justify passing the limit into the integral, take an arbitrary sequence $h_m \xrightarrow{\to 0} 0$ and consider

$$f_m(y) := H(h_m, y)$$

We want to consider

$$\begin{aligned} |H(h_m, y)| &\leq \Phi(y)\left[\frac{f(x + h_m e_i - y) - f(x - y)}{h}\right] \\ &\leq \Phi(y) f'(c) \end{aligned}$$

Where c is along the curve between $f(x + h_m e_i - y)$ and $f(x - y)$ and chosen by mean value theorem.

- Exercise

$$|H(h_m, y)| \leq \Phi(y)||f'||_{L^\infty} \chi_{B(x,R)}(y)$$

Note that

$$C \int_{\mathbb{R}^n} |\Phi(y)| \chi_{B(x,R)}(y)\, dy = \int_{B(x,R)} |\Phi(y)|\, dy < \infty$$

- Exercise

Using the fact that a continuous function is uniformly continuous on a compact set, show that $u \in C^2(\mathbb{R}^n)$.

Dominated Convergence Theorem

If $f_m(x)$ such that $f_m(x) \underset{\text{pointwise}}{\xrightarrow{m\to\infty}} f(x)$, and $|f_m(x)| \leq g(x)$ for $g \in L^1$, then f is integrable and

$$\lim_{m\to\infty} \int f_m(x)\, dx = \int f(x)\, dx$$

January 17, 2024

Recall: Averages

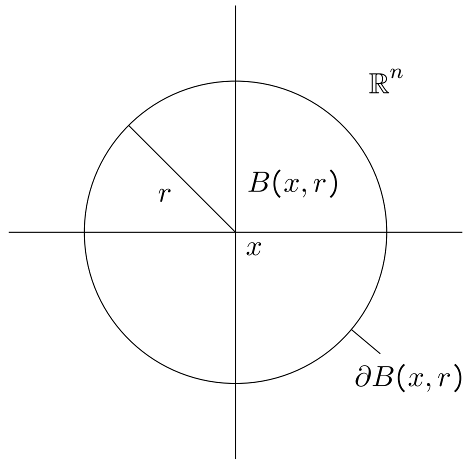
$f : \{1, \ldots, n\} \to \mathbb{R}$
$i \to a(i)$
The average is given as $\frac{a(i)+\cdots+a(n)}{n}$.
Then for $f : \Omega \to \mathbb{R}$, the average is given as

$$\frac{1}{|\Omega|} \int f(y)\, dy := ⨍_\Omega f\, d\mu$$

In our case, $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$

$$⨍_{B(x,n)} f\, d\mu \equiv \frac{1}{|B(x,n)|} ⨍_{B(x,n)} f\, d\mu$$

$$⨍_{\partial B(x,n)} f\, d\mu = \frac{1}{|\partial B(x,n)} ⨍_{\partial B(x,b)} f\, d\mu$$

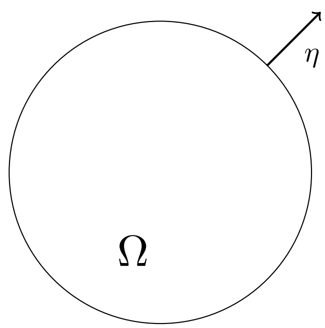


Theorem: Lebesgue Differentiation

$$u|x| = \lim_{r\to 0} \fint_{B(x,n)} u \, d\mu = \lim_{r\to 0} \fint_{\partial B(x,n)} u \, d\mu$$

Integration by Parts

$$\int_\Omega u \Delta v = -\int_\Omega \nabla u \cdot \nabla v + \int_{\partial\Omega} u \cdot \frac{\partial v}{\partial \eta}$$



Recall: Poisson's PDE

$f \in C_c^2|\mathbb{R}^n|$, $u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) \, dy$.

$$\Phi(x) = \left\{ \frac{1}{n(n-2)|\alpha(n)|} \frac{1}{|X|(n-2)} \right.$$

$$u(x) \text{``} = \text{``} \int_{\mathbb{R}^n} f(x-y)\Phi(y) \, dy$$

Part A

$u \in C^2(\mathbb{R}^n)$
Then

$$\frac{\partial u}{\partial x_1} = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_1}(x-y)\Phi(y) \, dy$$

$$\frac{\partial^2 u}{\partial x_1 x_T} = \int_{\mathbb{R}^n} \frac{\partial^2 f}{\partial x_1 x_T}(x-y)\Phi(y) \, dy$$

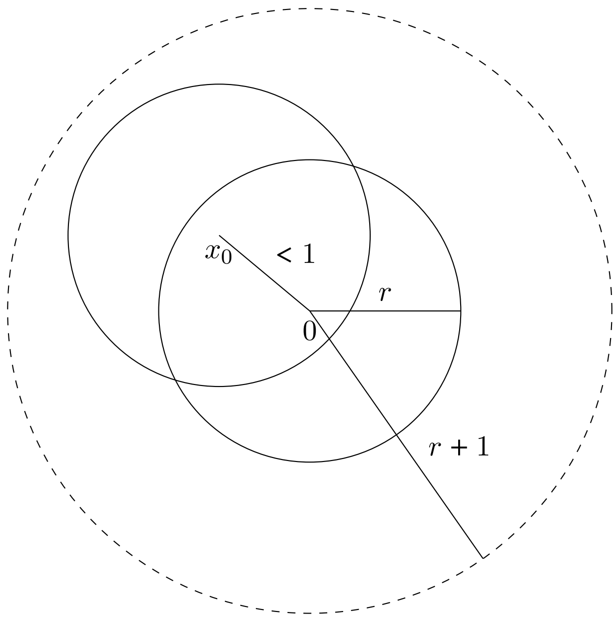
Notice that if we prove that

$$g(x) = \int_{\mathbb{R}^n} h(x-y)\Phi(y)\,dy$$

– where h is continuous with compact support – is continuous then we are done.
Let us prove that g is continuous.
Let $\varepsilon > 0$,

$$|g(x) - g(x_0)| \leq \int_{\mathbb{R}^n} \Phi(y)|h(x-y) - h(x_0 - y)|\,dy$$

Without loss of generality, h has compact support on $B(0, r)$ for some radius r.
Therefore $h(x, y)$ has compact support on $B(x, r)$ and $h(x_0, y)$ has compact support on $B(x_0, r)$.



Consider $|x - x_0| < 1$, then $|h(x-y) - h(x_0 - y)|$ has compact support on $B(x_0, r+1)$. Then

$$|g(x) - g(x_0)| \leq \int_{B(x_0, r+1)} \Phi(y)|h(x-y) - h(x_0 - y)|\,dy$$

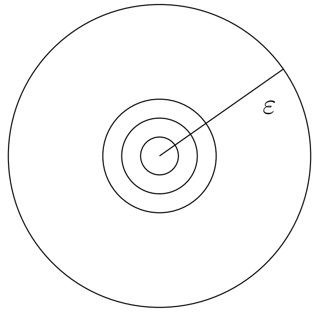
Since h is continuous on a compact domain, it is uniformly continuous.
Therefore $\exists \delta > 0$ such that $|w - z| < \delta \implies |h(w) - h(z)| < \epsilon$.
Set $w = x - y$ and $z = x_0 - y$ such that $|w - z| = |x - x_0| < \delta$, then $|h(x-y) - h(x_0 - y)| < \epsilon$. Thus,

$$|g(x) - g(x_0)| \leq \varepsilon \int_{B(x_0, r+1)} \Phi(y)\,dy$$

Part B

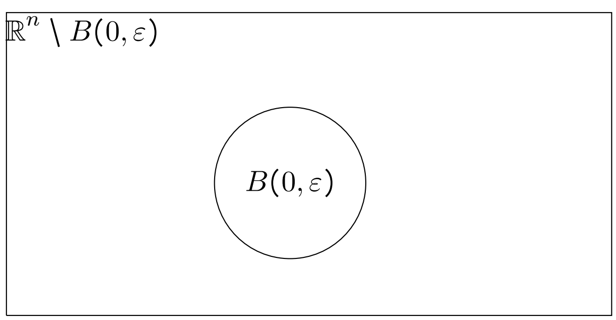
$-\Delta u = f$
Letting $\varepsilon > 0$ and taking the Laplacian of both sides,

$$\begin{aligned} \Delta_x u(x) &= \int_{\mathbb{R}^n} \Delta_x f(x-y)\Phi(y)\,dy \\ &= \overbrace{\int_{B(0,\varepsilon)} \Delta_x f(x-y)\Phi(y)\,dy}^{I_\varepsilon} + \overbrace{\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Delta_x f(x-y)\Phi(y)\,dy}^{J_\varepsilon} \end{aligned}$$



Then

$$
\begin{aligned}
|I_\varepsilon| &\le \int_{B(0,\varepsilon)} |\Delta_x f(x-y)| \Phi(y)\, dy \\
&\le ||\overset{2}{\nabla} f||_{L^\infty} \int_{B(0,\varepsilon)} \Phi(y)\, dy \\
&\le c \int_0^\varepsilon \int_{\partial B(0,r)} \Phi(y)\, dS(y)\, dr \\
&\le c \int_0^\varepsilon \int_{\partial B(0,r)} \frac{1}{|y|^{n-2}}\, dS(y)\, dr \\
&= c \int_0^\varepsilon \int_{\partial B(0,r)} \frac{1}{r^{n-2}}\, dS(y)\, dr \\
&= c \int_0^\varepsilon \frac{1}{r^{n-2}} \int_{\partial B(0,r)} dS(y)\, dr \\
&\le c \int_0^\varepsilon \frac{r^{n-1}}{r^{n-2}}\, dr \\
c &\int_0^\varepsilon r\, dr = c\varepsilon^2
\end{aligned}
$$



As an exercise, attempt the same argument with $n = 2$.
Therefore,

$$\Delta_x u = I_\varepsilon + J_\varepsilon$$

and $\lim_{\varepsilon \to 0} I_\varepsilon = 0$.
Now, we need to control J_ε.

$$J_\varepsilon = \int_{\mathbb{R}^n} \Delta_x f(x-y) \Phi(y)\, dy$$

$$\Delta_x f(x-y) = \sum \frac{\partial^2 f}{\partial x^2} f(x-y)$$

$$\frac{\partial f}{\partial x}(x-y) = \nabla f|_{z=(x-y)} \cdot e_i = \frac{\partial f}{\partial z_i}|z = (x-y)$$
$$\frac{\partial^2 f}{\partial x_i^2} = \frac{\partial^2 f}{\partial z_i^2}|_{z=(x-y)}$$

$$\Delta_y f(x-y) = \sum \frac{\partial f^2}{\partial y_i}(x-y)$$
$$\frac{\partial f}{\partial y_i}(x-y) = \nabla f|_{z=(x-y)} \cdot -e_i = -\frac{\partial f}{\partial z_i}|_{z=(x-y)}$$
$$\frac{\partial^2 f}{\partial y_i^2} = \frac{\partial^2}{\partial y_i^2}|_{z=x-y}$$

So

$$\begin{aligned} J_\varepsilon &= \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Delta_y f(x-y)\Phi(y)\, dy \\ &= \overbrace{-\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \nabla_y f(x-y) \nabla \Phi(y)\, dy}^{K_\varepsilon} + \overbrace{\int_{\partial(\mathbb{R}^n \setminus B(0,\varepsilon))} \frac{\partial f}{\partial \eta}\Phi(y)\, dS(y)}^{L_\varepsilon} \end{aligned}$$

and

$$\Delta_x u = I_\varepsilon + K_\varepsilon + L_\varepsilon$$

To control L_ε, since

$$\begin{aligned} |L_\varepsilon| &\leq \int_{\partial B(0,\varepsilon)} \frac{|\partial f|}{\partial \eta}\Phi(y)\, dy \\ &\leq \int_{\partial B(0,\varepsilon)} |\nabla f|\Phi(y)\, dy \\ &\leq ||\nabla f||_{L^\infty} \int_{\partial B(0,\varepsilon)} \Phi(y)\, dy \\ &\leq c \int_{\partial B(0,\varepsilon)} \frac{1}{|y|^{n-2}}\, dy \\ &= \frac{c}{\varepsilon^{n-2}} \int_{\partial B(0,\varepsilon)} dy \\ &\leq \frac{c\varepsilon^{n-1}}{\varepsilon^{n-2}} \\ &= c\varepsilon \end{aligned}$$

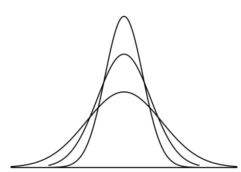
and K_ε, since $\frac{\partial \Phi}{\partial \eta} = \nabla \phi \cdot \eta = \frac{-y}{n\alpha(n)|y|^n} \cdot \eta = \frac{|y|^2}{n\alpha(n)|y|^n|y|} = \frac{1}{n\alpha(n)|y|^{n-1}}$

$$
\begin{aligned}
|K_\varepsilon| &= -\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \nabla_y f(x-y) \nabla_y \Phi(y)\, dy \\
&= \overbrace{\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} f(x-y) \Delta_y \Phi(y)\, dy}^{0} - \int_{\partial(\mathbb{R}^n \setminus B(0,\varepsilon))} f(x-y) \frac{\partial \Phi}{\partial \eta} \\
&= -\int_{\partial B(0,\varepsilon)} f(x-y) \frac{\partial \Phi}{\partial \eta} \\
&= -\int_{\partial B(0,\varepsilon)} f(x-y) \frac{1}{n\alpha(n)|y|^{n-1}}\, dS(y) \\
&= -\frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\delta B(0,\varepsilon)} f(x-y)\, dS(y) \\
&= \underbrace{-\overset{\text{volume}}{\frac{1}{n\alpha(n)\varepsilon^{n-1}}}} \int_{\partial B(0,\varepsilon)} f(z)\, dS(z) \\
&= \frac{1}{|\partial B(0,\varepsilon)|} \int_{\partial B(0,\varepsilon)} f(z)\, dz \\
&= -\fint_{\partial B(x,\varepsilon)} f(z)\, dz
\end{aligned}
$$

Laplacian as a Distribution

$-\Delta \Phi(y) = \delta(y)$
Define the Dirac delta "function" as

$$
\delta(y) = \begin{cases} \infty & y = 0 \\ 0 & y \neq 0 \end{cases}
$$



such that $\int_{\mathbb{R}^n} \delta = 1$.
Translate the Dirac delta as

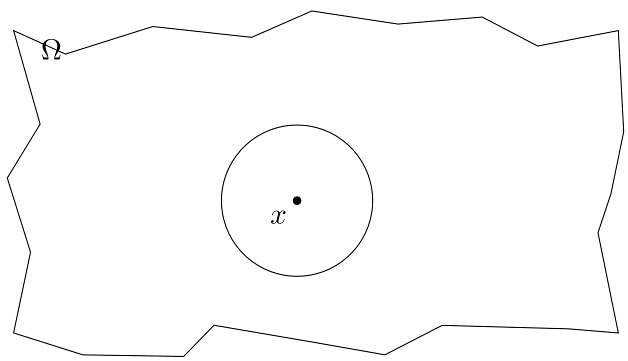
$$
\delta_x = \begin{cases} \infty & y = x \\ 0 & y \neq x \end{cases}
$$

and recall that

$$
\begin{aligned}
\Delta u(x) &= \Delta\left(\int_{\mathbb{R}^n} \Phi(x-y)f(y)\,dy\right) \\
&= \int_{\mathbb{R}^n} \overbrace{\Delta\Phi(x-y)}^{-\delta_x(y)} f(y)\,dy \\
&= -\int_{\mathbb{R}^n} \delta_x(y)f(y)\,dy \\
&= -\int_{\mathbb{R}^n} \delta_x(y)f(x)\,dy \\
&= -f(x)\overbrace{\int_{\mathbb{R}^n} \delta_x(y)\,dy}^{1} \\
&= -f(x)
\end{aligned}
$$

Harmonic Functions

Suppose u is harmonic



$u : \Omega \to \mathbb{R}^n$ harmonic.

Mean-value Formulas

Let U be an open set in $\mathbb{R}^n$, $u : U \to \mathbb{R}$ such that $\Delta u = 0$ in U. Then

$$
\begin{aligned}
u(x) &= \fint_{\partial B(0,r)} -u(y)\,dS(y) \\
&= \fint_{B(x,r)} u(y)\,dy
\end{aligned}
$$

where $B(x,r) \subseteq U$.
IMAGE HERE

Proof

Consider $\phi(r) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y)\,dS(y)$.
If $\phi'(r) = 0$, when we are done since that would make ϕ constant and $\phi(r) = \lim_{s\to 0}\phi(s) = u(x)$. Then

$$\begin{aligned}
\phi(r) &= \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y)\ dS(y) \\
&\underset{y=x+rz}{=} \frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B(0,1)} u(x+rz)r^{n-1}\ dS(z) \\
&= \frac{1}{n\alpha(n)}\int_{\partial B(0,1)} u(x+rz)\ dS(z)
\end{aligned}$$

Therefore

$$\begin{aligned}
\phi'(r) &= \frac{1}{n\alpha(n)}\int_{\partial B(0,1)} \nabla u(x+rz)\cdot z\ dS(z) \\
&\underset{y=x+z}{=} \frac{1}{|\partial B(0,r)|}\int_{\partial B(x,r)} \nabla u(y)\cdot \frac{y-x}{r}\ dS(y) \\
&= \frac{1}{|\partial B(0,r)|}\int_{\partial B(x,r)} \nabla u(y)\cdot \eta\ dS(y) \\
&= \frac{1}{|\partial B(0,r)|}\int_{\partial B(x,r)} \frac{\partial y}{\partial \eta}\ dS(y) \\
&= \frac{1}{|\partial B(0,r)|}\int_{B(x,r)} \Delta u \\
&= 0
\end{aligned}$$

January 22, 2024

Mean Value Formula

For $U \subseteq \mathbb{R}^n$, U open with $u : U \to \mathbb{R}$ such that $u \in C^2(U)$, $\Delta u = 0$, we have

$$u(x) \underbrace{=}_{\text{(a)}} ⨍_{\partial B(x,r)} u \underbrace{=}_{\text{(b)}} ⨍_{B(x,r)} u$$

for all $B(x,r) \subseteq U$.
Recall that (a) was proven above by setting $\phi(r) = ⨍_{\partial B(r)} u(y)\ dS(y)$ and showing $\phi'(r) = 0$.
For (b), we again apply spherical coordinates such that

$$\begin{aligned}
\int_{B(x,r)} u(y)\ dy &= \int_0^r \int_{\partial B(x,s)} u(y)\ dS(y) ds \\
&= \int_0^r |\partial B(x,s)| \overbrace{⨍_{\partial B(x,s)} u(y)\ dS(y)}^{u(x)} ds \\
&= u(x)\int_0^r |\partial B(x,s)|\ ds \\
&= u(x)\int_0^r n\alpha(n)S^{n-1}\ ds \\
&= \frac{u(x)n\alpha(n)S^n}{n}\Big|_0^r \\
&= u(x)\overbrace{\alpha(n)r^n}^{|B(x,r)|}
\end{aligned}$$

Converse

Recall that

$$\phi'(r) = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u(y)\, dy$$

Suppose then that we do not know that $\Delta u = 0$ but we have

$$u(x) = \fint_{\partial B(x,r)} u, \quad \forall B(x,r) \subseteq U$$

Then, necessarily, $\Delta u = 0$ in U.

- Proof
  Suppose, for sake of contradiction, that $\Delta u \neq 0$. Then, without loss of generality, there exists $y \in U$ such that $\Delta u(x) > 0$ for $x \in B(y,n) \subseteq U$.
  IMAGE HERE

  $$\phi(r) = \fint_{\partial B(x,r)} u(x)\, dS(x)$$

  and

  $$\phi'(r) = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(y,r)} \Delta u(x)\, dS(x) > 0$$

  which contradicts $\phi'(x) = 0$.

Strong Maximum Principle

Let $U \subseteq \mathbb{R}^n$ be a bounded open set, $u \in C^2(U) \cap C(\overline{U})$, $\Delta u = 0$ on U. Then

1. $\max_{\overline{U}}(u) = \max_{\partial U}(u)$.

2. If U is connected and u has its maximum in an interior point, then u is constant on $\overline{U}$.

IMAGE HERE - 2

Proof of A

Since $\partial U \subseteq \overline{U}$, $\max_{\partial U}(u) \leq \max_{\overline{U}}(u)$.
Let $x_0 \in \overline{U}$ such that $u(x_0) = \max_{\overline{U}}(u)$.
IMAGE HERE - 4
So either $x_0 \in \partial U$ or $x_0 \in U$.
Let U' be the connected component which contains x_0. Then $x_0 \in U'$, so by part (b) u is constant on $\overline{U'}$. So

$$\max_{\overline{U}}(u) = u(x_0) = \max_{\partial U'}(u) \leq \max_{\partial U}(u)$$

Proof of B

Then there exists $x_0 \in U$ such that $\max_{\overline{U}}(u) = u(x_0) = M$.
Let us define $\Omega = \{y \in U \mid u(y) = M\}$. Then

1. $\Omega \neq \varnothing$, $B \setminus x_0 \in \Omega$.

2. Ω open set.

IMAGE HERE - 3

1. Ω is closed, since $\Omega = u^{-1}(\{M\})$.

It follows that $\Omega = U$ and, therefore, $u(y) = M$, $\forall y \in U$.

- Proof of 2
  Let $y \in \Omega$, $y \in U$, $u(y) = M$. Then there exists $B(y,r) \subseteq U$, and

$$M = u(y) = \fint_{B(y,r)} u(x)\, dS(x) \leq M$$

  Then

$$\fint_{B(y,r)} u(x)\, dx = M$$

  so $u(x) = M$, $\forall x \in B(y,r)$ and, therefore $B(y,r) \subseteq \Omega$ and Ω is open.

Remark: Boundedness Is Important

1. Consider $f(x) = x$ on $\mathbb{R}_{\geq 0}$.

2. IMAGE HERE - 5

Remark: Strong Minimum Principle Is Equivalent

Consequences

1. Positivity of harmonic functions.

2. Uniqueness of the Poisson problem.

Corollary: Positivity of Harmonic Functions

Suppose that U is connected and $u : U \to \mathbb{R}$, $u \in C^2(U) \cap C(\overline{U})$ solves the following problem

$$\begin{cases} \Delta u = 0 & \text{on } U \\ u = g & \text{on } \partial U \end{cases}$$

If $g \geq 0$ on ∂U, then u is positive on U as long as g is positive in some point.

Proof

Assume $\exists x_0 \in \partial U$ where x_0 is the minimum. Then $u(x_0) = \min_{\overline{U}}(u)$ and, $\forall x \in U$,

$$0 \leq u(x_0) = \min_{\overline{U}}(u) \leq u(x)$$

so u is non-negative. If $u(x) = 0$, then $u(x_0) = 0$ and the minimum is achieved in the interior. That would mean $u(x) = 0$, $\forall x \in \overline{U} \supseteq \partial U$ and $g(x) = 0$, $\forall x \in \partial U$ which would be a contradiction.

Theorem: Uniqueness of the Poisson Problem

Suppose $U \subseteq \mathbb{R}^n$ is open, connected and bounded. Then, there exists at most one solution to

$$(*) \begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

where $u \in C^2(U) \cap C(\overline{U})$.

Proof

Let u_1 and u_2 be two solutions of $*$.
Consider $w = u_1 - u_2$ and observe that

$$\Delta w = \Delta u_1 - \Delta u_2 = -f + f = 0, \quad \text{in } U$$

and $w|_{\partial U} = g - g = 0$ on ∂U. Therefore

$$\begin{cases} \Delta w = 0 & \text{in } U \\ w = 0 & \text{on } \partial U \end{cases}$$

By applying the minimum and maximum principles,

$$w(x_0) = \min_{\overline{U}}(w) \leq w(x) \leq \max_{U}(w) = w(x)$$

so $w(x) = 0$, $\forall x \in \overline{U}$ and therefore $u_1 = u_2$.

Example

Let's consider $f : \mathbb{C} \to \mathbb{C}$ analytic (i.e. $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $a_n, z \in \mathbb{C}$). Then

$$f(z) = u(z) + v(z)$$

If $\mathbb{C} \cong \mathbb{R}^2$,

$$f(x + y) = u(x, y) + v(x, y)$$

for $u : \mathbb{R}^2 \to \mathbb{R}$ and $v : \mathbb{R}^2 \to \mathbb{R}$.
Claim: u and v are Harmonic.

$$u(x,y) + v(x,y) = \sum_{n=0}^{\infty} a_n (x+iy)^n$$

and

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \sum_{n=1}^{\infty} a_n n |x+iy|^{n-1}$$
$$\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = \sum_{n=1}^{\infty} a_n n |x+iy|^{n-1} i$$

So

$$i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

Recall: Convolution and smoothing

Let $U \subseteq \mathbb{R}^n$ be an open set.
For $\varepsilon > 0$, define $U_\varepsilon = \{x \in U \mid d(x, \partial U) > \varepsilon\}$.
IMAGE HERE - 6
Define

$$\eta(x) \begin{cases} ce\left(\frac{1}{|x|^2-1}\right) & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

with c such that $\int_{\mathbb{R}^n} \eta(x)\, dx = 1$, $\eta \in C^\infty(\mathbb{R}^n)$
IMAGE HERE - 7
Note that $\operatorname{supp}(\eta) = B(0,1)$ and take

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right), \quad \eta_\varepsilon \in C^\infty(\mathbb{R}^n)$$

Then

$$\int_{\mathbb{R}^n} \eta_\varepsilon(x)\, dx = 1$$

and $\operatorname{supp}(\eta_\varepsilon) \subseteq B(0,\varepsilon)$.
If f is locally integrable on U, define its mollification

$$f^\varepsilon = (x) = \int_U \eta_\varepsilon(x-y) f(y)\, dy \quad \forall x \in U_\varepsilon$$