The Kernel is a Subgroup

Let $g_1, g_2 \in \ker(\phi)$. Then

$$\begin{aligned} \phi(g_1 g_2) &= \phi(g_1)\phi(g_2) && \phi \text{ is a homomorphism} \\ &= 1_H 1_H && g_1, g_2 \in \ker(\phi) \\ &= 1_H && g_1, g_2 \in \ker(\phi) \end{aligned}$$

Similarly, $1_G \in \ker(\phi)$ and $g^{-1} \in \ker(\phi)$ if $g \in \ker(\phi)$. ■

Alternating Group

Let X be a set, $|X| = n \leq \infty$.
The alternating group on X is the $\mathrm{Alt}(X) = \ker(\mathrm{sign} : \mathrm{Sym}(X) \to \{\pm 1\})$.

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Group Action

Let G be a group and X a set.
A (left) action of G on X is a function $\alpha : G \times X \to X$ which satisfies two conditions:

1. $\alpha(1_G, x) = x$ for all $x \in X$.

2. $\alpha(g_1, \alpha(g_2, x)) = \alpha(g_1 g_2, x)$ for all $g_1, g_2 \in G$ and $x \in X$.

Notation

Write $\alpha(g, x) = g * x = g \cdot x = gx$.

Example A

Let X be any set, and let $G = \mathrm{Sym}(X) = \{f : X \to X \text{ bijections}\}$ where the group operation $\circ$ is the composition of functions.
Then G acts (on the left) on X by $f * x \underset{\text{def}}{=} f(x)$.
Then the features

1. $\mathrm{Id}_X(x) = x, \ \forall x \in X$

2. $g_1 * (g_2 * x) = (g_1 \circ g_2) * x, \ \forall g_1, g_2 \in G, \ \forall x \in X$

    - Or $g_1(g_2(x)) = (g_1 \circ g_2)(x)$

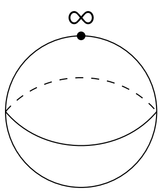
are satisfied.

Example B

Let $G = \mathrm{Sym}(\{B,\ P,\ W,\ Y\})$ which acts on $X = \{B,\ P,\ W,\ Y\}$.
If $H \le G$, then H acts on X as well, define $h * x = \dot{h} * x$ (where $\dot{h}$ is regarded as in the alternating group of G).
In particular, $\mathrm{Alt}(\{B,\ P,\ W,\ Y\})$ acts on X by rotations.

Example C*

This example is not required for this class.
From complex Analysis we have the Riemann sphere $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$.



Let $G = \mathrm{SL}_2(\mathbb{C})$.
Define G-action on $X = \mathbb{P}^1(\mathbb{C})$ by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} z := \frac{\alpha z + \beta}{\gamma z + \delta} \qquad (\infty \text{ if } \gamma z + \delta = 0)$$

This is called the Möbius group action on $\mathbb{P}^1(\mathbb{C})$.
Exercise: show that 1. and 2. are satisfied.

Definitions

Let G act on X. (Say X is a (left) G-set)

Stabilizer

Let $x \in X$. The stabilizer of x in G is $\mathrm{Stab}_G(x) = \{g \in G | g * x = x\} \subseteq G$.

- Example 1 Let G be any group and X a G-set.
  Then for any $x \in X$, $\mathrm{Stab}_G(x) \le G$.

  - Proof

    1. $1_G \in \mathrm{Stab}_G(x)$ since, by definition, $1_G * x = x$.
       Therefore the identity is present.

    2. If $g_1, g_2 \in \mathrm{Stab}_G(x)$ are such that $g_1 * x = x$ and $g_2 * x = x$, then $(g_1 g_2) * x \underset{\text{2nd Axiom}}{=} g_1 * (g_2 * x) = g_1 * x = x$.
       Therefore the stabilizer is closed under composition.

    3. Say $g \in \mathrm{Stab}_G(x)$ and $g * x = x$. Apply g^{-1} to both sdies to get

    $$x \underset{\text{1st Axiom}}{=} 1_G * x = (g^{-1} g) * x \underset{\text{2nd Axiom}}{=} g^{-1} * (g * x) = g^{-1} * x$$

    Therefore the stabilizer is closed under inverse.

- Example 2 Let $G = \mathrm{Alt}(\{B,\ P,\ W,\ Y\})$ and consider $H = \mathrm{Stab}_G(W) = \{\mathrm{Id}, (B\ P\ Y), (B\ Y\ P)$.
  Fact: H does not act transitively on X, since W is fixed and no element $g \in H$ satisfies $g * W = B$.

Orbit

Let $x \in X$. The G-orbit of x in X is $G \cdot x = \{g * x | g \in G\} \subseteq X$.
Let G act on X and $x, y \in X$. Either $G \cdot x = G \cdot Y$ or $G \cdot x \cap G \cdot y = \varnothing$.
So X is the disjoint union of G-orbits.
e.g. $\{B,\ P,\ W,\ Y\} = \{W\} \coprod \{B,\ P,\ Y\}$ gives the $\mathrm{Stab}_G(W)$-orbits.

- Example 1 When $G = \mathrm{Alt}(X)$, for $X = \{B,\ P,\ W,\ Y\}$, there is only one orbit since $\forall x \in X$, $G \cdot x = X$.

- Example 2 When $G = \mathrm{Stab}_G(W)$, for $X = \{B,\ P,\ W,\ Y\}$, then $G \cdot W = \{W\}$ while

$$\begin{aligned} G \cdot B &= \{\mathrm{Id}(B),\ (B\ P\ Y)(B),\ (B\ Y\ P)(B)\} = \{B,\ P,\ Y\} \\ &= G \cdot P = \{\mathrm{Id}(P),\ (B\ P\ Y)(P),\ (B\ Y\ P)(P)\} = \{P,\ Y,\ B\} \\ &= G \cdot Y \end{aligned}$$

Transitivity

Say G acts transitively on X (or the action is transitive) if, for any pair $x, y \in X$, there exists $g \in G$ (depending on x and y) such that $g * x = y$.

- Example $G = \mathrm{Alt}(\{B,\ P,\ W,\ Y\}) \circlearrowright \{B,\ P,\ W,\ Y\}$ is transitive.

  - Proof Let $x, y \in X$ be arbitrary.
    If $x = y$, then take $g = \mathrm{Id}_X$ and we have $g * x = y$.
    Suppose $x \neq y$, then write $X = \{x,\ y,\ z,\ w\}$ and take $g = (x\ y)(z\ w)$. We have $g * x = y$.
    e.g. $x = P$, $y = Y$, $z = B$ and $w = W$ gives $g = (P\ Y)(B\ W)$.

- Exercise * This exercise is not required for the course.
  Prove that $\mathrm{SL}_2(\mathbb{C})$ acts transitively on $\mathbb{P}^1(\mathbb{C})$.
  Say $\mathbb{P}^1(\mathbb{C})$ is a homogeneous space under $\mathrm{SL}_2(\mathbb{C})$.

Group Action Gives Group Homomorphisms

($\longrightarrow$) Let G act on X. Then

1. For any $g \in G$, the function $\pi_g : X \to X$ defined by $\pi_g(x) = g * x$ is a bijection of X, hence $\pi_G \in \mathrm{Sym}(X)$.

2. The function $G \xrightarrow{\phi} \mathrm{Sym}(X)$ given by $\phi(g) = \pi_g$ is a group homomorphism.

Proof of 1

Need to show that π_g is injective and surjective.
(Inj) Let $x, y \in X$ and assume $\pi_g(x) = \pi_g(y)$ (i.e. $g * x = g * y$).
Apply $g^{-1}*$ on both sides, such that $x = g^{-1} * (g * x) = g^{-1} * (g * y) = y$.
(Sur) Let $x \in X$ be arbitrary. Need to find $y \in X$ such that $\pi_g(y) = x$.
Take $y = g^{-1} * x$, and $\pi_g(y) = g * (g^{-1} * x) = x$.

Proof of 2

Need to show that $\forall g_1, g_2 \in G$, $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$.
$\phi(g_1 g_2) \in \mathrm{Sym}(X)$ is characterized by $[\phi(g_1 g_2)](x) = \pi_{g_1 g_2}(x) = (g_1 g_2) * x$.
On the other hand, $\phi(g_1)\phi(g_2) \in \mathrm{Sym}(X)$ is characterized by $[\phi(g_1)\phi(g_2)](x) = \phi(g_1)[\phi(g_2)(x)] = g_1 * (g_2 * x)$.
By the second group action axiom, these must be the same.

Group Homomorphism Admits Group Action

($\longleftarrow$) Let $G \xrightarrow{\rho} \mathrm{Sym}(X)$ be a group homomorphism.
Then, by letting $g * x = \rho(g)(x) \in X$ we get a left G-action on X.

Proof

1. $1_G * x = \rho(1_G)(x) = \mathrm{Id}_X(x) = x$.

2. Let $g_1, g_2 \in G$ and $x \in X$. Then $(g_1 g_2) * x = [\rho(g_1 g_2)](x) = [\rho(g_1) \circ \rho(g_2)](x) = \rho(g_1)[\rho(g_2)(x)] = g_1 * (g_2 * x)$.

Right Group Actions

Let G be a group and X be a set. A right G-action on X is a function $\beta : X \times G \to X$ such that

1. $\beta(x, 1_G) = x,\ \forall x \in X$.

2. $\beta(x, g_1 g_2) = \beta(\beta(x, g_1), g_2),\ \forall g_1, g_2 \in G,\ \forall x \in X$.

Notation

$\beta(x, g) = x * g = x \cdot g = xg$

Remark

If $\alpha : G \times X \to X$ is a left action, we get a right action $\beta : X \times G \to X$ by $\beta(x, g) = \alpha(g^{-1}, x)$ and vice versa.
That is $x * g = g^{-1} * x$.
Proof recommended as an exercise.

Analogues

Stability, orbit and transitivty all have analogues which can be demonstrated by converting to left actions.

Cosets

Let $H \leq G$, and let $X = G$.
We have left action $H \times X \to X$ and $h * x = hx$ (taken in G).
As well as right action $X \times H \to X$ where $x * h = xh$.
A (left) H-coset is an orbit xH for some $x \in X$.
A (right) H-coset is an orbit Hx for some $x \in X$.

Example

Let $G = \mathrm{Alt}(4)$, $H = \mathrm{Stab}_G(W) = \{\mathrm{Id},\ (B\ P\ Y),\ (B\ Y\ P)\}$.

1. Take any $x \in H$, $xH = H$.

2. Take $x = (B\ P)(W\ Y)$, and $xH = \{(B\ P)(W\ Y), (B\ P)(W\ Y)(B\ P\ Y) = (P\ W\ Y),\ (B\ P)(W\ Y)(B\ Y\ P) = (B\ W\ Y)\}$.

3. There are two more; what are they?