

# Topics in Analysis (F24)

September 30, 2024

## Chapter 1: Banach Algebras

### 1.1: Definitions and Basic Properties

#### Definition: Banach Space

A Banach space  $X$  (over  $\mathbb{C}$ ) is a normed vector space with algebraic operations

$$\begin{array}{ll} (x, y) \mapsto x + y & \text{addition} \\ (\lambda, y) \mapsto \lambda y & \text{scalar multiplication} \end{array}$$

and a norm

$$x \mapsto \|x\|$$

which is complete (i.e. every Cauchy sequence converges).

#### Definition: (Complex) Banach Algebra

A (complex) Banach algebra  $B$  is a Banach space in which there is multiplication

$$B \times B \ni (x, y) \mapsto xy \in B$$

such that

1.  $x(yz) = (xy)z$
2.  $(x+y)z = xz + yz$  and  $x(y+z) = xy + xz$
3.  $\lambda(xy) = (\lambda x)y = x(\lambda y)$
4.  $\|xy\| \leq \|x\| \cdot \|y\|$

#### Definition: Unital Banach Algebra

$B$  is called a unital Banach algebra if  $\exists e \in B$  such that

$$xe = ex = x \quad \text{and} \quad \|e\| = 1.$$

If  $e$  exists, it is unique.

### 1.2: Examples

#### Example 1

If  $X$  is a Banach space, then  $B = \mathcal{L}(X)$  (the set of all bounded linear operators  $A : X \rightarrow X$ ) equipped with algebraic operations

$$\begin{aligned}
(A+B)x &= Ax + Bx \\
(\lambda A)x &= \lambda(Ax) \\
(AB)x &= A(Bx)
\end{aligned}$$

and the operator norm

$$\|A\|_{\mathcal{L}(X)} = \sup_{x \neq 0} \frac{\|Ax\|_X}{\|x\|_X}.$$

$B = \mathcal{L}(X)$  is complete because  $X$  is complete.

The unit element is given by  $I_X x = x$ .

### Example 2

If  $X = \mathbb{C}^n$ , then  $B = \mathcal{L}(\mathbb{C}^n) \cong \mathbb{C}^{n \times n}$ .

$$A = (a_{ij})_{i,j=1}^n \quad Ax = y \quad \sum_{j=1}^n a_{ij} x_j = y_i.$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

The norm in  $\mathbb{C}^n$  leads to a norm in  $\mathbb{C}^{n \times n}$

$$\begin{aligned}
\|(x_i)\| &= \left( \sum |x_i|^2 \right)^{1/2} & \|A\| &= \\
\|(x_i)\| &= \sum |x_i| & \|A\| &= \max_j \sum_i |a_{ij}| \\
\|(x_i)\| &= \max |x_i| & \|A\| &= \max_i \sum_j |a_{ij}|
\end{aligned}$$

All norms are equivalent.

### Example 3

Take  $B = C(K)$  with  $K$  a compact Hausdorff space,  $f : K \rightarrow \mathbb{C}$  continuous and  $\|f\| = \max_{t \in K} |f(t)|$ .

### Example 4

Take  $B = A(K)$ ,  $K \subseteq \mathbb{C}$  compact with  $\text{int}(K) \neq \emptyset$ ,  $f : K \rightarrow \mathbb{C}$  continuous where  $f$  is holomorphic on  $\text{int}(K)$  and

$$\|f\| = \max_{t \in K} |f(t)| = \max_{t \in K \setminus \text{int}(K)} |f(t)|$$

e.g.  $K = \overline{\mathbb{D}} = \{t \in \mathbb{C} : |t| \leq 1\}$ . Then  $A(K) \subseteq C(K)$ .

### Example 5

Take  $B = \ell^\infty(\mathbb{N})$  or  $B = L^\infty(S, \sigma, \mu)$  with  $(S, \sigma, \mu)$  a measure space,  $f : S \rightarrow \mathbb{C}$  essentially bounded functions and

$$\|f\| = \text{ess sup}_{t \in S} |f(t)| = \inf_{\substack{N \subseteq S \\ \mu(N)}} \left( \sup_{t \in S \setminus N} |f(t)| \right)$$

### Example 6

Take  $B = \ell^1(\mathbb{Z})$  or  $B = L^1(\mathbb{R}^d)$  with  $\|\{x_n\}\| = \sum |x_n|$  and  $\|f\| = \int_{\mathbb{R}^d} |f(t)| dt$  respectively. Multiplication is given by the convolution. e.g.

$$fg = (f * g)(x) = \int_{\mathbb{R}^d} f(x-t)g(t) dt$$

$\ell^1(\mathbb{Z})$  is unital, but  $L^1(\mathbb{R}^d)$  is non-unital (since the unit of convolution is the Dirac delta; see Example 7).

### Example 7

Take  $B = M(\mathbb{R}^d)$  the complex measures on  $\mathbb{R}^d$  with bounded variation.

Then multiplication is given as

$$(\mu * \nu)(A) = \int_{\mathbb{R}^d} \mu(A-x) d\nu(x)$$

and norm

$$\|\mu\| = \sup_{\substack{\mathbb{R}^d = \bigcup A_i \\ \text{disjoint}}} \sum_{i=1}^n |\mu(A_i)| < +\infty.$$

Then,  $f dm = d\mu$  gives  $L^1(\mathbb{R}^d) \rightarrow M(\mathbb{R}^d)$ .

### Example 8

Take  $B = C^{n \times n}[K]$  with  $K$  compact and Hausdorff, continuous functions  $f : K \rightarrow \mathbb{C}^{n \times n}$  and norm

$$\|f\|_B = \max_{t \in K} \|f(t)\|_{C^{n \times n}}.$$

Then  $B \cong (C(K))^{n \times n}$  the  $n \times n$  matrices with entries from  $C(K)$ .

### 1.3: Remarks

- If  $B$  does not have a unit element, consider  $B_1 = B \times \mathbb{C}$  with operations

$$\begin{aligned} (b_1, \lambda_1) + (b_2, \lambda_2) &= (b_1 + b_2, \lambda_1 + \lambda_2) \\ \alpha(b, \lambda) &= (\alpha b, \alpha \lambda) \\ (b_1, \lambda_1)(b_2, \lambda_2) &= b_1 b_2 + \lambda_1 b_2 + \lambda_2 b_1, \lambda_1 \lambda_2) \end{aligned}$$

and norm

$$||(b, \lambda)|| = ||b|| + |\lambda|.$$

Then  $B_1$  is a unital Banach algebra with  $e = (0, 1)$ . One writes  $(b, \lambda) = (b, 0) + \lambda(0, 1) = b + \lambda \cdot e$ . In some sense,  $B \subseteq B_1$  where  $b \in B \mapsto (b, 0) \in B_1$ .

## 1.4: Definitions

### Definition: Commutative Banach Algebra

$B$  is called commutative if  $xy = yx$ .

### Definition: Banach Subalgebra

A subset  $B_0$  of a  $B$ -algebra is called a subalgebra if it is closed with respect to the algebraic operations

$$x, y \in B_0, \lambda \in \mathbb{C} \rightsquigarrow x + y, xy, \lambda x \in B$$

### Definition: Closed Subalgebra

$B_0$  is a closed subalgebra or Banach subalgebra if it is norm-closed.

- Proposition:  $B_0$  is a Banach algebra.

### Definition: Generated Subalgebra

Let  $M \neq \emptyset$  be a subset of a Banach algebra  $B$ .

The Banach subalgebra generated by  $M$  is the smallest closed subalgebra containing  $M$ .

$$\text{alg } M = (\text{clos alg}_B M)$$

- Remark

$\text{alg } M$  is the intersection of all closed subalgebras containing  $M$ .

$\text{alg } M = \text{clos} \left\{ \sum_{i=1}^N \lambda_i a_1^{(i)} a_2^{(i)} \cdots a_{n_i}^{(i)} \right\}$  is the norm-closure of finite linear combinations of finite products of  $a_j^{(i)} \in M$ .

## 1.5: Examples

### Example 1

Take  $B$  unital,  $b \in B$ . Then

$$\text{alg}\{e, b\} = \text{clos}_B \left\{ \sum_{i=0}^N \lambda_i b^i : \lambda_i \in \mathbb{C}, N \in \mathbb{N} \right\}$$

where  $b^0 = e$ .

## 1.6 Definitions

### Definition: Banach Algebra Homomorphism

A Banach algebra homomorphism is a map  $\phi : B_1 \rightarrow B_2$  between Banach algebras  $B_1$  and  $B_2$  such that

- $\phi$  is linear
- $\phi$  is bounded (continuous)
- $\phi$  is multiplicative

$$\phi(b_1 b_2) = \phi(b_1) \cdot \phi(b_2)$$

- $\phi$  is unital if both  $B_1, B_2$  have units and  $\phi(e_{B_1}) = e_{B_2}$ .

### Definition: Banach Algebra Isomorphism

A Banach algebra homomorphism which is bijective is called a Banach algebra isomorphism.

Then  $\phi^{-1} : B_2 \rightarrow B_1$  is an isomorphism as well.

### Definition: Banach Algebra Isometry

$\phi$  is an isometry if  $||\phi(x)|| = ||x||$ .

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### Recall

Given  $M \subseteq \mathcal{L}(X)$  with  $X$  a Banach space (and  $\mathcal{L}(X)$  itself a Banach algebra), we may construct  $B = \text{alg}_{\mathcal{L}(X)} M$ .

### 1.7 Proposition

Let  $B$  be a unital Banach algebra. Then the map

$$\phi : B \ni x \rightarrow L_x \in \mathcal{L}(B)$$

is an isometric isomorphism onto a closed subalgebra of  $\mathcal{L}(B)$  where

$$L_x : B \ni z \mapsto xz \in B$$

is the left-representation of  $x$ .

### Proof

$L_x$  is in  $\mathcal{L}(B)$  since  $L_x z = xz$

- is linear in  $z$  and
- $||L_x z|| = ||xz|| \leq ||x|| \cdot ||z||$  implies  $||L_x|| \leq ||x||$  (i.e.  $L_x$  is a bounded).

The map  $\phi : x \mapsto L_x$  is linear

$$L_{x_1+x_2}z = (x_1 + x_2)z = x_1z + x_2z = L_{x_1}z + L_{x_2}z = (L_{x_1} + L_{x_2})z$$

$\phi$  is multiplicative

$$L_{x_1x_2}z = (x_1x_2)z = x_1(x_2z) = L_{x_1}(L_{x_2}z)$$

From the above, we conclude that  $\phi$  is a homomorphism.

To show that  $\phi$  is an isometry,

$$\|L_x\| = \sup_{z \neq 0} \frac{\|L_x z\|}{\|z\|} \geq \frac{\|L_x e\|}{\|e\|} = \frac{\|x\|}{1} = \|x\|.$$

Then also  $\phi$  is injective and  $\text{im } \phi$  is closed. Since  $\text{im } \phi$  is a Banach algebra, it is therefore a closed subalgebra.

### 1.7 Remark: Right-Regular Representation

Every unital Banach algebra is isometrically isomorphic to a Banach algebra of operators.

Right-regular representation:

$$R_x = z \mapsto zx$$

## Section 1.2: Group of Invertible Elements in a Banach Algebra

### 2.1 Definition: Invertible Element

Let  $B$  be a unital Banach algebra. An element  $x \in B$  (in  $B$ ) if there exists  $y \in B$  such that  $xy = yx = e$ .

Note that  $y = x^{-1}$  is uniquely determined.

Write  $GB$  for the set of all invertible elements of  $B$ .

#### Remark

$GB$  is a (multiplicative group).

- $x, y \in GB \implies xy \in GB$  and  $(xy)^{-1} = y^{-1}x^{-1}$ ,
- $x \in GB \implies x^{-1} \in GB$  and  $(x^{-1})^{-1} = x$ , and
- $e \in GB$ .

### 2.2 Lemma

If  $x \in B$  and  $\|x\| < 1$ , then  $e - x \in GB$ .

#### Proof

Take the Neumann series

$$e + x + x^2 + x^3 + \dots$$

which converges to some  $s \in B$

$$s_n = e + x + \cdots + x^n$$

where  $s_n$  are Cauchy:

$$||s_{n+k} - s_n|| = ||x^{n+1} + \cdots + x^{n+k}|| \leq ||x||^{n+1} + ||x||^{n+2} + \cdots = \frac{||x||^{n+1}}{1 - ||x||}.$$

So  $s_n \rightarrow S$ ,

$$(e - x)s_n = s_n(e - x)e - x^{n+1}.$$

Taking  $n \rightarrow \infty$

$$(e - x)s = s(e - x) = e.$$

## 2.3 Proposition

The group  $GB$  is open in  $B$  and the map  $\Lambda : GB \ni x \mapsto x^{-1} \in GB$  is continuous (in the norm).

### Proof

Take  $x \in GB$  and consider  $y \in B$  with  $||y|| < \frac{1}{||x^{-1}||} = \varepsilon$ .

Then  $x + y \in B_\varepsilon(x)$  is invertible,

$$x + y = x(e + x^{-1}y),$$

and

$$||x^{-1}y|| \leq ||x^{-1}|| \cdot ||x|| < 1.$$

Therefore  $GB$  is open, since  $B_\varepsilon(X) \subseteq GB$ . The inverse

$$(x + y)^{-1} = (e + x^{-1}y)^{-1}x^{-1} = \sum_{n=0}^{\infty} (-x^{-1}y)^n x^{-1} = x^{-1} + \sum_{n=1}^{\infty} (-x^{-1}y)^n x^{-1}$$

so

$$||(x + y)^{-1} - x^{-1}|| \leq \sum_{n=1}^{\infty} ||x^{-1}||^{n+1} ||y||^n = \frac{||x^{-1}||^2 ||y||}{1 - ||x^{-1}|| \cdot ||y||}.$$

This converges to zero as  $||y|| \rightarrow 0$ .

## 2.4 Examples

### Example 1

$B = C(K)$ ,  $K$  compact Hausdorff,  $f : K \rightarrow \mathbb{C}$  continuous.

$GB = \{f \in C(K) : f(t) \neq 0, \forall t \in K\}$ .

## Example 2

$$B = \mathbb{C}^{n \times n}.$$

$$GB = \{A \in \mathbb{C}^{n \times n} : \det A \neq 0\}.$$

## 2.5 Definition:

Let  $G_0 B$  stand for the connected component of  $GB$  containing  $e$ .

### Remarks

- the  $\varepsilon$ -neighborhoods  $B_\varepsilon(x) \subseteq B$  are (path-)connected.

$$B_\varepsilon(x) = \{y \in B : ||x - y|| < \varepsilon\}$$

For  $y_1, y_2 \in B_\varepsilon(x)$ , there is a continuous path

$$\sigma : [0, 1] \ni \lambda \mapsto y_1 \lambda + y_2 (1 - \lambda) \in B_\varepsilon(x)$$

- Because  $GB$  is open and  $B_\varepsilon(x)$  is path-connected,  $GB$  is locally (path-)connected (i.e. every  $x \in GB$  has a (path-)connected open neighborhood in  $GB$ ).
- In this context, connectedness and path-connectedness are equivalent. Therefore the components of  $GB$  are the path-components of  $GB$ .
- $GB$  is the union of disjoint (path-)components where each component is both open and closed in  $GB$ .
- $x, y \in GB$  belong to the same path-component if there exists a continuous path  $\gamma : [0, 1] \rightarrow GB$  such that  $\gamma(0) = x$  and  $\gamma(1) = y$ . Here,  $x \sim y$  is an equivalence relation.
- $G_0 B = \{x \in GB : \exists \text{ a path in } GB \text{ connecting } e \text{ and } x\}$ .

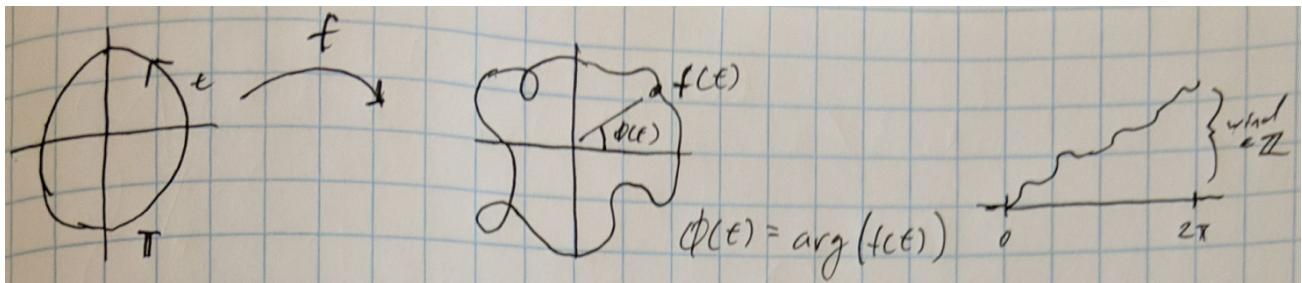
## 2.6 Examples

### Example 1

Take  $B = C(\mathbb{T})$  with  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$  and continuous functions  $f : \mathbb{T} \rightarrow \mathbb{C}$ .

$GB$  is the non-vanishing continuous functions  $f : \mathbb{T} \rightarrow \mathbb{C}$  ( $f(t) \neq 0, \forall t \in \mathbb{T}$ ).

For  $f \in GB$  one can define a winding number.



We have  $\frac{1}{2\pi} \arg f(e^{ix})$  a continuous function with

$$\text{wind}(t) = \left[ \frac{1}{2\pi} \arg f(e^{ix}) \right]_{x=0}^{2\pi} = \phi(2\pi) - \phi(0)$$

and  $\text{wind}(t) \in \mathbb{Z}$ .

The map  $GB \ni f \mapsto \text{wind}(f) \in \mathbb{Z}$  is continuous, hence locally constant (i.e. constant on each connected component).

Therefore  $G_0 C(\mathbb{T}) \subseteq \{f \in GC(\mathbb{T}) : \text{wind}(f) = 0\}$ . In fact, we will see that we have equality.

That is,  $f$  can be contracted (in  $GB$ ) to the constant function  $e(t) = 1$ .

## 2.7 Proposition

$G_0 B$  is a normal subgroup of  $GB$ .

### Proof

- $G_0 B$  is a group.

For any  $x, y \in G_0 B$ , there exist paths  $\gamma_1 : [0, 1] \rightarrow GB$  and  $\gamma_2 : [0, 1] \rightarrow GB$  with  $\gamma_1(0) = \gamma_2(0) = e$ ,  $\gamma_1(1) = x$  and  $\gamma_2(1) = y$ .

Define  $\gamma(t) = \gamma_1(t)\gamma_2(t)$  a path in  $GB$  such that  $\gamma(0) = e$  and  $\gamma(1) = xy$ . Then  $xy \in G_0 B$ .

Following from Lemma 2.2,  $\hat{\gamma} = (\gamma_1(t))^{-1}$  is a continuous path with  $\hat{\gamma}_1(0) = e$ ,  $\hat{\gamma}_1(1) = x^{-1}$  and  $x^{-1} \in GB$ .

- $G_0 B$  is a normal subgroup of  $GB$ .

For every  $y \in GB$ ,  $yG_0By^{-1} \subseteq G_0B$  if and only if  $yG_0B = G_0By$ .

Take  $x \in G_0 B$  with path  $\gamma$ , then

$$\delta(t) = y\gamma(t)y^{-1}, \quad \delta(0) = yey^{-1} = e, \quad \text{and} \quad \delta(1)yxy^{-1} \in G_0 B.$$

## 2.8 Definition: Abstract Index Group

The quotient group  $GB/G_0 B$  is called the abstract index group of  $B$ .

### Remark

$GB/G_0 B$  is in 1-to-1 correspondence with the set of connected components of  $GB$ .

Indeed, the (path-)connected components of  $GB$  are given by  $yG_0 B = G_0 B y$  (for  $y \in GB$ ).

$$y_1 G_0 B = y_2 G_0 B \iff y_2^{-1} y_1 G_0 B = G_0 B \iff y_2^{-1} y_1 \in G_0 B \iff [y_2] = [y_1] \text{ in } GB/G_0 B.$$

## 2.9 Definition: Exponential Map

For  $x \in B$ , we define the exponential map  $B \ni x \mapsto \exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$ .

### 2.10 Lemma

The exponential map  $B \ni x \mapsto \exp(x) \in GB$  is well-defined and continuous.

For  $xy = yx$ , we have  $\exp(x+y) = \exp(x)\exp(y)$ .

In particular,  $(\exp(x))^{-1} = \exp(-x)$ .

### Proof

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$  is absolutely convergent.

$$\sum_{n=0}^{\infty} \frac{||x||^n}{n!} < +\infty.$$

It follows that  $s_n = \sum_{k=0}^n \frac{x^k}{k!}$  is a Cauchy sequence and therefore converges.  
 Continuity left as an exercise. Need to show:

$$\left| \left| \sum \frac{x^n}{n!} - \sum \frac{y^n}{n!} \right| \right| \leq ||x - y|| \cdot M_{x,y}$$

The fact that  $\exp(x + y) = \exp(x)\exp(y)$  follows from multiplying terms and the binomial formula.

## October 7, 2024

### Recall

$GB$   $e + x$ .

$G_0B$  connected component of  $GB$  containing  $e$ .

$GB/G_0B$  is the abstract index group.

$B = C(\mathbb{T}) \rightsquigarrow f \in GC(\mathbb{T}) \rightsquigarrow \text{ind}(f)$ .

### Definition: Exponential Map

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \in GB$$

### Lemma:

For  $y \in B$ ,  $||y|| < 1$ , there exists  $x \in B$  such that  $\exp(x) = e + y$ .

### Proof

Define

$$\log(e + y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \dots \in B.$$

This converges absolutely ( $||y|| < 1$ ), therefore it converges in  $B$  by completeness.

### Identities

$$\exp(\log(e + y)) = \sum_{n=0}^{\infty} \frac{\left( \sum_k \frac{y^k}{k} (-1)^{k-1} \right)^n}{n!} = e + y$$

### Proof

$G_0B$  is equal to the set of all finite products of exponentials of elements in  $B$ .

$$G_0B = \bigcup_{n=0}^{\infty} \Gamma_n = \bigcup_{n=0}^{\infty} \{ \exp(a_1) \exp(a_2) \cdots \exp_{a_n} \in B \}$$

## Proof

Call  $\Gamma = \bigcup_{n=0}^{\infty} \Gamma^n$ .

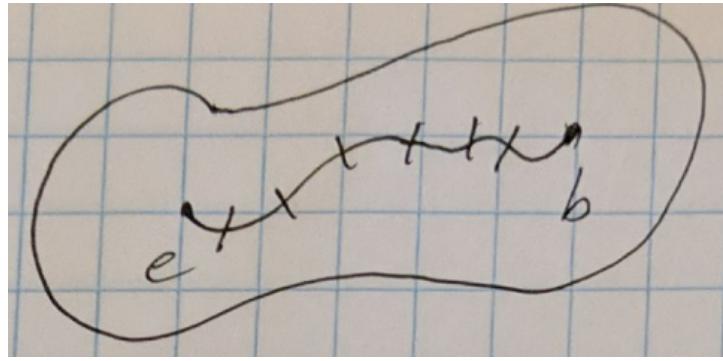
Then observe that each  $\Gamma_n$  is path-connected and contains  $e$ .

For  $b = \exp(a_1) \cdots \exp(a_n) \in \Gamma_n$ , define a path

- $\sigma : [0, 1] \rightarrow \Gamma_n$
- $\sigma(t) = \exp(ta_1) \cdots \exp(ta_n)$  is continuous with  $\sigma(0) = e$  and  $\sigma(1) = b$ .

Therefore,  $\Gamma$  is path-connected and contains  $e$ . It follows that  $\Gamma \subseteq G_0 B$ .

To prove that  $G_0 B \subseteq \Gamma$ , take  $b \in G_0 B$  and show that there exists a path in  $GB$   $\gamma : [0, 1] \rightarrow GB$  continuous with  $\gamma(0) = e$  and  $\gamma(1) = b$ .



We have that  $(\gamma(t))^{-1}$  is continuous and bounded in the norm. Then  $\gamma(t)$  is uniformly continuous.

$$\|\gamma^{-1}(t)\| \leq M.$$

$$(\exists N) : |t - s| \leq \frac{1}{N} \implies \|\gamma(t) - \gamma(s)\| \leq \frac{1}{M} \cdot \frac{1}{2}. \text{ Write}$$

$$b = \gamma(1) \cdot \gamma^{-1}(0) = \gamma(1) \gamma^{-1}\left(\frac{N-1}{N}\right) \gamma\left(\frac{N-1}{N}\right) \gamma^{-1}\left(\frac{N-2}{2}\right) \cdots \gamma\left(\frac{1}{N}\right) \gamma^{-1}\left(\frac{1}{N}\right) \gamma(0) = \prod_{k=1}^N \gamma^{-1}\left(\frac{k}{N}\right) \gamma\left(\frac{k-1}{N}\right).$$

Therefore, with  $s_k = \gamma^{-1}\left(\frac{k}{N}\right) \gamma\left(\frac{k-1}{N}\right)$ ,  $b = \prod_{k=1}^N \exp(\log(s_k))$ .

$$\|s_k - e\| \leq \|\gamma^{-1}\left(\frac{k}{N}\right)\| \cdot \|\gamma\left(\frac{k-1}{N}\right) - \gamma\left(\frac{k}{N}\right)\| \leq M \cdot \frac{1}{2M} \leq \frac{1}{2}.$$

## Corollary

If  $B$  is commutative,  $G_0 B = \{\exp(a) : a \in B\}$ .

## Remark

Special case:  $B = C(K)$  ( $K$  compact Hausdorff space).

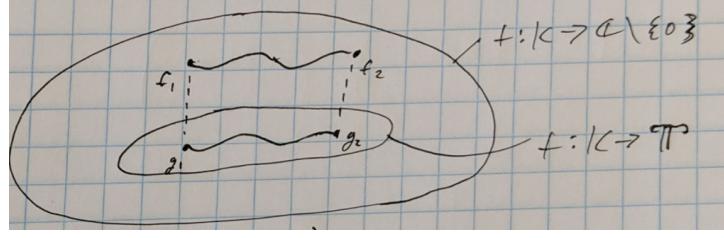
$$G_0 B = \{\exp(a) : a \in C(K)\}.$$

$GB/G_0 B$  is an equivalence class of functions  $f : K \rightarrow \mathbb{C} \setminus \{0\}$  with respect to path-connectedness.

That is,  $f_1 \sim f_2$  if and only if there exists continuous  $F(t, x) : [0, 1] \times K \rightarrow \mathbb{C} \setminus \{0\}$  with  $F(0, x) = f_1(x)$  and  $F(1, x) = f_2(x)$ .

These are the homotopy classes of continuous functions  $f : K \rightarrow \mathbb{C} \setminus \{0\}$ .

This corresponds to homotopy classes of continuous functions  $f : K \rightarrow \mathbb{T}$  (with  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ ) called the 1st co-homotopy group of  $K$   $\pi_1^*(K)$ .



$f : K \rightarrow \mathbb{C} \setminus \{0\}$  and  $\frac{f}{|f|} : K \rightarrow \mathbb{C} \setminus \{0\}$  are path-connected by  $\sigma(s) = \frac{f}{|f|^s}$ ,  $s \in [0, 1]$ .

$f_1 \sim f_2$  in  $K \rightarrow \mathbb{C} \setminus \{0\}$  implies that  $\frac{f_1}{||f_1||} \sim \frac{f_2}{||f_2||}$  in  $K \rightarrow \mathbb{T}$  by  $F(s, x)$  and  $\frac{F(s, x)}{|F(s, x)|}$ .

We conclude that  $\pi^1(K) \cong GC(K)/G_0C(K)$ .

### Example

Let  $B = C(\mathbb{T})$ .

$$G_0B = \{\exp(a) : a \in C(\mathbb{T}) = \{f \in GC(\mathbb{T}) : \text{wind}(f) = 0\}\}$$

For  $f \in GC(\mathbb{T})$ ,  $\text{wind}(f) = 0$  implies that  $f = \exp(a)$  has a logarithm.

This implies that  $f \in G_0B$  which itself implies that  $\text{wind}(f) = 0$ , since  $\text{wind}(f)$  is continuous on  $GC(\mathbb{T})$  and therefore constant on the component.

Therefore,  $GB/G_0B \cong \mathbb{Z}$  via the winding number.

For connected components of  $GB$ , define  $\chi_n(t) = t^n$ ,  $|t| = 1$ , where  $\text{wind}(\chi_n) = n$ .

### Remark: Closed Subalgebras and Invertibility

Let  $A$  be a closed subalgebra of  $B$  (both being unital,  $e \in A$ ,  $e \in B$ ).

Obviously, if  $a \in A$  is invertible in  $A$  (i.e.  $a^{-1} \in A$ ) then  $a$  is invertible in  $B$ . Then  $GA \subseteq GB \cap A \subseteq GB$ .

### Example

Take  $B = C(\mathbb{T})$  and  $A = \{f \in C(\mathbb{T}) : f_n = 0, \forall n < 0\} = C_+(\mathbb{T})$  where  $f_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{ix}) e^{-inx} dx$  is the  $n$ th Fourier coefficient.

Formally:  $f(t) \cong \sum_{n=-\infty}^{\infty} f_n t^n$  in  $B = C(\mathbb{T})$ ,  $|t| = 1$ .

$f \in A : f(t) = \sum_{n=0}^{\infty} f_n t^n$ ,  $|t| = 1$  has an analytic extension into the unit disk  $|t| < 1$ .

More precisely,  $\phi : A(\overline{\mathbb{D}}) \rightarrow C_+(\mathbb{T}) \subseteq C(\mathbb{T})$  by  $f \mapsto f|_{\mathbb{T}}$ .

Where  $A(\overline{\mathbb{D}}) = \{f \in \overline{\mathbb{D}} \rightarrow \mathbb{C} \text{ continuous, holomorphic on } \mathbb{D}\}$  and  $\mathbb{D} = \{t \in \mathbb{C} : |t| \leq 1\}$ .

Then, for  $f \in A(\overline{\mathbb{D}})$  with  $n \in \{-1, -2, -3, \dots\}$ ,

$$f_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{ix}) e^{-inx} dx = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(z)}{z^{n+1}} dz = \lim_{r \rightarrow 1^-} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(rz)}{z^{n+1}} dz = 0$$

- In fact,  $\phi$  is an isometry.

$$\|f\|_{A(\overline{\mathbb{D}})} = \sup_{|z| \leq 1} |f(z)| = \max_{|z|=1} |f(z)| = \|f\|_{\mathbb{T}} \|f\|_{C(\mathbb{T})}$$

By maximum modulus principle of holomorphic functions, since  $\phi$  is not constant.

- $\phi$  is linear and multiplicative.

- $C_+(\mathbb{T})$  is a closed subset of  $C(\mathbb{T})$ .

$$\Lambda_n : C(\mathbb{T}) \ni f \mapsto f_n \in \mathbb{C}$$

is a continuous linear functional.

$$C_+(\mathbb{T}) = \bigcap_{n=0} \ker \Lambda_n$$

- Less trivially,  $\phi$  is surjective and  $C_+(\mathbb{T})$  is an algebra.

### Example

$\chi_1(t) = t$  is invertible in  $C(\mathbb{T}) = B$ .  
 $\chi_1^{-1}(t) = \frac{1}{t} = x_{-1}(t) \notin C_+(\mathbb{T})$  while  $\chi_1(t) \in C_+(\mathbb{T})$ .  
Therefore  $GA \subseteq GB \cap A$  may not be equal.

### Definition: Boundary

The boundary of a subset  $U$  of a topological space  $X$  is  $\partial U = \overline{U} \setminus \text{int}(U)$ .

### Remark

For  $U \subseteq X$ ,  $X = \text{int}(U) \cup \partial U \cup \text{int}(X \setminus U)$  a union of disjoint sets.

### Lemma:

1. if  $a \in \partial GA$ , then  $a \notin GA$  and there exists a sequence  $a_n \in GA$  such that  $a_n \rightarrow a$ .
2. if  $a \in \partial a$  and  $a_n \in GA$  such that  $a_n \rightarrow a$ , then  $\|a_n^{-1}\| \rightarrow +\infty$ .

### Proof of 1

$a \in GA$  would imply  $a \in \text{int}(GA)$  and not a boundary point.

### Proof of 2

Otherwise, there would exist a bounded subsequence  $\|a_{n_i}^{-1}\| \leq M$ .

$$\|a_{n_i}^{-1} - a_{n_j}^{-1}\| \leq \|a_{n_i}^{-1}\| \cdot \|a_{n_j} - a_{n_i}\| \cdot \|a_{n_j}^{-1}\| \leq M^2 \|a_{n_i} - a_{n_j}\|$$

Since  $a_n$  converges,  $\{a_n\}$  is Cauchy which implies  $a_{n_i}^{-1}$  is Cauchy.

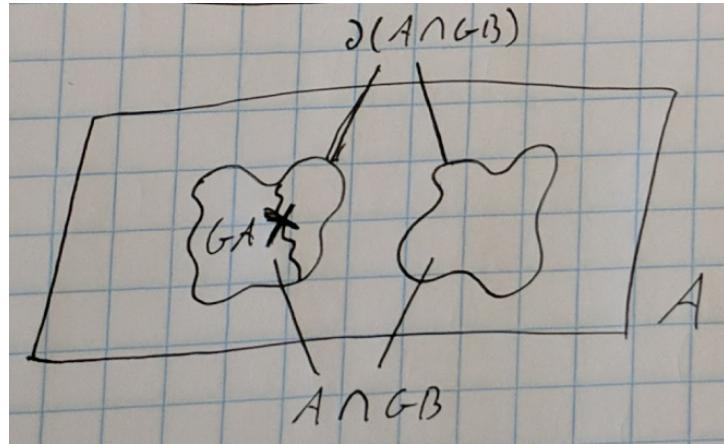
Then  $a_{n_i}^{-1} \rightarrow b \in A$ .  $e = a_{n_i} a_{n_i}^{-1} \rightarrow ab$  implies  $a^{-1} = b$  and  $a \in GA$ . However  $a \notin GA$ .

### Proposition

Let  $A$  be a closed subalgebra of  $B$  ( $e \in A$ ,  $e \in B$ ). Then  $\partial GA \subseteq \partial(A \cap GB)$  (both boundaries are considered in  $A$ ).

## Remark

Both  $GA$  and  $A \cap GB$  are open subsets of  $A$ .



## Proof

Take  $a \in \partial GA$  and suppose  $a \notin \partial(A \cap GB)$ .

Take  $a \in \partial GA$ :  $a_n \in GA$ ,  $a \notin GA$ ,  $a_n \rightarrow a$ ,  $\|a_n^{-1}\| \rightarrow +\infty$ .

**October 9, 2024**

## Recall

$A \subseteq B$ ,  $GA \subseteq A \cap GB$ .

If  $A = C_+(\mathbb{T}) \cong A(\overline{\mathbb{D}})$  and  $B = C(\mathbb{T})$ .

## Recall: Theorem

For  $GA$ ,  $A \cap GB$  open sets in  $A$ ,  $U \subseteq X$ ,  $\partial U = \overline{U} \setminus \text{int } U$ , we have that  $\partial GA \subseteq \partial(A \cap GB)$ .

## Proof

Take  $a \in \partial GA$ ,  $a_n \rightarrow a$ ,  $a \notin GA$ ,  $a \in A$ .

Since  $a_n \in GA$ ,  $\|a_n^{-1}\| \rightarrow +\infty$ .

However,  $a \notin GB$  otherwise  $a \in GB$ ,  $a_n \rightarrow a$  implies  $a_n^{-1} \rightarrow a^{-1}$  (in  $GB$ ) and, consequently,  $\sup \|a_n^{-1}\| < +\infty$ , a contradiction.

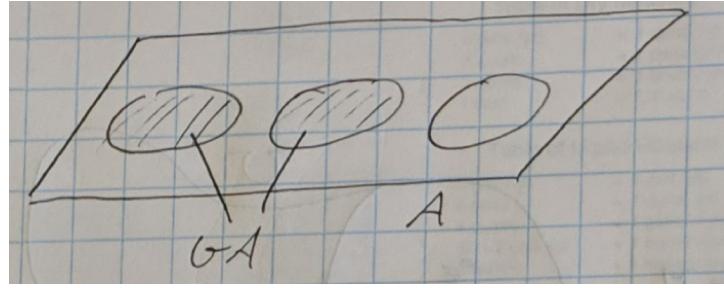
Therefore  $a \notin A \cap GB$  and, consequently,  $a \in \partial(A \cap GB) = \overline{(A \cap GB)} \setminus (A \cap GB)$ .

## Theorem

Let  $A$  be a closed subalgebra of  $B$ .

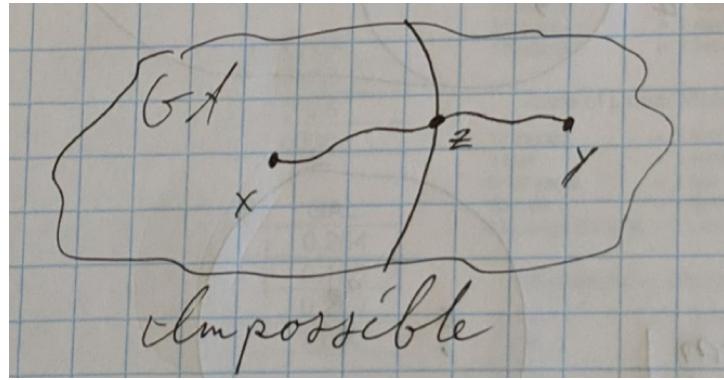
$GA$  is equal to the union of some components of  $A \cap GB$ .

## Proof



Let  $U$  be a component of  $A \cap GB$ .

We want to show that either  $U \cap GA \neq \emptyset$  or  $U \subseteq GA$ .



The above cannot occur since, by path-connectedness, for  $x, y \in U$ ,  $x \in GA$ ,  $y \notin GA$ , there would need to be some  $z \in \partial GA$  with  $z \notin A \cap GB$  a contradiction.

Alternatively, take  $A \cap GB$  open in  $A$ .

Then  $A \cap GB \cap \partial(A \cap GB) = \emptyset$  and  $(A \cap GB) \cap \partial GA = \emptyset$  by the previous theorem.

Write  $A = GA \cup \partial GA \cup \text{int}(A \setminus GA)$ . Then

$$A \cap GB = GA \cup \emptyset \cup \text{int}(A \setminus GA) \cap (A \cap GB)$$

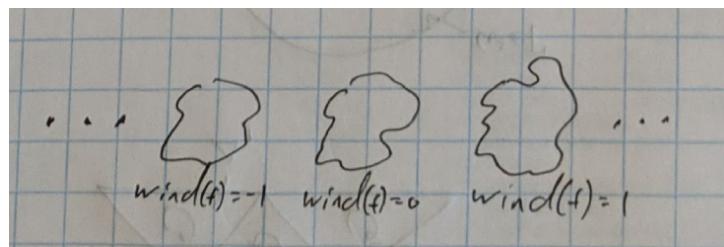
and  $U = (GA \cap U) \cup \text{int}(A \setminus GB) \cap U$  where  $(GA \cap U) \cap \text{int}(A \setminus GA) = \emptyset$  and open in  $U$ .

Therefore either  $GA \cap U = \emptyset$  or  $GA \cap U = U$  which implies that  $U \subseteq GA$ .

## Example

Take  $B(\mathbb{T})$  and  $A = C_+(\mathbb{T}) \cong A(\overline{D})$ .

Then  $GB = \{f: \mathbb{T} \rightarrow \mathbb{C} : f(t) \neq 0\}$ .



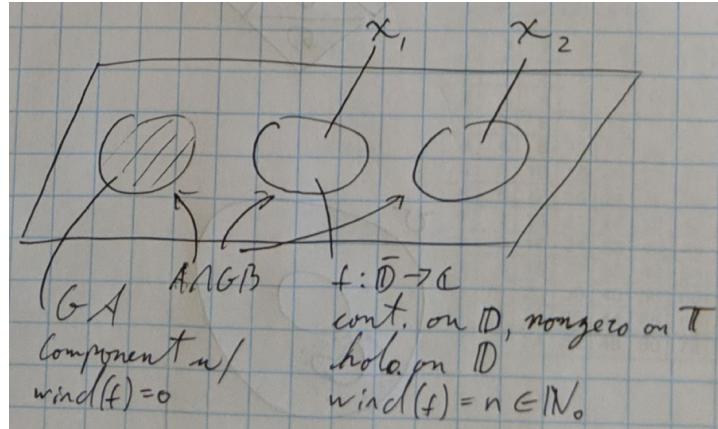
Then take

$$A \cap GB = \{f: \mathbb{T} \rightarrow \mathbb{C} \text{ continuous, } f(t) \neq 0, |t| = 1 \text{ with analytic continuation into } |t| < 1\}$$

such that  $f \in A \cap GB$  which implies  $\text{wind}(f) \in \{0, 1, 2, 3, \dots\}$  gives the number of zeroes of  $f$  inside  $\mathbb{D}$ .

$$\begin{aligned}\text{wind}(f) &= \frac{1}{2\pi i} \left[ \log f(e^{ix}) \right]_{x=0}^{2\pi} \\ &= \frac{1}{2\pi i} \lim_{r \rightarrow 1^-} \left[ \log f(re^{ix}) \right]_{x=0}^{2\pi} \\ &= \frac{1}{2\pi i} \lim_{r \rightarrow 1} \int_0^\pi \frac{f'(re^{ix})}{f(re^{ix})} ire^{ix} dx \\ &= \frac{1}{2\pi i} \lim_{z=re^{ix}} \int_{|z|=r} \frac{f'(z)}{f(z)} dz\end{aligned}$$

Which gives the number of zeros of  $f(z)$  inside  $|z| < 1$



## Section 1.3: Holomorphic Vector-Valued Functions

### Goal

Define the notion of holomorphic/analytic functions  $f : \Omega \rightarrow X$  where  $\Omega \subseteq \mathbb{C}$  open and  $X$  a (complex) Banach space.

### Summary

- Basically all classical results remain true.
- There is a strong and a weak version of holomorphy, but they are equivalent.

### Theorem

For a function  $f : \Omega \rightarrow X$ ,  $\Omega \subseteq \mathbb{C}$  open and  $X$  Banach, the following are equivalent

1.  $f$  is differentiable at every  $z_0 \in \Omega$ , i.e. there exists  $f'(z_0) \in X$  such that

$$\lim_{z \rightarrow z_0} \left\| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right\|_X = 0$$

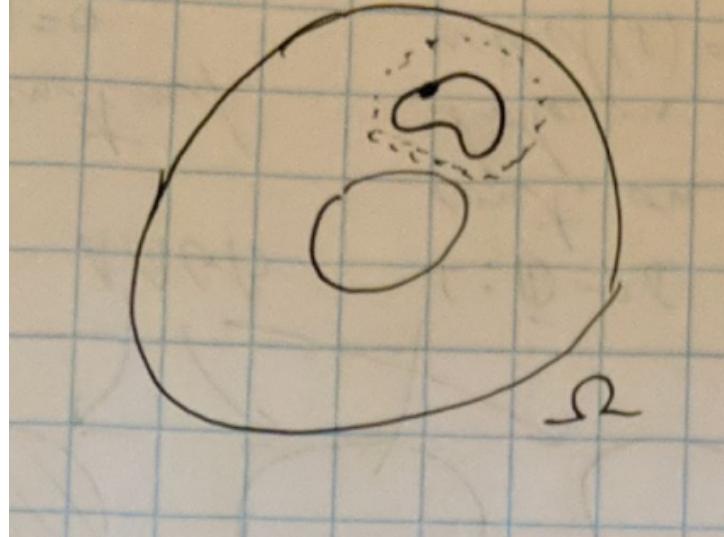
2.  $f$  is analytic at each point  $z_0 \in \Omega$ , i.e.  $f$  has a convergent power series at  $z_0$  with radius of convergence  $R_{z_0} > 0$ .

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, |z - z_0| < R_{z_0}, a_n \in X$$

which converges in the norm of  $X$ .

3.  $f : \Omega \rightarrow X$  is continuous (in the norm) and for every piecewise smooth closed contour  $\Gamma$  contained in a disk  $D$  ( $\Gamma \subseteq D \subseteq \Omega$ ).

$$\int_{\Gamma} f(z) dz = 0$$



### Definition: (Strongly) Holomorphic Function

If (1)-(3) hold, then  $f$  is (strongly)-holomorphic.

### Remarks: Integration of Vector-Valued Functions

A piecewise smooth contour  $\Gamma$  can be parameterized by  $\sigma : [0, 1] \rightarrow \Omega$ .

$$\int_{\Gamma} f(z) dz = \int_0^1 \underbrace{f(\sigma(t))\sigma'(t)}_{h(t) \text{ continuous}} dt$$

This is independent of the choice of parameterization.

Now  $I = \int_0^1 h(t) dt$  can be defined via Riemann sums. Given a partition  $P$ ,  $h : [0, 1] \rightarrow X$  continuous.

$$\lim_{\text{mesh}(P) \rightarrow 0} \|S(h, P, \xi) - I\|_X = 0$$

where  $S(h, P, \xi) = \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1})$ ,  $P = \{x_0, x_1, \dots, x_n\}$ ,  $\xi_i \in [x_{i-1}, x_i]$ .

Note that  $h$  is uniformly continuous and  $\forall \varepsilon > 0$ ,  $\exists \delta > 0$  such that  $\text{mesh}(P_1) < \delta$ ,  $\text{mesh}(P_2) < \delta$  implies

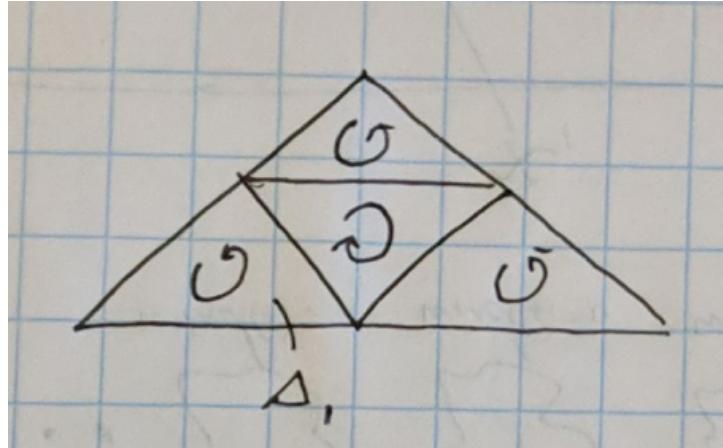
$$\|S(f, P_1, \xi^{(1)}) - S(f, P_2, \xi^{(2)})\| < \varepsilon$$

All usual properties of integrals hold.

- linear in integrand
- $\|\int_{\Gamma} f(z) dz\| \leq \int_{\Gamma} \|f(z)\| |dz| \leq (\text{length}(\Gamma)) \sup_{z \in \Gamma} \|f(z)\|$ .

### Sketch of Proof (1) to (3)

To show:  $\int_{\Delta} f(z) dz = x_0 = 0$  by contradiction that  $x_0 \neq 0$ .

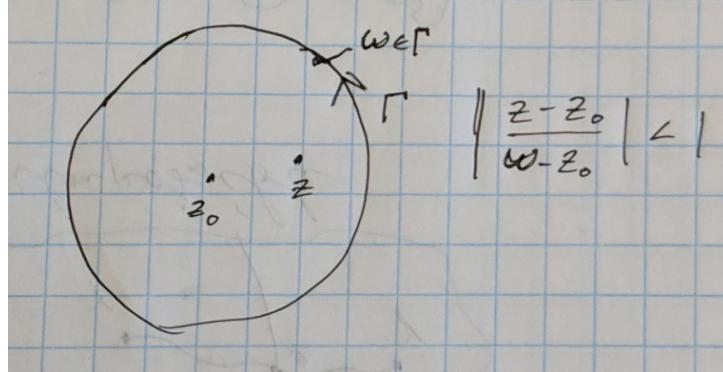


We have  $\left| \int_{\Delta_1} f dz \right| \geq \frac{\|x_0\|}{4}$ ,  $\left| \int_{\Delta_n} f dz \right| \geq \frac{\|x_0\|}{4^n}$ .

### Sketch of Proof (3) to (2)

$\int_{\Gamma} f dz = 0$  implies the Cauchy integral formula. Take

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\omega)}{\omega - z} d\omega$$



$$\frac{1}{\omega - z} = \frac{1}{(\omega - z_0) - (z - z_0)} = \frac{1}{w - z_0} \sum_{n=0}^{\infty} \left( \frac{z - z_0}{w - t} \right)^n$$

Therefore

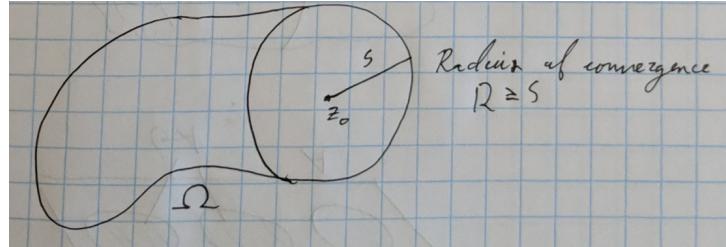
$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} f(\omega) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\omega - z)^{n+1}} d\omega = \sum_{n=0}^{\infty} (z - z_0)^n \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\omega)}{(\omega - z)^{n+1}} d\omega = \sum_{n=0}^{\infty} (z - z_0)^n a_n$$

with the sequence converging (in  $X$ ) on  $|z - z_0| < |\omega - z_0|$ .

- Radius of Convergence

$$R^{-1} = \limsup_{n \rightarrow \infty} \|a_n\|^{1/n}$$

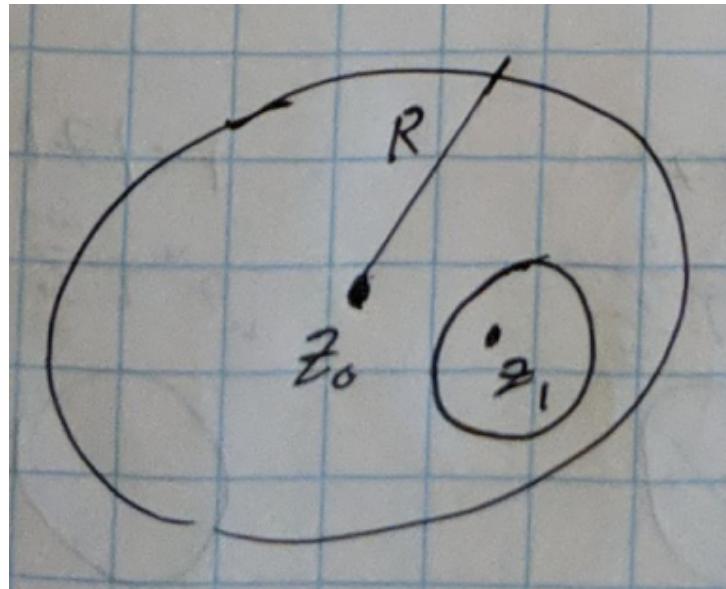
(Root Test:  $|z - z_0| < R$  convergence;  $|z - z_0| > R$  divergence)



### Sketch of Proof (2) to (1)

One can show that a function defined by convergent power series is differentiable,  $f(z) = \sum a_n(z - z_0)^n$ , then  $f'(z) = \sum a_n \cdot n(z - z_0)^{n-1}$ .

The radius of convergence is the same. This also implies that  $f$  is infinitely differentiable.



Take  $z - z_0 = (z - z_1) + (z_1 - z_0)$  and, by the binomial theorem,

$$f(z) = \sum_{k=0}^{\infty} (z - z_1)^k \left( \sum_{n=k}^{\infty} a_n \binom{n}{k} (z_1 - z_0) \right)$$

which converges for at least  $|z - z_1| < R - |z_1 - z_0|$ .

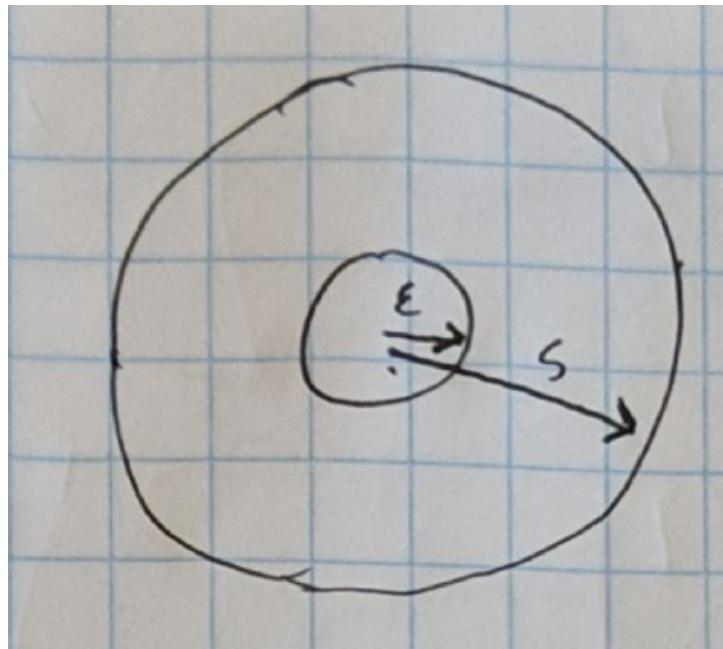
**October 14, 2024**

### Theorem

Let  $f : D_\varepsilon(z_0) \rightarrow X$  ( $D_\varepsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$ ) be holomorphic.

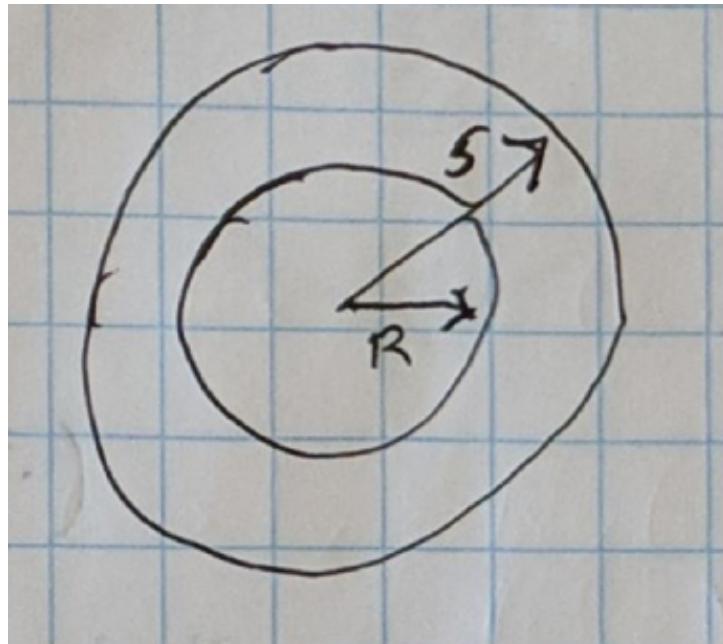
Then  $R = S$  where

1.  $R$  is the radius of convergence of  $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$  ( $R^{-1} = \limsup_{n \rightarrow \infty} \|a_n\|^{\frac{1}{n}}$ ).
2.  $S$  is the radius of the largest open disk  $D_S(z_0)$  such that there exists an analytic extension of  $f$  from  $D_\varepsilon(z_0)$  to  $D_S(z_0)$ .



### Proof

By definition,  $\sum_{n=0}^{\infty} a_n(z-z_0)^n$  converges for  $|z-z_0| < R$ . Then  $|z-z_0| < R$  if and only if  $\limsup_{n \rightarrow \infty} ||a_n(z-z_0)^n||^{\frac{1}{n}} < 1$  if and only if  $\sum_{n=0}^{\infty} a_n(z-z_0)^n$  converges. Therefore, it converges to a holomorphic function on  $R \leq S$ . If  $f(z)$  has an analytic extension to  $D_S(z_0)$ , see step (3)  $\implies$  (2) of previous theorem.



Then  $f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\omega)}{\omega \cdot z} d\omega = \sum_{n=0}^{\infty} a_n(z-z_0)^n$  converges for  $|z-z_0| < r < S$  with  $a_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\omega)}{(z-\omega)^{n+1}} d\omega$ . From this, we conclude  $R \geq S$ .

### Definition: (Weakly) Holomorphic Function

A function  $f : \Omega \rightarrow X$  ( $\Omega \subseteq \mathbb{C}$  open,  $X$  Banach) is called weakly holomorphic if  $\phi \circ f : \Omega \rightarrow \mathbb{C}$  is holomorphic,  $\forall \phi \in X^* = \mathcal{L}(X; \mathbb{C})$  bounded linear functionals.

A function  $f : \Omega \rightarrow \mathcal{L}(X, Y)$  ( $X, Y$  Banach) is weakly-operator holomorphic if  $h_{\phi, X} : \Omega \rightarrow \mathbb{C}$  is holomorphic for all  $\phi \in Y^*$ ,  $x \in X$  where  $h_{\phi, X}(z) = \phi(f(z)x)$ .

## Remarks

Obviously:  $f$  strongly holomorphic  $\implies f$  weakly holomorphic.

$$\left\| \frac{\phi(f(z+h)) - \phi(f(z))}{h} - \phi(f'(z)) \right\| \leq \|\phi\| \cdot \left\| \frac{f(z+h) - f(z)}{h} - f'(z) \right\|$$

For  $f : \Omega \rightarrow \mathcal{L}(X, Y)$ :  $f$  strongly holomorphic  $\implies f$  weakly holomorphic  $\implies f$  weakly operator holomorphic.

For  $x \in X$ ,  $\phi \in Y^*$ ,  $\Lambda_{x,\phi} : \mathcal{L}(X, y) \ni A \mapsto \phi(Ax) \in \mathbb{C}$  and  $\Lambda_{x,\phi} \in (\mathcal{L}(X, y))^*$ .

All the converses are also true.

## Theorem (Dunford)

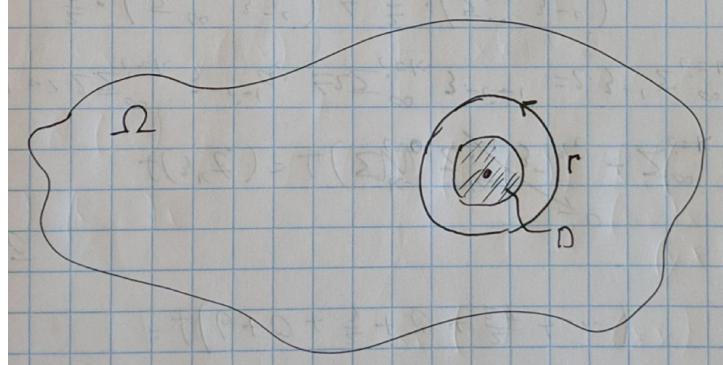
Take  $X$  Banach,  $\Omega \subseteq \mathbb{C}$  open.

If  $f : \Omega \rightarrow X$  is weakly holomorphic, then it is strongly holomorphic.

## Proof

We want to show that for any  $z_0 \in \Omega$ ,  $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$  exists in  $X$ .

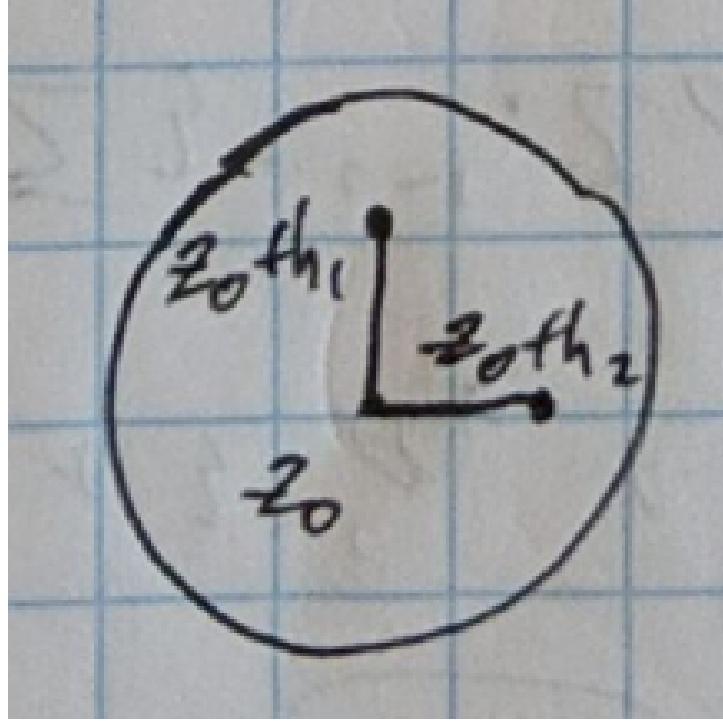
Choose  $\varepsilon > 0$  such that the disk  $D_\varepsilon(z_0)$  and circle  $C_{2\varepsilon}(z_0) = \Gamma$  are in  $\Omega$ .



For  $\phi \in X^*$ ,  $\phi(f(z))$  is holomorphic in  $\Omega$ .

$$\phi(f(z)) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\phi(f(\omega))}{z - \omega} d\omega, z \in D$$

Apply this to  $z = z_0$ ,  $z = z_0 + h_1$  and  $z = z_0 + h_2$  with  $0 < |h_1| < \varepsilon$ ,  $0 < |h_2| < \varepsilon$ ,  $h_1 \neq h_2$ .



$$\begin{aligned}
A_{h_1, h_2} &= \frac{1}{h_1 - h_2} \left\{ \frac{f(z_0 + h_1) - f(z_0)}{h_1} - \frac{f(z_0 + h_2) - f(z_0)}{h_2} \right\} \\
\phi(A_{h_1, h_2}) &= \frac{1}{h_1 - h_2} \left\{ \frac{\phi(f(z_0 + h_1)) - \phi(f(z_0))}{h_1} - \frac{\phi(f(z_0 + h_2)) - \phi(f(z_0))}{h_2} \right\} \\
&= \frac{1}{2\pi i} \int_{\Gamma} \phi(f(\omega)) \frac{1}{h_1 - h_2} \left\{ \frac{1}{h_1} \left( \frac{1}{z_0 + h_1 - \omega} - \frac{1}{z_0 - \omega} \right) - \frac{1}{h_2} \left( \frac{1}{z_0 + h_2 - \omega} - \frac{1}{z_0 - \omega} \right) \right\} d\omega \\
&= \frac{1}{2\pi i} \int_{\Gamma} \phi(f(\omega)) \frac{1}{h_1 - h_2} \left\{ \frac{1}{(z + h_1 - \omega)(z_0 - \omega)} - \frac{1}{(z + h_2 - \omega)(z_0 - \omega)} \right\} d\omega \\
&= \frac{1}{2\pi i} \int_{\Gamma} \phi(f(\omega)) \frac{1}{(z_0 + h_1 - \omega)(z_0 + h_2 - \omega)(z_0 - \omega)} d\omega
\end{aligned}$$

Observe that the denominator is at least  $\varepsilon^3$ , therefore  $|\phi(A_{h_1, h_2})| \leq \frac{\varepsilon^3}{2\pi} \sup_{\omega \in \Gamma} |f(\omega)| \cdot |\phi|$  (so long as  $f$  continuous, which will be proven).

Therefore  $\forall \phi \in X^*$ ,

$$\sup_{\substack{0 < |h_1| < \varepsilon \\ 0 < |h_2| < \varepsilon \\ h_1 \neq h_2}} |\phi(A_{h_1, h_2})| < +\infty.$$

By the uniform boundedness principle, identify  $A_{h_1, h_2} \in X$  with  $X^{**} = \mathcal{L}(X^*, \mathbb{C})$ .

Then  $\sup_{h_1, h_2} \|A_{h_1, h_2}\| < +\infty$  and

$$\left\| \frac{f(z_0 + h_1) - f(z_0)}{h_1} - \frac{f(z_0 + h_2) - f(z_0)}{h_2} \right\| \leq C \cdot |h_1 - h_2|.$$

Now, for any sequence  $\{h_n\}_{n=3}^{\infty}$ ,  $0 < |h_n| < \varepsilon$ ,  $h_n \rightarrow 0$ ,

$$\frac{f(z_0 + h_n) - f(z_0)}{h_n}$$

is a cauchy sequence. Therefore  $\lim_{n \rightarrow \infty} \frac{f(z_{0+h_n}) - f(z_0)}{h_n}$  exists in  $X$  independent of choice of  $\{h_n\}$ . That is

$$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists in  $X$ .

## Section 1.4: Spectrum and Resolvent

Consider a unital Banach algebra  $B$ .

### Definition: Spectrum

For  $b \in B$ , the spectrum of  $b$  in  $B$   $\sigma_B(b) = \{\lambda \in \mathbb{C} : \lambda e - b \text{ is not invertible in } B\}$ .

### Definition: Resolvent

The resolvent is a function  $R(b; \lambda) = (\lambda e - b)^{-1}$ .  $R(b, \cdot) : \mathbb{C} \setminus \sigma_B(b) \rightarrow B$ .  
 $\mathbb{C} \setminus \sigma_B(b)$  is the resolvent set.

### Theorem

1. The spectrum  $\sigma_B(b)$  is a non-empty, compact subset of  $\mathbb{C}$ .
2. The resolvent  $R(b, \lambda)$  is an analytic, Banach valued function on  $\mathbb{C} \setminus \sigma_B(b)$ .

### Proof of (a)

$\sigma_B(b)$  is bounded, because  $\lambda e - b$  is invertible for  $|\lambda| > \|b\|$ .

$$\lambda e - b = \lambda \left( e - \frac{1}{\lambda} b \right)$$

has  $\left\| \frac{1}{\lambda} b \right\| < 1$  for sufficiently large  $\lambda$ . Therefore,  $\sigma_B(b) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|b\|\}$ .

To show that  $\sigma_B(b)$  is closed, if  $\lambda \notin \sigma_B(b)$  then  $\forall \mu$  such that  $|\lambda - \mu| < \varepsilon$  we have that  $\mu \notin \sigma_B(b)$ .

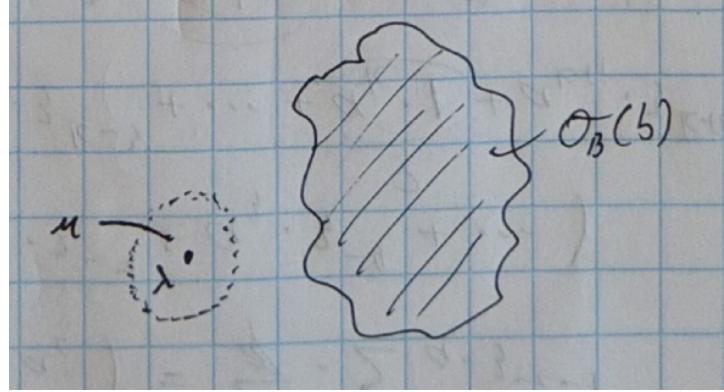
$$\mu e - b = \lambda e - b + (\mu - \lambda)e = (\lambda e - b) \underbrace{\left[ e + \frac{(\mu - \lambda)(\lambda e - b)^{-1}}{\|\cdot\| < 1} \right]}_{\text{closed set}}$$

when  $|\mu - \lambda| < \frac{1}{\|(\lambda e - b)^{-1}\|}$ .

Therefore  $\mathbb{C} \setminus \sigma_B(b)$  is open.

### Proof of (b)

Take  $\lambda \notin \sigma_B(b)$



$$\begin{aligned}
 \frac{R(b, \mu) - R(b, \lambda)}{\mu - \lambda} &= \frac{1}{\mu - \lambda} \left( (\mu e - b)^{-1} - (\lambda e - b)^{-1} \right) \\
 &= \frac{1}{-\mu - \lambda} (\mu e - b)^{-1} \{(\lambda e - b) - (\mu e - b)\} (\lambda e - b)^{-1} \\
 &= -(\mu e - b)^{-1} (\lambda e - b)^{-1}
 \end{aligned}$$

Using continuity with  $GB \ni a \mapsto a^{-1} \in GB$  in the norm,  $-(\mu e - b)^{-1} (\lambda e - b)^{-1} \rightarrow -((\lambda e - b)^{-1})^2$  as  $\mu \rightarrow \lambda$ . Therefore  $R^1(b, \lambda) = -(R(b, \lambda))^2$  and  $R(b, \lambda)$  is analytic.

### Proof of non-empty in (a)

Take  $\sigma_B(b) \neq 0$ , otherwise  $R(b, \lambda)$  is analytic on  $\mathbb{C}$  and bounded

$$(\lambda e - b)^{-1} = \frac{1}{\lambda} \left( e - \frac{1}{\lambda} b \right)^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\lambda^{n+1}} b^n$$

We can estimate

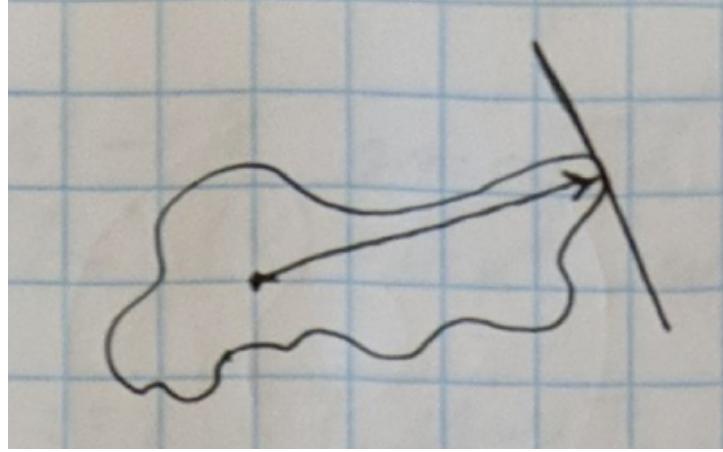
$$\|\cdot\| \leq \frac{1}{|\lambda| \left( 1 - \frac{\|b\|}{|\lambda|} \right)} = \frac{1}{|\lambda| - \|b\|}$$

so  $\lim_{\lambda \rightarrow \infty} \|(\lambda e - b)^{-1}\| = 0$ .

By Liouville's theorem, bounded and entire functions are constant. But we may also proceed by weak analyticity. If  $\phi(R(b, \lambda))$  is analytic and bounded on  $\mathbb{C}$ ,  $\forall \phi \in B^*$ , it follows that  $\phi(R(b, \lambda)) \equiv 0$ ,  $\forall \lambda$ ,  $\forall \phi \in B^*$  and that  $R(b, \lambda) \equiv 0$  for any  $\lambda$  a contradiction.

### Definition: Spectral Radius

For  $b \in B$ , the spectral radius  $r(b) = \max\{|\lambda| : \lambda \in \sigma_B(b)\}$ .



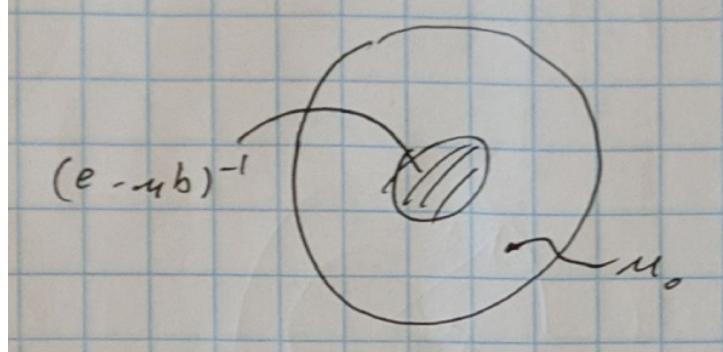
### Remark

Write  $\frac{1}{r(b)} = \min\{|\lambda|^{-1} : \lambda e - b \text{ is not invertible}\} = \min\{|\mu| : e - \mu b \text{ is not invertible}\}$  with  $\mu = \frac{1}{\lambda}$ .

$$\underbrace{(e - \mu b)^{-1}}_{\text{analytic in } |\mu| < \frac{1}{||b||}} = \sum_{n=0}^{\infty} \mu^n b^n$$

converges for  $|\mu| < \frac{1}{||b||}$ .

Then the radius of convergence  $R^{-1} = \limsup_{n \rightarrow \infty} ||b^n||^{\frac{1}{n}}$  gives us that  $R$  is equal to the largest disk where  $(e - \mu b)^{-1}$  has an analytic extension. Therefore  $S = \frac{1}{r(b)}$ .



Suppose we have an analytic extension  $f(\mu)$  beyond  $S$ .

$$f(\mu)(e - \mu b) = (e - \mu b)f(\mu) = e$$

implies that and, if  $(e - \mu_0 b)$  not invertible,  $f(\mu_0)(e - \mu_0 b) = \dots = e$  a contradiction.

### Theorem

$$r(b) = \lim_{n \rightarrow \infty} ||b^n||^{\frac{1}{n}} = \inf_{n \in \mathbb{N}} ||b^n||^{\frac{1}{n}}$$

## Proof

To demonstrate existence, fix  $n_0 \in \mathbb{N}$ ,  $n = q \cdot n_0 + r$ ,  $0 \leq r < n_0$ .

$$\begin{aligned} ||b^n|| &\leq ||b^{n_0}||^q \cdot ||b||^r \\ ||b^n||^{\frac{1}{n}} &\leq ||b^{n_0}||^{\frac{q}{n}} \cdot ||b||^{\frac{r}{n}} \\ \limsup_{n \rightarrow \infty} ||b^n||^{\frac{1}{n}} &\leq ||b^{n_0}||^{\frac{1}{n_0}} \cdot 1 \end{aligned}$$

Since  $1 = \frac{q}{n} \cdot n_0 + \frac{r}{n}$ . Take  $n \rightarrow \infty$ . Write

$$\limsup_{n \rightarrow \infty} ||b^n||^{\frac{1}{n}} \leq \inf_{n_0 \in \mathbb{N}} ||b^{n_0}||^{\frac{1}{n_0}} \leq \liminf_{n \rightarrow \infty} ||b^n||^{\frac{1}{n}}$$

**October 16, 2024**

## Note: Closed Subalgebras

Assume  $A$  is a closed subalgebra of  $B$  ( $e \in A \subseteq B$ ).

Take  $b \in A \subseteq B$ .

Obviously,  $b - \lambda e$  being invertible in  $A$  implies  $b - \lambda e$  is invertible in  $B$ . We also have

$$\mathbb{C} \setminus \text{sp}_A(b) \subseteq \mathbb{C} \setminus \text{sp}_B(b)$$

(confer.  $GA \subseteq GB$  with  $\partial GA = \partial(A \cap GB)$ ) and, equivalently,

$$\text{sp}_B(b) \subseteq \text{sp}_A(b).$$

One can show similarly that

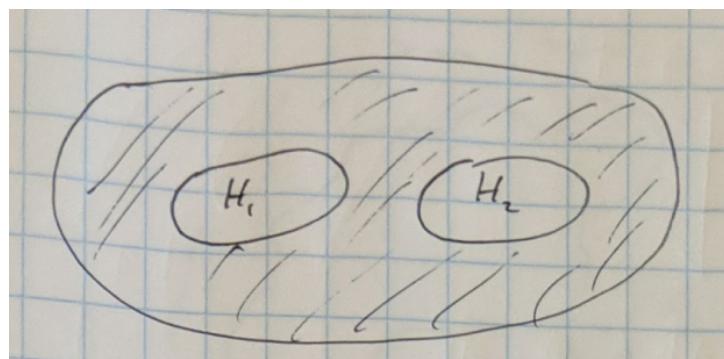
$$\begin{aligned} \partial(\mathbb{C} \setminus \text{sp}_A(b)) &\subseteq \partial(\mathbb{C} \setminus \text{sp}_B(b)) \\ &= \dots = \\ \partial \text{sp}_A(b) &\subseteq \partial \text{sp}_B(b) \end{aligned}$$

## Proposition

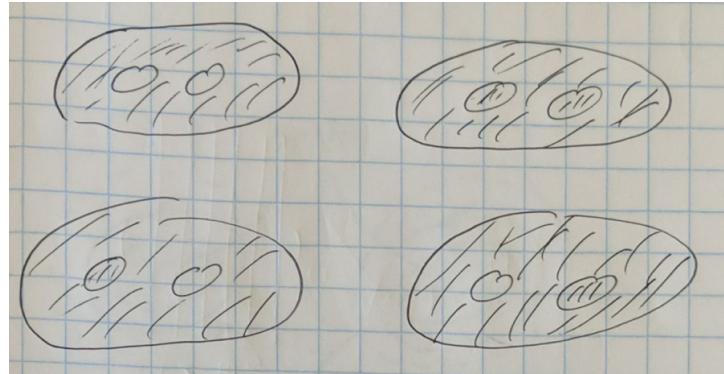
1.  $\mathbb{C} \setminus \text{sp}_A(b)$  is the union of some components of  $\mathbb{C} \setminus \text{sp}_B(b)$ .
2.  $\text{sp}_A(b) = \text{sp}_B(b) \cup \bigcup_{\omega} H_{\omega}$  where  $H_{\omega}$  are some components of  $\mathbb{C} \setminus \text{sp}_B(b)$ .

## Example 1

Suppose  $\text{sp}_B(b)$  looks like

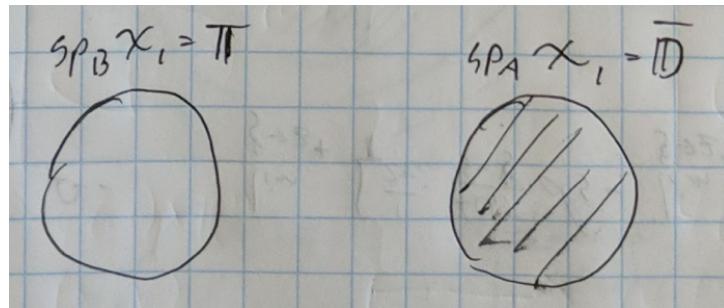


Now  $\text{sp}_A(b)$  can only be one of the 4 possibilities.



### Example 2

$B = C(\mathbb{T})$ ,  $A = C_+(\mathbb{T}) \simeq A(\mathbb{D})$ ,  $\chi_1(t) = t$ ,  $\text{sp}_B \chi_1 = \mathbb{T}$ .



### Theorem: Spectral Mapping Theorem (Simple Version)

For a polynomial  $p(z) = \sum_{n=0}^N p_n z^n$  we define  $p(b) = \sum_{n=0}^N p_n b^n$  for  $b \in B$  where  $b^0 = e$ .

Let  $p$  be a polynomial and  $b \in B$  with  $B$  a unital Banach algebra, then  $\text{sp}(p(b)) = p(\text{sp}(b)) := \{p(z) : z \in \text{sp}(b)\}$ .

### Proof

For  $\lambda \in \mathbb{C}$ , consider  $q(z) = p(z) - \lambda = c \prod_{i=1}^N (z - \gamma_i)$ .

Now,  $q(b) = p(b) - \lambda e = c \prod_{i=1}^N (b - \gamma_i e)$ . It follows that

$$\lambda \notin \text{sp}(p(b)) \iff p(b) - \lambda e \text{ is invertible.}$$

a commuting product

$$\iff \overbrace{\prod_{i=1}^N (b - \gamma_i e)}^{\text{a commuting product}} \text{ is invertible.}$$

$$\iff \forall i, b - \gamma_i e \text{ is invertible.}$$

$$\iff \forall i, \gamma_i \notin \text{sp}(b)$$

$$\iff \forall z \in \text{sp}(b), q(z) = c \prod_{i=1}^N (z - \gamma_i) \neq 0$$

$$\iff \forall z \in \text{sp}(b), p(z) \neq \lambda$$

$$\iff \lambda \notin p(\text{sp}(b))$$

## Applications

If  $p(b) = 0$ , then  $\text{sp}(b) \subseteq \{z \in \mathbb{C} : p(z) = 0\}$ , because

$$\{0\} = \text{sp } 0 = \text{sp } p(b) \stackrel{\text{SMT}}{=} p(\text{sp } b).$$

It follows that if  $b$  is nilpotent, such that  $b^n = 0$  for some  $n$  ( $p(z) = z^2$ ), then  $\text{sp}(b) = \{0\}$ .

If  $b$  is idempotent, such that  $b^2 = b$  ( $p(z) = z^2 - z$ ), then  $\text{sp}(b) \subseteq \{0, 1\}$ .

If  $b$  is unipotent (or flip), such that  $b^2 = e$ , then  $\text{sp}(b) = \{\pm 1\}$ .

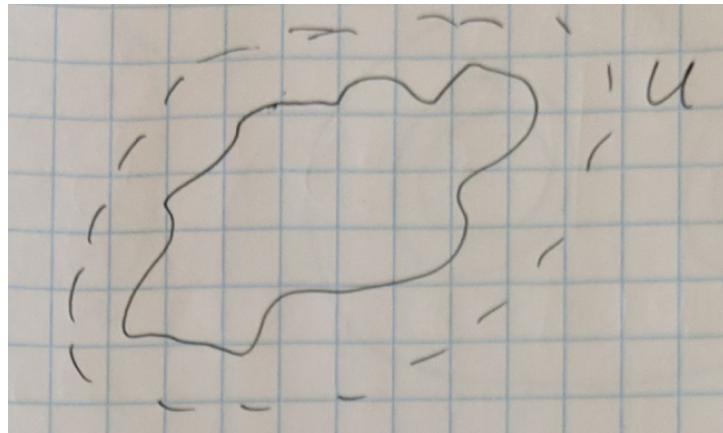
## Section 1.5: Riesz Functional Calculus

### Question:

Can one define  $f(b)$  for  $b \in B$  a unital Banach algebra for more general functions  $f$ ?

### Definition: Set of Functions Holomorphic on the Spectrum

For a unital Banach algebra  $B$  and  $b \in B$ , let  $A[\text{sp}(b)]$  stand for the set of all functions  $f$  which are holomorphic on some open neighborhood  $U$  of  $\text{sp}(b)$ .



### Lemma

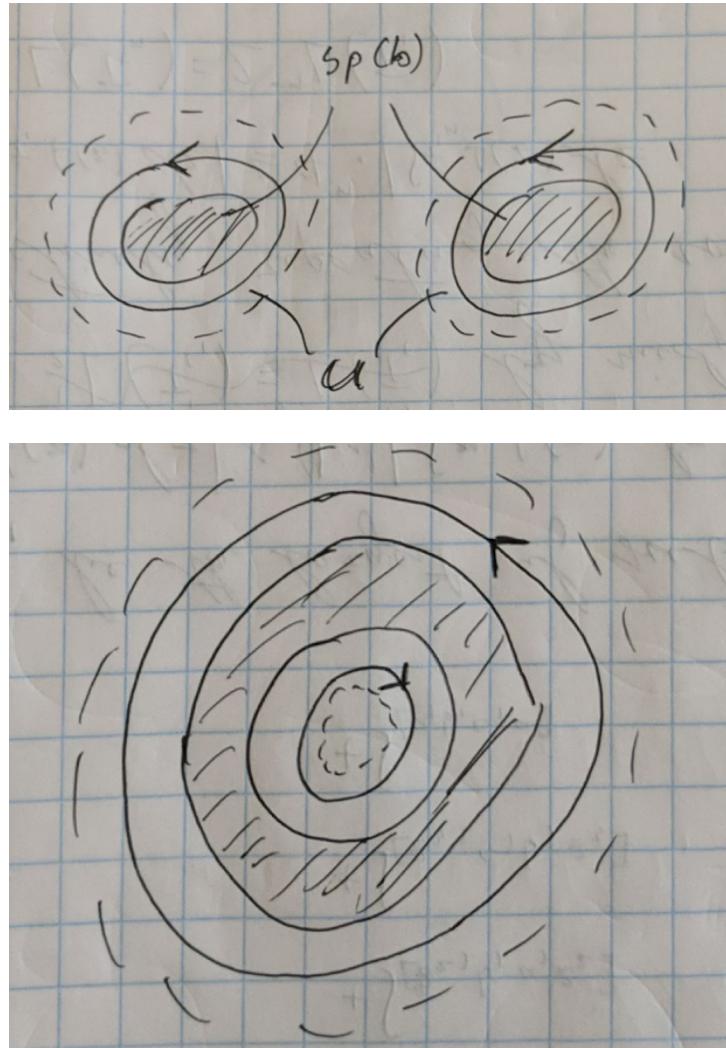
Let  $f \in A[\text{sp}(b)]$ , i.e.  $f : U \rightarrow \mathbb{C}$  holomorphic. Then there exists an open set  $W$  with (piece-)smooth boundary such that

$$\text{sp}(f) \subseteq W \subseteq \overline{W} \subseteq U$$

(i.e.  $\partial \overline{W} \subseteq U \setminus \text{sp}(b)$ ) and

$$\frac{1}{2\pi} \int_{\partial W} \frac{d\omega}{\omega - z} = \begin{cases} 1 & z \in \text{sp}(b) \\ 0 & z \notin U \end{cases}.$$

### Example



- Proof

IMAGE 7

SQUARES

**Definition:**

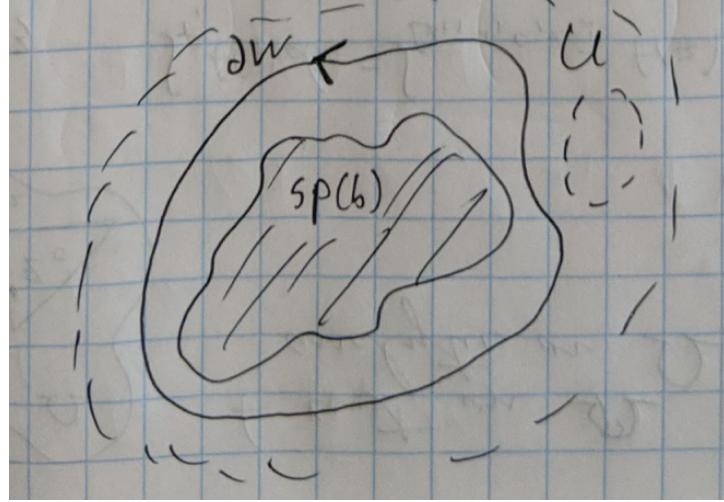
Using the lemma, we define for  $f \in A[\text{sp}(b)]$

$$f(b) := \frac{1}{2\pi i} \int_{\partial W} f(\lambda)(\lambda e - b)^{-1} d\lambda$$

(where  $\text{sp}(b) \subseteq W \subseteq \overline{W} \subseteq U$ ).

One can show that this is independent of choice of  $W$  (and also of  $U$ ).

Note  $f(\lambda)(\lambda e - b)^{-1}$  is holomorphic on  $U \setminus \text{sp}(b)$ .



### Remark

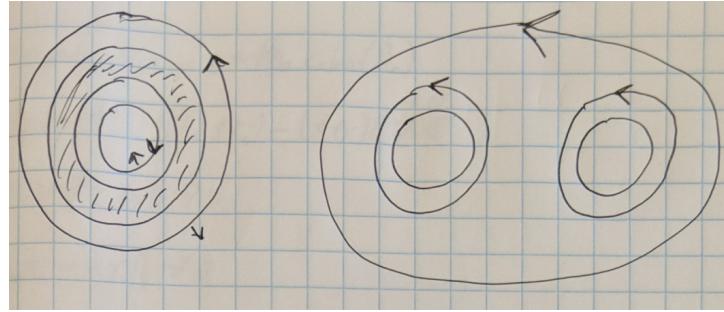
$f_1, f_2 \in A[\text{sp}(b)]$  implies  $f_1 + f_2 \in A[\text{sp}(b)]$  and  $(f_1 + f_2)(b) = f_1(b) + f_2(b)$ .

### Proposition

For a polynomial  $f(z) = p(z) = \sum p_i z^i$ , we get  $f(b) = p(b) = \sum p_i b^i$ .

### Proof

$$\frac{1}{2\pi i} \int_{\partial W} (\lambda e - b)^{-1} d\lambda = \frac{1}{2\pi i} \int_{|\lambda|=R} (\lambda e - b)^{-1} d\lambda = \frac{1}{2\pi i} \int_{|\lambda|=R} \sum_{n=0}^{\infty} \frac{b^n}{\lambda^{n+1}} d\lambda = e$$



Therefore,  $p(b) = \frac{1}{2\pi i} \int_{\partial W} p(b)(\lambda e - b)^{-1} d\lambda$ , and

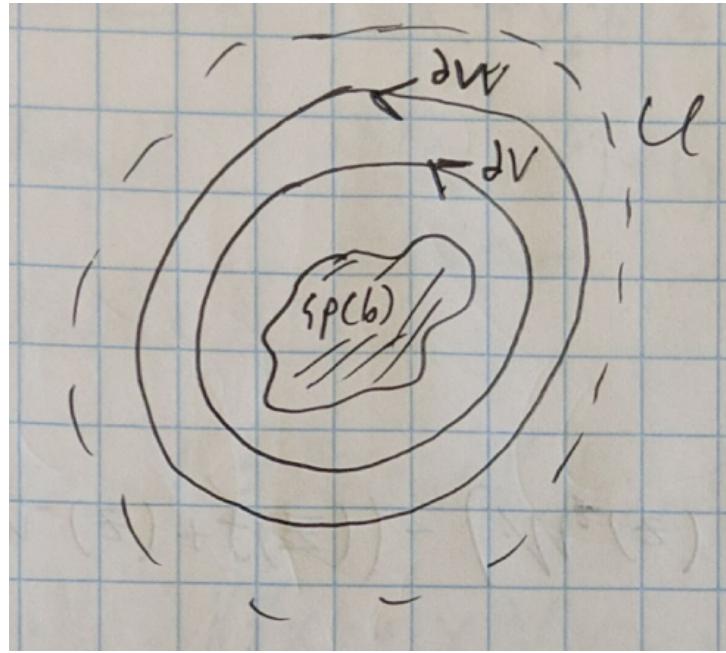
$$\begin{aligned} f(b) - p(b) &= \frac{1}{2\pi i} \int_{\partial W} (f(\lambda)e - p(b))(\lambda e - b)^{-1} d\lambda \\ &= \frac{1}{2\pi i} \int_{\partial W} \sum_{n=0}^N \underbrace{(\lambda^n e - b^n)}_{(\lambda^{n-1}e + \dots + b^{n-1})(\lambda e - b)} (\lambda e - b)^{-1} d\lambda \\ &= \frac{1}{2\pi i} \int_{\partial W} \text{"polynomial in } \lambda, b \text{" } d\lambda = 0 \end{aligned}$$

### Proposition

If  $f_1, f_2 \in A[\text{sp}(b)]$ , then  $f_1 f_2 \in A[\text{sp}(b)]$ .

$$(f_1 f_2)(b) = f_1(b) \cdot f_2(b)$$

## Proof



We assume  $\partial V$  is inside  $\partial W$ .

$$f_1(b) = \frac{1}{2\pi i} \int_{\partial W} f_1(\lambda)(\lambda e - b)^{-1} d\lambda$$

$$f_2(b) = \frac{1}{2\pi i} \int_{\partial V} f_2(\xi)(\xi e - b)^{-1} d\xi$$

Then

$$f_1(b)f_2(b) = \frac{1}{(2\pi i)^2} \int_{\partial W} \int_{\partial V} f_1(\lambda)f_2(\xi)(\lambda e - b)^{-1}(\xi e - b)^{-1} d\xi d\lambda$$

Recall that

$$(\lambda e - b)^{-1}(\xi e - b)^{-1} = (\lambda e - b)^{-1} \left[ \frac{(\lambda e - b) - (\xi e - b)}{\lambda - \xi} \right] (\xi e - b)^{-1} = \frac{(\xi e - b)^{-1} - (\lambda e - b)^{-1}}{\lambda - \xi}$$

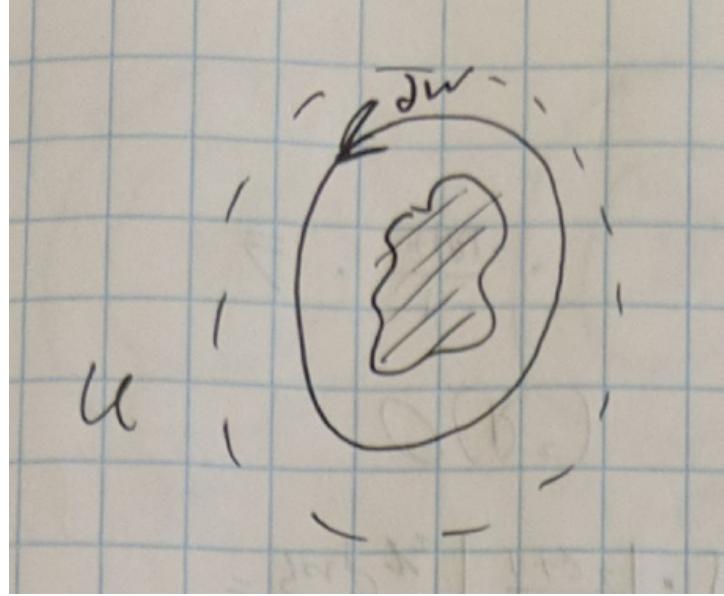
Therefore

$$\begin{aligned} f_1(b)f_2(b) &= \frac{1}{(2\pi i)^2} \int_{\partial V} \int_{\partial W} f_1(\lambda)f_2(\xi)(\xi e - b)^{-1} \frac{1}{\lambda - \xi} d\lambda d\xi - \frac{1}{(2\pi i)^2} \int_{\partial W} \int_{\partial V} f_1(\lambda)f_2(\xi)(\lambda e - b)^{-1} \frac{1}{\lambda - \xi} d\xi d\lambda \\ &= \frac{1}{(2\pi i)^2} \int_{\partial V} f_2(\xi)(\xi e - b)^{-1} \underbrace{\int_{\partial W} \frac{f_1(\lambda)}{\lambda - \xi} d\lambda}_{\equiv f_2(\xi)} d\xi - \frac{1}{(2\pi i)^2} \int_{\partial W} f_1(\lambda)(\lambda e - b)^{-1} \underbrace{\int_{\partial V} \frac{f_2(\xi)}{\lambda - \xi} d\xi}_{\equiv 0} d\lambda \\ &= \frac{1}{2\pi i} \int_{\partial V} f_2(\xi)f_1(\xi)(\xi e - b)^{-1} d\xi \\ &= (f_1f_2)(b) \end{aligned}$$

**Recall**

$f \in A[\text{sp } b]$ ,  $f : U \rightarrow \mathbb{C}$ ,  $\text{sp}(b) \subseteq U$  open. Define

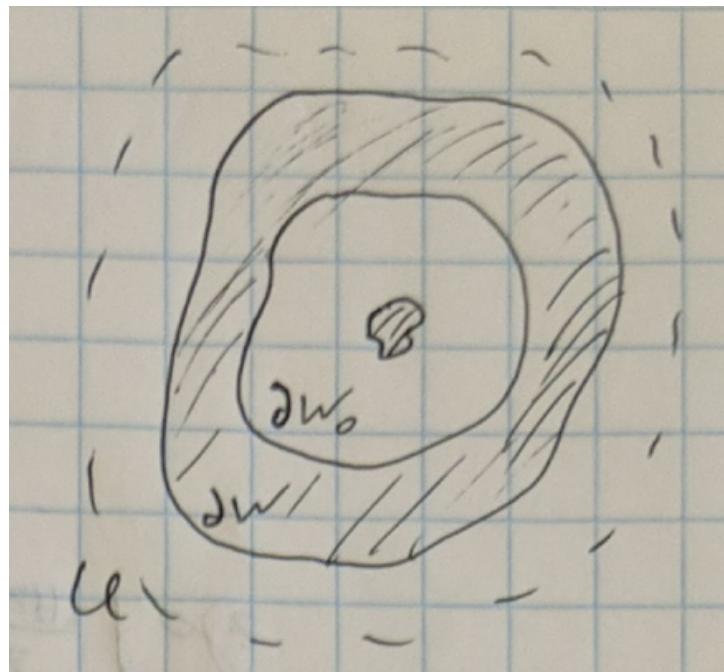
$$(1) \quad f(b) = \frac{1}{2\pi i} \int_{\partial W} \underbrace{f(\lambda)(\lambda e - b)^{-1}}_{\text{analytic in } U \setminus \text{sp } b} dz$$



with  $\text{sp } b \subseteq W \subseteq \overline{W} \subseteq U$  and  $\partial W$  piecewise smooth.

From the above lemma, applied to  $W$ , we get  $W_0$  such that  $\text{sp } b \subseteq W_0 \subseteq \overline{W}_0 \subseteq W \subseteq \overline{W} \subseteq U$ . Then

$$(2) \quad \frac{1}{2\pi i} \int_{\partial W_0} f(\lambda)(\lambda e - b)^{-1} dz$$



with  $V = W \setminus W_0$ ,  $\partial V = \partial W \cup \partial W_0$ .

$$(1) - (2) = \frac{1}{2\pi i} \int_{\partial V} \underbrace{\frac{f(\lambda)(\lambda e - b))^{-1}}{\text{holomorphic on } V}} dz = 0$$

and  $V \subseteq \overline{V} \subseteq U \setminus \text{sp}(b)$ .

## Results

$$f_1, f_2 \in A[\text{sp } b] \implies f_1 + f_2 \in A[\text{sp } b]$$

$$f_1(b) + f_2(b) = (f_1 + f_2)(b).$$

For  $f$  polynomial,  $\sum_{n=0}^N f_n t^n$ ,  $f(b) = \sum_{n=0}^N f_n b^n$ .

## Proposition

$$f_1(b)f_2(b) = (f_1 f_2)(b).$$

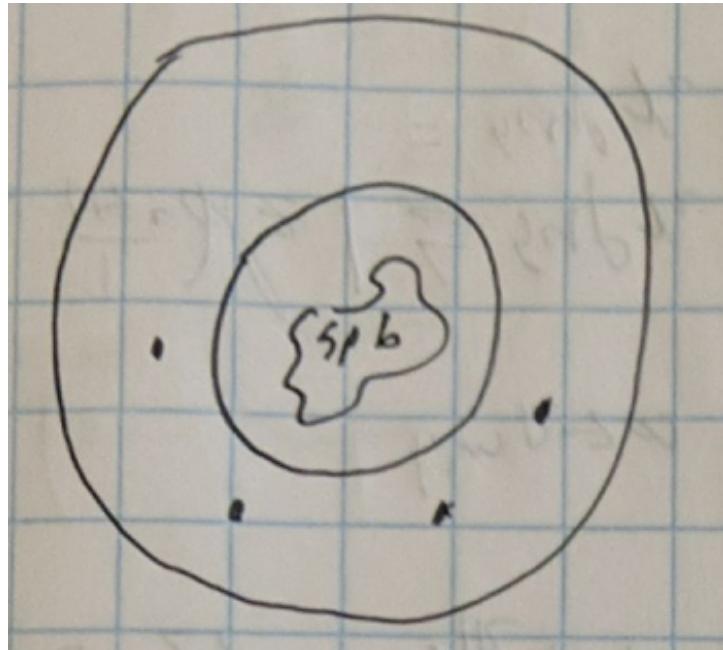
## Theorem: Spectral Mapping Theorem

Let  $b \in B$  and  $f \in A[\text{sp } b]$ . Then  $\text{sp}(f(b)) = f(\text{sp } b) := \{f(z) : z \in \text{sp } b\}$ .

## Proof

1. take  $\mu \notin f(\text{sp } b)$ .

Then  $\mu \notin f(z)$ ,  $\forall z \in \text{sp } b$  and  $\mu - f(z) \neq 0$ .



Therefore, there exist an open  $U_1 \ni \text{sp}(b)$ ,  $U_1 \subseteq U$ , such that  $\mu - f(z) \neq 0$ ,  $\forall z \in U_1$ .

Define  $g(z) = \frac{1}{\mu - f(z)}$  holomorphic on  $U_1$ , and

$$g(z) \cdot (\mu - f(z)) = 1 \implies g(b) \cdot (\mu e - f(b)) = e$$

by the previous proposition and the polynomial result. So  $\mu e - f(b)$  is invertible, and  $\mu \notin \text{sp}(f(b))$ .

- Remark

$$(\mu e - f(b))^{-1} = \frac{1}{2\pi i} \int_{\partial W_1} \frac{1}{\mu - f(z)} (ze - b)^{-1} dz$$

for  $\text{sp } b \subseteq W_1 \subseteq \overline{W}_1 \subseteq U_1$ .

- take  $\mu \notin \text{sp}(f(b))$  and, for contradiction, assume  $\mu \in f(\text{sp } b)$ .

Then  $\mu e - f(b)$  is invertible,  $\mu = f(\lambda)$  for some  $\lambda \in \text{sp } b$ .

- Idea

$$\mu e - f(b) = f(\lambda)e - f(b) = (\lambda e - b) \cdot g_\lambda(b)$$

We define

$$g_\lambda(z) = \begin{cases} \frac{f(\lambda)e - f(z)}{\lambda - z} & z \in U \supseteq \text{sp}(b) \\ f'(\lambda) & z = \lambda \end{cases}$$

such that  $g_\lambda(z)$  is holomorphic on  $U$ . Therefore  $g_\lambda(b) \in B$ ,

$$(\lambda - z)g_\lambda(z) = f(\lambda) - f(z), \quad \forall z \in U$$

and  $(\lambda e - b)g_\lambda(b) = f(\lambda)e - f(b) = g_\lambda(b)(\lambda e - b)$ . Since this is invertible,  $(\lambda e - b)$  is left and right invertible.

### Remark

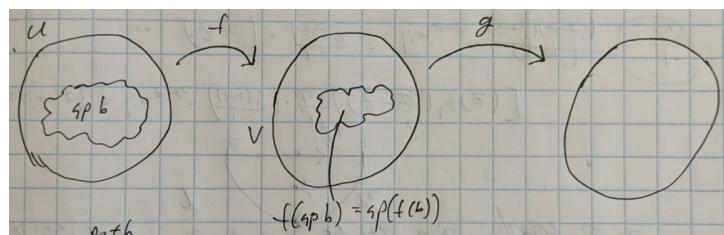
$$g_\lambda(b) = \frac{1}{2\pi i} \int_{\partial W} \frac{f(\lambda) - f(z)}{\lambda - z} (ze - b)^{-1} dz$$

### Theorem: Composition of Functions

Let  $b \in B$  unital,  $f \in A[\text{sp } b]$ , and  $g \in A[\text{sp}(f(b))] = A[f(\text{sp } b)]$ .

Then  $h = g \circ f \in A[\text{sp } b]$  and  $h(b) = g(f(b))$ .

### Remark



$f$  is an open mapping and maps  $U$  to the open set  $V \supseteq \text{sp}(f(b))$ .

## Applications

- Exponentials

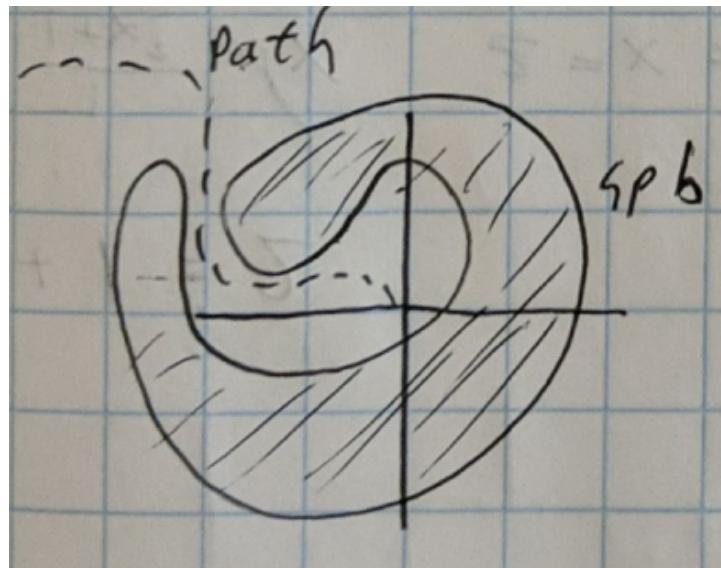
$$\exp(b) = \sum_{n=0}^{\infty} \frac{b^n}{n!} = \frac{1}{2\pi i} \int_{|z|=R} e^z (ze - b)^{-1} dz$$

- Logarithms

$\log b$ ,  $b \in B$  under the assumption that

- $0 \notin \text{sp } b$
- There exists a path connecting 0 to  $\infty$  in  $\mathbb{C} \setminus \text{sp } b$ .

This gives us that  $\log z$  is analytic on  $U \supseteq \text{sp } b$ .



$\mathbb{C} \setminus \text{path}$  is simply connected, so there exists an analytic  $\log z$  on  $\mathbb{C} \setminus \text{path}$ .

- if  $\log b$  is well-defined, then  $\exp(\log b)) = b$  (via composition)
- likewise, one can define powers  $f(z) = z^\alpha$  ( $\alpha \in \mathbb{C}$ )

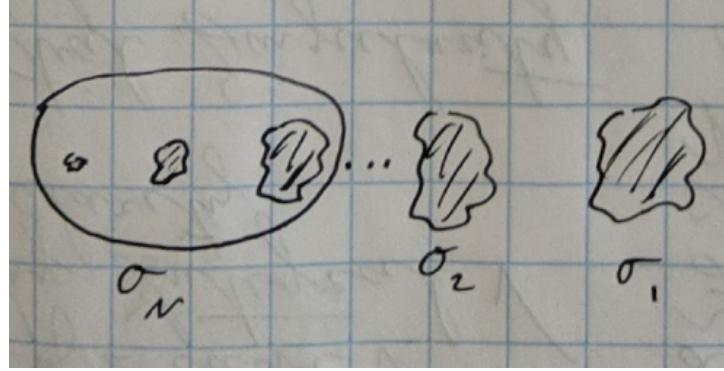
## Application: Spectral Idempotents (Riesz Idempotents)

$p$  is idempotent if  $p^2 = p$ .

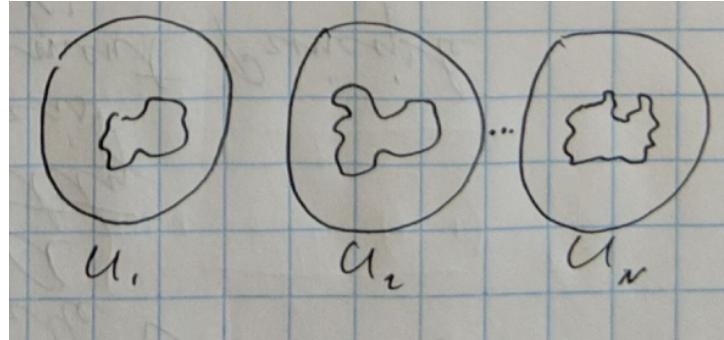
Assume that  $b \in B$  and that  $\text{sp } b$  is not connected.

$$\text{sp } b = \sigma_1 \cup \sigma_2 \cup \dots \cup \sigma_N$$

with  $\sigma_i$  closed and disjoint subsets of  $\text{sp } b$ .



Now let  $U_1, \dots, U_n$  be open neighborhoods of  $\sigma_1, \dots, \sigma_n$  which are themselves disjoint.



Write  $U = U_1 \cup \dots \cup U_n \supseteq \text{sp } b$ , and consider

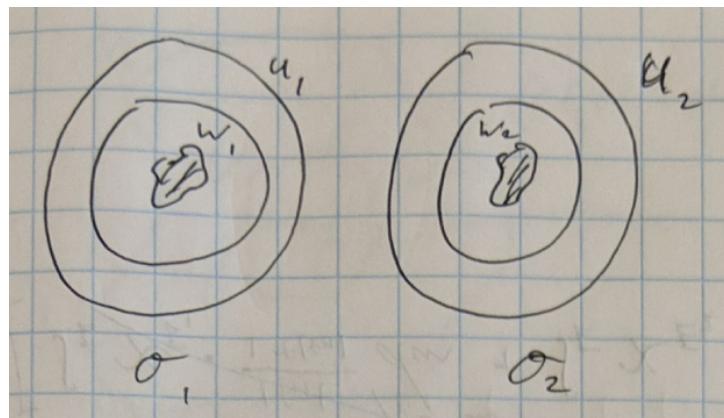
$$\chi_i(x) = \begin{cases} 1 & x \in U_i \\ 0 & x \in U_j, j \neq i \end{cases}$$

Then  $\chi_i$  is analytic on  $U \supseteq \text{sp}(b)$ .

Put  $p_i = \chi_i(b)$  the spectral or Riesz idempotents.

### Properties / Remarks

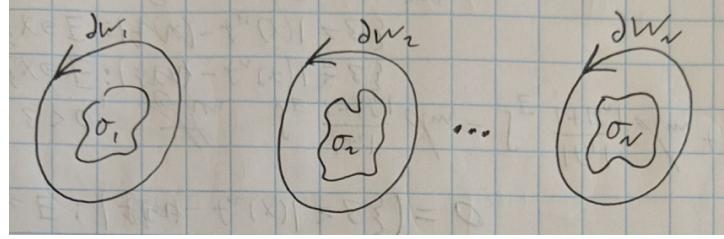
- $p_i^2 = p_i$  because  $(\chi_i)^2 = \chi_i$ .
- $e = p_1 + \dots + p_N$ , mutually orthogonal such that  $p_i p_j = 0, \forall i \neq j$ , because  $\chi_1 + \dots + \chi_N = 1$  and  $\chi_i \chi_j = 0$ .
- $p_i b = b p_i$ , because  $\chi_i f = f \chi_i$  for  $f(z) = z$ .
- $p_i = \frac{1}{2\pi i} \int_{\partial W} \chi_i(z) (ze - b)^{-1} dz$  where  $\text{sp } b \subseteq W \subseteq \overline{W} \subseteq U$ .



$W_i = W \cap U_i$ ,  $W_1 \cup \dots \cup W_N = W$ . Therefore

$$p_i = \frac{1}{2\pi i} \sum_{j \neq i}^N \int_{\partial W_j} \chi_i(z) (ze - b)^{-1} dz = 0, \quad i \neq j$$

Then  $p_i = \frac{1}{2\pi i} \int_{\partial W_i} (ze - b)^{-1} dz$ .



- Write

$$b = (p_1 + p_2 + \dots + p_N)b(p_1 + p_2 + \dots + p_N) = p_1bp_1 + p_2bp_2 + \dots + p_Nbp_N$$

since  $p_i bp_j = bp_i p_j = 0$ .

- For an idempotent  $p \neq 0$ ,

$$B_p = \{pap : a \in B\}$$

and, therefore,  $B_p$  has a unit element  $p$ .

### Lemma

Assume  $b \in B$  with Riesz idempotents  $p_1, \dots, p_N \neq 0$ .

Then  $b$  is invertible if and only if  $p_i bp_i$  is invertible in  $B_{p_i}$  for all  $i$ .

### Proof

$b^{-1} = c$ ,  $bc = e$ , then

$$\begin{aligned} (p_1 + \dots + p_N)b(p_1 + \dots + p_N)c &= e \\ \sum p_i b(p_i p_i)c &= e \\ (p_i b_i)(p_i c p_i) &= p_i \end{aligned}$$

Suppose  $p_i bp_i$  invertible in  $B_{p_i}$ . Then  $p_i bp_i c = p_i$ ,  $c_i = p_i cp_i$  and  $b^{-1} = c = \sum_{i=1}^N p_i c_i p_i$ .

### Remark

$$B = \begin{pmatrix} B_1 & & 0 \\ & B_2 & \\ 0 & & \ddots & & B_N \end{pmatrix} \quad P_1 = \begin{pmatrix} I & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \quad P_2 = \begin{pmatrix} 0 & & & \\ & I & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

Therefore  $B$  invertible if and only if  $B_i$  are invertible.  $B_i \cong P_i B P_i$ .

$$\begin{pmatrix} 0 & & & \\ & B_i & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

**October 23, 2024**

### Lemma

Let  $b \in B$  and  $p_1, \dots, p_n \in B$  satisfying  $p_i^2 = p_i$ ,  $p_i p_j = 0$  ( $i \neq j$ ),  $p_1 + \dots + p_n = e$ ,  $b p_i = p_i b$ . Then  $b$  is invertible in  $B$  if and only if for each  $i$ ,  $p_i b p_i$  are invertible in  $B_{p_i}$  and

$$\text{sp}_B(b) = \bigcup_{i=1}^N \text{sp}_{B_{p_i}}(p_i b p_i)$$

where  $B_{p_i} = \{p_i a p_i : a \in B\}$  is a unital Banach algebra with unit  $p_i$ .

### Theorem

Let  $p_1, \dots, p_N$  be spectral idempotents of  $b$  with respect to  $\text{sp}(b) = \sigma_1 \cup \sigma_2 \cup \dots \cup \sigma_N$  (closed and disjoint). If  $\sigma_1, \dots, \sigma_N \neq \emptyset$ , then  $p_1, \dots, p_N \neq 0$  and  $\text{sp}_{B_{p_i}}(p_i b p_i) = \sigma_i$ .

Note: if  $\sigma_i = \emptyset$  then

$$p_i = \chi_{U_i}(b) = \frac{1}{2\pi i} \int_{\partial W_i} \underbrace{|ze - b|}_{\text{analytic}}^{-1} dz.$$

### Proof

Without loss of generality, we may assume  $p_1, \dots, p_M \neq 0$  ( $M \geq 1$ ) and  $p_{M+1} = \dots = p_N = 0$ . Then by the above lemma  $p_1 + \dots + p_M = e$  and

$$\text{sp}_B(b) = \bigcup_{i=1}^M \text{sp}_{B_{p_i}}(p_i b p_i)$$

Assuming  $\text{sp}_{B_{p_i}}(p_i b p_i) = \sigma_i$  is proven for  $j = 1, \dots, M$ , then

$$\text{sp}_B(b) = \bigcup_{i=1}^M \sigma_i = \bigcup_{i=1}^N \sigma_i$$

and therefore that  $M = N$ .

To prove that  $\text{sp}_{B_{p_i}}(p_i b p_i) = \sigma_i$  for each  $p_i \neq 0$ ,

$$\text{sp}_{B_{p_i}}(p_i b p_i) = \{\lambda \in \mathbb{C} : p_i(b - \lambda)p_i + e - p_i \text{ not invertible in } B\}$$

For fixed  $\lambda$ ,  $f_\lambda(z) = \chi_i(z)(z - \lambda)\chi_i(z) + (1 - \chi_i(z))$  is analytic in a neighborhood of  $\text{sp}(b)$ .

$$f_\lambda(b) = p_i(b - \lambda e)p_i + (1 - p_i)$$

Then  $\lambda \in \text{sp}_{B_{p_i}}(p_i b p_i)$  if and only if  $f_\lambda(b)$  is not invertible in  $B$ .

Equivalently that  $0 \in \text{sp}(f_\lambda(b))$  or, by spectral mapping theorem,  $0 \in f_\lambda(\text{sp } b)$ .

This is further equivalent to there existing some  $\xi \in \text{sp } b : 0 = f_\lambda(\xi)$

$$f_\lambda(z) = \begin{cases} 1 & z \in \sigma_j \subseteq U_j, i \neq j \\ z - \lambda & z \in \sigma_i \end{cases}$$

That is, if  $\xi \in \text{sp } b : \xi \in \sigma_i$  and  $\xi = \lambda$  or, simply,  $\lambda \in \sigma_i$ .

## Chapter 2: Commutative Banach Algebras

### Section 2.1: Homomorphisms, Ideals and Quotient Algebras.

$A$  need not be commutative.

#### Definition: Banach Algebra Homomorphisms

$\phi : A \rightarrow B$  is a Banach algebra homomorphism if it is linear, multiplicative and bounded.

#### Definition: Banach Algebra Ideal

A (two-sided) ideal  $J$  of a Banach algebra is a linear subspace  $J \subseteq A$  such that  $\forall a \in A, \forall j \in J, aj, ja \in J$ .

#### Remark

If  $\phi : A \rightarrow B$  is a Banach algebra homomorphism then  $\ker \phi$  is a closed two-sided ideal of  $A$ .

#### Proof

Put  $J \in \ker \phi$ ,  $a \in A, j \in J$ . Then  $\phi(j) = 0, \phi(aj) = \phi(a)\phi(j) = 0 = \phi(j)\phi(a) = \phi(ja)$  and  $aj, ja \in J$ .

#### Definition: Quotient Algebra

If  $J$  is a closed, two-sided ideal of  $A$  ( $J \neq A$ ), then  $A/J$  is a Banach algebra.  $A/J$  is a Banach algebra  $[a] = a + J$ .

$A/J$  is a vector space, a normed space ( $J$  closed) with  $\|[a]_J\| = \inf_{j \in J} \|a + j\|$ , and a Banach space because  $A$  is complete.

$$[a_1] + [a_2] = [a_1 + a_2] \text{ and } [a_1] \cdot [a_2] = [a_1 \cdot a_2]$$

$$(a_1 + j_1)(a_2 + j_2) = a_1 a_2 + \underbrace{a_1 j_2 + a_2 j_1 + j_1 j_2}_{\in J}$$

#### Definition: Quotient Map

Take  $\pi : A \rightarrow A/J$  by  $a \mapsto [a]$ .

This is a Banach algebra homomorphism which is surjective with  $\ker \pi = J$ .

#### Proposition

Let  $\phi : A \rightarrow B$  be a Banach algebra homomorphism and  $J \subseteq \ker \phi$  a closed, two-sided ideal of  $A$ .

Then there exists a Banach algebra homomorphism  $\phi^J : A/J \rightarrow B$  such that  $\phi = \phi^J \circ \pi$

$$\begin{array}{ccc}
 A & \xrightarrow{\phi} & B \\
 & \searrow \pi & \swarrow \phi^J \\
 & A/J &
 \end{array}$$

Write  $\phi^J([a]_J) = \phi(a)$ , and  $[a_1] = [a_2]$  implies  $a_1 - a_2 \in J \subseteq \ker \phi$  and subsequently that  $\phi(a_1) = \phi(a_2)$ .

### Remark

$J = \{0\}$  and  $J = A$  are always closed, two-sided ideals of  $A$ .

### Examples

- $A = \mathbb{C}^{n \times n}$ . Only ideals are  $\{0\}$  and  $A$ .
- $A = L(X)$  (continuous operators) for  $X$  a Banach space. Then at least  $\{0\}$ ,  $K(X)$  (compact operators), and  $A$  are ideals.
- $X$  a separable hilbert space. Only  $\{0\}$ ,  $K(X)$  and  $A$ .
- $A = \mathbb{C}_{\text{upper}}^{n \times n}$  upper triangular matrices. Then there are many (one sided) ideals for  $n = 2$ .
- $A = C(X)$  for  $X$  compact Hausdorff spaces. Then every closed set  $E \subseteq X$  generates a closed ideal

$$J_E = \{f \in C(x) : f|_E \equiv 0\}$$

In particular,  $E = \{x_0\}$ ,  $J_{x_0} = \{f \in C(X) : f(x_0) = 0\}$ ,  $\dim(A/J_{x_0}) = 1$  implies  $A/J_{x_0} \cong \mathbb{C}$ .

### Remark

Every closed (2-sided) ideal is a closed subalgebra of  $A$  but not vice versa.

For a set  $S \subseteq A$ , let  $J = \text{clos id}_A(S)$  be the smallest closed 2-sided ideal containing  $S$  (i.e. the ideal generated by  $S$  or the intersection of all ideals containing  $S$ ). One can show that

$$J = \text{clos}_A \left\{ \sum_{i=1}^N a_i j_i b_i : a_i, b_i \in A, j_i \in S \right\}$$

## Section 2.2: Maximal Ideals and Multiplicative Linear Functionals

From now on,  $B$  is a unital, commutative Banach algebra.

### Definition: Multiplicative Linear Functional

A multiplicative linear functional on  $B$  is a linear map  $\phi : B \rightarrow \mathbb{C}$  such that  $\phi(ab) = \phi(a)\phi(b)$  ( $\phi \neq 0$ ).

### Proposition

A multiplicative linear functional on  $B$  is bounded. In fact  $\phi \in B^*$ ,  $\|\phi\| = 1$ ,  $\phi(e) = 1$ .

### Proof

$\phi \neq 0$  means that there exists  $a \in B$  such that  $\phi(a) \neq 0$ .

Then  $\phi(e)\phi(a) = \phi(ea) = \phi(a)$  so  $\phi(e) = 1$  and consequently that  $\|\phi\| \geq 1$ .

If  $|\phi(a)| \leq ||a||$ , then  $||\phi|| \leq 1$ . If this were not the case,

$$|\phi(a)| > ||a|| \iff \left\| \frac{a}{\phi(a)} \right\| < 1$$

and  $e - \frac{a}{\phi(a)}$  is invertible. Call the inverse  $b$ . Then

$$\begin{aligned} b \left( e - \frac{a}{\phi(a)} \right) = e &\implies \underbrace{\phi(b) \phi \left( e - \frac{a}{\phi(a)} \right)}_{= \phi(e) - \frac{1}{\phi(a)} \phi(a) = 0} = \phi(e) = 1 \end{aligned}$$

which is a clear contradiction.

### Definition: Maximal Ideal

A (two-sided) ideal  $I$  of  $B$  is called maximal if

- $I \neq B$  ( $I$  is a proper ideal)
- if  $J$  is another ideal of  $B$  such that  $I \subseteq J \subseteq B$ , then either  $I = J$  or  $J = B$ .

### Proposition

A maximal ideal  $I$  is closed and  $B/I$  is a field.

### Proof (Closed)

We have  $I \subseteq \bar{I} \subseteq B$  with  $\bar{I}$  an ideal. Since  $I$  is maximal, either  $I = \bar{I}$  or  $\bar{I} = B$ . But then  $e \in \bar{I}$ , and there exists  $a \in I$  such that  $||a - e|| < 1$ . Then  $a = e + (a - e)$  is invertible and for each  $b \in B$ ,  $b = ba^{-1}a \in I$  and  $I = B$  a contradiction.

**October 28, 2024**

### Recall: Multiplicative Linear Functionals

$$\phi : B \rightarrow \mathbb{C}$$

- linear
- $\phi(ab) = \phi(a)\phi(b)$ ,  $\phi \neq 0$ .

Then  $\phi(e) = 1$ ,  $||\phi|| = 1$

### Recall: Maximal Ideals

$I$  is a maximal ideal of  $B$  if

- $I \subsetneq B$
- If  $J$  is an ideal with  $I \subseteq J \subseteq B$ , then either  $I = J$  or  $J = B$ .

## Proposition

Every maximal ideal is closed and, in a commutative Banach algebra  $B$ ,  $B/I$  is a field.

### Proof (Field)

We know that  $B/I$  is a Banach algebra. We need to show that every nonzero  $[a] \in B/I$  is invertible.

Consider  $[a] \in B/I$  with  $[a] \neq 0$ . Then  $a \notin I$ . Define

$$J = \{i + ax : i \in I, x \in B\}$$

Then  $J$  is a linear subspace and an ideal in  $B$  since for any  $y \in B$

$$\begin{aligned} y(i + ax) &= yia(yx) \in J \\ (i + ax)y &= \underbrace{iy}_{\in I} + a(xy) \in J \end{aligned}$$

Since  $I \subseteq J \subseteq B$  and  $I$  is maximal, it cannot be that  $I = J$  since  $0 + a \cdot e \in J$  implies  $a \in I$  a contradiction.

If  $J = B$ , then  $e \in J$  and  $e = i + ax$  for some  $i \in I$  and  $x \in B$ . Therefore, in the quotient,

$$[e] = [a] \cdot [x] = [x] \cdot [a]$$

and  $[a]$  is invertible in  $B/I$ .

## Theorem: Gelfand/Mazur

Any (complex) Banach algebra which is a field is isomorphic to  $\mathbb{C}$ .

### Proof

Let  $A$  be a Banach algebra which is a field with unit  $e \in A$ , and consider the map

$$\Lambda : \mathbb{C} \ni \lambda \mapsto \lambda e \in A$$

This Banach algebra homomorphism is isometric ( $\|\lambda e\| = |\lambda|$ ) and injective.

If  $\Lambda$  is surjective, then it is a Banach algebra isomorphism.

Take  $a \in A$ . We know that  $\text{sp}(a) \neq \emptyset$ , therefore  $\exists \lambda \in \mathbb{C}$  such that  $\lambda e - a$  is not invertible.

It follows that  $\lambda e - a = 0$  and, consequently,  $a = \lambda e = \Lambda(\lambda)$ .

## Corollary

Let  $B$  be a unital, commutative Banach algebra and  $I$  be a maximal ideal. Then  $B/I \cong \mathbb{C}$ .

## Theorem: 1-1 Correspondence Between Maximal Ideals and Multiplicative Linear Functionals

Let  $B$  be a unital, commutative Banach algebra.

1. If  $\phi$  is a multiplicative linear functional on  $B$ , then  $\ker \phi$  is a maximal ideal in  $B$ .
2. If  $I$  is a maximal ideal in  $B$ , then there exists a unique multiplicative linear functional  $\phi$  such that  $\ker \phi = I$ .

## Proof of 1

If  $\phi$  is a multiplicative linear functional, it is bounded and  $\ker \phi$  is a closed, two-sided ideal.

We know  $I \subseteq B$  because  $\phi \not\equiv 0$  ( $\phi(e) = 1$ ).

We have that  $B = \ker \phi + \mathbb{C} \cdot e$  because

$$b = \underbrace{b - \phi(b) \cdot e}_{\in \ker \phi} + \phi(b) \cdot e$$

$$\phi(b - \phi(b) \cdot e) = \phi(b) - \phi(b) \cdot \phi(e) = 0$$

and therefore  $\dim B/I = 1$ .

If  $I \subseteq J \subseteq B$ , then  $J/I \subseteq B/I$  and either  $\dim(J/I) = 0$  ( $J = I$ ) or  $\dim(J/I) = 1$  ( $J = B$ ).

## Proof of 2

Let  $I$  be a maximal ideal in  $B$ .

By Gelfand/Mazur (corollary)  $B/I \cong \mathbb{C}$ , so there exists a Banach algebra isomorphism  $\psi : B/I \rightarrow \mathbb{C}$ .

$$\begin{array}{ccc} B & \xrightarrow{\phi} & \mathbb{C} \\ & \searrow \pi & \nearrow \psi \\ & B/I & \end{array}$$

where  $\pi$  and  $\psi$  are Banach algebra homomorphisms.

Put  $\phi = \psi \circ \pi : B \rightarrow \mathbb{C}$  by  $\phi(b) = \psi([b])$ . Then  $\phi$  is a Banach algebra homomorphism that is also a multiplicative linear functional with  $\phi(e) = 1$ . Since  $\psi$  is injective,

$$\ker \phi = \ker \pi = I$$

Suppose  $\ker \phi_1 = \ker \phi_2 = I$ . Then

$$b - \phi_1(b) \cdot e \in \ker \phi_1 = I$$

$$b - \phi_2(b) \cdot e \in \ker \phi_2 = I$$

and  $(\phi_1(b) - \phi_2(b))e \in I$ . But a proper ideal cannot contain invertible elements. Therefore  $\phi_1(b) = \phi_2(b)$ .

Since this holds for all  $b$ ,  $\phi_1 \equiv \phi_2$ .

## Definition: Maximal Ideal Space

For a unital (complex) commutative Banach algebra  $B$ ,  $M(B)$  denotes the maximal ideal space (i.e. the set of all multiplicative linear functionals on  $B$ ).

### Remark

$$M(B) \subseteq B^*$$

$$M(B) \subseteq \{\phi \in B^* : \|\phi\| \leq 1\}$$

The multiplicative linear functionals are a proper subset of bounded linear functionals.

### Example

$B = C(X)$  (the set of continuous functions  $f : X \rightarrow \mathbb{C}$ ) for  $X$  a compact Hausdorff space.

Every  $x_0 \in X$  determines a multiplicative linear functional  $\phi_{x_0} : B \rightarrow \mathbb{C}$  by  $\phi_{x_0}(f) = f(x_0)$  for  $f \in B$ .

The corresponding maximal ideals are of the form  $I_{x_0} = \ker \phi_{x_0} = \{f \in C(X) : f(x_0) = 0\}$ . Then

$$\begin{aligned} C(X) &= I_{x_0} + \mathbb{C} \cdot e, \quad e(x) \equiv 1 \\ C(X)/I_{x_0} &\cong \mathbb{C} \end{aligned}$$

Therefore  $\phi_{x_0} \in M(C(X))$ . In fact one can show that  $M(C(X)) = \{\phi_x : x \in X\}$ . So  $M(C(X)) \cong X$ .

### Remark

$$M(C(X)) \subseteq C(X)^*$$

$C(X)^*$  is isomorphic to the set of all complex Borel measures on  $X$  by

$$\phi(f) = \int_X f(x) d\mu(x)$$

with bounded linear functional  $\mu = \delta_{x_0} \rightsquigarrow \delta_{x_0} \in M(C(X))$ .

## Section 2.3: Gelfand Theory and Gelfand Transform

### Setting

Unital commutative (complex) Banach algebra  $B$ .

### Lemma

Every proper (two-sided) ideal  $I_0$  of  $B$  is contained in some maximal ideal of  $B$ .

### Proof

Zorn's lemma applied to the collection  $S$  of all proper ideals.

### Lemma

Every non-invertible element  $a \in B$  is contained in at least one maximal ideal.

### Proof

Consider  $a \in B$  and  $I_0 \in \{ax : x \in B\}$  an ideal ( $y(ax) = a(yx) \in I_0$  and  $(ax)y = a(xy) \in I_0$ ).

$I_0$  is proper, otherwise  $I_0 = B$ ,  $e \in I$  and  $ax = xa = e$  a contradiction.

Then, by the previous lemma,  $a = ae \in I_0 \subset I$  for some maximal ideal  $I$ .

## Theorem: Gelfand Theory

Let  $B$  be a unital commutative Banach algebra and  $b \in B$  arbitrary. Then  $b$  is invertible in  $B$  if and only if  $\phi(b) \neq 0$ ,  $\forall \phi \in M(B)$ .

### Proof

( $\implies$ ) if  $b$  is invertible,

$$1 = \phi(e) = \phi(b^{-1}b) = \phi(b^{-1}) \cdot \phi(b)$$

so  $\phi(b) \neq 0$  for all  $\phi$ .

( $\impliedby$ ) if  $b$  is not invertible, then there exists  $\phi \in M(B)$  such that  $\phi(b) = 0$ .

If  $b$  is not invertible, then  $b$  is contained in some maximal ideal  $I$  and  $I = \ker \phi$  for some  $\phi \in M(B)$ . Therefore  $b \in I = \ker \phi$  implies  $\phi(b) = 0$ .

### Definition/Notation: Gelfand Transform

The Gelfand transform of an element  $b \in B$  is the function  $\hat{b} : M(B) \rightarrow \mathbb{C}$  defined by  $\hat{b}(\phi) := \phi(b)$ .

#### Remark

$$\begin{aligned}\widehat{a+b} &= \hat{a} + \hat{b} \\ \widehat{ab} &= \hat{a} \cdot \hat{b}\end{aligned}$$

$$\sup_{\phi \in M(B)} |\hat{a}(\phi)| = \sup_{\phi \in M(B)} |\phi(a)| \leq \|a\|.$$

Later we will consider  $M(B)$  with topology (a compact Hausdorff space).

### Definition: Gelfand Transform of B

$$\Lambda : B \ni b \mapsto \hat{b} \in C(M(B))$$

It is a Banach algebra homomorphism.

Gelfand's theorem states that  $b$  is invertible in  $B$  if and only if  $\Lambda(b)$  is invertible in  $C(M(B))$ .

#### Note

$b$  is invertible if and only if  $\phi(b) = \hat{b}(\phi) \neq 0, \forall \phi \in M(B)$ .

Equivalently,  $\hat{b}$  is invertible in  $C(M(B))$ .

A continuous functional is invertible within the set of continuous functions if and only if it is non-zero everywhere.

### Purpose of Gelfand Theory

Invertibility in  $B$  corresponds to invertibility in  $C(M(B))$ .

We need to determine  $M(B)$ .

#### Remark

If  $B = C(X)$  ( $X$  compact Hausdorff), then  $M(B) = M(C(X))$  is homeomorphic to  $X$ .

$M(B) \cong X$  (homeomorphic) implies that  $C(M(B)) \cong C(X) = B$  isometric Banach algebra isomorphisms.

## October 30, 2024

#### Recall

$M(B)$  the multiplicative linear functionals or the maximal ideal space.

$b$  invertible if and only if  $\phi(b) \neq 0, \forall \phi \in M(B)$ .

$\hat{b} : M(B) \rightarrow \mathbb{C}$  where  $\hat{b}(\phi) = \phi(b)$  the Gelfand transform of  $b$ .

$\Lambda : B \rightarrow C(M(B))$  where  $b \mapsto \hat{b}$  is the Gelfand transform of  $B$ .

### Section 2.4: The Topology of the Maximal Ideal Space

Since  $M(B) \subseteq \{\phi \in B^* : \|\phi\| \leq 1\} \subseteq B^*$ ,  $M(B)$  is a topological space with the subspace topology with respect to the weak\*-topology of  $B^*$ .

A base for the topology in  $M(B)$  is given by

$$U_{\varepsilon; b_1, \dots, b_n}[\phi] = \{\psi \in M(B) : |\psi(b_i) - \phi(b_i)| < \varepsilon, i = 1, \dots, n\}$$

with  $\varepsilon > 0$ ,  $b_1, \dots, b_n \in B$  and  $\phi \in M(B)$ .

### Theorem

$M(B)$  is a compact Hausdorff space.

### Proof

$M(B)$  is Hausdorff because it is a subspace of the Hausdorff space  $B^*$ .

By Banach-Alaoglu, the unit ball is compact in the weak\*-topology. We need that  $M(B)$  is a closed subset of the unit ball.

Let  $\phi$  be in the closure of  $M(B)$  with respect to the unit ball such that  $\phi \in B^*$  and  $\|\phi\| \leq 1$ . To show that  $\phi(ab) = \phi(a)\phi(b)$ , consider

$$U_{\varepsilon;a,b,ab}[\phi] = \{\psi \in B^* : |\psi(b_i) - \phi(b_i)| < \varepsilon, i = 1, \dots, n\}$$

Then  $\psi \in M(B) \cap U_{\varepsilon;a,b,ab}[\phi]$ , so we have

$$|\psi(a) - \phi(a)| < \varepsilon, \quad |\psi(b) - \phi(b)| < \varepsilon, \quad \text{and} \quad |\psi(ab) - \phi(ab)| < \varepsilon.$$

We know that  $\psi(ab) = \psi(a)\psi(b)$ . Therefore

$$\begin{aligned} |\phi(ab) - \phi(a)\phi(b)| &= |\phi(ab) - \psi(ab) - \phi(a)\phi(b) + \psi(a)\psi(b)| \\ &\leq |\phi(ab) - \psi(ab)| + |\phi(a) - \psi(a)| \cdot |\phi(b)| + |\psi(a)| \cdot |\phi(b) - \psi(b)| \\ &\leq \varepsilon \|b\| + \varepsilon \|a\| \end{aligned}$$

Taking  $\varepsilon \rightarrow 0$ ,  $\phi(ab) = \phi(a)\phi(b)$ ,  $\forall a, b \in B$ . Similarly,  $\phi(e) = 1$ .

Thus  $\phi \in M(B)$  and  $M(B)$  is closed in the unit ball of  $B^*$ .

### Proposition

For  $b \in B$ , the Gelfand transform  $\hat{b} : M(B) \rightarrow \mathbb{C}$  is continuous and  $\|\hat{b}\| := \max_{\phi \in M(B)} |\hat{b}(\phi)| \leq \|b\|$ . In other words,  $\hat{b} \in C(M(B))$ .

### Proof

We need to show that  $\hat{b}$  is continuous at each  $\phi_0 \in M(B)$ .

Consider  $U = B_\varepsilon(\hat{b}(\phi_0))$  an  $\varepsilon$ -neighborhood in  $\mathbb{C}$ . Then the preimage is

$$\hat{b}^{-1}(U) = \{\phi \in M(B) : \hat{b}(\phi) \in B_\varepsilon(\hat{b}(\phi_0))\} = \{\phi \in M(B) : |\hat{b}(\phi) - \hat{b}(\phi_0)| < \varepsilon\} = \{\phi \in M(B) : |\phi(b) - \phi_0(b)| < \varepsilon\} = U_{\varepsilon;b}[\phi_0]$$

with  $[\phi_0]$  open in  $M(B)$ .

Also note that

$$|\hat{b}(\phi)| = |\phi(b)| \leq \|\phi\| \cdot \|b\| = \|b\|$$

### Corollary

The Gelfand transform of  $B$ ,  $\Lambda : B \rightarrow C(M(B))$  by  $b \mapsto \hat{b}$  is a Banach algebra homomorphism with  $\|\Lambda\| = 1$

## Proof

$\Lambda$  is linear and multiplicative with

$$\begin{aligned}\widehat{a+b} &= \hat{a} + \hat{b} \\ \widehat{ab} &= \hat{a}\hat{b}\end{aligned}$$

$||\Lambda|| = 1$  because  $||\hat{b}|| \leq ||b||$  and  $||\hat{e}|| = ||e|| = 1$ .

Then  $\hat{e}(\phi) = \phi(e) = 1$ .

It follows also that

$$(\widehat{ab})(\phi) = \phi(ab) = \phi(a)\phi(b) = \hat{a}(\phi) \cdot \hat{b}(\phi) = (\hat{a} \cdot \hat{b})(\phi)$$

## Corollary

For  $b \in B$ ,  $b \in GB$  if and only if  $\hat{b} = \Lambda(b) \in GC(M(B))$ .

As a consequence,  $\text{sp}_B(b) = \text{sb}_{C(M(B))} \Lambda(b)$  ( $\Lambda$  preserves spectrum).

## Proof

$b \in GB$  implies that  $ab = e$  and that  $\hat{a} \cdot \hat{b} = \hat{e} = 1$ . Therefore  $(\hat{b})^{-1} = \hat{a} = \widehat{b^{-1}}$ .

We have also that  $\hat{b} \in GC(M(B))$  implies  $\hat{b}(\phi) \neq 0$ ,  $\forall \phi \in M(B)$  and  $\phi(b) \neq 0$  similarly. Therefore  $b \in GB$ .

## Gelfand Theory

Two problems for a given Banach algebra  $B$ :

1. How to determine  $M(B)$ .
2. How to determine its topology.

## Theorem

Let  $B$  be a commutative Banach algebra with maximal ideal space  $M(B) = X$  and topology  $\tau$  on  $X$ .

Now assume we have another topology  $\rho$  on  $X$  such that

1.  $(X, \rho)$  is a compact topological space.
2.  $\forall b \in B$ ,  $\hat{b} : X \rightarrow \mathbb{C}$  is continuous in the  $(X, \rho)$  topology.

Then  $\tau = \rho$ .

## Proof

First show that  $\tau \subseteq \rho$ .

Take  $U \in \tau$  from the base of the topology

$$U = U_{\varepsilon; b_1, \dots, b_n}[\phi] = \{\psi \in X : |\psi(b_i) - \phi(b_i)| < \varepsilon, \forall i\}$$

Then

$$U = \bigcap_{i=1}^n \{\psi \in X : |\hat{b}_i(\psi) - \hat{b}_i(\phi)| < \varepsilon\} = \bigcap_{i=1}^n (\hat{b}_i)^{-1}(B_\varepsilon(\hat{b}_i(\phi)))$$

which is open in the  $(X, \rho)$  topology because  $\hat{b}_i$  is continuous. Therefore  $U \in \rho$ .

We have that  $\text{id} : (X, \rho) \rightarrow (X, \tau)$  where  $(X, \rho)$  is compact and  $(X, \tau)$  is Hausdorff is continuous ( $\tau \leq \rho$ ).

Then as an open map, we map closed sets to closed sets.

$A \subset X$  closed  $\implies A$  compact  $\implies \text{i}(A)$  compact  $\implies \text{id}(A)$  closed.

Therefore  $(\text{id})^{-1}$  is continuous and  $\rho = \tau$ .

## Theorem

Let  $X$  be a compact Hausdorff space and  $B = C(X)$ .

Then  $M(B)$  is homeomorphic to  $X$  by the map

$$\tau : X \ni x \mapsto \phi_x \in M(B)$$

where  $\phi_x(b) = b(x)$  is the point evaluation for  $b \in B = C(X)$ .

## Proof

We have that  $\phi_x$  is indeed in  $M(B)$  easily.

Then to show  $\tau$  is injective,  $\phi_x = \phi_y$  implies that  $\phi_x(b) = \phi_y(b)$ ,  $\forall b \in B$ . Then  $b(x) = b(y)$ ,  $\forall b \in C(X)$  and  $x = y$ .

Because  $x \neq y$  implies that there exists  $b \in C(X)$  such that  $b(x) \neq b(y)$ .

Since  $X$  is a normed space,  $\{x\}$  and  $\{y\}$  are closed. By Urysohn's Lemma, there exists a continuous  $b$  such that  $b|_{\{x\}} = 0$  and  $b|_{\{y\}} = 1$ .

## IMAGE 1

To see that  $\tau$  is surjective, otherwise there would exist  $\phi \in M(B)$  such that  $\phi \neq \phi_x$ ,  $\forall x \in X$ .

That implies that there exists  $b_x \in B$  such that  $\phi(b_x) \neq \phi_x(b_x) = b_x(x)$ .

Put  $a_x = b_x - \phi(b_x) \cdot e$ . Then  $\phi(a_x) = 0$ . So

$$\phi_x(a_x) = \phi_x(b_x) - \phi(b_x) \neq 0$$

## IMAGE 2

With  $a_x$  continuous, we find a neighborhood  $U_x \ni x$  where  $a_x(t) \neq 0$ ,  $\forall t \in U_x$ .

For  $\{U_x\}_{x \in X}$  an open cover of  $X$ , there exists a finite subcover  $\{U_{x_i}\}_{i=1,\dots,N}$ . Consider

$$a(t) = \sum_{i=1}^N |a_{x_i}(t)|^2$$

which is itself continuous and  $a(t) > 0$  for each  $t \in X$ .

So  $a \in B = C(X)$ ,  $a$  is invertible in  $B$  and

$$\phi(a) = \sum_{i=1}^N \phi(\overline{a_{x_i}}) \underbrace{\phi(a_{x_i})}_{=0}$$

which would imply that  $a$  is not invertible, a contradiction.

To show that  $\tau : X \ni x \mapsto \phi_x \in M(B)$  is continuous, take  $X \ni x_0 \mapsto \phi_{x_0}$  and consider

$$U = U_{\varepsilon; b_1, \dots, b_N}[\phi_{x_0}] = \{\phi_x : |\phi_x(b_i) - \phi_{x_0}(b_i)| < \varepsilon\} = \{\phi_x : |b_i(x) - b_i(x_0)| < \varepsilon\}$$

Then

$$\tau^{-1}(U) = \{x \in X : |b_i(x) - b_i(x_0)| < \varepsilon\}$$

which is the open preimage of  $b_i : X \rightarrow \mathbb{C}$  with  $b_i$  continuous.

Then  $\tau : X \rightarrow M(B)$  is a continuous map between compact, Hausdorff spaces and  $\tau^{-1}$  is continuous.

## Section 2.5: Commutative Banach Algebras Generated by Single Elements

Consider the Banach algebra  $A \ni B$  and

$$B = \text{alg}_A\{e, b\} = \text{clos}_A \left\{ \sum_{n=0}^N \lambda_n b^n \right\}$$

### Theorem

The maximal ideal space  $M(B)$  of  $B = \text{alg}_A\{e, b\}$  is homeomorphic to  $\text{sp}_B(b)$ .

**November 4, 2024**

### Recall

Let  $A$  be a unital Banach algebra,  $b \in A$  and  $B = \text{alg}_A\{e, b\} = \text{clos}_A\{\sum \lambda_i b^i : \lambda_i \in \mathbb{C}, N \in \mathbb{N}\}$  the smallest closed subalgebra containing  $e$  and  $b$ .

Then  $B$  is a commutative Banach algebra and the closure of all polynomials in  $b$ ,  $p(b)$ .

### Theorem

The maximal ideal space of  $B = \text{alg}\{e, b\}$  is homeomorphic to  $\text{sp}_B(b)$ . The map

$$\tau : M(B) \ni \phi \mapsto \phi(b) \in \text{sp}_B(b)$$

is a homeomorphism.

### Proof

- $\tau : M(B) \rightarrow \mathbb{C}$  maps into  $\text{sp}_B(b)$ .

Otherwise,  $z = \phi(b) \notin \text{sp}_B(b)$  (for some  $\phi$ ) and  $b - z \cdot e$  is invertible in  $B$ . So

$$\phi(b - z \cdot e) = \phi(b) - z \cdot \phi(e) = 0$$

and, therefore,

$$\phi((b - ze)^{-1}(b - ze)) = \phi(e) = 1$$

a contradiction.

- $\tau$  is injective.

Assume that  $\phi_1(b) = \phi_2(b)$ .  
Then  $\phi_1(b^n) = \phi_2(b^n)$  for  $n = 0, 1, \dots$  and, consequently,

$$\phi_1\left(\sum_{n=0}^N \lambda_n b^n\right) = \phi_2\left(\sum_{n=0}^N \lambda_n b^n\right)$$

Because  $B = \text{clos}\{\sum \lambda_n b^n\}$  and  $\phi_i$  is continuous,  $\phi_1(a) = \phi_2(a)$  for each  $a \in B$  and  $\phi_1 = \phi_2$ .

- $\tau$  is surjective.

Take  $z \in \text{sp}_B(b)$ . Then  $b - z \cdot e$  is not invertible in  $B$ .  
It follows from Gelfand theorem that there exists some  $\phi \in M(B)$  such that  $\phi(b - ze) = 0$  and  $\phi(b) = z$ .

- We know  $\tau : M(B) \rightarrow \text{sp}_B(b)$  is an injection with the natural topology (weak\*-topology of  $B^*$ ) on  $M(B)$  and  $\text{sp}_B(b)$  Hausdorff compact.

Define (another) topology on  $M(B)$  via  $\tau$ .  
To show that both topologies are the same, we need that for each  $b \in B$ ,  $\hat{b} : M(B) \rightarrow \mathbb{C}$  is a continuous function.

$$\text{sp}_B(b) \xrightarrow{\tau^{-1}} M(B) \xrightarrow{\hat{a}} \mathbb{C}$$

equivalently,  $\tilde{a} = \hat{a} \circ \tau^{-1} : \text{sp}_B(b) \rightarrow \mathbb{C}$  is continuous.

Let  $\phi = \tau^{-1}(z)$ ,  $\tau(\phi) = z$  and  $\hat{b}(\phi) = \phi(b) = z$ .

Then if  $a = e$ ,  $\hat{e}(\phi) = \phi(e) = 1$ ,  $\tilde{e}(z) = 1$

If  $a = b$ ,  $\hat{b}(\phi) = \phi(b) = z$  and  $\tilde{b}(z) = z$  continuous.

For  $a = b^n$ ,  $\hat{b}^n = (\hat{b})^n$  and  $\tilde{b}^n = (\tilde{b})^n$  so  $\tilde{b}^n(z) = z^n$ .

When  $a = p(b)$ ,  $p(z) = \sum_{i=1}^N \lambda_i z^i$  a polynomial,

$$\begin{aligned}\hat{a} &= \sum_{i=1}^N \widehat{\lambda_i b^i} = \sum_{i=1}^N \lambda_i \hat{b}_i \\ \tilde{a} &= \sum_{i=1}^N \lambda_i \tilde{b}^i \\ \tilde{a}(t) &= \sum_{i=1}^N \lambda_i t^i = p(z)\end{aligned}$$

For  $a \in B$ ,  $\|a - a_n\| \rightarrow 0$ ,  $a_n = p_n(b)$ , we have  $\hat{a} - \hat{a}_n = \widehat{a - a_n}$  and take

$$\max_{\phi \in M(B)} |\hat{a}(\phi) - \hat{a}_n(\phi)| \leq \|a - a_n\|_B = \max_{z \in \text{sp}_B(b)} |\tilde{a}(z) - \tilde{a}_n(z)|$$

Therefore  $\tilde{a}_n \Rightarrow \tilde{a}$  uniformly on  $\text{sp}_B(b)$  with  $\tilde{a}$  continuous.

Both topologies on  $M(B)$  coincide. Therefore  $\tau : \text{sp}_B(b) \rightarrow M(B)$  is a homeomorphism.

## Theorem

Let  $A$  be a unital Banach algebra, and  $b \in A$  an invertible element. Then for

$$B = \text{alg}_A\{e, b, b^{-1}\} = \text{clos}_A \left\{ \sum_{i=-N}^N \lambda_i b^i : \lambda_i \in \mathbb{C}, N \in \mathbb{N} \right\}$$

This is the closure of all trigonometric polynomials. The map

$$\tau : M(B) \ni \phi \mapsto \phi(b) \in \text{sp}_B(b)$$

is a homomorphism.

The proof follows similarly to that for the previous theorem.

## Example: Wiener Algebra

Let  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$  with  $C(\mathbb{T})$  continuous functions  $\mathbb{T} \rightarrow \mathbb{C}$ .

$W \subseteq C(\mathbb{T})$  is the set of all functions with an absolutely convergent Fourier series.

$$a(t) = \sum_{n=-\infty}^{\infty} a_n t^n$$

such that  $\|a\|_W := \sum_{n=-\infty}^{\infty} |a_n| < +\infty$ .

This is a Banach algebra with addition and multiplication defined pointwise and  $\|ab\| \leq \|a\| \cdot \|b\|$  (verify!).

Further, for  $a \in W$ ,  $\|a\|_{C(\mathbb{T})} \leq \|a\|_W$  and  $W \subseteq C(\mathbb{T})$  is a continuous embedding.

## Remark

$W$  is a Banach algebra.

$W$  is isometrically isomorphic to  $\ell^1(\mathbb{Z})$  (as a Banach algebra)

$$\begin{aligned} c(t) &= a(t) \cdot b(t) & \{c_n\} &= \{a_n\} * \{b_n\} \\ (\sum c_n t^n) &= (\sum a_n t^n)(\sum b_n t^n) & c_n &= \sum_{k \in \mathbb{Z}} a_{n-k} b_k \end{aligned}$$

Consider  $\chi_n(t) = t^n$  for  $n \in \mathbb{Z}$ .  $\chi_0 = e$ ,  $\chi_n = (\chi_1)^n$ ,  $(\chi_1)^{-1} = \chi_{-1}$ . Note that

$$W = \text{alg}_W\{X_0, X_1, X_{-1}\}$$

Write

$$W = \text{clos}_W \left\{ p(\chi_1) = \sum_{i=-N}^N \lambda_i (\chi_1)^i \right\}$$

and, to show that the trigonometric polynomials are dense in  $W$ , for  $a(t) = \sum_{n=-\infty}^{\infty} a_n t^n$ , consider  $a^{(N)}(t) = \sum_{n=-N}^N a_n t^n$  and

$$\|a(t) - a^{(N)}(t)\| = \sum_{|n|>N} |a_n|$$

which converges to 0 as  $N \rightarrow +\infty$  because  $\sum_{n \in \mathbb{N}} |a_n| < +\infty$ .

Claim:  $\text{sp}_W(\chi_1) = \mathbb{T}$ .

Indeed, for  $|z| > 1$ ,

$$(\chi_1 - z\chi_0)^{-1} = \frac{1}{t-z} = -\frac{1}{z} \frac{1}{1-t/z} = -\frac{1}{z} \sum_{n=0}^{\infty} \left(-\frac{t}{z}\right)^n \in W$$

For  $|z| < 1$

$$(\chi_1 - z\chi_0)^{-1} = \frac{1}{t-z} = \frac{1}{t} \frac{1}{1-z/t} = \frac{1}{t} \sum_{n=-\infty}^0 \left(-\frac{t}{z}\right)^n \in W$$

However, for  $|z| = 1$ ,  $\chi_1 - z\chi_0 = t - z$  vanishes at  $t = z \in \mathbb{T}$ . Thus  $\chi_1 - z\chi_0 \notin GW$ .

Therefore, the maximal ideal space of  $W$  is homeomorphic to  $\mathbb{T}$ .

If we identify  $M(W) \cong \mathbb{T}$  via  $\tau$  from above,

$$\phi(\chi_0) = t_0 \in \mathbb{T}$$

$$\mathbb{T} \xrightarrow{\tau^{-1}} M(W) \xrightarrow{\hat{a}} \mathbb{C}$$

$$\text{For } a = \sum a_n t^n = \sum a_n \chi_n,$$

$$\phi(a) = \sum a_n \phi(\chi_n) = \sum a_n \phi(\chi_1)^n = \sum a_n t_0^n$$

Then  $\hat{a}(\phi) = \sum a_n t_0^n$  and  $\hat{a}(t^{-1}(t_0)) = \sum a_n t_0^n$ .

Finally  $\tilde{a}(t_0) = \sum a_n t_0^n$  implies that  $\tilde{a} = a$ .

### Theorem (Wiener)

Let  $a \in W$ . Then  $a$  is invertible in  $W$  if and only if  $a(t) \neq 0$ ,  $\forall t \in \mathbb{T}$  (i.e. if  $a$  is invertible in  $C(\mathbb{T})$ ).

### Remark

$a \in W$  and  $a(t) \neq 0$ ,  $\forall t \in \mathbb{T}$  implies that  $\frac{1}{a} \in W$ .

That is, if  $a$  has an absolutely convergent Fourier series (under these conditions) then  $\frac{1}{a}$  has an absolutely convergent Fourier series.

### Example

Let  $A = C(\mathbb{T})$  and  $B = \text{alg}_A\{X_0, X_1\} = \text{clos}_{C(\mathbb{T})} \{p(t) = \sum_{n=0}^N \lambda_n t^n\}$ .

One can show that  $B = C_+(\mathbb{T}) = \iota(A(\mathbb{D}))$  where

$$C_+(\mathbb{T}) = \left\{ a \in C(\mathbb{T}) : \int_0^{2\pi} e^{inx} a(e^{ix}) dx = 0, \forall n < 0 \right\}$$

and

$$\begin{aligned} A(\mathbb{D}) &= \{a \in C(\overline{\mathbb{D}}) : a|_{\mathbb{D}} \text{ holomorphic}\} \\ \iota : A(\mathbb{D}) &\rightarrow a|_{\mathbb{T}} \in C(\mathbb{T}) \end{aligned}$$

### Claim

$\text{sp}_B(\chi_1) = \overline{\mathbb{D}}$ .  
For  $|z| < 1$ ,

$$\frac{1}{\chi_1 - z} = \frac{1}{t - z} \in C(\mathbb{T})$$

and (for  $|t| \leq 1$ ),  $\frac{1}{t - z} \notin A(\mathbb{D})$ .

So  $M(B) \cong \overline{\mathbb{D}}$ . For  $|t_0| = 1$ ,  $\phi_{t_0}(a) = a(t_0)$  ( $a \in B$ ).

For  $|t_0| < 1$ ,

$$\phi_{t_0}(a) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{a(t)}{t - t_0} dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{a(e^{ix})}{1 - e^{-ix} t_0} dx$$

or  $\phi_{t_0}(a) = a(t_0)$  (if  $a$  is holomorphically extended into  $\mathbb{D}$ ).

## November 6, 2024

### Theorem

Let  $A$  be a unital Banach algebra and  $b_1, \dots, b_n \in A$  be commuting elements.

Then the maximal ideal space of  $B = \text{alg}_A\{e, b_1, \dots, b_n\}$  is homeomorphic to some compact subset of

$$K \subseteq \text{sp}_B(b_1) \times \cdots \times \text{sp}_B(b_n) \subseteq \mathbb{C}^n$$

That is, the map

$$\tau : M(B) \ni \phi \mapsto (\phi(b_1), \dots, \phi(b_n)) \in K \subseteq \mathbb{C}^n$$

is a homeomorphism.

### Remark

$$K = \text{im } \tau = \tau(M(\phi)).$$

$$\phi(b_i) \subseteq \text{sp}_B(b_i).$$

$$\text{For } a \in B, \hat{A} : M(B) \rightarrow \mathbb{C}, \tilde{a} = \hat{a} \circ \tau^{-1} : K \rightarrow \mathbb{C},$$

$$\begin{aligned} \tilde{e}(x) &= 1 & x &= (x_1, \dots, x_n) \\ \tilde{b}_i(x) &= x_i \end{aligned}$$

$$\Lambda : b \mapsto \hat{b} \in C(M(B)), \tilde{b} \in C(K).$$

## Section 2.6: Shilov Idempotent and Arens-Royden Theorems

For  $B$  a unital, commutative Banach algebra,

$$\Lambda : B \rightarrow \underbrace{C(M(B))}_{M(B)}$$

Obviously, if  $p \in B$  is idempotent then  $(\hat{p})^2 = \hat{p}$ .

## Theorem: Shilov

Let  $\chi \in C(M(B))$  be an idempotent.

Then there exists a unique idempotent  $p \in B$  such that  $\hat{p} = \chi$ .

## Note

Idempotents  $\chi \in C(M(B))$  correspond (uniquely) to clopen (closed and open) subsets of  $K \subseteq M(B)$ .

$$\chi(\phi) = \begin{cases} 0 & \emptyset \notin K \\ 1 & \emptyset \in K \end{cases}$$

## Theorem: Arens-Royden

The abstract index group  $\kappa(B) = GB/G_0B$  is group-isomorphic to the abstract index group  $\kappa(C(M(B)))$  via the map

$$\iota : \kappa(B) \ni [b] \mapsto [\hat{b}] \in \kappa(C(M(B)))$$

## Remark

For  $\hat{B} = C(M(B))$  and  $\Lambda : B \ni b \mapsto \hat{b} \in \hat{B}$ ,

- $GB \ni b \mapsto \hat{b} \in G\hat{B}$ .
- $G_0B \ni b \mapsto \hat{b} \in G_0\hat{B}$

$$G_0B = \{\exp(a) : a \in B\}$$

$$\exp(a) = b \mapsto \hat{b} = \exp(\hat{a}) \in G_0\hat{B}$$

- $\iota[b] \mapsto [\hat{b}]$  is well defined and

$$[b] = \{b \exp(a) : a \in B\}$$

$$[\hat{b}] = \{\hat{b} \exp(\alpha) : \alpha \in \hat{B}\}$$

- Easy:  $\iota$  is a group homomorphism.
- $\iota$  is injective and surjective (non-trivial)
  - $\iota$  injective means that if  $b \in GB$  is such that  $\hat{b}$  has a logarithm in  $\hat{B}$ ,  $\hat{b} = \exp(\alpha)$ ,  $\alpha \in \hat{B}$ , then  $b$  has a logarithm in  $B$ ,  $b = \exp(a)$  for some  $a \in B$  (and  $\hat{a} = \alpha$ ). Or if  $b_1, b_2 \in GB$  such that  $\hat{b}_1, \hat{b}_2 \in \hat{B}$  are homotopic, continuous functions  $M(B) \rightarrow \mathbb{C} \setminus \{0\}$ . Then  $b_1$  and  $b_2$  are connected by a path in  $GB$  (i.e.  $b_1 = b_2 \exp(a)$ )
  - $\iota$  surjective means that if  $\gamma \in G\hat{B}$  (i.e.  $\gamma : M(B) \rightarrow \mathbb{C} \setminus \{0\}$ ) is continuous, then there exist a  $c \in GB$  such that  $\hat{c} = \gamma \cdot \exp(\alpha)$  with  $\alpha \in \hat{B}$ .

## Chapter 3: C\*-Algebras

### Section 3.1: Operators on Hilbert Space

- Inner-product space: a complex vector space  $H$  with inner product  $H \times H \ni (x, y) \mapsto \langle x, y \rangle \in \mathbb{C}$  such that

1.  $\langle x, y \rangle$  is linear in  $x$  and anti-linear in  $y$

2.  $\langle x, y \rangle = \overline{\langle y, x \rangle}$ .

3.  $\langle x, x \rangle \geq 0$  and  $\langle x, x \rangle = 0 \iff x = 0$ .

- Norm:  $\|x\| = \sqrt{\langle x, x \rangle}$

- Cauchy-Schwarz Inequality:  $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$ .

- Triangle Inequality:  $\|x + y\| \leq \|x\| + \|y\|$ .

## Definition: Hilbert Space

A Hilbert space is a complete inner-product space.

IMAGE 1

## Examples

$$H = \mathbb{C}^n, \langle x, y \rangle = \sum_{i=1}^n x_i \bar{y}_i$$

$$H = \ell^2(\Omega) (\Omega \text{ possibly uncountable}), \langle x, y \rangle = \sum_{\omega \in \Omega} x_\omega \bar{y}_\omega \text{ where } \|x\|^2 = \sum_{\omega \in \Omega} |x_\omega|^2 < +\infty$$

$$H = L^2(S, d\mu) \text{ with } (S, B_s, d\mu) \text{ a measure space.}$$

## Theorem: Riesz Representation

For every  $\phi \in H^*$ , there exists a unique  $y \in H$  such that  $\phi(x) = \langle x, y \rangle$ .

The map  $\tau : H \ni y \mapsto \phi \in H^*$  is an isometric antilinear (almost) isomorphism.

$$\tau(\lambda y) = \bar{\lambda} \tau(y)$$

$$\langle x, \lambda y \rangle = \bar{\lambda} \langle x, y \rangle$$

We will consider  $L(H)$  the Banach algebra of bounded linear operators  $A : H \rightarrow H$  equipped with the norm

$$\|A\|_{L(H)} = \sup_{\substack{x \in H \\ x \neq 0}} \frac{\|Ax\|}{\|x\|}$$

## Definition: Adjoint Operator (for Hilbert Spaces)

For  $A \in L(H)$ , its adjoint  $A^* \in L(H)$  is given by  $\langle Ax, y \rangle = \langle x, A^* y \rangle$  for any  $x, y \in H$ .

## Remark: Well-Defined

$A^*$  is well-defined. For  $y \in H$ , consider  $\phi(x) = \langle Ax, y \rangle$  which is a bounded, linear functional ( $\phi \in H^*$ ).

By Cauchy-Schwarz,  $|\langle Ax, y \rangle| \leq \|Ax\| \cdot \|y\| \leq \|A\| \cdot \|x\| \cdot \|y\|$ , so  $\|\phi\| \leq \|A\| \cdot \|y\|$ .

By the Riesz Representation Theorem, there exists  $z_y \in H$  such that  $\phi(x) = \langle x, z_y \rangle$ .

Put  $A^*(y) = z_y$ ,  $A^* : H \ni y \mapsto z_y \in H$  such that  $\langle Ax, y \rangle = \langle x, A^* y \rangle$ .

## Remark: Linearity

$A^*$  is linear.

### Remark: Boundedness

$A^*$  is bounded,  $\|A^*y\| = \|z_y\| = \|\phi\| \leq \|A\| \cdot \|y\|$ .  
 Therefore,  $A^* \in L(H)$ .

### Properties

- $(A^*)^* = A$ .
- $\|A^*\| = \|A\|$
- $(A+B)^* = A^* + B^*$  and  $(\lambda A)^* = \bar{\lambda} A^*$ .
- $(AB)^* = B^* A^*$
- $A$  is invertible if and only if  $A^*$  is invertible and  $(A^{-1})^* = (A^*)^{-1}$ .
- $\|A^*A\| = \|A\|^2$ .

$$\begin{aligned}\langle A^*Ax, x \rangle &= \langle Ax, Ax \rangle = \|Ax\|^2 \\ \|Ax\|^2 &\leq \|A^*Ax\| \cdot \|x\| \leq \|A^*A\| \cdot \|x\|^2 \\ \|A\|^2 &= \sup_{x \neq 0} \frac{\|Ax\|^2}{\|x\|^2} \leq \|A^*A\| \\ \|A^*A\| &\leq \|A^*\| \cdot \|A\| = \|A\|^2\end{aligned}$$

### Definitions

$A \in L(H)$  is called

- Self-adjoint if  $A^* = A$ .
- Unitary if  $A^*A = AA^* = I$
- Normal if  $A^*A = AA^*$
- Positive ( $A \geq 0$ ) if  $\langle Ax, x \rangle \geq 0, \forall x \in H$

Positive implies self-adjoint which implies normal.

Unitary implies normal.

### Proposition

$A \in L(H)$  is self-adjoint if and only if  $\langle Ax, x \rangle \in \mathbb{R}$ .

### Proof

$A = A^*$  implies  $\langle Ax, x \rangle = \langle x, A^*x \rangle = \langle x, Ax \rangle = \overline{\langle Ax, x \rangle}$ . Therefore  $\langle Ax, x \rangle \in \mathbb{R}$ .  
 $\langle Ax, x \rangle \in \mathbb{R}$  implies  $\langle Ax, x \rangle = \overline{\langle x, Ax \rangle} = \langle x, Ax \rangle = \langle A^*x, x \rangle$ . Therefore  $\langle (A - A^*)x, x \rangle = 0$  so (exercise)  $\langle (A - A^*)x, y \rangle = 0$ .  
 Use with  $\phi(x, y) = \langle (A - A^*)x, y \rangle$ ,  $\phi(x, y) = \frac{1}{4}(\phi(x+y, x+y) - \phi(x-y, x-y) + i\phi(x+iy, x+iy) - i\phi(x-iy, x-iy))$ .

## Proposition

For  $A \in L(H)$ ,  $A^* A \geq 0$ .

### Proof

$$\langle A^* Ax, x \rangle = \langle Ax, Ax \rangle = \|Ax\|^2 \geq 0.$$

**November 13, 2024**

### Recall: Propositions

$A$  is self adjoint if and only if  $\langle Ax, x \rangle \in \mathbb{R}$ .

$A^* A$  is positive (i.e.  $\langle A^* Ax, x \rangle \geq 0$ ).

### Theorem

Let  $A \in L(H)$

1. if  $A = A^*$ , then  $\text{sp}(A) \subseteq \mathbb{R}$ .
2. if  $A \geq 0$ , then  $\text{sp}(A) \subseteq [0, +\infty)$ .

### Lemma

If  $K \in L(H)$ ,  $\|Kx\| \geq \delta \|x\|$ , and  $\|K^* x\| \geq \delta \|x\|$  for  $\delta > 0$ , then  $K$  is invertible.

- Proof

$K$  is injective ( $\ker K = \{0\}$ ).

The image  $\text{im } K$  is closed (use cauchy sequences with  $Kx_n \rightarrow y$ ,  $\{Kx_n\}$  Cauchy,  $\{x_n\}$  Cauchy,  $x_n \rightarrow x$ . Then  $Kx_n \rightarrow Kx \implies Kx = y$ ).

$\ker K^* = \{0\}$  since

$$\begin{aligned} (\ker K^*)^\perp &= (\text{im } K)^\perp \\ &= \{y \in H : \langle y, z \rangle = 0, \forall z \in \text{im } K\} \\ &= \{y \in H : \langle y, Kx \rangle = 0, \forall x \in H\} \\ &= \{y \in H : \langle K^* y, x \rangle = 0, \forall x \in H\} \\ &= \{y \in H : Ky^* = 0\} \end{aligned}$$

and  $\{0\} = (\text{im } K)^\perp$  implies that  $\text{im } K$  is dense.

Therefore,  $\text{im } K = H$ ,  $K$  is surjective and injective, and ultimately  $K$  is invertible.

### Proof of a

Take  $\lambda = \alpha + i\beta \in \mathbb{C} \setminus \mathbb{R}$  ( $\beta \neq 0$ ), and write

$$A - \lambda I = A - \alpha I - i\beta I = \beta \left( \frac{\overbrace{A - \alpha I}^{\stackrel{\text{:=} T}{}}}{\beta} - i \right) = \beta \underbrace{(T - iI)}_{\stackrel{\text{:=} K}}$$

Then  $K = T - iI$  and  $K^* = T + iI$ . So

$$\|(T - iI)x\|^2 \langle (T - iI)x, (T - iI)x \rangle = \|Tx\|^2 + \|x\|^2 - i\langle x, Tx \rangle + i\langle Tx, x \rangle$$

Since  $\overline{\langle x, Tx \rangle} = \langle Tx, x \rangle$ , we get  $\|Tx\|^2 + \|x\|^2 \geq \|x\|^2$ .

Likewise,  $\|(T + iI)x\|^2 \geq \|x\|^2$  so  $T - iI$  is invertible and  $A - \lambda I$  is invertible.

### Proof of b

Recall that positive implies self-adjoint.

For  $\lambda \in (-\infty, 0)$ ,  $\lambda = -\alpha$ ,  $\alpha > 0$ , we have  $A - \lambda I = A + \alpha I =: K$ . Then

$$\|Kx\|^2 = \langle (A + \alpha I)x, (A + \alpha I)x \rangle = \|Ax\|^2 + \alpha^2 \|x\|^2 + \alpha(\langle Ax, x \rangle + \langle x, Ax \rangle)$$

By assumption,  $\langle Ax, x \rangle \geq 0$  and  $\langle x, Ax \rangle \geq 0$ , so  $\|Kx\|^2 \geq \alpha^2 \|x\|^2$ .

### Remark

For  $A \in L(H)$  where  $A = A^*$ , the following are equivalent

1.  $A$  is positive.
2.  $\text{sp}(A) \subseteq [0, +\infty)$ .
3.  $A = B^*B$  for some  $B \in L(H)$ .

## Section 3.2: C\* Algebras

### Definition:

A  $C^*$ -algebra is a Banach algebra  $B$  which has a map

$$*: B \ni a \mapsto a^* \in B$$

(called an involution) such that

- $(a^*)^* = a$ ,  $(ab)^* = b^*a^*$ , and  $e^* = e$ .
- $(a+b)^* = a^* + b^*$ ,  $(\lambda a)^* = \bar{\lambda}a^*$ .
- $\|a^*a\| = \|a\|^2$ .

### Remark

If  $B$  has a unit,  $e^* = e$ .

For invertible elements,  $b$  invertible if and only if  $b^*$  is invertible and  $(b^*)^{-1} = (b^{-1})^*$ .

$\|a^*\| = \|a\|$ . Indeed  $\|a\|^2 = \|a^*a\| \leq \|a\| \cdot \|a^*\|$  so  $\|a\| \leq \|a^*\|$  (for  $a \neq 0$ ).

Since  $a \mapsto a^*$ ,  $\|a^*\| \leq \|a^{**}\| = \|a\|$ .

## Examples

- $B = L(H)$  bounded linear operators on Hilbert spaces.
- $B = C(X)$  with  $X$  a compact Hausdorff space given by  $a : X \rightarrow \mathbb{C}$  continuous and  $a^*(x) := \overline{a(x)}$  (complex conjugate).
- $B = L^\infty(S, d\mu)$  essentially bounded functions on a measure space  $(S, \mathcal{B}_S, \mu)$  again with  $a^*(x) := \overline{a(x)}$ .

## Non-examples

$B = W = \{\sum_{n \in \mathbb{Z}} a_n t^n : \sum |a_n| < +\infty\}$ . We have  $\|a\| = \sum |a_n|$  and  $\|a^*\| = \|a\|$  but not  $\|a^* a\| = \|a\|^2$ .  
 $B = C^1[0, 1]$ .

## Definitions:

An element  $b \in B$  is called

- self adjoint if  $b^* = b$
- unitary if  $b^* b = b b^* = e$
- normal if  $b^* b = b b^*$
- positive if  $b^* = b$  and  $\text{sp}(b) \subseteq [0, +\infty]$ .

## Proposition

For  $b \in B$  normal, the spectral radius  $r(b) := \max\{|\lambda| : \lambda \in \text{sp}(b)\} = \|b\|$ .

### Proof

We know that  $r(b) = \lim_{n \rightarrow \infty} \|b^n\|^{1/n} = \inf_{n \in \mathbb{N}} \|b^n\|^{1/n}$ . Therefore,  $r(b) \leq \|b\|$ .

For any element  $a$  which is self adjoint, we have  $\|a^* a\| = \|a^2\| = \|a\|^2$ . By induction,  $\|a^{2^k}\| = \|a\|^{2^k}$  so  $\|a^n\|^{1/n} = \|a\|$  for  $n = 2^k$ .

For a normal element  $b$ ,

$$\|b^n\|^2 = \|(b^n)^* b^n\| = \|(b^* b)^n\|$$

Then, since  $b^* b$  is self-adjoint,  $\|b^n\|^2 = \|b^* b\|^n$  for  $n = 2^k$  and  $\|b^n\|^{1/n} = \|b^* b\|^{1/2}$ .

Therefore  $r(b) = \|b^* b\|^{1/2} = \|b\|$ .

## Corollary

The norm in a  $C^*$ -algebra is uniquely determined (i.e. there are no different equivalent norms).

### Proof

Let  $a \in B$  be arbitrary. Then  $r(a^* a) = \|a^* a\| = \|a\|^2$ .

Therefore  $\|a\| = \sqrt{r(a^* a)}$ . The spectral radius (and the spectrum) are determined in terms of algebraic properties.

## Proposition

The spectrum of

1. a unitary element is contained in  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ .
2. a self-adjoint element is contained in  $\mathbb{R}$  (in fact, it is a subset of  $[-||a||, ||a||]$ ).

### Proof of 1

For a unitary element,  $||a||^2 = ||a^*a|| = ||e|| = 1$ , so  $||a|| = 1$  and  $\text{sp}(a) \subseteq \mathbb{T}$ . Since  $a^{-1} = a^*$ ,  $(a^{-1})^*a^{-1} = a^{-1}(a^{-1})^* = e$ ,  $a^{-1}$  is also unitary and  $\text{sp}(a^{-1}) \subseteq \mathbb{T}$ . Observe that  $\text{sp}(a) = \left\{ \frac{1}{\lambda} : \lambda \in \text{sp}(a^{-1}) \right\}$ , then  $\text{sp}(a) \subseteq \mathbb{T}$  and  $\text{sp}(a^{-1}) \subseteq \mathbb{T}$ .

### Proof of 2

Recall that  $\exp(c) = \sum_{n=0}^{\infty} \frac{c^n}{n!}$ . Then  $(\exp(c))^* = \exp(c^*)$ .

Let  $a$  be a self adjoint element, then  $u = \exp(ia)$  is unitary since  $u^* = \exp((ia)^*) = \exp(-ia) = u^{-1}$ . By the spectral mapping theorem,

$$\exp(i\text{sp}(a)) = \exp(\text{sp}(ia)) = \text{sp}(\exp(ia)) \subseteq \mathbb{T}$$

Therefore  $\text{sp}(a) \subseteq \mathbb{R}$ .

## Proposition

Each  $b \in B$  can be written uniquely as  $b = b_1 + ib_2$  with  $b_1$  and  $b_2$  self-adjoint.

### Proof

Define  $b_1 = \frac{b+b^*}{2}$  and  $b_2 = \frac{b-b^*}{2i}$  which are self-adjoint and compute

$$b_1 + ib_2 = b \quad \text{and} \quad b_1 - ib_2 = b^*$$

For uniqueness, assume that  $0 = b_1 + ib_2$  with  $b_1$  and  $b_2$  self-adjoint. Then  $b_1 = -ib_2$  and, since  $\text{sp}(b_1) \subseteq \mathbb{R}$  and  $\text{sp}(b_2) \subseteq \mathbb{R}$ . Then  $\text{sp}(b_1) = \text{sp}(b_2) = \{0\}$ . Therefore  $||b_1|| = r(b_1) = 0$ , and, since  $b_1$  is self-adjoint,  $b_1 = 0$  (similarly for  $b_2$ ).

## Theorem

Let  $B \subseteq A$  be a unital,  $C^*$ -subalgebra of a  $C^*$  subalgebra  $A$ . Then  $B$  is inverse closed in  $A$  (i.e.  $GA \cap B = GB$ ).

### Remarks

$B$  is inverse-closed if and only if  $\forall b \in B$ , if  $b$  invertible in  $A$  then  $b^{-1} \in B$ .

Equivalently,  $\text{sp}_A(b) = \text{sp}_B(b)$  for every  $b \in B$ .

### Proof

Let  $b \in B$  be invertible in  $A$ . Then  $c = b^*b$  is invertible in  $A$  ( $c^{-1} = b^{-1}(b^{-1})^* \in A$ ). Now  $c$  is a self-adjoint, therefore  $\text{sp}_A(c) \subseteq [-||c||, ||c||] \subseteq \mathbb{R}$ .

For subalgebras,  $\text{sp}_B(c) \supseteq \text{sp}_A(c)$  and  $\partial \text{sp}_B(c) \subseteq \partial \text{sp}_A(c)$ . In fact,  $\text{sp}_B(c) = \text{sp}_A(c) \cup \bigcup_{\omega \in \Omega} H_\omega$  where  $H_\omega$  are (bounded) connected components of  $\mathbb{C} \setminus \text{sp}_A(c)$ .

However, because  $\text{sp}_A(c) \subseteq \mathbb{R}$  there are no holes. Therefore  $\text{sp}_B(c) = \text{sp}_A(c)$ .

We know that  $0 \notin \text{sp}_A(c)$  so  $0 \notin \text{sp}_B(c)$ . So  $c$  is invertible in  $B$ , so  $b$  is left invertible in  $B$  since

$$e = c^{-1}c = \underbrace{c^{-1}b^*b}_{\in B}$$

Similarly,  $b$  is right invertible by repeating the argument with  $d = bb^*$ . Therefore  $b$  is invertible in  $B$ .

**November 18, 2024**

## Section 3.3: Commutative C\*-Algebras

### Proposition

For a unital commutative  $C^*$ -algebra  $B$  and  $\phi \in M(B)$ , we have  $\phi(b^*) = \overline{\phi(b)}$ ,  $\forall b \in B$ .

### Proof

Write  $b = b_1 + ib_2$  with  $b_1$  and  $b_2$  self-adjoint. Then  $b^* = b_1 - ib_2$ .

Then  $\text{sp}(b_i) \subseteq \mathbb{R}$  implies that  $\phi(b_i) \in \mathbb{R}$ , otherwise  $\phi(b_i) = z \in \mathbb{C} \setminus \mathbb{R}$  gives  $\phi(b_1 - ze) = 0$  which implies  $b_1 - ze$  is not invertible and  $z \in \text{sp}(b_i)$ .

Then  $\phi(b) = \phi(b_1) + i\phi(b_2)$  and  $\phi(b^*) = \phi(b_1) - i\phi(b_2)$  with  $\phi(b_i)$  real.

### Remarks

A map  $\phi : B \rightarrow \mathbb{C}$  is a  $*$ -homomorphism if  $\phi(b^*) = \phi(b)^* = \overline{\phi(b)}$  (involution in  $\mathbb{C}$ ).

The Gelfand transform is also a  $*$ -homomorphism

$$\Lambda : B \ni b \mapsto \hat{b} \in C(M(B))$$

with  $\Lambda(b^*) = (\Lambda(b))^*$  and  $\widehat{b^*} = (\hat{b})^*$ . So

$$\widehat{b^*}(\phi) = \phi(b^*) = \overline{\phi(b)} = \overline{\hat{b}(\phi)} = (\hat{b})^*(\phi)$$

### Theorem: Stone-Weierstrass

Let  $X$  be a compact Hausdorff space and  $\mathfrak{a} \subseteq C(X)$  a subalgebra of  $C(X)$  such that

- $\mathfrak{a}$  is norm-closed in  $C(X)$
- $I \in \mathfrak{a}$
- if  $f \in \mathfrak{a}$  then  $\bar{f} \in \mathfrak{a}$
- $\forall x, y \in X$  with  $x \neq y$ , there exists  $f \in \mathfrak{a}$  such that  $f(x) \neq f(y)$ .

$C_r(X)$  the set of all continuous functions  $f : X \rightarrow \mathbb{R}$  and  $\mathfrak{a}_r = \mathfrak{a} \cap C_r(X)$ .

## Proof

It is enough to show that  $\mathfrak{a}_r = c_R(X)$  for  $f \in C(X)$ ,  $f = \operatorname{Re} f + i \operatorname{Im} f$  then  $f \in \mathfrak{a}$ .  
If  $f \in \mathfrak{a}_r$ , then  $|f| \in \mathfrak{a}_r$ . Without loss of generality,  $\|f\| \leq 1$ ,  $|f(x)| \leq 1$ . Note that

$$\phi(t) = \sqrt{1-t} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} t^n$$

which converges uniformly on  $[-1, 1]$ . Write  $|f| = \sqrt{|f|^2} = \sqrt{f^2} = \sqrt{1 - (1 - f^2)}$ . Then for  $t = 1 - f^2(x) \in [0, 1]$ ,

$$|f(x)| = \left| \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (1 - f^2(x))^n \right| = \sum_{n=0}^N (\dots) + R_N(1 - f^2(x))$$

where the first term is polynomial in  $f^2 = \bar{f}f \in \mathfrak{a}_r$  and we may estimate the remainder

$$\sup_{x \in X} |R_n(\dots)| \leq \sup_{t \in [0, 1]} |R_N(t)|$$

which converges to 0 as  $N \rightarrow \infty$ . Therefore  $|f|$  may be approximated by elements in  $\mathfrak{a}_r$ .  
Then  $\mathfrak{a}_r$  is a lattice. Given  $f, g \in \mathfrak{a}_r$ , we have

$$f \vee g = \max\{f, g\} = \frac{1}{2}\{f + g + |f - g|\} \quad \text{and} \quad f \wedge g = \min\{f, g\} = \frac{1}{2}\{f + g - |f - g|\}$$

and  $f \vee g, f \wedge g \in \mathfrak{a}_r$ .

Now,  $\forall \alpha, \beta \in \mathbb{R}$ , there exists  $f \in \mathfrak{a}_r$  such that  $f(x) = \alpha$  and  $f(y) = \beta$ .

First, we obtain a function  $h \in \mathfrak{a}$  with  $h(x) \neq h(y)$  and  $1 \in \mathfrak{a}$ , then put  $f = \gamma 1 + \delta h$ . We need

$$\alpha = \gamma + \delta h(x) \quad \text{and} \quad \beta = \gamma + \delta h(y)$$

which can be found by solving  $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & h(x) \\ 1 & h(y) \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$ .

Now we replace  $f$  by  $\operatorname{Re} f = \frac{1}{2}(f + \bar{f}) \in \mathfrak{a}$  so  $\operatorname{Re} f \in \mathfrak{a}_r$ .

Let  $f \in C_r(X)$ . We want to show that  $f$  can be approximated by functions in  $\mathfrak{a}_r$ .

Let  $\varepsilon > 0$ , fix  $x_0 \in X$  and take  $x \in X$  arbitrary. Find  $g_x \in \mathfrak{a}_r$  such that  $g_x(x_0) = f(x_0)$  and  $g_x(x) = f(x)$ .

Since these functions are continuous, we have that  $g_x(y) \leq f(y) + \varepsilon$  in some open neighborhood  $U_x \ni x$ . Then  $U_x$  covers  $X$  as we range across  $x \in X$ . We have a finite subcover  $U_{x_1}, \dots, U_{x_N}$  and  $h_{x_0} = g_{x_1} \wedge g_{x_2} \wedge \dots \wedge g_{x_N}$ .

Then  $h_{x_0}(y) \leq f(y) + \varepsilon$  for every  $y \in X$  where  $h_{x_0}(x_0) = f(x_0)$ .

Doing this for all  $x_0 \in X$  gives a collection  $\{h_{x_0}\}_{x_0 \in X}$ . For some  $V_{x_0} \ni x_0$ ,  $h_{x_0}(y) \geq f(y) - \varepsilon$ .

Then  $\{V_{x_0}\}_{x_0 \in X}$  covers  $X$  and admits a finite subcover  $V_{x_1}, \dots, V_{x_n}$  which gives rise to  $h = h_{x_1} \vee h_{x_2} \vee \dots \vee h_{x_n}$ .

Therefore  $f(y) - \varepsilon \leq h(y) \leq f(y) + \varepsilon$ . Therefore  $|f(y) - h(y)| \leq \varepsilon$  for every  $y \in X$  and  $h \in \mathfrak{a}_r$  (by the lattice property).

## Theorem

Let  $B$  be a unital  $C^*$ -algebra.

Then the Gelfand transform  $\Lambda : B \ni b \mapsto \hat{b} \in C(M(B))$  is an isometric  $*$ -isomorphism.

## Proof

We know that  $\Lambda$  is a Banach algebra homomorphism (i.e. multiplicative, linear and  $\|\Lambda\| = 1$ ).  
We know also that  $\Lambda$  is a  $*$ -homomorphism with  $\Lambda(b^*) = (\Lambda(b))^*$ .

The Gelfand transform also preserves spectrum where  $\text{sp}_B(b) = \text{sp}_{C(M(B))}(\Lambda(b))$  because  $b \in B$  is invertible if and only if  $\phi(b) = \hat{b}(\phi)$  is non-zero for each  $\phi \in M(B)$  and, equivalently,  $\hat{b} = \Lambda(b)$  is invertible in  $C(M(B))$ . Therefore the spectral radius is the same  $r(b) = r(\Lambda(b))$ .

Take  $b = a^*a$  for arbitrary  $a \in B$ . Then  $\|a^*a\| = \|\Lambda(a^*a)\| = \|\Lambda(a)^*\Lambda(a)\|$ , so  $\|a\|^2 = \|\Lambda(a)\|^2$  and  $\|a\| = \|\Lambda(a)\|$  an isometry. It follows that  $\Lambda$  is injective ( $\Lambda(a) = 0 \implies a = 0$ ).

To show that  $\Lambda$  is surjective ( $\text{im } \Lambda = C(M(B))$ ), put  $\mathfrak{a} = \text{im } \Lambda$ . Then

- $\mathfrak{a}$  is an algebra
- $\mathfrak{a}$  is isometrically isomorphic to  $B$  ( $\Lambda : B \rightarrow \mathfrak{a} \subseteq C(M(B))$ )
- $\mathfrak{a}$  is norm closed (via Cauchy sequences)
- $1 \in \mathfrak{a}$
- $f \in \mathfrak{a}$  implies  $\bar{f} \in \mathfrak{a}$  since  $f = \Lambda(b)$  gives  $\bar{f} = \Lambda(b)^* = \Lambda(b^*)$ .
- for  $\phi_1, \phi_2 \in M(B)$ ,  $\phi_1 \neq \phi_2$  implies there exists  $b \in B$  such that  $\phi_1(b) \neq \phi_2(b)$ , so  $\hat{b}(\phi_1) \neq \hat{b}(\phi_2)$ .

Therefore, we may apply Stone-Weierstrass to see that  $\mathfrak{a} = C(M(B))$  implies  $\Lambda$  is surjective.

## Examples

- $B = \ell^\infty(\mathbb{N}) = \{\{x_n\}_{n=0}^\infty : \|x\| = \sup_n |x_n| < +\infty\}$  is a commutative  $C^*$ -algebra. Therefore  $\ell^\infty$  is isometrically  $*$ -isomorphic to  $C(M(\ell^\infty))$ . Some maximal ideals can be identified with  $\mathbb{N}$ .

$$\begin{aligned}\phi_N : x = \{x_n\}_{n=0}^\infty &\mapsto X_N \\ \mathbb{N} \cong \{\phi_N\}_{N=0}^\infty &\subseteq M(\ell^\infty(\mathbb{N})) = X\end{aligned}$$

where  $\phi_N$  are dense in  $X$ . All other functionals are given by axiom of choice.  $X$  may be identified with the ech-compcatification of  $\mathbb{N}$  which is uncountable.

- $B = L^\infty([0,1])$ .  $X = M(B)$  is an exotic space. It is "totally disconnected" and uncountable, but  $B \cong C(X)$ .
- $B = PC(\mathbb{T})$  gives a maximal ideal space that can be roughly identified as  $X \cong \mathbb{T} \times \{0,1\}$  with a non-trivial topology.
- Finitely generated algebras. Caution: if  $\mathcal{A}$  is a  $C^*$ -algebra and  $a \in \mathcal{A}$ ,  $B = \text{alg}\{e, a\}$  may not be a  $C^*$ -algebra since  $a^*$  may not be in  $B$ . If  $B = \text{alg}\{e, a, a^*\}$ , then it is a  $C^*$ -algebra but may not be commutative.
- Take  $\mathcal{A}$  a  $C^*$ -algebra,  $a \in A$  with  $B = \text{alg}\{e, a\}$  with  $a = a^*$  or  $B = \text{alg}\{e, a, a^*\}$  with  $aa^* = a^*a$ . Then these are both unital, commutative  $C^*$ -algebras. In these cases,  $B \cong C(X)$  where  $X \cong \text{sp}_{\mathcal{A}}(a) \text{sp}_B(a)$ . We have a homeomorphism

$$\tau : M(B) \ni \phi \mapsto \phi(a) \in \text{sp}(a)$$

with  $\phi(a^*) = \overline{\phi(a)}$ .

## Recall

Given a unital, commutative  $C^*$ -algebra  $B$ ,  $B \cong C(X)$  where  $X = M(B)$ .

## Proposition

Let  $B$  be a  $C^*$ -algebra such that  $B = \text{alg}\{e, b, b^*\}$  where  $b^*b = bb^*$  (i.e.  $b$  is normal). Then  $B$  is commutative and  $M(B)$  is homeomorphic to  $X = \text{sp } b$  by

$$\tau : M(B) \rightarrow \phi(b) \in X = \text{sp } b.$$

In particular,  $B$  is isometrically  $*$ -isomorphic to  $C(x)$  by

$$\tilde{\Lambda} : B \ni a \mapsto \hat{a} \circ \tau^{-1} = \tilde{a} \in C(X)$$

## Remarks

- $\text{sp } b^* = \overline{\text{sp } b}$ ;  $\phi(b^*) = \overline{\phi(b)}$ .
- $a \in B \xrightarrow{\Lambda} \hat{a} \in C(M(B)) \rightarrow \tilde{a} = (\hat{a} \circ \tau^{-1}) \in C(X)$
- $\tilde{b}(x) = \hat{b}(\tau^{-1}(x)) = \hat{b}(\phi) = \phi(b) = x$ .
- $b \mapsto \tilde{b}(x) = x$ ,  $\tilde{b} \in C(\text{sp } b)$ .
- $\tilde{b}^*(x) = \widehat{b^*}(\tau^{-1}(x)) = \hat{B}^*(\phi) = \phi(b^*) = \overline{\phi(b)} = \overline{x}$ .

## Section 3.3.5: Functional Calculus for Normal Elements in $C^*$ -Algebras

For a continuous function  $g \in C(X)$  where  $X = \text{sp}(b)$ ,  $b \in B$  normal (i.e.  $b^*b = bb^*$ , define  $\widetilde{g}(b)$  such that  $\widetilde{g}(b)(x) = g(x)$  for  $g(b) \in \text{alg}\{e, b, b^*\}$  (i.e.  $g(b) = \tilde{\Lambda}^{-1}(g)$  where  $\tilde{\Lambda} : B = \text{alg}\{e, b, b^*\} \rightarrow C(\text{sp } b)$ ).

Then  $b \mapsto \hat{b}(x) = x$ ,  $b^* \mapsto \widehat{b^*} = \overline{x}$  and  $g(b) \mapsto g(x)$ .

In particular, if  $p(x) = \sum p_i x^i$ , then  $p(b) = \sum p_i b^i$ .

$q(x) = \sum_{i,j} p_{ij} x^i \overline{x}^j$  and  $q(b) = \sum_{i,h} p_{ij} b^i (b^*)^j$ .

## Spectral Mapping Theorem

If  $b$  is normal and  $g \in C(\text{sp } b)$ , then  $\text{sp}_B(g(b)) = g(\text{sp}_B b)$  ( $= \text{im } g$ ).

## Section 3.4: Positive Elements in $C^*$ -Algebras

## Recall

$a \geq 0$  ( $a$  positive) if and only if  $a^* = a$  and  $\text{sp } a \subseteq [0, +\infty)$ .

This generalizes to  $A \in L(H)$  where  $A \geq 0$  if and only if  $\forall x \in H$ ,  $\langle Ax, x \rangle \geq 0$ .

For  $a \in C(X)$ ,  $a \geq 0$  if and only if  $a(x) \geq 0$ ,  $\forall x \in X$ .

## Proposition

If  $a \geq 0$  and  $b \geq 0$ , then  $a + b \geq 0$ .

### Proof

If  $a \geq 0$  and  $b \geq 0$ , then both  $a$  and  $b$  are self-adjoint and  $(a + b)^* = (a + b)$ .

Then  $\text{sp } a \subseteq [0, \|a\|]$  and  $\text{sp } b \subseteq [0, \|b\|]$  implies that  $\text{sp}(a - \frac{\|a\|}{2}e) \subseteq [-\frac{\|a\|}{2}, \frac{\|a\|}{2}]$  and  $\text{sp}(b - \frac{\|b\|}{2}e) \subseteq [-\frac{\|b\|}{2}, \frac{\|b\|}{2}]$ . Since they are self adjoint,

$$\left\| a - \frac{\|a\|}{2}e \right\| = r\left(a - \frac{\|a\|}{2}e\right) \leq \frac{\|a\|}{2} \quad \text{and} \quad \left\| b - \frac{\|b\|}{2}e \right\| = r\left(b - \frac{\|b\|}{2}e\right) \leq \frac{\|b\|}{2}$$

So

$$\left\| a + b - \frac{\|a\| + \|b\|}{2}e \right\| \leq \left\| a - \frac{\|a\|}{2}e \right\| + \left\| b - \frac{\|b\|}{2}e \right\| \leq \frac{\|a\| + \|b\|}{2}$$

Therefore  $r(a + b - \frac{\|a\| + \|b\|}{2}e) \leq \frac{\|a\| + \|b\|}{2}$  and

$$\text{sp}\left(a + b - \frac{\|a\| + \|b\|}{2}e\right) \subseteq \left[-\frac{\|a\| + \|b\|}{2}, \frac{\|a\| + \|b\|}{2}\right]$$

We conclude that  $\text{sp}(a + b) \subseteq \left[0, \frac{\|a\| + \|b\|}{2}\right] \subseteq [0, +\infty)$ .

## Theorem: Square Roots of Positive Operators

Let  $a \geq 0$ . Then there exists a unique element  $b \geq 0$  such that  $a = b^2$  (notation:  $b = \sqrt{a}$ ).

### Proof: Existence

(Using functional calculus)

Consider  $B = \text{alg}\{e, a\}$  with  $a = a^*$  a  $C^*$ -algebra. Then  $B \cong C(X)$  for  $X = \text{sp } a \subseteq [0, +\infty)$ .

$\tilde{a}(x)$  is continuous and positive, so  $\sqrt{\tilde{a}(x)}$  is also continuous and positive and therefore an element of  $C(X)$ .

Then there must exist  $b \in B$  such that  $\tilde{b}(x) = \sqrt{\tilde{a}(x)}$ . So  $(\tilde{b})^2 = a$ ,  $\tilde{b}(x) \geq 0$  and  $b^2 = a$  with  $b \geq 0$ .

### Proof: Uniqueness

Assume  $a = b^2 = c^2$  where  $b = \sqrt{a}$  as above and  $c \geq 0$ . We have  $b \in \text{alg}\{e, a\}$ .

Obviously,  $ca = ac$  implies  $ca^n = a^n c$  and  $cx = xc$  for all  $x \in \text{alg}\{e, a\}$ . So  $cb = bc$ .

Now consider  $B_0 = \text{alg}\{e, a, c\}$ ,  $a^* = a$ ,  $c^* = c$  and  $ac = ca$  a commutative  $C^*$ -algebra. Then  $b = \sqrt{a} \in B_0$ ,  $a = b^2 = c^2$ ,  $b \geq 0$  and  $c \geq 0$  implies that  $\hat{a} = (\hat{b})^2 = (\hat{c})^2$ ,  $\hat{b} \geq 0$ ,  $\hat{c} \geq 0$  (continuous functions on  $M(\cdots)$ ). Therefore  $\hat{b} = \hat{c}$  and  $b = c$ .

## Lemma

For  $a = a^*$ , there exist  $a_+, a_- \geq 0$  such that  $a = a_+ - a_-$ ,  $a_+ a_- = a_- a_+ = 0$ .

### Proof

For  $B = \text{alg}\{e, a\}$  a commutitve  $C^*$ -algebra, we apply Gelfand Theory / Functional Calculus. For  $x \in M(B)$ , define

$$\hat{a}_+ = \begin{cases} \hat{a}(x) & \hat{a}(x) \geq 0 \\ 0 & \hat{a}(x) < 0 \end{cases} \quad \text{and} \quad \hat{a}_- = \begin{cases} -\hat{a}(x) & \hat{a}(x) \leq 0 \\ 0 & \hat{a}(x) > 0 \end{cases}$$

So  $a_+, a_- \in B$  and  $\hat{a}_\pm C(M(B))$ .

### Theorem:

Let  $b \in B$  be an arbitrary element in a unital  $C^*$ -algebra. Then  $b^* b \geq 0$ .

### Remarks

- Obviously true for  $B = L(H)$  or  $B = C(X)$ .
- It was open for quite some time for general  $C^*$ -algebras. It would be trivial if every  $C^*$ -algebra were isomorphic to some  $*$ -subalgebra of  $L(H)$ . This turns out to be true.
- Recall that  $b^* b \geq 0$  boils down to whether  $e + b^* b$  is invertible.

### Proof

$b^* b$  is self-adjoint. Therefore  $b^* b = c - d$  for  $c \geq 0$  and  $d \geq 0$  with  $cd = dc = 0$ .

To show:  $d = 0$ . Consider  $(bd)^*(bd) = d(b^* b)d = -d^3 \leq 0$  (i.e.  $d^3 \geq 0$ ).

Then take  $bd = s + it$  where  $s$  and  $t$  are self-adjoint. Since  $s$  and  $t$  are self-adjoint,  $s^2 \geq 0$ ,  $t^2 \geq 0$  and  $(s^2 + t^2) \geq 0$ . So  $(bd)^*(bd) + (bd)(bd)^* = 2(s^2 + t^2) \geq 0$ .

Therefore  $(bd)(bd)^* = 2(s^2 + t^2) + d^3 \geq 0$ . We know that  $(bd)(bd)^*$  and  $(bd)^*(bd)$  have the same spectrum (except possibly  $\{0\}$ ). Recall from the homework that for  $\lambda \neq 0$ ,  $\lambda - xy$  is invertible if and only if  $\lambda - yx$  is invertible.

Therefore the spectrum of  $(bd)(bd)^*$  and  $(bd)^*(bd)$  is  $\{0\}$  and

$$\|bd\|^2 = \|(bd)^*(bd)\| = r((bd)^*(bd)) = 0$$

so  $bd = 0$  implies  $d^3 = 0$ ,  $0 = r(d^3) = (r(d))^3 = \|d\|^3$  and therefore  $d = 0$ .

### Corollary

An element  $a \in B$  is positive ( $a \geq 0$ ) if and only if  $a = b^* b$  for some  $b \in B$ .

### Proof

( $\Leftarrow$ ) by previous theorem.

( $\Rightarrow$ )  $a \geq 0$  ( $a$  self-adjoint) means  $b = \sqrt{a}$ . Therefore  $b \geq 0$  and  $b^2 = a$  which implies  $b = b^*$  and  $a = b^* b$ .

### Theorem: Polar Decomposition

Let  $a \in B$  be invertible. Then there exists a unique unitary element  $u \in B$  and positive element  $r \in B$  ( $r \geq 0$ ) such that  $a = u \cdot r$ .

### Proof: Existence

Define  $r = \sqrt{a^* a}$ . Since  $a$  and  $a^*$  are invertible,  $\text{sp}(a^* a) \subseteq [\delta, +\infty)$  and  $r$  is also invertible.

Put  $u = a \cdot r^{-1}$  such that  $u^* u = r^{-*} a a^* r^{-1} = r^{-1} r^2 r^{-1} = e$  and  $u u^* = a r^{-1} a^{-*} a^* = a r^{-2} a^* = a(a^* a)^{-1} a = e$ .

### Proof: Uniqueness

Write  $a = u_1 r_1 = u_2 r_2$  and consider  $a^* a = (r_1)^2 = (r_2)^2$  for  $r_1, r_2 \geq 0$  which implies  $r_1 = r_2$ . It follows that  $u_i = a r_i^{-1}$  is likewise unique.

## Remarks

- left / right polar decompositions  $a = ur = sv$  for  $r, s \geq 0$  and  $u, v$  unitary.
- for  $A \in L(H)$  (not in general, not even for  $C(X)$ ) (but it does work for  $L^p(S)$ ),  $A = U \cdot R$  for  $R \geq 0$  and  $U$  a partial isometry ( $UU^*U = U$  or equivalently that  $(U^*U)^2 = U^*U$ ).  $\ker A$  and  $\ker A^*$  may not be trivial.

## Corollary

If two Hilbert spaces are isomorphic as Banach spaces, then they are isomorphic as Hilbert spaces.

### Proof

Take  $H_1$  and  $H_2$  Hilbert spaces with  $A : H_1 \rightarrow H_2$  an invertible bounded linear operator.

Define  $R = (A^*A)^{1/2} : H_1 \rightarrow H_1$ . Then  $A = U \cdot R$  for  $U : H_1 \rightarrow H_2$  unitary gives  $U^*U = I_{H_1}$  and  $UU^* = I_{H_2}$ . So

$$H_1 \xrightarrow{R} H_1 \xrightarrow{U} H_2$$

where  $U$  is a Hilbert space isomorphism and isometry.  $U^*U = I_{H_1}$  gives

$$\langle x, y \rangle_{H_1} = \langle U^*Ux, y \rangle_{H_1} = \langle UX, Uy \rangle_{H_2}$$

so  $U$  preserves  $\langle \cdot, \cdot \rangle$  and  $\|x\| = \|UX\|$ .

**November 25, 2024**

## Section 3.5: -ideals, \*-homomorphisms and Quotients in C-algebras

### Definition: \*-homomorphism

A homomorphism  $\phi : A \rightarrow B$  such that  $\phi(a^*) = \phi(a)^*$ .

### Definition: \*-ideal

An ideal  $J \subseteq A$  such that  $a \in J$  implies that  $a^* \in J$ .

### Definition: \*-subalgebra

$A \subseteq B$  is a subalgebra if it is an algebra and  $a \in A$  implies  $a^* \in A$ .

### Proposition

If  $\phi : A \rightarrow B$  is a \*-homomorphism between  $C^*$ -algebras, then  $\ker \phi$  is a \*-ideal and  $\text{im } \phi$  is a \*-subalgebra.

### Theorem

Let  $I \subseteq A$  be a closed, two-sided ideal in a  $C^*$ -algebra. Then  $I$  is a \*-ideal.

### Proof

Take  $a \in I$ , consider  $b = a^*a \in I$ . Consider  $u_n(t) = \frac{t}{1/n+t}$  for  $t \geq 0$ .

## IMAGE 1

Then  $u_n(b) = b \cdot \left(\frac{1}{n}e + b\right)^{-1} \in I$ .

## IMAGE 2

For  $0 \leq u_n(t) \leq 1$ ,

$$0 \leq (u_n(t) - 1)^2 \cdot t = \left( \frac{\frac{1}{n}}{\frac{1}{n} + t} \right)^2 \cdot t \leq \frac{1}{4n}$$

With  $u_n \in C([0, ||b||])$  and  $\text{sp } b \subseteq [0, ||b||]$ ,  $||(u_n(t) - 1)^2 t||_{C([0, ||b||])} \rightarrow 0$  and, equivalently,  $||(u_n(b) - e)^2 b||_A \rightarrow 0$ . Since  $(u_n(b) - e)^* = u_n(b) - e$ , it follows that

$$\begin{aligned} ||(u_n(t) - 1)^2 t||_{C([0, ||b||])} &= ||(u_n(b) - e)^2 b||_A \\ &= ||(u_n(b) - e)b(u_n(b) - e)|| \\ &= ||(u_n(b) - e)^* a^* a(u_n(b) - e)|| \\ &= ||a(u_n(b) - e)||^2 \\ &= ||(u_n(b) - e)a^*||^2 \rightarrow 0 \end{aligned}$$

Since  $u_n(b) \in I$ ,  $u_n(b)a^* \in I$  and since  $I$  is closed and  $u_n(b)a^* \rightarrow a^*$ ,  $a^* \in I$ .

### Theorem

Let  $A$  be a  $C^*$ -algebra and  $I \subseteq A$  be a closed  $*$ -ideal. Then  $A/I$  is a  $C^*$ -algebra.

### Proof

$A/I$  is a Banach algebra with operations  $[a] + [b] = [a + b]$  and  $[a] \cdot [b] = [a \cdot b]$  where  $[a] = a + I = \{a + i : i \in I\}$ . We may further define  $[a]^* = [a^*]$  and a norm  $||[a]|| = \inf_{i \in I} ||a + i||_A$ . We want to show that  $||[a]||^2 = ||[a]^*[a]||^2$ .

$$||[a]_I|| = \inf_{i \in I} ||a + i|| = \inf_{i \in I} ||a^* + i^*|| = \inf_{i \in I} ||a^* + i|| = ||[a^*]|| = ||[a]^*||$$

By the sub-multiplicativity of the norm,  $||[a]^*[a]|| \leq ||[a^*]|| \cdot ||[a]|| \leq ||[a]||^2$ . In the other direction, write  $0 \leq z \leq e$  to mean  $z \geq 0$  and  $e - z \geq 0$  (i.e.  $z = z^*$  and  $\text{sp } z \subseteq [0, 1]$ ). Then for  $z \in I$ ,

$$||[a]|| = \inf_{i \in I} ||a + i|| \geq \inf_{0 \leq z \leq e} ||a + z|| \geq ||[a]||$$

Since  $||e - z|| \leq 1$ ,

$$\begin{aligned} ||[a]||^2 &= \inf \{ ||a - az||^2 : 0 \leq z \leq e, z \in I \} \\ &= \inf \{ ||(e - z)a^* a(e - z)|| : 0 \leq z \leq e, z \in I \} \\ &\leq \{ ||a^* a(e - z)|| : 0 \leq z \leq e, z \in I \} \\ &= ||[a^* a]|| \end{aligned}$$

## Theorem

Let  $A, B$  be  $C^*$ -algebras and  $\phi : A \rightarrow B$  a  $*$ -homomorphism. Then  $\text{im } \phi = \phi(A)$  is a closed  $*$ -subalgebra of  $B$ . If  $\phi$  is injective, then it is an isometric  $*$ -isomorphism to its image.

## Proof

Assume  $\phi$  is injective.

Obviously,  $\text{sp}(\phi(a)) \subseteq \text{sp } a$  since  $a - \lambda e$  invertible implies  $\phi(a) - \lambda e$  is invertible.

Claim: for  $a = a^*$ ,  $\text{sp}(\phi(a)) = \text{sp}(a)$ . Otherwise, there would exist  $\lambda_0 \in \text{sp}(a)$  with  $\lambda_0 \notin \text{sp}(\phi(a))$ .

### IMAGE 3

So there must exist  $f : \text{sp}(a) \rightarrow [0, 1]$  continuous with  $f|_{\text{sp}(\phi(a))} = 0$  and  $f(\lambda_0) = 1$ .

### IMAGE 4

Therefore  $\text{sp } f(\phi(a)) = f(\text{sp}(\phi(a))) = 0$  and, since  $f$  is real valued and  $\phi(a)$  is self adjoint  $f(\phi(a))$  suffices. It follows that  $f(\phi(a)) = 0 = \phi(f(a))$  and that  $f(a) = 0$ . However, this contradicts our construction of  $f$  which set  $f(\lambda_0) = 1$ . Hence,  $\text{sp}(\phi(a)) = \text{sp}(a)$  for any  $a = a^*$ . Further,  $\|\phi(a)\| = r(\phi(a)) = r(a) = \|a\|$  for any such element. For an arbitrary element  $b$ ,

$$\|\phi(b)\|^2 = \|\phi(b)^* \phi(b)\| = \|\phi(b^* b)\| = \|b^* b\| = \|b\|^2$$

We conclude that  $\phi$  is an isometry. Therefore, its image is closed since it is an isometry.

In the general case, assume that  $\phi$  is not injective.

Consider the quotient algebra with  $I = \ker \phi$  and  $\psi([a]) = \phi(a)$ .

$$\begin{array}{ccc} A & \xrightarrow{\phi} & \phi(A) \subseteq B \\ & \searrow \pi & \swarrow \psi \\ & A/I & \end{array}$$

We have that  $\pi$  is a surjective  $*$ -homomorphism,  $\psi$  is an injective  $*$ -homomorphism, and  $A/I$  is a  $C^*$ -algebra. Therefore  $A/I \cong \phi(A)$  is an isometric isomorphism as above.

## Corollary

If  $B \subseteq A$  are  $C^*$ -algebras with  $I$  a closed  $*$ -ideal of  $A$ , then  $B + I$  is a  $*$ -subalgebra of  $A$ .

$$(B + I)/I \cong B/(B \cap I)$$

## Proof

Consider the quotient map  $\pi : A \rightarrow A/I$  restricted to  $B$ ,  $\pi|_B : B \rightarrow (B + I)/I$  which is closed in  $A/I$ . Therefore  $B + I$  is closed in  $A$  because if  $b_n + i_n \rightarrow a$ ,  $[b_n] \rightarrow [a]$ . Therefore  $[a] \in (B + I)/I$  and  $a = b + i \in B + I$ .

We have that  $B + I$  is a  $*$ -subalgebra of  $A$  where  $b \mapsto [b]$  is surjective ( $[b + i] = [b]$ )  $*$ -homomorphism.

Then  $\ker \pi|_B = B \cap I$ , and  $\hat{\pi} : B/(B \cap I) \rightarrow (B + I)/I$  is an isometric  $*$ -isomorphism.

## Section 3.6: Positive Linear Functionals

### Definition: Positive Linear Functional

A linear functional  $\phi$  (on a  $C^*$ -algebra) is called positive ( $\phi \geq 0$ ) if for all positive ( $a \geq 0$ )  $a \in A$ ,  $\phi(a) \geq 0$ . If, in addition,  $\phi(e) = 1$ , then  $\phi$  is called a state.

## Remark

Positive functionals satisfy  $\phi(a^*) = \overline{\phi(a)}$ .

## Proof

For  $a = a^*$ ,  $a = a_+ - a_-$ ,  $a_{\pm} \geq 0$ . Therefore  $\phi(a) = \phi(a_+) - \phi(a_-) \in \mathbb{R}$ .

For general  $a = s + it$ ,  $s$  and  $t$  self-adjoint,

$$\phi(a^*) = \phi(s - it) = \phi(s) - i\phi(t) = \overline{\phi(s) + i\phi(t)} = \overline{\phi(a)}$$

## Remarks

- $\phi_1, \phi_2 \geq 0$  implies  $\phi_1 + \phi_2 \geq 0$
- $\phi_1 \geq \phi_2$  if and only if  $\phi_1 - \phi_2 \geq 0$

## Examples

$B = C(X)$  with  $X$  compact Hausdorff.  $M(X)$  regular borel measures.

- All bounded linear functionals,  $\mu$  a complex regular borel measure.

$$\phi_\mu(f) = \int_X f d\mu$$

- Positive linear functionals,  $\mu$  a (positive) regular Borel measure

$$\phi_\mu(f) = \int_X f d\mu$$

- State  $\phi(e) = 1$  if and only if  $\mu(x) = \int_X d\mu = 1$ .
- Multiplicative linear functionals,  $\delta_{x_0}$  the Dirac point measure.

$$\phi_\mu(f) = f(x_0) = \int f d\delta_{x_0}$$

- $[f, g] = \int_X \overline{f}g d\mu$  semi-inner product (not a norm since  $[f, f] = 0$  does not imply  $f = 0$ , and no completeness).

**November 27, 2024**

## Recall: Positive Linear Functionals

$C^*$ -algebra  $A$ ,  $\phi \geq 0$  if  $\phi(a) \geq 0$ ,  $\forall a \in A$ ,  $a \geq 0$ .

- $\phi \geq 0$  gives  $\overline{\phi(a)} = \phi(a^*)$
- $\phi_1, \phi_2 \geq 0$  implies  $\phi_1 + \phi_2 \geq 0$
- $\phi_1 \geq \phi_2$  if and only if  $\phi_1 - \phi_2 \geq 0$

## Example

$A = C(X)$  positive linear functions,  $\phi(f) = \int_X f d\mu$ ,  $\mu$  regular Borel measure.  
 $\phi$  is a state if  $\phi(e) = 1$

## Example

$A = L(\mathbb{C}^n) = \mathbb{C}^{n \times n}$  linear functionals

$$\phi_B(A) = \text{trace}(B^T A) = \sum_{i,j=1}^n b_{ij} a_{ij}$$

Positive linear functionals correspond to  $B \geq 0$ . For  $A \geq 0$ , this implies

$$\phi_B(A) = \text{trace}(A^{1/2} B^T A^{1/2}) = \text{trace}(B^T A)$$

where  $A = A^{1/2} A^{1/2}$ .  $B \geq 0$  implies that

$$\phi_B(A) = \text{trace}(A^{1/2} (B^T)^{1/2} (B^T)^{1/2} A^{1/2}) = \text{trace}(C^* C) = \sum_{j,k} \overline{C_{jk}} C_{jk} \geq 0$$

Conversely, assume  $\phi_B \geq 0$  and choose

$$A = xx^* = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} \overline{x_1} & 7 \cdots & \overline{x_n} \end{pmatrix}$$

Then

$$\phi_B(A) = \text{trace}(B^T xx^*) = x^* B^T x = \langle B^T x, x \rangle$$

for any  $x \in \mathbb{C}^n$ . Therefore  $B^T \geq 0$  and  $B \geq 0$ .

## Example

$A = L(H)$ . A subset of positive functionals is described by positive trace class operators  $\mathcal{C}_1(H)$ .  
For  $K \in \mathcal{C}_1(H)$ ,  $\phi_K(A) = \text{trace}(KA)$ .

## Finite Rank Operators

Finite rank operators are a subset of trace class operators. For  $x \in H$ ,  $K = x \otimes x^*$ .  
 $kz = x \cdot \langle z, x \rangle$ .

## Proposition: Schwarz Inequality

For a positive linear functional  $\phi \geq 0$ ,  $|\phi(x^* y)| \leq \phi(x^* x) \cdot \phi(y^* y)$ ,  $\forall x, y \in A$ .

## Proof

$x^*x \geq 0$  and  $y^*y \geq 0$ , so  $\phi(x^*x) \geq 0$  and  $\phi(y^*y) \geq 0$ .

Define  $[x, y] = \phi(y^*x)$  which has

- linearity in  $x$  and anti-linearity in  $y$ .
- $[y, x] = \overline{[x, y]}$  since  $\phi(x^*y) = \phi((y^*x)^*) = \overline{\phi(y^*x)}$ .
- $[x, x] = \phi(x^*x) \geq 0$ .

Then  $[x - \lambda y, x - \lambda y] \geq 0$  for all  $\lambda$ . Assume, without loss of generality, that  $[x, y]$  is real (otherwise we may replace  $([\tau x, y]$  for  $|\tau| = 1$ ) If  $[x, x] = 0$  or  $[y, y] = 0$ , then  $[x, y] = 0$ . Otherwise, put  $\lambda = -\frac{[x, y]}{[y, y]}$ .

If  $[x, x] = [y, y] = 0$ , then  $-\lambda[x, y] + \bar{\lambda}[y, x] = 0$  implies that  $[x, y] = 0$ .

## Remark

$p(x) = \sqrt{[x, x]}$  is only a seminorm, not a norm. That is  $p(x) = 0 / x = 0$ .

## Proposition:

If  $\phi \geq 0$ , then  $\phi$  is bounded and  $||\phi|| = \phi(e)$ .

## Proof

$$|\phi(e^*x)|^2 \leq \phi(x^*x) \cdot \phi(e^*e)$$

We know that  $r(x^*x) = ||x^*x|| = ||x||^2$  and  $\text{sp}(x^*x) \subseteq [0, ||x||^2]$ .  
So  $||x||^2 \cdot e - x^*x \geq 0$  ( $x^*x \leq ||x||^2 \cdot e$ ) and  $\phi(||x||^2 \cdot e - x^*x) \geq 0$ .

$$\phi(e) \cdot ||x||^2 \geq \phi(x^*x)$$

Therefore  $|\phi(x)|^2 \leq ||x||^2 \cdot \phi(e)^2$  implies that  $\phi(e) \leq ||\phi|| \leq \phi(e)$  which shows also that  $\phi$  is bounded.

## Remark

$\phi \geq 0$  and  $\phi(e) = 0$  implies that  $\phi \equiv 0$ .

## Proposition

If  $\phi$  is a bounded linear functional on  $A$  and  $\phi(e) = ||\phi||$  then  $\phi \geq 0$ .

## Proof

Without loss of generality, consider  $\phi(e) = 1$ .

Assume that  $a \geq 0$  and  $\phi(a) = \alpha + i\beta < 0$ .

If  $\beta \neq 0$ , then consider  $\phi(a + i\lambda e) = \alpha + i(\beta + \lambda)$  for  $\lambda \in \mathbb{R}$ .

$$\begin{aligned}
|\beta + \lambda| &\leq |\phi(a + i\lambda e)| \\
&\leq ||a + i\lambda e|| \\
&= ||(a + i\lambda e)^*(a + i\lambda e)||^{1/2} \\
&= ||a^* a + a^* i\lambda - ai\lambda + \lambda^2 e||^{1/2} \\
&\leq (||a||^2 + \lambda^2)^{1/2}
\end{aligned}$$

So  $(\beta + \lambda)^2 \leq ||a||^2 + \lambda^2$  which means  $\beta^2 + 2\beta\lambda \leq ||a||^2$  which cannot hold for  $\lambda \rightarrow \pm\infty$ .  
If  $\beta = 0$  and  $\alpha < 0$ , then

$$\phi(||a||e - a) = ||a|| - \alpha > ||a||$$

On the other hand  $\phi(||a||e - a) \leq ||||a||e - a|| \leq ||a||$  since  $\text{sp}(a) \subseteq [0, ||a||]$  implies that  $\text{sp}(||a||e - a) \subseteq [0, ||a||]$  and  $||||a||e - a|| = r(||a||e - a||) \leq ||a||$  which is also a contradiction.

### Theorem:

Let  $B$  be a  $C^*$ -subalgebra of a  $C^*$ -algebra  $A$  (all unital) and  $\phi$  be a positive linear functional on  $B$ . Then  $\phi$  has an extension to a positive linear functional on  $A$ .

### Proof

$\phi \geq 0$  (on  $B$ ), so  $\phi \in B^*$  and  $||\phi|| = \phi(e)$ .

Hahn-Banach tells us that there exists an extension  $\psi \in A^*$  onto  $A$  with  $\psi|_B = \phi$  such that  $||\psi|| = ||\phi||$ . So  $||\psi|| = \phi(e) = \psi(e)$ . By the preceding proposition,  $\psi \geq 0$  on  $A$ .

## Section 3.7: Representations of $C^*$ -Algebras and GNS-Construction

### Definition: Representation of a $C^*$ -Algebra

A representation  $(\pi, H)$  of a  $C^*$ -algebra  $A$  is a  $*$ -homomorphism  $\pi : A \rightarrow L(H)$  such that  $\pi(e) = I$ .

### Example

$A = L(H)$ ,  $\pi = \text{id}$  is trivial  $[\text{id}, H]$ .

### Definition: Inflation

If  $(\pi, H)$  is a representation of  $A$ , the inflation  $\pi^n$  is given by the block matrix

$$\pi^n(a) = \begin{pmatrix} \pi(A) & & & \\ & \pi(A) & & \\ & & \ddots & \\ & & & \pi(A) \end{pmatrix}$$

with  $H_n = H \oplus H \oplus \cdots \oplus H (= H \times H \times \cdots \times H)$ .

### Example

$A = C(X)$  (arbitrary  $\mu$  Borel measure) or  $A = L^\infty(X, d\mu)$ .

Take  $H = L^2(X, d\mu)$ , then  $\pi : C(X) \ni f \mapsto M_f \in L(H)$  by  $M_f g = fg$  where  $g \in L^2(X, d\mu)$ .

## Definition: Direct Sum of Representations

The direct sum of representations  $(\pi_n, H_n)$  by  $\pi = \bigoplus_n \pi_n$  and  $H = \bigoplus_n H_n$ .  
For a finite or countable direct sum, we may think of this as

$$\pi(a) = \begin{pmatrix} \pi_1(a) & & & \\ & \pi_2(a) & & \\ & & \ddots & \\ & & & \pi_n(a) \end{pmatrix}$$

## Definition: Equivalence of Representations

Two representations  $(\pi_1, H_1)$  and  $(\pi_2, H_2)$  are said to be equivalent if there exists unitary  $U : H_1 \rightarrow H_2$  ( $U^* U = I_{H_1}$  and  $U U^* = I_{H_2}$ ) such that  $U\pi_1(a)U^* = \pi_2(a)$ .

## Definition: Cyclic Representation

A representation  $(\pi, H)$  is cyclic if there exists some  $v \in H$  such that  $\text{clos}(\pi(A)v) = H$  (i.e.  $\text{clos}\{\pi(a)v : a \in A\} = H$ ).

### Example

Take

$$A = \left\{ \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \right\}$$

with  $\pi(a) = a \in L(\mathbb{C}^2)$  since by  $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$  for  $v_1, v_2 \neq 0$ .

If instead we take

$$A = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \right\}$$

and  $\pi(a) = a$  then this is not cyclic since  $\pi(A)v = \text{lin}\{v\}$ .

## Theorem

Each representation is equivalent to a direct sum  $\bigoplus_n \pi_n$  of cyclic representations.

### Proof

If  $\pi$  is not cyclic, show that  $\pi \cong \pi_1 \oplus \pi_2$ .

Apply Zorn's lemma.