Partial Differential Equations I

January 8, 2024

Homework

Assigned exercises and concept maps. Graded by completion.

Presentations

Assigned topics; responsible for giving a class.

Definition: Partial Differential Equation(s) (PDE)

An identity relating an uknown function, its partial derivatives and its variables.

$$F(D^k u, \ldots, D^2 u, Du, u, x) = 0, \quad x \in U \subseteq \mathbb{R}^n$$

where U is an open subset of $\mathbb{R}^n$, $u : U \subset \mathbb{R}^n \to \mathbb{R}$, $Du = (\partial_{x_1} u_1, \ldots, \partial_{x_n} u)$.

Then $F : \mathbb{R}^{n^k} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, where F is given.

$x = (x_1 \ldots, x_n)$ is (are) the independent variable(s).

u is the unknown function or dependent variable.

k is the order of the PDE.

Goal

Given a PDE, we consider

- Existence

- Uniqueness

- Stability

Recall: Multiindex Notation

$\alpha = (\alpha_1, \ldots, \alpha_n)$ a vector such that $\alpha_i \in \mathbb{Z}_{\geq 0}$.

Then we say that α is a multiindex with order $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Notation

$u : U \subseteq \mathbb{R}^n \to \mathbb{R}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$.

$u^\alpha := D^\alpha u = \partial_{x_n}^{\alpha_n} \cdots \partial_{x_1}^{\alpha_1} u$, where $\partial^0 u = u$.

Definition: Linear Partial Differential Equation

A linear PDE of order k is of the form

$$(*) \sum_{|\alpha| = k} a_\alpha(x) D^\alpha u = f(x)$$

Remark

This means that F is multilinear in the first $n^k + n^{k-1} + \cdots$ variables.

Definition: Homogeneous Linear Partial Differential Equation

A linear given by $(*)$ is homogeneous if $f(x) \equiv 0$.
Otherwise, it is non-homogeneous.

Example 1: Linear Transport Equation

$$\nabla u \cdot (1, b) = u_t + b \cdot Du = f(t, x)$$

This is a linear PDE of order 1 on $\mathbb{R} \times \mathbb{R}^n \equiv \mathbb{R}^{n+1}$ where (t, x) are independent variables and u is dependent.
Here, x is the spatial variable while t is time and Du represents the gradient.
$\nabla u = (\partial_t u, \nabla u)$, $b \cdot Du = \sum_{i=1}^{n} b_i \partial_{x_i} u$, $(b_1, \ldots, b_n) \in \mathbb{R}^n$ is fixed.

Example 2: Laplace Equation

$$\Delta u := \sum_{i=1}^{n} \partial_{x_i} u = 0$$

This is a linear, homogeneous PDE of order 2.

Example 3: Poisson Equation

$-\Delta u := f(u)$
This is a nonlinear PDE of order 2.
Consider $f(u) = u^2$.

Example 4: Heat Equation (Diffusion Equation)

$$u_t - \Delta u = 0$$

This is a linear, homogeneous PDE of order 2.

Example 5: Wave Equation

$$u_{tt} - \Delta u = 0$$

This is a linear, homogeneous PDE of order 2.

Transport Equation

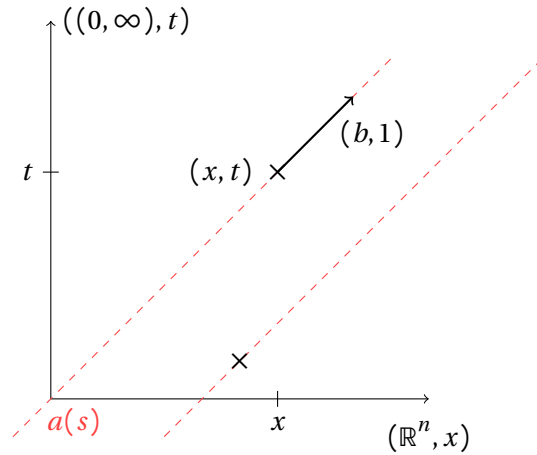$u : \mathbb{R}^n (0, \infty) \to \mathbb{R}$ given by

$$u_t + b \cdot Du = 0, \quad b \in \mathbb{R}^n$$

In order to get a solution, first assume that ther exists a "nice" (e.g. smooth, C^1, differentiable, etc.) solution.

Step 1

Notice that the PDE is equivalent to

$$\nabla u \cdot (b, 1) = 0$$

Step 2

Consider a curve on $\mathbb{R}^{n+1}$ with velocity $(1, b)$ which passes through (x, t). i.e.

$$\alpha(s) = (x + sb, t + s)$$

Notice $\alpha'(s) = (b, 1)$.
Then, let us study u along the curve $\alpha(s)$.

$$z(s) := u(\alpha(s))$$

Taking the derivative with respect to s,

$$z'(s) := \frac{d}{ds}(u \circ \alpha(s)) = \nabla u|_{\alpha(s)} \cdot \alpha'(s) = \nabla u|_{\alpha(s)} \cdot (b, 1) = 0$$

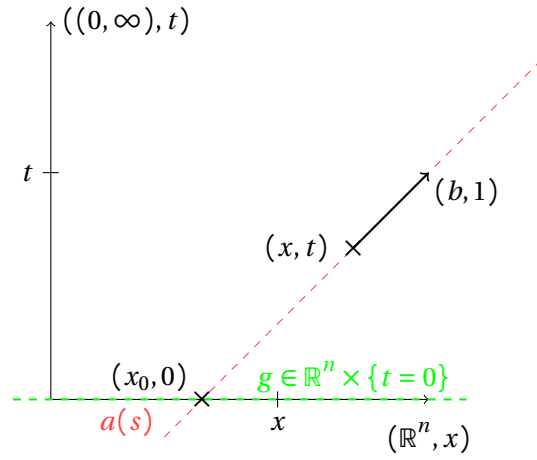That is $z'(s) = 0$, $z(s)$ is constant, and u along $\alpha(s)$ is constant.

Conclusion

If we know some value of u along $\alpha(s)$, then we know all values along $\alpha(s)$.
If we have some value of u along every $\alpha(s)$, then we know u on $\mathbb{R}^n \times (0, \infty)$.

Transport Equation - Homogeneous Initial Value Problem

$$(*) \begin{cases} \nabla u \cdot (b, 1) = 0, & \mathbb{R}^n \times (0, \infty) \\ u = g, & \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Here, $g : \mathbb{R}^n \to \mathbb{R}$ is given.

Consider (x, t); we want to find $(x_0, 0)$.

We know $\alpha(s) = (x + sb, t + s) = (x_0, 0)$, therefore

$$\begin{cases} x + sb = x_0 & (1) \\ t + s = 0 \implies s = -t & (2) \end{cases}$$

Then, by replacing (2) in (1),

$$x_0 = x - tb$$

Then from the conclusion

$$u(x, t) = u(x_0, 0) = g(x_0) = g(x - tb)$$

Therfore, $u(x, t) := g(x - tb)$ ($\heartsuit$).

Remark

1. If there exists a regular (differentiable or C^1) solution u for $*$, then u should look like $\heartsuit$.

2. If g is (differentiable or C^1), then u defined by $\heartsuit$ is a (differentiable or C^1) solution for my problem.

Homework

Show that $\heartsuit$ satisfies $*$.

Transport Equation - Non-homogeneous Initial Value Problem

$$(*) \begin{cases} \nabla u \cdot (b, 1) = f(x, t), & \mathbb{R}^n \times (0, \infty) \\ u = g, & \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Where $g : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ are given.

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Solution

Notice that the PDE is equivalent to

$$\nabla u \cdot (b, 1) = f(x, t)$$

Define the "characteristic curve"

$$\alpha(s) = (x + sb, t + s)$$

and

$$z(s) := u(\alpha(s))$$

Taking $\frac{d}{ds}$,

$$z'(s) = \nabla u|_{\alpha(s)} \cdot (b, 1) = f(\alpha(s)) \implies z'(s) = f(x + sb, t + s) \ (c)$$

Notice that c is an ordinary differential equation. Integrating from $-t$ to 0.

$$\int_{-t}^{0} z'(s) \, ds = \int_{-t}^{0} f(x + sb, t + s) \, ds$$
$$z(0) - z(-t) = \int_{-t}^{0} f(x + sb, t + s) \, ds$$

Notice that $z(0) = u(x, t)$ and $z(-t) = u(\alpha(-t)) = u(x - tb, 0)$.

$$u(x, t) = u(x - tb, 0) + \int_{-t}^{0} f(x + sb, t + s) \, ds$$

Then

$$u(x, t) = g(x - tb) + \int_{-t}^{0} f(x + sb, t + s) \, ds$$
$$\underset{\bar{s} = s + t}{=} g(x - tb) + \int_{0}^{t} f(x + (\bar{s} - t)b, \bar{s}) \, d\bar{s}$$
$$= g(x - tb) + \int_{0}^{t} f(x + (s - t)b, s) \, ds$$

Remark: Method of Characteristics

Try to vert the PDE into an ODE and solve using characteristic curves.

January 10, 2024

Definition: Harmonic Function

If $u \in C^2$ such that $\Delta u = 0$, then u is a harmonic function.

Laplace Equation

Consider $u : U \subseteq \mathbb{R}^n \to \mathbb{R}$ with U open, then the homogeneous (Laplace) form is given by

$$\Delta u = 0$$

and the non-homogeneous (Poisson) form is

$$-\Delta u = f$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is given.

Remark / Exercise

The Laplace equation is invariant under translation and rotation.
That is, if $\Delta u(x) = 0$ and $v(x) = u(x - y)$, then $\Delta v = 0$.
Similarly, if $w(x) = u(O(x))$ then $\Delta w = 0$ where O is an orthogonal matrix.

Fundamental Solution of the Laplace Equation

Remark: since the Laplace equation is invariant under rotation, we can consider a function in terms of the radius $v(x) = v(|x|)$.
Recall $r = |x| = (x_1^2 + \cdots + x_n^2)^{1/2}$.
Because of this remark, assume that $u(x) = v(|x|) = v(r(x))$ $(*)$ where $v : (0, \infty) \to \mathbb{R}$.
Therefore, we need

$$\frac{\partial r}{\partial x_i} = \frac{1}{2} \cdot \frac{2x_i}{(x_1^2 + \cdots + x_n^2)^{1/2}} = \frac{x_i}{r}$$

Replace $(*)$ in the PDE

$$\frac{\partial u}{\partial x_i} = \frac{\partial}{\partial x_i} v(r(x)) = v'(r(x)) \cdot \frac{\partial r}{\partial x_i} = v'(r(x)) \cdot \frac{x_i}{r}$$

and

$$\frac{\partial^2 u}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(v'(r(x)) \cdot \frac{x_i}{r} \right)$$

$$= \frac{\partial}{\partial x_i}(v'(r(x))) \cdot \frac{x_i}{r} + v'(r(x)) \cdot \frac{\partial}{\partial x_i}\left(\frac{x_i}{r} \right)$$

$$= v''(r(x)) \cdot \frac{x_i^2}{r^2} + v'(r(x))\left[\frac{1}{r} + x_i \frac{\partial}{\partial x_i}(r) \right]$$

$$= v'' \frac{x_i^2}{r^2} + v'\left[\frac{1}{r} - \frac{x_i^2}{r^3} \right]$$

Thenm, summing across i,

$$\Delta u = v'' + v'\left[\frac{n}{r} + \frac{1}{r} \right] = 0$$

Then the PDE is equivalent to

$$v''(r) + \frac{v'}{r}(n-1) = 0 \ (\square)$$

We need to find a solution for $\square$.

$$v''(r) = \frac{(1-n)v'}{r}$$

Assume, without loss of generality, that $v' \neq 0$ such that

$$\frac{v''(r)}{v'(r)} = \frac{1-n}{r} \implies (\log(|v'|))' = \frac{1-n}{r}$$

Then, integrating,

$$\log(|v'|) = (1-n)\log(r) + C = \log\left(r^{1-n}\right) + C$$

such that

$$|v'| = Cr^{1-n} \implies v' = Cr^{1-n} \implies v(r) = Cr^{1-n+1} + D = Cr^{2-n} + D$$

Definition: Fundamental Solution of the Laplace Equation

The function Φ given by

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x|, & n = 2 \\ \frac{1}{n(n-2)\alpha(n)} \cdot \frac{1}{|x|^{n-2}}, & n \geq 3 \end{cases}$$

where $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$ is called the fundamental solution.

Remark

Φ solves the Laplace equation away from 0.

Lemma: Estimates for the Fundamental Solution

- First Estimate
  $|D\Phi(x)| \leq \frac{C}{|x|^{n-1}}$, for $x \neq 0$.

$$\frac{\partial \Phi}{\partial x_i} = C\frac{\partial}{\partial x_i}\left(|x|^{2-n}\right) = \frac{C(2-n)}{1-n}|x|^{2-n-1}\frac{\partial|x|}{\partial x_i} = |x|^{1-n} \cdot \frac{x_i}{|x|} = Cx_i|x|^{-n}$$

  Therefore

$$|D\Phi(x)| \leq C|x||x|^{-n} \implies |D\Phi(x)| \leq C|X|^{1-n}$$

  - Exercise
    Compute for $n = 2$.

- Second Estimate
  $|D^2\Phi(x)| \le \frac{C}{|x|^n}$, for $x \neq 0$.

$$\frac{\partial^2}{\partial x_J \partial x_i}\Phi = C\frac{\partial}{\partial x_J}\left(x_i|x|^{-n}\right)$$

$$= C\left[\delta_{iJ}|x|^{-n} + x_i\frac{\partial}{\partial x_J}|x|^{-n}\right]$$

$$= C\left[\delta_{iJ}|x|^{-n} + (-n)\cdot\frac{x_i|x|^{-n-1}x_J}{|x}\right]$$

$$= C\left[\frac{\delta_{iJ}|x|}{|x|^n} + \frac{Cx_ix_J}{|x|^{n+1}}\right]$$

Then

$$\left|\frac{\partial\Phi}{\partial x_i\partial x_J}\right| \le \frac{C}{|x|^n} + \frac{C|x_i||x_J|}{|x|^{n+2}} \le \frac{2C}{|x|^n} = \frac{C}{|x|^n}$$

Then, we are done since

$$|D^2\Phi(x)| = \sqrt{\sum_J\left(\frac{\partial\Phi}{\partial x_i\partial x_J}\right)}$$

Poisson Equation

Motivation

Suppose we have $\Phi(x)$, the fundamental solution.
Then for an arbitrary, fixed element $y \in \mathbb{R}^n$, then we have $x \to \Phi(x-y)$ harmonic for $x \neq y$.
Consider $f : \mathbb{R}^n \to \mathbb{R}$ such that $y \to f(y)$ then $x \to f(y)\Phi(x-y)$ is similarly harmonic for $x \neq y$.
Now, if given $\{y_1, \dots, y_m\}$ where $y_i \in \mathbb{R}^n$, then $x \to \sum_{i=1}^m f(y_i)\Phi(x-y_i)$ is harmonic $\forall x \neq \{y_1, \dots, y_m\}$.
Then, what happens if we consider

$$u(x) := \int_{\mathbb{R}^n} f(y)\Phi(x-y)\,dy \quad (\square_3)$$

Is u harmonic? No, since $\Delta\Phi(x-y)$ is not summable in $\mathbb{R}^n$ we may not pass the limit into the integral.
(to be covered later) However, since $\Delta\Phi(x-y)$ acts as δ_{xy} in distribution, this may solve the Poisson equation.

Remark / Exercise

Assume that $f \in C_C^2(\mathbb{R}^n)$ (i.e f is twice continuously differentiable with compact support on $\mathbb{R}^n$).
The function Φ is integrable near the singularity on compact sets.
Prove using spherical coordinates.
Therefore, u defined by $\square_3$ is well defined

$$|u| = \left|\int_{\mathbb{R}^n} f(y)\Phi(x-y)\,dy\right| = \left|\int_K \Phi(x-y)\,dy\right| < \infty$$

Theorem: Solving the Poisson Equation

If $f \in C_C^2(\mathbb{R}^n)$ and u is defined by $\square_3$, then

1. $u \in C^2(\mathbb{R}^n)$

2. $-\Delta u = f$, in $\mathbb{R}^n$

- Proof of 1
  Since Φ presents a problem at $x = y$ but f is well behaved, we will change variables such that $\overline{y} = x - y$, $y = x - \overline{y}$, and $\frac{dy}{d\overline{y}}(-1)I_{m \times m}$ and then redefine $\overline{y} = y$.

$$u(x) = \int_{\mathbb{R}^n} f(y)\Phi(x - y)\, dy = \int_{\mathbb{R}^n} f(x - \overline{y})\Phi(\overline{y})\, d\overline{y} = \int_{\mathbb{R}^n} f(x - y)\Phi(y)\, dy$$

In short, we have sent the problem from Φ to f.
Now, let us consider $e_i = (0, \dots, 1, \dots, 0)$.
Then for $h > 0$,

$$\frac{u(x + he_i) - u(x)}{h} = \frac{1}{h} \int_{\mathbb{R}^n} \Phi(y)\left[f(x + he_i - y) - f(x - y) \right] dy$$

Now, the limit as $h \to 0$

$$\lim_{h \to 0} \frac{u(x + he_i) - u(x)}{h} = \lim_{h \to 0} \int_{\mathbb{R}^n} \Phi(y)\overbrace{\left[\frac{f(x + he_i - y) - f(x - y)}{h} \right]}^{H(h,y)} dy$$

$$= \int_{\mathbb{R}^n} \Phi(y) \cdot \frac{\partial f(x - y)}{\partial x_i}\, dy$$

To justify passing the limit into the integral, take an arbitrary sequence $h_m \xrightarrow{m \to 0} 0$ and consider

$$f_m(y) := H(h_m, y)$$

We want to consider

$$|H(h_m, y)| \le \Phi(y)\left[\frac{f(x + h_m e_i - y) - f(x - y)}{h} \right]$$

$$\le \Phi(y)f'(c)$$

Where c is along the curve between $f(x + h_m e_i - y)$ and $f(x - y)$ and chosen by mean value theorem.

  - Exercise

$$|H(h_m, y)| \le \Phi(y)||f'||_{L^\infty} \chi_{B(x,R)}(y)$$

Note that

$$C \int_{\mathbb{R}^n} |\Phi(y)|\chi_{B(x,R)}(y)\, dy = \int_{B(x,R)} |\Phi(y)|\, dy < \infty$$

- Exercise

Using the fact that a continuous function is uniformly continuous on a compact set, show that $u \in C^2(\mathbb{R}^n)$.

Dominated Convergence Theorem

If $f_m(x)$ such that $f_m(x) \xrightarrow[\text{pointwise}]{m \to \infty} f(x)$, and $|f_m(x)| \le g(x)$ for $g \in L^1$, then f is integrable and

$$\lim_{m \to \infty} \int f_m(x) \, dx = \int f(x) \, dx$$

January 17, 2024

Recall: Averages

$f : \{1, \dots, n\} \to \mathbb{R}$
$i \to a(i)$
The average is given as $\frac{a(i) + \cdots + a(n)}{n}$.
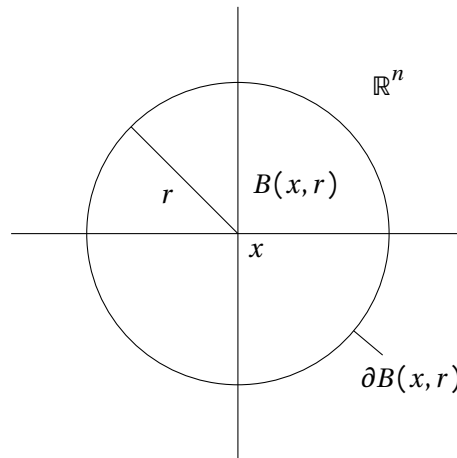Then for $f : \Omega \to \mathbb{R}$, the average is given as

$$\frac{1}{|\Omega|} \int f(y) \, dy := \fint_\Omega f \, d\mu$$

In our case, $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$

$$\fint_{B(x,n)} f \, d\mu \equiv \frac{1}{|B(x,n)|} \fint_{B(x,n)} f \, d\mu$$

$$\fint_{\partial B(x,n)} f \, d\mu = \frac{1}{|\partial B(x,n)|} \fint_{\partial B(x,b)} f \, d\mu$$
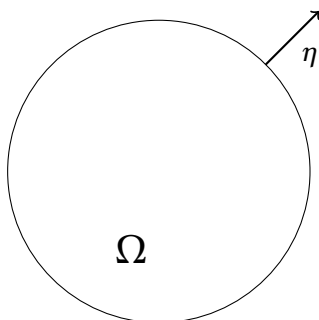

Theorem: Lebesgue Differentiation

$$u|x| = \lim_{r \to 0} \fint_{B(x,n)} u \, d\mu = \lim_{r \to 0} \fint_{\partial B(x,n)} u \, d\mu$$

Integration by Parts

$$\int_{\Omega} u \Delta v = -\int_{\Omega} \nabla u \cdot \nabla v + \int_{\partial \Omega} u \cdot \frac{\partial v}{\partial \eta}$$

$$\Omega$$

$$\eta$$

Recall: Poisson's PDE

$f \in C_c^2 |\mathbb{R}^n|$, $u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\, dy$.

$$\Phi(x) = \left\{ \frac{1}{n(n-2)|\alpha(n)|} \frac{1}{|X|(n-2)} \right.$$

$$u(x) \text{``} = \text{''} \int_{\mathbb{R}^n} f(x-y)\Phi(y)\, dy$$

Part A

$u \in C^2(\mathbb{R}^n)$
Then

$$\frac{\partial u}{\partial x_1} = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_1}(x-y)\Phi(y)\, dy$$

$$\frac{\partial^2 u}{\partial x_1 x_T} = \int_{\mathbb{R}^n} \frac{\partial^2 f}{\partial x_1 x_T}(x-y)\Phi(y)\, dy$$

Notice that if we prove that

$$g(x) = \int_{\mathbb{R}^n} h(x-y)\Phi(y)\, dy$$

– where h is continuous with compact support – is continuous then we are done.
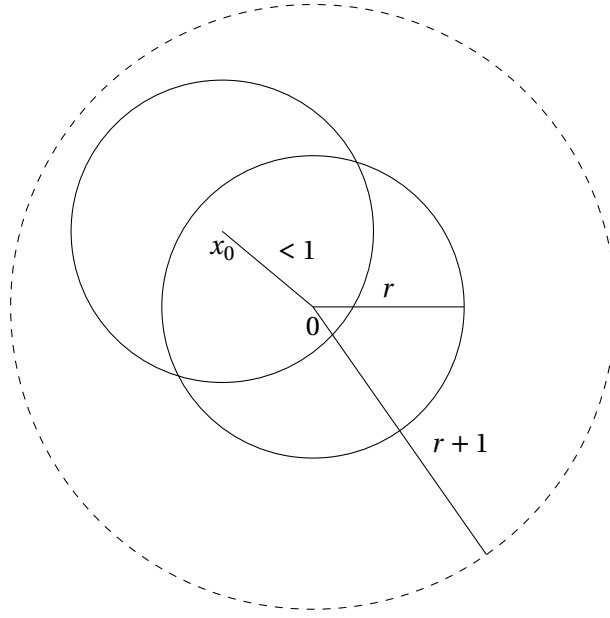Let us prove that g is continuous.
Let $\varepsilon > 0$,

$$|g(x) - g(x_0)| \le \int_{\mathbb{R}^n} \Phi(y)|h(x-y) - h(x_0 - y)|\, dy$$

Without loss of generality, h has compact support on $B(0, r)$ for some radius r.
Therefore $h(x, y)$ has compact support on $B(x, r)$ and $h(x_0, y)$ has compact support on $B(x_0, r)$.

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Consider $|x - x_0| < 1$, then $|h(x - y) - h(x_0 - y)|$ has compact support on $B(x_0, r + 1)$. Then

$$|g(x) - g(x_0)| \leq \int_{B(x_0, r+1)} \Phi(y)|h(x - y) - h(x_0 - y)| \, dy$$

Since h is continuous on a compact domain, it is uniformly continuous.
Therefore $\exists \delta > 0$ such that $|w - z| < \delta \implies |h(w) - h(z)| < \epsilon$.
Set $w = x - y$ and $z = x_0 - y$ such that $|w - z| = |x - x_0| < \delta$, then $|h(x - y) - h(x_0 - y)| < \epsilon$. Thus,

$$|g(x) - g(x_0)| \leq \varepsilon \int_{B(x_0, r+1)} \Phi(y) \, dy$$

Part B

$-\Delta u = f$
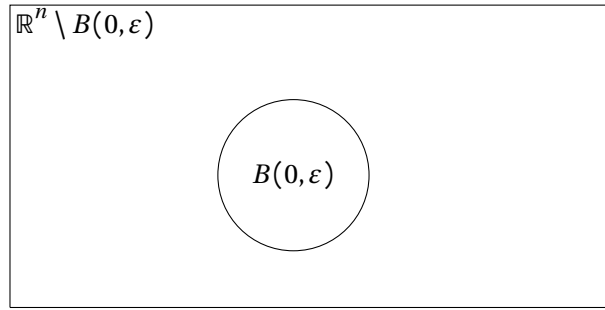Letting $\varepsilon > 0$ and taking the Laplacian of both sides,

$$\Delta_x u(x) = \int_{\mathbb{R}^n} \Delta_x f(x - y)\Phi(y) \, dy$$

$$= \overbrace{\int_{B(0,\varepsilon)} \Delta_x f(x - y)\Phi(y) \, dy}^{I_\varepsilon} + \overbrace{\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Delta_x f(x - y)\Phi(y) \, dy}^{J_\varepsilon}$$
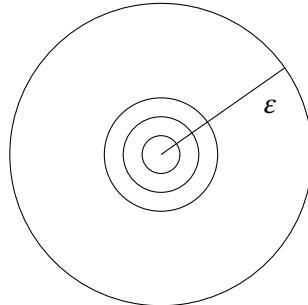
Then

$$|I_\varepsilon| \le \int_{B(0,\varepsilon)} |\Delta_x f(x-y)| \Phi(y) \, dy$$

$$\le ||\overset{2}{\nabla} f||_{L^\infty} \int_{B(0,\varepsilon)} \Phi(y) \, dy$$

$$\le c \int_0^\varepsilon \int_{\partial B(0,r)} \Phi(y) \, dS(y) \, dr$$

$$\le c \int_0^\varepsilon \int_{\partial B(0,r)} \frac{1}{|y|^{n-2}} \, dS(y) \, dr$$

$$= c \int_0^\varepsilon \int_{\partial B(0,r)} \frac{1}{r^{n-2}} \, dS(y) \, dr$$

$$= c \int_0^\varepsilon \frac{1}{r^{n-2}} \int_{\partial B(0,r)} \, dS(y) \, dr$$

$$\le c \int_0^\varepsilon \frac{r^{n-1}}{r^{n-2}} \, dr$$

$$c \int_0^\varepsilon r \, dr = c\varepsilon^2$$

$\mathbb{R}^n \setminus B(0,\varepsilon)$

$B(0,\varepsilon)$

As an exercise, attempt the same argument with $n = 2$.
Therefore,

$$\Delta_x u = I_\varepsilon + J_\varepsilon$$

and $\lim_{\varepsilon \to 0} I_\varepsilon = 0$.
Now, we need to control J_ε.

$$J_\varepsilon = \int_{\mathbb{R}^n} \Delta_x f(x-y) \Phi(y) \, dy$$

$$\Delta_x f(x-y) = \sum \frac{\partial^2 f}{\partial x^2} f(x-y)$$

$$\frac{\partial f}{\partial x}(x-y) = \nabla f|_{z=(x-y)} \cdot e_i = \frac{\partial f}{\partial z_i}|z = (x-y)$$

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{\partial^2 f}{\partial z_i^2}|_{z=(x-y)}$$

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$$\Delta_y f(x-y) = \sum \frac{\partial f^2}{\partial y_i}(x-y)$$

$$\frac{\partial f}{\partial y_i}(x-y) = \nabla f|_{z=(x-y)} \cdot -e_i = -\frac{\partial f}{\partial z_i}|_{z=(x-y)}$$

$$\frac{\partial^2 f}{\partial y_i^2} = \frac{\partial^2}{\partial y_i^2}|_{z=x-y}$$

So

$$J_\varepsilon = \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Delta_y f(x-y)\Phi(y)\, dy$$

$$= \overbrace{-\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \nabla_y f(x-y)\nabla\Phi(y)\, dy}^{K_\varepsilon} + \overbrace{\int_{\partial(\mathbb{R}^n \setminus B(0,\varepsilon))} \frac{\partial f}{\partial\eta}\Phi(y)\, dS(y)}^{L_\varepsilon}$$

and

$$\Delta_x u = I_\varepsilon + K_\varepsilon + L_\varepsilon$$

To control L_ε, since

$$|L_\varepsilon| \le \int_{\partial B(0,\varepsilon)} \frac{|\partial f|}{\partial\eta}\Phi(y)\, dy$$

$$\le \int_{\partial B(0,\varepsilon)} |\nabla f|\Phi(y)\, dy$$

$$\le ||\nabla f||_{L^\infty} \int_{\partial B(0,\varepsilon)} \Phi(y)\, dy$$

$$\le c \int_{\partial B(0,\varepsilon)} \frac{1}{|y|^{n-2}}\, dy$$

$$= \frac{c}{\varepsilon^{n-2}} \int_{\partial B(0,\varepsilon)} dy$$

$$\le \frac{c\varepsilon^{n-1}}{\varepsilon^{n-2}}$$

$$= c\varepsilon$$

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and K_ε, since $\frac{\partial \Phi}{\partial \eta} = \nabla \phi \cdot \eta = \frac{-y}{n\alpha(n)|y|^n} \cdot \eta = \frac{|y|^2}{n\alpha(n)|y|^n|y|} = \frac{1}{n\alpha(n)|y|^{n-1}}$
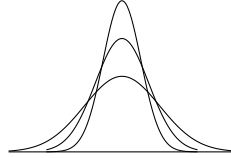
$$|K_\varepsilon| = - \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \nabla_y f(x-y) \, \nabla_y \Phi(y) \, dy$$

$$= \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} f(x-y) \overbrace{\Delta_y \Phi(y)}^{0} \, dy - \int_{\partial(\mathbb{R}^n \setminus B(0,\varepsilon))} f(x-y) \frac{\partial \Phi}{\partial \eta}$$

$$= - \int_{\partial B(0,\varepsilon)} f(x-y) \frac{\partial \Phi}{\partial \eta}$$

$$= - \int_{\partial B(0,\varepsilon)} f(x-y) \frac{1}{n\alpha(n)|y|^{n-1}} \, dS(y)$$

$$= - \frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\delta B(0,\varepsilon)} f(x-y) \, dS(y)$$

$$= - \underbrace{\frac{1}{n\alpha(n)\varepsilon^{n-1}}}_{\text{volume}} \int_{\partial B(0,\varepsilon)} f(z) \, dS(z)$$

$$= \frac{1}{|\partial B(0,\varepsilon)|} \int_{\partial B(0,\varepsilon)} f(z) \, dz$$

$$= - \fint_{\partial B(x,\varepsilon)} f(z) \, dz$$

Laplacian as a Distribution

$-\Delta \Phi(y) = \delta(y)$
Define the Dirac delta "function" as

$$\delta(y) = \begin{cases} \infty & y = 0 \\ 0 & y \neq 0 \end{cases}$$

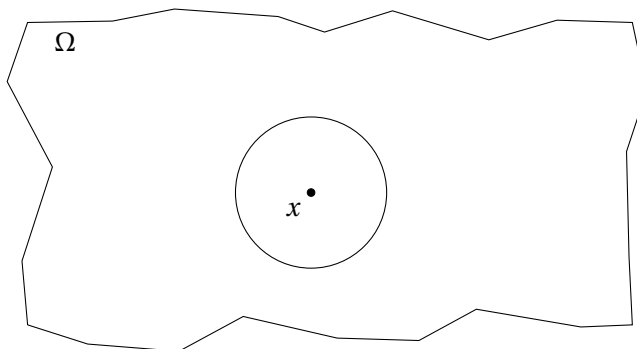such that $\int_{\mathbb{R}^n} \delta = 1$.
Translate the Dirac delta as

$$\delta_x = \begin{cases} \infty & y = x \\ 0 & y \neq x \end{cases}$$

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and recall that

$$\Delta u(x) = \Delta\left(\int_{\mathbb{R}^n} \Phi(x-y)f(y)\,dy\right)$$

$$= \int_{\mathbb{R}^n} \overbrace{\Delta\Phi(x-y)}^{-\delta_x(y)}f(y)\,dy$$

$$= -\int_{\mathbb{R}^n} \delta_x(y)f(y)\,dy$$

$$= -\int_{\mathbb{R}^n} \delta_x(y)f(x)\,dy$$

$$= -f(x)\overbrace{\int_{\mathbb{R}^n} \delta_x(y)\,dy}^{1}$$

$$= -f(x)$$

Harmonic Functions

Suppose u is harmonic

$u : \Omega \to \mathbb{R}^n$ harmonic.

Mean-value Formulas

Let U be an open set in $\mathbb{R}^n$, $u : U \to \mathbb{R}$ such that $\Delta u = 0$ in U. Then

$$u(x) = \fint_{\partial B(0,r)} -u(y)\,dS(y)$$

$$= \fint_{B(x,r)} u(y)\,dy$$

where $B(x,r) \subseteq U$.
IMAGE HERE

Proof

Consider $\phi(r) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y)\,dS(y)$.
If $\phi'(r) = 0$, when we are done since that would make ϕ constant and $\phi(r) = \lim_{s\to 0}\phi(s) = u(x)$. Then

$$\phi(r) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y)\, dS(y)$$

$$\underset{y=x+rz}{=} \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(0,1)} u(x+rz)r^{n-1}\, dS(z)$$

$$= \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} u(x+rz)\, dS(z)$$

Therefore

$$\phi'(r) = \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} \nabla u(x+rz) \cdot z\, dS(z)$$

$$\underset{y=x+rz}{=} \frac{1}{|\partial B(0,r)|} \int_{\partial B(x,r)} \nabla u(y) \cdot \frac{y-x}{r}\, dS(y)$$

$$= \frac{1}{|\partial B(0,r)|} \int_{\partial B(x,r)} \nabla u(y) \cdot \eta\, dS(y)$$

$$= \frac{1}{|\partial B(0,r)|} \int_{\partial B(x,r)} \frac{\partial y}{\partial \eta}\, dS(y)$$

$$= \frac{1}{|\partial B(0,r)|} \int_{B(x,r)} \Delta u$$

$$= 0$$

January 22, 2024

Mean Value Formula

For $U \subseteq \mathbb{R}^n$, U open with $u : U \to \mathbb{R}$ such that $u \in C^2(U)$, $\Delta u = 0$, we have

$$u(x) \underset{(a)}{=} \fint_{\partial B(x,r)} u \underset{(b)}{=} \fint_{B(x,r)} u$$

for all $B(x,r) \subseteq U$.
Recall that (a) was proven above by setting $\phi(r) = \fint_{\partial B(r)} u(y)\, dS(y)$ and showing $\phi'(r) = 0$.
For (b), we again apply spherical coordinates such that

$$\int_{B(x,r)} u(y)\, dy = \int_0^r \int_{\partial B(x,s)} u(y)\, dS(y)\, ds$$

$$= \int_0^r |\partial B(x,s)| \overbrace{\fint_{\partial B(x,s)} u(y)\, dS(y)}^{u(x)}\, ds$$

$$= u(x) \int_0^r |\partial B(x,s)|\, ds$$

$$= u(x) \int_0^r n\alpha(n)s^{n-1}\, ds$$

$$= \frac{u(x)n\alpha(n)s^n}{n} \bigg|_0^r$$

$$= u(x) \overbrace{\alpha(n)r^n}^{|B(x,r)|}$$

Converse

Recall that

$$\phi'(r) = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u(y)\, dy$$

Suppose then that we do not know that $\Delta u = 0$ but we have

$$u(x) = \fint_{\partial B(x,r)} u, \quad \forall B(x,r) \subseteq U$$

Then, necessarily, $\Delta u = 0$ in U.

- Proof
  Suppose, for sake of contradiction, that $\Delta u \neq 0$. Then, without loss of generality, there exists $y \in U$ such that $\Delta u(x) > 0$ for $x \in B(y,n) \subseteq U$.
  IMAGE HERE

$$\phi(r) = \fint_{\partial B(x,r)} u(x)\, dS(x)$$

  and

$$\phi'(r) = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(y,r)} \Delta u(x)\, dS(x) > 0$$

  which contradicts $\phi'(x) = 0$.

Strong Maximum Principle

Let $U \subseteq \mathbb{R}^n$ be a bounded open set, $u \in C^2(U) \cap C(\overline{U})$, $\Delta u = 0$ on U. Then

1. $\max_{\overline{U}}(u) = \max_{\partial U}(u)$.

2. If U is connected and u has its maximum in an interior point, then u is constant on $\overline{U}$.

IMAGE HERE - 2

Proof of A

Since $\partial U \subseteq \overline{U}$, $\max_{\partial U}(u) \leq \max_{\overline{U}}(u)$.
Let $x_0 \in \overline{U}$ such that $u(x_0) = \max_{\overline{U}}(u)$.
IMAGE HERE - 4
So either $x_0 \in \partial U$ or $x_0 \in U$.
Let U' be the connected component which contains x_0. Then $x_0 \in U'$, so by part (b) u is constant on $\overline{U'}$. So

$$\max_{\overline{U}}(u) = u(x_0) = \max_{\partial U'}(u) \leq \max_{\partial U}(u)$$

Proof of B

Then there exists $x_0 \in U$ such that $\max_{\overline{U}}(u) = u(x_0) = M$.
Let us define $\Omega = \{ y \in U : u(y) = M \}$. Then

1. $\Omega \neq \emptyset$, $B \setminus x_0 \in \Omega$.

2. Ω open set.

IMAGE HERE - 3

1. Ω is closed, since $\Omega = u^{-1}(\{M\})$.

It follows that $\Omega = U$ and, therefore, $u(y) = M$, $\forall y \in U$.

- Proof of 2
  Let $y \in \Omega$, $y \in U$, $u(y) = M$. Then there exists $B(y, r) \subseteq U$, and

$$M = u(y) = \fint_{B(y,r)} u(x) \, dS(x) \leq M$$

Then

$$\fint_{B(y,r)} u(x) \, dx = M$$

so $u(x) = M$, $\forall x \in B(y, r)$ and, therefore $B(y, r) \subseteq \Omega$ and Ω is open.

Remark: Boundedness Is Important

1. Consider $f(x) = x$ on $\mathbb{R}_{\geq 0}$.

2. IMAGE HERE - 5

Remark: Strong Minimum Principle Is Equivalent

Consequences

1. Positivity of harmonic functions.

2. Uniqueness of the Poisson problem.

Corollary: Positivity of Harmonic Functions

Suppose that U is connected and $u : U \to \mathbb{R}$, $u \in C^2(U) \cap C(\overline{U})$ solves the following problem

$$\begin{cases} \Delta u = 0 & \text{on } U \\ u = g & \text{on } \partial U \end{cases}$$

If $g \geq 0$ on ∂U, then u is positive on U as long as g is positive in some point.

Proof

Assume $\exists x_0 \in \partial U$ where x_0 is the minimum. Then $u(x_0) = \min_{\overline{U}}(u)$ and, $\forall x \in U$,

$$0 \le u(x_0) = \min_{\overline{U}}(u) \le u(x)$$

so u is non-negative. If $u(x) = 0$, then $u(x_0) = 0$ and the minimum is achieved in the interior. That would mean $u(x) = 0$, $\forall x \in \overline{U} \supseteq \partial U$ and $g(x) = 0$, $\forall x \in \partial U$ which would be a contradiction.

Theorem: Uniqueness of the Poisson Problem

Suppose $U \subseteq \mathbb{R}^n$ is open, connected and bounded. Then, there exists at most one solution to

$$(*)\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

where $u \in C^2(U) \cap C(\overline{U})$.

Proof

Let u_1 and u_2 be two solutions of $*$.
Consider $w = u_1 - u_2$ and observe that

$$\Delta w = \Delta u_1 - \Delta u_2 = -f + f = 0, \quad \text{in } U$$

and $w|_{\partial U} = g - g = 0$ on ∂U. Therefore

$$\begin{cases} \Delta w = 0 & \text{in } U \\ w = 0 & \text{on } \partial U \end{cases}$$

By applying the minimum and maximum principles,

$$w(x_0) = \min_{\overline{U}}(w) \le w(x) \le \max_{U}(w) = w(x)$$

so $w(x) = 0$, $\forall x \in \overline{U}$ and therefore $u_1 = u_2$.

Example

Let's consider $f : \mathbb{C} \to \mathbb{C}$ analytic (i.e. $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $a_n, z \in \mathbb{C}$). Then

$$f(z) = u(z) + v(z)$$

If $\mathbb{C} \cong \mathbb{R}^2$,

$$f(x + y) = u(x, y) + v(x, y)$$

for $u : \mathbb{R}^2 \to \mathbb{R}$ and $v : \mathbb{R}^2 \to \mathbb{R}$.
Claim: u and v are Harmonic.

$$u(x, y) + v(x, y) = \sum_{n=0}^{\infty} a_n (x + iy)^n$$

and

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \sum_{n=1}^{\infty} a_n n |x + iy|^{n-1}$$

$$\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = \sum_{n=1}^{\infty} a_n n |x + iy|^{n-1} i$$

So

$$i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

Recall: Convolution and Smoothing

Let $U \subseteq \mathbb{R}^n$ be an open set.
For $\varepsilon > 0$, define $U_\varepsilon = \{x \in U : d(x, \partial U) > \varepsilon\}$.
IMAGE HERE - 6
Define

$$\eta(x)\begin{cases} ce\left(\frac{1}{|x|^2 - 1}\right) & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

with c such that $\int_{\mathbb{R}^n} \eta(x)\, dx = 1$, $\eta \in C^\infty(\mathbb{R}^n)$
IMAGE HERE - 7
Note that $\text{supp}(\eta) = B(0, 1)$ and take

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right), \quad \eta_\varepsilon \in C^\infty(\mathbb{R}^n)$$

Then

$$\int_{\mathbb{R}^n} \eta_\varepsilon(x)\, dx = 1$$

and $\text{supp}(\eta_\varepsilon) \subseteq B(0, \varepsilon)$.
If f is locally integrable on U, define its mollification

$$f^\varepsilon = (x) = \int_U \eta_\varepsilon(x - y) f(y)\, dy \quad \forall x \in U_\varepsilon$$

January 24, 2024

Recall: Mollifiers

Define

$$\eta(x) = \begin{cases} ce\left(\frac{1}{|x|^2 - 1}\right) & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

where $\eta \in C^\infty(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \eta(x) = 1$ and supp$(\eta) \subseteq B(0,1)$.
Then for $\varepsilon > 0$, $\eta_\varepsilon(x) = \frac{1}{\varepsilon}\left(\frac{x}{\varepsilon}\right)$ where $\eta_\varepsilon \in C^\infty(\mathbb{R}^n)$.
So $\int_{\mathbb{R}^n} \eta_\varepsilon(x) = 1$ and supp$(\eta_\varepsilon) \subseteq B(0,\varepsilon)$
Given f locally summable; $f : U \to \mathbb{R}$,

$$f^\varepsilon(x) := \int_U \eta_\varepsilon(x-y)f(y)\,dy \quad x \in U_\varepsilon$$

$$= \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)f(y)\,dy \quad x \in U_\varepsilon$$

Properties

1. $f^\varepsilon \in C^\infty(U_\varepsilon)$.

2. $f^\varepsilon \xrightarrow[\varepsilon\to 0]{} f$ a.e.

3. If f continuous, then $f^\varepsilon \xrightarrow[\varepsilon\to 0]{} f$ uniformly on compact sets of U.

Theorem 6:

Let $u \in C(U)$ with $U \in \mathbb{R}^n$ open and such that u satisfies the mean-value property (i.e. $u(x) = \fint_{\partial B(x,r)} u(y)\,dS(y)$, $\forall B(x,r) \subseteq U$), then $u \in C^\infty$.

Corollary

If $u \in C^2(U)$ is harmonic, then $u \in C^\infty(U)$.

Proof

Let us take $x_0 \in U$
IMAGE HERE - 1
Notice, that if we prove that $u = u_\varepsilon$ on U_ε then we are done.
Let $x \in U_\varepsilon$, and noticing that $\eta(x) = \eta(|x|)$,

$$u_\varepsilon(x) = \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)u(y)\,dy$$

$$= \frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} \eta \frac{|x-y|}{\varepsilon} u(y)\,dy$$

$$\underset{\text{spherical}}{=} \frac{1}{\varepsilon^n} \int_0^\varepsilon \int_{\partial B(x,r)} \eta \overbrace{\frac{|x-y|}{\varepsilon}}^{r} u(y)\,dS(y)dr$$

$$= \frac{1}{\varepsilon^n} \int_0^\varepsilon \eta \frac{r}{\varepsilon} \int_{\partial B(x,r)} u(y)\,dS(y)dr$$

$$= \frac{1}{\varepsilon n} \int_0^\varepsilon \eta \frac{r}{\varepsilon} \underbrace{|\partial B(x,r)|}_{|\partial B(0,r)|} u(x)\,dr$$

$$= \frac{u(x)}{\varepsilon^n} \int_0^r \eta \frac{r}{\varepsilon} \int_{\partial B(0,r)} dS(y)dr$$

$$= u(x) \int_0^\varepsilon \frac{1}{\varepsilon^n} \eta \frac{r}{\varepsilon}\,dS(y)dr$$

$$= u(x) \overbrace{\int_{B(0,\varepsilon)} \eta_\varepsilon(y)\,dy}^{1} = u(x)$$

Theorem 7: Local Estimates of Harmonic Functions

Suppose $u \in C^2(U)$ a harmonic function.
Then $|D^\alpha u(x_0)| \le \frac{C_k}{r^{n+k}}||u||_{L^1(B(x_0,r))}$, $B(x_0,r) \subseteq U$, where α is a multiindex of order k, $C_0 = \frac{1}{\alpha(n)}$ and $C_k = \frac{(2^{n+1}nk)^k}{\alpha(n)}$ for $k = 1,2,\dots$.
We may take α since, by previous theorem, $u \in C^\infty(U)$.

Proof

By induction.
Consider $k = 0$.

$$u(x_0) = \fint_{B(x_0,r)} u(y)\,dy$$

$$= \frac{1}{\alpha(n)r^n} \int_{B(x_0,r)} u(y)\,dy$$

$$|u(x_0)| \le \frac{1}{\alpha(n)r^n} \int_{B(x_0,r)} |u(y)|\,dy$$

$$= \frac{C_0}{r^n}||u||_{L^1(B(x_0,r))}$$

For $k = 1$, if $|\alpha| = k = 1$ then $D^\alpha u(X_0) = \frac{\partial u}{\partial x_i}(x)$ for $i = 1,2,\dots$.
Notice that $\frac{\partial u}{\partial x_i}$ is also harmonic.

$$\Delta \frac{\partial u}{\partial x_i} = \sum_{t=1}^n \frac{\partial^2}{\partial x_t^2} \frac{\partial u}{\partial x_i}$$

$$= \frac{\partial}{\partial x_i} \underbrace{\sum_{t=1}^\infty \frac{\partial^2 u}{\partial x_t^2}}_{0}$$

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Applying the mean-value formula to $\frac{\partial u}{\partial x_i}(x_0)$ in $B(x, r/2)$.

IMAGE HERE - 2

$$\frac{\partial u}{\partial x_i}(x_0) = \fint_{B(x_0, r/2)} \frac{\partial u}{\partial x_i}(y)\, dy$$

$$= \frac{2^n}{\alpha(n) r^n} \int_{B(x_0, r/2)} \frac{\partial u}{\partial x_i}(y)\, dy$$

Recall $\int_\Omega w \Delta v = -\int_\Omega \nabla w \cdot \nabla v + \int_\Omega w \frac{\partial v}{\partial \eta}$.

$$\underset{e_i = \nabla y_i}{=} \frac{2^n}{\alpha(n) r^n} \int_{B(x_0, r/2)} \underbrace{\nabla u(y)}_{w} \cdot \underbrace{\nabla y_i}_{v}\, dy$$

$$\underset{IBP}{=} \frac{2^n}{\alpha(n) r^n} \left[-\int_{B(x_0, r/2)} u(y) \Delta y_i\, dy + \int_{\partial B(x_0, r/2)} u(y) \frac{\partial y_i}{\partial \eta} \right]$$

Note that

$$\frac{\partial y_i}{\partial \eta} = \nabla y_i \cdot \eta = e_i \cdot \eta = \eta_i$$

and

$$\left| \frac{\partial y_i}{\partial \eta} \right| = |\eta_i| \le |\eta| = 1$$

So,

$$\left| \frac{\partial u}{\partial x_i} x_0 \right| \le \frac{2^n}{\alpha(n) r^n} \int_{\partial B(x_0, r/2)} |u(y)|\, dS(y)$$

$$= \frac{2^n n \alpha(n) \left(\frac{r}{2}\right)^{n-1}}{\alpha(n) r^n} ||u||_{L^\infty(\partial B(x_0, r/2))}$$

$$= \frac{2n}{r} \underbrace{||u||_{L^\infty(\partial B(x_0, r/2))}}_{*}$$

Let's analyze $*$.

Let $x \in \partial B(x_0, r/2)$, then $B(x, r/2) \subseteq B(x_0, r)$.

IMAGE HERE - 3

Then we may apply $k = 0$.

$$|u(x)| \le \frac{C_0}{\left(\frac{r}{2}\right)^n} ||u||_{L^1(B(x, r/2))}$$

$$\le \frac{C_0}{\left(\frac{r}{2}\right)^n} ||u||_{L^1(B(x_0, r))}$$

Then

$$\left| \frac{\partial u}{\partial x_i}(x_0) \right| \le \frac{2n}{r} \frac{C_0}{\left(\frac{r}{2}\right)^n} ||u||_{L^1(B(x_0, r))}$$

$$= \frac{2^{n+1} n}{r^{n+1} \alpha(n)} ||u||_{L^1(B(x_0, r))}$$

HOMEWORK: Induct for arbitrary k.

Theorem 8: Liouville's Theorem

Suppose $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic and bounded. Then u is constant.

Proof

$$|D^\alpha u(x)| = \sqrt{\sum_{i=1}^n \left[\frac{\partial u}{\partial x_i}\right]^2} \leq \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}\right|$$

Let $r > 0$, $B(x, r) \subseteq \mathbb{R}^n$. Then, using estimates

$$\left|\frac{\partial u}{\partial x_i}(x)\right| \leq \frac{C_1}{r^{n+1}} ||u||_{L^1(B(x,r))}$$

Therefore,

$$
\begin{aligned}
|D^\alpha u(x)| &\leq \frac{nC_1}{r^{n+1}} ||u||_{L^1(B(x,r))} \\
&= \frac{nC_1}{r^{n+1}} \int_{B(x,r)} |u(y)|\, dy \\
&\leq \frac{nC_1}{r^{n+1}} ||u||_{L^\infty(B(x,r))} \alpha(n) r^n \\
&= \frac{C ||u||_{L^\infty(B(x,r))}}{r}
\end{aligned}
$$

and

$$\left|\frac{\partial u}{\partial x_i}(x)\right| \leq \frac{C ||u||_{L^\infty(B(x,r))}}{r}$$

$$\left|\frac{\partial u}{\partial x_i}(x)\right| \leq C ||u||_{L^\infty(B(x,r))} \lim_{r \to \infty} \frac{1}{r} \implies \frac{\partial u}{\partial x_i}(x) = 0 \implies u = Ck$$

Theorem: Representation Formula

Recall: $f \in C_c^2(\mathbb{R}^n)$, $(*) - \Delta u = f$ in $\mathbb{R}^n$, we saw that

$$\tilde{u}(x) : \int_{\mathbb{R}^n} \Phi(x - y) f(y)\, dy$$

solves $*$.
Let us consider $u \in C^2(\mathbb{R}^n)$ solving $-\Delta u = f$ for $n \geq 3$ where $f \in C_c^2(\mathbb{R}^n)$ and u is bounded.
Then $u(x) = \tilde{u}(x) + C = \int_{\mathbb{R}^n} \Phi(x - y) f(y)\, dy + C$.

Proof

Notice that if $\tilde{u}$ is bounded, then we are done. Because we may consider $w = u - \tilde{u}$ on $\mathbb{R}^n$ where

$$\Delta w = \Delta u - \Delta \tilde{u} = -f - (-f) = 0$$

Therefore w is bounded and, by Liouville's Theorem, $w = C$ and $u - \tilde{u} = c \iff u = \tilde{u} + C$.

$$|\tilde{u}(x)| \leq \int_{B(0,k)} \Phi(x - y) f(y)\, dy$$

$$\leq ||f||_{L^\infty(\mathbb{R}^n)} \int_{B(0,k)} \Phi(x - y)\, dy$$

If this is less than some C which does not depend on x, we are done.

Since $\Phi(x) \to 0$ for $|x| \to \infty$, for any C_1 $\exists \alpha$ such that if $|x| > \alpha$ then $|\Phi(x)| < C_1$.

IMAGE HERE - 4

$\text{dist}(x, B(0, k)) = \text{dist}(x, y_0)$ where $y_0 \in \overline{B(0, k)}$.

IMAGE HERE - 5

There are two cases.

- Case 1
  $\text{dist}(x, B(0, k)) \leq \alpha$
  $B(x, k) \subseteq B(0, \alpha + Ck)$
  Let $y \in B(x, k)$, then $|y - x| < k$ so $|x - y_0| < \alpha$.
  Therefore $|y - y_0| \leq k + \alpha \implies |y| \leq k + \alpha + |y_0| = 2k + \alpha \implies y \in B(0, 2k + \alpha)$. Then

$$||f||_{L^\infty(\mathbb{R}^n)} \int_{B(x,k)} \Phi(y)\, dy \leq ||f||_{L^\infty(\mathbb{R}^n)} \int_{B(0,\alpha+2k)} \Phi(y)\, dy$$

  HOMEWORK - Consider the other case.

January 29, 2024

Recall: Representation Formula

For $n \geq 3$.

$$\tilde{u}(x) : \int_{\mathbb{R}^n} \Phi(x - y) f(y)\, dy$$

It is sufficient to show that $\tilde{u}$ is bounded. Then

$$|\tilde{u}| \leq C \int_{B(0,k)} \Phi(x - y)\, dy$$

$\forall C_1$, $\exists \alpha$ such that $|z| \geq \alpha \implies |\Phi(z)| \leq C_1$.

Case 2

For $\text{dist}(x, B(0, k)) \geq \alpha$, $\text{dist}(x, y) \geq \alpha$, $\forall y \in B(0, k)$. Then

$$|x - y| \geq \alpha$$
$$\frac{1}{|x - y|} \leq \frac{1}{\alpha}$$
$$\frac{1}{|x - y|^{n-2}} \leq \frac{1}{\alpha^{n-1}}$$

and

$$|\tilde{u}(x)| \leq C \int_{B(0,k)} \frac{1}{|x - y|^{n-2}}\, dy \leq \frac{C}{\alpha^{n-2}} \int_{B(0,k)} dy$$

Theorem 10: Harmonic Implies Analytic

Let $U \subseteq \mathbb{R}^n$ open, $u \in C^2(U)$ harmonic. Then u is analytic in U.

Proof

Let $x_0 \in U$. We want to prove that the power series converges to $u(x)$ for x in a neighborhood around x_0.
Let $r = \text{dist}\left(x_0, \frac{\partial U}{4}\right)$, $M = \frac{1}{\alpha(n)r^n}||u||_{L^1(B(x_0,r))} \subset U$.
IMAGE HERE - 1
We want to analyze $x \in B(x_0, r)$.
Notice that $B(x, r) \leq B(x_0, 2r)$, and $z \in B(x, r)$ gives $|z - x| < r$ so

$$|z - x_0| \leq \underbrace{|z - x|}_{\leq r} + \underbrace{|x - x_0|}_{\leq r} \leq 2r$$

Applying estimates on $B(x, r)$, $|\alpha| = k$,

$$|D^\alpha u(x)| \leq \frac{C_k}{r^{n+k}}||u||_{L^1(B(x,r))}$$
$$\leq \frac{C_k}{r^{n+k}}||u||_{L^1(B(x_0,2r))}$$

and

$$\sup_{x \in B(x_0,r)} |D^\alpha u(x)| \leq \frac{\left(2^{n+1}nk\right)^k}{\alpha(n)r^{n+k}}||u||_{L^1(B(x_0,2r))}$$

Notice, by Stirling's approximation or Taylor expansion, $\frac{k^k}{k!} < e^k$, $\forall k \geq 1$. So

$$|\alpha|^{|\alpha|} < e^{|\alpha|}|\alpha|!$$

and

$$n^k = \underbrace{(1 + \cdots + 1)}_{n-\text{times}}^k = \sum_{|\beta|=k} \frac{|\beta|!}{\beta!} \geq \frac{|\alpha|!}{\alpha!}$$

where $|\alpha|! \leq \alpha!n^k$, $\beta = (\beta_1, \ldots, \beta_2)$ and $\beta! := \beta_1!\beta_2!\cdots\beta_n!$. Therefore

$$|\alpha|^{|\alpha|} \leq e^{|\alpha|}|\alpha|! \leq e^{|\alpha|}\alpha!n^k$$

and finally

$$(*) \quad k^k \leq e^k \alpha!n^k$$

Applying $*$ to the above inequality,

$$\sup_{X \in B(x_0,r)} |D^\alpha u(x)| \leq \frac{\left(2^{n+1}n\right)^k e^k \alpha!n^k}{\alpha(n)r^n r^k}||u||_{L^1(B(x_0,2r))}$$
$$\leq \left(\frac{2^{n+1}n^2 e}{r}\right)^k \cdot \alpha!M$$

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Let us analyze the Taylor expansion

$$\sum_{k=0}^{N} \sum_{|\alpha|=k} \frac{D^\alpha u(x_0)}{\alpha!} (x - x^0)^\alpha$$

Where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $y \in \mathbb{R}^n$ and $y^\alpha = y_1^{\alpha_1} \cdots y_n^{\alpha_n}$.

Pick $|x - x_0| \leq \frac{r}{2^{n+2} n^3 e}$. We want to prove that the remainder $R_N(x) \xrightarrow[N \to \infty]{} 0$.

$$R_N(x) = u(x) - \sum_{k=0}^{N-1} \sum_{|\alpha|=k} \frac{D^\alpha u(x_0)}{\alpha!} (x - x_0)^\alpha$$

$$= \sum_{|\alpha|=N} \frac{D^\alpha u(x_0 + t(x - x_0))(x - x_0)^\alpha}{\alpha!}, \quad \text{for some } |t| \leq 1$$

Using the remainder of the Taylor expansion with $g(t) = u(x_0 + t(x - x_0))$ for $g : I \to \mathbb{R}$.

Homework: show this around $t = 0$ at $t = 1$.

Note that $u(x_0 + t(x - x_0))$ describes a straight long with $t = 0 \implies u(x_0)$ and $t = 1 \implies u(x)$.

Notice also that $x_0 + t(x - x_0) \in B(x_0, r)$. Then, considering the superemum of the remainder,

$$|R_n(x)| \leq \sum_{|\alpha|=N} \left(\frac{2^{n+1} n^2 e}{r} \right)^N \cdot M\alpha! \cdot \frac{|(x - x_0)^\alpha|}{\alpha!}$$

Remark: for $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $y = (y_1, \ldots, y_n)$,

$$|y^\alpha| = |y_1^{\alpha_1} \cdots y_n^{\alpha_n}| \leq |y_1|^{\alpha_1} \cdots |y_n|^{\alpha_n}$$
$$\leq |y|^{\alpha_1} \cdots |y|^{\alpha_n}$$
$$= |y|^{\alpha_1 + \alpha_2 + \cdots + \alpha_n}$$
$$= |y|^\alpha$$

Therefore

$$|R_n(x)| \leq \sum_{|\alpha|=N} \left(\frac{2^{n+1} n^2 e}{r} \right)^N \cdot M |x - x_0|^N$$

$$\leq M \cdot \sum_{|\alpha|=N} \left(\frac{2^{n+1} n^2 e}{r} \right)^N \left(\frac{r}{2^{n+2} n^3 e} \right)^N$$

$$= M \cdot \sum_{|\alpha|=N} \left(\frac{1}{2n} \right)^N$$

$$\leq M \left(\frac{1}{2n} \right)^N \sum_{|\alpha|=N}$$

$$\leq M \left(\frac{1}{2n} \right)^N n^N$$

$$= M \left(\frac{1}{2} \right)^N$$

Note that $\sum_{|\alpha|=N} (1) \leq n^N$ since

$$\alpha = (\alpha_1, \ldots, \alpha_n) = (\alpha_{1_N}, \ldots, \alpha_{i_N}) = n^N$$

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Theorem 11: Harnack's Inequality

Define $V \subset\subset U$ as "V totally contained in U" meaning $\overline{V}$ compact and $V \subseteq \overline{V} \subseteq U$.
IAMGE HERE - 2
Let U open and $u \in C^2(U)$ harmonic and non-negative.
Then for each connected open set $V \subset\subset U$

$$\sup_V u \leq C \inf_V u$$

for some C that depends on V.

Remark

Then

$$\frac{1}{C} u(y) \leq u(x) \leq C u(y), \quad \forall x, y \in V$$

Since

$$u(x) \leq \sup_V u \leq C \inf_V u \leq C u(y)$$

and

$$\frac{1}{C} u(y) \leq \frac{1}{C} \sup_V u \leq \inf_V u \leq u(x).$$

Proof

Take $r = \frac{\text{dist}(v, \partial U)}{4} > 0$.

- Case 1
  Let us suppose that $x, y \in V$ such that $|x - y| < r$.
  IMAGE HERE - 3
  Notice $B(x, 2r) \subseteq U$. Applying mean-value formulas,

$$u(x) = \fint_{B(x,2r)} u = \frac{1}{\alpha(n)(2r)^n} \int_{B(x,2r)} u$$

  But notice that $B(y, r) \subseteq B(x, 2r)$, so

$$u(x) \geq \frac{1}{\alpha(n)2^n r^n} \int_{B(y,r)} u = \frac{1}{2^n} \fint_{B(y,r)} u = \frac{1}{2^n} u(y)$$

  That is, if $x, y \in V$ such that $|x - y| < r$, then $u(x) \geq \frac{1}{2^n} u(y)$ and, mutatis mutandis, $u(y) \geq \frac{1}{2^n} u(x)$.

- Case 2
  Let us cover $\overline{V}$ by an open covering of balls $\{B_i\}_{i=1}^N$ such that the radius of each ball is $\frac{r}{2}$ and $B_i \cap B_{i-1} \neq \emptyset$.
  IMAGE HERE - 4
  Then $u(x) \geq \frac{1}{2^n} u(z) \frac{1}{2^n 2^n} u(y)$, so $u(x) \geq \frac{1}{2^{2n}} u(y)$.

In the same way, $u(y) \geq \frac{1}{2^{2n}} u(x)$.

IMAGE HERE - 5

For three balls, $u(x) \geq \frac{1}{2^{3n}} u(y)$ and $u(y) \geq \frac{1}{2^{3n}} u(x)$.

Since we have a finite covering of N balls, the same strategy gives

$$u(x) \geq \frac{1}{2^{Nn}} u(y) \qquad\qquad u(y) \geq \frac{1}{2^{Nn}} u(x)$$

and

$$\frac{1}{2^{Nn}} \leq u(x)$$

Taking the supremum $y \in V$;

$$\sup_{y \in V} u(y) \leq 2^{Nn} u(x)$$

taking the infemum $x \in V$

$$\inf_{x \in V} u(y)$$

Recap: Laplace Equation

- Fundemental Solution

    - Poisson Equation in $\mathbb{R}^n$

- Mean-value Formulas

- Properties

    - Strong Maximum / Minimum Principles

        * Uniqueness of the Poisson Equation on Bounded Domains

    - Regularity

    - Derivative Estimates

    - Liouville's Theorem

        * Representation Formula

            · Uniqueness of the Poisson Equation up to a Constant on $\mathbb{R}^n$ for Bounded Functions

    - Analyticity

    - Harnack's Inequality

Green's Functions

For U open and bounded, $\partial U \in C^1$.
Goal: We want to solve $-\Delta u = f$ on U and $u = g$ on ∂U.

Recall: Green's Formula

$$\int_\Omega u\Delta v - v\Delta u = \int_{\partial\Omega} u\frac{\partial v}{\partial\eta} - v\frac{\partial u}{\partial\eta}$$

Obtaining Green's Formula

Let $x \in U$ and consider $u(y)$, $\Phi(y-x)$ as functions of y.
Let $\varepsilon > 0$ and consider $V_\varepsilon = U \setminus B_\varepsilon(x)$. Applying Green's formula; $\Omega = V_\varepsilon$,

$$\int_{V_\varepsilon} u(y)\underbrace{\Delta_y\Phi(y-x)}_{=0} - \Phi(y-x)\Delta_y u = \int_{\partial V_\varepsilon} u(y)\frac{\partial\Phi(y-x)}{\partial\eta} - \Phi(y-x)\frac{\partial u(y)}{\partial\eta}$$

IMAGE HERE - 6

January 31, 2024

Green's Functions

Goal is to solve for $U \subseteq \mathbb{R}^n$ open and bounded,

$$\begin{cases} -\Delta u = f, & \text{in } U \\ \quad u = g, & \text{on } \partial U \end{cases}$$

by obtaining Green's function.
Let $x \in U$ and assume $u \in C^2(U)$, and consider $u(y)$ and $\Phi(y-x)$.
Recall Green's formula $\int_\Omega u\Delta v - v\Delta u = \int_{\partial\Omega} u\frac{\partial u}{\partial\eta} - v\frac{\partial v}{\partial\eta}$.
Then, let $\varepsilon > 0$ and define $V_\varepsilon U \setminus B(x,\varepsilon)$.
IMAGE HERE - 1
By applying Green's Formula,

$$\int_{V_\varepsilon} u(y)\underbrace{\Delta\Phi(y-x)}_{0} - \Phi(y-x)\Delta u(y) = \int_{\partial V_\varepsilon} \overbrace{u\underbrace{\frac{\partial\Phi(y-x)}{\partial\eta}}_{\square_1}}^{\square} - \underbrace{\Phi(y-x)\frac{\partial u}{\partial\eta}}_{\square_2}$$

Notice that $\partial V_\varepsilon = \partial U \cup \partial B(x,\varepsilon)$.
Let us analyze $\square$ along $\partial B(x,\varepsilon)$
For $\square_2$ along $\partial B(x,\varepsilon)$,

$$\left| \int_{\partial B(x,\varepsilon)} \Phi(y-x)\frac{\partial u(y)}{\partial\eta} \right| \leq \sup_{\overline{U}} |\nabla U| \int_{\partial B(x,\varepsilon)} \Phi(y-x)\, dS(y)$$

$$= \frac{C}{\varepsilon^{n-2}} \int_{\partial B(x,\varepsilon)} dS(y)$$

$$= \frac{C\varepsilon^{n-1}}{\varepsilon^{-2}}$$

$$= c\varepsilon$$

Then $\lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} \square_2 = 0$.

Now, for $\square_1$ along $\partial B(x,\varepsilon)$ and recalling $\nabla \Phi(z) = \frac{-z}{n\alpha(n)|z|^n}$ while $\eta(z) = \frac{-z}{|z|}$ such that

$$\frac{\partial \Phi}{\partial \eta}(z) = \nabla \Phi \cdot \eta = \frac{|z|^2}{n\alpha(n)|z|^{n+1}} = \frac{1}{n\alpha(n)|z|^{n-1}}$$

we have

$$
\begin{aligned}
\int_{\partial B(x,\varepsilon)} u(y)\frac{\partial \Phi(y-x)}{\partial \eta}\, dS(y) &\underset{z=y-x}{=} \int_{\partial U(0,\varepsilon)} u(z+x)\frac{\partial \Phi(z)}{\partial \eta}|z|\, ds(z) \\
&= \frac{1}{n\alpha(n)} \int_{\partial B(0,\varepsilon)} \frac{u(z+x)}{|z|^{n-1}}\, dS(z) \\
&= \frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B(0,\varepsilon)} u(z+x)\, dS(z) \\
&\underset{y=z+x}{=} \frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B(x,\varepsilon)} u(y)\, dS(y) \\
&= \fint_{\partial B(x,\varepsilon)} u(y)\, dS(y)
\end{aligned}
$$

Then $\lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} \square_1 = u(x)$. It follows, then, that

$$
\int_U -\Phi(y-x)\Delta u(y) = \int_{\partial U} \overbrace{u\frac{\partial \Phi(y-x)}{\partial \eta}}^{\square_1} \underbrace{- \Phi(y-x)\frac{\partial u}{\partial \eta}}_{\square_2} + u(x)
$$

That is

$$
u(x) \underset{\square_4}{=} - \int_U \Phi(y-x)\Delta u + \int_{\partial u} \Phi(y-x)\frac{\partial u}{\partial \eta} - u\frac{\partial \Phi(y-x)}{\partial \eta}
$$

Notice that we have $-\Delta u = f$ in U and $u = g$ on ∂U, but we will need $\frac{\partial u}{\partial \eta}|_{\partial U}$.

Definition: Corrector Function

Given a domain $U \subseteq \mathbb{R}^n$ open and bounded with $x \in U$, define the function $\phi^x(y)$ that satisfies the following

$$
\begin{cases}
\Delta \phi^x(y) = 0, & \text{in } U \\
\phi^x(y) = \Phi(y-x), & \text{on } y \in \partial U
\end{cases}
$$

Note that we do not know that such a function exists.

Green's Function Continued

Suppose that we have $\phi^x(y)$. Then, applying green's formula for $u(y)$ and $\phi^x(y)$,

$$
\int_U u\underbrace{\Delta \phi^x(y)}_{0} - \phi^x(y)\Delta u = \int_{\partial U} u\frac{\partial \phi^x(y)}{\partial \eta} - \underbrace{\phi^x(y)\frac{\partial u}{\partial \eta}}_{\Phi(y-x)\frac{\partial u}{\partial \eta}}
$$

Then

$$\int_{\partial U} \Phi(y-x)\frac{\partial u}{\partial \eta} \overset{\square_3}{=} \int_{\partial U} u\frac{\partial \phi^x(y)}{\partial \eta} + \int_U \phi^x(y)\Delta u$$

Replacing $\square_3$ in $\square_4$,

$$u(x) = -\int_U \Phi(y-x)\Delta u + \int_{\partial U} u\frac{\partial \phi^x(y)}{\partial \eta} + \int_U \phi^x(y)\Delta u - \int_{\partial U} u\frac{\partial \Phi(y-x)}{\partial \eta}$$

and, therefore,

$$u(x) = -\int_U \Delta u\left[\Phi(y-x) - \phi^x(y)\right] - \int_{\partial U} u\frac{\partial}{\partial \eta}\left[\Phi(y-x) - \phi^x(y)\right]$$

Definition: Green's Function

Given a domain $U \subseteq \mathbb{R}^n$, the Green's function for $x \in U$ is defined by

$$G(x,y) := \Phi(y-x) - \phi^x(y)$$

Theorem: Representation Formula

Suppose $U \subseteq \mathbb{R}^n$, and $u \in C^2(U)$ that solves

$$\begin{cases} -\Delta u = f, & \text{in } U \\ \quad u = g, & \text{on } \partial U \end{cases}$$

Then,

$$u(x) = \int_U fG(x,y) - \int_{\partial U} g\frac{\partial G(x,y)}{\partial \eta}$$

Interpretation of the Green's Functions

$$\Delta_y G(x,y) = \Delta_y \Phi(y-x) - \underbrace{\Delta_y \phi^x(y)}_{0} = \delta^x(y)$$

and

$$G(x,y) = \Phi(y-x) - \phi^x(y) = 0, \quad y \in \partial U$$

That is, it is the Dirac delta on the interior which vanishes at the boundary.

Theorem: Symmetry of the Green's Function

For all $x, y \in U$, $x \neq y$, we want to show that $G(x,y) = G(y,x)$.

Proof

Let $x, y \in U$, $x \neq y$.

Define $V(z) := G(x, z)$ and $W(z) := G(y, z)$.

Notice that $\Delta_z V = 0$ for $z \neq x$ and $\Delta_z W = 0$ for $w \neq y$ and $V(z) = W(z) = 0$ for $z \in \partial U$.

IMAGE HERE - 2

Then, let us consider $\varepsilon > 0$ and

$$\Omega_\varepsilon := U \setminus \left[B(x, \varepsilon) \bigsqcup B(y, \varepsilon) \right]$$

Then

$$0 = \int_{\Omega_\varepsilon} W \underbrace{\Delta V}_{0} - V \underbrace{\Delta W}_{0} = \int_{\partial \Omega_\varepsilon} W \frac{\partial V}{\partial \eta} - V \frac{\partial W}{\partial \eta}$$

$$= \int_{\partial U} W \frac{\partial V}{\partial \eta} - V \frac{\partial W}{\partial \eta} + \int_{\partial B(x,\varepsilon)} W \frac{\partial V}{\partial \eta} - V \frac{\partial W}{\partial \eta} + \int_{\partial B(y,\varepsilon)} W \frac{\partial V}{\partial \eta} - V \frac{\partial W}{\partial \eta}$$

It follows that

$$\underbrace{\int_{\partial B(x,\varepsilon)} W \overbrace{\frac{\partial V}{\partial \eta}}^{(a)} - V \overbrace{\frac{\partial W}{\partial \eta}}^{(b)}}_{\heartsuit_1} = \underbrace{\int_{\partial B(y,\varepsilon)} V \frac{\partial W}{\partial \eta} - W \frac{\partial V}{\partial \eta}}_{\heartsuit_2}$$

Let us analyze (b), fixing $\varepsilon_0 > 0$ such that $\varepsilon < \varepsilon_0$

$$\left| \int_{B(x,\varepsilon)} V \frac{\partial W}{\partial \eta} \right| \leq \sup_{z \in \partial B(x,\varepsilon)} |V(z)| \int_{B(x,\varepsilon)} \left| \frac{\partial W}{\partial \eta}(z) \right| dS(z)$$

$$\leq \sup_{z \in \partial B(x,\varepsilon_0)} |\nabla W(z)| \int_{\partial B(x,\varepsilon)} dS(z)$$

$$\leq C\varepsilon^{n-1} \sup_{z \in \partial B(x,\varepsilon)} |V(z)|$$

$$\leq C\varepsilon^{n-1} \left(\frac{C}{\varepsilon^{n-2} + C} \right)$$

$$= C\varepsilon + C\varepsilon^{n-1}$$

Since, given $z \in \partial B(x, \varepsilon)$,

$$V(z) = G(x, z) = \Phi(z - x) - \phi^x(z)$$

we have

$$|V(z)| \leq |\Phi(z - x)| + |\phi^x(z)|$$

$$\leq \frac{C}{\varepsilon^{n-2}} + \sup_{z \in B(x,\varepsilon_0)} |\phi^x(z)|$$

Thus, we have $\lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} V \frac{\partial W}{\partial \eta} = 0$.

Let us analyze (a),

$$\int_{\partial B(x,\varepsilon)} W(z) \frac{\partial V}{\partial \eta}(z) \, dS(z) = \int_{\partial B(x,\varepsilon)} W(z) \left[\frac{\Phi(z-x)}{\partial \eta} - \frac{\partial \phi^x(z)}{\partial \eta} \right] dS(z)$$

$$= \int_{\partial B(x,\varepsilon)} \overbrace{W(z) \frac{\partial \Phi(z-x)}{\partial \eta}}^{(e)} - \overbrace{W(z) \frac{\partial \phi^x(z)}{\partial \eta}}^{(h)} \, dS(z)$$

Analyzing (h),

$$\left| \int_{\partial B(x,\varepsilon)} W(z) \frac{\partial \phi^x(z)}{\partial \eta} \right| \leq \sup_{\partial B(x,\varepsilon_0)} |\nabla \phi^x(z)||W(z)| \int_{\partial B(x,\varepsilon)} dS(z)$$

$$= C\varepsilon^{n-1}$$

Then $\lim_{\varepsilon \to 0} h = 0$ and

$$\lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} W(z) \frac{\partial \Phi(z-x)}{\partial \eta} = W(x)$$

So $\lim_{\varepsilon \to 0}(a) = W(x)$. Then

$$\lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} W \frac{\partial V}{\partial \eta} - V \frac{\partial W}{\partial \eta} = W(x)$$

Applying the same process,

$$\lim_{\varepsilon \to 0} \int_{\partial B(y,\varepsilon)} V \frac{\partial W}{\partial \eta} - W \frac{\partial V}{\partial \eta} = V(y)$$

Therefore $W(x) = V(y)$ and $G(y,x) = G(x,y)$.

Definition: Half Space

Define the half space $\mathbb{R}_+^n = \{(x_1,\ldots,x_n) : x_n > 0$.
IMAGE HERE - 3

Definition: Reflection

For a $x = (x_1,\ldots,x_n) \in \mathbb{R}_+^n$, define its reflection $\tilde{x} = (x_1,\ldots,-x_n)$.

Green's Function in the Half Space

We want to find $\phi^x(y)$ that solves

$$(*) \begin{cases} \Delta \phi^x(y) = 0, & \text{in } \mathbb{R}_+^n \\ \phi^x(y) = \Phi(y-x), & y \in \partial \mathbb{R}_+^n \end{cases}$$

Let us consider $\phi^x(y) := \Phi(y - \tilde{x})$, $x, y \in \mathbb{R}_+^n$. Then $\phi^x(y)$ satisfies $*$.
Then we can see that $\Delta \phi^x(y) = 0$.

Let $y \in \partial \mathbb{R}^n_+$ such that $y = (y_1, \ldots, y_{n-1}, 0)$. So

$$
\begin{aligned}
\phi^x(y) &= \Phi(y - \tilde{x}) \\
&= \Phi(|y - \tilde{x}|) \\
&= \Phi\left(\sqrt{(y_1 - x_1)^2 + \cdots + (y_{n-1} - x_{n-1})^2 + (0 + x^n)^2}\right) \\
&= \Phi\left(|y - x|^2\right) \\
&= \Phi(y - x)
\end{aligned}
$$

February 5, 2024

Recall: Green's Function

$G(x, y) = \Phi(y - x) - \phi^x(y)$.
For $U \subset \mathbb{R}^n_+$, when

$$
G(x, y) = \Phi(y - x) - \Phi(y - \tilde{x})
$$

we proved that if $u \in C^2(\overline{U})$,

$$
\begin{cases}
-\Delta u = f, & U \\
u = g, & \partial U
\end{cases}
$$

then

$$
u(x) = \int_U f G(x, y) \, dy - \int_{\partial U} g \frac{\partial G(x, y)}{\partial \eta} \, dS(y)
$$

Let us consider

$$
\begin{cases}
\Delta u = 0, & \mathbb{R}^n_+ \\
u = g, & \partial \mathbb{R}^n_+
\end{cases}
$$

such that

$$
u(x) = - \int_{\partial \mathbb{R}^n_+} g \frac{\partial G(x, y)}{\partial \eta} \, dS(y)
$$

Let us compute $\frac{\partial G}{\partial \eta}$.
IMAGE HERE - 1 UPPER HALF SPACE WITH NORMAL VECOTR ETA
Recall

$$
\nabla \Phi(z) = \frac{-z}{n\alpha(n)|z|^n}
$$
$$
\frac{\partial \Phi(z)}{\partial z_n} = \frac{-z_n}{n\alpha(n)|z|^n}
$$

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so, since $y - \tilde{x}_n = y_n + x_n$,

$$\frac{\partial G}{\partial \eta} = \nabla G(x, y) \cdot \eta$$

$$= -\frac{\partial G(x, y)}{\partial y_{n+1}}$$

$$= -\frac{\partial}{\partial y_{n+1}} (\Phi(y - x) - \Phi(y - \tilde{x}))$$

$$= -\left[\frac{-(y_n - x_n)}{n\alpha(n)|y - x|^n} - \frac{-(y_n + x_n)}{n\alpha(n)|x - \tilde{x}|^n} \right]$$

But recall that if $y \in \partial\mathbb{R}_+^n$, $|y - x| = |y - \tilde{x}|$. Then $y \in \partial\mathbb{R}_+^n$,

$$\frac{\partial G(x, y)}{\partial \eta} = -\frac{1}{n\alpha(n)|y - x|^n} [-y_n + x_n + y_n + x_n] = -\frac{2x_n}{n\alpha(n)|y - x|^n}$$

Then

$$u(x) = \int_{\partial\mathbb{R}_+^n} \frac{g(y)2x_n}{n\alpha(n)|y - x|^n} \, dS(y)$$

Definition: Poisson Kernel

$$K(x, y) = \frac{2x_n}{n\alpha(n)|y - x|^n} = \frac{\partial G}{\partial y_n}$$

is called the Poisson Kernel and

$$u(x) \int_{\partial\mathbb{R}_+^n} g(x) K(x, y) \, dS(y)$$

is called the Poisson Formula.
Notice (HW): $\int_{\partial\mathbb{R}_+^n} K(x, y) \, dy = 1$, $\forall x \in \mathbb{R}_+^n$ (hint: apply spherical coordinates).

Theorem 14:

Define

$$(*) \quad u(x) = \int_{\partial\mathbb{R}_+^n} K(x, y) g(y) \, dS(y)$$

Suppose that $g \in C^\infty(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$.
Then

1. $u \in C^\infty(\mathbb{R}_+^n) \cap L^\infty(\mathbb{R}_+^n)$.

2. $\Delta u = 0$, $\mathbb{R}_+^n$.

3. $\lim_{x \to x^0} u(x) = g(x^0)$, $x \in \mathbb{R}_+^n$, $x^0 \in \partial\mathbb{R}_+^n$.

Proof

We know $G(x, y)$ satisfies

$$\Delta_y G(x, y) = \delta^x(y).$$

Notice that $y \to G(x, y)$ is harmonic for $x \neq y$.
Recall that $G(x, y) = G(y, x)$, so $x \to G(x, y)$ is harmonic for $x \neq y$.
Then $x \to \frac{\partial G(x,y)}{\partial y_n}$ is harmonic $(*_2)$ for $x \neq y$ and for $y \in \partial \mathbb{R}^n_+$.
Homework: compute this directly.
Noticing that K is smooth when $x \neq y$, then

$$\frac{\partial u}{\partial x_i} = \int_{\partial \mathbb{R}^n_+} \frac{\partial}{\partial x_i} K(x, y) g(y) \, dS(y)$$

Homework: justify puting the limit inside the integral.
Homework: prove that $\frac{\partial u}{\partial x_i}$ is continuous.
By repeatedly taking derivaties, we see $u \in C^\infty(\mathbb{R}^n_+)$.
Moreover,

$$\Delta_x u = \int_{\partial \mathbb{R}^n} \underbrace{\Delta_x K(x, y)}_{=0} g(y) \, dS(y) = 0$$

by $*_2$. Then

$$|u(x)| \leq \int_{\partial \mathbb{R}^n_+} |K(x, y)| |g(y)| \, dS(y) \leq ||g||_{L^\infty(\mathbb{R}^{n-1})} \underbrace{\int_{\partial \mathbb{R}^n_+} K(x, y) \, dS(y)}_{=1} < \infty$$

For part c, consider $x^0 \in \partial \mathbb{R}^n_+$ and $\varepsilon > 0$.
Since $g \in C^\infty(\mathbb{R}^{n-1})$, let $\delta > 0$ such that $|y - x^0| < \delta \implies |g(y) - g(x^0)| < \varepsilon$ for $y \in \partial \mathbb{R}^n_+$.
IMAGE HERE - 2 DELTA BALL AROUND X0 HALF DELTA BALL WITH X INSIDE
Now, let us consider $|x - x_0| < \frac{\delta}{2}$.

$$
\begin{aligned}
|u(x) - g(x^0)| &= \left| \int_{\partial \mathbb{R}^n_+} K(x, y) g(y) - K(x, y) g(x^0) \, dS(y) \right| \\
&\leq \int_{\partial \mathbb{R}^n_+} K(x, y) \left| g(y) - g(x^0) \right| \, dS(y) \\
&= \underbrace{\int_{\partial \mathbb{R}^n_+ \cap B(x^0, \delta)} K(x, y) |g(y) - g(x^0)| \, dS(y)}_{I} + \underbrace{\int_{\partial \mathbb{R}^n_+ \cap B^c(x^0, \delta)} K(x, y) |g(y) - g(x^0)| \, dS(y)}_{II}
\end{aligned}
$$

Then

$$I \leq \varepsilon \int_{\partial \mathbb{R}^n_+ \cap B(x^0, \delta)} K(x, y) \leq \varepsilon$$

Now, we want to control II

$$
\begin{aligned}
\int_{\partial \mathbb{R}^n_+ \cap B^c(x^0, \delta)} K(x, y) |g(y) - g(x^0)| \, dS(y) &\leq C ||g||_{L^\infty} \int_{\partial \mathbb{R}^n_+ \cap B^c(x^0, \delta)} \\
&= \frac{2C}{n\alpha(n)} \int_{\partial \mathbb{R}^n_+ \cap B^c(x^0, \delta)} \frac{x_n}{|x - y|^n} \, dS(y)
\end{aligned}
$$

We want to control $|x^0 - y|$ with something related to $|x - y|$.
We know $|y - x^0| > \delta$ and we will consider $|x - x^0| < \frac{\delta}{2}$. So

$$|y - x^0| \le |y - x| + |x - x^0| \le |y - x| + \frac{\delta}{2} \le |y - x| + \frac{|y - x^0|}{2}$$

So $\frac{|y - x_0|}{2} \le |y - x|$ implies that $\frac{1}{|y - x|^n} \le \frac{2^n}{|y - x_0|^n}$. Therefore

$$\int_{\partial \mathbb{R}^n_+ \cap B^c(x^0, \delta)} K(x, y)|g(y) - g(x^0)| \, dS(y) \le C x_n \int_{\partial \mathbb{R}^n_+ \cap B^c(x^0, \delta)} \frac{1}{|y - x^0|^n} \, dS(y)$$

$$= \int_\delta^\infty \int_{\partial B^{n-1}(x^0, r)} \frac{1}{r^n} \, dS(y) dr$$

$$= C \int_\delta^\infty \frac{1}{r^n} r^{n-2} \, dr$$

$$= C \int_\delta^\infty \frac{1}{r^2} \, dr$$

$$= C(\frac{1}{r})|_\delta^\infty$$

$$= \frac{C}{\delta}$$

Then $II \le \frac{C x_n}{\delta}$. Now let us consider $|x - x^0| < \frac{\delta}{J}$ where $\frac{1}{J} < \varepsilon$. Then

$$II \le \frac{C|x - x^0|}{\delta} \le C \frac{\delta}{\delta J} \le C\varepsilon$$

Energy Methods: Uniqueness

Consider the boundary value problem

$$(*) \quad \begin{cases} -\Delta u = f, & U, \, f \in C(U) \\ u = g, & \partial U, \, g \in C(\partial U) \end{cases}$$

with U open and bounded in $\mathbb{R}^n$, $u \in C^2(\overline{U})$ and $\partial U \in C^1$.

Theorem 16: Uniquness

There exists at most one solution $u \in C^2(\overline{U})$ for $*$.

Proof

Let us suppose that $\tilde{u}$ is another solution.
Then $w := u - \tilde{u}$ solves

$$\begin{cases} \Delta w = 0, & U, \, w \in C^2(\overline{U}) \\ w = 0, & \partial U \end{cases}$$

where

$$0 = \int_U w \Delta w = -\int_U |\nabla W|^2 + \int_{\partial U} w \frac{\partial w}{\partial \eta}$$

so

$$0 = \int_U |\nabla w|^2 \implies \nabla w = 0 \implies w = 0 \implies u = \tilde{u}$$

Definition: Energy Functional

Let us consider

$$A = \left\{ w \in C^2(\overline{U}) \ : \ W|_{\partial U} = g \right\}$$

for $g \in C(\partial U)$ and $f \in C(U)$.

Define the energy functional $I : A \to \mathbb{R}$ given by $I(w) := \int_U \frac{|\nabla w|^2}{2} - f w$.

Energy Methods: Dirichlet Principle

Calculus of variations applied to the Laplace equation.

Theorem:

Suppose $u \in C^2(\overline{U})$ is a solution to the problem

$$\square \quad \begin{cases} -\Delta u = f, & U \\ u = g, & \partial U \end{cases}$$

Then,

$$(*) \quad I(u) = \min_{w \in A}\{I(w)\}$$

Moreover, if $u \in A$ such that $*$ happens, then u satisfies $\square$.

Proof

$(\implies)$ For $w \in A$,

$$\begin{aligned}
0 &= \int_U \underbrace{(-\Delta u - f)}_{=0}(u - w) \\
&= \int_U -\Delta u (u - w) - \int_U f(u - w) \\
&= \int_U \nabla(u - w) \cdot \nabla u - \underbrace{\int_{\partial U} (u - w) \cdot \frac{\partial u}{\partial \eta}}_{=0} - \int_U f(u - w) \\
&= \int_U |\nabla u|^2 - \int_U \nabla w \cdot \nabla u - \int_U f u + \int_U f w
\end{aligned}$$

Notice that, since $|a - b|^2 \geq 0$ implies $\frac{a^2 + b^2}{2} \geq ab$,

$$\int_U \nabla w \cdot \nabla u \leq \int_U |\nabla w||\nabla u| \leq \frac{1}{2}\int_U |\nabla w|^2 + \frac{1}{2}\int_U |\nabla u|^2$$

Therefore

$$\int_U |\nabla u|^2 - \int_U fu = \int_U \nabla w \cdot \nabla u - \int_U fw$$

$$\leq \int_U \frac{|\nabla w|^2}{2} + \int_U \frac{|\nabla u|^2}{2} - \int_U fw$$

$$\int_U \frac{|\nabla u|^2}{2} - fu \leq \int_U \frac{|\nabla w|^2}{2} - fw$$

Then

$$I(u) \leq I(w), \quad \forall w \in A$$

$B / u \in A$.

February 7, 2024

Recall: Energy Functional

For $U \subseteq \mathbb{R}^n$ bounded, $g \in C(\partial U)$, $f \in C(\overline{U})$

$$A = \left\{ w \in C^2(\overline{U}) \;:\; w|_{\partial U} = g \right\}$$

we have

$$I(w) := \int_U \frac{1}{2}|\nabla w|^2 - fw$$

Theorem:

Suppose $u \in A$ such that $I(u) = \min\{I(w) \;:\; w \in A\}$. Then u satisfies

$$\begin{cases} -\Delta u = f, & \text{in } U \\ \quad u = g, & \text{on } \partial U \end{cases}$$

Proof

Consider $v \in C_c^\infty(U)$.
Define $i : \mathbb{R} \to \mathbb{R}$ such that $\tau \mapsto I(\tau) := I(u + \tau v)$.
Notice that $u + \tau v$ is a perturbation of u and, since $u + \tau v \in C^2(\overline{U})$ while $u + \tau v|_{\partial U} = u|_{\partial U} = g$, $u + \tau v \in A$. Then

$$i(0) = I(u) \leq I(u + \tau v) = i(\tau)$$

so i has a minimum point at $\tau = 0$. Compute

$$i(\tau) = I(u + \tau v)$$

$$= \int_U \frac{|\nabla(u + \tau v)|^2}{2} - f(u + \tau v)$$

$$= \int_U \frac{|\nabla u|^2}{2} + \tau\langle \nabla u, \nabla v \rangle + \frac{\tau^2 |\nabla v|^2}{2} - \int_U fu - \tau \int_U fv$$

$$= \int_U \frac{|\nabla u|^2}{2} + \tau \int_U \langle \nabla u, \nabla v \rangle + \frac{\tau^2}{2} \int_U |\nabla v|^2 - \int_U fu - \tau \int_U fv$$

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So i is a polynomial in τ, and

$$i'(0) = i'(\tau)_{\tau=0} = \left(\int_U \langle \nabla u, \nabla v \rangle + \tau \int_U |\nabla v|^2 - \int_U fv \right)_{\tau=0}$$

So

$$
\begin{aligned}
0 &= i'(0) \\
&= \int_U \langle \nabla u, \nabla v \rangle - \int_U fv \\
&= \int_U -\Delta u \cdot v + \underbrace{\int_{\partial U} \frac{\partial u}{\partial \eta} \cdot v}_{=0} - \int_U fv \\
&= \int_U \underbrace{(-\Delta u - f)}_{=0} v
\end{aligned}
$$

Since $0 = \int gv$, $\forall v \in C_c^\infty(U)$ requires $g \equiv 0$.
Then $-\Delta u - f = 0$.

Heat Equation (Diffusion Equation)

The equations

$$(*) \begin{cases} u_t - \Delta u = 0, & \text{homogeneous case} \\ u_t - \Delta u = f, & \text{non-homogeneous case} \end{cases}$$

(note that $\Delta u = \Delta_x u$)
subject to some boundary and initial conditions $t \geq 0$ time and $x \in \mathbb{R}^n$, space variable, $x \in U$ and opsen set of $\mathbb{R}^n$.
$u : U \times (0, \infty) \to \mathbb{R}$ defined as $(x, t) \mapsto u(x, t)$ with u unknown.
IMAGE HERE - 1

Motivation: Fundamental Solution of the Heat Equation

We would like to have the following:
If u solves

$$\begin{cases} u_t - \Delta u = 0, & \mathbb{R}^n \times (0, \infty) \\ u(x, 0) = g(x), & \mathbb{R}^n \times \{0\} \end{cases}$$

then

$$u(x, t) = \int_{\mathbb{R}^n} G(x - y, t) g(y) \, dy$$

How do we get G?
Let us suppose that $u(\tilde{x}, \tilde{t})$ solves

$$\begin{cases} u_{\tilde{t}} - \Delta_{\tilde{x}} u = 0 \\ u(\tilde{x}, 0) = g(\tilde{x}) \end{cases}$$

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We would like to have invariance under dilation.

$$v(x,t) := u(\lambda x, \lambda^2 t)$$

Such that

$$v_t = \nabla U|_{(\lambda x, \lambda^2 t)} - \frac{\partial}{\partial t} \begin{bmatrix} \lambda x \\ \lambda^2 t \end{bmatrix}$$

$$= \lambda^2 u_{\tilde{t}}(\lambda x, \lambda^2 t)$$

$$v_{x_i} = \lambda u_{\tilde{x}_i}(\lambda x, \lambda^2 t)$$

$$v_{x_i x_i} = \lambda^2 u_{\tilde{x}_i \tilde{x}_i}(\lambda x, \lambda^2 t)$$

Therefore

$$v_t - \Delta_x v = \lambda^2 u_{\tilde{t}} - \lambda^2 \Delta_{\tilde{x}} u = \lambda^2 \underbrace{(u_{\tilde{t}} - \Delta_{\tilde{x}} u)}_{=0} = 0$$

with

$$v(x,0) = u(\lambda x, 0) = g(\lambda x)$$

Then, applying the motivation,

$$v(x,t) = \int_{\mathbb{R}^n} G(x-y,t) g(\lambda y)\, dy \underset{z=\lambda y}{=} \int_{\mathbb{R}^n} G\left(x - \frac{z}{\lambda}, t\right) g(z) \frac{dz}{\lambda^n}$$

On the other hand,

$$v(x,t) = u(\lambda x, \lambda^2 t) = \int_{\mathbb{R}^n} G(\lambda x - z, \lambda^2 t) g(z)\, dz$$

It follows that

$$\frac{1}{\lambda^n} G\left(\overbrace{x - \frac{z}{\lambda}}^{w}, t\right) = G(\lambda x - z, \lambda^2 t)$$

$$\frac{1}{\lambda^n} G(w,t) = G(\lambda w, \lambda^2 t)$$

If $\lambda^2 t = 1$, then

$$G(w,t) = \frac{1}{t^{n/2}} G\left(\frac{1}{\sqrt{t}} w, 1 \right)$$

If we call $G\left(\frac{w}{\sqrt{t}}, 1\right) = v\left(\frac{w}{\sqrt{t}} \right)$, then we are looking at $G(w,t) = \frac{1}{t^{n/2}} v\left(\frac{w}{t^{1/2}} \right)$.
So, we have motivation to define

$$u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta} \right)$$

for α, β appropriate and $v(y) : \mathbb{R}^n \to \mathbb{R}$.

Obtaining a Fundamental Solution to the Heat Equation

Let us compute u_t and $\Delta_x u$.

$$u_t = \frac{\partial}{\partial t}\left(\frac{1}{t^\alpha}v\left(\frac{x}{t^\beta}\right)\right)$$

$$= \frac{(-\alpha)}{t^{\alpha+1}}v\left(\frac{x}{t^\beta}\right) + \frac{1}{t^\alpha}\frac{\partial}{\partial t}\left(v\left(\frac{x}{t^\beta}\right)\right)$$

$$= \frac{(-\alpha)}{t^{\alpha+1}}v\left(\frac{x}{t^\beta}\right) + \frac{1}{t^\alpha}\cdot\nabla v|_{\frac{x}{t^\beta}}\cdot\frac{\partial}{\partial t}\left(\frac{x}{t^\beta}\right)$$

$$u_t = \frac{(-\alpha)}{t^{\alpha+1}}v\left(\frac{x}{t^\beta}\right) + \frac{(-\beta)}{t^\alpha t^{\beta+1}}\nabla v|_{\frac{x}{t^\beta}}\cdot x \quad \square_1$$

and

$$\frac{\partial u}{\partial x_i} = \frac{1}{t^\alpha}\frac{\partial}{\partial x_i}\left(v\left(\frac{x}{t^\beta}\right)\right)$$

$$= \frac{1}{t^\alpha}\nabla v|_{\frac{x}{t^\beta}}\cdot\frac{\partial}{\partial x_i}\left(\frac{x}{t^\beta}\right)$$

$$= \frac{1}{t^{\alpha+\beta}}\frac{\partial v}{\partial x_i}|_{\frac{x}{t^\beta}}$$

while

$$\frac{\partial^2 u}{\partial x_i x_i} = \frac{1}{t^{\alpha+2\beta}}\frac{\partial^2 v}{\partial x_i x_i}|_{\frac{x}{t^\beta}} \quad \square_2$$

Then, replacing $\square_1$ and $\square_2$ in $*$,

$$-\frac{\alpha}{t^{\alpha+1}}v\left(\frac{x}{t^\beta}\right) - \frac{\beta}{t^{\alpha+\beta+1}}\nabla v|_{\frac{x}{t^\beta}}\cdot x - \frac{1}{t^{\alpha+2\beta}}\Delta v|_{\frac{x}{t^\beta}} \stackrel{?}{=} 0$$

Set $y := \frac{x}{t^\beta}$

$$-\frac{\alpha}{t^{\alpha+1}}v(y) - \frac{\beta}{t^{\alpha+1}}\nabla v(y)\cdot y - \frac{1}{t^{\alpha+2\beta}}\Delta v(y) = 0$$

Multiplying through by $-t^{\alpha+1}$,

$$\alpha v(y) + \beta \nabla v(y)\cdot y + \frac{1}{t^{2\beta-1}}\Delta v(y) = 0$$

Let us assume that $2\beta - 1 = 0$ such that $\beta = \frac{1}{2}$, giving

$$\alpha v(y) + \frac{1}{2}\nabla v(y)\cdot y + \Delta v(y) = 0$$

Since the Laplacian is rotationally invariant, assume $v(y) = w(|y|)$ for $w : \mathbb{R}^+ \to \mathbb{R}$.
Recall that $\frac{\partial}{\partial y_i}|y| = \frac{\partial}{\partial y_i}\left(\sqrt{y_1^2 + \cdots + y_n^2}\right) = \frac{y_i}{|y|}$. Now

$$\frac{\partial}{\partial y_i}v(y) = \frac{\partial}{\partial y_i}(w(|y|)) = w'(|y|)\cdot\frac{\partial}{\partial y_i}(|y|) = w'(|y|)\cdot\frac{y_i}{|y|}$$

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$$\frac{\partial^2 v(y)}{\partial y_i y_i} = \frac{\partial}{\partial y_i}\left(w'(|y|)\right)\frac{y_i}{|y|} + w'(|y|)\cdot\frac{\partial}{\partial y_i}\left(\frac{y_i}{|y|}\right)$$

$$= w''(|y|)\cdot\frac{y_i^2}{|y|^2} + w'(|y|)\left[\frac{1}{|y|} + y_i\frac{\partial}{\partial y_i}\left(\frac{1}{|y|}\right)\right]$$

$$= w''(|y|)\frac{y_i^2}{|y|^2} + w'(|y|)\left[\frac{1}{|y|} - \frac{y_i^2}{|y|^3}\right]$$

Replacing in the PDE of v,

$$0 = \alpha w(|y|) + \frac{1}{2}\frac{w'(|y|)y}{|y|}\cdot y + \sum_{i=1}^{n} w''(|y|)\frac{y_i^2}{|y|^2} + w'(|y|)\left[\frac{1}{|y|} - \frac{y_i^2}{|y|^3}\right]$$

$$= \alpha w(|y|) + \frac{1}{2}w'(|y|)|y| + w''(|y|) + w'(|y|)\left[\frac{n}{|y|} - \frac{1}{|y|}\right]$$

If $|y| = r$

$$0 = \alpha w(r) + \frac{1}{2}w'(r)r + w''(r) + w'(r)\frac{n-1}{r}$$

Take $\alpha = \frac{n}{2}$ and multiply through by r^{n-1},

$$0 = \frac{nr^{n-1}}{2}w(r) + \frac{r^n}{2}w'(r) + w''(r)r^{n-1} + w'(r)(n-1)r^{n-2}$$

$$= \frac{1}{2}\left[w(r)r^n\right]' + \left[w'(r)r^{n-1}\right]'$$

Then by the fundamental theorem of calculus, $w'(r)r^{n-1} + \frac{w(r)r^n}{2} = C$.
We would like $w, w' \xrightarrow[r\to\infty]{} 0$. Then $C = 0$, so

$$w'(r)r^{n-1} = -\frac{w(r)r^n}{2}$$

Which gives

$$w' = \frac{-wr}{2} \iff \frac{w'}{w} = -\frac{r}{2} \iff (\ln(w))' = \frac{-r}{2} \iff \ln(w) = -\frac{r^2}{4} + d$$

and, finally,

$$w(r) = be^{-\frac{r^2}{4}}$$

Then define

$$u(x,t) := \frac{1}{t^{n/2}}v\left(\frac{x}{t^{1/2}}\right)$$

$$= \frac{1}{t^{n/2}}w\left(\left|\frac{x}{t^{1/2}}\right|\right)$$

$$= \frac{b}{t^{n/2}}e^{-\frac{1}{4}\left|\frac{x}{t^{1/2}}\right|^2}$$

$$= \frac{b}{t^{n/2}}e^{-\frac{1}{4t}|x|^2}$$

Where b is chosen such that the expression integrates to 1.

Definition: Fundamental Solution of the Heat Equation

The fundamental solution for the heat equation is given by

$$\begin{cases} \Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, & x \in \mathbb{R}^n, \ t > 0 \\ \qquad\qquad 0, & x \in \mathbb{R}^n, \ t < 0 \end{cases}$$

where we have chosen $b = \frac{1}{(4\pi)^{n/2}}$.

IMAGE HERE - 2

Notice that these match in the limit away from the origin ($\lim_{(x,t)\to(x_0,0)} \Phi(x,t) = 0$).

Remark: $\Phi(x,t)$ has a unique singularity at $(0,0)$.

February 12, 2024

Recall: Heat Equation

$$\Phi(x,t) = \begin{cases} \frac{b}{(t)^{n/2}} e^{\frac{-|x|^2}{4t}}; & t > 0, x \in \mathbb{R}^n \\ 0; & t < 0 \end{cases}$$

Remark: Φ is radial such that $\Phi(x,t) = \Phi(|x|,t)$.

Lemma:

For each $t > 0$,

$$\int_{\mathbb{R}^n} \Phi(x,t)\,dx = 1$$

Proof

$$\begin{aligned}
\int_{\mathbb{R}^n} \Phi(x,t)\,dx &= \frac{b}{t^{n/2}} \int_{\mathbb{R}^n} e^{\frac{-|x|^2}{4t}}\,dx \\
&= \frac{b}{t^{n/2}} \int_{\mathbb{R}^n} e^{-\left|\frac{x}{2\sqrt{t}}\right|^2} \\
&\underset{z=\frac{x}{2\sqrt{t}}}{=} \frac{b}{t^{n/2}} \int_{\mathbb{R}^n} e^{-|z|^2} (2\sqrt{t})^n\,dz \\
&= b2^n \int_{\mathbb{R}^n} e^{-|z|^2}\,dz \\
&= 2^n b \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-z_1^2 - \cdots - z_n^2}\,dz_1 \cdots dz_n \\
&= 2^n b \left[\int_{-\infty}^{\infty} e^{-x}\,dx \right]^n
\end{aligned}$$

We need

$$A = \int_{-\infty}^{\infty} e^{-x^2} \, dx$$

$$A^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} \, dx dy$$

$$= \int_{\mathbb{R}^n} e^{-|z|^2} \, dz$$

$$= \int_0^{\infty} \int_{\partial B_r^2} e^{-r^2} \, dS(z) dr$$

$$= 2\pi \int_0^{\infty} e^{-r^2} r \, dr$$

$$\underset{s=r^2}{=} \pi \int_0^{\infty} e^{-s} \, ds$$

$$= -\pi(e^{-s})|_0^{\infty} = -\pi(0-1) = \pi$$

Therefore $A^2 = \pi$ and $A = \sqrt{\pi}$. So, picking $b = \frac{1}{(4\pi)^{n/2}}$,

$$\int_{\mathbb{R}^n} \Phi(x,t) \, dx = b2^n A^n = b2^n \pi^{n/2} = 1$$

Remark:

Φ solves the Heat Equation, except at the point $(x,t) = (0,0)$.

Remark:

Φ is infinitely differentiable on $\mathbb{R}^n \times (\delta, \infty)$, $\forall \delta > 0$.

Cauchy Problem (Initial Value Problem)

$$\begin{cases} u_t - \Delta_x u = 0, & \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x), & x \in \mathbb{R}^n \end{cases}$$

Recall $y \in \mathbb{R}^n$,

$$(x,t) \to \Phi(x-y)$$

solves the heat equation except at $(y,0)$.
Define, $x \in \mathbb{R}^n$, $t > 0$,

$$(*) \quad u(x,t) = \int_{R^n} \Phi(x-y,t)g(y) \, dy = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{\frac{-|x-y|^2}{4t}} g(y) \, dt$$

Theorem (#?): Solution to the Cauchy Problem

Assume $g \in C(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$. Then u defined by $*$ satisfies

1. $u \in C^{\infty}(\mathbb{R}^n \times (0,\infty))$.

2. $u_t(x, t) - \Delta_x(x, t) = 0$, $(x, t) \in \mathbb{R}^n \times (0, \infty)$.

3. $\lim_{\substack{(x,t) \to (x_0, 0) \\ x \in \mathbb{R}^n, \, t > 0}} u(x, t) = g(x_0)$, $x_0 \in \mathbb{R}^n$.

Proof

Homework: justify putting the limit inside to prove (1).
For (2), observe that

$$u_t - \Delta_x u(x, t) = \int_{\mathbb{R}^n} \underbrace{[\Phi_t(x - y, t) - \Delta_x \Phi(x - y, t)]}_{=0} g(y) \, dy$$

For (3), let $\varepsilon > 0$. Let $\delta > 0$ such that $|x - x_0| < \delta \implies |g(x) - g(x_0)| < \varepsilon$ (since g continuous). Then

$$|u(x, t) - g(x_0)| = \left| \int_{\mathbb{R}^n} \Phi(x - y, t) g(y) - g(x_0) \underbrace{\int_{\mathbb{R}^n} \Phi(x - y, t) \, dy}_{=1} \right|$$

$$\leq \int_{\mathbb{R}^n} \Phi(x - y, t) |g(y) - g(x_0)| \, dy$$

$$= \underbrace{\int_{B(x_0, \delta)} \Phi(x - y, t) |g(y) - g(x_0)| \, dy}_{I} + \underbrace{\int_{\mathbb{R}^n \setminus B(x_0, \delta)} \Phi(x - y, t) |g(y) - g(x_0)| \, dy}_{J}$$

Bounding I, $|y - x_0| < \delta \implies |g(y) - g(x_0)| < \varepsilon$ gives

$$I \leq \varepsilon \underbrace{\int_{B(x_0, \delta)} \Phi(x - y, t) \, dy}_{\leq 1} \leq \varepsilon$$

Bounding J, assume $|x - x_0| < \frac{\delta}{2}$. Then

$$|J| \leq ||g||_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n \setminus B(x_0, \delta)} \Phi(x - y, t) \, dy$$

Now we want to compare $|x - y|$ with $|x_0 - y|$. Then, for $|x - x_0| < \frac{\delta}{2}$ and $|y - x_0| > \delta$,

$$|y - x_0| < |y - x| + |x - x_0| < |y - x| + \frac{\delta}{2} < |y - x| + \frac{|y - x_0|}{2}$$

so $\frac{|y - x_0|}{2} < |y - x|$. It follows that

$$\frac{|y - x_0|^2}{4} \leq |y - x|^2$$

$$-\frac{|y - x|^2}{4t} \leq -\frac{|y - x_0|^2}{16t}$$

$$e^{-\frac{|y - x|^2}{4t}} \leq e^{-\frac{|y - x_0|^2}{16t}}$$

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Then

$$|J| \le 2||g||_{L^\infty} \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n \setminus B(x_0,\delta)} e^{-\frac{|y-x_0|^2}{16t}} \, dy$$

$$= \frac{C}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n \setminus B(x_0,\delta)} e^{-\frac{1}{16}\left|\frac{y-x_0}{\sqrt{t}}\right|^2} \, dy$$

Letting $z = \frac{y-x_0}{\sqrt{t}}$ such that $\sqrt{t}\, dz = dy$,

$$|J| \le \frac{C}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n \setminus B(0,\delta/\sqrt{t})} e^{-\frac{|z|^2}{16}} \underbrace{(\sqrt{t})^n \, dz}_{dy}$$

$$= \frac{C}{(4\pi)^{n/2}} \int_{\mathbb{R}^n \setminus B(0,\delta/\sqrt{t})} e^{-\frac{|z|^2}{16}} \, dz$$

Let $\delta_2 > 0$ such that $\delta_2 = \max\left\{\frac{\delta}{2}, \delta^3\right\}$.
If $|(x,t) - (x_0,0)| < \delta_2$,

$$t < \delta_2 < \delta^3$$
$$\sqrt{t} < \delta^{3/2}$$
$$\frac{1}{\delta^{3/2}} < \frac{1}{\sqrt{t}}$$
$$\frac{1}{\delta^{1/2}} < \frac{\delta}{\sqrt{t}}$$

so

$$B(0,1/\delta^{1/2}) \subseteq B(0,\delta/\sqrt{t}) \quad \text{and} \quad \mathbb{R}^n \setminus B(0,\delta/\sqrt{t}) \subseteq \mathbb{R}^n \setminus B(0,1/\delta^{1/2})$$

Therefore,

$$|u| \le C \int_{\mathbb{R}^n \setminus B(0,1/\sqrt{\delta})} e^{-\frac{|z|^2}{16}} \, dz \to 0$$

Intepretation of Fundamental Solution for the Heat Equation

$$\begin{cases} \Phi_t - \Delta_x \Phi(x,t) = 0, & x \in \mathbb{R}^n, \ t > 0 \\ \Phi(x,0) = \delta_0(x), & x \in \mathbb{R}^n \end{cases}$$

Then

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y) \, dy$$

if $t = 0$,

$$u(x,0) = \int_{\mathbb{R}^n} \Phi(x-y,0)g(y)\,dy$$

$$= \int_{\mathbb{R}^n} \underbrace{\delta^x(y)g(y)}_{y=x}\,dy$$

$$= \int_{\mathbb{R}^n} \delta^x(y)g(x)\,dy$$

$$= g(x)\underbrace{\int_{\mathbb{R}^n} \delta^x(y)\,dy}_{=1} = g(x)$$

Remark: Infinite Propagation Speed

Let $g \in C(\mathbb{R}^n \cap L^\infty(\mathbb{R}^n))$, $g \geq 0$, $g \neq 0$. Then

$$u(x,t)\frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy > 0, \quad \forall x \in \mathbb{R}^n,\ \forall t > 0$$

IMAGE HERE - 1
That is, the heat equation forces infinite propagation speed for disturbances.

Non-Homogeneous Heat Problem

$$(*_2) \quad \begin{cases} u_t - \Delta_x u = f, & f: \mathbb{R}^n \times (0,\infty) \to \mathbb{R} \\ u(x,0) = 0, & x \in \mathbb{R}^n \end{cases}$$

Motivation

Let $y \in \mathbb{R}^n$, $s > 0$. Then $(x,t) \to \Phi(x-y, t-s)$ solves the heat equation except at $x = y$ and $t = s$.
That is, it satisfies the equation on $\mathbb{R}^n \times (s,\infty)$.
Then for s fixed, define

$$(\square) \quad u(x,t;s) := \int_{\mathbb{R}^n} \Phi(x-y, t-s)f(y;s)\,dy$$

which solves

$$\begin{cases} u_t(x,t;s) - \Delta_x u(x,t;s) = 0, & \mathbb{R}^n \times (s,\infty) \\ u(x,s;s) = f(x;s), & \mathbb{R}^n \times \{s\} \end{cases}$$

which is the IVP with $t = 0 \iff t = s$ and $g(y) \iff f(y;s)$.

Definition: Duhamel's Principle

If we integrate $\square$ from 0 to t,

$$u(x,t) := \int_0^t u(x,t;s)\,ds$$

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Let us consider,

$$(\square_2) \quad u(x,t) := \int_0^t \int_{\mathbb{R}^n} \Phi(x - y, t - s) f(y; s) \, dy \, ds$$

as a candidate solution for $*_2$.

Theorem: Solution to the Non-Homogeneous Heat Equation

Suppose $f \in C_c^2((\mathbb{R}^n \times (0, \infty)))$ with compact support.
If we define u by $\square_2$, then

1. $u \in C_c^2(\mathbb{R}^n \times (0, \infty))$.

2. $u_t(x,t) - \Delta_x u(x,t) = f(x,t); \ x \in \mathbb{R}^n, \ t > 0$.

3. $\lim_{\substack{(x,t) \to (x_0, 0) \\ x \in \mathbb{R}^n, \ t > 0}} u(x,t) = 0, \ \forall x_0 \in \mathbb{R}^n$.

February 14, 2024

Recall: Non-Homogeneous Heat Equation

Given

$$\begin{cases} u_f - \Delta_x u = f(x,t), & f : \mathbb{R}^n \times (0, \infty) \to \mathbb{R} \\ \quad\quad u(x, 0) = 0 \end{cases}$$

we have a candidate solution from Duhamel's Principle.

$$\begin{aligned} (*) \quad u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(x - y, t - s) f(y, s) \, dy \, ds \\ &= \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi(t-s))^{n/2}} e^{-\frac{|x-y|^2}{4(t-s)}} f(y, s) \, dy \, ds \end{aligned}$$

Note that unlike the homogeneous case, the integral approaches the singularity at $(0,0)$ and we cannot pass a limit inside.

Theorem: Differentiation Under Moving Regions

Take $\Omega(t) \subseteq \mathbb{R}^n$ a nice region with nice boundaries ($\partial \Omega(t) \in C^1$ and $t \in \mathbb{R}$) and $F(z, t)$ smooth.

$$\frac{d}{dt} \left(\int_{\Omega(t)} F(x,t) \, dz \right) = \int_{\partial \Omega(t)} F v \eta \, ds(z) + \int_{\partial \Omega(t)} F_t \, dz$$

where v is the velocity vector on $\partial \Omega(t)$ and η is the unit outer normal.

Theorem:

Supopse $f \in C_1^2(\mathbb{R}^n \times (0, \infty))$ with compact support.
Then, if u is defined by $*$,

1. $u \in C_1^2(\mathbb{R}^n \times (0, \infty))$.

2. $u_t - \Delta_x u = f(x, t);\ x \in \mathbb{R}^n,\ t > 0$

3. $\lim_{(x,t) \to (x_0, 0)} u(x, t) = 0$ for $x \in \mathbb{R}^n,\ t > 0,\ \forall x_0 \in \mathbb{R}^n$.

Proof of 1

Since Φ has a singularity at $(0,0)$, we cannot differentiate under the integral sign.
Define $\overline{y} = x - y$ and $\overline{s} = t - s$, then $\frac{d\overline{s}}{ds} = -1$, $-d\overline{s} = ds$, and $\frac{d\overline{y}}{dy} = (-1)$. So

$$u(x, t) = -\int_t^0 \int_{\mathbb{R}^n} \Phi(\overline{y}, \overline{s}) f(x - \overline{y}, t - \overline{s})\, d\overline{y}\, d\overline{s}$$

Then, rewrite

$$u(x, t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y, s) f(x - y, t - s)\, dy\, ds$$

We may now justify passing the derivative of the space variable inside

$$\frac{\partial u}{\partial x_i} = \int_0^t \int_{\mathbb{R}^n} \Phi(y, s) \frac{\partial}{\partial x_i} f(x - y, t - s)\, dy\, ds$$

In the same way, justifying putting the limit inside, we have $\frac{\partial u}{\partial x_i}$ is continuous.
Now, apply the Differentiation Theorem for Moving Regions (above) where $\Omega(t) = \mathbb{R}^n \times [0, t]$.
Define $F(y, s, t) := \Phi(y, s) f(x - y, t - s)$.
IMAGE HERE - 1
Then,

$$\frac{\partial}{\partial t} u(x, t) = \int_{\partial \Omega(t)} F(\overset{z}{\overbrace{y, s}}, t) v\eta\, dS(y, s) + \int_0^t \int_{\mathbb{R}^n} \partial_t F(y, s, t)\, dy\, ds$$

$$= \int_{\mathbb{R}^n \times \{t\}} F(\overset{z}{\overbrace{y, s}}, t)\, dS(y, s) + \int_0^t \int_{\mathbb{R}^n} \partial_t F(y, s, t)\, dy\, ds$$

$$= \int_{\mathbb{R}^n} F(y, t, t)\, dy + \int_0^t \int_{\mathbb{R}^n} \partial_t F(y, s, t)\, dy\, ds$$

Therefore

$$\frac{\partial u}{\partial t}(x, t) = \int_{\mathbb{R}^n} \Phi(y, t) f(x - y, 0)\, dy + \int_0^t \int_{\mathbb{R}^n} \Phi(y, s) \partial_t f(x - y, t - s)\, dy\, ds$$

Homework: Prove that $\frac{\partial u}{\partial t}$ is continuous to complete the proof.

Proof of 2

$$u_t - \Delta_x u = \overset{K}{\overbrace{\int_{\mathbb{R}^n} \Phi(y, t) f(x - y, 0)\, dy}} + \int_0^t \int_{\mathbb{R}^n} \Phi(y, s) [f_{(t}(x - y, t - s) - \Delta_x f(x - y, t - s)]\, dy\, ds$$

Since Φ has a signularity, let $\varepsilon > 0$ and isolate

$$u_t - \Delta_x u = K + \underbrace{\int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,s) \left[f_t(x-y, t-s) - \Delta_x f(x-y, t-s) \right] dy ds}_{J_\varepsilon}$$

$$+ \underbrace{\int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s) \left[f_t(x-y, t-s) - \Delta_x f(x-y, t-s) \right] dy ds}_{I_\varepsilon}$$

Controlling J_ε,

$$|J_\varepsilon| \leq (||f_t||_{L^\infty} + ||\nabla_x f||_{L^\infty}) \overbrace{\int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,s) \, dy \, ds}^{1}$$

$$\leq C\varepsilon$$

So $J_\varepsilon \to 0$ as $\varepsilon \to 0$.

Controlling I_ε, using symmetry of t and s and x and y,

$$I_\varepsilon = -\int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s) \partial_s f(x-y, t-s) \, dy ds - \int_\varepsilon^t \Phi(y,s) \Delta_y f(x-y, t-s) \, dy ds$$

Recall that

$$\int_U u_{x_i} v = -\int_U u v_{x_i} + \int_{\partial U} u v \eta^i$$

where η^{-i} is the ith component of η. and

$$\int_\Omega u\Delta v - v\Delta u = \int_{\partial\Omega} u \frac{\partial v}{\partial \eta} - v \frac{\partial u}{\partial \eta}$$

So, integrating by parts,

$$I_\varepsilon = -\left[-\int_\varepsilon^t \int_{\mathbb{R}^n} \partial_s \Phi(y,s) f(x-y, t-s) \, dy ds + \int\int_{\partial(\mathbb{R}^n \times [\varepsilon, t])} \Phi(y,s) f(x-y, t-s) \eta^{n+1} \, dy ds \right]$$

$$- \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s) \Delta_y f(x-y, t-s) \, dy ds$$

Since $\eta^{n+1} = 1$ and f has compact support, this gives

$$I_\varepsilon = \int_0^t \int_{\mathbb{R}^n} \partial_s \Phi(y,s) f(x-y, t-s) \, dy ds - K + \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y, t-\varepsilon) \, dy$$

$$- \int_\varepsilon^t \Delta_y \phi(y,s) f(x-y, t-s) \, dy ds$$

Notice that the first and last summands solve the heat equation on $\mathbb{R}^n \times [\varepsilon, t]$. So

$$I_\varepsilon = -K + \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y, t-\varepsilon) \, dy$$

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Therefore

$$u_t - \Delta_x u = \lim_{\varepsilon \to 0} K + 0 - K + \int_{\mathbb{R}^n} \Phi(y, \varepsilon) f(x - y, t - \varepsilon) \, dy$$

Homework: prove that we may pass the limit inside.

$$
\begin{aligned}
u_t - \Delta_x u &= \int_{\mathbb{R}^n} \Phi(y, 0) f(x - y, t) \, dy \\
&= \int_{\mathbb{R}^n} \delta^0(y) f(x - y, t) \, dy \\
&= \int_{\mathbb{R}^n} \delta^0(y) f(x, t) \, dy \\
&= f(x, t) \int_{\mathbb{R}^n} \delta^0(y) \, dy \\
&= f(x, t)
\end{aligned}
$$

Proof of 3

Write

$$|u(x, t)| \le ||f||_{L^\infty} \int_0^t \overbrace{\int_{\mathbb{R}^n} \Phi(y, s) \, dy \, ds}^{=1} \le ct$$

General Solution to the Heat Equation

If $f \in C_1^2(\mathbb{R}^n \times (0, \infty))$ and $g \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, then

$$u(x, t) := \int_0^t \int_{\mathbb{R}^n} \Phi(x - y, t - s) f(y, s) \, dy \, ds + \int_{\mathbb{R}^n} \Phi(x - y, t) g(y) \, dy$$

is a solution for

$$
\begin{cases}
u_t - \Delta_x u = f(x, t) \\
u(x, 0) = g(x)
\end{cases}
$$

Mean-Value Formulas for the Heat Equation

Definition: Parabolic Cylinder

Let $U \subseteq \mathbb{R}^n$ be an open set and $T > 0$.
The parabolic cylinder U_T is given by

$$U_T := U \times (0, T]$$

and the parabolic boundary is

$$\Gamma_T = \overline{U}_T - U_T$$

IMAGE HERE - 2

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Motivation for Mean-Formulas

In the harmonic case,

$$\Phi(x) = \frac{c1}{|x|^{n-2}}; \quad n \geq 3$$

for x fixed and r fixed

$$\phi : \mathbb{R}^n \to \mathbb{R}$$
$$y \to \Phi(x - y)$$

Then the balls $B(x, r)$ are the level surface of ϕ. See that

$$
\begin{aligned}
\phi^{-1}(c_0) &= \{y \in \mathbb{R}^n : \Phi(x - y) = c_0\} \\
&= \{y \in \mathbb{R}^n : \frac{C}{|x - y|^{n-2}} = c_0\} \\
&= \{y \in \mathbb{R}^n : |x - y|^{n-2} = \sqrt[n-2]{\frac{c}{c_0}}\} \\
&= \partial B\left(x, \sqrt[n-2]{\frac{c}{c_0}}\right)
\end{aligned}
$$

Then to get the mean-value formula, it is worth it to pay attention to the level surface of the fundemental solution of the heat equation.

February 26, 2024

Recall: Mean-Value Formula for Heat Equation

$$\Phi(x, t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{\frac{|x|^2}{4t}} & , x \in \mathbb{R}^n, t > 0 \\ 0 & , x \in \mathbb{R}^n, t < 0 \end{cases}$$

For $U \subset \mathbb{R}^n$ open and bounded, $T > 0$, $U_t = U \times (0, T]$, $\Gamma_T = \overline{U}_T - U_T$.

Definition: Heat Balls

Let $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $r \in \mathbb{R}_+$. Defint the heat ball E as

$$E(x, t; r) = \{(y, s) \in \mathbb{R}^{n+1} : s \leq t, \Phi(x - y, t - s) \geq 1/r^n\}$$

Remark 1

$$\frac{1}{4r^n} \int_{E(x,t;r)} \frac{|x - y|^2}{|t - s|^2} \, dy \, ds = 1$$

Do as homework.

Remark 2

$\partial E(x, t; n)$, Φ is constant.

Theorem: Mean-Value Formulas

Let $u \in C_1^2(U_T)$ solves the heat equation. Then

$$u(x, t) = \frac{1}{4r^n} \iint\limits_{E(x,t;r)} u(y, s) \frac{|x - y|^2}{|t - s|^2} \, dy \, ds$$

for all $E(x, t; r) \subseteq U_T$.

Proof

Define

$$\phi(r) := \frac{1}{4r^n} \iint\limits_{E(x,t;r)} u(y, s) \frac{|x - y|^2}{|t - s|^2} \, dy \, ds$$

We want to prove ϕ constant with $\phi' = 0$.
Without loss of generality, set $x = 0$, $t = 0$ such that $E(r) := E(0, 0, r)$. Then

$$\phi(r) = \frac{1}{4r^n} \iint\limits_{E(r)} u(y, s) \frac{|y|^2}{s^2} \, dy \, ds$$

Rescaling by $y = r\overline{y}$ and $s = r^2\overline{s}$,

$$\phi(r) = \frac{1}{4r^n} \iint\limits_{E(1)} u(r\overline{y}, r^2\overline{s}) \frac{r^2 |\overline{y}|^2}{r^4 \overline{s}^2} r^n r^2 \, d\overline{y} \, d\overline{s}$$

$$= \frac{1}{4} \iint\limits_{E(1)} \frac{|\overline{y}|^2}{\overline{s}^2} \, d\overline{y} \, d\overline{s}$$

Where we have $E(1)$ because $(y, s) \in E(r) = E(0, 0, r)$, $s \le 0$, $\frac{1}{(4\pi(-s))^{n/2}} e^{\frac{-|-y|^2}{4(-s)}} \ge \frac{1}{r^n}$.

So $r^2\overline{s} \le 0$ and $\frac{1}{4\pi(-r^2\overline{s})^{n/2}} e^{\frac{-|-r\overline{y}|^2}{4(-r^2\overline{s})}} \ge \frac{1}{r^n}$.
Therefore

$$\overline{s} \le 0 \quad \text{and} \quad \frac{1}{4\pi(-\overline{s}))^{n/2}} e^{\frac{-|-\overline{y}|^2}{4(-\overline{s})}} \ge 1$$

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Reindexing $\overline{y} = y$ and $\overline{s} = s$,

$$4\phi'(r) = \int\int_{E(1)} \left[Du|_{(ry,r^2s)} \cdot \left(\frac{y}{2rs} \right) \right] \frac{|y|^2}{s^2} \, dyds$$

$$= \int\int_{E(1)} \left[\sum_{i=1}^{n} \frac{\partial u}{\partial y_i}|_{(ry,r^2s)} y_i + \frac{\partial u}{\partial s} 2rs \right] \frac{|y|^2}{s^2} \, dyds$$

$$= \int\int_{E(1)} \frac{|y|^2}{s^2} \sum_{i=1}^{n} \frac{\partial u}{\partial y_i}|_{(ry,r^2s)} y_i \, dyds + 2 \int\int_{E(1)} \frac{\partial u}{\partial s} r \frac{|y|^2}{s} \, dyds$$

Then, again applying the change of variables,

$$4\phi'(r) = \int\int_{E(r)} \frac{|\overline{y}|^2 r^4}{r^2 \overline{s}^2} \sum_{i=1}^{n} \frac{\partial u}{\partial y_i} \frac{\overline{y}_i}{r} \frac{d\overline{y}}{r^n} \frac{d\overline{s}}{r^2} + 2 \int\int_{E(r)} \frac{\partial u}{\partial s} \frac{r|\overline{y}|^2}{r^2} \frac{r^2}{\overline{s}} \frac{d\overline{y}}{r^n} \frac{d\overline{s}}{r^2}$$

$$= \underbrace{\frac{|y|^2}{r^{n+1}} \int\int_{E(r)} \frac{1}{s^2} \sum_{i=1}^{n} \frac{\partial u}{\partial y_i} y_i \, dyds}_{A} + \underbrace{\frac{2}{r^{n+1}} \int\int_{E(r)} \frac{\partial u}{\partial s} \frac{|y|^2}{s} \, dyds}_{B}$$

We want to analyze B.
Let us introduce the notation

$$\psi(y,s) = \frac{-n}{2} \log(-4\pi s) + \frac{|y|^2}{4s} + n\log(n)$$

- Lemma 1
  $\psi(y,s) = 0$, $(y,s) \in \partial E(r)$.

    – Proof

  If $(y,s) \in \partial E(r)$, $\Phi(-y,-s) = \frac{1}{r^n}$, $\frac{1}{(4\pi(-s))^{n/2}} e^{\frac{-|-y|^2}{4(-s)}} = \frac{1}{r^n}$. Therefore

$$r^n = (4\pi(-s))^{n/2} e^{\frac{|-y|^2}{4(-s)}} = e^{\log((4\pi(-s))^{n/2} e^{\frac{|-y|^2}{4(-s)}})}$$

  So

$$n\log(r) = \frac{n}{2} \log(4\pi(-s)) - \frac{-|-y|^2}{4s}$$

- Lemma 2

$$\frac{\partial \psi}{\partial y_i} = \frac{2y_i}{4s} = \frac{y_i}{2s}.$$

    – Proof

$$4 \sum_{i} \frac{\partial \psi}{\partial y_i} y_i = \frac{2|y|^2}{s}$$

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- Analyzing B

Then, integrating by parts,

$$B = \frac{4}{r^{n+1}} \int\int_{E(r)} \frac{\partial u}{\partial s} \sum_i \frac{\partial \psi}{\partial y_i} y_i \, dy ds$$

$$= \frac{4}{r^{n+1}} \sum_i \left[\int\int_{E(r)} \frac{\partial}{\partial y_i}\left(\frac{\partial u}{\partial s} y_i\right) \psi \, dy ds + \overbrace{\int\int_{\partial E(r)} \frac{\partial u}{\partial s} y_i \psi \eta^i}^{=0} \right]$$

$$= \frac{4}{r^{n+1}} \sum_i \int\int_{E(r)} \psi \left[\frac{\partial u}{\partial s} + y_i \frac{\partial^2 u}{\partial y_i \partial s} \right] dy ds$$

Then, again integrating by parts,

$$B = \underbrace{-\frac{4}{r^{n+1}} \sum_i \int_{E(r)} \psi \frac{\partial u}{\partial s} \, dy ds}_{C} - \frac{4}{r^{n+1}} \sum_i \int_{E(r)} \psi y_i \frac{\partial^2 u}{\partial y_i \partial s} \, dy ds$$

$$= C - \frac{4}{r^{n+1}} \sum_i \left[-\int_{E(r)} y_i \frac{\partial \psi}{\partial s} \frac{\partial u}{\partial y_i} \, dy s + \overbrace{\int_{\partial E(r)} \psi y_i \frac{\partial u}{\partial y_i} \eta^s}^{=0} \right]$$

$$= C - \frac{4}{r^{n+1}} \sum_i \int_{E(r)} y_i \frac{\partial u}{\partial y_i} \left[\frac{n}{2s} + \frac{|y|^2}{4s^2} \right] dy ds$$

Since $-\int_{E(r)} \sum_i y_i \frac{\partial u}{\partial y_i} \frac{|y|^2}{4s^2} = -A$, we have

$$B = -\frac{4n}{r^{n+1}} \int_{E(r)} \psi \frac{\partial u}{\partial s} \, dy ds - \frac{4}{r^{n+1}} \sum_i \int_{E(r)} y_i \frac{\partial u}{\partial y_i} \frac{n}{2s} \, dy ds - A$$

So, since u solves the heat equation, we have $\Delta u = \frac{\partial u}{\partial s}$ and may integrate by parts

$$4\phi'(r) = -\frac{4n}{r^{n+1}} \int_{E(r)} \psi \frac{\partial u}{\partial s} \, dy ds - \frac{4}{r^{n+1}} \sum_i \int_{E(r)} y_i \frac{\partial u}{\partial y_i} \frac{n}{2s} \, dy ds$$

$$= -\frac{4n}{r^{n+1}} \left[\int_{E(r)} \nabla \psi \cdot \nabla u \, dy ds + \overbrace{\int_{\partial E(r)} \psi \frac{\partial u}{\partial \eta}}^{=0} \right] - \frac{4}{r^{n+1}} \sum_i \int_{E(r)} y_i \frac{\partial u}{\partial y_i} \frac{n}{2s} \, dy ds$$

$$= 0$$

Then we have
$)\phi'(r) = 0$ and $\phi(r)$ constant. We know

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$$\phi(r) = \lim_{t \to 0} \phi(t)$$

$$= \lim_{t \to 0} \frac{1}{4} \iint_{E(1)} u(ty, ts^2) \frac{|y|^2}{s^2} \, dy \, ds$$

$$= \frac{1}{4} \int_{E(1)} u(0,0) \frac{|y|^2}{s^2} \, dy \, ds$$

$$= u(0,0) \frac{1}{4} \overbrace{\int_{E(1)} \frac{|y|^2}{|s|^2} \, dy \, ds}^{=1}$$

$$= u(0,0)$$

Theorem: Strong Maximum Principle for Heat Equation

Let U be bounded, $u \in C_1^2(U_T) \cap C(\overline{U}_T)$ that satisfies the heat equation.

1. $\max_{\overline{U}_T} u = \max_{\Gamma_T} u$.

2. If U is connected and $(x_0, t_0) \in U_T$ such that $u(x_0, t_0) = \max_{\overline{U}_T} u$, then u is constant in $\overline{U}_{t_0}$.
   IMAGE HERE - 1 CYLINDER to Ut0

Proof of 2

Let $(x_0, t_0) \in U_T$ such that $u(x_0, t_0) = \max_{\overline{U}_T} u := M$.
Pick r small enough such that $E(x_0, t_0; r) \subseteq U_T$.
IMAGE HERE - 2 BALL IN CYLINDER
Then, applying the mean-value formula,

$$M = \frac{1}{4r^n} \int_{E(x_0,t_0;r)} u(y,s) \frac{|x_0 - y|^2}{|t_0 - s|^2} \, dy \, ds$$

$$\leq \frac{M}{4r^n} \int_{E(x_0,t_0;r)} \frac{|x_0 - y|^2}{|t_0 - s|} \, dy \, ds$$

$$= M$$

Therefore $u(y,s) = M, \ \forall (y,s) \in E(x_0, t_0; r)$.

- Part A
  Let $(y_0, s_0) \in U_T$ such that we may connect (x_0, t_0) and (y_0, s_0) with a line L where $L \subseteq U_T$.
  Then $u = M$ on L.

February 28, 2024

Recall: Strong Maximum Principle

Let $u \in C_1^2(U_T) \cap C(\overline{U}_T)$, where $U_T = U \times [0, T]$, solve the heat equation. Then

1. $\max_{\overline{U}_T} u = \max_{\Gamma_T} u$

2. If U is connected and if $\exists (x_0, t_0) \in U_T$ such that $\max_{\overline{U}_T} u = u(x_0, t_0)$, then u constant on $\overline{U}_{t_0}$

Proof of 2

Let $(x_0, t_0) \in U_T$, $M = \max_{\overline{U}_T} = u(x_0, t_0)$.
Using mean-value formula, we proved $\exists r > 0$ such that $u = M$ is constant on $E(x_0, t_0; r)$.

- Part A
  Let (y_0, s_0), $s_0 < t_0$, such that (y_0, s_0) and (x_0, t_0) are connected by a line $L \subseteq U_T$.
  So $\Omega = \{s \geq s_0 : u(x, t) = M, \forall (x, t) \in L, s \leq t \leq t_0\}$ is nonempty since $t_0 \in \Omega$
  We know $\inf(\Omega)$ exists and, since u is continuous, $\min(\Omega)$ exists.
  Set $r_0 := \min\{\Omega\}$. From the construction, $s_0 \leq r_0$.
  We want to show that $s_0 = r_0$.
  Suppose $s_0 < r_0$. Then $\exists z_0 \in U$ such that $M = u(z_0, r_0) \in L \subset U_T$.
  IMAGE HERE - 1
  Applying the argument from the beginning, $\exists r$ such that $u = M$ on $E(z_0, r_0; r)$.
  But $E(z_0, r_0; r)$ contains points on $L \cap \{r_0 - \sigma \leq t \leq r_0\}$, for some $\sigma > 0$.
  This implies that $r_0 - \sigma \in \Omega$ which contradicts the assumption that r_0 was the minimum of Ω.
  Therefore, $u(y_0, s_0) = M = \max_{\overline{U}_T} u$.

- Part B
  Let $x \in U$, $t < t_0$.
  Since U is connected, there exists a finite set of points $x_0, \ldots, x_m = x$ such that the line connected x_i with x_{i-1} is contained in U.
  Then we may define a finite set of times, $t_0 > t_1 > \cdots > t_m = t$ such that the straight line L_i connecting (x_i, t_i) and (x_{i-1}, t_{i-1}) is totally contained in U_T.
  Then, applying Part A on each L_i, we have $u(x, t) = M$.

Proof of 1

Trivially, $\max_{\Gamma_T} u \leq \max_{\overline{U}_T} u$.
Assume that U is connected, and let $(x_0, t_0) \in \overline{U}_T$ be such that $u(x_0, t_0) = \max_{\overline{U}_T} u$.
If $(x_0, t_0) \in \Gamma_T$, then $\max_{\overline{U}_T} = u(x_0, t_0) \leq \max_{\Gamma_t} u$.
If $(x_0, t_0) \in U_T$, then, using 2, $u = \max_{\overline{U}_T} u$ is constant on $\overline{U}_{t_0}$.
Then we may pick $(x_1, t_0) \in \overline{U}_{t_0}$ and $x_1 \in \partial U$ such that

$$M = u(x_0, t_0) = u(x_1, t_0) \leq \max_{\Gamma_{t_0}} u \leq \max_{\Gamma_T} u$$

If U is not connected, we may take $U = \bigcup_{i \in \Lambda} U_i$,

$$\max_{\Gamma_T} u = \max_{i \in \Lambda}\{\max_{\Gamma_T^i} u\} = \max_{i \in \Lambda}\{\max_{\overline{U}} u\} = \max_{\overline{U}} u$$

Remark: Strong Minimum

Given that strong maximum principle, we have also the strong minimum principle.

Remark: Infinite Propagation Speed for Disturbances on Bounded Domains

Let U be bounded and connected, and $u \in C_1^2(U_T) \cap C(\overline{U}_T)$ which solves

$$\begin{cases} u_t - \Delta u = 0 \\ \qquad u \geq 0 \quad \text{on } \partial U \times [0, T] \\ \qquad u = g \quad \text{on } U \times \{0\} \end{cases}$$

If g is postive, where $g(x) \geq 0$, $\forall x$ and $\exists x_1$ for $g(x_1) > 0$, then $u(x, t) > 0$, $\forall (x, t) \in U_T$.

Proof

By the strong minimum principle,

$$u(x, t) \geq \min_{\overline{U}_T} u = \min_{\Gamma_T} u \geq 0$$

If $u(x, t) = 0 = \min_{\overline{U}_T} u$, then u is constant on $\overline{U}_t$ which contradicts the assumption that g is positive.

Theorem 5: Uniquness on Bounded Domains

Let $g \in C(\Gamma_T)$ and $f \in C(U_T)$ with U bounded and connected.
Then there exists at most one solution $u \in C_1^2(U_T) \cap C(\overline{U}_T)$ satisfying

$$(*) \begin{cases} u_t - \Delta u = f & U_T \\ u = g & \Gamma_T \end{cases}$$

Proof

Suppose that $u, \tilde{u}$ solve $*$. Then

$$(u - \tilde{u})_t - \Delta(u - \tilde{u}) = (u_t - \Delta u) - (\tilde{u}_t - \Delta \tilde{u}) = f - f = 0$$

Then $u - \tilde{u} \equiv 0$ on Γ_T. Applying the strong maximum and minimum principles to extend to $\overline{U}_T$, we have

$$u - \tilde{u} \equiv 0 \iff u = \tilde{u}$$

Theorem 6: Strong Maximum (Supremum) Principle for Unbounded Domains

Let $u \in C_1^2(\mathbb{R}^n \times [0, T]) \cap C(\overline{\mathbb{R}^n \times (0, T]})$ satisfy

$$\begin{cases} u_t - \Delta u = 0, & \mathbb{R}^n \times (0, T] \\ \qquad u = g, & \mathbb{R}^n \times \{0\} \end{cases}$$

with $|u(x, t)| \leq Ae^{a|x|^2}$ for some $A, a \geq 0$.
Then, $\sup_{\overline{\mathbb{R}^n \times (0, T]}} u = \sup g$.

Proof

Trivially, $\sup g \leq \sup_{\overline{\mathbb{R}^n \times [0,T)}} u$.

- Part 1
  Assume $4aT < 1$, then for some $\varepsilon > 0$ $4a(T + \varepsilon) < 1$.
  For $y \in \mathbb{R}^n$, $\mu > 0$,

$$v(x,t) := u(x,t) - \frac{\mu e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}}}{(T+\varepsilon-t)^{n/2}}; \quad x \in \mathbb{R}^n, t > 0$$

Notice that $v_t - \Delta v = 0$.
IMAGE HERE - 2
Then let $r > 0$ and let us consider $U = B_r(y)$ bounded.
Then we may apply the strong maximum principle for bounded domains to the function v.

$$U_T = B_r(y) \times (0, T]$$
$$\Gamma_T = (\partial B_r(y) \times (0, T]) \cup (B_r(y) \times \{0\})$$

Then $\max_{\Gamma_T} v = \max_{\overline{U}_T} u$. We need to analyze v on Γ_T.
Consider $B_r(y) \times \{0\}$ and $v(x,0)$ where $x \in B_r(y)$.

$$u(x,0) - \frac{\mu e^{\frac{|x-y|^2}{4(T+\varepsilon)}}}{(T+\varepsilon)^{n/2}} \leq u(x,0) = g(x)$$

Let $|x - y| = r$,

$$v(x,t) = u(x,t) - \frac{\mu e^{\frac{r^2}{4(T+\varepsilon-t)}}}{(t+\varepsilon-t^{n/2}}$$

$$\leq Ae^{a(|y|+r)^2} - \frac{\mu e^{\frac{r^2}{4(T+\varepsilon-t)}}}{(T+\varepsilon-t)^{n/2}}$$

We know $T - \varepsilon - t \leq T + \varepsilon$, so

$$(T+\varepsilon-t)^{n/2} \leq (T+\varepsilon)^{n/2}$$
$$-\frac{\mu}{(T+\varepsilon-t)^{n/2}} \leq -\frac{\mu}{(T+\varepsilon)^{n/2}} \leq 0$$

and

$$\frac{4(T+\varepsilon-t)}{r^2} \leq \frac{4(T+\varepsilon)}{r^2}$$
$$e^{\frac{r^2}{4(T+\varepsilon-t)}} \geq e^{\frac{r^2}{4(T+\varepsilon)}}$$

Therefore

$$\frac{-\mu e^{\frac{r^2}{4(T+\varepsilon-t)}}}{(T+\varepsilon-t)^{n/2}} \leq \frac{-\mu e^{\frac{r^2}{4(T+\varepsilon)}}}{(T+\varepsilon)^{n/2}}$$

and

$$v(x,t) \leq A e^{a(|y|+r)^2} - \frac{\mu e^{\frac{r^2}{4(T+\varepsilon)}}}{(T+\varepsilon)^{n/2}}$$

Then for $a < \frac{1}{4(T+\varepsilon)}$, there exists γ such that $a + \gamma = \frac{1}{4(T+\varepsilon)}$. So

$$v(x,t) \leq A e^{a(|y|+r)^2} - \mu e^{r^2(a+\gamma)} (4(a+\gamma))^{n/2}$$

If $\sup g = \infty$, we are done. Otherwise, we claim that $\exists r$ big enough such that $v(x,t) \leq \sup g$. Idea: we want $r^2(a+\gamma) >> a(|y|+r)^2$, for r big enough. Write

$$(a+\gamma) > a\left(\frac{|y|}{r} + 1\right)^2 \geq a\left(\frac{|y|}{r} + 1\right)$$

and

$$\gamma > \frac{a|y|}{r}$$

March 4, 2024

Notation

The disjoint union between A and B is denoted $A \sqcup B$.
The interior of U is denoted $\overset{\circ}{U}$.

Recall: Strong Maximum Principle of the Cauchy Problem

Let $u \in C_1^2(\mathbb{R}^n \times (0, t]) \cap C(\mathbb{R}^n \times [0, t])$ satisfy

$$\begin{cases} u_t - \Delta u = 0 & \mathbb{R}^n \times (0, t] \\ u = g & \mathbb{R}^n \times \{0\} \end{cases}$$

with $u(x, t) \leq A e^{a|x|^2}$, $A, a > 0$ constants. Then

$$\sup_{\mathbb{R}^n \times (0, T]} = \sup g$$

Proof

Trivially, $\sup g \leq \sup_{\mathbb{R}^n \times [0, T]} u$.

- Part 1
  Let us assume $4aT < 1$ and, for ε small enough, $4a(T+\varepsilon) < 1$.
  Then for $y \in \mathbb{R}^n$, $\mu > 0$, define

$$v(x,t) := u(x,t) - \frac{\mu e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}}}{(T+\varepsilon-t)^{n/2}}; \quad x \in \mathbb{R}^n, t > 0$$

Notice that v satisfies $v_t - \Delta v = 0$.

For $r > 0$, define $U = B_r(y)$. Consider U_{T_i} and appy the maximum principle for bounded domains

$$\max_{\overline{U}_T} v = \max_{\Gamma_T} v$$

- Part 2

  Analyzing v on Γ_T. Note that

$$\Gamma_T = (\partial B(y, r) \times [0, T]) \cup \underbrace{(B(y, r) \times \{0\})}_{v(x,0) \le g(x)}$$

If $|x - y| = r$, we proved that for r big enough such that $v(x, t) \le \sup_{\mathbb{R}^n} g$,

$$v(y, t) \le \max_{\overline{U}_T} v = \max_{\Gamma_T} v \le \sup_{\mathbb{R}^n} g, \quad \forall t \in [0, T]$$

Then if $\mu \to 0$,

$$v(y, t) \le \sup_{\mathbb{R}^n} g, \quad \forall t \in [0, T]$$

Therefore,

$$\sup_{\overline{U}_T} u(y, t) \le \sup_{\mathbb{R}^n} g$$

That is, if $T < \frac{1}{4a}$, the maximum is achieved at $T = 0$.

- Part 3

  If $4aT \ge 1$, we will divide $[0, T]$ into subintervals such that each subinterval has length smaller than $\frac{1}{4a}$. Then

$$\begin{aligned} \sup_{\mathbb{R}^n \times [0,T]} u &= \sup\{ \sup_{\mathbb{R}^n \times [0,T_1]} u, \sup_{\mathbb{R}^n \times [T_1,T_2]} u, \ldots, \sup_{\mathbb{R}^n \times [T_{n-1},T_n]} u\} \\ &= \sup\{ \sup_{\mathbb{R}^n \times \{0\}} u, \sup_{\mathbb{R}^n \times \{T_1\}} u, \ldots, \sup_{\mathbb{R}^n \times \{T_{n-1}\}} u\} \\ &\le \sup\{ \sup_{x \in \mathbb{R}^n} g, \sup_{\mathbb{R}^n \times [0,T_1]} u, \ldots, \sup_{\mathbb{R}^n \times [T_{n-2},T_{n-1}]} u\} \\ &\le \sup g \end{aligned}$$

Theorem: Uniqueness of the Cauchy Problem

Let $g \in C(\mathbb{R}^n)$, $f \in C(\mathbb{R}^n \times [0, T])$. Then there is at most one solution to

$$\begin{cases} u_t - \Delta u, & C(\mathbb{R}^n \times [0, T]) \\ u = g, & \mathbb{R}^n \times \{0\} \end{cases}$$

such that $|u(x, t)| \le A e^{a|x|^2}$ for constants $a > 0$, $A > 0$.

Proof

Homework.

Homework

Show that the general solution satisfies this uniqueness property.

Theorem: Smoothness of the Heat Equation

Let $u \in C_1^2(U_T)$ satisfy the heat equation. Then $u \in C^\infty(U_T)$ $(u \in C^\infty(\mathring{U}_T))$.

Proof: Step 1

IMAGE HERE - 2
Take

$$c(x, t; r) = \{(y, s) : |x - y| \leq r, t - r^2 \leq s \leq t\}$$

for $(x_0, t_0) \in \mathring{U}_T$. Then

$$C := C(x_0, t_0; r)$$
$$C' := C\left(x_0, t_0; \frac{3}{4}r\right)$$
$$C'' := C\left(x_0, t_0 \frac{r}{2}\right)$$

IMAGE HERE - 3
Let $\zeta \in C^\infty$ be a cutoff function such that

$$\begin{cases} 0 \leq \zeta \leq 1, & C \\ \zeta = 1, & C' \\ \zeta = 0 & \text{near parabolic boundary of } C \end{cases}$$

IMAGE HERE - 4
We may extend $\zeta = 0$ outside of C.
Remark: $\zeta_t, \nabla\zeta, \Delta\zeta, \mathbb{R}^{n+1}$ vanishes outside C.

Proof: Step 2

Suppose $w \in C^\infty(U_T)$ and define

$$v(x, t) := w(x, t)\zeta(x, t), \quad x \in \mathbb{R}^n, 0 \leq t \leq t_0$$

We have

$$v_t = w_t \zeta + w\zeta_t$$
$$\frac{\partial v}{\partial x_i} = w_{x_i}\zeta + w\zeta_i$$
$$\frac{\partial v^2}{\partial^2 x_i} = w_{x_i x_i}\zeta + w_{x_i}\zeta_{x_i} + w_{x_i}\zeta_{x_i} + w\zeta_{x_i x_i}$$

and

$$\Delta v = \zeta \Delta w + 2\langle \nabla w, \nabla \zeta \rangle + w\Delta\zeta$$

So define

$$w_t\zeta + w\zeta_t - \zeta\Delta w - 2\langle \nabla w, \nabla \zeta \rangle - w\Delta\zeta := \tilde{f}$$

such that

$$(*) \quad \begin{cases} v_t - \Delta v = \tilde{f} \\ v(x,0) = 0, \quad \mathbb{R}^n \times \{0\} \end{cases}$$

Notice that $\tilde{f}$ has compact support on $\mathbb{R}^n \times [0, t_0]$. Then by Theorem 2 (existence), we have

$$\tilde{v}(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s)\tilde{f}(y,s)\, dy ds$$

also solves $(*)$.

Claim: $|v|, |\tilde{v}| \leq A$ for some constant A.

$$|v(x,t)| \leq |w(x,t)||\zeta(x,t)| \leq |w(x,t)|\chi_C(x,t) \leq A'$$

$$|\tilde{v}(x,t)| \leq \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s)|\tilde{f}(y,s)|\, dy ds \leq \tilde{A}\int_0^1 \Phi(x-y, t-s)\, dy ds \leq \tilde{A}t_0 \leq A''$$

Set $A = \max\{A', A''\}$. Then $A \leq Ae^{|x|^2}$ and, trivially, v and $\tilde{v}$ satisfy the growth control.

By the strong maximum principle, we have uniqueness of solutions and conclude $v = \tilde{v}$. So

$$v(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s)\tilde{f}(y,s)\, dy ds$$

Proof: Step 3

For $(x,t) \in C'' \subset C'$ given $w \in C^\infty(U_T)$ solves the heat equation on C, $\zeta = 1$ while $\zeta, \zeta_t, \Delta\zeta$ have support in C. Therefore

$$\begin{aligned} w(x,t) &= v(x,t) \\ &= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s)\tilde{f}(y,s)\, dy ds \\ &= \int_C \Phi(x-y, t-s)\left[w_t\zeta + w\zeta_t - \zeta\Delta w - 2\langle \nabla w, \nabla \zeta \rangle - w\Delta\zeta\right] dy ds \end{aligned}$$

If W solves the heat equation, $w_t\zeta - \zeta\Delta w = \zeta(w_t - \Delta w) = 0$. So for $(x,t) \in C''$.

$$w(x,t) = \int_C \Phi(x-y, t-s)\left[w\zeta_t - 2\langle \nabla w, \nabla \zeta \rangle - w\Delta\zeta\right] dy ds$$

Notice that we do not have problems around the singularity (x,t), because $w\zeta_t - 2\langle \nabla w, \nabla \zeta\rangle - w\Delta\zeta$ vanishes around (x,t) since $\zeta = 1$ on C'.

Let us analyze

$$\int_C \Phi(x-y, t-s)\langle \nabla w, \nabla \zeta\rangle \, dyds = \sum_{i=1}^n \int_{t-r^2}^t \int_{B(x_0,r)} \Phi \frac{\partial \zeta}{\partial y_i} \frac{\partial w}{\partial y_i} \, dyds$$

$$\underset{\text{IBP}}{=} \sum_{i=1}^n \int_{t-r^2}^t \left[-\int_{B(x_0,r)} \frac{\partial}{\partial y_i}\left(\Phi \frac{\partial \zeta}{\partial y_i}\right) w \, dy + \overbrace{\int_{\partial B(x_0,r)} \Phi \frac{\partial \zeta}{\partial y_i} w \eta^i \, dy}^{=0} \right] ds$$

Where the latter term is zero since $\zeta = 0$ near the parabolic boundary.

$$\int_C \Phi(x-y, t-s)\langle \nabla w, \nabla \zeta\rangle \, dyds = \sum_{i=1}^n \int_{t-r^2}^t \int_{B(x_0,r)} w \left[\frac{\partial \Phi}{\partial y_i} \frac{\partial \zeta}{\partial y_i} - \Phi \frac{\partial^2 \zeta}{\partial y_i^2} \right] \, dyds$$

$$= \int_C w\langle \nabla \Phi, \nabla \zeta\rangle - w\Phi\Delta\zeta \, dyds$$

So

$$w(x,t) = \int_C \Phi w\zeta_t - \phi w\Delta\zeta - 2w\langle\nabla\Phi, \nabla\zeta\rangle + 2w\Phi\Delta\zeta \, dyds$$

$$= \int_C \Phi w\zeta_t + \phi w\Delta\zeta - 2w\langle\nabla\Phi, \nabla\zeta\rangle \, dyds$$

Proof: Step 4

Then, define

$$u^\varepsilon := \eta_\varepsilon * u, \quad (U_T)_\varepsilon$$

the convolution on $\mathbb{R}^{n+1}$.

IMAGE HERE - 5

We know u^ε is smooth. Moreover, by properties of convolution, u^ε satisfies the heat equation. Applying Step 3 to u^ε,

$$u^\varepsilon(x,t) = \int_C u^\varepsilon(y,s) \overbrace{\left[\Phi\zeta_t + \Phi\Delta\zeta - 2\langle\nabla\Phi, \nabla\zeta\rangle\right]}^{K(x,t,y,s)} \, dyds$$

When $\varepsilon \to 0$,

$$u(x,t) = \int_C u(y,s)K(x,t,y,s) \, dyds$$

To be continued.

March 6, 2024

Theorem: Smoothness Continued

If $u \in C_1^2(U_T)$ solves the heat equation, then $u \in C^\infty(U_T)$ $(u \in C^\infty(\mathring{U}_T))$.

Proof

Let $(x_0, t_0) \in \overset{\circ}{U}_T$ and ζ a cutoff function satisfying

$$\begin{cases} 0 \leq \zeta \leq 1, & C \\ \zeta = 1, & C' \\ \zeta = 0, & \text{near the boundary} \end{cases}$$

Notice that $\zeta_s, \nabla\zeta, \Delta\zeta = 0$ on C'.
For $\varepsilon > 0$

$$u^\varepsilon(x, t) = \int_C u^\varepsilon(y, s) K(x, t, y, s)\, dy ds, \quad \forall(x, t) \in C''$$

where

$$K(x, t, y, s) = \Phi(x - y, t - s)(\zeta_s - \Delta_y\zeta) - 2\nabla\Phi(x - y, t - s)\nabla\zeta$$

Let $\varepsilon > 0$,

$$u(x, t) = \int_C u(y, s) K(x, t, y, s)\, dy ds, \quad \forall(x, t) \in C''$$

$$= \int_{C-C'} u(y, s) K(x, t, y, s)\, dy ds, \quad \forall(x, t) \in C''$$

Notice that u is smooth on C''. If we can prove that $K(\cdot, \cdot, y, s)$ is smooth on C'' for each $(y, s) \in C \setminus C'$
IMAGE HERE - 1
But that's true because (y, s) is far from the neighborhood around (x, t), $\forall(x, t) \in C''$
We have proven than $\forall(x_0, t_0) \in \overset{\circ}{U}_T$, u is smooth on $C\left(x_0, t_0; \frac{r}{2}\right) := C''$.
Then we are done, because $\forall(x_1, t_1) \in \overset{\circ}{U}_T$, $\exists(x_0, t_0) \in \overset{\circ}{U}_T$ such that $(x_1, t_1) \in C\left(x_0, t_0; \frac{r}{2}\right)$.

Theorem 9: Estimates on Derivatives

There exist constants C_{kl} for every pair of integers $k, l = 0, 1, 2, \ldots$ such that

$$\max_{C\left(x, t; \frac{r}{2}\right)} |D_x^k D_t^l u(x, t)| \leq \frac{C_{kl}}{r^{k+2l+n+2}} ||u||_{L^1(C(x,t;r))}$$

for all cylinders $C(x, t; r) \subseteq U_T$ and u any solution of the heat equation on U_T.
Note that k should be understood as the order of the appropriate multiindex.

Proof

Without loss of generality, let us assume that $(x, t) = (0, 0)$ and $C(1) := C(0, 0; 1) \subseteq U_T$.
If $C\left(\frac{1}{2}\right) = C\left(0, 0; \frac{1}{2}\right)$, using the same technique as in the proof of smoothness,

$$u(x, t) = \int_{C(1)} K(x, t, y, s) u(y, s)\, dy ds, \quad \forall(x, t) \in C\left(\frac{1}{2}\right)$$

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For K a smooth function,

$$|D_x^k D_t^l| \le C_{kl} \int\int_{C(1)} |u(y,s)| \, dy ds, \quad \forall (\tilde{x}, \tilde{t}) \in C\left(\frac{1}{2}\right)$$

Taking the maximum gives

$$\max_{C\left(\frac{1}{2}\right)} |D_x^k D_t^l u(x,t)| \le C_{kl} ||u||_{L^1(C(1))}$$

Let u satisfy the heat equation on U_T, $C(r) \subseteq U_T$. Then define

$$v(x,t) := u(\overbrace{rx}^{y}, \overbrace{r^2 t}^{s})$$

where v satisfies the heat equation on $C(1) \subseteq \tilde{U}_T$. Then by the above computation

$$\max_{C\left(\frac{1}{2}\right)} |D_x^k D_t^l v| \le C_{kl} ||v||_{L^1(C(1))}$$

We may analyze

$$\begin{aligned}
D_x^k D_t^l v(x,t) &= D_x^k D_t^l u(rx, r^2 t) \\
&= D_x^k \left[(r^2)^l D_s^l u(rx, r^2 t) \right] \\
&= r^{2l} r^k D_y^k D_s^l u|_{(rx, r^2 t)}
\end{aligned}$$

and

$$\begin{aligned}
||v||_{L^1(C(1))} &= \int_{C(1)} |v(x,t)| \, dx dt \\
&= \underset{\substack{\tilde{y}=rx \\ \tilde{s}=r^2 t}}{=} \int_{C(r)} \left| v\left(\frac{\tilde{y}}{r}, \frac{\tilde{s}}{r^2}\right) \right| \frac{d\tilde{y} d\tilde{s}}{r^n r^2} \\
&= \frac{1}{r^{n+2}} \int_{C(r)} |u(\tilde{y}, \tilde{s})| \, d\tilde{y} d\tilde{s}
\end{aligned}$$

therefore

$$(*) \quad \max_{C\left(\frac{r}{2}\right)} |D_x^k D_t^l u| \le \frac{C_{kl}}{r^{2l+k+n+2}} ||u||_{L^1(C(r))}$$

Remark:

Recall that if u was harmonic,

$$|D^k u(x_0)| \le \frac{C_k}{r^{n+k}} ||u||_{L^1(B(x_0, r))}$$

such that $B(x_0, r) \subseteq U$.
Moreover, recall that $\frac{1}{r^{n+k}}$ was important in proving that u was analytic.

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Let us examine

$$n + 1 + k + l \le k + 2l + n + 2$$

For r small, $\frac{1}{r}$ is big, so

$$\frac{1}{r^{n+1+k+l}} \le \frac{1}{r^{k+2l+n+2}}$$

Then the estimate $(*)$ is not good enough to expect that u is analytic when u solves the heat equation.

Remark:

Let us instead consider t_0 fixed and $\phi_u : x \to u(x, t_0)$ where u satisfies the heat equation. Then

$$|D_x^k u| \le \frac{C_{k0}}{r^{k+n+2}} ||u||_{L^1(C(x,t;r))}$$

Heuristically, we may consider replacing

$$||u||_{L^1(C(x,t;r))} \sim r^2 ||u||_{L^1(B(x,r))}$$

IMAGE HERE - 2
Then we can expect ϕ_u is analytic.

Energy Methods for the Heat Equation

Take $f \in C(U_T)$, $g \in C(\Gamma_T)$, and consider

$$(*) \begin{cases} u_t - \Delta u = f, & U_T \\ u = g, & \Gamma_T \end{cases}$$

with U bounded and open in $\mathbb{R}^n$ and $\partial U \in C^1$.

Theorem: Uniqueness in Energy Methods

With the previous conditions, there exists at most one solution $u \in C_1^2(\overline{U}_T)$ of the problem $(*)$.

Proof

Suppose there are two solutions u_1 and u_2 of $(*)$, then $w = u_1 - u_2$ solves

$$\begin{cases} w_t - \Delta w = 0, & U_T \\ w = 0, & \Gamma_T \end{cases}$$

where $W \in C_1^2(\overline{U}_T)$.
Then define $e : [0, T] \to \mathbb{R}^+$ as

$$0 \le e(t) := \int_U w^2(x, t) \, dx$$

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where

$$e(0) = \int_U w^2(x,0)\,dx = 0$$

Taking the derivative

$$e'(t) = \int_U 2w(x,t)w_t(x,t)\,dx$$

$$= 2\int_U w\Delta_x w(x,t)\,dx$$

$$\underset{\text{IBP}}{=} 2\left[-\int_U |\nabla w|^2\,dx + \overbrace{\int_{\partial U} w\frac{\partial w}{\partial \eta}\,dy}^{=0}\right]$$

therefore $e'(t) \le 0$ and e is nonincreasing such that $e(0) \ge e(t)$.
Then it must be the case that $e(t) = 0$, $\forall t \in [0,T]$ and $w = 0$.

Method of Characteristics

We want to study $F(Du, u, x) = 0$ for $u : U \to \mathbb{R}$ with u unknown and where $F : \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}$ is smooth given some boundary data $u = g$ on $\Gamma \subseteq \partial U$.
We will refer to this problem as $(*)$ below.
Note that F is a first order, nonlinear PDE.
The idea is to convert the PDE into an ODE by analyzing u along appropriate curves.

Recall: Transport Equation

$$Du \cdot (b,1) = f(x,t)$$

we found $\alpha(s) = (x + sb, t + s)$ were nice because it converts the PDE to

$$u(\alpha(s)) = z'(s) = f(x + sb, t + s)$$

Notation

Write $F(p,z,x) : \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}$, then

$$D_p F = (F_{p_1}, \dots, F_{p_n})$$
$$D_z F = F_z$$
$$D_x F = (F_{x_1}, \dots, F_{x_n})$$

Let $x \in U$ and $x_0 \in \Gamma$. Then let C be a curve linking x with x_0 parameterized by

$$x(s) = (x_1(s), \dots, x_n(s)) : [0,1] \to U \subseteq \mathbb{R}^n$$

If $u \in C^2$ is a solution of $(*)$, then write

$$z(s) := u(x(s))$$
$$p(s) := Du(x(s))$$
$$p_i(s) := \frac{\partial u}{\partial x_i} u(x(s))$$

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Finding the Characteristic Equation

We want to find an appropriate $x(s)$. Take

$$
\begin{aligned}
\frac{d}{ds} p_i(s) &= \frac{d}{ds}\left(\frac{\partial}{\partial x_i} u(x(s))\right) \\
&= D\left(\frac{\partial}{\partial x_i}\right) \cdot \frac{d}{ds} x(s) \\
&= \left[\frac{\partial^2 u}{\partial x_1 x_i}, \ldots, \frac{\partial^2 u}{\partial x_n x_i}\right] \cdot [x_1(s), \ldots, x_n(s)] \\
&= \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j \partial x_i}(x(s)) x_j'(s)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial x_i} F(Du, u, x) &= DF|_{(Du,u,x)} \cdot \frac{\partial}{\partial x_i}[Du, u, x] \\
&= \begin{bmatrix} F_{p_1} \\ \vdots \\ F_{p_n} \\ F_z \\ F_{x_1} \\ \vdots \\ F_{x_n} \end{bmatrix} \cdot \left[\frac{\partial^2 u}{\partial x_i x_1}, \ldots, \frac{\partial^2 u}{\partial x_i x_n}, \frac{\partial u}{\partial x_i}, e_i\right] \\
&= \sum_{j=1}^{n} F_{p_j}(Du, u, x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + F_z(Du, u, x)\frac{\partial u}{\partial x_i} + F_{x_i}(Du, u, x) \\
&= 0
\end{aligned}
$$

March 11, 2024

Recall

(P, z, x)
$F : \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}$, $U \subseteq \mathbb{R}^n$ open

$$
\begin{cases}
F(Du, u, x) = 0 \\
\qquad\qquad u = g \quad \text{on } \Gamma
\end{cases}
$$

$\Gamma \subseteq \partial U$.
IMAGE HERE - 1
We consider

$$
\begin{aligned}
x(s) &= (x_1(s), \ldots, x_n(s)) \\
P(s) &= Du(x(s)) \\
z(s) &= u(x(s))
\end{aligned}
$$

and have

$$\implies \frac{dP_i}{ds} = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j \partial x_i}(x(s)) x_j'(s)$$

$$\implies \sum_{j=1}^{n} F_{P_j}(Du, u, x)\frac{\partial^2 u}{\partial x_i \partial x_j} + F_z(Du, u, x)\frac{\partial u}{\partial x_i} + F_{x_i}(Du, u, x) = 0$$

Evaluating along the curve,

$$\sum_{j=1}^{n} F_{P_j}(P(s), z(s), x(s))\frac{\partial^2 u}{\partial x_i \partial x_j}(x(s)) + F_z(P(s), z(s), x(s))\frac{\partial u}{\partial x_i}(x(s)) + F_{x_i}(P(s), z(s), x(s)) = 0$$

If we assume that $x_j'(s) = F_{p_j}(P(s), z(s), x(s))$, then

$$\sum_{j=1}^{n} \frac{dP_i}{ds} + F_z(P(s), z(s), x(s))\frac{\partial u}{\partial x_i}(x(s)) + F_{x_i}(P(s), z(s), x(s)) = 0$$

Taking $\frac{d}{ds}$,

$$\frac{dz}{ds} = \frac{d}{ds}u(x(s)) = \nabla u|_{x(s)} \cdot x'(s) = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\Big|_{x(s)} \cdot x_i'(s) = \sum_{i=1}^{n} P_i(s) \cdot F_{p_i}(P(s), z(s), x(s))$$

Definition: Characteristic Equations

$$
\begin{aligned}
(a) \quad & \dot{P}(s) = -D_z F(P(s), z(s), x(s)) \cdot P(s) - D_x F(P(s), z(s), x(s)) \\
(b) \quad & \dot{z}(s) = P(s) \cdot D_P F(P(s), z(s), x(s)) \\
(c) \quad & \dot{x}(s) = D_P F(P(s), z(s), x(s)) \\
(d) \quad & F(P(s), z(s), x(s)) = 0
\end{aligned}
$$

$(P(s), z(s), x(s))$ are called the characteristics.

Example: Linear Homogeneous PDE

Linear with respect to to P and z variables. Let

$$F(P, z, x) = b(x) \cdot P + c(x)z$$

with $b(x)$ and $c(x)$ given.
Then the PDE looks like

$$b(x) \cdot Du(x) + c(x)u(x) = 0$$

First: Find Characteristic Curve

Since $D_P F = b(x)$,

$$
\begin{aligned}
(b) \quad & \dot{z}(s) = P(s) \cdot b(x(s)) \\
(d) \quad & b(x(s)) \cdot P(s) + c(x)z(x) = 0 \\
(c) \quad & \dot{x}(s) = b(x(s)) \cdot P(s) = -c(x(s))z(s)
\end{aligned}
$$

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Therefore

$$\dot{x}(s) = b(x(s))$$
$$\dot{z}(s) = -c(x(s))z(s)$$

Example 1

Solve

$$\begin{cases} (*) \quad x_1 u_{x_2} - x_2 u_{x_1} = u; \quad U \\ \qquad\qquad\qquad\qquad u = g; \quad \Gamma \end{cases}$$

with $U = \{x_1 > 0, x_2 > 0\}$, $\Gamma = \{x_1 > 0, x_2 = 0\} \subseteq \partial U$.

(1) Identify Parts

Notice

$$b(x) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad c(x) = -1$$

then $*$ is equivalent to

$$\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \cdot Du(x) - u(x) = 0$$

(2) Obtaining Characteristic ODEs and Solving

Examining $\dot{x}(s) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x(s)$,

$$\begin{cases} \dot{x}(s) = \begin{pmatrix} -x_2(s) \\ x_1(s) \end{pmatrix} \\ \dot{z}(s) = z(s) \end{cases} \implies \begin{cases} \dot{x}_1(s) = -x_2(s) \\ \dot{x}_2(s) = x_1(s) \\ \dot{z} = z(s) \end{cases} \implies \begin{cases} x_1(s) = x_0 \cos(s) \\ x_2(s) = x_0 \sin(s) \\ z(s) = z_0 e^s \end{cases}$$

(3) Linking Solution with Boundary Data

$$\begin{cases} x(0) = (x_1(0), x_2(0)) = (x_0, 0) \\ z(0) = z_0 = z(0) = u(x(0)) = u(x_0, 0) = g(x_0) \end{cases} \implies \begin{cases} x_1(s) = x_0 \cos(s) \\ x_2(s) = x_0 \sin(s) \\ z(s) = g(x_0) e^s \end{cases}$$

(4) Finding the Characteristic Curve

We need the curve which passes through (x_1, x_2) and the time at which it does so.
Let $(x_1, x_2) \in U$.
We want to find x_0 and s such that $x(s) = (x_0 \cos(s), x_0 \sin(s)) = (x_1, x_2)$, so

$$\begin{cases} x_1 = x_0 \cos(s) \\ x_2 = x_0 \sin(s) \end{cases} \implies x_1^2 + x_2^2 = x_0^2 \implies x_0 = \sqrt{x_1^2 + x_2^2}$$

Then $\frac{x_2}{x_1} = \tan(s)$ and $s = \arctan\left(\frac{x_2}{x_1}\right)$.

Solution

Then

$$u(x_1, x_2) = u(x(s)) = z(s) = g\left(\sqrt{x_1^2 + x_2^2}\right) e^{\arctan\left(\frac{x_2}{x_1}\right)}$$

Example 2

Solve

$$\begin{cases} u_{x_1} + x_2 u_{x_2} = 0 \\ u(0, x_0 = g(x_0)) \end{cases}$$

(1) Identify Parts

Notice that

$$b(x) = \begin{pmatrix} 1 \\ x_2 \end{pmatrix}, \quad c(x) = 0$$

such that

$$\begin{pmatrix} 1 \\ x_2 \end{pmatrix} \cdot Du(x) = 0$$

(2) Obtaining Characteristic ODEs and Solving

$$\begin{cases} \dot{x}(s) = \begin{pmatrix} 1 \\ x_2 \end{pmatrix} \\ \dot{z}(s) = 0 \end{cases} \implies \begin{cases} x_1(s) = s + c_1 \\ x_2(s) = c_2 e^s \\ z(s) = c_3 \end{cases}$$

(3) Linking Solutions with Boundary Data

$$\begin{cases} x_1(0) = 0 \\ x_2(0) = x_2 \\ z(0) = g(x_0) \end{cases} \implies \begin{cases} x_1(s) = s \\ x_2 \implies x_2(s) = x_0 e^s \\ z(s) = g(x_0) \end{cases}$$

(4) Finding the Characteristic Curve

$$\begin{cases} x_1(s) = s \\ x_2(s) = x_0 e^s \\ z(s) = g(x_0) \end{cases}$$

Let $(x_1, x_2) \in U$. We want x_0 and s which makes the curve pass through the point.

$$(x_1, x_2) = (x_1(s), x_2(s)) = (s, x_0 e^s)$$

So

$$\begin{cases} x_1 = s \\ x_2 = x_0 e^s \end{cases} \implies \begin{cases} s = x_1 \\ x_0 = x_2 e^{-s} = x_2 e^{-x_1} \end{cases}$$

Solution

$$u(x_1, x_2) = u(x_1(s), x_2(s)) = z(s) = g(x_2 e^{-x_1})$$

Example: Quasilinear Homogeneous PDE

Quasilinear with respect to variable P.

$$\begin{cases} F(P, z, x) = b(x, z) \cdot P + c(x, z) \\ b(x, u(x)) \cdot Du(x) + C(x, u(x)) = 0 \end{cases}$$

Obtaining Fundamental Equations

$$\begin{aligned} (c) \quad & \dot{x}(s) = b(x(s), z(s)) \\ (d) \quad & b(x(s), z(s)) \cdot P(s) + c(x(s), z(s)) = 0 \\ (b) \quad & \dot{z}(s) = P(s) \cdot b(x(s), z(s)) = -c(x(s), z(s)) \end{aligned}$$

therefore

$$\begin{cases} \dot{x}(s) = b(x(s), z(s)) \\ \dot{z}(s) = -c(x(s), z(s)) \end{cases}$$

Example 3

$$\begin{cases} u_{x_1} + u_{x_2} = u^2; & U \\ u = g; & \Gamma \end{cases}$$

with U the half space and $\Gamma = \{x_2 = 0\} = \partial U$.

(1) Identify Parts

We have

$$b(x, z) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad c(x, z) = -z^2$$

so the PDE is equivalent to

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot Du(x) - u^2 = 0$$

(2) Obtaining Characteristic ODEs and Solving

$$\begin{cases} \dot{x}_1(s) = 1 \\ \dot{x}_2(s) = 1 \\ \dot{z}(s) = z^2 \end{cases} \implies \begin{cases} x_1(s) = s + \alpha_1 \\ x_2(s) = s + \alpha_2 \\ z(s) = -\frac{1}{s + \alpha_3} \end{cases}$$

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(3) Linking Solution with Boundary Data

$$\begin{cases} x(0) = (x_1(0), x_2(0)) = (x_0, 0) \\ z(0) = u(x_0, 0) = g(x_0) \end{cases} \implies \begin{cases} x_1(s) = s + x_0 \\ x_2(s) = s \\ z(s) = -\dfrac{1}{s - \frac{1}{g(x_0)}} \end{cases}$$

(4) Finding the Characteristic Curve

Given (x_1, x_2),

$$\begin{cases} x_1(s) = x_1 \\ x_2(s) = x_2 \end{cases} \implies \begin{cases} s + x_0 = x_1 \\ s = x_2 \end{cases} \implies \begin{cases} x_0 = x_1 - s = x_1 - x_2 \\ x_2 = s \end{cases}$$

Solution

Therefore,

$$u(x_1, x_2) = z(s) = -\dfrac{1}{x_2 - \frac{1}{g(x_1 - x_2)}}$$