Analysis II

January 9, 2024

(Real) Analysis

- Calculus
    - Differential
    - Integral (Riemann)
- Functions and Maps
    - Measure Theory
    - (Lebesgue) Integration
- Topology
    - Completeness (as a metric space)
    - Compactness (Bolzano-Weierstrass theorem [real]) (Arzela-Ascoli)
    - Paracompactness / Metrizable / Baire Category Theorem
    - Algebraic / Combinatoric (continuous maps or functions)

Definition: Cardinality

For sets A, B, $\mathrm{Card}(A) = \mathrm{Card}(B)$ if there exists a one-to-one correspondence $q : A \leftrightarrow B$.
Counting, labelling, indexing, etc.
$\mathrm{Card}(A) \leq \mathrm{Card}(B)$ if $A \subset B$ or there exists a one-to-one mapping $A \to B$.

Definition: Countable

If $A \hookrightarrow \mathbb{N}$, then A is countable.

Theorem

The countable union of countable sets is countable.

Proof

Let $A_i = \{a_j\}_{j=1}^{\infty}$, $i = 1, 2, \ldots$.

$$\begin{matrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ \vdots & & & \\ a_{k1} & a_{k2} & a_{k3} & \cdots \end{matrix}$$

Index by diagonalization.

Theorem

The cartesian product of countable sets is countable.

Proof

$X \times Y = \{(x_i, y_j \mid x_i \in X, y_j \in Y\}$

$$\begin{matrix} (x_1,y_1) & (x_1,y_2) & (x_1,y_3) & \cdots \\ (x_2,y_1) & (x_2,y_2) & (x_2,y_3) & \cdots \\ \vdots & & & \\ (x_k,y_1) & (x_k,y_2) & (x_k,y_3) & \cdots \end{matrix}$$

Theorem

$\mathrm{Card}\left(2^X\right) > \mathrm{Card}(X)$, where $2^X = \{A \subset X\}$ is the power set of X.

Proof

For all $x \in X$, $\{x\} \subset 2^X$, so $\mathrm{Card}(X) \leq \mathrm{Card}\left(2^X\right)$.
Assume, for sake of contradiction, that $\mathrm{Card}(X) = \mathrm{Card}\left(2^X\right)$.
Then, by definition, there exists a one-to-one correspondence $\phi : X \leftrightarrow 2^X$.
Set $A = \{x \subset X \mid x \notin \phi(x)\}$, and let $a = \phi^{-1}(A)$ (i.e. $A = \phi(a)$).
If $a \in A$, then $a \notin A \subset \phi(a)$; but if $a \notin A$, then $a \in A$, a contradiction.

Theorem

$\mathrm{Card}(\mathbb{R}) = \mathrm{Card}\left(2^{\mathbb{N}}\right)$.

Topology of the Real Line

Completeness (as a metric space)

$d(a,b) = |a - b|, \quad \forall a, b \in \mathbb{R}$.

1. $x_i \to x$ if $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ such that $|x_i - x| < \varepsilon$, $\forall i \geq n$.

2. $\{x_i\}$ is Cauchy if $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ such that $|x_i - x_j| < \varepsilon$, $\forall i, j \geq n$.

Definition: Open Inteval

(a,b) is an open set on the real line.
There exist interior points for any subset A of real numbers.
$\forall x \in A$, x is interior if $\exists (a,b)$ such that (1) $x \in (a,b)$ and (2) $(a,b) \subset A$.

- Theorem
  The union of open sets is open.
  The intersection of finitely many open sets is open.
  $\varnothing$ and $\mathbb{R}$ are open.

Definition: Limit Point

A limit point $x \in \mathbb{R}$ of a subset A is a limit point in A if for every open neighborhood U of X, $(U \setminus \{x\}) \cap A \neq \varnothing$.

Definition: Closed

A is closed if A contains all of its limit points.

- Theorem
  A is closed if and only if $A^c = \mathbb{R} \setminus A$ is open.

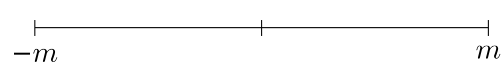
  – Proof
    A closed $\implies$ A^c open.
    Otherwise, $\exists x \in A^c$ such that for every neighborhood U of X, $(U \setminus \{x\}) \cap A = \varnothing$ which would make it a limit point of A not in A. By assumption, A contains all its limit points so this is a contradiction.
    A^c open $\implies$ A closed.
    For any x a limit point of A, assume otherwise that $x \in A^c$.
    Then there exists some neighborhood U of x such that $U \subset A^c$ (since A^c is open).
    It follows that $(U \setminus \{x\}) \cap A = \varnothing$ and x is not a limit point of A, which is a contradiction.

Definition: Sequential Compactness

A is compact if $\forall \{x_i\}$, $x_i \in A$ there exists a convergent subsequence $\{x_{i_k}\}$ and $x_{i_k} \to x \in A$.

- Theorem: Bolzano-Weierstrass
  For $A \subseteq \mathbb{R}$, A is compact if and only if A is closed and bounded.

  – Proof
    A compact $\implies$ A closed and bounded.
    Assume that A is not bounded from abvove.
    Then there exists a sequence $\{x_i\}$, $x_i \in A$ where $x_{i+1} > x_i + 1$ and $\{x_i\}$ has no convergent subsequences.
    Then compactness implies closedness.
    A closed and bounded $\implies$ A (sequentially) compact.
    Let any $\{x_i\}$, $x_i \in A$.
    Claim: $\forall \{x_i\}$ of reals, if there exists $m \in \mathbb{R}$ such that $|x_i| \leq m$, $\forall m$ then there is some convergent subsequence.



    Divide and conquer: dividing the interval in half necessitates that at least one half contains infinitely many points. Repeat indefinitely.

- Theorem: Heine-Borel)
  $A \subseteq \mathbb{R}$ is (sequentially) compact if and only if any open cover has a finite subcover.

  – Proof
    Heine-Borel Property $\implies$ closed and bounded.
    Assume that A is unbounded, $U_n = (-n, n)$ and $\{U_n\}_{n=1}^{\infty}$ an open cover for $A \subseteq \mathbb{R}$ has no finite subcover.
    Assume A is not closed, then $x \in \dot{A}$ (where $\dot{A}$ is the limit set of A) and $x \notin A$, $U_n \left\{\left(-\infty, x - \frac{1}{n}\right) \cup \left(x + \frac{1}{n}, +\infty\right)\right\}$.
    Then $\{U_n\}$ covers $\mathbb{R} \setminus \{x\} \supset A$ has no finite subcover of A.
    A is bounded and closed $\implies$ A is Heine-Borel

Divide and conquer: using open sets with respect to open covers.

Definition: Cantor Set

$C = \{x \in [0,1] \mid \text{the ternary expansion of } x \text{ has only the digits } \{0,2\}\}$.
Equivalenetly, let $C_0 = [0,1]$, $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$.
Then $C_n = \bigcup_{k=1}^{2^n} C_n^k$ and $C = \bigcap_{n=1}^{\infty} C_n$.
$|C_n| = 2^n \left(\frac{1}{3}\right)^n \to 0$.

Definition: Perfectly Symmetric Sets

Let $\{\xi_n\}$ where $\xi_n \in \left(0, \frac{1}{2}\right)$.
$E_0 = [0,1]$, $E_1 = [0, \xi_1] \cup [1 - \xi_1, 1]$, $E_2 = [0, \xi_1\xi_2] \cup [\xi_1 - \xi_1\xi_2, \xi_1] \cup [1 - \xi_1, 1 - \xi_1 + \xi_1\xi_2] \cup [1 - \xi_1\xi_2, 1]$.
Then the cantor set is given by $\xi_n = \frac{1}{3}$.
$E_n = \bigcup_{k=1}^{2^n} E_n^k$, $|E_n^k| = \xi_1\xi_2\cdots\xi_n$, and $|E_n| = \sum |E_n^k| = 2^n \xi_1\xi_2\cdots\xi_n$.
Therefore, $E = \bigcap_{n=1}^{\infty} E_n$ and we define $|E| = \lim_{n\to\infty} |E_n| = \lim_{n\to\infty} \left(2^n \xi_1\xi_2\cdots\xi_n\right) = \lambda$ where $\lambda \in [0,1)$.
Let

$$2\xi_n = \frac{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n-1}\right)^{n-1}}{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n}\right)^{n}} < 1$$

, then

$$2^n \xi_1\cdots\xi_n = \frac{1}{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n}\right)^{n}} \to \lambda.$$

Proof

$\lim_{n\to\infty} \left(\left(1 + \frac{x}{n}\right)^{n/x}\right)^x = e^x$, then $\lim_{y\to 0} (1+y)^{1/y} = e$, $\log(1+y)^{1/y} = \frac{\log(1+y)}{y} \underset{y\to 0}{\to} 1$.
Observe that

$$\left(\frac{\log(1+y)}{y}\right)' = \frac{\frac{y}{1+y} - \log(1+y)}{y^2} = \left(1 + \frac{1}{1+y} - \log(1+y)\right)' = \frac{1}{(1+y)^2} - \frac{1}{1+y} = -\frac{y}{(1+y)^2} < 0$$

Theorem

Cantor sets and perfect symmetric sets are closed, perfect, uncountable, and nowhere dense.

January 11, 2024

Last Week

Cardinality.
Topology of the reals.

- Cantor (perfect symmetric sets)

$C_0 = [0,1]$
$C_1 = [0,1/3] \cup [2/3,1]$
$C_2 = [0,1/9] \cup [2/9,3/9] \cup [6/9,7/9] \cup [8/9,1]$
$C_n = \bigcup_{n=1}^{2^n} C_n^k$
$|C_n^k| = \left(\frac{1}{3}\right)^n$
$C = \bigcap_{n=1}^{\infty} C_n$
$|C_n| = 2^n \frac{1}{3^n} = \left(\frac{2}{3}\right)^n \implies |C| = \lim_{n\to\infty} |C_n| = 0$
Closed, no interior points and uncountable.

- Perfect Symmetric Sets

  $\{\xi_k\} \in \left(0, \frac{1}{2}\right)$
  $E_0 = [0,1]$
  $E_1 = [0,\xi_1] \cup [1-\xi_1, 1]$
  $E_2 = [0,\xi_1\xi_2] \cup [\xi_1 - \xi_1\xi_2, \xi_1] \cup [1-\xi_1, 1-\xi_1+\xi_1\xi_2] \cup [1-\xi_1\xi_2, 1]$
  $E_n = \bigcup_{n=1}^{2^n} E_n^k$
  $|E_n^k| \xi_1\xi_2\cdots\xi_n$
  $|E_n| = 2^n\xi_1\xi_2\cdots\xi_n$
  $2\xi_n = \dfrac{\left(1+\frac{\log\left(\frac{1}{n}\right)}{n-1}\right)^{n-1}}{\left(1+\frac{\log\left(\frac{1}{n}\right)}{n}\right)^{n}} < 1$
  $|E_n| = \dfrac{1}{\left(1+\frac{\log\left(\frac{1}{n}\right)}{n}\right)^{n}}$
  $|E| = \lim_{n\to\infty} |E_n| = \frac{1}{e^{\log\left(\frac{1}{n}\right)}} = \lambda, \quad \lambda \in (0,1)$

Volterra's Function

$$\phi(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

IMAGE HERE - graph of phi(x)

$$\phi'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$f(x) = \begin{cases} 0 & x \in E \\ \phi(x-a) & x \in (a, a+y) \\ -\phi(b-x) & x \in (b-y, b) \\ \phi(y) & x \in (a+y, b-y) \end{cases}, \quad (a,b) \in E^c$$

IMAGE HERE - f interval (a,b)

Propositions

1. $f'(x) = 0$ for $x \in E$.

2. $f'(x)$ discontinuous on E.

3. f' exists on $[0,1]$ and is bounded.

Since $|E| > 0$, $f'(x)$ is not Riemann integrable and, therefore, the fundamental theorem of calculus does not apply.

Lebesgue Outer Measure

$|(a,b)| = b - a$.
Let $A \subseteq \mathbb{R}$, then $m^*(A) = \inf\left\{\sum_{n=1}^{\infty} I_n \mid A \subseteq \bigcup_{n=1}^{\infty}\right\}$
Question: $m^*(A \cup B) \stackrel{?}{=} m^*(A) + m^*(B)$ for $A \cap B \neq \varnothing$?

Properties

1. $A \subseteq B \implies m^*(A) \leq m^*(B)$.

2. $m^*(\varnothing) = 0$.

3. If I is an interval, then $m^*(I) = |I|$.

4. If $\{A_i\}$ is countable, $m^*\left(\bigcup A_i\right) \leq \sum m^*(A_i)$.

- Proof of 4
  $\forall A_i$, $\exists \{I_n\}$ open intervals such that $\sum_n |I_n| < m^*(A_i) + \frac{\varepsilon}{2^i}$.
  Then $\bigcup_i \bigcup_n I_n^i \supset \bigcup_i A_i$, and $\sum_{n,i} |I_n^i| = \sum_i \left(\sum_n |I_n^i|\right) \leq \sum_i \left(m^*(A_i) + \frac{\varepsilon}{2^i}\right)$.

  - Corollary

    If A is countable, then $m^*(A) = 0$.
    Thus, by contraposition, every interval is uncountable.

Proposition

For $A \subseteq \mathbb{R}$, $\forall \varepsilon > 0$, $\exists U$ open such that $A \subseteq U$ and $m^*(U) \leq m^*(A) + \varepsilon$.

Corollary

There exists G in the intersection of countable open sets such that $m^*(G) = m^*(A)$ and $G \supseteq A$.

Caratheodory Criteria

If $\forall E$, $m^*(E \cap A) + m^*(E \cap A^c) = m^*(E)$, then A is Lebesgue measurable.

- Remark: $m^*(E \cap A) + m^*(E \cap A^c) \leq m^*(E) \leq +\infty$

Propositions

1. If A is measurable, then A^c is measurable.

2. $m^*(A) = 0$, then A is measurable.

3. If $A,\ B$ are measurable, then $A \cup B,\ A \cap B,\ A \setminus B$ are measurable.

4. If $\{A_i\}_{i=1}^k$ are disjoint and measurable, then $m^*\left(\bigcup_{i=1}^k A_i\right) = \sum_{i=1}^k m^*(A_i)$.

- Proof of 3

$$\begin{aligned} m^*(E \cap (A \cup B)) + m^*(E \cap (A \cup B)^c) &= m^*((E \cap A) \cup (E \cap B)) + m^*(E \cap A^c \cap B^c) \\ &= m^*(E \cap A) + m^*((E \cap A^c) \cap B) + m^*((E \cap A^c) + B^c) \\ &\le m^*(E) \end{aligned}$$

Since o$(A \cap B)^C = A^c \cup B^c$, this holds from before; similarly, $A \setminus B = A \cap B^c = A^c \cup B$.
If $A,\ B$ disjoint, then

$$\begin{aligned} m^*(A \cup B) &= m^*(E \cap A) + m^*(E \cap A^c) \\ &= m^*(A) + m^*(B) \end{aligned}$$

Theorem

If $\{A_i\}$ is a countable collection of disjoint and measurable sets, then

1. $\bigcup_i A_i$ is measurable.

2. $m^*\left(\bigcup_i A_i\right) = \sum_i m^*(A_i)$.

Proof of 1

Want to show:

$$m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) + m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right) \le m^*(E)$$

By assumption, since the measure of E is finite, $m^*\left(E \cap \bigcup_{i=1}^{\infty} A_i\right) < +\infty$.
Claim: $\forall \varepsilon > 0,\ \exists k$ such that
Therefore $m^*\left(E \cap \bigcup_{i=1}^{k} A_i\right) \ge m^*\left(E \cap \bigcup_{i=1}^{\infty} A_i\right) - \varepsilon$.

$$m^*(E) \le m^*\left(E \cap \bigcup_{i=1}^{k} A_i\right) + \varepsilon + m^*\left(E \cap \left(\bigcup_{i=1}^{k} A_i\right)^c\right) \le m^*(E) + \varepsilon$$

Proof of 2

We have shown $m^*\left(\bigcup_i A_i\right) \le \sum_{i=1}^{\infty} m^*(A_i)$.
Assume $m^*\left(\bigcup_i A_i\right) < +\infty$, then

$$\sum_{i=1}^{k} m^*(A_i) = m^*\left(\bigcup_{i=1}^{k} A_i\right) \le m^*\left(\bigcup_{i}^{\infty} A_i\right) \implies \sum_{i=1}^{\infty} m^*(A_i) \le m^*\left(\bigcup_{i}^{\infty} A_i\right)$$

January 16, 2024

Office Hours Tuesday / Thursday 10 AM - 11:30 AM
A note on notation: Latin characters are to be understood as countable indecies; greek as possible uncountable.

Lebesgue Outer Measure

$A \subset \mathbb{R}$
$m^*(A) = \inf\left\{\sum_{i=1}^{\infty} |I_i| \mid \bigcup_{i=1}^{\infty} I_i \supset A,\ I_i \text{ open intervals}\right\}$

Properties

1. $A \subset B \implies m^*(A) \leq m^*(B)$.
2. $m^*(\varnothing) = 0$.
3. $m^*(I) = |I|$ for I an interval.
4. Countable Subadditivity: $\{A_i\}_{i=1}^{\infty} \implies m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} m(A_i)$.
5. $\forall A \subset \mathbb{R},\ \forall \varepsilon > 0,\ \exists$ open neighborhood $U \supseteq A$ such that $m^*(U) \leq m^*(A) + \varepsilon$.
6. $\exists G \in \bigcap_{n=1}^{\infty} U_n$, U_n open, $U_n \supseteq A \implies G \supseteq A$, such that $m^*(G) = m^*(A)$.

Measurable (Caratheodory Criterion)

$\forall A \subseteq \mathbb{R}$ is Lebesgue measurable if

$$m^*(A) = m^*(E \cap A) + m^*(E \cap A^c)$$

Essentially, $m^*(E \cap A) + m^*(E \cap A^c) \leq m^*(E) \leq +\infty$.

- Propositions
  1. A measurable $\implies A^c$ measurable.
  2. $m^*(A) = 0 \implies A$ measurable.
  3. $\{A_i\}_{i=1}^{\infty}$ countable with A_i measurable, then
     (a) $\bigcap_{i=1}^{\infty} A_i$ are measurable.
     (b) Moreover, $A_i \cap A_j = \varnothing \implies m^*\left(\bigcup_{i=1}^{\infty}(A_i) = \sum_{i=1}^{\infty} m^* A_i\right)$.
     (c) A, B measurable $\implies A \cup B$, $A \cap B$, $A \setminus B$ measurable.
     (d) $A \cap B = \varnothing \implies m^*(A \cup B) = m^*(A) + m^*(B)$.
     (e) $\{A_i\}_i^{\infty}$ with A_i measurable, then $\bigcup_{i=1}^{\infty} A_i$ is measurable and $A_i \cap A_j \varnothing \implies m^*\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m^*(A_i)$.

  – Proof of e $\forall E \subset \mathbb{R}$, $m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) + m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right)$.

Claim: $m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) = \sum_{i=1}^{\infty} m^*(E \cap A_I)$ for $A_i \cap A_j = \varnothing$.
Then, $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$,

$$m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) = \sum_{i=1}^{\infty} m^*(E \cap A_i) \le \sum_{i=1}^{n} m^*(E \cap A_i) + \varepsilon$$
$$\implies m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) + m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right) \le m^*\left(E \cap \left(\bigcup_{i=1}^{n} A_i\right)\right) + m^*\left(E \cap \left(\bigcup_{i=1}^{n} A_i\right)^c\right) + \varepsilon \le m^*(E) + \varepsilon$$
$$\implies \bigcup_{i=1}^{\infty} A_i \text{ measurable}$$

Proof of Claim:
Step 1: A, B measurable and $A \cap B = \varnothing$. Since A is measurable,

$$\begin{aligned} m^*(E \cap (A \cup B)) &= m^*((E \cap (A \cup B)) \cap A) + m^*((E \cap (A \cup B)) \cap A^c) \\ &= m^*(E \cap A) + m^*(E \cap A^c) \end{aligned}$$

For $\{A_i\}_{i=1}^{\infty}$, $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} A_i'$ with $A_1 = A_1'$ and $A_i' = A_i \setminus \bigcup_{k=1}^{i-1} A_k$, $\forall i \ge 2$.
Therefore $A_i' \cap A_j' = \varnothing$ and A_i' is measurable.

$$m^*\left(\bigcup_{i=1}^{n} A_i\right) \le m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} m^*(A_i)$$
$$m^*\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} m^*(A_i) \le m^*\left(\bigcup_{k=1}^{\infty} A_k\right) < +\infty \implies \sum_{i=1}^{\infty} m^*(A_i) \le m^*\left(\bigcup_{k=1}^{\infty} A_k\right) \le \sum_{i=1}^{\infty} m^*(A_i)$$

Sigma Algebra and Borel Sets

Definition: Sigma Algebra

Let $S \subset 2^X$ for some set X. Then S is said to be a σ-algebra if

1. $\varnothing \in S$.
2. $A^c \in S$ if A^c.
3. $\bigcup_{i=1}^{\infty} A_i \in S$ if $A_i \in S$.
    - Equivalently, $\bigcap_{i=1}^{\infty} A_i \in S$ if $A_i \in S$.

Theorem:

The collection $\mathcal{L}$ of all Lebesgue measurable sets is a σ-algebra.

Definition: Borel Set

Let B be the σ-algebra generated by open sets of reals (i.e. the smallet σ-algebra containing all open sets of reals).
Then $b \in B$ is called a Borel set.

Remark

B is generated by $\{(a,+\infty) \mid a \in \mathbb{R}\}$.

1. $(a,+\infty)^c = (-\infty, a]$.
2. $\bigcap_{n=1}^{\infty}\left(a - \frac{1}{n}, +\infty\right) = [a, +\infty)$.
3. $[a,+\infty)^c = (-\infty, a)$.
4. $(-\infty, b) \cap (a, +\infty) = (a, b)$.
5. $(-\infty, b] \cap [a, +\infty) = [a, b]$.

Theorem:

Any Borel set is Lebesgue measurable.

Proof

It suffices to demonstrate that $(a,+\infty)$ is measurable $\forall a \in \mathbb{R}$.
$\forall E \subset \mathbb{R}$, we want to show that $m^*(E \cap (a,+\infty)) + m^*(-\infty, a]) \leq m^*(E)$.
Then, $\forall \varepsilon > 0$, $\exists \mathcal{C} = \{I_i\}$ with I_i open intervals such that $\sum_{I_i \in \mathcal{C}} |I_i| \leq m^*(E) + \varepsilon/2$. Set

$$\begin{aligned}
\mathcal{C}^{\ell} &= \{I \in \mathcal{C} \mid x < a, \forall x \in I\} \\
\mathcal{C}^{r} &= \{I \in \mathcal{C} \mid x > a, \forall x \in I\} \\
\mathcal{C}^{m} &= \{I \in \mathcal{C} \mid a \in I\} = \{I_k\}
\end{aligned}$$

Then A$\mathcal{C} = \mathcal{C}^{\ell} \cup \mathcal{C}^{r} \cup \mathcal{C}^{m}$.
$\forall I_k \in \mathcal{C}^m = \{I_k\}$, $I_k = (c_k, d_k)$ for some $c_k, d_k \in \mathbb{R}$, define

$$\begin{aligned}
I_k^{\ell} &= \left(c_k, a + \frac{\varepsilon}{2^{k+1}}\right) \\
I_k^{r} &= (a, d_k)
\end{aligned}$$

Let $\mathcal{C}^m = \{I_k^{\ell}\} \cup \{I_k^{r}\} = \overline{\mathcal{C}}^{m\ell} \cup \overline{\mathcal{C}}^{mr}$. Then

$$\begin{aligned}
&\mathcal{C}^{\ell} \cup \overline{\mathcal{C}}^{m\ell} \text{ covers } E \cap (-\infty, k] \\
&\mathcal{C}^{r} \cup \overline{\mathcal{C}}^{mr} \text{ covers } E \cap (k, +\infty) \\
&\mathcal{C}^{\ell} \cup \mathcal{C}^{r} \cup \mathcal{C}^{m} \text{ covers } E
\end{aligned}$$

Observe that

$$|I_k^{\ell}| + |I_k^{r}| \leq |I_k| + \frac{\varepsilon}{2^{k+1}}$$

Therefore

$$
\begin{aligned}
m^*(E \cap (a, +\infty)) &\leq \sum_{I \in \mathcal{C}^R + \overline{\mathcal{C}}^{mr}} |I| \\
m^*(E \cap [-\infty, a]) &\leq \sum_{I \in \mathcal{C}^\ell + \overline{\mathcal{C}}^{m\ell}} |I|
\end{aligned}
$$

Therefore

$$
\begin{aligned}
m^*(E \cap (a, +\infty)) + m^*(E \cap (-\infty, a]) &\leq \sum_{I \in \mathcal{C}^r \cup \overline{\mathcal{C}}^{mr}} |I| + \sum_{I \in \mathcal{C}^\ell |I| \cup \overline{\mathcal{C}}^{m\ell}} |I| \\
&= \sum_{I \in \mathcal{C}^r} |I| + \sum_{I \in \mathcal{C}^\ell} |I| + \sum_k \left(|I_k^\ell| + |I_k^r| \right) \\
&\leq \sum_{I \in \mathcal{C}} |I| + \sum_{k=1}^{+\infty} \frac{\varepsilon}{2^{k+1}} \\
&\leq m^*(E) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&\leq m^*(E) + \varepsilon
\end{aligned}
$$

Lebesgue Measurable vs Borel

Theorem

The following statements are equivalent

1. A is measurable.
2. $\forall \varepsilon > 0$, $\exists U$ open, $U \supset A$ such that $m(U \setminus A) < \varepsilon$.
3. $\forall \varepsilon > 0$, $\exists C$ closed, $C \subset A$ such that $m(A \setminus C) < \varepsilon$.
4. $\forall A \in \mathbb{R}$, $\exists \bigcap_{n=1}^\infty U_n = F \in B$, U_n open, $U_n \supset A$ such that $F \supset A$ and $m(F \setminus A) = 0$.
5. $\exists \{C_n\}$, C_n closed and $C_n \subset A$ such that $G = \bigcup_{n=1}^\infty C_n \subset A$ and $m(A \setminus G) = 0$.

Corollary

Every measurable set is the union of a Borel set and a measure zero set.

Proof 1 Implies 2

Step 1: if $m(A) < \infty$, then for $\varepsilon > 0$, $\exists U$ open and $U \supset A$, then

$$
m(U) \leq m(A) + \varepsilon \iff m(U \setminus A) = m(U) - m(A) \leq \varepsilon
$$

Step 2: let $A_n = A \cap (-n, n)$, $n \in \mathbb{N}$.
Then $m(A_n) \leq 2n < +\infty$.
For ech A_n, $\exists U_n$ open with $U_n \supset A_n$ and $m(U_n \setminus A_n) < \frac{\varepsilon}{2^n}$
Let $U = \bigcup_{n=1}^\infty U_n$ and $A = \bigcup_{n=1}^\infty A_n$.
Now verify that

$$m(U \setminus A) = m\left(\bigcup_{n=1}^{\infty} U_n \setminus \bigcup_{n=1}^{\infty} A_n\right) = m\left(\bigcup_{n=1}^{\infty} U_n \setminus A_n\right) \leq \sum_{n=1}^{\infty} m(U_n \setminus A_n) \leq \varepsilon$$

Proof 2 Implies 3

Write

$$A \setminus C = A \cap C^c = C^c \cap A = C^c \setminus A^c$$

Apply (2).

Proof 3 Implies 4

U_n comes from 2.

Proof 4 Implies 5

Follows from 4.

Proof 5 Implies 1

$A = G \cup (A \setminus G) \implies A$ is measurable.

Example: Non-measurable Set

Define $x \sim y$ if $x - y \in \mathbb{Q}$, $\forall x, y \in \mathbb{R}$.
Let $A = \{x \in (0,1) \mid x \text{ is a representative of each class } \mathbb{R}/\sim\} \subset (0,1) \subset \mathbb{R}$.
Claim: A is not Lebesgue measurable.
Let $(-1,1) \supset S = \bigcup_{r \in \mathbb{Q} \cap (0,1)} (A + r) \supset (0,1)$, and observe that $\mathbb{Q} \cap (0,1)$ is countable.
So $(A + r) \cap (A + s) = \varnothing$ for $s \neq r$.
Then $1 < m(S) < 2$, so $m(A) = 0$ and $m(A) > 0$ are both contradictions.

January 18, 2024

Abstract measure theory.

Definition: Topological Space

A set X equipped with a collection of subsets $\tau \subset 2^X$ where τ is a topology if

1. $\varnothing, X \in \tau$
2. Union of subsets in τ remains in τ.
3. Intersection of finitely many subsets in τ remains in τ.

Any subset of τ is called an open set of X.

Definition: Measure Space

For a set X with $\Lambda \subset 2^X$ a σ-algebra such that

1. $\varnothing \in \Lambda$
2. $A^c \in \Lambda$ if $A \in \Lambda$.
3. $\bigcup_{i=1}^{\infty} A_i \in \Lambda$ if $A_i \in \Lambda$.
4. Remark: Borel Sigma Algebra

The σ-algebra generated by τ for a topological space (X, τ).
The measure space (X, Λ, μ), $\Lambda \subset 2^X$ a σ-algebra equipped with set function $\mu : \Lambda \to [0, +\infty]$ such that

1. $\mu(\varnothing) = 0$
2. $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m(A_i)$ for $A_i \in \Lambda$ and $A_i \cap A_j = \varnothing$ for all $i \neq j$ (countable additivity).

Proposition: Monotonicity

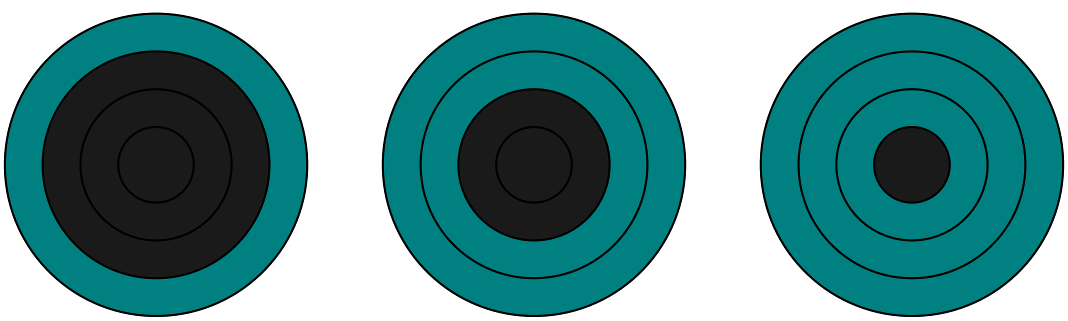
$A, B \in \Lambda,\ A \subseteq B \implies \mu(A) \leq \mu(B)$.

Proposition: Countable Subadditivity

$\mu\left(\bigcup A_i\right) \leq \sum \mu(A_i)$ if $A_i \in \Lambda$

Proposition: Monotone Convergence

Given $A_i \subset \Lambda$ such that $A_i \subset A_{i+1}$ where $A = \bigcup_{i=1}^{\infty} A_i$, then $\mu(A_i) \to \mu(A)$.
Similarly, if $A_i \supset A_{i+1}$ such that $A = \bigcap_{i=1}^{\infty} A_i$, then $\mu(A_i) \to \mu(A)$ if $\mu(A_k) < +\infty$ for some $k = 1, 2, 3, \ldots$.



Given $A_i' = \begin{cases} A_1 & i = 1 \\ A_i \setminus \bigcup_{j=1}^{i-1} A_j & i > 1 \end{cases}$, $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} A_i'$ and

$$\mu(A) \sum_{i=1}^{\infty} A_i' = \lim_{n\to\infty} \sum_{i=1}^{\infty} \mu(A_i')$$

and

$$\sum_{i=1}^{n} \mu(A_i') = \mu(A_1) + (\mu(A_2) - \mu(A_1)) + (\mu(A_3) - \mu(A_2)) + \cdots + (\mu(A_n) - \mu(A_{n-1})) = \mu(A_n)$$

Similarly, $A_1 \setminus A = \bigcup_{i=2}^{\infty} (A_i \setminus A_{i-1})$ where $\mu(A_1) < +\infty$ gives

$$\mu(A_1) - \mu(A) = \mu(A_1) + \sum_{i=2}^{\infty} (\mu(A_i) - \mu(A_{i-1})) = \lim_{n\to\infty} \mu(A_n)$$

Definition: Complete Measure Space

A measure space (X, Λ, μ) is complete if $\forall A \in \Lambda$ with $\mu(A) = 0$, then $\forall B \subset A$ and $B \in \Lambda$.

Example

The Lebesgue measure space on the reals $(\mathbb{R}, \mathcal{L}, m)$ is complete.

Theorem: Completion of a Measure Space

Given a measure space (X, Λ, μ), then there exists $(X, \overline{\Lambda}, \overline{\mu})$ such that

1. $\Lambda \subset \overline{\Lambda}$.
2. If $A \in \Lambda$, then $\overline{\mu}(A) = \mu(A)$.
3. $(X, \overline{\Lambda}, \overline{\mu})$ is complete.

Proof (Construction)

Let $\overline{\Lambda} = \{A \cup Z \mid A \in \Lambda, \exists D \in \Lambda, m(D) = 0, Z \subset D\}$ and $\overline{\mu}(A \cup Z) := \mu(A)$.
Verify:

1. $\overline{\Lambda}$ is a σ-Algebra.
   (a) If $A \cup Z \in \overline{\Lambda}$, then $(A \cup Z)^c \in \overline{\Lambda}$.
   (b) If $A_i \cup Z_i \in \overline{\Lambda}$, then $\bigcup (A_i \cup Z_i) \in \overline{\Lambda}$.
2. $\overline{\mu}$ is a well-defined measure on $\overline{\Lambda}$.
3. $(X, \overline{\Lambda}, \overline{\mu})$ is complete.

- Proof of 1
  Given $A \in \Lambda$ and $Z \subset D$ where $\mu(D) = 0$ and $D \in \Lambda$, we know $D^c \subset Z^c$ and $Z^c = D^c \cup (Z^c \cap D)$. Therefore

$$(A \cup Z)^C = A^c \cap Z^c = A^c \cap (D^c \cup (Z^c \cap D)) = (A^c \cap D^c) \cup (A^c \cap Z^c \cap D) \in \overline{\Lambda}$$

  Since $A^c \cap D^c \in \Lambda$ and $A^c \cap Z^c \cap D \in D$
  Since $\bigcup A_i \in \Lambda$ and $\bigcup Z_i \subset \bigcup D_i$,

$$\bigcup_{i=1}^{\infty} (A_i \cup Z_i) = \left(\bigcup_{i=1}^{\infty} A_i\right) \cup \left(\bigcup_{i=1}^{\infty} Z_i\right) \in \overline{\Lambda}$$

- Proof of 2
  Given $A_1 \cup Z_1 = A_2 \cup Z_2$, $A_1 \subset A_2 \cup Z_2 \subset A_2 \cup D_2$ implies $\mu(A_1) \le \mu(A_2)$.
  Then, $\mu(A_2) \le \mu(A_1) \implies \mu(A_1) = \mu(A_2)$. So $\overline{\mu}$ is well defined.
  Given $\{A_i \cup Z_i\}$ with $(A_i \cup Z_i) \cap (A_j \cup Z_j) = \varnothing$ for all $i \neq j$,

$$\overline{\mu}\left(\bigcup_{i=1}^{\infty}(A_i \cup Z_i)\right) = \overline{\mu}\left(\left(\bigcup_{i=1}^{\infty} A_i\right) \cup \bigcup_{i=1}^{\infty} Z_i\right) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty} \overline{\mu}(A_i \cup Z_i)$$

  So $\overline{\mu}$ is countably additive and therefore a measure.

Borel Measure and Radon Measure

Given a measure space (X, Λ, μ) and an underlying topology (X, τ),

Definition: Borel Measure

μ is a Borel measure if all borel sets $\tau \subset \Lambda$.

Definition: Locally Finite Measure

μ is locally finite if $\forall x \in X$, $\exists U \subset X$ a neighborhood such that $\mu(U) < +\infty$.

Definition: Borel Regularity

μ is Borel regular if $\forall A \in \Lambda$, $\exists B$ a Borel set such that $B \supseteq A$ and $\mu(B) = \mu(A)$.

Definition: Radon Measure

μ is a Radon measure if

1. it is a Borel measure.
2. $\mu(K) \le +\infty$ for K compact.
3. $\mu(V) = \sup\{\mu(K) \mid K \subset V, K \text{ compact}\}$, V open.
4. $\mu(A) = \inf\{\mu(V) \mid A \subset V, V \text{ open}\}$, $\forall A \in \Lambda$.

- Example 1
  Lebesgue measure.

- Example 2
  Point charge: $\mu(\{x\}) = 1$ and $\mu(A) = 0$ if $x \notin A$.

Theorem:

Let (X, Λ, μ) be a Borel regular measure space where the underlying topology (X, τ) is a metric space. Then

1. For $A \in \Lambda$ with $\mu(A) < +\infty$, $\forall \varepsilon > 0$, $\exists C \subseteq A$ closed such that $\mu(A \setminus C) < \varepsilon$.
2. For $A \in \Lambda$, $\exists\{V_i\}$ open sets such that $A \subset \bigcup_{i=1}^{\infty} V_i$ and $\mu(V_i) < +\infty$. Then $\forall \varepsilon > 0$, $\exists U$ open with $A \subset U$ and $\mu(U \setminus A) < \varepsilon$.

Proof

Given $\mu(A) < +\infty$, $\nu(B) = \mu(B \cap A) < +\infty$, $\forall B \in \Lambda$ and (X, Λ, ν).
Let $F = \{B \in \Lambda \mid \forall \varepsilon > 0, \exists C \subset B$ closed, with $\nu(B \setminus C) < \varepsilon\}$.
Note that closed sets are in F.
Claim 1: the Borel σ-algebra is in F.
Claim 2: if $A_i \in F$, $\bigcup A_i, \bigcap A_i \in F$.
Given claim 2, $\forall U$ open, U^c is closed. Then $U_\varepsilon = \{x \in U \mid \operatorname{dist}(x, U^c) \leq \varepsilon\}$ is closed and, therefore, $U = \bigcup_{i=1}^{\infty} U_{1/i}$.
So, given $A_i \in F$, $\exists C_i \subset A_i$ closed where $\nu(A_i \setminus C_i) < \varepsilon/2^{i+1}$. We want to show that $\nu\left(\bigcap A_i \setminus \bigcap C_i\right) < \varepsilon$.
Then, for $x \in \bigcap A_i \setminus \bigcap C_i$, $x \in A_i$ for all i and $x \notin C_{i_0}$ for some i_0.
Therefore $x \in A_{i_0}$, $x \notin C_{i_0}$, and $x \in A_{i_0} \setminus C_{i_0}$. It follows that

$$\bigcap_{i=1}^{\infty} A_i \setminus \bigcap_{i=1}^{\infty} C_i \subset \bigcup_{i=1}^{\infty} (A_i \setminus C_i)$$
$$\nu\left(\bigcap_{i=1}^{\infty} A_i \setminus \bigcap_{i=1}^{\infty} C_i\right) \leq \sum_{i=1}^{\infty} \nu(A_i \setminus C_i) < \varepsilon$$

Therefore

$$\nu\left(\bigcup_{i=1}^{\infty} A_i \setminus \bigcup_{i=1}^{n} C_i\right) \to \nu\left(\bigcup_{i=1}^{\infty} A_i \setminus \bigcup_{i=1}^{\infty} C_i\right) \leq \nu\left(\bigcup_{i=1}^{\infty} (A_i \setminus C_i\right) < \frac{\varepsilon}{2}$$

so $\exists N >> 1$ such that $\nu\left(\bigcup_{i=1}^{\infty} \setminus \bigcup_{i=1}^{N} C_i < \varepsilon\right)$ with $\bigcup_{i=1}^{N} C_i$ closed.

Restatement

For A Borel,

$$\varepsilon > \nu(A \setminus C) = \mu((A \setminus C) \cap A) = \mu(A \setminus C)$$

January 23, 2024

Review - Abstract Measure

Given (X, Λ, μ) where $\Lambda \subseteq 2^X$ is a σ-algebra, $\mu : \Lambda \to [0, +\infty]$

1. $\mu(\varnothing) = 0$.
2. $m\left(\bigcup A_i\right) = \sum \mu(A_i)$, $A_i \cap A_j = \varnothing$.

Properties of a Measure

Monotonicity

$\mu(A) \subseteq \mu(B)$, $A, B \in \Lambda$, $A \subseteq B$

Countable Subadditivity

$\mu\left(\bigcup A_i\right) \leq \sum \mu(A_i)$

Monotone Convergence

$A_i \subset A_{i+1},\ A_i \to \bigcup A_i \implies \mu(A) = \mu\left(\bigcup A_i\right)$.
$A_i \supset A_{i+1},\ A_i \to \bigcap A_i \implies \mu(A_i) \to \mu\left(\bigcap A_i\right)$ if $\mu(A_1) < \infty$

- Example
  $A_n = (n, +\infty)$ gives $\bigcap A_n = \varnothing$

Completeness of a Measure

(X, Λ, μ) is complete if $\forall A \in \Lambda$ with $\mu(A) = 0$, then $\forall B \in \Lambda$ if $B \subseteq A$.

Theorem:

Given (X, Λ, μ), there exists $(X, \overline{\Lambda}, \overline{\mu})$ such that $\Lambda \subset \overline{\Lambda}$ and $\overline{\mu}(A) = \mu(A)$ if $A \in \Lambda$.

$$\overline{\Lambda} = \{A \cup Z \mid A \in \Lambda, Z \subset D, D \in \Lambda \text{ with } \mu(D) = 0\}$$

$$\overline{\mu}(A \cup Z) = \mu(A)$$

$(X, \overline{\Lambda}, \overline{\mu})$ is complete.

Measure Space with Topology

Given a topological space (X, τ), a measure space (X, Λ, μ)

Definition: Locally Finite

The measure μ is locally finite if $\forall x \in X$, there exists an open neighborhood U of x such that $U \in \Lambda$ and $\mu(U) < +\infty$.

Definition: Borel Measure

μ is a Borel measure if the Borel σ-algebra generated by τ, $\mathcal{B}$, is a subset of Λ.

Definition: Borel Regular

$\forall A \in \Lambda$, $\exists B \in \mathcal{B}$ such that $B \supset A$ and $\mu(B) = \mu(A)$.

Definition: Radon Measure

1. Borel.
2. $\mu(K) < +\infty$ for K compact.
3. $\mu(V) = \sup\{\mu(K) \mid K \text{ compact}, K \subset V\}$, $\forall V$ open.
4. $\mu(A) = \inf\{\mu(V) \mid V \text{ open}, A \subset V\}$, $\forall A \in \Lambda$.

Theorem:

If X is a metric space equipped with a Borel regular (X, Λ, μ), then

1. $\forall A \in \Lambda$, $\mu(A) < +\infty$, $\forall \varepsilon > 0$, $\exists C$ closed where $C \subset A$ and $\mu(C \setminus A) < \varepsilon$.

2. If $\exists \{V_i\}$, V_i open and $\mu(V_i) < +\infty$, and $A \in \Lambda$ with $A \subset \bigcup V_i$, then $\exists U$ open such that $A \subset U$ and $\mu(U \setminus A) < \varepsilon$.

Proof of 1

Define $\nu(B) = \mu(B \cap A)$ such that (X, Λ, ν) is a new measure space.
Define $F = \{B \in \Lambda \mid \forall \varepsilon > 0, \exists C \text{ closed}, C \subset B, \nu(B \setminus C) < \varepsilon\}$, all closed sets in F.
Claim 1: $\bigcap A_i, \bigcap A_i \in F$ if $A_i \in F$.
Claim 2: U is open.
$U = \bigcup U_i$, $U_i = \left\{x \in U \mid \text{dist}(x, U^c) \leq \frac{1}{i}\right\}$, therefore $\mathcal{B} \subset F$.
IMAGE HERE - 1
If A is Borel, then $\forall \varepsilon > 0$, $\exists C$ closed with $C \subset A$ and $\mu(A \setminus C) < \varepsilon$.
To finish, $\forall A \subset \Lambda$ by Borel Regularity of μ, $\exists B \in \mathcal{B}$ such that $B \supset A$ and $\mu(B) = \mu(A)$.
Note also that this requires $\mu(B \setminus A) = 0$ since $\mu(A) < +\infty$.
IMAGE HERE - 2
Then $B \setminus A \in \Lambda$, $\exists D \in \mathcal{B}$ such that $DB \setminus A$ and $\mu(D) = \mu(B \setminus A) = 0$. Then

$$\begin{aligned} B \cap A^c = B \setminus A &\subset D \\ (B \cap A^c)^c &\supset D^c \\ B \cap (B^c \cup A) &\supset D^c \cap B \\ A &\supset B \setminus D \end{aligned}$$

$$A \setminus (B \setminus D) = A \cap (B \cap D^c)^c = A \cap (B^c \cup D = \overbrace{(A \cap B^c)}^{\varnothing} \cup A \cap D = A \cap D \subset D$$

Therefore $B \setminus D \subset A$, and $\mu(A \setminus (B \setminus D)) = 0$.
$B \setminus D \in \mathcal{B}$, $\forall \varepsilon > 0$, $\exists C$ closed such that $C \subset B \setminus D \subset A$, $\mu((B \setminus D) \setminus C) < \varepsilon$.
This implies that $\mu(A \setminus C) = \mu(A \setminus (B \setminus D)) + \mu((B \setminus D) \setminus C) < \varepsilon$.

Proof of 2

Consider $V_i \setminus A$ where $\mu(V_i \setminus A) \leq \mu(V_i) < +\infty$.
By (1), $\exists C_i$ closed with $C_i \subset V_i \setminus A$ and $\mu((V_i \setminus A) \setminus C_i) < \varepsilon / 2^{i+1}$. Write

$$(V_i \setminus A) \setminus C_i = (V_i \setminus A) \cap C_i^c = V_i \cap A^c \cap C_i^c = (V_i \cap C_i^c) \cap A^c = (V_i \setminus C_i) \setminus A$$

Note that $V_i \setminus C_i$ is open, since C_i is closed.
Define $U = \bigcup (V_i \setminus C_i) \supset A$. Then,

$$U \setminus A = \left(\bigcup (V_i \setminus C_i)\right) \setminus A = \bigcup ((V_i \setminus C_i) \setminus A)$$

Therefore $\mu(U \setminus A) \leq \varepsilon \frac{\varepsilon}{2^{1+1}} = \varepsilon$.

Remark

$X = \bigcup V_i$, V_i open and $\mu(V_i) < +\infty$.
Then $\forall A \in \Lambda$, $\forall \varepsilon > 0$, $\exists U$ open such that $U \supset A$ and $\mu(U \setminus A) < \varepsilon$.
For A^c, $\exists U \supset A^c$ ($\implies U^c \subset A$), $\mu(U \setminus A^c) < \varepsilon$. So

$$U \cap A = U \setminus A^c = A \setminus U^c = A \cap U$$

and $\mu(A \setminus U^c) < \varepsilon$, $U^c \subset A$ with U^c closed.

Corollary

For $\mathbb{R}^n$, a measure is Radon if and only if it is locally finite and Borel regular.

- Proof
  $(\Longrightarrow)$
  Let $B(r, x_0) = \{x \in \mathbb{R}^n \mid |x - x_0| < r\}$ and $\overline{B(r, x_0)} = \{x \in \mathbb{R}^n \mid |x - x_0| \leq r, \text{ compact}\}$.
  Then $\mu(B(r, x_0)) \leq \mu(\overline{B(r, x_0)}) < +\infty$. So μ is locally finite.
  For $A \in \Lambda$, we may assume without loss of generality that $\mu(A) < +\infty$.
  Then $\forall i$, $\exists U_i$ open where $U_i \supset A$ and $\mu(A) \leq \mu(U_i) \leq \mu(A) + \frac{1}{i} < +\infty$.
  Set $G = \bigcap U_i \in \mathcal{B}$, then $\mu(G) = \mu(A)$.
  $(\Longleftarrow)$

  1. Borel regular implies Borel.

  2. For K compact, $\forall x \in K \ni U_x$ open where $\mu(U_x) < +\infty$.

     $\{U_\lambda\}_{\lambda \in k}$ is an open cover. Therefore there is a finite subcover $\{U_{\lambda_i}\}_{i=1}^{\lambda}$ where

     $$\mu(K) \leq \mu\left(\bigcup_{i=1}^{k} U_{\lambda_i}\right) \leq \sum_{i=1}^{k} \mu\left(U_{x_i}\right) < +\infty$$

  3. $\forall V$ open, $B(i) = B(i, 0)$, $V \cap B(i)$, $\mu(V \cap B(i)) < +\infty$, $\exists C_i$ closed where $C_i \subset V_{\cap B(i)}$ so C_i is bounded and therefore compact.

     So $\mu(C_i) \leq \mu\left((V \cap B(i)) \setminus C_i\right) < \frac{1}{i}$ and $\mu(V \cap B(i)) \leq \mu(C_i) + \frac{1}{i}$.
     Then $\mu(V) = \lim_{i\to\infty} \mu(V \cap B(i)) = \lim_{i\to\infty} \mu(C_i)$, and $C_i \subset V \cap B(i) \subset V$ compact.
     Therefore $\mu(V) = \sup\{\mu(K) \mid K \text{ compact}, K \subset V\}$.

  4. $\forall A \in \Lambda$, $\forall i$, $\exists U_i$ open where $U_i \supset A$ and $\mu(U_i \setminus A) < \frac{1}{i}$

     This implies that $\mu(A) \leq \mu(U_i) \leq \mu(A) + \frac{1}{i}$ and therefore $\mu(A) = \inf\{\mu(U) \mid U \supset A, U \text{ open}\}$.

Caratheodory Construction

Definition: Outer Measure

$\mu^*(A)$, $\forall A \in 2^X$

1. $\mu^*(\varnothing) = 0$.

2. $\mu^*(A) \leq \mu^*(B)$ if $A \subseteq B$.

3. $\mu^*\left(\bigcup A_i\right) \leq \sum \mu^*(A_i)$, $\forall A_i \in 2^X$ (countable subadditivity)

Define $\Lambda = \{A \in 2^x \mid \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c), \forall E \in 2^X\}$.
Then $\mu(A) = \mu^*(A)$ if $A \in \Lambda$.
(X, Λ, μ) is complete.

January 25, 2024

Theorem: Caratheodory Construction

Outer Measure

$u^* : 2^X \to [0, +\infty]$.

1. $\mu^*(\varnothing) = 0$
2. Monotonicity: $\mu^*(A) \leq \mu^*(B)$, $A \subseteq B$
3. Countable Subadditivity: $\mu^*\left(\bigcup_i A_i\right) \leq \sum_i \mu^*(A_i)$.

Caratheodory Criterion

$A \subset X$ is measurable if $\forall E \in X$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

Theorem

The collection Λ of all measurable sets is a σ-algebra.
(X, Λ, μ) is a complete measure space (cf. proof of Lebesgue completeness).

Hausdorff Measure

$\forall A \subseteq \mathbb{R}^n$, $\forall s \geq 0$, $H_s^\delta(A) = \inf\left\{\sum_i (d(E_i))^s \mid \bigcup_i E_i \supset A,\ d(E_i) \leq \delta\right\}$ where $d(E_i)$ is the diameter of E_i.
Notice that $H_s^{\delta_1}(A) \leq H_s^{\delta_2}(A)$ if $\delta_2 \leq \delta_1$.
Let $H_s^*(A) = \lim_{\delta \to 0} H_s^\delta(A)$, $\forall A \in 2^{\mathbb{R}^n}$.
Claim: H_s^* is an outer measure.

- Verify
  1. $H_s^*(\varnothing) = 0$.
  2. $H_s^*(A) \leq H_s^*(B)$, $\forall A \subseteq B \subseteq \mathbb{R}^n$.
  3. Given $A_i \subset \mathbb{R}^N$,

     $\exists \delta_0 > 0$ such that $\forall \delta < \delta_0$, $H_s^*\left(\bigcup_i A_i\right) \leq H_s^\delta\left(\bigcup_i A_i\right) + \frac{\varepsilon}{2}$.
     Then $\forall \delta < \delta_0$ fixed, $\forall A_i$, $\exists \{E_i^j\}$ such that $\bigcup_j E_i^j \supset A_i$, $\sum_j (d(E_i^j))^s \leq H_s^\delta(A_i) + \frac{\varepsilon}{2^{i+1}}$, and $d(E_j^j) \leq \delta$. So

$$
\begin{aligned}
H_s^\delta\left(\bigcup_i A_i\right) &\leq \sum_{i,j}(d(E_i^j))^s \\
&= \sum_i \left(\sum_j (d(E_i^j)^s\right) \\
&= \sum_i \left(H_s^\delta(A_i) + \frac{\varepsilon}{2^{i+1}}\right) \\
&= \sum_i H_s^\delta(A_i) + \frac{\varepsilon}{2}
\end{aligned}
$$

and

$$
H_s^*\left(\bigcup_i A_i\right) \leq \sum_i H_s^\delta(A_i) + \varepsilon \leq \sum_i H_s^*(A_i) + \varepsilon, \quad \forall \varepsilon > 0.
$$

Then, since H_s^* is an outer measure, it is a measure by the Caratheodory construction.

Definition: Hausdorff Measure

The Hausdroff Measure $H_s : \Lambda \to [0, +\infty)$ on a σ-algebra $\Lambda \subset 2^{\mathbb{R}^n}$.

Not Locally Finite

Consider $B(0,1) = \{x \mid |x| < 1\}$.
Then $H_s(B(0,1)) = \infty$ for $s < n$.
That is, the Hausdorff measure is not locally finite for $s < n$.

Complete

The Hausdorff measure, by the Caratheodory construction, is complete.

Symmetry

1. Translation Invariance: $H_s(A + x) = H_s(A)$.
2. Rotation Invariance: $H_s(RA) = H_s(A)$.
3. Scaling: $H_s(\lambda A) = \lambda^s H_s(A)$.

Open Balls Measurable

What about $B(0,1) \subset \mathbb{R}^n$. For $\delta > 0$,

$$
H_s^*(E \cap B(0,1)) + H_s^*(E \cap B(0,1)^c) \leq H_s^*(E \cap B(0,1-\delta)) + H_s^*(E \cap (B(0,1) \setminus B(0,1-\delta))) + H_s^*(E \cap B(0,1)^c)
$$

Want to show that for all $\varepsilon > 0$, this is $\leq H_s^*(E) + \varepsilon$.

- Lemma 1

$$
\begin{aligned}
H_s^*(E \cap B(0,1-\delta)) + H_s^*(E \cap B(0,1)^c) &= H_s^*(E \cap (B(0,1-\delta) \cup B(0,1)^c)) \\
&\leq H_s^*(E)
\end{aligned}
$$

- Lemma 2

$$H_s^*(E \cap (B(0,1) \setminus B(0,1-\delta)) < \varepsilon.$$

- Lemma 1'

  If $A, B \subset \mathbb{R}^n$, $\operatorname{dist}(A,B) > 0$, then $H_s^*(A \cup B) = H_s^*(A) + H_s^*(B)$.
  Since $\{E_i\}$ covering $A \cup B$, $d(E_i) < \frac{1}{4}\operatorname{dist}(A,B)$ gives

$$\delta < \frac{1}{4}\operatorname{dist}(A,B) \iff \{E_j^A\} \cup \{E_k^B\}$$

  if and only if $\{E_j^A\}$ covers A and $\{E_k^B\}$ covers B. Therefore,

$$\sum_i (d(E_i))^s = \sum_j (d(E_j^A))^s + \sum_k (d(E_k^B))^s$$
$$\inf\left\{\sum_i (d(E_i))^s\right\} = \inf\left\{\sum_j (d(E_j^A))^s\right\} + \inf\left\{\sum_k (d(E_k^B))^s\right\}$$

  and $H_s^\delta(A \cup B) = H_s^\delta(A) + H_s^\delta(B)$.
  Thus $H_s^*(A \cup B) = H_s^*(A) + H_s^*(B)$.

Let $T_i = E \cap \left(B\left(0, 1 - \frac{1}{i+1}\right)\right) \setminus B\left(0, 1 - \frac{1}{i}\right)$.
IMAGE HERE - 1 CONCENTRIC RINGS
We want to show that $H_s^*\left(E \cap \left(B(0,1) \setminus B\left(0, \frac{1}{i}\right)\right)\right) < \varepsilon$ for $i >> 1$. Then

$$\bigcup_{k=1} T_k = (B(0,1) \setminus \{0\}) \cap E$$
$$\bigcup_{k=i} T_k = \left(B(0,1) \setminus B\left(0, 1 - \frac{1}{i}\right)\right) \cap E$$

Claim: $\sum_i H_s^*(T_i) < +\infty$. It suffices to prove this claim.

$$\sum_{i \text{ even}}^{2k} H_s^*(T_i) = H_s^*\left(\bigcup_{i \text{ even}}^{2k}\right) \leq H_s^*(E) < +\infty$$

$$\sum_{i \text{ odd}}^{2k+1} H_s^*(T_i) = H_s^*\left(\bigcup_{i \text{ odd}}^{2k+1}\right) \leq H_s^*(E) < +\infty$$

Then $\sum_i^k H_s^*(T_i) <= \infty$.

Borel

Take a countable, dense set $\{q_i\} \subset \mathbb{R}^n$ and $\left\{B\left(q_i, \frac{1}{k}\right)\right\}_{i,k}$.
Claim: $\forall V \subseteq \mathbb{R}^n$ open, then $V = \bigcup_l B\left(q_{i_l}, \frac{1}{k_l}\right)$.
Then $\mathcal{B} \subseteq \Lambda$ and the Hausdorff measure is Borel.

Borel Regular

$\forall A \subset \Lambda$, $\exists B \in \mathcal{B}$ such that $B \supset A$ and $H_s(B) = H_s(A)$.
$\forall \delta = \frac{1}{j}$, $\{E_i^j\}$ E_i^j closed balls with $d(E_i^j) < \frac{1}{j}$,

$$\sum_i (d(E_i))^s \le H_s^{\frac{1}{j}}(A) + \frac{1}{j}$$

Take $B = \bigcap_j \left(\bigcup_i E_i^j\right) \in \mathcal{B}$ since $B = \bigcap_j \bigcup_i E_i^j \supset A$. Then

$$\begin{aligned} H_s^{\frac{i}{j}}(B) &\le H_s^{\frac{1}{j}}\left(\bigcup_i E_i^j\right) \\ &\le \sum_i H_s^{\frac{1}{j}}\left(E_i^j\right) \\ &\le \sum_i \left(d(E_i^j)\right)^s \\ &\le H_s^{\frac{1}{j}}(A) + \frac{1}{j} \end{aligned}$$

and in the limit as $j \to \infty$

$$H_s^*(A) \le H_s^*(B) \le H_s^*(A)$$

Fractional or Hausdorff Dimension

Theorem:

1. $H_s^*(A) < +\infty \implies H_t^*(A) = 0$, $\forall t > s \ge 0$.
2. $H_t^s > 0 \implies H_s(A) = \infty$, $\forall 0 \le s < t$

Proof

$$\begin{aligned} H_s^\delta(A) &\sim \sum_i (d(E_i))^s \\ &= \sum_i (d(E_i))^t (d(E_i))^{s-t} \end{aligned}$$

So $s < t$ gives $\ge \delta^{s-t}$.
In the other direction, when $s < t$

$$\begin{aligned} \sum_i (d(E_i))^t &= \sum (d(E_i))^s (d(E_i))^{t-s} \\ &\le \delta^{t-s} \sum_i (d(E_i))^s \end{aligned}$$

Definition: Hausdorff Dimension

Given $A \subset \mathbb{R}^n$,

$$\begin{aligned}
\dim_H(A) &= \sup\left\{s \mid H_s^*(A) = \infty\right\} \\
&= \sup\left\{s \mid H_s^*(A) > 0\right\} \\
&= \inf\left\{s \mid H_s^*(A) = 0\right\} \\
&= \inf\left\{s \mid H_s^*(A) < +\infty\right\}
\end{aligned}$$

Example 1

$\mathbb{R}^n$ has n Hausdorff dimension.
Consider the n-cube with sides d, $C(d)$. Then

$$H_s(C(d)) = C(n,s)d^s$$

So $C(n,s) = C(n,s)2^{nk}\frac{1}{(2^k)^s} = C(n,s)2^{(n-1)k}$.
If $s < n$, this tends to infinity as $k \to \infty$.
Is $s > n$ it tends to 0.

Example 2

Cantor set has Hausdorff dimension $\frac{\log(2)}{\log(3)}$.

$$\bigcup_{k=1}^{2^n} C_n^k = \frac{\log(2)}{\log(3)}$$

where $|C_n^k| = \frac{1}{3^n}$, so $H_s^\delta(C^n) \sim \frac{2^n}{(3^n)^s} = \left(\frac{2}{3^s}\right)^n$.

Example 3

The Koch snowflake has dimension $\frac{\log(4)}{\log(3)}$.