

# Analysis III

## Homework

Exercises (homework) can be found in the script at the end of each Chapter.

Chapter I: # 3 (only for convex sets), # 4 due Th 4-25

Chapter II: # 2,4,5,6 due Th 5-2

Chapter III: # 3c, 4 due Th 5-9

Chapter IV: # 2b, 3, 4, 6 due Th 5-16

Chapter V: # 2,4,6 due Th 5-25

Chapter VI: # 2,3,4 due Th 6-1

## Key Dates

Instruction begins: Mo, April 1

Instruction ends: Fr, June 7

Final's week: June 10-13 (Mo-Th)

Holiday: Mo, May 27

## April 2, 2024

No class Thursday, April 04.

Makeup class (tentatively) on Friday, April 12 at 10:30.

Discussion sections on Fridays (tentatively) at 11:40.

## Topological Vector Spaces

### Definition: Vector Spaces

$V$  over a field  $\mathbb{F}$  ( $\mathbb{R}$  or  $\mathbb{C}$ ).

### Definition: Topological Spaces

$(X, \tau)$  where  $\tau \subseteq \mathcal{P}(X)$  satisfying

1.  $\emptyset, X \in \tau$
2.  $A, B \in \tau \implies A \cap B \in \tau$
3.  $A_\omega \in \tau \implies \bigcup_\omega A_\omega \in \tau$

Recall:  $A \in \tau \iff A$  open  $\iff X \setminus A$  closed.

$A^\circ = \bigcup_{\substack{U \subseteq A \\ U \text{ open}}} U$  the set of interior points of  $A$ .

$\overline{A} = \bigcap_{\substack{F \supseteq A \\ F \text{ closed}}} F$  the closure of  $A$ .

$A'$  limit points of  $A$ .

Compact sets.

Locally compact sets.

Recall:  $X$  is Hausdorff iff  $\forall x, y \in X, \exists U, V \in \tau$  such that  $x \in U, y \in V$  and  $U \cap V = \emptyset$ .

## Definition: Bases for Topological Spaces

Definition: Let  $(X, \tau)$  be a topological space.  $\sigma \subseteq \tau$  is called a base for topology  $\tau$  if  $\forall x \in X, \forall U \in \tau, x \in U, \exists W \in \sigma$  such that  $x \in W \subseteq U$ .

### Proposition

$\sigma \subseteq \tau$  is a base for  $\tau$  if and only if every  $U \in \tau$  is the union of certain sets taken from  $\sigma$ .

$$(*) \quad \tau = \left\{ \bigcup_{\omega \in \Omega} W_\omega : \{W_\omega\}_{\omega \in \Omega} \subseteq \sigma, \Omega \right\}$$

### Proof

$(\Leftarrow)$   $\checkmark$

$(\Rightarrow)$  Take  $U \in \tau$  and let  $x \in U$ ,  $\leadsto$  find  $W_x \in \sigma, x \in W_x \subseteq U$ .

$$U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} W_x \subseteq U$$

Therefore  $\bigcup_{x \in U} W_x = U$ .

### Proposition

If  $\sigma$  is a base for some topology  $\tau$  on  $X$ , then

1.  $\forall x \in X, \exists W \in \sigma$  such that  $x \in W$ .
2.  $\forall U, V \in \sigma, \forall x \in U \cap V, \exists W \in \sigma$  such that  $x \in W \subseteq U \cap V$ .

Conversely, if  $\sigma \in \mathcal{P}(X)$  ( $\emptyset \notin \sigma$ ) satisfies (1) and (2), then  $\sigma$  is the base for a topology  $\tau$  (and  $\tau$  is given by  $(*)$ ).

Note that  $U, V \in \tau \implies U \cap V \in \tau$  (requires (2)).

If  $U = \bigcup U_\alpha$  and  $V = \bigcup V_\beta$ , then  $U \cap V = \bigcup_{\alpha, \beta} (U_\alpha \cap V_\beta) = \bigcup_{\alpha, \beta} \bigcup_{x \in U_\alpha \cap V_\beta} W_{\alpha, \beta, x}$ .

## Example: Metric Spaces

$(X, d)$  is a metric space if  $d : X \times X \rightarrow [0, +\infty)$  satisfies

1.  $d(x, y) = 0$  if and only if  $x = y$ .
2.  $d(x, y) = d(y, x)$ .
3.  $d(x, z) \leq d(x, y) + d(y, z)$  (triangle inequality).

## Definition: Epsilon Neighborhoods

$$B_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$$

$A \subseteq X$  is open if and only if  $\forall x \in A, \exists \varepsilon > 0$  such that  $B_\varepsilon(x) \subseteq A$ .  $x \in B_\varepsilon(x)$ .

$\tau$  = set of all open sets.

$$\sigma_1 = \{B_\varepsilon(x) : x \in X, \varepsilon > 0\}$$

is a base for the topology  $\tau$ .

$$\sigma_2 = \{B_{1/n}(x) : x \in X, n \in \mathbb{N}\}$$

is also a base for  $\tau$ .

## Definition: Direct Product - Product Topology

Let  $(X_1, \tau_1)$  and  $(X_2, \tau_2)$  be topological spaces.

Consider  $X = X_1 \times X_2$ . The product topology  $\tau$  on  $X$  is given by the base

$$\sigma = \{U_1 \times U_2 : U_1 \in \tau_1, U_2 \in \tau_2\}$$

More explicitly,

$$\tau = \left\{ \bigcup_{\omega} U_{1,\omega} \times U_{2,\omega} : U_{i,\omega} \in \tau_i \right\}$$

$(X_\omega, \tau_\omega)$  topological spaces  $(\omega \in \Omega)$

$$X = \prod_{\omega \in \Omega} X_\omega = \{(x_\omega)_{\omega \in \Omega} : x_\omega \in X_\omega\}$$

Formally,  $f \cong (x_\omega)_{\omega \in \Omega}$ ,  $x_\omega = f(\omega)$ ,  $f : \Omega \rightarrow \bigcup_{\omega \in \Omega} X_\omega$  such that  $f(\omega) \in X_\omega$ .

[ $x \neq \emptyset \iff X_\omega \neq \emptyset$  axiom of choice]

$$\sigma = \left\{ \prod_{\omega \in \Omega} U_\omega : U_\omega \in \tau_\omega \text{ and all but finitely many } U_\omega = X_\omega \right\}$$

## Definition: Subspace Topology

Given  $(X, \tau)$  and  $Y \subseteq X$ , then  $(Y, \tau_Y)$  is also a topological space where

$$\tau_Y \{U \cap Y : U \in \tau\}$$

## Definition: Local Bases for Topological Spaces

A collection  $\gamma \subseteq \tau$  is called a local base at  $x \in X$  if

1.  $\forall U \in \tau, x \in U, \exists W \in \gamma$  such that  $x \in W \subseteq U$ .
2.  $\forall W \in \gamma, x \in W$

## Example

Let  $(X, d)$  be a metric space.

$$\gamma_x = \{B_\varepsilon(x) : \varepsilon > 0\}$$

is a local base at  $x$ . Similarly,

$$\tilde{\gamma}_x = \{B_{1/n} : n \in \mathbb{N}\}$$

is a countable local base.

## Proposition

If  $\gamma_x$  ( $x \in X$ ) are local bases for  $\tau$  at  $X$ , then

$$\sigma = \bigcup_{x \in X} \gamma_x$$

is a bse for  $\tau$ .

## Proposition

$\{\gamma_x\}_{x \in X}$  are local bases at  $x$  for some topology  $\tau$  if and only if

1.  $\forall x \in X$ ,  $\gamma_x$  is a non-empty collection of subsets containing  $x$ .
2. If  $U \in \gamma_x$ ,  $V \in \gamma_y$ , and  $z \in U \cap V$ , then  $\exists W \in \gamma_z$  such that  $z \in W \subseteq U \cap V$ .

## Definition: Topological Vector Spaces

Suppose  $V$  is a vector space over  $\mathbb{F} = \mathbb{R}, \mathbb{C}$  and let  $\tau$  be a topology on  $V$ . Then  $V$  is a topological vector space (TVS) if

1.  $\forall x \in V$ ,  $\{x\}$  is closed.
2. The functions  $f, g$  (i.e. algebraic operations) are continuous.

$$\begin{aligned} f : V \times V &\rightarrow V, f(x, y) = x + y \\ g : \mathbb{F} \times V &\rightarrow V, g(\lambda, x) = \lambda \cdot x \end{aligned}$$

## Notation

For  $A_1, A_2 \subseteq V$  and  $B \subseteq \mathbb{F}$ ,

$$\begin{aligned} A_1 + A_2 &= \{a_1 + a_2 : a_1 \in A_1, a_2 \in A_2\} \\ a + A_1 &= \{a + \alpha : \alpha \in A_1\} \\ B \cdot A &= \{\beta \cdot a : \beta \in B, a \in A\} \\ \alpha \cdot A &= \{\alpha \cdot a : a \in A\} \end{aligned}$$

## Lemma

Let  $V$  be a TVS. Then

1.  $\forall x, y \in V, \forall \text{ open } U_{x+y} \ni x + y, \exists \text{ open } U_x \ni x, \text{ open } U_y \ni y \text{ such that } U_x + U_y \subseteq U_{x+y}.$
2.  $\forall x \in V, \alpha \in \mathbb{F}, \forall \text{ open } U_{\alpha x} \ni \alpha x, \exists \text{ open } U_x \ni x, U_\alpha \ni \alpha \text{ such that } U_\alpha \cdot U_x \subseteq U_{\alpha x}.$

### Proof of 1

Given  $x, y \in X, x + y \in U_{x+y}$  open.

$$f(x, y) = x + y \in U_{x+y}$$

and  $(x, y) \in f^{-1}(U_{x+y})$  open. In the product topology

$$(x, y) \in U_x \times U_y \subseteq f^{-1}(U_{x+y})$$

which implies  $x \in U_x$  and  $y \in U_y$ , both open, and  $U_x + U_y \subseteq U_{x+y}$ .

## April 9, 2024

Office Hours: Th 11:30 AM - 1:00 PM

Makeup Class: Friday, April 12 at 11:00 AM.

## Lemma 1

Let  $V$  be a TVS

1.  $\forall x, y \in V, \forall U_{x+y} \ni x + y \text{ open}, \exists U_x \ni x, U_y \ni y \text{ such that } U_x + U_y \subseteq U_{x+y}.$
2.  $\forall \alpha \in F, \forall U_{\alpha x} \ni \alpha x \text{ open}, \exists U_\alpha \ni \alpha \text{ open in } F, U_x \ni x \text{ such that } U_\alpha \cdot U_x \subseteq U_{\alpha x}.$

For 2. with  $\alpha = 0, \forall x \in X, \forall U \ni 0 \text{ open}, \exists \delta > 0, U_\delta \ni x \text{ open such that } B_\delta(0) \cdot U_x \subseteq U. \text{ That is, } \beta U_x \subseteq U, \forall |\beta| < \delta.$

## Proposition

In a TVS, the maps

1. Translation:  $T_a : x \in V \mapsto x + a \in V (a \in V)$
2. Multiplication:  $M_\lambda : x \in V \mapsto \lambda \cdot x \in V (\lambda \in \mathbb{F}, \lambda \neq 0)$

are continuous (in fact, homeomorphic).

### Proof

We know  $(x, y) \mapsto x + y$  and  $(\lambda, x) \mapsto \lambda \cdot x$  are continuous.

## Inversions

$T_a \circ T_{-a} = \text{id}$ ,  $T_{-a} \circ T_a = \text{id}$ ,  $M_\lambda \circ M_{1/\lambda} = \text{id}$ , and  $M_{1/\lambda} \circ M_\lambda = \text{id}$ .  
Therefore they are bijective and the inverses are continuous.

## Remark

If  $U$  is open, then  $a + U$  is also open.

If  $\gamma_0$  is a local base at 0, then  $\gamma_x = \{x + U : U \in \gamma_0\} = x + \gamma_0$  is a local base at  $x$ .

Recall that  $\gamma_x$  is a local base at  $x$  if  $\forall W \ni x$  open,  $\exists U \in \gamma_x$  such that  $x \in U \subseteq W$ .

That is, in a TVS only local bses at 0 are needed. We may interpret “local base” as “local base at 0”.

$$\sigma = \bigcup_{x \in V} (x + \gamma_0)$$

is a base for the TVS.

## Types of Topological Vector Spaces

### Normed Spaces / Banach Spaces

A normed space is a vector space over  $\mathbb{F}$  together with a norm  $|| \cdot ||$ , i.e. a map  $|| \cdot || : x \in V \mapsto ||x|| \in [0, \infty)$  such that

1.  $||x|| = 0 \iff x = 0$ .
2.  $||x + y|| \leq ||x|| + ||y||$ .
3.  $||\lambda x|| = |\lambda| \cdot ||x||$ .

### Remarks

A normed space is a metric space with  $d(x, y) = ||x - y||$ .

A local base (at 0) is given by  $\varepsilon$ -neighborhoods:

$$\gamma_0 = \{B_\varepsilon(0) : \varepsilon > 0\}$$

where

$$B_\varepsilon(0) = \{x \in V : ||x|| < \varepsilon\}$$

(open ball with radius  $\varepsilon > 0$ ).

### Convergence in Normed Space

A sequence  $\{x_n\}$  ( $x_n \in V$ ) converges to  $\lambda \in V$  if  $\lim_{n \rightarrow \infty} ||x_n - \lambda|| = 0$ .

A sequence  $\{x_n\}$  is Cauchy if  $\forall \varepsilon > 0$ ,  $\exists N$  such that  $\forall j, k \geq N$ ,  $||x_j - x_k|| < \varepsilon$ .

A normed space is complete if  $\{x_n\}$  Cauchy implies  $\exists x \in V$  such that  $x_n \rightarrow x$ .

Complete normed spaces are called Banach spaces.

### Example 1

$\ell^p(\mathbb{N})$ ,  $1 \leq p < \infty$ , the set of all sequences  $\{x_n\}_{n=1}^\infty = x$  such that

$$||x|| = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} < +\infty$$

Recall  $\{x_n\} + \{y_n\} = \{x_n + y_n\}$  and  $\lambda\{x_n\} = \{\lambda x_n\}$ .

$\ell^p$  spaces are complete and therefore Banach.

If  $\{x_n\} \in \ell^p$  and  $\{y_n\} \in \ell^q$ , then  $\{x_n y_n\} \in \ell^r$ ,  $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \in [0, 1]$  (e.g.  $\ell^2 \cdot \ell^2 \leq \ell^1$ )

### Example 2

$\ell^\infty(\mathbb{N})$ , the set of all bounded sequences

$$||x|| = \sup_{n \in \mathbb{N}} |x_n| < +\infty$$

### Example 3

$C_0(\mathbb{N}) \subseteq \ell^p(\mathbb{N})$ , the set of all sequences  $\{x_n\}$

$$\lim_{n \rightarrow \infty} x_n = 0$$

$C_0$  is a closed subspace, and both are Banach.

### Example 4

$L^p(\Omega)$ ,  $1 \leq p < \infty$ ,  $\Omega \subseteq \mathbb{R}^d$  a Lebesgue measurable set with  $m(\Omega) > 0$ , the space of all equivalence classes of Lebesgue measurable functions  $f : \Omega \rightarrow \mathbb{F}$  such that

$$||f|| = \left( \int_{\Omega} |f(x)|^p dx \right)^{1/p} < +\infty$$

### Example 5

$L^\infty(\Omega)$ , the measurable and essentially bounded functions

$$\begin{aligned} ||f|| &= \inf_{\substack{N \subseteq \Omega \\ m(N)=0}} \sup_{x \in \Omega \setminus N} |f(x)| < +\infty \\ &= \text{ess sup}_{x \in \Omega} |f(x)| \end{aligned}$$

$L^p(\Omega)$  spaces,  $1 \leq p \leq \infty$ , are Banach.

### Example 6

For  $\Omega \neq \emptyset$ , let  $B(\Omega)$  the set of all bounded functions  $f : \Omega \rightarrow \mathbb{F}$  with

$$||f|| = \sup_{x \in \Omega} |f(x)|$$

is a Banach space.

$f_n \rightarrow f$  in  $B(\Omega)$  if and only if  $f_n$  converges uniformly on  $\Omega$  to  $f$ .

### Example 7

Let  $\Omega$  be a topological space and  $BC(\Omega)$  the set of all bounded, continuous functions  $f : \Omega \rightarrow \mathbb{F}$ . Then  $BC(\Omega) \subseteq B(\Omega)$  is a closed Banach subspace under the same norm. That is, the uniform limit of continuous functions is a continuous function.

$$\lim_{f_n \in BC(\Omega)} f_n \rightarrow f \implies f \in BC(\Omega)$$

### Example 8

Let  $K$  be a compact, Hausdorff space.

Then  $C(K)$  is the set of all continuous functions  $f : K \rightarrow \mathbb{F}$  and  $C(K) = BC(K)$ .

### F Spaces / pre-F Spaces

A pre- $F$ -space is a TVS where the topology is given by some invariant metric  $d(x+z, y+z) = d(x, y)$  or  $d(x, y) = d(x-y, 0)$ .

An  $F$ -space is a complete pre- $F$ -space.

A local base (at 0) is given by

$$\gamma_0 = \{B_\varepsilon(0) : \varepsilon > 0\}, \quad B_\varepsilon(x) = \{y \in V : d(x, y) < \varepsilon\}$$

### Example 1

$\ell^p(\mathbb{N})$ ,  $0 < p < 1$ , the set of all  $\{x_n\}_{n=1}^\infty$  such that

$$\sum_{n=1}^{\infty} |x_n|^p < +\infty$$

with

$$d(x, y) = \sum_{n=1}^{\infty} |x_n - y_n|^p$$

Note that this obeys the triangle inequality specifically because it has not been raised to  $1/p$ .

$$d(\lambda x, \lambda y) = |\lambda|^p \cdot d(x, y)$$

means that  $d(z, 0)$  is not a norm.

Here,  $B_\varepsilon(x)$  are not convex sets.

### Side Remark

Given  $\mathbb{R}^2$ , the  $\ell^p$  norm for  $1 \leq p \leq \infty$  is given by

$$|| (x_1, x_2) || = (|x_1|^p + |x_2|^p)^{1/p}$$

and the distance for  $0 < p < 1$  by

$$d((x_1, x_2)) = |x_1|^p + |x_2|^p$$



The  $\varepsilon$  neighborhoods for  $p = 1$  are diamonds,  $p = 2$  circles,  $p = \infty$  squares with smooth transition between them. However, for  $0 < p < 1$ , we have concave diamond shapes. These norms and metrics are all equivalent on  $\mathbb{R}^2$  in the sense that they give the same topology.

## Locally Convex TVS

A TVS which has a local base  $\gamma$  at 0 consisting of open neighborhoods of 0 which are all convex.

### Definition: Convex Set

A set  $A \subseteq V$  is convex if  $\forall x, y \in A, \lambda \in [0, 1]$ , then  $\lambda x + (1 - \lambda)y \in A$ .  
Alternatively, the line segment between  $x$  and  $y$  is contained in  $A$  ( $[x, y] \subseteq A$ ).

## Fréchet Spaces and pre-Fréchet Spaces

A pre-Fréchet space is locally convex.  
A Fréchet space is a locally convex  $F$ -space.

**April 11, 2024**

## Fréchet Spaces

### Example

$\mathcal{S} = \{\{x_n\}_{n=1}^{\infty} \mid \text{the space of all sequences } x_n \in \mathbb{F}\}$ .

$$d(\{x_n\}, \{y_n\}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|x_n - y_n|}{1 + |x_n - y_n|} < +\infty$$

Triangle inequality:

$$\frac{a+b}{1+a+b} \leq \frac{a}{1+a} + \frac{b}{1+b}, \quad a, b \geq 0$$

invariant metric, complete.

$\gamma_0 = \{B_\varepsilon(0) : \varepsilon > 0\}$  is a local base.

$\hat{\gamma}_0 = \{U_{\varepsilon, N} : \varepsilon > 0, N \in \mathbb{N}\}$ .

$U_{\varepsilon, N} = \{\{x_n\}_{n=1}^{\infty} : |x_n| < \varepsilon, \forall n = 1, \dots, N\}$ .

$\forall \varepsilon > 0, \exists \hat{\varepsilon} > 0, N$  such that  $U_{\hat{\varepsilon}, N} \subseteq B_\varepsilon(0)$ .

$\forall \hat{\varepsilon} > 0, N, \exists \varepsilon > 0$  such that  $B_\varepsilon(0) \subseteq U_{\hat{\varepsilon}, N}$ .

$x^{(m)} \rightarrow x$  in metric of  $\mathcal{S}$  as  $m \rightarrow \infty$ .

$x^{(m)} = \{x_n^{(m)}\}_{n=1}^{\infty}, x = \{x_n\}_{n=1}^{\infty}$  if and only if  $\forall n \in \mathbb{N}, x_n^{(m)} \rightarrow x_n$  as  $m \rightarrow \infty$  (pointwise, componentwise convergence).

### Example

$C(\mathbb{R}^d)$ , the set of continuous functions  $f : \mathbb{R}^d \rightarrow \mathbb{F}$ .

$$|||f|||_N = \sup_{\substack{x \in \mathbb{R}^d \\ ||x|| \leq N}} |f(x)|$$

$$d(f, g) = \sum_{N=1}^{\infty} \frac{1}{2^N} \cdot \frac{|||f - g|||_N}{1 + |||f - g|||_N}$$

Complete Fréchet space.

“Locally uniform convergence” such that  $f_n \rightarrow f$  in metric of  $C(\mathbb{R}^d)$  if and only if  $\forall$  compact set  $K \subseteq \mathbb{R}^d$ ,  $f_n$  converges to  $f$  uniformly on  $K$ .

### Example

$C^\infty[0,1]$  the set of infinitely differentiable functions  $f : [0,1] \rightarrow \mathbb{F}$ .

$$|||f|||_n = \sup_{x \in [0,1]} |f^{(n)}(x)|$$

$$d(f, g) = \sum_{n=0}^{\infty} \frac{1}{2^n} \cdot \frac{|||f-g|||_n}{1 + |||f-g|||_n}$$

Fréchet space.

$f_m \rightarrow f$  in  $C^\infty[0,1]$  as  $m \rightarrow \infty$  if and only if for every  $m \in \{0,1,\dots\}$ ,  $f_m^{(n)} \rightarrow f^{(n)}$  uniformly on  $[0,1]$  as  $m \rightarrow \infty$ .

### Proposition

Every TVS is Hausdorff.

### Proof

Let  $x, y \in V$ ,  $x \neq y$ .

For  $U = V \setminus \{0\}$ , and open set,  $x - y \in U$ .

Using the continuity of  $(x^2, y^2) \mapsto x^2 - y^2$  and Lemma 1, there exist  $U_x \ni x$  and  $U_y \ni y$  open such that  $U_x - U_y \subseteq U$ .

Note that  $U_x \cap U_y = \emptyset$ , otherwise there would exist  $z \in U_x \cap U_y$  such that  $0 = z - z \in U_x - U_y \subseteq U$  a contradiction.

### Definition: Balancedness

A subset  $U$  of a vector space  $V$  is called balanced if  $\forall \lambda \in \mathbb{F}$ ,  $|\lambda| \leq 1$ ,  $\lambda U \subseteq U$ .

### Example

For  $V = \mathbb{R}^2$ ,  $\mathbb{F} = \mathbb{R}$ , an ellipse is convex and balanced.

Note that since  $\lambda = 0$  is a valid choice,  $0$  is always in a balanced set.

A rectangle, offset from the origin, is convex but not balanced.

A concave diamond centered at  $0$  may be balanced.

An annulus is neither.

### Exercise

Show that for  $V = \mathbb{C}$ ,  $\mathbb{F} = \mathbb{C}$ , the balanced, convex sets are the open and closed disks along with the entire plane.

### Proposition

1. Every TVS has a balanced, local base.
2. Every locally convex TVS has a balanced and convex local base.

## Proof of A

e.g.  $\gamma = \{U : U \text{ open}, 0 \in U\}$ .

For every  $U \in \gamma$ , construct another  $\hat{U}$  open,  $0 \in \hat{U} \subseteq U$  balanced.

Then  $\hat{\gamma} = \{\hat{U} : U \text{ taken from } \gamma\}$  is a local base.

Use Lemma 1 again and the continuity of  $(\lambda, x') \mapsto \lambda \cdot x'$  at  $\lambda = 0, x' = 0$ .

Given open  $U \ni 0$ , find  $\delta > 0$  and open  $U_0 \ni 0$  such that  $B_{2\delta}(0) \cdot U_0 \subseteq U$ .

Then for  $\alpha \in \mathbb{F}, |\alpha| \leq \delta, \alpha \cdot U_0 \subseteq U$ . Take

$$\hat{U} = \bigcup_{\substack{\alpha \in \mathbb{F} \\ |\alpha| \leq \delta}} \alpha \cdot U_0$$

Therefore  $\hat{U}$  is a union of open sets and  $0 \in \hat{U} \subseteq U$ . Finally, for  $|\lambda| \leq 1$ ,

$$\lambda \cdot \hat{U} = \bigcup_{\substack{\alpha \in \mathbb{F} \\ |\alpha| < \delta}} \lambda \cdot \alpha \cdot U_0 = \bigcup_{\substack{\beta \\ |\beta| \leq |\lambda| \cdot \delta \leq \delta}} \beta U_0 = \hat{U}$$

## Proof of B

We have a local base  $\gamma = \{U_\omega\}$ ,  $U_\omega \ni 0$  open and convex.

We want to construct  $\hat{\gamma} = \{\hat{U}_\omega\}$ ,  $\hat{U}_\omega \ni 0$  open, convex and balanced.

Given  $U$  convex, define

$$\hat{U} = \bigcap_{|\alpha| \leq \delta} \alpha U$$

convex and balanced.

Need to show that  $\hat{U} \ni 0$  is an open neighborhood.

Rest of the owl left to the reader.

## Recall

The intrinsic characterization of a base for a topological space or a local base for a topological space  $X$ ,  $\{\gamma_x\}_{x \in X}$ .

- $\forall x \in X, U \in \gamma_x, x \in U$
- $\forall x, y \in X, U \in \gamma_x, V \in \gamma_y, z \in U \cap V, \exists W \in \gamma_z, W \subseteq U \cap V$ .

## Proposition

A balanced, local base  $\gamma$  (at 0) of a TVS  $V$  has the following properties:

1.  $\gamma$  is a nonempty collection of subsets of  $V$  containing 0.
2.  $\forall U_1, U_2 \in \gamma, \exists U \in \gamma$  such that  $U \subseteq U_1 \cap U_2$ .
3.  $\forall U \in \gamma, x \in U, \exists W \in \gamma$  such that  $x + W \subseteq U$ .

4.  $\forall U \in \gamma, \exists W \in \gamma$  such that  $W + W \subseteq U$  (continuity of  $(x, y) \mapsto x + y$  at  $(x = y = 0)$ ).
  5.  $\forall U \in \gamma, \forall x \in V, \exists t > 0, x \in t \cdot U$  (continuity of scalar multiplication  $(\lambda, x') \mapsto \lambda x'$  at  $\lambda = 0, x' = x$ ).
- $$\frac{1}{t} \cdot x \in U, \frac{\delta}{2} \cdot x \subset B_\delta(0) \cdot \hat{U} \subseteq U.$$
6.  $\forall x \in V, x \neq 0, \exists U \in \gamma, x \notin U$  ( $\{x\}$  closed;  $0 \in V \setminus \{x\}$  open;  $0 \in U \subseteq V \setminus \{x\}$ ). (Hausdorff)

## Converse

Conversely, if  $\gamma$  satisfies properties 1-6, then there exists a unique topology on  $V$  such that  $\gamma$  is a balanced, local base for  $V$  and  $V$  with this topology is a TVS.

## Theorem:

Any two TVS of finite dimension  $d$  (over  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{F} = \mathbb{C}$ ) are homeomorphic to each other.

## Proof

Let  $V$  be a TVS with  $\dim(V) = d$ .

We want to show that  $V \cong \mathbb{F}^d$ . We have

$$V = \text{lin}\{v_1, \dots, v_d\}$$

a basis and

$$f : (\lambda_1, \dots, \lambda_n) \in \mathbb{F}^d \mapsto \sum_{i=1}^d \lambda_i v_i \in V$$

an isomorphism between  $\mathbb{F}^d$  and  $V$  as vector spaces. Further,  $f$  is continuous. Consider  $\mathbb{F}^d$  equipped with the product topology and the continuity of addition and scalar multiplication.

We need only that  $f^{-1}$  is continuous at 0 which is equivalent to  $\forall U \ni 0$  open in  $\mathbb{F}^d, \exists W \ni 0$  open in  $V$  such that  $W \subseteq f(U) ((f^{-1})^{-1}(U))$ .

**April 12, 2024**

## Lemma

$\forall U \ni 0$  open in  $\mathbb{F}^d, \exists W \ni 0$  open such that  $f(U) \supseteq W$ .

That is, 0 is an interior point of  $f(U)$ .

## Proof

$f : \mathbb{F}^d \rightarrow V$ , continuous.

We may assume without loss of generality that  $U = B_1(0)$ .

Let  $S = \{\lambda \in \mathbb{F}^d : \|\lambda\| = 1\}$ , a compact set.

Since  $f$  continuous,  $f(S)$  is compact in  $V$ . Since  $V$  is Hausdorff,  $f(S)$  is closed.

Take  $\hat{U} = V \setminus f(S) \ni 0$  open (because  $0 \notin f(S)$  else  $f(\lambda) = 0$  would imply  $\|\lambda\| = 1$ )

Now, there exists a balanced, open set  $0 \in W \subseteq \hat{U}$ . Therefore,  $W \subseteq f(U)$ .

Otherwise,  $x \in W, x \notin f(U), x = f(\lambda), \lambda \notin U, \|\lambda\| \geq 1$  would give  $\frac{x}{\|\lambda\|} = \frac{1}{\|\lambda\|} \cdot f(\lambda) = f\left(\frac{\lambda}{\|\lambda\|}\right) \in f(S)$ .

But,  $\frac{x}{\|\lambda\|} \in W \subseteq \hat{U}$  because  $x \in W, \frac{1}{\|\lambda\|} \in [0, 1]$  and  $W$  is balanced shows a contradiction.

## Theorem

Any finite-dimensional subspace in a TVS is closed.

## Theorem

Every locally compact TVS is finite-dimensional.

## Definition: Locally Compact

$V$  is locally compact if  $\forall x \in V, \exists U \ni x$  open and  $K \subseteq V$  such that  $U \subseteq K$ .  
For Hausdorff spaces,  $\forall x \in V, \exists U \ni x$  open such that  $\overline{U}$  compact.

## Example

Let  $V$  be a normed space,  $\dim(V) = +\infty$ .  
Then  $\overline{B_1(0)} = \{x \in V : \|x\| \leq 1\}$  is not compact.

## Definition: Semi-norm

A semi-norm on a metric space  $V$  (over  $\mathbb{F} = \mathbb{R}, \mathbb{C}$ ) is a map

$$p : V \rightarrow [0, +\infty)$$

such that

1.  $p(x + y) \leq p(x) + p(y)$
2.  $p(\lambda x) = |\lambda| \cdot p(x)$ .

Note that  $p(0) = 0$  and  $(p(x - y) \geq |p(x) - p(y)|$ .

$$N_p = \{x \in V : p(x) = 0\}$$

is a linear subspace of  $V$ :  $x, y \in N$  such that  $p(x + y) \leq p(x) + p(y) = 0$ ,  $p(\lambda x) = 0$ .  
A semi-norm on  $V$  induces a norm on the quotient space  $V/N_p$ .

$$\|[x]_{N_p}\| = p(x) \quad [x]_N = \{x + z : z \in N\} = x + N$$

## Definition: Absorbing

A set  $A \subseteq V$  is called absorbing if  $\forall x \in V, \exists \lambda > 0$  such that  $\lambda x \in A$ .

Equivalently,  $\bigcup_{\lambda > 0} \frac{1}{\lambda} A = V$ .

There is a relationship between semi-norms on  $V$  and balanced, convex and absorbing subsets of  $V$ .

## Proposition

If  $p$  is a semi-norm on a vector space  $V$ , then

$$A = \{x \in V : p(x) < 1\}$$

is balanced, convex and absorbing.

## Proof

Convex:  $x, y \in A, p(x) < 1, p(y) < 1,$

$$p(\lambda x + (1 - \lambda)y) \leq \lambda \cdot p(x) + (1 - \lambda)p(y) < 1$$

Balanced:  $x \in A, |\lambda| \leq 1, p(x) < 1,$

$$p(\lambda x) = |\lambda| \cdot p(x) < 1$$

Absorbing:  $x \in V$ . If  $p(x) = 0$ , then  $x \in A$  ( $\lambda = 1$ ).

If  $p(x) > 0$ ,  $\lambda = \frac{1}{2p(x)}$  gives  $p(\lambda x) = |\lambda| \cdot p(x) = \frac{1}{2} < 1$ .

## Example

Let  $V = \mathbb{R}^2$  and  $\mathbb{F} = \mathbb{R}$ .

An ellipse is balanced, convex and absorbing.

A band is also balanced, convex and absorbing.

An open interval along the axis is balanced and convex, but it is not absorbing.

## Proposition

Each open neighborhood of 0 in a TVS is absorbing.

## Proof

Continuity of the map  $(\lambda, x) \mapsto \lambda x'$  at  $\lambda = 0$  and  $x' = x$ .

Given  $x \in V$ ,  $U \ni 0$  open,  $\exists \delta > 0$ ,  $W \ni x$  such that  $B_r(0) \cdot W \subseteq U$  and  $\frac{\delta}{2} \cdot x \in U$ .

## Definition: Minkowski Functional

Let  $A$  be a subset in a vector space  $V$ .

If  $A$  is absorbing, one can define the Minkowski functional

$$\mu_A(x) = \inf \left\{ \lambda > 0 : \frac{x}{\lambda} \in A \right\} = \inf \{ \lambda > 0 : x \in \lambda \cdot A \}$$

## Proposition

If  $A$  is convex, balanced and absorbing, then  $\mu_A$  is a semi-norm.

## Proof

Absorbing  $\leadsto \mu_A$  is well defined,  $\mu_A(x) \in [0, +\infty)$ . For  $\alpha \neq 0$ ,

$$\begin{aligned} \mu_A(\lambda x) &= \inf \left\{ \lambda > 0 : \frac{\alpha \cdot x}{\lambda} \in A \right\} \\ &= \inf \left\{ |\alpha| \cdot \lambda > 0 : \frac{\alpha \cdot x}{|\alpha| \cdot \lambda} \in A \right\} \quad (\lambda \mapsto |\alpha| \cdot \lambda) \\ &= |\alpha| \cdot \inf \left\{ \lambda > 0 : \frac{x}{\lambda} \in \frac{|\alpha|}{\alpha} A \right\} \\ &= |\alpha| \cdot \inf \left\{ \lambda > 0 : \frac{x}{\lambda} \in A \right\} \\ &= |\alpha| \cdot \mu_A(x) \end{aligned}$$

since  $A$  is balanced,  $\frac{\alpha}{|\alpha|}A = A$ .

Note that  $\mu_A(0) = 0$  since  $0 \in A$  balanced.

Given  $x, y \in V$  and  $\varepsilon > 0$ , let  $s = \mu_A(x) + \varepsilon$  and  $t = \mu_A(y) + \varepsilon$ . Then, since  $A$  is balanced,  $\frac{x}{s}, \frac{y}{t} \in A$ . By convexity,

$$\frac{x+y}{t+s} = \underbrace{\frac{s}{t+s}}_{(1-\sigma)} \cdot \underbrace{\frac{x}{s}}_{\in A} + \underbrace{\frac{t}{t+s}}_{\sigma} \cdot \underbrace{\frac{y}{t}}_{\in A} \in A$$

Therefore,  $\mu_A(x+y) \leq t+s$  which implies  $\mu_A(x+y) \leq \mu_A(x) + \mu_A(y) + 2\varepsilon$  for all  $\varepsilon > 0$ .

## Equivalence between Semi-norm and ABC Sets

$p \rightsquigarrow A = \{x : p(x) < 1\} \rightsquigarrow \mu_A = p$ .

$A$  bounded, convex, absorbing  $\rightsquigarrow \mu_A \rightsquigarrow \tilde{A} = \{x : \mu_A(x) < 1\}$  where  $\tilde{A} \subseteq A$  differing possibly by the boundary.

## Question: which TVS are normable?

That is a norm such that the topology is given by this norm.

## Definition: Bounded Sets

A subset  $A$  in a TVS is bounded if  $\forall U \ni 0$  open,  $\exists \delta > 0$  such that  $A \subseteq t \cdot U$ ,  $\forall t > \delta$ .

## Theorem:

A TVS is normable if and only if there exists a bounded, convex, balanced, open neighborhood of 0.

## Proof (Sketch)

Suppose  $V$  is a normed space with norm  $\|\cdot\|$ .

$$B = \{x \in V : \|x\| < 1\} = B_1(0)$$

is convex, balanced, and an open neighborhood of 0.

$B$  is bounded, since given  $U \ni 0$  open,  $B_\varepsilon(0) \subseteq U$ , so  $B = \frac{1}{\varepsilon} \cdot B_\varepsilon(0) \subseteq \lambda B_\varepsilon(0) \subseteq \lambda \cdot U$  for  $\lambda \geq \frac{1}{\varepsilon}$ .

Now, let  $B$  be a bounded, convex, balanced, open neighborhood of 0 in a TVS.

Therefore  $B$  is absorbing (as an open neighborhood of 0).

It follows that the semi-norm  $\mu_B(x)$  may be defined.

Then  $\mu_B(x) = 0 \implies x = 0$  since  $B$  is bounded, otherwise  $0 \in U = V \setminus \{x\}$  open gives  $B \subseteq t \cdot U$ ,  $\forall t > \delta$  and  $\frac{1}{t}B \subseteq U$ ,  $\forall t > \delta$ .

Thus,  $\|x\| = \mu_B(x)$  is a norm on  $V$ .

One need only demonstrate that the norm topology is the same as the original topology on  $V$ .

That is,  $\forall U \ni 0$  open,  $\exists \varepsilon > 0$  such that  $\varepsilon \cdot B \subseteq U$ .

$\forall \varepsilon > 0$ ,  $\exists \hat{U} \ni 0$  open such that  $\hat{U} \subseteq \varepsilon B$ .

**April 16, 2024**

## Recall

Given  $p$  a semi-norm,

$$A = \{x \in V : p(x) < 1\}$$

a convex, balanced absorbing set gives the Minkowski formula for a seminorm  $\mu_a$ .

The TVS  $V$  is normable if and only if there exist bounded, convex, balanced, open  $U \ni 0$ .

## Definition: Separating Family of Semi-norms

Let  $V$  be a vector space.

A family of semi-norms  $\{p_\omega\}_{\omega \in \Omega}$  is called separating if  $\forall x \in V, x \neq 0, \exists \omega \in \Omega$  such that  $p_\omega(x) \neq 0$ .

Equivalently,

$$\{x \in V : \forall \omega \in \Omega, p_\omega(x) = 0\} = \{0\}$$

or

$$\bigcap_{\omega \in \Omega} N_{p_\omega} = \bigcap_{\omega \in \Omega} \{x \in V : p_\omega(x) = 0\} = \{0\}.$$

For a given separating family of semi-norms, define

$$U_{n,\omega} = \left\{x \in V : p_\omega(x) < \frac{1}{n}\right\}$$

$$U_{n,\omega_1,\dots,\omega_N} = \left\{x \in V : p_{\omega_i}(x) < \frac{1}{n}, i = 1, \dots, N\right\} = \bigcap_{i=1}^N U_{n,\omega_i}$$

and put

$$\gamma = \{U_{n,\omega_1,\dots,\omega_N} : n \in \mathbb{N}, \omega_1, \dots, \omega_N \in \Omega, N \in \mathbb{N}\}.$$

One can show by earlier propositions that  $\gamma$  is a local base at 0 for some topology  $\tau$ .

Perhaps unsurprisingly, if  $\{p_\omega\}$  is separating, then this locally convex TVS is Hausdorff.

## Theorem:

Let  $\{p_\omega\}$  be a separating family of semi-norms on a vector space  $V$ . Then with local base  $\gamma$  defined above,  $V$  becomes a locally convex TVS, and all  $p_\omega : V \rightarrow [0, +\infty)$  continuous.

## Example

$$\mathcal{S} = \{\{x_n\}_{n=1}^\infty \text{ all sequences}\}$$

$$\text{with } p_n(x) = |x_n|, x = \{x_n\}_{n=1}^\infty, d(x, y) = \sum_{n=1}^\infty \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$

## Remark

Local base at  $x$

$$\gamma_x = x + \gamma = \{U_{n,\omega_1,\dots,\omega_N}[x] : n, N \in \mathbb{N}, \omega_1, \dots, \omega_N \in \Omega\}$$

$$U_{n,\omega_1,\dots,\omega_N}[x] = \left\{y \in V : p_{\omega_i}(x - y) < \frac{1}{n}, i = 1, \dots, N\right\}$$



## Theorem:

Let  $V$  be a locally convex TVS. Then there exists a separating family of semi-norms  $\{p_\omega\}_{\omega \in \Omega}$  on  $V$  such that the topology defined by  $\{p_\omega\}$  coincides with the original topology.

## Proof (Sketch)

$V$  is locally convex, so it has a convex, balanced, local base at 0

$$\hat{\gamma} = \{U_\omega\}_{\omega \in \Omega}$$

where  $U_\omega \ni 0$  are open, convex, balanced, and absorbing.

Put  $p_\omega = \mu_{U_\omega}$  (i.e. set the Minkowski functional to be the semi-norm).

Then we need to show that these are separating.

Then define  $U_{n,\omega_1,\dots,\omega_N}$ ,  $\gamma = \{U_{n,\omega_1,\dots,\omega_N}\}$ ,  $U_\omega = U_{1,\omega}$ ,  $\hat{\gamma} \subseteq \gamma$  and show that  $\gamma$  and  $\hat{\gamma}$  induce the same topology.

## Theorem:

A TVS  $V$  is a pre-Fréchet space if and only if  $V$  has a countable, convex, balanced local base.

## Proof

( $\implies$ ) Assume that  $V$  is a pre-Fréchet space.

Then we have an invariant metric  $d$  and

$$B_\varepsilon(x) = \{y \in V : d(x, y) < \varepsilon\}.$$

It follows that  $\gamma_1 = \{B_{1/n}(0) : n \in \mathbb{N}\}$  is a local base.

The fact that  $V$  is locally convex means that  $\gamma_2 = \{U_\omega : \omega \in \Omega\}$  with  $U_\omega \ni 0$  open, convex and balanced is a convex, balanced local base.

To every  $n \in \mathbb{N}$ ,  $B_{1/n}(0)$  is an open neighborhood of 0, and there exists  $\omega_n \in \Omega$ ,  $0 \in U_{\omega_n} \subseteq B_{1/n}(0)$ . Put

$$\gamma_3 = \{U_{\omega_n} : n \in \mathbb{N}\}$$

a countable, convex, balanced collection.

Then, for any  $U \ni 0$  open,  $\exists n$  such that  $U_{\omega_n} \subseteq B_{1/n}(0) \subseteq U$ . So  $\gamma_3$  is a local base.

( $\impliedby$ ) Assume a TVS  $V$  has a countable, convex, balanced local base:

$$\gamma = \{U_n : n \in \mathbb{N}\}$$

Without loss of generality, we may assume that  $U_{n+1} \subseteq U_n$ . Otherwise, we may take  $\hat{U}_n = U_1 \cap \dots \cap U_n \subseteq U_n$  such that  $\{\hat{U}_n : n \in \mathbb{N}\}$  is also a local base where  $\hat{U}_{n+1} \subseteq \hat{U}_n$ .

Then, since  $U_n$  are open, they are absorbing and  $p_n = \mu_{U_n}$  gives a separating semi-norm from the Minkowski functional. Define a metric

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x-y)}{1 + p_n(x-y)}$$

where  $d(x, y) = 0 \implies x = y$  since  $\{p_n\}$  are separating.

Claim: the metric topology (local base  $\tilde{\gamma}$ ) is the same as the original topology (local base  $\gamma$ ). Write

$$\tilde{\gamma} = \{B_{1/2^m}(0) : m \in \mathbb{N}\}$$

Then for all  $m \in \mathbb{N}$ ,

$$\frac{1}{2^{m+1}}U_{m+1} \subseteq B_{1/2^m}(0)$$

there exists  $n \in \mathbb{N}$  such that  $U_n \subseteq \frac{1}{2^{m+1}}U_{m+1}$ .

Also,  $\forall n, B_{1/2^{n+1}}(0) \subseteq U_n$ . Then  $V$  is locally convex ( $\gamma$ ) and has an invariant metric ( $\tilde{\gamma}$ ). That is,  $V$  is pre-Fréchet space.

### Corollary

In a pre-Fréchet space, the metric can always be defined by

$$\forall n, B_{1/2^{n+1}}(0) \subseteq U_n$$

where  $\{p_n\}$  are separating family of semi-norms.

A separating family of semi-norms leads to a locally convex TVS.

A countable, separating family of semi-norms leads to a pre-Fréchet space.

A finite, separating family of semi-norms leads to a normed space.

### Quotient Spaces

For a vector space  $X$  and a linear subspace  $N \subseteq X$ ,  $X/N = \{[x]_N : x \in X\}$ ,  $[x]_N = x + N$ .

$\pi : X \rightarrow X/N$  is the quotient map to the vector space  $X/N$ .

For a TVS  $X$ ,  $N \subseteq X$  a subspace,  $\pi : X \rightarrow X/N$  where  $\tau$  is the topology of  $X$  and  $\hat{\tau}$  is the topology of  $X/N$  given by

$$\hat{\tau} = \{\pi(U) : U \in \tau\}.$$

$N$  is closed if and only if  $X/N$  is Hausdorff.

### Thoeerem:

For  $X$  a TVS and  $N \subseteq X$  a linear subspace,  $X/N$  is a TVS and  $\pi : X \rightarrow X/N$  is open and continuous.

### Normed / Banach

For  $X$  a normed (Banach) space,  $X/N$  is a normed (Banach) space where  $||[x]||_{X/N} = \inf_{z \in N} ||x + z||$ .

### Pre-Fréchet / Fréchet

For  $X$  a (pre-)Fréchet space,  $X/N$  is a (pre-)Fréchet space where  $d_{X/N}(x, y) = \inf_{z \in N} d(x + z, y) = \inf_{z_1, z_2} d(x + z_1, y + z_2)$ .

### Definition: Linear Operator

A map  $T : V \rightarrow W$  between vector spaces  $V, W$  is linear (or a linear operator) if

$$T(x + y) = Tx + Ty \quad \text{and} \quad T(\alpha x) = \alpha(Tx)$$

## Notation

$M(V, W)$  is the set of all linear operators.

$M(V, V) = M(V)$ .

$V' = M(V, \mathbb{F})$  (linear functionals) is the algebraic dual of  $V$ .

Note that  $M(V, W)$  is a vector space.

$$(T_1 + T_2)(x) := T_1x + T_2x \quad \text{and} \quad (\lambda T)(x) := \lambda(Tx)$$

If  $T_1, T_2$  are linear, then  $T_1 + T_2$  is linear; likewise,  $\lambda T$  is linear precisely when  $T$  is linear.

## Definition: Continuous Linear Operator

For  $V, W$  TVS,  $T$  is a continuous linear operator if  $T \in M(V, W)$  and  $T$  is continuous with respect to the topologies.

## Notation

$L(V, W)$  is the set of all continuous linear operators.

$L(V, V) = L(V)$ .

$V^* = L(V, \mathbb{F})$ , the set of continuous linear functionals on  $V$ , is the dual space of  $V$ .

## Example

Let  $V = \mathbb{R}^n, W = \mathbb{R}^m$ .

$M(V, W) = L(V, W)$ .

To an  $m \times n$  matrix  $A = (a_{ij})_{i=1, j=1}^{m, n}$ , one associates the linear operator  $T_A$

$$T_A : (x_j)_{j=1}^n \mapsto (y_i)_{i=1}^m, \quad y_i = \sum_{j=1}^n a_{ij} x_j$$

$V' = V^*$ . Given  $\phi = (\phi_1, \dots, \phi_n) \in \mathbb{R}^n$  (a row vector),

$$\phi(x) = \phi \cdot x = \sum_{j=1}^n \phi_j x_j$$

In this case,  $V^* \cong \mathbb{R}^n$ .

## Definition: Image or Range

For  $T \in M(V, W)$ ,  $T : V \rightarrow W$ ,

$$\text{im } T = R(T) = \{Tx : x \in V\}$$

## Definition: Kernel or Nullspace

$$\ker T = N(T) = \{x \in V : Tx = 0\}$$

## Remarks

$R(T)$  is a linear subspace of  $W$  while  $N(T)$  is a linear subspace of  $V$ .

$T$  is injective if and only if  $N(T) = \{0\}$ .

If  $T$  is injective, then one has an inverse map  $T^{-1} : R(T) \rightarrow V$ .  $T^{-1}$  is linear.

$T$  is invertible if and only if  $T$  is injective and surjective if and only if  $N(T) = \{0\}$  and  $R(T) = W$ .

**April 18, 2024**

## Proposition

Let  $V, W$  be TVS.

1. a linear operator  $T : V \rightarrow W$  is continuous if and only if  $T$  is continuous at some  $x_0 \in V$ .
2. if  $T$  is a continuous linear operator, then  $N(T) = \ker(T)$  is a closed, linear subspace of  $V$ .

### Proof of A

( $\implies$ ) continuous at all points imply continuous at  $x_0$ .

( $\impliedby$ ) Write  $f(x) = T(x + x_0 - x_1) - T(x_0 - x_1)$  and assume  $T$  is continuous at  $x = x_0$ .

Then  $T(x + x_0 - x_1)$  is continuous at  $x = x_1$ .

### Proof of B

We have that  $\ker(T) = \{x \in V : Tx = 0\} = T^{-1}(\{0\})$  where  $\{0\}$  is closed and so must be its preimage.

## Definition: Bounded Linear Operator

Let  $V, W$  be normed spaces with norms  $\|\cdot\|_V, \|\cdot\|_W$ .

A linear operator  $T : V \rightarrow W$  is called bounded if there exists some  $c \geq 0$  such that

$$\|Tx\|_W \leq c \cdot \|x\|_V, \quad \forall x \in V$$

## Proposition:

A linear operator  $T : V \rightarrow W$  ( $V, W$  normed spaces) is continuous if and only if it is bounded.

### Proof

( $\impliedby$ ) We know that  $\|Tx\|_W \leq c \cdot \|x\|_V, \forall x$ .

Consider  $\{x_n\}, x_n \rightarrow a$  in  $V$ . Then

$$\lim_{n \rightarrow \infty} \|x_n - a\|_V = 0$$

so  $\|Tx_n - Ta\|_W \leq c \cdot \|x_n - a\|_V, \|Tx_n - Ta\|_W = 0$ , and  $Tx_n \rightarrow Ta$  in  $W$ .

( $\implies$ ) For every  $n \in \mathbb{N}$ , find  $x_n \in V$  such that

$$\|Tx_n\|_W > n \cdot \|x_n\|_V$$

Then  $y_n = \frac{x_n}{\|Tx_n\|}$ , since  $\|y_n\| = \frac{\|x_n\|}{\|Tx_n\|} < \frac{1}{n}$  it must be  $y_n \rightarrow 0$ .

Hence,  $Ty_n \rightarrow T0 = 0$  ( $T$  continuous)  $\implies Ty_n = \frac{Tx_n}{\|Tx_n\|}$ .

But  $\|Ty_n\| = 1$ , so  $Ty_n \not\rightarrow 0$  a contradiction.

## Remark

The following statements are equivalent

- $T$  is continuous.
- $T$  is bounded.
- $Tx_n \rightarrow 0$  whenever  $x_n \rightarrow 0$ .
- $\{Tx_n\}$  is bounded whenever  $\{x_n\}$  is bounded.

## Definition: Operator Norm

For  $V, W$  normed spaces.

For  $T : V \rightarrow W$  a bounded linear operator, we define

$$\|T\| = \sup_{\substack{x \in V \\ x \neq 0}} \frac{\|Tx\|_W}{\|x\|_V}$$

the operator norm of  $T$ .

## Remark

$\|T\| \in [0, +\infty)$  and it is equal to the smallest  $c \geq 0$  such that  $\|Tx\|_W \leq c \cdot \|x\|_V, \forall x \in V$ .

Indeed, if this holds for some  $c \geq 0$ , then  $\|T\| \leq c$ .

Conversely, from the definition  $\|Tx\|_W \leq \|T\| \cdot \|x\|_V$ .

That is,  $\|T\| = \min\{c \geq 0 : \|Tx\|_W \leq c \cdot \|x\|_V, \forall x\}$ .

## Remark

$$\|T\| = \sup_{\substack{x \in V \\ \|x\|=1}} \|Tx\| = \sup_{\substack{x \in V \\ \|x\| \leq 1}} \|Tx\|$$

Note that

$$\sup_{x \neq 0} \frac{\|Tx\|_W}{\|x\|_V} = \sup_{x \neq 0} \left\| T \left( \frac{x}{\|x\|_V} \right) \right\|_W = \sup_{\|z\|_V=1} \|Tz\|_W$$

## Remark

$M(V, W)$  and  $L(V, W)$  are linear spaces,

$$(T + S)(x) = Tx + TS$$

$$(\lambda T)(x) = \lambda(Tx)$$

If  $T, S$  are continuous, linear operators, then  $T + S$  and  $\lambda T$  are continuous linear operators.

## Further Properties

- $||T|| = 0$  if and only if  $T = 0$  (i.e.  $Tx = 0, \forall x \in V$ ).
- $||T + S|| \leq ||T|| + ||S||$ , because

$$||(T + S)x||_W = ||Tx + Sx||_W \leq ||Tx||_W + ||Sx||_W \leq ||T|| \cdot ||x||_V + ||S|| \cdot ||x||_V \leq \underbrace{(||T|| + ||S||)}_c \cdot ||x||_V$$

Since  $T + S$  is bounded.  $\frac{||(T+S)x||_W}{||x||_V} \leq ||T|| + ||S||$ , etc.

- $||\alpha T|| = |\alpha| \cdot ||T||$ .
- if  $T \in L(U, V)$  and  $S \in L(V, W)$ , then  $ST \in L(U, W)$  and

$$||ST|| \leq ||S|| \cdot ||T||$$

## Proposition

Let  $V, W$  be normed spaces.

Then  $L(V, W)$  is a normed space with the operator norm.

If, in addition,  $W$  is Banach, then  $L(V, W)$  is also Banach.

### Proof

#### Part A

$|| \cdot ||$  is a norm.

#### Part B

Let  $W$  be a Banach space, and let  $T_n \in L(V, W)$  be such that  $\{T_n\}$  is a Cauchy sequence in the operator norm.

Then,  $\forall \varepsilon > 0, \exists N \in \mathbb{N}$  such that  $\forall j, k \geq N, ||T_j - T_k|| < \varepsilon$ .

So  $\forall x \in V, \{T_n x\}$  is Cauchy in  $W$ .

$$||T_j x - T_k x|| = ||(T_j - T_k)x|| \leq ||T_j - T_k|| \cdot ||x|| \leq \varepsilon \cdot ||x||$$

By completeness, for every  $x \in V, T_n x$  converges in  $W$ . Define

$$Tx = \lim_{n \rightarrow \infty} T_n x$$

such that  $||Tx - T_n x|| \rightarrow 0$  as  $n \rightarrow \infty$ .

We need to show that  $T$  is a linear operator:

$$T(x + y) = \lim_{n \rightarrow \infty} T_n(x + y) = \lim_{n \rightarrow \infty} T_n x + \lim_{n \rightarrow \infty} T_n y = Tx + Ty.$$

$$T(\lambda x) = \lambda \cdot Tx.$$

We need also show that  $T$  is bounded:

$$\frac{||Tx||_W}{||x||_V} = \lim_{n \rightarrow \infty} \frac{||T_n x||_W}{||x||_V} = \liminf_{n \rightarrow \infty} ||T_n||$$

Since  $\{T_n\}$  is Cauchy, it is bounded and  $\liminf_{n \rightarrow \infty} \|T_n\| \leq c$  for some  $c$ .

We have that  $\lim_{n \rightarrow \infty} \|Tx - T_n x\| = 0$  such that  $T_n$  converges pointwise.

We need that  $\lim_{n \rightarrow \infty} \|T - T_n\| = 0$ .

For given  $\varepsilon > 0$ , we find  $N$  such that  $\forall j, k \geq N, x \in V$ :

$$\|T_j x - T_k x\| \leq \varepsilon \cdot \|x\|$$

Then

$$\|T_j x - Tx\| = \|T_j x - T_k x + T_k x - Tx\| \leq \varepsilon \cdot \|x\| + \|T_k x - Tx\|$$

and sending  $k \rightarrow \infty$  sends  $T_k x - Tx$  to 0.

Therefore,  $\|T_j x - Tx\| \leq \varepsilon \cdot \|x\|, \forall j \geq N, \forall x \in V$ . It follows that

$$\frac{\|T_j x - Tx\|}{\|x\|} \leq \varepsilon$$

and, taking the supremum over  $x$ , that  $\|T_j - T\| \leq \varepsilon, \forall j \geq N, \forall x \in V$ .

Hence,  $\lim_{n \rightarrow \infty} \|T_n - T\| = 0$ .

That is,  $L(V, W)$  is complete.

## Corollary

The dual space of a normed space is a Banach space. Recall  $V^* = L(V, \mathbb{F})$ , and both  $\mathbb{R}$  and  $\mathbb{C}$  are complete.

## Notation

Read  $\dot{+}$  as a direct sum implied to be between components of a larger space.

Read  $\text{lin}\{v_1, \dots, v_n\}$  as the linear combinations of  $v_1, \dots, v_n$ .

## Definition: Codimension

If  $V$  is a vector space and  $W$  is a subspace, we say that  $W$  has codimension  $n$  in  $V$  if there exists a subspace  $\hat{W} \subseteq V$  such that

$$V = W \dot{+} \hat{W}$$

and  $\dim(\hat{W}) = n$ .

Equivalently,  $\dim(V/W) = n, V/W = \text{lin}\{[e_1], \dots, [e_n]\}$  basis and  $\hat{W} = \text{lin}\{e_1, \dots, e_n\}$  implies  $V = W \dot{+} \hat{W}$ .

## Proposition:

Let  $V$  be a vector space and  $\phi \neq V^I, \phi \neq 0$ . Then  $\ker(\phi)$  is a subspace of  $V$  of codimension 1.

## Proof

$\phi \neq 0$ . Find  $x_0 \in V$  such that  $\phi(x_0) = 1$ .

Claim:  $V = \ker(\phi) \dot{+} \text{lin}\{x_0\}$ .

Indeed, for  $x \in V$  write

$$x = \underbrace{x - \phi(x) \cdot x_0}_{\in \ker(\phi)} + \underbrace{\phi(x) \cdot x_0}_{\in \text{lin}\{x_0\}}$$

so

$$\phi(x - \phi(x) \cdot x_0) = \phi(x) - \phi(\phi(x) \cdot x_0) = \phi(x) - \phi(x) \cdot \phi(x_0) = 0$$

and

$\ker(\phi) \cap \text{lin}\{x_0\} = \{0\}$  which means  $z = \lambda \cdot x_0 \in \ker(\phi)$ . Therefore

$$0 = \phi(\lambda x_0) = \lambda \cdot 1$$

so  $\lambda = 0$  and  $z = 0$ .

## Proposition:

Let  $V$  be a normed space and  $\phi \in V'$ .

Then  $\phi$  is bounded if and only if  $\ker(\phi)$  is closed in  $V$ .

### Proof

( $\implies$ )  $\phi$  continuous, as a linear operator, implies  $\ker(\phi) = \phi^{-1}(\{0\})$  is closed.

( $\impliedby$ ) assume that  $\ker(\phi)$  is closed. Then

$$V = \ker(\phi) \dot{+} \text{lin}\{x_0\}$$

for some  $x_0 \in V$  and  $x_0 \notin \ker(\phi)$ .

Without loss of generality, we may assume  $\phi(x_0) = 1$ .

Claim:  $\inf_{x \in \ker(\phi)} \|x_0 - x\| = \text{dist}(\ker(\phi), x_0) > 0$ .

Otherwise, there would exist some sequence  $\{x_n\} \subseteq \ker(\phi)$  such that  $\|x_0 - x_n\| \rightarrow 0$ .

From the assumption of closure, this would mean  $x_0 \in \ker(\phi)$  a contradiction.

Therefore,  $\exists c > 0$  such that  $\|x_0 - x\| \geq c, \forall x \in \ker(\phi)$ . So

$$\|\lambda x_0 - \lambda x\| \geq c \cdot |\lambda|$$

$$\|\lambda x_0 - u\| \geq c \cdot |\lambda|, \quad \forall u \in \ker(\phi)$$

Write  $y \in V$  as  $y = \underbrace{-u}_{\in \ker(\phi)} + \underbrace{\lambda x_0}_{\in \text{lin}\{x_0\}}$ . So  $\phi(y) = 0 + \lambda \cdot \phi(x_0) = \lambda$ .

Thus,  $\forall x \in V, \|x\| \geq c \cdot |\phi(x)|$  and  $|\phi(x)| \leq \frac{1}{c} \cdot \|x\|$  and  $\phi$  is bounded.

**April 23, 2024**

## Proposition:

A linear functional  $\phi$  on a TVS  $V$  is continuous if and only if  $\ker(\phi)$  is closed in  $V$ .

### Proof

( $\impliedby$ ) Difficult.

( $\implies$ )  $\ker(\phi) = \phi^{-1}(\{0\})$ .



## Recall:

$V'$  is the set of linear functionals on  $V$   $\phi : V \rightarrow \mathbb{F}$  linear.

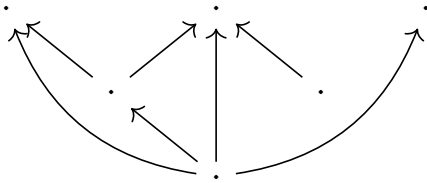
$V^*$  is the set of continuous linear functionals on  $V$   $\phi : V \rightarrow \mathbb{F}$  linear and continuous.

On a normed  $V$ , continuous and bounded are equivalent.

## Zorn's Lemma

A non-empty partially ordered set  $(S, \leq)$  has a maximal element if every totally ordered subset has an upper bound.

- $(S, \leq)$  reflexive, transitive and anti-symmetric.
- $S_0 \subseteq S$  is totally (or linearly) ordered if  $\forall a, b \in S$  either  $a \leq b$  or  $b \leq a$ .
- $S_0$  has an upper bound if  $\exists b \in S$  such that  $\forall x \in S_0, x \leq b$ .
- $m$  is a maximal element of  $S$  if  $\forall x \geq m, x = m$ .



## Theorem:

Let  $V$  be a vector space,  $W_0 \subseteq V$  a subspace, and a linear functional  $\phi_0$  on  $W_0$  (i.e.  $\phi_0 \in W_0'$ ). Then there exists an extension, i.e. a linear functional,  $\phi \in V'$  such that  $\phi|_{W_0} = \phi_0$ .

### Proof

Let  $S$  be the set of all pairs  $(W, \phi)$  such that

- $W_0 \subseteq W \subseteq V$  is a linear subspace and
- $\phi \in W'$ ,  $\phi|_{W_0} = \phi_0$ .

Say that  $(W_1, \phi_1) \leq (W_2, \phi_2)$  if and only if  $W_1 \subseteq W_2$  and  $\phi_2|_{W_1} = \phi_1$ .

Since  $\leq$  is reflexive, transitive and anti-symmetric, it is an order relation.

A totally ordered subset has an upper bound. Given

$$S_0 = \{(W_\omega, \phi_\omega)\}$$

totally ordered, the upper bound is given by  $(W, \phi)$  where

$$W = \bigcup_{\omega} W_{\omega}$$
$$\phi(x) = \phi_{\omega}(x) \quad \text{if } x \in W_{\omega}$$

such that for  $x \in W_{\omega_1} \cap W_{\omega_2}$  we have  $\phi_{\omega_1}(x) = \phi_{\omega_2}(x)$  and consequently  $(W_{\omega_1}, \phi_{\omega_1}) \leq (W_{\omega_2}, \phi_{\omega_2})$ . Then, by Zorn's Lemma, we have that  $S$  has a maximal element  $(\hat{W}, \hat{\phi})$ .

Claim:  $\hat{W} = V$ ,  $\hat{\phi} \in V'$ , and  $\hat{\phi}|_{W_0} = \phi_0$ .

Otherwise, there exists  $(\hat{W}, \hat{\phi}) > (\hat{W}, \hat{\phi})$ .

Namely,  $\hat{W} = \hat{W} + \text{lin}\{x_0\} = \{\hat{w} + \lambda x_0 : \hat{w} \in \hat{W}, \lambda \in \mathbb{F}\}$ ,  $x_0 \in V \setminus \hat{W}$  with  $\hat{W} \subsetneq V$ .

Then  $\hat{W} \subsetneq \hat{W} \subseteq V$ .

Define  $\hat{\hat{\phi}}$  on  $\hat{W}$  as

$$\hat{\hat{\phi}}(\hat{W} + \lambda x_0) = \hat{\phi}(\hat{W}) + \lambda \cdot \hat{\hat{\phi}}(x_0) = \hat{\phi}(\hat{W}) + \lambda \cdot c$$

with  $c$  an arbitrary choice. Then  $\hat{\hat{\phi}}$  is linear.

## Conclusion

Each infinite dimensional, normed space has an unbounded linear functional.

For  $(V, || \cdot ||)$  a normed space, there exist  $\{e_1, e_2, \dots\}$  linearly independent and

$$W_0 = \text{lin}\{e_1, e_2, \dots\}$$

is the set of all finite linear combinations. So

$$\phi_0\left(\sum \lambda_k e_k\right) = \sum \lambda_k \cdot k \cdot ||e_k||$$

where  $\phi_0 \in W_0'$  and  $\phi_0$  is unbounded. Take  $\phi_0(e_k) = k \cdot ||e_k||$ . Then

$$\sup_{\substack{x \in W_0 \\ x \neq 0}} \frac{|\phi_0(x)|}{||x||} \geq \sup \frac{k ||e_k||}{||e_k||} = +\infty$$

Then extend  $\phi_0$  to a linear functional on  $V$ ,  $\phi|_{W_0} = \phi_0$ ,  $\phi \in V'$ ,  $\phi$  unbounded.

## Preliminaries: Hahn-Banach

On normed space, given  $\phi_0 \in W_0^*$  bounded we have a bounded extension  $\phi \in V^*$  where  $||\phi|| = ||\phi_0||$ .

On locally convex TVS, continuous  $\phi_0 \in W^*$  implies a continuous extension  $\phi \in V^*$ .

Equivalently, given  $p(x)$  a seminorm,  $|\phi_0(x)| \leq p(x)$  implies  $|\phi(x)| \leq p(x)$ .

## Lemma:

Let  $V$  be a vector space and  $p$  a seminorm on  $V$ .

Let  $W$  be a subspace of codimension 1,

$$V = W + \text{lin}\{x_0\}$$

Let  $\phi$  be a real linear functional on  $W$  such that

$$\phi(x) \leq p(x) \quad \forall x \in W$$

Then there exists an extension  $\hat{\phi}$  (a real linear functional on  $V$ ) such that

$$\hat{\phi}(x) \leq p(x) \quad \forall x \in V$$

## Proof

Write  $V = W \dot{+} \text{lin}\{x_0\}$  such that

$$\hat{\phi}(W + \lambda x_0) := \phi(W) + \lambda \cdot c$$

with a suitable choice  $c$ .

We know already that  $\hat{\phi} \in V'$ . For  $u, v \in W$ ,

$$\begin{aligned}\phi(u) - \phi(v) &= \phi(u - v) \\ &\leq p(u - v) \\ &= p((u + x_0) - (v + x_0)) \\ &\leq p(u + x_0) + p(v + x_0)\end{aligned}$$

Therefore

$$-p(v + x_0) - \phi(v) \leq p(u + x_0) - \phi(u)$$

and  $\exists c \in \mathbb{R}$  such that

$$-p(v + x_0) - \phi(v) \leq c \leq p(u + x_0) - \phi(u)$$

(e.g. take inf or sup). So

$$\begin{array}{ll} -p(v + x_0) \leq \phi(v) + c & \phi(u) + c \leq p(u + x_0) \\ -p(v + x_0) \leq \hat{\phi}(v + x_0) & \hat{\phi}(u + x_0) \leq p(u + x_0) \\ v = \frac{w}{\lambda}, \lambda < 0 & u = \frac{w}{\lambda}, \lambda > 0 \\ p(w + \lambda x_0) \geq \hat{\phi}(w + \lambda x_0) & \hat{\phi}(w + \lambda x_0) \leq p(w + \lambda x_0) \end{array}$$

and

$$\hat{\phi}(w + \lambda x_0) \leq p(w + \lambda x_0) \quad \forall \lambda \in \mathbb{R}, w \in W$$

## Lemma

Take  $\mathbb{F} = \mathbb{C}$ , let  $W$  be a subspace of  $V$  and

$$V = W \dot{+} \text{lin}\{e_0\}$$

such that  $\phi \in W'$

$$|\phi(x)| \leq p(x) \quad \forall x \in W$$

Then there exists an extension  $\hat{\phi} \in V'$  on,  $\hat{\phi}|_W = \phi$  such that

$$|\hat{\phi}(x)| \leq p(x) \quad \forall x \in V$$

## Proof

Given  $\phi$  on  $W$ , define the real linear functional

$$\psi(x) = \Re(\phi(x))$$

Note that

$$\psi(ix) = \Re(i\phi(x)) = -\Im(\phi(x))$$

Therefore

$$\phi(x) = \psi(x) - i\psi(ix)$$

So by extending  $\hat{\psi}$  on  $V$  we can construct an extension  $\hat{\phi}$  on  $V$ . We know

$$\psi(x) = |\phi(x)| \leq p(x) \quad \forall x \in W$$

therefore  $\hat{\psi}(x) \leq p(x)$  for all  $x \in V$ .

Now define  $\hat{\phi}$  on  $V$  by

$$\hat{\phi}(x) := \hat{\psi}(x) - i\hat{\psi}(ix)$$

1.  $\hat{\phi}$  is a real linear functional on  $V$

$$\hat{\phi}|_W = \phi$$

1.  $\hat{\phi}$  is a complex linear functional on  $V$

$$\hat{\phi}(\alpha x) = \alpha \hat{\phi}(x)$$

$$\alpha = \alpha_1 + i\alpha_2$$

$$\hat{\phi}(ix) = i\hat{\phi}(x)$$

$$\hat{\psi}(ix) - i\hat{\psi}(i^2 x) = i(\hat{\psi}(x) - i\hat{\psi}(ix))$$

1.  $|\hat{\phi}(x)| \leq p(x), \forall x \in V$

We know that  $\hat{\psi}(x) \leq p(x)$ .

For any  $x \in V$ , find  $\alpha \in \mathbb{C}$ ,  $|\alpha| = 1$  such that  $0 \leq \alpha \hat{\phi}(x)$ . Then

$$\begin{aligned} 0 \leq \alpha \hat{\phi}(x) &= \hat{\phi}(\alpha x) \\ &= \underbrace{\hat{\psi}(\alpha x)}_{\text{real}} - \underbrace{i\hat{\psi}(i\alpha x)}_{\substack{\text{imaginary} \\ =0}} \\ &= \hat{\psi}(\alpha x) \leq p(\alpha x) = |\alpha|p(x) = p(x) \end{aligned}$$

Therefore  $0 \leq \alpha \hat{\phi}(x) \leq p(x)$  and  $|\hat{\phi}(x)| \leq p(x)$ .

## Corollary

Let  $V$  be a normed space with the seminorm  $p$  and  $W_0 \subseteq V$  a subspace with  $\phi_0 \in W_0'$  such that

$$|\phi_0(x)| \leq p(x), \quad x \in W_0$$

Then there exists  $\hat{\phi} \in V'$  such that  $\hat{\phi}|_{W_0} = \phi_0$  and

$$|\hat{\phi}(x)| \leq p(x), \quad x \in V$$

## Proof

Apply the two lemmas and Zorn's lemma.

**April 25, 2024**

## Recall:

Take  $W_0 \subseteq V$ ,  $p$  a seminorm, and  $\phi_0 \in W_0'$  such that

$$|\phi_0(x)| \leq p(x), \quad x \in W_0$$

Then there exists an extension  $\hat{\phi} \in V'$ ,  $\hat{\phi}|_{W_0} = \phi_0$  where

$$|\hat{\phi}(x)| \leq p(x), \quad x \in V$$

## Theorem: Hahn-Banach for Normed Spaces

Let  $V$  be a normed space,  $W_0 \subseteq V$  a linear subspace, and  $\phi_0 \in (W_0)^*$ .

Then there exist  $\hat{\phi} \in (V)^*$  such that  $\hat{\phi}|_{W_0} = \phi_0$  and

$$||\hat{\phi}|| = ||\phi_0||$$

## Proof:

From the previous result with

$$p(x) = ||x|| \cdot ||\phi_0||$$

it is obvious that  $|\phi_0(x)| \leq p(x)$ ,  $x \in W_0$ .

Then there is an extension  $\hat{\phi} \in V'$  where

$$|\hat{\phi}(x)| \leq p(x) = ||x|| \cdot ||\phi_0||, \quad x \in V$$

It follows that  $\hat{\phi} \in V^*$  is bounded and

$$\sup \frac{|\hat{\phi}(x)|}{||x||} \leq ||\phi_0||$$

Consequently  $||\hat{\phi}|| \leq ||\phi_0||$ .

We have also that  $||\hat{\phi}|| \geq ||\phi_0||$  because  $\hat{\phi}$  is an extension of  $\phi_0$ .

## Corollary

$\forall x_0 \in V, V$  a normed space,  $x_0 \neq 0, \exists \hat{\phi} \in V^*$  such that  $\hat{\phi}(x_0) = \|x_0\|$  and  $\|\hat{\phi}\| = 1$ .

## Definition:

For  $\mathcal{F} \subseteq V'$ , we say that  $\mathcal{F}$  separates the points of  $V$  is

$$\forall x_0 \in V, x_0 \neq 0, \exists \phi \in \mathcal{F} : \phi(x_0) \neq 0$$

## Remark

- $V'$  separates the points of  $V$  on any vector space  $V$ .
- $V^*$  separates the points of  $V$  on any normed space.

## Theorem: Hahn-Banach for Locally Convex TVS

Let  $V$  be a locally convex TVS,  $W_0 \subseteq V$  a linear subspace, and  $\phi_0 \in (W_0)^*$  a continuous linear functional. Then there exist  $\hat{\phi} \in V^*$  continuous linear functionals such that  $\hat{\phi}|_{W_0} = \phi_0$ . Consequently,  $V^*$  separates the points of  $V$ .

## Proof

$\phi_0 : W_0 \rightarrow \mathbb{F}$  continuous gives

$$U = \{x \in W_0 : |\phi_0(x)| < 1\}$$

open with respect to the subspace topology in  $W_0$ .

That is,  $U = \hat{U} \cap W_0$  with  $\hat{U}$  open in  $V$  and  $0 \in \hat{U}$ .

Therefore, there exists some  $\tilde{U}$  convex, balanced, and open such that  $0 \in \tilde{U} \subseteq \hat{U}$ .

Let  $p(x) = \mu_{\tilde{U}}(x)$ , the Minkowski Functional and a seminorm on  $V$ .

It follows that  $|\phi_0(x)| \leq p(x), x \in W_0$ .

Equivalently,  $p(x) < 1 \implies |\phi_0(x)| < 1, x \in W_0$ .

$$\begin{array}{ccc} p(x) < 1 & \implies & |\phi_0(x)| < 1 \\ \downarrow & & \uparrow \\ x \in \tilde{U} & \implies & x \in \hat{U} \implies x \in U \end{array}$$

Therefore there exists an extension  $\hat{\phi} \in V'$  such that

$$|\hat{\phi}(x)| \leq p(x), x \in V$$

We have

$$\underbrace{\{x \in V : p(x) < 1\}}_{\tilde{U} \ni 0 \text{ open}} \subseteq \underbrace{\{x \in V : |\hat{\phi}(x)| < 1\}}_{\hat{\phi}^{-1}(B_r(0))}$$

Therefore  $\hat{\phi}$  is continuous at  $x_0 = 0$  and  $\hat{\phi}$  is continuous.

## Theorem:

Let  $0 < p < 1$ ,  $V = L^p[0, 1]$ . Then  $V^* = \{0\}$ .

## Remark

The  $F$ -space  $L^p[0, 1]$  is not a locally convex TVS.

## Definition: (Nowhere) Dense Subset

Let  $X$  be a topological space and  $A \subseteq X$ .

Then  $A$  is called dense in  $X$  if  $\text{clos}(A) = X$ .

$A$  is called nowhere dense in  $X$  if  $\text{int}(\text{clos}(A)) = \emptyset$ .

One can say  $A$  is dense at  $x_0 \in X$  if  $x_0 \in \text{int}(\text{clos}(A))$ .

## Examples

$X = \mathbb{R}$  and  $A = \mathbb{Q}$ , then  $A$  is dense in  $\mathbb{R}$ .

$X = \mathbb{R}^n$  and  $A$  a proper linear subspace, then  $A$  is nowhere dense.

$X = \mathbb{R}$  and  $A = [0, 1] \cap \mathbb{Q}$ , then  $A$  is dense at points in  $(0, 1)$ .

## Lemma:

If  $A$  is open:  $A$  is dense if and only if  $X \setminus A$  is nowhere dense.

If  $B$  is closed:  $X \setminus B$  is dense if and only if  $B$  is nowhere dense.

$$\begin{aligned} B \text{ nowhere dense} &\iff \text{int}(\text{clos}(B)) = \emptyset \\ &\iff \text{int}(B) = \emptyset \\ &\iff X \setminus \text{int}(B) = \emptyset \\ &\iff \text{clos}(X \setminus B) = \emptyset \\ &\iff X \setminus B \text{ dense in } X \end{aligned}$$

## Proposition:

Any closed proper linear subspace  $W$  of a TVS  $V$  is nowhere dense in  $V$ .

## Proof

Let  $\text{clos}(W) = W$ ,  $W \subsetneq V$ .

Find  $x_0 \in V$ ,  $x_0 \neq 0$

$$V \supseteq V_1 = W + \text{lin}\{x_0\}$$

To show:  $\text{int}(W) = \emptyset$ .

Otherwise,  $v \in \text{int}(W)$ ,  $U$  open,  $V \in U \subseteq W$ .

Now  $\lambda \in \mathbb{F} \mapsto v + \lambda x_0$  continuous,  $\lambda = 0 \mapsto v \in U$ .

Then there exists some  $\delta > 0$  such that  $|\lambda| < \delta \implies v + \lambda x_0 \in U$ .

For some  $\lambda \neq 0$ ,  $v + \lambda x_0 \in U \subseteq W$ ,  $v \in U \subseteq W$  linear.

Then  $\lambda x_0 \in W$  and  $x_0 \in W$  a contradiction.

## Definition: First and Second Category (Meager)

A topological space  $X$  is called of

- first category (meager) if  $X$  is the countable union of nowhere dense subsets.
- second category (nonmeager) otherwise.

## Examples

$X = \mathbb{Q}$  is first category.  $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ .

$X = \ell^1 = \{\{x_k\}_{k=1}^{\infty} : \sum |x_k| < +\infty\}$  is Banach of second category.

$X_n = \{\{x_k\}_{k=1}^{\infty} = x : x = \{x_1, x_2, \dots, x_n, 0, 0, \dots\}\} \subseteq X$  an  $n$ -dimensional subspace. Take

$$\hat{X} = \bigcup_{n=1}^{\infty} X_n$$

Then  $\hat{X}$  is of first category.  $X_n \subseteq \hat{X}$  a closed, proper subspace which is nowhere dense.

## Theorem: Baire Category Theorem

Every complete metric space is of second category.

All Banach spaces or  $F$ -spaces (Fréchet spaces) are of second category.

## Remark: Uniform Bounded Principle

For normed spaces / Banach spaces (more general; see notes for  $F$ -spaces).

## Theorem: (Uniform Bounded Norm)

Let  $X, Y$  be normed spaces and let  $\{T_{\omega}\}_{\omega \in \Omega}$  be a collection of bounded linear operators  $T_{\omega} \in L(X, Y)$ . Suppose that the set  $E$  of all  $x \in X$  such that

1.  $\sup_{\omega \in \Omega} \|T_{\omega}x\| < +\infty$  is of second category.

Then

2.  $\sup_{\omega \in \Omega} \|T_{\omega}\| < +\infty$ .

## Remark

If (2) holds, then (1) holds for all  $x \in X$ .

$$\|T_{\omega}x\| \leq \|T_{\omega}\| \cdot \|x\|$$

so  $\sup \|T_{\omega}x\| \leq \sup \|T_{\omega}\| \cdot \|x\|$  and  $E = X$ .



## Proof

Define

$$E_n := \{x \in X : \sup_{\omega \in \Omega} ||T_\omega x|| \leq n\}$$

Then  $E = \bigcup_{n=1}^{\infty} E_n$ .

If  $E$  is of second category, then there exists  $n_0$  such that  $E_{n_0}$  is not nowhere dense.

We know that  $E_n$  is closed since

$$E_n = \bigcap_{\omega \in \Omega} \{x \in X : ||T_\omega x|| \leq n\}$$

which are preimages with respect to  $T_\omega$  of closed balls  $\overline{B_n(0)} \subseteq Y$  and therefore closed in  $X$ .

Then  $\text{int}(\text{clos}(E_n)) = \text{int}(E_n) \neq \emptyset$ , so there exists  $x_0 \in X$ ,  $\varepsilon > 0$  such that

$$B_\varepsilon(x_0) \subseteq E_{n_0}$$

Consider  $x \in X$ ,  $||x|| \leq 1$ . Then  $x_0 + \frac{\varepsilon}{2}x \in B_\varepsilon(x_0) \subseteq E_{n_0}$  and  $x_0 \in B_\varepsilon(x_0) \subseteq E_{n_0}$ .

It follows that

$$\begin{aligned} ||T_\omega(x_0 + \frac{\varepsilon}{2}x)|| &\leq n, \forall \omega \\ ||T_\omega(x_0)|| &\leq n, \forall \omega \end{aligned}$$

and

$$\begin{aligned} ||T_\omega(\frac{\varepsilon}{2}x)|| &\leq ||T_\omega(x_0 + \frac{\varepsilon}{2}x)|| + ||T_\omega x_0|| \\ ||T_\omega x|| &\leq \frac{4n_0}{\varepsilon} = C \end{aligned}$$

holds for all  $x$  with  $||x|| < 1$ . Therefore

$$||T_\omega|| = \sup_{x \neq 0} \frac{||T_\omega x||}{||x||} = \sup_{x \neq 0} \left\| T_\omega \frac{x}{||x||} \right\| = \sup_{||x||=1} ||T_\omega x|| \leq C$$

**April 30, 2024**

## Recall: Uniform Boundedness Principle

$X, Y$  normed spaces.

$\{T_\omega\}$ ,  $T_\omega \in L(X, Y)$  bounded.

If the set  $E$  of all  $x \in X$

1.  $\sup ||T_\omega x|| < +\infty$  is of second category, then

2.  $\sup ||T_\omega|| < +\infty$ .

## Theorem: Banach-Steinhaus

Let  $X, Y$  be Banach spaces and  $\{T_\omega\}$  a collection of bounded linear operators ( $T_\omega \in L(X, Y)$ ).  
If

1.  $\forall x \in X: \sup_\omega ||T_\omega x|| < +\infty$ , then
2.  $\sup_\omega ||T_\omega|| < +\infty$ .

### Proof

$E = X$  a Banach space, which is complete and therefore second category by Baire Category Theorem.

### Remark

If  $X$  is not complete, then the conclusion may fail.

### Example

Let  $\hat{X} = \ell^1(\mathbb{N})$  (sequences  $\{x_n\}_{n=1}^\infty$  such that  $\sum |x_n| < +\infty$ ).  
Take  $X = \{x \in \{x_n\}_{n=1}^\infty \in \hat{X} : \exists N, \forall n \geq N, x_n = 0\}$ .

$$X = \underbrace{\bigcup_{N=1}^{\infty} X_N}_{\text{1st Category}} \quad \text{and} \quad X_N = \{\{x_1, \dots, x_N, 0, 0, \dots\}\}$$

Then for  $T_n \in L(X, \mathbb{F}) = X^*$ ,  $T_n x = n \cdot x_n$  for  $x = \{x_n\}$ .  
 $T_n$  linear and bounded, since

$$|T_n x| = n \cdot |x_n| \leq n \cdot \sum_{k=1}^{\infty} |x_k| = n \cdot ||x||$$

and therefore  $||T_n|| \leq n$ . In fact  $||T_n|| = n$  because  $x = \{0, \dots, 0, \underbrace{1}_{n\text{th}}, 0, \dots\}$  gives  $T_n x = n$ ,  $||x|| = 1$ .

Therefore, 2 fails  $\sup_n ||T_n|| = +\infty$ .

However, 1 holds for all  $x \in X$ . For  $x = \{x_1, \dots, x_N, 0, \dots\}$  take

$$\sup_n |T_n x| = \sup_n n \cdot |x_n| = \max_{1 \leq n \leq N} n \cdot |x_n| < +\infty$$

## Definition: Strong Convergence

Let  $X$  and  $Y$  be normed spaces and  $T_n, T \in L(X, Y)$ .

1.  $T_n$  is said to converge strongly on  $X$  to  $T$  if  $\forall x \in X: \lim_{n \rightarrow \infty} ||T_n x - T x|| = 0$ .
2.  $T_n$  is said to be strongly convergent on  $X$  if  $\forall x \in X, \exists y \in Y: \lim_{n \rightarrow \infty} ||T_n x - y|| = 0$ .

Obviously (1)  $\implies$  (2).

Suppose (2) holds. Then one can define

$$Tx := \lim_{n \rightarrow \infty} T_n x$$

such that  $\|T_n x - Tx\| \rightarrow 0$ .

One can show that  $T$  is a linear operator, but  $T$  does not need to be bounded.

### Example

$\hat{X} \subseteq \ell^1$ ,  $X = \{x \in \{x_n\}_{n=1}^\infty \in \hat{X} : \exists N, \forall n \geq N, x_n = 0\}$ . Take

$$S_n x = \{1 \cdot x_1, 2 \cdot x_2, 3 \cdot x_3, \dots, n \cdot x_n, 0, 0, \dots\}$$

then  $S_n : X \rightarrow X$  is linear, and bounded where

$$\|S_n x\| = \sum_{k=1}^n k \cdot |x_k| = n \cdot \sum_{k=1}^n |x_k| \leq n \cdot \|x\|_{\ell^1}$$

implies  $\|S_n\| = n$ . Define

$$Sx = \{1 \cdot x_1, 2 \cdot x_2, \dots, k \cdot x_k, \dots\}$$

which is a linear operator  $S : X \rightarrow X$  but is not bounded since

$$x = e_k = \{0, \dots, \underbrace{1}_{k\text{th}}, 0, \dots\}$$

gives  $Se_k = k \cdot e_k$  implies  $\frac{\|Se_k\|}{\|e_k\|} = k$  so  $\sup \frac{\|Sx\|}{\|x\|} = +\infty$ .

Yet  $\|S_n x - Sx\| \rightarrow 0, \forall x \in X$  since for

$$x = \{x_1, \dots, x_N, 0, 0, \dots\}$$

we have that  $S_n x = Sx$  for  $n \geq N$ .

We conclude that  $S_n$  is strongly convergent on  $X$ ; it converges to  $S$  but  $S$  is not bounded.

Note  $X$  not of second category.

### Theorem:

Let  $X$  and  $Y$  be Banach spaces and  $T_n \in L(X, Y)$ . If  $T_n$  converges strongly on  $X$ , then

$$\sup_n \|T_n\| < +\infty$$

and there exists an operator  $T \in L(X, Y)$  such that  $Tx = \lim_{n \rightarrow \infty} T_n x$  (i.e.  $\lim_{n \rightarrow \infty} \|T_n x - Tx\| = 0, \forall x \in X$ ).

Moreover,

$$\|T\| \leq \liminf_{n \rightarrow \infty} \|T_n\| \leq \sup_n \|T_n\| < +\infty$$

## Proof

For all  $x \in X$ ,  $T_n x$  converges to some  $y \in Y$ .

Since convergent sequences are bounded in normed spaces, this implies  $\sup_n \|T_n x\| < +\infty$ .

By the Banach-Steinhaus theorem,  $C = \sup_n \|T_n\| < +\infty$ .

Now define  $Tx = \lim_{n \rightarrow \infty} T_n x = y$ . So  $T : X \rightarrow Y$  is a linear map

$$\lim_{n \rightarrow \infty} \|T_n x - Tx\| = 0, \forall x \in X$$

Then  $T$  is bounded since

$$\|T_n x\| \leq \|T_n\| \cdot \|x\| \leq C \|x\|$$

or equivalently, taking the limit,

$$\lim_{n \rightarrow \infty} \|Tx\| \leq \lim_{n \rightarrow \infty} \|-T_n x + Tx\| + \|T_n x\| \leq \lim_{n \rightarrow \infty} \|Tx - T_n x\| + C \|x\|$$

implies that  $\|Tx\| < C \|x\|$ .

Take  $\alpha = \liminf_{n \rightarrow \infty} (\|T_n\|)$  and find  $\{T_{n_k}\}$  such that  $\alpha = \lim_{k \rightarrow \infty} T_{n_k}$ . Then

$$\|Tx\| \leq \underbrace{\|Tx - T_{n_k} x\|}_{\rightarrow 0} + \underbrace{\|T_{n_k}\|}_{\rightarrow \alpha} \cdot \|x\|$$

implies that  $\|Tx\| \leq \alpha \cdot \|x\|$  and  $\|T\| \leq \alpha$ .

## Remark

For  $X$  and  $Y$  normed spaces and  $T_n \in L(X, Y)$ ,

Convergence in the operator norm:  $\|T_n - T\|_{L(X, Y)} \rightarrow 0$ .

Strong convergence of operators:  $\forall x \in X : \|T_n x - Tx\|_Y \rightarrow 0$ .

The former implies the latter, but not vice versa.

Strong convergence of operators is analogous to pointwise convergence.

## Example

$Q_n : \ell^p \rightarrow \ell^p$ ,  $1 \leq p < \infty$ .

$Q_n : \{x_k\} \mapsto \{0, \dots, 0, x_{n+1}, x_{n+2}, \dots\}$ .

$\|Q_n x\| \leq \|x\|$  implies that  $\|Q_n\| \leq 1$  and, for

$$e_{n+1} = \{0, \dots, 0, \underbrace{1}_{n+1}, 0, \dots\}$$

we have  $\|Q_n e_{n+1}\| = 1$  and  $\|e_{n+1}\| = 1$  which implies  $\|Q_n\| = 1$ .

Therefore  $Q_n \not\rightarrow 0$  in operator norm. But  $Q_n \rightarrow 0$  strongly.

For  $x \in \ell^p$ ,

$$\|Q_n x\| = \left( \sum_{k=n+1}^{\infty} |x_k|^p \right)^{1/p} \xrightarrow{n \rightarrow \infty} 0$$

because  $\sum_{k=1}^{\infty} |x_k|^p < +\infty$ .

## Divergence of Fourier Series

$X = C_{\text{per}}[-\pi, \pi] \ni f$  (continuous, periodic functions)

$f : [-\pi, \pi] \rightarrow \mathbb{C}$  continuous,  $f(-\pi) = f(\pi)$ .

Define Fourier coefficients

$$f_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx, \quad n \in \mathbb{Z}$$

and consider the formal Fourier series

$$\sum_{n=-\infty}^{\infty} f_n e^{inx}$$

Consider the partial sums

$$F_n(x) = \sum_{k=-n}^n f_k e^{-ikx}$$

### Theorem

There exists an  $f \in X = C_{\text{per}}[-\pi, \pi]$  such that  $f_n(0)$  does not converge (i.e. we do not even have pointwise convergence).

### Proof

Write

$$\begin{aligned} F_n(x) &= \sum_{k=-n}^n f_k e^{ikx} \\ &= \sum_{k=-n}^n \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} f(t) e^{-itx} dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{\left( \sum_{k=-n}^n e^{i(x-t)k} \right)}_{D_n(x-t)} f(t) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(x-t) f(t) dt \end{aligned}$$

where

$$D_n(t) = \sum_{k=-n}^n e^{itx} = \frac{\sin((n+1/2)t)}{\sin(t/2)}$$

is the Dirichlet kernel. Note that  $D_n(t) = D_n(-t)$ .

Define a map  $L_n : f \in X \rightarrow \mathbb{C}$  as

$$L_n(f) = F_n(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(t) f(t) dt$$

By contradiction, assume that  $F_n(0) = L_n(f)$  converges for every  $f \in X$ .

We have that  $L_n$  is a linear operator (as an integral).

Then given

$$|L_n(f)| \leq \sup_{t \in [-\pi, \pi]} |f(t)| \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)| dt \leq \|D_n\|_{L^1} \cdot \|f\|_X$$

since  $D_n(t)$  is continuous on  $[-\pi, \pi]$  we have that  $L_n$  is bounded and  $\|L_n\|_{X^*} \leq \|D_n\|_{L^1}$ .

Therefore,  $L_n$  is strongly convergent on  $X$  and  $L_n \in L(X, \mathbb{C}) = X^{*j}$ .

So, by Banach-Steinhaus  $\sup_{n \in \mathbb{N}} \|L_n\| < +\infty$ .

But  $\|L_n\|_{X^*} = \|D_n\|_{L^1}$  and  $\|D_n\|_{L^1} \rightarrow +\infty$ . (See below)

We have that  $D_n(0) = 2n + 1$  and that the Dirichlet kernel oscillates as a sinusoidal. We want to find  $f \in C_{\text{per}}[-\pi, \pi]$  such that

$$|L_n(f)| = \|D_n\|_{L^1} \cdot \|f\|_{C(-\pi, \pi)}$$

That is

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(t) f(t) dt \right| = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)| dt \cdot \sup_t |f(t)|$$

which is satisfied by

$$g = \begin{cases} +1 & \text{if } D_n(t) > 0 \\ -1 & \text{if } D_n(t) < 0 \end{cases}.$$

If we approximate  $g(t)$  by suitable continuous functions, calling that function  $f_\varepsilon$ , then

$$|L_n(g - f_\varepsilon)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(t)(g - f_\varepsilon) dt \right| \leq \|D_n\|_{C[-\pi, \pi]} \cdot \|g - f_\varepsilon\|_{L^1}$$

We can show (see lecture notes) that

$$\int_{-\pi}^{\pi} \left| \frac{\sin(n + 1/2)t}{\sin(t/2)} \right| dt \geq \alpha_n$$

where  $\alpha_n \rightarrow +\infty$ .

**May 2, 2024**

**Recall:**

$$f \in C_{\text{per}}[-\pi, \pi]$$

$$f_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$$

$$F_n(x) = \sum_{k=-n}^n f_k e^{ikx} \xrightarrow[\times]{?} f(x)$$

$$F_n(x) = \int_{-\pi}^{\pi} f(x-t) D_n(t) dt$$

with

$$D_n(t) = \sum_{k=-n}^n e^{-nt} = \frac{\sin(n+1/2)t}{\sin(t/2)}$$

the Dirichlet kernel.

## Fejér-Cesàro Means

$$\begin{aligned}\sigma_n &= \frac{1}{n} \sum_{k=0}^{n-1} F_k(x) = \sum_{k=-n}^n \left(1 - \frac{|k|}{n}\right) f_k e^{ikx} \\ &= \int_{-\pi}^{\pi} f(x-t) s_n(t) dt\end{aligned}$$

with

$$s_n(t) = \sum_{k=-n}^n \left(1 - \frac{|k|}{n}\right) e^{ikt} \frac{1}{n} \left(\frac{\sin(nt/2)}{\sin(t/2)}\right)^2$$

the Fejér kernel.

Note that  $\int_{-\pi}^{\pi} s_n(t) dt = 1$  and  $s_n(t) \rightarrow 0$  for  $\delta \leq |t| \leq \pi$ .

## Theorem

For  $f \in C_{\text{per}}[-\pi, \pi]$ ,  $\sigma_n(x) \rightarrow f(x)$  uniformly on  $[-\pi, \pi]$  as  $n \rightarrow \infty$ .

## Proof (Sketch)

$$\begin{aligned}\sigma_n(x) - f(x) &= \int_{-\pi}^{\pi} (f(x-t) - f(x)) s_n(t) dt \\ &= \int_{|t| < \delta} (f(x-t) - f(x)) s_n(t) dt + \int_{\pi \geq |t| \geq \delta} (f(x-t) - f(x)) s_n(t) dt \\ |\sigma_n(x) - f(x)| &= \sup_{|t| \leq \delta} |f(x-t) - f(x)| \cdot \|s_n\|_{L^1} + 2\|f\|_{\infty} \cdot 2\pi \cdot \sup_{\pi \geq |t| \geq \delta} |s_n(t)|\end{aligned}$$

Given  $\varepsilon$ , by the uniform continuity of  $f$ , find  $\delta > 0$  such that

$$\sup_x \sup_{|t| \leq \delta} |f(x-t) - f(x)| < \varepsilon$$

Then  $\|s_n\|_{L^1} = 1$ ,  $s_n(t) \geq 0$ ,  $\int_{-\pi}^{\pi} s_n(t) dt = 1$  and, for fixed  $\delta$ ,

$$\lim_{n \rightarrow \infty} \sup_{\pi \geq |t| \geq \delta} |s_n(t)| = 0$$

It follows that

$$\sup_x |\sigma_n(x) - f(x)| \leq \varepsilon + c \cdot \sup_{|t| \geq \delta} |s_n(t)|$$

Taking  $n \rightarrow \infty$ ,

$$\limsup_{n \rightarrow \infty} \sup_x |\sigma_n(x) - f(x)| \leq \varepsilon, \forall \varepsilon > 0$$

and

$$\lim_{n \rightarrow \infty} \sup_x |\sigma_n(x) - f(x)| = 0$$

## Operator Interpretation

One can define  $A_n : f \in C_{\text{per}}[-\pi, \pi] \rightarrow \sigma_n(x) \in C_{\text{per}}[-\pi, \pi]$  where  $\sigma_n \rightrightarrows f$  means  $A_n \rightarrow I$  strongly on  $C_{\text{per}}[-\pi, \pi]$ . Since  $\forall f \in C_{\text{per}}[-\pi, \pi]$ , we have  $\sigma_n = A_n f \rightarrow f$  in the norm of  $C_{\text{per}}[-\pi, \pi]$ .

## Theorem: Open Mapping Theorem

Let  $V$  be an  $F$ -space,  $W$  be a TVS, and let  $T : V \rightarrow W$  be a continuous linear operator such that  $\text{im } V$  is of 2nd category in  $W$ .

Then  $T$  is open,  $\text{im } T = W$  and  $W$  is an  $F$ -space.

### Remark

$\text{im } T$  is of 2nd category in  $W$  means  $\text{im } T$  is not a countable union of nowhere dense subsets in  $W$ .

### Definition: Open Map

$T$  open means  $T$  maps open sets into open sets.

### Proof

Have to show: for each open neighborhood  $U \ni 0$  in  $V$ ,  $T(U)$  contains an open neighborhood of 0.

Consider  $V_n = \{x \in V : d(x, 0) < r/2^n\}$  and  $r > 0$  such that  $V_0 \subseteq U$ .

Idea:  $\overline{TV_1} \subseteq TV_0 \subseteq TU$  and  $\overline{TV_1}$  contains an open neighborhood of 0.

### Step 1

$\overline{TV_n}$  contains an open neighborhood of 0.

Note that  $d(x, 0) < r/2^{n+1}$  and  $d(y, 0) < r/2^{n+1}$  implies

$$d(x - y, 0) = d(x, y) \leq d(x, 0) + d(0, y) < 2 \cdot r/2^{n+1}$$

Take  $V_n \supseteq V_{n+1} - V_{n+1}$  such that  $TV_n \subseteq T(V_{n+1} - V_{n+1}) = TV_{n+1} - TV_{n+1}$ . Then

$$\overline{TV_{n+1}} \supseteq \overline{TV_{n+1} - TV_{n+1}} \supseteq \overline{TV_{n+1}} - \overline{TV_{n+1}}$$

Obviously,

$$V = \bigcup_{k=1}^{\infty} k \cdot V_{n+1}$$

because  $V_{n+1}$  is an open neighborhood of zero and absorbing. Hence



$$TV = \bigcup_{k=1}^{\infty} kTv_{n+1} \quad \text{and} \quad TV \subseteq \bigcup_{k=1}^{\infty} k\overline{TV_{n+1}}$$

Since  $TV$  is of second category, there exists some  $k$  such that  $k\overline{TV_{n+1}}$  is not nowhere dense. Then  $\text{int}(k\overline{TV_{n+1}}) \neq \emptyset$  which implies  $\text{int}(\overline{TV_{n+1}}) \neq \emptyset$ . That is,  $\overline{TV_{n+1}}$  contains an interior point, say  $x_0$ . Then there exists an open neighborhood  $\hat{U} \ni 0$  such that  $x_0 + \hat{U} \subseteq \overline{TV_{n+1}}$ .

$$\hat{U} = (x_0 + \hat{U}) - x_0 \subseteq \overline{TV_{n+1}} - \overline{TV_{n+1}} \subseteq \overline{TV_n}$$

## Step 2

$$\overline{TV_1} \subseteq TV_0.$$

Let  $y_1 \in \overline{TV_1}$ ,  $y_1 - \overline{TV_2}$  contains some neighborhood of  $y_1$ .

Then  $(y_1 - \overline{TV_2}) \cap TV_1 \neq \emptyset$ . Choose  $w_1 = y_1 - y_2$ ,  $y_2 \in \overline{TV_2}$ ,  $w_1 = Tx_1$ ,  $x_1 \in V_1$ .

By the same argument, choose  $w_2 = y_2 - y_3$ ,  $y_3 \in \overline{TV_3}$ ,  $w_2 = Tx_2$ ,  $x_2 \in V_2$ .

Continuing gives  $y_1, y_2, y_3, \dots$ ,  $x_1, x_2, x_3, \dots$ ,  $w_1, w_2, w_3, \dots$

Where  $x_n \in V_n$ ,  $y_n \in \overline{TV_n}$ ,  $w_n = y_n - y_{n+1} = Tx_n$  or, equivalently,  $y_{n+1} = y_n - Tx_n$ .

It follows that  $y_{n+1} = y_1 - T(x_1 + \dots + x_n)$ .

Because  $x_n \in V_n$  ( $d(x_n, 0) < r/2^n$ ),  $x_1 + \dots + x_n$  is a Cauchy sequence.

That is, by completeness,  $v = \sum_{n=1}^{\infty} x_n$  with  $d(v, 0) \leq \sum_{k=1}^{\infty} d(x_k, 0) < r$  and  $v \in V_0 \subseteq V$ .

Taking  $n \rightarrow \infty$ ,  $\lim_{n \rightarrow \infty} y_n = y_1 - Tv$ .

Claim:  $y := \lim_{n \rightarrow \infty} y_n = 0$ . Otherwise,  $y \neq 0$ ,  $y \in W$  where  $W$  is Hausdorff, there exists open neighborhoods of 0 and  $y$  where

$$W_0 \cap W_y = \emptyset$$

But as a continuous linear operator,  $T^{-1}(W_0)$  has an open neighborhood of 0.

So there exists some  $n$  such that  $V_n \subseteq T^{-1}(W_0)$  which implies that  $TV_n \subseteq W_0 \subseteq W \setminus W_y$  closed.

Then  $\overline{TV_n} \subseteq W \setminus W_y$  but  $W \setminus W_y$  which implies  $y \notin \overline{TV_n}$ .

For  $N \geq n$ ,  $y_n \in \overline{TV_N} \subseteq \overline{TV_n}$ . So  $y_n \rightarrow y$ ,  $y_n \in \overline{TV_n}$  ( $N \geq n$ ),  $y \notin \overline{TV_n}$  a contradiction.

Therefore  $y = 0$ ,  $y_1 = TV$ ,  $v \in V_0$ ,  $y_1 \in TV_0$  and finally  $\overline{TV_1} \subseteq TV_0$ .

## To Show

The above demonstrates that  $T$  is open.

We still need that  $\text{im } T = W$  and  $W$  is an  $F$ -space.

## Part 3

We have that

$$\text{im } T = T(V)$$

open in  $W$ . Since open neighborhoods of 0 are absorbing,

$$\bigcup_{k=1}^{\infty} kTV = W = \bigcup_{k=1}^{\infty} T(kV) = \bigcup_{k=1}^{\infty} TV = TV$$

so  $TV = W$ .

## Part 4 (Sketch)

We have that  $T : V \rightarrow W$  open, surjective, and continuous.

Define  $\hat{T} : V/\ker(T) \rightarrow W$  as

$$\begin{aligned}\hat{T}:[x] &\rightarrow Tx \\ [x] &= x + \ker(T)\end{aligned}$$

a continuous linear operator with  $\ker(T)$  a closed subspace. Then

$$\begin{array}{ccc} V & \xrightarrow{\pi} & V/\ker(T) \\ & \searrow T & \downarrow \hat{T} \\ & & W \end{array}$$

We have that  $V/\ker(T) = F$ -space,  $\hat{d}([x], [y]) = \inf_{z \in \ker(T)} d(x+z, y)$ .

Per the commutative diagram,  $\hat{T}$  is open and continuous (a linear homeomorphism). Take

$$\begin{aligned}\hat{T} : V/\ker(T) &\rightarrow W \\ \hat{d} &\leadsto d_W\end{aligned}$$

With  $d_W(Tx_1, Tx_2) = \hat{d}([x_1], [x_2]) = \inf_z d(x_1 + z, x_2)$ .

Then  $\hat{T}$  is an isometry and, with  $d_W$ , an  $F$ -space.

The topology induced by  $d_W$  is equivalent to the original topology.

**May 7, 2024**

## Theorem: Open Mapping Theorem

Let  $V$  and  $W$  be  $F$ -spaces, and let  $T : V \rightarrow W$  be a continuous linear operator which is surjective. Then  $T$  is open.

### Proof

$\text{im } T = W$  is of second category since it is an  $F$ -space.

## Corollary: Banach's Theorem About the Inverse Operator

Let  $V, W$  be  $F$ -spaces, and let  $T : V \rightarrow W$  be a continuous linear operator which is bijective (invertible). Then the inverse  $T^{-1} : W \rightarrow V$  is continuous.

### Remark

This result implies:

- Each (pre-)  $F$ -space of dimension  $n$  is topologically isomorphic to  $\mathbb{F}^n$ .

### Proof

For  $V$  a pre- $F$ -space,  $T : \mathbb{F}^n \rightarrow V$  a linear bijection and  $V$  complete,  $T^{-1}$  is continuous.

## Corollary

Let  $V$  be a vector space with two topologies  $\tau_1, \tau_2$  such that  $(V, \tau_1)$  and  $(V, \tau_2)$  become  $F$ -spaces. If  $\tau_1 \subseteq \tau_2$ , then  $\tau_1 = \tau_2$ .

### Proof

For  $I: \underset{(\tau_2)}{V} \rightarrow \underset{(\tau_1)}{V}$  the identity map  $Ix = x$ ,  $I$  is continuous.

Then  $I^{-1}: \underset{(\tau_1)}{V} \rightarrow \underset{(\tau_2)}{V}$  is continuous and  $\tau_2 \subseteq \tau_1$ .

## Corollary

Let  $V, W$  be Banach spaces and  $T: V \rightarrow W$  be a bounded linear operator which is bijective (invertible). Then  $\exists a, b > 0$  such that

$$a \cdot \|x\|_V \leq \|Tx\|_W \leq b \cdot \|x\|_V$$

### Proof

Since  $T: V \rightarrow W$  is bounded (continuous),

$$\|Tx\| \leq \underbrace{\|T\|_{L(V,W)}}_b \cdot \|x\|$$

and since  $T^{-1}: W \rightarrow V$  is bounded

$$\|x\| = \|T^{-1}Tx\| \leq \underbrace{\|T^{-1}\|_{L(W,V)}}_{1/a} \cdot \|Tx\|$$

## Corollary

Let  $V$  be a vector space with two norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$  such that both  $(V, \|\cdot\|_1)$  and  $(V, \|\cdot\|_2)$  are Banach spaces. Assume that there exists some  $M$  such that  $(1) \|x\|_1 \leq M \cdot \|x\|_2, \forall x \in V$ .

Then both norms are equivalent, and there exists  $m > 0$  such that

$$(2) \|x\|_2 \leq m \cdot \|x\|_1, \quad \forall x \in V$$

### Proof

For  $I$  the identity operator,  $I: (V, \|\cdot\|_2) \rightarrow (V, \|\cdot\|_1)$ ,

(1) implies that  $I$  is bounded which implies  $I^{-1}$  is bounded which finally implies (2).

## Examples

### Counter-Example 1

For  $\ell^1 \subseteq \ell^\infty$ , take  $I: \ell^1 \rightarrow \ell^\infty$  the identity map  $Ix = x$ .

Take  $V = (\ell^1, \|\cdot\|_1)$  where  $\|x\|_1 = \sum_{n=1}^{\infty} |x_n|$  and  $W = (\ell^1, \|\cdot\|_\infty)$  where  $\|x\|_\infty = \sup_{n \geq 1} |x_n|$ .

$V$  is complete while  $W$  is not complete (completion  $c_0 = \{x \in \ell^\infty : \lim_{n \rightarrow \infty} x_n = 0\}$ ).

$I$  is bounded, so

$$||Ix||_{\infty} = ||x||_{\infty} = \sup_{n \geq 1} |x_n| \leq \sum_{n=1}^{\infty} |x_n| = ||x||_1$$

However,  $I^{-1}$  is not bounded otherwise for some constant  $b > 0$ ,

$$||x||_1 \leq b ||x||_{\infty}, \quad \forall x \in \ell^1$$

and

$$\sum_{n=1}^{\infty} |x_n| \leq b \cdot \sup_{n \geq 1} |x_n|$$

If we choose

$$x = (\underbrace{1, 1, \dots, 1}_{n \text{ times}}, 0, 0, 0, \dots)$$

Then  $n \geq b \cdot 1$  and sending  $n \rightarrow \infty$  causes a contradiction.

## Counter-Example 2

Let  $V$  be an infinite dimensional Banach space with norm  $|| \cdot ||$ .

Choose an unbounded linear functional  $\phi \in V'$  ( $\phi \notin V^*$ ),  $\phi : V \rightarrow \mathbb{F}$ .

Define a new norm  $||x||_* = ||x|| + |\phi(x)|$ . Then take the identity map

$$I : \underset{\text{not complete}}{(V, || \cdot ||_*)} \rightarrow \underset{\text{complete}}{(V, || \cdot ||)}$$

Obviously  $||x|| \leq ||x||_*$ , so  $I$  is bounded. But it is not true that  $||x||_* \leq C \cdot ||x||$ ,  $\forall x \in V$ .

Otherwise we would have that  $|\phi(x)| \leq C ||x||$  which would make  $\phi$  bounded, a contradiction.

By previous corollary, this implies that  $(V, || \cdot ||_*)$  is not complete.

## Definition: Graph of a Function

Given  $f : X \rightarrow Y$ , the graph of  $f$ :  $G(f) = \{(x, f(x)) : x \in X\} \subseteq X \times Y$ .

Sometimes,  $f : D(f) \subseteq X \rightarrow Y$  where  $D(f)$  is the domain and  $G(f) = \{(x, f(x)) : x \in D(f)\} \subseteq X \times Y$ .

## Definition: Closed Graph of a Function

Let  $x, Y$  be topological spaces and  $f$  be a function from  $X$  (or  $D(f) \subseteq X$ ) into  $Y$ .

Then  $f$  is of closed graph if  $G(f)$  is a closed subset in  $X \times Y$ .

## Examples

$f(x) = \frac{1}{x}$ ,  $D = \mathbb{R} \setminus \{0\}$ ,  $X = Y = \mathbb{R}$  is continuous on  $D$  and has a closed graph. Contrarily

$$f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$D(f) = X = Y = \mathbb{R}$  is of closed graph but not continuous. Finally

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

is neither continuous nor of closed graph.

### Lemma:

Let  $X, Y$  be metric spaces and  $f : D(f) \subseteq X \rightarrow Y$ .

Then  $f$  is of closed graph if and only if whenever  $x_n \rightarrow x$  with  $x_n \in D(f)$  and  $f(x_n) \rightarrow y$ ,  $x \in D(f)$  and  $f(x) = y$ .

### Proof

For  $G(f)$  closed in  $X \times Y$ , we have that whenever  $(x_n, f(x_n)) \in G(f)$  converges  $(x_n, f(x_n)) \rightarrow (x, y)$ , then  $(x, y) \in G(f)$ .

Then whenever  $x_n \in D(f)$  converges  $x_n \rightarrow x$  and  $f(x_n) \rightarrow y$ , then  $x \in D(f)$  and  $y = f(x)$ .

### Proposition:

If  $f : X \rightarrow Y$  is continuous,  $X$  a topological space and  $Y$  Hausdorff, then  $f$  is of closed graph.

### Proof

Take  $U = (X \times Y) \setminus G(f)$ ,  $(x_0, y_0) \in U$ .

Then  $(x_0, y_0) \notin G(f)$ , so  $y_0 \neq f(x_0)$ . Since  $Y$  is Hausdorff, there exist open sets  $U_{f(x_0)} \ni f(x_0)$  and  $U_{y_0} \ni y_0$  with  $U_{y_0} \cap U_{f(x_0)} = \emptyset$ .

$U_{x_0} = f^{-1}(U_{f(x_0)})$  is open in  $X$  with  $x_0 \in U_{x_0}$ .

Claim:  $U_{x_0} \times U_{y_0} \subseteq U$  a neighborhood of  $(x_0, y_0)$  so  $(x_0, y_0)$  is an interior point of  $U$ .

We have that  $(U_{x_0} \times U_{y_0}) \cap G(f) = \emptyset$  with  $(x, y) \in G(f)$ .

But  $y = f(x) \in U_{f(x_0)}$ ,  $x \in U_{x_0} = f^{-1}(U_{f(x_0)})$ , and  $f(x) \in U_{f(x_0)}$  contradicts the fact that they are disjoint.

### Theorem: Closed Graph Theorem

Let  $X, Y$  be  $F$ -spaces and  $A : X \rightarrow Y$  be a linear operator which is of closed graph.

Then  $A$  is continuous.

### Proof

$X \times Y$  is an  $F$ -spaces equipped with a metric  $d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$ .

Then  $\{(x, Ax) : x \in X\} = G(A) \subseteq X \times Y$  is a linear subspace and closed by assumption.

$$\begin{aligned} (x_1, Ax_1) + (x_2, Ax_2) &= (x_1 + x_2, A(x_1 + x_2)) \\ \lambda(x, Ax) &= (\lambda x, A(\lambda x)) \end{aligned}$$

Further,  $G(A)$  is an  $F$ -space (complete). Take the projection

$$\begin{aligned} \pi : G(A) &\rightarrow X \\ (x, Ax) &\mapsto x \end{aligned}$$

a continuous linear operator since

$$(x_n, Ax_n) \rightarrow (x, Ax) \implies x_n \rightarrow x$$

We have also that  $\pi$  is bijective, since

$$\pi((x, Ax)) = x \quad \text{and} \quad \pi((x, Ax)) = 0 \implies x = 0 \implies Ax = 0$$

Applying the open mapping theorem and the Banach theorem for inverse operators,

$$\begin{aligned} \pi^{-1} : X &\rightarrow G(A) \\ x &\mapsto (x, Ax) \end{aligned}$$

is also continuous. If  $x_n \rightarrow x$ , then  $\pi^{-1}(x_n) \rightarrow \pi^{-1}(x)$  ( $(x_n, Ax_n) \rightarrow (x, Ax)$ ) gives  $Ax_n \rightarrow Ax$  and  $A$  is continuous.

## For Banach Spaces

$X, ||x||$ .

$$||x||_* = ||\pi^{-1}(x)|| = ||x|| + ||Ax||$$

$||x|| \neq |\phi(x)|$ .