Manifolds III

March 31, 2025

Review

If X, Y are topological spaces and $f, g : X \to Y$ continuous maps, we say f and g are homotopic (written $f \simeq g$) if there is a homotopy $H : X \times I \to Y$ (where $I = [0,1]$) such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$.
We say that f is null-homotopic if it is homotopic to a constant map.

Proposition

Homotopy is an equivalence relation on the collection of continuous maps.

1. $f \simeq f$ by $H(x,t) := f(x)$.
2. $f \overset{\tilde{H}}{\simeq} g \implies g \simeq f$ by defining $\tilde{H}(x,t) := H(x, 1-t)$.
3. $(f \overset{F}{\simeq} g \wedge g \overset{G}{\simeq} h) \implies f \simeq h$ by

$$H(x,t) := \begin{cases} F(x,2t) & 0 \le t \le 1/2 \\ G(x,2t-1) & 1/2 \le t \le 1 \end{cases}.$$

Proposition

For $f_0, f_1 : X \to Y$ and $g_0, g_1 : Y \to Z$, if $f_0 \overset{F}{\simeq} f_1$ and $g_0 \overset{G}{\simeq} g_1$, then $g_0 \circ f_0 \simeq g_1 \circ f_1$.

Proof

Define $H(x,t) := G(F(x,t),t)$ such that $H(x,0) = G(F(x,0),0) = G(f_0(x),0) = g_0 \circ f_0(x)$.
Similarly, $H(x,1) = g_1 \circ f_1(x)$.

Definition: Homotopic Spaces

We say that two spaces X and Y are homotopic to each other ($X \simeq Y$) if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.

Example

$\mathbb{R}^n$ is homotopic to $\{0\}$ (or any single point) by $\iota : 0 \to \mathbb{R}^n$ and $r : \mathbb{R}^n \to 0$. Then $r \circ \iota : 0 \to 0$ is id_0 and $\iota \circ r : \mathbb{R}^n \ni x \mapsto 0 \in \mathbb{R}^n$ is homotopic to $\mathrm{id}_{\mathbb{R}^n}$. In fact, consider $H : \mathbb{R}^n \times I \to \mathbb{R}^n$ where $H(x,t) = tx$, $H(x,1) = x = \mathrm{id}_{\mathbb{R}^n}(x)$ and $H(x,0) = 0$.

Definition: Path

A path in X from p to q is a continuous map $f : I \to X$ such that $f(0) = p$ and $f(1) = q$.

Definition: Path Homotopic

Let $f, g : I \to X$ be two paths in X from p to q.
We say that f and g are path homotopic (write $f \sim g$) if there is a homotopy $H : I \times I \to X$ such that $H(s,0) = f(s)$, $G(s,1) = g(s)$, $H(0,t) = p$ and $H(1,t) = q$.

Proposition

Path homotopy is an equivalence relation on the collection of paths from p to q.
Write $[f]$, the equivalence class of f in the quotient.

Definition: Loop

In the special case that $p = q$, we say that $f : I \to X$ is a loop

Definition: Fundamental Group

Given (X, p), $\pi_1(X, p)$ (the fundamental group of X at the point p) is the set of all loops at p modulo the path homotopy.

$$\{\text{loops at } p\}/\sim$$

Equivalently, $(S^1, 1)$, $\{\text{loops at } p\} = \{\text{continuous maps } f : (S^1, 1) \to (X, p)\}$ with $f(1) = p$. We say this is the homotopy "relative to $1 \in S^1$". We have $H : S^1 \times I \to X$ such that $H(s, 0) = f(s)$, $H(s, 1) = g(s)$ and $H(1, t) = p$.

Definition: Free Homotopy

For two loops $f, g : S^1 \to X$, we say that f and g are free homotopic if $f \simeq g$.

Lemma

When $f : I \to X$ is a path from p to q, if $f \circ \varphi$ is a reparameterization of f then $(f \circ \varphi) \sim f$ where $\varphi : I \to I$ satisfies $\varphi(0) = 0$ and $\varphi(1) = 1$.

Proof

Note that φ is homotopic to the identity map id_I through $H(s, t) = ts + (1 - t)\varphi(s)$ since $H(s, 0) = \varphi(s)$ and $H(s, 1) = s = \mathrm{id}_I(s)$.
Then consider $f \circ H : I \times I \to X$ which is a path homotopy between f and $f \circ \varphi$.

Fundamental Group

Let $f, g : I \to X$ be two paths with $f(1) = g(0)$.
Then we can "compose" (concatenate) f and g together $(f \cdot g) : I \to X$ by

$$(f \cdot g)(s) := \begin{cases} f(2s) & 0 \le s \le 1/2 \\ g(2s - 1 & 1/2 \le s \le 1 \end{cases}.$$

Lemma

If $f_0 \overset{F}{\sim} f_1$, $g_0 \overset{G}{\sim} g_1$ and $f_0(1) = f_1(1) = g_0(0) = g_1(0)$, then $f_0 \cdot g_0 \sim f_1 \cdot g_1$.

Proof

Define

$$H(s, t) := \begin{cases} F(2s, t) & 0 \le s \le 1/2 \\ G(2s - 1, t) & 1/2 \le s \le 1 \end{cases}.$$

Then

$$H(s,0) = \begin{cases} F(2s,0) = f_0(2s) & 0 \le s \le 1/2 \\ G(2s-1,0 = g_0(2s-1) & 1/2 \le s \le 1 \end{cases}.$$

Similarly $H(s,1) = (f_1 \cdot g_1)(s)$, hence $f_0 \cdot g_0 \sim f_1 \cdot g_1$.
With this, we have a well-defined $[f] \cdot [g] := [f \cdot g]$.

Simple Properties

For f from p to q where c_p is the constant map at p,

1. $[c_p] \cdot [f] = [f] = [f] \cdot [c_q]$ since $c_p \cdot f$ is a reparameterization of f.

2. Let $\overline{f}$ be the inverse path of f (i.e. $\overline{f}(s) = f(1-s)$). Then $[f] \cdot [\overline{f}] = [c_p]$ and $[\overline{f}] \cdot [f] = [c_q]$.

$$H(s,t) := \begin{cases} f(2s) & 0 \le s \le t/2 \\ f(t) & t/2 \le s \le 1 - t/2 \\ f(2-2s) & 1 - t/2 \le s \le 2 \end{cases}.$$

1. $([f] \cdot [g]) \cdot [h] = [f] \cdot ([g] \cdot [h])$, since these are reparameterizations of the same path.

Group Structure

$\pi_1(X,p) = \{\text{loops at } p\}/\sim$.
Define $[f] \cdot [g] := [f \cdot g]$.
It has an identity element $[c_p] = e$.
For any $f \in \pi_1(X,p)$, it has an inverse $[\overline{f}]$ such that $[f] \cdot [\overline{f}] = [\overline{f}] \cdot [f] = [c_p]$.
Finally, it is associative by (3) above.

Proposition

Suppose $p, q \in X$ with X path-connected.
Then $\pi_1(X,p)$ is isomorphic to $\pi_1(X,q)$.
Remark: this isomorphism is not canonical.

Proof

We define a path γ from q to p and $\Phi_\gamma : \pi_1(X,p) \to \pi_1(X,q)$ by $[f] \mapsto [\gamma \cdot f \cdot \overline{\gamma}]$.
Φ_γ is a group homomorphism.

$$\begin{aligned} \Phi_\gamma[f] \cdot \Phi_\gamma[g] &= [\gamma \cdot f \cdot \overline{\gamma}] \cdot [\gamma \cdot g \cdot \overline{\gamma}] \\ &= [\gamma \cdot f \cdot \overline{\gamma} \cdot \gamma \cdot g \cdot \overline{\gamma}] \\ &= [\gamma \cdot f] \cdot \overbrace{[\overline{\gamma} \cdot \gamma]}^{=e} \cdot [g \cdot \overline{\gamma}] \\ &= [\gamma \cdot (f \cdot g) \cdot \overline{\gamma}] \\ &= \Phi_\gamma[f \cdot g]. \end{aligned}$$

Φ_γ has an inverse, $\Phi_{\overline{\gamma}} : \pi_1(X,q) \to \pi_1(X,p)$.

$$\Phi_{\overline{\gamma}} \circ \Phi_\gamma[f] = \Phi_{\overline{\gamma}}[\gamma \cdot f \cdot \overline{\gamma}] = [\overline{\gamma} \cdot \gamma \cdot f \cdot \overline{\gamma} \cdot \gamma] = [f].$$

Induced Homomorphism

$\varphi : (X, p) \to (Y, q)$ induces

$$\begin{aligned} \varphi_* : \pi_1(X, p) &\to \pi_1(Y, q) \\ [f] &\mapsto [\varphi \circ f]. \end{aligned}$$

φ_* is a homomorphism.

$$(\varphi_*[f]) \cdot (\varphi_*[g]) = [\varphi \circ f] \cdot [\varphi \circ g] = [(\varphi \circ f) \cdot (\varphi \circ g)] = [\varphi \circ (f \cdot g)] = \varphi_*[f \cdot g].$$

Proposition

If $\varphi, \psi : (X, p) \to (Y, q)$ are homotopic, then $\varphi_* = \psi_* : \pi_1(X, p) \to \pi_1(Y, q)$.

Proof

Let $[f] \in \pi_1(X, p)$, $\varphi_*[f] = [\varphi \circ f]$ and $\psi_*[f] = [\psi \circ f]$ and $H : X \times I \to Y$ a homotopy between φ and ψ.
Then define $\tilde{H} := I \times I \to Y$ by $\tilde{H}(s, t) = H(f(s), t)$ such that

$$\begin{aligned} \tilde{H}(s, 0) &= H(f(s), 0) = \varphi \circ f(s) \\ \tilde{H}(s, 1) &= H(f(s), 1) = \psi \circ f(s). \end{aligned}$$

Corollary

If $X \simeq Y$, then $\pi_1(X) \simeq \pi_1(Y)$.

Examples (*)

$\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(S^n) = 0$ for $n \geq 2$.
For $n \geq 2$, write $S^n = A_+ \cup A_-$ where A_+ and A_- are large balls centered at the north and south pole respectively.
Then A_+ and A_- are both homeomorphic to $\mathbb{R}^n$ and $A_+ \cap A_-$ (their intersection about the equator) is homemorphic to $S^{n-1} \times \mathbb{R}$.
We fix a base point $p \in A_+ \cap A_-$ and let $f : I \to S^n$ be a loop based at p.
There exists a partition of I, $0 = s_0 < s_1 < \cdots < s_k = 1$, such that $f|_{[s_i, s_{i+1}]}$ is contained in A_- or A_+.
Draw a path γ_i from p to $f(s_i)$ such that $\gamma_i \subseteq A_+ \cap A_-$. Let $f_i = f|_{[s_i, s_{i+1}]}$ such that $f = f_0 \cdot f_1 \cdots f_k$. Then this is path homotopic to

$$(f_0 \cdot \overline{\gamma}_1) \cdot (\gamma_1 \cdot f \cdot \overline{\gamma}_2) \cdots (\gamma_{k-1} \cdot f_{k-1} \cdot \overline{\gamma}_k) \cdot (\gamma_k \cdot f_k).$$

Each $\gamma_i \cdot f_i \cdot \overline{\gamma}_i$ is contained in A_- or A_+, hence $\gamma_i \cdot f_i \overline{\gamma}_{i+1} \sim c_p$, $f \simeq c_p$ and $[f] = e$.

April 2, 2025

Correction

For $\varphi, \psi : (X, x_0) \to (Y, y_0)$ where $\varphi \simeq \psi$, we say a homotopy H between φ and ψ is base point preserving if $H(x_0, t) = y_0$ for all $t \in [0, 1]$.

Proposition

If $\varphi \simeq \psi$ through a base point preserving homotopy, then $\varphi_* = \psi_*$, $\pi_1(X, x_0) \to \pi_1(Y, y_0)$.
For $X \simeq Y$, $\varphi : X \to Y$ and $\psi : Y \to X$ where $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$, in general $\psi \circ \varphi(x_0) \neq x_0$ and $\varphi \circ \psi(y_0) \neq y_0$.
Set up: $\varphi_0, \varphi_1 : X \to Y$ with $\varphi_0 \simeq \varphi_1$ through a homotopy H.
Write $\varphi_t = H(\cdot, t) : X \to Y$ and fix a base point $x_0 \in X$ and set $\gamma(t) = \varphi_t(x_0)$ for $t \in [0, 1]$.

Proposition 1

$(\varphi_0)_* = \Phi_\gamma \circ (\varphi_1)_* : \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$.

Proof

Let f be a loop at x_0.

IMAGE 1

Let γ_t be $\gamma|_{[0,t]}$ and then, by rescaling the domain $[0, t]$ to $[0, 1]$ i.e.

$$\begin{aligned} \gamma_t : [0,1] &\to Y \\ s &\mapsto \gamma(ts). \end{aligned}$$

from $\varphi_0(x_0)$ to $\gamma(t) = \varphi_t(x_0)$. Then $\gamma_t \cdot (\phi_t \circ f) \cdot \overline{\gamma}_t$ is a homotopy between $(\varphi_0 \circ f)$ and $\gamma \cdot (\varphi_1 \circ f) \cdot \overline{\gamma}$. Hence

$$(\varphi_0)_*[f] = [\varphi_0 \circ f] = [\gamma] \cdot [\varphi_1 \circ f] \cdot [\overline{\gamma}] = \Phi_\gamma \circ (\varphi_1)_*[f].$$

Proposition 2

If $X \simeq Y$, then $\pi_1(X) \cong \pi_1(Y)$.

Proof

Since $(\psi \circ \varphi) \simeq \mathrm{id}_X$, by Proposition 1

$$\psi_* \circ \varphi_* = (\psi \circ \varphi)_* = \Phi_\gamma \circ (\mathrm{id}_x)_* = \Phi_\gamma.$$

Hence $\psi_* \circ \varphi_*$ is an isomorphism (as is $\varphi_* \circ \psi_*$). Therefre φ_* and ψ_* are isomorphisms.

Recall: Covering Map

For $X, \tilde{X}$ connected, $\pi : \tilde{X} \to X$ is a covering map if for each $p \in X$ there exists a neighborhood $U \subset X$ such that $\pi^{-1}(U)$ is a disjoint union

$$\pi^{-1}(U) = \dot{\bigcup}_{\alpha \in A} U_\alpha$$

such that $\pi|_{U_\alpha} : U_\alpha \to U$ Is a homeomorphism.

Lifting Properties

A lift is a map $\tilde{f}$ such that $f = \pi \circ \tilde{f}$.

1. Path Lifting: Let $f : I \to X$ be a path from x_0. Then, for any $\tilde{x}_0 \in \pi^{-1}(x_0)$, there is a unique lift $\tilde{f}$ of f with $\tilde{f}(0) = \tilde{x}_0$.
2. Homotopy Lifting: Let $f_0, f_1 : I \to X$ be paths in X with $f_0(0) = f_1(0) = x_0$ and $f_0(1) = f_1(1)$. Suppose H is a path homotopy between f_0 and f_1. Then for any $\tilde{x}_0 \in \pi^{-1}(x_0)$, there is a unique lift $\tilde{H} : I \times I \to \tilde{X}$ of H. In particular, $\tilde{H}$ is a path homotopy between $\tilde{f}_0$ and $\tilde{f}_1$. That is if $H(0,t) = x_0$ then $\tilde{H}(0,t) \in \pi^{-1}(x_0)$ for all t. Hence $\tilde{H}(0,t) = \tilde{x}_0$, $\forall\, t \in [0,1]$. Similarly, $\tilde{H}(1,t)$ is identically constant. In particular, $\tilde{f}_0(1) = \tilde{H}(1,0) = \tilde{H}(1,1) = \tilde{f}_1(1)$.

Fundamental Group of the Circle

$\pi_1(S^1) = \mathbb{Z}$.

Example

$\pi : \mathbb{R} \to S^1$ by $s \mapsto e^{2\pi i \cdot s}$.

Proof

Take as a base point $1 = x_0 \in S^1 \subseteq \mathbb{C}$. For each $n \in \mathbb{Z}$, we define a loop $\omega_n : [0,1] \to S^1$ by $s \mapsto e^{2\pi i \cdot ns}$. Let f be a loop at $x_0 \in S^1$. We can lift f to $\tilde{f} : I \to \mathbb{R}$ at $0 \in \mathbb{R}$. Then $\tilde{f}(1) \in \pi^{-1}(x_0) = \mathbb{Z} \subseteq \mathbb{R}$. This defines a map φ that sends a loop f to $\tilde{f}(1) \in \mathbb{Z}$. This φ induces $\varphi : \pi_1(S^1, x_0) \to \mathbb{Z}$ well-defined. If $f_0, f_1 : I \to S^1$ at x_0 are path homotopic via H, then we may lift H to $\tilde{H} : I \times I \to \mathbb{R}$ which implies $\tilde{f}_0(1) = \tilde{f}_1(1)$.
φ is surjective, since for any $n \in Z$ we may consider the loop ω_n where $\tilde{\omega}_n(1) = n$.
φ is a group homorphism since $\varphi[f \cdot g] = \widetilde{f \cdot g}(1) = \tilde{g} + \tilde{f}(1) = \varphi[f] + \varphi[g]$.
φ is injective, since if $\varphi[f] = 0$ (i.e. $\tilde{f}(0) = 0$) then $\tilde{f}$ is a loop in $\mathbb{R}$ and $\tilde{f}$ is null-homotopic to c_0 by H. Therefore $\pi \circ \tilde{H}$ is a path-homotopy between f and c_{x_0} (i.e. $[f] = e$).

Path-Lifting

For $f : I \to X$, we have a special case where $\operatorname{im} f \subseteq U$ evenly covered. Write $\pi^{-1}(U) = \dot{\bigcup} \tilde{U}_\alpha$ and pick the $\tilde{U}_\alpha$ which contains $\tilde{x}_0$. Since $\pi|_{\tilde{U}_\alpha} : \tilde{U}_\alpha \to U$ is a homemorphism, $\tilde{f} := (\pi|_{\tilde{U}_\alpha})^{-1} \circ f$ is the unique lift of f at $\tilde{x}_0$.
In general, pick a partititon of $I = [0,1]$, $0 = t_0 < t_1 < \cdots < t_m = 1$, such that $\operatorname{im} f|_{[t_i, t_{i+1}]} \subseteq U_i$ evenly covered. We can lift $f|_{[0,t_1]}$ at $\tilde{x}_0$, giving $\tilde{f} : [0,t] \to \tilde{X}$. Next, we lift $f|_{t_1, t_2]}$ at $\tilde{f}(t_1) \in \tilde{X}$. Since the partition is finite, we may repeat the process until f is entirely lifted. This lift is unique.

Homotopy Lifting

For each fixed $(y_0, t_0) \in I \times I$, by continuity, there is a neighborhood $N(y_0) \times (t_0 - \varepsilon, t_0 + \varepsilon)$ such that H sends $N(y_0) \times (t_0 - \varepsilon, t_0 + \varepsilon)$ inside an evenly covered neighborhood. By compactness of $\{y_0\} \times [0,1]$, there is a finite collection of $N_{t_i}(y_0) \times (t_i - \varepsilon_i, t_i + \varepsilon_i)$ such that they cover $\{y_0\} \times I$ and the image of each under H is contained in an evenly covered neighborhood. Set $N = \bigcap_i N_{t_i}(y_0)$, a neighborhood of y_0, and construct a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ such that $H(N \times [t_i, t_{i+1}] \subseteq U_i$ evenly covered. Then we can start with $H|_{N \times [0,t_1]}$ and lift it at $\tilde{x}_0$ by some $(\pi|_{\tilde{U}_\alpha})^{-1}$. Then lift each $H|_{N \times [t_i, t_{i+1}]}$ one by one. Eventually, we have $\tilde{H} : N \times [0,1] \to \tilde{X}$ that lifts $H : N \times [0,1] \to \tilde{X}$ at $\tilde{x}_0$. This lift holds for any $y_0 \in I$ and, if two strips overlap, then the lift must agree there by the uniqueness of path lifting. This assures that $\tilde{H} : I^2 \to \tilde{X}$ is continuous.

Remark

Given a continuous map $F : Y \times I \to X$ and a covering $\pi : \tilde{X} \to X$, suppose that we have a map $\tilde{F} : Y \times \{0\} \to \tilde{X}$ that lifts $F|_{Y\times\{0\}} : Y \times \{0\} \to X$. Then there is a unique lift $\tilde{F} : Y \times I \to \tilde{X}$ of F which extends $\tilde{F} : Y \times \{0\} \to \tilde{X}$.

Theorem: Fundamental Theorem of Algebra

A polynomial $p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ (with $a_i \in \mathbb{C}$) has a root in $\mathbb{C}$.

Proof

Suppose otherwise. Then $p(z) \neq 0$, $\forall z \in \mathbb{C}$. Consider $f_r : [0,1] \to S^1$ $(r \geq 0)$ by

$$f_r(s) = \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}.$$

Then $f_0(s) \equiv 1$ is a constant loop at $1 \in \mathbb{C}$, and $f_r \simeq f_0$ for each $r \geq 0$. Consider $R \geq 1$ large such that $R \gg \sum_{i=1}^n |a_i|$. On $\{z \,:\, |z| = R\}$, we have

$$|z^n| > \left(\sum_{i=1}^n |a_i|\right) \cdot |z^{n-1}| \geq \sum_{i=1}^n |a_i| \cdot |z^{n-i}| = \left|\sum_{i=1}^n |a_i z^{n-i}\right|.$$

This implies that p does not have any roots on $\{|z| = R\}$. Moreover, for $p_t(z) = z^n + t(a_1 z^{n-1} + \cdots a_{n-1} z + a_n)$ with $0 \leq t \leq 1$, p_t does not have any roots on $\{|z| = R\}$. Consider

$$f_{R,t}(s) = \frac{p_t(Re^{2\pi i s})/p_t(R)}{|p_t(Re^{2\pi i s})/p_t(R)|}.$$

Then

$$f_{R,0}(s) = \frac{(Re^{2\pi i s})^n/R^n}{|(Re^{2\pi i s})^n/R^n|} = (e^{2\pi i s})^n = \omega_n(s).$$

Therefore $f_{R,1}(s) \simeq f_R(s)$ and $f_R \simeq \omega_n$. But since $\omega_n \not\simeq$ constant so this is a contradiction.

April 7, 2025

Definition: Retraction

Let X be a space and $A \subseteq X$ be a subset. We say that a continuous map $r : X \to A$ is a retraction if $r|_A = \mathrm{id}_A$. In particular, becasue $r \circ \iota_A = \mathrm{id_A}$, for $x_0 \in A$

$$r_* \circ (\iota_A)_* : \pi_1(A, x_0) \to \pi_1(A, x_0)$$

is an isomorphism. Hence $r_* : \pi(X, x_0) \to \pi(A, x_0)$ is surjective.

Corollary

There is no retraction $r : D^2 \to S^1 (= \partial D^2)$.

Proof

Suppose there is such a map r, then

$$r_* : \overbrace{\pi_1(D^2, x_0)}^{=0} \to \overbrace{\pi_1(S^1, x_0)}^{=\mathbb{Z}}$$

is surjective which is a contradiction.

Corollary

Every continuous map $h : D^2 \to D^2$ has a fixed point.

Proof

Suppose $\exists h : D^2 \to D^2$ without fixed points.

IMAGE 1

Define $r : D^2 \to D^2$ as the ray pictured from $h(x)$ through x to the boundary. If $x \in \partial D^2$, then by construction $r(x) = x$. Hence $r : D^2 \to S^1$ is a retraction which is a contradiction.

Corollary (Borsuk-Ulam)

Let $f : S^2 \to \mathbb{R}^2$. Then there exists a pair of antipodal points x and $-x$ on S^2 such that $f(x) = f(-x)$. This carries analogously to higher dimensions.

Proof

Suppose that $f(x) \neq f(-x)$ for all $x \in S^2$. We define $g : S^2 \to S^1$ by $g(x) = \frac{f(x)-f(-x)}{||f(x)-f(-x)||}$. On $S^2 \subseteq \mathbb{R}^3$, we consider a loop γ at the equator by $\gamma(s) = (\cos(2\pi s), \sin(2\pi s), 0)$ for $s \in [0,1]$. Because S^2 is simply connected, $g \circ \gamma : [0,1] \to S^1$ is path-homotopic to a constant loop in S^1. On the other hand, we lift $h := g \circ \gamma$ to $\tilde{h} : [0,1] \to \mathbb{R}$ with $\tilde{h}(0) = 0 \in \mathbb{R}$. Note

$$h(s+1/2) = g \circ \gamma(s+1/2) = g(\cos(2\pi s + \pi), \sin(2\pi s + \pi), 0) = g(-\gamma(s)) = -g(\gamma(s)) = -h(s).$$

Hence $\tilde{h}(s+1/2) \in \pi^{-1}(-h(s))$ where $\pi : \mathbb{R} \to S^1$ is the covering map. Since $\pi^{-1}(-h(s)) = \frac{1}{2} + \tilde{h}(s) + \mathbb{Z}$, for each $s \in [0,1/2]$ there is an integer q_s such that $\tilde{h}(s+1/2) = \frac{1}{2} + \tilde{h}(s) + q_s$ and

$$\tilde{h}(s+1/2) - \tilde{h}(s) = \frac{1}{2} + q_s.$$

The left hand side depends continuously on s and, by continuity, q_s is a constant (call it q). This gives

$$\tilde{h}(1) = \tilde{h}(1/2) + \frac{1}{2} + q = \tilde{h}(0) + 1 + 2q = 1 + 2q \neq 0$$

which contradicts the assertion that h is homotopic to a constant loop.

Corollary (Large Fiber Lemma)

If $f : [0,1]^{n+1} \to \mathbb{R}^n$ is a continuous map, then there exist $a, b \in [0,1]^{n+1}$ such that $f(a) = f(b)$ and $|a-b| \geq 1$. Remark: if $z = f(a) = f(b)$, then the lemma says that $\operatorname{diam} f^{-1}(z) \geq 1$.

Proof

Take the sphere of radius $1/2$ in $[0,1]^{n+1}$, then by Borsuk-Ulam there exist a pair of antipodal points $a,b \in S^1$ such that $f(a) = f(b)$ and $|a-b| \geq 1$.

Proposition

$\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$

Proof

Write $F : \pi_1(X \times Y) \to \pi_1(X) \times \pi_1(Y)$ by $[f] \mapsto ([g],[h])$. Then $f : [0,1] \to X \times Y$ is a loop at (x_0, y_0), $f(s) = (g(s), h(s))$, and $g : [0,1] \to X$ and $h : [0,1] \to Y$ are loops at x_0 and y_0 respectively.

Definition: Wedge Sum

Let X and Y be path-connected topological spaces. Then $X \vee Y = (X \coprod Y)/x_0 \sim y_0$
Let $\{X_\alpha\}$ be a family of such spaces. Then $\bigvee_\alpha X_\alpha = \coprod_\alpha X_\alpha / \sim$.

Sketch

$$\begin{aligned} \pi_1(S^1_-, x_0) &\to \pi_1(X, x_0) \quad \text{gen} \mapsto \alpha \\ \pi_1(S^1_+, x_0) &\to \pi_1(X, x_0) \quad \text{gen} \mapsto \beta \end{aligned}$$

with $\alpha \neq \beta$, $\alpha\beta \neq \beta\alpha$. Then $\pi_1(X, x_0)$ should be $\langle \alpha, \beta \rangle$.

Definition: Free Product

Let $\{G_\alpha\}_\alpha$ be a family of groups. $*_\alpha G_\alpha = \{g_1 g_2 \cdots g_k$: each g_i is a word in some $A_\alpha\}$.

Proposition

If for each α, there is a group homomorphism $\phi_\alpha : G_\alpha \to H$ then $\{\phi_\alpha\}$ induces a group homomorphism $\Phi : *_\alpha G_\alpha \to H$ by $g_1 \cdots g_k \mapsto \phi_{\alpha_1}(g_1) \cdots \phi_{\alpha_k}(g_k)$.

Van-Kapen Theorem

Setup

Let $X = \bigcup_\alpha A_\alpha$, each A_α open and connected where $\{A_\alpha\}$ have a common point x_0. Assume also that each $A_\alpha \cap A_\beta$ is path connected. Then $j_\alpha : A_\alpha \hookrightarrow X$ induces $j_\alpha : \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$. $\{j_\alpha\}_\alpha$ induces $\Phi : *_\alpha \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$ which is surjective by a similar argument as was used above for Example (*) ($S^2 = A_- \cup A_+$) applied to $X = \bigcup_\alpha A_\alpha$. Now, what is the kernel of Φ?

$$\begin{tikzcd} & A_\alpha \arrow[dr, hook] & \\ A_\alpha \cap A_\beta \arrow[ur, hook] \arrow[dr, hook] & & X \\ & A_\beta \arrow[ur, hook] & \end{tikzcd}$$

$$\begin{tikzcd}
 & \pi_1(A_\alpha) \arrow[dr, hook] & \\
\pi_1(A_\alpha \cap A_\beta) \arrow[ur, hook, "i_{\alpha\beta}"] \arrow[dr, hook, "i_{\beta\alpha}"'] & & \pi_1(X) \\
 & \pi_1(A_\beta) \arrow[ur, hook] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \underline{\quad} \arrow[dr, hook] & \\
w \arrow[ur, hook] \arrow[dr, hook] & & \underline{\quad} \\
 & \underline{\quad} \arrow[ur, hook] &
\end{tikzcd}$$

Then $i_{\beta\alpha}(w)i_{\alpha\beta}(w)^{-1}$ is NOT id in $*_\alpha \pi_1(A_\alpha)$. But through Φ, it hsould be $\mathrm{id} \in \pi_1(X, x_0)$. Hence every element in $*_\alpha \pi_1(A_\alpha)$ of the form $i_{\beta\alpha}(w)i_{\alpha\beta}(w)^{-1}$ where $w \in \pi_1(A_\alpha \cap A_\beta$ is in the kernel of Φ.

Theorem (Van-Kampen)

If every $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected, $\ker \Phi$ is the normal subgroup N generated by $\{i_{\beta\alpha}(w)i_{\alpha\beta}(w)^{-1} \ : \ \alpha, \beta \in A,\ w \in \pi_1(A_\alpha \cap A_\beta)\}$. Hence $\pi_1(X, x_0) \cong (*_\alpha \pi_1(A_\alpha, x_0))/N$.

Remarks

1. In the case that $X = A_0 \cup A_1$ with $A_0 \cap A_1$ path connected, then the intersection condition holds.

2. If $X = A_0 \cup A_1$ and $A_0 \cap A_1$ is simply connected, then $N = \{\mathrm{id}\}$ and $\pi_1(X) = \pi_1(A_0) * \pi_1(A_1)$.

3. If $X = A_0 \cup A_1$ and A_1 is simply connected, then $\pi_1(X) = \pi_1(A_0)/N$ and N is the normal subgroup generated by

$$i_{01}(w)\overbrace{i_{10}(w)^{-1}}^{\in \pi_1(A_1, x_0)} = i_{01}(w)$$

   i.e. N is the normal closure of $i_{01}(\pi_1(A_0 \cap A_1))$.

Example

IMAGE 2

For each $\alpha \in \{1, \ldots, 5\}$, let A_α be a small neighborhood of $T \cup e_1$. Every double/triple intersection is a neighborhood of T. Hence it is path continuous and we have that $\pi_1(A_\alpha) = \mathbb{Z}$. Thus $\pi_1(A_\alpha \cap A_\beta) = \mathrm{id}$, and $\pi_1(X) = *_\alpha \pi_1(A_\alpha)/N = *_1^5 \mathbb{Z}$.

Example

IMAGE 3

By Van-Kampen, $\pi_1(X) = \pi_1(A_0)$ modulo the normal closure of $i(\pi_1(A_0 \cap A_1))$. That is

$$\left\langle a, b \mid aba^{-1}b^1 \right\rangle = \mathbb{Z}^2.$$

Remark

In general, orientable M_g is the connected sum of g many toruses.

April 9, 2025

Recall: Van-Kampen Theorem

Write $\pi_1(X) = (\pi_1(A) * \pi_1(B))/N$ where N is the normal closure of $i_{\alpha\beta}(w)i_{\beta\alpha}(w)^{-1}$ for $w \in \pi_1(A \cap B)$, $i_{\alpha\beta} : \pi_1(A \cap B) \to \pi_1(A)$ and $i_{\beta\alpha} : \pi_1(A \cap B) \to \pi_1(B)$.

Example

M_g is the connected sum of g many tori, and $\pi_1(M_g) = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1 b_1] \cdots [a_g b_g] \rangle$.

Example

N_g is the connected sum of g many $\mathbb{RP}^2$ (e.g. N_2 is the Klein bottle). N_g has a polygon-representation by the $2g$-gon with boundary identified through $a_1 a_1 a_2 a_2 \cdots a_g a_g$. Therefore $\pi_1(N_g) = \left\langle a_1 \cdots a_g \mid a_1^2 \cdots a_g^2 \right\rangle$.

Abelianiztion

1. $\mathrm{Ab}(\pi_1(M_g))$ is the free abelian group generated by $\{a_1, b_1, \ldots, a_g, b_g\} = \mathbb{Z}^{2g}$.

2. $\mathrm{Ab}(\pi_1(N_g)) = \mathrm{Ab}(\left\langle a_1 \cdots a_{g-1} a_1 a_2 \cdots a_g \mid a_1^2 \cdots a_g^2 \right\rangle = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$.

Corollary

None of the surfaces in $\{S^2, M_1, \ldots, M_g, \ldots, N_1, \ldots, N_g, \ldots\}$ are homotopic to each other.

Definition: Cell Complex

0-cells are points; 1-cells, e^1, are intervals; 2-cells, e^2, are disks; n-cells, e^n, are $\overline{B}^n$.
A cell complex for space X is a decomposition (assuming finite dimensions) $X = X^0 \cup X^1 \cup \cdots \cup X^n$ where X^0 is the discrete set of points (i.e. 0-cells), X^1 is the space obtained by gluing 1-cells to X^0 ($\varphi_\alpha : \partial e_\alpha^1 \to X^0$), X^2 is the space obtained by gluing 2-cells to X^1 ($\varphi_\alpha : \partial e_\alpha^2 \to X^1$), and in general X^n is obtained by gluing n-cells $\{e_\alpha^n\}_\alpha$ to X^{n-1} by $\varphi_\alpha : \partial e_\alpha^n = S^{n-1} \to X^n$.

Examples

Cell complexes need not be unique. $S^2 = X^1 \cup_\alpha e_+^2 \cup_\alpha e_-^2$ and $S^2 = \{e^0\} \cup_\alpha \{e^2\}$.
$\mathbb{RP}^2 = \{e^1\} \cup_\alpha \{e^2\}$ where φ_α is given by $z \mapsto z^2$.
$\mathbb{T}^2$ is gluing e^2 to $S^1 \vee S^1$.

Theorem (Computing Fundamental Group)

Set up

Let X be a path-connected space, $Y = X \cup_\alpha e_\alpha^2$ (i.e. X is created by gluing 2-cells $\{e_\alpha^2\}_\alpha$ to X via $\phi_\alpha : \partial e_\alpha^2 \to X$). The inclusion $\iota : X \to Y$ induces $\iota_* : \pi_1(X) \to \pi_1(Y)$. Fix a base point $s_0 \in S^1$. For each α we draw a path γ_α from x_0 to $\varphi_\alpha(s_0)$. Then $\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha$ is a loop based at x_0. Thus $\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha$ is null-homotopic in Y (because φ_α is null-homotopic in e_α^2). That is $\iota_*[\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha] = \mathrm{id}$ in $\pi_1(Y)$ and is therefore in the kernel.

Theorem

Let N be the normal subgroup in $\pi_1(X)$ generated by elements of the form $[\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha]$. Then $\pi_1(Y) \cong \pi_1(X)/N$.

IMAGE 1

Example

$\mathbb{RP}^2$ is X^1 with e^2 glued to it by the map $\varphi : z \mapsto z^2$. Then $\pi_1(\mathbb{RP}^2) = \pi_1(S^1)/N = \langle \gamma \mid \gamma^2 \rangle$ where N is generated by φ. Similarly, the theorem applies to any M_g or N_g.

Definition: Deformation Retraction

For $X \subseteq Z$, $r : Z \to X$ is a retraction if $r|_X = \mathrm{id}_X$ implies $r \circ \iota = \mathrm{id}_X$. If $\iota \circ r : Z \to Z$ is homotopic to id_X, then $r_* : \pi_1(Z) \to \pi_1(X)$ is an isomorphism.

Proof

For each α, we glue a strip S_α along γ_α. We set the base at z_0 above x_0, $Z = Y \cup_\alpha S_\alpha$. Y is a deformation retraction of Z $(\pi_1(Y) = \pi_1(Z))$.

IMAGE 2

Set $A = Z - \bigcup_\alpha \{y_\alpha\}$, where y_α is a point in e^2_α not intersecting S_α. $B = Z - X$. A deformation retracts to X $\pi_1(A) = \pi_1(X)$. B is the union of some S_α (removing r_α) and some e^2_α (removing ∂e^2_α). B is contractible, $\pi_1(B) = \mathrm{id}$ and $A \cap B$ is the union of strips S_α and open disks punctured at y_α. Therefore

$$\pi_1(Y) = \pi_1(Z) = (\pi_1(A) * \pi_1(B))/N = \pi_1(A)/\iota_*(\pi_1(A \cap B)) \cong \pi_1(X)/\iota_*(\pi_1(A \cap B)).$$

Consider the loop $\delta_\alpha \cdot \gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha \cdot \overline{\delta}_\alpha$ where δ_α runs from z_0 to x_0, call this λ_α. It suffices to show that these generate $\pi_1(A \cap B, z_0)$. Cover $A \cap B$ by $A_\alpha = (A \cap B) - \bigcup_{\beta \neq \alpha} e^2_\beta$. Then A_α is a union of strips (with trivia fundamental group) and a single punctured, open disk $e^2_\alpha - \{y_\alpha\}$ and $\pi_1(A_\alpha) = \mathbb{Z} = \langle \lambda_\alpha \rangle$. So $A_\alpha \cap A_\beta$ is the union of strips, equal to $A_\alpha \cap A_\beta \cap A_\gamma$ and simply connected. By Van-Kampen,

$$\pi_1(A \cap B) = (*_\alpha \pi_1(A_\alpha))/N = *_\alpha \pi_1(A_\alpha)$$

is the free group generated by $\{\lambda_\alpha\}_\alpha$. This completes the proof.

Generalization (Theorem: Part 2)

If $Y = X \cup_\alpha e^n_\alpha$ for $n \geq 3$, then $\pi_1(Y) \cong \pi_1(X)$.
This follows from the same argument where instead A_α is the union of strips and a single punctured ball $B^n - \{y_\alpha\} \simeq S^{n-1}$. So $\pi_1(A_\alpha) = \mathrm{id}$, $\pi_1(A \cap B) = \mathrm{id}$, and $\pi_1(X) \cong \pi_1(Y)$.

Theorem: Part 3

Suppose X has a cell complex $X = X^0 \cup X^1 \cup \cdots \cup X^n$. Then $\pi_1(X) \cong \pi_1(X^2)$.
The proof follows directly from part 2.

Corollary

Given any group represented by generators and relations $G = \langle g_\alpha \mid r_\beta \rangle$, there is a cell complex X_G, of dimension 2, such that $\pi_1(X_G) \cong G$.

Proof

For each g_α, we draw a circle S^1_α. Then $X^1 = \bigvee_\alpha S^1_\alpha$ has fundamental group $*_\alpha \pi_1(S_\alpha) = \langle g_\alpha \rangle_\alpha$. To construct X_G, for each r_β glue a 2-cell e^2_α along r_β (think of r_β as a loop in X^1). Then in $X_G := X^1 \cup_\beta e^2_\beta$ we have $\pi_1(X_G) = \langle g_\alpha \mid r_\beta \rangle$.

April 14, 2025

Recall: Covering Spaces

Let $p : \tilde{X} \to X$, both X and $\tilde{X}$ path-connected.

1. Path-lifting: let $f : I \to X$ starting at $f(0) = x_0$. There is a unique lifting $\tilde{f}$ of f at $\tilde{x}_0 \in p^{-1}(x_0)$.

2. Homotopy-lifting: let $f_0, f_1 : I \to X$ be two paths with $f_0(0) = f_1(0) = x_0$ and $f_0(1) = f_1(1)$. Suppose f_t is a path-homotopy between f_0 and f_1. Then there exists a unique lift $\tilde{f}_t$ between $\tilde{f}_0$ and $\tilde{f}_1$ at $\tilde{x} \in p^{-1}(x)$.

These come from the following: let $f_t : Y \to X$ be a homotopy between f_0 and f_1. Given $\tilde{f}_0 : Y \to \tilde{X}$ that lifts f_0, there exists a unique lifting $\tilde{f}_t$. For path-lifting, we take Y a point; for homotopy-lifting, $Y = [0,1]$.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ I \arrow[ur, "f"] \arrow[r, "p \circ f"'] & X \end{tikzcd}$$

Proposition 1.31 (in Hatcher)

The covering map $p : \tilde{X} \to X$ induces $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x)$.

1. p_* is injective.

2. $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ are exactly loops at x_0 that lift to loops at $\tilde{x}_0$.

Proof of 1

Suppose $p_*[f] = \mathrm{id} \in \pi_1(X, x_0)$ where $[f] \in \pi_1(\tilde{X}, \tilde{x}_0)$. Then $[p \circ f] = \mathrm{id}$, and $[p \circ f]$ is path-homotopic to the constant loop c_{x_0}. Hence the lifting $\widetilde{p \circ f} = f$ is path-homotopic to a constant loop $c_{\tilde{x}_0}$.

Proof of 2

Let $[f] \in \pi_1(\tilde{X}, \tilde{x}_0)$. $p_*[f] = [p \circ f]$, $p \circ f$ lifts to f at $\tilde{x}_0$ which is a loop at $\tilde{x}_0$.
Let f be a loop at x_0. Suppose f lifts to a loop $\tilde{f}$ at $\tilde{x}_0$ (i.e. $p \circ \tilde{f} = f$). Hence $[f] = [p \circ \tilde{f}] = p_*[\tilde{[}f] \in p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

Example

If $p : S^1 \to S^1$ by $z \to z^2$, then $p_*(\pi_1(S^1, 1)) = 2\mathbb{Z} \leq \mathbb{Z} = \pi_1(S^1, 1)$.

Remark

If $p : \tilde{X} \to X$ connected, then $p^{-1}(x)$ has the same cardinality for all $x \in X$.

Proof

Fix $x_0 \in X$. Consider $\mathcal{A} = \{x \in X \ : \ p^{-1}(x)$ and $p^{-1}(x_0)$ have the same cardinality$\} \neq \varnothing$. Then $\mathcal{A}$ is open since for each $x \in \mathcal{A}$, there is a neighborhood U of x such that U is evenly covered by p (i.e. $p^{-1}(U) = \dot{\bigcup}_{\alpha \in I} V_\alpha$ where $V_\alpha \overset{p}{\cong} U$). Then $p^{-1}(x')$ has cardinality $|I|$ for all $x' \in U$. It follows, since $\mathcal{A}^c$ is open, that $\mathcal{A}$ is also closed.

Proposition

The number of sheets is given by $[\pi_1(X, x_0) : p_*(\pi_1(\tilde{X}, \tilde{x}_0)]$.

Proof

Write $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Define $\Phi : \{H\text{-cosets in } \pi_1(X, x_0)\} \to p^{-1}(x_0)$ by $H[g] \mapsto \tilde{g}(1)$ where $\tilde{g}$ is a lift of g at $\tilde{x}_0$.
This map is well defined, since for $[h \cdot g]$ with $h \in H$, $\widetilde{h \cdot g}(1) = \tilde{g}(1)$ (because $\tilde{h}(1) = \tilde{x}_0$).
Φ is surjective. Let $\tilde{x}_1 \in p^{-1}(x_0)$

IMAGE 1

and let $\tilde{g}$ be a path from $\tilde{x}_0$ to $\tilde{x}_1$. Define $g = p \circ \tilde{g}$, a loop at x_0. Then $\Phi(H[g]) = \tilde{g}(1) = \tilde{x}_1$.
Φ is injective. Suppose $\Phi(H[g_1]) = \Phi(H[g_2])$ (i.e. $\tilde{g}_1(1) = \tilde{g}_2(1)$.

IMAGE 2

Consider the loop $g_1\overline{g}_2$ in X at x_0. It lifts to $\tilde{g}_1\overline{\tilde{g}}_2$, which is a loop at $\tilde{x}_0$. This shows that $[g_1\overline{g}_2] \in H$ (i.e. $H[g_1] = H[g_2]$).

Recall (Manifolds 2)

If a smooth manifold M is non-orientable, then there is a double cover (2 sheets) $p : \hat{M} \to M$ ($\hat{M}$ connected). Consequently, $\pi_1(M)$ has a subgroup of index 2.

Definition: Locally Path-Connected

A topological space is called locally path-connected if for each $x \in X$ and every neighborhood $U \ni X$, there is a neighborhood $V \ni x$ such that $V \subseteq U$ and V is path-connected (i.e. $\forall x \in X$, there exists a local basis $\{U_\alpha\}$ at x such that each U_α is path-connected). For example, the Topologist's sine curve with endpoints identified is path-connected but not locally path-connected.

Proposition: Lifting Criterion

Let Y be path-connected and locally path-connected. Given a covering map $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and a map $f : (Y, y_0) \to (X, x_0)$, f has a lift $\tilde{f}$ at $\tilde{x}_0$ ($\tilde{f}(y_0) = \tilde{x}_0$) if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

Proof

$(\Longrightarrow)$

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & X \end{tikzcd} \qquad \begin{tikzcd} & \pi(\tilde{X}) \arrow[d, "p_*"] \\ \pi_1(Y) \arrow[ur, "\tilde{f}_*"] \arrow[r, "f_*"] & \pi_1(X) \end{tikzcd}$$

$$f_*\pi_1(Y) = (p_* \circ \tilde{f}_*)(\pi_1(Y)) \leq p_*\pi_1(\tilde{X}).$$

$(\Longleftarrow)$ Let $y \in Y$, and draw a path γ from y_0 to y.

IMAGE 3

We lift $f \circ \gamma$ to a path in $\tilde{X}$ starting at $\tilde{x}_0$ and define $\tilde{f}(y)$ as the endpoint (i.e. $\tilde{f}(y) = \widetilde{f \circ \gamma}(a)$).
This is well-defined, since $(f \circ \gamma) \cdot (f \circ \overline{\gamma}')$ is a loop at x_0 and $[(f \circ \gamma) \cdot (f \circ \overline{\gamma}')] = f_*[\gamma \cdot \overline{\gamma}'] \in f_* \pi_1(Y, y_0) \leq p_* \pi_1(\tilde{X}, \tilde{x}_0)$. Hence $(f \circ \gamma) \cdot (f \circ \overline{\gamma}')$ lifts to a loop at $\tilde{x}_0$.

IMAGE 4

Therefore $\widetilde{f \circ \gamma}(1) = \widetilde{f \circ \gamma'}(1)$.
$\tilde{f}$ is continuous. Fix $f(y) \in X$ and let U be a neighborhood of $f(y)$ that is evenly covered by p. Choose a path-connected neighborhood V of y such that $f(V) \subseteq U$. We check $\tilde{f}|_V$.

IMAGE 5

Because V is path-connected, we may draw a path η in V from y to y'. Then $\tilde{f}(y') = \widetilde{f \circ \gamma \circ \eta}(1)$, and $\widetilde{\gamma \cdot \eta}$ is first lifting $f \circ \gamma$ at $\tilde{x}_0$ followed by lifting $f \circ \eta$ at $\tilde{\gamma}(1)$. Let $\tilde{U} \subseteq \tilde{X}$ such that $p|_{\tilde{U}} : \tilde{U} \to U$ is a homeomorphism and $\widetilde{f \circ \gamma}(1) \in \tilde{U}$. Then $\widetilde{f \circ \eta}(1) = (p^{-1})|_U \circ f(y')$. Hence $\tilde{f}(y') = f \circ \widetilde{(\gamma \circ \eta)}(1) = \widetilde{f \circ \eta}(1) = (p^{-1})|_U \circ f(y')$ (i.e. $\tilde{f} = (p^{-1})|_U = f$ on V). Hence $\tilde{f}$ is continuous at y.
$\tilde{f}$ is a lift of f. In fact, $(p \circ \tilde{f})(y) = p \circ (\widetilde{f\gamma}(1)) = f(y)$.

Corollary

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

If Y is simply connected, then $f_* \pi_1(Y) \leq p_* \pi_1(\tilde{X})$ always holds (i.e. we can always lift f to $\tilde{f} : Y \to \tilde{X}$ in this case).

Proposition: Unique Lifting

Given $p : \tilde{X} \to X$ and $f : Y \to X$, if two lifts $\tilde{f}_1$ and $\tilde{f}_2$ of f agree at one point, then the agree everywhere on Y.

Proof

Take $\mathcal{A} = \{y \in Y \ : \ \tilde{f}_1(y) = \tilde{f}_2(y)\} \neq \varnothing$. Locally for each $y \in Y$ there exists a neighborhood V of y such that $\tilde{f} = (p^{-1})|_U \circ f$. If $y \in \mathcal{A}$, then $\tilde{f}_1(y) = \tilde{f}_2(y)$. Take a neighborhood U of $f(y)$ that is evenly covered and $\tilde{U}$ of $\tilde{f}_1(y) = \tilde{f}_2(y)$ such that $p|_{\tilde{U}} : \tilde{U} \to U$ is a homeomorphism. Then on V, a path-connected neighborhood such that $f(V) \subseteq U$, $\tilde{f}_i = (p^{-1})|_U \circ f$ (i.e. $\tilde{f}_1 = \tilde{f}_2$ on V). If $y \in \mathcal{A}^c$, $\tilde{f}_1(y) \neq \tilde{f}_2(y)$. Then $\tilde{U}_i \ni \tilde{f}_i(y)$ with $\tilde{U}_1 \cap \tilde{U}_2 = \varnothing$. Then on V, $\tilde{f}_i = (p^{-1})|_{\tilde{U}_i} \circ f$ (ie $\tilde{f}_1$ and $\tilde{f}_2$ never agree on V). Hence $\mathcal{A} = Y$.

Remark

If $p : \tilde{X} \to X$ is a covering map, recall that a covering transformation is a map $f : \tilde{X} \to \tilde{X}$ such that

$$\begin{tikzcd} \tilde{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

commutes. This $f : \tilde{X} \to \tilde{X}$ is a lift of $p : \tilde{X} \to X$. If we fix $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$, the lifting criterion says that $p_* \pi_1(\tilde{X}, \tilde{x}_1) \leq p_* \pi_1(\tilde{X}, \tilde{x}_2)$. In particular, if $\pi_1(\tilde{X})$ is, then this holds. Hence there is a unique lift of p (i.e. covering transformation) f such that $f(\tilde{x}_1) = \tilde{x}_2$.

April 16, 2025

Question

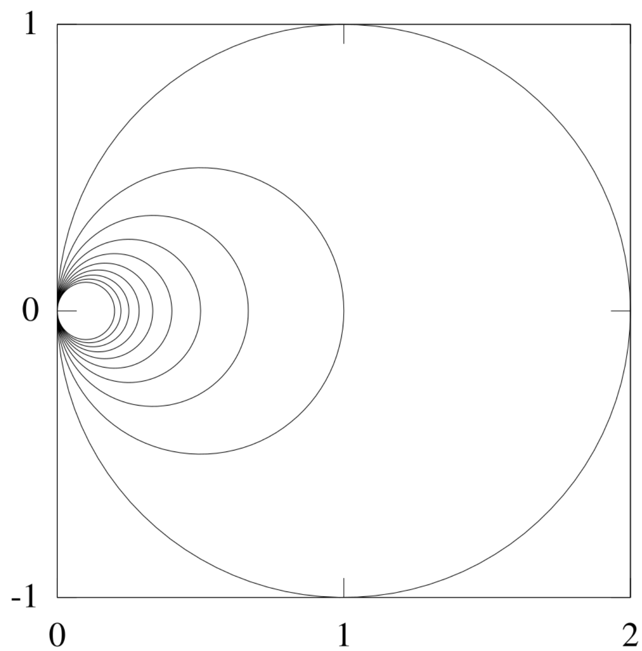
Given X path-connected and locally path-connected, when does X admit a simply connected covering space $p : \tilde{X} \to X$?

Definition: Semi-locally Simply Connected

We say that X is semi-locally simply connected if for any $x \in X$ there exists a neighborhood U such that every loop in U is null-homotopic in X. That is $\mathrm{Im}(\pi_1(U) \to \pi_1(X))$ is trivial.

Non-example

The Hawaiian earing in $\mathbb{R}^2$.



Example

The cones over the Hawaiian earing.

IMAGE 1

In fact, this is simply connected.

Example

The double Hawaiian earing with cones.

IMAGE 2

Theorem

X has a simply connected covering space (i.e. a universal covering) if and only if X is semi-locally simply connected.

Proof

($\Longrightarrow$) Let $x \in X$ and pick a neighborhood U of x that is evenly covered by p. Let f be a loop at x in U. f lifts to $\tilde{f}$ at $\tilde{x}_0$, which is a loop. Retract $\tilde{f}$ to $c_{\tilde{x}_0}$ by a path-homotopy H. Then $p \circ H$ shows that f is null-homotopic in X.
($\Longleftarrow$) We construct $\tilde{X}$ as follows: fix $x_0 \in X$ and set $\tilde{X} = \{[\gamma] \text{ path homotopies } : \gamma \text{ is a path starting at } x_0\}$. Let $\mathcal{U} = \{U : \operatorname{Im}(\pi_1(U) \to \pi_1(X)) \text{ is trivial}\}$. By assumption $\mathcal{U}$ is a basis for X. For each $u \in \mathcal{U}$ and each γ from x_0 to a point in U, we define $U_{[\gamma]} = \{\gamma \cdot \eta] : \eta \text{ starting at } \gamma(1) \text{ stays in } U\}$. Then $p : \tilde{X} \to X$ by $[\gamma] \to \gamma(1)$.
We need to check that $\{U_{[\gamma]} : U \in \mathcal{U}, \gamma \text{ a path from } x_0 \text{ to a point in } U\}$ generates a topology on $\tilde{X}$.
We need also to check that $p : U_{[\gamma]} \to U$ is bijective. It is clearly surjective, and if $p[\gamma \cdot \eta] = p[\gamma \cdot \delta]$ with η, δ paths starting at $\gamma(1)$ and staying in U. Then $\eta(1) = \delta(1)$ and, since η, δ share the same endpoints and they stay in $U_{[\gamma]}$, then $[\eta] = [\delta]$. Hence $[\gamma \cdot \eta] = [\gamma \cdot \delta]$ and p is injective.
Further, we need to check that $p : U_{[\gamma]} \to U$ is a homemorphism and that $p^{-1}(U) = \dot{\bigcup}_{[\gamma]} U_{[\gamma]}$. Hence p is a covering map.
Finally, we need to check that $\tilde{X}$ is simply connected. Recall that $p : \tilde{X} \to X$ induces an injective homomorphism $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$. It suffices to show that $p_* \pi_1(\tilde{X}, \tilde{x}_0) = \mathrm{id}$. We se t$\tilde{x}_0 = [C_{x_0}] \in \tilde{X}$. Recall also that elements in $p_* \pi_1(\tilde{X}, \tilde{x}_0)$ are exactly the loops in X at x_0 such that they lift to loops at $\tilde{x}_0$. Suppose $[\gamma] \in p_* \pi_1(\tilde{X}, \tilde{x}_0)$. Then γ lifts to a loop $\tilde{\gamma}$ at $\tilde{x}_0 = [C_{x_0}]$. For $t \in [0,1]$, consider the path γ_t which follows γ on $[0,t]$ then stays stationary at $\gamma(t)$ for the remaining time. Then $t \mapsto [\gamma_t]$ is a path on $\tilde{X}$, $p([\gamma_t]) = \gamma_t(1) = \gamma(t)$, and $t \mapsto [\gamma_t]$ is a lift of γ at $\tilde{x}_0 = [C_{x_0}]$. Then $t \mapsto [\gamma_t]$ is a loop (i.e. $[\gamma] = [\gamma_1] = \tilde{x}_0 = [C_{x_0}]$) and γ is null-homotopic. This shows that $p_* \pi_1(\tilde{X}, \tilde{x}_0) = \mathrm{id}$ (i.e. $\tilde{X}$ is simply connected).

Group Actions on Fibers (Monodromy Action)

Given $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ a covering map, $\pi_1(X, x_0)$ acts on p^{-1} as follows: $p^{-1}(x_0) \times \pi_1(X, x_0) \to p^{-1}(x_0)$ by $(e, [f]) \mapsto \tilde{f}_e(1)$ where $\tilde{f}_e$ is the (unique) lift of f at $e \in p^{-1}(x_0)$. This is a right $\pi_1(X, x)$ action.
We want to check that $(e \cdot [f]) \cdot [g] = e \cdot [f \cdot g]$. We have that $e \cdot [f \cdot g] = \widetilde{(f \cdot g)}_e(1)$, but $\widetilde{(f \cdot g)}_e$ is the lift of f at e followed by the lift of g at the endpoint of $\tilde{f}_e$, call it $\tilde{f}_e(1) = z$. Then $\widetilde{(f \cdot g)}_e(1) = \tilde{g}_z(1) = z \cdot [g] = (e \cdot [f]) \cdot [g]$.
This action is transitive. Given e and e', draw a path connecting them $\tilde{g}$. Under the map p, we have that $p \circ \tilde{g} = g$ which is a loop at x_0. Then $e \cdot [g] = \tilde{g}(1) = e'$.
Recall: Given a right G-set S, $G_S = \{g \in G : s \cdot g = s\}$ is the isotropy subgroup at $s \in S$.
Given $e \in p^{-1}(x_0)$, the isotropy subgroup at e is all the loops such that their lfts at e are loops (i.e. the isotropy subgroup at e is precisely $p_* \pi_1(\tilde{X}, e)$).
Recall: $G \cdot S = G / G_s$. Here, this tells us that $p^{-1}(x_0) = \pi_1(X, x_0) / p_* \pi_1(\tilde{X}, e)$. This recovers the fact that the number of sheets is equal to the index of $\operatorname{im}(p_*)$.
In particular, if $\tilde{X}$ is simply connected, then

- $\pi_1(X, x_0)$ acts freely on $p^{-1}(x_0)$ and
- the number of sheets equals the cardinality of $\pi_1(X, x_0)$.

Definition: Universal Cover

A covering space $p : \tilde{X} \to X$ is called universal if it has the universal property (i.e. for any covering space $q : Y \to X$, there is a covering map $\tilde{p} : \tilde{X} \to Y$ such that the associated diagram commutes).

$$\begin{tikzcd}
\tilde{X} \arrow[r, dashed, "\tilde{p}"] \arrow[d, "p"'] & Y \arrow[dl, "q"] \\
X
\end{tikzcd}$$

Definition: Covering Homomorphism

Let $p_1 : X_1 \to X$ and $p_2 : X_2 \to X$ be two covering spaces. A covering homomorphism is a map $\varphi : X_1 \to X_2$ such that the associated diagram commutes

$$\begin{tikzcd}
X_1 \arrow[rr, "\varphi"] \arrow[dr, "p_1"] & & X_2 \arrow[dl, "p_2"] \\
& X &
\end{tikzcd}$$

By definition, φ is a lift of p_1.

Proposition

1. A covering homomorphism φ is uniquely determined by its value at one point.
2. For each $x \in X$, $\varphi|_{p_1^{-1}(x)} : p_1^{-1}(x) \to p_2^{-1}(x)$ is $\pi_1(X, x_0)$-equivariant.
3. A covering homomorphism $\varphi : X_1 \to X_2$ is a covering map. Assuming this, the universal cover is unique.

Recall: if S_1, S_2 are right G-sets, a G-equivariant map $\varphi : S_1 \to S_2$ is a map such that the associated diagram commutes

$$\begin{tikzcd}
S_1 \arrow[r, "\varphi"] \arrow[d, "\cdot g"] & S_2 \arrow[d, "\cdot g"] \\
S_1 \arrow[r, "\varphi"] & S_2
\end{tikzcd}$$

Proof of 2

Let $e \in p_1^{-1}(x)$. We need to show taht $\varphi(e)\cdot g = \varphi(e\cdot g)$. We have that $g \in \pi_1(X, x_0)$ is represented by a loop f at x_0. So $e\cdot g = e\cdot[f] = \tilde{f}_e(1) \in X_1$, and $\varphi(e\cdot g) = \varphi(\tilde{f}_e(1))$. On the left hand side, we have that $\varphi(e)\cdot g = f_{\varphi(e)}(1) \in X_2$. We need to verify that $\varphi(\tilde{f}_e) = \tilde{f}_{\varphi(e)}$ which are both lifts of f at $\varphi(e)$. But since the diagram commutes, $p_2(\varphi \circ \tilde{f}_e) = p_1 \circ \tilde{f}_e = f$.

Uniqueness in 3

Suppose we have

$$\begin{tikzcd}
X_1 \arrow[rr, shift right, "\varphi"'] \arrow[dr, "p_1"] & & X_2 \arrow[ll, shift right, "\psi"'] \arrow[dl, "p_2"] \\
& X &
\end{tikzcd}$$

with $\varphi(e_1) = e_2$ and $\psi(e_2) = e_1$. Then $\psi \circ \varphi(e_1) = e_1$. Hence $\psi \circ \varphi = \mathrm{id}$ and, similarly, $\psi \circ \varphi = \mathrm{id}$. Hence φ is a bijection and a homemorphism.

Proof of 3

φ is surjective. Given any $e' \in X_2$, set $x_0 = p_2(e)$ and let $e \in p_1^{-1}(x_0)$ so $\varphi(e) \in p_2^{-1}(x_0)$. Since $\pi_1(X, x_0)$ acts transitively on $p_2^{-1}(x_0)$, there exists $g \in \pi_1(X, x_0)$ such that $e' = \varphi(e)\cdot g = \varphi(e \cdot g)$
φ is a covering map. Let V be a neighborhood of $x_0 \in X$ such that V is evenly covered by both p_1 and p_2. Let U be a component in $p_2^{-1}(V)$ that contains e_2. Then $p_1^{-1}(X) = \dot{\bigcup} U_\alpha$. U as a component in $p_2^{-1}(V)$ is both open and closed.

Hence $\varphi^{-1}(U)$ is open and closed in $p_1^{-1}(V) = \dot{\bigcup} U_\alpha$. It follows that $\varphi^{-1}(U)$ is the disjoint union of several components of $\{U_\alpha\}_\alpha$, and each component is homemorphic to V and consequently homeomorphic to U. This shows that φ is a covering map.

April 21, 2025

Recall: Lifting Criterion

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[r, "f"] \arrow[ur, dashed, "\tilde{f}"] & (X, x_0) \end{tikzcd}$$

There exists a lift $\tilde{f}$ of f at $\tilde{x}_0$ if and only if $f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)$.

If $(\tilde{X}, \tilde{x}_0) \xrightarrow{p} (Y, y_0)$, $\pi_1(X, x_0)$ acts transitively on $p^{-1}(x_0)$ by path lifting (a right action where $e \in p^{-1}(x_0)$ by $e \cdot [\gamma] = \tilde{\gamma}_e(1)$). The isotropy subgroup at e is $p_*\pi_1(\tilde{X}, e)$.

Covering Transformations

$$\begin{tikzcd} \tilde{X} \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

Write $\mathrm{Aut}(\tilde{X} \xrightarrow{p} X)$ for the covering group $\{\varphi : \tilde{X} \to \tilde{X} \text{ covering transformations}\}$.

1. $\varphi : \tilde{X} \to \tilde{X}$ is uniquely determined by its value at one point.

2. Given $e_1, e_2 \in p^{-1}(x)$, there is $\varphi \in \mathrm{Aut}(\tilde{X} \xrightarrow{p} X)$ if and only if $p_*\pi_1(\tilde{X}, e_1) = p_*\pi_1(\tilde{X}, e_2)$. In fact, for $\varphi \in \mathrm{Aut}(\tilde{X} \xrightarrow{p} X)$ with $p_*\pi_1(\tilde{X}, e_1) \subseteq p_*\pi_1(\tilde{X}, e_2)$.

3. $\varphi|_{p^{-1}(x)} : p^{-1}(x) \to p^{-1}(x)$ is $\pi_1(X, x)$-equivariant (i.e. $\varphi(e) \cdot \gamma = \varphi(e \cdot \gamma)$.

Example

Given $p : \mathbb{R} \to S^1$, what is $\mathrm{Aut}(\mathbb{R} \xrightarrow{p} S^1)$?
$1 \in S^1$, $p^{-1}(1) = \mathbb{Z} \subseteq \mathbb{R}$, $\forall \varphi \in \mathrm{Aut}(\mathbb{R} \xrightarrow{p} S^1)$, $\varphi(0) = k \in \mathbb{Z}$. Then $\varphi(x) = x + k$. In fact, the map $x \mapsto x + k$ is a covering transofmratio that agrees with φ at $0 \in \mathbb{R}$. Hence they agree everywhere (i.e. $\varphi(x) = x + k$ for all x).

Example

Given $p : S^2 \to \mathbb{RP}^2$, then $\mathrm{Aut}(S^2 \xrightarrow{p} \mathbb{RP}^2) = \{\mathrm{id}, A\}$ with A the antipodal map.

Proposition: Normal Covering

Let $\tilde{X} \xrightarrow{p} X$ be a covering map. The following are equivalent

1. There exists $x \in X$ such that $p_*\pi_1(\tilde{X}, e)$ is normal for one (thus for all) $e \in p^{-1}(x)$.

2. For every $x \in X$ and each $e \in p^{-1}(x)$, $p_*\pi_1(\tilde{X}, e)$ is normal.

3. $\mathrm{Aut}(\tilde{X} \xrightarrow{p} X)$ acts transitively on some (thus all) fiber $p^{-1}(x)$.

If any of these hold, we say that $p : \tilde{X} \to X$ is a normal covering.

Proof

Suppose $e, e' \in p^{-1}(x)$ with $p_*\pi_1(\tilde{X}, e)$ and $p_*\pi_1(\tilde{X}, e')$. These are the isotropy subgroups at e and e' respectively. We know also $\pi_1(X, x)$ acts transitively on $p^{-1}(x)$.
Fact: If S is a right G-set, then $G_s = \{h \in G : s \cdot h = s\}$ and $G_{sg} = \{h \in G : s \cdot g \cdot h = s \cdot g\} = \{h \in S : s \cdot g \cdot h \cdot g^{-1} = s\}$. So $g \cdot G_{sg} \cdot g^{-1} \in G_s$ which implies that $G_{sg} = g^{-1} \cdot G_s \cdot g$. So if G_s is normal then so is G_{sg}.

IMAGE 1

$$\begin{tikzcd}
\pi_1(\tilde{X}, e_0) \arrow[r, "\Phi_{\tilde{h}}"] \arrow[d, "p_*"] & \pi_1(\tilde{X}, e) \arrow[d, "p_*"] \\
\pi_1(X, x_0) \arrow[r, "\Phi_h"] & \pi_1(X, x)
\end{tikzcd}$$

commutes. Hence Φ_h maps $p_*\pi_1(\tilde{X}, e_0)$ to $p_*\pi_1(\tilde{X}, e)$, and $\Phi_h : \pi_1(X, x_0) \xrightarrow{\sim} \pi_1(X, x)$ preserves normal subgroups.

(3) implies (1)

Finally, for every $e_1, e_2 \in p^{-1}(x)$, there exists $\varphi \in \mathrm{Aut}(\tilde{X} \xrightarrow{p} X)$ such that $\varphi(e_1) = e_2$. This holds if and only if $p_*\pi_1(\tilde{X}, e_1) = p_*\pi_1(\tilde{X}, e_2)$ for every $e_1, e_2 \in p^{-1}(x)$. That is, $e_2 = e_1 \cdot \gamma$ for some $\gamma \in \pi_1(X, x)$ and $H = \gamma^{-1} H \gamma$ for every $\gamma \in \pi_1(X, x)$. So H is normal.

Remark

The (simply connected) universal cover is always normal because $\{\mathrm{id}\}$ is normal in $\pi_1(X, x)$.

Theorem

Let $p : \tilde{X} \to X$ be a covering map with $x \in X$ and $e \in p^{-1}(x)$. Then $\mathrm{Aut}(\tilde{X} \xrightarrow{p} X) \cong \frac{N_G(H)}{H}$ where $G = \pi_1(X, x)$, $H = p_*\pi_1(\tilde{X}, e)$, and $N_G(H) = \{g \in G : g^{-1} H g = H\}$.

Special Case 1

If $p : \tilde{X} \to X$ is a normal covering, then H is normal in G. Then also $\mathrm{Aut}(\tilde{X} \xrightarrow{p} X) \cong G/H$.

Special Case 2

If $p : \tilde{X} \to X$ is the (simply connected) universal covering, then $\mathrm{Aut}(\tilde{X} \xrightarrow{p} X) \cong \pi_1(X, x)$.

Proof

Let S be a right G-set with transitive action and $\mathrm{Aut}_G(S)\{\varphi : S \to S$ G-equivariant bijections$\}$. Fix $s \in S$. Then $\mathrm{Aut}_G(S) \cong \frac{N_G(H)}{H}$ where $h = G_s$.
Define $\Phi : N_G(H) \to \mathrm{Aut}_G(S)$ by $\gamma \mapsto \Phi(\gamma) = \varphi_\gamma$ with $\varphi_\gamma : S \to S$ defined by

$$G_{s \cdot \gamma} = \gamma^{-1} H \gamma = H = G_s.$$

Then there exists a unique $\varphi_\gamma \in \mathrm{Aut}_G(S)$ such that $\varphi_\gamma(s) = s\cdot\gamma$.

- Lemma

  For each $s' \in S$, $s' = s\cdot\gamma'$ for some $\gamma' \in G$. Then $\varphi_\gamma(s') = \varphi_\gamma(s\cdot\gamma') = \varphi_\gamma(s)\cdot\gamma' = s\cdot\gamma\gamma'$. This is well defined. If $s' = s\cdot\gamma''$, then $s = s(\gamma\cdot\gamma''\cdot(\gamma')^{-1}\cdot\gamma^{-1})$ which implies that $\gamma\cdot\gamma''(\gamma')^{-1}\cdot\gamma^{-1} \in G_s$ and $\gamma''\cdot(\gamma')^{-1} \in G_s$.

Φ is a group homomorphism since

$$\varphi_{\gamma_1}\circ\varphi_{\gamma_2}(s) = \varphi_{\gamma_1}(s\cdot\gamma_2) = \varphi_{\gamma_1}(s)\cdot\gamma_2 = s\cdot\gamma_1\cdot\gamma_2.$$

Φ is surjective since letting $\varphi \in \mathrm{Aut}_G(S)$, it maps s to some $\varphi(s) = s' = s\cdot\gamma$ and hence $\varphi = \varphi_\gamma$.
If $\varphi_\gamma = \mathrm{id}$, then $\varphi_\gamma(s) = s$ and $\gamma \in G_s = H$. So Φ induces $\frac{N_G(H)}{H} \cong \mathrm{Aut}_G(S)$.
Take $G = \pi_1(X,x)$ and $\mathrm{Aut}(\tilde{X} \overset{p}{\to} X) \to \mathrm{Aut}_G(p^{-1}(x)) \cong \frac{N_G(H)}{H}$ by $\varphi \mapsto \varphi|_{p^{-1}(x)}$ where H is the isotropy subgroup of the $\pi_1(X,x)$ action at e $(p_*\pi_1(\tilde{X},e))$. Then $\varphi \mapsto \varphi|_{p^{-1}(x)}$ is injective because it is uniquely determined by its value at one point.
$\varphi \mapsto \varphi|_{p^{-1}(x)}$ is surjective. Letting $\eta \in \mathrm{Aut}_g(p^{-1}(x))$ and $e_1 \in p^{-1}(x)$, we set $e_2 = \eta(e_1)$ and see that $p_*\pi_1(\tilde{X},e_1) = G_{e_1} = G_{e_2} = p_*\pi_1(\tilde{X},e_2)$. By the lifting criterion, there exists $\varphi \in \mathrm{Aut}(\tilde{X} \overset{p}{\to} X)$ such that $\varphi(e_1) = e_2$. Then $\varphi|_{p^{-1}(x)} = \eta$ since both are in $\mathrm{Aut}_G(p^{-1}(x))$ and they agree at one point (hence everywhere). Thus we conclude that the map is a bijection and

$$\mathrm{Aut}(\tilde{X} \overset{p}{\to} X) \cong \mathrm{Aut}_G(p^{-1}(x)) \cong \frac{N_G(H)}{H}.$$

Definition: Covering Space Action

Let X be connected and locally path connected with a group action Γ acting by homeomorphism. The quotient map $p : X \to X/\Gamma$ will be a covering map if we impose $(*)$ for all $x \in X$, there exists a neighborhood U of x such that $U \cap (g\cdot U) = \varnothing$ for each $g \in \Gamma - \{\mathrm{id}\}$. In particular, G acts freely on X. We say that a Γ-action on X is a covering space action if $(*)$ if fulfilled.

Counter-example

Consider an $\mathbb{R}$ action on $\mathbb{R}^2$ by translation. Then $U \cap (g\cdot U) \neq \varnothing$.

IMAGE 2

Remark

Assuming $(*)$, $\{g\cdot U : g \in \Gamma\}$ is a disjoint family of open sets.

Example

Take a $\mathbb{Z}$-action by $\mathbb{R}^2$ given by $\gamma(x,y) = (x+1,-y)$.

IMAGE 3

Example

S^2 with $\mathbb{Z}_2$-action ($\{\mathrm{id}, A\}$).

Theorem

If Γ acts on X as a covering space action, then $q : X \to X/\Gamma$ is a normal covering map.

Proof

Let $\overline{x} \in X/\Gamma$ and pick $x \in q^{-1}(\overline{x})$. By $(*)$, we have a neighborhood U such that $\{g \cdot U \,:\, g \in \Gamma\}$ is a disjoint collection. Let $V = q(U)$, an open neighborhood of $\overline{x}$ in X/Γ. Then $q^{-1}(V) = \{g \cdot U \,:\, g \in \Gamma\}$. Moreover, $g \cdot U \to V$ is a homeomorphism. If there exist $x', g'x' \in g \cdot U$, then $x' = h_1 \cdot u_1$ and $g' \cdot x' = h_2 \cdot u_2$. So $h_1^{-1}x' \in U$ and $h_2^{-1}g' \cdot x' \in U$ but this holds only for the identity map. So the covering map is injective.

Classifications of Covering Spaces

Take X path-connected, locally path-connected and semi-locally simply path connected (guaranteeing a simply connected universal cover). Then there is a $1-1$ correspondence between

$$\left\{\begin{array}{c}\text{isomorphism classes of covering}\\ \text{spaces } p{:}\hat{X}\to X\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{conjugacy classes of}\\ \text{subgroups in } \pi_1(X,x)\end{array}\right\}$$
$$(p : \hat{X} \to X) \leftrightarrows [p_* \pi_1(\hat{X}, \hat{e})]$$

April 23, 2025

Recall: Theorem

For X path-connected, locally path-connected and semi-locally simply path connected, Γ acts on X as a covering group action (i.e. $\forall x \in X$, there exists a neighborhood U of x such that $U \cap (g \cdot U) = \varnothing$ for all $g \in \Gamma \setminus \{e\}$).
Then $p : X \to X/\Gamma$ is a normal covering map. Moreover $\mathrm{Aut}(X \overset{p}{\to} X/\Gamma) = \Gamma$.

Proof

$(\supseteq)$ this follows from

$$\begin{tikzcd} X \arrow[rr, "g\cdot"] \arrow[dr, "p"'] & & X \arrow[dl, "p"] \\ & X/\Gamma & \end{tikzcd}$$

$(\subseteq)$ Let $\varphi \in \mathrm{Aut}(X \overset{p}{\to} X/\Gamma)$. That is

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & X \arrow[dl, "p"] \\ & X/\Gamma & \end{tikzcd}$$

commutes with φ a homeomorphism. Now let $x \in p^{-1}(\overline{x})$ where $\overline{x} \in X/\Gamma$, and let $x' = \varphi(x)$. Then $p(x) = \overline{x} = p(x')$, hence $x, x' \in p^{-1}(\overline{x})$. Hence there is $g \in \Gamma$ such that $gx = x'$. So we have

$$\varphi : X \to X \varphi(x) = x'$$
$$g : X \to X g(x) = x'$$

so φ is equivalent to an action by g.

Theorem

Take X path-connected, locally path-connected and semi-locally simply path connected (guaranteeing a simply connected universal cover). Then there is a $1-1$ correspondence between

$$\left\{\begin{matrix}\text{isomorphism classes of}\\ \text{covering maps } p: \hat{X} \to X\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{conjugacy classes of}\\ \text{subgroups in } \pi_1(X, x_0)\end{matrix}\right\}$$

→ Assign a subgroup $H = p_*(\hat{X}, \hat{e})$ for $\hat{e} \in p^{-1}(x_0)$.
← Given a conjugacy class of subgroups, pick a subgroup H in the class.

$$H \leq \pi_1(X, x_0) \cong \mathrm{Aut}(\tilde{X} \xrightarrow{p} X)$$

Hence H acts naturally on $\tilde{X}$ as covering transformations. Consider $q : \tilde{X} \to \tilde{X}/H =: \hat{X}$, a normal covering map.

$$\begin{tikzcd} \tilde{X} \arrow[dr, "/H"] \arrow[dd, "p"'] & \\ & \hat{X} \arrow[dl, dashed] \\ X & \end{tikzcd}$$

Since $\tilde{X}/\pi_1(X, x_0) = x$, we have an induced map $\hat{p} : \hat{X} \to X$. We need to show that $\hat{p} : \hat{X} \to X$ is a covering map with $\hat{p}_*\pi_1(\hat{X}, \hat{e}) = H$ for some $\hat{e} \in \hat{p}^{-1}(x)$. Let U be a neighborhood of x such that $p^{-1}(U) = \dot{\bigcup}_\alpha \tilde{U}_\alpha$. Then $\{\tilde{U}_\alpha\}$ is a collect iof disjoint open sets and identical to $\{g \cdot \tilde{U} \,:\, g \in \pi_1(X, x)\}$ where $\tilde{U}$ is a component of $p^{-1}(U)$. The H-action permutes the copies in $\{g \cdot \tilde{U}\} = \{\tilde{U}_\alpha\}$. Hence $q|_{\tilde{U}_\alpha} : \tilde{U}_\alpha \to \hat{X}$ is a homeomorphism. Let $\hat{U}$ be a component in $\hat{p}^{-1}(U)$. Then $q^{-1}(\hat{p}^{-1}(U)) = p^{-1}(U) = \dot{\bigcup}_\alpha \tilde{U}_\alpha$ where $q^{-1}(\hat{U})$ is a union of components in $\dot{\bigcup}_\alpha \tilde{U}_\alpha$. Hence $\hat{U}$ is homeomorphic to U, and $\hat{p}^{-1}(U)$ is a union of components that are homemorphic to U.
Lastly, we show that $\hat{p}_*\pi_1(\hat{X}, \hat{e}_0) = H$. This is the isotropy subgroup of $\pi_1(X, x_0)$-actions at $\hat{e}_0$. $q|_{p^{-1}(x_0)} : p^{-1}(x_0) \to \hat{p}^{-1}(x_0)$ is $\pi_1(X, x_0)$-equivariant (i.e. $q(e \cdot \gamma) = q(e) \cdot \gamma$, $q(e) = \hat{e}$ for $e \in \tilde{X}$). Hence γ fixes $q(e) = \hat{e}$ if and only if $q(e \cdot \gamma) = q(e)$, if and only if $e \cdot \gamma$ and e are in the same H-orbit, if and only if $\gamma \in H$.

Example 1

$X = S^1$ with $\pi_1(S^1) = \mathbb{Z}$.
$\mathbb{Z}$ has subgroups $\mathbb{Z}$, $2\mathbb{Z}$, $3\mathbb{Z}$, $\ldots$, $k\mathbb{Z}$, $\ldots$ where $k\mathbb{Z}$ corresponds to the covering map $p_k : z \mapsto z^k$.

Example 2

X the Mobius strip with $\pi_1(X) = \mathbb{Z}$ with $\pi_1(X) = \langle\gamma\rangle$ and $\gamma(x, y) = (x + 1, -y)$.
Take $H = 2\mathbb{Z} = \langle 2\gamma\rangle \leq \mathbb{Z}$. Then $2\gamma(x, y) = (x + 2, y)$ and $\mathbb{R}^2/H$ is the cylinder while the cylinder modulo $\mathbb{Z}_2$ is the mobius strip.

Example 3

The Klein bottle, $K = \mathbb{R}^2/\Gamma$ with Γ genereted by $g(x, y) = (x + 1, -y)$ and $h(x, y) = (x, y + 1)$.
So $\pi_1(K) = \langle g, h\rangle$. $g^2(x, y) = (x + 2, y)$ commutes with h, so $\mathbb{Z}^2 \cong \langle g^2, h\rangle \leq \pi_1(K)$ and $\mathbb{R}^2/\langle g^2, h\rangle = \mathbb{T}^2$ covers K.

Simplexes

IMAGE 1

The standard n-simplex is

$$\Delta^n = \left\{(t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \,:\, \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0, \forall i\right\}$$

$$\Delta^1 = \left\{(t_0, t_1) \in \mathbb{R}^2 \,:\, t_0 + t_1 = 1,\ t_0, t_1 \geq 0\right\}$$

IMAGE 2

$$\Delta^2 = \left\{(t_0, t_1, t_2) \in \mathbb{R}^3 \ : \ t_0 + t_1 + t_2 = 1, \ t_0, t_1, t_2 \geq 0\right\}$$

IMAGE 3

Δ^n has $(n+1)$-many faces ($(n+1)$-simplex) where the ith face is $\Delta^{n-1} \to \Delta^n$ by $(t_0, \dots, t_{n-1}) \mapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$.
Let X be a topological space. A Δ-complex structure on X is a family of maps $\sigma_\alpha : \Delta^n \to X$ (n may depend on α) such that

1. $\sigma_\alpha|_{\mathring{\Delta}^n} : \mathring{\Delta}^n \to X$ is injective and each point is in the image of at most one of $\sigma_\alpha|_{\circ\Delta^n}$.
2. $\sigma_\alpha|_{\text{a face of } \Delta^N}$ is some $\sigma_\beta : \Delta^{n-1} \to X$ in the family.
3. $A \subseteq X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for all α.

σ_β is $\Delta^{n-1} \overset{i\text{th face}}{\to} \Delta^n \overset{\sigma}{\to} X$.

Example

S^1 is the following iwht 1-simplex

IMAGE 4

Then the “body” of $\Delta^1 \overset{\sigma}{\to} X$ is

IMAGE 5

with $\sigma \circ \delta_0 : \Delta^0 \to X$ and $\sigma \circ \delta_1 : \Delta^0 \to X$. The boundary $\partial\sigma = \sum_{i=0}^n (-1)^i \sigma \circ \delta_i$. They define $\delta : C_n(X) \to C_{n-1}(X)$. For this example, we have $\partial\sigma = \sigma \circ \delta_0 + (-1)\sigma \circ \delta_1 = 0$.
The ith face is $\delta_i : \Delta^{n-1} \to \Delta^n$ by $(t_0, \dots, t_{n-1}) \mapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$.
In Hatcher’s notation, the boundary is $\partial\sigma = \sum_{i=0}^n (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$ where we shoild think of $[v_0, \dots, \hat{v}_i, \dots, v_n]$ as the ith face. So $\sigma : \Delta^n = [v_0, \dots, v_n] \to X$. Now we have

$$\cdots \overset{\partial}{\to} C_n(X) \overset{\partial}{\to} C_{n-1}(X) \cdots$$

where $\partial^2 = 0$.

Proof

$$\begin{aligned}
\partial(\partial\sigma) &= \partial\left(\sum_{i=0}^n (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}\right) \\
&= \sum_{i=0}^n (-1)^i \partial(\sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}) \\
&= \sum_{j<i} (-1)^i (-1)^j \sigma_{[v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n]} + \sum_{j>i} (-1)^i (-1)^{j-1} \sigma_{[v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]} \\
&= 0
\end{aligned}$$

Homoology Associated to the Delta Complex

We have $\ker\partial \supseteq \operatorname{im}\partial$ where $\ker\partial$ are the n-cycles and $\operatorname{im}\partial$ are the n-bodies, and

$$H_n^{\delta}(X) = \frac{\{n\text{-cycles}\}}{\{n\text{-bodies}\}} = \frac{\ker\partial}{\operatorname{im}\partial}$$

Example

For the circle, $C_1(X) = \mathbb{Z} = \langle\sigma\rangle$ and $C_0(X) = \mathbb{Z} = \langle v\rangle$. Therefore

$$\overbrace{C_2(X)}^{=0} \to \overbrace{C_1(X)}^{=\mathbb{Z}} \xrightarrow{0} \overbrace{C_0(X)}^{=\mathbb{Z}} \to 0$$

Then $H_1^{\Delta}(X) = \frac{\ker\partial}{\operatorname{im}\partial} = \mathbb{Z}/\{0\} = \mathbb{Z}$ and $H_0^{\Delta}(X) = \frac{\ker\partial}{\operatorname{im}\partial} = \mathbb{Z}$.

An Aside

IMAGE 7

$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

Example

For the torus, draw

IMAGE 6

So $C_0(X) = \langle v\rangle = \mathbb{Z}$, $C_1(X) = \langle a, b, c\rangle = \mathbb{Z}^3$ and $C_2(X) = \langle U, L\rangle = \mathbb{Z}^2$. Then also $\partial U = a + b - c$ and $\partial L = a + b - c$, so $\partial(U - L) = 0$ and $\ker\partial_2 = \langle U - L\rangle \cong \mathbb{Z}$. That is $H_2^{\Delta}(X) = \frac{\ker\partial}{\operatorname{im}\partial} \cong \mathbb{Z}$. Now $\partial a = 0 = \partial b = \partial c$, so $\ker\partial_1 = \langle a, b, c\rangle$ and $H_1^{\Delta}(X) = \frac{\ker\partial_1}{\operatorname{im}\partial_2} = \frac{\langle a,b,a+b-c\rangle}{\langle a+b-c\rangle} \cong \mathbb{Z}^2$. Finally we have that $H_0^{\Delta} = \frac{\ker\partial_0}{\operatorname{im}\partial_1} = \frac{\langle v\rangle}{\{0\}} \cong \mathbb{Z}$.

Example

For $\mathbb{RP}^2$, draw

IMAGE 8

A$C_0(X) = \langle v, w\rangle \cong \mathbb{Z}^2$, $C_1(X) = \langle a, b, c\rangle \cong \mathbb{Z}^3$, and $C_2(X) = \langle U, L\rangle \cong \mathbb{Z}^2$. Then $\partial U = a + b + c$ while $\partial L = a + b - c$, so $\ker\partial_2 = \{0\}$ and $H_2^{\Delta}\frac{\ker\partial_2}{\operatorname{im}\partial_3} = \{0\}$. $\partial_1(a) = w - v$, $\partial_1(b) = v - w$ and $\partial_1(c) = 0$, so $\ker\partial_1 = \langle c, a - b\rangle$ and

$$H_1^{\Delta}(X) = \frac{\ker\partial_1}{\operatorname{im}\partial_2} = \frac{\langle c, a+b\rangle}{\langle a+b+c, a+b-c\rangle} = \langle a+b+c, c\rangle/\langle a+b+c, 2c\rangle \cong \langle c\rangle/\langle 2c\rangle \cong \mathbb{Z}^2.$$

April 28th, 2025

Recall:

For X with a Δ-complex structure, we have $H_n^{\Delta}(X)$.

Definition: Singular Simplex

A singular n-simplex is a continuous map $\sigma : \Delta^n \to X$.
The singular chain $C_n(X)$ is the free Abelian group generated by singular n-simplecies. Write

$$C_n(X) = \left\{ \sum n_i \sigma_i \ : \ |\sum n_i \sigma_i| < \infty,\ n_i \in \mathbb{Z},\ \sigma_i : \Delta^n \to X \right\}$$

$$\cdots \xrightarrow{\partial} C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial} \cdots$$

$$\partial(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}$$

While $\partial^2 = 0$ and $H_n(X) = \ker \partial_n / \operatorname{Im} \partial_{n+1}$ is the singular homology.

Proposition

If $X = \coprod_\alpha X_\alpha$ with X_α connected components of X, then $H_n(X) \cong \oplus_\alpha H_n(X_\alpha)$.

Proof

$\sigma : \Delta^n \to x$, $\operatorname{im}\sigma \subseteq X_\alpha$ for some α. So $C_n(X) = \oplus_\alpha C_n(X_\alpha)$ and $\partial : C_n(X) \to C_{n-1}(X)$ maps $C_n(X_\alpha)$ to $C_{n-1}(X_\alpha)$. Therefore $\ker \partial_n = \oplus_\alpha \ker(\partial|_{C_n(X_\alpha)})$ and $\operatorname{im} \partial_{n+1} = \oplus_\alpha \operatorname{im}(\partial|_{C_{n+1}(X_\alpha)})$. Then $H_n(X) \cong \oplus_\alpha \ker(\partial|_{C_n(X_\alpha)}) / \oplus_\alpha \operatorname{im}(\partial|_{C_{n+1}(X_\alpha)}) \cong \oplus_\alpha H_n(X_\alpha)$.

Proposition

Let X be a point. Then

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \geq 1 \end{cases}$$

Proof

For each n, $C_n(X)$ is generated by a single element $\sigma_n : \Delta^n \to p$ so $C_n(X) \cong \mathbb{Z}$. Then

$$\partial \sigma_n = \sum_{i=0}^{n} (-1)^i \sigma_n|_{[v_0,\dots,\hat{v}_i,\dots,v_n]} = \sum_{i=0} (-1)^i \sigma_{n-1} = \begin{cases} 0 & n \text{ odd} \\ \sigma_{n-1} & n \text{ even} \end{cases}$$

$$\cdots \longrightarrow C_{n+1}(X) \longrightarrow C_n(X) \longrightarrow C_{n-1}(X) \longrightarrow \cdots$$

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \cdots$$ We see that

$$\partial_n = \begin{cases} \cong & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Therefore $\ker / \operatorname{im} = 0$ or $\ker / im = \mathbb{Z}/\mathbb{Z} = 0$. Because

$$C_1(X) \xrightarrow{0} C_0(X) \xrightarrow{0} 0$$ we have thata $H_0(X) = \ker / \operatorname{im} = \mathbb{Z}/\{0\} = \mathbb{Z}$.

Proposition

If X is path connected, then $H_0(X) \cong \mathbb{Z}$.

Proof

Define a map $\epsilon : C_0(X) \to Z$ by $\sum n_i \sigma_i \mapsto \sum n_i$ given that $\sigma_i : \{v\} \to X$. Then ϵ is surjective. Also,

$$H_0(X) = \ker \partial_0 / \operatorname{im} \partial_1 = C_0(X)/\operatorname{im}\partial_1 = C_0(X)/\ker\epsilon \cong \operatorname{im}\epsilon \cong \mathbb{Z}$$

We claim that $\ker \epsilon = \operatorname{im}\delta$.
$(\supseteq)$ Let $\sigma : \Delta^1 \to X$, $\epsilon(\delta_1(\sigma)) = \epsilon(v_1 - v_0) = 1 - 1 = 0$.
$(\subseteq)$ Let $\sum n_i \sigma_i \in C_0(X)$ such that $0 = \epsilon(\sum n_i \sigma_i) = \sum n_i$. We fix a point $x_0 \in X$. Because X is path-connected, we can draw paths τ_i from x_0 to σ_i. Consider $\sum n_i \tau_i \in C_1(X)$, then

$$\partial(\sum n_i \tau_i) = \sum n_i(\partial \tau_i) = n_i(\sigma_i - x_0) = \sum n_i \sigma_i - \overbrace{\sum n_i}^{=0} x_0 = \sum n_i \sigma_i$$

Reduced Homology

$$\cdots \longrightarrow C_n(X) \longrightarrow \cdots \longrightarrow C_1(X) \longrightarrow C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

Usually written as $\tilde{H}_n(X)$, and $\tilde{H}_n(X) = H_n(X)$ if $n \geq 1$. We have that $\tilde{H}_0(X) = \ker\epsilon / \operatorname{im}\partial_1$ and $\epsilon|_{\operatorname{im}\partial_1} = 0$ so ϵ induces a map $\tilde{\epsilon} H_0(X) \hookrightarrow \mathbb{Z}$. Then $\ker\tilde{\epsilon} = \tilde{H}_0(X)$. It follows that

$$0 \longrightarrow \tilde{H}_0(X) \longrightarrow H_0(X) \longrightarrow \mathbb{Z} \longrightarrow 0$$

is a split exact sequence since $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$. In particualr, $\tilde{H}(\text{pt}) = \{0\}$.

Remark

$$\pi_1/[\pi_1, \pi_1] \cong H_1$$

Homotopy Invariance

Suppose we have $f : X \to Y$ continuous. It induces $f_\sharp : C_n(X) \to C_n(Y)$ by $\sigma \mapsto f \circ \sigma$. $f_\sharp$ is called a chain map and the following diagram commutes

$$\begin{array}{ccccccc}
\cdots \longrightarrow & C_n(X) & \xrightarrow{\partial} & C_{n-1} & \xrightarrow{\partial} & \cdots \\
& \downarrow f_\sharp & & \downarrow f_\sharp & & \\
\cdots \longrightarrow & C_n(X) & \xrightarrow{\partial} & C_{n-1} & \xrightarrow{\partial} & \cdots
\end{array}$$

Let $\sigma \in C_n(X)$ and

$$f_\sharp(\partial\sigma) = f_\sharp\left(\sum_{i=0}^{n}(-1)^i \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}\right) = \sum_{i=0}^{n}(-1)^i (f \circ \sigma)|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]} = \partial(f_\sharp \sigma)$$

Then $f_\sharp$ maps cycles to cycles ($\partial c = 0$, $\partial(f_\sharp c) = f_\sharp(\sigma c) = 0$) and boundaries to boundaries ($f_\sharp(\partial c) = \partial(f_\sharp c)$). So $f_\sharp$ induces $f_* : H_n(X) \to H_n(Y)$.

Theorem

If $f, g : X \to Y$ are homotopic, then $f_* = g_* : H_n(X) \to H_n(Y)$ for all n.

Corollary

If $X \simeq Y$ are homotopic, then $H_n(X) \cong H_n(Y)$. $g \circ f \simeq \mathrm{id}_X$, $f \circ g \simeq \mathrm{id}_Y$,

$$g_* \circ f_* = (g \circ f)_* = (\mathrm{id}_X)_* = \mathrm{id}$$

and similarly $g_* \circ f_* = \mathrm{id}$. So f_* and g_* are isomorphisms.

Definition

Let $f, g : C.(X) \to C.(Y)$ be two chain maps. We say that f and g are chain homotopic if there is a map $p : C_n(X) \to C_{n+1}(Y)$ such that $\partial P + P\partial = g - f$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_n(X) \arrow[r, "\partial"] \arrow[d, "f,g"] \arrow[dl, "P"'] & C_{n-1} \arrow[r, "\partial"] \arrow[d, "f,g"] \arrow[dl, "P"'] & \cdots \\
\cdots \arrow[r] & C_n(X) \arrow[r, "\partial"] & C_{n-1} \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

Theorem

If $f \simeq g$ are homotopic, then

1. $f_\sharp$ and $g_\sharp$ are chain homotopic,
2. $f_* = g_*$ on homoology
3. For any n-cycle, $c \in C_n(X)$, $g(c) - f(c) = \partial P(c) + \overbrace{P(\partial c)}^{=0}$. Hence $g_*[c] = f_*[c]$.

Proof

Consider $\Delta^n \times I$, and set $\Delta^n \times \{0\} = [v_0, \ldots, v_n]$ and $\Delta^n \times \{1\} = [w_0, \ldots, w_n]$. Then the following are all n-simplicies

$$\begin{aligned}
[v_0, v_1, \ldots, v_{n-1}, v_n] \\
[v_0, v_1, \ldots, v_{n-1}, w_n] \\
[v_0, v_1, \ldots, w_{n-1}, w_n] \\
\vdots \\
[v_0, w_1, \ldots, w_{n-1}, w_n] \\
[w_0, w_1, \ldots, w_{n-1}, w_n]
\end{aligned}$$

They divide $\Delta^n \times I$ into $(n+1)$-simplicies, $\{[v_0, \ldots, v_i, w_i, \ldots, w_n] \,:\, i = 0, \ldots, n\}$. Now let $F : X \times I \to Y$ be a homotopy between f and g. Consider

$$\Delta^n \times I \xrightarrow{\sigma \times \mathrm{id}} X \times I \xrightarrow{F} Y$$

and define $P : C_N(X) \to C_{n+1}(Y)$ by $\sigma \mapsto \sum_{i=0}^{n} (-1)^i F \circ (\sigma \times \mathrm{id})|_{[v_0, \ldots, v_i, w_i, \ldots, w_n]}$. We need to check that $\partial P + P\partial = g_\sharp - f_\sharp$.

Short Exact Sequences of Chain Complexes Induce Long Exact Sequences of Homology Groups

Applications

1. Rleative homology group.

2. Meyer-Vietoris sequence.

Short Exact Sequences Induce Long Exact Sequences

Suppose we have sequences

$$
\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
A_* \arrow[r] \arrow[d, "i"] & \cdots \arrow[r] & A_{n+1} \arrow[r, "\partial"] \arrow[d] & A_n \arrow[r, "\partial"] \arrow[d] & A_{n-1} \arrow[r] \arrow[d] & \cdots \\
B_* \arrow[r] \arrow[d, "j"] & \cdots \arrow[r] & B_{n+1} \arrow[r, "\partial"] \arrow[d] & B_n \arrow[r, "\partial"] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & \cdots \\
C_* \arrow[r] \arrow[d] & \cdots \arrow[r] & C_{n+1} \arrow[r, "\partial"] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
0 & & 0 & 0 & 0 &
\end{tikzcd}
$$

So H induces a long exact sequence

$$\xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C)$$

$$\xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} \cdots$$

where $\partial : H_n(C) \to H_{n-1}(A)$ by $[c] \mapsto [a]$, our connecting homomorphism, for $c \in C_n$. Then we have that the following commutes

$$
\begin{tikzcd}
 & a \arrow[r, mapsto, "\partial"] \arrow[d, mapsto, "i"] & {} \arrow[d, "i"] \\
b \arrow[r, mapsto] \arrow[d, mapsto, "j"] & \partial b \arrow[r, mapsto] \arrow[d, mapsto, "j"] & 0 \\
c \arrow[r, mapsto] & 0 &
\end{tikzcd}
$$

So a is a cycle. We need to show that $\partial a = 0$. Note that $i(\partial a) = \partial(ia) = \partial(\partial b) = 0$. Because i is injective, $\partial a = 0$. ∂ is well defined since

- choice of a: i is injective

- choice of b: suppose $b' \in B_n$ such tha t$j(b') = j(b) = c$. Then $b - b'$ satisfies $j(b - b') = 0$ and $b - b' \in \ker j = \operatorname{im} i$ (i.e. there exists $a' \in A_n$ such that $i(a') = b - b'$, so $b' = b + i(a')$. Then

$$\begin{tikzcd} a' \arrow[r, mapsto] \arrow[d, mapsto] & \partial a' \\ b-b' \arrow[d, mapsto] \\ 0 \end{tikzcd}$$

So $a+\partial a'$ satisfies

$$i(a+\partial a') = i(a) + i(\partial a') = \partial b + \partial(ia') = \partial b'$$

and

$$\begin{tikzcd} & a+\partial a' \arrow[d, mapsto] \\ b' \arrow[r, mapsto] \arrow[d, mapsto] & \partial b' \\ c \end{tikzcd}$$

since $[a+\partial a'] = [a]$.

- We need to check choice of c, but we will skip this.
- We need to check that δ is a homomorphism, which follows from the definitions.
- Finally, check that the induced long sequence is exact. We will check only exactness about $H_n(C)$ (i.e. $\operatorname{im} j_* = \ker \delta$).

  $\operatorname{im} j_* \subseteq \ker \delta$: $\delta(j_*[b]) = 0$ because

$$\begin{tikzcd} & 0 \arrow[d, mapsto] \\ b \arrow[r, mapsto, "\partial"] \arrow[d, mapsto, "j"] & 0 \\ j(b) \end{tikzcd}$$

$\ker \delta \subseteq \operatorname{im} j_*$: Suppose $[c] \in H_n(C)$ such that $\partial[c] = 0$, then

$$\begin{tikzcd} a' \arrow[r, mapsto, "\partial"] & a = \partial a' \arrow[d, mapsto] \\ b \arrow[r, mapsto] \arrow[d, mapsto, "j"] & \partial b \\ c \end{tikzcd}$$

Consider $b - i(a')$, then $j(b-i(a')) = j(b) - \overbrace{j \circ i(a')}^{=0} = j(b) = c$. So $[c] = j_*[b-i(a')] \in \operatorname{im} j_*$. This is a cycle, since $\partial(b) - \partial(i(a')) = \partial b - i(\partial a') = \partial b - \partial b = 0$.

April 30, 2025

Recall

1. if $f, g : X \to Y$ are homotopic, then $f_* = g_* : H_n(X) \to H_n(Y)$.

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[dl, "P"'] \arrow[d, "f_\sharp = g_\sharp"] & C_n(X) \arrow[r, "\partial"] \arrow[dl, "P"'] \arrow[d, "f_\sharp = g_\sharp"] & \cdots \\
\cdots \arrow[r, "\partial"] & C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

$\partial P + P\partial = f_\sharp - g_\sharp.$

1. Short exact sequence of chain complexes

$$\begin{tikzcd} 0 \arrow[r] & A_* \arrow[r, "i"] & B_* \arrow[r, "j"] & C_* \arrow[r] & 0 \end{tikzcd}$$

induces a long exact sequence of homology groups

$$\begin{tikzcd} \cdots \arrow[r] & H_n(A) \arrow[r] & H_n(B) \arrow[r] & H_n(C) \arrow[r] & \cdots \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[r, "\partial"] & H_{n-1}(A) \arrow[r] & \cdots \end{tikzcd}$$

Relative Homoology Group

Setup: $A \subseteq X$, A closed and non-empty. Then

$$C_n(A) = \{c \in C_n(X) \,:\, c = \sum n_i \sigma_i,\ \operatorname{im}\sigma_i \subseteq A\}.$$

Define $C_n(X, A) = C_n(X)/C_n(A)$ such that

$$\begin{tikzcd} 0 \arrow[r] & C_n(A) \arrow[r, "i"] & C_n(X) \arrow[r, "j"] & C_n(X, A) \arrow[r] & 0 \end{tikzcd}$$

is a short exact sequence. Then $C_*(X, A)$ is a chain complex

$$\begin{tikzcd} \cdots \arrow[r, "\partial"] & C_n(X, A) \arrow[r, "\partial"] & C_{n-1}(X, A) \arrow[r, "\partial"] & \cdots \end{tikzcd}$$

with $\partial^2 = 0$. Note that $\partial : C_n(X) \to C_{n-1}(X)$ maps $C_n(A)$ to $C_{n-1}(A)$. Hence it induces $\partial : C_n(X)/C_n(A) \to C_{n-1}(X)/C_{n-1}(A)$. It gives homoology groups $H_n(X, A) = \ker\partial_n / \operatorname{im}\partial_{n+1}$ and induces a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & H_n(A) \arrow[r, "i_*"] & H_n(X) \arrow[r, "j_*"] & H_n(X, A) \arrow[r] & \cdots \end{tikzcd}$$

$$\begin{tikzcd} {} \arrow[r, "\partial"] & H_{n-1}(A) \arrow[r] & \cdots \end{tikzcd}$$

Remarks

1. the elements in $H_n(X, A)$ are represented by relative cycles (i.e. $\alpha \in C_n(X)$ such that $\partial\alpha \in C_{n-1}(A)$).

2. A relative cycle α is trivial in $H_n(X, A)$ means α is a “relative boundary” (i.e. $\alpha = \partial\beta + \gamma$ for $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$).

$\partial : H_n(X, A) \to H_{n-1}(A)$ is defined by $[\alpha] \mapsto [\partial\alpha]$

$$\begin{tikzcd}
 & \partial\alpha \arrow[d, "i"] \\
\alpha \in C_n(X) \arrow[r, mapsto, "\partial"] \arrow[d, "j"] & \partial\alpha \in C_{n-1}(A) \\
\alpha \in C_n(X, A) &
\end{tikzcd}$$

We can also define the relative version.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r, "\partial"] & C_0(A) \arrow[r, "\epsilon"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "\cong"] & 0 \\
\cdots \arrow[r, "\partial"] & C_0(X) \arrow[r, "\epsilon"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "0"] & 0 \\
\cdots \arrow[r, "\partial"] & C_0(X,A) \arrow[r, "0"] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
\sum n_i\sigma_i \arrow[r] \arrow[d] & \sum n_i \arrow[d, "="] \\
\sum n_i\sigma_i \arrow[r, mapsto] & \sum n_i
\end{tikzcd}$$

$$\tilde{H}_n(A) \longrightarrow \tilde{H}_n(X) \longrightarrow H_n(X,A)$$

$$\xrightarrow{\partial}$$

$$\longrightarrow \tilde{H}_0(A) \longrightarrow \tilde{H}_0(X) \longrightarrow H_0(X,A)$$

$$\longrightarrow 0$$

Example

$H_n(X,X) = 0$ for all n, because $C_n(X,X) = C_n(X)/C_n(X) = \{0\}$. So $H_n(X,X_0) \cong \tilde{H}_n(X)$

$$\overbrace{\tilde{H}_n(X_0)}^{=0} \longrightarrow \tilde{H}_n(X) \xrightarrow{\cong} H_n(X,X_0)$$

$$\xrightarrow{\partial} \overbrace{\tilde{H}_{n-1}(X_0)}^{=0} \longrightarrow \cdots$$

Fact

$H_n(X,A) \cong \tilde{H}_n(X/A)$ if (X,A) is a "good" pair (i.e. there exists a neighborhood V of A which deformation retracts to A).

Example

$(X,A) = (D^n, \partial D^n)$ is a good pair, so $H_i(X,A) \cong \tilde{H}_i(D^n/\partial D^n) = \tilde{H}_i(S^n)$. This give a long exact sequence

$$\tilde{H}_i(S^{n-1}) \longrightarrow \overbrace{\tilde{H}_i(D^n)}^{=0} \longrightarrow H_i(X,A)$$

$$\xrightarrow{\partial} \tilde{H}_{i-1}(S^{n-1}) \longrightarrow \overbrace{\tilde{H}_{i-1}(D^n)}^{=0} \longrightarrow \cdots$$

and $\tilde{H}_{i-1}(S^{n-1}) \cong H_i(D^n, \partial D^n) \cong \tilde{H}_i(S^n)$. We conclude that $\tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1})$.
For $n = 0$, S^0 is two points, $\tilde{H}_0(S^0) = \mathbb{Z}$, and $\tilde{H}_i(S^0) = \tilde{H}_i(\text{pt}) \oplus \tilde{H}_i(\text{pt}) = 0$ for each $i \geq 1$.
For $n = 1$, $\tilde{H}_1(S^1) \cong \tilde{H}_0(S^0) \cong \mathbb{Z}$ and $\tilde{H}_0(S^1) = 0$.
For $n = 2$, $\tilde{H}_2(S^2) \cong \tilde{H}_1(S^1) \cong \mathbb{Z}$, $\tilde{H}_1(S^2) \cong \tilde{H}_0(S^1) = 0$ and $\tilde{H}_0(S^2) = 0$.
So $\tilde{H}_i(S^n$ is $\mathbb{Z}$ when $i = n$ and 0 otherwise.

Induced Maps on Pairs

Write $f : (X,A) \to (Y,B)$ for a continuous map $f : X \to Y$ such that $f(A) \subseteq B$. Then $f_\sharp : C_n(X) \to C_n(Y)$ ($f_\sharp : C_n(A) \to C_n(B)$) induces $f_\sharp : C_n(X,A) \to C_n(Y,B)$ a chain map $\partial f_\sharp = f\sharp\partial$. This induces $f_* : H_n(X,A) \to H_n(Y,B)$.

Proposition

Given $f, g : (X,A) \to (Y,B)$ which are homotopic through maps between pairs $(X,A) \to (Y,B)$, then $f_* = g_* : H_n(X,A) \to H_n(Y,B)$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X,A) \arrow[r] \arrow[d] \arrow[dl, "p"'] & C_n(X,A) \arrow[r] \arrow[d] \arrow[dl, "p"'] & \cdots \\
\cdots \arrow[r] & C_{n+1}(Y,B) \arrow[r] & C_n(Y,B) \arrow[r] & \cdots
\end{tikzcd}$$

such that $\partial P + P\partial = g_\sharp - f_\sharp$ (i.e. $f_* = g_*$). $P : C_n(X) \to C_{n+1}(Y)$ maps $C_n(A)$ to $C_{n+1}(B)$. P defined by $P(\sigma)\sum(-1)^i F \circ (0 \times \mathrm{id})|_{[v_0,\ldots,v_i,w_i,\ldots,w_j]}$

$$\Delta^n \times I \xrightarrow{0\times\mathrm{id}} X \xrightarrow{F} Y$$

If $\sigma : \Delta^n \to A$, then $P(\sigma) : \Delta^{n+1} \to B$.

Excision

Given a good pair (X,A), $H_n(X,A) \cong \tilde{H}_n(X/A)$.
Suppose we have $Z \subseteq A \subseteq X$ such that $\overline{Z} \subseteq A^\circ$ (i.e. the closure of Z is in the interior of A). Then $H_n(X,A) \cong H_n(X-Z, A-Z)$. Equivalently, if $B = X - Z$ then $A \cap B = A - Z$ and $\overline{Z} \subseteq A^\circ \implies A^\circ \cup B^\circ = X$. If A and B satisfy $A^\circ \cup B^\circ = X$, then by excision $H_n(X,A) \cong H_n(B, A\cap B)$.

Remark

If X has a Δ-complex structure such that A, $X-Z$ and $A-Z$ are subcomplexes, then we claim that $C_n^\Delta(X,A) = C_n^\Delta(X-Z, A-Z)$ (and $H_n^\Delta(X,A) = H_n^\Delta(X-Z,A-Z)$. In fact, consider $\varphi : C_n^\Delta(X-Z) \to C_n^\Delta(X)/C_n^\Delta(A)$ which factors through

$$C_n^\Delta(X-Z) \xhookrightarrow{\iota} C_n^\Delta(X) \longrightarrow C_n^\Delta(X,A) = C_n^\Delta(X)/C_n^\Delta(A)$$

Then φ is surjective, $\ker\varphi = C_n^\Delta(A-Z)$ and

$$C_n^\Delta(X,A) = C_n^\Delta(X)/C_n^\Delta(A) = C_n^\Delta(X-Z)/\ker\varphi = C_n^\Delta(X-Z,A-Z)$$

Proof

Let $\{U_\alpha\}_\alpha = \mathcal{U}$ be a collection of subsets such that $\{U_\alpha^\circ\}_\alpha$ is an open cover of X (it will suffices to consider $\mathcal{U} = \{A,B\}$). Write

$$C_n^{\mathcal{U}}(X) = \left\{\sum n_i\sigma_i \in C_n(X) \,:\, \operatorname{im}\sigma_i \subseteq U_j^\circ \text{ for some } j\right\}.$$

Then $\partial : C_n(X) \to C_{n-1}(X)$ maps $C_n^{\mathcal{U}}(X)$ to $C_{n-1}^{\mathcal{U}}(X)$. The chain complex $C_*^{\mathcal{U}}(X)$ gives homoology groups $H_*^{\mathcal{U}}(X)$.

Proposition

$\iota : C_n^{\mathcal{U}} \to C_n(X)$ induces an isomorphism $H_n^{\mathcal{U}}(X) \cong H_n(X)$.
The sketch of this proof is to construct a map $\rho : C_n(X) \to C_n^{\mathcal{U}}(X)$ by subdivision. That is, if the simplex $\sigma : \Delta^n \to X$ does not sit inside any U_α we may subdivide into further simplices that do. Then $\rho\circ\iota = \mathrm{id}$ and $\iota\circ\rho$ is chain homotopic to the identity.

$$\begin{tikzcd}
\cdots \arrow[r] & C_n(X) \arrow[r] \arrow[d, "\iota\circ\rho"] \arrow[dl, "D"'] & C_{n-1}(X) \arrow[r] \arrow[d] \arrow[dl, "D"'] & \cdots \\
\cdots \arrow[r] & C_n(X) \arrow[r] & C_{n-1}(X) \arrow[r] & \cdots
\end{tikzcd}$$

where $D : C_{n-1}(X) \to C_n(X)$ such that $\partial D + D\partial = \mathrm{id} - \iota\circ\rho$ which implies $(\iota\circ\rho)_* : H_n(X) \to H_n(X)$ is the identity map. There also exists a relative version. For simplicty, say $\mathcal{U} = \{A,B\}$ and denote $C_n^{\mathcal{U}}(X) \stackrel{\Delta}{=} C_n(A+B)$ so we have $H_n(A+B,A) \cong H_n(X,A)$.

Proof Continued

We have that $H_n(A+B,A) \cong H_n(X,A)$ (proof in Hatcher).
The left hand side comes from the chain complex of

$$C_n(A+B,A) = C_n(A+B)/C_n(A) = C_n(B)/C_n(A\cap B) = C_n(B, A\cap B)$$

so $H_n(A+B,A) = H_n(B, A\cap B)$.

Proposition

Let (X,A) be a good pair. Then the quotient map $q : (X,A) \to (X/A, A/A)$ induces an isomorphism $q_* : H_n(X,A) \xrightarrow{\sim} H_n(X/A, \text{pt}) \cong \tilde{H}_n(X/A)$.

Proof

Let V be a neighborhood of A which deformation retracts to A.

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "(1)"] \arrow[d, "q_*"] & H_n(X,V) \arrow[r, "\sim"] & H_n(X-A, V-A) \arrow[d, "\sim"] \\
H_n(X/A, A/A) \arrow[r, "(2)"] & H_n(X/A, V/A) & \substack{H_n((X-A)/A, (V-A)/A) \\ = \\ H_n(X/A - A/A, V/A - A/A)} \arrow[l, "\text{excision}"']
\end{tikzcd}$$

It remains to show that (1) and (2) are isomorphisms. For (2), V/A deformation retracts to A/A in X/A. So consider the triple $A \subseteq V \subseteq X$. It induces a short exact sequence

$$0 \longrightarrow \substack{C_n(V,A) \\ = \\ C_n(V)/C_n(A)} \xrightarrow{\ i\ } \substack{C_n(X,A) \\ = \\ C_n(X)/C_n(A)} \xrightarrow{\ j\ } \substack{C_n(X,V) \\ = \\ C_n(X)/X_n(V)} \longrightarrow 0$$

So $\ker j = \operatorname{im} i$, and this induces a long exact sequence

$$\longrightarrow \overbrace{H_n(V,A)}^{=0} \longrightarrow H_n(X,A) \xrightarrow{\sim} H_n(X,V)$$

$$\xrightarrow{\partial} \overbrace{H_{n-1}(V,A)}^{=0} \longrightarrow$$

where the terms zero since V deformation retracts to A.

May 5, 2025

Recall

For $A \subseteq X$, we have

$$0 \longrightarrow C_*(A) \longrightarrow C_*(X) \longrightarrow C_*(X,A) = C_*(X)/C_*(A) \longrightarrow 0$$

which induces

$$\cdots \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X,A) \longrightarrow \cdots$$

$$\xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots$$

Also, we have excision where

1. if $Z \subseteq A \subseteq X$ such that $\overline{Z} \subseteq A^\circ$, then $H_n(X-Z, A-Z) = H_n(X,A)$.

2. if (X,A) is a good pair, i.e. A has a neighborhood V such that V deformation retracts to A, then $H_n(X,A) = \tilde{H}_n(X/A)$.

Simplicial and Singular Homology

Goal: given X with Δ-complex structure, $H_n^{\Delta}(X) \cong H_n(X)$.

Example

$H_n(D^n,\partial D^n) \cong \tilde{H}_n(D^n/\partial D^n) = \tilde{H}_n(S^n) \cong \mathbb{Z}$. We can construct a generator for this $\mathbb{Z}$. We consider $H_n(\Delta^n,\partial\Delta^n)$ and claim that it is generated by $i_n : \Delta^n \to \Delta^n$ as the identity map. We prove by induction, first observing that $n=0$ is good. Then suppose $n-1$ and let $\Lambda \subseteq \Delta^n$ be the space obtained by removing a face from the boundary $\partial\Delta^n$. Then take

$$H_n(\Delta^n,\partial\Delta^n) \xrightarrow{\partial} H_n(\partial\Delta^n,\Lambda) \xleftarrow[(2)]{} H_{n-1}(\Delta^{n-1},\partial\Delta^{n-1})$$

Consider the triple $\Lambda \subseteq \partial\Delta^n \subseteq \Delta^n$ and the short exact sequence on the chain level

$$0 \longrightarrow C_\bullet(\partial\Delta^n,\Lambda) \xrightarrow{i} C_\bullet(\Delta^n,\Lambda) \xrightarrow{j} C_\bullet(\Delta^n,\partial\Delta^n) \longrightarrow 0$$

which induces the long exact sequence

$$\cdots \longrightarrow H_n(\partial\Delta^n,\Lambda) \longrightarrow \overbrace{H_n(\Delta^n,\Lambda)}^{=0} \longrightarrow H_n(\Delta^n,\partial\Delta^n) \longrightarrow \cdots$$

$$\xrightarrow{\partial} H_{n-1}(\partial\Delta^n,\Lambda) \longrightarrow \overbrace{H_{n-1}(\Delta^n,\lambda)}^{=0} \longrightarrow \cdots$$

since Δ^n deformation retracts to Λ, $H_*(\Delta^n,\Lambda) = 0$. Hence $H_n(\Delta^n,\partial\Delta^n) \underset{\partial}{\cong} H_{n-1}(\partial\Delta^n,\Lambda)$.

For (2), let Δ^{n-1} be the face that is not in Λ. Then $\Delta^{n-1} \hookrightarrow \partial\Delta^n$ induces a homeomorphism $\Delta^{n-1}/\partial\Delta^{n-1} \cong \partial\Delta^n/\Lambda$. Hence $(\partial\Delta^n,\Lambda)$ Is a good pair, and

$$H_{n-1}(\partial\Delta^n,\Lambda) \cong \tilde{H}_{n-1}(\partial\Delta^n/\Lambda) \cong \tilde{H}_{n-1}(\Delta^{n-1}/\partial\Delta^{n-1}) \cong H_{n-1}(\Delta^{n-1},\partial\Delta^{n-1})$$

We have

$$\begin{tikzcd}
 & \partial i_n \in C_{n-1}(\partial\Delta^n,\Lambda) \arrow[d] \\
i_n \in C_n(\Delta^n,\Lambda) \arrow[r, "\partial"] \arrow[d] & \partial i_n \in C_{n-1}(\Delta^n,\partial\Delta^n) \\
i_n \in C_n(\Delta^n,\partial\Delta^n) &
\end{tikzcd}$$

so $\delta^{-1} : [\partial i_n] \mapsto [i_n]$. Through the isomorphism $H_{n-1}(\Delta^{n-1},\partial\Delta^{n-1}) \cong H_n(\Delta^n,\partial\Delta^n)$, $[i_n]$ is identified with $[\partial i_n]$ for $i_n : \Delta^n \to \Delta^n$. Hence $[\partial i_n]$ is $[\pm i_{n-1}]$ in $H_{n-1}(\Delta^{n-1},\partial\Delta^{n-1})$.

Corollary

Let $\bigvee_\alpha X_\alpha$ by identifying $x_\alpha \in X_\alpha$ for each α. Suppose (X_α, x_α) is a good pair for each α. Then $\bigoplus_\alpha \tilde{H}_n(X_\alpha) \cong \tilde{H}_n(\bigvee_\alpha X_\alpha)$.

Proof

Consider the good pair $(X,A) := \left(\coprod_\alpha X_\alpha, \coprod_\alpha \{x_\alpha\}\right)$ where $X/A = \bigvee_\alpha X_\alpha$ such that

$$\tilde{H}_n\left(\bigvee_\alpha X_\alpha\right) \cong H_n(X,A) \cong \bigoplus_\alpha H_n(X_\alpha, x_\alpha) = \bigoplus_\alpha \tilde{H}_n(X_\alpha).$$

Theorem

Let $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ be open sets. If U and V are homeomorphic, then $m = n$.

Proof

Let $x \in U$. By excision,

$$H_i(U, U - \{x\}) \cong H_i(\mathbb{R}^m, \mathbb{R}^m - \{x\}$$

where we note that $(\mathbb{R}^m, \mathbb{R}^m - \{x\})$ is not a good pair. However, it still induces a long exact sequence

$$\longrightarrow \tilde{H}_i(\mathbb{R}^m - \{x\}) \longrightarrow \overbrace{\tilde{H}_i(\mathbb{R}^m)}^{=0} \longrightarrow H_i(\mathbb{R}^m, \mathbb{R}^m - \{x\})$$

$$\longrightarrow \tilde{H}_{i+1}(\mathbb{R}^m - \{x\}) \longrightarrow \overbrace{\tilde{H}_{i-1}(\mathbb{R}^m)}^{=0} \longrightarrow \cdots$$

Hence

$$H_i(\mathbb{R}^m, \mathbb{R}^m - \{x\}) \cong \tilde{H}_{i-1}(\mathbb{R}^m - \{x\}) \cong \tilde{H}_{i-1}(S^{m-1}) = \begin{cases} \mathbb{Z} & i = m \\ 0 & i \neq m \end{cases}.$$

If U and V are homemorphisms, then $H_i(U, U - \{x\}) \cong H_i(V, V - \{\varphi(x)\})$ and $m = n$.

Naturality of Long Exact Sequences of Pairs

$f : (X,A) \to (Y,B)$ with $f(A) \subseteq B$,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_\bullet(A) & \longrightarrow & C_\bullet(X) & \longrightarrow & C_\bullet(X,A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_\sharp} & & \downarrow{\scriptstyle f_\sharp} & & \downarrow{\scriptstyle f_\sharp} & & \\ 0 & \longrightarrow & C_\bullet(B) & \longrightarrow & C_\bullet(Y) & \longrightarrow & C_\bullet(Y,B) & \longrightarrow & 0 \end{array}$$

commutes. Then the long exact sequence

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,A) & \xrightarrow{\delta} & H_{n-1}(A) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\ \cdots & \longrightarrow & H_n(B) & \longrightarrow & H_n(Y) & \longrightarrow & H_n(Y,B) & \xrightarrow{\delta} & H_{n-1}(B) & \longrightarrow & \cdots \end{array}$$

$$\begin{array}{ccc} & & \partial\alpha \in C_{n-1}(A) \\ & & \downarrow \\ \alpha \in C_n(X) & \xrightarrow{\partial} & \partial\alpha \in C_{n-1}(X)) \\ \downarrow & & \\ \alpha \in C_n(X, A) & & \end{array}$$

So $\delta : [\alpha] \to [\partial\alpha]$ and

$$f_*(\delta[\alpha]) = f_*[\partial\alpha] = [f_*(\partial\alpha)] = [\partial f_*(\alpha)] = \delta(f_*[\alpha]).$$

Recall: the Five Lemma

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E & \longrightarrow & \cdots \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \downarrow\cong & & \\ \cdots & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' & \longrightarrow & \cdots \end{array}$$

implies that $C \cong C'$.

Equivalence Between Simplicial and Singular Homology

Given X with a finite dimensional Δ-complex structure, then $C_n^\Delta(X) \hookrightarrow C_n(X)$ induces an isomorphism $H_n^\Delta(X) \cong H_n(X)$.

Proof

Suppose it holds for all (X, Δ) with dimension less than $k-1$. We condier the k-dimensional case.
Let X^i be the i-skeleton of X. Note that $X^k = X$, so the pair (X^k, X^{k-1}) induces a long exact sequence

$$\begin{array}{ccccccccc} H_{n+1}^\Delta(X^k, X^{k-1}) & \longrightarrow & H_n^\Delta(X^{k-1}) & \longrightarrow & H_n^\Delta(X^k) & \longrightarrow & H_n^\Delta(X^k, X^{k-1}) & \longrightarrow & H_{n-1}^\Delta(X^{k-1}) \\ \downarrow(1) & & \downarrow(2) & & \downarrow(3) & & \downarrow(4) & & \downarrow(5) \\ H_{n+1}(X^k, X^{k-1}) & \longrightarrow & H_n(X^{k-1}) & \longrightarrow & H_n(X^k) & \longrightarrow & H_n(X^k, X^{k-1}) & \longrightarrow & H_{n-1}(X^{k-1}) \end{array}$$

We have that (2) and (5) are isomorphisms per our inductive assumption. Note also that $C_n^\Delta(X^k) = 0$ for $n \geq k$, so

$$C_n^\Delta(X^k, X^{k-1}) = C_n^\Delta(X^k)/C_n^\Delta(X^{k-1}) = \begin{cases} C_n^\Delta(X^k) & k = n \\ 0 & n < k \end{cases}.$$

So the chain complex $C_\bullet^\Delta(X^k, X^{k-1})$ is

$$0 \longrightarrow 0 \longrightarrow C_n^\Delta(X^k, X^{k-1}) = C_n^\Delta(X^k) \longrightarrow 0 \longrightarrow 0$$

and $H_n^\Delta(X^k, X^{k-1}) \cong \begin{cases} C_k^\Delta(X^k) & k = n \\ 0 & k \neq n \end{cases}$. Now consider $\Phi : \left(\coprod_\alpha \Delta_\alpha^k, \coprod_\alpha \partial\Delta_\alpha^k\right) \to (X^k, X^{k-1})$. It induces a homemorphism $X^k/X^{k-1} \cong \left(\coprod_\alpha \Delta_\alpha^k\right)/\left(\coprod_\alpha \partial\Delta_\alpha^k\right)$. So \(

$$H_n(X^k, X^{k-1}) \cong \tilde{H}_n(X^k, X^{k-1}) \cong \tilde{H}_n\left(\left(\coprod_\alpha \Delta_\alpha^n\right)/\left(\coprod_\alpha \partial\Delta_\alpha^k\right)\right) \cong H_n\left(\coprod_\alpha \Delta_\alpha^k, \coprod_\alpha \partial\Delta_\alpha^k\right) \cong \bigoplus_\alpha H_n(\Delta_\alpha^k, \partial\Delta_\alpha^k)$$

where each $H_n(\Delta_\alpha^k, \partial\Delta_\alpha^k)$ is generated by $i_\alpha^k : \Delta_\alpha^k \to \Delta_\alpha^k$ (the identity map) if $n = k$ or $H_n(\Delta_\alpha^k, \partial\Delta_\alpha^k)$ when $n \neq k$. Finally, we observe that

$$C_k^\Delta(X^k) \cong \bigoplus_\alpha \left\langle i_\alpha^k \right\rangle \cong \bigoplus_\alpha H_n(\Delta_\alpha^k, \partial\Delta_\alpha^k).$$

So (1) and (4) are isomorphisms and, by the five lemma, (3) is an isomorphy as well.

Remark

$H_n^\Delta(X,A) \cong H_n(X,A)$ if X has a Δ-complex structure and $A \subseteq X$ is a sub-complex.

$$\begin{array}{ccccccccc} H_n^\Delta(A) & \longrightarrow & H_n^\Delta(X) & \longrightarrow & H_n^\Delta(X,A) & \longrightarrow & H_{n-1}^\Delta(A) & \longrightarrow & H_{n-1}^\Delta(X) \\ \downarrow{\scriptstyle(1)} & & \downarrow{\scriptstyle(2)} & & \downarrow{\scriptstyle(3)} & & \downarrow{\scriptstyle(4)} & & \downarrow{\scriptstyle(5)} \\ H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,A) & \longrightarrow & H_{n-1}(A) & \longrightarrow & H_{n-1}(X) \end{array}$$

where (1), (2), (4), (5) are isomorphisms, so we have the conclusion by the five lemma.

May 7, 2025

Definition: Degree

Let $f: S^n \to S^n$ which induces $f_*: H_n(S^n) \to H_n(S^n)$ (i.e. $\mathbb{Z} \to \mathbb{Z}$). Hence f_* is multiplication by some integer $d \in \mathbb{Z}$. Define $\deg(f) = d$.

Properties

1. $\deg(\mathrm{id}) = 1$.

2. If $f, g: S^n \to S^n$ are homotopic, then $f_* = g_*$ thus $\deg(f) = \deg(g)$.

3. $\deg(f \circ g) = \deg(f) \cdot \deg(g)$, because $(f \circ g)_* = f_* \circ g_*$. In particular, if $f \circ g \simeq \mathrm{id}_{S^n}$ then $\deg(f) \cdot \deg(g) = \deg(f \circ g) = 1$ and $\deg(f) = \pm 1$.

4. Suppose $f: S^n \to S^n$ is not surjective, say $x_0 \in S^n \backslash \operatorname{im} f$. Then $f: S^n \to S^n \backslash \{x_0\} \cong \mathbb{R}^n$. So f is $S^n \xrightarrow{f} S^n \backslash \{x_0\} \xrightarrow{\iota} S^n$ and

$$H_n(S^n) \longrightarrow \overbrace{H_n(S^n \setminus \{x_0\})}^{=0} \longrightarrow H_n(S^n)$$

   So $f_*: H_n(S^n) \to H_n(S^n)$ is the zero map (i.e. $\deg(f) = 0$).

1. $f: S^n \to S^n$ a reflection has degree -1. In general, if we take two copies of Δ^n glued along corresponding edges by the identity map then we get S^n. Then $H_n^\Delta(S^n)$ has a generator $U - L$, and reflection of f maps $U - L$ to $L - U$ (i.e. $f_*: \mathbb{Z} \to \mathbb{Z}$ is $1 \mapsto -1$).

2. $f: S^n \to S^n$ an antipodal map $(-\mathrm{id})$ which sends $(x^1, \ldots, x^{n+1}) \mapsto (-x^1, \ldots, -x^{n+1})$ has $\deg(-1\,\mathrm{id}) = (-1)^{n+1}$.

3. Theorem (Hopf) if $f, g: S^n \to S^n$ have the same degree, then $f \simeq g$.

4. If $f: S^n \to S^n$ has no fixed points, then $f \simeq -\mathrm{id}$ and $\deg(f) = (-1)^{n+1}$. Proof: if $x \neq f(x)$, then the segment $(1-t)f(x) + t(-x)$ does not pass through $0 \in \mathbb{R}^{n+1}$. Consider $f_t(x) = \frac{(1-t)f(x)+t(-x)}{||(1-t)f(x)+t(-x)||}$ where $f_0(x) = f(x)$ and $f_1(x) = -x$ show that $f_t(x)$ gives a homotopy between f and $-\mathrm{id}$.

5. S^n has a continuous, non-vanishing vector field if and only if n is odd. Proof: $(\Longleftarrow)$ say $n = 2k-1$ such that $S^n \subseteq \mathbb{R}^{2k}$. Define $V(x_1, \ldots, x_{2k}) = (-x_2, x_1, -x_4, x_3, \ldots)$. Then $V(\vec{x}) \perp \vec{x}$. $(\Longrightarrow)$ Think of $V(\vec{x})$ starting at $\vec{x}$

and without loss of generality that $||V(\vec{x})|| = 1$. Consider $f_t(x) = (\cos t)\vec{x} + (\sin t)V(\vec{x})$ where $f_\pi(x) = -x$ and $f_0(x) = x$ such that $\{f_t\}$ is a homotopy between id and $-\mathrm{id}$. Hence $1 = \deg(\mathrm{id}) = \deg(-\mathrm{id}) = (-1)^{n+1}$ and n is odd.

6. If n is even, then $\mathbb{Z}_2$ is the only non-trivial group that can act freely on S^n. For example, S^1 acts on S^3 freely if we consider $(z_1, z_2) \in S^3 \subseteq C^2$ and $\theta(z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2)$. Proof: suppose $G \neq \mathrm{id}$ acts freely on S^n. Consider $\deg : G \to \mathbb{Z}$ where $\mathrm{im}(\deg) \subseteq \{\pm 1\} \subseteq \mathbb{Z}$ and for $g \neq e$ then $\deg(g) = (-1)^{n+1} = -1$. Then $G/\ker \cong \mathrm{im} = \{-1, 1\}$ since $\ker = \{e\}$. Hence $G \cong \mathrm{im} = (\{\pm 1, \cdot\}) = \mathbb{Z}_2$.

Theorem

Below, we assume that S^n has a point y such that $f^{-1}(y) = \{x_1, \dots, x_m\}$ is a finite set. If f is smooth, then by Sard's theorem we may pick a regular point y. Then $f^{-1}(y)$ is an embedded submanifold of dimension zero (i.e. $f^{-1}(y)$ is a collection of finitely many points). That is, when f is smooth this assumption holds automatically.
For each $i = 1, \dots, m$, we choose a small ball U_i about x_i and a ball V about y such that $f(U_i) \subseteq V$. The pair $(S^n, S^n \setminus \{x\})$ induces

$$\cdots \longrightarrow \overbrace{H_n(S^n \setminus \{x\})}^{=0} \longrightarrow H_n(S^n) \xrightarrow{j} H_n(S^n, S^n \setminus \{x\}) \longrightarrow \overbrace{H_{n-1}(S^n \setminus \{x\})}^{=0} \longrightarrow \cdots$$

The pair $(U, U \setminus \{x\})$ gives

$$\cdots \longrightarrow H_n(U \setminus \{x\}) \longrightarrow \overbrace{H_n(U)}^{=0} \longrightarrow H_n(U, U \setminus \{x\}) \xrightarrow{\delta} H_{n-1}(U \setminus \{x\}) \longrightarrow \overbrace{H_{n-1}(U)}^{=0} \longrightarrow \cdots$$

and we observe that $H_n(S^n, S^n \setminus \{x\}) \cong H_n(U, U \setminus \{x\})$ by excision.

$$\begin{tikzcd}
& \overbrace{H_n(U_i, U_i \setminus \{x_i\})}^{=\mathbb{Z}} \arrow[r, "f_*"] \arrow[d, "k_i"] \arrow[dl, "excision"'] & \overbrace{H_n(V, V \setminus \{y\})}^{=\mathbb{Z}} \arrow[d, "excision"] \\
H_n(S^n, S^n \setminus \{x_i\}) & \overbrace{H_n(S^n, S^n - f^{-1}(y))}^{=\mathbb{Z}^m} \arrow[l, "p_i"] \arrow[r, "f_*"] & H_n(S^n, S^n \setminus \{y\}) \\
& H_n(S^n) \arrow[u, "j"] \arrow[ul, "j"] \arrow[r, "f_*"] & H_n(S^n) \arrow[u, "j"]
\end{tikzcd}$$

We have that $f_* : H_n(U_i, U_i \setminus \{x_i\}) \to H_n(V, V \setminus \{y\})$ is $\mathbb{Z} \to \mathbb{Z}$ and hence it gives an integer. We call this the local degree $\deg(f|_{x_i})$.
Theorem: $\deg(f) = \sum_{i=1}^m \deg(f|_{x_i})$.
Write

$$H_n(S^n, S^n - f^{-1}(y)) \underset{\text{excision}}{\cong} H_n\left(\coprod_i U_i, \coprod_i (U_i \setminus \{x_i\}\right) \cong \bigoplus_i H_n(U_i, U_i \setminus \{x_i\}) \cong \mathbb{Z}^m.$$

then $k_i : H_n(U_i, U_i \setminus \{x_i\}) \to \bigoplus_i H_n(U_i, U_i \setminus \{x_i\})$ by $1 \mapsto (0, \dots, 0, 1, 0, \dots, 0) =: e_i$. Consider the triple $S^n - f^{-1}(y) \subseteq S^n \setminus \{x_i\} \subseteq S^n$ which induces

$$0 \longrightarrow C_\bullet(S^n \setminus \{x_i\}, S^n \setminus f^{-1}(y)) \longrightarrow C_\bullet(S^n, S^n \setminus f^{-1}(y)) \longrightarrow C_\bullet(S^n, S^n \setminus \{x_i\}) \longrightarrow 0$$

So we have $p_i : H_n(S^n, S^n \setminus f^{-1}(y)) \to H_n(S^n, S^n \setminus \{x_i\})$. Then

$$\begin{tikzcd} & \mathbb{Z} \arrow[dl, "\mathrm{id}"'] \arrow[d, "k_i"] \\ \mathbb{Z} & \mathbb{Z}^m \arrow[l, "p_i"] \end{tikzcd}$$

commutes and $1 = p_i(k_i(1)) = p_i(e_i)$, hence p_i is the projection to the i-th component. Similarly

$$\begin{tikzcd} \mathbb{Z} & \mathbb{Z}^m \arrow[l, "p_i"'] \\ & \mathbb{Z} \arrow[ul, "\mathrm{id}"] \arrow[u, "j"] \end{tikzcd}$$

commutes so $1 = p_i(j(1))$ and the i-th component of $j(1)$ is 1 (i.e. $j(1) = (1, 1, \ldots, 1) \in \mathbb{Z}^m$. Then $\deg(f|_{x_i}) = f_*(k_i(1)) = f_*(e_i)$. Finally,

$$\deg f = f_*(1) = f_*(j(1)) = f_*\left(\sum e_i\right) = \sum f_*(e_i) = \sum \deg(f_*|_{x_i})$$

Remark

If f is smooth and y is a regular value, then we can pick U_i and V such that each $f|_{U_i} : U_i \to V$ is a diffeomorphism. Hence $\deg(f|_{x_i}) = \pm 1$.

Example

If $f : S^1 \to S^1$ by $z \mapsto z^k$, $f^{-1}(1)$ has k many points (viz. the roots of unity). $f|_{U_i} : U_i \to V$ is diffeomorphic (by rotation and scaling) and $\deg(f|_{x_i}) = 1$. $\deg(f) = \sum \deg(f|_{x_i}) = k$.

IMAGE 1

Definition: Suspension of a Space

Recall that the cone of X is $C(X) = X \times I / X \times \{1\}$.

IMAGE 2

The suspension of X is $S(X) = C(X)/X \times \{0\}$.

IMAGE 3

Examples

$S(S^1) = S^2$. In general $S(S^n) = S^{n+1}$.

Definition: Suspension of a Map

$f : X \to Y$ induces $f : X \times I \to Y \times I$ by $(x, t) \mapsto (f(x), t)$. This induces $Cf : C(X) \to C(Y)$ and $Sf : S(X) \to S(Y)$.

Examples

$f : S^n \to S^n$ induces a map $Sf : S^{n+1} \to S^{n+1}$. $f : S^1 \to S^1$ by $z \mapsto z^2$ induces $Sf : S^2 \to S^2$

IMAGE 4

Proposition

$\deg(Sf) = \deg(f)$.

Proof

Consider the pair $(C(S^n), S^n \times \{0\})$ which induces

$$\overbrace{\tilde{H}_{n+1}(S^N)}^{=0} \longrightarrow \overbrace{\tilde{H}_{n+1}(C(S^n))}^{=0} \longrightarrow H_{n+1}(C(S^n), S^n \times \{0\}) \xrightarrow{\delta} \overbrace{\tilde{H}_n(S^n)}^{=\mathbb{Z}} \longrightarrow \overbrace{\tilde{H}_n(C(S^n))}^{=0} \longrightarrow$$

Hence $\mathbb{Z} \cong H_{n+1}(C(S^n), S^n \times \{0\}) \cong \tilde{H}_{n+1}(S(S^n)) = \tilde{H}_{n+1}(S^{n+1})$. Therefore

$$\begin{tikzcd}
\tilde{H}_{n+1}(S^{n+1}) \arrow[r, "\sim"] \arrow[d, "(Sf)_*"] & H_{n+1}(C(S^n), S^n \times \{0\}) \arrow[r, "\delta"] \arrow[d, "(Cf)_*"] & \tilde{H}_n(S^N) \arrow[d, "f_*"] \\
H_{n+1}(S^{n+1}) \arrow[r, "\sim"] & H_{n+1}(C(S^n), S^n \times \{0\}) \arrow[r] & \tilde{H}_n(S^n)
\end{tikzcd}$$

So $\deg(Sf) = \deg(f)$.

Remark

For any $k, n \in \mathbb{Z}_+$, by iterated suspension of the map $z \mapsto z^k$, we can construct $f : S^n \to S^n$ of degree k.

Remark

$Sf : S^{n+1} \to S^{n+1}$, pick $p \in S^{n+1}$ a pole, then $(Sf)^{-1}(p) = \{p\}$.

IMAGE 5

Hence $\deg(Sf|_p) = \deg(Sf) = k$.

May 12, 2025

Recall

Let X be a CW-Complex of finite dimension $X = X^0 \cup X^1 \cup \cdots \cup X^{\dim X}$.
X^0 is a discrete set of points.
X^1 is a gluing of $\{e_\alpha^1\}_{\alpha \in A}$ to X^0, where $e^1 = [-1, 1]$, by the attaching map $\varphi_\alpha : \partial e_\alpha^1 \to X^0$.
X^{k+1} is the gluing of $\{e_\alpha^{k+1}\}_{\alpha \in A}$, where $e^{k+1} \cong D^{k+1}$, by $\varphi_\alpha : \partial e_\alpha^{k+1} \cong S^k \to X^k$.

Lemma

(a)

Let X be a CW-Complex of $\dim X$. Then

$$H_k(X^n, X^{n-1}) = \begin{cases} 0 & k \neq n \\ \text{free abelian with a basis in 1-1 correspondence to} \{n\text{-cells}\} & \end{cases}$$

Proof

(X^n, X^{n-1}) is a good pair. So

$$H_k(X^n, X^{n-1}) \cong \tilde{H}_k(X^n / X^{n-1}) = \tilde{H}_k\left(\bigvee_\alpha S^n_\alpha\right) = \bigoplus_\alpha \tilde{H}_k(S^n_\alpha).$$

If $k \neq n$, then $\tilde{H}_k(S^n_\alpha) = 0$.
If $k = n$, then $\tilde{H}_k(S^n_2) = \mathbb{Z}$ and $H_n(X^n, X^{n-1}) \cong \bigoplus_\alpha \mathbb{Z}$.

(b)

$H_k(X^n) = 0$ if $k > n$.

Proof

The pair (X^n, X^{n-1}) gives a long exact sequence.

$$\cdots \longrightarrow H_{k+1}(X^n, X^{n-1}) \xrightarrow{\delta} H_k(X^{n-1}) \longrightarrow H_k(X^n)$$

$$\longrightarrow H_k(X^n, X^{n-1}) \xrightarrow{\quad\delta\quad} \cdots$$

Supposing both $k \neq n$ and $k+1 \neq n$, the first and last terms are zero and $H_k(X^{n-1}) \cong H_k(X^n)$. Then

$$H_k(X^n) \cong H_k(X^{n-1}) \cong H_k(X^{n-2}) \cong \cdots \cong H_k(X^0) = 0$$

(c)

$i : X^n \hookrightarrow X$ induces an isomorphism $i_* : H_k(X^n) \to H_k(X)$ if $k < n$.

Proof

If $k < n$, then

$$H_k(X^n) \cong H_k(X^{n+1}) \cong \cdots \cong H_k(X^{\dim X}) = H_k(X)$$

Chain Complexes

$$
\begin{tikzcd}
 & & & \overbrace{H_n(X^{n+1},X^n)}^{=0} & \\
0 \arrow[dr] & & \overbrace{H_{n+1}(X^{n+1})}^{\cong H_n(X)} \arrow[ur] & & \\
 & H_{n+1}(X^n) \arrow[ur, "i"] \arrow[dr, "j"] & & & \\
H_{n+1}(X^{n+1},X^n) \arrow[ur, "\delta_n"] \arrow[rr, dashed, "d_{n+1}"] & & H_n(X^n,X^{n-1}) \arrow[rr, dashed, "d_n"] \arrow[dr, "\delta_{n+1}"] & & H_{n-1}(X^{n-1},X^{n-2}) \\
 & & & H_{n-1}(X^{n-1}) \arrow[ur, "j"] & \\
 & & 0 \arrow[ur] & &
\end{tikzcd}
$$

This give a cellular chain complex $\{H_n(X^n,X^{n-1}),d_n\}$ with $d_n \circ d_{n+1} = 0$ because $\xrightarrow{j} \cdot \xrightarrow{\delta} = 0$. This defines a cellular homology $H_k^{CW}(X)$. We claim that $H_n^{CW}(X) \cong H_n(X)$.

Proof

$$
\begin{aligned}
H_n(X) &\cong H_n(X^{n+1} & & \\
&\cong H_n(X^n)/\ker i & &\text{because } i \text{ is surjective} \\
&= H_n(X^n)/\operatorname{im}\delta_{n+1} & &\text{because } \overset{\delta_{n+1}}{\to} \cdot \overset{i}{\to} \text{ is exact} \\
&\cong j(H_n(X^n))/j(\operatorname{im}\delta_{n+1}) & &\text{because } j \text{ is injective} \\
&= \ker(\delta_n)/\operatorname{im}(d_{n+1}) & & j(\operatorname{im}\delta_{n+1}) = \operatorname{im}(j\circ\delta_{n+1}) = \operatorname{im}(d_{n+1}) \\
&= \ker(d_n)/\operatorname{im}(d_{n+1}) & & \ker(\delta_n) = \ker(j\circ\delta_n) = \ker d_n \\
&= H_n^{CW}(X) & &
\end{aligned}
$$

Applications

For

$$\cdots \longrightarrow H_n(X^n,X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1},X^{n-2}) \xrightarrow{d_{n-1}} \cdots$$

where $H_n(X^n,X^{n-1}) = \bigoplus_\alpha \mathbb{Z}$

(1)

If a CW-Complex does not have any n-cells, then $H_n(X^n,X^{n-1}) = 0$ and $H_n(X) \cong H_n^{CW}(X) = 0$.

(2)

If a CW-Complex X has k-many n-cells, then $H_n(X^n,X^{n-1}) \cong \mathbb{Z}^k$. Then $H_n(X) \cong H_n^{CW}(X) = ker d_n/\operatorname{im} d_{n-1}$. $\ker d_n \le H_n(X^n,X^{n-1}) = \mathbb{Z}^k$. Hence $\ker d_n$ and $H_n(X)$ can be generated by at most k many elements.

(3)

If X and Y are CW-complexes with $\{\varphi_\alpha : e_\alpha^n \to X^{n-1}\}$ and $\{\psi_\beta : e_\beta^n \to Y^{n-1}\}$ respectively, then $X \times Y$ has $\{\varphi_\alpha \times \psi_\beta : e_\alpha^m \times e_\beta^n \to (X \times Y)^{m+n-1}\}$ where $e_\alpha^m \times e_\beta^n \cong e^{m+n}$.
Consider $S^n \times S^n$ (for $n \geq 2$) where S^n is constructed by one 0-cell and one n-cell. Then $S^n \times S^n$ has one 0-cell ($\mathbb{Z}^1$), two n-cells ($\mathbb{Z}^2$) and one $2n$-cell ($\mathbb{Z}^1$).

$$0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots \longrightarrow \mathbb{Z}^2 \longrightarrow 0 \longrightarrow \cdots \longrightarrow \mathbb{Z} \longrightarrow 0$$

so

$$H_k(S^n \times S^n) = \begin{cases} \mathbb{Z} & k = 0, 2n \\ \mathbb{Z}^2 & k = n \\ 0 & \text{otherwise} \end{cases}.$$

(4)

Take $\mathbb{CP}^n$ as $\mathbb{C}^{n+1}/\sim$ or as $S^{2n+1}/\sim$ where $v \sim \lambda v$ and $\lambda v = (e^{i\theta} z_1, \ldots, e^{i\theta} z_{n+1})$. Consider the set of vectors in S^{2n+1} whose last component is real and nonnegative. $D_+^{2n} = \{(w, \sqrt{1-|w|^2}) \in C^{n+1} \ : \ w \in C^n, \ |w| \leq 1\}$ is the graph of the function $w \mapsto \sqrt{1-|w|^2}$ defined on $\{w \ : \ |w| \leq 1\} \subseteq C^n$. So D_+^{2n} is homeomorphic to a disk $\{|w| \leq 1\} = D^{2n} \subseteq \mathbb{C}^n$. For any vector $v \in S^{2n+1}$, $v = (z_1, \ldots, z_{n+1})$ if $z_{n+1} \neq 0$, then v is equivalent to a unique vector in D_+^{2n}. If $z_{n+1} = 0$, $\{(z_1, \ldots, z_n, 0) \in S^{2n-1} \times \{0\}\} = S^{2n-1}$. So $q : S^{2n+1} \to \mathbb{CP}^n$ has that $q|_{D_+^{2n}}$ is a homeomorphism. Then $S^{2n-1}/\sim$ is exactly $\mathbb{CP}^{n-1}$. Therefore, we may view $\mathbb{CP}^n$ as gluing e^{2n} to $\mathbb{CP}^{n-1}$ by the attaching map $\partial e^{2n} = S^{2n-1} \to \mathbb{CP}^{n-1}$.
So $\mathbb{CP}^n$ has cells e^0, e^2, ..., e^{2n} and the cellular chain complex is

$$\mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

$$H_k(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & k = 0, 2, \ldots, 2n \\ 0 & \text{otherwise} \end{cases}.$$

Recall that $\mathbb{RP}^n$ by $\S^n/\sim$ with $S^n \subseteq \mathbb{R}^{n+1}$ and $v \sim -v$, we may take the upper hemisphere D_+^n. For every $v \in S^n = (x_1, \ldots, x_n)$, if $x_{n+1} \neq 0$ then v is equivaent to a unique vector in D_+^n where $q|_{D_+^n} : D_+^n \to \mathbb{RP}^n$ homemorphic to its image. If $x_{n+1} = 0$, then $\{(x_1, \ldots, x_n, 0) \in S^n\}/\sim$ and $\mathbb{RP}^n$ is gluing e^n to $\mathbb{RP}^{n-1}$ via the attaching map $\varphi : \partial e^n = S^{n-1} \to \mathbb{RP}^{n-1}$ as the quotient map.

Computation

We want $d_n : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$. For $n = 1$ we have

$$\begin{tikzcd}
& 0 \arrow[dr] & & & \\
& & H_0(X^0) \arrow[dr, "j"] & & \\
H_1(X^1, X^0) \arrow[urr, "\delta"] \arrow[rrr, "d_1"] & & & H_0(X^0) \arrow[r] \arrow[dr] & 0 \\
& & & & 0
\end{tikzcd}$$

where $d_1 = \delta : H_1(X^1, X^0) \to H_0(X^0)$. If X is connected, and $X^0 = \{v\}$, then $H_0(X^0) = \mathbb{Z}$ and $H_0(X) = H_0(X^0)/\operatorname{im} d_1$ implies that $\operatorname{im} d_1 = 0$.
For $n \geq 2$, $H_n(X^n, X^{n-1})$ is $\bigoplus_\alpha \mathbb{Z}$ and the generators are in one-to-one correspondence with $\{e_\alpha^n\}_\alpha$. We have a cellular boundary formula

$$d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1}$$

where $d_{\alpha\beta} = \deg(\Delta_{\alpha\beta})$ and $\Delta_{\alpha\beta} : S^{n-1} = \partial e_\alpha^n \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{q_\beta} S_\beta^{n-1}$. $q_\beta : X^{n-1} \to S_\beta^{n-1}$ is obtained by collapsing everything in X^{n-1} except $(e_\beta^{n-1})^\circ$. For every n-cel e_α^n and every $(n-1)$-cell e_β^{n-1}, we obtain $d_{\alpha\beta} = \deg(\Delta_{\alpha\beta})$.

Example

Suppose we have M_g, an orientable surface of genus g. M_g has one 0-cell, $2g$ 1-cells $(a_1, b_1, \ldots, a_g, b_{g\_}$ and one 2-cell. Then $d_1 = 0$, and $d_2(e_2)$ comes from $\Delta_{\alpha\beta} : S^2 = \partial e^2 \xrightarrow{\alpha} X^1 = \bigvee S^1 \xrightarrow{q_\beta} S_\beta^1$ which glues S^1 to S^1 by $a \cdot a^{-1}$. So $\deg(\Delta_{\alpha\beta}) = 0$ and $d_2(e_2) = 0$.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

so $H_2 = \mathbb{Z}$, $H_1 = \mathbb{Z}^{2g}$ and $H_0 = \mathbb{Z}$.

Example

N_g is a non-orientable surface of genus g. N_g has one 0-cell, g 1-cells $(a_1^2 a_2^2 \cdots a_g^2)$, and one 2-cell. We know that $d_1 = 0$. Consider $\Delta_{\alpha\beta} : S_\alpha^1 \to X^1 \to S_\beta^1$ which glues S^1 to S^1 by a^2 (i.e. $z \mapsto z^2$) and $\deg(\Delta_{\alpha\beta}) = 2$. So $d_2(e_2) = \sum_\beta 2e_\beta^1 = (2, 2, \ldots, 2) \in \mathbb{Z}^g$ and

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{g} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

So $H_0 = \mathbb{Z}$, $H_1 = \mathbb{Z}^g/\operatorname{im} d_2 = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$ and $H_2 = \ker d_2/0 = 0$.