Topics in Analysis (F24)

September 30, 2024

Chapter 1: Banach Algebras

1.1: Definitions and Basic Properties

Definition: Banach Space

A Banach space X (over $\mathbb{C}$) is a normed vector space with algebraic operations

$$\begin{aligned} (x, y) &\mapsto x + y && \text{addition} \\ (\lambda, y) &\mapsto \lambda y && \text{scalar multiplication} \end{aligned}$$

and a norm

$$x \mapsto ||x||$$

which is complete (i.e. every Cauchy sequence converges).

Definition: (Complex) Banach Algebra

A (complex) Banach algebra B is a Banach space in which there is multiplication

$$B \times B \ni (x, y) \mapsto xy \in B$$

such that

1. $x(yz) = (xy)z$
2. $(x + y)z = xz + yz$ and $x(y + z) = xy + xz$
3. $\lambda(xy) = (\lambda x)y = x(\lambda y)$
4. $||xy|| \leq ||x|| \cdot ||y||$

Definition: Unital Banach Algebra

B is called a unital Banach algebra if $\exists e \in B$ such that

$$xe = ex = x \quad \text{and} \quad ||e|| = 1.$$

If e exists, it is unique.

1.2: Examples

Example 1

If X is a Banach space, then $B = \mathcal{L}(X)$ (the set of all bounded inear operators $A : X \to X$) equipped with algebraic operations

$$
\begin{aligned}
(A+B)x &= Ax + Bx \\
(\lambda A)x &= \lambda(Ax) \\
(AB)x &= A(Bx)
\end{aligned}
$$

and the operator norm

$$||A||_{\mathcal{L}(X)} = \sup_{x \neq 0} \frac{||Ax||_X}{||x||_X}.$$

$B = \mathcal{L}(X)$ is complete because X is complete.
The unit element is given by $I_X x = x$.

Example 2

If $X = \mathbb{C}^n$, then $B = \mathcal{L}(\mathbb{C}^n) \cong \mathbb{C}^{n \times n}$.

$$A = (a_{ij})_{i,j=1}^n \qquad Ax = y \qquad \sum_{j=1}^n a_{ij} x_j = y_i.$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

The norm in $\mathbb{C}^n$ leadsto a norm in $\mathbb{C}^{n \times n}$

$$
\begin{aligned}
||(x_i)|| &= \left(\sum |x_i|^2\right)^{1/2} & ||A|| &= \\
||(x_i)|| &= \sum |x_i| & ||A|| &= \max_j \sum_i |a_{ij}| \\
||(x_i)|| &= \max |x_i| & ||A|| &= \max_i \sum_j |a_{ij}|
\end{aligned}
$$

All norms are quivalent.

Example 3

Take $B = C(K)$ with K a compact Hausdorff space, $f : K \to \mathbb{C}$ continuous and $||f|| = \max_{t \in K} |f(t)|$.

Example 4

Take $B = A(K)$, $K \subseteq \mathbb{C}$ compact with $\operatorname{int}(K) \neq 0$, $f : K \to \mathbb{C}$ continuous where f is holomorphic on $\operatorname{int}(K)$ and

$$||f|| = \max_{t \in K} |f(t)| = \max_{t \in K \setminus \operatorname{int}(K)} |f(t)|$$

e.g. $K = \overline{\mathbb{D}} = \{t \in \mathbb{C} : |t| \leq 1\}$. Then $A(K) \subseteq C(K)$.

Example 5

Take $B = \ell^\infty(\mathbb{N})$ or $B = L^\infty(S, \sigma, \mu)$ with (S, σ, μ) a measure space, $f : S \to \mathbb{C}$ essentially bounded functions and

$$||f|| = \text{ess sup}_{t \in S} |f(t)| = \inf_{\substack{N \subseteq S \\ \mu(N)}} \left(\sup_{t \in S \setminus N} |f(t)| \right)$$

Example 6

Take $B = \ell^1(\mathbb{Z})$ or $B = L^1(\mathbb{R}^d)$ with $||\{x_n\}|| = \sum |x_n|$ and $||f|| = \int_{\mathbb{R}^d} |f(t)| \, dt$ respectively.
Multiplication is given by the convolution. e.g.

$$fg = (f * g)(x) = \int_{\mathbb{R}^d} f(x-t) g(t) \, dt$$

$\ell^1(\mathbb{Z})$ is unital, but $L^1(\mathbb{R}^d)$ is non-unital (since the unit of convolution is the Dirac delta; see Example 7).

Example 7

Take $B = M(\mathbb{R}^d)$ the complex measures on $\mathbb{R}^d$ with bounded variation.
Then multiplications is given as

$$(\mu * \nu)(A) = \int_{\mathbb{R}^d} \mu(A - x) d\nu(x)$$

and norm

$$||\mu|| = \sup_{\substack{\mathbb{R}^d = \bigcup A_i \\ \text{disjoint}}} \sum_{i=1}^{n} |\mu(A_i)| < +\infty.$$

Then, $f \, dm = d\mu$ gives $L^1(\mathbb{R}^d) \to M(\mathbb{R}^d)$.

Example 8

Take $B = C^{n \times n}[K]$ with K compcat and Hausdorff, continuous functions $f : K \to \mathbb{C}^{n \times n}$ and norm

$$||f||_B = \max_{t \in k} ||f(t)||_{C^{n \times n}}.$$

Then $B \cong (C(K))^{n \times n}$ the $n \times n$ matrices with entries from $C(K)$.

1.3: Remarks

- If B does not have a unit element, consider $B_1 = B \times \mathbb{C}$ with operations

$$\begin{aligned} (b_1, \lambda_1) + (b_2, \lambda_2) &= (b_1 + b_2, \lambda_1 + \lambda_2) \\ \alpha(b, \lambda) &= (\alpha b, \alpha \lambda) \\ (b_1, \lambda_1)(b_2, \lambda_2) &= b_1 b_2 + \lambda_1 b_2 + \lambda_2 b_1, \lambda_1 \lambda_2) \end{aligned}$$

and norm

$$||(b,\lambda)|| = ||b|| + |\lambda|.$$

Then B_1 is a unital Banach algebra with $e = (0,1)$. One writes $(b,\lambda) = (b,0) + \lambda(0,1) = b + \lambda \cdot e$.
In some sense, $B \subseteq B_1$ where $b \in B \mapsto (b,0) \in B_1$.

1.4: Definitions

Definition: Commutative Banach Algebra

B is called commutative if $xy = yx$.

Definition: Banach Subalgebra

A subset B_0 of a B-algebra is called a subalgebra if it is closed with respect to the algebraic operations

$$x, y \in B_0,\ \lambda \in \mathbb{C} \rightsquigarrow x + y, xy, \lambda x \in B$$

Definition: Closed Subalgebra

B_0 is a closed subalgebra or Banach subalgebra if it is norm-closed.

- Proposition: B_0 is a Banach algebra.

Definition: Generated Subalgebra

Let $M \neq \emptyset$ be a subset of a Banach algebra B.
The Banach subalgebra generated by M is the smallest closed subalgebra containing M.

$$\operatorname{alg} M = (\operatorname{clos} \operatorname{alg}_B M)$$

- Remark

  $\operatorname{alg} M$ is the intersection of all closed subalgebras containing M.
  $\operatorname{alg} M = \operatorname{clos}\left\{\sum_{i=1}^N \lambda_i a_1^{(i)} a_2^{(i)} \cdots a_{n_i}^{(i)}\right\}$ is the norm-closure of finite linear combinations of finite products of $a_j^{(i)} \in M$.

1.5: Examples

Exammple 1

Take B unital, $b \in B$. Then

$$\operatorname{alg}\{e, b\} = \operatorname{clos}_B \left\{\sum_{i=0}^N \lambda_i b^i \ : \ \lambda_i \in \mathbb{C},\ N \in \mathbb{N}\right\}$$

where $b^0 = e$.

1.6 Definitions

Definition: Banach Algebra Homomorphism

A Banach algebra homomorphism is a map $\phi : B_1 \to B_2$ between Banach algebras B_1 and B_2 such that

- ϕ is linear
- ϕ is bounded (continuous)
- ϕ is multiplicative

  $\phi(b_1 b_2) = \phi(b_1) \cdot \phi(b_2)$
- ϕ is unital if both B_1, B_2 have units and $\phi(e_{B_1}) = e_{B_2}$.

Definition: Banach Algebra Isomorphism

A Banach algebra homomorphism which is bijective is called a Banach algebra isomorphism.
Then $\phi^{-1} : B_2 \to B_1$ is an isomorphism as well.

Definition: Banach Algebra Isometry

ϕ is an isometry if $||\phi(x)|| = ||x||$.

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Recall

Given $M \subseteq \mathcal{L}(X)$ with X a Banach space (and $\mathcal{L}(X)$ itself a Banach algebra), we may construct $B = \mathrm{alg}_{\mathcal{L}(X)} M$.

1.7 Proposition

Let B be a unital Banach algebra. Then the map

$$\phi : B \ni x \to L_x \in \mathcal{L}(B)$$

is an isometric isomorphism onto a closed subalgebra of $\mathcal{L}(B)$ where

$$L_x : B \ni z \mapsto xz \in B$$

is the left-representation of x.

Proof

L_x is in $\mathcal{L}(B)$ since $L_x z = xz$

- is linear in z and
- $||L_x z|| = ||xz|| \leq ||x|| \cdot ||z||$ implies $||L_x|| \leq ||x||$ (i.e. L_x is a bounded).

The map $\phi : x \mapsto L_x$ is linear

$$L_{x_1+x_2} z = (x_1 + x_2)z = x_1 z + x_2 z = L_{x_1} z + L_{x_2} z = (L_{x_1} + L_{x_2})z$$

ϕ is multiplicative

$$L_{x_1 x_2} z = (x_1 x_2)z = x_1(x_2 z) = L_{x_1}(L_{x_2} z)$$

From the above, we conclude that ϕ is a homomorphism.
To show that ϕ is an isometry,

$$||L_x|| = \sup_{z \neq 0} \frac{||L_x z||}{||z||} \geq \frac{||L_x e}{||e||} = \frac{||x||}{1} = ||x||.$$

Then also ϕ is injective and $\mathrm{im}\,\phi$ is closed. Since $\mathrm{im}\,\phi$ is a Banach algebra, it is therefore a closed subalgebra.

1.7 Remark: Right-Regular Representation

Every unital Banach algebra is isometrically isomorphic to a Banach algebra of operators.
Right-regular representation:

$$R_x = z \mapsto zx$$

Chapter 2: Group of Invertible Elements in a Banach Algebra

2.1 Definition: Invertible Element

Let B be a unital Banach algebra. An element $x \in B$ (in B) if there exists $y \in B$ such that $xy = yx = e$.
Note that $y = x^{-1}$ is uniquely determined.
Write GB for the set of all invertible elements of B.

Remark

GB is a (multiplicative group).

- $x, y \in GB \implies xy \in GB$ and $(xy)^{-1} = y^{-1}x^{-1}$,
- $x \in GB \implies x^{-1} \in GB$ and $(x^{-1})^{-1} = x$, and
- $e \in GB$.

2.2 Lemma

If $x \in B$ and $||x|| < 1$, then $e - x \in GB$.

Proof

Take the Neumann series

$$e + x + x^2 + x^3 + \cdots$$

which converges to some $s \in B$

$$s_n = e + x + \cdots + x^n$$

where s_n are Cauchy:

$$||s_{n+k} - s_n|| = ||x^{n+1} + \cdots x^{n+k}|| \leq ||x||^{n+1} + ||x||^{n+2} + \cdots = \frac{||x||^{n+1}}{1 - ||x||}.$$

So $s_n \to S$,

$$(e - x)s_n = s_n(e - x)e - x^{n+1}.$$

Taking $n \to \infty$

$$(e - x)s = s(e - x) = e.$$

2.3 Proposition

The group GB is open in B and the map $\Lambda : GB \ni x \mapsto x^{-1} \in GB$ is continuous (in the norm).

Proof

Take $x \in GB$ and consider $y \in B$ with $||y|| < \frac{1}{||x^{-1}||} = \varepsilon$.
Then $x + y \in B_\varepsilon(x)$ is invertible,

$$x + y = x(e + x^{-1}y),$$

and

$$||x^{-1}y|| \leq ||x^{-1}|| \cdot ||x|| < 1.$$

Therefore GB is open, since $B_\varepsilon(X) \subseteq GB$. The inverse

$$(x + y)^{-1} = (e + x^{-1}y)^{-1}x^{-1} = \sum_{n=0}^{\infty}(-x^{-1}y)^n x^{-1} = x^{-1} + \sum_{n=1}^{\infty}(-x^{-1}y)^n x^{-1}$$

so

$$||(x + y)^{-1} - x^{-1}|| \leq \sum_{n=1}^{\infty} ||x^{-1}||^{n+1}||y||^n = \frac{||x^{-1}||^2||y||}{1 - ||x^{-1}|| \cdot ||y||}.$$

This converges to zero as $||y|| \to 0$.

2.4 Examples

Example 1

$B = C(K)$, K compact Hausdroff, $f : K \to \mathbb{C}$ continuous.
$GB = \{f \in C(K) \,:\, f(t) \neq 0,\ \forall\, t \in K\}$.

Example 2

$B = C^{n\times n}$.
$GB = \{A \in \mathbb{C}^{n\times n} \,:\, \det A \neq 0\}$.

2.5 Definition:

Let G_0B stand for the connected componet of GB containing e.

Remarks

- the ε-neighborhoods $B_\varepsilon(x) \subseteq B$ are (path-)connected.

$$B_\varepsilon(x) = \{y \in B \,:\, ||x-y|| < \varepsilon\}$$

  For $y_1, y_2 \in B_\varepsilon(x)$, there is a continuous path

$$\sigma : [0,1] \ni \lambda \mapsto y_1\lambda + y_2(1-\lambda) \in B_\varepsilon(x)$$

- Because GB is open and $B_\varepsilon(x)$ is path-connected, GB is locally (path-)connected (i.e. every $x \in GB$ has a (path-)connected open neighborhood in GB).

- In this context, connectedness and path-connectedness are equivalent. Therefore the components of GB are the path-components of GB.

- GB is the union of disjoint (path-)components where each component is both open and closed in GB.

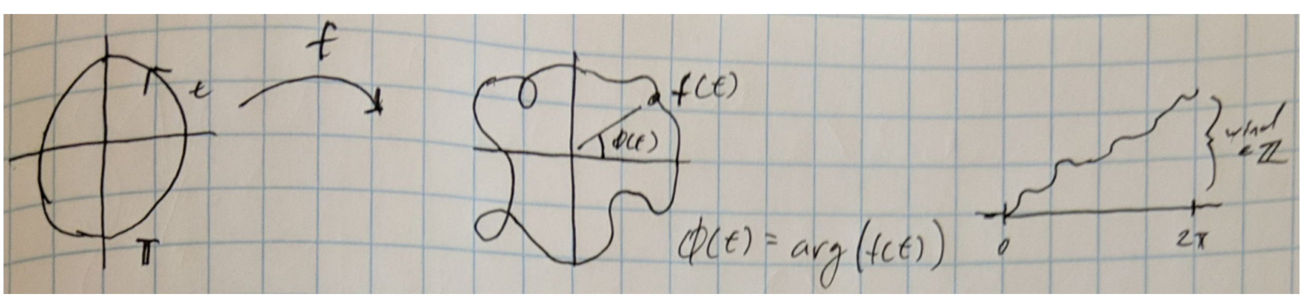
- $x, y \in GB$ belong to the same path-component if there exists a continuous path $\gamma : [0,1] \to GB$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Here, $x \sim y$ is an equivalence relation.

- $G_0B = \{x \in GB \,:\, \exists$ a path in GB connecting e and $x\}$.

2.6 Examples

Example 1

Take $B = C(\mathbb{T})$ with $\mathbb{T} = \{z \in \mathbb{C} \,:\, |z| = 1\}$ and continuous functions $f : \mathbb{T} \to \mathbb{C}$.
GB is the non-vanishing continuous functions $f : \mathbb{T} \to \mathbb{C}$ ($f(t) \neq 0,\ \forall t \in \mathbb{T}$).
For $f \in GB$ one can define a winding number.



We have $\frac{1}{2\pi}\arg f(e^{ix})$ a continuous function with

$$\text{wind}(t) = \left[\frac{1}{2\pi}\arg f(e^{ix})\right]_{x=0}^{2\pi} = \phi(2\pi) - \phi(0)$$

and $\mathrm{wind}(t) \in \mathbb{Z}$.
The map $GB \ni f \mapsto \mathrm{wind}(t) \in \mathbb{Z}$ is continuous, hence locally constant (i.e. constant on each connected component).
Therefore $G_0C(\mathbb{T}) \subseteq \{f \in GC(\mathbb{T}) : \mathrm{wind}(f) = 0\}$. In fact, we will see that we have equality.
That is, f can be contracted (in GB) to the constant function $e(t) = 1$.

2.7 Proposition

G_0B is a normal subgroup of GB.

Proof

- G_0B is a group.

  For any $x, y \in G_0B$, there exist paths $\gamma_1 : [0,1] \to GB$ and $\gamma_2 : [0,1] \to GB$ with $\gamma_1(0) = \gamma_2(0) = e$, $\gamma_1(1) = x$ and $\gamma_2(1) = y$.
  Define $\gamma(t) = \gamma_1(t)\gamma_2(t)$ a path in GB such that $\gamma(0) = e$ and $\gamma(1) = xy$. Then $xy \in G_0B$.
  Following from Lemma 2.2, $\hat{\gamma} = (\gamma_1(t))^{-1}$ is a continuous path with $\hat{\gamma}_1(0) = e$, $\hat{\gamma}_1(1) = x^{-1}$ and $x^{-1} \in GB$.

- G_0B is a normal subgroup of GB.

  For every $y \in GB$, $yG_0By^{-1} \subseteq G_0B$ if and only if $yG_0B = G_0By$.
  Take $x \in G_0B$ with path γ, then

$$\delta(t) = y\gamma(t)y^{-1}, \quad \delta(0) = yey^{-1} = e, \quad \text{and} \quad \delta(1)yxy^{-1} \in G_0B.$$

2.8 Definition: Abstract Index Group

The quotient group GB/G_0B is called the abstract index group of B.

Remark

GB/G_0B is in 1-to-1 correspondence with the set of connected components of GB.
Indeed, the (path-)connected components of GB are given by $yG_0B = G_0By$ (for $y \in GB$).

$$y_1G_0B = y_2G_0B \iff y_2^{-1}y_1G_0B = G_0B \iff y_2^{-1}y_1 \in G_0B \iff [y_2] = [y_1] \text{ in } GB/G_0B.$$

2.9 Definition: Exponential Map

For $x \in B$, we define the exponential map $B \ni x \mapsto \exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

2.10 Lemma

The exponential map $B \ni x \mapsto \exp(x) \in GB$ is well-defined and continuous.
For $xy = yx$, we have $\exp(x+y) = \exp(x)\exp(y)$.
In particular, $(\exp(x))^{-1} = \exp(-x)$.

Proof

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergent.

$$\sum_{n=0}^{\infty} \frac{||x||^n}{n!} < +\infty.$$

It follows that $s_n = \sum_{n=0}^{k} \frac{x^k}{k!}$ is a Cauchy sequence and therefore converges.
Continuity left as an exercise. Need to show:

$$\left|\left|\sum \frac{x^n}{n!} - \sum \frac{y^n}{n!}\right|\right| \leq ||x - y|| \cdot M_{x,y}$$

The fact that $\exp(x + y) = \exp(x)\exp(y)$ follows from multiplying terms and the binomial formula.

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Recall

$GB \ e + x$.
$G_0 B$ connected component of GB containing e.
$GB / G_0 B$ is the abstract index group.
$B = C(\mathbb{T}) \rightsquigarrow f \in GC(\mathbb{T}) \rightsquigarrow \operatorname{ind}(f)$.

Definition: Exponential Map

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \in GB$$

Lemma:

For $y \in B$, $||y|| < 1$, there exists $x \in B$ such that $\exp(x) = e + y$.

Proof

Define

$$\log(e + y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \cdots \in B.$$

This converges absolutely ($||y|| < 1$), therefore it converges in B by completeness.

Identities

$$\exp(\log(e + y)) = \sum_{n=0}^{\infty} \frac{\left(\sum_k \frac{y^k}{k}(-1)^{k-1}\right)^n}{n!} = e + y$$

Proof

$G_0 B$ is equal to the set of all finite products of exponentials of elements in B.

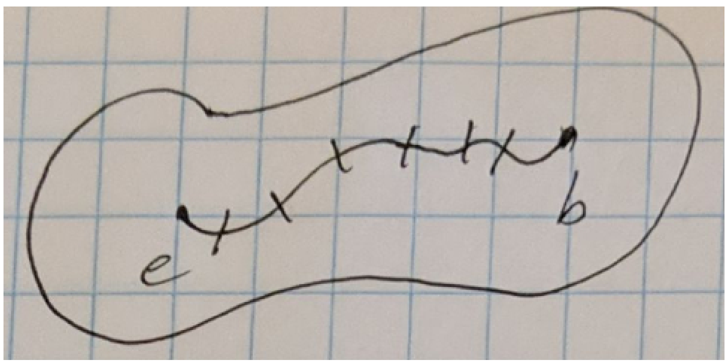
$$G_0 B = \bigcup_{n=0}^{\infty} \Gamma_n = \bigcup_{n=0}^{\infty} \{\exp(a_1)\exp(a_2)\cdots\exp_{a_n} \in B\}$$

Proof

Call $\Gamma = \bigcup_{n=0}^{\infty} \Gamma^n$.
Then observe that each Γ_n is path-connected and contains e.
For $b = \exp(a_1)\cdots\exp(a_n) \in \Gamma_n$, define a path

- $\sigma : [0,1] \to \Gamma_n$
- $\sigma(t) = \exp(ta_1)\cdots\exp(ta_n)$ is continuous with $\sigma(0) = e$ and $\sigma(1) = b$.

Therefore, Γ is path-connected and contains e. It follows that $\Gamma \subseteq G_0 B$.
To prove that $G_0 B \subseteq \Gamma$, take $b \in G_0 B$ and show that there exists a path in GB $\gamma : [0,1] \to GB$ continuous with $\gamma(0) = e$ and $\gamma(1) = b$.



We have that $(\gamma(t))^{-1}$ is continuous and bounded in the norm. Then $\gamma(t)$ is uniformly continuous.
$||\gamma^{-1}(t)|| \leq M$.
$(\exists N) : |t-s| \leq \frac{1}{N} \implies ||\gamma(t) - \gamma(b)|| \leq \frac{1}{M} \cdot \frac{1}{2}$. Write

$$b = \gamma(1)\cdot\gamma^{-1}(0) = \gamma(1)\gamma^{-1}\left(\frac{N-1}{N}\right)\gamma\left(\frac{N-1}{N}\right)\gamma^{-1}\left(\frac{N-2}{2}\right)\cdots\gamma\left(\frac{1}{N}\right)\gamma^{-1}\left(\frac{1}{N}\right)\gamma(0) = \prod_{k=1}^{N}\gamma^{-1}\left(\frac{k}{N}\right)\gamma\left(\frac{k-1}{N}\right).$$

Therefore, with $s_k = \gamma^{-1}\left(\frac{k}{N}\right)\gamma\left(\frac{k-1}{N}\right)$, $b = \prod_{k=1}^{N}\exp(\log(s_k))$.

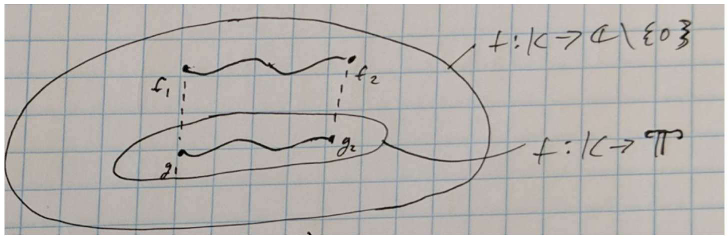
$$||s_k - e|| \leq ||\gamma^{-1}\left(\frac{k}{N}\right)|| \cdot ||\gamma\left(\frac{k-1}{N}\right) - \gamma\left(\frac{k}{N}\right)|| \leq M \cdot \frac{1}{2M} \leq \frac{1}{2}.$$

Corollary

If B is commutative, $G_0 B = \{\exp(a) \,:\, a \in B\}$.

Remark

Special case: $B = C(K)$ (K compact Hausdorff space).
$G_0 B = \{\exp(a) \,:\, a \in C(K)\}$.
$GB/G_0 B$ is an equivalence class of functions $f : K \to \mathbb{C}\setminus\{0\}$ with respect to path-connectedness.
That is, $f_1 \sim f_2$ if and only if there exists continuous $F(t,x) : [0,1]\times K \to \mathbb{C}\setminus\{0\}$ with $F(0,x) = f_1(x)$ and $F(1,x) = f_2(x)$.
These are the homotopy classes of continuous functions $f : K \to \mathbb{C}\setminus\{0\}$.
This corresponds to homotopy classes of continuous functions $f : K \to \mathbb{T}$ (with $\mathbb{T} = \{z \in \mathbb{C} \,:\, |z| = 1\}$) called the 1st co-homotopy group of K $\pi^1(K)$.



$f : K \to \mathbb{C} \setminus \{0\}$ and $\frac{f}{|f|} : K \to \mathbb{C} \setminus \{0\}$ are path-connected by $\sigma(s) = \frac{f}{|f|^s}$, $s \in [0, 1]$.
$f_1 \sim f_2$ in $K \to \mathbb{C} \setminus \{0\}$ implies that $\frac{f_1}{||f_1||} \sim \frac{f_2}{||f_2||}$ in $K \to \mathbb{T}$ by $F(s, x)$ and $\frac{F(s,x)}{|F(s,x)|}$.
We conclude that $\pi^1(K) \cong GC(K)/G_0C(K)$.

Example

Let $B = C(\mathbb{T})$.

$$G_0B = \{\exp(a) \,:\, a \in C(\mathbb{T}) = \{f \in GC(\mathbb{T}) \,:\, \mathrm{wind}(f) = 0\}$$

For $f \in GC(\mathbb{T})$, $\mathrm{wind}(f) = 0$ implies that $f = \exp(a)$ has a logarithm.
This implies that $f \in G_0B$ which itself implies that $\mathrm{wind}(f) = 0$, since $\mathrm{wind}(f)$ is continuous on $GC(\mathbb{T})$ and therefore constant on the component.
Therefore, $GB/G_0B \cong \mathbb{Z}$ via the winding number.
For connected components of GB, define $\chi_n(t) = t^n$, $|t| = 1$, where $\mathrm{wind}(\chi_n) = n$.

Remark: Closed Subalgebras and Invertibility

Let A be a closed subalgebra of B (both being unital, $e \in A$, $e \in B$).
Obviously, if $a \in A$ is invertible in A (i.e. $a^{-1} \in A$) then a is invertible in B. Then $GA \subseteq GB \cap A \subseteq GB$.

Example

Take $B = C(\mathbb{T})$ and $A = \{f \in C(\mathbb{T}) \,:\, f_n = 0,\ \forall n < 0\} = C_+(\mathbb{T})$ where $f_n = \frac{1}{2\pi}\int_0^{2\pi} f(e^{ix})e^{-inx}\,dx$ is the nth Fourier coefficient.
Formally: $f(t) \cong \sum_{n=-\infty}^{\infty} f_n t^n$ in $B = C(\mathbb{T})$, $|t| = 1$.
$f \in A : f(t) = \sum_{n=0}^{\infty} f_n t^n$, $|t| = 1$ has an analytic extension into the unit disk $|t| < 1$.
More precisely, $\phi : A(\overline{\mathbb{D}}) \to C_+(\mathbb{T}) \subseteq C(\mathbb{T})$ by $f \mapsto f|_{\mathbb{T}}$.
Where $A(\overline{\mathbb{D}}) = \{f \in \overline{D} \to \mathbb{C}$ continuous, holomorphic on $\mathbb{D}\}$ and $\mathbb{D} = \{t \in \mathbb{C} \,:\, |t| \leq 1\}$.
Then, for $f \in A(\overline{\mathbb{D}})$ with $n \in \{-1, -2, -3, \ldots\}$,

$$f_n = \frac{1}{2\pi}\int_0^{2\pi} f(e^{ix})e^{-inx}\,dx = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(z)}{z^{n+1}}\,dz = \lim_{r \to 1^-} \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(rz)}{z^{n+1}}\,dz = 0$$

- In fact, ϕ is an isometry.

$$||f||_{A(\overline{\mathbb{D}})} = \sup_{|z| \leq 1} |f(z)| = \max_{|z|=1} |f(z)| = ||f|_{\mathbb{T}}||_{C(\mathbb{T})}$$

By maximum modulus principle of holomorphic functions, since ϕ is not constant.

- ϕ is linear and multiplicative.

- $C_+(\mathbb{T})$ is a closed subset of $C(\mathbb{T})$.

$$\Lambda_n : C(\mathbb{T}) \ni f \mapsto f_n \in \mathbb{C}$$

  is a continuous linear functional.

$$C_+(\mathbb{T}) = \bigcap_{n=0} \ker \Lambda_n$$

- Less trivaially, ϕ is surjective and $C_+(\mathbb{T})$ is an algebra.

Example

$\chi_1(t) = t$ is invertible in $C(\mathbb{T}) = B$.
$x_1^{-1}(t) = \frac{1}{t} = x_{-1}(t) \notin C_+(\mathbb{T})$ while $\chi_1(t) \in C_+(\mathbb{T})$.
Therefore $GA \subseteq GB \cap A$ may not be equal.

Definition: Boundary

The boundary of a subset U of a topological space X is $\partial U = \overline{U} \setminus \mathrm{int}(U)$.

Remark

For $U \subseteq X$, $X = \mathrm{int}(U) \cup \partial U \cup \mathrm{int}(X \setminus U)$ a union of disjoint sets.

Lemma:

1. if $a \in \partial GA$, then $a \notin GA$ and there exists a sequence $a_n \in GA$ such that $a_n \to a$.
2. if $a \in \partial a$ and $a_n \in GA$ such that $a_n \to a$, then $||a_n^{-1}|| \to +\infty$.

Proof of 1

$a \in GA$ would imply $a \in \mathrm{int}(GA)$ and not a boundary point.

Proof of 2

Otherwise, there would exist a bounded subsequence $||a_{n_i}^{-1}|| \le M$.

$$||a_{n_i}^{-1} - a_{n_j}^{-1}|| \le ||a_{n_i}^{-1}|| \cdot ||a_{n_j} - a_{n_i}|| \cdot ||a_{n_j}^{-1}|| \le M^2 ||a_{n_i} - a_{n_j}||$$

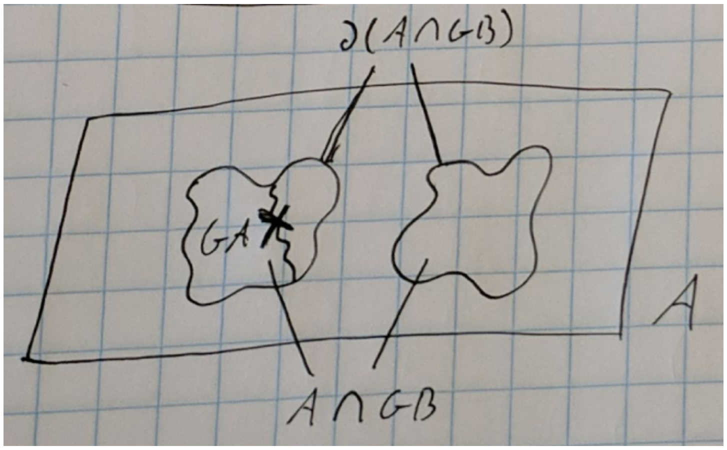
Since a_n converges, $\{a_n\}$ is Cauchy which implies $a_{n_i}^{-1}$ is Cauchy.
Then $a_{n_i}^{-1} \to b \in A$. $e = a_{n_i} a_{n_i}^{-1} \to ab$ implies $a^{-1} = b$ and $a \in GA$. However $a \notin GA$.

Proposition

Let A be a closed subalgebra of B ($e \in A$, $e \in B$). Then $\partial GA \subseteq \partial(A \cap GB)$ (both boundaries are considered in A).

Remark

Both GA and $A \cap GB$ are open subsets of A.



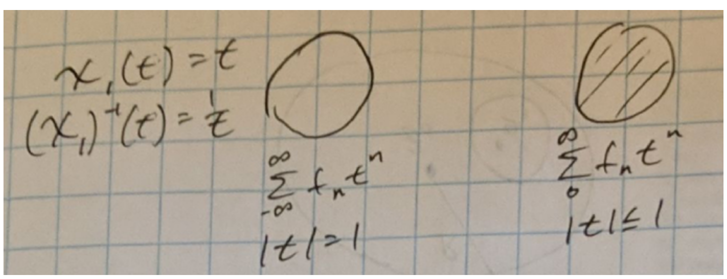
Proof

Take $a \in \partial GA$ and suppose $a \notin \partial(A \cap GB)$.
Take $a \in \partial GA$: $a_n \in GA$, $a \notin GA$, $a_n \to a$, $||a_n^{-1}|| \to +\infty$.

October 9, 2024

Recall

$A \subseteq B$, $GA \subseteq A \cap GB$.
If $A = C_+(\mathbb{T}) \cong A(\overline{\mathbb{D}})$ and $B = C(\mathbb{T})$.



Recall: Theorem

For GA, $A \cap GB$ open sets in A, $U \subseteq X$, $\partial U = \overline{U} \setminus \operatorname{int} U$, we have that $\partial GA \subseteq \partial(A \cap GB)$.

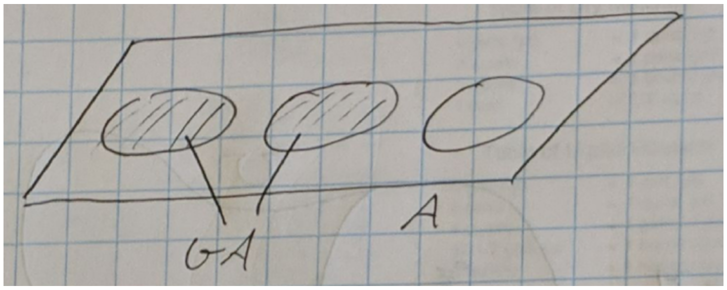
Proof

Take $a \in \partial GA$, $a_n \to a$, $a \notin GA$, $a \in A$.
Since $a_n \in GA$, $||a_n^{-1}|| \to +\infty$.
However, $a \notin GB$ otherwise $a \in GB$, $a_n \to a$ implies $a_n^{-1} \to a^{-1}$ (in GB) and, consequently, $\sup ||a_n^{-1}|| < +\infty$, a contradiction.
Therefore $a \notin A \cap GB$ and, consequently, $a \in \partial(A \cap GB) = \overline{(A \cap GB} \setminus (A \cap GB)$.

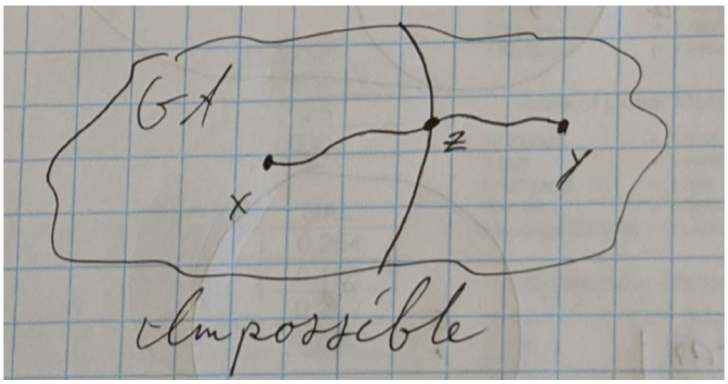
Theorem

Let A be a closed subalgebra of B.
GA is equal to the union of some components of $A \cap GB$.

Proof



Let U be a component of $A \cap GB$.
We want to show that either $U \cap GA \neq \varnothing$ or $U \subseteq GA$.



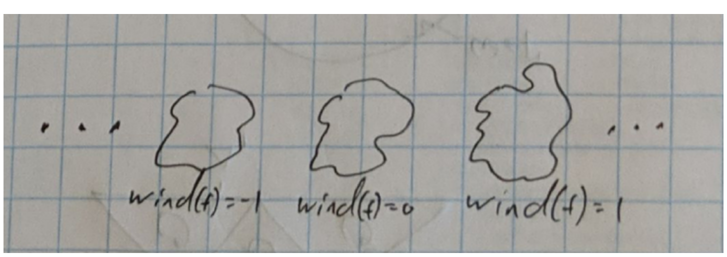
The above cannot occur since, by path-connectedness, for $x, y \in U,\ x \in GA,\ y \neq\in GA$, there would need to be some $z \in \partial GA$ with $z \notin A \cap GB$ a contradiction.
Alternatively, take $A \cap GB$ open in A.
Then $A \cap GB \cap \partial(A \cap GB) = \varnothing$ and $(A \cap GB) \cap \partial GA = \varnothing$ by the previous theorem.
Write $A = GA \cup \partial GA \cup \operatorname{int}(A \setminus GA)$. Then

$$A \cap GB = GA \cup \varnothing \cup \operatorname{int}(A \setminus GA) \cap (A \cap GB)$$

and $U = (GA \cap U) \cup \operatorname{int}(A \setminus GB) \cap U$ where $(GA \cap U) \cap \operatorname{int}(A \setminus GA) = \varnothing$ and open in U.
Therefore either $GA \cap U = \varnothing$ or $GA \cap U = U$ which implies that $U \subseteq GA$.

Example

Take $B(\mathbb{T})$ and $A = C_+(\mathbb{T}) \cong A(\overline{D})$.
Then $GB = \{f : \mathbb{T} \to \mathbb{C} \ :\ f(t) \neq 0$.



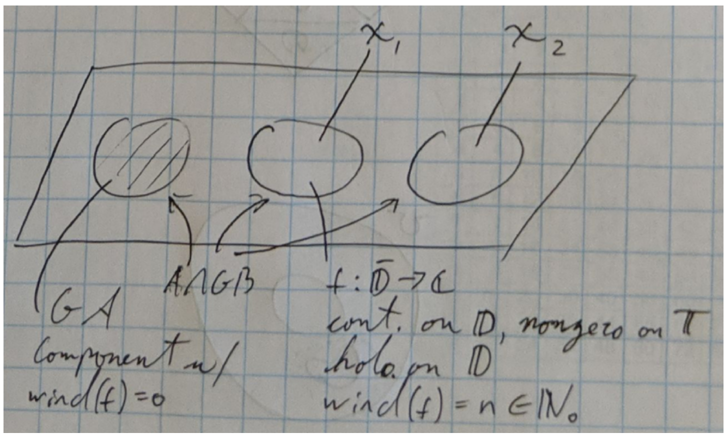
Then take

$$A \cap GB = \{f : \mathbb{T} \to \mathbb{C} \text{ continuous, } f(t) \neq 0,\ |t| = 1 \text{ with analytic continuation into } |t| < 1\}$$

such that $f \in A \cap GB$ which implies $\mathrm{wind}(f) \in \{0, 1, 2, 3, \cdots\}$ gives the number of zeroes of f inside $\mathbb{D}$.

$$\begin{aligned}
\mathrm{wind}(f) &= \frac{1}{2\pi i}\left[\log f(e^{ix})\right]_{x=0}^{2\pi} \\
&= \frac{1}{2\pi i}\lim_{r\to 1^-}\left[\log f(re^{ix})\right]_{x=0}^{2\pi} \\
&= \frac{1}{2\pi i}\lim_{r\to 1}\int_0^{\pi}\frac{f'(re^{ix})}{f(re^{ix})}ire^{ix}\,dx \\
&\underset{z=re^{ix}}{=} \frac{1}{2\pi i}\lim_{r\to 0}\int_{|z|=r}\frac{f'(z)}{f(z)}\,dz
\end{aligned}$$

Which gives the number of zeros of $f(z)$ inside $|z| < 1$



Chapter 3: Holomorphic Vector-Valued Functions

Goal

Define the notion of holomorphic/analytic functions $f : \Omega \to X$ where $\Omega \subset \mathbb{C}$ open and X a (complex) Banach space.

Sumary

- Basically all classical results remain true.
- There is a strong and a weak version of holomorphy, but they are equivalent.

Theorem

For a function $f : \Omega \to X$, $\Omega \subseteq \mathbb{C}$ open and X Banach, the following are equivalent

1. f is differentiable at every $z_0 \in \Omega$, i.e. there exists $f'(z_0) \in X$ such that

$$\lim_{z\to z_0}\left|\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right|\right|_X = 0$$

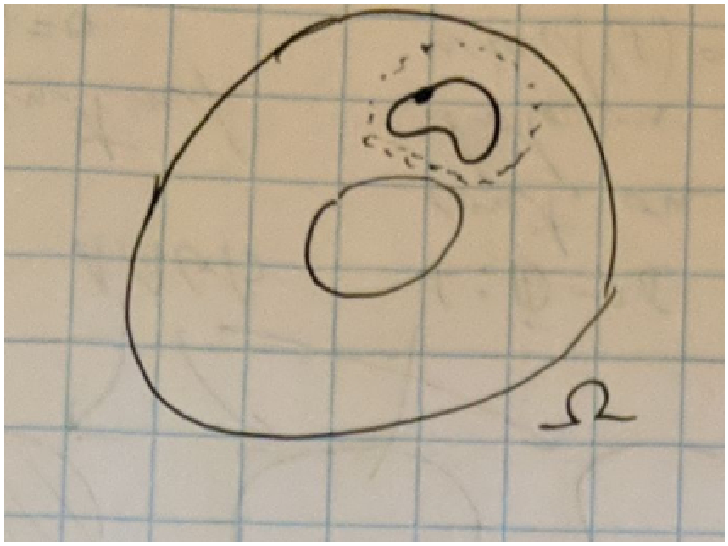
2. f is analytic at each point $z_0 \in \Omega$, i.e. f has a convergent power series at z_0 with radius of convergence $R_{z_0} > 0$.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,\ |z - z_0| < R_{z_0},\ a_n \in X$$

which converges in the norm of X.

3. $f : \Omega \to X$ is continuous (in the norm) and for every piecewise smooth closed contour Γ contained in a disk D ($\Gamma \subseteq D \subseteq \Omega$).

$$\int_\Gamma f(z)\,dz = 0$$



Definition: (Strongly) Holomorphic Function

If (1)-(3) hold, then f is (strongly)-holomorphic.

Remarks: Integration of Vector-Valued Functions

A piecewise smooth contour Γ can be parameterized by $\sigma : [0,1] \to \Omega$.

$$\int_\Gamma f(z)\,dz = \int_0^1 \underbrace{f(\sigma(t))\sigma'(t)}_{h(t)\text{ continuous}}\,dt$$

This is independent of the choice of parameterization.
Now $I = \int_0^1 h(t)\,dt$ can be defined via Riemann sums. Given a partition P, $h : [0,1] \to X$ continuous.

$$\lim_{\text{mesh}(P)\to 0} ||S(h,P,\xi) - I||_X = 0$$

where $S(h,P,\xi) = \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1})$, $P = \{x_0, x_1, \ldots, x_n\}$, $\xi_i \in [x_{i-1}, x_i]$.
Note that h is uniformly continuous and $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\text{mesh}(P_1) < \delta$, $\text{mesh}(P_2) < \delta$ implies

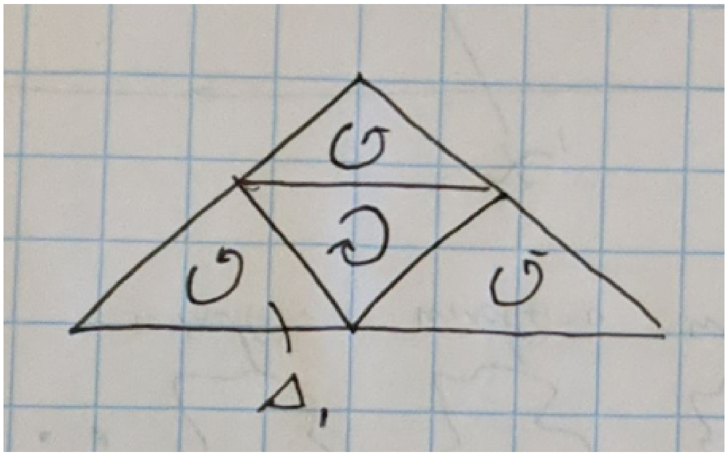
$$\left|\left|S(f,P_1,\xi^{(1)}) - S(f,P_2,\xi^{(2)})\right|\right| < \varepsilon$$

All usual properties of integrals hold.

- linear in integrand
- $||\int_\Gamma f(z)\,dz|| \le \int_\Gamma ||f(z)||\,|dz| \le (\text{length}(\Gamma)) \sup_{z\in\Gamma} ||f(z)||$.

Sketch of Proof (1) to (3)

To show: $\int_\Delta f(z)\, dz = x_0 = 0$ by contradiction that $x_0 \neq 0$.

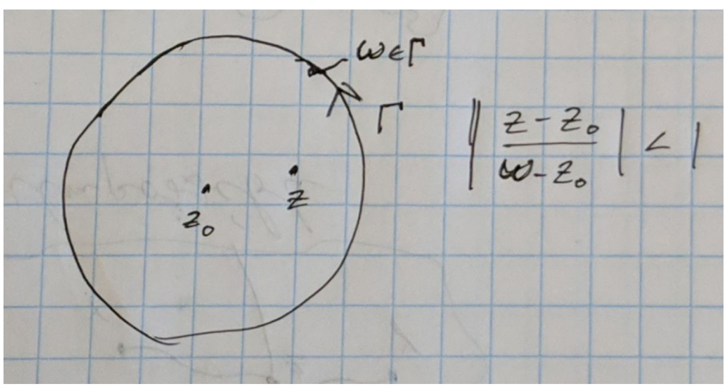


We have $\left|\left|\int_{\Delta_1} f\, dz\right|\right| \geq \frac{||x_0||}{4}$, $\left|\left|\int_{\Delta_n} f\, dz\right|\right| \geq \frac{||x_0||}{4^n}$.

Sketch of Proof (3) to (2)

$\int_\Gamma f\, dz = 0$ implies the Cauchy integral formula. Take

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\omega)}{\omega - z}\, d\omega$$



$$\frac{1}{\omega - z} = \frac{1}{(\omega - z_0) - (z - z_0)} = \frac{1}{w - z_0}\sum_{n=0}^{\infty}\left(\frac{z - z_0}{w - t}\right)^n$$

Therefore

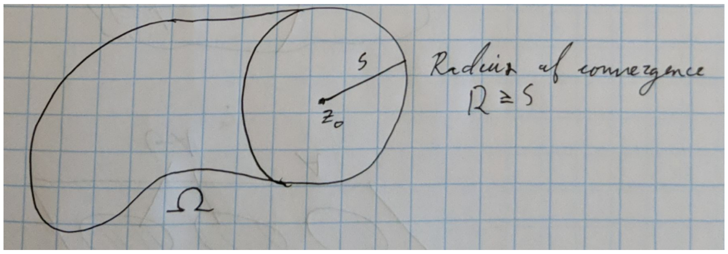
$$f(z) = \frac{1}{2\pi i}\int_\Gamma f(\omega)\sum_{n=0}^{\infty}\frac{(z - z_0)^n}{(\omega - z)^{n\pi}}\, d\omega = \sum_{n=0}^{\infty}(z - z_0)^n \frac{1}{2\pi i}\int_\Gamma \frac{f(\omega)}{(\omega - z)^{n\pi}}\, d\omega = \sum_{n=0}^{\infty}(z - z_0)^n a_n$$

with the sequence converging (in X) on $|z - z_0| < |\omega - z_0|$.

- Radius of Convergence

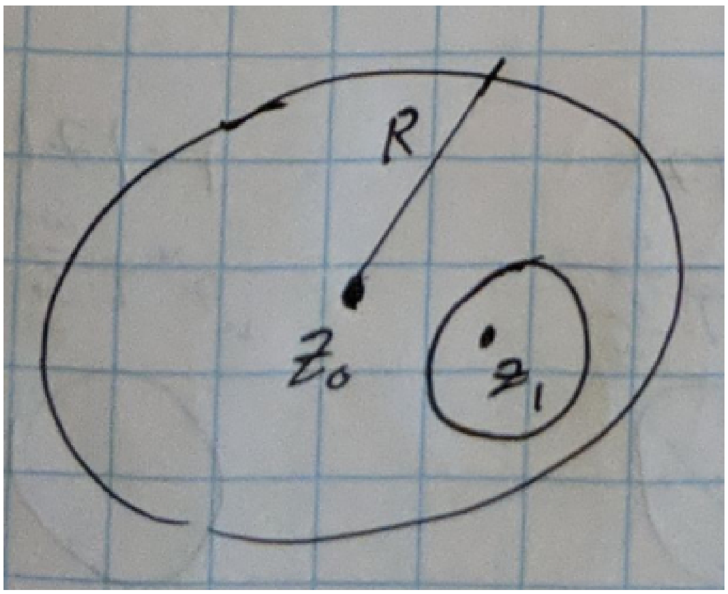
$$R^{-1} = \limsup_{n\to\infty} ||a_n||^{1/n}$$

(Root Test: $|z - z_0| < R$ convergence; $|z - z_0| > R$ divergence)



Sketch of Proof (2) to (1)

One can show that a function defined by convergent power series is differentiable, $f(z) = \sum a_n (z - z_0)^n$, then $f'(z) = \sum a_n \cdot n(z - z_0)^{n-1}$.
The radius of convergence is the same. This also implies that f is infinitely differentiable.



Take $z - z_0 = (z - z_1) + (z_1 - z_0)$ and, by the binomial theorem,

$$f(z) = \sum_{k=0}^{\infty} (z - z_1)^k \left(\sum_{n=k}^{\infty} a_n \binom{n}{k} (z_1 - z_0) \right)$$

which converges for at least $|z - z_1| < R - |z_1 - z_0|$.