

Random Matrix Theory

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Preliminaries

Let ξ_{ij}, η_{ij} be normal random variables (i.e. Gaussian, mean 0, variance 1).

e.g. $\mathbb{P}(\xi_{11} < s) = \int_{-\infty}^s \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$.

$\int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ is the variance.

$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is the Probability Density Function (PDF).

$\frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ is the probability measure on our probability space (i.e. totally finite measure space).

We build matrices

$$\begin{bmatrix} \xi_{11} & \frac{\xi_{12} + i\eta_{12}}{\sqrt{2}} & \frac{\xi_{13} + i\eta_{13}}{\sqrt{2}} & \dots \\ \frac{\xi_{21} + i\eta_{21}}{\sqrt{2}} & \xi_{22} & \frac{\xi_{22} + i\eta_{22}}{\sqrt{2}} & \\ \frac{\xi_{31} + i\eta_{31}}{\sqrt{2}} & \frac{\xi_{32} + i\eta_{32}}{\sqrt{2}} & \xi_{33} & \\ \vdots & & & \ddots \end{bmatrix}$$

Computing Random Matrices in Matlab

Gaussian, real valued 1x1 matrix.

```
randn
```

Gaussian, real valued 2x2 matrix.

```
randn(2)
```

Gaussian, complex valued 2x2 matrix.

```
randn(2)+sqrt(-1)*randn(2)
```

Gaussian, complex valued, self-adjoint 2x2 matrix.

Note that appending ' to a matrix takes the conjugate transpose, and matlab reserves i for the imaginary unit.

```
m = randn(2)+i*randn(2);  
(m+m')/2
```

Producing eigenvalues.

```
m = randn(2)+i*randn(2);  
l=(m+m')/2;  
eig(l)
```

Running tests to see how many hits we get within the interval $[0, 2]$.

```

edges=[0,2];
H=zeros(1,length(edges)-1);
trials=10;
for j=1:trials
m = randn(2)+i*randn(2);
l=(m+m')/2;
ev=eig(l);
H=H+histcount(ev,edges)
end

```

Homework

Is the PDF of $\frac{a+b}{2}$ the same as $\frac{\xi_{12}}{\sqrt{2}}$ for normal RVs a, b, ξ_{12} ?

i.e. $\mathbb{P}\left(\frac{a+b}{2} < s\right) \stackrel{?}{=} \mathbb{P}\left(\frac{\xi_{12}}{\sqrt{2}} < s\right)$

2x2 Random Matrix

Our matrix L corresponds to eigenvalues λ_1, λ_2 which are random variables determined by $\{\xi_{ij}, \eta_{ij}\}$. Then the number of evaluations in the interval B is given by $\sum_{j=1}^2 \chi_B(\lambda_j)$. We may take the average by

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \sum_{j=1}^2 \chi_B(\lambda_j) \frac{1}{\sqrt{2\pi}} e^{-\xi_{11}^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\xi_{22}^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\xi_{12}^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\eta_{12}^2} d\xi_{11} d\xi_{22} d\xi_{12} d\eta_{12}.$$

Expected Evaluations

We have that the expectation of the number of evaluations in the interval (a, b) is given by $\int_a^b G(s) ds$ where

$$G(s) = e^{-\frac{s^2}{2}} \sum_{\ell=0}^2 P_{\ell}(s)^2$$

and $P_{\ell}(s)$ is the Hermite polynomial of degree d .