Analogues

Stability, orbit and transitivty all have analogues which can be demonstrated by converting to left actions.

Definition: Cosets

Let $H \leq G$, and let $X = G$.
We have left action $H \times X \to X$ and $h * x = hx$ (taken in G).
As well as right action $X \times H \to X$ where $x * h = xh$.
A (left) H-coset is an orbit xH for some $x \in X$.
A (right) H-coset is an orbit Hx for some $x \in X$.

Example

Let $G = \mathrm{Alt}(4)$, $H = \mathrm{Stab}_G(W) = \{\mathrm{Id},\ (B\ P\ Y),\ (B\ Y\ P)\}$.

1. Take any $x \in H$, $xH = H$.

2. Take $x = (B\ P)(W\ Y)$, and $xH = \{(B\ P)(W\ Y), (B\ P)(W\ Y)(B\ P\ Y) = (P\ W\ Y),\ (B\ P)(W\ Y)(B\ Y\ P) = (B\ W\ Y)\}$.

3. There are two more; what are they?

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Cosets Revisited

Let G be a group, $H \leq G$. Then a (left) H-coset in G is a set of the form

$$gH = \{gh | h \in H\}$$

, where $g \in G$

Coset Space

G/H is the set of H-cosets.

- Example For $G = \mathrm{Alt}(4)$, given $C_1 = H = \mathrm{Stab}_G(B) = \{1, (P\ W\ Y), (P\ Y\ W)\}$, we have
  $C_2 = (B\ P\ W)H = \{(B\ P\ W), (B\ P\ )(W\ Y), (B\ P\ Y)\}$
  $(B\ P\ W) \circ (P\ W\ Y) = (B\ P)(W\ Y)$

  P ⟵ B ⟵ B
  B ⟵ W ⟵ P
  Y ⟵ Y ⟵ W
  W ⟵ P ⟵ Y

  $(B\ P\ W) \circ (P\ Y\ W) = (B\ P\ Y)$

P ⟵ B ⟵ B
Y ⟵ Y ⟵ P
W ⟵ P ⟵ W
B ⟵ W ⟵ Y

$C_3 = (B\ W\ P)H = \{(B\ W\ P), (B\ W\ Y), (B\ W)(P\ Y)\}$
$C_4 = (B\ Y\ P)H = \{(B\ Y\ P), (B\ Y\ )(P\ W), (B\ Y\ W)\}$
Then $G/H = \{C_1, C_2, C_3, C_4\}$.

- Q: What do the 3 elements in C3 have in common in geometric terms? C_3 sends B to W. Similarly, the cosets send B to all other vertices (including to itself).

Definition: Transporter

Let G be a group and X a G-set.
For two points, $x, y \in X$, the transporter $\mathrm{Trsp}_G(x, y) = \{g \in G | gx = y\}$.

Example

$G/H = \{\mathrm{Trsp}_G(B, B), \mathrm{Trsp}_G(B, P), \mathrm{Trsp}_G(B, W), \mathrm{Trsp}_G(B, Y)\}$

Note

When $x = y$, we recover $\mathrm{Trsp}_G(x, x) = \mathrm{Stab}_G(x)$.
For general G and H, there may not be a nice geometric action associated with it.
But G/H is still a G-set since $g^{'}(gH) = (g^{'}g)H$.

Proposition (B)

Let $H \leq G$ be a subgroup and let $g \in G$.
Then the map $H \xrightarrow{f} gH$ defined by $h \mapsto f(h) = gh$ is a bijection.

Proof

(Surjective) Any element x in gH is, by definition, of the form gh for some $h \in H$. So $x = f(h)$.
(Injective) Say $h_1, h_2 \in H$ satisfy $f(h_1) = f(h_2)$. That is $gh_1 = gh_2$. Multiplying g^{-1} on the left, we get $h_1 = h_2$.

Proposition (C)

Let G act on X, $x \in X$, and $g \in G$.
Take $y := gx$ and $H = \mathrm{Stab}_G(x)$. Then $gH = \mathrm{Trsp}_G(x, y)$.

Proof

$(\subseteq)$ Let $gh \in gH$ be arbitrary. Then

$$(gh) * x \underset{\text{Axiom 2}}{=} g * (h * x) \underset{h \in \mathrm{Stab}_G(x)}{=} g * x \underset{y = g*x}{=} y$$

Therefore $gh \in \mathrm{Trsp}_G(x,y)$.
($\supseteq$) Suppose $g' \in \mathrm{Trsp}_G(x,y)$. Consider $g^{-1}g'$. Then

$$(g^{-1}g') * x = g^{-1} * (g' * x) \underset{g' \in \mathrm{Trsp}_G(x,y)}{=} g^{-1}(y) = x$$

Therefore $(g^{-1}g') \in \mathrm{Stab}_G(x)$. Setting $g^{-1}g' := h$, so $g' = gh \in gH$.

Theorem: Orbit-Stabilizer Theorem

Let G act transitively on a set X (so that there is only one orbit in X, namely X itself).
If $|G| < \infty$, then for any $x \in X$ we have

$$|X| \cdot |\mathrm{Stab}_G(x)| = |G|$$

Proof

Let us count $|G|$ by partitioning G into transporters.

$$G = \coprod_{y \in X} \mathrm{Trsp}_G(x,y)$$

Therefore

$$|G| = \sum_{y \in X} |\mathrm{Trsp}_G(x,y)| \underset{B+C}{=} \sum_{y \in X} |\mathrm{Stab}_G(x)| = |X||\mathrm{Stab}_G(x)| \quad \blacksquare$$

Theorem: Lagrange

If G is a finite group and $H \leq G$, then $|G| = |H| \cdot |G/H|$.

Proof (Sketch)

Apply the Orbit-Stabilizer Theorem to $X = G/H$.
This action is transitive as $g(1H) = gH$.
Note $gH = H \iff g \in H$ and $g1 \in H$.
Therefore $\mathrm{Stab}_G(1H) = \{g \in G \mid g(1H) = 1H\} = H$.

Corollary

If $H \leq G$ and $|G| < \infty$, then $|H| \mid |G|$.
The converse is not true. No subgroup of order 6 in $\mathrm{Alt}(4)$ (where $|\mathrm{Alt}(4)| = 12$).

Definition: Conjugate

Let G be a group, $H \leq G$, $g \in G$.

1. For $x \in G$ the g-conjugate of x is $gxg^{-1} = {}^g x$.

2. The g-conjugate of H is $gHg^{-1} = {}^g H = \{gxg^{-1} \mid x \in H\}$.

Example

Let $G = \mathrm{Alt}(4)$ and $H = \mathrm{Stab}_G(B) = \{1, (P\ W\ Y), (P\ Y\ W)\}$. Then, for $g = (B\ Y\ P)$

$$gHg^{-1} = \{1, (B\ W\ P), (B\ P\ W)\} = \mathrm{Stab}_G(Y)$$

$(B\ Y\ P)1(B\ P\ Y) = 1$
$(B\ Y\ P)(P\ W\ Y)(B\ P\ Y) = (B\ W\ P)$

W ← W ⟵ P ⟵ B
B ⟵ P ⟵ Y ⟵ P
P ⟵ Y ← W ← W
Y ⟵ B ⟵ B ⟵ Y

- Note: Shortcut $(gxg^{-1})^{-1} = (g^{-1})^{-1}x^{-1}g^{-1} = gx^{-1}g^{-1}$.
  Applying this to $g = (B\ Y\ P)$ with $x = (P\ W\ Y)$
  Therefore, from the previous calculation, $gx^{-1}g^{-1} = (gxg^{-1})^{-1} = (B\ P\ W)$.

Proposition: Geometric Meaning of Conjugate

Let G act on a set X, $x \in X$, $g \in G$, and define $y := g * x$.
Then for $H = \mathrm{Stab}_G(x)$, we have

$$gHg^{-1} = \mathrm{Stab}_G(y)$$

That is, the conjugate of a a stabilizer is a stabilizer.

Proof

($\subseteq$) Let $ghg^{-1} \in gHg^{-1}$ be arbitrary.
Then .

$$(ghg^{-1}) * y = g * (h * (g^{-1} * y))g * (h * x) = g * x = y$$

Therefore $ghg^{-1} \subseteq \mathrm{Stab}_G(y)$.
($\supseteq$) Let $g' \in \mathrm{Stab}_G(y)$ be arbitrary.
Consider $g^{-1}g'g$. Then

$$(g^{-1}g'g) * x = g^{-1} * (g' * (g * x)) = g^{-1} * (g' * y) = g^{-1} * y = x$$

Therefore $h := g^{-1}g'g \in H$. Then by multiplying g on the left and g^{-1} on the right, we get

$$g' = ghg^{-1} \in gHg^{-1}$$

Orbit-Stablizer Theorem and Lagrange

1. If G acts transitively on X, then all the stabilizers have the same cardinality because they are all conjugates. So the Orbit-Stabilizer Theorem is consistent.

2. If $X = G/H$, then $\mathrm{Stab}_G(1H) = H$. What about $\mathrm{Stab}_G(gH) = gHg^{-1}$?