Manifolds I

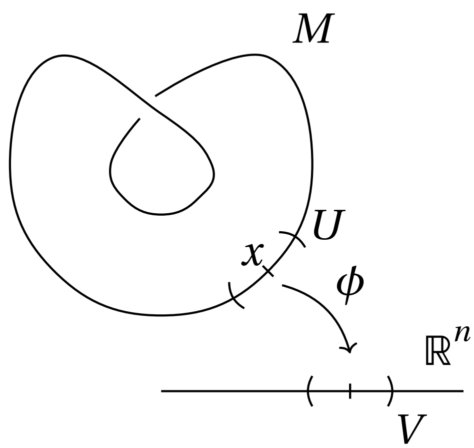
September 26, 2024

Class Organization

1 Takehome Midterm
1 Takehome Final
Homeworks assigned, but not graded.
`https://ginzburg.math.ucsc.edu/teaching/208manifolds1-2024/syl.html`

Definition: Topological Manifolds

For M a topological space, M is a topological manifold if $\forall x \in M$, $\exists M \supset U \ni x$ and homeomorphism $\phi : U \to V \subset \mathbb{R}^n$ for V open.
To avoid problems (see below), further assume that M is Hausdorff and second countable.

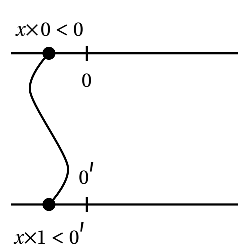


Exercise

We can require V to be an open ball.

Problems

- M need not be Hausdorff.



With $(\mathbb{R} \times 0 \coprod \mathbb{R} \times 1)/\sim$.

- M need not be second countable.

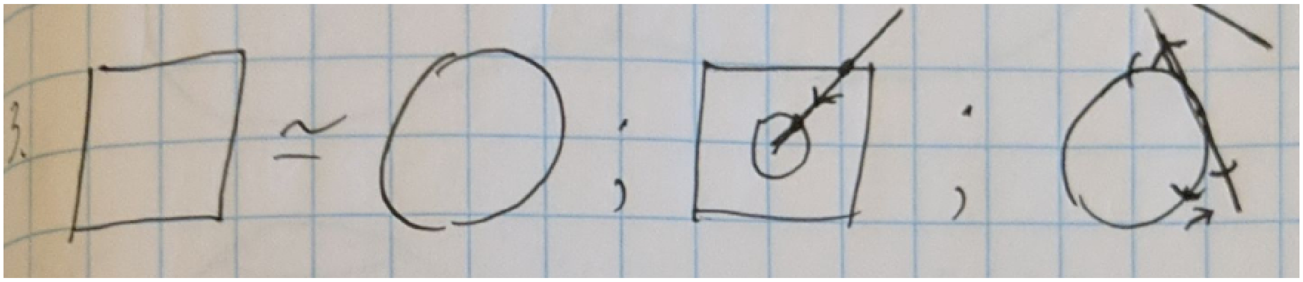
Take $\coprod_S \mathbb{R}_S$ where S is an uncountable index.

Examples

Example 1

If $N \underset{\text{homeo}}{\simeq} M$, this implies N is a manifold.

Example 2



Example 3

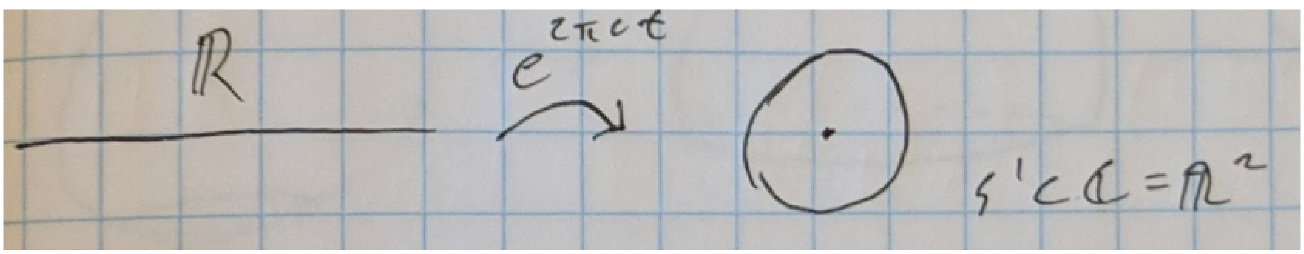
An open subset of a manifold is a manifold.

Example 4

M, N manfiolds implies $M \times N$ is a manifold.

Example 5

Take $\mathbb{R}/\mathbb{Z}$ by the equivalence relation $t \sim t'$ iff $t' - t \in \mathbb{Z}$.



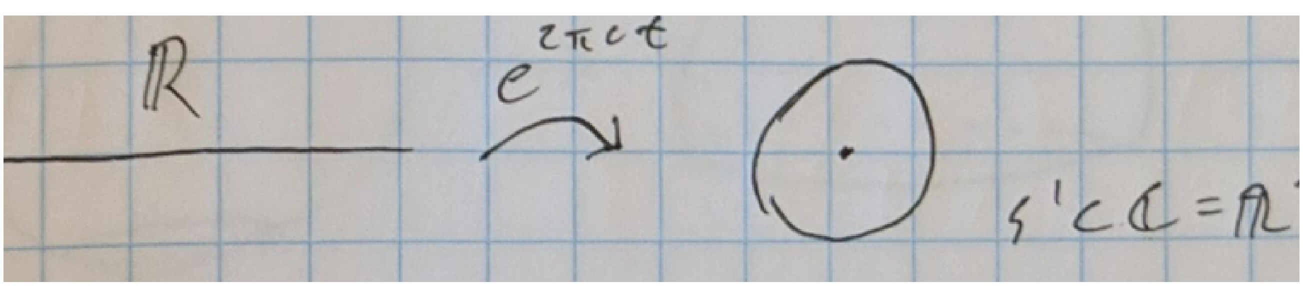
Then $C^0(S^1)$ relates to periodic functions with period 1.

Example 6

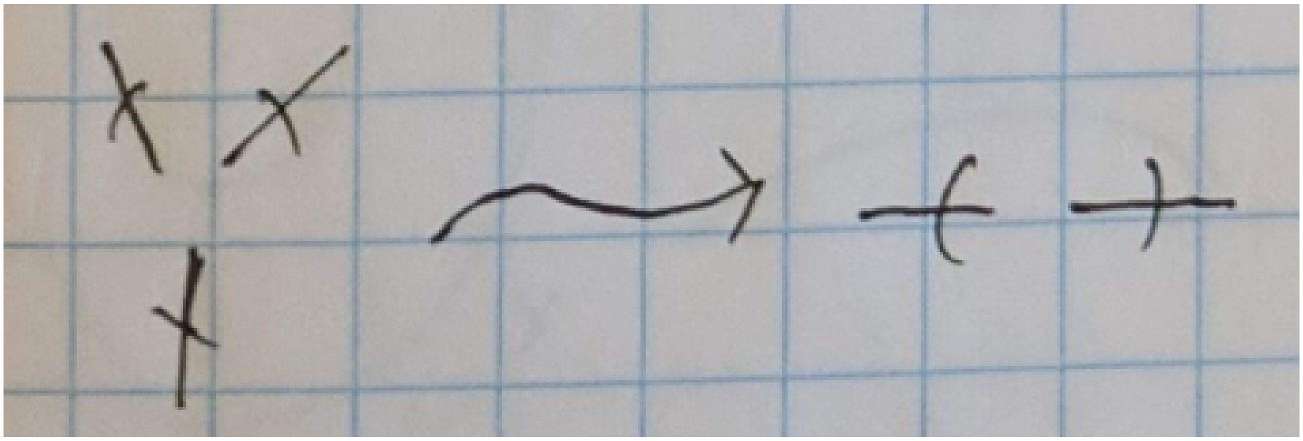
$\mathbb{T}^n = S^1 \times \cdots \times S^1$.

Counterexample 1

$[0, 1]$ is not a manifold.

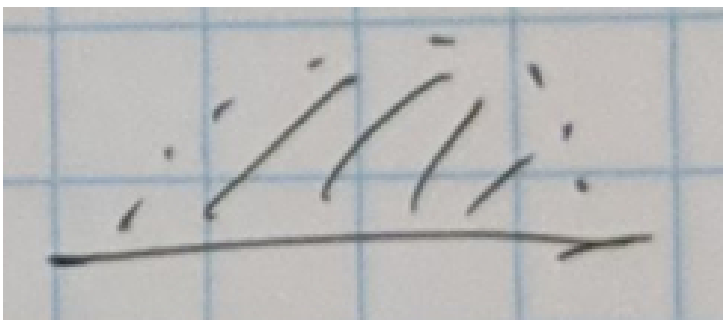


Since 0 must map somewhere in the open interval, its deletion results in a connected space in the former case but a disconnected one in the latter. Similarly, the following breaks into three and two connected components respectively.



Definition: Manifold with Boundary

There exists a neighborhood $\forall x \in M$ homeomorphic to either the open ball or the half-closed half-ball.



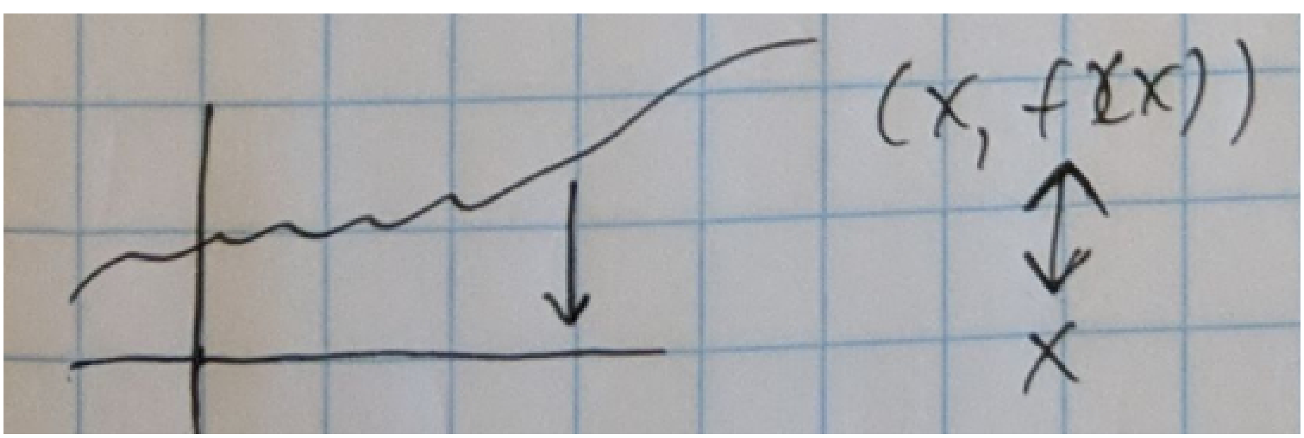
Exercise

A connected manifold is path-connected.

Examples

Example 7

Take $f : \mathbb{R}^n \xrightarrow{C^0} \mathbb{R}$ with graph $\Gamma_f = \{(x, f(x)) \, : \, x \in \mathbb{R}^n\} \subset \mathbb{R}^n \times \mathbb{R}$.

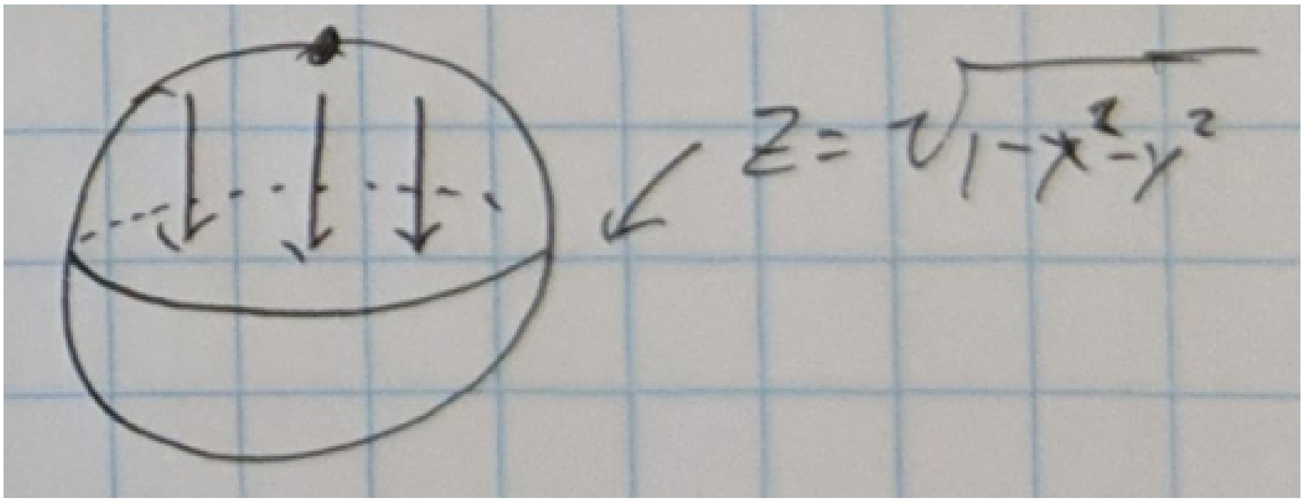


Example 8

Take $f : M \to N$ between manfiolds, then $M \simeq \Gamma_f \subseteq M \times N$.

Example 9

$S^n \subset \mathbb{R}^{n+1}$.

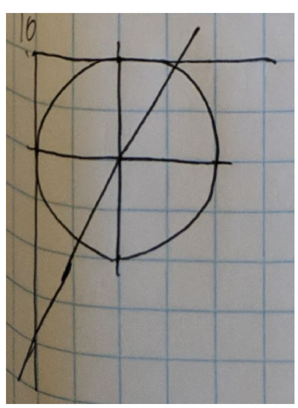


Definition: Real Projective Spaces

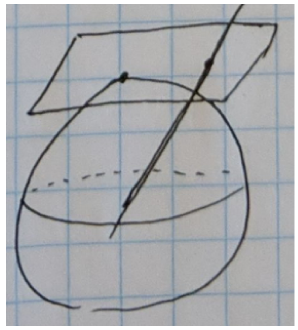
Take $\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim$ where $x \sim y \iff x = \lambda y$ for $\lambda \neq 0$.
Informally, the collection of lines through the origin.
Alternatively, $\mathbb{RP}^n = S^n / \sim$ where $x \sim -x$.
That is, identifiying the antipodal points of the unit sphere.
We may also consider $\mathbb{RP}^n = SO(n+1)/SO(n)$.

Claim

$\mathbb{RP}^n$ is a manifold.

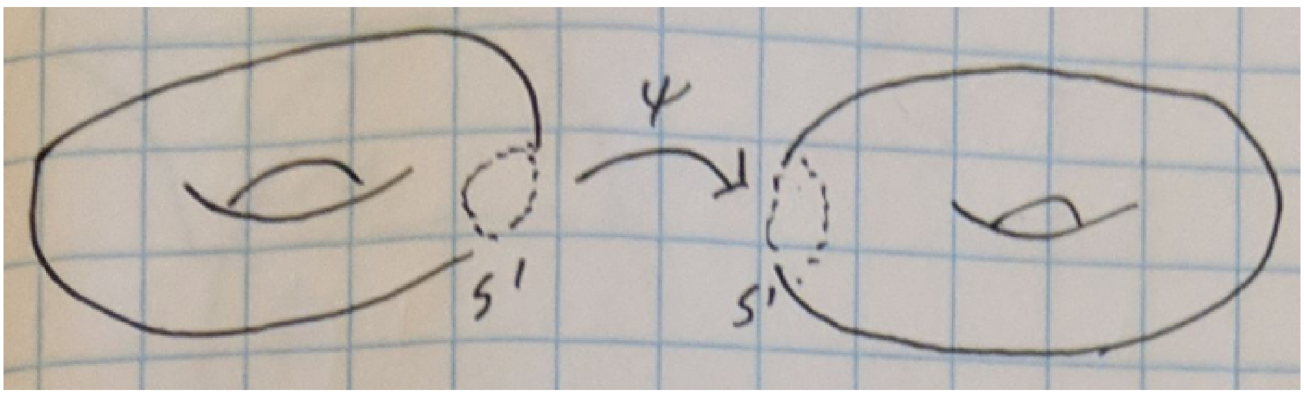


$\mathbb{RP}^1 \setminus \{x\text{-axis}\} \overset{\text{homeo}}{\to} \mathbb{R}$.

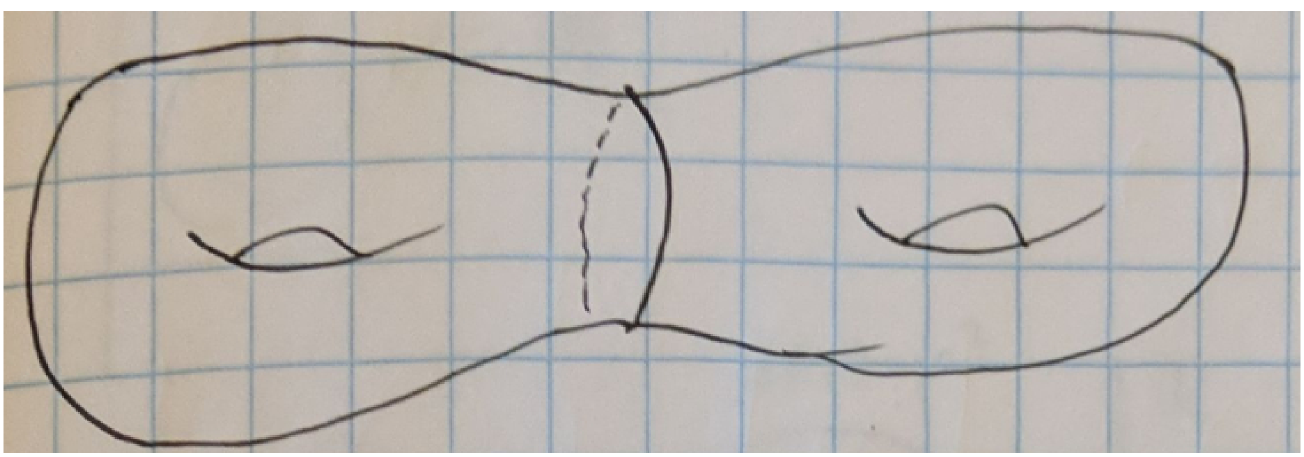


$\mathbb{RP}^2 \setminus \mathbb{RP}^1 \overset{\text{homeo}}{\to} \mathbb{R}^2$
We have that $\mathbb{RP}^1$ is homeomorphic to the circle, and $\mathbb{RP}^n = \mathbb{RP}^{n-1} \cup B^n$.
Take $x = (x_0, \dots, x_n)$, $y = (y_0, \dots, y_n) = (\lambda x_0, \dots, \lambda x_n)$ and $[x] = [x_0 : x_1 : \cdots : x_n]$.
Then for $U_k \subset \mathbb{RP}^n$ with $U_k = \{[x] \, : \, x_k \neq 0\}$, we have that $U_0, \dots, U_n$ covers $\mathbb{RP}^n$.
Then define $U_k \to \mathbb{R}^n$ by $[x_0 : \cdots : x_n] \to \left(\frac{x_0}{x_k}, \dots, \frac{x_k}{x_k}, \dots, \frac{x_n}{x_k}\right)$.

Connected Sum of Manfiolds

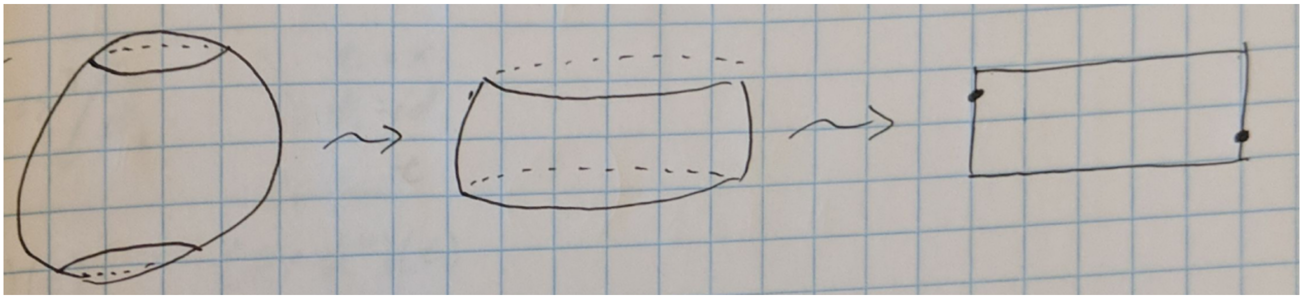


$M \setminus B^n \coprod N \setminus B^n$



$M \# N$.

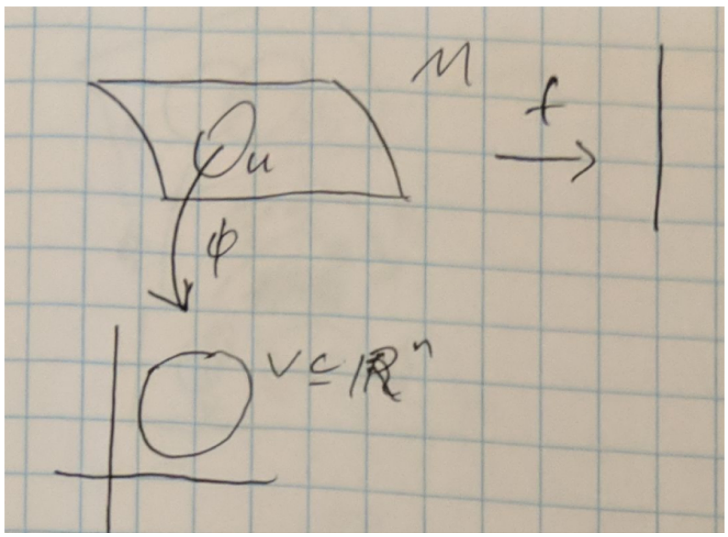
Mobius Band



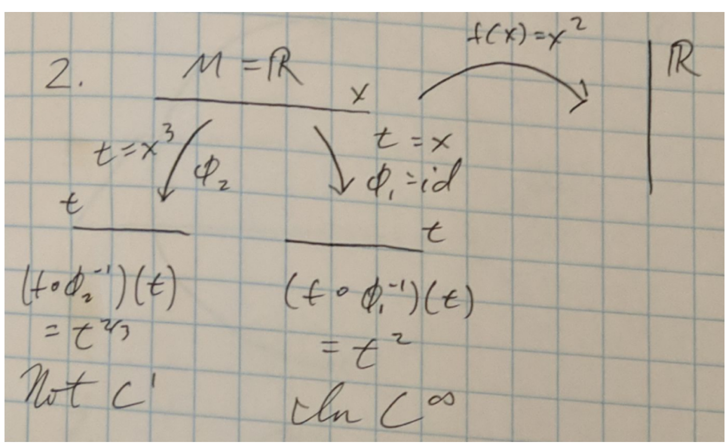
October 1, 2024

A Failed Definition

$f \in C^{r \geq 1}$; $f \circ \phi^{-1} : V \xrightarrow{C^r} \mathbb{R}$.



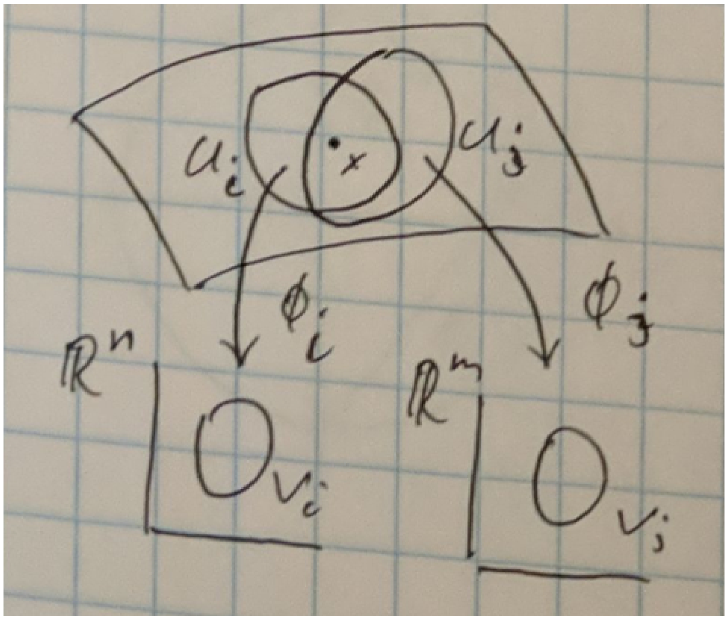
Example



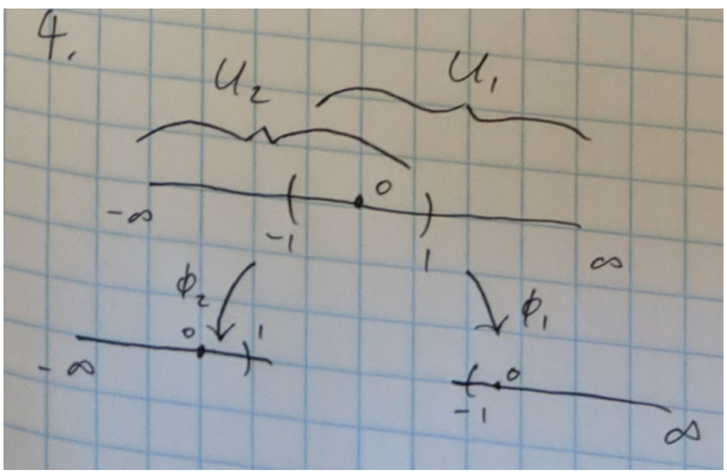
Definition: Charts

Say there exists a cover U_i by open sets and $U_i \xrightarrow{\phi_i} V_i \subseteq \mathbb{R}^n$ fixed.
Then the pair (U_i, ϕ_i) is a chart.

What if a point belongs to two charts?



With f smooth at x, $f \circ \phi_i^{-1}$ smooth at $\phi_i(x)$ and $f \circ \phi_j^{-1}$ smooth at $\phi_j(x)$.

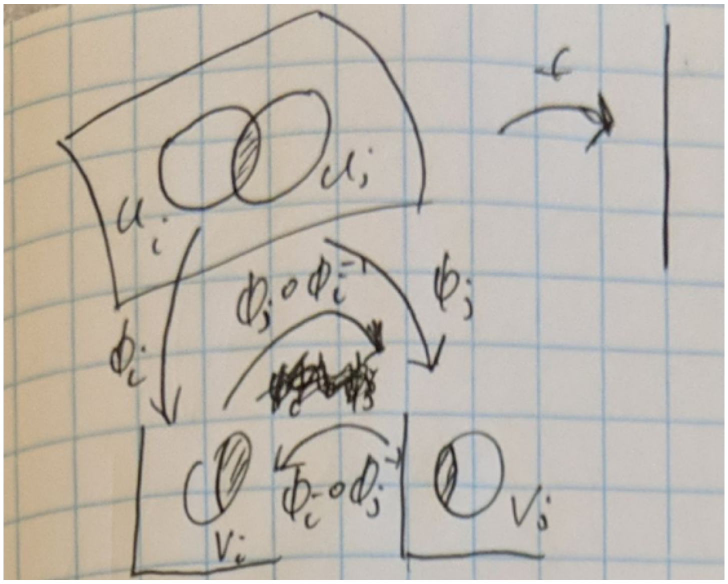


Notation

The notation C^r will be used interchangably with the term smooth.

Definition: Smooth Atlas

Let M be a topological manifold. A smooth atlas on M is a cover $(U_i, \phi_i : U_i \xrightarrow{\sim} V_i \subset \mathbb{R}^n)$ where $\phi_j \circ \phi_i^{-1}$ and $\phi_i \circ \phi_j^{-1}$ are smooth for every i and j.



Say that the charts are (smooth) compatible.

Definition: Smooth Function

Say that f is smooth at $x \in M$ if there exists a chart $U_i \ni x$ such that $f \circ \phi_i$ is smooth at $\phi_i(x)$.
Equivalently, if for every chart $U_i \ni x$ we have that $f \circ \phi_i$ is smooth at $\phi_i(x)$.

- Proof

$$f \circ \phi_j^{-1} = (f \circ \phi_i^{-1}) \circ \underbrace{(\phi_i \circ \phi_j^{-1})}_{C^r}$$

Definition: Compatibility (Equivalence) of Atlases

Atlases A_1 and A_2 are compatible or equivalent if every chart in A_1 is compatible with every chart in A_2.
Equivalently, $A_1 \cup A_2$ is also an atlas.

- Claim: This is an equivalence relation.

Example

Consider $\mathbb{R}$.
Atlas 1: $U = \mathbb{R}$ and $\phi = \mathrm{id}$.
Atlas 2: $U_1 = (1, \infty)$, $\phi_1 = (x) = x^2$, $U_2 = (-\infty, 2)$ and $\phi_2(x) = x$.

Definition: Diffeomorphism

$\mathbb{R}^n \supset V \xrightarrow{F} W \subset \mathbb{R}^n$ is a diffeomorphism if

- F is C^r,
- F is invertible, and
- F^{-1} is C^r

Counterexample

$y = x^3$ is a smooth homemorphism but not a diffeomorphism.

Definition: Smooth Structure / Maximal Atlas

Given an atlas, we may take all compatible atlases and define a smooth structure by the union of all such objects (i.e. the maximal atlas).

Lemma:

Every smooth manifold has a countable, locally finite atlas of precompact charts.

Examples

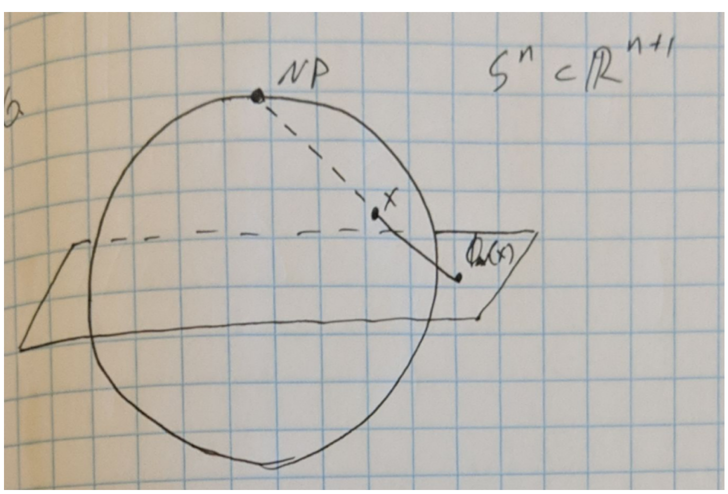
- Zero dimensional manfiolds (i.e. a point).
- $\mathbb{R}^n$ and open subsets of $\mathbb{R}^n$.
- If M, N are smooth manfiolds, then $M \times N$ is a a smooth manfiold.

  That is, if we have atlases (U_i, ϕ_i) and (W_j, ψ_j), we may generate $(U_i \times W_j, \phi_i \times \psi_j)$.
- Take $F : M \overset{\text{homeo}}{\to} N$ with N a smooth manfiold. Then M is smooth.

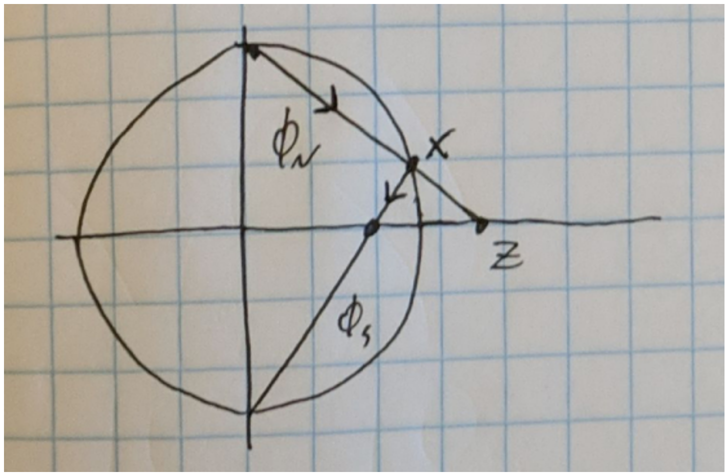
  Take an atlas A on N and the pullback $F^{-1}A = \{(F^{-1}(U_i), \phi_i \circ F)\}$.
- An open subset of a smooth M is a smooth manifold.
- $\mathrm{GL}(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$.

The n-Sphere

- S^n is a manifold



$$U_N = S^n \setminus NP \overset{\phi_N}{\to} \mathbb{R}^n$$
$$U_S = S^n \setminus SP \underset{\phi_S}{\to} \mathbb{R}^n$$



$\phi_S\phi_N^{-1}(z) = \frac{z}{|z|^2}$.

- A different construction for S^n.

  Take hemispheres $U \xrightarrow{\text{orthogonal projection}} B^n$.

Projective Space

$\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus 0/\sim$ where $x \sim \lambda x$ for $\lambda \neq 0$.
$[x] = [x_0 : x_2 : \cdots : x_n] = [\lambda x_0 : \lambda x_2 : \cdots : \lambda x_n]$.
Take $U_i = \{x_i \neq 0\}$ and open cover, and maps $U_i \to \mathbb{R}^n$ given by $[x_0 : \cdots : x_n] \mapsto \left(\frac{x_0}{x_i}, \ldots, \frac{x_i}{x_i}, \ldots, \frac{x_n}{x_i}\right)$. Then for $j < i$ take

$$\phi_j\phi_i^{-1}(y_1, \ldots, y_n) = \left(\frac{y_0}{y_j}, \ldots, \frac{y_{j-1}}{y_j}, \frac{y_{j+1}}{y_j}, \ldots, \frac{y_{i-1}}{y_1}, \frac{1}{y_i}, \frac{y_i}{y_i}, \ldots, \frac{y_n}{y_1}\right)$$

Definition: Diffeomorphism

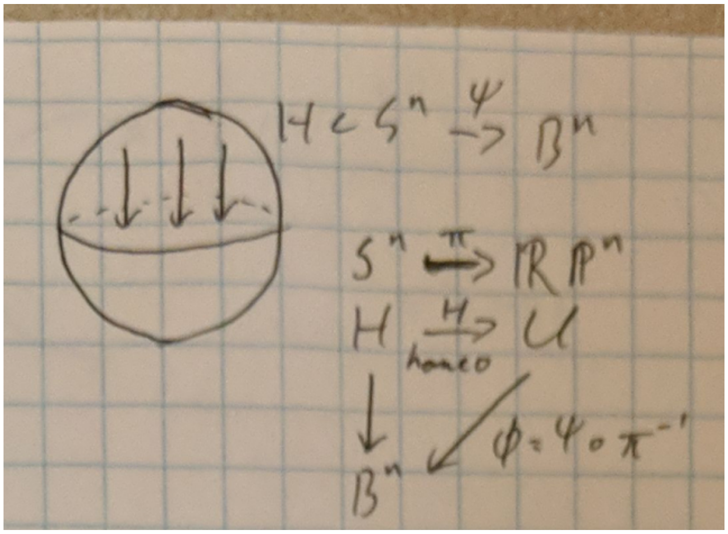
$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ B \subset B_{\text{max}} & & A \supset A_{\text{max}} \end{array}$$

F is a diffeomorphism if F is a homoeomorphism and $F^{-1}A_{\text{max}} = B_{\text{max}}$ ($F^{-1}A \sim B$).

October 3, 2024

Recall

$$\mathbb{RP}^n = \begin{cases} \mathbb{R}^{n+1} \setminus 0/\sim & x \mapsto \lambda x \\ S^n/x \sim -x \end{cases}$$



Note

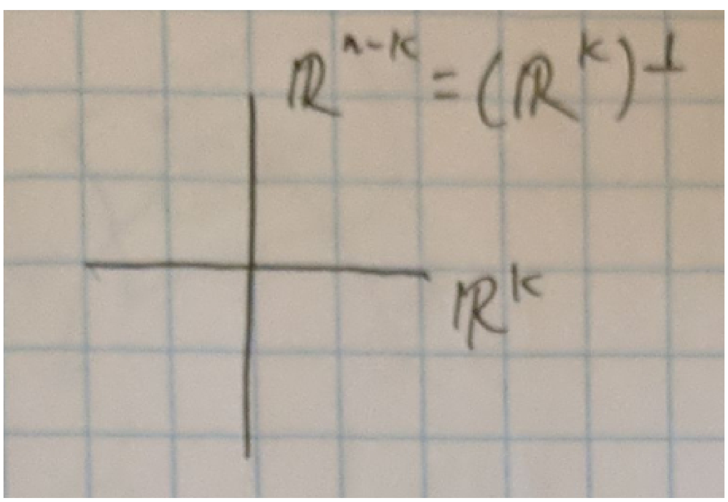
Given a manifold M and A a smooth atlas, we generate a continuum of smooth atlases not equivalent to each other. That is, given $M \underset{\text{homeo}}{\overset{F}{\to}} M$, $F^{-1}A \nsim A$.

Confer With Groups

$G \overset{F}{\to} G$, $a * b = F^{-1}(F(a)F(b))$.

Definition: Grassmannians

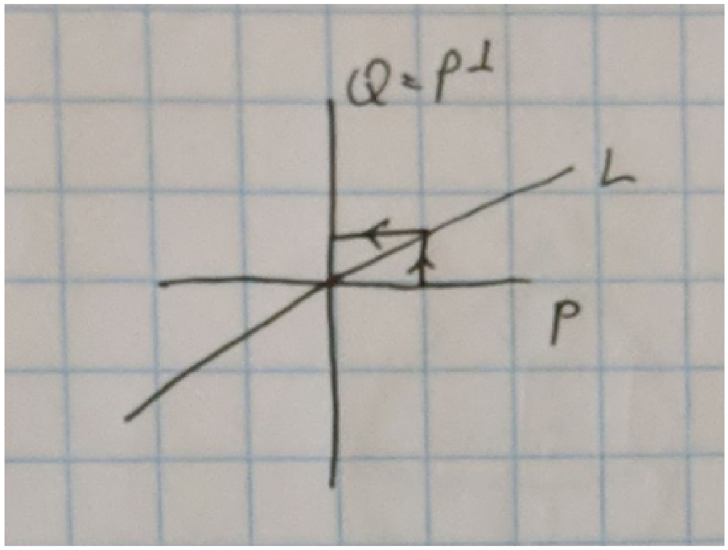
Write $G_k(n)$, the collection of all k-dimensional subspace L in $\mathbb{R}^n$.



Observe that if $O(i)$ is the collection of orthogonal transformations in dimension i,

$$G_k(n) = \frac{O(n)}{O(k) \times O(n-k)}$$

with $X \sim Y$ when $Y = XA = X(O(K) \times O(n-k))$.

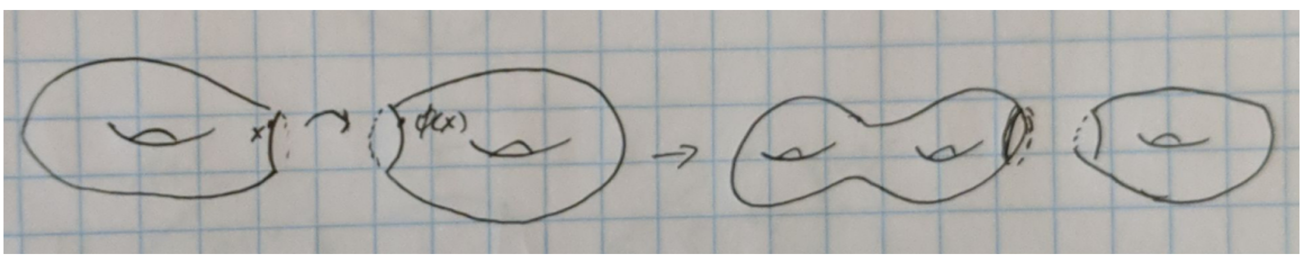


Where $\dim(L) = k$, $U_p = \{L : L \cap Q = \{0\}\}$, $L = \mathrm{graph}(A : P \to Q)$, and we have a homemorphism

$$U_p \xrightarrow{\phi} \underbrace{\{\text{linear maps } P \to Q\}}_{\mathbb{R}^{k \times (n-k)}}.$$

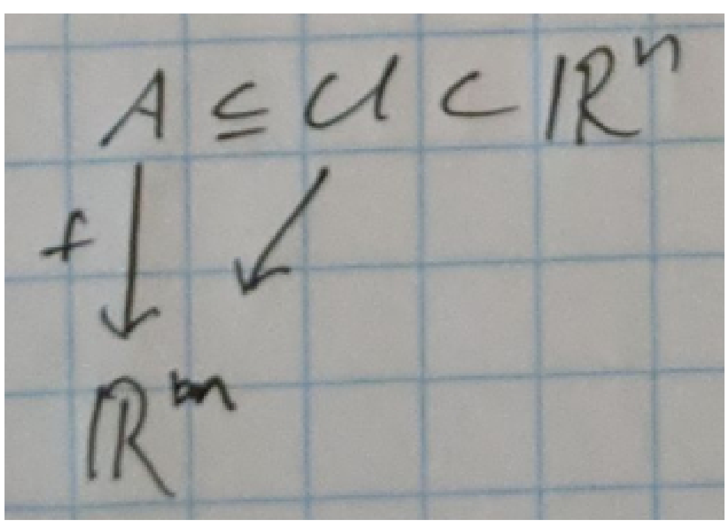
Surfaces

We have explored S^2, $\mathbb{RP}^2$, $\mathbb{T}^2 = S^1 \times S^1$. We have also connected sums.



Terminological Remark

Let $\mathbb{R}^N \supset A \xrightarrow{f} \mathbb{R}^m$.
Then f is smooth if it extends to a smooth map



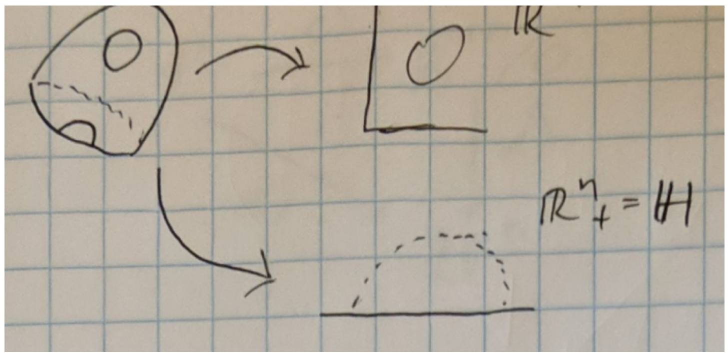
Exercise

Let $A = [0, \infty) \subset \mathbb{R}$. $f : A \to \mathbb{R}$ is smooth if and only if it is infinitely differentiable.
Construct $(-\varepsilon, \infty)$.

Definition: Smooth Manifold with Boundary

A smooth manifold with boundary is a topological space along with an atlas A with charts of two types

$$\phi : U \to B^n \quad \text{(open ball)}$$
$$\phi : U \to B^n \cap H$$



As before, $\phi_i \circ \phi_j^{-1}$ must be smooth.

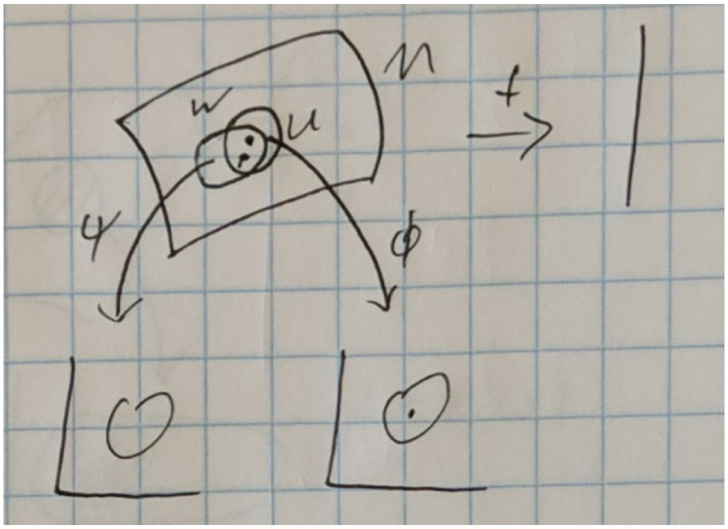
Examples

- $M \backslash$open ball.
- The upper half space.

Definition-Lemma: Smooth Function

A function $f : M \to \mathbb{R}$ is smooth at $p \in M$ if either of the following equivalent conditions is satisfied

1. $\exists$ a chart $(U, \phi) \ni p$ such that $f \circ \phi^{-1}$ is smooth at $\phi(p)$.
2. $\forall$ a chart $(U, \phi) \ni p$ such that $f \circ \phi^{-1}$ is smooth at $\phi(p)$.



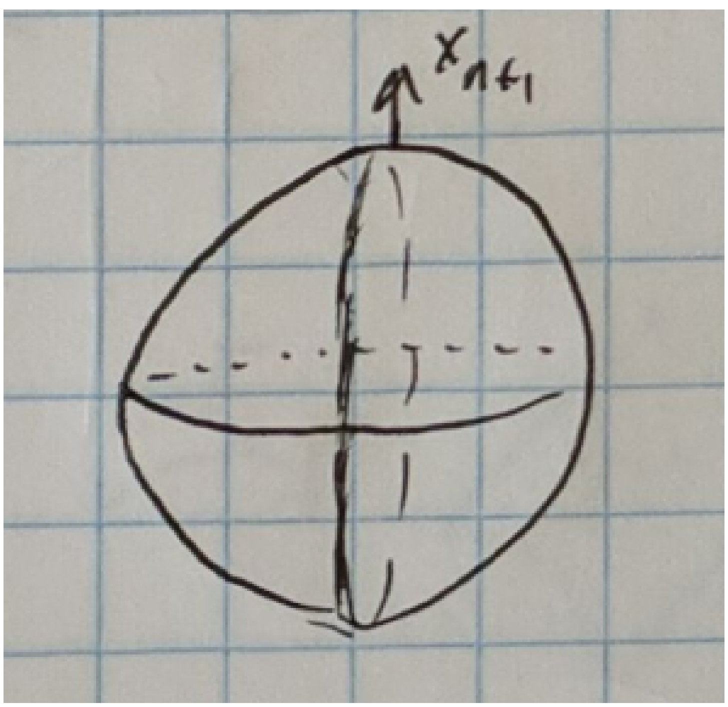
Where $f \circ \phi^{-1} = f \circ \psi^{-1}(\psi \circ \phi^{-1})$.
If the above hold for each $p \in M$, then f is smooth.

Remark

f smooth implies f is C^0

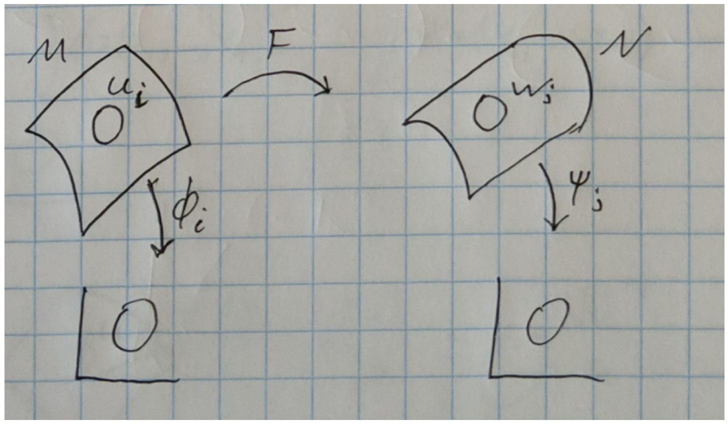
Exercise / Sketch

The height function on S^n is smooth.



$\phi : (x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_n)$.
$f \circ \phi^{-1} = \pm\sqrt{1 - x_1^2 - \cdots - x_n^2}$.
Note that handling the equator requires examining the Eastern and Western hemisphere.
The stereographic projection leads to a simpler proof.

Definition: Smooth Function Between Manifolds



$F: M \to N$ is smooth if F is C^0 and one of the following equivalent conditions is satisfied

1. $\exists$ an atlas $A \subset A_{\max}$ on M and an atlas $B \subset B_{\max}$ on N such that $\psi_j \circ F \circ \phi_i^{-1}$ is smooth on $F^{-1}(W_j) \cap U_i$.

2. The same as a., but for $A_{\max}$ and $B_{\max}$.

Consider as an example $S^n \to \mathbb{RP}^n$.

Properties of Smooth Maps

$C^\omega \implies C^\infty \implies C^r \implies C^{r-1} \implies C^1 \implies C^0$.
The sum and product of smooth functions is smooth.

Exercise

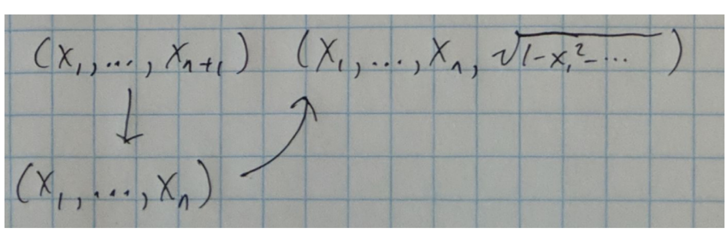
The composition of smooth maps is smooth.

Examples

1. $M \times N \overset{pr}{\to} M$ is smooth.

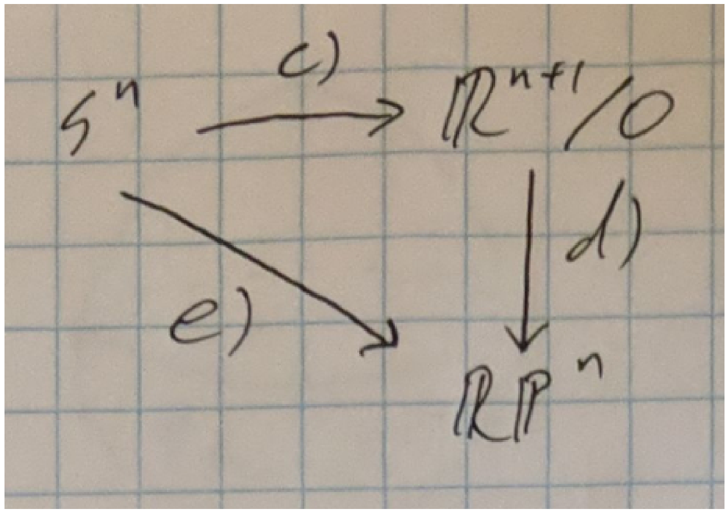
2. $\underbrace{M \overset{(F_1,F_2)}{\to} N_1 \times N_2}_{\text{smooth}}$ if and only if F_1 and F_2 are smooth.

3. $S^n \hookrightarrow \mathbb{R}^{n+1}$ is smooth.



1. $\mathbb{R}^{n+1} \setminus 0 \to \mathbb{RP}^n$ is smooth with $[x_0 : \cdots : x_n] \mapsto \left(\frac{x_0}{x_i}, \ldots, \frac{\hat{x}_i}{x_i}, \ldots, \frac{x_n}{x_i}\right)$.

2. $S^n \to \mathbb{RP}^n$.



Definition: Diffeomorphism

$F: \underset{A_{\max}}{M} \xrightarrow{\text{diffeo}} \underset{B_{\max}}{N}$ if

- F is smooth.
- F is invertible.
- F^{-1} is smooth.

Previous Definition

$F^{-1}(B_{\max}) = A_{\max}$.

Exercise

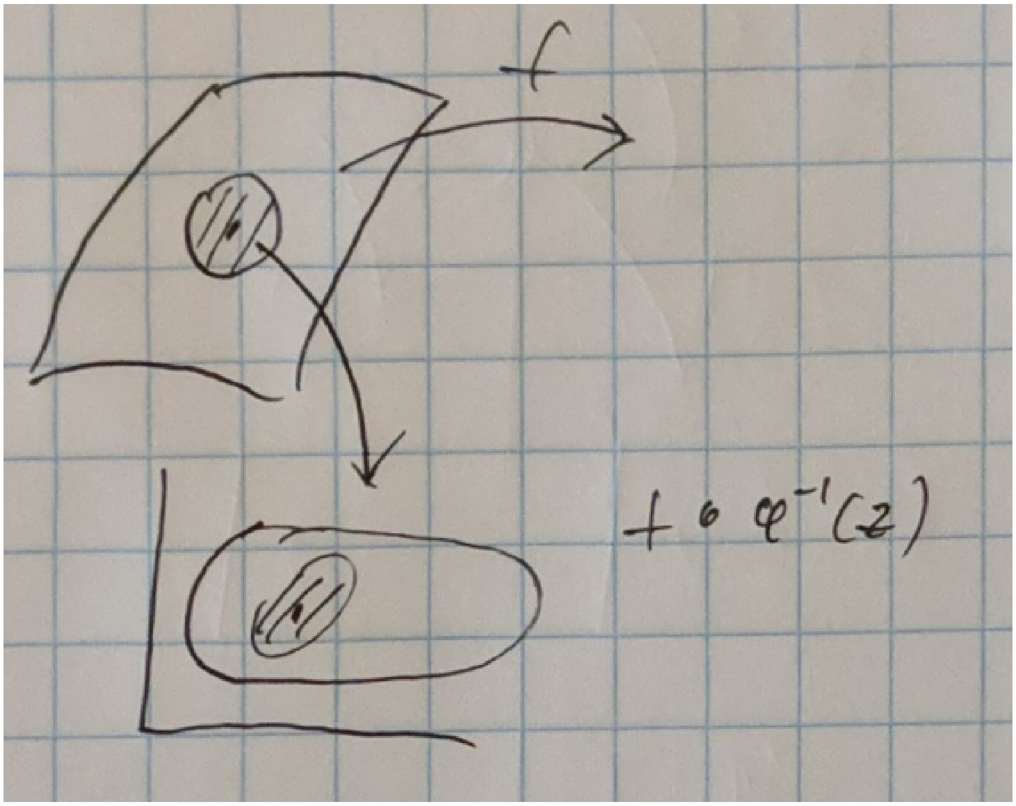
Prove that the definitions are equivalent.

Examples

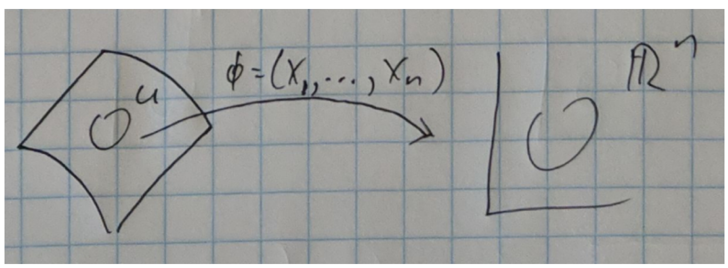
1. $\tan: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \xrightarrow{\text{diffeo}} \mathbb{R}$.
2. $x \mapsto x^3$, $\mathbb{R} \to \mathbb{R}$ is not a diffeomorphism.
3. $G_k(n) \leftrightarrow G_{n-k}(n)$ with $P \leftrightarrow P^{\perp}$.

Example 4

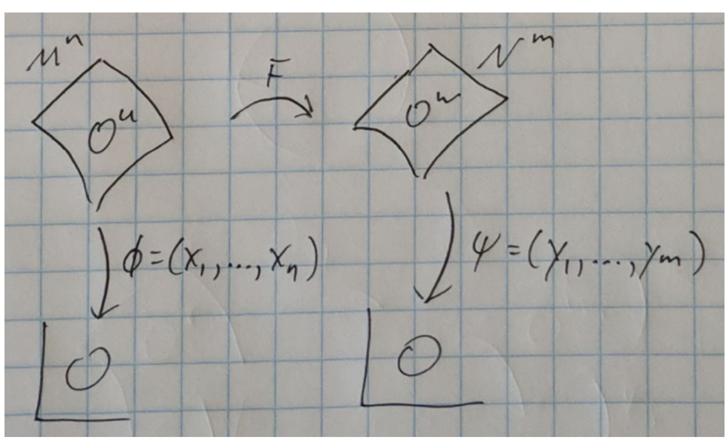
A compact, analytic manifold admits only constant smooth functions by the maximum modulus principle.



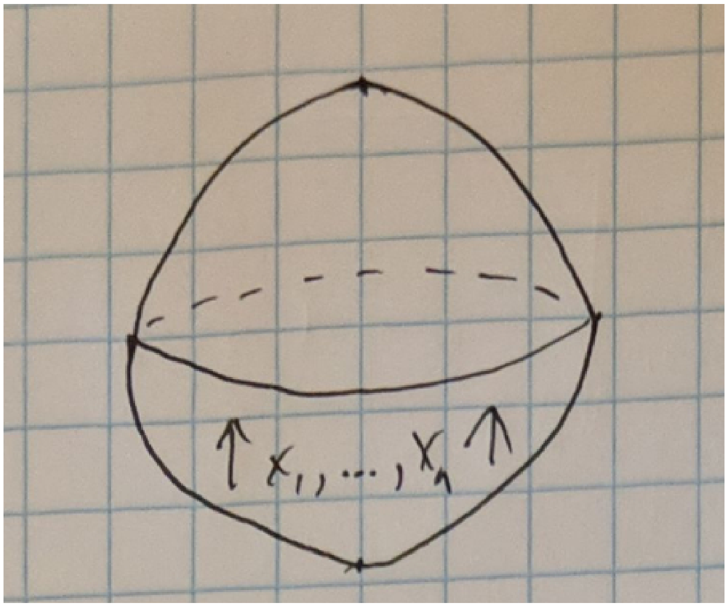
Example 5



Where $\phi = (x_1, \ldots, x_n)$ and each x_i is a real-valued function.



$\psi \circ F \circ \phi^{-1} = (y_1(x_1, \ldots, x_n), \ldots, y_m(x_1, \ldots, x_n))$.
Then $S^n \hookrightarrow \mathbb{R}^{n+1}$



where $y_1 = x_1, \dots, y_n = x_n$, and $y_{n+1} = -\sqrt{1 - x_1^2 - \cdots - x_n^2}$.

Example 6

$\mathbb{R}^{n+1} \setminus 0 \xrightarrow{F} \mathbb{RP}^n$.
Need to check that $\psi_j \circ F$ is smooth.
$[t_0 : \cdots : t_n] \xrightarrow{\psi_0} \left(\frac{t_1}{t_0}, \dots, \frac{t_n}{t_0}\right)$ with $U_0 : t_0 \neq 0$

October 8, 2024

Questions

Question 1

Given M smooth and $x \neq y \in M$, does there exist $f \in C^r(M)$ such that $f(x) = 0$ and $f(y) = 1$.

Question 2

Given $K \subset U \subset M$ with K compact and U open and $g : K \overset{C^r}{\to} \mathbb{R}$, does there exist a C^r extension f of g on M such that $\operatorname{supp} f \subset U$.

Definition: Partitions of Unity

Let W_i be a locally finite open cover.
A partition of unity subordinated to W_i, is a collection of functions $f_i : M \overset{C^r}{\to} \mathbb{R}$ satisfying

- $0 \leq f_i \leq 1$
- $\operatorname{supp} f_i \subset W_i$
- $\sum f_i \equiv 1$

Definition: Refinement

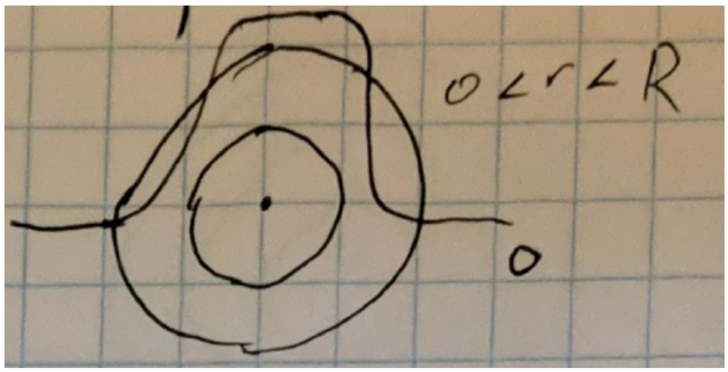
Given covers U_j and W_i, U_j is a refinement if for each j we may find i such that $U_j \subset W_i$.

Theorem

There exists a partition of unity subordinated to W_i.

Lemma 1

Take $B(r) \subset B(R) \subset \mathbb{R}^n$.



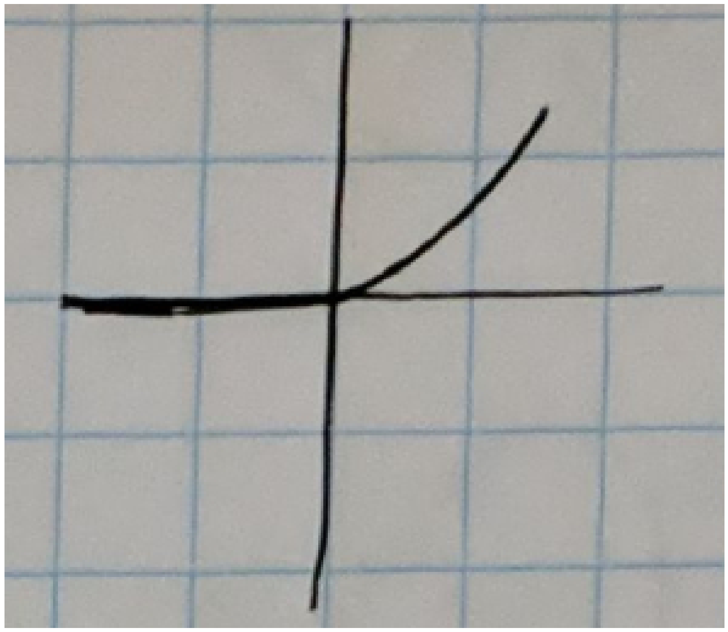
Then there exists $g \in C^\infty$ such that

- $0 \leq g \leq 1$
- $g|_{\overline{B(r)}} \equiv 1$
- $\operatorname{supp} g \subset B(R)$

Proof

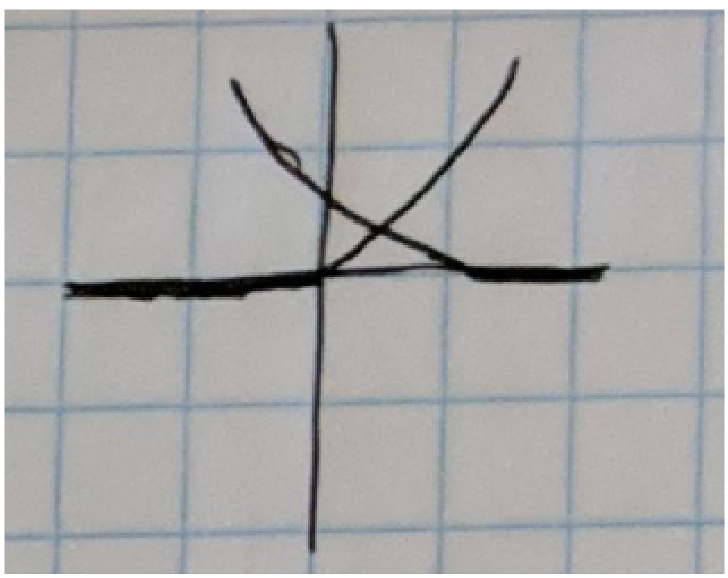
Take

$$h_0(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$



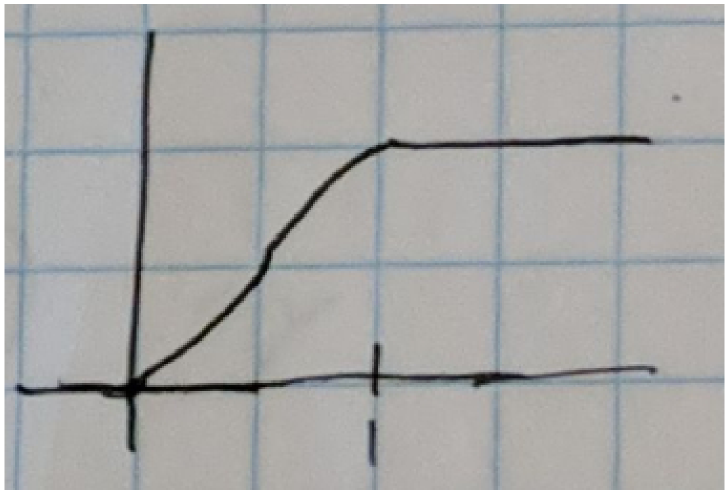
- Exercise: show that $h_0 \in C^\infty(\mathbb{R})$ (Hint: show that derivatives agree at zero from the right.)

Then take $h_1(x) = h_0(x) \cdot h_0(1-x)$.

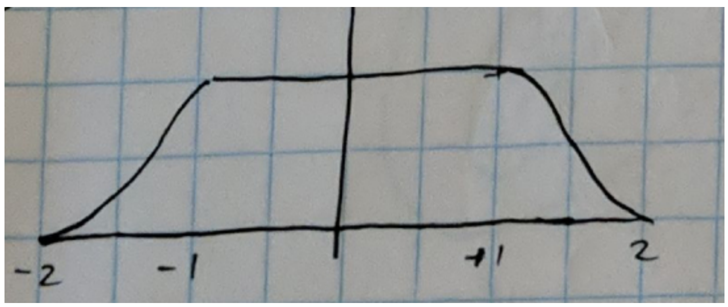


Then define

$$h_2(x) = \frac{1}{\int_{-\infty}^{\infty} h_1(t)\,dt} \int_{-\infty}^{x} h_1(t)\,dt$$



Finally, we define $h(x) = h_2(x+2) \cdot h_2(2-x)$.



Then $g(x) = h(||x||)$ satisfies our requirements.

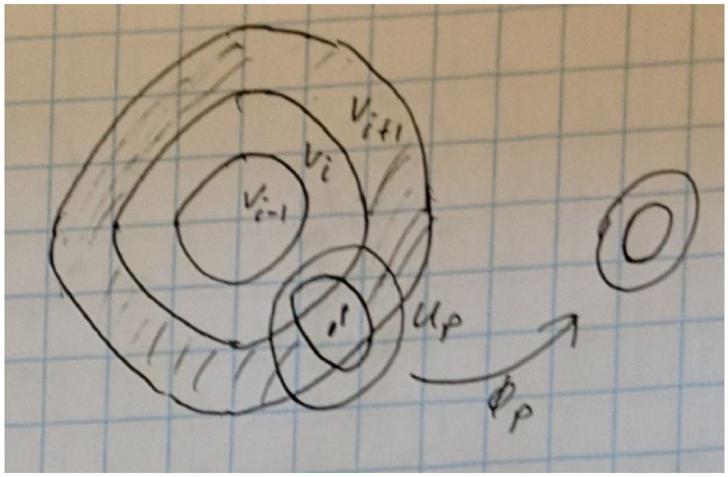
Lemma 2

There exists a refinement (U_j, ϕ_j) by coordinate charts such that

1. U_j is locally finite.
2. $\phi_i : U_j \xrightarrow{\text{diffeo}} B^n(2)$.
3. $\phi_j^{-1}(B(1))$ is also a cover.

Proof

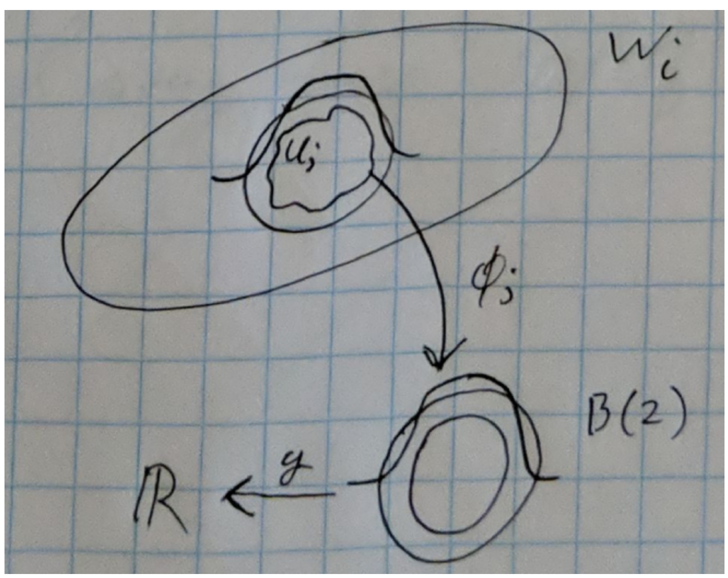
There exists a compact exhaustion of M, $C_1 \subset C_2 \subset C_3 \subset \cdots$ where $\bigcup C_i = M$.
There exists also an open exhaustion by precompact open sets $\varnothing = V_0 \subset V_1 \subset V_2 \subset \cdots$ where $\overline{V_i} \subset V_{i+1}$.
Then define $V_i := \mathrm{nbhd}(C_i \cup \overline{V}_{i-1})$.



Take $A_i = \overline{V}_{i+1} \setminus V_i$ compact, $p \in A_i$. Then we have a map $U_p \xrightarrow{\phi_p} B(2)$.

- $U_p \subset W_i$ for some i. (Refinement)
- $U_p \cap V_{i-1} = \varnothing$. (Locally Finite)
- There exists a finite subcover such that $\phi_p^{-1}(B(1))$ is also an open cover.

Proof of Theorem



Set $g_j = g \circ \phi_j \in C^r$, extended to 0 outside of U_j.

- $\operatorname{supp} g_j \subset U_j$
- $\forall p \in M, \exists g_j(p) = 1 \neq 0 \implies \sum g_j > 0$
- $\operatorname{supp} g_j$ is locally finite.

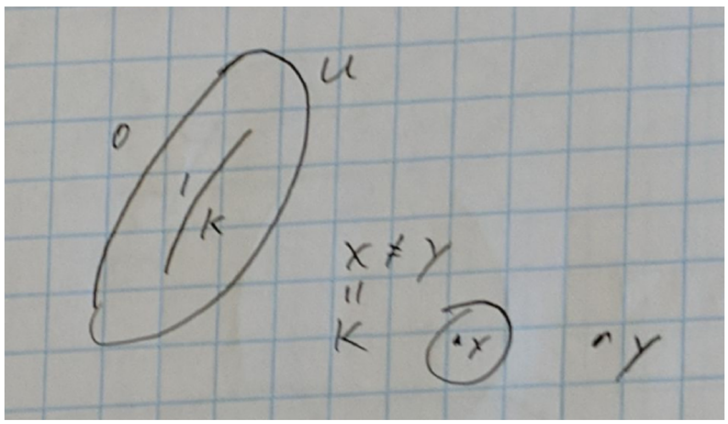
Then we have

$$f_j = \frac{g_j}{\sum g_j}$$

Corollary 1

For $K \subset U$, K compact and U open, there exists $h \in C^r(M)$ such that

- $h|_K \equiv 1$
- $\operatorname{supp} h \subset U$



Proof

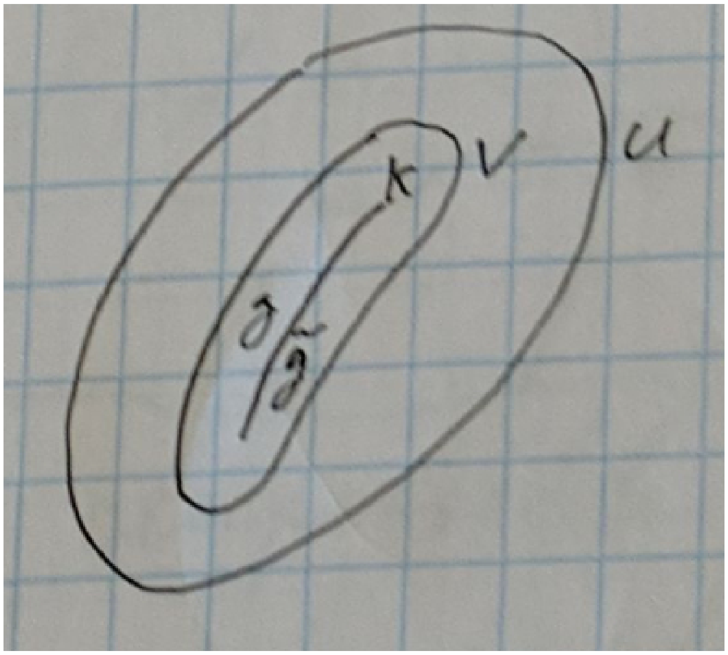
Take $U_0 = U$ and $U_1 = M \setminus K$.
Then there exists a partition of unity f_0 and f_1 where $f_0 + f_1 = 1$. Therefore f_0 has support in U and f_1 has support in $M \setminus K$ this occurs if and only if $f_1|_K = 0$.

Corollary 2

For $K \subset U$, K compact and U open, and $g : K \overset{C^r}{\to} \mathbb{R}$, there exists an extension $f : M \to \mathbb{R}$ of g: $f|_K = g$ such that $\operatorname{supp} f \subset U$.

Proof

For $g \in C^r(K)$, there exists a neighborhood V of K and a function $\tilde{g} : V \to \mathbb{R}$ such that $\tilde{g}|_K = g$.



By corollary 1, there exists $h \in C^r(M)$ such that

- $\operatorname{supp} h \subset V$
- $h|_K = 1$

Therefore $f = h \cdot \tilde{g}$.

Corollary 3

There exists $f : M \overset{C^r}{\to} \mathbb{R}$ bounded from below and proper.

Definition: Proper Function

f is proper if and only if $f^{-1}(K)$ is compact for K compact.

Proof

See Textbook.

Consequence

Take $\{x \, : \, f(x) \leq C_i\} =: E_i$ as $i \to \infty$. Then we get a compact exhaustion $E_1 \subset E_2 \subset E_3 \subset \cdots$.
e.g. $f(x) = ||x||^2$.

October 10, 2024

Algebra = Analysis (Geometry)

Take M to be either a compact metrizable space ($A = C^0(M)$) or a copact manifold ($A = C^\infty(M)$).

$$M \leftrightarrow A$$

Let $I \subset A$ be an ideal and take $V(I) = \{x : f(x) = 0, \forall f^n \in I\}$.
Take also $Y \subset M$ closed and consider $I_Y = \{f : f|_Y = 0\}$ which is also an ideal.

$$Y \to I_y$$
$$V(I) \leftarrow I$$

Denote $I_x = \{f : f(x) = 0\}$.

Theorem

- I_x is a maximal ideal.
- Every maximal ideal is of this form, and x is unique.

$$M \leftrightarrow \text{Maximal Ideals of } A$$

Proof

Maximal Ideal

Take I_x and $g \notin I_x$.
We want to show that g with I_x generates A.
Then take $f \in A$ defined as $f = h + ag$ for some $h \in I_x$.
Since $g \notin I_x$, $g(x) \neq 0$. Take $a = \frac{f(x)}{g(x)}$ and define $h := f - ag$.
Then $f(x) - \frac{f(x)}{g(x)} g(x) = 0 \in I_x$.

Every Maximal Ideal

Let I be a proper ideal such that $V(I) \neq 0$, $\exists x$ such that $f(x) = 0$, $\forall f \in I$.
Maximal $\implies V(I) = \{x\}$.
By contradiction, assume not: $\forall x \in M$, $\exists f_x \in I$, $f_x(x) \neq 0$.
It follows that $f_x|_{U_x} \neq 0$ where $U_x \ni x$ is a neighborhood from an open cover.
Then, by compactness, we have a finite subcover
$I \ni f_i = f_{x_i} \neq 0$ on U_i.
$I \ni \underbrace{\sum f_i^2}_{g>0} > 0$ on M. But then $1 = g^{-1} g \in I$.

Uniqueness

$I_{x_1} = I_{x_2} \iff x_1 = x_2$
($\Longleftarrow$) is obvious.
($\Longrightarrow$) $x_1 \neq x_2$ implies that there exists $f \in A$ such that $f(x_1) = 1$ and $f(x_2) = 0$.
Then $f \in I_{x_2}$ while $f \notin I_{x_1}$ implies that $I_{x_1} \neq I_{x_2}$.

Reading the Topology / Smooth Structure

We have a correspondence

$$\text{closed sets} \leftrightarrow \text{ideals}$$

Algebra Homomorphisms

Take $\phi : A \to \mathbb{R}$. Then $\ker\phi$ is a maximal ideal, and

$$\begin{aligned} A/I_x &\xrightarrow{\delta_x} \mathbb{R} \\ x &\mapsto \pm f(x) \end{aligned}$$

Then

$$\begin{aligned} M \leftrightarrow \text{Alg. Hom. } A &\to \mathbb{R} \\ 1 &\mapsto 1 \\ A > 0 &\mapsto \text{pos. \#} \end{aligned}$$

Counterexample

Take instead $M = \mathbb{R}$. Claim: there exist a maximal ideals other than I_x.

Proof

Take J to be the ideal of all compactly supported functions such that $J \subset M$ for some maximal ideal M. However, $\forall x \in \mathbb{R}$ there exists a compactly supported function $f(x) \neq 0$. So $M \neq I_x$.

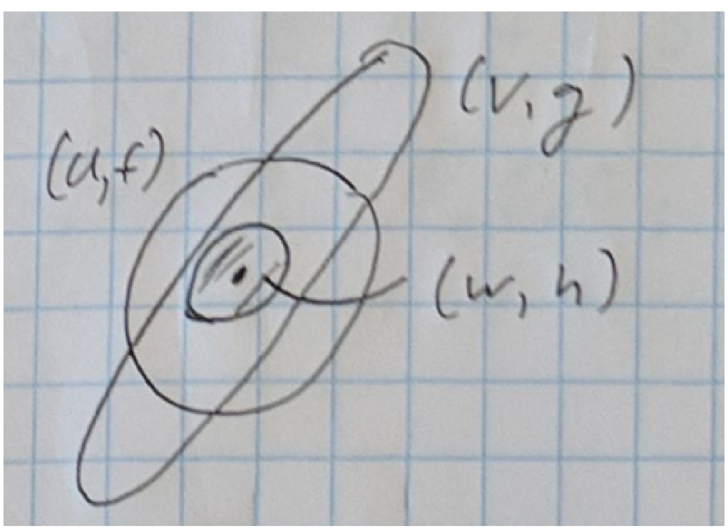
Functorial Considerations

Take $F : M \to N$, M and N compact. Then

$$\begin{aligned} F &: M \to N \\ C^{0/\infty}(M) &\underset{F^A}{\leftarrow} C^0(N) \\ f \circ F &\leftarrow\!\shortmid f \end{aligned}$$

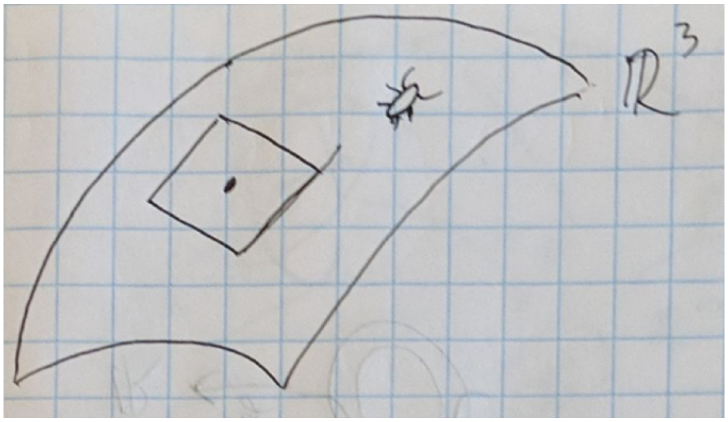
Definition: Germ

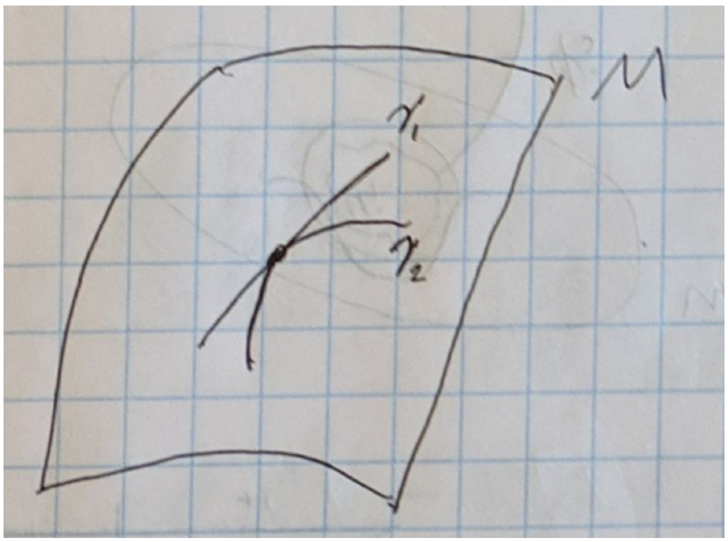
Take $M \ni P$.



Where $(U, f) \sim (V, g)$ if $f \equiv g$ on some neighborhood of p.

Definition: Tangent Spaces



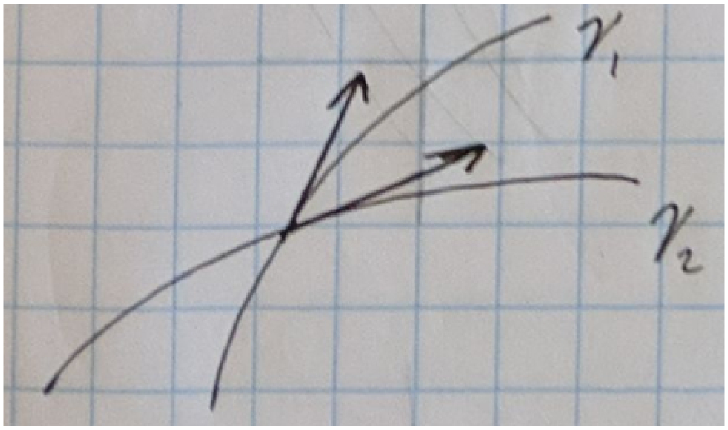


With $\gamma_1 : (-\varepsilon, \varepsilon) \to M$, $\gamma_2(-\varepsilon, \varepsilon) \to M$, and $\gamma_1 \sim \gamma_2 \iff \gamma_1'(0) = \gamma_2'(0)$.
$\gamma(t) = ((x(t), y(t), z(t))$ and $\gamma'(0) = (x'(0), y'(0), z'(0))$.
The tangent space to M at p, written T_pM, is the set of equivalence classes.

Remarks: Tangent Vectors

Take $V \ni p$ a finite-dimensional vector space ($= \mathbb{R}^n$).

$$0 \mapsto p$$
$$\gamma_1, \gamma_2 : (-\varepsilon, \varepsilon) \to V \quad \text{(germs)}$$
$$\gamma_1' \sim \gamma_2' : \gamma_1'(0) = \gamma_2'(0)$$



Write

$$\gamma'(0) = \lim_{t\to 0} \frac{\gamma(t) - \gamma(0)}{t}$$

$\gamma = (x_1, \ldots, x_n)$ and $\gamma'(0) = (x_1'(0), \ldots, x_n'(0))$.
A tangent vector to V at p is an equivalence class T_pV.
Claim: T_pV is a vector space.

Operations

Take $p = 0$. Write

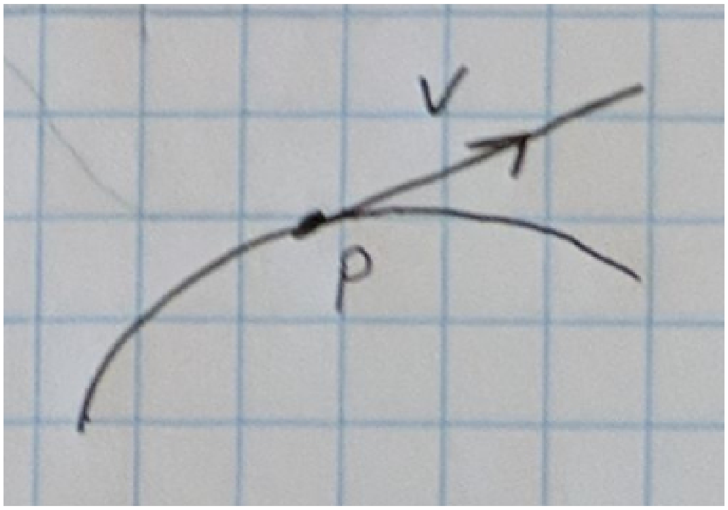
$$[\gamma_1] + [\gamma_2] = [\gamma_1 + \gamma_2]$$
$$\lambda[\gamma] = [\lambda\gamma]$$

When $p \neq 0$, instead

$$[\gamma_1] + [\gamma_2] = [\gamma_1 + \gamma_2 - p]$$
$$\lambda[\gamma] = [\lambda\gamma + (1-\lambda)p]$$

Claim: T_pV is canonically isomorphic to V

$$[\gamma] \mapsto \gamma'(0)$$
$$[\gamma = (x_1, \ldots, x_n)] \mapsto (x_1'(0), \ldots, x_n'(0))$$
$$T_pV \to V$$
$$p + tv \leftarrow\!\shortmid v$$



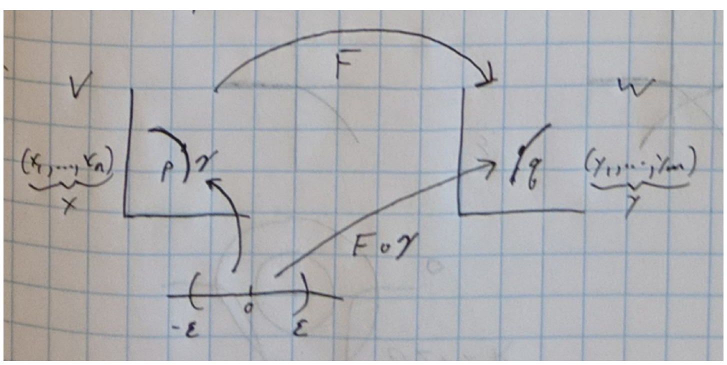
Proposition

Take

$$\underset{\mathbb{R}^n}{V} \xrightarrow{C^r} \underset{\mathbb{R}^m}{W}$$
$$p \mapsto q$$

Then

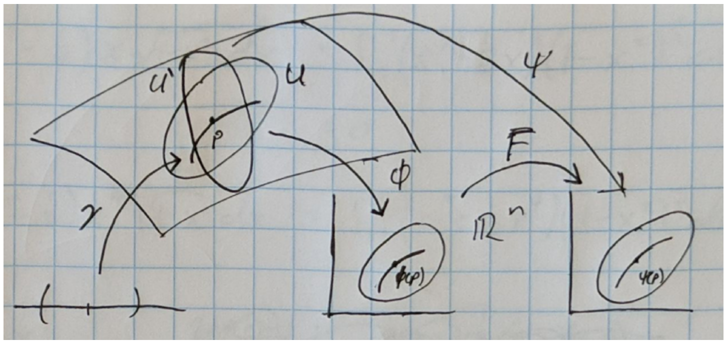
$$V \xrightarrow{\sim} W$$
$$DF_p = F_* : T_pV \to T_qW$$
$$[\gamma] \mapsto [F \circ \gamma]$$



Then F is well-defined and linear.
$F = (F_1, \dots, F_m)$ and $F \circ \gamma(F_1(\gamma_1, \dots, \gamma_n), \dots, F_m(\gamma_1, \dots, \gamma_n)$.
We have that $[\gamma] = \gamma'(0)$ and $[F \circ \gamma] = \frac{d}{dt} F \cdot \gamma(t)|_{t=0}$. By chain rule,

$$\frac{d}{dt}(F \circ \gamma)|_{t=0} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{pmatrix} \begin{pmatrix} \gamma_1'(0) \\ \vdots \\ \gamma_n'(0) \end{pmatrix}$$

Tangent Space



$\gamma_1 \sim \gamma_2 \iff \phi \circ \gamma_1 \sim \phi \circ \gamma_2$ and $\gamma_1 \sim \gamma_2 \iff \psi \circ \gamma_1 \sim \psi \circ \gamma_2$, so

$$(\psi \circ \phi^{-1})(\phi \circ \gamma_1) \sim (\psi \circ \phi^{-1})(\phi \circ \gamma_2)$$

Now, take $\{[\gamma]\} = T_pM$. Claim: this is a vector space.

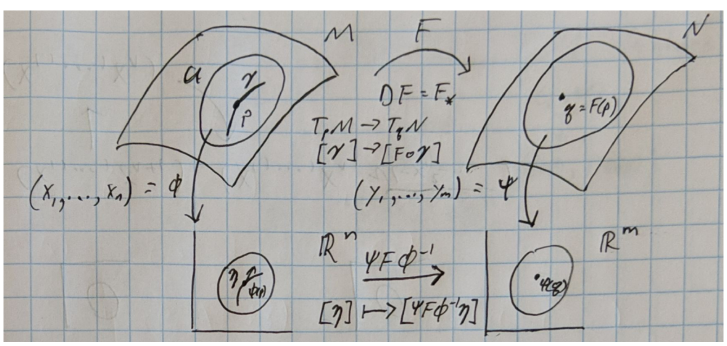
$$\begin{tikzcd} T_pM \arrow[r, "D\phi"] \arrow[dr, "D\psi"'] & T_{\phi(p)}\mathbb{R}^n \arrow[d, "D(\overbrace{\psi\circ\phi^{-1}}^{F})"] \\ & T_{\psi(p)}\mathbb{R}^n \end{tikzcd}$$

$$[\gamma_1] + [\gamma_2] = \left[\phi^{-1}(\phi \circ \gamma_1 + \phi \circ \gamma_2)\right].$$

October 15, 2024

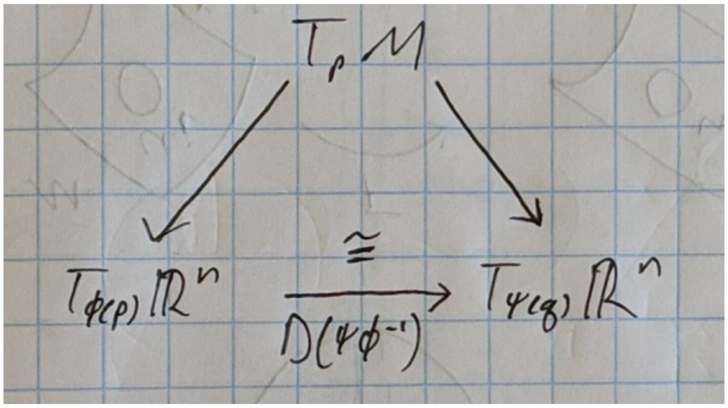
Recall: Tangent Space by Equivalence Classes

$\gamma : (-\varepsilon, \varepsilon) \to M$, $\gamma(0) = p$, $\gamma_1 \sim \gamma_2 \iff \phi \circ \gamma_1' \sim \phi \circ \gamma_2' \iff (\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0)$.
$T_pM = \{[\gamma]\} \overset{D\phi = \phi_*}{\to} T_{\phi(p)}\mathbb{R}^n = \mathbb{R}^n$.



Then for $(F_1, \ldots, F_m)$, we have $D(\psi F \phi^{-1}) : \underbrace{T_{\phi(p)}\mathbb{R}^n}_{\mathbb{R}^n} \to \underbrace{T_{\psi(q)}\mathbb{R}^m}_{\mathbb{R}^m}$ where

$$D(\psi F \phi^{-1}) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$



Chain Rule

We have that $D(FG) = DF \circ DG$ since $(FG) \circ \gamma = F(G \circ \gamma)$.

Example

Take

$$f : \underbrace{M}_{(x_1,\ldots,x_n)} \to \mathbb{R}$$

and $Df = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$.

Definition: Directional Derivatives

Take $v = [\gamma] \in T_pM$ and $f \in C^r(M)$.
The directional derivative is given by

$$L_v f = \frac{d}{dt} f(\gamma(t))|_{t=0} = \frac{d}{dt} \underbrace{(f \circ \phi^{-1})}_{g} \underbrace{(\phi \circ \gamma)}_{\eta}(t)|_{t=0}$$

Then

$$\frac{d}{dt} g(\eta(t))|_{t=0} = \sum \frac{\partial g}{\partial x_i}(\phi)\eta_i'(0)$$

which is determined by $\left(\eta_1'(0), \ldots, \eta_n'(0)\right) = \eta'(0)$.

Properties

$L_v : C^\infty(M) \to \mathbb{R}$

1. Linear over $\mathbb{R}$
2. $L_v(fg) = (L_v f)\, g(\phi) + f(\phi)\, (L_v g)$ (product rule)
3. Linear in v.

Derivations

The collection $\mathcal{D}_p = \{C^r(M) \to \mathbb{R} :\ (1) \text{ and } (2) \text{ hold}\}$ is called the derivations at p.

Algebraic Aside

$\delta_p : A \to \mathbb{R}$ given by $f \mapsto f(p)$ yields

$$D(fg) = Df\delta_p(g) + \delta_p(f)Dg$$

Theorem

$T_pM \to \mathcal{D}_p$ given by $v = [\gamma] \mapsto L_v$ is a linear isomorphism.

Recall: Germ

Take $(f, U) \ni p$ and $(g, V) \ni p$. Then $(f, U) \sim (g, V)$ if and only iff $f \equiv g$ on $W \subset U \cap V$. The equivalence classes of this relation are germs.

Hadamard's Lemma

Take $f \in C^r(\mathbb{R}^n, 0)$ on $\mathbb{R}^n$ with $f(0) = 0$.
There exists $g_1, \ldots, g_n \in C^{r-1}(\mathbb{R}^n, 0)$ such that

$$f(x) = \sum x_i g_i(x)$$

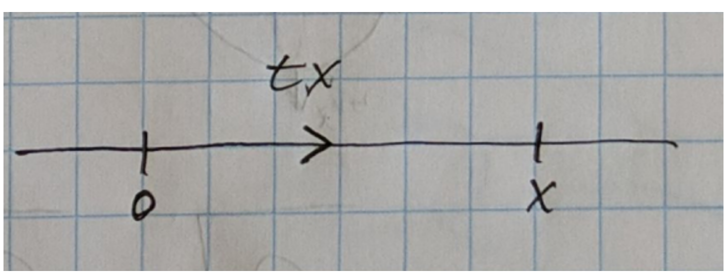
and $g_i(0) = \frac{\partial f}{\partial x_i}(0)$.

Example

For $n = 1$ and $f : \mathbb{R} \xrightarrow{C^r} \mathbb{R}$ with $f(0) = 0$. Then we have $f(x) = x\overbrace{g(x)}^{c^{r-1}}$ given by

$$g(x) = \begin{cases} \frac{f(x)}{x} & x \neq 0 \\ f'(0) & x = 0 \end{cases}.$$

Proof of Example



Take

$$\int_0^1 \underbrace{\frac{d}{dt} f(tx)}_{f'(tx)\cdot x}\, dt = x \underbrace{\int_0^1 f'(tx)\, dt}_{g(x)}$$
$$= f(1 \cdot x) - f(0 \cdot x)$$

Proof of Lemma

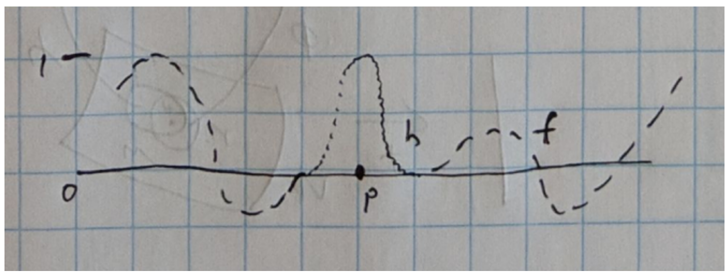
$$\int_0^1 \underbrace{\frac{d}{dt} f(tx)\,dt}_{\sum \frac{\partial f}{\partial x_i}(tx)\cdot x_i} = \sum x_i \underbrace{\int_0^1 \frac{\partial f}{\partial x_i}(tx)\,dt}_{g_i}$$

Lemma

For $D \in \mathcal{D}_p$, Df depends only on the germ of f.

Proof

Need to show that if $f \equiv 0$ near p, $Df = 0$.



Where $\operatorname{supp} h \subset \{x : f(x) = 0\}$, $h(p) = 1$ and, consequently, $f \circ h = 0$.
So $D(0 = f \circ h)$, $D0 = Df \cdot h(p) + f(p)Dh$ and $0 = Df$.

Lemma

$D(k) = 0$ for k constant.

Proof

$1 \cdot 1 = 1 \implies D(1) = D(1 \cdot 1) = D(1) \cdot 1 + 1 \cdot D(1) = 2 \cdot D(1)$, so $D(1) = 0$.
Arbitrary constants follow from the fact that D is linear.

Proof: One-to-One

Let $L_v f = L_w f$ for all $f = x_i$, then

$$\sum v_i \frac{\partial f}{\partial x_i} = \sum w_i \frac{\partial f}{\partial x_i}$$

for each f and $v_i = w_i$.

Lemma

$$\operatorname{Der}(C^r(m)@p) = \operatorname{Der}(C^r(M, p))$$

Proof

Take $M = \mathbb{R}^n$ and $p = 0$.
Given D, we need $v = \sum v_i \frac{\partial}{\partial x_i}$ with

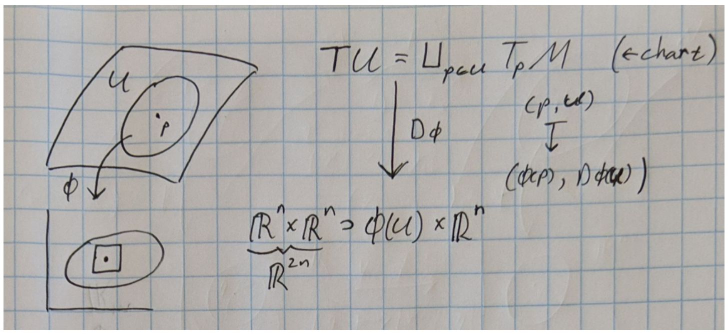
$$L_v f = \sum v_i \frac{\partial f}{\partial x_i} = Df$$

for every f.
By Hadamard's lemma, write $f = \sum x_i g_i(x)$. Then

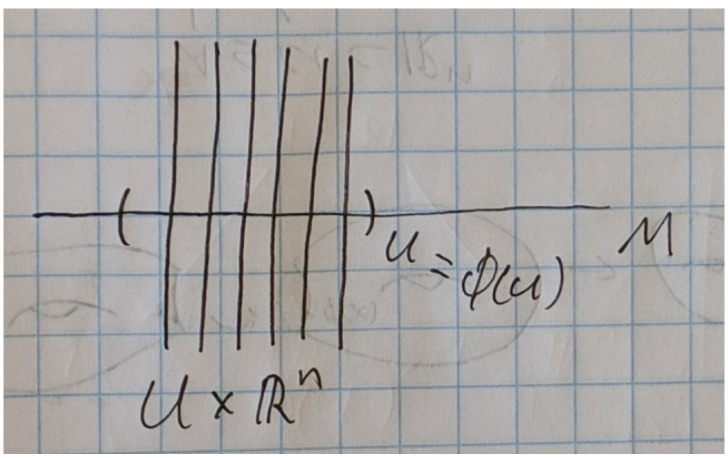
$$\begin{aligned} Df &= \sum (Dx_i) g_i(0) + \overbrace{x_i(0)}^{=0} Dg_i \\ &= \sum \underbrace{(Dx_i)}_{v_i} \frac{\partial f}{\partial x_i}(0) \end{aligned}$$

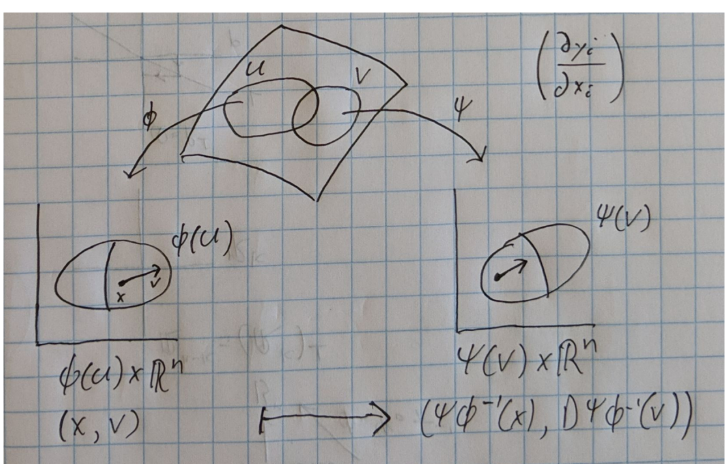
That is, since $v_i = L_v x_i$, we have $v_i = Dx_i$.

Definition: Tangent Bundle



For every $p \in M$, we have T_pM.
The tangent bundle $TM = \coprod_{p\in M} T_pM$.





When M is C^r, TM is C^{r-1}.

October 17, 2024

Chapter 7 of Lee (Lie Groups) will not be covered in class, but is highly recommended reading.

Preliminary Definition: Vector Fied

Take $M \overset{C^\infty}{\to} TM$ by $p \overset{v}{\mapsto} v(p) \in T_pM$.

Space of Vector Fields

Write $\mathcal{X}(M)$ to be the collection of all C^∞ vector fields.

- This is a module over $C^\infty(M)$.
- $\mathcal{X}(M)$ acts on $C^\infty(M)$ by $v \mapsto L_v$.

Smooth

1. v is C^∞
2. In local coordinates, for $p \in U$ and $(U, \phi) = (x_1, \ldots, x_n)$ with $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ a basis in T_pM.

$$v = \sum \underbrace{v_i(x)}_{\substack{\text{functions}\\ \text{on } U}} \frac{\partial}{\partial x_i}$$

3. $f \in C^\infty(M) \implies L_v f \in C^\infty$

2 Implies 3

$$L_v f = \sum v_i \frac{\partial f}{\partial x_i}$$

IMAGE 1

With $\phi|_W \equiv 1$, $\operatorname{supp} \phi = U$, $x_i\phi \in C^\infty(M)$, and $x_i\phi|W \equiv W_i$.
Then $L_v(x_i\phi) \underset{\text{on } W}{=} v_i$.

Definition: Lie Algebra

Take A a vector space equipped with (a Lie bracket) $[\cdot, \cdot] : A \times A \to A$ such that

- $[a, b] = -[b, a]$ (Skew Symmetric)
- $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$ (Jacobi Identity)

Example 1

Take A to be an algebra and define $[a,b] = ab - ba$. Then if A is associative, it satisfies the Jacobi identity.

- $gl(n)$
- $so(n)$ (skew symmetric matrices)
  - $(ab-ba)^T = b^T a^T - a^T b^T = ba - ab = -[a,b]$
- $su(n)$ $(A^T = A^\dagger)$

Theorem:

The space of vector fields $\mathcal{X}(M)$ is a Lie algebra:

- $\forall V, W \in \mathcal{X}(M)$, $\exists! U \in \mathcal{X}(M)$ such that $L_U f = L_W L_V f - L_V L_W f$. Write $U = [V, W]$.

Lemma

$L_V f = L_W f$, $\forall f$ implies $V = W$.

Proof

$$V = \sum v_i \frac{\partial}{\partial x_i} \stackrel{?}{=} \sum w_i \frac{\partial}{\partial x_i} = W$$

Pick $p \in U$. We want to find $v_i(p) = w_i(p)$.

IMAGE 1

With $\phi|_W \equiv 1$, $\operatorname{supp}\phi \subset U$ and $f = x_i\phi \in C^\infty(M)$.

$$L_V f = L_W f$$
$$\sum v_j \frac{\partial \overbrace{(x_i\phi)}^{x_i}}{\underbrace{\partial x_j}_{\partial_{ij}}} = \sum w_j \frac{\partial \overbrace{(x_i\phi)}^{x_i}}{\underbrace{\partial x_j}_{\delta_{ij}}}$$

on W. Therefore $v_i = w_i$ on W.

Variant

For W_i an open cover, $L_V f = L_W f$ for all $f \in C^\infty(W_i)$ implies that $V = W$.

Compute

$$L_V L_W f = L_V\left(\sum w_j \frac{\partial f}{\partial x_j}\right) = \sum v_i \frac{\partial w_j}{x_i}\frac{\partial f}{\partial x_j} + \sum v_i w_j \frac{\partial^2 f}{\partial x_i \partial x_j}$$

$$L_W L_V f = \sum w_i \frac{\partial v_j}{\partial x_i}\frac{\partial f}{\partial x_j} + \sum w_i v_j \frac{\partial^2 f}{\partial x_j \partial x_i}$$

$$L_V L_W f - L_W L_V f = \sum_{i,j}\left(v_i \frac{\partial w_j}{\partial x_i} - w_i \frac{\partial v_j}{x_i}\right)\frac{\partial f}{\partial x_j}$$

$$u = \sum_{i,j}\left(v_i \frac{\partial w_j}{\partial x_i} - w_i \frac{\partial v_i}{\partial x_i}\right)\frac{\partial}{\partial x_j}$$

Therefore $L_V L_W f - L_W L_V f = L_U f$.

Remark

Consider $U \ni u$ and $U' \ni u'$

IMAGE 2

Properties

- Lie algebra: Skew symmetric an dsatisfying the Jacobi identity
- Product rule: $[V, fW] = (L_V f)W + f[V, W]$.

Example

Let V be a finite dimensional vector space (e.g. $\mathbb{R}^n$).
Recall that $T_p V \cong V$ by $[p + tv] \leftarrow v$. Then $TV = V \times V\ (p, v)$.
Take $A \in \mathrm{End}(V)$ given by

$$v(x) = Ax = \sum v_i \frac{\partial}{\partial x_i} = \begin{pmatrix} & & \\ & a_{ij} & \\ & & \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Take also $w(x) = Bx$. Then $[V, W] = -(AB - BA)x = -[A, B]$.

Exercise

Given the example with W constant, determine $[V, W]$.

Theorem (Midterm Problem)

Take $A = C^\infty(M)$ and the derivations $D \in \mathrm{Der}(A)$ with $D : A \xrightarrow{\text{lin.}} A$ over $\mathbb{R}$ such that $D(fg) = Df \cdot g + f \cdot Dg$.
There exists a linear isomorphism $\mathcal{X}(M) \xrightarrow{\cong} \mathrm{Der}(A)$ given by $v \mapsto L_v$.

Lemma

Take $D \in \mathrm{Der}(A)$. Then $D_p \in \mathcal{D}_p$ where $D_p f := (Df)(p)$.

$$D_p(fg) = (D_p f)g(p) + f(p)(D_p g)$$

We know from above that $\mathcal{D}_p \cong T_p m$. Therefore $L_v f = Df$.

Definition: Push Forward

Take $F : M \to N$ which gives rise to $F_* : T_p M \to T_{F(p)} N$ (equivalent to $F_* : \mathcal{D}_p \to \mathcal{D}_{F(p)}$) given by $[\gamma] \mapsto [F \circ \gamma]$.
We see that $(F_* D)(g) := D(g \circ F)$.

You Cannot Push Forward Vector Fields

- if F is not surjective

IMAGE 3

- if F is not injective

IMAGE 4

- it is possible that $F_* V$ would fail to be smooth. Take $(F_* v)(y) = 3y^{2/3} \frac{\partial}{\partial y}$

IMAGE 5

Remark

Take a diffeomorphism $F : M \to N$.
Then $F : \mathcal{X}(M) \to \mathcal{X}(N)$ is given by $F_*[V, W] = [F_* V, F_* W]$.