Math 11B Discussion Section

Evaluate each of the following integrals.

(1)

$$\int 2y^2 \cos(9y)\, dy$$

We notice that our integrand is the product of two functions, one of which is polynomial and disappears under repeated differentiation, so we expect to integrate by parts.

We may compute directly in multiple steps. . . . . . or by constructing the table

$$u = 2y^2 \qquad v = \frac{1}{9}\sin(9y)$$

$$du = 4y\, dy \qquad dv = \cos(9y)\, dy$$

±	D	I
+	$2y^2$	$\cos(9y)$
−	$4y$	$\frac{1}{9}\sin(9y)$
+	4	$-\frac{1}{81}\cos(9y)$

$$\int 2y^2 \cos(9y)\, dy$$

$$= (2y^2)\left(\frac{1}{9}\sin(9y)\right) - \int \left(\frac{1}{9}\sin(9y)\right)(4y)\, dy$$

$$u = 4y \qquad v = -\frac{1}{81}\cos(9y)$$

$$du = 4\, dy \qquad dv = \frac{1}{9}\sin(9y)\, dy$$

$$\int \left(\frac{1}{9}\sin(9y)\right)(4y)\, dy$$

$$= (4y)\left(-\frac{1}{81}\cos(9y)\right) - \int \left(-\frac{1}{81}\cos(9y)\right)(4)\, dy$$

In either case, we have

$$\begin{aligned}\int 2y^2 \cos(9y)\, dy &= \frac{2}{9}y^2\sin(9y) + \frac{4y}{81}\cos(9y) - \frac{4}{81}\int \cos(9y)\, dy \\ &= \frac{2}{9}y^2\sin(9y) + \frac{4y}{81}\cos(9y) - \frac{4}{729}\sin(9y)\, dy + C\end{aligned}$$

(2)

$$\int_0^4 \frac{8y-1}{2y^2-15y-8}\,dy$$

We see a rational function and a denominator which factors into linear terms. We expect that we have some partial fraction decomposition.

$$\frac{8y-1}{2y^2-15y-8} = \frac{8y-1}{(2y+1)(y-8)} = \frac{A}{2y+1} + \frac{B}{y-8}$$
$$8y-1 = A(y-8) + B(2y+1) = y(A+2B) + (-8A+B)$$

So we have a system of linear equations where $A+2B=8$ and $-8A+B=-1$. Then $A=8-2B$, and $-8(8-2B)+B=-1$. So $B=\frac{63}{17}$ and $A=\frac{10}{17}$. We compute

$$\begin{aligned}
\int_0^4 \frac{8y-1}{2y^2-15y-8}\,dy &= \frac{10}{17}\int_0^4 \frac{1}{2y+1}\,dy + \frac{63}{17}\int_0^4 \frac{1}{y-8}\,dy \\
&= \frac{10}{17}\int_0^4 \frac{1}{2y+1}\,dy + \frac{63}{17}\int_0^4 \frac{1}{8-y}\,dy \\
&= \frac{10}{17}\left[\frac{1}{2}\log|2y+1|\right]_0^4 + \frac{63}{17}\left[\log|8-y|\right]_0^4 \\
&= \frac{10}{17}\left(\frac{1}{2}\log(9)\right) - \frac{10}{17}\left(\frac{1}{2}\overbrace{\log(1)}^{=0}\right) + \frac{63}{17}\log(4) - \frac{63}{17}\log(8) \\
&= \frac{5}{17}\log(9) + \frac{63}{17}(2\log(2)) - \frac{63}{17}(3\log(2)) \\
&= \frac{5}{17}\log(9) - \frac{63}{17}\log(2)
\end{aligned}$$

(3)

$$\int_0^3 \frac{w^3}{\sqrt{9-w^2}}\,dw$$

We observe that the integrand is undefined at $w = 3$, so this is an improper integral. Choosing $u = w^2$ such that $du = 2w\,dw$ and $v = 9 - u$ such that $dv = -du$, we compute

$$\begin{aligned}
\lim_{b\to 3}\int_0^b \frac{uw}{\sqrt{9-u}}\cdot\frac{1}{2w}\,du &= \lim_{b\to 3}\frac{1}{2}\int_{w=0}^{w=b}\frac{u}{\sqrt{9-u}}\,du \\
&= \lim_{b\to 3} -\frac{1}{2}\int_{w=0}^{w=b}\frac{9-v}{\sqrt{v}}\,dv \\
&= -\lim_{b\to 3}\frac{9}{2}\int_{w=0}^{w=b} v^{-1/2}\,dv + \lim_{b\to 3}\frac{1}{2}\int_{w=0}^{w=b} v^{1/2}\,dv \\
&= -\lim_{b\to 3}\frac{9}{2}\left[\frac{1}{2}v^{1/2}\right]_{w=0}^{w=b} + \lim_{b\to 3}\frac{1}{2}\left[\frac{2}{3}v^{3/2}\right]_{w=0}^{w=b} \\
&= -9\lim_{b\to 3}\left[(9-u)^{1/2}\right]_{w=0}^{w=b} + \frac{13}{}\lim_{b\to 3}\left[(9-u)^{3/2}\right]_{w=0}^{w=b} \\
&= -9\lim_{b\to 3}\left[(9-w^2)^{1/2}\right]_0^b + \frac{1}{3}\lim_{b\to 3}\left[(9-w^2)^{3/2}\right]_0^b \\
&= -9(9-3^2)^{1/2} + 9(9-0^2)^{1/2} + \frac{1}{3}(9-3^2)^{3/2} - \frac{1}{3}(9-0^2)^{3/2} = 18
\end{aligned}$$

(4)

$$\int \frac{6x^2 - 10x^4}{x^5 - x^3}\, dx$$

Here, we recognize that the numerator may be rewritten as $-2(5x^4-3x^2)$ and do a u-substitution by choosing $u = x^5-x^3$ such that $du = 5x^4 - 3x^2\, dx$ and

$$\begin{aligned}
\int \frac{6x^2 - 10x^4}{x^5 - x^3}\, dx &= \int \frac{-2(5x^4 - 3x^2)}{x^5 - x^3}\, dx \\
&= \int \frac{-2(5x^4 - 3x^2)}{u} \cdot \frac{1}{5x^4 - 3x^2}\, du \\
&= -2 \int \frac{1}{u}\, du \\
&= -2 \log(u) \\
&= -2 \log(x^5 - x^3)
\end{aligned}$$

(5)

$$\int_{\frac{3\pi}{4}}^{\pi} \sec^6(10t)\tan^4(10t)\,dt$$

Here we see a large mass of secants and tangents and hope to simplify by recalling that $\frac{d}{dt}\tan(t) = \sec^2(t)$ and $1+\tan^2(t) = \sec^2(t)$ (the Pythagorean identity!). We can rewrite our expression as

$$\begin{aligned}
\int_{\frac{3\pi}{4}}^{\pi} \sec^6(10t)\tan^4(10t)\,dt &= \int_{\frac{3\pi}{4}}^{\pi} \sec^2(10t)\sec^4(10t)\tan^4(10t)\,dt \\
&= \int_{\frac{3\pi}{4}}^{\pi} \sec^2(10t)(\sec^2(10t))^2\tan^4(10t)\,dt \\
&= \int_{\frac{3\pi}{4}}^{\pi} \sec^2(10t)(1+\tan^2(t))^2\tan^4(10t)\,dt
\end{aligned}$$

However, we have a problem: $\cos(t) = 0$ for $t = n\pi - \frac{\pi}{2}$ with n an integer, so $\cos(10t) = 0$ for $t = \frac{n\pi}{10} - \frac{\pi}{20}$. Therefore $\tan(10t)$ is undefined at $\frac{9\pi}{10} - \frac{\pi}{20}$ and $\frac{10\pi}{10} - \frac{\pi}{20}$ which are both within our bounds of integration.
We could break the integral into pieces and evaluate the limits

$$\begin{aligned}
&\lim_{b\to\frac{9\pi}{10}-\frac{\pi}{20}} \int_{\frac{3\pi}{4}}^{b} \sec^2(10t)(1+\tan^2(t))^2\tan^4(10t)\,dt \\
&+ \lim_{a\to\frac{9\pi}{10}-\frac{\pi}{20}} \int_{a}^{\pi-\frac{\pi}{10}} \sec^2(10t)(1+\tan^2(t))^2\tan^4(10t)\,dt \\
&+ \lim_{b\to\pi-\frac{\pi}{20}} \int_{\pi-\frac{\pi}{10}}^{b} \sec^2(10t)(1+\tan^2(t))^2\tan^4(10t)\,dt \\
&+ \lim_{a\to\pi-\frac{\pi}{20}} \int_{a}^{\pi} \sec^2(10t)(1+\tan^2(t))^2\tan^4(10t)\,dt
\end{aligned}$$

However, we will find that these limits do not converge. Indeed, attempting the intended substitution of $u = \tan(10t)$ only moves the undefined terms into the bounds of integration. We conclude that the integral diverges.

(6) Estimate the net area between $h(x) = 5 + x - x^2$ on $[0,4]$ and the x-axis given $n = 8$ subintervals and using the midpoints for the height of the rectangles.

We are evenly dividing the interval $[0,4]$ into eight pieces, so the width of our rectangles are $\frac{1}{2}$. We are choosing the midpoint of the interval to determine the height of our rectangles, so these are the points $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, etc. Therefore, we compute

$$\begin{aligned}
\frac{1}{2}\sum_{n=0}^{7} f\left(\frac{1}{4}+\frac{n}{2}\right) &= \frac{1}{2}\left(f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)+f\left(\frac{5}{4}\right)+f\left(\frac{7}{4}\right)+f\left(\frac{9}{4}\right)+f\left(\frac{11}{4}\right)+f\left(\frac{13}{4}\right)+f\left(\frac{15}{4}\right)\right)\\
&= \frac{1}{2}\left(\left(\frac{80}{16}+\frac{4}{16}-\frac{1}{16}\right)+\left(\frac{80}{16}+\frac{12}{16}-\frac{9}{16}\right)+\left(\frac{80}{16}+\frac{20}{16}-\frac{25}{16}\right)+\left(\frac{80}{16}+\frac{28}{16}-\frac{49}{16}\right)\right)\\
&\quad+\frac{1}{2}\left(\left(\frac{80}{16}+\frac{36}{16}-\frac{81}{16}\right)+\left(\frac{80}{16}+\frac{44}{16}-\frac{121}{16}\right)+\left(\frac{80}{16}+\frac{52}{16}-\frac{169}{16}\right)+\left(\frac{80}{16}+\frac{60}{16}-\frac{225}{16}\right)\right)\\
&= \frac{1}{32}(8(80)+64(4)-680) = \frac{1}{32}(216) = \frac{27}{4}
\end{aligned}$$

(7) Determine the area of the region bounded by the curves $y = x^2 - 6x + 10$ and $y = 5$.

We start by identifying all points where the two curves meet:

$$x^2 - 6x + 10 = 5 \iff x^2 - 6x + 5 = 0 \iff (x-5)(x-1) = 0.$$

So the curves meet at $x = 1$ and $x = 5$, and we will be integrating with respect to x on $[1,5]$. Next, we need to identify which curve is "above" the other. We recognize $y = x^2 - 6x + 10$ to be a parabola opening upwards. If it intersects twice with the constant curve $y = 5$, then the vertex must lie below $y = 5$. We could reach the same conclusion by trying test points in $[1,5]$ and seeing that $x^2 - 6x + 10 \leq 5$ for any choice in that range.

Finally, we integrate the difference across the appropriate range

$$\begin{aligned}
\int_1^5 5 - (x^2 - 6x + 10)\, dx &= \int_1^5 -x^2 + 6x - 5\, dx \\
&= -\int_1^5 x^2\, dx + \int_1^5 6x\, dx - \int_1^5 5\, dx \\
&= -\left[\frac{1}{3}x^3\right]_1^5 + \left[3x^2\right]_1^5 - [5x]_1^5 \\
&= -\left(\frac{125}{3} - \frac{1}{3}\right) + \left(\frac{225}{3} - \frac{9}{3}\right) - \left(\frac{75}{3} - \frac{15}{3}\right) = \frac{32}{3}
\end{aligned}$$

(8) Find f_{avg}, the average value, of $f(x) = 10 - 4x - 6x^2$ on $[2, 6]$, and determine the value c for which $f(c) = f_{\text{avg}}$.

First, we compute

$$\begin{aligned} f_{\text{avg}} &= \frac{1}{6-2}\int_2^6 -6x^2 - 4x + 10\,dx \\ &= \frac{1}{4}\left(-\left[2x^3\right]_2^6 - \left[2x^2\right]_2^6 + \left[10x\right]_2^6\right) \\ &= \frac{1}{4}\left(-(432-16) - (72-8) + (60-20)\right) = -\frac{440}{4} = -110 \end{aligned}$$

Now, by the Mean Value Theorem, we know that

$$f_{\text{avg}} = f(c) \iff -110 = -6c^2 - 4c + 10 \iff 6c^2 + 4c - 120 = 0.$$

Applying the quadratic formula, we get

$$c = \frac{-4 \pm \sqrt{4^2 - 4(6)(-120)}}{12} = -\frac{1}{3} \pm \frac{\sqrt{16(1+180)}}{12} = -\frac{1}{3} \pm \frac{\sqrt{181}}{3}.$$

Since the square root of 181 must be close to $13^2 = 169$, we can ball-park this as $-\frac{1}{3} + \frac{13}{3} = 4$ and $-\frac{1}{3} - \frac{13}{3} = -\frac{14}{3}$. Since the latter term is nowhere near our interval, we conclude that the unique value of c for which $f_{\text{avg}} = f(c)$ is $-\frac{1}{3} + \frac{\sqrt{181}}{3}$.

(9) Using both the method of cylinders and the method of rings, determine the volume of the solid obtained by rotating the region bounded by $x = y^3$, $x = 8$ and the x-axis about the y-axis. Show that both methods agree.

We have some flat area (our surface) interior to the curves $x = y^3$, $x = 8$, and $y = 0$. We imagine creating a volume by fanning out infinitely many of these in a circle centered at $x = 0$. To calculate the volume, we recreate this construction in the integral. This is the method by cylinders. We will compute $2\pi \int_a^b (\text{Radius})(\text{Height})\, dx$.

We know that our horizontal bounds are given by $x = 0$ and $x = 8$, so we will integrate along $[0, 8]$. To get our radius, we ask how far x can be from the axis of rotation (in this case the y-axis) to get $\text{Radius} = x - 0$. Finally, we know that the total vertical height of the shape is bounded above and below by $y = \sqrt[3]{x}$ and $y = 0$, so $\text{Height} = \sqrt[3]{x} - 0$. We integrate

$$2\pi \int_0^8 (x)(\sqrt[3]{x})\, dx = 2\pi \int_0^8 x^{4/3}\, dx = 2\pi \left[\frac{3}{7} x^{7/3} \right]_0^8 = \frac{6\pi}{7} \cdot 8^{7/3} = \frac{6\pi}{7} \cdot 128$$

Instead, we could imagine calculating the volume of the same structure by stacking up rings with appropriately sized holes removed. This is the method of rings. We will compute $\int_a^b \pi(R^2 - r^2)\, dy$ where R is the radius of the largest ring and r is the radius of the hole. Here we see that the shape achieves its greatest radius at the constant $x = 8$, but the hole changes size with respect to $x = y^3$. The shape begins at $y = 0$ and reaches its apex when $y^3 = 8$, so we will integrate on $[0, 2]$.

$$\begin{aligned}
\pi \int_0^2 ((8)^2 - (y^3)^2)\, dy &= 64\pi \int_0^2 dy - \pi \int_0^2 y^6\, dy \\
&= 64\pi [y]_0^2 - \pi \left[\frac{1}{7} y^7 \right]_0^2 \\
&= 64\pi(2) - 64\pi(0) - \pi\left(\frac{128}{7}\right) + \pi\left(\frac{0}{7}\right) \\
&= 7 \cdot \frac{128\pi}{7} - \frac{128\pi}{7} \\
&= 6 \cdot \frac{128\pi}{7}
\end{aligned}$$

Therefore, we conclude that the volume is $\frac{768\pi}{7}$ and both methods agree.