

# Manifolds III

March 31, 2025

## Review

If  $X, Y$  are topological spaces and  $f, g : X \rightarrow Y$  continuous maps, we say  $f$  and  $g$  are homotopic (written  $f \simeq g$ ) if there is a homotopy  $H : X \times I \rightarrow Y$  (where  $I = [0, 1]$ ) such that  $H(x, 0) = f(x)$  and  $H(x, 1) = g(x)$  for all  $x \in X$ . We say that  $f$  is null-homotopic if it is homotopic to a constant map.

## Proposition

Homotopy is an equivalence relation on the collection of continuous maps.

1.  $f \simeq f$  by  $H(x, t) := f(x)$ .
2.  $f \stackrel{\tilde{H}}{\simeq} g \implies g \simeq f$  by defining  $\tilde{H}(x, t) := H(x, 1 - t)$ .
3.  $(f \stackrel{F}{\simeq} g \wedge g \stackrel{G}{\simeq} h) \implies f \simeq h$  by

$$H(x, t) := \begin{cases} F(x, 2t) & 0 \leq t \leq 1/2 \\ G(x, 2t - 1) & 1/2 \leq t \leq 1 \end{cases}.$$

## Proposition

For  $f_0, f_1 : X \rightarrow Y$  and  $g_0, g_1 : Y \rightarrow Z$ , if  $f_0 \stackrel{F}{\simeq} f_1$  and  $g_0 \stackrel{G}{\simeq} g_1$ , then  $g_0 \circ f_0 \simeq g_1 \circ f_1$ .

## Proof

Define  $H(x, t) := G(F(x, t), t)$  such that  $H(x, 0) = G(F(x, 0), 0) = G(f_0(x), 0) = g_0 \circ f_0(x)$ . Similarly,  $H(x, 1) = g_1 \circ f_1(x)$ .

## Definition: Homotopic Spaces

We say that two spaces  $X$  and  $Y$  are homotopic to each other ( $X \simeq Y$ ) if there are continuous maps  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  such that  $f \circ g \simeq \text{id}_Y$  and  $g \circ f \simeq \text{id}_X$ .

## Example

$\mathbb{R}^n$  is homotopic to  $\{0\}$  (or any single point) by  $\iota : 0 \rightarrow \mathbb{R}^n$  and  $r : \mathbb{R}^n \rightarrow 0$ . Then  $r \circ \iota : 0 \rightarrow 0$  is  $\text{id}_0$  and  $\iota \circ r : \mathbb{R}^n \ni x \mapsto 0 \in \mathbb{R}^n$  is homotopic to  $\text{id}_{\mathbb{R}^n}$ . In fact, consider  $H : \mathbb{R}^n \times I \rightarrow \mathbb{R}^n$  where  $H(x, t) = tx$ ,  $H(x, 1) = x = \text{id}_{\mathbb{R}^n}(x)$  and  $H(x, 0) = 0$ .

## Definition: Path

A path in  $X$  from  $p$  to  $q$  is a continuous map  $f : I \rightarrow X$  such that  $f(0) = p$  and  $f(1) = q$ .

## Definition: Path Homotopic

Let  $f, g : I \rightarrow X$  be two paths in  $X$  from  $p$  to  $q$ .

We say that  $f$  and  $g$  are path homotopic (write  $f \sim g$ ) if there is a homotopy  $H : I \times I \rightarrow X$  such that  $H(s, 0) = f(s)$ ,  $H(s, 1) = g(s)$ ,  $H(0, t) = p$  and  $H(1, t) = q$ .

## Proposition

Path homotopy is an equivalence relation on the collection of paths from  $p$  to  $q$ .  
Write  $[f]$ , the equivalence class of  $f$  in the quotient.

## Definition: Loop

In the special case that  $p = q$ , we say that  $f : I \rightarrow X$  is a loop

## Definition: Fundamental Group

Given  $(X, p)$ ,  $\pi_1(X, p)$  (the fundamental group of  $X$  at the point  $p$ ) is the set of all loops at  $p$  modulo the path homotopy.

$$\{\text{loops at } p\} / \sim$$

Equivalently,  $(S^1, 1)$ ,  $\{\text{loops at } p\} = \{\text{continuous maps } f : (S^1, 1) \rightarrow (X, p)\}$  with  $f(1) = p$ . We say this is the homotopy “relative to  $1 \in S^1$ ”. We have  $H : S^1 \times I \rightarrow X$  such that  $H(s, 0) = f(s)$ ,  $H(s, 1) = g(s)$  and  $H(1, t) = p$ .

## Definition: Free Homotopy

For two loops  $f, g : S^1 \rightarrow X$ , we say that  $f$  and  $g$  are free homotopic if  $f \simeq g$ .

## Lemma

When  $f : I \rightarrow X$  is a path from  $p$  to  $q$ , if  $f \circ \varphi$  is a reparameterization of  $f$  then  $(f \circ \varphi) \sim f$  where  $\varphi : I \rightarrow I$  satisfies  $\varphi(0) = 0$  and  $\varphi(1) = 1$ .

## Proof

Note that  $\varphi$  is homotopic to the identity map  $\text{id}_I$  through  $H(s, t) = ts + (1 - t)\varphi(s)$  since  $H(s, 0) = \varphi(s)$  and  $H(s, 1) = s = \text{id}_I(s)$ .

Then consider  $f \circ H : I \times I \rightarrow X$  which is a path homotopy between  $f$  and  $f \circ \varphi$ .

## Fundamental Group

Let  $f, g : I \rightarrow X$  be two paths with  $f(1) = g(0)$ .

Then we can “compose” (concatenate)  $f$  and  $g$  together  $(f \cdot g) : I \rightarrow X$  by

$$(f \cdot g)(s) := \begin{cases} f(2s) & 0 \leq s \leq 1/2 \\ g(2s - 1) & 1/2 \leq s \leq 1 \end{cases}.$$

## Lemma

If  $f_0 \stackrel{F}{\sim} f_1$ ,  $g_0 \stackrel{G}{\sim} g_1$  and  $f_0(1) = f_1(1) = g_0(0) = g_1(0)$ , then  $f_0 \cdot g_0 \sim f_1 \cdot g_1$ .

## Proof

Define

$$H(s, t) := \begin{cases} F(2s, t) & 0 \leq s \leq 1/2 \\ G(2s - 1, t) & 1/2 \leq s \leq 1 \end{cases}.$$

Then

$$H(s, 0) = \begin{cases} F(2s, 0) = f_0(2s) & 0 \leq s \leq 1/2 \\ G(2s - 1, 0) = g_0(2s - 1) & 1/2 \leq s \leq 1 \end{cases}.$$

Similarly  $H(s, 1) = (f_1 \cdot g_1)(s)$ , hence  $f_0 \cdot g_0 \sim f_1 \cdot g_1$ .

With this, we have a well-defined  $[f] \cdot [g] := [f \cdot g]$ .

## Simple Properties

For  $f$  from  $p$  to  $q$  where  $c_p$  is the constant map at  $p$ ,

1.  $[c_p] \cdot [f] = [f] = [f] \cdot [c_q]$  since  $c_p \cdot f$  is a reparameterization of  $f$ .
2. Let  $\bar{f}$  be the inverse path of  $f$  (i.e.  $\bar{f}(s) = f(1 - s)$ ). Then  $[f] \cdot [\bar{f}] = [c_p]$  and  $[\bar{f}] \cdot [f] = [c_q]$ .

$$H(s, t) := \begin{cases} f(2s) & 0 \leq s \leq t/2 \\ f(t) & t/2 \leq s \leq 1 - t/2 \\ f(2 - 2s) & 1 - t/2 \leq s \leq 2 \end{cases}.$$

1.  $([f] \cdot [g]) \cdot [h] = [f] \cdot ([g] \cdot [h])$ , since these are reparameterizations of the same path.

## Group Structure

$\pi_1(X, p) = \{\text{loops at } p\} / \sim$ .

Define  $[f] \cdot [g] := [f \cdot g]$ .

It has an identity element  $[c_p] = e$ .

For any  $f \in \pi_1(X, p)$ , it has an inverse  $[\bar{f}]$  such that  $[f] \cdot [\bar{f}] = [\bar{f}] \cdot [f] = [c_p]$ .

Finally, it is associative by (3) above.

## Proposition

Suppose  $p, q \in X$  with  $X$  path-connected.

Then  $\pi_1(X, p)$  is isomorphic to  $\pi_1(X, q)$ .

Remark: this isomorphism is not canonical.

## Proof

We define a path  $\gamma$  from  $q$  to  $p$  and  $\Phi_\gamma : \pi_1(X, p) \rightarrow \pi_1(X, q)$  by  $[f] \mapsto [\gamma \cdot f \cdot \bar{\gamma}]$ .

$\Phi_\gamma$  is a group homomorphism.

$$\begin{aligned} \Phi_\gamma[f] \cdot \Phi_\gamma[g] &= [\gamma \cdot f \cdot \bar{\gamma}] \cdot [\gamma \cdot g \cdot \bar{\gamma}] \\ &= [\gamma \cdot f \cdot \bar{\gamma} \cdot \gamma \cdot g \cdot \bar{\gamma}] \\ &= [\gamma \cdot f] \cdot \overbrace{[\bar{\gamma} \cdot \gamma]}^{=e} \cdot [g \cdot \bar{\gamma}] \\ &= [\gamma \cdot (f \cdot g) \cdot \bar{\gamma}] \\ &= \Phi_\gamma[f \cdot g]. \end{aligned}$$

$\Phi_\gamma$  has an inverse,  $\Phi_{\bar{\gamma}} : \pi_1(X, q) \rightarrow \pi_1(X, p)$ .

$$\Phi_{\bar{\gamma}} \circ \Phi_\gamma[f] = \Phi_{\bar{\gamma}}[\gamma \cdot f \cdot \bar{\gamma}] = [\bar{\gamma} \cdot \gamma \cdot f \cdot \bar{\gamma} \cdot \gamma] = [f].$$

## Induced Homomorphism

$\varphi : (X, p) \rightarrow (Y, q)$  induces

$$\begin{aligned}\varphi_* : \pi_1(X, p) &\rightarrow \pi_1(Y, q) \\ [f] &\mapsto [\varphi \circ f].\end{aligned}$$

$\varphi_*$  is a homomorphism.

$$(\varphi_*[f]) \cdot (\varphi_*[g]) = [\varphi \circ f] \cdot [\varphi \circ g] = [(\varphi \circ f) \cdot (\varphi \circ g)] = [\varphi \circ (f \cdot g)] = \varphi_*[f \cdot g].$$

## Proposition

If  $\varphi, \psi : (X, p) \rightarrow (Y, q)$  are homotopic, then  $\varphi_* = \psi_* : \pi_1(X, p) \rightarrow \pi_1(Y, q)$ .

## Proof

Let  $[f] \in \pi_1(X, p)$ ,  $\varphi_*[f] = [\varphi \circ f]$  and  $\psi_*[f] = [\psi \circ f]$  and  $H : X \times I \rightarrow Y$  a homotopy between  $\varphi$  and  $\psi$ . Then define  $\tilde{H} : I \times I \rightarrow Y$  by  $\tilde{H}(s, t) = H(f(s), t)$  such that

$$\begin{aligned}\tilde{H}(s, 0) &= H(f(s), 0) = \varphi \circ f(s) \\ \tilde{H}(s, 1) &= H(f(s), 1) = \psi \circ f(s).\end{aligned}$$

## Corollary

If  $X \simeq Y$ , then  $\pi_1(X) \simeq \pi_1(Y)$ .

## Examples (\*)

$\pi_1(S^1) \cong \mathbb{Z}$  and  $\pi_1(S^n) = 0$  for  $n \geq 2$ .

For  $n \geq 2$ , write  $S^n = A_+ \cup A_-$  where  $A_+$  and  $A_-$  are large balls centered at the north and south pole respectively.

Then  $A_+$  and  $A_-$  are both homeomorphic to  $\mathbb{R}^n$  and  $A_+ \cap A_-$  (their intersection about the equator) is homeomorphic to  $S^{n-1} \times \mathbb{R}$ .

We fix a base point  $p \in A_+ \cap A_-$  and let  $f : I \rightarrow S^n$  be a loop based at  $p$ .

There exists a partition of  $I$ ,  $0 = s_0 < s_1 < \dots < s_k = 1$ , such that  $f|_{[s_i, s_{i+1}]}$  is contained in  $A_-$  or  $A_+$ .

Draw a path  $\gamma_i$  from  $p$  to  $f(s_i)$  such that  $\gamma_i \subseteq A_+ \cap A_-$ . Let  $f_i = f|_{[s_i, s_{i+1}]}$  such that  $f = f_0 \cdot f_1 \cdots f_k$ . Then this is path homotopic to

$$(f_0 \cdot \bar{\gamma}_1) \cdot (\gamma_1 \cdot f_1 \cdot \bar{\gamma}_2) \cdots (\gamma_{k-1} \cdot f_{k-1} \cdot \bar{\gamma}_k) \cdot (\gamma_k \cdot f_k).$$

Each  $\gamma_i \cdot f_i \cdot \bar{\gamma}_i$  is contained in  $A_-$  or  $A_+$ , hence  $\gamma_i \cdot f_i \cdot \bar{\gamma}_i \sim c_p$ ,  $f \simeq c_p$  and  $[f] = e$ .

**April 2, 2025**

## Correction

For  $\varphi, \psi : (X, x_0) \rightarrow (Y, y_0)$  where  $\varphi \simeq \psi$ , we say a homotopy  $H$  between  $\varphi$  and  $\psi$  is base point preserving if  $H(x_0, t) = y_0$  for all  $t \in [0, 1]$ .

## Proposition

If  $\varphi \simeq \psi$  through a base point preserving homotopy, then  $\varphi_* = \psi_*$ ,  $\pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ .

For  $X \simeq Y$ ,  $\varphi : X \rightarrow Y$  and  $\psi : Y \rightarrow X$  where  $\psi \circ \varphi = \text{id}_X$  and  $\varphi \circ \psi = \text{id}_Y$ , in general  $\psi \circ \varphi(x_0) \neq x_0$  and  $\varphi \circ \psi(y_0) \neq y_0$ .

Set up:  $\varphi_0, \varphi_1 : X \rightarrow Y$  with  $\varphi_0 \simeq \varphi_1$  through a homotopy  $H$ .

Write  $\varphi_t = H(\cdot, t) : X \rightarrow Y$  and fix a base point  $x_0 \in X$  and set  $\gamma(t) = \varphi_t(x_0)$  for  $t \in [0, 1]$ .

## Proposition 1

$$(\varphi_0)_* = \Phi_\gamma \circ (\varphi_1)_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, \varphi(x_0)).$$

### Proof

Let  $f$  be a loop at  $x_0$ .

IMAGE 1

Let  $\gamma_t$  be  $\gamma|_{[0, t]}$  and then, by rescaling the domain  $[0, t]$  to  $[0, 1]$  i.e.

$$\begin{aligned} \gamma_t : [0, 1] &\rightarrow Y \\ s &\mapsto \gamma(ts). \end{aligned}$$

from  $\varphi_0(x_0)$  to  $\gamma(t) = \varphi_t(x_0)$ . Then  $\gamma_t \cdot (\phi_t \circ f) \cdot \bar{\gamma}_t$  is a homotopy between  $(\varphi_0 \circ f)$  and  $\gamma \cdot (\varphi_1 \circ f) \cdot \bar{\gamma}$ . Hence

$$(\varphi_0)_*[f] = [\varphi_0 \circ f] = [\gamma] \cdot [\varphi_1 \circ f] \cdot [\bar{\gamma}] = \Phi_\gamma \circ (\varphi_1)_*[f].$$

## Proposition 2

If  $X \simeq Y$ , then  $\pi_1(X) \cong \pi_1(Y)$ .

### Proof

Since  $(\psi \circ \varphi) \simeq \text{id}_X$ , by Proposition 1

$$\psi_* \circ \varphi_* = (\psi \circ \varphi)_* = \Phi_\gamma \circ (\text{id}_X)_* = \Phi_\gamma.$$

Hence  $\psi_* \circ \varphi_*$  is an isomorphism (as is  $\varphi_* \circ \psi_*$ ). Therefore  $\varphi_*$  and  $\psi_*$  are isomorphisms.

## Recall: Covering Map

For  $X, \tilde{X}$  connected,  $\pi : \tilde{X} \rightarrow X$  is a covering map if for each  $p \in X$  there exists a neighborhood  $U \subset X$  such that  $\pi^{-1}(U)$  is a disjoint union

$$\pi^{-1}(U) = \bigcup_{\alpha \in A} U_\alpha$$

such that  $\pi|_{U_\alpha} : U_\alpha \rightarrow U$  is a homeomorphism.

## Lifting Properties

A lift is a map  $\tilde{f}$  such that  $f = \pi \circ \tilde{f}$ .

1. Path Lifting: Let  $f : I \rightarrow X$  be a path from  $x_0$ . Then, for any  $\tilde{x}_0 \in \pi^{-1}(x_0)$ , there is a unique lift  $\tilde{f}$  of  $f$  with  $\tilde{f}(0) = \tilde{x}_0$ .
2. Homotopy Lifting: Let  $f_0, f_1 : I \rightarrow X$  be paths in  $X$  with  $f_0(0) = f_1(0) = x_0$  and  $f_0(1) = f_1(1)$ . Suppose  $H$  is a path homotopy between  $f_0$  and  $f_1$ . Then for any  $\tilde{x}_0 \in \pi^{-1}(x_0)$ , there is a unique lift  $\tilde{H} : I \times I \rightarrow \tilde{X}$  of  $H$ . In particular,  $\tilde{H}$  is a path homotopy between  $\tilde{f}_0$  and  $\tilde{f}_1$ . That is if  $H(0, t) = x_0$  then  $\tilde{H}(0, t) \in \pi^{-1}(x_0)$  for all  $t$ . Hence  $\tilde{H}(0, t) = \tilde{x}_0$ ,  $\forall t \in [0, 1]$ . Similarly,  $\tilde{H}(1, t)$  is identically constant. In particular,  $\tilde{f}_0(1) = \tilde{H}(1, 0) = \tilde{H}(1, 1) = \tilde{f}_1(1)$ .

## Fundamental Group of the Circle

$$\pi_1(S^1) = \mathbb{Z}.$$

### Example

$$\pi : \mathbb{R} \rightarrow S^1 \text{ by } s \mapsto e^{2\pi i \cdot s}.$$

### Proof

Take as a base point  $1 = x_0 \in S^1 \subseteq \mathbb{C}$ . For each  $n \in \mathbb{Z}$ , we define a loop  $\omega_n : [0, 1] \rightarrow S^1$  by  $s \mapsto e^{2\pi i \cdot ns}$ . Let  $f$  be a loop at  $x_0 \in S^1$ . We can lift  $f$  to  $\tilde{f} : I \rightarrow \mathbb{R}$  at  $0 \in \mathbb{R}$ . Then  $\tilde{f}(1) \in \pi^{-1}(x_0) = \mathbb{Z} \subseteq \mathbb{R}$ . This defines a map  $\varphi$  that sends a loop  $f$  to  $\tilde{f}(1) \in \mathbb{Z}$ . This  $\varphi$  induces  $\varphi : \pi_1(S^1, x_0) \rightarrow \mathbb{Z}$  well-defined. If  $f_0, f_1 : I \rightarrow S^1$  at  $x_0$  are path homotopic via  $H$ , then we may lift  $H$  to  $\tilde{H} : I \times I \rightarrow \mathbb{R}$  which implies  $\tilde{f}_0(1) = \tilde{f}_1(1)$ .

$\varphi$  is surjective, since for any  $n \in \mathbb{Z}$  we may consider the loop  $\omega_n$  where  $\tilde{\omega}_n(1) = n$ .

$\varphi$  is a group homomorphism since  $\varphi[f \cdot g] = \tilde{f \cdot g}(1) = \tilde{g} + \tilde{f}(1) = \varphi[f] + \varphi[g]$ .

$\varphi$  is injective, since if  $\varphi[f] = 0$  (i.e.  $\tilde{f}(0) = 0$ ) then  $\tilde{f}$  is a loop in  $\mathbb{R}$  and  $\tilde{f}$  is null-homotopic to  $c_0$  by  $H$ . Therefore  $\pi \circ \tilde{H}$  is a path-homotopy between  $f$  and  $c_{x_0}$  (i.e.  $[f] = e$ ).

## Path-Lifting

For  $f : I \rightarrow X$ , we have a special case where  $\text{im } f \subseteq U$  evenly covered. Write  $\pi^{-1}(U) = \dot{\bigcup} \tilde{U}_\alpha$  and pick the  $\tilde{U}_\alpha$  which contains  $\tilde{x}_0$ . Since  $\pi|_{\tilde{U}_\alpha} : \tilde{U}_\alpha \rightarrow U$  is a homeomorphism,  $\tilde{f} := (\pi|_{\tilde{U}_\alpha})^{-1} \circ f$  is the unique lift of  $f$  at  $\tilde{x}_0$ .

In general, pick a partition of  $I = [0, 1]$ ,  $0 = t_0 < t_1 < \dots < t_m = 1$ , such that  $\text{im } f|_{[t_i, t_{i+1}]} \subseteq U_i$  evenly covered. We can lift  $f|_{[0, t_1]}$  at  $\tilde{x}_0$ , giving  $\tilde{f} : [0, t_1] \rightarrow \tilde{X}$ . Next, we lift  $f|_{[t_1, t_2]}$  at  $\tilde{f}(t_1) \in \tilde{X}$ . Since the partition is finite, we may repeat the process until  $f$  is entirely lifted. This lift is unique.

## Homotopy Lifting

For each fixed  $(y_0, t_0) \in I \times I$ , by continuity, there is a neighborhood  $N(y_0) \times (t_0 - \varepsilon, t_0 + \varepsilon)$  such that  $H$  sends  $N(y_0) \times (t_0 - \varepsilon, t_0 + \varepsilon)$  inside an evenly covered neighborhood. By compactness of  $\{y_0\} \times [0, 1]$ , there is a finite collection of  $N_{t_i}(y_0) \times (t_i - \varepsilon_i, t_i + \varepsilon_i)$  such that they cover  $\{y_0\} \times I$  and the image of each under  $H$  is contained in an evenly covered neighborhood. Set  $N = \bigcap_i N_{t_i}(y_0)$ , a neighborhood of  $y_0$ , and construct a partition  $0 = t_0 < t_1 < \dots < t_m = 1$  such that  $H(N \times [t_i, t_{i+1}]) \subseteq U_i$  evenly covered. Then we can start with  $H|_{N \times [0, t_1]}$  and lift it at  $\tilde{x}_0$  by some  $(\pi|_{\tilde{U}_\alpha})^{-1}$ . Then lift each  $H|_{N \times [t_i, t_{i+1}]}$  one by one. Eventually, we have  $\tilde{H} : N \times [0, 1] \rightarrow \tilde{X}$  that lifts  $H : N \times [0, 1] \rightarrow \tilde{X}$  at  $\tilde{x}_0$ . This lift holds for any  $y_0 \in I$  and, if two strips overlap, then the lift must agree there by the uniqueness of path lifting. This assures that  $\tilde{H} : I^2 \rightarrow \tilde{X}$  is continuous.

## Remark

Given a continuous map  $F : Y \times I \rightarrow X$  and a covering  $\pi : \tilde{X} \rightarrow X$ , suppose that we have a map  $\tilde{F} : Y \times \{0\} \rightarrow \tilde{X}$  that lifts  $F|_{Y \times \{0\}} : Y \times \{0\} \rightarrow X$ . Then there is a unique lift  $\tilde{F} : Y \times I \rightarrow \tilde{X}$  of  $F$  which extends  $\tilde{F} : Y \times \{0\} \rightarrow \tilde{X}$ .

## Theorem: Fundamental Theorem of Algebra

A polynomial  $p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$  (with  $a_i \in \mathbb{C}$ ) has a root in  $\mathbb{C}$ .

### Proof

Suppose otherwise. Then  $p(z) \neq 0, \forall z \in \mathbb{C}$ . Consider  $f_r : [0, 1] \rightarrow S^1$  ( $r \geq 0$ ) by

$$f_r(s) = \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}.$$

Then  $f_0(s) \equiv 1$  is a constant loop at  $1 \in \mathbb{C}$ , and  $f_r \simeq f_0$  for each  $r \geq 0$ . Consider  $R \geq 1$  large such that  $R \gg \sum_{i=1}^n |a_i|$ . On  $\{z : |z| = R\}$ , we have

$$|z^n| > \left( \sum_{i=1}^n |a_i| \right) \cdot |z^{n-1}| \geq \sum_{i=1}^n |a_i| \cdot |z^{n-i}| = \left| \sum_{i=1}^n a_i z^{n-i} \right|.$$

This implies that  $p$  does not have any roots on  $\{|z| = R\}$ . Moreover, for  $p_t(z) = z^n + t(a_1 z^{n-1} + \cdots + a_{n-1} z + a_n)$  with  $0 \leq t \leq 1$ ,  $p_t$  does not have any roots on  $\{|z| = R\}$ . Consider

$$f_{R,t}(s) = \frac{p_t(Re^{2\pi i s})/p_t(R)}{|p_t(Re^{2\pi i s})/p_t(R)|}.$$

Then

$$f_{R,0}(s) = \frac{(Re^{2\pi i s})^n / R^n}{|(Re^{2\pi i s})^n / R^n|} = (e^{2\pi i s})^n = \omega_n(s).$$

Therefore  $f_{R,1}(s) \simeq f_R(s)$  and  $f_R \simeq \omega_n$ . But since  $\omega_n \neq \text{constant}$  so this is a contradiction.

**April 7, 2025**

## Definition: Retraction

Let  $X$  be a space and  $A \subseteq X$  be a subset. We say that a continuous map  $r : X \rightarrow A$  is a retraction if  $r|_A = \text{id}_A$ . In particular, because  $r \circ \iota_A = \text{id}_A$ , for  $x_0 \in A$

$$r_* \circ (\iota_A)_* : \pi_1(A, x_0) \rightarrow \pi_1(A, x_0)$$

is an isomorphism. Hence  $r_* : \pi(X, x_0) \rightarrow \pi(A, x_0)$  is surjective.

### Corollary

There is no retraction  $r : D^2 \rightarrow S^1 (= \partial D^2)$ .

## Proof

Suppose there is such a map  $r$ , then

$$r_* : \overbrace{\pi_1(D^2, x_0)}^{=0} \rightarrow \overbrace{\pi_1(S^1, x_0)}^{=\mathbb{Z}}$$

is surjective which is a contradiction.

## Corollary

Every continuous map  $h : D^2 \rightarrow D^2$  has a fixed point.

## Proof

Suppose  $\exists h : D^2 \rightarrow D^2$  without fixed points.

IMAGE 1

Define  $r : D^2 \rightarrow D^2$  as the ray pictured from  $h(x)$  through  $x$  to the boundary. If  $x \in \partial D^2$ , then by construction  $r(x) = x$ . Hence  $r : D^2 \rightarrow S^1$  is a retraction which is a contradiction.

## Corollary (Borsuk-Ulam)

Let  $f : S^2 \rightarrow \mathbb{R}^2$ . Then there exists a pair of antipodal points  $x$  and  $-x$  on  $S^2$  such that  $f(x) = f(-x)$ . This carries analogously to higher dimensions.

## Proof

Suppose that  $f(x) \neq f(-x)$  for all  $x \in S^2$ . We define  $g : S^2 \rightarrow S^1$  by  $g(x) = \frac{f(x)-f(-x)}{\|f(x)-f(-x)\|}$ . On  $S^2 \subseteq \mathbb{R}^3$ , we consider a loop  $\gamma$  at the equator by  $\gamma(s) = (\cos(2\pi s), \sin(2\pi s), 0)$  for  $s \in [0, 1]$ . Because  $S^2$  is simply connected,  $g \circ \gamma : [0, 1] \rightarrow S^1$  is path-homotopic to a constant loop in  $S^1$ . On the other hand, we lift  $h := g \circ \gamma$  to  $\tilde{h} : [0, 1] \rightarrow \mathbb{R}$  with  $\tilde{h}(0) = 0 \in \mathbb{R}$ . Note

$$h(s + 1/2) = g \circ \gamma(s + 1/2) = g(\cos(2\pi s + \pi), \sin(2\pi s + \pi), 0) = g(-\gamma(s)) = -g(\gamma(s)) = -h(s).$$

Hence  $\tilde{h}(s + 1/2) \in \pi^{-1}(-h(s))$  where  $\pi : \mathbb{R} \rightarrow S^1$  is the covering map. Since  $\pi^{-1}(-h(s)) = \frac{1}{2} + \tilde{h}(s) + \mathbb{Z}$ , for each  $s \in [0, 1/2]$  there is an integer  $q_s$  such that  $\tilde{h}(s + 1/2) = \frac{1}{2} + \tilde{h}(s) + q_s$  and

$$\tilde{h}(s + 1/2) - \tilde{h}(s) = \frac{1}{2} + q_s.$$

The left hand side depends continuously on  $s$  and, by continuity,  $q_s$  is a constant (call it  $q$ ). This gives

$$\tilde{h}(1) = \tilde{h}(1/2) + \frac{1}{2} + q = \tilde{h}(0) + 1 + 2q = 1 + 2q \neq 0$$

which contradicts the assertion that  $h$  is homotopic to a constant loop.

## Corollary (Large Fiber Lemma)

If  $f : [0, 1]^{n+1} \rightarrow \mathbb{R}^n$  is a continuous map, then there exist  $a, b \in [0, 1]^{n+1}$  such that  $f(a) = f(b)$  and  $|a - b| \geq 1$ .

Remark: if  $z = f(a) = f(b)$ , then the lemma says that  $\text{diam } f^{-1}(z) \geq 1$ .



## Proof

Take the sphere of radius  $1/2$  in  $[0, 1]^{n+1}$ , then by Borsuk-Ulam there exist a pair of antipodal points  $a, b \in S^1$  such that  $f(a) = f(b)$  and  $|a - b| \geq 1$ .

## Proposition

$$\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$$

## Proof

Write  $F : \pi_1(X \times Y) \rightarrow \pi_1(X) \times \pi_1(Y)$  by  $[f] \mapsto ([g], [h])$ . Then  $f : [0, 1] \rightarrow X \times Y$  is a loop at  $(x_0, y_0)$ ,  $f(s) = (g(s), h(s))$ , and  $g : [0, 1] \rightarrow X$  and  $h : [0, 1] \rightarrow Y$  are loops at  $x_0$  and  $y_0$  respectively.

## Definition: Wedge Sum

Let  $X$  and  $Y$  be path-connected topological spaces. Then  $X \vee Y = (X \amalg Y) / x_0 \sim y_0$   
Let  $\{X_\alpha\}$  be a family of such spaces. Then  $\bigvee_\alpha X_\alpha = \bigamalg_\alpha X_\alpha / \sim$ .

## Sketch

$$\pi_1(S_-^1, x_0) \rightarrow \pi_1(X, x_0) \quad \text{gen} \mapsto \alpha$$

$$\pi_1(S_+^1, x_0) \rightarrow \pi_1(X, x_0) \quad \text{gen} \mapsto \beta$$

with  $\alpha \neq \beta$ ,  $\alpha\beta \neq \beta\alpha$ . Then  $\pi_1(X, x_0)$  should be  $\langle \alpha, \beta \rangle$ .

## Definition: Free Product

Let  $\{G_\alpha\}_\alpha$  be a family of groups.  $*_\alpha G_\alpha = \{g_1 g_2 \cdots g_k : \text{each } g_i \text{ is a word in some } A_\alpha\}$ .

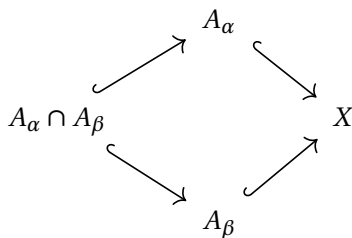
## Proposition

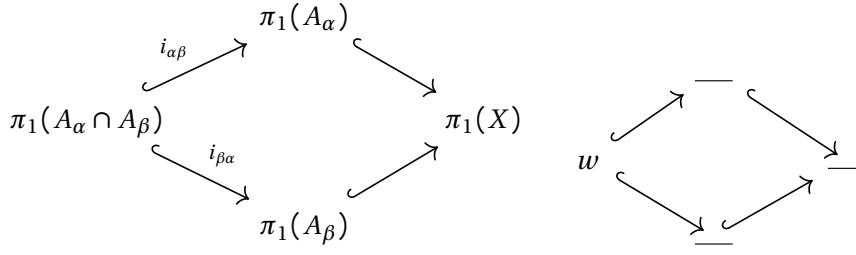
If for each  $\alpha$ , there is a group homomorphism  $\phi_\alpha : G_\alpha \rightarrow H$  then  $\{\phi_\alpha\}$  induces a group homomorphism  $\Phi : *_\alpha G_\alpha \rightarrow H$  by  $g_1 \cdots g_k \mapsto \phi_{\alpha_1}(g_1) \cdots \phi_{\alpha_k}(g_k)$ .

## Van-Kapen Theorem

### Setup

Let  $X = \bigcup_\alpha A_\alpha$ , each  $A_\alpha$  open and connected where  $\{A_\alpha\}$  have a common point  $x_0$ . Assume also that each  $A_\alpha \cap A_\beta$  is path connected. Then  $j_\alpha : A_\alpha \hookrightarrow X$  induces  $j_\alpha : \pi_1(A_\alpha, x_0) \rightarrow \pi_1(X, x_0)$ .  $\{j_\alpha\}_\alpha$  induces  $\Phi : *_\alpha \pi_1(A_\alpha, x_0) \rightarrow \pi_1(X, x_0)$  which is surjective by a similar argument as was used above for Example (\*) ( $S^2 = A_- \cup A_+$ ) applied to  $X = \bigcup_\alpha A_\alpha$ . Now, what is the kernel of  $\Phi$ ?





Then  $i_{\beta\alpha}(w)i_{\alpha\beta}(w)^{-1}$  is NOT id in  $*_{\alpha}\pi_1(A_{\alpha})$ .

But through  $\Phi$ , it should be  $\text{id} \in \pi_1(X, x_0)$ . Hence every element in  $*_{\alpha}\pi_1(A_{\alpha})$  of the form  $i_{\beta\alpha}(w)i_{\alpha\beta}(w)^{-1}$  where  $w \in \pi_1(A_{\alpha} \cap A_{\beta})$  is in the kernel of  $\Phi$ .

### Theorem (Van-Kampen)

If every  $A_{\alpha} \cap A_{\beta} \cap A_{\gamma}$  is path connected,  $\ker \Phi$  is the normal subgroup  $N$  generated by  $\{i_{\beta\alpha}(w)i_{\alpha\beta}(w)^{-1} : \alpha, \beta \in A, w \in \pi_1(A_{\alpha} \cap A_{\beta})\}$ . Hence  $\pi_1(X, x_0) \cong (*_{\alpha}\pi_1(A_{\alpha}, x_0))/N$ .

### Remarks

1. In the case that  $X = A_0 \cup A_1$  with  $A_0 \cap A_1$  path connected, then the intersection condition holds.
2. If  $X = A_0 \cup A_1$  and  $A_0 \cap A_1$  is simply connected, then  $N = \{\text{id}\}$  and  $\pi_1(X) = \pi_1(A_0) * \pi_1(A_1)$ .
3. If  $X = A_0 \cup A_1$  and  $A_1$  is simply connected, then  $\pi_1(X) = \pi_1(A_0)/N$  and  $N$  is the normal subgroup generated by

$$i_{01}(w) \overbrace{i_{10}(w)^{-1}}^{\in \pi_1(A_1, x_0)} = i_{01}(w)$$

i.e.  $N$  is the normal closure of  $i_{01}(\pi_1(A_0 \cap A_1))$ .

### Example

IMAGE 2

For each  $\alpha \in \{1, \dots, 5\}$ , let  $A_{\alpha}$  be a small neighborhood of  $T \cup e_1$ . Every double/triple intersection is a neighborhood of  $T$ . Hence it is path continuous and we have that  $\pi_1(A_{\alpha}) = \mathbb{Z}$ . Thus  $\pi_1(A_{\alpha} \cap A_{\beta}) = \text{id}$ , and  $\pi_1(X) = *_{\alpha}\pi_1(A_{\alpha})/N = *_1^5 \mathbb{Z}$ .

### Example

IMAGE 3

By Van-Kampen,  $\pi_1(X) = \pi_1(A_0)$  modulo the normal closure of  $i(\pi_1(A_0 \cap A_1))$ . That is

$$\langle a, b \mid aba^{-1}b^1 \rangle = \mathbb{Z}^2.$$

### Remark

In general, orientable  $M_g$  is the connected sum of  $g$  many toruses.

April 9, 2025

## Recall: Van-Kampen Theorem

Write  $\pi_1(X) = (\pi_1(A) * \pi_1(B))/N$  where  $N$  is the normal closure of  $i_{\alpha\beta}(w)i_{\beta\alpha}(w)^{-1}$  for  $w \in \pi_1(A \cap B)$ ,  $i_{\alpha\beta} : \pi_1(A \cap B) \rightarrow \pi_1(A)$  and  $i_{\beta\alpha} : \pi_1(A \cap B) \rightarrow \pi_1(B)$ .

### Example

$M_g$  is the connected sum of  $g$  many tori, and  $\pi_1(M_g) = \langle a_1, b_1, \dots, a_g, b_g \mid [a_1 b_1] \cdots [a_g b_g] \rangle$ .

### Example

$N_g$  is the connected sum of  $g$  many  $\mathbb{RP}^2$  (e.g.  $N_2$  is the Klein bottle).  $N_g$  has a polygon-representation by the  $2g$ -gon with boundary identified through  $a_1 a_1 a_2 a_2 \cdots a_g a_g$ . Therefore  $\pi_1(N_g) = \langle a_1 \cdots a_g \mid a_1^2 \cdots a_g^2 \rangle$ .

## Abelianization

1.  $\text{Ab}(\pi_1(M_g))$  is the free abelian group generated by  $\{a_1, b_1, \dots, a_g, b_g\} = \mathbb{Z}^{2g}$ .
2.  $\text{Ab}(\pi_1(N_g)) = \text{Ab}(\langle a_1 \cdots a_{g-1} a_1 a_2 \cdots a_g \mid a_1^2 \cdots a_g^2 \rangle) = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$ .

### Corollary

None of the surfaces in  $\{S^2, M_1, \dots, M_g, \dots, N_1, \dots, N_g, \dots\}$  are homotopic to each other.

## Definition: Cell Complex

0-cells are points; 1-cells,  $e^1$ , are intervals; 2-cells,  $e^2$ , are disks;  $n$ -cells,  $e^n$ , are  $\overline{B}^n$ .

A cell complex for space  $X$  is a decomposition (assuming finite dimensions)  $X = X^0 \cup X^1 \cup \dots \cup X^n$  where  $X^0$  is the discrete set of points (i.e. 0-cells),  $X^1$  is the space obtained by gluing 1-cells to  $X^0$  ( $\varphi_\alpha : \partial e_\alpha^1 \rightarrow X^0$ ),  $X^2$  is the space obtained by gluing 2-cells to  $X^1$  ( $\varphi_\alpha : \partial e_\alpha^2 \rightarrow X^1$ ), and in general  $X^n$  is obtained by gluing  $n$ -cells  $\{e_\alpha^n\}_\alpha$  to  $X^{n-1}$  by  $\varphi_\alpha : \partial e_\alpha^n = S^{n-1} \rightarrow X^{n-1}$ .

### Examples

Cell complexes need not be unique.  $S^2 = X^1 \cup_\alpha e_+^2 \cup_\alpha e_-^2$  and  $S^2 = \{e^0\} \cup_\alpha \{e^2\}$ .

$\mathbb{RP}^2 = \{e^1\} \cup_\alpha \{e^2\}$  where  $\varphi_\alpha$  is given by  $z \mapsto z^2$ .

$\mathbb{T}^2$  is gluing  $e^2$  to  $S^1 \vee S^1$ .

## Theorem (Computing Fundamental Group)

### Set up

Let  $X$  be a path-connected space,  $Y = X \cup_\alpha e_\alpha^2$  (i.e.  $X$  is created by gluing 2-cells  $\{e_\alpha^2\}_\alpha$  to  $X$  via  $\phi_\alpha : \partial e_\alpha^2 \rightarrow X$ ). The inclusion  $\iota : X \rightarrow Y$  induces  $\iota_* : \pi_1(X) \rightarrow \pi_1(Y)$ . Fix a base point  $s_0 \in S^1$ . For each  $\alpha$  we draw a path  $\gamma_\alpha$  from  $x_0$  to  $\varphi_\alpha(s_0)$ . Then  $\gamma_\alpha \cdot \varphi_\alpha \cdot \bar{\gamma}_\alpha$  is a loop based at  $x_0$ . Thus  $\gamma_\alpha \cdot \varphi_\alpha \cdot \bar{\gamma}_\alpha$  is null-homotopic in  $Y$  (because  $\varphi_\alpha$  is null-homotopic in  $e_\alpha^2$ ). That is  $\iota_*[\gamma_\alpha \cdot \varphi_\alpha \cdot \bar{\gamma}_\alpha] = \text{id}$  in  $\pi_1(Y)$  and is therefore in the kernel.

## Theorem

Let  $N$  be the normal subgroup in  $\pi_1(X)$  generated by elements of the form  $[\gamma_\alpha \cdot \varphi_\alpha \cdot \bar{\gamma}_\alpha]$ . Then  $\pi_1(Y) \cong \pi_1(X)/N$ .

IMAGE 1

## Example

$\mathbb{RP}^2$  is  $X^1$  with  $e^2$  glued to it by the map  $\varphi : z \mapsto z^2$ . Then  $\pi_1(\mathbb{RP}^2) = \pi_1(S^1)/N = \langle \gamma \mid \gamma^2 \rangle$  where  $N$  is generated by  $\varphi$ . Similarly, the theorem applies to any  $M_g$  or  $N_g$ .

## Definition: Deformation Retraction

For  $X \subseteq Z$ ,  $r : Z \rightarrow X$  is a retraction if  $r|_X = \text{id}_X$  implies  $r \circ \iota = \text{id}_X$ . If  $\iota \circ r : Z \rightarrow Z$  is homotopic to  $\text{id}_Z$ , then  $r_* : \pi_1(Z) \rightarrow \pi_1(X)$  is an isomorphism.

## Proof

For each  $\alpha$ , we glue a strip  $S_\alpha$  along  $\gamma_\alpha$ . We set the base at  $z_0$  above  $x_0$ ,  $Z = Y \cup_\alpha S_\alpha$ .  $Y$  is a deformation retraction of  $Z$  ( $\pi_1(Y) = \pi_1(Z)$ ).

IMAGE 2

Set  $A = Z - \bigcup_\alpha \{y_\alpha\}$ , where  $y_\alpha$  is a point in  $e_\alpha^2$  not intersecting  $S_\alpha$ .  $B = Z - X$ . A deformation retracts to  $X$   $\pi_1(A) = \pi_1(X)$ .  $B$  is the union of some  $S_\alpha$  (removing  $r_\alpha$ ) and some  $e_\alpha^2$  (removing  $\partial e_\alpha^2$ ).  $B$  is contractible,  $\pi_1(B) = \text{id}$  and  $A \cap B$  is the union of strips  $S_\alpha$  and open disks punctured at  $y_\alpha$ . Therefore

$$\pi_1(Y) = \pi_1(Z) = (\pi_1(A) * \pi_1(B))/N = \pi_1(A)/\iota_*(\pi_1(A \cap B)) \cong \pi_1(X)/\iota_*(\pi_1(A \cap B)).$$

Consider the loop  $\delta_\alpha \cdot \gamma_\alpha \cdot \varphi_\alpha \cdot \bar{\gamma}_\alpha \cdot \bar{\delta}_\alpha$  where  $\delta_\alpha$  runs from  $z_0$  to  $x_0$ , call this  $\lambda_\alpha$ . It suffices to show that these generate  $\pi_1(A \cap B, z_0)$ . Cover  $A \cap B$  by  $A_\alpha = (A \cap B) - \bigcup_{\beta \neq \alpha} e_\beta^2$ . Then  $A_\alpha$  is a union of strips (with trivial fundamental group) and a single punctured, open disk  $e_\alpha^2 - \{y_\alpha\}$  and  $\pi_1(A_\alpha) = \mathbb{Z} = \langle \lambda_\alpha \rangle$ . So  $A_\alpha \cap A_\beta$  is the union of strips, equal to  $A_\alpha \cap A_\beta \cap A_\gamma$  and simply connected. By Van-Kampen,

$$\pi_1(A \cap B) = (*_\alpha \pi_1(A_\alpha))/N = *_\alpha \pi_1(A_\alpha)$$

is the free group generated by  $\{\lambda_\alpha\}_\alpha$ . This completes the proof.

## Generalization (Theorem: Part 2)

If  $Y = X \cup_\alpha e_\alpha^n$  for  $n \geq 3$ , then  $\pi_1(Y) \cong \pi_1(X)$ .

This follows from the same argument where instead  $A_\alpha$  is the union of strips and a single punctured ball  $B^n - \{y_\alpha\} \simeq S^{n-1}$ . So  $\pi_1(A_\alpha) = \text{id}$ ,  $\pi_1(A \cap B) = \text{id}$ , and  $\pi_1(X) \cong \pi_1(Y)$ .

## Theorem: Part 3

Suppose  $X$  has a cell complex  $X = X^0 \cup X^1 \cup \dots \cup X^n$ . Then  $\pi_1(X) \cong \pi_1(X^2)$ .

The proof follows directly from part 2.

## Corollary

Given any group represented by generators and relations  $G = \langle g_\alpha \mid r_\beta \rangle$ , there is a cell complex  $X_G$ , of dimension 2, such that  $\pi_1(X_G) \cong G$ .

## Proof

For each  $g_\alpha$ , we draw a circle  $S_\alpha^1$ . Then  $X^1 = \bigvee_\alpha S_\alpha^1$  has fundamental group  $*_a \pi_1(S_\alpha) = \langle g_\alpha \rangle_\alpha$ . To construct  $X_G$ , for each  $r_\beta$  glue a 2-cell  $e_\alpha^2$  along  $r_\beta$  (think of  $r_\beta$  as a loop in  $X^1$ ). Then in  $X_G := X^1 \cup_\beta e_\beta^2$  we have  $\pi_1(X_G) = \langle g_\alpha \mid r_\beta \rangle$ .

**April 14, 2025**

## Recall: Covering Spaces

Let  $p : \tilde{X} \rightarrow X$ , both  $X$  and  $\tilde{X}$  path-connected.

1. Path-lifting: let  $f : I \rightarrow X$  starting at  $f(0) = x_0$ . There is a unique lifting  $\tilde{f}$  of  $f$  at  $\tilde{x}_0 \in p^{-1}(x_0)$ .
2. Homotopy-lifting: let  $f_0, f_1 : I \rightarrow X$  be two paths with  $f_0(0) = f_1(0) = x_0$  and  $f_0(1) = f_1(1)$ . Suppose  $f_t$  is a path-homotopy between  $f_0$  and  $f_1$ . Then there exists a unique lift  $\tilde{f}_t$  between  $\tilde{f}_0$  and  $\tilde{f}_1$  at  $\tilde{x} \in p^{-1}(x)$ .

These come from the following: let  $f_t : Y \rightarrow X$  be a homotopy between  $f_0$  and  $f_1$ . Given  $\tilde{f}_0 : Y \rightarrow \tilde{X}$  that lifts  $f_0$ , there exists a unique lifting  $\tilde{f}_t$ . For path-lifting, we take  $Y$  a point; for homotopy-lifting,  $Y = [0, 1]$ .

$$\begin{array}{ccc} & \tilde{X} & \\ f \nearrow & & \downarrow p \\ I & \xrightarrow{p \circ f} & X \end{array}$$

## Proposition 1.31 (in Hatcher)

The covering map  $p : \tilde{X} \rightarrow X$  induces  $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x)$ .

1.  $p_*$  is injective.
2.  $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$  are exactly loops at  $x_0$  that lift to loops at  $\tilde{x}_0$ .

## Proof of 1

Suppose  $p_*[f] = \text{id} \in \pi_1(X, x_0)$  where  $[f] \in \pi_1(\tilde{X}, \tilde{x}_0)$ . Then  $[p \circ f] = \text{id}$ , and  $[p \circ f]$  is path-homotopic to the constant loop  $c_{x_0}$ . Hence the lifting  $\tilde{p \circ f} = f$  is path-homotopic to a constant loop  $c_{\tilde{x}_0}$ .

## Proof of 2

Let  $[f] \in \pi_1(\tilde{X}, \tilde{x}_0)$ .  $p_*[f] = [p \circ f]$ ,  $p \circ f$  lifts to  $f$  at  $\tilde{x}_0$  which is a loop at  $\tilde{x}_0$ .

Let  $f$  be a loop at  $\tilde{x}_0$ . Suppose  $f$  lifts to a loop  $\tilde{f}$  at  $\tilde{x}_0$  (i.e.  $p \circ \tilde{f} = f$ ). Hence  $[f] = [p \circ \tilde{f}] = p_*[\tilde{f}] \in p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ .

## Example

If  $p : S^1 \rightarrow S^1$  by  $z \rightarrow z^2$ , then  $p_*(\pi_1(S^1, 1)) = 2\mathbb{Z} \leq \mathbb{Z} = \pi_1(S^1, 1)$ .

## Remark

If  $p : \tilde{X} \rightarrow X$  connected, then  $p^{-1}(x)$  has the same cardinality for all  $x \in X$ .

## Proof

Fix  $x_0 \in X$ . Consider  $\mathcal{A} = \{x \in X : p^{-1}(x) \text{ and } p^{-1}(x_0) \text{ have the same cardinality}\} \neq \emptyset$ . Then  $\mathcal{A}$  is open since for each  $x \in \mathcal{A}$ , there is a neighborhood  $U$  of  $x$  such that  $U$  is evenly covered by  $p$  (i.e.  $p^{-1}(U) = \bigcup_{\alpha \in I} V_\alpha$  where  $V_\alpha \stackrel{p}{\cong} U$ ). Then  $p^{-1}(x')$  has cardinality  $|I|$  for all  $x' \in U$ . It follows, since  $\mathcal{A}^c$  is open, that  $\mathcal{A}$  is also closed.

## Proposition

The number of sheets is given by  $[\pi_1(X, x_0) : p_*(\pi_1(\tilde{X}, \tilde{x}_0))]$ .

## Proof

Write  $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ . Define  $\Phi : \{H\text{-cosets in } \pi_1(X, x_0)\} \rightarrow p^{-1}(x_0)$  by  $H[g] \mapsto \tilde{g}(1)$  where  $\tilde{g}$  is a lift of  $g$  at  $\tilde{x}_0$ . This map is well defined, since for  $[h \cdot g]$  with  $h \in H$ ,  $\overline{h \cdot g}(1) = \tilde{g}(1)$  (because  $\tilde{h}(1) = \tilde{x}_0$ ).  $\Phi$  is surjective. Let  $\tilde{x}_1 \in p^{-1}(x_0)$

IMAGE 1

and let  $\tilde{g}$  be a path from  $\tilde{x}_0$  to  $\tilde{x}_1$ . Define  $g = p \circ \tilde{g}$ , a loop at  $x_0$ . Then  $\Phi(H[g]) = \tilde{g}(1) = \tilde{x}_1$ .  $\Phi$  is injective. Suppose  $\Phi(H[g_1]) = \Phi(H[g_2])$  (i.e.  $\tilde{g}_1(1) = \tilde{g}_2(1)$ ).

IMAGE 2

Consider the loop  $g_1 \bar{g}_2$  in  $X$  at  $x_0$ . It lifts to  $\tilde{g}_1 \bar{\tilde{g}_2}$ , which is a loop at  $\tilde{x}_0$ . This shows that  $[g_1 \bar{g}_2] \in H$  (i.e.  $H[g_1] = H[g_2]$ ).

## Recall (Manifolds 2)

If a smooth manifold  $M$  is non-orientable, then there is a double cover (2 sheets)  $p : \hat{M} \rightarrow M$  ( $\hat{M}$  connected). Consequently,  $\pi_1(M)$  has a subgroup of index 2.

## Definition: Locally Path-Connected

A topological space is called locally path-connected if for each  $x \in X$  and every neighborhood  $U \ni x$ , there is a neighborhood  $V \ni x$  such that  $V \subseteq U$  and  $V$  is path-connected (i.e.  $\forall x \in X$ , there exists a local basis  $\{U_\alpha\}$  at  $x$  such that each  $U_\alpha$  is path-connected). For example, the Topologist's sine curve with endpoints identified is path-connected but not locally path-connected.

## Proposition: Lifting Criterion

Let  $Y$  be path-connected and locally path-connected. Given a covering map  $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$  and a map  $f : (Y, y_0) \rightarrow (X, x_0)$ ,  $f$  has a lift  $\tilde{f}$  at  $\tilde{x}_0$  ( $\tilde{f}(y_0) = \tilde{x}_0$ ) if and only if  $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ .

## Proof

( $\Rightarrow$ )

$$\begin{array}{ccc} & \tilde{X} & \\ \tilde{f} \nearrow & \downarrow p & \\ Y & \xrightarrow{f} & X \end{array} \quad \begin{array}{ccc} & \pi(\tilde{X}) & \\ \tilde{f}_* \nearrow & \downarrow p_* & \\ \pi_1(Y) & \xrightarrow{f_*} & \pi_1(X) \end{array}$$

$$f_*\pi_1(Y) = (p_* \circ \tilde{f}_*)(\pi_1(Y)) \subseteq p_*\pi_1(\tilde{X}).$$

( $\Leftarrow$ ) Let  $y \in Y$ , and draw a path  $\gamma$  from  $y_0$  to  $y$ .

### IMAGE 3

We lift  $f \circ \gamma$  to a path in  $\tilde{X}$  starting at  $\tilde{x}_0$  and define  $\tilde{f}(y)$  as the endpoint (i.e.  $\tilde{f}(y) = \widetilde{f \circ \gamma}(a)$ ).

This is well-defined, since  $(f \circ \gamma) \cdot (f \circ \gamma')$  is a loop at  $x_0$  and  $[(f \circ \gamma) \cdot (f \circ \gamma')] = f_*[\gamma \cdot \gamma'] \in f_*\pi_1(Y, y_0) \leq p_*\pi_1(\tilde{X}, \tilde{x}_0)$ . Hence  $(f \circ \gamma) \cdot (f \circ \gamma')$  lifts to a loop at  $\tilde{x}_0$ .

### IMAGE 4

Therefore  $\widetilde{f \circ \gamma}(1) = \widetilde{f \circ \gamma'}(1)$ .

$\tilde{f}$  is continuous. Fix  $f(y) \in X$  and let  $U$  be a neighborhood of  $f(y)$  that is evenly covered by  $p$ . Choose a path-connected neighborhood  $V$  of  $y$  such that  $f(V) \subseteq U$ . We check  $\tilde{f}|_V$ .

### IMAGE 5

Because  $V$  is path-connected, we may draw a path  $\eta$  in  $V$  from  $y$  to  $y'$ . Then  $\tilde{f}(y') = \widetilde{f \circ \gamma \circ \eta}(1)$ , and  $\widetilde{\gamma \cdot \eta}$  is first lifting  $f \circ \gamma$  at  $\tilde{x}_0$  followed by lifting  $f \circ \eta$  at  $\tilde{\gamma}(1)$ . Let  $\tilde{U} \subseteq \tilde{X}$  such that  $p|_{\tilde{U}} : \tilde{U} \rightarrow U$  is a homeomorphism and  $\widetilde{f \circ \gamma}(1) \in \tilde{U}$ . Then  $\widetilde{f \circ \eta}(1) = (p^{-1})|_U \circ f(y')$ . Hence  $\tilde{f}(y') = f \circ (\gamma \circ \eta)(1) = \widetilde{f \circ \eta}(1) = (p^{-1})|_U \circ f(y')$  (i.e.  $\tilde{f} = (p^{-1})|_U = f$  on  $V$ ). Hence  $\tilde{f}$  is continuous at  $y$ .

$\tilde{f}$  is a lift of  $f$ . In fact,  $(p \circ \tilde{f})(y) = p \circ (\tilde{f}\gamma(1)) = f(y)$ .

### Corollary

$$\begin{array}{ccc} & \tilde{X} & \\ & \downarrow p & \\ Y & \xrightarrow{f} & X \end{array}$$

If  $Y$  is simply connected, then  $f_*\pi_1(Y) \leq p_*\pi_1(\tilde{X})$  always holds (i.e. we can always lift  $f$  to  $\tilde{f} : Y \rightarrow \tilde{X}$  in this case).

### Proposition: Unique Lifting

Given  $p : \tilde{X} \rightarrow X$  and  $f : Y \rightarrow X$ , if two lifts  $\tilde{f}_1$  and  $\tilde{f}_2$  of  $f$  agree at one point, then they agree everywhere on  $Y$ .

### Proof

Take  $\mathcal{A} = \{y \in Y : \tilde{f}_1(y) = \tilde{f}_2(y)\} \neq \emptyset$ . Locally for each  $y \in Y$  there exists a neighborhood  $V$  of  $y$  such that  $\tilde{f} = (p^{-1})|_U \circ f$ . If  $y \in \mathcal{A}$ , then  $\tilde{f}_1(y) = \tilde{f}_2(y)$ . Take a neighborhood  $U$  of  $f(y)$  that is evenly covered and  $\tilde{U}$  of  $\tilde{f}_1(y) = \tilde{f}_2(y)$  such that  $p|_{\tilde{U}} : \tilde{U} \rightarrow U$  is a homeomorphism. Then on  $V$ , a path-connected neighborhood such that  $f(V) \subseteq U$ ,  $\tilde{f}_i = (p^{-1})|_U \circ f$  (i.e.  $\tilde{f}_1 = \tilde{f}_2$  on  $V$ ). If  $y \in \mathcal{A}^c$ ,  $\tilde{f}_1(y) \neq \tilde{f}_2(y)$ . Then  $\tilde{U}_i \ni \tilde{f}_i(y)$  with  $\tilde{U}_1 \cap \tilde{U}_2 = \emptyset$ . Then on  $V$ ,  $\tilde{f}_i = (p^{-1})|_{\tilde{U}_i} \circ f$  (ie  $\tilde{f}_1$  and  $\tilde{f}_2$  never agree on  $V$ ). Hence  $\mathcal{A} = Y$ .

### Remark

If  $p : \tilde{X} \rightarrow X$  is a covering map, recall that a covering transformation is a map  $f : \tilde{X} \rightarrow \tilde{X}$  such that

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{f} & \tilde{X} \\ & \searrow p & \swarrow p \\ & X & \end{array}$$

commutes. This  $f : \tilde{X} \rightarrow \tilde{X}$  is a lift of  $p : \tilde{X} \rightarrow X$ . If we fix  $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$ , the lifting criterion says that  $p_*\pi_1(\tilde{X}, \tilde{x}_1) \leq p_*\pi_1(\tilde{X}, \tilde{x}_2)$ . In particular, if  $\pi_1(\tilde{X})$  is trivial, then this holds. Hence there is a unique lift of  $p$  (i.e. covering transformation)  $f$  such that  $f(\tilde{x}_1) = \tilde{x}_2$ .

April 16, 2025

## Question

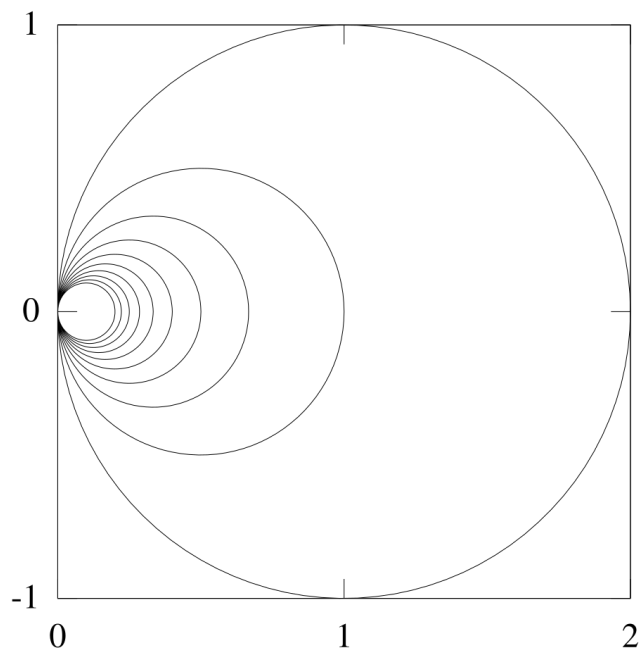
Given  $X$  path-connected and locally path-connected, when does  $X$  admit a simply connected covering space  $p : \tilde{X} \rightarrow X$ ?

## Definition: Semi-locally Simply Connected

We say that  $X$  is semi-locally simply connected if for any  $x \in X$  there exists a neighborhood  $U$  such that every loop in  $U$  is null-homotopic in  $X$ . That is  $\text{Im}(\pi_1(U) \rightarrow \pi_1(X))$  is trivial.

## Non-example

The Hawaiian earring in  $\mathbb{R}^2$ .



## Example

The cones over the Hawaiian earring.

IMAGE 1

In fact, this is simply connected.

## Example

The double Hawaiian earring with cones.

IMAGE 2



## Theorem

$X$  has a simply connected covering space (i.e. a universal covering) if and only if  $X$  is semi-locally simply connected.

## Proof

( $\implies$ ) Let  $x \in X$  and pick a neighborhood  $U$  of  $x$  that is evenly covered by  $p$ . Let  $f$  be a loop at  $x$  in  $U$ .  $f$  lifts to  $\tilde{f}$  at  $\tilde{x}_0$ , which is a loop. Retract  $\tilde{f}$  to  $c_{\tilde{x}_0}$  by a path-homotopy  $H$ . Then  $p \circ H$  shows that  $f$  is null-homotopic in  $X$ .

( $\impliedby$ ) We construct  $\tilde{X}$  as follows: fix  $x_0 \in X$  and set  $\tilde{X} = \{[\gamma] \text{ path homotopies} : \gamma \text{ is a path starting at } x_0\}$ . Let  $\mathcal{U} = \{U : \text{Im}(\pi_1(U) \rightarrow \pi_1(X)) \text{ is trivial}\}$ . By assumption  $\mathcal{U}$  is a basis for  $X$ . For each  $u \in \mathcal{U}$  and each  $\gamma$  from  $x_0$  to a point in  $U$ , we define  $U_{[\gamma]} = \{\gamma \cdot \eta : \eta \text{ starting at } \gamma(1) \text{ stays in } U\}$ . Then  $p : \tilde{X} \rightarrow X$  by  $[\gamma] \rightarrow \gamma(1)$ .

We need to check that  $\{U_{[\gamma]} : U \in \mathcal{U}, \gamma \text{ a path from } x_0 \text{ to a point in } U\}$  generates a topology on  $\tilde{X}$ .

We need also to check that  $p : U_{[\gamma]} \rightarrow U$  is bijective. It is clearly surjective, and if  $p[\gamma \cdot \eta] = p[\gamma \cdot \delta]$  with  $\eta, \delta$  paths starting at  $\gamma(1)$  and staying in  $U$ . Then  $\eta(1) = \delta(1)$  and, since  $\eta, \delta$  share the same endpoints and they stay in  $U_{[\gamma]}$ , then  $[\eta] = [\delta]$ . Hence  $[\gamma \cdot \eta] = [\gamma \cdot \delta]$  and  $p$  is injective.

Further, we need to check that  $p : U_{[\gamma]} \rightarrow U$  is a homeomorphism and that  $p^{-1}(U) = \bigcup_{[\gamma]} U_{[\gamma]}$ . Hence  $p$  is a covering map.

Finally, we need to check that  $\tilde{X}$  is simply connected. Recall that  $p : \tilde{X} \rightarrow X$  induces an injective homomorphism  $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$ . It suffices to show that  $p_*\pi_1(\tilde{X}, \tilde{x}_0) = \text{id}$ . We set  $\tilde{x}_0 = [C_{x_0}] \in \tilde{X}$ . Recall also that elements in  $p_*\pi_1(\tilde{X}, \tilde{x}_0)$  are exactly the loops in  $X$  at  $x_0$  such that they lift to loops at  $\tilde{x}_0$ . Suppose  $[\gamma] \in p_*\pi_1(\tilde{X}, \tilde{x}_0)$ . Then  $\gamma$  lifts to a loop  $\tilde{\gamma}$  at  $\tilde{x}_0 = [C_{x_0}]$ . For  $t \in [0, 1]$ , consider the path  $\gamma_t$  which follows  $\gamma$  on  $[0, t]$  then stays stationary at  $\gamma(t)$  for the remaining time. Then  $t \mapsto [\gamma_t]$  is a path on  $\tilde{X}$ ,  $p([\gamma_t]) = \gamma_t(1) = \gamma(t)$ , and  $t \mapsto [\gamma_t]$  is a lift of  $\gamma$  at  $\tilde{x}_0 = [C_{x_0}]$ . Then  $t \mapsto [\gamma_t]$  is a loop (i.e.  $[\gamma] = [\gamma_1] = \tilde{x}_0 = [C_{x_0}]$ ) and  $\gamma$  is null-homotopic. This shows that  $p_*\pi_1(\tilde{X}, \tilde{x}_0) = \text{id}$  (i.e.  $\tilde{X}$  is simply connected).

## Group Actions on Fibers (Monodromy Action)

Given  $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$  a covering map,  $\pi_1(X, x_0)$  acts on  $p^{-1}$  as follows:  $p^{-1}(x_0) \times \pi_1(X, x_0) \rightarrow p^{-1}(x_0)$  by  $(e, [f]) \mapsto \tilde{f}_e(1)$  where  $\tilde{f}_e$  is the (unique) lift of  $f$  at  $e \in p^{-1}(x_0)$ . This is a right  $\pi_1(X, x)$  action.

We want to check that  $(e \cdot [f]) \cdot [g] = e \cdot [f \cdot g]$ . We have that  $e \cdot [f \cdot g] = (\widetilde{f \cdot g})_e(1)$ , but  $(\widetilde{f \cdot g})_e$  is the lift of  $f$  at  $e$  followed by the lift of  $g$  at the endpoint of  $\tilde{f}_e$ , call it  $\tilde{f}_e(1) = z$ . Then  $(\widetilde{f \cdot g})_e(1) = \tilde{g}_z(1) = z \cdot [g] = (e \cdot [f]) \cdot [g]$ .

This action is transitive. Given  $e$  and  $e'$ , draw a path connecting them  $\tilde{g}$ . Under the map  $p$ , we have that  $p \circ \tilde{g} = g$  which is a loop at  $x_0$ . Then  $e \cdot [g] = \tilde{g}(1) = e'$ .

Recall: Given a right  $G$ -set  $S$ ,  $G_s = \{g \in G : s \cdot g = s\}$  is the isotropy subgroup at  $s \in S$ .

Given  $e \in p^{-1}(x_0)$ , the isotropy subgroup at  $e$  is all the loops such that their lifts at  $e$  are loops (i.e. the isotropy subgroup at  $e$  is precisely  $p_*\pi_1(\tilde{X}, e)$ ).

Recall:  $G \cdot S = G/G_s$ . Here, this tells us that  $p^{-1}(x_0) = \pi_1(X, x_0)/p_*\pi_1(\tilde{X}, e)$ . This recovers the fact that the number of sheets is equal to the index of  $\text{im}(p_*)$ .

In particular, if  $\tilde{X}$  is simply connected, then

- $\pi_1(X, x_0)$  acts freely on  $p^{-1}(x_0)$  and
- the number of sheets equals the cardinality of  $\pi_1(X, x_0)$ .

## Definition: Universal Cover

A covering space  $p : \tilde{X} \rightarrow X$  is called universal if it has the universal property (i.e. for any covering space  $q : Y \rightarrow X$ , there is a covering map  $\tilde{p} : \tilde{X} \rightarrow Y$  such that the associated diagram commutes).

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{p}} & Y \\ p \downarrow & \swarrow q & \\ X & & \end{array}$$

## Definition: Covering Homomorphism

Let  $p_1 : X_1 \rightarrow X$  and  $p_2 : X_2 \rightarrow X$  be two covering spaces. A covering homomorphism is a map  $\varphi : X_1 \rightarrow X_2$  such that the associated diagram commutes

$$\begin{array}{ccc} X_1 & \xrightarrow{\varphi} & X_2 \\ & \searrow p_1 & \swarrow p_2 \\ & X & \end{array}$$

By definition,  $\varphi$  is a lift of  $p_1$ .

## Proposition

1. A covering homomorphism  $\varphi$  is uniquely determined by its value at one point.
2. For each  $x \in X$ ,  $\varphi|_{p_1^{-1}(x)} : p_1^{-1}(x) \rightarrow p_2^{-1}(x)$  is  $\pi_1(X, x_0)$ -equivariant.
3. A covering homomorphism  $\varphi : X_1 \rightarrow X_2$  is a covering map. Assuming this, the universal cover is unique.

Recall: if  $S_1, S_2$  are right  $G$ -sets, a  $G$ -equivariant map  $\varphi : S_1 \rightarrow S_2$  is a map such that the associated diagram commutes

$$\begin{array}{ccc} S_1 & \xrightarrow{\varphi} & S_2 \\ \downarrow \cdot g & & \downarrow \cdot g \\ S_1 & \xrightarrow{\varphi} & S_2 \end{array}$$

## Proof of 2

Let  $e \in p_1^{-1}(x)$ . We need to show that  $\varphi(e) \cdot g = \varphi(e \cdot g)$ . We have that  $g \in \pi_1(X, x_0)$  is represented by a loop  $f$  at  $x_0$ . So  $e \cdot g = e \cdot [f] = \tilde{f}_e(1) \in X_1$ , and  $\varphi(e \cdot g) = \varphi(\tilde{f}_e(1))$ . On the left hand side, we have that  $\varphi(e) \cdot g = f_{\varphi(e)}(1) \in X_2$ . We need to verify that  $\varphi(\tilde{f}_e) = \tilde{f}_{\varphi(e)}$  which are both lifts of  $f$  at  $\varphi(e)$ . But since the diagram commutes,  $p_2(\varphi \circ \tilde{f}_e) = p_1 \circ \tilde{f}_e = f$ .

## Uniqueness in 3

Suppose we have

$$\begin{array}{ccc} X_1 & \xleftarrow{\psi} & X_2 \\ & \searrow p_1 & \swarrow p_2 \\ & X & \end{array}$$

with  $\varphi(e_1) = e_2$  and  $\psi(e_2) = e_1$ . Then  $\psi \circ \varphi(e_1) = e_1$ . Hence  $\psi \circ \varphi = \text{id}$  and, similarly,  $\varphi \circ \psi = \text{id}$ . Hence  $\varphi$  is a bijection and a homomorphism.

## Proof of 3

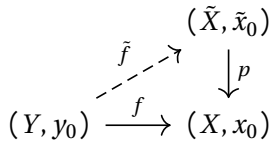
$\varphi$  is surjective. Given any  $e' \in X_2$ , set  $x_0 = p_2(e)$  and let  $e \in p_1^{-1}(x_0)$  so  $\varphi(e) \in p_2^{-1}(x_0)$ . Since  $\pi_1(X, x_0)$  acts transitively on  $p_2^{-1}(x_0)$ , there exists  $g \in \pi_1(X, x_0)$  such that  $e' = \varphi(e) \cdot g = \varphi(e \cdot g)$ .

$\varphi$  is a covering map. Let  $V$  be a neighborhood of  $x_0 \in X$  such that  $V$  is evenly covered by both  $p_1$  and  $p_2$ . Let  $U$  be a component in  $p_2^{-1}(V)$  that contains  $e_2$ . Then  $p_1^{-1}(V) = \bigcup U_\alpha$ .  $U$  as a component in  $p_2^{-1}(V)$  is both open and closed.

Hence  $\varphi^{-1}(U)$  is open and closed in  $p_1^{-1}(V) = \bigcup U_\alpha$ . It follows that  $\varphi^{-1}(U)$  is the disjoint union of several components of  $\{U_\alpha\}_\alpha$ , and each component is homemorphic to  $V$  and consequently homeomorphic to  $U$ . This shows that  $\varphi$  is a covering map.

**April 21, 2025**

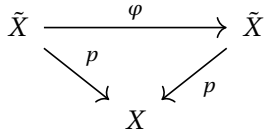
## Recall: Lifting Criterion



There exists a lift  $\tilde{f}$  of  $f$  at  $\tilde{x}_0$  if and only if  $f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)$ .

If  $(\tilde{X}, \tilde{x}_0) \xrightarrow{p} (Y, y_0)$ ,  $\pi_1(X, x_0)$  acts transitively on  $p^{-1}(x_0)$  by path lifting (a right action where  $e \in p^{-1}(x_0)$  by  $e \cdot [\gamma] = \tilde{\gamma}_e(1)$ ). The isotropy subgroup at  $e$  is  $p_*\pi_1(\tilde{X}, e)$ .

## Covering Transformations



Write  $\text{Aut}(\tilde{X} \xrightarrow{p} X)$  for the covering group  $\{\varphi : \tilde{X} \rightarrow \tilde{X} \text{ covering transformations}\}$ .

1.  $\varphi : \tilde{X} \rightarrow \tilde{X}$  is uniquely determined by its value at one point.
2. Given  $e_1, e_2 \in p^{-1}(x)$ , there is  $\varphi \in \text{Aut}(\tilde{X} \xrightarrow{p} X)$  if and only if  $p_*\pi_1(\tilde{X}, e_1) = p_*\pi_1(\tilde{X}, e_2)$ . In fact, for  $\varphi \in \text{Aut}(\tilde{X} \xrightarrow{p} X)$  with  $p_*\pi_1(\tilde{X}, e_1) \subseteq p_*\pi_1(\tilde{X}, e_2)$ .
3.  $\varphi|_{p^{-1}(x)} : p^{-1}(x) \rightarrow p^{-1}(x)$  is  $\pi_1(X, x)$ -equivariant (i.e.  $\varphi(e) \cdot \gamma = \varphi(e \cdot \gamma)$ ).

## Example

Given  $p : \mathbb{R} \rightarrow S^1$ , what is  $\text{Aut}(\mathbb{R} \xrightarrow{p} S^1)$ ?

$1 \in S^1$ ,  $p^{-1}(1) = \mathbb{Z} \subseteq \mathbb{R}$ ,  $\forall \varphi \in \text{Aut}(\mathbb{R} \xrightarrow{p} S^1)$ ,  $\varphi(0) = k \in \mathbb{Z}$ . Then  $\varphi(x) = x + k$ . In fact, the map  $x \mapsto x + k$  is a covering transofmratio that agrees with  $\varphi$  at  $0 \in \mathbb{R}$ . Hence they agree everywhere (i.e.  $\varphi(x) = x + k$  for all  $x$ ).

## Example

Given  $p : S^2 \rightarrow \mathbb{RP}^2$ , then  $\text{Aut}(S^2 \xrightarrow{p} \mathbb{RP}^2) = \{\text{id}, A\}$  with  $A$  the antipodal map.

## Proposition: Normal Covering

Let  $\tilde{X} \xrightarrow{p} X$  be a covering map. The following are equivalent

1. There exists  $x \in X$  such that  $p_*\pi_1(\tilde{X}, e)$  is normal for one (thus for all)  $e \in p^{-1}(x)$ .
2. For every  $x \in X$  and each  $e \in p^{-1}(x)$ ,  $p_*\pi_1(\tilde{X}, e)$  is normal.

3.  $\text{Aut}(\tilde{X} \xrightarrow{p} X)$  acts transitively on some (thus all) fiber  $p^{-1}(x)$ .

If any of these hold, we say that  $p : \tilde{X} \rightarrow X$  is a normal covering.

### Proof

Suppose  $e, e' \in p^{-1}(x)$  with  $p_*\pi_1(\tilde{X}, e)$  and  $p_*\pi_1(\tilde{X}, e')$ . These are the isotropy subgroups at  $e$  and  $e'$  respectively. We know also  $\pi_1(X, x)$  acts transitively on  $p^{-1}(x)$ .

Fact: If  $S$  is a right  $G$ -set, then  $G_s = \{h \in G : s \cdot h = s\}$  and  $G_{sg} = \{h \in G : s \cdot g \cdot h = s \cdot g\} = \{h \in S : s \cdot g \cdot h \cdot g^{-1} = s\}$ . So  $g \cdot G_{sg} \cdot g^{-1} \in G_s$  which implies that  $G_{sg} = g^{-1} \cdot G_s \cdot g$ . So if  $G_s$  is normal then so is  $G_{sg}$ .

IMAGE 1

$$\begin{array}{ccc} \pi_1(\tilde{X}, e_0) & \xrightarrow{\Phi_{\tilde{h}}} & \pi_1(\tilde{X}, e) \\ \downarrow p_* & & \downarrow p_* \\ \pi_1(X, x_0) & \xrightarrow{\Phi_h} & \pi_1(X, x) \end{array}$$

commutes. Hence  $\Phi_h$  maps  $p_*\pi_1(\tilde{X}, e_0)$  to  $p_*\pi_1(\tilde{X}, e)$ , and  $\Phi_h : \pi_1(X, x_0) \xrightarrow{\sim} \pi_1(X, x)$  preserves normal subgroups.

### (3) implies (1)

Finally, for every  $e_1, e_2 \in p^{-1}(x)$ , there exists  $\varphi \in \text{Aut}(\tilde{X} \xrightarrow{p} X)$  such that  $\varphi(e_1) = e_2$ . This holds if and only if  $p_*\pi_1(\tilde{X}, e_1) = p_*\pi_1(\tilde{X}, e_2)$  for every  $e_1, e_2 \in p^{-1}(x)$ . That is,  $e_2 = e_1 \cdot \gamma$  for some  $\gamma \in \pi_1(X, x)$  and  $H = \gamma^{-1}H\gamma$  for every  $\gamma \in \pi_1(X, x)$ . So  $H$  is normal.

### Remark

The (simply connected) universal cover is always normal because  $\{\text{id}\}$  is normal in  $\pi_1(X, x)$ .

### Theorem

Let  $p : \tilde{X} \rightarrow X$  be a covering map with  $x \in X$  and  $e \in p^{-1}(x)$ . Then  $\text{Aut}(\tilde{X} \xrightarrow{p} X) \cong \frac{N_G(H)}{H}$  where  $G = \pi_1(X, x)$ ,  $H = p_*\pi_1(\tilde{X}, e)$ , and  $N_G(H) = \{g \in G : g^{-1}Hg = H\}$ .

### Special Case 1

If  $p : \tilde{X} \rightarrow X$  is a normal covering, then  $H$  is normal in  $G$ . Then also  $\text{Aut}(\tilde{X} \xrightarrow{p} X) \cong G/H$ .

### Special Case 2

If  $p : \tilde{X} \rightarrow X$  is the (simply connected) universal covering, then  $\text{Aut}(\tilde{X} \xrightarrow{p} X) \cong \pi_1(X, x)$ .

### Proof

Let  $S$  be a right  $G$ -set with transitive action and  $\text{Aut}_G(S) = \{\varphi : S \rightarrow S \text{ } G\text{-equivariant bijections}\}$ . Fix  $s \in S$ . Then  $\text{Aut}_G(S) \cong \frac{N_G(H)}{H}$  where  $h = G_s$ .

Define  $\Phi : N_G(H) \rightarrow \text{Aut}_G(S)$  by  $\gamma \mapsto \Phi(\gamma) = \varphi_\gamma$  with  $\varphi_\gamma : S \rightarrow S$  defined by

$$G_{s \cdot \gamma} = \gamma^{-1}H\gamma = H = G_s.$$

Then there exists a unique  $\varphi_\gamma \in \text{Aut}_G(S)$  such that  $\varphi_\gamma(s) = s \cdot \gamma$ .

• Lemma

For each  $s' \in S$ ,  $s' = s \cdot \gamma'$  for some  $\gamma' \in G$ . Then  $\varphi_\gamma(s') = \varphi_\gamma(s \cdot \gamma') = \varphi_\gamma(s) \cdot \gamma' = s \cdot \gamma \gamma'$ . This is well defined. If  $s' = s \cdot \gamma''$ , then  $s = s(\gamma \cdot \gamma'' \cdot (\gamma')^{-1} \cdot \gamma^{-1})$  which implies that  $\gamma \cdot \gamma''(\gamma')^{-1} \cdot \gamma^{-1} \in G_s$  and  $\gamma'' \cdot (\gamma')^{-1} \in G_s$ .

$\Phi$  is a group homomorphism since

$$\varphi_{\gamma_1} \circ \varphi_{\gamma_2}(s) = \varphi_{\gamma_1}(s \cdot \gamma_2) = \varphi_{\gamma_1}(s) \cdot \gamma_2 = s \cdot \gamma_1 \cdot \gamma_2.$$

$\Phi$  is surjective since letting  $\varphi \in \text{Aut}_G(S)$ , it maps  $s$  to some  $\varphi(s) = s' = s \cdot \gamma$  and hence  $\varphi = \varphi_\gamma$ .

If  $\varphi_\gamma = \text{id}$ , then  $\varphi_\gamma(s) = s$  and  $\gamma \in G_s = H$ . So  $\Phi$  induces  $\frac{N_G(H)}{H} \cong \text{Aut}_G(S)$ .

Take  $G = \pi_1(X, x)$  and  $\text{Aut}(\tilde{X} \xrightarrow{p} X) \rightarrow \text{Aut}_G(p^{-1}(x)) \cong \frac{N_G(H)}{H}$  by  $\varphi \mapsto \varphi|_{p^{-1}(x)}$  where  $H$  is the isotropy subgroup of the  $\pi_1(X, x)$  action at  $e$  ( $p_*\pi_1(\tilde{X}, e)$ ). Then  $\varphi \mapsto \varphi|_{p^{-1}(x)}$  is injective because it is uniquely determined by its value at one point.

$\varphi \mapsto \varphi|_{p^{-1}(x)}$  is surjective. Letting  $\eta \in \text{Aut}_G(p^{-1}(x))$  and  $e_1 \in p^{-1}(x)$ , we set  $e_2 = \eta(e_1)$  and see that  $p_*\pi_1(\tilde{X}, e_1) = G_{e_1} = G_{e_2} = p_*\pi_1(\tilde{X}, e_2)$ . By the lifting criterion, there exists  $\varphi \in \text{Aut}(\tilde{X} \xrightarrow{p} X)$  such that  $\varphi(e_1) = e_2$ . Then  $\varphi|_{p^{-1}(x)} = \eta$  since both are in  $\text{Aut}_G(p^{-1}(x))$  and they agree at one point (hence everywhere). Thus we conclude that the map is a bijection and

$$\text{Aut}(\tilde{X} \xrightarrow{p} X) \cong \text{Aut}_G(p^{-1}(x)) \cong \frac{N_G(H)}{H}.$$

## Definition: Covering Space Action

Let  $X$  be connected and locally path connected with a group action  $\Gamma$  acting by homeomorphism. The quotient map  $p : X \rightarrow X/\Gamma$  will be a covering map if we impose  $(*)$  for all  $x \in X$ , there exists a neighborhood  $U$  of  $x$  such that  $U \cap (g \cdot U) = \emptyset$  for each  $g \in \Gamma - \{\text{id}\}$ . In particular,  $G$  acts freely on  $X$ . We say that a  $\Gamma$ -action on  $X$  is a covering space action if  $(*)$  is fulfilled.

### Counter-example

Consider an  $\mathbb{R}$  action on  $\mathbb{R}^2$  by translation. Then  $U \cap (g \cdot U) \neq \emptyset$ .

IMAGE 2

### Remark

Assuming  $(*)$ ,  $\{g \cdot U : g \in \Gamma\}$  is a disjoint family of open sets.

### Example

Take a  $\mathbb{Z}$ -action by  $\mathbb{R}^2$  given by  $\gamma(x, y) = (x + 1, -y)$ .

IMAGE 3

### Example

$S^2$  with  $\mathbb{Z}_2$ -action  $(\{\text{id}, A\})$ .

## Theorem

If  $\Gamma$  acts on  $X$  as a covering space action, then  $q : X \rightarrow X/\Gamma$  is a normal covering map.

### Proof

Let  $\bar{x} \in X/\Gamma$  and pick  $x \in q^{-1}(\bar{x})$ . By  $(*)$ , we have a neighborhood  $U$  such that  $\{g \cdot U : g \in \Gamma\}$  is a disjoint collection. Let  $V = q(U)$ , an open neighborhood of  $\bar{x}$  in  $X/\Gamma$ . Then  $q^{-1}(V) = \{g \cdot U : g \in \Gamma\}$ . Moreover,  $g \cdot U \rightarrow V$  is a homeomorphism. If there exist  $x', g'x' \in g \cdot U$ , then  $x' = h_1 \cdot u_1$  and  $g' \cdot x' = h_2 \cdot u_2$ . So  $h_1^{-1}x' \in U$  and  $h_2^{-1}g' \cdot x' \in U$  but this holds only for the identity map. So the covering map is injective.

## Classifications of Covering Spaces

Take  $X$  path-connected, locally path-connected and semi-locally simply path connected (guaranteeing a simply connected universal cover). Then there is a 1 – 1 correspondence between

$$\left\{ \begin{array}{l} \text{isomorphism classes of covering} \\ \text{spaces } p: \hat{X} \rightarrow X \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{conjugacy classes of} \\ \text{subgroups in } \pi_1(X, x) \end{array} \right\}$$

$$(p : \hat{X} \rightarrow X) \mapsto [p_*\pi_1(\hat{X}, \hat{x})]$$

April 23, 2025

### Recall: Theorem

For  $X$  path-connected, locally path-connected and semi-locally simply path connected,  $\Gamma$  acts on  $X$  as a covering group action (i.e.  $\forall x \in X$ , there exists a neighborhood  $U$  of  $x$  such that  $U \cap (g \cdot U) = \emptyset$  for all  $g \in \Gamma \setminus \{e\}$ ).

Then  $p : X \rightarrow X/\Gamma$  is a normal covering map. Moreover  $\text{Aut}(X \xrightarrow{p} X/\Gamma) = \Gamma$ .

### Proof

( $\supseteq$ ) this follows from

$$\begin{array}{ccc} X & \xrightarrow{g \cdot} & X \\ & \searrow p & \swarrow p \\ & X/\Gamma & \end{array}$$

( $\subseteq$ ) Let  $\varphi \in \text{Aut}(X \xrightarrow{p} X/\Gamma)$ . That is

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X \\ & \searrow p & \swarrow p \\ & X/\Gamma & \end{array}$$

commutes with  $\varphi$  a homeomorphism. Now let  $x \in p^{-1}(\bar{x})$  where  $\bar{x} \in X/\Gamma$ , and let  $x' = \varphi(x)$ . Then  $p(x) = \bar{x} = p(x')$ , hence  $x, x' \in p^{-1}(\bar{x})$ . Hence there is  $g \in \Gamma$  such that  $gx = x'$ . So we have

$$\begin{aligned} \varphi : X &\rightarrow X \varphi(x) = x' \\ g : X &\rightarrow X g(x) = x' \end{aligned}$$

so  $\varphi$  is equivalent to an action by  $g$ .

## Theorem

Take  $X$  path-connected, locally path-connected and semi-locally simply path connected (guaranteeing a simply connected universal cover). Then there is a 1 – 1 correspondence between

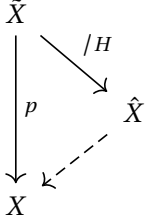
$$\{\text{isomorphism classes of covering maps } p: \tilde{X} \rightarrow X\} \leftrightarrow \{\text{conjugacy classes of subgroups in } \pi_1(X, x_0)\}$$

→ Assign a subgroup  $H = p_*(\hat{X}, \hat{e})$  for  $\hat{e} \in p^{-1}(x_0)$ .

← Given a conjugacy class of subgroups, pick a subgroup  $H$  in the class.

$$H \leq \pi_1(X, x_0) \cong \text{Aut}(\tilde{X} \xrightarrow{p} X)$$

Hence  $H$  acts naturally on  $\tilde{X}$  as covering transformations. Consider  $q: \tilde{X} \rightarrow \tilde{X}/H =: \hat{X}$ , a normal covering map.



Since  $\tilde{X}/\pi_1(X, x_0) = x$ , we have an induced map  $\hat{p}: \hat{X} \rightarrow X$ . We need to show that  $\hat{p}: \hat{X} \rightarrow X$  is a covering map with  $\hat{p}_*\pi_1(\hat{X}, \hat{e}) = H$  for some  $\hat{e} \in \hat{p}^{-1}(x)$ . Let  $U$  be a neighborhood of  $x$  such that  $p^{-1}(U) = \bigcup_{\alpha} \tilde{U}_{\alpha}$ . Then  $\{\tilde{U}_{\alpha}\}$  is a collect iof disjoint open sets and identical to  $\{g \cdot \tilde{U} : g \in \pi_1(X, x)\}$  where  $\tilde{U}$  is a component of  $p^{-1}(U)$ . The  $H$ -action permutes the copies in  $\{g \cdot \tilde{U}\} = \{\tilde{U}_{\alpha}\}$ . Hence  $q|_{\tilde{U}_{\alpha}}: \tilde{U}_{\alpha} \rightarrow \hat{X}$  is a homeomorphism. Let  $\hat{U}$  be a component in  $\hat{p}^{-1}(U)$ . Then  $q^{-1}(\hat{p}^{-1}(U)) = p^{-1}(U) = \bigcup_{\alpha} \tilde{U}_{\alpha}$  where  $q^{-1}(\hat{U})$  is a union of components in  $\bigcup_{\alpha} \tilde{U}_{\alpha}$ . Hence  $\hat{U}$  is homeomorphic to  $U$ , and  $\hat{p}^{-1}(U)$  is a union of components that are homemorphic to  $U$ .

Lastly, we show that  $\hat{p}_*\pi_1(\hat{X}, \hat{e}_0) = H$ . This is the isotropy subgroup of  $\pi_1(X, x_0)$ -actions at  $\hat{e}_0$ .  $q|_{p^{-1}(x_0)}: p^{-1}(x_0) \rightarrow \hat{p}^{-1}(x_0)$  is  $\pi_1(X, x_0)$ -equivariant (i.e.  $q(e \cdot \gamma) = q(e) \cdot \gamma$ ,  $q(e) = \hat{e}$  for  $e \in \tilde{X}$ ). Hence  $\gamma$  fixes  $q(e) = \hat{e}$  if and only if  $q(e \cdot \gamma) = q(e)$ , if and only if  $e \cdot \gamma$  and  $e$  are in the same  $H$ -orbit, if and only if  $\gamma \in H$ .

### Example 1

$X = S^1$  with  $\pi_1(S^1) = \mathbb{Z}$ .

$\mathbb{Z}$  has subgroups  $\mathbb{Z}, 2\mathbb{Z}, 3\mathbb{Z}, \dots, k\mathbb{Z}, \dots$  where  $k\mathbb{Z}$  corresponds to the covering map  $p_k: z \mapsto z^k$ .

### Example 2

$X$  the Mobius strip with  $\pi_1(X) = \mathbb{Z}$  with  $\pi_1(X) = \langle \gamma \rangle$  and  $\gamma(x, y) = (x + 1, -y)$ .

Take  $H = 2\mathbb{Z} = \langle 2\gamma \rangle \leq \mathbb{Z}$ . Then  $2\gamma(x, y) = (x + 2, y)$  and  $\mathbb{R}^2/H$  is the cylinder while the cylinder modulo  $\mathbb{Z}_2$  is the mobius strip.

### Example 3

The Klein bottle,  $K = \mathbb{R}^2/\Gamma$  with  $\Gamma$  generated by  $g(x, y) = (x + 1, -y)$  and  $h(x, y) = (x, y + 1)$ .

So  $\pi_1(K) = \langle g, h \rangle$ .  $g^2(x, y) = (x + 2, y)$  commutes with  $h$ , so  $\mathbb{Z}^2 \cong \langle g^2, h \rangle \leq \pi_1(K)$  and  $\mathbb{R}^2/\langle g^2, h \rangle = \mathbb{T}^2$  covers  $K$ .

## Simplexes

IMAGE 1

The standard  $n$ -simplex is

$$\Delta^n = \left\{ (t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^n t_i = 1, t_i \geq 0, \forall i \right\}$$

$$\Delta^1 = \left\{ (t_0, t_1) \in \mathbb{R}^2 : t_0 + t_1 = 1, t_0, t_1 \geq 0 \right\}$$

IMAGE 2

$$\Delta^2 = \{(t_0, t_1, t_2) \in \mathbb{R}^3 : t_0 + t_1 + t_2 = 1, t_0, t_1, t_2 \geq 0\}$$

IMAGE 3

$\Delta^n$  has  $(n+1)$ -many faces ( $(n+1)$ -simplex) where the  $i$ th face is  $\Delta^{n-1} \rightarrow \Delta^n$  by  $(t_0, \dots, t_{n-1}) \mapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$ . Let  $X$  be a topological space. A  $\Delta$ -complex structure on  $X$  is a family of maps  $\sigma_\alpha : \Delta^n \rightarrow X$  ( $n$  may depend on  $\alpha$ ) such that

1.  $\sigma_\alpha|_{\Delta^n} : \Delta^n \rightarrow X$  is injective and each point is in the image of at most one of  $\sigma_\alpha|_{\Delta^n}$ .
2.  $\sigma_\alpha|_{\text{a face of } \Delta^n}$  is some  $\sigma_\beta : \Delta^{n-1} \rightarrow X$  in the family.
3.  $A \subseteq X$  is open if and only if  $\sigma_\alpha^{-1}(A)$  is open in  $\Delta^n$  for all  $\alpha$ .

$$\sigma_\beta \text{ is } \Delta^{n-1} \xrightarrow{\text{ith face}} \Delta^n \xrightarrow{\sigma} X.$$

### Example

$S^1$  is the following 1-simplex

IMAGE 4

Then the “body” of  $\Delta^1 \xrightarrow{\sigma} X$  is

IMAGE 5

with  $\sigma \circ \delta_0 : \Delta^0 \rightarrow X$  and  $\sigma \circ \delta_1 : \Delta^0 \rightarrow X$ . The boundary  $\partial\sigma = \sum_{i=0}^n (-1)^i \sigma \circ \delta_i$ . They define  $\delta : C_n(X) \rightarrow C_{n-1}(X)$ . For this example, we have  $\partial\sigma = \sigma \circ \delta_0 + (-1)\sigma \circ \delta_1 = 0$ .

The  $i$ th face is  $\delta_i : \Delta^{n-1} \rightarrow \Delta^n$  by  $(t_0, \dots, t_{n-1}) \mapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$ .

In Hatcher’s notation, the boundary is  $\partial\sigma = \sum_{i=0}^n (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$  where we should think of  $[v_0, \dots, \hat{v}_i, \dots, v_n]$  as the  $i$ th face. So  $\sigma : \Delta^n = [v_0, \dots, v_n] \rightarrow X$ . Now we have

$$\cdots \xrightarrow{\partial} C_n(X) \xrightarrow{\partial} C_{n-1}(X) \cdots$$

where  $\partial^2 = 0$ .

### Proof

$$\begin{aligned} \partial(\partial\sigma) &= \partial\left(\sum_{i=0}^n (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}\right) \\ &= \sum_{i=0}^n (-1)^i \partial(\sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}) \\ &= \sum_{j < i} (-1)^i (-1)^j \sigma|_{[v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n]} + \sum_{j > i} (-1)^i (-1)^{j-1} \sigma|_{[v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]} \\ &= 0 \end{aligned}$$



## Homology Associated to the Delta Complex

We have  $\ker \partial \supseteq \operatorname{im} \partial$  where  $\ker \partial$  are the  $n$ -cycles and  $\operatorname{im} \partial$  are the  $n$ -boundaries, and

$$H_n^\Delta(X) = \frac{\{n\text{-cycles}\}}{\{n\text{-boundaries}\}} = \frac{\ker \partial}{\operatorname{im} \partial}$$

### Example

For the circle,  $C_1(X) = \mathbb{Z} = \langle \sigma \rangle$  and  $C_0(X) = \mathbb{Z} = \langle v \rangle$ . Therefore

$$\overbrace{C_2(X)}^{=0} \rightarrow \overbrace{C_1(X)}^{=\mathbb{Z}} \xrightarrow{0} \overbrace{C_0(X)}^{=\mathbb{Z}} \rightarrow 0$$

Then  $H_1^\Delta(X) = \frac{\ker \partial}{\operatorname{im} \partial} = \mathbb{Z} / \{0\} = \mathbb{Z}$  and  $H_0^\Delta(X) = \frac{\ker \partial}{\operatorname{im} \partial} = \mathbb{Z}$ .

### An Aside

IMAGE 7

$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

### Example

For the torus, draw

IMAGE 6

So  $C_0(X) = \langle v \rangle = \mathbb{Z}$ ,  $C_1(X) = \langle a, b, c \rangle = \mathbb{Z}^3$  and  $C_2(X) = \langle U, L \rangle = \mathbb{Z}^2$ . Then also  $\partial U = a + b - c$  and  $\partial L = a + b - c$ , so  $\partial(U - L) = 0$  and  $\ker \partial_2 = \langle U - L \rangle \cong \mathbb{Z}$ . That is  $H_2^\Delta(X) = \frac{\ker \partial}{\operatorname{im} \partial} \cong \mathbb{Z}$ . Now  $\partial a = 0 = \partial b = \partial c$ , so  $\ker \partial_1 = \langle a, b, c \rangle$  and  $H_1^\Delta(X) = \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \frac{\langle a, b, a+b-c \rangle}{\langle a+b-c \rangle} \cong \mathbb{Z}^2$ . Finally we have that  $H_0^\Delta(X) = \frac{\ker \partial_0}{\operatorname{im} \partial_1} = \frac{\langle v \rangle}{\{0\}} \cong \mathbb{Z}$ .

### Example

For  $\mathbb{RP}^2$ , draw

IMAGE 8

$AC_0(X) = \langle v, w \rangle \cong \mathbb{Z}^2$ ,  $C_1(X) = \langle a, b, c \rangle \cong \mathbb{Z}^3$ , and  $C_2(X) = \langle U, L \rangle \cong \mathbb{Z}^2$ . Then  $\partial U = a + b + c$  while  $\partial L = a + b - c$ , so  $\ker \partial_2 = \{0\}$  and  $H_2^\Delta(X) = \frac{\ker \partial_2}{\operatorname{im} \partial_3} = \{0\}$ .  $\partial_1(a) = w - v$ ,  $\partial_1(b) = v - w$  and  $\partial_1(c) = 0$ , so  $\ker \partial_1 = \langle c, a - b \rangle$  and

$$H_1^\Delta(X) = \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \frac{\langle c, a + b \rangle}{\langle a + b + c, a + b - c \rangle} = \langle a + b + c, c \rangle / \langle a + b + c, 2c \rangle \cong \langle c \rangle / \langle 2c \rangle \cong \mathbb{Z}^2.$$

April 28th, 2025

### Recall:

For  $X$  with a  $\Delta$ -complex structure, we have  $H_n^\Delta(X)$ .

## Definition: Singular Simplex

A singular  $n$ -simplex is a continuous map  $\sigma : \Delta^n \rightarrow X$ .

The singular chain  $C_n(X)$  is the free Abelian group generated by singular  $n$ -simplices. Write

$$C_n(X) = \left\{ \sum n_i \sigma_i : \left| \sum n_i \sigma_i \right| < \infty, n_i \in \mathbb{Z}, \sigma_i : \Delta^n \rightarrow X \right\}$$

$$\cdots \xrightarrow{\partial} C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial} \cdots$$

$$\partial(\sigma) = \sum_{i=0}^n (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

While  $\partial^2 = 0$  and  $H_n(X) = \ker \partial_n / \text{Im } \partial_{n+1}$  is the singular homology.

## Proposition

If  $X = \coprod_{\alpha} X_{\alpha}$  with  $X_{\alpha}$  connected components of  $X$ , then  $H_n(X) \cong \oplus_{\alpha} H_n(X_{\alpha})$ .

### Proof

$\sigma : \Delta^n \rightarrow x$ ,  $\text{im } \sigma \subseteq X_{\alpha}$  for some  $\alpha$ . So  $C_n(X) = \oplus_{\alpha} C_n(X_{\alpha})$  and  $\partial : C_n(X) \rightarrow C_{n-1}(X)$  maps  $C_n(X_{\alpha})$  to  $C_{n-1}(X_{\alpha})$ . Therefore  $\ker \partial_n = \oplus_{\alpha} \ker(\partial|_{C_n(X_{\alpha})})$  and  $\text{im } \partial_{n+1} = \oplus_{\alpha} \text{im}(\partial|_{C_{n+1}(X_{\alpha})})$ . Then  $H_n(X) \cong \oplus_{\alpha} \ker(\partial|_{C_n(X_{\alpha})}) / \oplus_{\alpha} \text{im}(\partial|_{C_{n+1}(X_{\alpha})}) \cong \oplus_{\alpha} H_n(X_{\alpha})$ .

## Proposition

Let  $X$  be a point. Then

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \geq 1 \end{cases}$$

### Proof

For each  $n$ ,  $C_n(X)$  is generated by a single element  $\sigma_n : \Delta^n \rightarrow p$  so  $C_n(X) \cong \mathbb{Z}$ . Then

$$\partial \sigma_n = \sum_{i=0}^n (-1)^i \sigma_n|_{[v_0, \dots, \hat{v}_i, \dots, v_n]} = \sum_{i=0}^n (-1)^i \sigma_{n-1} = \begin{cases} 0 & n \text{ odd} \\ \sigma_{n-1} & n \text{ even} \end{cases}$$

$$\cdots \longrightarrow C_{n+1}(X) \longrightarrow C_n(X) \longrightarrow C_{n-1}(X) \longrightarrow \cdots$$

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \cdots \quad \text{We see that}$$

$$\partial_n = \begin{cases} \cong & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Therefore  $\ker / \text{im} = 0$  or  $\ker / \text{im} = \mathbb{Z} / \mathbb{Z} = 0$ . Because

$$C_1(X) \xrightarrow{0} C_0(X) \xrightarrow{0} 0 \quad \text{we have that } H_0(X) = \ker / \text{im} = \mathbb{Z} / \{0\} = \mathbb{Z}.$$

## Proposition

If  $X$  is path connected, then  $H_0(X) \cong \mathbb{Z}$ .

### Proof

Define a map  $\epsilon : C_0(X) \rightarrow \mathbb{Z}$  by  $\sum n_i \sigma_i \mapsto \sum n_i$  given that  $\sigma_i : \{v\} \rightarrow X$ . Then  $\epsilon$  is surjective. Also,

$$H_0(X) = \ker \partial_0 / \operatorname{im} \partial_1 = C_0(X) / \operatorname{im} \partial_1 = C_0(X) / \ker \epsilon \cong \operatorname{im} \epsilon \cong \mathbb{Z}$$

We claim that  $\ker \epsilon = \operatorname{im} \partial_1$ .

( $\supseteq$ ) Let  $\sigma : \Delta^1 \rightarrow X$ ,  $\epsilon(\delta_1(\sigma)) = \epsilon(v_1 - v_0) = 1 - 1 = 0$ .

( $\subseteq$ ) Let  $\sum n_i \sigma_i \in C_0(X)$  such that  $0 = \epsilon(\sum n_i \sigma_i) = \sum n_i$ . We fix a point  $x_0 \in X$ . Because  $X$  is path-connected, we can draw paths  $\tau_i$  from  $x_0$  to  $\sigma_i$ . Consider  $\sum n_i \tau_i \in C_1(X)$ , then

$$\partial(\sum n_i \tau_i) = \sum n_i (\partial \tau_i) = \sum n_i (\sigma_i - x_0) = \sum n_i \sigma_i - \sum \overbrace{n_i}^{=0} x_0 = \sum n_i \sigma_i$$

## Reduced Homology

$\cdots \longrightarrow C_n(X) \longrightarrow \cdots \longrightarrow C_1(X) \longrightarrow C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$   
Usually written as  $\tilde{H}_n(X)$ , and  $\tilde{H}_n(X) = H_n(X)$  if  $n \geq 1$ . We have that  $\tilde{H}_0(X) = \ker \epsilon / \operatorname{im} \partial_1$  and  $\epsilon|_{\operatorname{im} \partial_1} = 0$  so  $\epsilon$  induces a map  $\tilde{\epsilon} : \tilde{H}_0(X) \hookrightarrow \mathbb{Z}$ . Then  $\ker \tilde{\epsilon} = \tilde{H}_0(X)$ . It follows that

$0 \longrightarrow \tilde{H}_0(X) \longrightarrow H_0(X) \longrightarrow \mathbb{Z} \longrightarrow 0$   
is a split exact sequence since  $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$ . In particular,  $\tilde{H}(\text{pt}) = \{0\}$ .

### Remark

$$\pi_1 / [\pi_1, \pi_1] \cong H_1$$

## Homotopy Invariance

Suppose we have  $f : X \rightarrow Y$  continuous. It induces  $f_\# : C_n(X) \rightarrow C_n(Y)$  by  $\sigma \mapsto f \circ \sigma$ .  $f_\#$  is called a chain map and the following diagram commutes

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_n(X) & \xrightarrow{\partial} & C_{n-1} & \xrightarrow{\partial} & \cdots \\ & & \downarrow f_\# & & \downarrow f_\# & & \\ \cdots & \longrightarrow & C_n(X) & \xrightarrow{\partial} & C_{n-1} & \xrightarrow{\partial} & \cdots \end{array}$$

Let  $\sigma \in C_n(X)$  and

$$f_\#(\partial \sigma) = f_\# \left( \sum_{i=0}^n (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]} \right) = \sum_{i=0}^n (-1)^i (f \circ \sigma)|_{[v_0, \dots, \hat{v}_i, \dots, v_n]} = \partial(f_\# \sigma)$$

Then  $f_\#$  maps cycles to cycles ( $\partial c = 0$ ,  $\partial(f_\# c) = f_\#(\partial c) = 0$ ) and boundaries to boundaries ( $f_\#(\partial c) = \partial(f_\# c)$ ). So  $f_\#$  induces  $f_* : H_n(X) \rightarrow H_n(Y)$ .

## Theorem

If  $f, g : X \rightarrow Y$  are homotopic, then  $f_* = g_* : H_n(X) \rightarrow H_n(Y)$  for all  $n$ .

## Corollary

If  $X \simeq Y$  are homotopic, then  $H_n(X) \cong H_n(Y)$ .  $g \circ f \simeq \text{id}_X$ ,  $f \circ g \simeq \text{id}_Y$ ,

$$g_* \circ f_* = (g \circ f)_* = (\text{id}_X)_* = \text{id}$$

and similarly  $g_* \circ f_* = \text{id}$ . So  $f_*$  and  $g_*$  are isomorphisms.

## Definition

Let  $f, g : C_*(X) \rightarrow C_*(Y)$  be two chain maps. We say that  $f$  and  $g$  are chain homotopic if there is a map  $p : C_n(X) \rightarrow C_{n+1}(Y)$  such that  $\partial p + p \partial = g - f$ .

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_n(X) & \xrightarrow{\partial} & C_{n-1} & \xrightarrow{\partial} & \cdots \\ & \swarrow p & \downarrow f, g & \swarrow p & \downarrow f, g & & \\ \cdots & \longrightarrow & C_n(X) & \xrightarrow{\partial} & C_{n-1} & \xrightarrow{\partial} & \cdots \end{array}$$

## Theorem

If  $f \simeq g$  are homotopic, then

1.  $f_\#$  and  $g_\#$  are chain homotopic,
2.  $f_* = g_*$  on homology
3. For any  $n$ -cycle,  $c \in C_n(X)$ ,  $g(c) - f(c) = \partial p(c) + \overbrace{p(\partial c)}^{=0}$ . Hence  $g_*[c] = f_*[c]$ .

## Proof

Consider  $\Delta^n \times I$ , and set  $\Delta^n \times \{0\} = [v_0, \dots, v_n]$  and  $\Delta^n \times \{1\} = [w_0, \dots, w_n]$ . Then the following are all  $n$ -simplices

$$\begin{aligned} & [v_0, v_1, \dots, v_{n-1}, v_n] \\ & [v_0, v_1, \dots, v_{n-1}, w_n] \\ & [v_0, v_1, \dots, w_{n-1}, w_n] \\ & \vdots \\ & [v_0, w_1, \dots, w_{n-1}, w_n] \\ & [w_0, w_1, \dots, w_{n-1}, w_n] \end{aligned}$$

They divide  $\Delta^n \times I$  into  $(n+1)$ -simplices,  $\{[v_0, \dots, v_i, w_i, \dots, w_n] : i = 0, \dots, n\}$ . Now let  $F : X \times I \rightarrow Y$  be a homotopy between  $f$  and  $g$ . Consider

$\Delta^n \times I \xrightarrow{\sigma \times \text{id}} X \times I \xrightarrow{F} Y$  and define  $P : C_n(X) \rightarrow C_{n+1}(Y)$  by  $\sigma \mapsto \sum_{i=0}^n (-1)^i F \circ (\sigma \times \text{id})|_{[v_0, \dots, v_i, w_i, \dots, w_n]}$ . We need to check that  $\partial P + P \partial = g_\# - f_\#$ .

# Short Exact Sequences of Chain Complexes Induce Long Exact Sequences of Homology Groups

## Applications

1. Relative homology group.
2. Meyer-Vietoris sequence.

## Short Exact Sequences Induce Long Exact Sequences

Suppose we have sequences

$$\begin{array}{ccccccc}
 0 & & 0 & & 0 & & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 A_* & \longrightarrow & \cdots & \longrightarrow & A_{n+1} & \xrightarrow{\partial} & A_n & \xrightarrow{\partial} & A_{n-1} & \longrightarrow & \cdots \\
 \downarrow i & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 B_* & \longrightarrow & \cdots & \longrightarrow & B_{n+1} & \xrightarrow{\partial} & B_n & \xrightarrow{\partial} & B_{n-1} & \longrightarrow & \cdots \\
 \downarrow j & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 C_* & \longrightarrow & \cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial} & C_n & \xrightarrow{\partial} & C_{n-1} & \longrightarrow & \cdots \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & & 0 & & 0 & & 0 & & 0
 \end{array}$$

So  $H$  induces a long exact sequence

$$\cdots \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} \cdots$$

$$\cdots \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} \cdots$$

where  $\partial : H_n(C) \rightarrow H_{n-1}(A)$  by  $[c] \mapsto [a]$ , our connecting homomorphism, for  $c \in C_n$ . Then we have that the following commutes

$$\begin{array}{ccccc}
 & a & \xrightarrow{\partial} & & \\
 & \downarrow i & & \downarrow i & \\
 b & \xrightarrow{\quad} & \partial b & \xrightarrow{\quad} & 0 \\
 \downarrow j & & \downarrow j & & \\
 c & \xrightarrow{\quad} & 0 & & 
 \end{array}$$

So  $a$  is a cycle. We need to show that  $\partial a = 0$ . Note that  $i(\partial a) = \partial(i a) = \partial(\partial b) = 0$ . Because  $i$  is injective,  $\partial a = 0$ .  $\partial$  is well defined since

- choice of  $a$ :  $i$  is injective
- choice of  $b$ : suppose  $b' \in B_n$  such that  $j(b') = j(b) = c$ . Then  $b - b'$  satisfies  $j(b - b') = 0$  and  $b - b' \in \ker j = \text{im } i$  (i.e. there exists  $a' \in A_n$  such that  $i(a') = b - b'$ , so  $b' = b + i(a')$ ). Then

$$\begin{array}{ccc}
a' & \xrightarrow{\quad} & \partial a' \\
\downarrow & & \\
b - b' & & \\
\downarrow & & \\
0 & & 
\end{array}$$

So  $a + \partial a'$  satisfies

$$i(a + \partial a') = i(a) + i(\partial a') = \partial b + \partial(i a') = \partial b'$$

and

$$\begin{array}{ccc}
& & a + \partial a' \\
& & \downarrow \\
b' & \xrightarrow{\quad} & \partial b' \\
\downarrow & & \\
c & & 
\end{array}$$

since  $[a + \partial a'] = [a]$ .

- We need to check choice of  $c$ , but we will skip this.
- We need to check that  $\delta$  is a homomorphism, which follows from the definitions.
- Finally, check that the induced long sequence is exact. We will check only exactness about  $H_n(C)$  (i.e.  $\text{im } j_* = \ker \delta$ ).

$\text{im } j_* \subseteq \ker \delta$ :  $\delta(j_*[b]) = 0$  because

$$\begin{array}{ccc}
& & 0 \\
& & \downarrow \\
b & \xrightarrow{\partial} & 0 \\
\downarrow j & & \\
j(b) & & 
\end{array}$$

$\ker \delta \subseteq \text{im } j_*$ : Suppose  $[c] \in H_n(C)$  such that  $\partial[c] = 0$ , then

$$\begin{array}{ccc}
a' & \xrightarrow{\partial} & a = \partial a' \\
& & \downarrow \\
b & \xrightarrow{\quad} & \partial b \\
\downarrow j & & \\
c & & 
\end{array}$$

Consider  $b - i(a')$ , then  $j(b - i(a')) = j(b) - \overbrace{j \circ i(a')}^{=0} = j(b) = c$ . So  $[c] = j_*[b - i(a')] \in \text{im } j_*$ .  
This is a cycle, since  $\partial(b - i(a')) = \partial b - i(\partial a') = \partial b - \partial b = 0$ .

**April 30, 2025**

## Recall

1. if  $f, g : X \rightarrow Y$  are homotopic, then  $f_* = g_* : H_n(X) \rightarrow H_n(Y)$ .

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{\partial} & C_{n+1}(X) & \xrightarrow{\partial} & C_n(X) & \xrightarrow{\partial} & \cdots \\
& \swarrow P & \downarrow f_{\#}=g_{\#} & \swarrow P & \downarrow f_{\#}=g_{\#} & & \\
\cdots & \xrightarrow{\partial} & C_{n+1}(Y) & \xrightarrow{\partial} & C_n(Y) & \xrightarrow{\partial} & \cdots
\end{array}$$

$\partial P + P\partial = f_{\#} - g_{\#}.$

1. Short exact sequence of chain complexes

$$0 \longrightarrow A_* \xrightarrow{i} B_* \xrightarrow{j} C_* \longrightarrow 0$$

induces a long exact sequence of homology groups

$$\cdots \longrightarrow H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(C) \longrightarrow \cdots$$

$$\xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots$$

## Relative Homology Group

Setup:  $A \subseteq X$ ,  $A$  closed and non-empty. Then

$$C_n(A) = \{c \in C_n(X) : c = \sum n_i \sigma_i, \text{ im } \sigma_i \subseteq A\}.$$

Define  $C_n(X, A) = C_n(X)/C_n(A)$  such that

$$0 \longrightarrow C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{j} C_n(X, A) \longrightarrow 0$$

is a short exact sequence. Then  $C_*(X, A)$  is a chain complex

$$\cdots \xrightarrow{\partial} C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \xrightarrow{\partial} \cdots$$

with  $\partial^2 = 0$ . Note that  $\partial : C_n(X) \rightarrow C_{n-1}(X)$  maps  $C_n(A)$  to  $C_{n-1}(A)$ . Hence it induces  $\partial : C_n(X)/C_n(A) \rightarrow C_{n-1}(X)/C_{n-1}(A)$ . It gives homology groups  $H_n(X, A) = \ker \partial_n / \text{im } \partial_{n+1}$  and induces a long exact sequence

$$\cdots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \longrightarrow \cdots$$

$$\xrightarrow{\partial} H_{n-1}(A) \longrightarrow \cdots$$

## Remarks

1. the elements in  $H_n(X, A)$  are represented by relative cycles (i.e.  $\alpha \in C_n(X)$  such that  $\partial\alpha \in C_{n-1}(A)$ ).
2. A relative cycle  $\alpha$  is trivial in  $H_n(X, A)$  means  $\alpha$  is a “relative boundary” (i.e.  $\alpha = \partial\beta + \gamma$  for  $\beta \in C_{n+1}(X)$  and  $\gamma \in C_n(A)$ ).

$\partial : H_n(X, A) \rightarrow H_{n-1}(A)$  is defined by  $[\alpha] \mapsto [\partial\alpha]$

$$\begin{array}{ccc}
& & \partial\alpha \\
& & \downarrow i \\
\alpha \in C_n(X) & \xrightarrow{\partial} & \partial\alpha \in C_{n-1}(A) \\
\downarrow j & & \\
\alpha \in C_n(X, A) & & 
\end{array}$$

We can also define the relative version.

$$\begin{array}{ccccccc}
& 0 & & 0 & & & \\
& \downarrow & & \downarrow & & & \\
\cdots & \xrightarrow{\partial} & C_0(A) & \xrightarrow{\epsilon} & \mathbb{Z} & \longrightarrow & 0 \\
& \downarrow & & \downarrow \cong & & & \\
\cdots & \xrightarrow{\partial} & C_0(X) & \xrightarrow{\epsilon} & \mathbb{Z} & \longrightarrow & 0 \\
& \downarrow & & \downarrow 0 & & & \\
\cdots & \xrightarrow{\partial} & C_0(X, A) & \xrightarrow{0} & 0 & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & & \\
& 0 & & 0 & & & 
\end{array}
\quad
\begin{array}{ccc}
\sum n_i \sigma_i & \longrightarrow & \sum n_i \\
\downarrow & & \downarrow = \\
\sum n_i \sigma_i & \longmapsto & \sum n_i
\end{array}
\quad
\begin{array}{ccccccc}
\tilde{H}_n(A) & \longrightarrow & \tilde{H}_n(X) & \longrightarrow & H_n(X, A) & & \\
& \xrightarrow{\partial} & & & & & \\
& & & & \tilde{H}_0(A) & \longrightarrow & \tilde{H}_0(X) \longrightarrow H_0(X, A) \\
& & & & & & \\
& & & & & & 0
\end{array}$$

### Example

$H_n(X, X) = 0$  for all  $n$ , because  $C_n(X, X) = C_n(X)/C_n(X) = \{0\}$ . So  $H_n(X, X_0) \cong \tilde{H}_n(X)$

$$\overbrace{\tilde{H}_n(X_0)}^{=0} \longrightarrow \tilde{H}_n(X) \xrightarrow{\cong} H_n(X, X_0)$$

$$\xrightarrow{\partial} \overbrace{\tilde{H}_{n-1}(X_0)}^{=0} \longrightarrow \cdots$$

### Fact

$H_n(X, A) \cong \tilde{H}_n(X/A)$  if  $(X, A)$  is a “good” pair (i.e. there exists a neighborhood  $V$  of  $A$  which deformation retracts to  $A$ ).

### Example

$(X, A) = (D^n, \partial D^n)$  is a good pair, so  $H_i(X, A) \cong \tilde{H}_i(D^n / \partial D^n) = \tilde{H}_i(S^n)$ . This give a long exact sequence

$$\tilde{H}_i(S^{n-1}) \longrightarrow \overbrace{\tilde{H}_i(D^n)}^{=0} \longrightarrow H_i(X, A)$$

$$\xrightarrow{\partial} \tilde{H}_{i-1}(S^{n-1}) \longrightarrow \overbrace{\tilde{H}_{i-1}(D^n)}^{=0} \longrightarrow \cdots$$

and  $\tilde{H}_{i-1}(S^{n-1}) \cong H_i(D^n, \partial D^n) \cong \tilde{H}_i(S^n)$ . We conclude

that  $\tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1})$ .

For  $n = 0$ ,  $S^0$  is two points,  $\tilde{H}_0(S^0) = \mathbb{Z}$ , and  $\tilde{H}_i(S^0) = \tilde{H}_i(\text{pt}) \oplus \tilde{H}_i(\text{pt}) = 0$  for each  $i \geq 1$ .

For  $n = 1$ ,  $\tilde{H}_1(S^1) \cong \tilde{H}_0(S^0) \cong \mathbb{Z}$  and  $\tilde{H}_0(S^1) = 0$ .

For  $n = 2$ ,  $\tilde{H}_2(S^2) \cong \tilde{H}_1(S^1) \cong \mathbb{Z}$ ,  $\tilde{H}_1(S^2) \cong \tilde{H}_0(S^1) = 0$  and  $\tilde{H}_0(S^2) = 0$ .

So  $\tilde{H}_i(S^n)$  is  $\mathbb{Z}$  when  $i = n$  and 0 otherwise.

## Induced Maps on Pairs

Write  $f : (X, A) \rightarrow (Y, B)$  for a continuous map  $f : X \rightarrow Y$  such that  $f(A) \subseteq B$ . Then  $f_{\#} : C_n(X) \rightarrow C_n(Y)$  ( $f_{\#} : C_n(A) \rightarrow C_n(B)$ ) induces  $f_{\#} : C_n(X, A) \rightarrow C_n(Y, B)$  a chain map  $\partial f_{\#} = f_{\#} \partial$ . This induces  $f_* : H_n(X, A) \rightarrow H_n(Y, B)$ .

## Proposition

Given  $f, g : (X, A) \rightarrow (Y, B)$  which are homotopic through maps between pairs  $(X, A) \rightarrow (Y, B)$ , then  $f_* = g_* : H_n(X, A) \rightarrow H_n(Y, B)$ .



$$\begin{array}{ccccccc}
\cdots & \longrightarrow & C_{n+1}(X, A) & \longrightarrow & C_n(X, A) & \longrightarrow & \cdots \\
& & \downarrow & & \downarrow & & \\
\cdots & \xrightarrow{p} & C_{n+1}(Y, B) & \longrightarrow & C_n(Y, B) & \longrightarrow & \cdots
\end{array}$$

such that  $\partial P + P\partial = g_{\#} - f_{\#}$  (i.e.  $f_* = g_*$ ).  $P : C_n(X) \rightarrow C_{n+1}(Y)$

maps  $C_n(A)$  to  $C_{n+1}(B)$ .  $P$  defined by  $P(\sigma) \sum (-1)^i F \circ (0 \times \text{id})|_{[v_0, \dots, v_i, w_i, \dots, w_j]}$

$\Delta^n \times I \xrightarrow{0 \times \text{id}} X \xrightarrow{F} Y$  If  $\sigma : \Delta^n \rightarrow A$ , then  $P(\sigma) : \Delta^{n+1} \rightarrow B$ .

## Excision

Given a good pair  $(X, A)$ ,  $H_n(X, A) \cong \tilde{H}_n(X/A)$ .

Suppose we have  $Z \subseteq A \subseteq X$  such that  $\overline{Z} \subseteq A^\circ$  (i.e. the closure of  $Z$  is in the interior of  $A$ ). Then  $H_n(X, A) \cong H_n(X - Z, A - Z)$ . Equivalently, if  $B = X - Z$  then  $A \cap B = A - Z$  and  $\overline{Z} \subseteq A^\circ \implies A^\circ \cup B^\circ = X$ . If  $A$  and  $B$  satisfy  $A^\circ \cup B^\circ = X$ , then by excision  $H_n(X, A) \cong H_n(B, A \cap B)$ .

## Remark

If  $X$  has a  $\Delta$ -complex structure such that  $A$ ,  $X - Z$  and  $A - Z$  are subcomplexes, then we claim that  $C_n^\Delta(X, A) = C_n^\Delta(X - Z, A - Z)$  (and  $H_n^\Delta(X, A) = H_n^\Delta(X - Z, A - Z)$ ). In fact, consider  $\varphi : C_n^\Delta(X - Z) \rightarrow C_n^\Delta(X)/C_n^\Delta(A)$  which factors through

$$C_n^\Delta(X - Z) \xhookrightarrow{\iota} C_n^\Delta(X) \longrightarrow C_n^\Delta(X, A) = C_n^\Delta(X)/C_n^\Delta(A)$$

Then  $\varphi$  is surjective,  $\ker \varphi = C_n^\Delta(A - Z)$  and

$$C_n^\Delta(X, A) = C_n^\Delta(X)/C_n^\Delta(A) = C_n^\Delta(X - Z)/\ker \varphi = C_n^\Delta(X - Z, A - Z)$$

## Proof

Let  $\{U_\alpha\}_\alpha = \mathcal{U}$  be a collection of subsets such that  $\{U_\alpha^\circ\}_\alpha$  is an open cover of  $X$  (it will suffice to consider  $\mathcal{U} = \{A, B\}$ ). Write

$$C_n^\mathcal{U}(X) = \left\{ \sum n_i \sigma_i \in C_n(X) : \text{im } \sigma_i \subseteq U_j^\circ \text{ for some } j \right\}.$$

Then  $\partial : C_n(X) \rightarrow C_{n-1}(X)$  maps  $C_n^\mathcal{U}(X)$  to  $C_{n-1}^\mathcal{U}(X)$ . The chain complex  $C_*^\mathcal{U}(X)$  gives homology groups  $H_*^\mathcal{U}(X)$ .

## Proposition

$\iota : C_n^\mathcal{U} \rightarrow C_n(X)$  induces an isomorphism  $H_n^\mathcal{U}(X) \cong H_n(X)$ .

The sketch of this proof is to construct a map  $\rho : C_n(X) \rightarrow C_n^\mathcal{U}(X)$  by subdivision. That is, if the simplex  $\sigma : \Delta^n \rightarrow X$  does not sit inside any  $U_\alpha$  we may subdivide into further simplices that do. Then  $\rho \circ \iota = \text{id}$  and  $\iota \circ \rho$  is chain homotopic to the identity.

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & C_n(X) & \longrightarrow & C_{n-1}(X) & \longrightarrow & \cdots \\
& & \downarrow \iota \circ \rho & & \downarrow & & \\
\cdots & \xrightarrow{D} & C_n(X) & \longrightarrow & C_{n-1}(X) & \longrightarrow & \cdots
\end{array}$$

where  $D : C_{n-1}(X) \rightarrow C_n(X)$  such that  $\partial D + D\partial = \text{id} - \iota \circ \rho$  which implies  $(\iota \circ \rho)_* : H_n(X) \rightarrow H_n(X)$  is the identity map. There also exists a relative version. For simplicity, say  $\mathcal{U} = \{A, B\}$  and denote  $C_n^\mathcal{U}(X) \triangleq C_n(A + B)$  so we have  $H_n(A + B, A) \cong H_n(X, A)$ .

## Proof Continued

We have that  $H_n(A + B, A) \cong H_n(X, A)$  (proof in Hatcher).  
The left hand side comes from the chain complex of

$$C_n(A + B, A) = C_n(A + B) / C_n(A) = C_n(B) / C_n(A \cap B) = C_n(B, A \cap B)$$

so  $H_n(A + B, A) = H_n(B, A \cap B)$ .

## Proposition

Let  $(X, A)$  be a good pair. Then the quotient map  $q : (X, A) \rightarrow (X/A, A/A)$  induces an isomorphism  $q_* : H_n(X, A) \xrightarrow{\sim} H_n(X/A, \text{pt}) \cong \tilde{H}_n(X/A)$ .

## Proof

Let  $V$  be a neighborhood of  $A$  which deformation retracts to  $A$ .

$$\begin{array}{ccccc} H_n(X, A) & \xrightarrow{(1)} & H_n(X, V) & \xrightarrow{\sim} & H_n(X - A, V - A) \\ \downarrow q_* & & & & \downarrow \sim \\ H_n(X/A, A/A) & \xrightarrow{(2)} & H_n(X/A, V/A) & \xleftarrow[\text{excision}]{H_n((X-A)/A, (V-A)/A)} & H_n(X/A - A/A, V/A - A/A) \end{array}$$

It remains to show that (1) and (2) are isomorphisms. For (2),  $V/A$  deformation retracts to  $A/A$  in  $X/A$ . So consider the triple  $A \subseteq V \subseteq X$ . It induces a short exact sequence

$$0 \longrightarrow \frac{C_n(V, A)}{C_n(V)/C_n(A)} \xrightarrow{i} \frac{C_n(X, A)}{C_n(X)/C_n(A)} \xrightarrow{j} \frac{C_n(X, V)}{C_n(X)/C_n(V)} \longrightarrow 0$$

So  $\ker j = \text{im } i$ , and this induces a long exact sequence

$$\longrightarrow \overbrace{H_n(V, A)}^{=0} \longrightarrow H_n(X, A) \xrightarrow{\sim} H_n(X, V)$$

$$\xrightarrow{\partial} \overbrace{H_{n-1}(V, A)}^{=0} \longrightarrow$$

where the terms zero since  $V$  deformation retracts to  $A$ .

May 5, 2025

## Recall

For  $A \subseteq X$ , we have

$$\begin{aligned} 0 &\longrightarrow C_*(A) \longrightarrow C_*(X) \longrightarrow C_*(X, A) = C_*(X)/C_*(A) \longrightarrow 0 \\ \dots &\longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \longrightarrow \dots \end{aligned}$$

which induces

$$\xrightarrow{\partial} H_{n-1}(A) \longrightarrow \dots$$

Also, we have excision where

1. if  $Z \subseteq A \subseteq X$  such that  $\bar{Z} \subseteq A^\circ$ , then  $H_n(X - Z, A - Z) = H_n(X, A)$ .

2. if  $(X, A)$  is a good pair, i.e.  $A$  has a neighborhood  $V$  such that  $V$  deformation retracts to  $A$ , then  $H_n(X, A) = \tilde{H}_n(X/A)$ .

## Simplicial and Singular Homology

Goal: given  $X$  with  $\Delta$ -complex structure,  $H_n^\Delta(X) \cong H_n(X)$ .

### Example

$H_n(D^n, \partial D^n) \cong \tilde{H}_n(D^n / \partial D^n) = \tilde{H}_n(S^n) \cong \mathbb{Z}$ . We can construct a generator for this  $\mathbb{Z}$ . We consider  $H_n(\Delta^n, \partial \Delta^n)$  and claim that it is generated by  $i_n : \Delta^n \rightarrow \Delta^n$  as the identity map. We prove by induction, first observing that  $n = 0$  is good. Then suppose  $n - 1$  and let  $\Lambda \subseteq \Delta^n$  be the space obtained by removing a face from the boundary  $\partial \Delta^n$ . Then take

$$H_n(\Delta^n, \partial \Delta^n) \xrightarrow{\partial} H_n(\partial \Delta^n, \Lambda) \xleftarrow{(2)} H_{n-1}(\Delta^{n-1}, \partial \Delta^{n-1})$$

Consider the triple  $\Lambda \subseteq \partial \Delta^n \subseteq \Delta^n$  and the short exact

sequence on the chain level

$$0 \longrightarrow C_\bullet(\partial \Delta^n, \Lambda) \xrightarrow{i} C_\bullet(\Delta^n, \Lambda) \xrightarrow{j} C_\bullet(\Delta^n, \partial \Delta^n) \longrightarrow 0$$

which induces the long exact sequence

$$\cdots \longrightarrow H_n(\partial \Delta^n, \Lambda) \longrightarrow \overbrace{H_n(\Delta^n, \Lambda)}^{=0} \longrightarrow H_n(\Delta^n, \partial \Delta^n) \longrightarrow \cdots$$

$$\xrightarrow{\partial} H_{n-1}(\partial \Delta^n, \Lambda) \longrightarrow \overbrace{H_{n-1}(\Delta^n, \Lambda)}^{=0} \longrightarrow \cdots$$

since  $\Delta^n$  deformation retracts to  $\Lambda$ ,  $H_*(\Delta^n, \Lambda) = 0$

0. Hence  $H_n(\Delta^n, \partial \Delta^n) \cong H_{n-1}(\partial \Delta^n, \Lambda)$ .

For (2), let  $\Delta^{n-1}$  be the face that is not in  $\Lambda$ . Then  $\Delta^{n-1} \hookrightarrow \partial \Delta^n$  induces a homeomorphism  $\Delta^{n-1} / \partial \Delta^{n-1} \cong \partial \Delta^n / \Lambda$ . Hence  $(\partial \Delta^n, \Lambda)$  is a good pair, and

$$H_{n-1}(\partial \Delta^n, \Lambda) \cong \tilde{H}_{n-1}(\partial \Delta^n / \Lambda) \cong \tilde{H}_{n-1}(\Delta^{n-1} / \partial \Delta^{n-1}) \cong H_{n-1}(\Delta^{n-1}, \partial \Delta^{n-1})$$

We have

$$\begin{array}{ccc} & \partial i_n \in C_{n-1}(\partial \Delta^n, \Lambda) & \\ & \downarrow & \\ i_n \in C_n(\Delta^n, \Lambda) & \xrightarrow{\partial} & \partial i_n \in C_{n-1}(\Delta^n, \partial \Delta^n) \\ & \downarrow & \\ i_n \in C_n(\Delta^n, \partial \Delta^n) & & \end{array}$$

so  $\delta^{-1} : [\partial i_n] \mapsto [i_n]$ . Through the isomorphism  $H_{n-1}(\Delta^{n-1}, \partial \Delta^{n-1}) \cong H_n(\Delta^n, \partial \Delta^n)$ ,  $[i_n]$  is identified with  $[\partial i_n]$  for  $i_n : \Delta^n \rightarrow \Delta^n$ . Hence  $[\partial i_n]$  is  $[\pm i_{n-1}]$  in  $H_{n-1}(\Delta^{n-1}, \partial \Delta^{n-1})$ .

### Corollary

Let  $\bigvee_\alpha X_\alpha$  by identifying  $x_\alpha \in X_\alpha$  for each  $\alpha$ . Suppose  $(X_\alpha, x_\alpha)$  is a good pair for each  $\alpha$ . Then  $\bigoplus_\alpha \tilde{H}_n(X_\alpha) \cong \tilde{H}_n(\bigvee_\alpha X_\alpha)$ .

## Proof

Consider the good pair  $(X, A) := (\coprod_{\alpha} X_{\alpha}, \coprod_{\alpha} \{x_{\alpha}\})$  where  $X/A = \bigvee_{\alpha} X_{\alpha}$  such that

$$\tilde{H}_n\left(\bigvee_{\alpha} X_{\alpha}\right) \cong H_n(X, A) \cong \bigoplus_{\alpha} H_n(X_{\alpha}, x_{\alpha}) = \bigoplus_{\alpha} \tilde{H}_n(X_{\alpha}).$$

## Theorem

Let  $U \subseteq \mathbb{R}^m$  and  $V \subseteq \mathbb{R}^n$  be open sets. If  $U$  and  $V$  are homeomorphic, then  $m = n$ .

## Proof

Let  $x \in U$ . By excision,

$$H_i(U, U - \{x\}) \cong H_i(\mathbb{R}^m, \mathbb{R}^m - \{x\})$$

where we note that  $(\mathbb{R}^m, \mathbb{R}^m - \{x\})$  is not a good pair. However, it still induces a long exact sequence

$$\longrightarrow \tilde{H}_i(\mathbb{R}^m - \{x\}) \longrightarrow \overbrace{\tilde{H}_i(\mathbb{R}^m)}^{=0} \longrightarrow H_i(\mathbb{R}^m, \mathbb{R}^m - \{x\})$$

$$\longrightarrow \tilde{H}_{i+1}(\mathbb{R}^m - \{x\}) \longrightarrow \overbrace{\tilde{H}_{i+1}(\mathbb{R}^m)}^{=0} \longrightarrow \dots$$

Hence

$$H_i(\mathbb{R}^m, \mathbb{R}^m - \{x\}) \cong \tilde{H}_{i-1}(\mathbb{R}^m - \{x\}) \cong \tilde{H}_{i-1}(S^{m-1}) = \begin{cases} \mathbb{Z} & i = m \\ 0 & i \neq m \end{cases}.$$

If  $U$  and  $V$  are homeomorphisms, then  $H_i(U, U - \{x\}) \cong H_i(V, V - \{\varphi(x)\})$  and  $m = n$ .

## Naturality of Long Exact Sequences of Pairs

$f : (X, A) \rightarrow (Y, B)$  with  $f(A) \subseteq B$ ,

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_{\bullet}(A) & \longrightarrow & C_{\bullet}(X) & \longrightarrow & C_{\bullet}(X, A) \longrightarrow 0 \\ & & \downarrow f_{\#} & & \downarrow f_{\#} & & \downarrow f_{\#} \end{array}$$

$$0 \longrightarrow C_{\bullet}(B) \longrightarrow C_{\bullet}(Y) \longrightarrow C_{\bullet}(Y, B) \longrightarrow 0$$

commutes. Then the long exact sequence

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A) \xrightarrow{\delta} H_{n-1}(A) \longrightarrow \dots \\ & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\ \dots & \longrightarrow & H_n(B) & \longrightarrow & H_n(Y) & \longrightarrow & H_n(Y, B) \xrightarrow{\delta} H_{n-1}(B) \longrightarrow \dots \end{array}$$

$$\begin{array}{ccc}
& \partial\alpha \in C_{n-1}(A) \\
& \downarrow \\
\alpha \in C_n(X) & \xrightarrow{\partial} & \partial\alpha \in C_{n-1}(X) \\
\downarrow & & \\
\alpha \in C_n(X, A) & & 
\end{array}$$

So  $\delta : [\alpha] \rightarrow [\partial\alpha]$  and

$$f_*(\delta[\alpha]) = f_*[\partial\alpha] = [f_*(\partial\alpha)] = [\partial f_*(\alpha)] = \delta(f_*[\alpha]).$$

## Recall: the Five Lemma

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E & \longrightarrow & \cdots \\
& & \downarrow \cong & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \downarrow \cong & & \\
\cdots & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' & \longrightarrow & \cdots
\end{array}$$

implies that  $C \cong C'$ .

## Equivalence Between Simplicial and Singular Homology

Given  $X$  with a finite dimensional  $\Delta$ -complex structure, then  $C_n^\Delta(X) \hookrightarrow C_n(X)$  induces an isomorphism  $H_n^\Delta(X) \cong H_n(X)$ .

### Proof

Suppose it holds for all  $(X, \Delta)$  with dimension less than  $k-1$ . We consider the  $k$ -dimensional case.

Let  $X^i$  be the  $i$ -skeleton of  $X$ . Note that  $X^k = X$ , so the pair  $(X^k, X^{k-1})$  induces a long exact sequence

$$\begin{array}{ccccccccc}
H_{n+1}^\Delta(X^k, X^{k-1}) & \longrightarrow & H_n^\Delta(X^{k-1}) & \longrightarrow & H_n^\Delta(X^k) & \longrightarrow & H_n^\Delta(X^k, X^{k-1}) & \longrightarrow & H_{n-1}^\Delta(X^{k-1}) \\
\downarrow (1) & & \downarrow (2) & & \downarrow (3) & & \downarrow (4) & & \downarrow (5) \\
H_{n+1}(X^k, X^{k-1}) & \longrightarrow & H_n(X^{k-1}) & \longrightarrow & H_n(X^k) & \longrightarrow & H_n(X^k, X^{k-1}) & \longrightarrow & H_{n-1}(X^{k-1})
\end{array}$$

(5) are isomorphisms per our inductive assumption. Note also that  $C_n^\Delta(X^k) = 0$  for  $n \geq k$ , so

We have that (2) and

$$C_n^\Delta(X^k, X^{k-1}) = C_n^\Delta(X^k) / C_n^\Delta(X^{k-1}) = \begin{cases} C_n^\Delta(X^k) & k = n \\ 0 & n < k \end{cases}.$$

So the chain complex  $C_\bullet^\Delta(X^k, X^{k-1})$  is

$$0 \longrightarrow 0 \longrightarrow C_n^\Delta(X^k, X^{k-1}) = C_n^\Delta(X^k) \longrightarrow 0 \longrightarrow 0 \quad \text{and} \quad H_n^\Delta(X^k, X^{k-1}) \cong \begin{cases} C_k^\Delta(X^k) & k = n \\ 0 & k \neq n \end{cases}.$$

Now consider

$\Phi : \left( \bigsqcup_\alpha \Delta_\alpha^k, \bigsqcup_\alpha \partial \Delta_\alpha^k \right) \rightarrow (X^k, X^{k-1})$ . It induces a homomorphism  $X^k / X^{k-1} \cong \left( \bigsqcup_\alpha \Delta_\alpha^k \right) / \left( \bigsqcup_\alpha \partial \Delta_\alpha^k \right)$ . So \(\backslash\)

$$H_n(X^k, X^{k-1}) \cong \tilde{H}_n(X^k, X^{k-1}) \cong \tilde{H}_n \left( \left( \bigsqcup_\alpha \Delta_\alpha^k \right) / \left( \bigsqcup_\alpha \partial \Delta_\alpha^k \right) \right) \cong H_n \left( \bigsqcup_\alpha \Delta_\alpha^k, \bigsqcup_\alpha \partial \Delta_\alpha^k \right) \cong \bigoplus_\alpha H_n(\Delta_\alpha^k, \partial \Delta_\alpha^k)$$

where each  $H_n(\Delta_\alpha^k, \partial \Delta_\alpha^k)$  is generated by  $i_\alpha^k : \Delta_\alpha^k \rightarrow \Delta_\alpha^k$  (the identity map) if  $n = k$  or  $H_n(\Delta_\alpha^k, \partial \Delta_\alpha^k)$  when  $n \neq k$ . Finally, we observe that

$$C_k^\Delta(X^k) \cong \bigoplus_\alpha \langle i_\alpha^k \rangle \cong \bigoplus_\alpha H_n(\Delta_\alpha^k, \partial \Delta_\alpha^k).$$

So (1) and (4) are isomorphisms and, by the five lemma, (3) is an isomorphism as well.

## Remark

$H_n^\Delta(X, A) \cong H_n(X, A)$  if  $X$  has a  $\Delta$ -complex structure and  $A \subseteq X$  is a sub-complex.

$$\begin{array}{ccccccccc}
 H_n^\Delta(A) & \longrightarrow & H_n^\Delta(X) & \longrightarrow & H_n^\Delta(X, A) & \longrightarrow & H_{n-1}^\Delta(A) & \longrightarrow & H_{n-1}^\Delta(X) \\
 \downarrow (1) & & \downarrow (2) & & \downarrow (3) & & \downarrow (4) & & \downarrow (5) \\
 H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A) & \longrightarrow & H_{n-1}(A) & \longrightarrow & H_{n-1}(X)
 \end{array}$$

where (1), (2), (4), (5) are isomorphisms,

so we have the conclusion by the five lemma.

**May 7, 2025**

## Definition: Degree

Let  $f : S^n \rightarrow S^n$  which induces  $f_* : H_n(S^n) \rightarrow H_n(S^n)$  (i.e.  $\mathbb{Z} \rightarrow \mathbb{Z}$ ). Hence  $f_*$  is multiplication by some integer  $d \in \mathbb{Z}$ . Define  $\deg(f) = d$ .

## Properties

1.  $\deg(\text{id}) = 1$ .
2. If  $f, g : S^n \rightarrow S^n$  are homotopic, then  $f_* = g_*$  thus  $\deg(f) = \deg(g)$ .
3.  $\deg(f \circ g) = \deg(f) \cdot \deg(g)$ , because  $(f \circ g)_* = f_* \circ g_*$ . In particular, if  $f \circ g \simeq \text{id}_{S^n}$  then  $\deg(f) \cdot \deg(g) = \deg(f \circ g) = 1$  and  $\deg(f) = \pm 1$ .
4. Suppose  $f : S^n \rightarrow S^n$  is not surjective, say  $x_0 \in S^n \setminus \text{im } f$ . Then  $f : S^n \rightarrow S^n \setminus \{x_0\} \cong \mathbb{R}^n$ . So  $f$  is  $S^n \xrightarrow{f} S^n \setminus \{x_0\} \xrightarrow{L} S^n$  and

$$H_n(S^n) \longrightarrow \overbrace{H_n(S^n \setminus \{x_0\})}^{=0} \longrightarrow H_n(S^n)$$

So  $f_* : H_n(S^n) \rightarrow H_n(S^n)$  is the zero map (i.e.  $\deg(f) = 0$ ).

1.  $f : S^n \rightarrow S^n$  a reflection has degree  $-1$ . In general, if we take two copies of  $\Delta^n$  glued along corresponding edges by the identity map then we get  $S^n$ . Then  $H_n^\Delta(S^n)$  has a generator  $U - L$ , and reflection of  $f$  maps  $U - L$  to  $L - U$  (i.e.  $f_* : \mathbb{Z} \rightarrow \mathbb{Z}$  is  $1 \mapsto -1$ ).
2.  $f : S^n \rightarrow S^n$  an antipodal map  $(- \text{id})$  which sends  $(x^1, \dots, x^{n+1}) \mapsto (-x^1, \dots, -x^{n+1})$  has  $\deg(-1 \text{id}) = (-1)^{n+1}$ .
3. Theorem (Hopf) if  $f, g : S^n \rightarrow S^n$  have the same degree, then  $f \simeq g$ .
4. If  $f : S^n \rightarrow S^n$  has no fixed points, then  $f \simeq -\text{id}$  and  $\deg(f) = (-1)^{n+1}$ . Proof: if  $x \neq f(x)$ , then the segment  $(1-t)f(x) + t(-x)$  does not pass through  $0 \in \mathbb{R}^{n+1}$ . Consider  $f_t(x) = \frac{(1-t)f(x) + t(-x)}{\|(1-t)f(x) + t(-x)\|}$  where  $f_0(x) = f(x)$  and  $f_1(x) = -x$  show that  $f_t(x)$  gives a homotopy between  $f$  and  $-\text{id}$ .
5.  $S^n$  has a continuous, non-vanishing vector field if and only if  $n$  is odd. Proof: ( $\Leftarrow$ ) say  $n = 2k - 1$  such that  $S^n \subseteq \mathbb{R}^{2k}$ . Define  $V(x_1, \dots, x_{2k}) = (-x_2, x_1, -x_4, x_3, \dots)$ . Then  $V(\vec{x}) \perp \vec{x}$ . ( $\Rightarrow$ ) Think of  $V(\vec{x})$  starting at  $\vec{x}$

and without loss of generality that  $\|V(\vec{x})\| = 1$ . Consider  $f_t(x) = (\cos t)\vec{x} + (\sin t)V(\vec{x})$  where  $f_\pi(x) = -x$  and  $f_0(x) = x$  such that  $\{f_t\}$  is a homotopy between  $\text{id}$  and  $-\text{id}$ . Hence  $1 = \deg(\text{id}) = \deg(-\text{id}) = (-1)^{n+1}$  and  $n$  is odd.

6. If  $n$  is even, then  $\mathbb{Z}_2$  is the only non-trivial group that can act freely on  $S^n$ . For example,  $S^1$  acts on  $S^3$  freely if we consider  $(z_1, z_2) \in S^3 \subseteq C^2$  and  $\theta(z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2)$ . Proof: suppose  $G \neq \text{id}$  acts freely on  $S^n$ . Consider  $\deg : G \rightarrow \mathbb{Z}$  where  $\text{im}(\deg) \subseteq \{\pm 1\} \subseteq \mathbb{Z}$  and for  $g \neq e$  then  $\deg(g) = (-1)^{n+1} = -1$ . Then  $G/\ker \cong \text{im} = \{-1, 1\}$  since  $\ker = \{e\}$ . Hence  $G \cong \text{im} = (\{\pm 1, \cdot\} = \mathbb{Z}_2$ .

## Theorem

Below, we assume that  $S^n$  has a point  $y$  such that  $f^{-1}(y) = \{x_1, \dots, x_m\}$  is a finite set. If  $f$  is smooth, then by Sard's theorem we may pick a regular point  $y$ . Then  $f^{-1}(y)$  is an embedded submanifold of dimension zero (i.e.  $f^{-1}(y)$  is a collection of finitely many points). That is, when  $f$  is smooth this assumption holds automatically.

For each  $i = 1, \dots, m$ , we choose a small ball  $U_i$  about  $x_i$  and a ball  $V$  about  $y$  such that  $f(U_i) \subseteq V$ . The pair  $(S^n, S^n \setminus \{x\})$  induces

$$\cdots \longrightarrow \overbrace{H_n(S^n \setminus \{x\})}^{=0} \longrightarrow H_n(S^n) \xrightarrow{j} H_n(S^n, S^n \setminus \{x\}) \longrightarrow \overbrace{H_{n-1}(S^n \setminus \{x\})}^{=0} \longrightarrow \cdots$$

The pair  $(U, U \setminus \{x\})$  gives

$$\cdots \longrightarrow H_n(U \setminus \{x\}) \longrightarrow \overbrace{H_n(U)}^{=0} \longrightarrow H_n(U, U \setminus \{x\}) \xrightarrow{\delta} H_{n-1}(U \setminus \{x\}) \longrightarrow \overbrace{H_{n-1}(U)}^{=0} \longrightarrow \cdots$$

and we observe that  $H_n(S^n, S^n \setminus \{x\}) \cong H_n(U, U \setminus \{x\})$  by excision.

$$\begin{array}{ccccc} & \overbrace{H_n(U_i, U_i \setminus \{x_i\})}^{=\mathbb{Z}} & \xrightarrow{f_*} & \overbrace{H_n(V, V \setminus \{y\})}^{=\mathbb{Z}} & \\ & \downarrow k_i & & \downarrow \text{excision} & \\ \text{excision} \swarrow & \overbrace{H_n(S^n, S^n - f^{-1}(y))}^{=\mathbb{Z}^m} & \xrightarrow{f_*} & H_n(S^n, S^n \setminus \{y\}) & \\ \swarrow p_i & \uparrow j & & \uparrow j & \\ H_n(S^n, S^n \setminus \{x_i\}) & \xleftarrow{j} & H_n(S^n) & \xrightarrow{f_*} & H_n(S^n) \end{array}$$

We have that  $f_* : H_n(U_i, U_i \setminus \{x_i\}) \rightarrow H_n(V, V \setminus \{y\})$  is  $\mathbb{Z} \rightarrow \mathbb{Z}$  and hence it gives an integer. We call this the local degree  $\deg(f|_{x_i})$ .

Theorem:  $\deg(f) = \sum_{i=1}^m \deg(f|_{x_i})$ .

Write

$$H_n(S^n, S^n - f^{-1}(y)) \underset{\text{excision}}{\cong} H_n\left(\bigsqcup_i U_i, \bigsqcup_i (U_i \setminus \{x_i\})\right) \cong \bigoplus_i H_n(U_i, U_i \setminus \{x_i\}) \cong \mathbb{Z}^m.$$

then  $k_i : H_n(U_i, U_i \setminus \{x_i\}) \rightarrow \bigoplus_i H_n(U_i, U_i \setminus \{x_i\})$  by  $1 \mapsto (0, \dots, 0, 1, 0, \dots, 0) =: e_i$ . Consider the triple  $S^n - f^{-1}(y) \subseteq S^n \setminus \{x_i\} \subseteq S^n$  which induces

$$0 \longrightarrow C_\bullet(S^n \setminus \{x_i\}, S^n \setminus f^{-1}(y)) \longrightarrow C_\bullet(S^n, S^n \setminus f^{-1}(y)) \longrightarrow C_\bullet(S^n, S^n \setminus \{x_i\}) \longrightarrow 0$$

So we have  $p_i : H_n(S^n, S^n \setminus f^{-1}(y)) \rightarrow H_n(S^n, S^n \setminus \{x_i\})$ . Then

$$\begin{array}{ccc} & \mathbb{Z} & \\ & \downarrow k_i & \\ \mathbb{Z} & \xleftarrow[p_i]{\text{id}} & \mathbb{Z}^m \end{array}$$

commutes and  $1 = p_i(k_i(1)) = p_i(e_i)$ , hence  $p_i$  is the projection to the  $i$ -th component. Similarly

$$\begin{array}{ccc} \mathbb{Z} & \xleftarrow[p_i]{\text{id}} & \mathbb{Z}^m \\ & \uparrow j & \\ & \mathbb{Z} & \end{array}$$

commutes so  $1 = p_i(j(1))$  and the  $i$ -th component of  $j(1)$  is 1 (i.e.  $j(1) = (1, 1, \dots, 1) \in \mathbb{Z}^m$ ). Then  $\deg(f|_{x_i}) = f_*(k_i(1)) = f_*(e_i)$ . Finally,

$$\deg f = f_*(1) = f_*(j(1)) = f_*\left(\sum e_i\right) = \sum f_*(e_i) = \sum \deg(f_*|_{x_i})$$

### Remark

If  $f$  is smooth and  $y$  is a regular value, then we can pick  $U_i$  and  $V$  such that each  $f|_{U_i} : U_i \rightarrow V$  is a diffeomorphism. Hence  $\deg(f|_{x_i}) = \pm 1$ .

### Example

If  $f : S^1 \rightarrow S^1$  by  $z \mapsto z^k$ ,  $f^{-1}(1)$  has  $k$  many points (viz. the roots of unity).  $f|_{U_i} : U_i \rightarrow V$  is diffeomorphic (by rotation and scaling) and  $\deg(f|_{x_i}) = 1$ .  $\deg(f) = \sum \deg(f|_{x_i}) = k$ .

IMAGE 1

## Definition: Suspension of a Space

Recall that the cone of  $X$  is  $C(X) = X \times I / X \times \{1\}$ .

IMAGE 2

The suspension of  $X$  is  $S(X) = C(X) / X \times \{0\}$ .

IMAGE 3

### Examples

$S(S^1) = S^2$ . In general  $S(S^n) = S^{n+1}$ .

## Definition: Suspension of a Map

$f : X \rightarrow Y$  induces  $f : X \times I \rightarrow Y \times I$  by  $(x, t) \mapsto (f(x), t)$ . This induces  $Cf : C(X) \rightarrow C(Y)$  and  $Sf : S(X) \rightarrow S(Y)$ .

### Examples

$f : S^n \rightarrow S^n$  induces a map  $Sf : S^{n+1} \rightarrow S^{n+1}$ .  $f : S^1 \rightarrow S^1$  by  $z \mapsto z^2$  induces  $Sf : S^2 \rightarrow S^2$

IMAGE 4



## Proposition

$$\deg(Sf) = \deg(f).$$

## Proof

Consider the pair  $(C(S^n), S^n \times \{0\})$  which induces

$$\overbrace{\tilde{H}_{n+1}(S^n)}^{=0} \longrightarrow \overbrace{\tilde{H}_{n+1}(C(S^n))}^{=0} \longrightarrow H_{n+1}(C(S^n), S^n \times \{0\}) \xrightarrow{\delta} \overbrace{\tilde{H}_n(S^n)}^{=\mathbb{Z}} \longrightarrow \overbrace{\tilde{H}_n(C(S^n))}^{=0} \longrightarrow$$

Hence  $\mathbb{Z} \cong H_{n+1}(C(S^n), S^n \times \{0\}) \cong \tilde{H}_{n+1}(S(S^n)) = \tilde{H}_{n+1}(S^{n+1})$ . Therefore

$$\begin{array}{ccccc} \tilde{H}_{n+1}(S^{n+1}) & \xrightarrow{\sim} & H_{n+1}(C(S^n), S^n \times \{0\}) & \xrightarrow{\delta} & \tilde{H}_n(S^n) \\ \downarrow (Sf)_* & & \downarrow (Cf)_* & & \downarrow f_* \\ H_{n+1}(S^{n+1}) & \xrightarrow{\sim} & H_{n+1}(C(S^n), S^n \times \{0\}) & \longrightarrow & \tilde{H}_n(S^n) \end{array}$$

So  $\deg(Sf) = \deg(f)$ .

## Remark

For any  $k, n \in \mathbb{Z}_+$ , by iterated suspension of the map  $z \mapsto z^k$ , we can construct  $f : S^n \rightarrow S^n$  of degree  $k$ .

## Remark

$Sf : S^{n+1} \rightarrow S^{n+1}$ , pick  $p \in S^{n+1}$  a pole, then  $(Sf)^{-1}(p) = \{p\}$ .

IMAGE 5

Hence  $\deg(Sf|_p) = \deg(Sf) = k$ .

**May 12, 2025**

## Recall

Let  $X$  be a CW-Complex of finite dimension  $X = X^0 \cup X^1 \cup \dots \cup X^{\dim X}$ .

$X^0$  is a discrete set of points.

$X^1$  is a gluing of  $\{e_\alpha^1\}_{\alpha \in A}$  to  $X^0$ , where  $e^1 = [-1, 1]$ , by the attaching map  $\varphi_\alpha : \partial e_\alpha^1 \rightarrow X^0$ .

$X^{k+1}$  is the gluing of  $\{e_\alpha^{k+1}\}_{\alpha \in A}$ , where  $e^{k+1} \cong D^{k+1}$ , by  $\varphi_\alpha : \partial e_\alpha^{k+1} \cong S^k \rightarrow X^k$ .

## Lemma

(a)

Let  $X$  be a CW-Complex of  $\dim X$ . Then

$$H_k(X^n, X^{n-1}) = \begin{cases} 0 & k \neq n \\ \text{free abelian with a basis in 1-1 correspondence to } \{n\text{-cells}\} & k = n \end{cases}$$

**Proof**

$(X^n, X^{n-1})$  is a good pair. So

$$H_k(X^n, X^{n-1}) \cong \tilde{H}_k(X^n/X^{n-1}) = \tilde{H}_k\left(\bigvee_{\alpha} S_{\alpha}^n\right) = \bigoplus_{\alpha} \tilde{H}_k(S_{\alpha}^n).$$

If  $k \neq n$ , then  $\tilde{H}_k(S_{\alpha}^n) = 0$ .

If  $k = n$ , then  $\tilde{H}_k(S_{\alpha}^n) = \mathbb{Z}$  and  $H_n(X^n, X^{n-1}) \cong \bigoplus_{\alpha} \mathbb{Z}$ .

**(b)**

$H_k(X^n) = 0$  if  $k > n$ .

**Proof**

The pair  $(X^n, X^{n-1})$  gives a long exact sequence.

$$\cdots \longrightarrow H_{k+1}(X^n, X^{n-1}) \xrightarrow{\delta} H_k(X^{n-1}) \longrightarrow H_k(X^n)$$

$$\longrightarrow H_k(X^n, X^{n-1}) \xrightarrow{\delta} \cdots$$

Supposing both  $k \neq n$  and  $k+1 \neq n$ , the first and last

terms are zero and  $H_k(X^{n-1}) \cong H_k(X^n)$ . Then

$$H_k(X^n) \cong H_k(X^{n-1}) \cong H_k(X^{n-2}) \cong \cdots \cong H_k(X^0) = 0$$

**(c)**

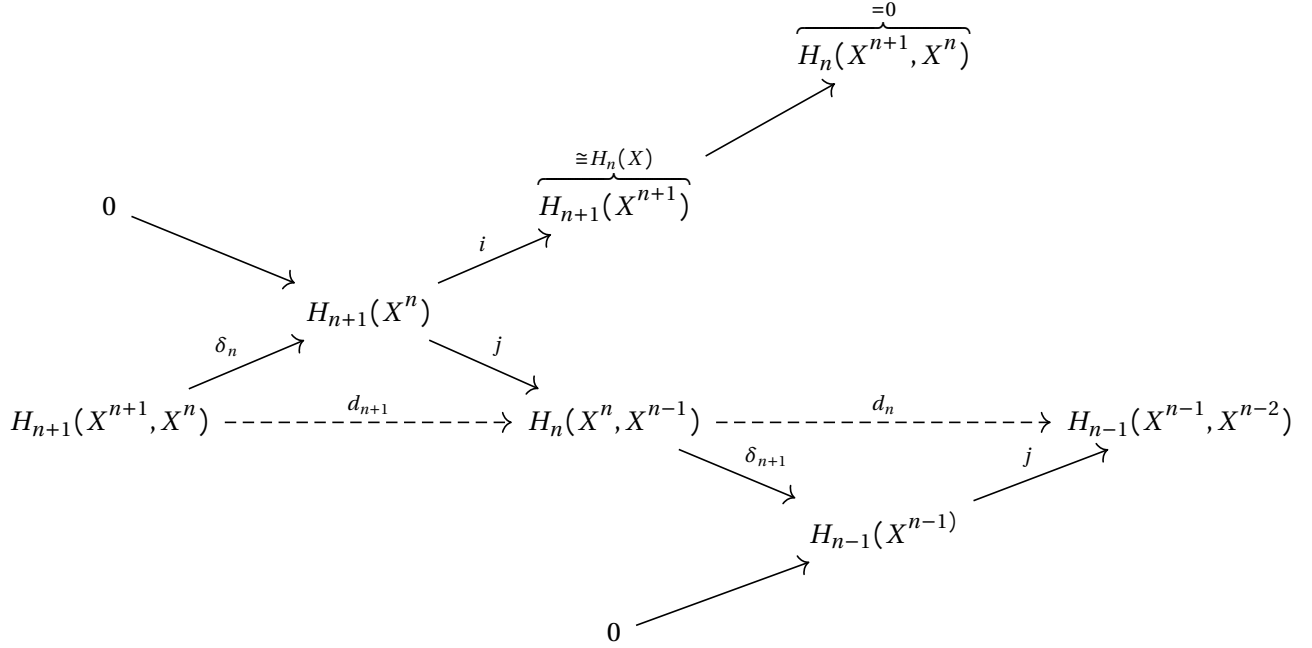
$i : X^n \hookrightarrow X$  induces an isomorphism  $i_* : H_k(X^n) \rightarrow H_k(X)$  if  $k < n$ .

**Proof**

If  $k < n$ , then

$$H_k(X^n) \cong H_k(X^{n+1}) \cong \cdots \cong H_k(X^{\dim X}) = H_k(X)$$

## Chain Complexes



This gives a cellular chain complex  $\{H_n(X^n, X^{n-1}), d_n\}$  with  $d_n \circ d_{n+1} = 0$  because  $\xrightarrow{j} \cdot \xrightarrow{\delta} = 0$ . This defines a cellular homology  $H_k^{CW}(X)$ . We claim that  $H_n^{CW}(X) \cong H_n(X)$ .

### Proof

$$\begin{aligned}
 H_n(X) &\cong H_n(X^{n+1}) \\
 &\cong H_n(X^n) / \ker i && \text{because } i \text{ is surjective} \\
 &= H_n(X^n) / \text{im } \delta_{n+1} && \text{because } \xrightarrow{\delta_{n+1}} \cdot \xrightarrow{i} \text{ is exact} \\
 &\cong j(H_n(X^n)) / j(\text{im } \delta_{n+1}) && \text{because } j \text{ is injective} \\
 &= \ker(\delta_n) / \text{im}(d_{n+1}) \\
 &= \ker(d_n) / \text{im}(d_{n+1}) \\
 &= H_n^{CW}(X)
 \end{aligned}$$

$j(\text{im } \delta_{n+1}) = \text{im}(j \circ \delta_{n+1}) = \text{im}(d_{n+1})$   
 $\ker(\delta_n) = \ker(j \circ \delta_n) = \ker d_n$

### Applications

For

$$\cdots \longrightarrow H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \xrightarrow{d_{n-1}} \cdots$$

$$\text{where } H_n(X^n, X^{n-1}) = \bigoplus_{\alpha} \mathbb{Z}$$

(1)

If a CW-Complex does not have any  $n$ -cells, then  $H_n(X^n, X^{n-1}) = 0$  and  $H_n(X) \cong H_n^{CW}(X) = 0$ .

(2)

If a CW-Complex  $X$  has  $k$ -many  $n$ -cells, then  $H_n(X^n, X^{n-1}) \cong \mathbb{Z}^k$ . Then  $H_n(X) \cong H_n^{CW}(X) = \ker d_n / \text{im } d_{n-1}$ .  $\ker d_n \leq H_n(X^n, X^{n-1}) = \mathbb{Z}^k$ . Hence  $\ker d_n$  and  $H_n(X)$  can be generated by at most  $k$  many elements.

(3)

If  $X$  and  $Y$  are CW-complexes with  $\{\varphi_\alpha : e_\alpha^n \rightarrow X^{n-1}\}$  and  $\{\psi_\beta : e_\beta^n \rightarrow Y^{n-1}\}$  respectively, then  $X \times Y$  has  $\{\varphi_\alpha \times \psi_\beta : e_\alpha^m \times e_\beta^n \rightarrow (X \times Y)^{m+n-1}\}$  where  $e_\alpha^m \times e_\beta^n \cong e^{m+n}$ .

Consider  $S^n \times S^n$  (for  $n \geq 2$ ) where  $S^n$  is constructed by one 0-cell and one  $n$ -cell. Then  $S^n \times S^n$  has one 0-cell ( $\mathbb{Z}^1$ ), two  $n$ -cells ( $\mathbb{Z}^2$ ) and one  $2n$ -cell ( $\mathbb{Z}^1$ ).

$$0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots \longrightarrow \mathbb{Z}^2 \longrightarrow 0 \longrightarrow \cdots \longrightarrow \mathbb{Z} \longrightarrow 0$$

so

$$H_k(S^n \times S^n) = \begin{cases} \mathbb{Z} & k = 0, 2n \\ \mathbb{Z}^2 & k = n \\ 0 & \text{otherwise} \end{cases}.$$

(4)

Take  $\mathbb{CP}^n$  as  $\mathbb{C}^{n+1} / \sim$  or as  $S^{2n+1} / \sim$  where  $v \sim \lambda v$  and  $\lambda v = (e^{i\theta} z_1, \dots, e^{i\theta} z_{n+1})$ . Consider the set of vectors in  $S^{2n+1}$  whose last component is real and nonnegative.  $D_+^{2n} = \{(w, \sqrt{1-|w|^2}) \in \mathbb{C}^{n+1} : w \in \mathbb{C}^n, |w| \leq 1\}$  is the graph of the function  $w \mapsto \sqrt{1-|w|^2}$  defined on  $\{w : |w| \leq 1\} \subseteq \mathbb{C}^n$ . So  $D_+^{2n}$  is homeomorphic to a disk  $\{|w| \leq 1\} = D^{2n} \subseteq \mathbb{C}^n$ . For any vector  $v \in S^{2n+1}$ ,  $v = (z_1, \dots, z_{n+1})$  if  $z_{n+1} \neq 0$ , then  $v$  is equivalent to a unique vector in  $D_+^{2n}$ . If  $z_{n+1} = 0$ ,  $\{(z_1, \dots, z_n, 0) \in S^{2n+1} \times \{0\}\} = S^{2n-1}$ . So  $q : S^{2n+1} \rightarrow \mathbb{CP}^n$  has that  $q|_{D_+^{2n}}$  is a homeomorphism. Then  $S^{2n+1} / \sim$  is exactly  $\mathbb{CP}^{n-1}$ . Therefore, we may view  $\mathbb{CP}^n$  as gluing  $e^{2n}$  to  $\mathbb{CP}^{n-1}$  by the attaching map  $\partial e^{2n} = S^{2n-1} \rightarrow \mathbb{CP}^{n-1}$ . So  $\mathbb{CP}^n$  has cells  $e^0, e^2, \dots, e^{2n}$  and the cellular chain complex is

$$\mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

$$H_k(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & k = 0, 2, \dots, 2n \\ 0 & \text{otherwise} \end{cases}.$$

Recall that  $\mathbb{RP}^n$  by  $\mathbb{S}^n / \sim$  with  $S^n \subseteq \mathbb{R}^{n+1}$  and  $v \sim -v$ , we may take the upper hemisphere  $D_+^n$ . For every  $v \in S^n = (x_1, \dots, x_n)$ , if  $x_{n+1} \neq 0$  then  $v$  is equivalent to a unique vector in  $D_+^n$  where  $q|_{D_+^n} : D_+^n \rightarrow \mathbb{RP}^n$  homomorphic to its image. If  $x_{n+1} = 0$ , then  $\{(x_1, \dots, x_n, 0) \in S^n\} / \sim$  and  $\mathbb{RP}^n$  is gluing  $e^n$  to  $\mathbb{RP}^{n-1}$  via the attaching map  $\varphi : \partial e^n = S^{n-1} \rightarrow \mathbb{RP}^{n-1}$  as the quotient map.

## Computation

We want  $d_n : H_n(X^n, X^{n-1}) \rightarrow H_{n-1}(X^{n-1}, X^{n-2})$ . For  $n = 1$  we have

$$\begin{array}{ccccccc} 0 & & & & & & \\ & \searrow & & & & & \\ & & H_0(X^0) & & & & \\ & \nearrow \delta & \searrow j & & & & \\ H_1(X^1, X^0) & \xrightarrow{d_1} & H_0(X^0) & \longrightarrow & 0 & & \\ & & & \searrow & & & \\ & & & & 0 & & \end{array}$$

where  $d_1 = \delta : H_1(X^1, X^0) \rightarrow H_0(X^0)$ . If  $X$  is connected, and  $X^0 = \{v\}$ , then  $H_0(X^0) = \mathbb{Z}$  and  $H_0(X) = H_0(X^0) / \text{im } d_1$  implies that  $\text{im } d_1 = 0$ .

For  $n \geq 2$ ,  $H_n(X^n, X^{n-1})$  is  $\bigoplus_{\alpha} \mathbb{Z}$  and the generators are in one-to-one correspondence with  $\{e_{\alpha}^n\}_{\alpha}$ . We have a cellular boundary formula

$$d_n(e_{\alpha}^n) = \sum_{\beta} d_{\alpha\beta} e_{\beta}^{n-1}$$

where  $d_{\alpha\beta} = \deg(\Delta_{\alpha\beta})$  and  $\Delta_{\alpha\beta} : S^{n-1} = \partial e_{\alpha}^n \xrightarrow{\varphi_{\alpha}} X^{n-1} \xrightarrow{q_{\beta}} S_{\beta}^{n-1}$ .  $q_{\beta} : X^{n-1} \rightarrow S_{\beta}^{n-1}$  is obtained by collapsing everything in  $X^{n-1}$  except  $(e_{\beta}^{n-1})^{\circ}$ . For every  $n$ -cell  $e_{\alpha}^n$  and every  $(n-1)$ -cell  $e_{\beta}^{n-1}$ , we obtain  $d_{\alpha\beta} = \deg(\Delta_{\alpha\beta})$ .

### Example

Suppose we have  $M_g$ , an orientable surface of genus  $g$ .  $M_g$  has one 0-cell,  $2g$  1-cells  $(a_1, b_1, \dots, a_g, b_g)$  and one 2-cell. Then  $d_1 = 0$ , and  $d_2(e_2)$  comes from  $\Delta_{\alpha\beta} : S^2 = \partial e^2 \xrightarrow{\alpha} X^1 = \bigvee S^1 \xrightarrow{q_{\beta}} S_{\beta}^1$  which glues  $S^1$  to  $S^1$  by  $a \cdot a^{-1}$ . So  $\deg(\Delta_{\alpha\beta}) = 0$  and  $d_2(e_2) = 0$ .

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

so  $H_2 = \mathbb{Z}$ ,  $H_1 = \mathbb{Z}^{2g}$  and  $H_0 = \mathbb{Z}$ .

### Example

$N_g$  is a non-orientable surface of genus  $g$ .  $N_g$  has one 0-cell,  $g$  1-cells  $(a_1^2, a_2^2, \dots, a_g^2)$ , and one 2-cell. We know that  $d_1 = 0$ . Consider  $\Delta_{\alpha\beta} : S_{\alpha}^1 \rightarrow X^1 \rightarrow S_{\beta}^1$  which glues  $S^1$  to  $S^1$  by  $a^2$  (i.e.  $z \mapsto z^2$ ) and  $\deg(\Delta_{\alpha\beta}) = 2$ . So  $d_2(e_2) = \sum_{\beta} 2e_{\beta}^1 = (2, 2, \dots, 2) \in \mathbb{Z}^g$  and

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^g \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

So  $H_0 = \mathbb{Z}$ ,  $H_1 = \mathbb{Z}^g / \text{im } d_2 = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$  and  $H_2 = \ker d_2 / 0 = 0$ .

**May 14, 2025**

## Remaining Homology Topics

1. More examples of  $H_*^{CW}$ .
2. Euler characteristic.
3. Homology with coefficients.
4. Mayer-Vietoris Sequence.
5.  $\pi_1 / [\pi_1, \pi_1] = H_1$ .

## Recall: Cellular Chain Complex

$(H_n(X^n, X^{n-1}), d_n)$  where  $H_n(X^n, X^{n-1})$  has a basis in one-to-one correspondence with the  $n$ -cells in  $X$  and  $d_n(e_{\alpha}^n) = \sum_{\beta} d_{\alpha\beta} e_{\beta}^{n-1}$  with  $d_{\alpha\beta} = \deg(\Delta_{\alpha\beta})$  for  $\Delta_{\alpha\beta} : \partial e_{\alpha}^n = S_{\alpha}^{n-1} \xrightarrow{\varphi_{\alpha}} X^{n-1} \xrightarrow{q_{\beta}} S_{\beta}^{n-1}$  where  $q_{\beta}$  collapses everything in  $X^{n-1}$  except

$\text{int}(e_\beta^{n-1})$  to a point.

### Example

$\mathbb{RP}^n = e^0 \cup e^1 \cup \dots \cup e^n$  with the attaching map  $S^{k-1} = \partial e^k \rightarrow \mathbb{RP}^{k-1} = X^{n-1}$  as the quotient map (2-sheet covering).

IMAGE 1

So the cellular chain complex is

$0 \longrightarrow \mathbb{Z} \longrightarrow \dots \longrightarrow \mathbb{Z} \longrightarrow 0$  with  $d_n(e^n) = \deg(\Delta)e^{n-1}$  for  $\Delta : S^{n-1} \xrightarrow{\varphi} X^{n-1} = \mathbb{RP}^{n-1} \xrightarrow{q} \mathbb{RP}^{n-1}/\mathbb{RP}^{n-2} = S^{n-1}$ . Then  $\Delta$  restricted to the upper (or lower) hemisphere is a homeomorphism to  $S^{n-1} - \{\text{pt}\}$ . So antipodal open sets differ only by the antipodal map

IMAGE 2

So  $\deg(\Delta) = 1 + (-1)^n = \begin{cases} 0 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}$ . If  $n$  is even, the chain complex is

$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \dots \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$  which means  $H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}_2 & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$ . Similarly for odd  $n$ ,

$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \dots \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$

so  $H_n(\mathbb{RP}^n) = \mathbb{Z}$  and we conclude

$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & k = n \text{ odd} \\ \mathbb{Z}_2 & 0 < k < n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

### Example

Recall for the torus  $\mathbb{T}^2$ ,  $0 \rightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z} \rightarrow 0$ , the degree is zero (see above). For  $\mathbb{T}^3$ ,

IMAGE 3

we have one 0-cell, three 1-cells, three 2-cells and one 3-cell and a chain complex

$0 \longrightarrow \mathbb{Z} \xrightarrow{d_3} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \longrightarrow 0$  So  $d_3(e^3) = \sum_\beta \deg(\Delta_{\alpha\beta})e_\beta^2$  and  $\Delta_{\alpha\beta} : \partial e^3 = S^2 \xrightarrow{\varphi} X^2 \xrightarrow{q_\beta} S_\beta^2$ .

IMAGE 4

$$H_k(\mathbb{T}^3) = \begin{cases} \mathbb{Z}^3 & k = 1, 2 \\ \mathbb{Z} & k = 0, 3 \\ 0 & \text{otherwise} \end{cases}.$$

## Example

Consider  $K \times S^1$  where  $K$  is the Klein bottle.

IMAGE 5

So  $d_2(A) = 2b$ ,  $d_2(B) = 0$  and  $d_2(C) = 0$ . However, for  $d_3$  the front face changes by reflection in the back

IMAGE 6

So  $d_3(e^3) = 2C$  and the degree is  $1 + (-1) \cdot (-1) = 2$  and

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_3} \mathbb{Z}^3 \xrightarrow{d_2} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

We have that  $\ker d_3 = 0$ ,  $H_3 = 0$ ;  $\ker d_2 = \langle B, C \rangle$ ,  $\text{im } d_3 = \langle 2C \rangle$ ,  
 $H_2 = \ker d_2 / \text{im } d_3 = \mathbb{Z} \oplus \mathbb{Z}_2$ ;  $\ker d_1 = \langle a, b, c \rangle$ ,  $\text{im } d_2 = \langle 2b \rangle$ ,  $H_1 = \mathbb{Z}^2 \oplus \mathbb{Z}_2$ .

## Homology with Coefficients (Moore Space)

Let  $G$  be a finitely generated abelian group and  $n \in \mathbb{N}$  with  $n \geq 1$ . We can construct a CW-complex  $X$  such that

$$\tilde{H}_k(X) = \begin{cases} G & k = n \\ 0 & \text{otherwise} \end{cases}.$$

First, let us consider  $G = 2m$ . We start with  $X^n = S^n$  (1 0-cell, 1  $n$ -cell). Then construct  $X = X^{n+1}$  by gluing in a  $(n+1)$ -cell  $e^{n+1}$  via the attaching map  $\varphi : S^n = \partial e^{n+1} \rightarrow X^n = S^n$ . Then  $d_{n+1}(e^{n+1}) = \deg(\Delta)e^n$  where  $\Delta : S^n = \partial e^{n+1} \rightarrow X^n = S^n$  is  $\varphi$ . If  $\varphi : S^n \rightarrow S^n$  has degree  $m$ , then  $d_{n+1}$  is multiplication by  $m$ .

$0 \longrightarrow \mathbb{Z} \xrightarrow{d_{n+1}} \mathbb{Z} \longrightarrow 0 \longrightarrow \dots$  So  $H_n(X) = \ker / \text{im} = \mathbb{Z} / m\mathbb{Z} = \mathbb{Z}_m = G$ . In general,  $G = \mathbb{Z}_{m_1} \oplus \dots \oplus \mathbb{Z}_{m_\ell} \oplus \mathbb{Z} \oplus \dots \oplus \mathbb{Z}$  a  $k$ -term sum. Use  $X^n = \bigvee_{i=1}^k S^n$ . For each  $\mathbb{Z}_{m_i}$ -factor, glue in  $e_\alpha^{n+1}$  by  $\varphi : \partial e_\alpha^{n+1} \rightarrow S_i^n$ . Consider  $X$  with  $\ell$ -many  $(n+1)$ -cells,  $k$ -many  $n$ -cells and a single 0-cell for  $\ell \leq k$ . Then

$$0 \longrightarrow \mathbb{Z}^\ell \longrightarrow \mathbb{Z}^k \longrightarrow 0$$

$$(1, 0, \dots, 0) \longmapsto (m_i, 0, \dots)$$

So  $\text{im } d_{n+1} = m_1\mathbb{Z} \oplus \dots \oplus m_\ell\mathbb{Z} \oplus 0 \subseteq \mathbb{Z}^k$ .

Let  $G$  be an abelian group ( $G = \mathbb{Z}$  or  $G = \mathbb{Z}_m$ ). Then  $C_n(X; G) = \{ \sum_i n_i \sigma_i : \text{finite formal sums of } \sigma_i : \Delta^n \rightarrow X \text{ with } n_i \in G \}$ . We can similarly define  $H_n(X, G)$ ,  $\tilde{H}_n(X; G)$ ,  $H_n(X, A; G)$ ,  $H_n^{CW}(X; G)$ , etc.

If we use  $\mathbb{Z}_m = G$  as the coefficient with  $\tilde{H}_k(X) = \begin{cases} \mathbb{Z}_m & k = m \\ 0 & k \neq m \end{cases}$ ,

$$0 \longrightarrow \mathbb{Z}_m \xrightarrow{\cdot m} \mathbb{Z}_m \longrightarrow 0$$

Consider the map  $f : X \rightarrow X / S^n = S^{n+1}$ . This induces  $f_* : H_k(X) \rightarrow H_k(S^{n+1})$  which is  $f_* = 0$  on all  $H_k(X)$ . If we use coefficients  $\mathbb{Z}_m$  instead, we still induce  $f_* : H_k(X; \mathbb{Z}_m) \rightarrow H_k(S^{n+1}; \mathbb{Z}_m)$ ,  $H^{n+1}(X; \mathbb{Z}_m) = \mathbb{Z}_m$  and  $H^n(X; \mathbb{Z}_m) = \mathbb{Z}_m$ . For  $k = n+1$ ,  $f_* : \mathbb{Z}_m \rightarrow \mathbb{Z}_m$  is the identity map which implies that  $f$  is not null-homotopic.

## Lemma

If  $f : S^k \rightarrow S^k$  is of degree  $m$ , then  $f_* : H_k(S^k; G) \rightarrow H_k(S^k; G)$  by  $g \mapsto mg$ . A homomorphism  $\varphi : G_1 \rightarrow G_2$  induces  $\varphi_\# : C_n(X; G_1) \rightarrow C_n(X; G_2)$  by  $\sum n_i \sigma_i \mapsto \sum \varphi(n_i) \sigma_i$ . Then  $\partial \varphi_\# = \varphi_\# \partial$  and  $\varphi_* : H_n(X; G_1) \rightarrow H_n(X; G_2)$ . If  $f : S^k \rightarrow S^k$

is of degree  $k$ , fix any  $g \in G$  and set  $\varphi : \mathbb{Z} \rightarrow G$  by  $1 \mapsto g$ .

$$\begin{array}{ccc} H_k(S^k; \mathbb{Z}) & \xrightarrow{f_*} & H_k(S^k; \mathbb{Z}) \\ \downarrow \varphi_* & & \downarrow \varphi_* \\ H_k(S^k; G) & \xrightarrow{f_*} & H_k(S^k; G) \end{array}$$

## Euler Characteristic

Let  $X$  be a finite CW-complex with  $c_i$  the number of  $i$ -cells. Then  $\chi(X) = \sum_{i=0}^n (-1)^i c_i$  with  $n = \dim X$ . For example,  $\chi(S^2) = 1 + 1 = 2$ ,  $\chi(\mathbb{T}^2) = 1 - 2 + 1 = 0$ ,  $\chi(\mathbb{RP}^2) = 1 - 1 + 1 = 1$ , and  $\chi(\mathbb{CP}^n) = 1 + 1 + \cdots + 1 = n$ .

### Theorem

$$\chi(X) = \sum_{i=0}^n (-1)^i \text{rank}(H_i(X)).$$

### Proof

$$0 \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_0 \longrightarrow 0 \quad \text{So } Z_i = \ker d_i, B_i = \text{im } d_{i+1} \text{ and } H_i = Z_i / B_i. \text{ Then}$$

$$0 \longrightarrow B_i \longrightarrow Z_i \longrightarrow H_i \longrightarrow 0$$

$$0 \longrightarrow Z_i = \ker d_i \longrightarrow C_i \longrightarrow B_{i-1} = \text{im } d_i \longrightarrow 0$$

So  $\text{rank } Z_i = \text{rank } B_i + \text{rank } H_i$  and  $\text{rank } C_i = \text{rank } Z_i + \text{rank } B_{i-1} = \text{rank } B_i + \text{rank } H_i + \text{rank } B_{i-1}$ . Therefore

$$\sum (-1)^i \overbrace{\text{rank } C_i}^{=c_i} = \sum (-1)^i \text{rank } H_i.$$

So  $\chi(M_g) = 2 - 2g$  and  $\chi(N_g) = 2 - g$ .

**May 19, 2025**

## Mayer-Vietoris Sequences

Given a space  $X$  and open sets  $A, B \subseteq X$  such that  $A \cup B = X$  (i.e.  $\mathcal{U} = \{A, B\}$  is an open cover of  $X$ ), we have the chain

$$C_n^{\mathcal{U}}(X) = \left\{ \sum_i n_i \sigma_i : \sigma_i : \Delta^n \rightarrow \text{some } U \in \mathcal{U} \right\}$$

Fact:  $H_n^{\mathcal{U}}(X) \cong H_n(X)$ .

For convenience, we write  $C_n^{\mathcal{U}}(X) = C_n(A + B)$  which induces a short exact sequence

$$0 \longrightarrow C_N(A \cap B) \xrightarrow{\varphi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A + B) \longrightarrow 0$$

$$\alpha \longmapsto (\alpha, -\alpha)$$

$$(\alpha, \beta) \longmapsto \alpha + \beta$$



## Verifying Exactness

$\varphi$  is injective, since if  $0 = \varphi(\alpha) = (\alpha, -\alpha)$ , then  $\alpha = 0$ .

$\psi$  is surjective, since any element in  $C_n(A + B)$  and for  $\sigma_i : \Delta^n \rightarrow A$  or  $B$  with  $\alpha_i : \Delta^n \rightarrow A$  and  $\beta_j : \Delta^n \rightarrow B$ , we have

$$\sum_i n_i \sigma_i = \sum_i n_i \alpha_i + \sum_i n_i \beta_i = \psi \left( \sum_i n_i \alpha_i, \sum_i n_i \beta_i \right)$$

$\text{im } \varphi \subseteq \ker \psi$  since  $\psi(\varphi(\alpha)) = \psi(\alpha, -\alpha) = 0$

$\ker \psi \subseteq \text{im } \varphi$ , since if we suppose  $(\alpha, \beta) \in C_n(A) \oplus C_n(B)$  such that  $\psi(\alpha, \beta) = 0$ . Then  $\alpha \in C_n(A \cap B)$  and  $\varphi(\alpha) = (\alpha, -\alpha) = (\alpha, \beta)$  (i.e.  $(\alpha, \beta) \in \text{im } \varphi$ ).

## Long Exact Sequence

The short exact sequence induces

$$\cdots \longrightarrow H_n(A \cap B) \xrightarrow{\varphi_*} H_n(A) \oplus H_n(B) \xrightarrow{\psi_*} H_n(X)$$

$$\xrightarrow{\partial} H_{n-1}(A \cap B) \longrightarrow \cdots$$

## Example: Klein Bottle

The Klein bottle is created by gluing two Mobius strips along their boundary

IMAGE 1

Let  $A, B$  be two Mobius strips as subsets of the Klein bottle. Let  $U, V$  be neighborhoods of  $A$  and  $B$  respectively that deformation retract to  $A$  and  $B$ . Then  $A \cap B$  is homotopic to the circle. Then

$$\begin{aligned} \overbrace{H_2(A \cap B)}^{=0} &\longrightarrow \overbrace{H_2(A) \oplus H_2(B)}^{=0} \longrightarrow H_2(X) \\ &\longrightarrow H_1(A \cap B) \longrightarrow H_1(A) \oplus H_1(B) \longrightarrow H_1(X) \\ &\longrightarrow \overbrace{\tilde{H}_0(A \cap B)}^{=0} \end{aligned}$$

Which leads to

$$0 \longrightarrow H_2(X) \longrightarrow H_1(A \cap B) \xrightarrow{\varphi_*} H_1(A) \oplus H_1(B) \xrightarrow{\psi_*} H_1(X) \longrightarrow 0$$

$$\alpha \longmapsto (\alpha, -\alpha)$$

$$0 \longrightarrow H_2(X) \longrightarrow \mathbb{Z} \xrightarrow{\varphi_*} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\psi_*} H_1(X) \longrightarrow 0$$

$$1 \longmapsto (2, -2)$$

In particular,  $\psi_*$  is injective. Hence  $H_2(X) \rightarrow \mathbb{Z}$  is the zero map and  $H_2(X) = 0$ . Then also

$$H_1(X) = (\mathbb{Z} \oplus \mathbb{Z}) / \ker \psi_* = (\mathbb{Z} \oplus \mathbb{Z}) / \text{im } \varphi_* = \langle a, b \rangle / \langle 2a - 2b \rangle = \langle a, a - b \rangle / \langle 2a - 2b \rangle \cong \mathbb{Z} \oplus \mathbb{Z}_2.$$

## Definition: Commutator

Let  $G$  be a group.  $[G, G]$  is the normal subgroup generated by elements of the form  $[g, h] = g^{-1}h^{-1}gh$ . Then  $G/[G, G]$  is Abelian.

## Commutator of the Fundamental Group

$$H_1 = \pi_1 / [\pi_1, \pi_1].$$

We define a map  $h : \pi_1(X, x_0) \rightarrow H_1(X)$ . Let  $f : \Delta^1 = [0, 1] \rightarrow X$  be a loop at  $x_0$ , such that  $f$  is also a singular 1-simplex (i.e. a cycle). So  $\partial f = f(1) - f(0) = x_0 - x_0 = 0$ . Hence we may assign  $[f] \in H_1(X)$  to the loop  $f$  (i.e.  $h(f) = [f]$ ). Write  $f \simeq g$  for path homotopy and  $f \sim g$  when  $f - g$  is a boundary.  $h$  is well defined, since

1. if  $f$  is homotopic to a constant loop, then  $f \sim 0$  (i.e.  $f$  is a boundary). We treat  $H : D^2 \rightarrow X$  as a singular 2-simplex, call it  $\sigma$ . Then  $\partial\sigma = f$ .
2. if  $f \simeq g$ , then  $f \sim g$ . Let  $H$  be a path homotopy between  $f$  and  $g$ .

IMAGE 2

Then  $\partial L = d - g - Cx_0$ ,  $\partial R = d - Cx_0 - f$  and  $\partial(L - R) = f - g$ . This shows that  $h : \pi_1(X, x_0) \rightarrow H_1(X)$  is well-defined.  $h$  is a group homomorphism. We need to show that  $h(f \cdot g) = h(f) + h(g)$  (i.e.  $f \cdot g \sim f + g$ ).

IMAGE 3

So  $\sigma : \Delta^2 \rightarrow X$  defined by the filling in of the 2-simplex has  $\partial\sigma = f \cdot g - f - g$ .  $h$  is surjective. For  $\sigma_i : \Delta^1 \rightarrow X$ , let  $\sum_i n_i \sigma_i \in C_1(X)$  be a 1-cycle. Then

$$0 = \partial \left( \sum_i n_i \sigma_i \right) = \sum_i n_i (\sigma_i(1) - \sigma_i(0)).$$

Let  $S$  be the set of distinct points in the list  $\{\sigma_i(0), \sigma_i(1) : |i = 1, \dots, k\}$  for  $m_p \in \mathbb{Z}$ . Then

$$\sum_i n_i (\sigma_i(1) - \sigma_i(0)) = \sum_{p \in S} m_p \cdot p.$$

and  $m_p = 0$  for all  $p \in S$ . For each  $\sigma_i$ , we consider a loop  $\eta_i$  at  $x_0$  by

IMAGE 4

For any  $p \in S$ ,  $\beta_p$  is a path from  $x_0$  to  $p$ . Then  $h(\eta_i) = \beta_{\sigma_i(0)} + \sigma_i - \beta_{\sigma_i(1)}$ . Now consider a loop  $\eta_1^{n_1} \cdots \eta_k^{n_k}$  at  $x_0$ .

$$h(\eta_1^{n_1} \cdots \eta_k^{n_k}) = \sum_i n_i h(\eta_i) = \sum_i n_i \sigma_i + \sum_i n_i (\beta_{\sigma_i(0)} - \beta_{\sigma_i(1)}) = \sum_i n_i \sigma_i + 0$$

$\ker(h) = [\pi_1, \pi_1]$ . Since  $[\pi_1, \pi_1]$  is generated by  $f^{-1}g^{-1}fg$ ,  $h(f^{-1}g^{-1}fg) = -h(f) - h(g) + h(f) + h(g) = 0$ . So  $[\pi_1, \pi_1] \subseteq \ker h$ .

### Lemma:

Let  $G$  be a group, and let  $w$  be a word in  $G$ . Suppose that for each  $g \in G$  its exponent in  $w$  adds up to zero, then  $w \in [G, G]$ .

### Proof

Let  $\pi : G \rightarrow G/[G, G]$  be the quotient map and  $\pi(w) = 0$  (i.e.  $w \in \ker \pi = [G, G]$ ) because if  $w = g^{k_1} \dots g^{k_2} \dots g^{k_3} \dots$  then  $\pi(w) = \pi(g)^{k_1+k_2+k_3+\dots} = 0$ .

### Commutator of the Fundamental Group Continued

Let  $f$  be a loop at  $x_0$  such that  $h(f) = 0$  (i.e. as a singular 1-simplex).  $f$  is a boundary. Hence there is  $\sum_i n_i \sigma_i \in C_2(X)$  such that  $(\sigma_i : \Delta^2 \rightarrow X)$

IMAGE 5

$$f = \partial \left( \sum_i n_i \sigma_i \right) = \sum n_i (\alpha_i + \beta_i + \gamma_i)$$

Let  $S$  be the set of disjoint edges in the list  $\{\alpha_i, \beta_i, \gamma_i : i = 1, \dots, k\}$ . Then

$$f = \sum n_i (\alpha_i + \beta_i + \gamma_i) = \sum_{e \in S} m_e \cdot e$$

in  $C_1(X)$ . Hence  $m_e = 1$  when  $e = f$  and  $m_e = 0$  otherwise. For each  $\sigma_i$ , we draw a loop  $\eta_i$  at  $x_0$  by joining

IMAGE 6

Then each  $\eta_i$  is homotopic to the boundary of  $\sigma_i$  (i.e. null-homotopic). Let us consider a loop  $\eta_1^{n_1} \dots \eta_k^{n_k} f^{-1} \simeq f^{-1}$ . Each  $\eta_i$  is a product of 3 loops. Hence  $\eta_1^{n_1} \dots \eta_k^{n_k} f^{-1}$  is a word in  $\pi_1$  in  $S' \cup \{f\}$  where  $S'$  is the collection of all the generated loops with basepoint at  $x_0$ . The exponent of such a loop in this word is either  $m_e$  when  $e \neq f$  or  $m_e - 1$  when  $e = f$ . So it is always zero by the precedeing. This shows that

$$[f^{-1}] = [\eta_1^{n_1} \dots \eta_k^{n_k} f^{-1}] \in [\pi_1, \pi_1].$$

### Mayer-Vietoris for Reduced Homology

$$\begin{array}{ccccc} 0 & & 0 & & \\ \downarrow & & \downarrow & & \\ C_0(A \cap B) & \xrightarrow{\epsilon} & \mathbb{Z} & \longrightarrow & 0 \\ \downarrow \varphi_* & & \downarrow \varphi & & \\ C_0(A) \oplus C_0(B) & \xrightarrow{\epsilon \oplus \epsilon} & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & 0 \\ \downarrow \psi_* & & \downarrow \psi & & \\ C_0(A + B) & \xrightarrow{\epsilon} & \mathbb{Z} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ 0 & & 0 & & \end{array}$$

Where  $\epsilon : C_0(X) \rightarrow \mathbb{Z}$  by  $\sum_i n_i \sigma_i \mapsto \sum n_i$  is surjective. Then  $\alpha \in \sum_i n_i \sigma_i$ , and we have  $\varphi(\sum_i n_i)$  and  $(\epsilon \oplus \epsilon)(\sum_i n_i \sigma_i, -\sum_i n_i \sigma_i) = (\sum_i n_i, -\sum_i n_i)$ . So assign  $\varphi : \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}$  by  $1 \mapsto (1, -1)$  and let  $(\sum_i n_i \alpha_i, \sum_i m_i \beta_i) \in C_0(A) \oplus C_0(B)$ . Then we have that  $\epsilon(\sum_i n_i \alpha_i, \sum_i m_i \beta_i) = \sum_i n_i + \sum_i m_i$  and  $\psi(\sum_i n_i, \sum_i m_i)$  is assigned to  $\psi : \mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z}$  by  $(m, n) \mapsto m + n$ . Then the above diagram commutes.

May 21, 2025

## Recall: de Rahm Cohomology

If  $M^n$  is a continuous manifold,  $\Omega^p(M)$  is the collection of  $k$ -forms on  $M$  and  $d : \Omega^p(M) \rightarrow \Omega^{p+1}(M)$  with  $d \circ d = 0$ . This gives a cochain

$$\dots \xrightarrow{d} \Omega^p(M) \xrightarrow{d} \Omega^{p+1}(M) \xrightarrow{d} \dots$$

It defines the de Rahm cohomology  $H_{\text{dR}}^p(M) = \ker d_p / \text{im } d_{p-1}$ . Our goal is the de Rahm Theorem:

$$H_{\text{dR}}^p(M) \cong H^p(M; \mathbb{R}) (= \text{Hom}(H_p(M), \mathbb{R})).$$

For  $p = 1$  we can construct a map  $I : H_{\text{dR}}^1(M) \rightarrow \text{Hom}(\pi_1(M, x), \mathbb{R})$  by  $[\omega] \mapsto I[\omega] : \pi_1(M, x) \rightarrow \mathbb{R}$  where  $(I[\omega])(\gamma) := \int_{\tilde{\gamma}} \omega$ . TO be precise, we pick a piecewise smooth  $\tilde{\gamma}$  at  $x_0$  such that  $[\tilde{\gamma}] = [\gamma]$  (well-defined because for  $\alpha, \beta$  piecewise smooth and “smoothly homotopic”,  $\int_{\alpha} \omega = \int_{\beta} \omega$ ).

Then  $I$  is well defined because it is independent of the choice of  $\tilde{\gamma}$  and  $\omega$ . In the latter case, if  $\omega = \omega' + df$  for  $f \in \Omega^0(M)$ , then

$$\int_{\tilde{\gamma}} \omega - \omega' = \int_{\tilde{\gamma}} df = f(\tilde{\gamma}(1)) - f(\tilde{\gamma}(0)) = 0.$$

$I$  is injective, because if we suppose  $I[\omega] = 0 \in \text{Hom}(\pi_1(M, x), \mathbb{R})$ , then  $\int_{\tilde{\gamma}} \omega = I[\omega](\gamma) = 0$  for all  $\gamma \in \pi_1(M, x)$ . Then we claim that  $\omega$  is conservative, because for any piecewise smooth loop  $\alpha$  we can let  $y$  be a point on  $\alpha$  and draw a composed curve with  $\beta$  from  $x$  to  $y$ . Then  $\beta\alpha\beta^{-1}$  is based at  $x$  and

$$0 = \int_{\beta\alpha\beta^{-1}} \omega = \int_{\beta} \omega + \int_{\alpha} \omega - \int_{\beta} \omega = \int_{\alpha} \omega$$

Then, recall that conservative implies exactness since  $f(y) = \int_{\gamma_{xy}} \omega$  holds for  $\gamma_{xy}$  from  $x$  to  $y$ . Therefore  $[\omega] = 0$ .

$I$  is surjective (to be proved later) since if  $\gamma \in \pi_1(M, x)$  such that  $\gamma^k = \text{id}$ , then for every  $\varphi \in \text{Hom}(\pi_1(M, x), \mathbb{R})$  we have that  $\varphi(\gamma) = 0$ .

### Corollary

If  $\pi_1(M)$  is torsion (in particular if  $\pi_1(M)$  is finite), then  $H_{\text{dR}}^1(M) = \{0\}$ .

## (A Little Bit of) Cohomology

In homology, we start with a chain complex

$$\dots \longrightarrow C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \longrightarrow \dots$$

Let  $G$  be an abelian group ( $G = \mathbb{R}$  or  $\mathbb{Z}$  or  $\mathbb{Z}_m$ ), and define  $C_n^* = \text{Hom}(C_n, G)$  and  $\partial^* : C_n^* \rightarrow C_{n+1}^*$  by  $\varphi \mapsto \partial^* \varphi$  where  $\partial^* \varphi$  is defined by  $\varphi \circ \partial$ . This gives a cochain

$$\dots \longleftarrow C_{n+1}^* \xleftarrow{\partial^*} C_n^* \xleftarrow{\partial^*} C_{n-1}^* \longleftarrow \dots$$

with  $\partial^* \circ \partial^* = 0$ .

## Definition: Cohomology Group

The cohomology group  $H^n(X, G)$  is  $\ker \partial^* / \text{im } \partial^*$ .

### Example

Given

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

we have a dual

$$0 \longleftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \longleftarrow 0$$

Then we have  $H_0 = \mathbb{Z}$ ,  $H_1 = \mathbb{Z}_2$ ,  $H_2 = 0$  and  $H_3 = \mathbb{Z}$  while  $H^0 = \mathbb{Z}$ ,  $H^1 = 0$ ,  $H^2 = \mathbb{Z}_2$  and  $H^3 = \mathbb{Z}$ . In general,  $H^n(X, G) \neq \text{Hom}(H_n(X), G)$ .

### Fact (1)

Let  $T_n$  be the torsion subgroup of  $H_n$ . Then  $H^n(X, \mathbb{Z}) \cong T_{n-1} \oplus (H_n / T_n)$ .

### Fact (2)

$H^n(X, \mathbb{R}) \cong \text{Hom}(H_n(X), \mathbb{R})$ . For example, we can dual by  $G = \mathbb{R}$

$$0 \longleftarrow \mathbb{R} \xleftarrow{0} \mathbb{R} \xleftarrow{\sim} \mathbb{R} \xleftarrow{0} \mathbb{R} \longleftarrow 0$$

with  $\mathbb{R}$  coefficients. Then  $H^0 = \mathbb{R}$ ,  $H^1 = 0$ ,  $H^2 = 0$  and  $H^3 = \mathbb{R}$ .

## Integration

Take  $H_p(M)$  and a singular  $n$ -simplex  $\sigma : \Delta^p \rightarrow M$ . We want  $I : H_{\text{dR}}^p(M) \rightarrow \text{Hom}(H_p(M), \mathbb{R})$  by  $[\omega] \mapsto I[\omega]$  where  $(I[\omega])(\sigma) = \int_\sigma \omega$ . To make this work, we consider smooth  $\sigma$  and the collection  $C_p^\infty(M) = \{\sigma : \Delta^p \rightarrow M \text{ smooth}\}$ . This gives a chain complex

$$\cdots \longrightarrow C_{p+1}^\infty \longrightarrow C_p^\infty \longrightarrow C_{p-1}^\infty \longrightarrow \cdots$$

Then it has homology group  $H_p^\infty(M)$ .

### Fact

The inclusion map  $\iota : C_p^\infty(M) \rightarrow C_p(M)$  induces an isomorphism  $\iota_* : H_p^\infty(M) \rightarrow H_p(M)$ . Then we can consider instead  $I : H_{\text{dR}}^p(M) \rightarrow \text{Hom}(H_p^\infty(M), \mathbb{R})$ . Then if  $\sigma : \Delta^p \rightarrow M$  is smooth, we can write  $\int_\sigma \omega := \int_{\Delta^p} \sigma^* \omega$  where  $\Delta^p \subseteq \mathbb{R}^p$  is formed by vertices  $[v_0, \dots, v_p]$  of the form  $v_0 = (0, \dots, 0)$  and  $v_i = (0, \dots, 1, \dots, 0)$ .

## Stoke's Theorem (for Integration over Smooth Chains)

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega$$

where  $\partial\sigma = \sum_i (-1)^i \sigma|_{i\text{-th face}}$ . More precisely,  $F_i : \Delta^{p-1} \rightarrow i\text{-th face of } \Delta^p$  by  $[v_0, \dots, v_{p-1}] \mapsto [v_0, \dots, \hat{v}_i, \dots, v_p]$ . So  $\partial\sigma = \sum_i (-1)^i \sigma \circ F_i$ . We need to check orientation, so write

$$\int_{\sigma} d\omega = \int_{\Delta^p} \sigma^*(d\omega) = \int_{\Delta^p} d(\sigma^*\omega) = \int_{\partial\Delta^p} \sigma^*\omega = \sum_i \int_{\partial_i\Delta^p} \sigma^*\omega$$

where  $\partial\Delta^p$  has outward orientation and  $\partial_i\Delta^p$  is the  $i$ -th face with outward orientation. On the right-hand side

$$\int_{\partial\sigma} \omega = \sum_i (-1)^i \int_{\sigma \circ F_i} \omega = \sum_i (-1)^i \int_{F_i(\Delta^{p-1})} \sigma^* \omega$$

where  $\Delta^{p-1} \subseteq \mathbb{R}^p$  has a standard orientation given by  $[e_1^{(p-1)}, \dots, e_{p-1}^{(p-1)}]$ . So  $F_i : \Delta^{p-1} \rightarrow \partial_i\Delta^p$ , where the domain has standard orientation and the image has outward orientation. Then  $F_i$  maps  $e_1^{(p-1)} \mapsto e_1^{(p)}, \dots, e_{i-1}^{(p-1)} \mapsto e_{i-1}^{(p)}, e_i^{(p-1)} \mapsto e_{i+1}^{(p)}, \dots, e_{p-1}^{(p-1)} \mapsto e_p^{(p)}$ . Then

$$((-e_i) \lrcorner d\text{vol}_p)[F_i(e_1^{(p-1)}, \dots, F_i(e_{p-1}^{(p-1)}]) = d\text{vol}_p[-e_i, e_1, \dots, e_i, \dots, e_p] = (-1)^i \sum_i (-1)^i \int_{F_i(\Delta^{p-1})} \sigma^* \omega = \sum_i (-1)^{2i} \int_{\partial_i\Delta^p} \sigma^* \omega$$

## Continuing...

For  $I : H_{\text{dR}}^p(M) \rightarrow \text{Hom}(H_p(M), \mathbb{R})$  by  $[\omega] \mapsto I[\omega]$  with  $I[\omega][c] = \int_c \omega$ , recall that  $c = \sum_i n_i \sigma_i$  and  $\sigma_i : \Delta^p \rightarrow M$  smooth. Then  $\int_c \omega := \sum_i n_i \int_{\sigma_i} \omega$ .

$I$  is well defined, since if  $\sigma' = \sigma + \partial\eta$  for  $\eta : \Delta^{p-1} \rightarrow M$  smooth,  $\int_{\sigma'} \omega = \int_{\sigma} \omega + \int_{\partial\eta} \omega$ . But by Stokes',  $\int_{\partial\eta} \omega = \int_{\eta} d\omega = 0$ .

Secondly, if  $\omega' = \omega + d\eta$  for  $\eta \in \Omega^{p-1}(M)$ , then  $\int_{\sigma} \omega' = \int_{\sigma} \omega + \int_{\sigma} d\eta$  and again  $\int_{\sigma} d\eta = \int_{\partial\sigma} \eta = 0$ .

## Naturality in de Rahm Cohomology

Given  $F : M \rightarrow N$  smooth,

$$\begin{array}{ccc} H_{\text{dR}}^p(N) & \xrightarrow{F^*} & H_{\text{dR}}^p(M) \\ \downarrow I & & \downarrow I \\ \text{Hom}(H_p(N), \mathbb{R}) & \xrightarrow{F^*} & \text{Hom}(H_p(M), \mathbb{R}) \end{array}$$

Let  $[\omega] \in H_{\text{dR}}^p(N)$  and  $[0] \in H^p(M)$ , then

$$(I \circ F^*[\omega])([0]) = (I \circ [F^*\omega])([0]) = \int_{\sigma} F^*\omega = \int_{F \circ \sigma} \omega = I[\omega](F \circ \sigma) = (F^* \circ I[\omega])(\sigma).$$

so the diagram commutes.

## Mayer-Vietoris for de Rahm Cohomology

If  $M = U \cup V$  for  $U, V$  open, then

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_{\text{dR}}^p(M) & \longrightarrow & H_{\text{dR}}^p(U) \oplus H_{\text{dR}}^p(V) & \longrightarrow & H_{\text{dR}}^p(U \cap V) \xrightarrow{\delta} H_{\text{dR}}^{p+1}(M) \longrightarrow \cdots \\ & & \downarrow I & & \downarrow I \oplus I & & \downarrow I \\ \cdots & \longrightarrow & H_{\text{dR}}^p(M, \mathbb{R}) & \longrightarrow & H_{\text{dR}}^p(U, \mathbb{R}) \oplus H_{\text{dR}}^p(V, \mathbb{R}) & \longrightarrow & H_{\text{dR}}^p(U \cap V, \mathbb{R}) \xrightarrow{\delta} H^{p+1}(M, \mathbb{R}) \longrightarrow \cdots \end{array}$$

Recall that

$$0 \longrightarrow \Omega^k(M) \xrightarrow{k^* \oplus l^*} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{i^* - j^*} \Omega^k(U \cap V) \longrightarrow 0$$

$$\omega \longmapsto (\omega|_U, \omega|_V) \longrightarrow 0$$

$$(\omega, \eta) \longmapsto (\omega|_{U \cap V} - \eta|_{U \cap V})$$

So we have a short exact sequence

$$0 \longrightarrow C_p(U \cap V) \xrightarrow{\alpha} C_p(U) \oplus C_p(V) \xrightarrow{\beta} C_p(M) \longrightarrow 0$$

$$\sigma \longmapsto (\sigma, -\sigma)$$

$$(\omega, \eta) \longmapsto \sigma + \eta$$

which we dualize to

$$0 \longrightarrow C_p^*(M) \xrightarrow{\beta^*} C_p^*(U) \oplus C_p^*(V) \xrightarrow{\alpha^*} C_p^*(U \cap V) \longrightarrow 0$$

$$\varphi \longmapsto \beta^* \varphi$$

$$(\varphi, \psi) \longmapsto \alpha^*(\varphi, \psi)$$

So ultimately we have

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega^p(M) & \longrightarrow & \Omega^p(U) \oplus \Omega^p(V) & \longrightarrow & \Omega^p(U \cap V) \longrightarrow 0 \\ & & \downarrow d & & \downarrow d & & \downarrow d \\ 0 & \longrightarrow & \Omega^{p+1}(M) & \longrightarrow & \Omega^{p+1}(U) \oplus \Omega^{p+1}(V) & \longrightarrow & \Omega^{p+1}(U \cap V) \longrightarrow 0 \end{array}$$

and

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_p^*(M) & \longrightarrow & C_p^*(U) \oplus C_p^*(V) & \longrightarrow & C_p^*(U \cap V) \longrightarrow 0 \\ & & \downarrow d & & \downarrow d & & \downarrow d \\ 0 & \longrightarrow & C_{p+1}^*(M) & \longrightarrow & C_{p+1}^*(U) \oplus C_{p+1}^*(V) & \longrightarrow & C_{p+1}^*(U \cap V) \longrightarrow 0 \end{array}$$

Where  $I$  maps from  $\Omega^p(M)$  to  $C_p^*(M)$  and the entire three dimensional diagram commutes.