

# Algebra I

**September 28, 2023**

## Grade Weights

50% Homework + 50% Final

Participation matters for pass/fail.

## Office Hours

Tuesday / Thursday 11:25 - 12:00

Or by appointment (jusuh@ucsc.edu)

## Recommended Text

Abstract Algebra (3e) - Dummit and Foote

Finite Groups: An Introduction (2nd revised) - Jean-Pierre Serre

Robert Boltje's Lecture Notes - (<https://boltje.math.ucsc.edu/courses/f17/f17m200notes.pdf>)

## Definition: Binary Operation

Let  $S$  be a set. A binary operation on  $S$  is a function  $f : S \times S \rightarrow S$ . We will almost never use  $f$  for the binary operation ( $f(s, t)$ ).

The usual notation for binary operations is  $s * t$ .

## Example

1.  $S = \mathbb{R}^3$ , define  $f : S \times S \rightarrow S$  as  $(\vec{x}, \vec{y}) \rightsquigarrow \vec{x} + \vec{y}$ .

2.  $S = \mathbb{R}^3$ , define  $S \times S \xrightarrow{f} S$  as  $(\vec{x}, \vec{y}) \rightsquigarrow \vec{x} + \vec{y}$ .

- Note that  $(\vec{x}, \vec{y}) \rightsquigarrow \vec{x} \cdot \vec{y}$  is not a binary operation.

3.  $S = \mathbb{Z}$  as  $(m, n) \mapsto m \cdot n$ .

4.  $S = \mathbb{R}^3$  as  $(\vec{x}, \vec{y}) \rightsquigarrow \frac{\vec{x} + \vec{y}}{2}$



5. Let  $n \geq 1$  be an integer and  $S = M_{n \times n}(\mathbb{R}) = \{n \times n \text{ real matrices}\}$ . Then  $(A, B) \rightsquigarrow AB$ .

## Observations

Examples 1,3,5 are associative; examples 2,4 are not.

Examples 1-4 are commutative; example 5 commutes only when  $n = 1$ .

$\vec{0}$  for example 1, 1 for example 3, and  $I_n$  for example 5.

## Q: What is a Group?

A group is a set equipped with a binary operation which satisfies three axioms.

Let  $*$  be a binary operation on a set  $S$ .

1. Say  $*$  is associative if  $\forall a, b, c \in S, (a * b) * c = a * (b * c)$ .
2. Say  $*$  is commutative if  $\forall a, b \in S, a * b = b * a$ .
3. An element  $e \in S$  is a neutral element (with respect to  $*$ ) if  $\forall a \in S, a * e = a = e * a$ .
  - If there exists a neutral element, then it is unique.
4. Suppose  $(S, *)$  has a neutral element  $e$ . Let  $a \in S$ . Then  $b \in S$  is called an inverse of  $a$  (with respect to  $*$ ) if  $a * b = e = b * a$ .

## Definition: Group

A group is a set  $G$  equipped with a binary operation  $*$  such that

1.  $*$  is associative.
2.  $*$  has a neutral element  $e$ .
3. Every  $g \in G$  has an inverse.

If, in addition,  $*$  is commutative, we say  $(G, *)$  is an abelian or commutative group.

## Examples

$(\mathbb{R}^3, +)$  is a commutative group.

$(\mathbb{R}^3, \times)$  has no neutral element.

$(\mathbb{Z}, \cdot)$  has no inverse (except  $\pm 1$ ).

$(\mathbb{R}^3, \text{mid})$  is not associative. (the midpoint)

$(M_{n \times n}(\mathbb{R}), \cdot)$  has no inverse of  $0_{n \times n}$ .

For  $n \geq 1$ ,  $(\mathbb{R}^n, +)$  and  $(\mathbb{C}^n, +)$  are abelian groups.

**Proof that the Neutral Element is unique.**

Let  $e, e'$  be neutral elements. Then  $e' = e * e' = e$ . ■

**Proof that the Inverse is unique.**

Left to the reader.

### Definition: Subgroup

Let  $G$  be a group, and let  $H$  be a subset of  $G$ . We say that  $H$  is a subgroup of  $G$  if

1.  $\forall h_1, h_2 \in H, h_1 * h_2 \in H$ .
2.  $e \in H$ .
3.  $\forall h \in H, h^{-1} \in H$ .

### Examples

$\mathbb{Z}^n \subseteq \mathbb{R}^n$  is a subgroup ( $*$  = +).

$G = \{A \in M_{n \times n} : \det(A) \neq 0\}$ . Then  $(G, \cdot)$  is a group.

- This is the General Linear Group on  $\mathbb{R}$ :  $\text{GL}_n(\mathbb{R})$ .
- Recall  $A^{-1} = \frac{1}{\det(A)} \left( (-1)^{ij} \det(M_{\alpha_i}) \right)$ .

### Definition: General Linear Subgroups

$S = \{A \in \text{GL}_n(\mathbb{R}) : a_{ij} \in \mathbb{Z}, \forall 1 \leq i, j \leq n\}$ .

$S$  is closed under  $\cdot$  and  $I_n \in S$ , but for example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \frac{1}{1 \cdot 4 - 2 \cdot 3} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

so  $S$  is not a subgroup.

However,  $T = \{A \in S : \det(A) = \pm 1\} \subseteq \text{GL}_n(\mathbb{R})$ .

- Note that if  $AA' = I_n$  then  $\det(A)\det(A') = 1$ .

### Definition: Additive Groups

For groups like  $\mathbb{Z}^n$ ,  $\mathbb{R}^n$  and  $\mathbb{C}^n$ , we will use  $+$  for the binary operation and say that they are additive groups. The Neutral Element is denoted as 0.

The inverse is denoted as  $-g$ .

For  $m \geq 1$  and  $g \in G$ ,  $mg = g + \dots + g$  and  $(-m)g = -(mg)$ .

## Definition: Multiplicative Groups

For groups like  $\text{GL}_n(\mathbb{C})$  or  $\text{GL}_n(\mathbb{Z})$ , we say that the group is multiplicative.

Denote the neutral element as 1.

Denote the inverse of  $g$  as  $g^{-1}$ .

For  $m \geq 1$ ,  $g^m = \underbrace{g \cdots g}_m$ .

$g^0 = 1$ .

$g^{-m} = (g^m)^{-1}$ .

## Definition: Group Element Order

Let  $G$  be a group,  $g \in G$ , and  $m \geq 1$ .

Say  $g$  has order  $m$  if  $g^m = 1$  and  $g^k \neq 1$ ,  $\forall k$  such that  $1 \leq k \leq m$ .

An element has infinite order if  $g^m \neq 1$ ,  $\forall m \in \mathbb{Z}^+$ .

## Examples

In  $D_{10}$ ,  $I_2$  has order 1, rotations have order 5 and reflections have order 2.

## Groups from Geometry

### Pentagon

Consider the regular pentagon  $P$ .



$$H = \{T \in \text{GL}_2(\mathbb{R}) : T(P) = P\}.$$

This is the symmetry group of  $P$  or  $D_{10}$  (sometimes  $D_5$ )

$$H \leq \text{GL}_2(\mathbb{R}).$$

- Proof of closure.

Suppose  $T_1, T_2 \in H$ . Then  $T_1(P) = P$ ,  $T_2(P) = P$  and  $(T_1 \circ T_2)(P) = T_1(T_2(P)) = T_1(P) = P$ .

Therefore  $H$  is closed under  $\circ$ .

- Proof of identity.

$\text{Id}_{\text{GL}_2} = I_2$  does satisfy  $I_2(P)$ .

- Proof of inverse.

If  $T \in H$  (i.e.  $T \in \text{GL}_2(\mathbb{R})$  and  $T(P) = P$ , apply  $T^{-1}$  and get  $T^{-1}(T(P)) = T^{-1}(P)$ . Therefore  $P = T^{-1}(P)$ .

### Tetrahedron

Let  $X$  be the regular tetrahedron and  $A = \{\text{rotational symmetries of } X\}$ .



Then  $A$  contains

- The identity: 1.
- $2 \cdot 4 = 8$  rotations by  $120^\circ$ .
- 3 rotations of  $180^\circ$ .

So we have a bijection  $r : \{B, P, W, Y\} \rightarrow \{B, P, W, Y\}$  where



### Definition: Symmetric Group

Let  $S$  be a set (e.g.  $E = \{B, P, W, Y\}$ ). The Symmetric Group  $\text{Sym}(E)$  is the set of bijections  $f : E \rightarrow E$  equipped with the binary operation  $\circ$  (composition).

**October 3, 2023**

### Homework

First homework should be released this Thursday, October 5th.  
Next lecture will be on group actions.

### Propositions: Symmetric Group

Let  $X$  be a set.

When  $|X| = n$  denote the elements  $\{1, 2, \dots, n\}$ .

$\text{Sym}(X) = \{f : X \rightarrow X \mid f \text{ is bijective}\}$ .

With  $\circ$  (composition of functions) as a binary operation,  $\text{Sym}(X)$  is a group.

### Symmetric Group Order

If  $|X| = n$ , then  $|\text{Sym}(X)| = n!$

- Proof

Let  $X = \{1, 2, \dots, n\}$ . A bijection  $f$  consists of  $f(1), f(2), \dots, f(n)$ .

For  $f(1)$ , we have  $n$  choices; for  $f(2)$  we have  $n - 1$  choices. This continues until only 1 choice remains for  $f(n)$ .

Therefore the choices are  $(n)(n - 1) \cdots (1) = n!$

### Example

For the symmetric group on four letters  $\{a, b, c, d\}$ ,  $|\text{Sym}(4)| = 4! = 24$

### Definition: Cycles

Let  $x = \{1, \dots, n\}$ ,  $m \geq 1$  be an integer and  $a_1, a_2, \dots, a_m$  distinct elements in  $X$ .

Then the  $m$ -cycle denoted by  $(a_1 a_2 \dots a_m)$  is the element of  $\text{Sym}(X)$  which maps  $a_1$  to  $a_2$ ,  $a_2$  to  $a_3, \dots, a_{m-1}$  to  $a_m$ , and  $a_m$  to  $a_1$ .

### Example

Let  $n = 7$  and  $m = 4$ . Then  $(2 \ 7 \ 1 \ 3)$  is a bijection.



### Degenerate Case

$m = 1$  gives  $\text{Id}_X$ .

### First Non-Degenerate Case

A transposition is, by definition a 2-cycle:  $(a_1 a_2)$ .

### Proposition: Symmetric Group as Cycle Composition

Every element in  $\text{Sym}(X)$  is the product (using  $\circ$ ) of  $m$ -cycles, where  $m$  can vary.

- Proof

Consider  $\text{Sym}(6)$ .



$6 \longrightarrow 6$  This gives a bijection  $\pi = (1 \ 4 \ 2)(3 \ 5)(6)$  which is the composition of cycles.

We say that this  $\pi$  has cycle type  $3 + 2 + 1$ .

- Cycle Type

If instead  $\pi = (1 \ 4 \ 2)(3 \ 5 \ 6)$  then the cycle type is given as  $3 + 3$ .

## Finite Symmetric Groups

For  $n = 2$ ,  $\text{Sym}(X) = \{\text{Id}, (1\ 2)\}$ .

This gives cycle types  $1 + 1$  and  $2$ .

For  $n = 3$ ,  $\text{Sym}(X) = \{\text{Id}, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$ .

This gives cycle types  $1 + 1$ ,  $2 + 1$  and  $3$ .

## Symmetric Group for Tetrahedron

For  $n = 4$  let  $X = \{B, P, W, Y\}$ .

Partitions of  $n = 4$  are

$4 = 4$	$6$	$(B\ P\ W\ Y)$	$\dots$					$\text{sign} = -1$
$= 3 + 1$	$4 \cdot 2 = 8$	$(P\ W\ Y)$	$\dots$					$\text{sign} = +1$
$= 2 + 2$	$\frac{\binom{4}{2}}{2} = 3$	$(B\ P)(W\ Y)$	$(B\ W)(P\ Y)$	$(B\ Y)(P\ W)$				$\text{sign} = +1$
$= 2 + 1 + 1$	$\binom{4}{2} = 6$	$(B\ P)$	$(B\ W)$	$(B\ Y)$	$(P\ W)$	$(P\ Y)$	$(W\ Y)$	$\text{sign} = -1$
$= 1 + 1 + 1 + 1$	$1$	$\text{Id}_X$						$\text{sign} = +1$

## Rotation Group for Tetrahedron

$$\begin{aligned}
 A &= \{\text{Rotational Symmetries}\} \\
 &= \{\text{Id}_X, 8\ 3\text{-cycles}, 3\ \text{of type } 2+2\}
 \end{aligned}$$

Note, from the sign, that  $A \leq \text{Sym}(4)$ .

## Symmetries Not in Rotation

Why, for example, is  $(B\ P)$  not in the rotation group?

If it were, it should be possible to swap vertices and then undo the switch with only rotation.

However, the two tetrahedra are mirror images across a plane.

Observe that the right hand rule with respect to  $P$ ,  $W$  and  $Y$  will give opposite, orthogonal vectors.

## Rotation as a Subgroup of Symmetry

Q: Is  $A$  a subgroup of  $\text{Sym}(4)$ ?

Following the definition, it would be necessary to verify

- $\text{Id} \in A$
- $A$  is closed under inverse.
- $A$  is closed under composition.

## Group Homomorphism

Let  $G$  and  $H$  be groups (whose binary operations are denoted by  $g_1 \cdot g_2$ ).  
A (group) homomorphism from  $G$  to  $H$  is a function  $\phi : G \rightarrow H$  such that

$$\bullet \phi(g_1 \cdot_G g_2) = \phi(g_1) \cdot_H \phi(g_2)$$

### Properties of Group Homomorphism

1.  $\phi(1_G) = 1_H$

2.  $\phi(g^{-1}) = [\phi(g)]^{-1}, \forall g \in G$

• Proof

By definition,  $\phi(1_G \cdot 1_G) = \phi(1_G) \cdot \phi(1_G)$ .

Letting  $e = \phi(1_G)$ , we get  $e = e \cdot e$ .

By multiplying both sides by  $e^{-1}$ , we get  $1_H = e$ .

Part two is left as an exercise.

### Example 1

Let  $n \geq 1$  and  $G = \text{GL}_n(\mathbb{R}) = \{A \in M_{n \times n}(\mathbb{R}) \mid \det(A) \neq 0\}$ .

In particular, when  $n = 1$ ,  $\text{GL}_1(\mathbb{R}) = \mathbb{R}^* = \{r \in \mathbb{R} \mid r \neq 0\}$  (with multiplication as the binary operation).

Then  $\det : G \rightarrow H$  is a group homomorphism.

That is  $\det(AB) = \det(A)\det(B)$  (as learned in MATH 21).

### Example 2

Let  $n \geq 1$ ,  $G = \text{Sym}(n)$ ,  $H = \text{GL}_n(\mathbb{R})$ .

Construct a group homomorphism  $\rho : G \rightarrow H$ .

Recall that a linear transformation  $A \in H$  is completely determined by  $Ae_1, Ae_2, \dots, Ae_n$

where  $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ .

For  $\pi \in G = \text{Sym}(n)$ ,  $\rho(\pi)$  is the linear transformation that maps  $e_i$  to  $e_j$  whenever  $\pi$  maps  $i$  to  $j$ .

This is a surjective linear transformation on a vector space and, therefore, invertible.

• Example

For  $n = 4$  and  $\pi = (2 \ 3 \ 4)$

$$\begin{array}{ccc} 1 & \longrightarrow & 1 \\ 2 & \searrow & 2 \\ 3 & \nearrow & 3 \\ 4 & \searrow & 4 \end{array}$$

$$\rho(\pi)$$



$$\begin{array}{ccc}
e_1 & \longrightarrow & e_1 \\
e_2 & \searrow & e_2 \\
e_3 & \nearrow & e_3 \\
e_4 & \searrow & e_4
\end{array}$$

Therefore

$$\rho(\pi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

– Is this a group homomorphism?

Let  $\pi_1, \pi_2 \in G$  be arbitrary elements.

Need to show:  $\rho(\pi_1 \circ \pi_2) = \rho(\pi_1) \circ \rho(\pi_2)$ .

Both sides are linear transformations and, hence, determined by their actions on  $e_i$  for  $i = 1, \dots, n$ .

$$\begin{aligned}
\rho(\pi_1 \circ \pi_2)e_i &= e_{\pi(i)} \\
&= e_{\pi_1(\pi_2(i))} \\
\rho(\pi_1)(\rho(\pi_2)e_i) &= \rho(\pi_1)(e_{\pi_2(i)})
\end{aligned}$$

## Composition of Group Homomorphisms

Let  $G, H$  and  $K$  be groups and  $G \xrightarrow{\phi} H$  and  $H \xrightarrow{\psi} K$  be homomorphisms.

Then the composite  $\psi \circ \phi : G \rightarrow K$  is a group homomorphism.

### Proof

Let  $g_1, g_2 \in G$  be arbitrary.

$$\begin{aligned}
(\psi \circ \phi)(g_1 g_2) &= \psi(\phi(g_1 g_2)) && \text{by definition of } \circ \\
&= \psi(\phi(g_1) \phi(g_2)) && \text{since } \phi \text{ is a group homomorphism} \\
&= \psi(\phi(g_1)) \psi(\phi(g_2)) && \text{since } \psi \text{ is a group homomorphism} \\
&= (\psi \circ \phi)(g_1) \circ (\psi \circ \phi)(g_2) && \text{by definition of } \circ
\end{aligned}$$

## Definition: Sign Homomorphism

Let  $n \geq 1$  and  $G = \text{Sym}(n)$ .

This sign homomorphism is the composition  $\text{sign}: G \xrightarrow{\rho} \text{GL}_n(\mathbb{R}) \xrightarrow{\det} \mathbb{R}^*$

## Sign of Symmetric Group

$$\text{sign}(\text{sym}(n)) \subseteq \{1, -1\} \leq \mathbb{R}^*$$

- Lemma

Let  $a_1, \dots, a_m$  be distinct numbers between 1 and  $n$ .

Then  $(a_1 \cdots a_m)$  is equal to  $(a_1 \cdots a_{m-1})(a_{m-1} a_m)$ .

This will be proven on homework.

- Corollary

Any  $m$  cycle is the composition of  $m - 1$  transpositions.

Namely,  $(a_1, \dots, a_m) = (a_1 a_2)(a_2 a_3) \cdots (a_{m-1} a_m)$ .

Easily check:  $\text{sign}((a_i a_{i+1})) = -1$ .

Now any  $g \in \text{Sym}(n)$  allows a cycle decomposition.

## Definition: Kernel of a Homomorphism

Let  $G \xrightarrow{\phi} H$  be a group homomorphism.

The kernel of  $\phi$  is  $\ker(\phi) := \{g \in G \mid \phi(g) = 1_H\}$ .

## The Kernel is a Subgroup

Let  $g_1, g_2 \in \ker(\phi)$ . Then

$$\begin{aligned}\phi(g_1 g_2) &= \phi(g_1) \phi(g_2) \\ &= 1_H 1_H \\ &= 1_H\end{aligned}$$

$\phi$  is a homomorphism

$$g_1, g_2 \in \ker(\phi)$$

$$g_1, g_2 \in \ker(\phi)$$

Similarly,  $1_G \in \ker(\phi)$  and  $g^{-1} \in \ker(\phi)$  if  $g \in \ker(\phi)$ . ■

## Definition: Alternating Group

Let  $X$  be a set,  $|X| = n \leq \infty$ .

The alternating group on  $X$  is the  $\text{Alt}(X) = \ker(\text{sign} : \text{Sym}(X) \rightarrow \{\pm 1\})$ .

## October 5, 2023

## Definition: Group Action

Let  $G$  be a group and  $X$  a set.

A (left) action of  $G$  on  $X$  is a function  $\alpha : G \times X \rightarrow X$  which satisfies two conditions:

1.  $\alpha(1_G, x) = x$  for all  $x \in X$ .
2.  $\alpha(g_1, \alpha(g_2, x)) = \alpha(g_1 g_2, x)$  for all  $g_1, g_2 \in G$  and  $x \in X$ .

## Notation

Write  $\alpha(g, x) = g * x = g \cdot x = gx$ .

### Example A

Let  $X$  be any set, and let  $G = \text{Sym}(X) = \{f : X \rightarrow X \text{ bijections}\}$  where the group operation  $\circ$  is the composition of functions.

Then  $G$  acts (on the left) on  $X$  by  $f * x \stackrel{\text{def}}{=} f(x)$ .

Then the features

1.  $\text{Id}_X(x) = x, \forall x \in X$
  2.  $g_1 * (g_2 * x) = (g_1 \circ g_2) * x, \forall g_1, g_2 \in G, \forall x \in X$
- Or  $g_1(g_2(x)) = (g_1 \circ g_2)(x)$

are satisfied.

### Example B

Let  $G = \text{Sym}(\{B, P, W, Y\})$  which acts on  $X = \{B, P, W, Y\}$ .

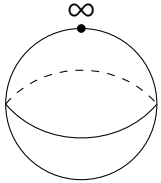
If  $H \leq G$ , then  $H$  acts on  $X$  as well, define  $h * x = \dot{h} * x$  (where  $\dot{h}$  is regarded as in the alternating group of  $G$ ).

In particular,  $\text{Alt}(\{B, P, W, Y\})$  acts on  $X$  by rotations.

### Example C\*

This example is not required for this class.

From complex Analysis we have the Riemann sphere  $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ .



Let  $G = \text{SL}_2(\mathbb{C})$ .

Define  $G$ -action on  $X = \mathbb{P}^1(\mathbb{C})$  by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} z := \frac{\alpha z + \beta}{\gamma z + \delta} \quad (\infty \text{ if } \gamma z + \delta = 0)$$

This is called the Möbius group action on  $\mathbb{P}^1(\mathbb{C})$ .

Exercise: show that 1. and 2. are satisfied.

### Definitions

Let  $G$  act on  $X$ . (Say  $X$  is a (left)  $G$ -set)

## Stabilizer

Let  $x \in X$ . The stabilizer of  $x$  in  $G$  is  $\text{Stab}_G(x) = \{g \in G \mid g * x = x\} \subseteq G$ .

- Example 1

Let  $G$  be any group and  $X$  a  $G$ -set.

Then for any  $x \in X$ ,  $\text{Stab}_G(x) \leq G$ .

– Proof

1.  $1_G \in \text{Stab}_G(x)$  since, by definition,  $1_G * x = x$ .

Therefore the identity is present.

2. If  $g_1, g_2 \in \text{Stab}_G(x)$  are such that  $g_1 * x = x$  and  $g_2 * x = x$ , then  $(g_1 g_2) * x \stackrel{\text{2nd Axiom}}{=} g_1 * (g_2 * x) = g_1 * x = x$ .

Therefore the stabilizer is closed under composition.

3. Say  $g \in \text{Stab}_G(x)$  and  $g * x = x$ . Apply  $g^{-1}$  to both sides to get

$$x \stackrel{\text{1st Axiom}}{=} 1_G * x = (g^{-1} g) * x \stackrel{\text{2nd Axiom}}{=} g^{-1} * (g * x) = g^{-1} * x$$

Therefore the stabilizer is closed under inverse.

- Example 2

Let  $G = \text{Alt}(\{B, P, W, Y\})$  and consider  $H = \text{Stab}_G(W) = \{\text{Id}, (B P Y), (B Y P)\}$ .

Fact:  $H$  does not act transitively on  $X$ , since  $W$  is fixed and no element  $g \in H$  satisfies  $g * W = B$ .

## Orbit

Let  $x \in X$ . The  $G$ -orbit of  $x$  in  $X$  is  $G \cdot x = \{g * x \mid g \in G\} \subseteq X$ .

Let  $G$  act on  $X$  and  $x, y \in X$ . Either  $G \cdot x = G \cdot y$  or  $G \cdot x \cap G \cdot y = \emptyset$ .

So  $X$  is the disjoint union of  $G$ -orbits.

e.g.  $\{B, P, W, Y\} = \{W\} \sqcup \{B, P, Y\}$  gives the  $\text{Stab}_G(W)$ -orbits.

- Example 1

When  $G = \text{Alt}(X)$ , for  $X = \{B, P, W, Y\}$ , there is only one orbit since  $\forall x \in X, G \cdot x = X$ .

- Example 2

When  $G = \text{Stab}_G(W)$ , for  $X = \{B, P, W, Y\}$ , then  $G \cdot W = \{W\}$  while

$$\begin{aligned} G \cdot B &= \{\text{Id}(B), (B P Y)(B), (B Y P)(B)\} = \{B, P, Y\} \\ &= G \cdot P = \{\text{Id}(P), (B P Y)(P), (B Y P)(P)\} = \{P, Y, B\} \\ &= G \cdot Y \end{aligned}$$

## Transitivity

Say  $G$  acts transitively on  $X$  (or the action is transitive) if, for any pair  $x, y \in X$ , there exists  $g \in G$  (depending on  $x$  and  $y$ ) such that  $g * x = y$ .

- Example

$G = \text{Alt}(\{B, P, W, Y\}) \subset \text{Sym}(\{B, P, W, Y\})$  is transitive.

- Proof

Let  $x, y \in X$  be arbitrary.

If  $x = y$ , then take  $g = \text{Id}_X$  and we have  $g * x = y$ .

Suppose  $x \neq y$ , then write  $X = \{x, y, z, w\}$  and take  $g = (x\ y)(z\ w)$ . We have  $g * x = y$ .

e.g.  $x = P, y = Y, z = B$  and  $w = W$  gives  $g = (P\ Y)(B\ W)$ .

- Exercise \*

This exercise is not required for the course.

Prove that  $\text{SL}_2(\mathbb{C})$  acts transitively on  $\mathbb{P}^1(\mathbb{C})$ .

Say  $\mathbb{P}^1(\mathbb{C})$  is a homogeneous space under  $\text{SL}_2(\mathbb{C})$ .

## Proposition: Group Action Gives Group Homomorphisms

( $\longrightarrow$ ) Let  $G$  act on  $X$ . Then

1. For any  $g \in G$ , the function  $\pi_g : X \rightarrow X$  defined by  $\pi_g(x) = g * x$  is a bijection of  $X$ , hence  $\pi_g \in \text{Sym}(X)$ .
2. The function  $G \xrightarrow{\phi} \text{Sym}(X)$  given by  $\phi(g) = \pi_g$  is a group homomorphism.

### Proof of 1

Need to show that  $\pi_g$  is injective and surjective.

(Inj) Let  $x, y \in X$  and assume  $\pi_g(x) = \pi_g(y)$  (i.e.  $g * x = g * y$ ).

Apply  $g^{-1} *$  on both sides, such that  $x = g^{-1} * (g * x) = g^{-1} * (g * y) = y$ .

(Sur) Let  $x \in X$  be arbitrary. Need to find  $y \in X$  such that  $\pi_g(y) = x$ .

Take  $y = g^{-1} * x$ , and  $\pi_g(y) = g * (g^{-1} * x) = x$ .

### Proof of 2

Need to show that  $\forall g_1, g_2 \in G, \phi(g_1 g_2) = \phi(g_1) \phi(g_2)$ .

$\phi(g_1 g_2) \in \text{Sym}(X)$  is characterized by  $[\phi(g_1 g_2)](x) = \pi_{g_1 g_2}(x) = (g_1 g_2) * x$ .

On the other hand,  $\phi(g_1) \phi(g_2) \in \text{Sym}(X)$  is characterized by  $[\phi(g_1) \phi(g_2)](x) = \phi(g_1)[\phi(g_2)(x)] = g_1 * (g_2 * x)$ .

By the second group action axiom, these must be the same.

## Proposition: Group Homomorphism Admits Group Action

( $\longleftarrow$ ) Let  $G \xrightarrow{\rho} \text{Sym}(X)$  be a group homomorphism.

Then, by letting  $g * x = \rho(g)(x) \in X$  we get a left  $G$ -action on  $X$ .

**Proof**

1.  $1_G * x = \rho(1_G)(x) = \text{Id}_X(x) = x$ .
2. Let  $g_1, g_2 \in G$  and  $x \in X$ . Then  $(g_1 g_2) * x = [\rho(g_1 g_2)](x) = [\rho(g_1) \circ \rho(g_2)](x) = \rho(g_1)[\rho(g_2)(x)] = g_1 * (g_2 * x)$ .

**Definition: Right Group Actions**

Let  $G$  be a group and  $X$  be a set. A right  $G$ -action on  $X$  is a function  $\beta : X \times G \rightarrow X$  such that

1.  $\beta(x, 1_G) = x, \forall x \in X$ .
2.  $\beta(x, g_1 g_2) = \beta(\beta(x, g_1), g_2), \forall g_1, g_2 \in G, \forall x \in X$ .

**Notation**

$$\beta(x, g) = x * g = x \cdot g = xg$$

**Remark**

If  $\alpha : G \times X \rightarrow X$  is a left action, we get a right action  $\beta : X \times G \rightarrow X$  by  $\beta(x, g) = \alpha(g^{-1}, x)$  and vice versa. That is  $x * g = g^{-1} * x$ .  
Proof recommended as an exercise.

**Analogues**

Stability, orbit and transitivity all have analogues which can be demonstrated by converting to left actions.

**Definition: Cosets**

Let  $H \leq G$ , and let  $X = G$ .

We have left action  $H \times X \rightarrow X$  and  $h * x = hx$  (taken in  $G$ ).

As well as right action  $X \times H \rightarrow X$  where  $x * h = xh$ .

A (left)  $H$ -coset is an orbit  $xH$  for some  $x \in X$ .

A (right)  $H$ -coset is an orbit  $Hx$  for some  $x \in X$ .

**Example**

Let  $G = \text{Alt}(4)$ ,  $H = \text{Stab}_G(W) = \{\text{Id}, (B P Y), (B Y P)\}$ .

1. Take any  $x \in H$ ,  $xH = H$ .
2. Take  $x = (B P)(W Y)$ , and  $xH = \{(B P)(W Y), (B P)(W Y)(B P Y) = (P W Y), (B P)(W Y)(B Y P) = (B W Y)\}$ .

3. There are two more; what are they?

## October 10, 2023

### Cosets Revisited

Let  $G$  be a group,  $H \leq G$ . Then a (left)  $H$ -coset in  $G$  is a set of the form

$$gH = \{gh | h \in H\}$$

, where  $g \in G$

### Coset Space

$G/H$  is the set of  $H$ -cosets.

- Example

For  $G = \text{Alt}(4)$ , given  $C_1 = H = \text{Stab}_G(B) = \{1, (P W Y), (P Y W)\}$ , we have  
 $C_2 = (B P W)H = \{(B P W), (B P)(W Y), (B P Y)\}$   
 $(B P W) \circ (P W Y) = (B P)(W Y)$

P  $\leftarrow$  B  $\leftarrow$  B  
 B  $\leftarrow$  W  $\leftarrow$  P  
 Y  $\leftarrow$  Y  $\leftarrow$  W  
 W  $\leftarrow$  P  $\leftarrow$  Y

$$(B P W) \circ (P Y W) = (B P Y)$$

P  $\leftarrow$  B  $\leftarrow$  B  
 Y  $\leftarrow$  Y  $\leftarrow$  P  
 W  $\leftarrow$  P  $\leftarrow$  W  
 B  $\leftarrow$  W  $\leftarrow$  Y

$C_3 = (B W P)H = \{(B W P), (B W Y), (B W)(P Y)\}$   
 $C_4 = (B Y P)H = \{(B Y P), (B Y)(P W), (B Y W)\}$   
 Then  $G/H = \{C_1, C_2, C_3, C_4\}$ .

- Q: What do the 3 elements in  $C_3$  have in common in geometric terms?  
 $C_3$  sends  $B$  to  $W$ .  
 Similarly, the cosets send  $B$  to all other vertices (including to itself).

### Definition: Transporter

Let  $G$  be a group and  $X$  a  $G$ -set.

For two points,  $x, y \in X$ , the transporter  $\text{Trsp}_G(x, y) = \{g \in G | gx = y\}$ .

**Example**

$$G/H = \{\text{Trsp}_G(B, B), \text{Trsp}_G(B, P), \text{Trsp}_G(B, W), \text{Trsp}_G(B, Y)\}$$

**Note**

When  $x = y$ , we recover  $\text{Trsp}_G(x, x) = \text{Stab}_G(x)$ .

For general  $G$  and  $H$ , there may not be a nice geometric action associated with it.

But  $G/H$  is still a  $G$ -set since  $g'(gH) = (g'g)H$ .

**Proposition (B)**

Let  $H \leq G$  be a subgroup and let  $g \in G$ .

Then the map  $H \xrightarrow{f} gH$  defined by  $h \mapsto f(h) = gh$  is a bijection.

**Proof**

(Surjective) Any element  $x$  in  $gH$  is, by definition, of the form  $gh$  for some  $h \in H$ . So  $x = f(h)$ .

(Injective) Say  $h_1, h_2 \in H$  satisfy  $f(h_1) = f(h_2)$ . That is  $gh_1 = gh_2$ . Multiplying  $g^{-1}$  on the left, we get  $h_1 = h_2$ .

**Proposition (C)**

Let  $G$  act on  $X$ ,  $x \in X$ , and  $g \in G$ .

Take  $y := gx$  and  $H = \text{Stab}_G(x)$ . Then  $gH = \text{Trsp}_G(x, y)$ .

**Proof**

( $\subseteq$ ) Let  $gh \in gH$  be arbitrary. Then

$$(gh) * x \underset{\text{Axiom 2}}{=} g * (h * x) \underset{h \in \text{Stab}_G(x)}{=} g * x \underset{y = g * x}{=} y$$

Therefore  $gh \in \text{Trsp}_G(x, y)$ .

( $\supseteq$ ) Suppose  $g' \in \text{Trsp}_G(x, y)$ . Consider  $g^{-1}g'$ . Then

$$(g^{-1}g') * x = g^{-1} * (g' * x) \underset{g' \in \text{Trsp}_G(x, y)}{=} g^{-1}(y) = x$$

Therefore  $(g^{-1}g') \in \text{Stab}_G(x)$ . Setting  $g^{-1}g' := h$ , so  $g' = gh \in gH$ .

**Theorem: Orbit-Stabilizer Theorem (OST)**

Let  $G$  act transitively on a set  $X$  (so that there is only one orbit in  $X$ , namely  $X$  itself).

If  $|G| < \infty$ , then for any  $x \in X$  we have

$$|X| \cdot |\text{Stab}_G(x)| = |G|$$



**Proof**

Let us count  $|G|$  by partitioning  $G$  into transporters.

$$G = \bigsqcup_{y \in X} \text{Trsp}_G(x, y)$$

Therefore

$$|G| = \sum_{y \in X} |\text{Trsp}_G(x, y)| \stackrel{B+C}{=} \sum_{y \in X} |\text{Stab}_G(x)| = |X| |\text{Stab}_G(x)| \quad \blacksquare$$

**Theorem: Lagrange's Theorem**

If  $G$  is a finite group and  $H \leq G$ , then  $|G| = |H| \cdot |G/H|$ .

**Proof (Sketch)**

Apply the Orbit-Stabilizer Theorem to  $X = G/H$ .

This action is transitive as  $g(1H) = gH$ .

Note  $gH = H \iff g \in H$  and  $g1 \in H$ .

Therefore  $\text{Stab}_G(1H) = \{g \in G \mid g(1H) = 1H\} = H$ .

**Corollary**

If  $H \leq G$  and  $|G| < \infty$ , then  $|H| \mid |G|$ .

The converse is not true. No subgroup of order 6 in  $\text{Alt}(4)$  (where  $|\text{Alt}(4)| = 12$ ).

**Definition: Conjugate**

Let  $G$  be a group,  $H \leq G$ ,  $g \in G$ .

1. For  $x \in G$  the  $g$ -conjugate of  $x$  is  $gxg^{-1} = {}^g x$ .
2. The  $g$ -conjugate of  $H$  is  $gHg^{-1} = {}^g H = \{gxg^{-1} \mid x \in H\}$ .

**Example**

Let  $G = \text{Alt}(4)$  and  $H = \text{Stab}_G(B) = \{1, (P W Y), (P Y W)\}$ . Then, for  $g = (B Y P)$

$$gHg^{-1} = \{1, (B W P), (B P W)\} = \text{Stab}_G(Y)$$

$$(B Y P)1(B P Y) = 1$$

$$(B Y P)(P W Y)(B P Y) = (B W P)$$

$$\begin{array}{l} W \leftarrow W \leftarrow P \leftarrow B \\ B \leftarrow P \leftarrow Y \leftarrow P \\ P \leftarrow Y \leftarrow W \leftarrow W \\ Y \leftarrow B \leftarrow B \leftarrow Y \end{array}$$

- Note: Shortcut

$$(gxg^{-1})^{-1} = (g^{-1})^{-1}x^{-1}g^{-1} = gx^{-1}g^{-1}.$$

Applying this to  $g = (B \ Y \ P)$  with  $x = (P \ W \ Y)$

Therefore, from the previous calculation,  $gx^{-1}g^{-1} = (gxg^{-1})^{-1} = (B \ P \ W)$ .

### Proposition: Geometric Meaning of Conjugate

Let  $G$  act on a set  $X$ ,  $x \in X$ ,  $g \in G$ , and define  $y := g * x$ .

Then for  $H = \text{Stab}_G(x)$ , we have

$$gHg^{-1} = \text{Stab}_G(y)$$

That is, the conjugate of a stabilizer is a stabilizer.

### Proof

( $\subseteq$ ) Let  $ghg^{-1} \in gHg^{-1}$  be arbitrary.

Then

$$(ghg^{-1}) * y = g * (h * (g^{-1} * y)) = g * (h * x) = g * x = y$$

Therefore  $ghg^{-1} \in \text{Stab}_G(y)$ .

( $\supseteq$ ) Let  $g' \in \text{Stab}_G(y)$  be arbitrary.

Consider  $g^{-1}g'g$ . Then

$$(g^{-1}g'g) * x = g^{-1} * (g' * (g * x)) = g^{-1} * (g' * y) = g^{-1} * y = x$$

Therefore  $h := g^{-1}g'g \in H$ . Then by multiplying  $g$  on the left and  $g^{-1}$  on the right, we get

$$g' = ghg^{-1} \in gHg^{-1}$$

### Orbit-Stabilizer Theorem and Lagrange

1. If  $G$  acts transitively on  $X$ , then all the stabilizers have the same cardinality because they are all conjugates.

So the Orbit-Stabilizer Theorem is consistent.

2. If  $X = G/H$ , then  $\text{Stab}_G(1H) = H$ . What about  $\text{Stab}_G(gH) = gHg^{-1}$ ?

**October 12, 2023**

### Recall: Conjugate

$h \in G$ ,  $H \leq G$ ,  $g \in G$

The conjugates  $ghg^{-1} = {}^g h$ ,  $gHg^{-1} = \{{}^g h \mid h \in H\}$ .

## Meaning

If  $G$  acts on  $X$ ,  $x \in X$ ,  $g \in G$ ,  $g * x =: y$ ,  $H = \text{Stab}_G(x)$ , then  $gHg^{-1} = \text{Stab}_G(y)$ .

- Example

$G = \text{Alt}(4)$ ,  $x = B$ ,  $H = \{1, (P W Y), (P Y W)\}$ ,  $g = (B P)(Y W)$   
gives  $gHg^{-1} = \text{Stab}_G(P) = \{1, (B W Y), (B Y W)\}$ .

## Definition: Cyclic Subgroup

The cyclic subgroup generated by  $g$  is given as  $\langle g \rangle = \{g^n \mid n \in \mathbb{Z}\} = \{\dots, g^{-2}, g^{-1}, 1, g, g^1, g^2, \dots\}$ .

## Definition: Normal Subgroup

Let  $H \leq G$ .

Say  $H$  is normal in  $G$  and write  $H \trianglelefteq G$  if for all  $g \in G$ ,  $gHg^{-1} = H$ .

Equivalently,  $gH = Hg$  for all  $g \in G$ .

## Trivial Example

If  $H = \{1\}$  or  $H = G$ , then  $H \trianglelefteq G$ .

## Example 1: Non-Normal Subgroup

$G = \text{Alt}(4)$ ,  $H = \text{Stab}_G(B)$  is not normal.

## Example 2: Alternating Group

Consider  $G = \text{Alt}(4)$ ,  $X = \{x_1, x_2, x_3\}$ ,  $H = \text{Stab}_G(x_1)$ .



Then we have the following facts.

- First: Transitivity  
 $G$  acts transitively.

- Second: Order of  $H$   
By the Orbit-Stabilizer Theorem,

$$|H||X| = |G| \implies |H| \cdot 3 = 12 \implies |H| = 4$$

- Third: Description of  $G$

We know

$$G = \left\{ 1, \begin{pmatrix} 8 \\ 3\text{-cycles} \end{pmatrix}, \begin{pmatrix} 3 \\ 2 + 2\text{-cycles} \end{pmatrix} \right\}$$

- Q: Can a 3-cycle belong to  $H$ ?

A: No. Lagrange's Theorem.

Suppose a 3-cycle  $g \in H$ . Then the cyclic subgroup  $\langle g \rangle = \{1, g, g^2\}$ , but  $\langle g \rangle \leq H$  and  $3 \nmid 4$ .

- Fourth: Description of  $H$

By Lagrange's Theorem,  $H \subseteq \left\{ 1, \begin{pmatrix} 3 \\ 2+2\text{-cycles} \end{pmatrix} \right\} \implies H = \left\{ 1, \begin{pmatrix} 3 \\ 2+2\text{-cycles} \end{pmatrix} \right\}$ .

- Fifth:  $H$  is a Normal Subgroup

Run the argument with  $x_1$  replaced with  $x_2$ , then  $\text{Stab}_G(x_2) = H = \text{Stab}_G(x_3)$ .

Therefore,  $H \trianglelefteq G$ .

### Example 3: Non-Normal

From Homework 1,  $\mathbb{P}_1(1)$ ,  $H = \{A \in \text{SL}_2\mathbb{C} \mid A^\dagger A = I_2\}$  is not normal.  
 $\mathbb{P}_1(2)$ ,  $S = \{A = A^\dagger\} \not\trianglelefteq G$ .

### Example 4: SL As Kernel

$$\text{SL}_n(\mathbb{R}) = \ker(\det \mid \text{GL}_n(\mathbb{R}) \rightarrow \mathbb{R}^\times) \trianglelefteq \text{GL}_n(\mathbb{R})$$

### Example 5: Alternating Group As Kernel

$$\text{Alt}(n) = \ker(\text{sign} \mid \text{Sym}(n) \rightarrow \{\pm 1\}) \trianglelefteq \text{Sym}(n)$$

- Proof

Let  $g \in G$  and  $k \in K$  be arbitrary.

We need to show that  $gkg^{-1} \in K$ .

$$\phi(gkg^{-1}) \underset{\phi \text{ homomorphism}}{=} \phi(g)\phi(k)\phi(g^{-1}) \underset{k \in K}{=} \phi(g)[\phi(g)]^{-1} = 1_H.$$

### Exercise 1

$$\text{Stab}_{\text{Alt}(4)}(1 \text{ of the 6 edges}) \stackrel{?}{\trianglelefteq} \text{Alt}(4).$$

### Exercise 2

Can you have a subset closed under conjugation?

On Homework 2 we will examine conjugate classes.

### Theorem: Quotient Group

Let  $N \trianglelefteq G$ . Then on  $G/N$ , we have a structure of a group, the quotient group, with the binary operation  $g_1N * g_2N := (g_1g_2)N$ .

The identity element  $1N = N$ .

The inverse  $(gN)^{-1} = g^{-1}N$ .

### Proof

The main difficulty is in demonstrating that  $*$  is well-defined.

That is, if  $g'_1, g'_2$  are other elements such that (1)  $g'_1N = g_1N$  and (2)  $g'_2N = g_2N$ , then we need to show that  $(g_1g_2)N = (g'_1g'_2)N$ .

But (1) means that  $g'_1 = g_1n_1$  for some  $n_1 \in N$ , while (2) similarly means  $g'_2 = g_2n_2$  for  $n_2 \in N$ .

It follows that  $g'_1g'_2 = g_1n_1g_2n_2$ .

Recall that  $N \trianglelefteq G$  implies  $Ng_2 = g_2N$ , so  $n_1g_2 = g_2n_3$  for some  $n_3 \in N$ .

Therefore  $g_1n_1g_2n_2 = g_1g_2n_3n_2 \in N$ .

Finally, multiplying  $N$  on the right side gives  $(g_1g_2)N = (g'_1g'_2)N$ .

- Associativity

Proof of associativity is left as an exercise.

### Remark

If  $H \leq G$  and  $g_1H * g_2H \stackrel{\text{def}}{=} g_1g_2H$  defines (well) a group, then  $H \trianglelefteq G$ .

### Proposition: Kernel Is Normal

Let  $G \xrightarrow{\phi} H$  be a group homomorphism, then  $K := \ker(\phi) = \{g \in G \mid \phi(g) = 1_H\}$  is a normal subgroup of  $G$ .

### Proposition: Abelian Subgroups are Normal

If  $G$  is abelian, and  $H \leq G$ , then  $H \trianglelefteq G$ .

### Proof

$$ghg^{-1} = h$$

### Proposition: Subgroup of Index 2

Let  $H \leq G$  of index 2 (i.e.  $[G : H] = |G/H| = 2$ ), then  $H \trianglelefteq G$ .

### Proof

Let  $g \in G$  be arbitrary.

Need to show that  $ogHg^{-1} = H$  or, equivalently,  $gH = Hg$ .

If  $g \in H$ , there is nothing to prove. So assume  $g \notin H$ .

But both  $gH$  and  $Hg$  are the (set) complement of  $H$  in  $G$ .



- Example

$$\text{Alt}(n) \trianglelefteq \text{Sym}(n)$$

If  $n = 1$ , there is nothing to prove.

$$\text{If } n \geq 2, \text{ then } |\text{Alt}(n)| = \frac{n!}{2} = \frac{|\text{Sym}(n)|}{2}.$$

### Definition: Simple Group

A group  $G$  is simple if  $G$  has exactly 2 normal subgroups, namely  $\{1\} \neq G$ .

### Non-Example

$$G = \text{Alt}(4) \supseteq H = \text{Stab}_G(x_1), \quad H \neq \{1\} \text{ or } G$$

So  $G$  is not simple.

### Proposition: Group of Prime Order

Let  $p$  be a prime number.

Any group  $G$  of order  $p$  is both cyclic and simple.

$$\mathbb{Z}/p\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \dots, \overline{p-1}\} \text{ with } +.$$

- Proof

Suppose  $g \in G$  is not the identity.

$$\text{Form } \langle g \rangle = \{g^n \mid n \in \mathbb{Z}\} =: H \leq G.$$

By Lagrange's Theorem,  $|H||G| = p$ . Since the only positive divisors of  $p$  are 1 and  $p$ ,  $\langle g \rangle = H = G$  for any  $g \neq 1_G$ .

If  $H \leq G$  is any normal subgroup,  $|H| \mid |G|$  so  $H = \{1\}$  or  $H = G$ .

### Remarks\*

For all  $n \geq 5$ ,  $\text{Alt}(n)$  is a simple group.

This may be asked as a homework exercise.

If  $p \geq 5$  is a prime, then  $\text{SL}_2(\mathbb{F}_p)/\{\pm I_2\}$  is a simple group.

Relatedly, if  $n \geq 4$  then  $\text{SL}_n(\mathbb{F}_p)/Z(\text{SL}_n(\mathbb{F}_p))$  is simple.

The point is that there are infinitely many finite simple groups.

Circa 1980, classification of finite simple groups was announced.

Requires  $\sim 10^4$  pages (not everyone has been convinced).

### Definition: Center

Let  $G$  be a group.

The center  $Z(G) = \{z \in G \mid zg = gz, \forall g \in G\}$  is a normal subgroup ( $gzg^{-1} = z, \forall g, z$ ).

## More Group Action Terminology

Let  $G$  act on a set  $X$ .

Recall that this corresponds to a group homomorphism  $G \xrightarrow{\rho} \text{Sym}(X)$ .

### Definition: Faithful

Say the action is faithful if  $\ker(\rho) = \{1_G\}$ .

$$\forall g \in G : gx = x, \forall x \in X \implies g = 1_G$$

### Definition: Free

Say the action is free if  $\forall g \in G, \forall x \in X, gx = x \implies g = 1_G$ .

$$\text{Stab}_G(x) = \{1_G\}, \forall x \in X$$

### Example 1

Let  $G$  act on  $X = G$  by left multiplication. Then this is free.

### Example 2

$$G = \text{Alt}(4)$$

$X_1 = \{B, P, W, Y\}$  is faithful but not free.

$X_2 = \{6 \text{ edges}\}$  is faithful but not free.

$X_3 = \{3 \text{ strings}\}$  is not faithful.

## October 17, 2023

### Recall: Normal Subgroup

A subgroup  $N \leq G$  is normal &  $N \trianglelefteq G$ , if  $gNg^{-1} = N$  for all  $g \in G$ .

### Notation

$$\begin{aligned} G &\xrightarrow{\pi} G/N \\ g &\mapsto gN \end{aligned}$$

### Recall: Quotient Group

We have constructed, for  $N \trianglelefteq G$ , the quotient group  $G/N$ , with  $g_1N * g_2N = g_1g_2N$ .

### Definition: Group Homomorphisms

For groups  $G, K$ ,  $\text{Hom}(G, K) = \{G \rightarrow K \mid \text{group homomorphisms}\}$ .

## Theorem: Universal Mapping Property of the Quotient Group

Let  $H$  be any group.

### Part 1

For any group homomorphism  $\bar{f} : G/N \rightarrow H$ , by composing, we get a homomorphism

$$\bar{f} \circ \pi = f : G \xrightarrow{\pi} G/N \xrightarrow{\bar{f}} H$$

such that  $f(N) = \{1_H\}$ .

- Proof

Let  $\bar{f} \in \text{Hom}(G/N, H)$  and  $f := \bar{f} \circ \pi \in \text{Hom}(G, H)$ .

Then

$$f(n) = \bar{f}(\pi(n)) = \bar{f}(nN) = \bar{f}(N) \underset{N \in G/N \text{ is the identity}}{=} 1_H$$

### Part 2

Conversely, for any group homomorphism  $f : G \rightarrow H$  such that  $f(N) = \{1_H\}$  we get a unique homomorphism  $\bar{f} : G/N \rightarrow H$  such that  $f = \bar{f} \circ \pi$ .

- Proof

- Well Defined

Let  $f \in \text{Hom}(G, H)$  satisfy  $f(N) = \{1_H\}$ .

Define  $\bar{f}(gN) = f(g)$  for every  $g \in G$ .

Need to Show: If  $g_1N = g_2N$ , then  $f(g_1) = f(g_2)$ .

But  $g_1N = g_2N$  implies, since  $g_1 1_G \in g_1N$ , that  $g_1 = g_2n$  for some  $n \in N$ .

Since  $f$  is a group homomorphism, we have

$$f(g_1) = f(g_2)f(n) \underset{f(N)=\{1_H\}}{=} f(g_2)1_H = f(g_2)$$

- Homomorphism

$\bar{f}$  is a homomorphism since for any  $g_1N, g_2N \in G/N$ , we have

$$\bar{f}(g_1N * g_2N) \underset{\text{definition of } *}{=} \bar{f}(g_1g_2N) \underset{\text{definition of } \bar{f}}{=} \bar{f}(g_1g_2) \underset{f \text{ is homomorphism}}{=} f(g_1)f(g_2) \underset{\text{definition of } \bar{f}}{=} \bar{f}(g_1N)\bar{f}(g_2N)$$

- Uniqueness

Suppose  $l \in \text{Hom}(G/N, H)$  is any element satisfying  $f = l \circ \pi$ .

Then for any  $g \in G$ , we have

$$\bar{f}(gN) = f(g) = l(\pi(g)) = l(gN)$$

Therefore  $\bar{f} = l$ .



## Rephrase: Universal Mapping Property of Quotient Group

$$\text{Hom}(G/N, H) \xrightarrow{1} \{f \in \text{Hom}(G, H) \mid f(N) = \{1_H\}\}$$

Equivalently,  $f(N) = \{1_H\} \iff N \leq \ker(f)$ .

Note:  $f(N) = \{1_H\} \iff N \subseteq f^{-1}(\{1_H\}) = \ker(f)$

## Loosely: Universal Mapping Property of Quotient Group

Giving a homomorphism  $G/N \rightarrow H$  is the same as giving a homomorphism  $G \rightarrow H$  that kills  $N$ .

## Definition: Group Generators

Let  $G$  be a group and let  $S \subseteq G$ .

## Definition: Word

A word in  $S$  of length  $l \geq 1$  is an expression  $x_1 x_2 \cdots x_l$  where  $x_i \in \{s, s^{-1}\}$  for some  $s \in S$  and  $i = 1, \dots, l$ .

The word of length zero is, by convention,  $1_G$ .

## Definition: Generated Subgroup

The subgroup generated by  $S$ ,  $\langle S \rangle$  is the subset of  $G$  consisting of all the words in  $S$  of all possible lengths.

Fact:  $\langle S \rangle \leq G$ .

- Example

$$S = \{A, C, I, T, L\}.$$

One word in  $S$  of length 3 is  $CA^{-1}T$ .

Inverse:  $(CA^{-1}T) = T^{-1}AC^{-1}$ .

Composition:  $CA^{-1}T * T^{-1}A^{-1}IL = CA^{-1}TT^{-1}A^{-1}IL = CA^{-2}IL$ . (Note that, strictly,  $A^{-2} \notin S$  but rather  $A^{-2} \equiv A^{-1}A^{-1}$ ).

## Common Usage

We usually do not use the bare definition to describe what  $\langle S \rangle$ .

- Example

Let  $G = \text{Sym}(n)$  and let  $S = \{\text{transpositions } (a \ b) \mid 1 \leq a < b \leq n\}$ .

Then  $\langle S \rangle = G$ .

- Proof

Step 1: Any  $\pi \in G$  has a cycle decomposition. So  $\pi = \gamma_1 \gamma_2 \cdots \gamma_k$ , where  $\gamma_i$  is an  $l_i$ -cycle for some  $l_i \geq 1$ .

Step 2: If  $\gamma$  is an  $l$ -cycle,  $l \geq 2$ , say  $\gamma = (a_1 \ a_2 \ \cdots \ a_l)$ , then  $\gamma \stackrel{\text{HW1}}{=} (a_1 \ a_2)(a_2 \ a_3) \cdots (a_{l-1} \ a_l)$ .

## Definition: Commutator

A commutator in  $G$  is an element of the form  $[g, h] := ghg^{-1}h^{-1}$  for some  $g, h \in G$ .

**Definition: Commutator Subgroup**

The commutator subgroup (or derived subgroup)  $D(G)$  (sometimes  $[G, G]$ ) is the subgroup of  $G$  generated by all the commutators.

That is, a typical element in  $D(G)$  is  $C_1 C_2 \cdots C_l$  where  $C_i = [g_i, h_i]$  for some  $g_i, h_i \in G, \forall i$ .

Note that  $[g, h]^{-1} = (ghg^{-1}h^{-1})^{-1} = hgh^{-1}g^{-1} = [h, g]$ .

**Proposition: Commutator Subgroup Is Normal**

$$D(G) \trianglelefteq G.$$

**Lemma:**

Let  $g_0 \in G$  be fixed.

The map  $G \rightarrow G$  defined as  $g \mapsto g_0 g g_0^{-1}$  – called the inner homomorphism and denoted  $\text{Int}(g_0)$  – is a group homomorphism.

- Proof

$$\text{Int}(g_0)(g_1 g_2) = g_0 g_1 g_2 g_0^{-1} = g_0 g_1 g_0^{-1} g_0 g_2 g_0^{-1} = [\text{Int}(g_0)(g_1)][\text{Int}(g_0)(g_2)]$$

**Proof**

Let  $g_0 \in G$  and  $C_1 \cdots C_l \in D(G)$  be arbitrary.

$$g_0(C_1 \cdots C_l)g_0^{-1} = \text{Int}(g_0)(C_1 \cdots C_l) = \underset{\text{lemma}}{=} [\text{Int}(g_0)(C_1)] \cdots [\text{Int}(g_0)(C_l)]$$

It suffices to prove that  $\text{Int}(g_0)(C)$  is a commutator for any commutator  $C$ .

$$\text{Int}(g_0)(C) = \text{Int}(g_0)(ghg^{-1}h^{-1}) = [\text{Int}(g_0)(g), \text{Int}(g_0)(h)] \quad \blacksquare$$

**Definition: Abelianization**

The quotient group  $G/D(G)$  is called the Abelianization  $G^{\text{ab}}$  of  $G$ .

**Property: Commutativity**

$G^{\text{ab}}$  is commutative.

- Proof

Let  $D = D(G)$  and let  $gD, hD \in G^{\text{ab}}$  be arbitrary.

Then

$$[gD, hD] = (gD)(hD)(gD)^{-1}(hD)^{-1} = ghg^{-1}h^{-1}D = D = 1_{G^{\text{ab}}}$$

Therefore, by multiplying on the right by  $(hD)(gD)$ , we get

$$(gD)(hD) = (hD)(gD)$$

So  $G^{\text{ab}}$  is Abelian.

## Note: Abelian Groups

$G$  is Abelian if and only if  $D(G) = \{1_G\}$ .

## Proposition: Minimal Normal Subgroup with Commutative Quotient

$D(G)$  is the smallest normal subgroup  $N \trianglelefteq G$  such that  $G/N$  is commutative.

### Proof

Suppose  $N \trianglelefteq G$  has Abelian  $G/N$ .

Need to Show:  $D(G) \leq N$ .

So let  $g, h \in G$  be arbitrary and let  $C = ghg^{-1}h^{-1}$ .

Then in  $G/N$ ,  $[gN, hN] = 1_{G/N} = N$ .

Therefore  $[g, h] \in N$  and  $[G, G] \leq N$ . ■

## Remark\*: Homology in Algebraic Topology

Let  $X$  be a connected manifold, say

IMAGE HERE - DOUBLE TORUS WITH CHUNK BEING REMOVED

In Algebraic Topology, we study loops up to homotopy  $G = \pi_1(X, x_0)$ .

Then  $G^{\text{ab}} = \underset{\text{Hurewicz}}{H_1(X; \mathbb{Z})}$  is a homology.

## Theorem: Universal Mapping Property of Abelianization

Let  $G$  be any group and  $H$  be any Abelian group.

Then we have a bijection  $\text{Hom}(G, H) \cong \text{Hom}(G^{\text{ab}}, H)$ .

### Proof

Since  $H$  is Abelian, any homomorphism  $f : G \rightarrow H$  will satisfy  $f([G, G]) = \{1_H\}$ .

So use UMP for quotient  $G/N$ . ■

October 19, 2023

## Review: Commutator

Let  $G$  be a group.

A commutator is an element of the form  $[g, h] = ghg^{-1}h^{-1}$ .

The derived (or commutator) subgroup  $DG = [G, G]$  is the subgroup generated by all the commutators.

## Properties of Commutators

$DG \trianglelefteq G$

$G^{\text{ab}} := G/DG$  is abelian.

$G^{\text{ab}}$  satisfies the universal mapping property. That is, for all abelian  $H$ .

$$\text{Hom}(G, H) \cong \text{Hom}(G^{\text{ab}}, H)$$

### Example: G Abelian

If  $G$  is abelian, then  $D(G) = \{1_G\}$  and  $G \xrightarrow{\sim} G^{\text{ab}} = G/\{1_G\}$  (usually written  $G = G^{\text{ab}}$ ).  
 Conversely,  $DG = \{1_G\}$  implies  $G$  is abelian.

### Example: G Simple and Non-abelian

If  $G$  is simple and non-abelian, then  $DG = G$ .

### Definition: Perfect Group

A group  $G$  is called perfect if  $D(G) = G$ .

cf. Poincaré Conjecture (Perelman circa 2002: Every closed 3-manifold  $M$  with  $\pi_1(M) = \{1\}$  is homeomorphic to  $S^3$ ).

### Example: G Symmetric Group 3

Let  $G = \text{Sym}(3)$ .

By Lagrange, if  $H \leq G$ , then  $|H| \mid |G| = 3! = 6$ . Consider

$$|H| = 1, 2, 3, \text{ or } 6$$

Then



Which one is  $DG$ ?

It cannot be  $\{1\}$  since  $G \neq G^{\text{ab}}$ .

It cannot be  $\langle \tau \rangle$ ,  $\tau^2 = 1$ , since they are not normal.

$G/\langle (1\ 2\ 3) \rangle$  is a group of order 2 and, therefore, abelian.

### Recall

$DG$  is the smallest normal subgroup  $N$  of  $G$  such that  $G/N$  is abelian.

Therefore,  $DG = D(\text{Sym}(3)) = \langle (1\ 2\ 3) \rangle$ .

### Note

For  $G = \text{Sym}(3)$ ,

$$G \supsetneq_{\neq} D(G) = \langle (1\ 2\ 3) \rangle \supsetneq_{\neq} D(D(G)) = \{1\}$$

## Definition: Solvable Group

Suppose  $G$  is a group such that

$$D(D(D(\dots D(G)\dots)) = \{1\}$$

Then  $G$  is solvable (or soluble).

## Definition: Conjugacy Class

The conjugacy class of  $\bar{x} \in G$  is the set of elements in  $G$  that are conjugate to  $\bar{x}$ .

Let  $G$  act on (the set)  $G = X$  by conjugation:  $g * x = gxg^{-1}$ .

Then the conjugacy class of  $x$  is the  $G$ -orbit of  $x \in X$ .

## Definition: Centralizer

The centralizer  $x \in G$  is the stabilizer of  $x$  in this conjugation action.

$$C_G(x) = \{g \in G \mid gxg^{-1} = x\} \iff gx = xg$$

## Example: G Alternating Group 4

Let  $G = \text{Alt}(4)$ .

Then, by Lagrange,  $H \leq G \implies |H| \mid |G| = 12$ .

So  $|H| = 1, 2, 3, 4, 6$ , or  $12$



Note:  $\text{Stab}(\text{String})$  refers to the stabilizer of  $x_1$  as depicted in Example 2 on page 19.

## Proposition: No Order 6 Subgroup

No subgroup  $H$  of order 6 in  $\text{AAlt}(4)$ .

- Proof

Suppose  $H \leq G$  has order six.

Then  $[G : H] = \frac{12}{6} = 2$  and, therefore,  $H \trianglelefteq G$ .

So  $H$  must be a union of certain conjugacy classes.

From Homework 1,  $G$  acts transitively on  $\{6 \text{ edges}\}$ , so  $\text{Stab}(\overline{BP})$ ,  $\text{Stab}(\overline{BW})$ , and  $\text{Stab}(\overline{BY})$  are conjugate to each other (HW2 P1(b)).

Therefore the 3 elements of order 2,  $T = \{(B P)(W Y), (B W)(P Y), (B Y)(P W)\}$  form a conjugacy class in  $G$

It follows that  $G = \{1\} \sqcup T \sqcup \{8 \text{ elements of order } 3\}$ .

By the following lemma, this resolves to

$$G = \{1\} \sqcup T \sqcup \{p\}^\# \sqcup \{p^2\}^\#$$

- Lemma (Key)

Every element  $p$  of order 3 in  $G = \text{Alt}(4)$  has conjugacy class of size 4.

– Proof

$|(\text{conjugacy class of } p)| \geq 4$  since  $G$  acts transitively on  $\{B, P, W, Y\}$  and  $\exists p \in \text{Stab}(P)$  such that, without loss of generality,  $gpg^{-1} = \text{Stab}(B)$ .

Since  $p = 1$  implies  $gpg^{-1} = 1$ , each  $g$  sends non-trivial  $p$  to either  $(P W Y)$  or  $(P Y W)$  in  $\text{Stab}(B)$ . Then, also

$$|(\text{conjugacy class of } p)| = |\text{orbit of } p \text{ under the conjugation action}| = |G|/|C_G(p)| \leq 12/3 = 4$$

Therefore no normal group of order 6 is in  $G = \text{Alt}(4)$ .

### Example Continued: 6 Alternating Group 4

$\{1\}$  is non-abelian.

$\text{Stab}(\overline{BP})$  inhabits a conjugate class.

Because  $|G/\text{Stab}(\text{String})| = 3$  it is cyclic and  $DG = \{1\} \coprod T$

Observe that  $DG$  is abelian.

#### Proof

What to Show: For  $t = (B P)(W Y)$  and every  $t \in T$ ,  $C_G(t) \supseteq DG$ .

Then

$$|C_G(T)| \stackrel{OST}{=} \frac{|G|}{|T|} = \frac{12}{3} = 4$$

Therefore,  $C_G(t) = DG$ .

### Remark: Enumerability of Subgroups

There are very few groups  $G$  such that we can enumerate all the subgroups of  $G$ .

### Theorem: Product of Integers is a Subgroup

Let  $G = (\mathbb{Z}, +)$ .

1. For any  $a \in \mathbb{Z}$ , the subset  $a\mathbb{Z} = \{an \mid n \in \mathbb{Z}\}$  is a subgroup of  $G$ .

2. Conversely, any subgroup  $H \leq \mathbb{Z}$  is of the form  $H = a\mathbb{Z}$  for some  $a \in G$ .

#### Proof of 1

Need to Show:  $a\mathbb{Z}$  contains 0, is closed under  $+$  and is closed under (additive) inverse.

But,  $0 = a0$ , and  $an + am = a(n + m)$  and  $-(an) = a(-n)$ , for  $n, m \in \mathbb{Z}$ .

## Proof of 2

Suppose  $H \leq (\mathbb{Z}, +)$ .

Since  $+$  is commutative, all subgroups will be normal.

If  $H = \{0\}$ , then  $h = 0\mathbb{Z}$ , and we're done.

If not, say  $H \ni x \neq 0$ , then  $h \ni -x$ .

Therefore  $S = \{x \in H \mid x > 0\}$  must be non-empty.

Let  $a$  be the smallest element in  $S$ .

Then we claim  $H = a\mathbb{Z}$ .

- Proof of the Claim

( $\supseteq$ ) Since  $H \ni a$  and  $H \leq G$ .

( $\subseteq$ ) Let  $x \in H$  be arbitrary. Apply Division Algorithm and get

$$x = aq + r$$

where  $q, r \in \mathbb{Z}$  and  $0 \leq r < a$ .

If  $r > 0$ , then

$$r = x - aq \in H$$

so  $r \in S$ .

However, since  $r < a$ , this contradicts the very choice of  $a$  as the smallest element.

So  $r = 0$ .

But then  $x = aq \in a\mathbb{Z}$ .

Therefore  $H \subseteq a\mathbb{Z}$ . ■

## Homework 2 Question

If  $H \leq G$ , then  $gHg^{-1} \leq G$ .

Consider the set  $X_G$  of all the subgroups  $H$  of  $G$ .

The group  $G$  acts on  $X_G$  by conjugation:  $g * H := gHg^{-1}$ .

## Definition: Normalizer

The normalizer of  $H$  in  $G$  is  $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$ .

## October 24, 2023

## Isomorphism Theorems

### Definition: Isomorphism

An isomorphism between two groups,  $G$  and  $H$ , is a group homomorphism  $f : G \rightarrow H$  that is bijective.

Two groups are said to be isomorphic if there is an isomorphism between them.

### Example

Let  $G$  be any group,  $N = \{1_G\}$ , and let  $\pi : G \rightarrow G/N$  be the canonical projection.

Then  $gN = \{g\}, \forall g \in G$ .

So  $\pi$  is an isomorphism.

e.g. for  $G = \{1, a, b, c\}$

$G/\{1_G\} = \{\{1\}, \{a\}, \{b\}, \{c\}\}$

1 -  $\{1\}$

a -  $\{a\}$

b -  $\{b\}$

c -  $\{c\}$

Here we will write  $G = G/\{1_G\}$ .

### Lemma:

Let  $G \xrightarrow{\phi} H$  be a group homomorphism.

Suppose  $\phi$  is surjective. Form  $K = \ker(\phi) \leq G$ .

Then we have an isomorphism, induced by  $\phi$ :

$$\bar{\phi} : G/K \rightarrow H$$

### Proof

Giving  $\bar{\phi}$  is the same as giving a homomorphism  $G \xrightarrow{\psi} H$  such that  $\psi(K) = \{1_H\}$  by the Universal Mapping Property.

So take  $\psi = \phi$ .

So have a group homomorphism  $\bar{\phi}(gK) = \phi(g), \forall g \in G$ .

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ & \searrow \pi & \nearrow \bar{\phi} \\ & K & \end{array}$$

### Set Theory

$\phi$  surjective  $\implies \bar{\phi}$  surjective.

Remains to show:  $\bar{\phi}$  is injective.

Let  $g_1K, g_2K \in G/K$  such that  $\bar{\phi}(g_1K) = \bar{\phi}(g_2K)$ .

Then, by definition of  $\bar{\phi}$ ,

$$\phi(g_1) = \phi(g_2)$$

So

$$\phi(g_1g_2^{-1}) \underset{\phi \text{ homomorphism}}{=} \phi(g_1\phi(g_2^{-1})) = 1_H.$$

Therefore  $g_1g_2^{-1} \in \ker(\phi) = K$  and  $g_1K = g_2K$ . ■



**Definition: Normalizer (Revisited)**

Let  $H \leq G$ . The normalizer  $N_G(H)$  is  $\{g \in G \mid gHg^{-1} = H\}$ .

**Note: Normalizer is a Stabilizer**

It is the stabilizer of  $H$  in the conjugation action of  $G$  on the set of all subgroups of  $G$ .

Since  $\forall h \in H$ , we have  $hHh^{-1} = H$ , we have  $H \leq N_G(H)$ .

Since  $N_G(H)$  is a stabilizer,  $N_G(H) \leq G$ .

Therefore  $H \leq N_G(H) \leq G$ .

$H \leq N_G(H) \leq G$ .

**Note: H is Normal If the Normalizer of H is G**

$H \trianglelefteq G \iff N_G(H) = G$ .

**Example 1**

Let  $G = \text{Alt}(4)$ ,  $K_1 = \text{Stab}_G(B) = \langle (P \ W \ Y) \rangle$ , and  $H_1 = \langle (B \ P)(W \ Y) \rangle$ .

Question: What are  $N_G(H_1)$  and  $N_G(K_1)$ ?

- H1

$H_1$  is not normal in  $G$ , so  $N_G(H_1) \neq G$ .

Since  $H_1 \leq N_G(H_1) \leq G$ .

Recall that  $H_1 \leq \text{Stab}_G(\text{String}) \leq G$  and  $\text{Stab}_G(\text{String})$  is abelian.

Therefore  $N_G(H_1) = \text{Stab}_G(\text{String})$ .

Note also:  $N_G(\text{Stab}_G(\text{String})) = G$ .

- K1

From homework,  $\forall g \in G$ ,  $gK_1g^{-1} = \text{Stab}_G(gB)$  and, if  $gB \neq B$ , then  $\text{Stab}_G(gB) \neq K_1$ .

So  $K_1 \leq N_G(K_1) \subseteq \text{Stab}_G(B) = K_1$ .

That is,  $N_G(K_1) = K_1$ . It is a “self-normalizer.”

**Example 2**

Let  $G = \text{GL}_2(\mathbb{R})$ ,  $H = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}$ .

What is  $N_G(H)$ ?

Say  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  belongs to  $N_G(H)$ .

Then  $g \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix} g^{-1} \in H$ . Say

$$g \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} g^{-1} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

for some  $x \in \mathbb{R}$ . Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} a & b+3a \\ c & d+3c \end{pmatrix} = \begin{pmatrix} a+xc & b+xd \\ c & d \end{pmatrix}$$

Therefore,  $c = 0$  (and  $x = \frac{3a}{d}$ ). So

$$N_G(H) \leq \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{R}^+, b \in \mathbb{R} \right\}$$

Are they equal?

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{-1} = \begin{pmatrix} a & b+ax \\ 0 & d \end{pmatrix} \frac{1}{ad} \begin{pmatrix} d & -b \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & ? \\ 0 & 1 \end{pmatrix} \in H$$

Therefore, for  $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ ,  $gHg^{-1} \in H$ .

Now, apply the same argument to  $g^{-1} = \begin{pmatrix} 1/a & -b/ad \\ 0 & 1/d \end{pmatrix}$  and get  $g^{-1}Hg \subseteq H$ .

That is to say,  $H \subseteq gHg^{-1}$ .

It follows that  $(A) + (B) \implies g \in N_G(H)$ , and

$$N_{\text{GL}_2(\mathbb{R})} \left( \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\} \right) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{R}^+, b \in \mathbb{R} \right\}$$

### Definition: Product

Let  $H, K \leq G$ .

Then  $HK := \{hk \mid h \in H, k \in K\}$

$HK$  need not be a subgroup, and may differ from  $KH$ .

### Example

Let  $G = \text{Alt}(4)$ ,  $H = \langle (B P)(W Y) \rangle$ , and  $K = \langle (P W Y) \rangle$ .

Note  $|H| = 2$  and  $|K| = 3$ . Then

$$HK = \{1 \cdot 1, 1(P W Y), 1(P Y W), (B P)(W Y)1, (B P)(W Y)(P W Y), (B P)(W Y)(P Y W)\}$$

### Fact

If  $H, K \leq G$  such that  $H \cap K = \{1_G\}$  and  $h_1, h_2 \in H$  and  $k_1, k_2 \in K$ , then

$$h_1 k_1 = h_2 k_2 \implies h_1 = h_2 \text{ and } k_1 = k_2$$

### Proof

Multiply  $h_2^{-1}$  on the left and  $k_1^{-1}$  on the right. Then

$$H \ni h_2^{-1} h_1 = k_2 k_1^{-1} \in H \quad \blacksquare$$

Therefore  $|HK| = 6$ .

Since  $G$  contains no subgroup of order 6,  $HK \not\leq G$ .

## Theorem: First Isomorphism Theorem

Note: Dummit and Foote lists this as the Second Isomorphism Theorem.  
Let  $H, K \leq G$  and assume that  $H \leq N_G(K)$ ,

1.  $HK = KH$  is a subgroup of  $G$ ,
2.  $K \trianglelefteq HK$  and  $H \cap K \trianglelefteq H$ , and
3.  $HK/K \cong H/H \cap K$  is an isomorphism of (quotient) groups.

### Proof of 1

- Equality

Need to prove  $HK \subseteq KH$  and  $KH \subseteq HK$ .

( $\implies$ ) Let  $h \in H$  and  $k \in K$ . We want  $hk \in KH$ .

Since  $h \in N_G(K)$ ,  $hKh^{-1} = K$ .

So, in particular,  $hkh^{-1} \in K$  for some  $k_2 \in K$ . Multiplying by  $h$  on the right,

$$hk = k_2h \in KH$$

( $\impliedby$ ) Let  $kh \in KH$ .

Since  $H \in N_G(K)$  and  $N_G(K)$  is a subgroup,  $h^{-1} \in N_G(K)$ .

So  $h^{-1}K(h^{-1})^{-1} = K$ .

In particular,  $h^{-1}kh = k_3$  for some  $k_3 \in K$ . So

$$kh = h_{k_3} \in HK$$

- Subgroup

Identity:  $1 = 1 \cdot 1 \in HK$

Product: Let  $h_1k_1$  and  $h_2k_2$  be arbitrary. Then for some  $h_4 \in H, k_4 \in K$

$$(h_1k_1)(h_2k_2) = h_1(k_1h_2)k_2 \stackrel{KH=HK}{=} h_1(h_4k_4)k_2 = \overset{\in H}{(h_1k_4)}\overset{\in K}{(k_4k_2)} \in HK$$

So  $HK$  is closed under the group operation.

Inverse:  $(hk)^{-1} = k^{-1}h^{-1} \in KH = HK$ .

### Proof of 2

- $K$  is Normal in  $HK$

Let  $hk \in HK$  be arbitrary.

Since  $h \in H \leq N_G(K)$ ,  $k \in K \leq N_G(K)$ , and  $N_G(K)$  is a subgroup,

$$hk \in N_G(K)$$

Therefore  $HK \leq N_G(K)$  and  $K$  is normal in  $HK$ .

- $H$  Meet  $K$  is Normal in  $H$

For  $h \in H$ ,  $hKh^{-1} = K$  by assumption, and  $hHh^{-1} = H$ .

Therefore  $h(H \cap K)h^{-1} = H \cap K$ , and  $K \cap H$  is normal in  $H$ .

### Proof of 3

Let us start with the inclusion homomorphism

$$\begin{array}{ccccc} h & \longmapsto & h \cdot 1 & \longmapsto & hK \\ H & \longrightarrow & HK & \longrightarrow & HK/K \\ & \searrow \phi & & \nearrow & \end{array}$$

$\phi$  is surjective.

A typical element in  $HK/K$  is  $hK = hK$  which is the image of  $h \in H$  under  $\phi$ .

Then  $\ker(\phi) = \{h \in H \mid hK = 1_K\} \iff h \in K$ . This is  $H \cap K$ .

Therefore, by lemma,  $H/H \cap K \xrightarrow[\sim]{\bar{\phi}} HK/K$ .

### Example

Let  $G = \text{GL}_3(\mathbb{R})$  and  $H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\} . H \leq G$  called the Heisenberg group of dimension .

**October 26, 2023**

### Heisenberg Groups

$$G = \text{Heis}_3(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\} \leq \text{GL}_3(\mathbb{R})$$

### Subgroup

1.  $I_3 \in G$  since we can take  $x = y = z = 0$ .

2. (Empty elements are read as zeros.)

$$\underbrace{\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix}}_{=g} \underbrace{\begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix}}_{=h} \begin{pmatrix} 1 & x+a & z+c+xb \\ & 1 & y+b \\ & & 1 \end{pmatrix}$$

So closed under matrix multiplication.

Note: (the  $(1,2)$ -component of  $gh$ ) = (the  $(1,2)$ -component of  $g$ ) + (the  $(1,2)$ -component of  $h$ ).

Ditto for the  $(2,3)$ -component.

1.  $G$  is closed under inverse.

Let  $a = -x$ ,  $b = -y$  and  $c = -z + xy$ . Then

$$\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -x & xy - x \\ & 1 & -y \\ & & 1 \end{pmatrix} \in G$$

### Example

Identify  $\mathbb{R}^3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\}$  with  $G$  and let  $G$  act on  $\mathbb{R}^3 = X$  by left multiplication. i.e.

$$g = \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightsquigarrow g * \vec{v} = \begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} = \dots$$

For example, if

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \leftrightarrow I_3 \quad \text{and} \quad g = \begin{pmatrix} 1 & 0 & 0 \\ & 1 & z \\ & & 1 \end{pmatrix}$$

then  $g$  maps  $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$  to  $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$  while

$$h = \begin{pmatrix} 1 & 3 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} \quad \text{maps} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}.$$

However, since this is a noncommutative group

$$\begin{aligned} gh &= \begin{pmatrix} 1 & 3 & 0 \\ & 1 & 2 \\ & & 1 \end{pmatrix} \quad \text{maps} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \\ hg &= \begin{pmatrix} 1 & 3 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ & 1 & 2 \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 6 \\ & 1 & 2 \\ & & 1 \end{pmatrix} \end{aligned}$$

So  $hg$  maps  $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$  to  $\begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$ .

So  $G$  acts on  $\mathbb{R}^3$  transitively and freely.

This is one of the eight “model geometries” of William Thurston.

## Heisenberg Group Center and Derived Group

### Recall

The center

$$Z(G) = \{g \in G \mid gh = hg, \forall h \in G\}$$

and the derived group

$$D(G) = \langle \{[g, h] \mid g, h \in G\} \rangle.$$

### Center of Heisenberg Group

If

$$g = \begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix} \in Z(G),$$

then for any  $x, y, z \in \mathbb{R}$ , we have

$$\begin{aligned} \begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix} \overbrace{\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix}}^{=h} &= \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & a+x & c+z+ay \\ & 1 & b+y \\ & & 1 \end{pmatrix} &= \begin{pmatrix} 1 & x+a & z+c+xb \\ & 1 & y+b \\ & & 1 \end{pmatrix} \end{aligned}$$

Therefore  $ay = xb$  for all  $x, y, z \in \mathbb{R}$ .

Take  $(x, y) = (1, 0)$ , and we get  $0 = b$ .

Take  $(x, y) = (0, 1)$ , and we get  $a = 0$ .

It follows that

$$Z(G) = \left\{ \begin{pmatrix} 1 & 0 & c \\ & 1 & 0 \\ & & 1 \end{pmatrix} \mid c \in \mathbb{R} \right\}$$

### Derived Group of Heisenberg Group

$[g, h] = ghg^{-1}h^{-1}$  has

$$(1, 2) - \text{component} = a + x + (-a) + (-x) = 0$$

$$(2, 3) - \text{component} = b + y + (-b) + (-y) = 0$$

So  $D(G) \leq Z(G)$  for this  $G$ .

- Q: Is the center a subgroup of the derived group?

Form

$$\begin{aligned} G/Z(G) &\xrightarrow[\phi]{\sim} (R^2, +) \\ \begin{pmatrix} 1 & a & 0 \\ & 1 & b \\ & & 1 \end{pmatrix} Z(G) &\longmapsto (a, b) \end{aligned}$$

$\phi$  is a group homomorphism since the  $(1,2)$ -component and  $(2,3)$  component of  $gh$  are the sum of those components in  $g$  and  $h$ .

$\phi$  is surjective, since

$$\begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix} Z(G) = \begin{pmatrix} 1 & a & 0 \\ & 1 & b \\ & & 1 \end{pmatrix} Z(G)$$

where the matrix on either side differ by only an element in  $Z(G)$ .

$\phi$  is injective, since if

$$\phi(a, b) = \begin{pmatrix} 1 & a & 0 \\ & 1 & b \\ & & 1 \end{pmatrix} \in Z(G)$$

, then  $a = b = 0$ . So  $(a, b) = 0$  in  $\mathbb{R}^2$ .

This is a cool isomorphism, but it only demonstrates  $D(G) \leq Z(G)$ .

- Q: Is the center a subgroup of the derived group? (Take Two)

$$\begin{aligned} & \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \\ & 1 & c \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & -c \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & c \\ & 1 & c \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & c \\ & 1 & -c \\ & & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & c \\ & 1 & 0 \\ & & 1 \end{pmatrix} \end{aligned}$$

Therefore  $Z(G) = D(G)$ .

### Theorem: Correspondence + 2nd Isomorphism Theorem

Let  $N \trianglelefteq G$ ,  $E_1 = \{H \leq G \mid H \geq N\}$  (subgroup of  $G$  containing  $N$ ),  $E_2 = \{K \leq G/N\}$  (subgroups of the quotient group).

Further, let  $\pi : G \rightarrow G/N$  be the canonical projection homomorphism.

Then  $E_1$  and  $E_2$  are in bijection by

$$\begin{aligned} \Phi : E_1 &\rightarrow E_2 \text{ given by } \Phi(H) = \pi(H) \\ \Psi : E_2 &\rightarrow E_1 \text{ given by } \Psi(K) = \pi^{-1}(K) \end{aligned}$$

1.  $\Phi$  and  $\Psi$  preserve inclusion (i.e.  $H_1 \leq H_2$  in  $E_1$  implies  $H_1/N \leq H_2/N$ , and vice versa).
2.  $\Phi$  and  $\Psi$  preserve normality (i.e.  $H \trianglelefteq G$  in  $E_1$  implies  $H/N \trianglelefteq G/N$ ).
3. For any  $H \in E_1$  that is normal in  $G$ ,  $G/H \cong (G/N)/(H/N)$  as groups.

### General Fact

If  $\phi : G_1 \rightarrow G_2$  is any homomorphism of groups, then 
$$\begin{cases} H_1 \leq G_1 \implies \phi(H_1) \leq G_2 \\ H_2 \leq G_2 \implies \phi^{-1}(H_2) \leq G_1 \end{cases}.$$

So  $\Phi$  and  $\Psi$  are well-defined.

#### • Proof of Second Fact

1.  $1_{G_1} \in \phi^{-1}(H_2)$ , since  $\phi(1_{G_1}) = 1_{G_2} \in H$ .
2. If  $g_1, g_1' \in \phi^{-1}(H_2)$ , then  $\phi(g_1 g_1') = \phi(g_1) \phi(g_1') \in H_2$  since  $H_2$  is closed under product.
3. If  $g_1 \in \phi^{-1}(H_2)$ , then  $g_1^{-1} \in \phi^{-1}(H_2)$  similarly.

### Proof of 1

Say  $H_1, H_2 \in E_1$  such that  $H_1 \leq H_2$ .

Then  $\Phi(H_1) = \{h_1 N \mid h_1 \in H_1\} \subseteq \{h_2 N \mid h_2 \in H_2\} = \Phi(H_2)$ .

Similarly,  $K_1 \leq K_2 \in H_2$  implies  $\Psi(K_1) \leq \Psi(K_2)$ .

### Proof of 2

Suppose  $H \in E_1$  and  $H \trianglelefteq G$ . We want to prove that  $\Phi(H) = H/N$  is normal in  $G/N$ .

So let  $gN$  be an arbitrary element in  $G/N$ , then  $\forall h \in H$  we have

$$(gN)(hN)(gN)^{-1} = ghg^{-1}N \leq H/N = \Phi(H)$$

Let  $h \in H$  vary, and we get

$$(gN)(H/N)(gN)^{-1} \subseteq H/N$$

Run the argument with  $g$  replaced by  $g^{-1}$ ,

$$(g^{-1}N)(H/N)(g^{-1}N)^{-1} = (gN)^{-1}(H/N)(gN) \subseteq H/N$$

hence

$$H/N \subseteq (gN)(H/N)(gN)^{-1}.$$

Therefore,  $gN$  normalizes  $H/N = \Phi(H)$ .

### Proof of 3

1.

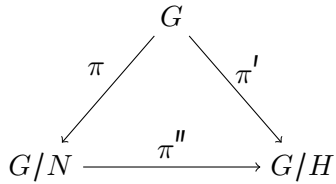
Start with  $G \xrightarrow{\pi'} G/H$ . As a canonical projection,  $\pi'$  is surjective.

That is  $g \mapsto \pi'(g) = gH$ , and  $\pi'(N) = \{1_{G/H}\}$  since for all  $n \in N$ ,  $n \in H$  so  $nH = H$ .



1.

By the Universal Mapping Property of  $G/N$ , we get a unique homomorphism



$$gN \longmapsto gH \quad \text{where } \pi'' \text{ is surjective.}$$

- Recall: 0th Isomorphism Theorem

If  $\phi : G_1 \rightarrow G_2$  is a surjective homomorphism of groups, then  $\bar{\phi} : G_1 / \ker(\phi) \xrightarrow{\sim} G_2$  is an isomorphism. Apply this to  $\pi'' = \phi$ .

$$\begin{aligned} \ker(\pi'') &= \{gN \in G/N \mid gH = H\} \iff g \in H \\ &= \{hN \mid h \in H\} \\ &= H/N \end{aligned}$$

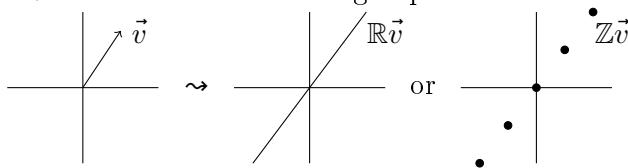
Therefore  $(G/N)/(H/N) \xrightarrow{\sim} G/H$ .

### Example 1

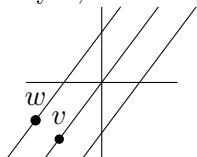
Let  $G = \text{Heis}_3(\mathbb{R})$  and  $N = Z(G) = D(G)$ . Some interesting subgroups of  $(\mathbb{R}^2, +)$  are

1.  $\{0\}$

2.  $\vec{v} \neq \vec{0} \rightsquigarrow \mathbb{R}\vec{v}$  and  $\mathbb{Z}\vec{v}$  are subgroups.



3. Say  $\vec{v}, \vec{w}$  are linear independent.  $\mathbb{Z}\vec{v} + \mathbb{R}\vec{w}$ .



4. IMAGE HERE - LEGIT NO IDEA. Collection of vertical lines in  $\mathbb{R}^3$  that form inverse of example 2?

October 31, 2023

### Definition: Subnormal Series

Let  $G$  be a group.

A subnormal series (or filtration) of  $G$  is a series

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_l = \{1\}$$

of subgroups of  $G$ , such that  $G_i \supseteq G_{i+1}$  for all  $i = 0, \dots, l-1$ .

The length of the filtration is  $l$ .

**Definition: Normal Series**

If each  $G_i$  is normal in  $G$ , we say the series is normal.

**Definition: Factor Group**

The quotient group  $G_i/G_{i+1}$  is called the  $i$ th factor group of the filtration.

This is often denoted  $\text{gr}_i(G.)$  meaning “graded.”

**Definition: Composition Series**

If, for each  $i = 0, \dots, l-1$ ,  $\text{gr}_i(G.)$  is simple, then we say the filtration is a composition series (or a Jordan-Hölder series).

**Example 1**

Let  $G = \text{Alt}(4)$ .

Take  $G_1 = D(G) = \text{Stab}_G(\text{String})$ . (Recall  $|G_1| = 4$ .)

Take  $G_2 = \{1\}$ .

Then  $G = G_0 = G_1 \supseteq G_2$  is a filtration.

It is a normal series.

$G_0/G_1$  is a cyclic group of order 3 and, hence, simple.

$G_1/G_2 = G_1$  (strictly isomorphic; this is an abuse of notation) has order 4, is abelian, and contains  $\langle (B\ P)(W\ Y) \rangle$  as a proper subgroup. So  $G.$  is not a JH series (composition series).

**Example 2**

Let  $G = \text{Alt}(4)$ ,  $H_1 = G_1 = D(G)$ ,  $H_2 = \langle (B\ P)(W\ Y) \rangle$  and  $H_3 = \{1\}$ .

Then,

$$G = H_0 \supseteq H_1 \supseteq H_2 \supseteq H_3$$

is a subnormal series.

It is not a normal series, since  $H_2$  is not normal in  $G$ .

It is a JH series, since  $H_0/H_1$ ,  $H_1/H_2$ , and  $H_2/H_3$  have orders 3, 2 and 2 respectively and, therefore, are all simple.

**Proposition:**

Every finite group  $G$  has at least one Jordan-Hölder Series.

## Proof

By induction on  $|G|$ .

If  $|G| = 1$ , then, by convention,  $(G)$  is a JH series of length 0.

Suppose  $|G| > 1$ .

If  $G$  is simple (i.e.  $G$  has exactly 2 normal subgroups, namely  $\{1\}$  and  $G$ ), then

$$(G = G_0 \supseteq \{1\} = G_1)$$

is a JH series of length 1.

Suppose  $G$  is not simple.

Then  $G$  contains a normal subgroup  $N$ ,  $N \neq \{1\}$  and  $G \neq N$ .

Among all such nontrivial proper normal  $N \trianglelefteq G$ , choose the one with maximal  $|N|$ .

Then  $Q := G/N$  is simple since, by the 2nd isomorphism theorem, if  $Q$  were not simple, it would contain a nontrivial proper normal subgroup  $Q_1 \triangleleft Q$ , which would correspond to a normal subgroup  $N \triangleleft N_1 \triangleleft G$ , but then  $|N_1| > |N|$ , which would contradict the choice of  $N$ .

By induction hypothesis,  $|N| < |G|$ , we have a JH series for  $N$ , say

$$N = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_l = \{1\}$$

Then take  $G_0 = G$  and  $G_{i+1} = N_i$  for  $i = 0, 1, 2, \dots, l$  to get

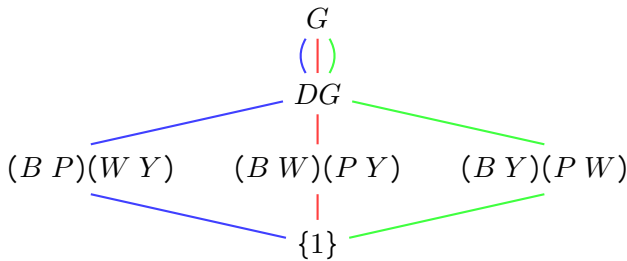
$$G = \underbrace{G_0 \supseteq N_0}_{Q} \supseteq N_1 \supseteq \cdots \supseteq N_l = \{1\}$$

which have simple factors.

Therefore,  $N$  is a JH series. ■

## Example 1

For  $G = \text{Alt}(4)$ ,



has 3 JH series.

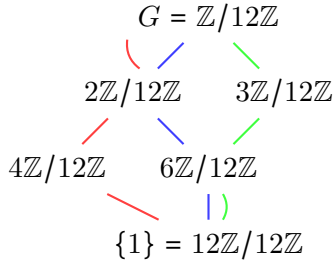
## Example 2

For  $G = \mathbb{Z}/12\mathbb{Z}$ .

Fact: by the 2nd isomorphism theorem, subgroups of  $G$  are exactly  $a\mathbb{Z}/12\mathbb{Z}$  where  $a \mid 12$ .

( $\iff$  subgroup  $a\mathbb{Z}$  of  $\mathbb{Z}$  containing  $12\mathbb{Z}$ )

12 has factors 1,2,3,4,6,12. Then



there are 3 JH series.

## Definition: Product Group

Let  $G$  and  $H$  be groups.

The product group  $G \times H$  is the cartesian product equipped with the component wise group operation.

$$(g_1, h_1) * (g_2, h_2) = (g_1 * g_2, h_1 * h_2)$$

## Example

Let  $p$  be a prime number.

Take  $G = H = \mathbb{Z}/p\mathbb{Z}$ . Then

$$(\mathbb{Z}/p\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z}) \underset{\text{notation}}{=} (\mathbb{Z}/p\mathbb{Z})^2$$

where  $|(\mathbb{Z}/p\mathbb{Z})^2| = p^2$  and

$$(a, b) + (a', b') = (a + a', b + b')$$

for  $a, a', b, b' \in \mathbb{Z}$ .

A subgroup  $K$  of  $\tilde{G} = (\mathbb{Z}/p\mathbb{Z})^2$ , by Lagrange's Theorem, must have order 1,  $p$  or  $p^2$ .

We know  $|\{1\}| = 1$  and  $|\tilde{G}| = p^2$ .

If  $|K| = p$ , then  $K$  is cyclic and generated by  $(a, b)$ .

If  $k \in \{1, 2, \dots, p-1\}$ , then note that for  $(a, b) \neq (0, 0)$

$$\langle (a, b) \rangle = \langle (ka, kb) \rangle$$

Because both have order  $p$ .

Conversely, if  $(a, b) \neq (0, 0)$  and  $(a', b') \neq (0, 0)$  satisfy

$$\langle (a, b) \rangle = \langle (a', b') \rangle$$

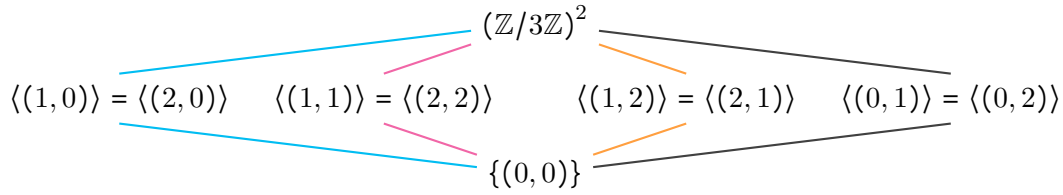
then for some  $k \in \{1, 2, \dots, p-1\}$  we have  $(a', b') = (ka, kb)$ .

- Upshot

There are exactly  $\frac{p^2-1}{p-1} = p+1$  subgroups  $K$  of order  $p$  in  $\tilde{G}$ .

– Example

For  $p = 3$ ,



there are 4 JH series.

## Definition: Automorphism

Let  $G$  be a group.

A (group) automorphism of  $G$  is a bijective group homomorphism

$$\phi : G \rightarrow G$$

$\text{Aut}(G) = \{\phi : G \rightarrow G \mid \text{automorphisms}\}.$

Fact:  $\text{Aut}(G)$  equipped with composition (of functions) is a group.

Justification:  $\phi \circ \psi \in \text{Aut}(G)$  if  $\phi, \psi \in \text{Aut}(G)$ ;  $\phi^{-1} = \phi^{-1}$  and  $1_{\text{Aut}(G)} = \text{Id}_G$ .

## Example 1

Let  $G = (\mathbb{Z}, +)$ .

A homomorphism  $G \xrightarrow{\phi} G$  is completely determined by  $\phi(1) = a_\phi$ .

Since  $\phi(n) = n\phi(1) = na_\phi$ .

Thus  $\{\text{group homomorphisms } \phi : G \rightarrow G\} \leftrightarrow (\mathbb{Z}, \times)$  and  $\phi \leftrightarrow a_\phi$ .

If  $(\phi(1) = a_\phi$  and  $\psi(1) = a_\psi$ , then note that

$$(\phi \circ \psi)(1) = \phi(\psi(1)) = \phi(a_\psi) = a_\phi a_\psi$$

So if  $\phi \in \text{Aut}(G)$ , then  $a_\phi \in \mathbb{Z}$  must have a multiplicative inverse in  $\mathbb{Z}$ .

Hence  $a_\phi = 1$  or  $a_\phi = -1$ .

Conversely  $\begin{cases} \phi_1(n) = n \\ \phi_{-1}(n) = -n \end{cases}, \forall n \in \mathbb{Z}$  are in  $\text{Aut}(G)$ .

Therefore  $\text{Aut}((\mathbb{Z}, +)) = \{\pm 1\}$ .

## Example 2

Let  $p$  be a prime and  $G = (\mathbb{Z}/p\mathbb{Z}, +)$ .

A group homomorphism  $\phi : G \rightarrow G$  is (again) determined by  $\phi(1) = a_\phi$ .

So  $\{\text{group homomorphisms } \phi : G \rightarrow G\} \leftrightarrow \mathbb{Z}/p\mathbb{Z}$ ,  $\phi \leftrightarrow a_\phi$  and  $\phi \circ \psi \leftrightarrow a_\phi a_\psi$ .

So if  $\phi$  is bijective, then  $a_\phi$  must have a multiplicative inverse in  $\mathbb{Z}/p\mathbb{Z}$ .

Fact: Any nonzero element  $a \in \mathbb{Z}/p\mathbb{Z}$  has a multiplicative inverse.

### • Proof

Look at  $\langle a \rangle = \{ka \mid k \in \mathbb{Z}\}$ .

Since  $\mathbb{Z}/p\mathbb{Z}$  is simple,  $\langle a \rangle = \mathbb{Z}/p\mathbb{Z}$ .

Hence  $\langle a \rangle \ni 1$ , so  $\exists k \in \mathbb{Z}$  such that  $ka = 1$  in  $\mathbb{Z}/p\mathbb{Z}$ .

Take  $b := k \pmod{p}$ , and we have  $ba = 1$ . ■

### Conclusion

$$\text{Aut}((\mathbb{Z}/p\mathbb{Z}, +)) = (\mathbb{Z}/p\mathbb{Z})^x = ((\mathbb{Z}/p\mathbb{Z}) \setminus \{0\}, \cdot)$$

### Example

Take  $G = (\mathbb{Z}/p\mathbb{Z})^2$ .

Compute  $|\text{Aut}(G)|$ .

$\text{Aut}(G)$  is also called  $\text{GL}_2(\mathbb{F}_p)$ .

### Definition: Semidirect Product (Left)

Let  $H$  and  $N$  be groups and

$$\psi : H \rightarrow \text{Aut}(N)$$

be a group homomorphism.

The semidirect product  $H \rtimes_{\psi} N$ , as a set is (simply) the cartesian product  $H \times N$  equipped with the binary operation

$$(h_1, n_1) * (h_2, n_2) = \left( h_1 h_2, n_1 \cdot \overbrace{(\underbrace{\psi(h_1)}_{\in \text{Aut}(N)} n_2)}^{\in N} \right)$$

This is a group.

$(1_H, 1_N)$  is the identity.

## November 2, 2023

### Definition: Semidirect Product (Right)

Let  $H$  and  $N$  be groups, and let

$$\phi : H \rightarrow \text{Aut}(N)$$

be a group homomorphism.

The semidirect product of  $H$  by  $N$  via  $\phi$  is

$$N \rtimes_{\phi} H \underset{\text{as a set}}{=} N \times H$$

equipped with the binary operation

$$(n_1, h_1) * (n_2, h_2) = (n_1 \cdot \overbrace{(\underbrace{\phi(h_1)}_{\in \text{Aut}(N)} n_2)}^{\in N}, h_1 h_2)$$

### Associativity

Given  $(n_1, h_1), (n_2, h_2), (n_3, h_3)$ , need to prove that

$$\underbrace{((n_1, h_1) * (n_2, h_2)) * (n_3, h_3)}_{n_1 \cdot [\phi(h_1)n_2], h_1 h_2} = (n_1, h_1) \underbrace{((n_2, h_2) * (n_3, h_3))}_{(n_2[\phi(h_2)n_3], h_2 h_3)}$$

Left Hand Side:

$$(n_1[\phi(h_1)n_2][\phi(h_1 h_2)n_3], h_1 h_2 h_3)$$

Since  $\phi(h_1)$  is a homomorphism,

$$\phi(h_1)(n_2[\phi(h_2)n_3]) = [\phi(h_1)n_2][\phi(h_1)(\phi(h_2)n_3)]$$

and

$$\phi(h_1)(\phi(h_2)n_3) = \phi(h_1 h_2)n_3.$$

Right Hand Side:

$$(n_1 \underbrace{\phi(h_1)(n_2[\phi(h_2)n_3])}_{[\phi(h_1)n_2][\phi(h_1)(\phi(h_2)n_3)]}, h_1 h_2 h_3) = (n_1[\phi(h_1)n_2][\phi(h_1 h_2)n_3], h_1 h_2 h_3)$$

### Identity

Use  $(1_N, 1_H)$ .

### Inverse

Given  $(n_1, h_1)$ , want to find  $(n_2, h_2)$  that is 2-sided inverse.

Use

$$(\phi(h_1^{-1})n_1^{-1}, h_1^{-1})$$

Need  $n_2 \in N$ , such that

$$\begin{aligned} n_1 \phi(h_1) n_2 &= 1_N \\ \phi(h_1) n_2 &= n_1^{-1} \\ n_2 &= \phi(h_1^{-1}) n_1^{-1} \end{aligned}$$

Left hand proof left as an exercise.

### Upshot

The external semidirect product produces a new group out of the old group.

If we take  $\phi$  to be  $\phi(h) = \text{Id}_N$ , we get the direct product  $N \times H$ .

### Proposition: Internal Semidirect Product

Let  $G$  be a group,  $H \leq G$ ,  $N \trianglelefteq G$ , so that we have a group homomorphism

$$H \leq G \xrightarrow{\phi} \text{Aut}(N)$$

$$g \rightsquigarrow \phi(g)n \stackrel{\text{defn.}}{=} gng^{-1}$$

Assume that  $H \cap N = \{1\}$  and  $HN \stackrel{\text{1st isomorphism}}{=} NH \stackrel{\text{assumption}}{=} G$ .

Then we have an isomorphism of groups

$$N \rtimes_{\phi} H \xrightarrow{f} G$$

$$(n, h) \rightsquigarrow nh$$

### Remark

In this situation, we say that  $G$  is the (internal) semidirect product of  $H$  and  $N$ .

### Proof

- $f$  is a group homomorphism.

$$\begin{aligned} f((n_1, h_1) * (n_2, h_2)) &= f(n_1 \phi(h_1) n_2, h_1 h_2) \\ &= n_1 \phi(h_1) n_2 h_1 h_2 \\ &= n_1 (h_1 n_2 h_1^{-1}) h_1 h_2 \\ &= n_1 h_1 n_2 h_2 \\ &= f(n_1, h_1) f(n_2, h_2) \end{aligned}$$

- $f$  is injective.

If  $f(n, h) = 1$ , then  $nh = 1$ . So

$$\underbrace{n}_{\in N} = \underbrace{h^{-1}}_{\in H}.$$

Therefore  $n = 1$  and  $h = 1$ .

- $f$  is surjective

Since  $NH = G$ . ■

### Example 1

Let  $N = \text{Stab}(\text{String})$ , of order 4, and  $H = \text{Stab}_G(P)$ , of order 3. Then  $H \cap N = \{1\}$  and  $HN = G$  gives  $|G| = 12$ .



**Example 2**

Let  $\widetilde{G} = \text{GL}_2(\mathbb{R})$  and

$$K = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}$$

Then

$$G = N_{\widetilde{G}}(K) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, d \in \mathbb{R}^\times, b \in \mathbb{R} \right\},$$

$N = K$ , and

$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{R}^\times \right\}$$

Observe

$$HN = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & ax \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{R}^\times, x \in \mathbb{R} \right\}$$

This is the “Borel subgroup.”

**Definition: Group Extension**

Let  $H$  and  $N$  be groups.

An extension of  $H$  by  $N$  is the data of

1. A group  $G$ .
2. An injective homomorphism of  $N \xhookrightarrow{\iota} G$  such that  $\iota(N) \trianglelefteq G$ .
3. An isomorphism of groups  $G/\iota(N) \xrightarrow[\sim]{\pi} H$ .

**Example**

All the semidirect products give extensions of  $H$  by  $N$ .

**Notation**

$$1 \longrightarrow N \xrightarrow{\sim} G \xrightarrow[\sim]{\pi} H \longrightarrow 1$$

(This is an exact sequence.)

## Recall: Derived Group

If  $G$  is a group, the derived group  $DG = \langle [g, h] \mid g, h \in G \rangle \trianglelefteq G$  is the smallest normal subgroup of  $G$  such that  $G/N$  is abelian.

We can apply  $D$  repeatedly.

$$\begin{array}{ccccccc} G & \supseteq & DG & \supseteq & D(DG) & \supseteq & D(D(DG)) & \supseteq & \dots \\ \parallel & & \parallel & & \parallel & & \parallel & & \\ G^{(0)} & & G^{(1)} & & G^{(2)} = D(G^{(1)}) & & G^{(3)} = D(G^{(2)}) = D(D(G^{(1)})) & & \end{array}$$

Inductive Definition:  $G^{(i+1)} = D(G^{(i)})$ ,  $i \geq 0$ ,  $G^{(0)} = G$ .

## Proposition:

Let  $G$  be a group.

$[\exists N \geq 0 \ G^{(N)} = \{1\}]$  if and only if there exists some filtration  $G$  of length  $l$  with  $G_0 = G$ ,  $G_l = \{1\}$  such that  $G_i/G_{i+1}$  is abelian.

## Definition: Solvable Group (Again)

A group  $G$  is solvable (or soluble) if the 2 equivalent conditions are satisfied.

## Proof

( $\implies$ ) Suppose  $G^{(n)} = \{1\}$ .

Then take  $G_i := G^{(i)}$  for  $i = 0, 1, \dots, N$  and  $G_i/G_{i+1} = G^{(i)}/D(G^{(i)})$  is abelian.

Hence a filtration required in the second condition.

( $\impliedby$ ) Suppose we have a filtration

$$G = G_0 \supseteq G_1 \supseteq \dots \supseteq G_l = \{1\}$$

such that  $G_i/G_{i+1}$  is abelian  $\forall i = 0, \dots, l-1$ .

Claim: for all  $i = 0, \dots, l-1$ , we have

$$G_i \supseteq G^{(i)}$$

If this is true, it finishes the proposition since  $G^{(l)} \subseteq G_l = \{1\}$ .

- Proof of Claim

By induction on  $i$ ,

$$i = 0 \quad G^{(0)} = G = G_0$$

Suppose, for  $i \geq 0$ , that  $G_i \supseteq G^{(i)}$ .

Then, since  $G_i/G_{i+1}$  is abelian, by the definition of  $DG$ ,  $G_{i+1} \supseteq D(G_i)$ .

Apply  $D$  to  $G_i \supseteq G^{(i)}$  and we get  $D(G_i) \supseteq D(G^{(i)})$ .

Since, in general, if  $G \supseteq H \supseteq K$ , then  $D(H) \supseteq D(K) = \langle k_1 k_2 k_1^{-1} k_2^{-1} \rangle$ .

Therefore,  $G_{i+1} \supseteq D(G^{(i)}) \stackrel{\text{defn.}}{=} G^{(i+1)}$ . ■

**Example 1**

$G = \text{Alt}(4)$  is solvable.

$D(G) = N$  of size 4,  $D(N) = \{1\}$  and  $D(DG) = \{1\}$ .

**Example 2**

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{R}^\times, b \in \mathbb{R} \right\}$$

is solvable since if

$$(\mathbb{R}, +) \cong N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}$$

then

$$G \triangleright_{\#} N \triangleright_{\#} \{1\}$$

Meets the condition that  $G_0 = G$ ,  $G_i = \{1\}$  such that  $G_i/G_{i+1}$  is abelian. So

$$G/N \cong H = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{R}^\times \right\} \text{ is abelian } \cong \mathbb{R}^\times \times \mathbb{R}^\times$$

**Nonexample**

$G = \text{GL}_2(\mathbb{R})$ .

Say  $A, B \in \text{GL}_2(\mathbb{R})$ . Then  $[A, B] = ABA^{-1}B^{-1}$  and  $\det(ABA^{-1}B^{-1}) = 1$ .

Therefore  $DG = \text{SL}_2(\mathbb{R})$ , but since  $\text{SL}_2(\mathbb{R})$  is perfect,  $D(\text{SL}_2(\mathbb{R})) = \text{SL}_2(\mathbb{R})$ .

Proof left as an exercise.

**Example 3**

Let  $p$  be a prime. The Heisenberg group of order  $p^3$  is given as

$$\text{Heis}_p = \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \mid x, y, z \in \mathbb{Z}/p\mathbb{Z} \right\} \quad |\text{Heis}_p| = p^3$$

(uses the same formula as with  $\mathbb{R}$ )

The exact same computation for  $\mathbb{R}$  gives:

$$D(\text{Heis}_p) = \mathbb{Z}(\text{Heis}_p) = \left\{ \begin{pmatrix} 1 & 0 & z \\ & 1 & 0 \\ & & 1 \end{pmatrix} \mid z \in \mathbb{Z}/p\mathbb{Z} \right\}$$

Therefore  $D(D(\text{Heis}_p)) = \{1\}$ .

So  $\text{Heis}_p$  is solvable as is  $\text{Heis}_3(\mathbb{R})$ .

## Next Week (Hopefully)

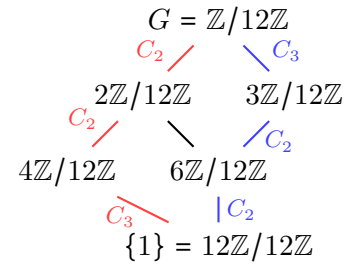
Any group of order prime power is solvable.

## To Read: Theorem

Let  $G$  be a finite group. Then the factor group in the JH series of  $G$  are the same as multisets independent of the choice of the JH series.

## Example

Let  $G = \mathbb{Z}/12\mathbb{Z}$



## November 7, 2023

## Final Exam

Thursday, Dec. 14  
12:00–2:00 p.m.

## Definition/Proposition: Solvable

A group  $G$  solvable if and only if it satisfies 2 equivalent conditions:

1.  $\exists N \geq 1, D^N(G) = \underbrace{D(D(\dots D(G)\dots))}_N = \{1\}$ .
2. There exists a filtration  $G. = (G_0 \supseteq \dots \supseteq G_l)$  with  $G_0 = G$  and  $G_l = \{1\}$  such that  $G_i/G_{i+1}$  is abelian  $\forall i = 0, \dots, l-1$ .

## Definition: Extension

If  $N \trianglelefteq G$ , we say  $G$  is an extension of  $G/N$  by  $N$ .

## Proposition: Solvability is Compatible with Extensions

Let  $N \trianglelefteq G$ , then

1. If  $N$  is solvable and  $G/N$  is solvable, then  $G$  is solvable.
2. If  $G$  is solvable, then so are  $N$  and  $G/N$ .

### Proof of 1

Let  $N. = (N_0 \triangleright \dots \triangleright N_l)$  and  $Q. = (Q_0 \triangleright \dots \triangleright Q_m)$  (where  $G/N = Q$ ) be filtrations as in the definition of solvability. Then we “graft” the two filtrations into one.

$$\underbrace{\pi^{-1}(Q_0) \triangleright \dots \triangleright \pi^{-1}(Q_{m-1}) \triangleright \pi^{-1}(Q_m)}_{\text{factors are abelian}} = \underbrace{N_0 \triangleright \dots \triangleright N_{l-1} \triangleright N_l}_{\text{factors are abelian}}$$

Where  $G \xrightarrow{\pi} Q = G/N$ .  
Since

$$\pi^{-1}(Q_i)/\pi^{-1}(Q_{i+1}) \overset{\text{correlation theorem}}{\cong} \overset{\text{2nd isomorphism theorem}}{Q_i/Q_{i+1}}$$

is abelian.

### Proof of 2

- Lemma

1. If  $H \leq G$ , then  $D(H) \leq D(G)$ .
2. If  $N \trianglelefteq G$ , then  $\pi(D(G)) = D(Q)$ .

– Proof of a

$$\begin{aligned} D(H) &= \langle hh'h^{-1}h'^{-1} \mid h, h' \in H \rangle \\ &\subseteq \langle gg'g^{-1}g'^{-1} \mid g, g' \in G \rangle = D(G) \end{aligned}$$

– Proof of b

$$\pi(gg'g^{-1}g'^{-1}) = \pi(g)\pi(g')\pi(g)^{-1}\pi(g')^{-1}$$

So  $\pi(D(G)) \subseteq D(Q)$ .

Conversely, any commutator like the right hand side is in the image of  $\pi$ , hence  $\pi(D(G)) \supseteq D(Q)$ .

- Back to the proof of 2

Suppose  $D^k(G) = \{1\}$ .

Note, by the above lemma and with an application of induction,

$$\begin{aligned} D(N) &\subseteq D(G) \\ D(D(N)) &\subseteq D(D(G)) \\ &\vdots \\ D^k(N) &\subseteq D^k(G) = \{1\} \end{aligned}$$

So  $N$  is solvable.

Then also

$$\begin{aligned} D(Q) &= \pi(D(G)) \\ D(D(Q)) &= \pi(D(D(G))) \\ &\vdots \\ D^k(Q) &= \pi(D^k(G)) \end{aligned}$$

So  $Q$  is solvable.

### Corollary

If  $A$  and  $B$  are solvable groups, then any semidirect product of  $A$  by  $B$  is solvable.

### Definition: Conjugacy Class

The conjugacy class of  $x \in G$

$$x^\# \{gxg^{-1} \mid g \in G\}$$

Recall that, by the orbit stabilizer theorem,  $|x^\#| = \frac{|G|}{|C_G(x)|}$ . So

$$C_G(X) = \{h \in G \mid hx = xh, h x h^{-1} = x\}$$

### Example 1

Let  $G = \text{Alt}(4)$ . Then

$$\begin{aligned} 1^\# &= \{1\} \\ |(B P)(W Y)^\#| &= 3 \quad \text{all the } 2 + 2 \text{ cycles} \end{aligned}$$

Consider  $g = (B P W)$ . Then  $\langle g \rangle = \text{Stab}_G(Y)$ ,  $\pi \text{Stab}(y) \pi^{-1} = \text{Stab}(\pi(y))$ , and

$$N_G(\langle g \rangle) = \langle g \rangle \underset{=}{\overset{\cong}{\supseteq}} C_G(g)$$

Therefore  $|C_G(g)| = 3$  and  $|(B P W)^\#| = 4$ .

Since  $\pi(B P W) \pi^{-1} = (\pi(B) \pi(P) \pi(W))$ ,

$$\begin{aligned} \pi &= (B P)(W Y) \implies \pi(B P W) \pi^{-1} = (P B Y) = (B Y P) \\ \pi &= (B P Y) \implies \pi(B P W) \pi^{-1} = (P Y W) \\ \pi &= (P W Y) \implies \pi(B P W) \pi^{-1} = (B W Y) \end{aligned}$$

So

$$(B P W)^{\#} = \{(B P W), (B Y P), (P Y W), (B W Y)\}$$

Therefore

$$\text{Alt}(4) = \begin{matrix} 1 & (B\ P)(W\ Y) & (B\ P\ W) & (B\ W\ P) \\ & (B\ W)(P\ Y) & (B\ Y\ P) & (B\ P\ Y) \\ & (B\ Y)(P\ W) & (P\ Y\ W) & (P\ W\ Y) \\ & & (B\ W\ Y) & (B\ Y\ W) \end{matrix} \quad \text{The class equation}$$

$$12 = 1 + 3 + 4 + 4$$

Consider  $\text{Sym}(4) = \widetilde{G}$ , which has conjugacy class and cycle types

$\widetilde{G} = 1 + \frac{3}{2+2\text{-cycles}} + \frac{8}{3\text{-cycles}} + \frac{\binom{4}{2}=6}{2\text{-cycles}} + \frac{\frac{4!}{4}=6}{4\text{-cycles}}$  Note that the 3 cycles in  $\text{Alt}(4)$  fuse into  
 $24 = 1 + 3 + 8 + 6 + 6$   
the same conjugacy class in  $\text{Sym}(4)$ .

### Definition: Fusion

Different conjugacy classes in  $G$  merge into 1 in  $\widetilde{G}$ .

## More Generally

If  $G = \coprod_{i=1}^h K_i$  into disjoint conjugacy classes, the class equation

$$|G| = \sum_{i=1}^h |K_i|$$

### Example 1

Let  $p$  be a prime and  $G = \text{Heis}_3(p) = \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & z \end{pmatrix} \mid x, y, z \in \mathbb{Z}/p\mathbb{Z} \right\}$ .

Recall

$$Z(G) = \left\{ \begin{pmatrix} 1 & 0 & z \\ & 1 & 0 \\ & & 1 \end{pmatrix} \mid z \in \mathbb{Z}/p\mathbb{Z} \right\}$$

and, for any  $g \in Z(G)$ ,  $g^\# = \{g\}$ .

So we have  $p$  conjugacy classes of size 1.

Let  $(x, y) \in (\mathbb{Z}/p\mathbb{Z})^2 \setminus \{(0, 0)\}$  and  $g := \begin{pmatrix} 1 & x & 0 \\ & 1 & y \\ & & 1 \end{pmatrix}$ .

Consider the centralizer,  $C_G(g) \geq Z(G)$  and  $C_G(g) \geq \langle g \rangle$ .

So  $|C_G(g)| \geq p + 1$  and, by Lagrange,  $|C_G(g)| \mid p^3$ .

Therefore  $|C_G(q)| = p^2$  and, by the orbit stabilizer theorem,  $|g^\#| = p^3/p^2 = p$ .

It follows that the class equation for  $\text{Heis}_3(p)$  is

$$p^3 = \underbrace{1 + \cdots + 1}_p + \underbrace{p + \cdots + p}_{p^2-1 \text{ one for each } (x,v)}$$

- Question

If  $(x, y) \neq (x', y')$ , then

$$\begin{pmatrix} 1 & x & \\ & 1 & y \\ & & 1 \end{pmatrix}^{\#} \neq \begin{pmatrix} 1 & x' & \\ & 1 & y' \\ & & 1 \end{pmatrix}^{\#}$$

Observe that

$$\begin{aligned} \begin{pmatrix} 1 & -a & ? \\ & 1 & -b \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix} &= \begin{pmatrix} 0 & -a & ? \\ & 0 & -b \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & x+a & ? \\ & 1 & y+b \\ & & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & x & ? \\ & 1 & y \\ & & 1 \end{pmatrix} \end{aligned}$$

That is,  $G \rightarrow G/Z(G) \cong (\mathbb{Z}/p\mathbb{Z})^2$ .

In particular,

$$\begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & 1 \end{pmatrix}$$

are not conjugate.

But they become conjugate in  $\text{GL}_3(\mathbb{Z}/p\mathbb{Z})$ , another instance of fusion.

## Definition: p-Group

Let  $p$  be a prime number.

A finite group  $G$  is called a  $p$ -group if  $|G| = p^n$  for some integer  $n$ .

## Examples

- $\mathbb{Z}/p^n\mathbb{Z}$  (the cyclic group of  $p^n$ ).
- $\text{Heis}_3(p)$ .
- Any repeated semi-direct products thereof.

## Theorem: Solvability of p-Groups.

Any  $p$ -group  $G \neq \{1\}$  has a nontrivial center  $Z(G)$  and, therefore, is solvable.



**Proof**

Let  $g \in G$ .

If  $g \in Z(G)$ , then  $g^\# = \{g\}$ .

If  $g \notin Z(G)$ , then  $C_G(g) \neq G$ , so  $|g^\#| = \frac{|G|}{|C_G(g)|}$  is divisible by  $p$ .

But then

$$p^n = |G| = \underbrace{1 + \cdots + 1}_{|Z(G)|} + (\text{integers divisible by } p)$$

Therefore  $p \mid |Z(G)|$  and  $|Z(G)| \geq p$ .

By induction on  $n$ , we may assume that any group of order  $p^m$  ( $0 \leq m \leq n-1$ ) is solvable.

But then

$$\underbrace{Z(G) \trianglelefteq G}_{\text{order } \geq p} \twoheadrightarrow \underbrace{G/Z(G)}_{\text{has order } p^m, m < n}$$

By the above proposition,  $G$  is solvable.

**Remark:**

In some ways, solvable groups are easier to understand.

So given a finite group  $G$ , one attempts to understand  $G$  by first studying solvable subgroups of  $G$ .

**Definition: p-Sylow Subgroups**

Let  $G$  be a finite group and  $p$  be prime.

Write  $|G| = p^n \cdot m$  where  $n \geq 0$ ,  $m \geq 1$  are integers and  $p \nmid m$ .

A  $p$ -Sylow subgroup  $H$  of  $G$ , is a subgroup of order  $|H| = p^n$ .

**Example 1**

Let  $G = \text{Alt}(4)$ .

$|G| = 12 = 2^2 \cdot 3$ , so the 2-Sylow subgroup of  $G$  is  $H = \text{Stab}(\text{String})$ .

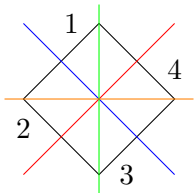
$|G| = 12 = 3^1 \cdot 4$ , so the 3-Sylow subgroups of  $G$  are the vertex stabilizers.

**Example 2**

Let  $G = \text{Sym}(4)$ .

$|G| = 4! = 24 = 2^3 \cdot 3 = 3^1 \cdot 8$ .

3-Sylows are the cyclic groups generated by (1 of the 8) 3-cycles, so there are 8 of them.



$$\begin{aligned}
 D_8 &\leq \text{Sym}(4) \\
 &= \begin{cases} 4 \text{ rotations} \\ 4 \text{ reflections} \end{cases} \\
 &= \{(1\ 2)(3\ 4), (2\ 4), (1\ 4)(2\ 3), (1\ 3), (1\ 2\ 3\ 4), (1\ 3)(2\ 4), (4\ 3\ 2\ 1), 1\}
 \end{aligned}$$

**November 9, 2023**

**Jean-Paul Serre**

The proofs of Sylow Theorems below are taken from Serre's Finite Groups, an Introduction.

### Definition: $p$ -Sylow Subgroups

Let  $G$  be a finite group,  $p$  be prime, and write  $|G| = p^n \cdot m$  where  $n \geq 0$  and  $m \geq 1$  are integers such that  $p \nmid m$ . A  $p$ -Sylow subgroup of  $G$  is a subgroup  $S \leq G$  such that  $|S| = p^n = q$ .

### Theorem: First $p$ -Sylow Theorem

There exist  $p$ -Sylow subgroups.

### Recall:

By Lagrange,  $H \leq G \iff |H| \mid |G|$ .

However, the converse is false.

For example,  $G = \text{Alt}(4)$  has no subgroup of order 6, even though  $6 \mid 12$ .

### Proof

Let  $X = \{S \subseteq G \mid |S| = p^n\}$ .

Note  $|X| = \binom{q \cdot m}{p^n}$ .

If  $p^n = 1$ , there is nothing to prove. So assume  $n \geq 1$ ,  $q \geq p$ .

- Lemma

$$\binom{q \cdot m}{p^n} \equiv m \pmod{p}; \text{ in particular, } p \nmid |X|.$$

– Proof

$R = (\mathbb{Z}/p\mathbb{Z})[T] = \{\text{polynomials in } T \text{ with } \mathbb{Z}\text{-coefficients (mod } p)\}$  is (going to be) a ring.

Consider  $(1 + T)^{q \cdot m}$ .

First, by the Binomial Theorem,

$$(1 + T)^p = \sum_{j=0}^p \binom{p}{j} T^j$$

If  $1 \leq j \leq p-1$ , then  $\binom{p}{j} = \frac{p!}{(p-j)!j!}$  is divisible by  $p$ . So

$$(1 + T)^p \equiv 1 + T^p \pmod{p}$$

Repeating the process,

$$(1 + T)^{p^2} \equiv ((1 + T)^p)^p \equiv (1 + T^p)^p \equiv 1 + T^{p^2} \pmod{p}$$

By induction,

$$(1 + T)^q \equiv 1 + T^q \pmod{p}$$

Using the Binomial Theorem again,

$$(\star) \quad (1 + T)^{q \cdot m} \equiv (1 + T^q)^m \equiv 1 + \binom{m}{1} T^q + \cdots \pmod{p}$$

On the other hand, by applying the Binomial Theorem directly,

$$(*) \quad (1 + T)^{q \cdot m} = \sum_{j=0}^{q \cdot m} \binom{q \cdot m}{j} T^j$$

Compare the coefficients of  $T^q$ , and we get  $m \equiv \binom{q \cdot m}{q} \pmod{p}$ .

Consider the (translation) action of  $G$  on  $X$ :

$$g * S := \{gs \mid s \in S\}$$

Now decompose  $X$  into disjoint  $G$ -orbits:

$$X = \bigsqcup_{j=1}^N O_j$$

Then

$$|X| = \sum_{j=1}^N |O_j|$$

So at least 1 orbit, say  $O_j = G \cdot S_0$  must have  $|O_j| \not\equiv 0 \pmod{p}$ .

Take  $H := \text{Stab}_G(S_0) = \{g \in G \mid gS_0 = S_0\}$ .

1. By Orbit Stabilizer Theorem,  $|H| = \frac{|G|}{|O_j|} = \frac{q \cdot m}{|O_j|} = q \frac{m}{|O_j|} \geq q$ .

2. On the other hand, we fix  $\sigma_0 \in S_0$  and we have a map

$$\begin{aligned} H &\xrightarrow{f} S_0 \\ h &\mapsto h\sigma_0 \end{aligned}$$

$f$  is injective, since  $h_1\sigma_0 = h_2\sigma_0$ , then  $h_1 = h_2$ .

Therefore  $|H| \leq |S_0| = q$ . ■

### Theorem: Second Sylow Theorem

Notation as before, let  $S_0$  be a  $p$ -Sylow subgroup.

1. Any  $p$ -subgroup,  $P$ , is contained in a conjugate of  $S_0$ .

$$P \leq gS_0g^{-1}$$

for some  $g \in G$ .

2. Any (other)  $p$ -Sylow subgroup  $S$  of  $G$  is conjugate to  $S_0$ .

$$S = gS_0g^{-1}$$

for some  $g \in G$ .

3. Let  $n_p(G) = \#\{p\text{-Sylow subgroups of } G\}$ . Then

$$n_p(G) \equiv 1 \pmod{p}$$

(This is sometimes listed as the third Sylow theorem)

- Lemma 2

Let  $P$  be a  $p$ -group acting on a finite set  $Y$ .

Define  $Y^P = \{y \in Y \mid gy = y, \forall g \in P\}$ , the set of fixed points. Then

$$|Y^P| \equiv |Y| \pmod{p}$$

– Remark (\*): in Algebraic Topology this leads to Smith Theory.

– Proof

Decompose  $Y$  into  $P$ -orbits.

$$Y = \underbrace{\{\ast_1\} \coprod \cdots \coprod \{\ast_a\}}_{Y^P} \coprod_{j=1}^M O_j$$

Therefore  $|O_j| = \frac{|P|}{|\text{Stab}_j|}$  is a power of  $p$  and  $\geq p$ .  
It follows that

$$|Y| = |Y^P| + \underbrace{\sum_{j=1}^M |O_j|}_{\text{divide by } p} \equiv |Y^P| \pmod{p}$$

### Proof (Second Sylow Theorem)

- Part 1

$y = G/S_0$ , the coset space, is not necessarily a group.  
 $G$  acts on  $y$  by translation:  $g \ast (g'S_0) = gg'S_0$ .  
Restrict the action to  $P$ , and

$$|y^P| \equiv |y| \pmod{p} = \frac{|G|}{|S_0|} = \frac{q \cdot m}{q} = m \not\equiv 0 \pmod{p}$$

Therefore  $y^P \neq \emptyset$ . Say  $gS_0 \in y^P$  or  $P \subseteq \text{Stab}_G(gS_0)$ .  
For any  $\pi \in P$ ,  $\pi \cdot gS_0 = gS_0$ , and

$$\begin{aligned} g^{-1}\pi g \underbrace{S_0}_{\ni 1} &= S_0 \\ g^{-1}\pi g &\in S_0 \\ \pi &\in gS_0g^{-1} \end{aligned}$$

Thus,  $P \leq gS_0g^{-1}$ . ■

- Part 2

If  $S$  is a  $p$ -Sylow and, by (1),  $S \subseteq gS_0g^{-1}$ , then, since both have  $q$  elements,

$$S = gS_0g^{-1} \quad \blacksquare$$

- Part 3

Write  $\mathcal{S} = \{p\text{-Sylows } S \text{ of } G\}$   
Let  $G$  act on  $\mathcal{S}$  by conjugation.

$$g \ast S = gSg^{-1}$$

By (2), the action is transitive.  
Note:  $\text{Stab}_G(S) = N_G(S)$ .

– Lemma 3

Suppose  $S, S' \in \mathcal{S}$ .

If  $S' \in N_G(S)$ , then  $S' = S$ .

\* Proof

Note that  $|N_G(S)| \mid |G| = q \cdot m$ , and  $S \subseteq N_G(S)$  where  $|S| = q$ .

Therefore  $S$  is a  $p$ -Sylow of  $N_G(S)$ .

Now, if  $S' \leq N_G(S)$ ,  $S'$  is a  $p$ -Sylow of  $N_G(S)$ .

So, by (2) applied to  $N_G(S)$  replacing  $G$ ,

$$\exists n \in N_G(S) : S' = nS'n^{-1} = S \quad \blacksquare$$

· Remark

$$n_p(G) = \frac{|G|}{|\text{Stab}_G(S)|} = \frac{|G|}{|N_G(S)|} \text{ for any } p\text{-Sylow } S.$$

Choose any  $S_0 \in \mathcal{S}$ , and let  $S_0$  act on  $\mathcal{S}$  by conjugation.

Then, by Lemma 2,  $|\mathcal{S}^{S_0}| \equiv |\mathcal{S}| \pmod{p}$ , and

$$\mathcal{S}^{S_0} = \{S' \in \mathcal{S} \mid S_0 \in \underbrace{\text{Stab}_G(S')}_{\substack{= N_G(S') \\ \iff S_0 = S'}}\} = \{S_0\} \quad \blacksquare$$

## Sylow Examples

### Example 1

Let  $G = \text{Alt}(4)$ ,  $|G| = 2^2 \cdot 3^1$ .

$$n_2(G) = 1 \equiv 1 \pmod{2}$$

$$n_3(G) = 4 \equiv 1 \pmod{3}$$

### Example 2

Let  $p$  be a prime.

Denote  $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ .

$$M_{2 \times 2}(\mathbb{F}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{F}_p \right\}$$

has addition and multiplication (not commutative).

$$|M_{2 \times 2}(\mathbb{F}_p)| = p^4$$

And  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2 \times 2}(\mathbb{F}_p)$  if and only if  $\det(A) = ad - bc$  is nonzero.

$$\left( A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \right)$$

## Definition: General Linear Group Over Finite Field

$$\mathrm{GL}_2(\mathbb{F}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2 \times 2}(\mathbb{F}_p) \mid ad - bc \neq 0 \text{ in } \mathbb{F}_p \right\}$$

equipped with matrix multiplication, is a group.

- Question:  $|\mathrm{GL}_2(\mathbb{F}_p)| = ?$   
 Conceptual answer will be given in Math 201.  
 An ad hoc answer is

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 0 \text{ in } \mathbb{F}_p \right\}$$

If  $a = 0$  (in  $\mathbb{F}_p$ ), then  $bc = 0$  so  $b = 0$  or  $c = 0$ . So consider

$$\underbrace{\{a = 0 \text{ and } b = 0\}}_{p^2} \cup \underbrace{\{a = 0 \text{ and } c = 0\}}_{p^2} : \cap \text{ has } p \text{ elements}$$

If  $a \neq 0$  (in  $\mathbb{F}_p$ ), then  $d = bc/a$ , then  $\underbrace{p-1}_a \underbrace{p}_b \underbrace{p}_c$ .

It follows that

$$\begin{aligned} |\mathrm{GL}_2(\mathbb{F}_p)| &= p^4 - \left[ \underbrace{(p^2 + p^2 - p)}_{a=0} + \underbrace{(p-1)pp}_{p \neq 0} \right] \\ &= p(p^3 - (2p-1) - p(p-1)) \\ &= p(p^3 - p^2 - p + 1) \\ &= p(p-1)(p^2 - 1) \\ &= p(p-1)^2(p+1) \end{aligned}$$

- Question: a  $p$ -Sylow of  $\mathrm{GL}_2(\mathbb{F}_p)$  has order  $p$ .

Example:  $H = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{F}_p \right\}$  is one. And

$$N_G(H) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{F}_p^\times, b \in \mathbb{F}_p \right\}$$

Therefore

$$n_p(G) = \frac{|G|}{|N_G(H)|} = \frac{p(p-1)^2(p+1)}{\underbrace{(p-1)}_a \underbrace{p}_b \underbrace{(p-1)}_d} = p+1 \equiv 1 \pmod{p}$$

**November 14, 2023**

## Overview

$p$ -syllow

Free Groups and Presentations (Computational Algebraic Topology)

## Recall: Sylow Theorems

Let  $G$  be a finite group where  $|G| = p^n m$ ,  $p$  prime,  $m \in \mathbb{Z}$  and  $p \nmid m$ . Then

1.  $p$ -Sylows exist:  $H \leq G$  with  $|H| = p^n$ .

2. All the  $p$ -sylows are conjugate in  $G$ .

3.

$$n_p(G) = \#\{p\text{-Sylows of } G\} = \frac{|G|}{|N_G(\underbrace{P}_{\text{any } p\text{-Sylow}})|} \equiv 1 \pmod{p}$$

Note:  $n_p(G) \mid m$  (from 3).

## Theorem\*: Feit-Thompson

All groups of odd order are solvable.

Proof too long to reproduce.

## Proposition: Order $pq$ Groups are Solvable

Let  $p, q$  be primes.

Any group  $G$  with order  $pq$  is solvable.

### Proof

If  $p = q$ , then  $G$  is a  $p$ -group. So it is solvable.

So let us assume that  $p \neq q$ .

We may assume, without loss of generality, that  $p < q$ . Then  $n_q(G) \equiv 1 \pmod{q}$  and  $n_q(G) \mid p$ .

Then  $p \not\equiv 1 \pmod{q}$ .

Therefore,  $n_q(G) = 1$ .

By Sylow 2, the unique  $q$ -Sylow, say  $Q$ , is normal in  $G$ .

Now  $|G/Q| = |G|/|Q| = p$ , hence  $G/Q$  is abelian. Similarly, since  $|Q| = q$ ,  $Q$  is abelian.

So, with the filtration

$$G \triangleright_{\neq} Q \triangleright_{\neq} 1,$$

$G$  is solvable.

## Proposition: Order $ppq$ Groups are Solvable

Let  $p, q$  be primes.

If  $|G| = p^2 q$ , then  $G$  is solvable.



### Proof

If  $n_p(G) = 1$ , then the  $p$ -Sylow  $P$  will be normal and have order  $p^2$ , hence by the previous proposition (or since  $P$  is a  $p$ -group),  $P$  is solvable and  $G/P$  is cyclic of order  $q$  (hence abelian). So  $G$  will be solvable.

If  $n_q(G) = 1$ , then the  $q$ -Sylow  $Q$  will be normal and have order  $q$  with  $|G/Q| = p^2$ , so  $G$  is solvable.

So we may assume, in the rest of the proof, that  $n_p(G) = q$  and  $n_q(G) \in \{p, p^2\}$ .

Then  $q \equiv 1 \pmod{p}$  by Sylow 3, and  $p|(q-1)$ , so  $p \leq q-1$ , so  $p < q$ .

But then  $p \not\equiv 1 \pmod{q}$  (because  $p \equiv 1 \pmod{q} \implies q|(p-1) \implies q < p$ ).

Therefore  $n_q(G) = p^2$ .

Let  $Q$  be a  $q$ -Sylow. Since  $n_q(G) = p^2 = \frac{|G|}{|N_G(Q)|}$ , we have  $|N_G(Q)| = q$ .

That is,  $N_G(Q) = Q$ .

Consider all the  $q$ -Sylows  $xQx^{-1}$  of  $G$ ,  $p^2$  in number.

For any  $g \in xQx^{-1} \setminus \{1\}$ ,  $\langle g \rangle = xQx^{-1}$ , so  $g$  has order  $q$ .

Conversely, any  $g \in G$  of order  $q$  a  $q$ -Sylow.

Consider

$$\Sigma = \bigcup_{xQ \in G/Q} x(Q \setminus \{1\})x^{-1}$$

a set consisting entirely of elements of order  $q$ .

The union is pairwise disjoint, since if

$$x(Q \setminus \{1\})x^{-1} \cap y(Q \setminus \{1\})y^{-1} \neq \emptyset$$

then  $\langle g \rangle = xQx^{-1} = yQy^{-1}$ .

$$\implies (y^{-1}x)Q(y^{-1}x)^{-1} = Q \implies y^{-1}x \in N_G(Q) = Q$$

So  $yQ = xQ$ .

Now

$$|\Sigma| = |G/Q|(|Q| - 1) = p^2(q - 1)$$

So

$$|G \setminus \Sigma| = |G| - |\Sigma| = p^2q - p^2(q - 1) = p^2$$

Now, let  $P$  be a  $p$ -Sylow of  $G$ . Then any element  $g \in P$  will have order 1,  $p$ , or  $p^2$  by Lagrange, so  $g \notin \Sigma$ . Therefore

$$\underbrace{P}_{\text{order } p^2} \subseteq \underbrace{G \setminus \Sigma}_{\text{order } p^2} \implies P = G \setminus \Sigma$$

Hence  $G \setminus \Sigma$  must be the unique  $p$ -Sylow of  $G$ , which contradicts  $n_p(G) = q$ .

### Remark\*: Burnside

Any group of order  $p^a q^b$  is solvable

**Definition: Word**

Let  $X$  be a set (of “letters”), e.g.  $X = \{x_1, \dots, x_n\}$ .

A word in  $X$  is an infinite sequence  $(a_1, a_2, a_3, \dots) = (a_n)_{n=1}^\infty$  such that for each  $i = 1, 2, 3, \dots$ ,

$$\begin{cases} a_i = 1 \\ a_i = x, \\ a_i = x^{-1} \end{cases} \quad x \in X$$

and, for all  $i \gg 0$ ,  $a_i = 1$ .

**Example**

$X = \{x, y, z\}$ .

$(x, x^{-1}, y, y, y, z, x, 1, 1, 1, \dots)$

$(y, 1, z, z, y^{-1}, 1, 1, 1, \dots)$

**Definition: Reduced Word**

A word  $(a_n)$  is reduced if  $\forall i \geq 1, \forall x \in X$ ,

1. If  $a_i = x$ , then  $a_{i+1} \neq x^{-1}$
2. If  $a_i = x^{-1}$ , then  $a_{i+1} \neq x$ .
3. If  $a_i = 1$ , then  $a_{i+1} = 1$ .

Note that neither of the previous examples are reduced.

The empty word  $(1, 1, 1, \dots)$  is always reduced.

**Fact/Construction: Word Reduction**

Given any word  $(a_n)$ , we construct the reduction,  $\text{red}(a_n)$ , in the following way:

1. If  $a_i = x$  and  $a_{i+1} = x^{-1}$  or  $a_i = x^{-1}$  and  $a_{i+1} = x$ , then skip both  $a_i$  and  $a_{i+1}$ .
2. If  $a_i = 1$  but  $a_{i+1} \neq 1$ , skip  $a_i$ .

**Example**

$(x, x^{-1}, y, y, 1, z, x, 1, \dots)$

$\rightsquigarrow (y, y, 1, z, x, 1, \dots)$

$\rightsquigarrow (y, y, z, x, 1, \dots)$  is the reduction.

### Definition: Free Group

The free group on  $X$ ,  $F(X)$ , is the set of reduced words on  $X$  equipped with the binary operation for  $\alpha = (a_n)$  and  $\beta = (b_n)$ :

$$\alpha * \beta = \text{red}(\text{concatenation of } \alpha \text{ and } \beta)$$

### Example

$$\alpha = (x, y, z, x^{-1}, 1, \dots) \text{ and } \beta = (x, z^{-1}, y^{-1}, x^{-1}, y, 1, \dots)$$

$$\begin{aligned} &\leadsto (\underbrace{x, y, z, x^{-1}}_{\alpha}, \underbrace{x, z^{-1}, y^{-1}, x^{-1}, y, 1, \dots}_{\beta}) \\ &\leadsto (y, 1, \dots) \end{aligned}$$

### Example

IMAGE HERE - TRIPLE TORUS

$\pi_1$  of the triple torus with piece removed is the free group on  $2 \times 3$  letters.

### Proposition (Universal Mapping Property)

Let  $X$  be a set and  $G$  be a group.

Then there is a natural bijection

$$\text{Fun}(X, G) \xrightarrow[\alpha]{\sim} \text{Hom}_{\text{group}}(F(X), G)$$

Where  $\text{Fun}(X, G)$  is the collection of functions from  $X$  to  $G$ .

### Proof

Given  $f \in \text{Fun}(X, G)$  (i.e. a function  $X \xrightarrow{f} G$ ) let  $\alpha(f)$  be defined

$$\alpha(f)(a_1, a_2, a_3, \dots, 1, 1, 1, \dots) = \underbrace{f(a_1)}_G \underbrace{f(a_2)}_G \underbrace{f(a_3)}_G \cdots \underbrace{1 \cdot 1 \cdot 1 \cdots}_{\text{ignored}}$$

$\alpha(f)$  is a homomorphism (skipped; see Robert Boltje's lecture notes).

Conversely, given  $\phi \in \text{Hom}_{\text{groups}}(F(X), G)$ , let

$$\beta(\phi)(x) = \phi((x, 1, 1, \dots))$$

Exercise:  $\alpha$  and  $\beta$  are inverses.

### Remark\*:

In categorical terms,  $F$  is adjoint functor to the forgetful functor from  $\text{Groups} \rightarrow \text{Sets}$ .

### Definition: Normal Generated Subgroup

Let  $G$  be a group and  $S \subseteq G$  a subset.

Recall  $\langle S \rangle$  is the subgroup generated by  $S$ .

Define  $\langle\langle S \rangle\rangle = \langle gsg^{-1} : s \in S, g \in G \rangle$ .

This is the normal subgroup generated by  $S$ .

### Definition: Presentation

Let  $x$  be a set (of letters) and  $S \subseteq F(X)$  a “relation”.

The quotient group  $F(X)/\langle\langle S \rangle\rangle$ , denoted by  $\langle X : S \rangle$ , is called a presentation of the group.

### Example

Let  $X = \{a, b, c\}$  and  $S = \{abba, aa, ac^{-1}\}$ .

$$\langle a, b, c : abba, aa, ac^{-1}, abac \rangle$$

refers to the quotient group  $F(x)/\langle\langle S \rangle\rangle$ .

### Example\*

IMAGE HERE - TRIPLE TORUS WITHOUT REMOVAL

$\pi_1$  of that is  $\langle a_1, b_1, a_2, b_2, a_3, b_3 : [a_1, b_1][a_2, b_2][a_3, b_3] \rangle$ .

### Proposition (Universal Mapping Property)

Let  $\langle X : S \rangle$  be a presentation of a group  $G$  (where  $G = X$ ) and a group  $H$ ,

Giving a group homomorphism  $G \longrightarrow H$  is equivalent to giving a function  $X \xrightarrow{f} H$  such that

$$S \subseteq \ker(\alpha(f) : F(X) \rightarrow H)$$

**November 16, 2023**

Rings.

### Definition: Monoid

A group is a set  $M$  equipped with a binary operation  $*$  satisfying

1.  $a * (b * c) = (a * b) * c, \quad \forall a, b, c \in M$

2.  $\exists e \in G : \quad a * e = a = e * a, \quad \forall a \in M$

3.  $\forall a \in G : \exists b \in G : ab = e = ba$

Say  $M$  is commutative if

$$1. \forall a, b \in M : a * b = b * a$$

### Examples

$\mathbb{N}_0 = \{0, 1, 2, 3, 4, \dots\}$  equipped with  $+$  is a monoid.

$\mathbb{N} = \{1, 2, 3, 4, \dots\}$  equipped with  $\times$  is a monoid.

$\mathbb{Z}$  with  $\times$  is a monoid.

### Definition: Ring

A ring is a set  $R$  equipped with 2 binary operations  $+$  and  $\cdot$  such that

1.  $(R, +)$  is a commutative group,

2.  $(R, \cdot)$  is a monoid and,

3.  $\forall a, b, c \in R$ :

- $a \cdot (b + c) = a \cdot b + a \cdot c$  and

- $(a + b) \cdot c = a \cdot c + b \cdot c$

### Notation

The additive neutral element is denoted by  $0_R = 0$ .

The multiplicative neutral element is denoted by  $1_R = 1$ .

### Definition: Rng

If, instead, we drop the multiplicative identity requirement, we construct a rng or “ring without identity”.

### Sources of Rings

Three main sources of rings (at least historically)

1. Numbers

2. Functions

3. Linear Operators

- Numbers

$(\mathbb{Z}, +, \cdot)$ ,  $(\mathbb{Q}, +, \cdot)$ ,  $(\mathbb{R}, +, \cdot)$ ,  $(\mathbb{C}, +, \cdot)$  are rings.

## Definition: Subring

A subset  $S \subseteq R$  is a subring if

1.  $(S, +)$  is a subgroup of  $(R, +)$  and,
2.  $(S, \cdot)$  is a submonoid of  $(R, \cdot)$ .

i.e.  $0_R \in S$ ,  $1_R \in S$ , and  $x, y \in S \implies x + y, x - y, xy \in S$ .

## Example: Gaussian Integers

Let  $R = \mathbb{C}$  (the operation notation is suppressed when understood)  
 $S := \{a + bi \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}$  (where  $i^2 = -1$ ) is a subring of  $\mathbb{C}$  since

1.  $0 = 0 + 0i$
2.  $1 = 1 + 0i$
3.  $z = a + bi$  and  $w = c + di$  are in  $S$ , then

$$\begin{aligned} z \pm w &= (\underbrace{a \pm c}_{\in \mathbb{Z}}) + (\underbrace{b \pm d}_{\in \mathbb{Z}})i \\ zw &= (\underbrace{ac - bd}_{\in \mathbb{Z}}) + (\underbrace{bc + ad}_{\in \mathbb{Z}})i \end{aligned}$$

This subring is the Gaussian integers, denoted by  $\mathbb{Z}[i]$ .

- Remark

To verify that  $S \subset R$  is a subring, we need NOT verify that  $x \in S \implies x^{-1} \in S$ .

e.g.  $\frac{1}{3+5i} = \frac{3-5i}{(3-5i)(3+5i)} = \frac{3-5i}{34}$ .

## Rings of Numbers

### Example: p-adics

Let  $m \geq 1$  be an integer,  $R := \mathbb{Z}/m\mathbb{Z}$  with  $+$  and  $\cdot$  is a ring.

\* (out of these, we can make “ $p$ -adic numbers”)

## Rings of Functions

### Example: Functions

Let  $X = \mathbb{R}$  and  $A = \text{Fun}(X, \mathbb{R}) = \{f : X \rightarrow \mathbb{R} \text{ functions}\}$ .

Equip  $A$  with pointwise  $+$  and  $\cdot$ : given  $f, g \in A$ , define

$$A \ni f + g : x \mapsto f(x) + g(x)$$

$$A \ni fg : x \mapsto f(x)g(x)$$

$0_A \in A$  is the constant function  $0_A(x) = 0_{\mathbb{R}}, \forall x \in A$  and  $1_A \in A$  is constant  $1_{\mathbb{R}}$ .  
Then  $\text{Fun}(X, \mathbb{R})$  is a ring.

### Example: Continuous Functions

Let  $X$  be a topological space.

Define  $C(X) = \{f \in \text{Fun}(X, \mathbb{R}) \mid f \text{ is continuous at every } x \in X\}$ , a subring of  $\text{Fun}(X, \mathbb{R})$ .

### Example: Infinitely Differentiable Functions

Let  $X$  be a manifold, e.g.  $X = \mathbb{R}$ .

Define  $C^\infty(X) = \{f \in C(X) \mid f \text{ is indefinitely differentiable at every } x \in X\}$  is a subring.

### Example: Periodic Functions

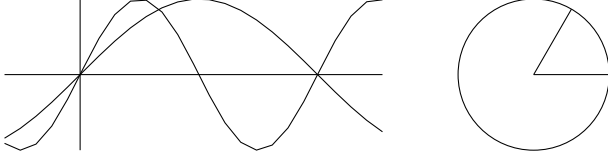
Let  $X = \mathbb{R}$ .

Define  $C_{2\pi}(\mathbb{R}) = \{f \in C(\mathbb{R}) \mid f(\theta + 2\pi) = f(\theta) \text{ for all } \theta \in \mathbb{R}\}$ .

e.g.  $0, 1 \in C_{2\pi}(\mathbb{R})$ .

For any  $n \in \mathbb{Z}$ ,  $\cos(n\theta), \sin(n\theta) \in C_{2\pi}(\mathbb{R})$ .

Note that this is the collection of all functions with period dividing  $2\pi$ .



### Example: Linear Combinations

Fix  $N \geq 1$  and let

$$P_N = \left\{ b_0 + \sum_{n=1}^N b_n \cos(n\theta) + \sum_{n=1}^N c_n \sin(n\theta) \mid b_0, \dots, b_n, c_0, \dots, c_n \in \mathbb{R} \right\} \subseteq C_{2\pi}(\mathbb{R})$$

Constants 0 and 1 are in  $P_N$  and  $f, g \in P_N \implies f \pm g \in P_N$ .

However, for  $f = g = \cos(N\theta) \in P_N$ ,

$$fg = \cos^2(N\theta) = \frac{1 + \cos(2N\theta)}{2}$$

Then  $fg \in P_N \iff \cos(2N\theta) \in P_N$ .

But,  $\forall f \in P_N, \int_0^{2\pi} f \cdot \cos(2N\theta) d\theta = 0$  while  $\int_0^{2\pi} \cos(2N\theta) \cdot \cos(2N\theta) d\theta \neq 0$ .

So  $P_N$  is not a subring.

But,

$$P_\infty = \bigcup_{N \geq 1} P_N$$

is a subring.

Need to show that  $P_\infty$  is closed under  $\cdot$ .

By linearity, it is enough to show that  $\forall m, n \geq 1$ ,

$$P_\infty \ni \begin{cases} \cos(m\theta) \cos(n\theta), \\ \cos(m\theta) \sin(n\theta), \\ \sin(m\theta) \sin(n\theta) \end{cases}$$

Since  $e^{i\theta} = \cos(\theta) + i \sin(\theta)$ ,

$$\begin{aligned} \cos(m\theta) \cos(n\theta) &= \left( \frac{e^{im\theta} + e^{-im\theta}}{2} \right) \left( \frac{e^{in\theta} + e^{-in\theta}}{2} \right) \\ &= \frac{1}{4} \left( e^{i(m+n)\theta} + e^{i(m-n)\theta} + e^{i(n-m)\theta} + e^{-i(m+n)\theta} \right) \\ &= \frac{1}{4} (2 \cos((m+n)\theta) + 2 \cos((m-n)\theta)) \\ &= \frac{1}{2} \cos((m+n)\theta) + \frac{1}{2} \cos((m-n)\theta) \end{aligned}$$

The rest follow similarly.

## Rings of Operators

Let  $V$  be a (real) vector space and  $A := \text{End}_{\mathbb{R}}(V) = \{T : V \rightarrow V \mid T \text{ is linear (MATH 21)}\}$ .

Then  $A$  is a ring with  $0_A =$  the zero transform and  $1_A =$  the identity transformation.

When  $V = \mathbb{R}^n$ , we can choose a basis (e.g.  $(e_1, \dots, e_n)$ ) and identify  $A = M_{n \times n}(\mathbb{R})$ .

### Example

For  $V = P_N$  or  $V = P_\infty$ ,

$$\frac{d}{d\theta} : V \rightarrow V$$

is a linear operator., so  $\frac{d}{d\theta} \in A$ .

Note: unless  $\dim(V) = 0, 1$ ,  $A$  is not a commutative ring.

### Definition: Commutator (Ring)

Let  $A$  be a ring and  $x, y \in A$ .

The commutator  $[x, y] = xy - yx$ .

So  $x$  and  $y$  commute if and only if  $[x, y] = 0$  in  $A$ .

### Example

Let  $V = C^\infty(\mathbb{R})$  with coordinates  $x$  and  $A = \text{End}_{\mathbb{R}}(V)$ .

For



$$V \ni f(x) \xrightarrow{T_1} x \cdot f(x) \in A$$

$$V \ni f(x) \xrightarrow{T_2} f'(x) = \frac{d}{dx}f(x) \in A$$

we have

$$\begin{aligned} [T_1, T_2]f(x) &= T_1(T_2(f(x))) - T_2(T_1(f(x))) \\ &= xf'(x) - (xf(x))' \\ &= -f(x) \end{aligned}$$

So  $[T_1, T_2] = -\text{Id} \in A$ .

### Definition: Zero Divisor

Let  $R$  be a ring.

An element  $r \in R$  is called a zero divisor if  $\exists s \neq 0 \in R$  such that  $rs = 0_R$  or  $sr = 0_R$ .

### Definition: Integral Domain

Say  $R$  is an integral domain if  $0_R \neq 1_R$  and  $\{\text{zero divisors of } R\} = \{0_R\}$ .

### Example

$R = \mathbb{Z}$  is an integral domain, since if  $n \neq 0 \in \mathbb{Z}$ , then  $\forall m \in \mathbb{Z} : mn = 0 \implies n = 0$ .

### Counterexample

$R = \mathbb{Z}/6\mathbb{Z}$  is not an integral domain, since  $\bar{2} \pmod{6}$  is a nonzero zero divisors.  
e.g.  $\bar{2} \cdot \bar{3} = \bar{0}$  and  $\bar{3} \neq \bar{0}$ .

### Counterexample

$R = M_{2 \times 2}(\mathbb{R})$ . Consider

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Then  $A_1 B_1 = 0 = B_1 A_1$  since  $A_1$  and  $B_1$  project onto the  $x$  and  $y$  axes respectively.

IMAGE HERE - PROJECTION OF V ONTO AXES

Similarly,  $A_2$  is a zero divisor since  $A_2 A_2 = 0$ .

### Definition: Nilpotency

Let  $R$  be a ring and  $r \in R$ .

Say  $r$  is nilpotent if  $\exists N \geq 1$  such that  $r^N = 0_R$  in  $R$ .

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  is nilpotent (with  $N = 2$ ), but

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  is not nilpotent.

### Proposition:

Let  $R$  be a ring and  $r \in R$  be a non zero divisor.

Then  $\forall a, b \in R, ra = rb \implies a = b$  and  $ar = br \implies a = b$ .

### Proof

By distributive law,  $ra = rb \implies r(a - b) = 0_R$ .

Let  $s := a - b$ . Since, by assumption,  $r$  is not a zero divisor,  $s = 0$  and  $a = b$ .

The proof is similar in the other case.

### Corollary

In an integral domain  $R$ , cancellation law  $ra = rb$  and  $r \neq 0_R$  implies  $a = b$ .

### Remarks

If  $S$  is a subring of  $R$  and  $R$  is an integral domain, then  $S$  is also an integral domain.

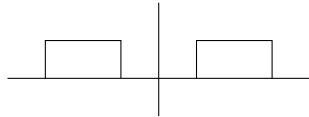
- Example

Since  $\mathbb{C}$  is an integral domain, so are  $\mathbb{Z}$  and  $\mathbb{Z}[i]$ .

### Examples of Zero Divisors

Let  $A = C(\mathbb{R})$

If  $f$  and  $g$  are zero at all overlap, they are zero divisors.



**November 21, 2023**

Main Topic Today: Units

### Definition: Ring

A ring is a set  $R$  equipped with two binary operations  $+$  and  $\cdot$  such that

1.  $(R, +)$  is an abelian group.
  - Additive neutral element:  $0_R = 0$ .
  - Additive Inverse:  $-x$ .
2.  $(R, \cdot)$  is a monoid.
  - Multiplicative neutral element:  $1_R = 1$ .
3. Distributive laws:
  - $a \cdot (b + c) = a \cdot b + a \cdot c$
  - $(b + c) \cdot a = b \cdot a + c \cdot a$

### Examples

1. Numbers
2. Functions
3. Operators

### Definition: Left/Right Inverse

Let  $R$  be a ring and  $x \in R$

A left inverse of  $x$  is an element  $l \in R$  such that  $lx = 1_R$ .

A right inverse of  $x$  is an element  $r \in R$  such that  $xr = 1_R$

### Proposition: Bilateral Inverse

Suppose  $x \in R$  has both a left inverse  $l$  and a right inverse  $r$ .

Then  $l = r$  is a 2-sided inverse.

### Proof

$$l = l1_R = l(xr) = (lx)r = 1_R r = r \quad \blacksquare$$

### Definition: Unit

1. An element  $x \in R$  is called invertible or a unit in  $R$  if  $x$  has a (2-sided) inverse.
2. The inverse (multiplicative) of  $x$  is denoted  $x^{-1}$ .

3.  $R^\times = \{\text{units in } R\}$  forms a group with  $\cdot$  as the binary operation called the unit group of  $R$ .

### Example 1: Numbers

Let  $R = \mathbb{Z}$ .

An integer  $n \in \mathbb{Z}$  if and only if  $\exists m \in \mathbb{Z} : mn = 1$  if and only if  $n = 1$  or  $n = -1$ .

Therefore  $\mathbb{Z}^\times = \{1, -1\}$  is a cyclic group of order 2.

### Example 2: Numbers

$R = \mathbb{Q}, \mathbb{R}, \text{ or } \mathbb{C} \implies R^\times = R \setminus \{0_R\}$ .

Recall  $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ .

### Example 3: Numbers

$R = \mathbb{Z}/p\mathbb{Z}$  where  $p$  is a prime.

Have shown that  $(\mathbb{Z}/p\mathbb{Z})^\times = (\mathbb{Z}/p\mathbb{Z}) \setminus \{\bar{0}\}$ .

### Definition: Division Ring

Let  $R$  be a nonzero ring (i.e.  $1_R \neq 0_R$ ).

Say  $R$  is a division ring if  $R^\times = R \setminus \{0_R\}$ .

### Definition: Field

A field is a commutative division ring.

### Examples

$R = \mathbb{Q}, \mathbb{R}, \mathbb{C}, \text{ and } \mathbb{Z}/p\mathbb{Z}$  are fields.

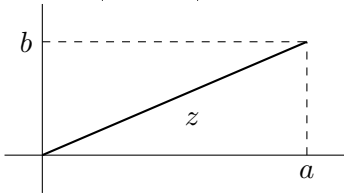
But  $\mathbb{Z}$  is not a field.

### Example 5

Let  $R = \mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$  is a subring of  $\mathbb{C}$ .

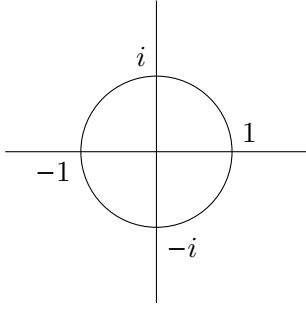
Q:  $R^\times = ?$

Define (a norm)  $N : R \rightarrow \mathbb{N}_0$  (natural numbers with 0) by  $N(a + bi) = (a + bi)(a - bi) = a^2 + b^2 (= |z|^2)$ .



- Proposition

For  $R = \mathbb{Z}[i]$ ,  $R^\times = \{1, -1, i, -i\} = \langle i \rangle$  is a cyclic subgroup of order 4.



– Proof

Suppose  $\alpha = a + bi \in R$  is a unit in  $R = \mathbb{Z}[i]$ .

Say  $\alpha\alpha' = 1$  where  $\alpha' = a' + b'i \in R$ .

Since  $N(\alpha\alpha') = N(\alpha)N(\alpha') = N(1) = 1$  and  $N(\alpha), N(\alpha') \in \mathbb{N}_0$ .

By Example 1 (Integers) above,  $N(\alpha) = 1$ .

### Example 6

Let  $A = \mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\} \subset \mathbb{R}$ .

Then  $A$  is a subring of  $\mathbb{R}$ , because  $0, 1 \in A$ ,

$(a + b\sqrt{2}) \pm (a' + b'\sqrt{2}) = (a \pm a') + (b \pm b')\sqrt{2}$ , and

$$(a + b\sqrt{2})(a' + b'\sqrt{2}) = (aa' + 2bb') + (ab' + a'b)\sqrt{2} \in \mathbb{Z}$$

if  $a, a', b, b' \in \mathbb{Z}$ .

Q:  $\mathbb{Z}[\sqrt{2}]^\times = ?$

Define (a norm)  $N'(a + b\sqrt{2}) = (a + b\sqrt{2})(a - b\sqrt{2}) = (a + b\sqrt{2})(a + b\sqrt{2})^* = a^2 - 2b^2$ .

So  $N' : \mathbb{Z}[\sqrt{2}] \rightarrow \mathbb{Z}$ .

Note: for  $\alpha = (a + b\sqrt{2}), \beta = (a' + b'\sqrt{2}) \in A$ ,

$$\begin{aligned} N'(\alpha)N'(\beta) &= \underbrace{(a + b\sqrt{2})}_{\alpha} \underbrace{(a - b\sqrt{2})}_{\alpha^*} \underbrace{(a' + b'\sqrt{2})}_{\beta} \underbrace{(a' - b'\sqrt{2})}_{\beta^*} \\ &= \alpha\beta\alpha^*\beta^* \end{aligned}$$

• Proposition

$$\mathbb{Z}[\sqrt{2}]^\times = \{a + b\sqrt{2} \mid a^2 - 2b^2 = \pm 1\}$$

– Proof

( $\subseteq$ ) Suppose  $\alpha = a + b\sqrt{2} \in A^\times$ , say  $\exists \beta \in A: \alpha\beta = 1$ .

Take  $N'$  of both sides, and get

$$\underbrace{N'(\alpha)N'(\beta)}_{\in \mathbb{Z}} = N'(1) = 1$$

Therefore  $N'(\alpha) \in \mathbb{Z}^\times = \{\pm 1\}$  and  $a^2 - 2b^2 = \pm 1$ .

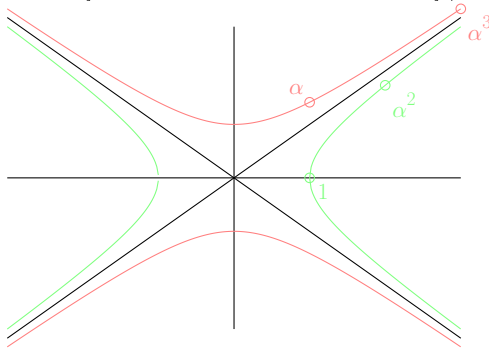
( $\supseteq$ ) If  $a^2 - 2b^2 = \pm 1$ , then for  $\alpha = a + b\sqrt{2}$  and  $\beta = a - 2b\sqrt{2}$ , we have  $\alpha\beta = \pm 1$ .

So either  $\alpha\beta = 1$  or  $\alpha(-\beta) = 1$ .

And either  $\beta$  or  $-\beta$  is the sought inverse of  $\alpha$ .

So  $\mathbb{Z}[\sqrt{2}]^\times \xleftrightarrow{1-1} (x, y) \in \mathbb{Z}^2$  such that  $x^2 - 2y^2 = \pm 1$ .

History: India in the 12th Century; later Pell's Equations



$$\alpha^2 = (1 + \sqrt{2})(1 + \sqrt{2}) = 3 + 2\sqrt{2}$$

$$\alpha^3 = (3 + 2\sqrt{2})(1 + \sqrt{2}) = 7 + 5\sqrt{2}.$$

– Fact (MATH 223A; Algebraic Number Theory)

Dirichlet's Unit Theorem

$$\mathbb{Z}[\sqrt{2}]^\times = \{\pm(1 + \sqrt{2})^n \mid n \in \mathbb{Z}\}$$

### Example 7: Operator

Let  $A$  be a commutative ring (e.g.  $A = \mathbb{Z}, \mathbb{R}, \mathbb{C}, \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ ) and  $n \geq 1$  integer.

Let

$$R = M_{n \times n}(A) = \left\{ \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \mid a_{ij} \in A \right\}$$

With “usual” matrix  $+$  and  $\cdot$ .  $R$  is a ring with

$$0 = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \quad 1_R = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

Q:  $R^\times = ?$

$B \in R$  is a unit if and only if  $\exists C \in R : BC = I_n = CB$ .

“Recall” (MATH 21 or 117)  $\det(B)$  can be defined as over  $\mathbb{R}$ , and

Define  $D \in R : d_{ij} = (-1)^{i+j} \det((j, i) - \text{minor of } B)$ .

And we have  $BD = \det(B) \cdot \det I_n = DB$  e.g.

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies D = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

So  $B \in R^\times$  if and only if  $\det(B) \in A^\times$  (take  $C = \frac{\overbrace{1}^{\in A^\times}}{\det(B)} \cdot D$ )

Define:

$$\text{GL}_n(A) = M_{n \times n}(A)^\times = \{B \in M_{n \times n}(A) \mid \det(B) \in A^\times\}$$

For example,

$$\mathrm{GL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = \pm 1 \right\}$$

$$\mathrm{GL}_2(\mathbb{F}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{F}_p, ad - bc \neq 0 \text{ in } \mathbb{F}_p \right\}$$

IMPORTANT FACT:  $|\mathrm{GL}_2(\mathbb{F}_p)| = p(p-1)^2(p+1)$  and

$$\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{F}_p \right\}$$

is a  $p$ -Sylow.

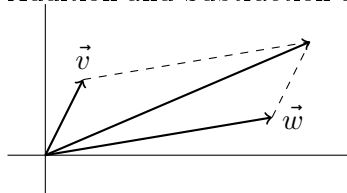
## Definition: Quaternions

(Numbers of Operators)

$\mathbb{H} = \mathbb{R}^4 = \{(a, b, c, d) \in \mathbb{R}^4\}$  but write  $a + b\hat{i} + c\hat{j} + d\hat{k}$ .

## Complex Arithmetic

Addition and Subtraction as vectors.



Multiplication:

1. “Force” distribution law.
2.  $r \in \mathbb{R}$  commutes  $i : ri = ir$ .
3.  $i^2 = -1$

## Quaternion Arithmetic

Addition and Subtraction as vectors.

Multiplication:

1. “Force” distribution law.
2.  $r \in \mathbb{R}$  commutes  $\hat{i}, \hat{j}, \hat{k} : r\hat{i} = \hat{i}r$
3.  $\hat{i}^2 = \hat{j}^2 = \hat{k}^2 = -1$

$$\begin{aligned} \hat{i}\hat{j} &= \hat{k} = -\hat{j}\hat{i} \\ \hat{j}\hat{k} &= \hat{i} = -\hat{k}\hat{j} \\ \hat{k}\hat{i} &= \hat{j} = -\hat{i}\hat{k} \end{aligned}$$

### Example

$$\begin{aligned} (2 + 3\hat{i})(5\hat{i} - 7\hat{j}) &\stackrel{1}{=} 2(5\hat{i}) + 2(-7\hat{j}) + (3\hat{i})(5\hat{i}) + (3\hat{i})(-7\hat{j}) \\ &= 10\hat{i} - 14\hat{j} + 15\hat{i}^2 - 21\hat{i}\hat{j} \\ &= -15 + 10\hat{i} - 14\hat{j} - 21\hat{k} \end{aligned}$$

### Definition: Quaternion Conjugate

Let  $q = a + b\hat{i} + c\hat{j} + d\hat{k} \in \mathbb{H}$ .

The quaternion conjugate  $q^* = a - b\hat{i} - c\hat{j} - d\hat{k} \in \mathbb{H}$

Note  $(q^*)^* = q$

Define  $N''(q) = qq^*a^2 + b^2 + c^2 + d^2 = q^*q \in \mathbb{R}_{\geq 0}$ .

### Proposition: Quaternions Form a Division Ring

$\mathbb{H}$  is a division ring (not a field, since  $\hat{i}\hat{j} \neq \hat{j}\hat{i}$ )

### Proof

Let  $q \in \mathbb{H}$  be nonzero, then  $N''(q) = r > 0$  is a real number.

Then  $qq^* = r = q^*q$ . Multiplying through by  $1/r$ , which commutes with both  $q$  and  $q^*$ ,

$$q \left( q^* \frac{1}{r} \right) = 1 \quad \text{and} \quad \left( \frac{1}{r} q^* \right) q = 1$$

So  $\frac{q^*}{N''(q)}$  is the 2-sided inverse.

### Example

$$(1 + 2\hat{i} + 3\hat{j} + 4\hat{k}) = \frac{1}{1^2+2^2+3^2+4^2}(1 - 2\hat{i} - 3\hat{j} - 4\hat{k}).$$

**November 28, 2023**

### Taking Stock

Have defined a ring  $(R, +, \cdot, 0_R, 1_R)$ .

Have defined and looked at the units  $R^\times$  (e.g.  $\mathbb{Z}$ ,  $\mathbb{F}_p$ ,  $\mathbb{C}$ ,  $\mathbb{Z}[i]$ ,  $\mathbb{Z}[\sqrt{2}]$ , etc.)



## Definition: Center of a Ring

Let  $R$  be a ring.

The center (or centre or zentrum)  $Z(R) = \{r \in R \mid rx = xr, \forall x \in R\}$ .

Fact:  $Z(R)$  is a subring of  $R$ .

Sketch of Proof:  $0_R, 1_R \in Z(R), x, y \in Z(R) \implies x \pm y, xy \in Z(R)$ .

### Example 0:

If  $R$  is commutative if and only if  $Z(R) = R$ .

### Example 1:

$$R = \mathbb{H} = \{a + b\hat{i} + c\hat{j} + d\hat{k} \mid a, b, c, d \in \mathbb{R}\}$$

What is  $Z(\mathbb{H})$ ?

By construction,  $Z(\mathbb{H}) \supseteq \mathbb{R} = \{a + 0\hat{i} + 0\hat{j} + 0\hat{k} \mid a \in \mathbb{R}\}$ .

Claim:  $Z(\mathbb{H}) = \mathbb{R}$ .

- Proof

( $\subseteq$ ) Let  $a + b\hat{i} + c\hat{j} + d\hat{k} \in Z(\mathbb{H})$ .

Then

1.  $(a + b\hat{i} + c\hat{j} + d\hat{k})\hat{i} = \hat{i}(a + b\hat{i} + c\hat{j} + d\hat{k})$
2.  $(a + b\hat{i} + c\hat{j} + d\hat{k})\hat{j} = \hat{j}(a + b\hat{i} + c\hat{j} + d\hat{k})$
3.  $(a + b\hat{i} + c\hat{j} + d\hat{k})\hat{k} = \hat{k}(a + b\hat{i} + c\hat{j} + d\hat{k})$

Expand 1.

$$-b + a\hat{i} + d\hat{j} - c\hat{k} = -b + a\hat{i} - d\hat{j} + c\hat{k}$$

Therefore  $c = 0$  and  $d = 0$ .

Left as an exercise: expand 2. and 3. to demonstrate  $b = 0$ . ■

## Definition: Ring Homomorphism

Let  $R$  and  $S$  be rings.

A function  $f : R \rightarrow S$  is called a ring homomorphism if

1.  $f(r_1 + r_2) = f(r_1) + f(r_2), \forall r_1, r_2 \in R$ .  
(implies that  $f(0_R) = 0_S$  and  $f(-r) = -f(r), \forall r \in R$ )
2.  $f(r_1 r_2) = f(r_1) f(r_2), \forall r_1, r_2 \in R$  and  $f(1_R) = 1_S$ .  
(implies  $f(r^{-1}) = (f(r))^{-1}, \forall r \in R^\times$ )

### Example 0:

$R \xrightarrow{\text{Id}} R$  is a ring homomorphism.

Fun exercise: If  $\mathbb{R} \xrightarrow{f} \mathbb{R}$  is a ring homomorphism, then  $f = \text{Id}$ .

**Example 1:**

If  $R'$  is a subring of  $R$ , then the inclusion  $i : R' \hookrightarrow R$  is a ring homomorphism.

e.g.  $\mathbb{Z}[\sqrt{2}] \xrightarrow{\text{incl.}} \mathbb{R}$

**Example 2:**

In general, a ring homomorphism does not need to be injective or surjective.

Let  $R = \mathbb{Z}[\sqrt{2}]$ .

Define  $f : R \rightarrow R$  by  $f(a + b\sqrt{2}) = a - b\sqrt{2}$ .

Then  $f$  is a ring homomorphism (automorphism).

Need to check that  $\forall a_1, a_2, b_1, b_2 \in \mathbb{Z}$ :

$$1. f((a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2})) = f(a_1 + b_1\sqrt{2}) + f(a_2 + b_2\sqrt{2}).$$

$$2. f((a_1 + b_1\sqrt{2})(a_2 + b_2\sqrt{2})) = f(a_1 + b_1\sqrt{2})f(a_2 + b_2\sqrt{2})$$

$$3. f(1 + 0\sqrt{2}) = 1 + 0\sqrt{2}$$

- Proof of 1

$$\begin{aligned} f((a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2})) &= f((a_1 + a_2) + (b_1 + b_2)\sqrt{2}) \\ &= (a_1 + a_2) - (b_1 + b_2)\sqrt{2} \\ &= (a_1 - b_1\sqrt{2}) + (a_2 - b_2\sqrt{2}) \\ &= f(a_1 + b_1\sqrt{2}) + f(a_2 + b_2\sqrt{2}) \end{aligned}$$

- Proof of 2

$$\begin{aligned} f((a_1 + b_1\sqrt{2})(a_2 + b_2\sqrt{2})) &= (a_1a_2 + 2b_1b_2) - (a_1b_2 + a_2b_1)\sqrt{2} \\ &= (a_1 - b_1\sqrt{2})(a_2 - b_2\sqrt{2}) \\ &= f(a_1 + b_1\sqrt{2})f(a_2 + b_2\sqrt{2}) \end{aligned}$$

- Proof of 3

$$1 + 0\sqrt{2} = 1 - 0\sqrt{2}$$

- Remark

Id and  $f$  are the only ring homomorphisms  $\mathbb{Z}[\sqrt{2}] \xrightarrow{g} \mathbb{Z}[\sqrt{2}]$ .

Prove:  $g(\sqrt{2}) = \sqrt{2}$  or  $-\sqrt{2}$ .

**Example 3**

Let  $X$  and  $Y$  be topological spaces (e.g.  $X \subseteq \mathbb{R}^m$ ,  $Y \subseteq \mathbb{R}^n$ )

Let  $\phi : X \rightarrow Y$  be a continuous map.

We have the map  $C(X) \xleftarrow{\phi^*} C(Y) = \{f : Y \rightarrow \mathbb{R} \mid \text{continuous everywhere}\}$ .

If  $f : Y \rightarrow \mathbb{R}$ , then  $f \circ \phi = \phi^*(f) : X \xrightarrow{\phi} Y \xrightarrow{f} \mathbb{R}$ .

Then  $\phi^*$  is a ring homomorphism.

- Proof of Addition

Left as an exercise.

- Proof of Multiplication

Let  $f_1, f_2 \in C(Y)$  and want to verify that  $\phi^*(f_1 f_2) = \phi^*(f_1) \phi^*(f_2)$ .

Both sides are elements in  $C(X)$ , so need to verify

$$[\phi^*(f_1 f_2)](x) = [\phi^*(f_1) \phi^*(f_2)](x), \forall x \in X$$

$$\begin{aligned} ((f_1 f_2) \circ \phi)(x) &= (f_1 f_2)(\underbrace{\phi(x)}_{\in Y}) \\ &= f_1(\phi(x)) f_2(\phi(x)) \\ &= (\underbrace{\phi^* f_1}_{\in C(X)})(x) (\underbrace{\phi^* f_2}_{\in C(X)})(x) \\ &= [(\phi^* f_1)(\phi^* f_2)](x) \end{aligned}$$

#### Example 4

IMAGE HERE - DOUBLE TORUS SYMMETRIC ABOUT AN AXIS

Then  $C(X) \xleftarrow{\phi^*} C(X)$ .

#### Non-example 4'

Let  $R = \mathbb{H}$  and let  $f : R \rightarrow R$  be defined by  $f(q) = q^*$ .

That is, if  $q = a + b\hat{i} + c\hat{j} + d\hat{k}$  then  $q^* = a - b\hat{i} - c\hat{j} - d\hat{k}$ .

- Proof of Addition

$$f(q_1 + q_2) = f(q_1) + f(q_2), \forall q_1, q_2 \in \mathbb{H} \text{ (cf. } \mathbb{Z}[\sqrt{2}])$$

- Failure of Multiplication

$f(q_1 q_2) \neq f(q_1) f(q_2)$  in general.

If  $q_1 = \hat{i}$  and  $q_2 = \hat{j}$ , then

$$(\hat{i}\hat{j})^* = (\hat{k})^* = -\hat{k} \neq \hat{k} = (-\hat{i})(-\hat{j})$$

So  $f = *$  is not a ring homomorphism.

#### Definition: Ring Anti-Homomorphism

Addition is unchanged, but  $f(r_1 r_2) = f(r_2) f(r_1)$ .

### Example

Claim  $\ast = f$  above is an antihomomorphism.

First,  $1^\ast = 1$ .

If  $q_1 = q_1' + q_1''$  and if we know  $(q_1' q_2)^\ast = q_2^\ast (q_1')^\ast$  and  $(q_1'' q_2)^\ast = q_2^\ast (q_1'')^\ast$ , then we conclude

$$(q_1 q_2)^\ast = q_2^\ast q_1^\ast, \forall q_1, q_2 \in \mathbb{H}$$

by adding the two and using the distributive law.

Similarly for  $q_2 = q_2' + q_2''$ .

Therefore, it is enough to show that  $q_1 \in \{a, b\hat{i}, c\hat{j}, d\hat{k}\}$  and  $q_2 \in \{t, x\hat{i}, y\hat{j}, z\hat{k}\}$ .

Many cases are trivial (e.g.  $a$  commutes with everything).

Consider  $q_1 = c\hat{j}$  and  $q_2 = z\hat{k}$ , then

$$((c\hat{j})(z\hat{k}))^\ast = (cz\hat{i})^\ast = (-cz)\hat{i} = cz\hat{k}\hat{j} = (-z\hat{k})(-c\hat{j}) = (z\hat{k})^\ast (c\hat{j})^\ast$$

### Proposition:

Let  $u \in R^\times$ .

Then the map

$$\begin{aligned} \text{Int}(u) : R &\rightarrow R \\ x &\mapsto uxu^{-1} \end{aligned}$$

is a ring homomorphism.

### Proof:

1. Addition

$$\text{Int}(u)(r_1 + r_2) = u(r_1 + r_2)u^{-1} = (ur_1 + ur_2)u^{-1} = ur_1u^{-1} + ur_2u^{-1} = \text{Int}(u)(r_1) + \text{Int}(u)(r_2)$$

2. Product

$$ur_1r_2u^{-1} = (ur_1u^{-1})(ur_2u^{-1})$$

$$u1u^{-1} = 1$$

Note  $\text{Int}(u^{-1})$  is the 2-sided inverse of  $\text{Int}(u)$ .

### Example 5

Let  $R = \mathbb{H}$ .

Recall

$$\begin{aligned} N : \mathbb{H}^\times &\rightarrow \mathbb{R}_{>0} \\ q &\mapsto N(q) = qq^\ast = q^\ast q \end{aligned}$$

$N : \mathbb{H}^\times \rightarrow \mathbb{R}_{>0}$  is a group homomorphism.

i.e.  $N(q_1 q_2) = N(q_1)N(q_2)$ ,  $\forall q_1, q_2 \in \mathbb{H}^\times$ .

**Proof**

$$\begin{aligned}
N(q_1 q_2) &= (q_1 q_2)(q_1 q_2)^* \\
&= (q_1 q_2)(q_2^* q_1^*) \\
&= q_1(q_2 q_2^*)q_1^* \\
&= q_1 \underbrace{N(q_2)}_{\in \mathbb{R} = Z(\mathbb{H})} q_1^* \\
&= q_1 q_1^* N(q_2) \\
&= N(q_1)N(q_2)
\end{aligned}$$

**Definition:**

$\mathbb{H}^{\times 1} = \ker(\mathbb{H}^{\times} \xrightarrow{N} \mathbb{R}_{>0}) = \{a + b\hat{i} + c\hat{j} + d\hat{k} \mid a^2 + b^2 + c^2 + d^2 = 1\} = S^3$  is a (mult) group.  
 Note, for  $u \in \mathbb{H}^{\times 1}$ ,  $u^{-1} = u^*$ .

**Definition:**

$$V = \{q \in \mathbb{H} \mid q + q^* = 0\} = \{x\hat{i} + y\hat{j} + z\hat{k} \mid x, y, z \in \mathbb{R}\} = \mathbb{R}^3.$$

**Proposition:**

For  $u \in \mathbb{H}^{\times 1}$ , write  $R(u) = \text{Int}(u)$ .

Then  $R(u) : \mathbb{H} \rightarrow \mathbb{H}$  is a linear map (MATH 21) and  $R(u)V \subseteq V$ .

**Proof**

Since  $\text{Int}$  is a ring automorphism, for  $a_1, a_2 \in \mathbb{R}$  and  $v_1, v_2 \in \mathbb{H}$ ,

$$R(u)(a_1 v_1 + a_2 v_2) = [R(u)(a_1)][R(u)(v_1)][R(u)(a_2)][R(u)(v_2)] = a_1 R(u)(v_1) + a_2 R(u)(v_2)$$

So  $R(u)$  is linear.

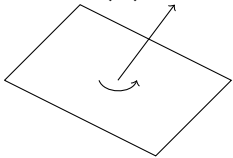
Let  $q \in V$  (i.e.  $q + q^* = 0$ ). Then

$$(R(u)q) + (R(u)q)^* = uqu^* + (uqu^*)^* = uqu^* + uq^*u^* = u(q + q^*)u^* = 0$$

**Proposition:**

Let  $u \in \mathbb{H}^{\times 1}$ , write  $u = (\cos(\theta)) + (\sin(\theta))\hat{v}$  where  $\theta \in \mathbb{R}$  and  $\hat{v} \in V \cap \mathbb{H}^{\times 1} = S^2$ .

Then  $R(u) : \mathbb{R}^3 = V \rightarrow V = \mathbb{R}^3$  is the rotation of  $\mathbb{R}^3$  with the axis  $\hat{v}$  by the angle  $2\theta$ .



**November 30, 2023**

### Previously

Have defined  $(R, +, \cdot, 0_R, 1_R)$ .

Looked at  $R^\times$  (e.g.  $\mathbb{Z}[\sqrt{2}]$ ,  $\mathbb{H}$ )

Homomorphisms and antihomomorphisms.

### Definition: Ideal

Let  $I$  be an additive subgroup of a ring  $R$ . Say that  $I$  is

1. a left ideal of  $R$  if  $\forall x \in I, \forall r \in R, rx \in I$  (i.e. closed under left multiplication by  $R$ )
2. a right ideal of  $R$  if  $\forall x \in I, \forall r \in R, xr \in I$  (i.e. closed under right multiplication by  $R$ )
3. a (bilateral) ideal of  $R$  if  $I$  is both a left ideal and a right ideal.

### Remarks

1. When  $R$  is commutative, the three notions coincide.
2. The only subring that is a left or a right ideal is  $R$ .

### Example 0

Let  $X \subseteq R$  be a subset.

The left ideal generated by  $X$

$$RX = {}_R(X) = \left\{ \sum_{i=1}^n r_i x_i \mid n \in \mathbb{N}_0, r_i \in R, x_i \in X \right\}$$

By convention, when  $n = 0$  nothing is summed.

The right ideal generated by  $X$

$$XR = (X)_R = \left\{ \sum_{i=1}^n x_i s_i \mid n \in \mathbb{N}_0, x_i \in X, s_i \in R \right\}$$

The (bilateral) ideal generated by  $X$

$$(X) = \left\{ \sum_{i=1}^n r_i x_i s_i \mid n \in \mathbb{N}_0, x_i \in X, r_i, s_i \in R \right\}$$

### Definition: Principle Ideal

If  $X = \{x\}$ , then write  $(X) = (\{x\}) = (x)$  and call it the principle ideal generated by  $X$ .

**Proposition:**

Let  $D$  be a division ring (e.g.  $\mathbb{R}, \mathbb{F}_p, \mathbb{H}$ ).

Then the only left (respectively right) ideals of  $D$  are  $\{0\}$  and  $D$ .

**Proof**

Let  $I$  be a nonzero left ideal, so  $\exists x \in D$  such that  $x \neq 0$  and  $x \in I$ .

Then since  $D$  is a division ring,  $x$  has an inverse  $y$  such that  $xy = 1_D = yx$ .

Since  $I$  is a left ideal, take  $r = y$  and  $x = x \in I$ .

In the definition, we get  $1_D \in I$ , but then for any  $r \in D$ ,  $r = r1_D \in I$ .

The argument for right ideals follows similarly. ■

**Proposition:**

Let  $R = \mathbb{Z}$ .

Then the ideals of  $\mathbb{Z}$  are exactly  $a\mathbb{Z}$  where  $a \in \mathbb{N}_0$ .

**Proof**

For any  $a \in \mathbb{Z}$ , the additive subgroup  $a\mathbb{Z} = \{na \mid n \in \mathbb{Z}\}$  is indeed an ideal since  $\forall r \in \mathbb{Z}, \forall na \in a\mathbb{Z}, r(na) = (rn)a \in a\mathbb{Z}$ .

Conversely, if  $I$  is any ideal of  $\mathbb{Z}$ , it is an additive subgroup of  $\mathbb{Z}$ ; in Group Theory we have shown that  $I = a\mathbb{Z}$  for some  $a \geq 0$ . ■

**Definition: Principle Ideal Domain (PID)**

Let  $R$  be an integral domain ( $0_R \neq 1_R$  and the only zero divisors of  $R$  is  $0_R$ ).

Say that  $R$  is a principle ideal domain (PID) if any ideal in  $R$  is a principle ideal.

**Example 1**

Following from the previous proposition,  $\mathbb{Z}$  is a PID.

**Example 2**

Let  $F$  be a field (e.g.  $\mathbb{C}$  or  $\mathbb{F}_p$ ) and  $R = M_{2 \times 2}(F) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in F \right\}$  is a noncommutative ring.

$$I_1 = \left\{ \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} \mid x, y \in F \right\} \quad I_2 = \left\{ \begin{bmatrix} 0 & 0 \\ x & y \end{bmatrix} \mid x, y \in F \right\}$$

are additive subgroups of  $R$ .

Then  $I_1$  is a left ideal but NOT a right ideal of  $R$  while  $I_2$  is only a right ideal.

Let  $r = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  and  $z = \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} \in I_1$  be arbitrary. Then  $rz = \begin{bmatrix} ax + by & 0 \\ cx + dy & 0 \end{bmatrix} \in I_1$ .

However, for  $z = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  and  $r = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ ,  $zr = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \notin I_1$  if  $b \neq 0_F$ .

Exercise: Prove  $I_2$  is a right ideal but not a left ideal.

**Proposition:**

The only (bilateral) ideals of  $M_{2 \times 2}(F)$  are  $\{0\}$  and  $R$ .

**Sketch of Proof**

Let  $I$  be a nonzero, bilateral ideal, say  $\exists \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0$  in  $I$ .

Note  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$  while  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$  and  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$ .

So, without loss of generality, we may show that  $a \neq 0$  gives

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \implies \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in I$$

Repeating this trick gives

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in I$$

Therefore,  $I = R$ .

**Definition: Quasisimple Ring**

Say a ring  $R$  is quasisimple if the only ideals in  $R$  are  $\{0\}$  and  $R$  (e.g. division rings,  $M_{2 \times 2}(F)$ ).

**Construction:**

Let  $I$  be an ideal of  $R$ . Then  $R/I$  (the quotient group for  $+$ ) has a natural ring structure:

$$(r_1 + I)(r_2 + I) := r_1 r_2 + I$$

Well-defined: If  $r'_1, r'_2 \in R$  satisfy (1)  $r'_1 + I = r_1 + I$  and (2)  $r'_2 + I = r_2 + I$ , then we want  $r'_1 + r'_2 + I = r_1 r_2 + I$ . (1) says  $r'_1 - r_1 = x \in I$  and (2) says  $r'_2 - r_2 = y \in I$ . So  $r'_1 = r_1 + x$  and  $r'_2 = r_2 + y$ . Then

$$r'_1 r'_2 = (r_1 + x)(r_2 + y) = r_1 r_2 + r_1 y + x r_2 + xy$$

$r_1 y \in I$ , since  $I$  is a left ideal;  $x r_2 \in I$  since  $I$  is a right ideal;  $xy \in I$  since it is an ideal.

Since  $I$  is an additive subgroup,  $r'_1 r'_2 + I = r_1 r_2 + I$ .

Need to verify the axioms for a ring:

$$((r_1 + I)(r_2 + I))(r_3 + I) = (r_1 + I)((r_2 + I)(r_3 + I))$$

This follows from  $(r_1 r_2) r_3 = r_1 (r_2 r_3)$ .

$1_R + I$  is the multiplicative neutral element.

Distribution laws: similarly follow from the distribution laws of  $R$  itself.



## Definition: Canonical Quotient Homomorphism

The canonical quotient homomorphism

$$\begin{aligned} R &\xrightarrow{\pi} R/I \\ r &\longmapsto r + I \end{aligned}$$

## Definition / Proposition: Kernel of a Ring Homomorphism

Let  $\phi : R \rightarrow S$  be a ring homomorphism.

The kernel of  $\phi$  is  $\ker(\phi) = \{r \in R \mid \phi(r) = 0_S\}$  is a bilateral ideal.

cf.  $G \xrightarrow{\phi} H$  group homomorphisms,  $\ker(\phi) \trianglelefteq G$ .

### Proof

$\ker(\phi)$  is an additive subgroup and if  $x \in \ker(\phi)$  and  $\forall r \in R$ , then  $\phi(rx) = \phi(r)\phi(x) = \phi(r)0 = 0$  and similarly  $\phi(xr) = 0$ . ■

## Theorem: Universal Mapping Property (Rings)

Let  $R$  be a ring and  $I$  a bilateral ideal.

Then for any ring  $S$ , there is a natural correspondence

$$\{f \in \text{Hom}_{\text{ring}}(R, S) \mid \ker(f) \supseteq I\} \leftrightarrow \text{Hom}_{\text{ring}}(R/I, S)$$

### Sketch of Proof

( $\longrightarrow$ ) Given such  $f$ , define the function  $\bar{f} : R/I \rightarrow S$  by  $r + I \mapsto f(r)$ . Show well-defined and is a ring homomorphism.

( $\longleftarrow$ ) given  $R/I \xrightarrow{\phi} S$ ,  $f = \phi \circ \pi$ .

- Recall: Universal Mapping Property (Groups)  
 $G \supseteq N$ , then for any group  $H$

$$\{f \in \text{Hom}_{\text{group}}(G, H) \mid \ker(f) \supseteq N\} \leftrightarrow \text{Hom}_{\text{group}}(G/N, H)$$

## Theorem: Correspondence & 2nd Isomorphism for Ideals

Let  $I$  be an ideal of  $R$ .

### Part 1

$$\{\text{bilateral ideals } J \text{ of } R \text{ that contain } I\} \leftrightarrow \{\text{bilateral ideals of } \bar{J} \text{ of } R/I\}$$

$$J \mapsto J/I \text{ and } \bar{J} \mapsto \pi^{-1}(\bar{J}).$$

## Part 2

$R/J \xrightarrow{\sim} (R/I)/(J/I)$  as rings.

Proof omitted.

## Operations on Ideals

Let  $I$  and  $J$  be ideals of  $R$ .

1.  $I \cap J$  is an ideal of  $R$ .
2.  $I + J = \{x + y \mid x \in I, y \in J\}$  is an ideal of  $R$ .
3.  $IJ = \{\sum_{k=1}^n x_k y_k \mid n \in \mathbb{N}_0, x_k \in I, y_k \in J\}$  is an ideal of  $R$ .

### Example 3

Let  $R = \mathbb{Z}$ ,  $I = a\mathbb{Z}$  and  $J = b\mathbb{Z}$  where  $a, b \geq 0$ .

Then  $I \cap J = \{n \in \mathbb{Z} \mid a \mid n \text{ and } b \mid n\} = \text{lcm}(a, b)\mathbb{Z}$ .

$IJ = \{\sum_{k=1}^n (ax_k)(by_k) \mid n \in \mathbb{N}_0, x_k, y_k \in \mathbb{Z}\} = (ab)\mathbb{Z}$ .

$I + J = \{ax + by \mid x, y \in \mathbb{Z}\} = \text{gcd}(a, b)\mathbb{Z}$  (MATH 110).

- Example 4: Euclidean Algorithm

If  $a = 8$  and  $b = 22$ , then  $22 = 2(8) + 6$  and  $8 = 1(6) + 2$ .

Then  $2 = 8 - 6 = 8 - (22 - 2(8)) = 3(8) + (-1)22$ .

## December 5, 2023

THIS WEEK: Rings are commutative (unless specified otherwise). Looking at ideals.

### Recall

For  $R = \mathbb{Z}$ ,

1. Let  $a, b \in \mathbb{Z}$ . Say  $a$  divides  $b$  (and write  $a \mid b$ ) if  $\exists c \in \mathbb{Z}$  such that  $b = ac$ .

Note that in noncommutative rings, one must specify left or right divisibility.

2. Let  $a, b \in \mathbb{Z}$  and  $m \in \mathbb{Z}$  (“modulus”). Write  $a \equiv b \pmod{m}$  if  $m \mid (a - b)$ .

$\equiv$  is an equivalence relation and, for any  $c \in \mathbb{Z}$ ,  $a \pm c \equiv b \pm c \pmod{m}$  and  $ac \equiv bc \pmod{m}$ .  
 $\mathbb{Z}/m\mathbb{Z}$  is a ring.

3.  $p > 0$  in  $\mathbb{Z}$  is a prime number if  $p \neq 1$  and satisfies the two equivalent conditions:

(a) if  $p = ab$  for some  $a, b \in \mathbb{Z}$ , then either  $a = \pm 1$  or  $b = \pm 1$ .

(b) if  $p|ab$  for any  $a, b \in \mathbb{Z}$ , then  $p|a$  or  $p|b$ .

The equivalence between (3a) and (3b) relies on

### Theorem:

Any nonzero integer  $n$  can be uniquely written:

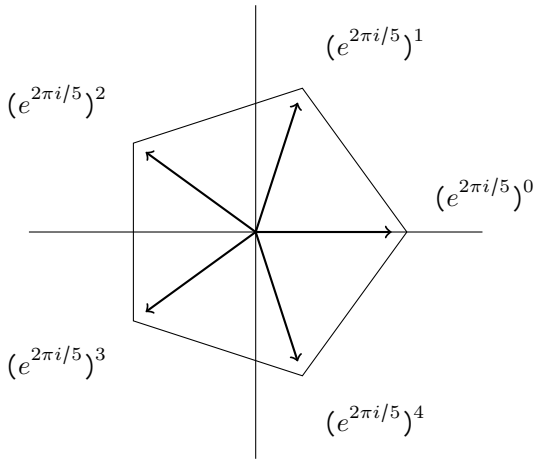
$$n = \pm 2^{e_2} \cdot 3^{e_3} \cdot 5^{e_5} \cdot \dots \cdot p^{e_p} \cdot \dots$$

where  $e_p \in \mathbb{N}_0$  are exponents, and all but finitely many equal zero.

This can be generalized to general rings, such as

:

$$\mathbb{Z} \left[ e^{2\pi i/p} \right]$$



If an analogue of the theorem holds for  $R_p$ , then we get Fermat's Last Theorem.

### On Commutative Rings

Let  $R$  be a commutative ring

1. Let  $a, b \in R$ . Say  $a$  divides  $b$  (and write  $a|b$ ) if  $\exists c \in R$  such that  $b = a \cdot c$ .

Note that in noncommutative rings, one must specify left or right divisibility.

2. Let  $a, b \in R$  and  $I \subseteq R$  an ideal ("modulus"). Write  $a \equiv b \pmod{I}$  if  $I \ni (a - b)$ .

$\equiv$  is an equivalence relation and, for any  $c \in R$ ,  $a \pm c \equiv b \pm c \pmod{I}$  and  $ac \equiv bc \pmod{I}$ .

3. Let  $\pi \in R \setminus \{0\}$  and suppose  $\pi \notin R^\times$ .

(a) Say  $\pi$  is an irreducible element if  $\pi = ab$  for some  $a, b \in R$ , then  $a \in R^\times$  or  $b \in R^\times$ .

(b) Say  $\pi$  is a prime element if  $\pi | ab$  for any  $a, b \in R$ , then  $\pi | a$  or  $\pi | b$ .

### Proposition:

Let  $R$  be an integral domain and let  $x \in R$  be a nonzero unit element.  
If  $x$  is a prime element, then  $x$  is irreducible.

### Proof

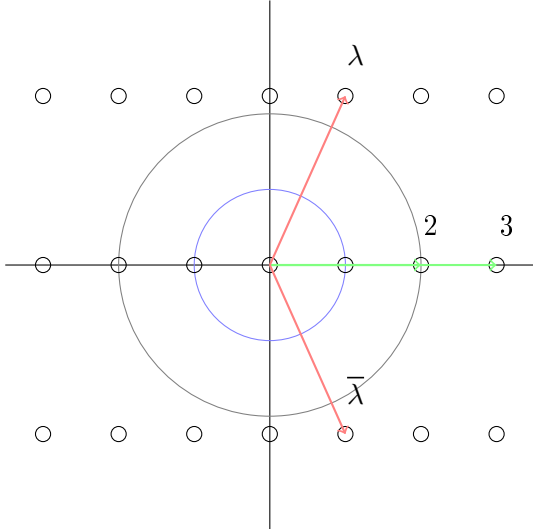
Suppose  $x$  is a prime element and that (1)  $x = ab$  for some two  $a, b \in R$ .  
Then  $x | ab$ , since  $ab = x \cdot 1$ . By assumption of primality,  $x | a$  or  $x | b$ .  
Without loss of generality, we may assume that  $x | a$ . Say (2)  $a = x \cdot c$  for some  $c \in R$ .  
Combine (1) and (2), and get  $x = (xc)b = x(cb)$ .  
Since  $x \neq 0$  and  $R$  is an integral domain, we can cancel  $x$  and get  $1 = cb = bc$ .  
Therefore  $b \in R^\times$ . ■

### Converse

The converse (irreducible implies prime) is generally false.

#### • Example

$R = \mathbb{Z}[\sqrt{5}i] = \{a + b\sqrt{5}i \mid a, b \in \mathbb{Z}\}$  is a subring of  $\mathbb{C} = \mathbb{Z}[i]$ .



Define  $N : R \rightarrow \mathbb{N}_0$  by  $N(a + b\sqrt{5}i) = |a + b\sqrt{5}i|^2 = a^2 + 5b^2$ .

Note that  $N(\alpha\beta) = N(\alpha)N(\beta)$ ,  $\forall \alpha, \beta \in R$ .

$\{\alpha \in R \mid N(\alpha) = 1\} = \{\pm 1\} = R^\times$

There are no elements of  $R$  with norm 2 or 3.

Then  $2, 3, \lambda, \bar{\lambda}$  are irreducible elements of  $R = \mathbb{Z}[\sqrt{5}i]$ , but not prime elements.

**Proof (Irreducibility)**

Start with  $\lambda$ . Suppose  $\lambda = \alpha\beta$  for some  $\alpha, \beta \in R$ . Take  $N$ .

$$\begin{aligned} N(\lambda) &= N(\alpha)N(\beta) \\ 6 &= \underbrace{N(\alpha)}_{\in \mathbb{N}_0} \underbrace{N(\beta)}_{\in \mathbb{N}_0} \end{aligned}$$

Therefore,  $(N(\alpha), N(\beta)) \in \{(1, 6), (2, 3), (3, 2), (6, 1)\}$ , but no elements of  $R$  are of norm 2 or 3. So  $N(\alpha) = 1$  or  $N(\beta) = 1$ , which implies  $\alpha \in R^\times$  or  $\beta \in R^\times$ . As an exercise, show that  $\bar{\lambda}, 2, 3$  are irreducible.

**Proof (Primacy)**

2 is not a prime element of  $R = \mathbb{Z}[\sqrt{5}i]$ .

$\lambda\bar{\lambda} = 6 = 2 \cdot 3$ , so  $2|\lambda\bar{\lambda}$ , but if  $2|\lambda$  or  $2|\bar{\lambda}$  then, by the multiplicativity of  $N$  we would have

$$4 = N(2)|N(\lambda) = N(\bar{\lambda}) = 6$$

in  $\mathbb{Z}$ , which is absurd.

Therefore 2 is not a prime element.

**Definition: Associates**

Let  $R$  be an integral domain.

Two elements  $x$  and  $y$  are associates in  $R$  and we write  $x \sim y$  if  $x|y$  and  $y|x$ .

Equivalently,  $x = y \cdot u$  for some  $u \in R^\times$ .

Equivalently,  $x, y \in R$  lie in the same orbit in the action of  $R^\times$  on  $R$  by multiplication.

In dealing with divisibility, an element is as good as any of its associates.

**Proposition:**

Let  $R$  be an integral domain,  $r, s \in \mathbb{N}_0$ , and  $p_1, \dots, p_r, q_1, \dots, q_s$  prime elements of  $R$ . Suppose

$$(*) \quad p_1 \cdots p_r = q_1 \cdots q_s$$

Then  $r = s$  and, after renumbering, we have  $p_i \sim q_i, i = 1, \dots, r$ .

**Remark**

The proposition becomes false when “prime” is replaced with “irreducible”.

**Proof**

Induction on  $r$ .

If  $r = 0$ ,  $1 = q_1 \cdots q_s$ , which implies (if  $s > 0$ ), that  $q_1$  is a unit which is absurd.

Suppose we know the proposition for all the values smaller than  $r$ .

Note  $p_1|q_1 \cdots q_s \implies p_1|q_i$  for some  $i = 1, \dots, s$ .

Then  $q_1 = p_r x$ ,  $\exists x \in R$ . But  $q_1$ , being prime, is irreducible. Hence  $x \in R^\times$ .

Therefore,  $p_r \sim q_i$  for some  $i$ . So reorder the  $q$ 's, and we may assume that  $p_r \sim q_s$  so  $p_r = q_s \cdot u$ . Substituting and canceling gives

$$\begin{aligned} p_1 \cdots p_{r-1} p_r &= q_1 \cdots q_{s-1} q_s \\ p_1 \cdots p_{r-1} q_s u &= q_1 \cdots q_{s-1} q_s \\ p_1 \cdots (p_{r-1} u) &= q_1 \cdots q_{s-1} \end{aligned}$$

Using the inductive hypothesis, we get  $r - 1 = s - 1$ . Therefore  $r = s$  and after renumbering,  $p_i \sim q_i$  for all  $i$ .

### Definition: Unique Factorization Domain (UFD)

Let  $R$  be an integral domain.

Say  $R$  is a unique factorization domain if every nonzero, nonunit  $x \in R$  can be written as some product of prime elements.

### Definition: Maximal Ideal

Let  $R$  be a commutative ring, and  $I \subsetneq R$  an ideal.

Say  $I$  is a maximal ideal if for any ideal  $J \supsetneq I$ , we necessarily have  $J = I$  or  $J = R$ .

Equivalently,  $R/I$  is a field.

### Definition: Prime Ideal

Let  $R$  be a commutative ring, and  $I \subsetneq R$  an ideal.

Say  $I$  is a prime ideal if  $R/I$  is an integral domain.

Equivalently,  $\forall a, b \in R, ab \in I \implies a \in I$  or  $b \in I$ .

### Definition: Ascending Chain of Ideals

Let  $R$  be a ring.

An ascending chain of ideals is

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots \subseteq I_n \subseteq \cdots$$

where each  $I_k$  is an ideal of  $R$ .

### Definition: Noetherian Ring

Say  $R$  is a Noetherian ring if every ascending chain of ideals is stationary.

$$\exists N \gg 0 : \quad I_{N+k} = I_N, \quad \forall k \geq 0$$

## Theorem: A Principle Ideal Domain is a Unique Factorization Domain

### Lemma

Let  $R$  be a PID,  $x \in R \setminus \{0\}$ ,  $x \notin R^\times$ . Then the following are equivalent

1.  $(x)$  is maximal.
2.  $(x)$  is a prime ideal.
3.  $x$  is a prime element.
4.  $x$  is an irreducible element.

- Proofs

(1)  $\implies$  (2) since a field is an integral domain.

(2)  $\implies$  (3) since  $ab \in (x) \iff x|ab$ ,  $a \in (x) \iff x|a$ , and  $b \in (x) \iff x|b$ .

(3)  $\implies$  (4) has been proven above.

(4)  $\implies$  (1)

Let  $x$  be an irreducible element and  $I := (x)$ .

Suppose  $J$  is an ideal containing  $I$ . Since  $R$  is a PID,  $J = (y)$  for some  $y \in R$ .

So  $x \in (x) \subseteq (y) = \{yr \mid r \in R\}$ .

Write  $x = yr$  for some  $r \in R$ .

Since  $x$  is irreducible, either  $y \in R^\times$  or  $r \in R^\times$ .

If  $y \in R^\times$ , then  $J = (y) = R$ ; if  $r \in R^\times$ , then  $x \sim y \implies I = (x) = (y) = J$ .

Therefore  $I$  is maximal.

### Proposition: A PID is Noetherian

Say  $I_1 \subseteq I_2 \subseteq \dots \subseteq I_n \subseteq \dots$  is an ascending chain.

Write it as  $(\alpha_1) \subseteq (\alpha_2) \subseteq \dots \subseteq (\alpha_n) \subseteq \dots$ .

Claim:  $\bigcup_{n=1}^{\infty} I_n$  is an ideal of  $R$ .

- Proof of Claim

Let  $x, y \in J$  and  $r \in R$  be arbitrary.

By definition,  $x \in I_n$  and  $y \in I_m$  for some  $m, n \in \mathbb{N}$

Take  $k := \max\{m, n\}$  and  $x, y \in I_k$ .

Since  $I_k$  is an ideal,  $x \pm y, rx \in I_k \subseteq J$ .

- Proof of Proposition So

$$J = \bigcup_{n=1}^{\infty} (\alpha_n) \underset{R=\text{PID}}{=} (\beta) \ni \beta$$

for some  $\beta \in R$ . But then  $\beta \in (\alpha_m)$  for some  $m \in \mathbb{N}_0$ .

Therefore  $J = (\beta) \subseteq (\alpha_m) = I_m$ , and  $I_{M+k} = I_m$ ,  $\forall k \geq 0$ .