

# Manifolds II

January 6, 2025

## Recall: Tangent Bundle

Given a chart  $(U, \phi)$  about a point  $p$ , we have coordinates  $(x^1, \dots, x^n)$  and a basis for  $T_q M$  of  $(\frac{\partial}{\partial x^1}|_q, \dots, \frac{\partial}{\partial x^n}|_q)$  for  $q \in U$ .

Then given  $TM \xrightarrow{\pi} M$ , we may write  $v_q = v^i \frac{\partial}{\partial x^i}|_q$ .

## Definition:

For  $M$  a topological manifold. A (real) vector bundle of rank  $k$  over  $M$  is a topological space  $E$  with a surjective continuous map  $\pi : E \rightarrow M$  such that

1.  $\forall p \in M$ , the fiber  $\pi^{-1}(p) =: E_p$  is endowed with the structure of a (real) vector space of dimension  $k$ .
2.  $\forall p \in M$ , there exists a neighborhood  $U$  of  $p$  in  $M$  and a homeomorphism  $\Phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$  called a local trivialization.

$$\begin{array}{ccc} \Phi : \pi^{-1}(U) & \xrightarrow{\quad} & U \times \mathbb{R}^k \\ & \searrow \pi & \swarrow \pi_U \\ & U & \end{array}$$

and  $\Phi|_{E_q} : E_q \rightarrow \{q\} \times \mathbb{R}^k$  is a linear isometry.

## Examples

1.  $TM \xrightarrow{\pi} M$
2.  $E = M \times \mathbb{R}^k$  with a global trivialization.
3. The Mobius bundle over  $S^1$ .  $\gamma : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by  $(x, y) \mapsto (x+1, (-1) \cdot y)$ . Then  $\langle \gamma \rangle \cong \mathbb{Z}$  a subgroup acting freely and isometrically on  $\mathbb{R}^2$ . Then  $E = \mathbb{R}^2 / \langle \gamma \rangle \xrightarrow{\pi} S^1 = \mathbb{R} / \mathbb{Z}$  by  $\overline{(x, y)} \mapsto \bar{x}$  is a vector bundle.

IMAGE 1

- We want to show that  $\pi^{-1}(U) \cong U \times \mathbb{R}$

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\gamma} & E \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{R} & \xrightarrow{\varepsilon} & S^1 \end{array} \quad \begin{array}{ccc} (x, y) & \mapsto & \overline{(x, y)} \\ \downarrow & & \downarrow \\ x & \mapsto & e^{(2\pi i)x} \end{array}$$

Then let  $p \in S^1$ . We choose  $U$  a neighborhood of  $p$  such that  $U$  is evenly covered by  $\varepsilon$ . This means  $\varepsilon^{-1}(U)$  is a disjoint union of open sets diffeomorphic to  $U$ .

IMAGE 2

Let  $\tilde{U}$  be a component in  $\pi^{-1}(U)$ . Then  $\pi|_{\tilde{U}} : \tilde{U} \rightarrow U$  is a diffeomorphism and  $\pi^{-1}(U)$  is diffeomorphic to  $U \times \mathbb{R}$ .

## Definition: Transition Function

Take  $E \xrightarrow{\pi} M$  with  $U, V \subseteq M$  admitting trivializations  $\phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$  and  $\Psi : \pi^{-1}(V) \rightarrow V \times \mathbb{R}^k$ . Let  $w = U \cap V (\neq \emptyset)$ .

$$\Phi \circ \Psi^{-1} : \begin{array}{ccccc} W \times \mathbb{R}^k & \longrightarrow & \pi^{-1}(W) & \longrightarrow & W \times \mathbb{R}^k \\ & \searrow & \downarrow & \swarrow & \\ & & W & & \end{array}$$

Then  $\Phi \circ \Psi^{-1}|_{\{p\} \times \mathbb{R}^k}$  by  $\{p\} \times \mathbb{R}^k \rightarrow \{p\} \times \mathbb{R}^k$  is a linear isomorphism.

$\Phi \circ \Psi^{-1}(p, v) = (p, \tau(p)v)$  by  $\tau : p \mapsto \tau(p)$  and  $\tau(p) \in GL(k, \mathbb{R})$  gives a smooth map  $W \rightarrow GL(k, \mathbb{R})$ .

## Definition:

Let  $\{E_1, \dots, E_k\}$  be a basis of  $\mathbb{R}^k$ . Then

$$\tau(p) \cdot E_i = \sum_j \tau(p)_i^j E_j$$

with  $\tau(p) = (\tau(p)_i^j)$  and  $\tau(p)_i^j \in \mathbb{R}$ . It suffices to show each  $\tau(p)_i^j$  mapping  $W \rightarrow \mathbb{R}$  and  $p \mapsto (\tau(p)_i^j)$  is smooth. Then if  $\sigma(p, v) := \Phi \circ \Psi^{-1}(p, v)$ ,  $\tau(p)_i^j = \pi_j(\sigma(p, E_i))$  and  $\pi_j$  is a projection to the  $j$ -th component in  $\mathbb{R}^k$ .

## Lemma 10.6 (Vector Bundle Chart Lemma)

Given  $M$  a smooth manifold, suppose that  $\forall p \in M$  we are given a vector space  $E_p$  of dimension  $k$ . Let  $E = \coprod_{p \in M} E_p$  (as a set) and  $\pi : E \rightarrow M$  a mapping  $E_p$  to  $p$ . Suppose also that we have

1.  $\{U_\alpha\}_{\alpha \in A}$  an open cover of  $M$  with a countable subcover.
2.  $\forall \alpha \in A$  we have a bijection  $\Phi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^k$  such that  $\Phi_\alpha|_{E_p} : E_p \rightarrow \{p\} \times \mathbb{R}^k$  is a linear isomorphism.
3.  $\forall \alpha, \beta \in A$  with  $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$  we have a smooth map  $\tau_{\alpha\beta} : U_{\alpha\beta} \rightarrow GL(k, \mathbb{R})$  such that  $\Phi_\alpha \circ \phi_\beta^{-1} : U_{\alpha\beta} \times \mathbb{R}^k \rightarrow U_{\alpha\beta} \times \mathbb{R}^k$  by  $(p, v) \mapsto (p, \tau(p)v)$ .

Then  $E \xrightarrow{\pi} M$  is a vector bundle.

## Example (Whitney Sum):

Suppose we have  $E' \xrightarrow{\pi'} M$  and  $E'' \xrightarrow{\pi''} M$  two vector bundles over  $M$ .

Define  $E = E' \oplus E''$  a new vector bundle over  $M$  by  $E_p = E'_p \oplus E''_p$ . Let  $\{U_\alpha\}_{\alpha \in A}$  be a countable open cover of  $M$  such that each  $U_\alpha$  admits trivializations for  $E'$  and  $E''$ . Then for  $\pi : E \rightarrow M$ , define  $\Phi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^{k'} \times \mathbb{R}^{k''}$  by  $(v', v'')_p \mapsto (p, \pi_2 \circ \Phi_\alpha^{-1}(v'), \pi_2 \circ \Phi_\alpha''(v''))$  where

$$\pi'(U_\alpha) \xrightarrow{\Phi_\alpha'} U_\alpha \times \mathbb{R}^{k'} \xrightarrow{\pi_2} \mathbb{R}^{k'}$$

Note that  $\pi_2$  is the projection into the second component. Then  $\tau : U_{\alpha\beta} \rightarrow G(k' + k'', \mathbb{R})$  by

$$p \mapsto \begin{pmatrix} \tau'(p) & 0 \\ 0 & \tau''(p) \end{pmatrix}$$

## Example

For  $\tau_{\alpha\beta} : U_{\alpha\beta} \rightarrow GL(k, \mathbb{R})$  by  $p \mapsto \tau_{\alpha\beta}(p)$ , we can write  $U_{\alpha\beta\gamma} = U_{\alpha} \cap U_{\beta} \cup U_{\gamma} (\neq \emptyset)$  and get  $\tau_{\alpha\beta} \cdot \tau_{\beta\gamma} = \tau_{\alpha\gamma}$ .

Note that this is  $\Phi_{\alpha} \circ (\phi_{\beta}^{-1} \circ \phi_{\beta}) \circ \Phi_{\gamma}^{-1}$ .

Without loss of generality, we assume each  $U_{\alpha}$  is a chart for  $M$ . Then we want to show that we satisfy Lemma 1.35 from Lee

$$\pi^{-1}(U_{\alpha}) \xrightarrow{\Phi_{\alpha}} U_{\alpha} \times \mathbb{R}^k \xrightarrow{\phi_{\alpha} \times \text{id}} \phi_{\alpha}(U_{\alpha}) \times \mathbb{R}^k \subseteq \mathbb{R}^{n+k}$$

$(\pi^{-1}(U_{\alpha}) \cdot \tilde{\phi}_{\alpha} = (\phi_{\alpha} \times \text{id}) \circ \Phi_{\alpha})_{\alpha \in A}$  which satisfies (1).

Since

$$\pi^{-1}(U_{\alpha}) \cap \pi^{-1}(U_{\beta}) = \pi^{-1}(U_{\alpha} \cap U_{\beta}) \rightarrow \phi_{\alpha}(U_{\alpha\beta}) \times \mathbb{R}^k$$

we have that (2) is satisfied.

Finally, for (3),

$$\tilde{\phi}_{\beta} \circ \tilde{\phi}_{\alpha}^{-1} = (\Phi_{\beta} \circ (\phi_{\beta} \times \text{id})) \circ ((\phi_{\alpha} \times \text{id})^{-1} \circ \Phi_{\alpha}^{-1}) = \Phi_{\beta} \circ ((\phi_{\beta} \circ \phi_{\alpha}) \times \text{id}) \circ \Phi_{\alpha}^{-1}$$

gives  $(x, c) \mapsto ((\phi_{\beta} \circ \phi_{\alpha}^{-1})x, (\Phi_{\beta} \circ \Phi_{\alpha}^{-1})c)$  a diffeomorphism.

(4) and (5) are trivial, and this is indeed a smooth manifold. Now we wish to show that it is a vector bundle. To show that  $\pi : E \rightarrow M$  is smooth,

$$\begin{array}{ccc} \pi^{-1}(U_{\alpha}) & \xrightarrow{\pi} & U_{\alpha} \\ \tilde{\phi}_{\alpha}^{-1} \uparrow & & \downarrow \phi_{\alpha} \\ \phi_{\alpha}(U_{\alpha}) \times \mathbb{R}^k & & \phi_{\alpha}(U_{\alpha}) \end{array}$$

We have  $\tilde{\phi}_{\alpha}^{-1} = (\phi_{\alpha} \times \text{id})^{-1} \circ \Phi_{\alpha}^{-1}$ .

$$\begin{array}{ccc} \pi^{-1}(U_{\alpha}) & \xrightarrow{\Phi_{\alpha}} & U_{\alpha} \times \mathbb{R}^k \\ \tilde{\phi}_{\alpha}^{-1} \uparrow & & \downarrow \phi_{\alpha} \times \text{id} \\ \phi_{\alpha}(U_{\alpha}) \times \mathbb{R}^k & & \phi_{\alpha}(U_{\alpha} \times \mathbb{R}^k) \end{array}$$

## Definition: Section of a Bundle

A (smooth) section of  $E \xrightarrow{\pi} M$  is a (smooth) map  $\sigma : M \rightarrow E$  such that  $\pi \circ \sigma = \text{id}_M$ .

$\Gamma(E) = \{\text{smooth sections of } E \xrightarrow{\pi} M\}$  and  $\Gamma(E)$  is a  $C^{\infty}(M)$ -module.

The zero section  $Z : M \rightarrow E$  is given by  $p \mapsto 0_p \in E_p$ .

If  $U$  has a local trivialization,  $\Phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$ .

$$\Phi : \begin{array}{ccccc} \pi^{-1}(U) & \xrightarrow{\quad} & U \times \mathbb{R}^k & \xleftarrow{\Phi^{-1}} & (p, e_i) \\ & \nwarrow \text{dashed} & \nearrow & \searrow \tilde{e}_i & \uparrow p \\ & U & & p & \end{array}$$

Define  $\sigma_i : U \rightarrow \pi^{-1}(U)$  by  $\sigma_i = \Phi^{-1} \circ \tilde{e}_i$  gives a local section that is non-zero on  $U$ .

$\{\sigma_1, \dots, \sigma_n\}$  form a local frame on  $U$  (i.e. form a basis in  $E_p$ ,  $\forall p \in U$ ).

January 8, 2025

## Recall

Last time we had a vector bundle  $E \xrightarrow{\pi} M$  of rank  $k$  satisfying

1.  $\pi^{-1}(p) = E_p$  has a (real) vector space structure of dimension  $k$ .
2. We have a local trivialization,  $\forall p \in M$  there exists a neighborhood  $U$  and a diffeomorphism  $\Phi$

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Phi} & U \times \mathbb{R}^k \\ & \searrow \pi & \swarrow \pi_U \\ & U & \end{array}$$

and  $\Phi|_{E_p} : E_p \rightarrow \{p\} \times \mathbb{R}^k$  is a linear isomorphism.

A section  $\sigma : M \rightarrow E$  is a smooth map such that  $\pi \circ \sigma = \text{id}_M$ .

We say that a collection of sections  $\{\sigma_1, \dots, \sigma_k : U \rightarrow E\}$  is linearly independent if  $\{\sigma_1(x), \dots, \sigma_k(x)\}$  is linearly independent for each  $x \in U$ . This is a (local) frame if it is a basis.

If  $U \subseteq M$  admits a trivialization

$$\Phi : \begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\quad} & U \times \mathbb{R}^k \\ & \searrow & \swarrow \\ & U & \end{array}$$

then there is a local frame  $\{\sigma_1, \dots, \sigma_k\}$  defined on  $U$ . Precisely, with  $\tilde{e}_i(x) = (x, e_i)$ ,  $\sigma_i = \Phi^{-1} \circ \tilde{e}_i$ .

## Proposition 10.19

If  $U \subseteq M$  admits a local frame, then  $\pi^{-1}(U)$  admits a local trivialization.

### Remember

If  $E \xrightarrow{\pi} M$  admits a global frame, then  $E = \pi^{-1}(M)$  has a trivialization. In other words,  $E$  is diffeomorphic to a trivial vector bundle  $M \times \mathbb{R}^k$ .

## Examples

### Example 1

Mobius bundle over  $S^1$ .

IMAGE 1

To check whether it is a trivial bundle of  $S^1$ , it suffices to check whether there exists a nowhere zero (global) section. This cannot happen (by intermediate value theorem), hence it is not  $S^1 \times \mathbb{R}$ .

### Example 2

$TS^2$  because there is no non-vanishing vector field over  $S^2$ , hence  $TS^2 \neq S^2 \times \mathbb{R}^2$ .

### Example 3

Let  $G$  be a Lie group. Every  $X \in T_e G (\cong \mathfrak{g})$  uniquely determines a (left-invariant) vector field  $\tilde{X} \in \mathfrak{X}(G)$ . Starting with a basis  $\{E_i\} \subseteq T_e G$  we get a global frame  $\{\tilde{E}_i\}$  for  $TG$ . Hence  $TG$  is a trivial vector bundle  $G \times \mathbb{R}^n$  ( $n = \dim G$ ). In particular,  $TS^1 = S^1 \times \mathbb{R}$ ,  $TS^3 = S^3 \times \mathbb{R}^3$ .

### Proof of Proposition

Define  $\Psi : (x, v^1, \dots, v^k) \in U \times \mathbb{R}^k \rightarrow \pi^{-1}(U) \ni v_x$  where  $v_x = v^i \sigma_i(x)$ .

$\Psi$  is a bijection. Note that  $\Psi|_{E_x} : E_x \rightarrow \{x\} \times \mathbb{R}^k$  is a linear isomorphism because  $\{\sigma_i(x)\}$  is a basis. Then to show that  $\Psi$  is a diffeomorphism, it suffices to show then that  $\Psi$  is a local diffeomorphism.

Let  $x \in U$  and let  $V$  be a neighborhood of  $x$  such that  $\pi^{-1}(V) \xrightarrow{\Phi} V \times \mathbb{R}^k$ .

$$V \times \mathbb{R}^k \xrightarrow{\Psi|_{V \times \mathbb{R}^k}} \pi^{-1}(V) \xrightarrow{\Psi} V \times \mathbb{R}^k$$

We show that this composition is a diffeomorphism. Since  $\Phi(\sigma_i(x)) = (x, \sigma_i^1(x), \dots, \sigma_i^k(x))$

$$\begin{aligned} \Phi \circ \Psi|_{V \times \mathbb{R}^k}(x, v^1, \dots, v^k) &= \Phi(v^i \sigma_i(x)) \\ &= (x, v^i \sigma_i^1(x), \dots, v^i \sigma_i^k(x)) \end{aligned}$$

Each  $\sigma_i^j(x)$  is smooth. Hence  $\Phi \circ \Psi|_{V \times \mathbb{R}^k}$  is smooth.

Let  $\vec{v} = (v^1, \dots, v^k)$  and  $\sum(x) = (\sigma_i^j(x))$ , then  $\Phi \circ \Psi(x, \vec{v}) = (x, \vec{v} \cdot \sum(x))$ . Its inverse

$$(\Phi \circ \Psi)^{-1}(x, \vec{w}) = (x, \vec{w} \cdot \sum(x))$$

is also smooth. This shows that  $\Phi \circ \Psi|_{V \times \mathbb{R}^k}$  is a diffeomorphism. Hence  $\Psi|_{V \times \mathbb{R}^k}$  is a diffeomorphism ( $V \subseteq U$ ) and  $\Psi : U \times \mathbb{R}^k \rightarrow \pi^{-1}(U)$  is also a diffeomorphism.

### Definition: Bundle Morphism

A bundle morphism between is a pair of smooth maps  $(f, F)$  such that this diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ \downarrow \pi & & \downarrow \pi' \\ M & \xrightarrow{f} & M' \end{array}$$

and  $F|_{E_p} : E_p \rightarrow E'_{f(p)}$  is a linear map ( $\forall p \in M$ ).

If it admits an inverse which is itself a bundle morphism, it is a unble isomorphism.

Remember that  $f$  is smooth because  $f = \pi' \circ F \circ Z$

$$p \xrightarrow{Z} 0_p \xrightarrow{F} 0_{f(p)} \xrightarrow{\pi'} f(p)$$

### Remark

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ & \searrow \pi & \swarrow \pi' \\ & M & \end{array}$$

commutes and  $F|_{E_p} : E_p \rightarrow E'_{f(p)}$  is linear ( $\forall p$ ).

## Remark

$\text{rank}(F|_{E_p})$  may depend on  $p \in M$ .

$$\begin{array}{ccc} TM & \xrightarrow{Df} & TR \\ \downarrow \pi & & \downarrow \pi \\ M & \xrightarrow{f} & \mathbb{R} \end{array}$$

e.g.  $M = \mathbb{R}^2$ ,  $E = E' = TR^2 (= \mathbb{R}^4)$ ,  $F((u, v)_{(x, y)}) = (u, xv)$ . For  $x \neq 0$ ,  $\text{rank}(F|_{(x, y)}) = 2$  but for  $x = 0$   $\text{rank}(F|_{(0, y)}) = 1$ .

## Proposition 10.26

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ \searrow \pi & & \swarrow \pi' \\ & M & \end{array}$$

If  $F$  is a bijective, smooth bundle homomorphism, then it is a bundle isomorphism. Proof left as an exercise. We need to show that  $F^{-1}$  is smooth.

## Definition: Fiber Bundle

$F \rightarrow E \xrightarrow{\pi} M$  with fiber  $F$  such that  $E_x = \pi^{-1}(x)$  is diffeomorphic to  $F$ . This diagram commutes.

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Phi} & U \times F \\ \searrow \pi & & \swarrow \pi_U \\ & U & \end{array}$$

## Fact

If  $N \xrightarrow{F} M$  is a submersion from compact manifolds, then  $F$  is a fiber bundle.

## Chapter 11: Cotangent Bundles

### Review: Linear Algebra

Suppose we have a real vector space  $V$  of dimension  $n$ . Then  $V^* = \{f : V \rightarrow \mathbb{R} \text{ linear}\}$ .

If  $V$  has a basis  $\{E_1, \dots, E_n\}$ , then we may define the dual basis for  $V^*$   $\{e^1, \dots, e^n\}$  by  $e^j(E_i) = \delta_i^j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ .

Remember  $V^{**} \cong V$  by  $\xi : V \rightarrow V^{**}$  by  $v \mapsto \xi(v) : V^* \rightarrow \mathbb{R}$  and  $\omega \mapsto \omega(v)$ .

Remember also that if  $A$  is a linear map  $V \rightarrow W$  then we may define  $A^* : W^* \rightarrow V^*$  by  $v \in V \rightarrow \mathbb{R} \ni \omega(Av)$  (ie.  $(A^*\omega)(v) = \omega(Av)$ ).

## Definition: Cotangent Bundle

Let  $M^n$  be a smooth manifold, and let  $(U, \phi)$  be a chart. Then  $T_p M$  has a basis

$$\left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\}$$

for every  $p \in U$ . Take its dual basis

$$\{\lambda^1|_p, \dots, \lambda^n|_p\}$$

for  $T_p^* M$ . The cotangent bundle  $T^* M = \coprod_{p \in M} T_p^* M$ .

Similar to the  $TM$  case, if  $T^* M \xrightarrow{\pi} M$ , then  $\omega|_p \in \pi^{-1}(U) \xrightarrow{\Phi} U \times \mathbb{R}^n \ni (p, a_1, \dots, a_n)$  where  $a_i$  is given by  $\omega|_p = a_i \lambda^i|_p$ .

In other words,  $a_i = \omega|_p \left( \frac{\partial}{\partial x^i} \Big|_p \right)$ .

## Computing Dual Transition

Suppose  $(U, (x^1, \dots, x^n))$  and  $(V, (y^1, \dots, y^n))$  are two charts ( $W = U \cap V \neq \emptyset$ ). Then  $\left\{ \frac{\partial}{\partial x^i} \Big|_p \right\}$  gives a dual  $\{\lambda^i|_p\}$  and  $\left\{ \frac{\partial}{\partial y^j} \Big|_p \right\}$  gives  $\{\mu^j|_p\}$ .

Then, recall,  $\frac{\partial}{\partial y^j} \Big|_p = \frac{\partial x^i}{\partial y^j} \frac{\partial}{\partial x^i} \Big|_p$  and  $x^j(y^1, \dots, y^n)$  is a  $j$ -component of  $(y^1, \dots, y^n) \rightarrow M \rightarrow (x^1, \dots, x^n)$ .

If  $\omega \in T_p^* M$ ,  $\omega = a_i \lambda^i|_p = b_j \mu^j|_p$

$$a_i = \omega|_p \left( \frac{\partial}{\partial x^i} \Big|_p \right) = \omega|_p \left( \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j} \Big|_p \right) = \frac{\partial y^j}{\partial x^i} \omega \left( \frac{\partial}{\partial y^j} \Big|_p \right) = \frac{\partial y^j}{\partial x^i} b_j$$

In particular,  $\mu^j = \omega$ , then  $a_i = \frac{\partial y^j}{\partial x^i} b_j = \frac{\partial y^j}{\partial x^i} \mu^j$ . Hence  $\mu^j = \omega = a_i \lambda^i = \frac{\partial y^j}{\partial x^i} \lambda^i$ .

## Definition: Smooth Covector Field

A smooth covector field is a smooth section of  $T^* M$ , call it  $\Omega^1(M) = \Gamma(T^* M)$ .

Given  $f \in C^\infty(M)$ , we can define a smooth covector field  $df \in \Omega^1(M)$  by  $df(v|_p) = (v_p)(f)$ .

$df(X) = Xf$  is smooth if  $X$  and  $f$  are smooth.

## Differential

Given a local chart  $(U, (x^1, \dots, x^n))$  and a smooth function  $f : U \rightarrow \mathbb{R}$ ,  $df_p = a_i(p) \lambda^i|_p$ .

$$\frac{\partial f}{\partial x^j} = df_p \left( \frac{\partial}{\partial x^j} \Big|_p \right) = a_i(p) \lambda^i|_p \left( \frac{\partial}{\partial x^j} \Big|_p \right) = a_i(p) \delta_j^i = a_j(p)$$

That is,  $df_p = \frac{\partial f}{\partial x^j}(p) \lambda^j|_p$ . In particular, if we consider the coordinate function  $x^i : U \rightarrow \mathbb{R}$ , then  $dx^i|_p = \frac{\partial x^i}{\partial x^j}(p) \lambda^j|_p = \lambda^i|_p$  for each  $p \in U$  (i.e.  $dx^i = \lambda^i$  on  $U$ ).

With this, we can write  $df = \frac{\partial f}{\partial x^i} dx^i$  and  $dy^j = \frac{\partial y^j}{\partial x^i} dx^i$ .

## Proposition 11.22

For  $f \in C^\infty(M)$ , then  $df = 0$  if and only if  $f$  is constant on every component of  $M$ .

### Proof

( $\Leftarrow$ ) is trivial.

( $\Rightarrow$ ) We assume  $M$  is connected. Fix  $p \in M$ , define  $\mathcal{A} = \{q \in M : f(p) = f(q)\}$  is closed.

Now let  $q \in \mathcal{A}$  and  $U$  a local chart around  $q$ . Then  $0 = df = \frac{\partial f}{\partial x^i} dx^i$  (i.e.  $\frac{\partial f}{\partial x^i} \equiv 0, \forall i$ ).

Hence  $f$  is constant on  $U$  and  $f(q) = f(p)$  for  $U \in \mathcal{A}$ .

## Proposition 11.23

Take  $\gamma : J \rightarrow M$  a smooth curve  $f \in C^\infty(M)$ . Then  $(df|_{\gamma(t)})(\gamma'(t)) = (\gamma'(t))f = (f \circ \gamma)'(t)$ .

IMAGE 2

Recall that if  $v \in T_p M$  and  $f \in C^\infty(M)$  then  $vf = (f \circ \gamma)'(0)$  where  $\gamma : (-\varepsilon, \varepsilon) \rightarrow M$ ,  $\gamma(0) = p$  and  $\gamma'(0) = v$  ( $f \circ \gamma : \mathbb{R} \rightarrow \mathbb{R}$ ).

**January 13, 2025**

### Recall

$T^*M$  and  $\Omega^1(M) = \Gamma(T^*M)$ . Let  $(U, (x^1, \dots, x^n))$  be a chart. Then inside  $U$ , we may write  $\omega = \omega_i dx^i$ .  $\{dx^i|_p\}$  is a dual basis of  $\{\frac{\partial}{\partial x^i} \subseteq T_p M\}$ .

They are also  $x^i : U \rightarrow \mathbb{R}$  coordinates functions where  $dx^i$  is the differential of  $x^i$ .

Given  $f \in C^\infty(M)$  or  $C^\infty(U)$ ,  $df \in \Omega^1(M)$  or  $\Omega^1(U)$  is defined by  $df(X_p) = (Xf)(p)$ .

Inside a chart,  $df = \frac{\partial f}{\partial x^i} dx^i$ .

We have a change of coordinates where  $(U, (x^1, \dots, x^n))$  and  $(V, (y^1, \dots, y^n))$  and  $W = U \cap V \neq \emptyset$  gives  $dy^j = \frac{\partial y^j}{\partial x^i} dx^i$ .

### Recall (Linear Algebra)

If  $A : V \rightarrow W$  is a linear map with  $w \in W^*$  and  $v \in V$ , then  $A^* : W^* \rightarrow V^*$  is the dual map defined by  $(A^* w)(v) := w(Av)$ .

### Dual of the Tangent Space

Let  $F : M \rightarrow N$  be a smooth map between manifolds.

$$\begin{aligned} DF_p : T_p M &\rightarrow T_{F(p)} N \\ (DF_p)^* : T_{F(p)}^* N &\rightarrow T_p^* M \end{aligned}$$

and  $(DF_p^* \omega)(v) = \omega(DF_p(v))$  for  $\omega \in T_{F(p)}^* N$  and  $v \in T_p M$ .

### Definition: Pullback

Given  $\omega \in \Omega^1(N)$ , we can define  $F^* \omega$ , a section of  $T^*M$ , by  $(F^* \omega)_p(v) = \omega(DF_p(v))$  or  $(F^* \omega)_p = DF_p^* \omega$ . We call this the pullback of  $\omega$  by  $F$ .



Recall that for  $u \in C^\infty(N)$ ,  $M \xrightarrow{F} N \xrightarrow{u} \mathbb{R}$ . Then we can define  $F^*u \in C^\infty(M)$  by  $F^*u = u \circ F$ .

## Proposition

If  $F : M \rightarrow N$  is smooth,  $u \in C^\infty(N)$  and  $\omega \in \Omega^1(N)$ , then

1.  $F^*(u\omega) = (F^*u)(F^*\omega)$ .
2.  $F^*(du) = d(F^*u)$ .

### Proof of 1

$\forall p \in M, \forall v \in T_p M$ ,

$$(F^*(u\omega))_p(v) = DF_p^*(u\omega)(v) = u_{F(p)}\omega_{F(p)}(DF_p(v)) = (u \circ F(p))\omega(DF_p(v)) = (F^*u)(F^*\omega)$$

### Proof of 2

$$(F^*(du))(v) = du(DF_p(v)) = (DF_p(v))u = (du)_{F(p)}DF_p(v) = d(u \circ F)(v) = d(F^*u)(v)$$

## Change of Coordinates

Locally,  $F : M \rightarrow N$ . Let  $(U, (x^1, \dots, x^n))$  be a chart around  $p$  and  $(V, (y^1, \dots, y^n))$  a chart around  $F(p)$ . For  $\omega \in \Omega^1(N)$ , in  $V$   $\omega = \omega_i dy^i$  and

$$F^*\omega = F^*(\omega_i dy^i) = (F^*\omega_i)(F^*dy^i) = (F^*\omega_i)d(F^*y^i) = (\omega_i \circ F)(dF^i)$$

where  $F^i = y^i \circ F$  is the  $i$ th component of  $F$ .

When  $F$  is smooth and  $\omega \in \Omega^1(N)$ , then  $F^*\omega \in \Omega^1(M)$ . In fact, locally,  $F^*\omega = (\omega_i \circ F)d(F^i)$ . Hence  $F^*\omega$  is smooth.

### Example 1

Take  $F : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  by  $(x, y, z) \mapsto (u(x, y, z), v(x, y, z)) = (x^2 y, y \sin(z))$ .

Then  $\omega = u dv + v du \in \Omega^1(\mathbb{R}^2)$ . So

$$\begin{aligned} F^*\omega &= F^*(u dv + v du) \\ &= (F^*u)d(F^*v) + (F^*v)d(F^*u) \\ &= x^2 y d(y \sin(z)) + (y \sin(z)) d(x^2 y) \\ &= x^2 y (\sin(z) dy + y \cos(z) dz) + y \sin(z) (2xy dx + x^2 dy) \end{aligned}$$

### Example 2

$M = \mathbb{R}^2 - \{0\}$  and  $\gamma : [0, 2\pi] \rightarrow M$  by  $t \mapsto (r \cos(t), r \sin(t))$  for  $r > 0$ . Take  $\omega = \frac{x dy - y dx}{x^2 + y^2} \in \Omega^1(M)$

$$\begin{aligned} \gamma^*\omega &= \frac{1}{r^2} (r \cos(t) d(r \sin(t)) - r \sin(t) d(r \cos(t))) \\ &= \cos(t)(\cos(t)) dt - \sin(t)(\sin(t)) dt \\ &= dt \end{aligned}$$

## Definition: Line Integral

If  $\eta \in \Omega'(\mathbb{R})$  or  $\Omega'(I)$  (where  $I \subseteq \mathbb{R}$  is an interval),  $\eta$  can be written as  $\eta(t) = f(t) dt$  and define

$$\int_I \eta = \int_a^b f(t) dt$$

Let  $\gamma : [a, b] \rightarrow M$  be a smooth curve on  $M$ . Let  $\omega \in \Omega'(I)$ . Define

$$\int_\gamma \omega = \int_a^b \gamma^* \omega$$

with  $\gamma^*(\omega) \in \Omega'([a, b])$ .

## Proposition 11.31

Take  $\phi : I \rightarrow J$  a diffeomorphism between intervals with  $\phi' > 0$ . Then

$$\int_J \phi^* \omega = \int_{\phi(I)} \omega$$

Write  $s$  for coordinates on  $J$  and  $t$  for coordinates on  $I$ . Then  $\omega = f(t) dt \in \Omega^1(I)$  and

$$\phi^* \omega = (\phi^* f) d(\phi^* t) = (f \circ \phi) d(t \circ \phi) = f(\phi(s)) d(\phi(s)) = f(\phi(s)) \phi'(s) ds$$

Then

$$\int_J \phi^* \omega = \int_J f(\phi(s)) \phi'(s) ds \stackrel{t=\phi(s)}{=} \int_I f(t) dt = \int_I \omega$$

## Proposition 11.37: Independence of Reparameterization

Suppose  $\gamma : I \rightarrow M$  is a smooth curve and  $\phi : J \rightarrow I$  is a diffeomorphism with  $\phi' > 0$ . Then  $\tilde{\gamma} := \gamma \circ \phi : J \rightarrow M$  is a reparameterization of  $\gamma$  and

$$\int_\gamma \omega = \int_{\tilde{\gamma}} \omega$$

If  $\phi' < 0$ , then  $\int_\gamma \omega = - \int_{\tilde{\gamma}} \omega$ .

### Proof

$$\int_\gamma \omega = \int_I \gamma^* \omega = \int_J \phi^* \gamma^* \omega = \int_J (\gamma \circ \phi)^* \omega = \int_{\tilde{\gamma}} \omega$$

### Example

Take  $\gamma : [0, 2\pi] \rightarrow M = \mathbb{R}^2 - \{0\}$  by  $t \mapsto (r \cos(t), r \sin(t))$  with  $r > 0$ . If  $\omega = \frac{x dy - y dx}{x^2 + y^2}$ , then  $\gamma^* \omega = dt$  and

$$\int_\gamma \omega = \int_0^{2\pi} \gamma^* \omega = \int_0^{2\pi} dt = 2\pi$$

## Proposition 11.38

For  $\gamma : I \rightarrow M$

$$\int_{\gamma} \omega = \int_I \omega_{\gamma(t)}(\dot{\gamma}(t)) dt$$

### Proof

In a local chart  $(U, (x^1, \dots, x^n))$ , we can write  $\omega = \omega_i dx^i$ . Then  $\gamma(t) = (\gamma^1(t), \dots, \gamma^n(t))$  and

$$\begin{aligned} \gamma^* \omega &= \gamma^*(\omega_i dx^i) \\ &= (\gamma^* \omega_i) d(\gamma^* x^i) \\ &= (\omega_i \circ \gamma) d\gamma^i \\ &= \omega_i(\gamma(t)) \frac{d\gamma^i}{dt} dt \\ &= \omega_i(\gamma(t)) \dot{\gamma}^i(t) dt \end{aligned}$$

Since  $\omega = \omega_i dx^i$  and  $\dot{\gamma}(t) = (\dot{\gamma}^1(t), \dots, \dot{\gamma}^n(t)) = \dot{\gamma}^i(t) \frac{\partial}{\partial x^i}$ ,  $\omega_{\gamma(t)}(\dot{\gamma}(t)) = \omega_i(\gamma(t)) \dot{\gamma}^i(t)$  and

$$\omega_i(\gamma(t)) \dot{\gamma}^i(t) dt = \omega_{\gamma(t)}(\dot{\gamma}(t)) dt$$

Hence  $\int_{\gamma} \omega = \int_I \gamma^* \omega = \int_I \omega_{\gamma(t)}(\dot{\gamma}(t)) dt$ .

### Corollary

Then, if  $f : M \rightarrow \mathbb{R}$  is a smooth function,

$$\int_{\gamma} df = \int_I (df)_{\gamma(t)}(\dot{\gamma}(t)) dt = \int_I (f \circ \gamma)'(t) dt = f(\gamma(b)) - f(\gamma(a))$$

Therefore  $\int_{\gamma} df$  only depends on the value of  $f$  at the endpoints of  $\gamma$ .

## Definition: Exact and Conservative Forms

Let  $\omega \in \Omega^1(M)$ . We say that  $\omega$  is...

1. exact if there exists  $f \in C^\infty(M)$  such that  $\omega = df$ .
2. conservative if  $\int_C \omega = 0$  for any closed, piecewise-smooth curve in  $M$

$f$  is called the potential of  $\omega$ .

### Remark

If  $\int_C \omega = 0$ , we may write  $C$  as the concatenation of curves  $\gamma$  then  $-\sigma$ . Then

$$0 = \int_C \omega = \int_{\gamma} \omega + \int_{-\sigma} \omega = \int_{\gamma} \omega - \int_{\sigma} \omega$$

## Remark

Exact implies conservative.

## Theorem

If  $\omega \in \Omega^1(M)$  is conservative, then it is exact.

## Proof

Fix a base point  $p_0 \in M$ .

We have that  $\int_p^q \omega = \int_\gamma \omega$  is well-defined by the conservative assumption, and we define  $f(p) = \int_{p_0}^p \omega$ .

Let  $q_0 \in M$  and let  $(U, (x^1, \dots, x^n))$  be a chart centered at  $q_0$ . Inside  $U$ ,  $\omega = \omega_i dx^i$  and  $df = \frac{\partial f}{\partial x^i} dx^i$ .

We need to show that  $\frac{\partial f}{\partial x^i} = \omega_i$  for each  $i$ . Fix an index  $i$  and consider a curve  $\sigma : (-\varepsilon, \varepsilon) \rightarrow U$  by  $t \mapsto (0, \dots, t, \dots, 0)$ .

IMAGE 1

Let  $q_- = \sigma(-\varepsilon)$ , then

$$f(q_0) = \int_{p_0}^q \omega = \int_{p_0}^{q_-} \omega + \int_{q_-}^q \omega =: \tilde{f}(q)$$

so  $f(q_0) = \text{constant} + \tilde{f}(q)$ . Hence  $\frac{\partial f}{\partial x^j} = \frac{\partial \tilde{f}}{\partial x^j}$  in  $U$ . Therefore

$$\begin{aligned} \tilde{f}(\sigma(s)) &= \int_{q_-}^{\sigma(s)} \omega \\ &= \int_{\sigma|_{[-\varepsilon, s]}} \omega \\ &= \int_{-\varepsilon}^s \omega_{\sigma(t)}(\dot{\sigma}(t)) dt \\ &= \int_{-\varepsilon}^s \omega_{\sigma(t)} \left( \frac{\partial}{\partial x^i} \right) dt \\ &= \int_{-\varepsilon}^s \omega_i(\sigma(t)) dt \end{aligned}$$

and

$$\left. \frac{\partial f}{\partial x^i} \right|_{q_0} = \left. \frac{\partial \tilde{f}}{\partial x^i} \right|_{q_0} = (\tilde{f} \circ \sigma)'(0) = \left. \frac{d}{ds} \right|_{s=0} \left( \int_{-\varepsilon}^s \omega_i(\sigma(t)) dt \right) = \omega_i(\sigma(0)) = \omega_i(q_0)$$

## Remark

Take  $\omega = df \in \Omega^1(M)$  which is  $\omega_i dx^i$  locally or  $\omega_i = \frac{\partial f}{\partial x^i}$  when exact.

$$\frac{\partial \omega_i}{\partial x^j} = \frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial \omega_j}{\partial x^i}$$

Note:  $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$  does not, in general, imply  $\omega = df$ .

January 15, 2025

## Recall

If  $\omega \in \Omega^1(M)$  and  $\gamma: \mathbb{R} \supseteq I \rightarrow M$  a (piecewise) smooth curve, then

$$\int_{\gamma} \omega = \int_I \gamma^* \omega$$

If  $df$  is the differential of a smooth function, then

$$\int_{\gamma} df = f(\gamma(b)) - f(\gamma(a))$$

only depends on endpoints. In particular, along a closed (piecewise) smooth curve

$$\int_C df = 0$$

We have that  $\omega$  is exact if  $\omega = df$  and conservative if  $\int_C \omega = 0$  for every closed curve.  $\omega$  is exact if and only if it is also conservative.

## Recall: Checking Exactness

Take  $\omega \in \Omega^1(M)$ ,

$$\omega_i dx^i = \omega = df = \frac{\partial f}{\partial x^i} dx^i$$

Then

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}$$

That is,  $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$ .

## Definition: Closed 1-Form

We say  $\omega \in \Omega^1(M)$  is closed if in every chart  $(U, (x^i))$ ,  $\omega = \omega_i dx^i$  satisfies  $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$ . Exact implies closed, however the converse is not true in general.

## Example

$\exists \omega \in \Omega^1(\mathbb{R}^2 - \{0\})$  such that  $\omega$  is closed but  $\int_C \omega = 2\pi$ .

## Corollary 11.50

If  $\omega \in \Omega^1(M)$  is closed, then  $\forall p \in M$  there exists a chart  $U$  at  $p$  such that  $\omega_U = df$  for some  $f \in C^\infty(U)$

## Proposition 11.45

For  $\omega \in \Omega^1(M)$ , the following are equivalent

1.  $\omega$  is closed.
2.  $\omega$  satisfies  $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$  in some chart at every point.
3. For every open  $U \subseteq M$  and  $X, Y \in \mathfrak{X}(U)$ , it holds that

$$X(\omega(Y)) - Y(\omega(X)) = \omega([X, Y])$$

The proof that 1 implies 2 is trivial.

### Proof 3 Implies 1

Pick  $U$  as a chart,  $X = \frac{\partial}{\partial x^i}$ , and  $Y = \frac{\partial}{\partial x^j}$ . Then, since  $\omega = \omega_i dx^i$ ,

$$X(\omega(Y)) = X(\omega_j) = \frac{\partial \omega_j}{\partial x^i}$$

Similarly,  $Y(\omega(X)) = \frac{\partial \omega_i}{\partial x^j}$ . Then  $[X, Y] = \left[ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] = 0$  and

$$\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} = 0$$

### Proof 2 Implies 3

Fix any  $p \in U$ . We have a chart  $(V, (x^i))$  at  $p$  such that  $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$ . Then

$$X(\omega(Y)) = X\left(\left(\omega_i dx^i\right)\left(Y^j \frac{\partial}{\partial x^j}\right)\right) = X(\omega_i Y^i) = (X\omega_i)Y^i + \omega_i(XY^i) = x^j \frac{\partial \omega_i}{\partial x^j} Y^i + \omega_i(XY^i) = X^j Y^i \frac{\partial \omega_j}{\partial x^i}$$

Similarly,

$$Y(\omega(X)) = Y(\omega_i x^i) = Y^j \frac{\partial \omega_i}{\partial x^j} x^i - \omega_i(YX^i) = X^i Y^j \frac{\partial \omega_i}{\partial x^j}$$

which is equivalent under a change of indicies. Hence

$$X(\omega(Y)) - Y(\omega(X)) = \omega_i(XY^i) - \omega_i(YX^i) = \omega_i(XY^i - YX^i) = \omega([X, Y])$$

## Lemma

Suppose  $F : M \rightarrow N$  is a local diffeomorphism. Then  $F^* : \Omega^1(N) \rightarrow \Omega^1(M)$  sends exact (or closed) 1-forms to exact (or closed) ones.

### Proof of Exact

If  $\omega = df \in \Omega^1(N)$ , then  $F^*\omega = F^*(df) = d(F^*f)$  is exact on  $M$ .

## Proof of Closed

If  $\omega \in \Omega^1(N)$  is closed, then  $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$  in every chart of  $N$ .  
For any  $p \in M$ , we consider a chart at  $p$  by  $(V, \phi \circ F)$

IMAGE 1

Therefore  $\phi \circ F \circ (\phi \circ F)^{-1} = \text{id}$  and  $F^* = \text{id}$  so  $F^* \omega$  is closed.

## Poincaré Lemma

Let  $\omega \in \Omega^1(M)$  be closed. Fix  $p \in M$ , and let  $(U, \phi)$  be a chart at  $p$  such that  $\phi(U) = B_1(0) \subseteq \mathbb{R}^n$ .

IMAGE 2

Assuming the above, every closed 1-form on  $B_1(0)$  is exact.  $(\phi^{-1})^*(\omega|_U) = df$  for some  $f \in C^\infty(B_1(0))$  where  $\omega|_U = \phi^*(df) = d(\phi^*f) \in C^\infty(U)$

## Definition: Star-Shaped Domain

We say that  $U \subseteq \mathbb{R}^n$  open is star-shaped with a center  $c \in U$  (wlog  $c = 0$ ) if for any  $x \in U$ , the segment  $\gamma_x$  from  $c$  to  $x$  is contained in  $U$ .

IMAGE 3

If  $x = (x^i)$ , then  $\gamma_x(t) = (tx^i)$ .

## Theorem 11.49 (Poincaré Lemma)

If  $U \subseteq \mathbb{R}^n$  is star-shaped, then every closed 1-form is exact.

### Recall

If  $\omega$  is an exact 1-form, then  $f(q) = \int_{p_0}^q \omega$  is a potential.

We also have that  $\int_\gamma \omega = \int_I \omega_{\gamma(t)}(\dot{\gamma}(t)) dt$ .

### Proof

Let  $\omega \in \Omega^1(U)$  be a closed 1-form.

We need to construct  $f \in C^\infty(U)$  such that  $df = \omega$ . That is, for all  $i$ ,  $\frac{\partial f}{\partial x^i} = \omega^i$ . Define

$$f(x) = \int_{\gamma_x} \omega = \int_0^1 \omega_{\gamma_x(t)}(\dot{\gamma}_x(t)) dt = \int_0^1 \omega_i|_{\gamma_x(t)} dx^i(x^1, \dots, x^n) dt = \int_0^1 \omega_i|_{tx} x^i dt$$

Since everything is smooth,

$$\begin{aligned}
\frac{\partial f}{\partial x^j}(x) &= \int_0^1 \frac{\partial}{\partial x^j}(\omega_i(tx) \cdot x^i) dt \\
&= \int_0^1 \frac{\partial \omega_i(tx)}{\partial x^j} \cdot x^i + \omega_i(tx) \frac{\partial x^i}{\partial x^j} dt \\
&= \int_0^1 \left( \frac{\partial \omega_i}{\partial x^j} \right) \Big|_{(tx)} tx^i + \omega_j(tx) dt \\
&= \int_0^1 \frac{\partial \omega_j}{\partial x^i} \Big|_{tx} tx^i + \omega_j(tx) dt \\
&= \int_0^1 \frac{d}{dt} (t \omega_j(tx)) dt \\
&= t \omega_j(tx) \Big|_0^1 \\
&= \omega_j(x)
\end{aligned}$$

## Tensors: Multilinear Maps

All vector spaces will be finite dimensional in our consideration.

$$F : V_1 \times \cdots \times V_k \rightarrow W$$

linear in every component. Denote  $L(V_1, \dots, V_k; W)$  to be the set of all such multilinear maps.

Given  $\omega \in L(V_1; \mathbb{R}) = V_1^*$  and  $\eta \in V_2^*$ , we can define  $\omega \otimes \eta \in L(V_1, V_2; \mathbb{R})$  by  $\omega \otimes \eta(v_1, v_2) = \omega(v_1) \cdot \eta(v_2)$ .

- Remark

$$(2\omega) \otimes \eta = \omega \otimes (2\eta). \text{ We assume } \otimes_{\mathbb{R}}.$$

Similarly, given  $\omega_i \in V_i^*$ , we can define  $\omega_1 \otimes \cdots \otimes \omega_k \in L(V_1, \dots, V_k; \mathbb{R})$ .

## Proposition

Let  $V_j$  with dimension  $n_j$  ( $j = 1, \dots, k$ ). Each  $V_j$  has a basis  $\{E_1^{(j)}, \dots, E_{n_j}^{(j)}\}$ .

Its dual basis  $\{\varepsilon_{(j)}^1, \dots, \varepsilon_{(j)}^{n_j}\} \subseteq V_j^*$ . Then  $L(V_1, \dots, V_k; \mathbb{R})$  has a basis

$$\mathcal{B} = \{\varepsilon_{(1)}^{i_1} \otimes \cdots \otimes \varepsilon_{(k)}^{i_k} : 1 \leq i_j \leq n_j\}$$

## Proof

For a multi-index  $I = (i_1, \dots, i_k)$  with  $i \leq i_j \leq n_j$ , we write  $\varepsilon^I = \varepsilon_{(1)}^{i_1} \otimes \cdots \otimes \varepsilon_{(k)}^{i_k}$ .

For any  $F \in L(V_1, \dots, V_k; \mathbb{R})$ , define  $F_I = F(E_{i_1}^{(1)}, \dots, E_{i_k}^{(k)})$ . We claim that  $F = F_I \varepsilon^I$ .

In fact, for  $(v_1, \dots, v_k) \in V_1 \times \cdots \times V_k$ ,  $v_j = v_j^i E_i^{(j)}$ . We may check that  $F(v_1, \dots, v_k) = F_I \varepsilon^I(v_1, \dots, v_k)$ .

Therefore  $\mathcal{B}$  spans  $L(V_1, \dots, V_k; \mathbb{R})$ .

Then, if  $F_I \varepsilon^I = 0$ , then applying it to  $(E_{i_1}^{(1)}, \dots, E_{i_k}^{(k)})$  gives  $F_I = 0$ . Therefore  $\mathcal{B}$  is linearly independent.

In particular,  $\dim L(V_1, \dots, V_k; \mathbb{R}) = \prod_{j=1}^k n_j = \prod_{j=1}^k \dim V_j$ .



## Definition: Formal Linear Combination

Let  $S$  be a set. Define

$$\mathcal{F}(S) = \left\{ \sum_{i=1}^m a_i s_i : a_i \in \mathbb{R}, s_i \in S \right\}$$

This is the free (real) vector space on  $S$  containing formal linear combinations of elements of  $S$ .

Define  $V_1 \otimes \cdots \otimes V_k = \mathcal{F}(V_1 \times \cdots \times V_k) / R$  where  $R$  is generated by

$$\begin{aligned} (v_1, \dots, v_j + v_j', \dots, v_k) &\sim (v_1, \dots, v_j, \dots, v_k) + (v_1, \dots, v_j', \dots, v_k) \\ (v_1, \dots, c v_j, \dots, v_k) &\sim c(v_1, \dots, v_k) \end{aligned}$$

In other words, in the quotient  $v_1 \otimes \cdots \otimes v_k = \prod (v_1, \dots, v_k)$ .

## Proposition

$V_1 \otimes \cdots \otimes V_k$  has a basis  $\{E_{i_1}^{(1)} \otimes \cdots \otimes E_{i_k}^{(k)} : 1 \leq i_j \leq n_j\}$ .

## Proposition

There exists a canonical isomorphism  $(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3)$  by sending  $(v_1 \otimes v_2) \otimes v_3 \mapsto v_1 \otimes (v_2 \otimes v_3)$ .

## Proposition

$$L(V_1, \dots, V_k; \mathbb{R}) \cong V_1^* \otimes \cdots \otimes V_k^*.$$

## Proof Sketch

Define  $\Phi : V_1^* \times \cdots \times V_k^* \rightarrow L(V_1, \dots, V_k; \mathbb{R})$  by  $(\omega^1, \dots, \omega^k) \mapsto \omega^1 \otimes \cdots \otimes \omega^k$ . By multilinearity, this induces an isomorphism

$$\Phi : V_1^* \otimes \cdots \otimes V_k^* \cong L(V_1, \dots, V_k; \mathbb{R})$$

## Recall

$V^{**} \cong V$  for finite dimensional vector spaces, so  $V_1 \otimes \cdots \otimes V_k = L(V_1^*, \dots, V_k^*; \mathbb{R})$ .

## Definition: Tensor

A tensor of  $(k, l)$ -type is an element in  $\underbrace{V \otimes \cdots \otimes V}_k \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_l$ .

The collection of such elements in  $T^{(k,l)}V$ . Most of the time we consider  $T^{(0,l)}V$ .

## Examples

A vector in  $V$  is a  $(1, 0)$ -tensor.

A covector in  $V^*$  is a  $(0, 1)$ -tensor.

A linear map  $A \in L(V)$  is a  $(1, 1)$ -tensor.

An inner product is a  $(0, 2)$ -tensor.

## Symmetric Tensor

We say that  $\alpha \in T^{(0,l)}V$  is symmetric if  $\alpha(\dots, v_i, \dots, v_j, \dots) = \alpha(\dots, v_j, \dots, v_i, \dots)$ .

## Alternating Tensor

We say that  $\alpha \in T^{(0,l)}V$  is alternating if  $\alpha(\dots, v_i, \dots, v_j, \dots) = -\alpha(\dots, v_j, \dots, v_i, \dots)$ .

**January 22, 2024**

## Alternating/Symmetric Tensors

Let  $\sigma \in S_l$  and  $\alpha \in T^{(0,l)}V$ .

Define  $\sigma_\alpha$  or  $(\sigma \cdot \alpha)$  as a new  $(0, l)$ -tensor by  $(\sigma \cdot \alpha)(v_1, \dots, v_l) := \alpha(v_{\sigma(1)}, \dots, v_{\sigma(l)})$ .

Then  $\alpha$  is symmetric if and only if  $\sigma \cdot \alpha = \alpha$ .

$\alpha$  is alternating if and only if  $\sigma \cdot \alpha = (\text{sign } \sigma) \cdot \alpha$ .

Define  $\text{Sym} : T^{(0,l)}V \rightarrow S^l V$  by

$$\text{Sym}(\alpha) = \frac{1}{l!} \sum_{\sigma \in S^l} (\sigma \cdot \alpha)$$

Then  $\text{Sym}(\alpha)$  is symmetric for all  $\alpha \in T^{(0,l)}V$ .

Define  $\text{Alt} : T^{(0,l)}V \rightarrow \Lambda^l V$ , the set of alternating (anti)-tensors by

$$\text{Alt}(\alpha) = \frac{1}{l!} \sum_{\sigma \in S^l} (\text{sign } \sigma)(\sigma \cdot \alpha)$$

## Definition: Tensor Bundles

Recall that  $T_p M \simeq T_p M$  and  $T_p^* M \simeq T_p^* M$ .

Then  $T^{(k,l)} T_p M \simeq T^{(k,l)} T_p M$  a tensor bundle.

Mostly, we will consider  $T^{(0,l)} T_p M$ .

Inside a chart  $(U, (x^1, \dots, x^n))$ ,  $T^{(k,l)} T_p M$  has a local frame

$$\left\{ \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_l} \right\}$$

## Definition: Smooth Tensor Field

A smooth tensor field of type  $(k, l)$  is a smooth section of  $T^{(k,l)} T_p M$ .

To check that a  $(k, l)$ -tensor field  $A$  is smooth, we can do either of the following

1. Write  $A$  in a local chart, then  $A = A_I dx^I$  where  $A_I$  are functions in  $U$  and  $dx^I = dx^{i_1} \otimes \dots \otimes dx^{i_l}$  with  $I = (i_1, \dots, i_l)$ . Then  $A$  is smooth if and only if  $A_I$  is smooth for all  $I$ .
2. Check  $A$  testing on any  $l$  many smooth vector fields results in a smooth function.

## Remark

Every  $(0, l)$ -tensor field  $A$  defines a map

$$\mathcal{A} = \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_l \rightarrow C^\infty(M)$$

by  $A(x_1, \dots, x_l)(p) = A_p(X_1(p), \dots, X_l(p))$ . This map  $\mathcal{A}$  is  $C^\infty(M)$ -multilinear.

## Lemma 12.24

Every  $C^\infty(M)$ -multilinear map  $\mathcal{A} : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \rightarrow C^\infty(M)$  defines a smooth  $(0, l)$ -tensor field

$$A_p(v_1, \dots, v_l) = (\mathcal{A}(X_1, \dots, X_l))(p)$$

## Example

Given  $\omega \in \Omega^1(M)$ , define  $\mathcal{A} : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \rightarrow C^\infty(M)$  by  $(X, Y) \mapsto \omega(L_X Y)$ . If  $X, Y$  and  $X', Y'$  only agree at a point  $p$ , then in general  $(L_X Y)(p) \neq (L_{X'} Y')(p)$ .

## Proof

$\mathcal{A}$  acts locally only depending on the value of  $X_1, \dots, X_l$  in a neighborhood of  $p$ , call it  $U$ .

It suffices to show that if  $X_i = 0$  for some  $i$  on  $U$ , then  $\mathcal{A}(X_1, \dots, X_l)(p) = 0$ .

Let  $\psi$  be a bump function with  $\text{supp } \psi \subseteq U$  and  $\psi(p) = 1$ . Let also  $V \subseteq U$  such that  $\bar{V} \subseteq U$ .

Then  $\psi X_i \equiv 0$  on  $M$ . Then

$$0 = \mathcal{A}(X_1, \dots, \psi X_i, \dots, X_l)(p) = \psi(p) \mathcal{A}(X_1, \dots, X_l)(p) = \mathcal{A}(X_1, \dots, X_l)(p)$$

Now  $\mathcal{A}$  acts pointwisely. Write  $X_i = a_i^j \frac{\partial}{\partial x^j}$  in  $U$ .

Extend each  $\frac{\partial}{\partial x^j} \Big|_V$  to  $E_j \in \mathfrak{X}(M)$  and each  $a_i^j|_V$  to  $f_i^j \in C^\infty(M)$ .

Then inside  $V$ ,

$$\mathcal{A}(X_1, \dots, X_l)(p) = \mathcal{A}(X_1, \dots, f_i^j E_j, \dots, X_l)(p) = f_i^j(p) \mathcal{A}(X_1, \dots, X_l)(p)$$

Now let  $v_1, \dots, v_l \in T_p M$ . Define  $A$  a  $(0, l)$ -tensor field by  $A_p(v_1, \dots, v_l) = \mathcal{A}(X_1, \dots, X_l)$  where  $X_i \in \mathfrak{X}(M)$  extends  $v_i$ .

By assumption,  $\mathcal{A}(X_1, \dots, X_l)$  is a smooth function if  $X_1, \dots, X_l \in \mathfrak{X}(M)$  hence  $A$  is a smooth  $(0, l)$ -tensor field.

## Definition:

Write  $\mathcal{T}^{(0, l)} M = \Gamma(T^{(0, l)} TM)$  where  $\Gamma$  is the section.

Then for  $F : M \rightarrow N$  a smooth map and  $A \in \mathcal{T}^{(0, l)} N$ , for  $v_i \in T_p M$  define  $F^* A \in \mathcal{T}^{(0, l)} M$  by

$$(F^* A)_p(v_1, \dots, v_l) := A_{F(p)}(DF_p(v_1), \dots, DF_p(v_l))$$

## Lie Derivatives

Recall that if  $X, Y \in \mathfrak{X}(M)$ , we define  $(L_X Y)_p$  where  $X$  generates a flow  $\phi_t : M \rightarrow M$

## IMAGE 1

$(\phi_{-t})_* Y_{\phi_t(p)} = ((\phi_{-t})_* Y)_p \in T_p M$  for  $Y_p \in T_p M$ . Then  $L_X Y = \frac{d}{dt} \Big|_{t=0} ((\phi_{-t})_* Y)_p$ .  
If  $A \in \mathcal{T}^{(0,l)} M$ ,

## IMAGE 2

$$(\phi_t^* A)_p = (\phi_t)^* (A_{\phi_t(p)}) \in T^{(0,l)} T_p M$$

$$\text{So } L_V A = \frac{d}{dt} \Big|_{t=0} (\phi_t^* A)_p.$$

### Properties

1.  $L_V f = Vf$  (where  $f \in C^\infty(M)$  can be thought of as a smooth  $(0,0)$ -tensor field). Then

$$(L_V f)(p) = \frac{d}{dt} \Big|_{t=0} (\phi_t^* f)_p = \frac{d}{dt} \Big|_{t=0} (f \circ \phi_t(p)) = (Vf)_p$$

$$1. L_V(fA) = (Vf)A + fL_V A.$$

$$2. L_V(A \otimes B) = (L_V A) \otimes B + A \otimes (L_V B).$$

$$3. L_V(A(X_1, \dots, X_l)) = (L_V A)(X_1, \dots, X_l) + A(L_V X_1, \dots, X_l) + \dots + A(X_1, \dots, L_V X_l) \text{ for } A \in \mathcal{T}^{(0,l)} M \text{ and } X_i \in \mathfrak{X}(M).$$

### Proof of 2

We have  $O := \{p \in M : V_p \neq 0\}$  open in  $M$  and  $\text{supp } V = \overline{\{p \in M : V_p \neq 0\}}$ .

1. (2) holds on  $O$ .

Recall that if  $V_p \neq 0$ , then there exists a local chart  $(U, (x^i))$  centered at  $p$  such that on  $U$ ,  $V = \frac{\partial}{\partial x^1}$ . In particular, its flow  $\phi_t$  is  $(x^1, \dots, x^n) \mapsto (x^1 + t, x^2, \dots, x^n)$ .

Then take some chart  $U \subseteq O$  centered at  $p$  such that  $V = \frac{\partial}{\partial x^1}$  in  $U$ . Inside  $U$ , write  $A = A_I dx^I$ , and

$$\begin{aligned} \phi_t^*(fA) &= (\phi_t^* f)(\phi_t^* A) \\ &= (f \circ \phi_t) \phi_t^*(A_I dx^I) \\ &= f(x^1 + t, x^2, \dots, x^n) A_I(x^1 + t, \dots, x^n) \phi_t^* dx^I \\ &= f(x^1 + t, x^2, \dots, x^n) A_I(x^1 + t, \dots, x^n) dx^I \end{aligned}$$

2. (2) holds on  $\text{supp } V$  by taking limits.

3. (2) holds outside  $\text{supp } V$ , since  $V \equiv 0$  on open  $M \setminus \text{supp } V$  and hence  $\phi_t \equiv \text{id}$ . So both sides are identically zero.

**January 27, 2025**

**Recall: Prop 12.32(2)**

$$L_V(fA) = (Vf)A + fL_V A$$

### Proof Step 1:

Show that the equality holds on  $\{p \in M : V(p) \neq 0\}$ .

Let  $p \in M$  with  $V(p) \neq 0$ .

Take any chart  $(U, x^i)$  centered at  $p$  such that  $V = \frac{\partial}{\partial x^1}$  on  $U$ . Then its flow is

$$\theta_t : (x^1, \dots, x^n) \mapsto (x^1 + t, x^2, \dots, x^n)$$

in  $U$ . In  $U$ , we write  $A = A_I dx^I$  (where  $dx^I = dx^{i_1} \otimes \dots \otimes dx^{i_l}$ ). Recall that

$$\theta_t^*(dx^i) = d(\theta_t^* x^i) = d(x^i \theta_t) = \begin{cases} d(x^1 + t) = dx^1 & i = 1 \\ d(x^i) & i \neq 1 \end{cases}$$

Write the pullback of  $\theta_t$

$$\begin{aligned} \theta_t^*(fA) &= (\theta_t^* f)(\theta_t^* A_I dx^I) \\ &= (f \circ \theta_t)(A_I \circ \theta_t)(dx^I) \\ &= f(x^1 + t, x^2, \dots, x^n) A_I(x^1 + t, \dots, x^n) dx^I \end{aligned}$$

So for  $p = (x^i)$ ,

$$\begin{aligned} (L_V(fA))_p &= \left. \frac{d}{dt} \right|_{t=0} f(x^1 + t, x^2, \dots, x^n) A_I(x^1 + t, \dots, x^n) dx^I \\ &= \underbrace{\frac{\partial f}{\partial x^1}(x^1, \dots, x^n)}_{Vf} \underbrace{A_I(x^1, \dots, x^n) dx^I}_{\theta_t^* A} + f(x^1, \dots, x^n) \frac{\partial A_I}{\partial x^1(x^1, \dots, x^n) dx^I} \end{aligned}$$

inside  $U$ . Hence  $Vf = \frac{\partial f}{\partial x^1}$ .

### Corollary

$L_V(df) = d(L_V f)$  for  $f \in C^\infty(M)$ .

- Proof

For all  $X \in \mathfrak{X}(M)$ ,

$$(L_V(df))(X) = V(df(X)) - df(L_V X) = VXf - [V, X]f = VXf - (VXf - XVf) = XVf$$

and

$$(d(L_V f))(X) = X(L_V f) = XVf.$$

### Proof Step 2:

Show that the equality holds on  $\overline{\{p \in M : V(p) \neq 0\}}$ .

### Proof Step 3:

Show that the equality holds elsewhere.

### Recall: Invariance

For two vector fields,  $X$  and  $Y$ ,  $Y$  is invariant under the flow of  $X$  if  $L_X Y \equiv 0$ .

We say a  $(0, l)$ -tensor field  $A$  is invariant under a map  $F : M \rightarrow M$  if  $F^* A = A$ . Equivalently, if under a flow  $\theta_t : M \rightarrow M$  if  $\theta_t^* A = A$  for all  $t$ .

### Theorem 12.37

$A$  is invariant under  $\theta_t$ ,  $\forall t$ , if and only if  $L_V A = 0$ .

#### Note

$$\left. \frac{d}{dt} \right|_{t=t_0} (\theta_t^* A)_p = (\theta_{t_0}^* (L_V A))_p = \theta_{t_0}^* (L_V A)_{\theta_{t_0}^*(p)}$$

So

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=t_0} (\theta_t^* A)_p &= \left. \frac{d}{dt} \right|_{t=t_0} (\theta_t^*) A_{\theta_t(p)} \\ &\stackrel{t=s+t_0}{=} \left. \frac{d}{ds} \right|_{s=0} \theta_{s+t}^* A_{\theta_{s+t_0}(p)} \\ &= \left. \frac{d}{ds} \right|_{s=0} \theta_{t_0}^* \circ \theta_s^* A_{\theta_{t_0}(\theta_s(p))} \\ &= \theta_{t_0}^* (L_V A)_{\theta_{t_0}(p)} \end{aligned}$$

Therefore, if  $A$  is invariant under  $\theta_t$ , then  $\theta_t^* A = A$  and

$$L_V A = \left. \frac{d}{dt} \right|_{t=0} (\theta_t^* A)_p = \left. \frac{d}{dt} \right|_{t=0} A_p = 0.$$

In the other direction, if  $L_V A \equiv 0$ , we show that  $(\theta_t^* A)_p = A_p$  for every  $p$  and each  $t$ . From above,

$$\left. \frac{d}{dt} \right|_{t=t_0} (\theta_t^* A)_p = \theta_{t_0}^* \underbrace{(L_V A)_{\theta_{t_0}(p)}}_{=0} = 0$$

Hence  $(\theta_t^* A)_p$  is a constant  $A_p$ .

### Special Tensors (for this course)

#### Riemannian Metric

$g$  a  $(0, 2)$ -tensor, symmetric and positive definite. That is, at each point  $p$

$$g_p : T_p M \times T_p M \rightarrow \mathbb{R}$$

which is bilinear, symmetric and positive definite. This is an inner product.

## K (Differential) Form

$\omega$  a  $(0, k)$ -tensor, alternating.

## Riemannian Metric

In a chart  $(U, (x^i))$ ,  $g = g_{ij} dx^i \otimes dx^j$ .

Since it is symmetric,  $g(\partial_i, \partial_j) = g(\partial_j, \partial_i)$  (i.e.  $g_{ij} = g_{ji}$ ). We write  $dx^i dx^j = \text{Sym}(dx^i \otimes dx^j)$ . In this case

$$\text{Sym}(dx^i \otimes dx^j) = \frac{1}{2} (dx^i \otimes dx^j + dx^j \otimes dx^i)$$

So we may write  $g = g_{ij} dx^i dx^j$  and, sometimes,  $(dx^1)^2 = dx^1 dx^1$ .

We have also that  $g_{ij}$  corresponds to a positive definite, symmetric  $n \times n$  matrix.

### Example

In  $\mathbb{R}^n$ ,  $g_E = \delta_{ij} dx^i dx^j$ . For  $v = v^k \partial_k$  and  $w = w^l \partial_l$ ,

$$g_E(v, w) = \delta_{ij} dx^i dx^j (v^k \partial_k w^l \partial_l) = v^k w^l \delta_{ij} \underbrace{dx^i(\partial_k)}_{\delta_k^i} \underbrace{dx^j(\partial_l)}_{\delta_l^j} = v^1 w^1 + \dots + v^n w^n$$

### Example

Consider  $S^2 \subseteq \mathbb{R}^3$  embedded such that  $T_p S^2 \hookrightarrow T_p \mathbb{R}^3 \cong \mathbb{R}^3$ .

Then  $g_p(v, w) = v \cdot w$  defines a Riemannian metric on  $S^2$ .

## Proposition

Any smooth manifold admits a Riemannian metric.

### Proof 1

Embed  $M$  into  $\mathbb{R}^N$  with  $N$  sufficiently large. Then  $M$  is an embedded submanifold in  $\mathbb{R}^N$  which induces a Riemannian metric on  $M$ .

### Proof 2

Let  $\{U_i\}$  be a countable cover of  $M$  (with each  $U_i$  a chart) and  $\{\psi_i\}$  be a partition of unity with respect to this cover.

IMAGE 1

So  $\phi_i^* g_E$  defines a Riemannian metric on  $U_i$  and we construct  $\sum_i \psi_i (\phi_i^* g_E)$ .

### Example: Metric Product

Take  $(M_1, g_1)$  and  $(M_2, g_2)$  and construct  $g_1 \oplus g_2$  on  $M_1 \times M_2$  by either

$$g_1 \oplus g_2 = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$$

or

$$(g_1 + g_2)((v_1, v_1), (w_1, w_2)) = g_1(v_1, w_1) + g_2(v_2, w_2)$$

e.g.  $S^1 \subseteq \mathbb{R}^2$  gives  $(S^1, g_1)$ , then on the  $n$ -torus we construct  $(\mathbb{T}^n, g_1 \oplus \cdots \oplus g_1)$ .

### Example: Warped Product

IMAGE 2

Take  $f : M \rightarrow \mathbb{R}^+$  smooth,  $(M, g)$  and  $(N, h)$ .

Define a new metric  $\tilde{g}$  on  $M \times N$  by

$$\tilde{g}_{(x,y)} = g_x + f(x)h_y$$

An example in polar coordinates is

$$(dx)^2 + (dy)^2 = (d(r \cos \theta))^2 + (d(r \sin \theta))^2 = (\cos \theta dr - r \sin \theta d\theta)^2 + (\sin \theta dr + r \cos \theta d\theta)^2 = dr^2 + r^2 d\theta^2$$

Imagine fixing a direction  $r$  and at each point attaching a circle of radius  $r$ .

IMAGE 3

### Recall: Gradient

If  $f \in C^\infty(\mathbb{R}^n)$ , then

$$\nabla f = \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}$$

Note that this violates our Einstein summation.

If  $f \in C^\infty(M)$ , its differential  $df$  is a 1-form and not a vector field. Why? Because in  $\mathbb{R}^n$  we are implicitly using the Euclidean metric.

If we have an inner product on a TVS, say  $(V, (\cdot, \cdot))$ , then we can construct an isomorphism  $V \cong V^*$  by  $v \mapsto (v, \cdot)$ .

On  $(M, g)$  we use  $g$  to construct a bundle isomorphism between  $TM$  and  $T^*M$  by  $(p, v) \mapsto g_p(v, \cdot)$ .

With this, given  $df \in \Omega^1(M)$ , we can define a vector field  $\nabla f \in \mathfrak{X}(M)$  by

$$g(\nabla f, X) = (df)(X) = Xf$$

In a chart  $(U, (x^i))$ , set  $\nabla f = b^i \frac{\partial}{\partial x^i}$ . Then

$$g\left(\nabla f, \frac{\partial}{\partial x^j}\right) = g\left(b^i \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = b^i g_{ij} = (df)\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^j}$$

Let  $g^{ij}$  be the inverse of  $g_{ij}$ , then

$$b^k = b^i \delta_i^k = b^i g_{ij} g^{jk} = \frac{\partial f}{\partial x^j} g^{jk}$$



so

$$\nabla f = b^k \frac{\partial}{\partial x^k} = \frac{\partial f}{\partial x^j} g^{jk} \frac{\partial}{\partial x^k}$$

Then from above, we actually have

$$\nabla f = \frac{\partial f}{\partial x^i} \delta_{ij} \frac{\partial}{\partial x^j}$$

which satisfies our summation convention.

### Example

If  $g_E = dr^2 + r^2 d\theta^2$  in polar coordinates,

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \quad \text{and} \quad (g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

So

$$\nabla f = \frac{\partial f}{\partial x^j} g^{jk} \frac{\partial}{\partial x^k} = \frac{\partial f}{\partial r} \frac{\partial}{\partial r} + \frac{\partial f}{\partial \theta} \frac{1}{r^2} \frac{\partial}{\partial \theta}$$

### Isometric Metrics

We say that  $(M, g)$  and  $(N, h)$  are isometric if there is a diffeomorphism  $F : M \rightarrow N$  such that  $F^* h = g$ .

With  $g$ , we can define (for  $v \in T_p M$ ),  $\|v\|_g = (g_p(v, v))^{1/2}$  and (for  $v, w \in T_p M$ )

$$\cos(v, w) = \frac{g_p(v, w)}{\|v\|_g \|w\|_g}$$

### Definition: Length

Let  $\gamma : I \rightarrow M$  be a (piecewise) smooth curve.

Define  $\text{length}_g(\gamma) = \int_I \|\gamma'(t)\|_g dt$ .

Remember that  $\text{length}_g(\gamma)$  is independent of reparameterization. That is

$$J \xrightarrow{\phi} I \xrightarrow{\gamma} M \quad \text{with } \tilde{\gamma} = \gamma \circ \phi \text{ we have}$$

$$\begin{aligned} \int_J \|\tilde{\gamma}'(t)\| dt &= \int_J \|(\gamma \circ \phi)'(t)\| dt \\ &= \int_J \|\gamma'(\phi(t)) \cdot \phi'(t)\| dt \\ &\stackrel{\phi' > 0}{=} \int_J \|\gamma'(\phi(t))\| |\phi'(t)| dt \\ &\stackrel{s=\phi(t)}{=} \int_I \|\gamma'(s)\| ds \end{aligned}$$

## Definition: Distance

Given  $(M, g)$ , define

$$d_g(p, q) = \inf \{ \text{length}_g(\gamma) : \gamma \text{ is piecewise smooth from } p \text{ to } q \}$$

## Theorem

$(M, d_g)$  is a metric space.

Moreover, it induces a metric topology that coincides with the manifold topology.

## Theorem: Hopf-Rinow

The following are equivalent.

1.  $(M, d_g)$  is a complete metric space.
2.  $\forall p, q \in M$ , there exists a length-minimizing curve (a geodesic) from  $p$  to  $q$ .

## Definition: Geodesic

A curve such that the second derivative along  $\gamma \equiv 0$ .

February 3, 2025

## Recall: Wedge Product

$$\bigwedge^k V^* \times \bigwedge^l V^* \rightarrow \bigwedge^{k+l} V^*$$

$$(\omega, \eta) \mapsto \omega \wedge \eta$$

By  $\frac{(k+l)!}{k!l!} \text{Alt}(\omega \otimes \eta) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\sigma \cdot (\omega \otimes \eta))$ .  
 $\epsilon^I \in \bigwedge^k V^*$ , so

$$\epsilon^I(v_1, \dots, v_k) = \det \begin{pmatrix} \epsilon^{i_1}(v_1) & \cdots & \epsilon^{i_1}(v_k) \\ \vdots & & \vdots \\ \epsilon^{i_k}(v_1) & \cdots & \epsilon^{i_k}(v_k) \end{pmatrix}$$

We have a  $V$  basis  $\{E_I\}$  and a  $V^*$  dual basis  $\{\epsilon^I\}$  with  $I = (i_1, \dots, i_k)$ . We also have that  $\epsilon^I(E_{j_1}, \dots, E_{j_k}) = \delta_J^I$ .  
 Then  $\mathcal{B} = \{\epsilon^I : I \text{ is strictly increasing}\}$  is a basis for  $\bigwedge^k V^*$ .

## Lemma 14.10

$$\epsilon^I \wedge \epsilon^J = \epsilon^{IJ}.$$

## Proof

We show that  $\epsilon^I \wedge \epsilon^J(E_{p_k}, \dots, E_{p_{k+l}}) = \epsilon^{IJ}(E_{p_1}, \dots, E_{p_{k+l}})$ ,  $P = (p_1, \dots, p_{k+l})$ .

If  $I \cup J \neq P$ , then both sides are zero.

If  $IJ$  or  $P$  has repeated index, then both sides are zero.

Then the only nontrivial case is when  $P = IJ$  without repeated indices. Write  $IJ = \{i_1, \dots, i_k, j_1, \dots, j_l\}$  such that we can apply a permutation  $\gamma \in S_{k+l}$  to generate a strictly increasing  $P = \{p_1, \dots, p_{k+l}\}$ . Then write  $P_1 = \{p_1, \dots, p_k\}$  and  $P_2 = \{p_{k+1}, \dots, p_{k+l}\}$ , and compute

$$\begin{aligned} \epsilon^P &= \epsilon^{P_1} \wedge \epsilon^{P_2} \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\text{sign } \sigma) \cdot (\sigma(\epsilon^{P_1} \otimes \epsilon^{P_2})) \\ &= \frac{1}{k!l!} \sum_{\sigma' \in S_{k+l}} (\text{sign } \sigma') (\text{sign } \gamma) ((\gamma \cdot \sigma')(\epsilon^{P_1} \otimes \epsilon^{P_2})) \\ &= \text{sign } \gamma (\epsilon^I \wedge \epsilon^J) \end{aligned}$$

## Proposition 14.11

1. If  $\omega^i \in V^*$  and  $v_j \in V$ , then  $\omega^1 \wedge \dots \wedge \omega^k(v_1, \dots, v_k) = \det(w^i(v_j))$ .

## Proof

It suffices to check (assuming  $I, J$  strictly increasing)

$$(\epsilon^{i_1} \wedge \dots \wedge \epsilon^{i_k})(E_{j_1}, \dots, E_{j_k}) = \epsilon^I(E_{j_1}, \dots, E_{j_k}) = \delta_J^I = \det(\epsilon^{i_p}(E_{j_q})).$$

## Definition: Graded Algebra

Write  $\bigwedge V^* = \bigoplus_{k=0}^n \bigwedge^k V^*$  with  $\dim \bigwedge^k V^* = 2^n$ .

Remember that  $\dim \bigwedge^k V^* = \binom{n}{k}$ .

It is graded if  $(\bigwedge^k) \wedge (\bigwedge^l) \subseteq \bigwedge^{k+l}$ .

## Differential Forms on Manifolds

Given a manifold  $M$ , a  $k$ -form on  $M$   $\bigwedge^k(T^*M) = \coprod_{p \in M} (\bigwedge^k T_p^*M)$  is a section of the bundle  $\bigwedge^k(T^*M) \rightarrow M$ .

$\Omega^k(M)$  is the collection of  $k$ -forms on  $M$ .

Locally,  $\omega \in \Omega^k(M)$  may be written  $\omega = \sum \omega_I dx^I$  for a chart  $(U, (x^i))$ .

Summing over strictly increasing  $I$ ,  $dx^I = dx^{i_1} \wedge \dots \wedge dx^{i_k}$  and  $\omega_I = \omega\left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}}\right)$ .

## Pullback

For  $F : M \rightarrow N$  and  $\omega \in \Omega^k(N)$ , we define  $(F^*\omega) \in \Omega^k(M)$  by

$$(F^*\omega)(v_1, \dots, v_k) = \omega(DF(v_1), \dots, DF(v_k)).$$

It follows that

$$F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$$

and

$$\begin{aligned}
F^* \left( \sum_I \omega_I dx^I \right) &= \sum_I (F^* \omega_I) F^* (dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \\
&= \sum_I (\omega_I \circ F) (d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F)) \\
&= \sum_I (\omega_I \circ F) dF^{i_1} \wedge \cdots \wedge dF^{i_k}
\end{aligned}$$

### Example

For  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^3$  by  $F(u, v) = (u, v, u^2 - v^2)$  and  $\omega = y dx \wedge dz \in \Omega^2(\mathbb{R}^3)$ .

$$F^* \omega = F^*(y dx \wedge dz) = v du \wedge d(u^2 - v^2) = v du \wedge (2u du - 2v dv) = -2v^2 du \wedge dv$$

### Proposition 14.20

For  $F : M^n \rightarrow N^n$  with local coordinates  $(x^i)$  and  $(y^i)$  respectively, if  $u \in C^\infty(N)$  then

$$F^*(u dy^1 \wedge \cdots \wedge dy^n) = (u \circ F) \det DF$$

### Proof

Write  $F$  in components  $(F^1, \dots, F^n)$  where  $F^i = y^i \circ F$

$$\begin{aligned}
F^*(u dy^1 \wedge \cdots \wedge dy^n) &= (u \circ F) dF^1 \wedge \cdots \wedge dF^n \left( \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right) \\
&= (u \circ F) \det \left( dF^i \left( \frac{\partial}{\partial x^j} \right) \right) \\
&= (u \circ F) \det(DF)
\end{aligned}$$

If  $(U, (x^i))$  and  $(\tilde{U}, (\tilde{x}^i))$  are local charts with  $U \cap \tilde{U} \neq \emptyset$ , then using  $F = \text{id}_{U \cap \tilde{U}}$  we have that  $F^* = \text{id}$

$$d\tilde{x}^i \wedge \cdots \wedge d\tilde{x}^n = \det \left( \frac{\partial \tilde{x}^j}{\partial x^i} \right) dx^1 \wedge \cdots \wedge dx^n$$

### Definition: Exterior Derivative

For  $\omega \in \Omega^k(U)$ ,  $U \subseteq \mathbb{R}^n$  open,  $\omega = \sum_I \omega_I dx^I$  define  $d : \omega^k(U) \rightarrow \omega^{k+1}(U)$  by  $\omega \mapsto d\omega$ . Then

$$d\omega = \sum_I \underbrace{d\omega_I}_{\in \Omega^1(U)} \wedge \underbrace{dx^I}_{\in \Omega^k(U)}$$

## Example

$$\omega \in \Omega^1(U), \omega = \sum_{i=1}^n \omega_i dx^i.$$

$$d\omega = \sum_{i=1}^n d\omega_i \wedge dx^i = \sum_{i,j=1}^n \frac{\partial \omega_i}{\partial x^j} dx^j \wedge dx^i = \sum_{i < j} \left( \frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} \right) dx^i \wedge dx^j$$

For  $\omega = df \in \Omega^1(M)$ ,  $d(df) = \sum_{i < j} \left( \frac{\partial^2 f}{\partial x^j \partial x^i} - \frac{\partial^2 f}{\partial x^i \partial x^j} \right) dx^i \wedge dx^j = 0$ . That is,  $(d \circ d)(f) = 0$  for any smooth function  $f \in C^\infty(M)$ .

## Proposition

1.  $d$  is  $\mathbb{R}$ -linear.
2.  $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge (d\eta)$  with  $k = \deg \omega$ .
3.  $d \circ d = 0$ .
4.  $F^*(d\omega) = d(F^*\omega)$ .

## Proof of b

Write  $\omega = u dx^I$  and  $\eta = v dx^J$ .

Claim:  $d(u dx^I) = du \wedge dx^I$  for any index (perhaps not strictly increasing)  $I$ .

If  $I$  has a repeated index, both sides are zero.

If not, let  $\sigma \in S_k$  such that  $I_\sigma = J$  strictly increasing.

$$d(u dx^I) = d((\text{sign } \sigma) u dx^J) = \text{sign } \sigma \cdot du \wedge dx^J = du \wedge (\text{sign } \sigma \cdot dx^J) = du \wedge dx^I$$

Then

$$d(\omega \wedge \eta) = d(u dx^I \wedge v dx^J) = d(uv dx^I \wedge dx^J) = d(uv dx^{IJ}) = d(uv) \wedge dx^{IJ} = (u dv + v du) \wedge (dx^I \wedge dx^J)$$

So

$$d\omega \wedge \eta + (-1)^k \omega \wedge d\eta = du \wedge dx^I \wedge (v dx^J) + (-1)^k u dx^I \wedge (dv \wedge dx^J)$$

and it suffices to show that  $dv \wedge dx^I \wedge dx^J = (-1)^k dx^I \wedge dv \wedge dx^J$ .

## Proof b Implies c

Write

$$d \circ d(\omega_I dx^I) = d(d\omega_I \wedge dx^I) = d(d\omega_I) \wedge dx^I + (-1)^1 d\omega_I \wedge d(dx^I) = 0$$

## Proof of d

Write  $\omega = u dx^I$  such that  $d\omega = du \wedge dx^I$ .

$$F^*(d\omega) = F^*(du \wedge dx^I) = d(u \circ F) \wedge dF^{i_1} \wedge \cdots \wedge dF^{i_k}$$

and

$$d(F^*\omega) = d((u \circ F)dF^{i_1} \wedge \cdots \wedge dF^{i_k}) = d(u \circ F) \wedge dF^{i_1} \wedge \cdots \wedge dF^{i_k}$$

**February 5, 2025**

## Theorem 14.24

There is a unique map  $d : \Omega^*(M) \rightarrow \Omega^*(M)$  with  $d(\Omega^k(M)) \subseteq \Omega^{k+1}(M)$  such that

1.  $d$  is  $\mathbb{R}$ -linear
2.  $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$
3.  $d \circ d = 0$
4.  $df(X) = Xf$  for all  $f \in \Omega^0(M) = C^\infty(M)$  and  $X \in \mathfrak{X}(M)$ .

## Proof: Existence

Let  $\omega \in \Omega^k(M)$ . Then  $\omega|_U \in \Omega^k(U)$ . We have that  $\varphi^{-1*}\omega \in \Omega^k(\varphi(U))$ ,  $d(\varphi^{-1*}\omega) \in \Omega^{k+1}(\varphi(U))$ , and  $d\omega := \varphi^*d(\varphi^{-1*}\omega) \in \Omega^{k+1}(U)$  on  $U$ .

IMAGE 1

## Proof: Well-defined

If  $(V, \psi)$  is another chart with  $U \cap V \neq \emptyset$ , we need to show that  $\psi^*(d(\psi^{-1*}\omega)) = \varphi^*(d(\varphi^{-1*}\omega))$ . This is equivalent to

$$\begin{aligned} \iff d(\psi^{-1*}\omega) &= \psi^{-1*}\varphi^*(d(\varphi^{-1*}\omega)) \\ \iff d(\psi^{-1*}\omega) &= (\varphi \circ \psi^{-1})^*d(\varphi^{-1*}\omega) \end{aligned}$$

where

$$(\varphi \circ \psi^{-1})^*d(\varphi^{-1*}\omega) = d((\varphi \circ \psi^{-1})^*\varphi^{-1*}\omega) = d(\psi^{-1*} \circ \varphi^* \circ \varphi^{-1*}\omega) = d(\psi^{-1*}\omega)$$

## Proof: Uni!

For any  $d : \Omega^*(M) \rightarrow \Omega^*(M)$  with the property  $(d\omega)_p$  only depends on  $\omega|_U$  where  $p \in U$ .

Suppose  $\omega_1 = \omega_2$  on  $U$ . We need to show that  $(d\omega_1)_p = (d\omega_2)_p$ .

So set  $\eta = \omega_1 - \omega_2$ . Then  $\eta \equiv 0$  on  $U$ , and we need to show that  $(d\eta)_p = 0$ .

Let  $\psi$  be a bump function such that  $\text{supp } \psi \subseteq U$  and  $\psi(p) = 1$ .

Then  $\psi\eta = 0 \in \Omega^k(M)$ .

$$0 = d(\psi\eta) = d\psi \wedge \eta + (-1)^0 \psi \wedge d\eta$$

At point  $p$ , it reads

$$0 = 0 \wedge \eta_p + \overbrace{\psi(p)}^{=1} \wedge d\eta_p$$

That is,  $0 = d\eta_p$ . Let  $p \in M$ ,  $U$  a chart around  $p$ , say  $(U, (x^i))$ , and  $\omega \in \Omega^k(U)$ . We know that  $(d\omega)_p$  only depends on  $\omega|_U = \sum_I \omega_I dx^I$ . Then for  $p \in V \subseteq \bar{V} \subseteq U$ ,  $\omega|_U$  extends functions  $\omega_I, x^I \in C^\infty(V)$  to globally defined functions  $\tilde{\omega}_I, \tilde{x}^I \in C^\infty(M)$ . Therefore

$$\begin{aligned} d(\omega|_U) &= \sum_I d(\omega_I dx^I) \\ &= \sum_I d(\tilde{\omega}_I \tilde{x}^I) \\ &= \sum_I (d\tilde{\omega}_I \wedge d\tilde{x}^I + \omega_I \wedge \overbrace{d(\tilde{x}^{i_1} \wedge \cdots \wedge \tilde{x}^{i_k})}^{=0}) \\ &= \sum_I d\omega_I \wedge dx^I \end{aligned}$$

which is the same formula for  $\mathbb{R}^n$ .

### Proposition: 14.26

$$F^*(d\omega) = d(F^*\omega).$$

### Proposition: 14.32

$$\mathcal{L}_V(\omega \wedge \eta) = (\mathcal{L}_V\omega) \wedge \eta + \omega \wedge (\mathcal{L}_V\eta).$$

### Corollary

$$\mathcal{L}_V(d\omega) = d(\mathcal{L}_V\omega).$$

### Definition: Interior Multiplication

Given  $\omega \in \bigwedge^k V^*$  and  $v \in V$ , define  $\iota_v \omega \in \bigwedge^{k-1} V^*$  (sometimes written  $V \lrcorner \omega$ ).

$$(\iota_v \omega)(u_1, \dots, u_{k-1}) = \omega(v, u_1, \dots, u_{k-1})$$

This defines  $\iota_v : \bigwedge^k V^* \rightarrow \bigwedge^{k-1} V^*$ , and we have  $\iota_v \circ \iota_v = 0$ .

$$\iota_v(\omega \wedge \eta) = (\iota_v \omega) \wedge \eta + (-1)^k \omega \wedge (\iota_v \eta)$$

## Proof

It suffices to show that for  $\omega^1, \dots, \omega^k \in V^*$

$$\iota_V(\omega^1 \wedge \dots \wedge \omega^k) = \sum_{i=1}^k (-1)^{i-1} \omega^i(v) \omega^1 \wedge \dots \wedge \hat{\omega}^i \wedge \dots \wedge \omega^k$$

Where  $\hat{\omega}^i$  is meant to denote “forgetting” a term in the wedge product. That is, the first term has no  $\omega^1$ , the second no  $\omega^2$ , etc.

Assuming this, it suffices to consider  $\omega = \omega^1 \wedge \dots \wedge \omega^k$  and  $\eta = \eta^1 \wedge \dots \wedge \eta^l$ . Then

$$\begin{aligned} \iota_V(\omega \wedge \eta) &= \iota_V(\omega^1 \wedge \dots \wedge \omega^k \wedge \eta^1 \wedge \dots \wedge \eta^l) \\ &= \sum_{i=1}^k (-1)^{i-1} \omega^i(v) \omega^1 \wedge \dots \wedge \hat{\omega}^i \wedge \dots \wedge \omega^k \wedge \eta^1 \wedge \dots \wedge \eta^l + \sum_{i=1}^l (-1)^{k+i-1} \eta^i(v) \omega^1 \wedge \dots \wedge \omega^k \wedge \eta^1 \wedge \dots \wedge \hat{\eta}^i \wedge \dots \wedge \eta^l \\ &= (\iota_V \omega) \wedge \eta + (-1)^k \omega \wedge (\iota_V \eta) \end{aligned}$$

Write  $v_1 = v$ , and apply both sides to  $(v_2, \dots, v_k)$ . The left hand side is

$$\omega^1 \wedge \dots \wedge \omega^k(v_1, \dots, v_k) = \det(\omega^i(v_j)) = \det \begin{pmatrix} \omega^1(v_1) & \dots & \omega^i(v_1) & \dots & \omega^k(v_1) \\ \vdots & & & & \vdots \\ \omega^1(v_k) & \dots & \omega^i(v_k) & \dots & \omega^k(v_k) \end{pmatrix}$$

The right hand side is given by

$$\sum_{i=1}^k (-1)^{i-1} \omega^i(v_1) (\omega^1 \wedge \dots \wedge \hat{\omega}^i \wedge \dots \wedge \omega^k)(v_2, \dots, v_k)$$

which, when expanded, gives  $\det(\omega^i(v_j))$  along the first row.

## Proposition 14.35 (Cartan)

If  $V \in \mathfrak{X}(M)$  and  $\omega \in \Omega^k(M)$ , then

$$\mathcal{L}_V \omega = V \lrcorner (d\omega) + d(V \lrcorner \omega)$$

## Corollary

$$\mathcal{L}_V(d\omega) = d(\mathcal{L}_V \omega)$$

## Proof

By assuming Cartan’s formula, the left hand side is

$$V \lrcorner \overbrace{(d \circ d\omega)}^{=0} + d(V \lrcorner d\omega)$$

and the right hand side is

$$d(V \lrcorner d\omega + d(V \lrcorner \omega)) = d(V \lrcorner d\omega) + \overbrace{d \circ d(v \lrcorner \omega)}^{=0}$$



## Proof (of Cartan's Formula)

We prove by induction on  $\deg(\omega)$ . When  $\omega$  is a function  $f \in C^\infty(M) = \Omega^0(M)$ , the left hand side is

$$\mathcal{L}_V f = Vf$$

and the right hand side is

$$\overbrace{V \lrcorner(df) + d(V \lrcorner f)}^{=0} = df(V) = Vf$$

since  $\iota_V$  maps  $\Omega^k$  to  $\Omega^{k-1}$ .

Assuming it holds for  $k-1$  forms, we consider  $\omega \in \Omega^k(M)$  and locally write  $\omega = \sum^I \omega_I dx^I$ .

It suffices to show that the formula holds for  $\omega = du \wedge \beta$ ,  $u \in C^\infty(M)$ ,  $\beta \in \Omega^{k-1}(M)$ .

$$(\omega_I dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \underbrace{dx^{i_1}}_{du} \wedge \underbrace{(\omega_I dx^{i_1} \wedge \cdots \wedge dx^{i_k})}_{\beta})$$

The left hand side is

$$\begin{aligned} \mathcal{L}_V(du \wedge \beta) &= \mathcal{L}_V du \wedge \beta + du \wedge \mathcal{L}_V \beta \\ &= d(\mathcal{L}_V u) \wedge \beta + du \wedge (V \lrcorner d\beta + d(V \lrcorner \beta)) \\ &= d(Vu) \wedge \beta + du \wedge (V \lrcorner d\beta) + du \wedge d(V \lrcorner \beta) \end{aligned}$$

and the right hand side is

$$\begin{aligned} V \lrcorner(d(du \wedge \beta)) + d(V \lrcorner(du \wedge \beta)) &= V \lrcorner(\overbrace{(d \circ du)}^{=0} \wedge \beta + (-1)du \wedge d\beta + \overbrace{d((V \lrcorner du) \wedge \beta + du \wedge (V \lrcorner \beta))}^{=Vu}) \\ &= (-1)(Vu)d\beta + d(Vu) \wedge \beta + (Vu)d\beta \end{aligned}$$

## Proposition 14.32

$$d\omega(X_1, \dots, X_{k+1}) = \sum_{1 \leq i \leq k+1} (-1)^{i-1} X_i(\omega(X_1, \dots, \hat{x}_i, \dots, X_{k+1})) + \sum_{1 \leq i \leq j \leq k+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{k+1})$$

When  $\omega \in \Omega^1$ , it reads

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$$

In particular, for  $\omega$  closed,

$$X(\omega(Y)) - Y(\omega(X)) = \omega([X, Y])$$

## Proof

It suffices to prove that for  $\omega = u dv$ ,  $u, v \in C^\infty(M)$  that

$$d(\omega) = d(udv) = du \wedge dv$$

The left hand side

$$(du \wedge dv)(X, Y) = \det \begin{pmatrix} du(X) & du(Y) \\ dv(X) & dv(Y) \end{pmatrix} = \det \begin{pmatrix} X_u & Y_u \\ X_v & Y_v \end{pmatrix}$$

and the right hand side

$$\begin{aligned} X(udv(Y)) - Y(udv(X)) - u(dv([X, Y])) &= X(u(Yv)) - Y(u(Xv)) - u([X, Y]v) \\ &= (Xu)(Yv) + u(XYv) - (Yu)(Xv) - u(YXv) - u([X, Y]v) \\ &= \det \begin{pmatrix} X_u & Y_u \\ X_v & Y_v \end{pmatrix} \end{aligned}$$

## Example

For  $f \in \Omega^*(\mathbb{R}^3)$  and  $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \in \Omega^{*+1}(\mathbb{R}^3)$ , write  $Pdx + Qdy + Rdz$  and

$$\begin{aligned} d(Pdx + Qdy + Rdz) &= \left( \frac{\partial P}{\partial y} dy + \frac{\partial P}{\partial z} dz \right) \wedge dx + \left( \frac{\partial Q}{\partial x} dx + \frac{\partial Q}{\partial z} dz \right) \wedge dy + \left( \frac{\partial R}{\partial x} dx + \frac{\partial R}{\partial y} dy \right) \wedge dz \\ &= \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dx \wedge dy \right) + \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} dy \wedge dz \right) + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} dz \wedge dx \right) \end{aligned}$$

Recall that for  $X = (P, Q, R) \in \mathfrak{X}(\mathbb{R}^3)$ , this is the curl of  $X$ .

Let  $\omega = u dx \wedge dy + v dy \wedge dz + w dz \wedge dx$ , then

$$\begin{aligned} d\omega &= \frac{\partial u}{\partial z} dz \wedge dx \wedge dy + \frac{\partial v}{\partial z} dx \wedge dy \wedge dz + \frac{\partial w}{\partial z} dy \wedge dz \wedge dx \\ &= \left( \frac{\partial V}{\partial x} + \frac{\partial W}{\partial y} + \frac{\partial U}{\partial z} \right) dx \wedge dy \wedge dz \end{aligned}$$

Recall that this is divergence. We can also look at the gradient

$$\text{grad } f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

we have

$$\text{grad } f \cdot X = Xf = df(X) = \sum \frac{\partial f}{\partial x^i} \cdot x^i$$

Putting this together,

$$\begin{array}{ccccccc} C^\infty(\mathbb{R}^3) & \xrightarrow{\text{grad}} & \mathfrak{X}(M) & \xrightarrow{\text{curl}} & \mathfrak{X}(M) & \xrightarrow{\text{div}} & C^\infty(\mathbb{R}^3) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Omega^0(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathbb{R}^3) \end{array}$$

**February 10, 2025**

Orientation. Lee pages 378 to 390.

**February 12, 2025**

## Recall

$$[E_1, E_2, \dots, E_n]$$

and  $\omega \in \Lambda^n V^* - \{0\}$

On a manifold, we say that  $\omega \in \Omega^n(M)$  is nonvanishing if and only if

- the manifold has an orientation if and only if
- the manifold admits an ordered atlas

For  $S^{n-1} \hookrightarrow M^n$ , if  $N$  is a vector field along  $S$  and nowhere tangent to  $S$  and  $M$  has an orientation given by  $\omega \in \Omega^n(M)$ , then  $S$  has an induced orientation  $(N \lrcorner \omega) \in \Omega^{n-1}(S)$ .  
In particular,  $\partial M \rightarrow M$  is oriented for  $N$  outwarding vector field along  $\partial M$ , we have induced orientation given by  $N \lrcorner \omega \in \Omega^{n-1}(\partial M)$ .

$$F : (M^n, O_M) \rightarrow (N^n, O_N)$$

is a local diffeomorphism and orientation preserving if  $F^* O_N = O_M$ . It is orientation reversing if  $F^* O_N = -O_M$ .  
 $F^* O_N$  is given the pullback  $F^* \omega$ , where  $\omega \in \Omega^n(N)$  is non-vanishing and matching with  $O_N$ .

### Example 1

For example,  $A : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$  by  $(x^i) \mapsto (-x^i)$  has orientation  $[E_1, \dots, E_{n+1}]$ . Then

$$[AE_1, \dots, AE_{n+1}] = [E_1, \dots, E_{n+1}] = (-1)^{n+1} [E_1, \dots, E_n]$$

and  $A$  is orientation preserving if and only if  $n$  is odd. Instead, if we consider forms  $\omega = \varepsilon^1 \wedge \dots \wedge \varepsilon^{n+1}$  then we have

$$A^* \omega(X_1, \dots, X_{n+1}) = \omega(AX_1, \dots, AX_{n+1}) = (\det A)(\omega(X_1, \dots, X_{n+1}))$$

so  $A^* \omega = (\det A) \omega = (-1)^{n+1} \omega$ .

### Example 2

Consider  $S^N \hookrightarrow \mathbb{R}^{n+1}$  and  $A : S^n \rightarrow S^n$  by  $x \mapsto -x$ .

IMAGE 1

$$A_*N = N.$$

Then  $S^n$  has an induced orientation  $(N \lrcorner \omega) \in \Omega^{n-1}(S)$  where  $\omega = \varepsilon^1 \wedge \cdots \wedge \varepsilon^{n+1} \in \Omega^{n+1}\mathbb{R}^{n+1}$ . Compute

$$\begin{aligned} A^*(N \lrcorner \omega)(X_1, \dots, X_n) &= (N \lrcorner \omega)(A_*X_1, \dots, A_*X_n) \\ &= \omega(N, A_*X_1, \dots, A_*X_n) \\ &= \omega(A_*N, A_*X_1, \dots, A_*X_n) \\ &= \det(DA)\omega(N, X_1, \dots, X_n) \\ &= (-1)^{n+1}(N \lrcorner \omega)(X_1, \dots, X_n) \end{aligned}$$

Therefore  $A^*(N \lrcorner \omega) = (-1)^{n+1}(N \lrcorner \omega)$  and  $A : S^n \rightarrow S^n$  is orientation preserving when  $n$  is odd.

## An aside about covering maps

Consider all  $\varphi$  such that this diagram

$$\begin{array}{ccc} \hat{M} & \xrightarrow{\varphi} & \hat{M} \\ & \searrow \pi & \swarrow \pi \\ & M & \end{array}$$

commutes. Then take  $\text{Aut}(\pi) = \{\varphi : \hat{M} \rightarrow \hat{M} \text{ diffeomorphic} : \pi = \pi \circ \varphi\}$ . Then  $\varphi \in \text{Aut}(\pi)$  preserves the preimage  $\pi^{-1}(x)$ .

IMAGE 2

IMAGE 3

So  $\text{Aut}(\pi) = \mathbb{Z}_2$ . For example,  $S^n \xrightarrow{\pi} \mathbb{R}P^n$ ,  $\text{Aut}(\pi) = \mathbb{Z}_2 = \{\text{id}, A\}$ . By theorem,  $\mathbb{R}P^n$  is orientable if and only if

- $A : S^n \rightarrow S^n$  is orientation perserving if and only if
- $n$  is odd.

In the case of the Mobius band,

IMAGE 4

$\text{Aut}(\pi) = \langle \gamma \rangle$  where  $\gamma : (x, y) \mapsto (x + 1, -y)$  is orientation reversing. This implies that  $M$  is not orientable.

## Theorem 15.36

Let  $\pi : \hat{M} \rightarrow M$  be a covering map.

1. If  $M$  is orientable, then  $\hat{M}$  is orientable.
2. If  $\hat{M}$  is orientable, what about  $M$ ?

$M$  is orientable if and only  $\text{Aut}(\pi)$  acts as an orientation preserving idffeomorphism on  $\hat{M}$ .

## Proof

( $\Leftarrow$ ) On  $M$ , we start with an atlas  $\{V_\beta\}$  such that each  $V_\beta$  is evenly covered by  $\pi$  with  $\pi^{-1}(V) = \bigcup_i U_i$

IMAGE 5

Each  $U_i$  carries an orientation (coming from  $O_{\hat{M}}$ ).

Define an orientation by  $V$  such that  $\pi|_{U_i} : U_i \rightarrow V$  is orientation preserving (i.e.  $\pi^* O_V = O_{U_i}$ ). For a different  $U_j$ ,

$$\pi^* O_V = (\pi \circ \varphi)^* O_V = \varphi^* \pi^* O_V = \varphi^* O_{U_i} = O_{U_j}$$

( $\Rightarrow$ ) As  $M$  is orientable, it has two orientations. Fix  $\hat{p} \in \hat{M}$ ,  $p = \pi(\hat{p}) \in M$ . Choose the orientation on  $M$  such that  $d\pi_{\hat{p}} : T_{\hat{p}}\hat{M} \rightarrow T_p M$  is orientation perserving. With this orientation  $O_M$ , we have

$$O_{\hat{M}} = \pi^* O_M = (\pi \circ \varphi)^* O_M = \varphi^* \pi^* O_M \varphi^* O_{\hat{M}}$$

so any  $\varphi \in \text{Aut}(\pi)$  is orientation preserving.

## Orientation Covering Space

If  $M$  is a connected un-orientable manifold, then there exists  $\pi : \hat{M} \rightarrow M$  a 2-folded (2-sheet) covering map – in the sense that  $\#\pi^{-1}(x) = 2$  (e.g.  $S^2 \rightarrow \mathbb{R}P^2$ ) – such that  $\hat{M}$  is orientable.

### Example: Mobius Band

We have  $\pi / \langle \gamma \rangle \rightarrow M$

IMAGE 6

and  $\gamma^2 : (x, y) \mapsto (x + 2, y)$  which gives a cylinder with  $\bar{\gamma} : (\theta, y) \mapsto (-\theta, -y)$ .

## Construction

Let  $M$  be connected. We construct

$$\hat{M} = \{(p, O_p) : p \in M, O_p \text{ is an orientation on } T_p M\}$$

where  $\pi : \hat{M} \rightarrow M$  is given by  $(p, O_p) \mapsto p$  which is 2-folded.

1.  $\hat{M}$  has a smooth structure.
2. with this smooth structure,  $\pi$  is a smooth covering map.
3.  $U \subseteq M$  (not necessariy a chart) is evenly covered by  $\pi$  if and only if  $U$  is orientable.

Given  $(U, O)$  where  $U$  is a chart in  $M$  and  $O$  is an orientation on  $U$ , we define  $\hat{U}_O \subseteq \hat{M}$  by

$$\hat{U}_O = \{(p, O_p) \in \hat{M} : p \in U \text{ and } O_p \text{ matches with } O\}$$

Consider a basis

$$\mathcal{B} = \{\hat{U}_O : U \subseteq M \text{ a chart, and } O \text{ an orienation on } U\}$$

1.  $\mathcal{B}$  covers  $\hat{M}$

2. For  $\hat{U}_O \cap \hat{U}_O^I \neq \emptyset$ , we have  $(p, O_p)$  such that  $p \in U \cap U^I$  and  $O_p$  matches with both  $O_{U^I}$  and  $O_U$ . Choose  $V \subseteq U \cap U^I$  and an orientation  $O_V$  such that  $O_V$  matches with  $O_p$ . Then  $O_V$  matches with both  $O_U$  and  $O_{U^I}$ ,  $\hat{V}_0 \subseteq \hat{U}_O \cap \hat{U}_O^I$ .

So  $\pi : \hat{U}_O \rightarrow U$  by  $(p, O_p) \mapsto p$  is a bijective homeomorphism, and it defines a smooth structure on  $\hat{M}$  such that  $\{\hat{U}_O\}$  is an atlas. Then  $\pi$  is a smooth covering map. In fact, every chart  $U \subseteq M$  is evenly covered by  $\hat{U}_O$  and  $\hat{U}_{-O}$ .

To show that  $\hat{M}$  is orientable, at each point  $\hat{p} = (p, O_p) \in \hat{M}$  we give an orientation at  $T_{\hat{p}}\hat{M}$  such that  $d\pi_{\hat{p}} : T_{\hat{p}}\hat{M} \rightarrow (T_p M, O_p)$  is orientation preserving. We need to show that this pointwise orientation is continuous.

We have that  $\hat{p} = (p, O_p) \in \hat{U}_O$  for the orientation of  $O$  on  $U$  matching with  $O_p$ . Then  $\pi : \hat{U}_O \rightarrow (U, O)$  is orientation preserving (i.e. the orientation on  $\hat{U}_O$  is  $\pi^* O$ ).

Finally, we need to show that if  $U \subseteq M$  is open and evenly covered, then  $U$  is orientable. In fact,  $\pi^{-1}(U) = V_1 \cup V_2 \subseteq \hat{M}$  where  $\pi : V_i \rightarrow U$  is a diffeomorphism. Since  $\hat{M}$  is orientable, it induces an orientation on  $V_1$ . Then we get an orientation on  $U$  through the diffeomorphism  $\pi$ .

Conversely, if  $U$  is orientable then it has two orientations – call them  $O$  and  $-O$ . So we can construct  $\hat{U}_O$  and  $\hat{U}_{-O}$  not necessarily charts where  $\pi^{-1}(U) = \hat{U}_O \cup \hat{U}_{-O}$ .

## Connectedness

So far, we have  $\pi : \hat{M} \rightarrow M$  a 2-folded covering with  $M$  connected.

1. if  $M$  is orientable, then  $\hat{M}$  is two copies of  $M$  (i.e.  $\hat{M}$  is not connected).

From above, we have that  $\pi^{-1}(M)$  is the disjoint union of two copies of  $M$ .

2. if instead  $M$  is un-orientable, then  $\hat{M}$  is connected.

Fact:  $\pi : \hat{M} \rightarrow M$  a covering map with  $M$  connected, then  $\#\pi^{-1}(x)$  is constant on  $M$ .

Suppose  $\hat{M}$  is not connected, then let  $W$  be components with  $\pi|_W : W \rightarrow M$  covering maps.  $\#(\pi|_W)^{-1}(x)$  is either one or two. If it is one, then  $\pi|_W : W \rightarrow M$  is a diffeomorphism. However  $W$  is orientable while  $M$  is not, a contradiction. If instead the cardinality is two, then  $W = \hat{M}$  and hence  $\hat{M}$  is connected.

## Corollary

If  $M$  is simply connected (i.e.  $\pi_1 = \{e\}$ ), then  $M$  is orientable. In fact, if  $M$  is orientable then  $\pi : \hat{M} \rightarrow M$  is a 2-folded covering with  $\hat{M}$  connected. If  $M$  is simply connected, then  $\hat{M} = M$  a contradiction.

## Remark

If  $\pi_1(M)$  does not have a subgroup of index 2, then  $M$  is orientable.

For example,  $\pi_1(\mathbb{R}P^2) = \text{Aut}(\pi) = \mathbb{Z}^2$  with  $\pi : S^2 \rightarrow \mathbb{R}P^2$  and, for the Mobius band  $M$ ,  $\pi_1(M) = \text{Aut}(\pi) = \mathbb{Z} = \langle \gamma \rangle$  has a subgroup  $\langle \gamma^2 \rangle$  and  $2\mathbb{Z} \leq \mathbb{Z}$  is a subgroup with index 2.

**February 19, 2025**

## Integration in $\mathbb{R}^n$

In  $\mathbb{R}^n$ , let  $\omega \in \Omega^n(\mathbb{R}^n)$  and suppose that a domain  $D$  is “good” and compact. Then  $\omega = f dx^1 \wedge \cdots \wedge dx^n$  and

$$\int_D \omega := \int_D f dx^1 \wedge \cdots \wedge dx^n.$$

## Proposition 16.1

Suppose we have domains  $D, E \in \mathbb{R}^n$  and a diffeomorphism  $G : \overline{D} \rightarrow \overline{E}$ . If  $\omega \in \Omega^n(\overline{E})$ , then  $G^* \omega \in \Omega^n(\overline{D})$  and

$$\int_D G^* \omega = \pm \int_E \omega$$

where  $\pm$  depends on whether  $G$  preserves orientations (i.e.  $\det(DG) > 0$  or  $\det(DG) < 0$ ).

### Proof

Write  $G : \overline{D} \rightarrow \overline{E}$  as  $(x^1, \dots, x^n) \mapsto (y^1, \dots, y^n)$  and  $\omega = f(y^1, \dots, y^n) dy^1 \wedge \dots \wedge dy^n$ . Then since

$$y^i = G^i(x^1, \dots, x^n) \quad \text{and} \quad dy^1 \wedge \dots \wedge dy^n = dG^1 \wedge \dots \wedge dG^n,$$

we have

$$\begin{aligned} \int_E \omega &= \int_E f(y^1, \dots, y^n) dy^1 \wedge \dots \wedge dy^n \\ &\stackrel{y^i = G^i(x^1, \dots, x^n)}{=} \int_D f \circ G(x^1, \dots, x^n) |\det(DG)| dx^1 \wedge \dots \wedge dx^n \\ &= \pm \int_D (f \circ G) \cdot \det(DG) dx^1 \wedge \dots \wedge dx^n \\ &= \pm \int_D G^* \omega \\ &= G^*(f dy^1 \wedge \dots \wedge dy^n) \\ &= (f \circ G) G^*(dy^1 \wedge \dots \wedge dy^n) \end{aligned}$$

### More Generally

If  $\omega \in \Omega^n(\mathbb{R}^n)$  with compact support, then we can pick a “good” domain  $D$  such that  $\text{supp } \omega \subseteq D$  and  $\overline{D}$  is compact. Define

$$\int_{\mathbb{R}^n} \omega := \int_D \omega$$

This works similarly on any open set  $U \supseteq \text{supp } \omega$ . Pick a good domain  $D$  such that  $\text{supp } \omega \subseteq D \subseteq U$  with  $\overline{D}$  compact. Then

$$\int_U \omega := \int_D \omega$$

where  $U$  may be chosen to be an open ball  $B_r^n(0)$ .

## Integration on Manifolds

On a manifold  $M^n$  with  $\omega \in \Omega^n(M)$ , we first consider the case where  $\text{supp } \omega \subseteq U$  for  $U$  a chart.

IMAGE 1

$$\int_M \omega := \pm_{\phi(U)} (\phi^{-1})^* \omega$$

where  $\pm$  depends on whether  $\phi : (U, O|_U) \rightarrow (\phi(U), O_E)$  is orientation preserving. This is well defined

IMAGE 2

Since  $\psi(W) = \psi \circ \phi^{-1}(\phi(W))$ ,

$$\int_{\psi(W)} (\psi^{-1})^* \omega = \int_{\psi \circ \phi^{-1}(\phi(W))} (\psi^{-1})^* \omega = \int_{\phi(W)} (\psi \circ \phi^{-1})^* (\psi^{-1})^* \omega = \int_{\phi(W)} (\phi^{-1})^* \omega$$

## General Case

Suppose  $M^n$  is oriented with  $\omega \in \Omega^n(M)$  having compact support.

Let  $\{U_i\}$  be a finite open cover of  $\text{supp } \omega$  such that each  $U_i$  is a chart, and  $\psi_i$  a partition of unity subordinated to  $U_i$  (i.e.  $\text{supp } \psi_i \subseteq U_i$ ). Assume further that  $\phi_i : (U_i, O|_{U_i}) \rightarrow (\phi_i(U_i), O_E)$  is orientation preserving (reversing introduces a sign). Define

$$\int_M \omega := \sum_{i=1}^n \int_M \psi_i \omega$$

This is well defined. Suppose  $\{\tilde{U}_j\}$  is another open cover and  $\tilde{\psi}_j$  another partition of unity with respect to  $\{\tilde{U}_j\}$ . Then

$$\int_M \psi_i \omega = \int_M \left( \sum_j \tilde{\psi}_j \right) \psi_i \omega = \sum_j \int_M \tilde{\psi}_j \psi_i \omega.$$

Summing over  $i$ ,

$$\sum_i \int_M \psi_i \omega = \sum_{i,j} \int_M \tilde{\psi}_j \psi_i \omega = \sum_j \int_M \tilde{\psi}_j \left( \sum_i \psi_i \right) \omega = \sum_j \int_M \tilde{\psi}_j \omega.$$

## Integration over Parameterizations

Take  $M^n$  oriented and  $\omega \in \Omega^n(M^n)$  with compact support. Suppose  $D_1, \dots, D_k$  are open domains in  $\mathbb{R}^n$  and  $F_i : \overline{D_i} \rightarrow M$  such that

1.  $F_i|_{D_i}$  is a diffeomorphism onto its image  $W_i := F_i(D_i)$ .
2.  $W_i \cap W_j = \emptyset$ ,  $\forall i, j$ , and
3.  $\bigcup_i \overline{W_i} = M$ .

Then

$$\int_M \omega = \sum_{i=1}^n \int_{W_i} \omega = \sum_{i=1}^n \int_{D_i} F_i^* \omega.$$



### Example

$$\omega = x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy$$

on  $S^2 \subseteq \mathbb{R}^3$ . Parameterize  $S^2$  by  $F : [0, \pi] \times [0, 2\pi] \rightarrow S^2$  by  $(\varphi, \theta) \mapsto (\sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi)$ .

IMAGE 3

Orient  $S^2$  by an outward normal vector field  $N$  (i.e. the induced orientation on  $S^2$  is  $N \lrcorner (e^1 \wedge e^2 \wedge e^3)$ ).

Then we need to show that  $(N \lrcorner (e^1 \wedge e^2 \wedge e^3)) \left( DF \left( \frac{\partial}{\partial \varphi} \right), DF \left( \frac{\partial}{\partial \theta} \right) \right) > 0$ .

$$\begin{aligned} DF \left( \frac{\partial}{\partial \varphi} \right) &= \frac{\partial F}{\partial \varphi} = (\cos \varphi \cos \theta, \cos \varphi \sin \theta, -\sin \varphi) \\ DF \left( \frac{\partial}{\partial \theta} \right) &= \frac{\partial F}{\partial \theta} = (-\sin \varphi \sin \theta, \sin \varphi \cos \theta, 0) \end{aligned}$$

At  $q = (0, 1, 0) \in S^2$ ,  $q = F \left( \frac{\pi}{2}, \frac{\pi}{2} \right)$  so

$$\begin{aligned} DF \left( \frac{\partial}{\partial \varphi} \right) &= (0, 0, -1) \\ DF \left( \frac{\partial}{\partial \theta} \right) &= (-1, 0, 0) \end{aligned}$$

while  $N = (0, 1, 0)$ . So we compute  $(e^1 \wedge e^2 \wedge e^3) \left( N, DF \left( \frac{\partial}{\partial \varphi} \right), DF \left( \frac{\partial}{\partial \theta} \right) \right)$  is

$$\det \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix} = 1$$

and preserves orientation. So  $\int_{S^2} \omega = \int_D F^* \omega$  and

$$F^* dx = d(F^* x) = d(x \circ F) = d(\sin \varphi \cos \theta) = \sin \varphi \, d \cos \theta + \cos \theta \, d \sin \varphi = -\sin \varphi \sin \theta \, d\theta + \cos \varphi \cos \theta \, d\varphi$$

Similarly,

$$F^* (dy) = d(\sin \varphi \sin \theta) = \sin \varphi \, d \sin \theta + \sin \theta \, d \sin \varphi = \sin \varphi \cos \theta \, d\theta + \cos \varphi \sin \theta \, d\varphi$$

Finally,  $F^* dz = d \cos \varphi = -\sin \varphi \, d\varphi$ , so

$$\begin{aligned} F^* \omega &= (\sin \varphi \cos \theta) \cdot (\sin^2 \varphi \cos \theta \, d\varphi \wedge d\theta) + (\sin \varphi \sin \theta) \cdot (\sin^2 \varphi \sin \theta \, d\varphi \wedge d\theta) \\ &\quad + \cos \varphi (\sin^2 \theta \sin \varphi \cos \varphi \, d\varphi \wedge d\theta) + \cos^2 \theta \sin \varphi \cos \varphi \, d\varphi \wedge d\theta \\ &= (\sin^3 \varphi \cos^2 \theta + \sin^3 \varphi \sin^2 \theta) \, d\varphi \wedge d\theta + (\cos^2 \varphi \sin \theta) \, d\varphi \wedge d\theta \\ &= \sin \varphi \, d\varphi \wedge d\theta \end{aligned}$$

We conclude

$$\int_{S^2} \omega = \int_D F^* \omega = \int_D \sin \varphi \, d\varphi \, d\theta = \int_0^\pi \sin \varphi \, d\varphi \int_0^{2\pi} d\theta = 2 \cdot 2\pi = 4\pi.$$

## Stokes' Theorem

For  $M^n$  with boundary  $\partial M$  ( $\dim \partial M = n - 1$ ),

$$\int_M d\omega = \int_{\partial M} \omega$$

for all  $\omega \in \Omega^{n-1}(M)$  where  $\partial M$  has outward orientation.

### Example

Take  $\omega \in \Omega^2(B_1^3)$ , then

$$d\omega = dx \wedge dy \wedge dz + dy \wedge dz \wedge dx + dx \wedge dx \wedge dy = 3dx \wedge dy \wedge dz.$$

Since  $S^2 = \partial B_1^3$ ,

$$\int_{S^2} \omega = \int_{\partial B_1^3} \omega = \int_{B_1^3} d\omega = \int_{B_1^3} 3dx \wedge dy \wedge dz = 3 \cdot \text{vol}(B_1^3) = 3 \cdot \frac{4}{3}\pi = 4\pi.$$

### Example

Take  $M = [a, b] \subseteq \mathbb{R}^1$  with orientation  $dt$

IMAGE 4

We have that  $\partial M = \{a\} \cup \{b\}$ . So, at  $a \left(-\frac{\partial}{\partial t}\right) \lrcorner(dt) = -1$  and at  $b \left(\frac{\partial}{\partial t}\right) \lrcorner(dt) = 1$ . So

$$\int_a^b f'(t) dt = \int_M d\omega = \int_{\partial M} \omega = -f(a) + f(b).$$

### Example

Take a line integral along  $\gamma : [0, 1] \rightarrow M$  with  $\omega \in \Omega^1(M)$ .

Suppose  $\omega = df$ . Then

$$\int_\gamma \omega = \int_\gamma df = \int_{\partial\gamma} f = f(\gamma(b)) - f(\gamma(a)).$$

## Consequences

If  $M^n$  is compact, oriented and without boundary, then

$$\int_M d\omega = \int_{\partial M} \omega = 0$$

for  $\omega \in \Omega^{n-1}(M)$ . That is to say integrating an exact form over a closed manifold returns zero.

If  $M^n$  is compact and oriented with  $\omega \in \Omega^{n-1}(M)$  satisfying  $d\omega = 0$  (i.e. closed), then

$$\int_{\partial M} \omega = \int_M d\omega = 0.$$

## Remark

If we write  $(M, \omega) := \int_M \omega$ , then Stokes' theorem says  $(\partial M, \omega) = (M, d\omega)$ .

## Proof

In the special case that  $M = \mathbb{R}^n$  with  $\omega \in \Omega^{n-1}(\mathbb{R}^n)$  having compact support. Cover the support of  $\omega$  by a large cube  $[-R, R]^n$ . Then

$$\begin{aligned}\omega &= \omega_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ d\omega &= \sum_i \frac{\partial \omega_i}{\partial x^i} dx^i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^n (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n.\end{aligned}$$

It follows that from Frobenius and the Fundamental Theorem of Calculus that

$$\begin{aligned}\int_{\mathbb{R}^n} d\omega &= \sum_{i=1}^n (-1)^{i-1} \int_{[-R, R]^n} \frac{\partial \omega_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^n (-1)^{i-1} \int_{-R}^R \cdots \left( \int_{-R}^R \frac{\partial \omega_i}{\partial x^i} dx^i \right) \cdots \\ &= \cdots (\omega_i(\cdots, R, \cdots) - \omega_i(\cdots, -R, \cdots)) \cdots \\ &= 0\end{aligned}$$

In the special case that  $M = \mathbb{H}^n$  with  $\omega \in \Omega^{n-1}(\mathbb{H}^n)$  having compact support. Covering the support of  $\omega$  by  $[-R, R]^{n-1} \times [0, R]$ ,

$$\begin{aligned}\int_{\mathbb{H}^n} d\omega &= \sum_{i=1}^n (-1)^{i-1} \int_{[-R, R]^{n-1} \times [0, R]} \frac{\partial \omega_i}{\partial x^i} dx^1 \cdots dx^n \\ &= (-1)^{n-1} \int_{[-R, R]^{n-1} \times [0, R]} \frac{\partial \omega_n}{\partial x^n} dx^1 \cdots dx^n \\ &= (-1)^{n-1} \int_{-R}^R \cdots \int_{-R}^R \left( \int_0^R \frac{\partial \omega_n}{\partial x^n} dx^n \right) dx^1 \cdots dx^{n-1} \\ &= (-1)^n \int_{-R}^R \cdots \int_{-R}^R \omega_n(x^1, \dots, x^{n-1}, 0) dx^1 \cdots dx^{n-1} \\ &= (-1)^n \int_{\partial \mathbb{H}^n \cap \text{supp } \omega} \omega_n dx^1 \wedge \cdots \wedge dx^{n-1}\end{aligned}$$

Recall that the induced orientation on the boundary  $\partial \mathbb{H}^n$  matches with the standard orientation on  $\mathbb{R}^{n-1}$  if and only if  $n$  is even. So

$$\begin{aligned}\int_{\partial \mathbb{H}^n} \omega &= \sum_i \int_{\partial \mathbb{H}^n} \omega_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \int_{\partial \mathbb{H}^n} \omega_n dx^1 \wedge \cdots \wedge dx^{n-1}\end{aligned}$$

which matches our previous calculation since  $(-1)^n = 1$  for  $n$  even.

## Green's Theorem

If  $D \subseteq \mathbb{R}^2$  is a domain with  $\overline{D}$  compact, then

$$\int_{\partial D} P dx + q dy = \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

and  $\omega = P dx + Q dy \in \Omega^1(\mathbb{R}^2)$  so

$$d\omega = \frac{\partial P}{\partial y} dy \wedge dx + \frac{\partial Q}{\partial x} dx \wedge dy = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$$

Therefore

$$\int_{\partial D} \omega = \int_D d\omega = \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

with  $\partial D$  outward oriented.

**February 24, 2025**

## Recall: Stoke's Theorem

For  $M^n$  a smooth manifold and  $\omega \in \Omega_C^{n-1}(M)$  with compact support,

$$\int_M d\omega = \int_{\partial M} \omega.$$

1.  $\omega \in \Omega_C^{n-1}(\mathbb{R}^n)$ ,  $\int_{\mathbb{R}^n} d\omega = 0$ .
2.  $\omega \in \Omega_C^{n-1}(\mathbb{H}^n)$ ,  $\int_{\mathbb{H}^n} d\omega = \int_{\partial \mathbb{H}^n} \omega$ .

## Special Case

If  $\text{supp } \omega \subseteq (U, \phi)$  a chart, then  $\text{supp}(d\omega) \subseteq U$ .

IMAGE 1

$$\int_M d\omega = \int_U \phi^* \omega = \int_{\phi(U)} (\phi^{-1})^* d\omega = \int_{\phi(U)} d(\phi^{-1*} \omega).$$

So

$$\int_{\mathbb{R}^n} d(\phi^{-1*} \omega) = 0$$

and

$$\int_{\mathbb{H}^n} d(\phi^{-1*} \omega) = \int_{\partial \mathbb{H}^n} \phi^{-1*} \omega = \int_{\partial \mathbb{H}^n \cap \phi(U)} \phi^{-1*} \omega = \int_{\partial M \cap U} \omega = \int_{\partial M} \omega$$

## Stoke's Theorem: General Case

In general,  $\omega \in \Omega_C^{n-1}(M)$ .

Let  $\{\psi_i\}_i$  be a partition of unity with respect to a countable cover of  $M$  by charts. Then, recalling that  $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge d\eta$  with  $k = \deg \omega$ ,

$$\int_{\partial M} \omega = \sum_i \int_{\partial M} \psi_i \omega = \sum_i \int_M d(\psi_i \omega) = \sum_i \int_M d\psi_i \wedge \omega + \psi_i d\omega = \int_M d\left(\sum_i \psi_i\right) \wedge \omega + \int_M \left(\sum_i \psi_i\right) d\omega = \int_M d\omega$$

## Integration on Riemannian Manifolds

### Recall

For  $(M^n, g)$  oriented, the volume form  $\omega_g$  is an  $n$ -form such that  $\omega_g(E_1, \dots, E_n) = 1$  for all positively oriented orthonormal frame  $\{E_1, \dots, E_n\}$ .

Inside a chart  $(U_i, (x^1, \dots, x^n))$  it has the formula

$$\omega_g = \sqrt{\det g} \cdot dx^1 \wedge \dots \wedge dx^n$$

where  $\det g = \det(g_{ij})$ .

### Definition:

Let  $f \in C_C^\infty(M)$ . Define

$$\int_M f = \int_M f \omega_g$$

### Remarks

1.  $\text{vol}(M) = \int_M 1$
2.  $\omega_g \in \Omega^n(M)$  is usually written as  $dV_g$  or  $d\text{vol}_g$ .

## Proposition

For  $(M, g)$  oriented and  $f \in C_C^\infty(M)$ , if  $f \geq 0$  then  $\int_M f \geq 0$ . Equality holds if and only if  $f \equiv 0$  on  $M$ .

### Proof

$$\int_M f = \int_M f \omega_g = \sum_i \int_{U_i} \psi_i f \omega_g = \sum_i \int_{U_i} \psi_i f \sqrt{\det g_{ij}} dx^1 \wedge \dots \wedge dx^n$$

where each term is greater than or equal to zero (assuming positive orientation on each  $U_i$ ).

## On Manifolds with Boundary

Take  $\partial M \subseteq M^n$  with outward orientation.

Recall that for  $N$  an outward pointing vector field along  $\partial M$ , if  $M$  has an orientation  $n$ -form  $\omega$ , then  $\partial M$  has an induced orientation given by

$$(N \lrcorner \omega) \in \omega^{n-1}(\partial M).$$

If  $(M, g)$  is an oriented Riemannian manifold with boundary  $\partial M$ ,  $\omega_g$  a volume form and  $N$  a unit outward pointing vector field orthogonal to  $\partial M$ .

Let  $\tilde{g}$  be the induced Riemannian metric on  $\partial M$ , we observe that

$$\omega_{\tilde{g}} = N \lrcorner \omega_g$$

Let  $\{E_1, \dots, E_{n-1}\}$  be a (locally defined) orthonormal frame on  $\partial M$ .  $\{E_1, \dots, E_n\}$  being positively oriented on  $\partial M$  means that

$$(N \lrcorner \omega_g)(E_1, \dots, E_n) = 1$$

### Lemma 16.30

For  $(M, g)$  oriented and  $(\partial M, \tilde{g})$ , if  $X \in \mathfrak{X}(\partial M)$ , then  $(X \lrcorner \omega_g)|_{\partial M} = g(X, N)\omega_{\tilde{g}}$ .

#### Proof

Decompose  $X = X^T + X^\perp$  where  $X^\perp = g(X, N)N$  and  $X^T = X - X^\perp$ . Write

$$(X^\perp \lrcorner \omega_g)|_{\partial M} = g(X, N)(N \lrcorner \omega_g)|_{\partial M} = g(X, N)\omega_{\tilde{g}}$$

and

$$(X^T \lrcorner \omega_g)|_{\partial M}(E_1, \dots, E_{n-1}) = \omega_g(X^T, E_1, \dots, E_{n-1}) = 0$$

## Generalized Stokes on Manifold with Boundary

Take  $X \in \mathfrak{X}(M)$ ,  $(X \lrcorner \omega_g) \in \Omega^{n-1}(M)$  and  $d(X \lrcorner \omega_g) \in \Omega^n(M)$ . Write

$$\int_M d(X \lrcorner \omega_g) = \int_{\partial M} X \lrcorner \omega_g = \int_{\partial M} g(X, N)\omega_{\tilde{g}} = \int_{\partial M} g(X, N).$$

### Definition: Divergence

Let  $\text{div } X \in C^\infty(M)$  defined by  $d(X \lrcorner \omega_g) = \text{div } X \cdot \omega_g$ . Then

$$\int_M d(X \lrcorner \omega_g) = \int_M \text{div } X \cdot \omega_g = \int_M \text{div } X$$

## Theorem: Divergence Theorem

$$\int_X \operatorname{div} X = \int_{\partial M} g(X, N)$$

### Remark

Inside  $\mathbb{R}^n$ ,  $X = X^i \frac{\partial}{\partial x^i} \in \mathfrak{X}(\mathbb{R}^n)$ , then  $\operatorname{div} X = \frac{\partial}{\partial x^i} (X^i)$ .

### Problem 16-11

$$\operatorname{div} \left( X^i \frac{\partial}{\partial x^i} \right) = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} (X^i \sqrt{\det g})$$

For  $(\mathbb{R}^n, g_E)$ ,  $g_{ij} = \delta_{ij}$  and  $\sqrt{\det g} = 1$ . Then  $\operatorname{div} \left( X^i \frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial x^i} (X^i)$ .

### Problem 16-9

$$\omega = |x|^{-n} \sum_{i=1}^n (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge \hat{dx}^i \wedge \cdots \wedge dx^n \in \Omega^{n-1}(\mathbb{R}^n - \{0\})$$

and

$$\omega|_{S^{n-1}} = \sum_{i=1}^n (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge \hat{dx}^i \wedge \cdots \wedge dx^n$$

For example

$$\begin{aligned} n=2 \quad \omega|_{S^1} &= x dy - y dx \\ n=3 \quad \omega|_{S^2} &= x dy \wedge dz - \underbrace{y dx \wedge dz + z dx \wedge dy}_{+y dz \wedge dx} \end{aligned}$$

Claim:  $\omega|_{S^{n-1}}$  is the standard volume form on  $S^{n-1}$  ( $S^{n-1} \hookrightarrow \mathbb{R}^n$  or  $S^{n-1} = \partial B_1^n$ ). We need to check that  $\omega_{S^{n-1}} = (N \lrcorner \omega_E)$ . We have that  $N$  is  $(x^1, \dots, x^n)$  at the point  $(x^1, \dots, x^n)$  (i.e.  $N = x^i \frac{\partial}{\partial x^i}$  on  $S^{n-1}$ ). Write

$$(N \lrcorner \omega_E) = \left( x^i \frac{\partial}{\partial x^i} \right) \lrcorner (dx^1 \wedge \cdots \wedge dx^n) = x^i \left( \frac{\partial}{\partial x^i} \lrcorner (dx^1 \wedge \cdots \wedge dx^n) \right)$$

Compute

$$\begin{aligned} \left( \frac{\partial}{\partial x^1} \lrcorner (dx^1 \wedge \cdots \wedge dx^n) \right) (E_1, \dots, E_{n-1}) &= dx^1 \wedge \cdots \wedge dx^n \left( \frac{\partial}{\partial x^1}, E_1, \dots, E_{n-1} \right) \\ &= \det \begin{pmatrix} dx^1 \left( \frac{\partial}{\partial x^1} \right) & \overbrace{dx^2 \left( \frac{\partial}{\partial x^1} \right)}^{=0} & \cdots & \overbrace{dx^n \left( \frac{\partial}{\partial x^1} \right)}^{=0} \\ dx^1(E_1) & \cdots dx^2(E_1) & \cdots & dx^n(E_1) \\ \vdots & & & \vdots \\ dx^1(E_{n-1}) & \cdots dx^2(E_{n-1}) & \cdots & dx^n(E_{n-1}) \end{pmatrix} \\ &= dx^2 \wedge \cdots \wedge dx^n (E_1, \dots, E_{n-1}) (-1)^{i-1} \end{aligned}$$

In general,

$$\frac{\partial}{\partial x^i} \lrcorner (dx^1 \wedge \cdots \wedge dx^n) = \frac{\partial}{\partial x^i} \lrcorner (dx^i \wedge dx^1 \wedge \cdots \wedge \hat{dx}^i \wedge \cdots \wedge dx^n) = (-1)^{i-1} dx^1 \wedge \cdots \wedge \hat{dx}^i \wedge \cdots \wedge dx^n$$

## Conclusion

$\omega|_{S^{n-1}}$  is the volume form on  $S^{n-1}$ ,  $0 < \int_{S^{n-1}} \omega|_{S^{n-1}}$ .

1.  $\omega|_{S^{n-1}} \in \Omega^{n-1}(S^{n-1})$  is not exact (if it is,  $\omega = d\eta$  and  $\int_{S^{n-1}} \omega = \int_{S^{n-1}} d\eta = 0$ )
2.  $\omega|_{S^{n-1}}$  is closed (By direct calculation on  $d\omega$  on  $\mathbb{R}^n - \{0\}$ ).

## Proposition 16.33

Let  $(M, g)$  be an oriented Riemannian manifold and  $X \in \mathfrak{X}(M)$  a complete vector field. Let  $\theta$  be the flow of  $X$ . Then  $\text{div } X \equiv 0$  if and only if  $\theta_t$  is volume preserving for all time.

## Proof

Let  $D \subseteq M$  be any compact domain.

$$\text{vol}(\theta_t(D)) = \int_{\theta_t(D)} \omega_g = \int_D \theta_t^* \omega_g$$

Recall Cartan's Formula:  $\mathcal{L}_X = i_X \circ d + d \circ i_X$ . So

$$\mathcal{L}_X(\omega_g) = X \lrcorner (\overbrace{d\omega_g}^{=0}) + d(X \lrcorner \omega_g) = \text{div } X \cdot \omega_g$$

Therefore

$$\begin{aligned} \frac{d}{dt} \Big|_{t=t_0} \text{vol}(\theta_t(D)) &= \int_D \frac{d}{dt} \Big|_{t=t_0} \theta_t^* \omega_g \\ &= \int_D \theta_{t_0}^* (\mathcal{L}_X \omega_g) \\ &= \int_D \theta_{t_0}^* (\text{div } X \cdot \omega_g) \\ &= \int_{\theta_{t_0}(D)} \text{div } X \cdot \omega_g \end{aligned}$$

If  $\text{div } X \equiv 0$  on  $M$ , then the right hand side is zero. Hence  $\text{vol}(\theta_t(D))$  is a constant function (i.e.  $\theta_t$  is volume preserving everywhere).

If instead  $\theta_t$  is assumed to be volume preserving, then the left hand side is zero for all times  $t_0$  and any domain  $D$ . Then, without loss of generality for  $t_0 = 0$ ,  $\int_D \text{div } X = 0$  (i.e.  $\text{div } X \equiv 0$ ).

## Remark

For  $f \in C^\infty(M)$ ,  $\text{grad } f \in \mathfrak{X}(M)$ ,  $\Delta f := \text{div}(\text{grad } f) \in C^\infty(M)$ .

In  $(\mathbb{R}^n, g_E)$ ,  $\text{grad } f = \frac{\partial f}{\partial x^i} \cdot \frac{\partial}{\partial x^i}$  and  $\Delta f := \text{div}(\text{grad } f) = \sum_{i=1}^n \frac{\partial^2 f}{\partial (x^i)^2}$ .



## Recall: Poincaré Lemma

Recall that if  $U \subseteq \mathbb{R}^n$  is star-shaped, then  $\omega \in \Omega^1(U)$  is closed if and only if  $\omega$  is exact. For ( $\Leftarrow$ ), this is always true; for ( $\Rightarrow$ ) we need star-shaped.

## Definition: Path-homotopic

$\gamma_0, \gamma_1 : I \rightarrow M$  continuous such that  $\gamma_0(a) = \gamma_1(a) = p$  and  $\gamma_0(b) = \gamma_1(b) = q$ .

IMAGE 2

A path-homotopy between  $\gamma_0$  and  $\gamma_1$  is a continuous map  $H : I \times [0, 1] \rightarrow M$  such that

$$\begin{aligned} H(\cdot, 0) &= \gamma_0 & H(a, \cdot) &= p \\ H(\cdot, 1) &= \gamma_1 & H(b, \cdot) &= q \end{aligned}$$

## Proposition

Let  $\gamma_0, \gamma_1 : [a, b] \rightarrow M$  be smooth path-homotopic, and let  $\omega \in \Omega^1(M)$  be closed. Then  $\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$ .

### Proof

Assume  $a = 0$  and  $b = 1$ , then noting that faces 2 and 4 (see above) collapse to points with integral zero,

$$\begin{aligned} 0 &= \int_{H(I)} \overbrace{d\omega}^{=0} = \int_{I^2} H^*(d\omega) \\ &= \int_{I^2} d(H^*\omega) \\ &= \int_{\partial I^2} H^*\omega \\ &= \int_{i=1}^4 \int_{F^i} H^*\omega \\ &= \sum_{i=1}^4 \int_{H(F_i)} \omega \\ &= \int_{H(F_1)} \omega + \int_{H(F_3)} \omega \\ &= \int_{\gamma_0} \omega - \int_{\gamma_1} \omega \end{aligned}$$

### Corollary

For  $M$  with  $\pi_1(M) = e$  (i.e. every closed curve is path-homotopic to a point), then every closed 1-form is exact.