Manifolds II

January 6, 2025

Recall: Tangent Bundle

Given a chart (U,ϕ) about a point p, we have coordinates $(x^1,\dots,x^n)$ and a basis for T_qM of $\left(\frac{\partial}{\partial x^1}|_q,\dots,\frac{\partial}{\partial x^n}|_q\right)$ for $q\in U$.
Then given $TM \xrightarrow{\pi} M$, we may write $v_q = v^i \frac{\partial}{\partial x^i}|_q$.

Definition:

For M a topological manifold. A (real) vector bndle of rank k over M is a topological space E with a surjective continuous map $\pi : E \to M$ such that

1. $\forall\, p\in M$, the fiber $\pi^{-1}(p) =: E_p$ is endowed with the structure of a (real) vector space of dimension k.

2. $\forall\, p\in M$, there exists a neighborhood U of p in M and a homeomorphism $\Phi : \pi^{-1}(U) \to U\times\mathbb{R}^k$ called a local trivialization.

$$\begin{tikzcd} \Phi : \pi^{-1}(U) \arrow[rr] \arrow[dr, "\pi"] & & U\times\mathbb{R}^k \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd}$$

and $\Phi|_{E_q} : E_q \to \{q\}\times\mathbb{R}^k$ is a linear isometry.

Examples

1. $TM \xrightarrow{\pi} M$

2. $E = M\times\mathbb{R}^k$ with a global trivialization.

3. The Mobius bundle over S^1. $\gamma : \mathbb{R}^2\to\mathbb{R}^2$ by $(x,y)\mapsto(x+1,(-1)\cdot y)$. Then $\langle\gamma\rangle\cong\mathbb{Z}$ a subgroup acting freely and isometrically on $\mathbb{R}^2$. Then $E=\mathbb{R}^2/\langle\gamma\rangle \xrightarrow{\pi} S^1 = \mathbb{R}/\mathbb{Z}$ by $\overline{(x,y)}\mapsto\overline{x}$ is a vector bundle.

IMAGE 1

- We want to show that $\pi^{-1}(U)\cong U\times\mathbb{R}$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "q"] \arrow[d, "\pi"] & E \arrow[d, "\pi"] \\ \mathbb{R} \arrow[r, "\varepsilon"] & S^1 \end{tikzcd} \qquad \begin{tikzcd} (x,y) \arrow[r, mapsto] \arrow[d, mapsto] & \overline{(x,y)} \arrow[d, mapsto] \\ x \arrow[r, mapsto] & e^{(2\pi i)x} \end{tikzcd}$$

Then let $p\in S^1$. We choose U a neighborhood of p such that U is evenly covered by ε. This means $\varepsilon^{-1}(U)$ is a disjoint union of open sets difeomorphic to U.

IMAGE 2

Let $\tilde{U}$ be a component in $\pi^{-1}(U)$. Then $\pi_1^{-1}(\tilde{U})\cong\tilde{U}\times\mathbb{R}$ and $\pi^{-1}(U)$ is diffeomorphic to $U\times\mathbb{R}$.

Definition: Transition Function

Take $E \xrightarrow{\pi} M$ with $U, V \subseteq M$ admitting trivializations $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ and $\Psi : \pi^{-1}(V) \to V \times \mathbb{R}^k$. Let $w = U \cap V (\neq \varnothing)$.

$$\Phi \circ \Psi^{-1} : \quad \begin{tikzcd} W \times \mathbb{R}^k \arrow[r] \arrow[dr] & \pi^{-1}(W) \arrow[r] \arrow[d] & W \times \mathbb{R}^k \arrow[dl] \\ & W & \end{tikzcd}$$

Then $\Phi \circ \Psi^{-1}|_{\{p\} \times \mathbb{R}^k}$ by $\{p\} \times \mathbb{R}^k \to \{p\} \times \mathbb{R}^k$ is a linear isomorphism.
$\Phi \circ \Psi^{-1}(p, v) = (p, \tau(p)v)$ by $\tau : p \mapsto \tau(p)$ and $\tau(p) \in GL(k, \mathbb{R})$ gives a smooth map $W \to GL(k, \mathbb{R})$.

Definition:

Let $\{E_1, \ldots, E_k\}$ be a basis of $\mathbb{R}^k$. Then

$$\tau(p) \cdot E_i = \sum_j \tau(p)_i^j E_j$$

with $\tau(p) = (\tau(p)_i^j)$ and $\tau(p)_j^i \in \mathbb{R}$. It suffices to show each $\tau(*)_i^j$ mapping $W \to \mathbb{R}$ and $p \mapsto (\tau(p)_i^j)$ is smooth. Then if $\sigma(p, v) := \Phi \circ \Psi^{-1}(p, v)$, $\tau(p)_i^j = \pi_j(\sigma(p, E_i))$ and π_j is a projection to the j-th component in $\mathbb{R}^k$.

Lemma 10.6 (Vector Bundle Chart Lemma)

Given M a smooth manifold, suppose that $\forall p \in M$ we are given a vector space E_p of dimension k. Let $E = \coprod_{p \in M} E_p$ (as a set) and $\pi : E \to M$ a mapping E_p to p. Suppose also that we have

1. $\{U_\alpha\}_{\alpha \in A}$ an open cover of M with a countable subcover.
2. $\forall \alpha \in A$ we hav ea bijection $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ such that $\Phi_\alpha|_{E_p} : E_p \to \{p\} \times \mathbb{R}^k$ is a linear isomorphism.
3. $\forall \alpha, \beta \in A$ with $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \varnothing$ we have a smooth map $\tau_{\alpha\beta} : U_{\alpha\beta} \to GL(k, \mathbb{R})$ such that $\Phi_\alpha \circ \phi_\beta^{-1} : U_{\alpha\beta} \times \mathbb{R}^k \to U_{\alpha\beta} \times \mathbb{R}^k$ by $(p, v) \mapsto (p, \tau(p)v)$.

Then $E \xrightarrow{\pi} M$ is a vector bundle.

Example (Whitney Sum):

Suppose we have $E' \xrightarrow{\pi'} M$ and $E'' \xrightarrow{\pi''} M$ two vector bundles over M.
Define $E = E' \oplus E''$ a new vector bundle over M by $E_p = E'_p \oplus E''_p$. Let $\{U_\alpha\}_{\alpha \in A}$ be a countable open cover of M such that each U_α admits trivializations for E' and E''. Then for $\pi : E \to M$, define $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{k'} \times \mathbb{R}^{k''}$ by $(v', v'')_p \mapsto (p, \pi_2 \circ \Phi_\alpha^{-1}(v'), \pi_2 \circ \Phi''_\alpha(v''))$ where

$$\pi'(U_\alpha) \xrightarrow{\Phi'_\alpha} U_\alpha \times \mathbb{R}^{k'} \xrightarrow{\pi_2} \mathbb{R}^{k'}$$

Note that π_2 is the projection into the second component. Then $\tau : U_{\alpha\beta} \to G(k' + k'', \mathbb{R})$ by

$$p \mapsto \begin{pmatrix} \tau'(p) & 0 \\ 0 & \tau''(p) \end{pmatrix}$$

Example

For $\tau_{\alpha\beta} : U_{\alpha\beta} \to GL(k,\mathbb{R})$ by $p \mapsto \tau_{\alpha\beta}(p)$, we can write $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cup U_\gamma (\neq \varnothing)$ and get $\tau_{\alpha\beta} \cdot \tau_{\beta\gamma} = \tau_{\alpha\gamma}$.
Note that this is $\Phi_\alpha \circ (\phi_\beta^{-1} \circ \phi_\beta) \circ \Phi_\gamma^{-1}$.
Without loss of generality, we assume each U_α is a chart for M. Then we want to show that we satisfy Lemma 1.35 from Lee

$$\pi^{-1}(U_\alpha) \overset{\Phi_\alpha}{\to} U_{\alpha\times\mathbb{R}^k} \overset{\phi_\alpha\times\mathrm{id}}{\to} \phi_\alpha(U_\alpha)\times\mathbb{R}^k \subseteq \mathbb{R}^{n+k}$$

$(\pi^{-1}(U_\alpha)\cdot\tilde{\phi}_\alpha = (\phi_\alpha\times\mathrm{id})\circ\Phi_\alpha)_{\alpha\in A}$ which satisfies (1).
Since

$$\pi^{-1}(U_\alpha)\cap\pi^{-1}(U_\beta) = \pi^{-1}(U_\alpha\cap U_\beta) \to \phi_\alpha(U_{\alpha\beta})\times\mathbb{R}^K$$

we have that (2) is satisfied.
Finally, for (3),

$$\tilde{\phi_\beta}\circ\tilde{\phi_\alpha}^{-1} = (\Phi_\beta\circ(\phi_\beta\times\mathrm{id}))\circ((\phi_\alpha\times\mathrm{id})^{-1}\circ\Phi_\alpha^{-1}) = \Phi_\beta\circ((\phi_\beta\circ\phi_\alpha)\times\mathrm{id})\circ\Phi_\alpha^{-1}$$

gives $(x,c)\mapsto((\phi_\beta\circ\phi_\alpha^{-1})x, (\Phi_\beta\circ\Phi_\alpha^{-1})v)$ a diffeomorphism.
(4) and (5) are trivial, and this is indeed a smooth manifold. Now we wish to show that it is a vector bundle. To show that $\pi : E \to M$ is smooth,

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[r, "\pi"] & U_\alpha \arrow[d, "\phi_\alpha"] \\
\phi_\alpha(U_\alpha)\times\mathbb{R}^k \arrow[u, "\tilde{\phi}_\alpha^{-1}"] & \phi_\alpha(U_\alpha)
\end{tikzcd}$$

We have $\tilde{\phi}_\alpha^{-1} = (\phi_\alpha\times\mathrm{id})^{-1}\circ\Phi_\alpha^{-1}$.

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[r, "\Phi_\alpha"] & U_\alpha\times\mathbb{R}^k \arrow[d, "\phi_\alpha\times\mathrm{id}"] \\
\phi_\alpha(U_\alpha)\times\mathbb{R}^k \arrow[u, "\tilde{\phi}_\alpha^{-1}"] & \phi_\alpha(U_\alpha\times\mathbb{R}^k)
\end{tikzcd}$$

Definition: Section of a Bundle

A (smooth) section of $E \overset{\pi}{\to} M$ is a (smooth) map $\sigma : M \to E$ such that $\pi\circ\sigma = \mathrm{id}_M$.
$\Gamma(E) = \{$smooth sections of $E \overset{\pi}{\to} M\}$ and $\Gamma(E)$ is a $C^\infty(M)$-module.
The zero section $Z : M \to E$ is given by $p \mapsto 0_p \in E_p$.
If U has a local trivialization, $\Phi : \pi^{-1}(U) \to U\times\mathbb{R}^k$.

$$\begin{tikzcd}
\Phi: & \pi^{-1}(U) \arrow[rr] & & U\times\mathbb{R}^k & & (p,e_i) \arrow[ll, "\Phi^{-1}"'] \\
& & U \arrow[ul, dashed] \arrow[ur] & & p \arrow[ur, mapsto, "\tilde{e}_i"] &
\end{tikzcd}$$

Define $\sigma_i : U \to \pi^{-1}(U)$ by $\sigma_i = \Phi^{-1}\circ\tilde{e}_i$ gives a local section that is non-zero on U.
$\{\sigma_1,\ldots,\sigma_n\}$ form a local frame on U (i.e. form a basis in E_p, $\forall\, p\in U$).

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Recall

Last time we had a vector bundle $E \xrightarrow{\pi} M$ of rank k satisfying

1. $\pi^{-1}(p) = E_p$ has a (real) vector space structure of dimension k.

2. We have a local trivialization, $\forall\, p \in M$ there exists a neighborhood U and a diffeomorphism Φ

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^k \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd}$$

   and $\Phi|_{E_p} : E_p \to \{p\} \times \mathbb{R}^k$ is a linear isomorphism.

A section $\sigma : M \to E$ is a smooth map such that $\pi \circ \sigma = \mathrm{id}_M$.
We say that a collection of sections $\{\sigma_1, \ldots, \sigma_k : U \to E\}$ is linearly independent if $\{\sigma_1(x), \ldots, \sigma_k(x)\}$ is linearly independent for each $x \in U$. This is a (local) frame if it is a basis.
If $U \subseteq M$ admits a trivialization

$$\Phi : \quad \begin{tikzcd} \pi^{-1}(U) \arrow[rr] \arrow[dr] & & U \times \mathbb{R}^k \arrow[dl] \\ & U & \end{tikzcd}$$

then there is a local frame $\{\sigma_1, \ldots, \sigma_k\}$ defined on U. Precisely, with $\tilde{e}_i(x) = (x, e_i)$, $\sigma_i = \Phi^{-1} \circ \tilde{e}_i$.

Proposition 10.19

If $U \subseteq M$ admits a local frame, then $\pi^{-1}(U)$ admits a local trivialization.

Remember

If $E \xrightarrow{\pi} M$ admits a global frame, then $E = \pi^{-1}(M)$ has a trivialization. In other words, E is diffeomorphic to a trivial vector bundle $M \times \mathbb{R}^k$.

Examples

Example 1

Mobius bundle over S^1.

IMAGE 1

To check whether it is a trivial bundle of S^1, it suffices to check whether there exists a nowhere zero (global) section. This cannot happen (by itermediate value theorem), hence it is not $S^1 \times \mathbb{R}$.

Example 2

TS^2 becasue there is no non-vanishing vector field over S^2, hence $TS^2 \neq S^2 \times \mathbb{R}^2$.

Example 3

Let G be a Lie group. Every $X \in T_eG(\cong \mathfrak{g})$ uniquely determines a (left-invariant) vector field $\tilde{X} \in \mathfrak{X}(G)$.
Starting with a basis $\{E_i\} \subseteq T_eG$ we get a global frame $\{\tilde{E}_i\}$ for TG. Hence TG is a trivial vector bundle $G \times \mathbb{R}^n$ ($n = \dim G$). In particular, $TS^1 = S^1 \times \mathbb{R}$, $TS^3 = S^3 \times \mathbb{R}^3$.

Proof of Proposition

Define $\Psi : (x, v^1, \dots, v^k) \in U \times \mathbb{R}^k \to \pi^{-1}(U) \ni v_x$ where $v_x = v^i \sigma_i(x)$.
Ψ is a bijection. Note that $\Psi|_{E_x} : E_x \to \{x\} \times \mathbb{R}^k$is a linear isomorphism because $\{\sigma_i(x)\}$ is a basis.
Then to show that Ψ is a diffeomorphism, it suffices to show then that Ψ is a local diffeomorphism.
Let $x \in U$ and let V be a neighborhood of x such that $\pi^{-1}(V) \overset{\Phi}{\to} V \times \mathbb{R}^k$.

$$V \times \mathbb{R}^k \overset{\Psi|_{V\times\mathbb{R}^k}}{\to} \pi^{-1}(V) \overset{\Psi}{\to} V \times \mathbb{R}^k$$

We show that this composition is a diffeomorphism. Since $\Phi(\sigma_i(x)) = (x, \sigma_i^1(x), \dots, \sigma_i^k(x))$

$$\begin{aligned}\Phi \circ \Psi|_{V\times\mathbb{R}^k}(x, v^1, \dots, v^k) &= \Phi(v^i \sigma_i(x)) \\ &= (x, v^i \sigma_i^1(x), \dots, v^i \sigma_i^k(x))\end{aligned}$$

Each $\sigma_i^j(x)$ is smooth. Hence $\Phi \circ \Psi|_{V\times\mathbb{R}^k}$ is smooth.
Let $\vec{v} = (v^1, \dots, v^k)$ and $\sum(x) = (\sigma_i^j(x))$, then $\Phi \circ \Psi(x, \vec{v}) = (x, \vec{v} \cdot \sum(x))$. Its inverse

$$(\Phi \circ \Psi)^{-1}(x, \vec{w}) = \left(x, \vec{w} \cdot \sum(x)\right)$$

is also smooth. This shows that $\Phi \circ \Psi|_{V\times\mathbb{R}^k}$ is a diffeomorphism. Hence $\Psi|_{V\times\mathbb{R}^k}$ is a diffeomorphism ($V \subseteq U$) and $\Psi : U \times \mathbb{R}^k \to \pi^{-1}(U)$ is also a diffeomorphism.

Definition: Bundle Morphism

A bundle morphism between is a pair of smooth maps (f, F) such that this diagram commutes

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d, "\pi"] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "f"] & M' \end{tikzcd}$$

and $F|_{E_p} : E_p \to E'_{f(p)}$ is a linear map ($\forall p \in M$).
If it admits an inverse which is itself a bundle morphism, it is a unble isomorphism.
Remember that f is smooth because $f = \pi' \circ F \circ Z$

$$p \overset{Z}{\mapsto} 0_p \overset{F}{\mapsto} 0_{f(p)} \overset{\pi'}{\mapsto} f(p)$$

Remark

$$\begin{tikzcd} E \arrow[rr, "F"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

commutes and $F|_{E_p} : E_p \to E'_p$ is linear ($\forall p$).

Remark

$\operatorname{rank}(F|_{E_p})$ may depend on $p \in M$.

$$\begin{tikzcd} TM \arrow[r, "Df"] \arrow[d, "\pi"] & TR \arrow[d, "\pi"] \\ M \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

e.g. $M = \mathbb{R}^2$, $E = E' = TR^2 (= \mathbb{R}^4)$, $F((u,v)_{(x,y)}) = (u, xv)$. For $x \neq 0$, $\operatorname{rank}(F|_{(x,y)}) = 2$ but for $x = 0$ $\operatorname{rank}(F|_{(0,y)}) = 1$.

Proposition 10.26

$$\begin{tikzcd} E \arrow[rr, "F"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

If F is a bijective, smooth bundle homomorphism, then it is a bundle isomorphism.
Proof left as an exercise. We need to show that F^{-1} is smooth.

Definition: Fiber Bundle

$F \to E \xrightarrow{\pi} M$ with fiber F such that $E_x = \pi^{-1}(x)$ is diffeomorphic to F. This diagram commutes.

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd}$$

Fact

If $N \xrightarrow{F} M$ is a submersion from compact manifolds, then F is a fiber bundle.

Chapter 11: Cotangent Bundles

Review: Linear Algebra

Suppose we have a real vector space V of dimension n. Then $V^* = \{f : V \to \mathbb{R} \text{ linear}\}$.

If V has a basis $\{E_1, \ldots, E_n\}$, then we may define the dual basis for V^* $\{\epsilon^1, \ldots, \epsilon^n\}$ by $\epsilon^j(E_i) = \delta_i^j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$.

Remember $V^{**} \cong V$ by $\xi : V \to V^{**}$ by $v \mapsto \xi(v) : V^* \to \mathbb{R}$ and $\omega \mapsto \omega(v)$.
Remember also that if A is a linear map $V \to W$ then we may define $A^* : \omega \in W^* \to V^* \ni A^*\omega$ by $v \in V \to \mathbb{R} \ni \omega(Av)$ (ie. $(A^*\omega)(v) = \omega(Av)$).

Definition: Cotangent Bundle

Let M^n be a smooth manifold, and let (U, ϕ) be a chart. Then T_pM has a basis

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \dots, \frac{\partial}{\partial x^n}\bigg|_p \right\}$$

for every $p \in U$. Take its dual basis

$$\left\{ \lambda^1|_p, \dots, \lambda^n|_p \right\}$$

for T_p^*M. The cotangent bundle $T^*M = \coprod_{p\in M} T_p^*M$.
Similar to the TM case, if $T^*M \xrightarrow{\pi} M$, then $\omega|_p \in \pi^{-1}(U) \xrightarrow{\Phi} U \times \mathbb{R}^n \ni (p, a_1, \dots, a_n)$ where a_i is given by $\omega|_p = a_i\lambda^i|_p$.
In other words, $a_i = \omega|_p\left(\frac{\partial}{\partial x^i}\Big|_p\right)$.

Computing Dual Transition

Suppose $(U, (x^1, \dots, x^n))$ and $(V, (y^1, \dots, y^n))$ are two charts $(W = U \cap V \neq \varnothing)$. Then $\left\{\frac{\partial}{\partial x^i}\Big|_p\right\}$ gives a dual $\{\lambda^i|_p\}$ and $\left\{\frac{\partial}{\partial y^i}\Big|_p\right\}$ gives $\{\mu^i|_p\}$.
Then, recall, $\frac{\partial}{\partial y^i}\Big|_p = \frac{\partial x^j}{\partial y^i}\frac{\partial}{\partial x^j}\Big|_p$ and $x^j(y^1, \dots, y^n)$ is a j-component of $(y^1, \dots, y^n) \to M \to (x^1, \dots, x^n)$.
If $\omega \in T_p^*M$, $\omega = a_i\lambda^i|_p = b_j\mu^j|_p$

$$a_i = \omega|_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = \omega_p\left(\frac{\partial y^j}{\partial x_i}\frac{\partial}{\partial y^j}\right) = \frac{\partial y^j}{\partial x^i}\omega\left(\frac{\partial}{\partial y^j}\right) = \frac{\partial y^j}{\partial x^i}b_j$$

In particular, $\mu^j = \omega$, then $a_i = \frac{\partial y^k}{\partial x^i}b_k = \frac{\partial y^j}{\partial x^i}$. Hence $\mu^j = \omega = a_i\lambda^i = \frac{\partial y^i}{\partial x^i}\lambda^i$.

Definition: Smooth Covector Field

A smooth covector field is a smooth section of T^*M, cal it $\Omega^1(M) = \Gamma(T^*M)$.
Given $f \in C^\infty(M)$, we can define a smooth covector field $df \in \Omega^1(M)$ by $df(v|_p) = (v_p)(f)$.
$df(X) = Xf$ is smooth if X and f are smooth.

Differential

Given a local chart $(U, (x^1, \dots, x^n))$ and a smooth function $f : U \to \mathbb{R}$, $df_p = a_i(p)\lambda^i|_p$.

$$\frac{\partial f}{\partial x^j} = df_p\left(\frac{\partial}{\partial x^j}\bigg|_p\right) = a_i(p)\lambda^i|_p\left(\frac{\partial}{\partial x^j}\bigg|_p\right) = a_i(p)\delta^i_j = a_j(p)$$

That is, $df_p = \frac{\partial f}{\partial x^j}(p)\lambda^j|_p$. In particular, if we consider the coordinate function $x^i : U \to \mathbb{R}$, then $dx^i|_p = \frac{\partial x^i}{\partial x^j}(p)\lambda^j|_p = \lambda^i|_p$ for each $p \in U$ (i.e. $dx^i = \lambda^i$ on U).
With this, we can write $df = \frac{\partial f}{\partial x^i}dx^i$ and $dy^j = \frac{\partial y^j}{\partial x^i}\partial x^i$.

Proposition 11.22

For $f \in C^\infty(M)$, then $df = 0$ if and only if f is constant on every compnent of M.

Proof

($\Longleftarrow$) is trivial.
($\Longrightarrow$) We assume M is connected. Fix $p \in M$, define $\mathcal{A} = \{q \in M : f(p) = f(q)\}$ is closed.
Now let $q \in A$ and U a local chart around q. Then $0 = df = \frac{\partial f}{\partial x^i} dx^i$ (i.e. $\frac{\partial f}{\partial x^i} \equiv 0$, $\forall i$).
Hence f is constant on U and $f(q) = f(p)$ for $U \in \mathcal{A}$.

Proposition 11.23

Take $\gamma : J \to M$ a smooth curve $f \in C^\infty(M)$. Then $(df|_{\gamma(t)})(\gamma'(t)) = (\gamma'(t))f = (f \circ \gamma)'(t)$.

IMAGE 2

Recall that if $v \in T_pM$ and $f \in C^\infty(M)$ then $vf = (f \circ \gamma)'(0)$ where $\gamma : (-\varepsilon, \varepsilon) \to M$, $\gamma(0) = p$ and $\gamma'(0) = v$ ($f \circ \gamma : \mathbb{R} \to \mathbb{R}$).

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Recall

T^*M and $\Omega'(M) = \Gamma(T^*M)$. Let $(U, (x^1, \dots, x^n))$ be a chart. Then inside U, we may write $\omega = \omega_i dx^i$.
$\{dx^i|_p\}$ is a dual basis of $\left\{\frac{\partial}{\partial x^i} \subseteq T_pM\right\}$.
They are also $x^i : U \to \mathbb{R}$ coordinates functions where dx^i is the differential of x^i.
Given $f \in C^\infty(M)$ or $C^\infty(U)$, $df \in \Omega'(M)$ or $\Omega'(U)$ is defined by $df(X_p) = (Xf)(p)$.
Inside a chart, $df = \frac{\partial f}{\partial x^i} dx^i$.
We have a change of coordinates where $(U, (x^1, \dots, x^n))$ and $(V, (y^1, \dots, y^n))$ and $W = U \cap V \neq \varnothing$ gives $dy^j = \frac{\partial y^j}{\partial x^i} dx^i$.

Recall (Linear Algebra)

If $A : V \to W$ is a linear map with $w \in W^*$ and $v \in V$, then $A^* : W^* \to V^*$ is the dual map defined by $(A^*w)(v) := w(Av)$.

Dual of the Tangent Space

Let $F : M \to N$ be a smooth map between manifolds.

$$DF_p : T_pM \to T_{F(p)}N$$
$$(DF_p)^* : T^*_{F(p)}M \to T^*_pN$$

and $(DF_p^*\omega)(v) = \omega(DF_p(v))$ for $\omega \in T^*_{F(p)}N$ and $v \in T_pM$.

Definition: Pullback

Given $\omega \in \Omega'(N)$, we can define $F^*\omega$, a section of T^*M, by $(F^*\omega)_p(v) = \omega(DF_p(v))$ or $(F^*\omega)_p = DF_p^*\omega$.
We call this the pullback of ω by F.

Recall that for $u \in C^\infty(N)$, $M \xrightarrow{F} N \xrightarrow{u} \mathbb{R}$. Then we can define $F^* u \in C^\infty(M)$ by $F^* u = u \circ F$.

Proposition

If $F : M \to N$ is smooth, $u \in C^\infty(N)$ and $\omega \in \Omega^I(N)$, then

1. $F^*(u\omega) = (F^* u)(F^* \omega)$.
2. $F^*(du) = d(F^* u)$.

Proof of 1

$\forall\, p \in M$, $\forall\, v \in T_p M$,

$$(F^*(u\omega))_p(v) = DF_p^*(u\omega)(v) = u_{F(p)}\omega_{F(p)}(DF_p(v)) = (u \circ F(p))\omega(DF_p(v)) = (F^* u)(F^* \omega)$$

Proof of 2

$$(F^*(du))(v) = du(DF_p(v)) = (DF_p(v))u = (du)_{F(p)} DF_p(v) = d(u \circ F)(v) = d(F^* u)(v)$$

Change of Coordinates

Locally, $F : M \to N$. Let $(U, (x^1, \dots, x^n))$ be a chart around p and $(V, (y^1, \dots, y^n))$ a chart around $F(p)$. For $\omega \in \Omega^I(N)$, in V $\omega = \omega_i dy^i$ and

$$F^* \omega = F^*(\omega_i dy^i) = (F^* \omega_i)(F^* dy^i) = (F^* \omega_i) d(F^* y^i) = (\omega_i \circ F)(dF^i)$$

where $F^i = y^i \circ F$ is the ith component of F.
When F is smooth and $\omega \in \Omega^I(N)$, then $F^* \omega \in \Omega^I(M)$. In fact, locally, $F^* \omega = (\omega_i \circ F) d(F^i)$. Hence $F^* \omega$ is smooth.

Example 1

Take $F : \mathbb{R}^3 \to \mathbb{R}^2$ by $(x, y, z) \mapsto (u(x, y, z), v(x, y, z)) = (x^2 y, y \sin(z))$.
Then $\omega = u\, dv + v\, du \in \Omega^I(\mathbb{R}^2)$. So

$$\begin{aligned}
F^* \omega &= F^*(u\, dv + v\, du) \\
&= (F^* u) d(F^* v) + (F^* v) d(F^* u) \\
&= x^2 y\, d(y \sin(z)) + (y \sin(z))\, d(x^2 y) \\
&= x^2 y(\sin(z)\, dy + y \cos(z)\, dz) + y \sin(z)(2xy\, dx + x^2\, dy)
\end{aligned}$$

Example 2

$M = \mathbb{R}^2 - \{0\}$ and $\gamma : [0, 2\pi] \to M$ by $t \mapsto (r\cos(t), r\sin(t))$ for $r > 0$. Take $\omega = \frac{x\,dy - y\,dx}{x^2 + y^2} \in \Omega^I(M)$

$$\begin{aligned}
\gamma^* \omega &= \frac{1}{r^2}(r\cos(t)\, d(r\sin(t)) - r\sin(t)\, d(r\cos(t)) \\
&= \cos(t)(\cos(t))\, dt - \sin(t)(\sin(t))\, dt \\
&= dt
\end{aligned}$$

Definition: Line Integral

If $\eta \in \Omega'(\mathbb{R})$ or $\Omega'(I)$ (where $I \subseteq \mathbb{R}$) is an interval), η can be written as $\eta(t) = f(t)\,dt$ and define

$$\int_I \eta = \int_a^b f(t)\,dt$$

Let $\gamma : [a,b] \to M$ be a smooth curve on M. Let $\omega \in \Omega'(t)$. Define

$$\int_\gamma \omega = \int_a^b \gamma^* \omega$$

with $\gamma^*(\omega) \in \Omega'([a,b])$.

Proposition 11.31

Take $\phi : I \to J$ a diffeomorphism between intervals with $\phi' > 0$. Then

$$\int_J \phi^* \omega = \int_{\phi(J)} \omega$$

Write s for coordinates on J and t for coordinates on I. Then $\omega = f(t)\,dt \in \Omega^1(I)$ and

$$\phi^*\omega = (\phi^* f)\,d(\phi^* t) = (f \circ \phi)\,d(t \circ \phi) = f(\phi(s))\,d(\phi(s)) = f(\phi(s))\phi'(s)\,ds$$

Then

$$\int_J \phi^*\omega = \int_J f(\phi(s))\phi'(s)\,ds \overset{t=\phi(s)}{=} \int_I f(t)\,dt = \int_I \omega$$

Proposition 11.37: Independence of Reparameterization

Suppose $\gamma : I \to M$ is a smooth curve and $\phi : J \to I$ is a diffeomorphism with $\phi' > 0$. Then $\tilde{\gamma} := \gamma \circ \phi : J \to M$ is a reparameterization of γ and

$$\int_\gamma \omega = \int_{\tilde{\gamma}} \omega$$

If $\phi' < 0$, then $\int_\gamma \omega = -\int_{\tilde{\gamma}} \omega$.

Proof

$$\int_\gamma \omega = \int_I \gamma^*\omega \int_J \phi^*\gamma^*\omega = \int_J (\gamma \circ \phi)^*\omega = \int_{\tilde{\gamma}} \omega$$

Example

Take $\gamma : [0, 2\pi] \to M = \mathbb{R}^2 - \{0\}$ by $t \mapsto (r\cos(t), r\sin(t))$ with $r > 0$. If $\omega = \frac{x\,dy - y\,dx}{x^2+y^2}$, then $\gamma^*\omega = dt$ and

$$\int_\gamma \omega = \int_0^{2\pi} \gamma^*\omega = \int_0^{2\pi} dt = 2\pi$$

Proposition 11.38

For $\gamma : I \to M$

$$\int_\gamma \omega = \int_I \omega_{\gamma(t)}(\gamma'(t))\, dt$$

Proof

In a local chart $(U, (x^1, \ldots, x^n))$, we can write $\omega = \omega_i dx^i$. Then $\gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t))$ and

$$\begin{aligned}
\gamma^* \omega &= \gamma^*(\omega_i dx^i) \\
&= (\gamma^* \omega_i) d(\gamma^* x^i) \\
&= (\omega_i \circ \gamma) d\gamma^i \\
&= \omega_i(\gamma(t)) \frac{d\gamma^i}{dt} dt \\
&= \omega_i(\gamma(t)) \dot{\gamma}^i(t) dt
\end{aligned}$$

Since $\omega = \omega_i dx^i$ and $\dot{\gamma}(t) = (\dot{\gamma}^1(t), \ldots, \dot{\gamma}^n(t)) = \dot{\gamma}^i(t) \frac{\partial}{\partial x^i}$, $\omega_{\gamma(t)}(\dot{\gamma}(t)) = \omega_i(\gamma(t)) \dot{\gamma}^i(t)$ and

$$\omega_i(\gamma(t)) \dot{\gamma}^i(t) dt = \omega_{\gamma(t)}(\dot{\gamma}(t)) dt$$

Hence $\int_\gamma \omega = \int_I \gamma^* \omega = \int_I \omega_{\gamma(t)}(\dot{\gamma}(t))\, dt$.

Corollary

Then, if $f : M \to \mathbb{R}$ is a smooth function,

$$\int_\gamma df = \int_I (df)_{\gamma(t)}(\dot{\gamma}(t))\, dt = \int_I (f \circ \gamma)'(t)\, dt = f(\gamma(b)) - f(\gamma(a))$$

Therefore $\int_\gamma df$ only depends on the value of f at the endpoints of γ.

Definition: Exact and Conservative Forms

Let $\omega \in \Omega^1(M)$. We say that ω is...

1. exact if there exists $f \in C^\infty(M)$ such that $\omega = df$.
2. conservative if $\int_C \omega = 0$ for any closed, piecewise-smooth curve in M

f is called the potential of ω.

Remark

If $\int_C \omega = 0$, we may write C as the concatenation of curves γ then $-\sigma$. Then

$$0 = \int_C \omega = \int_\gamma \omega + \int_{-\sigma} \omega = \int_\gamma \omega - \int_\sigma \omega$$

Remark

Exact implies conservative.

Theorem

If $\omega \in \Omega^1(M)$ is conservative, then it is exact.

Proof

Fix a bse point $p_0 \in M$.
We have that $\int_p^q \omega = \int_\gamma \omega$ is well-defined by the conservative assumption, and we define $f(p) = \int_{p_0}^p \omega$.
Let $q_0 \in M$ and let $(U, (x^1, \dots, x^n))$ be a chart centered at q_0. Inside U, $\omega = \omega_i dx^i$ and $df = \frac{\partial f}{\partial x^i} dx^i$.
We need to show that $\frac{\partial f}{\partial x^i} = \omega_i$ for each i. Fix an index i and consider a curve $\sigma : (-\varepsilon, \varepsilon) \to U$ by $t \mapsto (0, \dots, t, \dots, 0)$.

IMAGE 1

Let $q_- = \sigma(-\varepsilon)$, then

$$f(q_0) = \int_{p_0}^{q} \omega = \int_{p_0}^{q_-} \omega + \int_{q_-}^{q} \omega =: \tilde{f}(q)$$

so $f(q_0) = \text{constant} + \tilde{f}(q)$. Hence $\frac{\partial f}{\partial x^j} = \frac{\partial \tilde{f}}{\partial x^j}$ in U. Therefore

$$\begin{aligned}
\tilde{f}(\sigma(s)) &= \int_{q_-}^{\sigma(s)} \omega \\
&= \int_{\sigma|_{[-\varepsilon,s]}} \omega \\
&= \int_{-\varepsilon}^{s} \omega_{\sigma(t)}(\dot{\sigma}(t))\, dt \\
&= \int_{-\varepsilon}^{s} \omega_{\sigma(t)}\left(\frac{\partial}{\partial x^i}\right) dt \\
&= \int_{-\varepsilon}^{s} \omega_i(\sigma(t))\, dt
\end{aligned}$$

and

$$\left.\frac{\partial f}{\partial x^i}\right|_{q_0} = \left.\frac{\partial \tilde{f}}{\partial x^i}\right|_{q_0} = (\tilde{f} \circ \sigma)'(0) = \left.\frac{d}{ds}\right|_{s=0} \left(\int_{-\varepsilon}^{s} \omega_i(\sigma(t))\, dt \right) = \omega_i(\sigma(0)) = \omega_i(q_0)$$

Remark

Take $\omega = df \in \Omega^1(M)$ which is $\omega_i dx^i$ locally or $\omega_i = \frac{\partial f}{\partial x^i}$ when exact.

$$\frac{\partial \omega_i}{\partial x^j} = \frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial \omega_j}{\partial x^i}$$

Note: $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$ does not, in general, imply $\omega = df$.

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Recall

If $\omega \in \Omega^1(M)$ and $\gamma : \mathbb{R} \supseteq I \to M$ a (piecewise) smooth curve, then

$$\int_\gamma \omega = \int_I \gamma^* \omega$$

If df is the differential of a smooth function, then

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a))$$

only depends on endpoints. In particular, along a closed (piecewise) smooth curve

$$\int_C df = 0$$

We have that ω is exact if $\omega = df$ and conservative if $\int_C \omega = 0$ for every closed curve.
ω is exact if and only if it is also conservative.

Recall: Checking Exactness

Take $\omega \in \Omega^1(M)$,

$$\omega_i dx^i = \omega = df = \frac{\partial f}{\partial x^i} dx^i$$

Then

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2}{\partial x^j \partial x^i}$$

That is, $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$.

Definition: Closed 1-Form

We say $\omega \in \Omega^1(M)$ is closed if in every chart $(U, (x^i))$, $\omega = \omega_i dx^i$ satisfies $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$.
Exact implies closed, however the converse is not true in general.

Example

$\exists \omega \in \Omega^1(\mathbb{R}^2 - \{0\})$ such that ω is closed but $\int_C \omega = 2\pi$.

Corollary 11.50

If $\omega \in \Omega^1(M)$ is closed, then $\forall p \in M$ there exists a chart U at p such that $\omega_U = df$ for some $f \in C^\infty(U)$

Proposition 11.45

For $\omega \in \Omega^1(M)$, the following are equivalent

1. ω is closed.
2. ω satisfies $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$ in some chart at every point.
3. For every open $U \subseteq M$ and $X, Y \in \mathfrak{X}(U)$, it holds that

$$X(\omega(Y)) - Y(\omega(X)) = \omega([X, Y])$$

The proof that 1 implies 2 is trivial.

Proof 3 Implies 1

Pick U as a chart, $X = \frac{\partial}{\partial x^i}$, and $Y = \frac{\partial}{\partial x^j}$. Then, since $\omega = \omega_i dx^i$,

$$X(\omega(Y)) = X(\omega_j) = \frac{\partial w_j}{\partial x^i}$$

Similarly, $Y(\omega(X)) = \frac{\partial \omega_i}{\partial x^j}$. Then $[X, Y] = \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$ and

$$\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} = 0$$

Proof 2 Implies 3

Fix any $p \in U$. We have a chart $(V, (x^i))$ at p such that $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$. Then

$$X(\omega(y)) = X\left((\omega_i dx^i)\left(Y^j \frac{\partial}{\partial x^j}\right)\right) = X(\omega_i Y^i) = (X\omega_i)Y^i + \omega_i(XY^i) = x^j \frac{\partial w_i}{\partial x^j} Y^i + \omega_i(XY^i) = X^j Y^i \frac{\partial \omega_j}{\partial x^i}$$

Similarly,

$$Y(\omega(X)) = Y(\omega_i x^i) = Y^j \frac{\partial \omega_i}{\partial x^j} x^i - \omega_i(YX^i) = X^i Y^j \frac{\partial \omega_i}{\partial x^j}$$

which is equivalent under a change of indicies. Hence

$$X(\omega(Y)) - Y(\omega(X)) = \omega_i(XY^i) - \omega_i(YX^i) = \omega_i(XY^i - YX^i) = \omega([X, Y])$$

Lemma

Suppose $F : M \to N$ is a local diffeomorphism. Then $F^* : \Omega^1(N) \to \Omega^1(M)$ sends exact (or closed) 1-forms to exact (or closed) ones.

Proof of Exact

If $\omega = df \in \Omega^1(N)$, then $F^*\omega = F^*(df) = d(F^* f)$ is exact on M.

Proof of Closed

If $\omega \in \Omega^1(N)$ is closed, then $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$ in every chart of N.
For any $p \in M$, we consider a chart at p by $(V, \phi \circ F)$

IMAGE 1

Therefore $\phi \circ F \circ (\phi \circ F)^{-1} = \mathrm{id}$ and $F^* = \mathrm{id}$ so $F^* \omega$ is closed.

Poincaré Lemma

Let $\omega \in \Omega^1(M)$ be closed. Fix $p \in M$, and let (U, ϕ) be a chart at p such that $\phi(U) = B_1(0) \subseteq \mathbb{R}^n$.

IMAGE 2

Assuming the above, every closed 1-form on $B_1(0)$ is exact. $(\phi^{-1})^*(\omega|_U) = df$ for some $f \in C^\infty(B_1(0))$ where $\omega|_U = \phi^*(df) = d(\phi^* f) \in C^\infty(U)$

Definition: Star-Shaped Domain

We say that $U \subseteq \mathbb{R}^n$ open is star-shaped with a center $c \in U$ (wlog $c = 0$) if for any $x \in U$, the segment γ_x from c to x is contained in U.

IMAGE 3

If $x = (x^i)$, then $\gamma_x(t) = (tx^i)$.

Theorem 11.49 (Poincaré Lemma)

If $U \subseteq \mathbb{R}^n$ is star-shaped, then every closed 1-form is exact.

Recall

If ω is an exact 1-form, then $f(q) = \int_{p_0}^{p} \omega$ is a potential.
We also have that $\int_\gamma \omega = \int_I \omega_{\gamma(t)}(\dot{\gamma}(t))\, dt$.

Proof

Let $\omega \in \Omega^1(U)$ be a closed 1-form.
We need to construct $f \in C^\infty(U)$ such that $df = \omega$. That is, for all i, $\frac{\partial f}{\partial x^i} = \omega^i$. Define

$$f(x) = \int_{\gamma_x} \omega = \int_0^1 \omega_{\gamma_x(t)}(\dot{\gamma}_x(t))\, dt = \int_0^1 \omega_i|_{\gamma_x(t)} dx^i(x^1, \ldots, x^n)\, dt = \int_0^1 \omega_i|_{tx} x^i\, dt$$

Since everything is smooth,

$$\begin{aligned}\frac{\partial f}{\partial x^j}(x) &= \int_0^1 \frac{\partial}{\partial x^j}(\omega_i(tx)\cdot x^i)\,dt \\ &= \int_0^1 \frac{\partial \omega_i(tx)}{\partial x^j}\cdot x^i + \omega_i(tx)\frac{\partial x^i}{\partial x^j}\,dt \\ &= \int_0^1 \left(\frac{\partial w_i}{\partial x^j}\right)\Big|_{(tx)} tx^i + \omega_j(tx)\,dt \\ &= \int_0^1 \frac{\partial \omega_j}{\partial x^i}\Big|_{tx} tx^i + \omega_j(tx)\,dt \\ &= \int_0^1 \frac{d}{dt}\big(t\omega_j(tx)\big)\,dt \\ &= t\omega_j(tx)\big|_0^1 \\ &= \omega_j(x)\end{aligned}$$

Tensors: Multilinear Maps

All vector spaces will be finite dimensional in our consideration.

$$F : V_1 \times \cdots \times V_k \to W$$

linear in every component. Denote $L(V_1,\ldots,V_k;W)$ to be the set of all such multilinear maps.
Given $\omega \in L(V_1;\mathbb{R}) = V_1^*$ and $\eta \in V_2^*$, we can define $\omega\otimes\eta \in L(V_1,V_2;\mathbb{R})$ by $\omega\otimes\eta(v_1,v_2) = \omega(v_1)\cdot\eta(v_2)$.

- Remark

  $(2\omega)\otimes\eta = \omega\otimes(2\eta)$. We assume $\otimes_{\mathbb{R}}$.

  Similarly, given $\omega_i \in V_i^*$, we can define $\omega_1\otimes\cdots\otimes\omega_k \in L(V_1,\ldots,V_K;\mathbb{R})$.

Proposition

Let V_j with dimension n_j $(j=1,\ldots,k)$. Each V_j has a basis $\{E_1^{(j)},\ldots,E_{n_j}^{(j)}\}$.
Its dual basis $\{\varepsilon_{(j)}^1,\ldots,\varepsilon_{(j)}^{n_j}\} \subseteq V_j^*$. Then $L(V_1,\ldots,V_k;\mathbb{R})$ has a basis

$$\mathcal{B} = \left\{\varepsilon_{(1)}^{i_1}\otimes\cdots\otimes\varepsilon_{(k)}^{i_k} \;:\; 1 \le i_j \le n_j\right\}$$

Proof

For a multi-index $I = (i_1,\ldots,i_k)$ with $i \le i_j \le n_j$, we write $\varepsilon^I = \varepsilon_{(1)}^{i_1}\otimes\cdots\otimes\varepsilon_{(k)}^{i_k}$.
For any $F \in L(V_1,\ldots,V_k;\mathbb{R})$, define $F_I = F(E_{i_1}^{(1)},\ldots,E_{i_k}^{(k)})$. We claim that $F = F_I\varepsilon^I$.
In fact, for $(v_1,\ldots,v_k) \in V_1\times\cdots\times V_k$, $v_j = v_j^i E_i^{(j)}$. We may check that $F(v_1,\ldots,v_k) = F_I\varepsilon^I(v_1,\ldots,v_k)$.
Therefore $\mathcal{B}$ spans $L(V_1,\ldots,V_k;\mathbb{R})$.
Then, if $F_I\varepsilon^I = 0$, then applying it to $(E_{i_1}^{(1)},\ldots E_{i_k}^{(k)})$ gives $F_I = 0$. Therefore $\mathcal{B}$ is linearly independent.
In particular, $\dim\, L(V_1,\ldots,V_k;\mathbb{R}) = \prod_{j=1}^k n_j = \prod_{j=1}^k \dim\, V_j$.

Definition: Formal Linear Combination

Let S be a set. Define

$$\mathcal{F}(S) = \left\{ \sum_{i=1}^{m} a_i s_i \; : \; a_i \in \mathbb{R}, \; s_i \in S \right\}$$

This is the free (real) vector space on S containing formal linear combinations of elements of S.
Define $V_1 \otimes \cdots \otimes V_k = \mathcal{F}(V_1 \times \cdots \times V_k)/R$ where R is generated by

$$(v_1, \ldots, v_j + v_j', \ldots, v_k) \sim (v_1, \ldots, v_j, \ldots, v_k) + (v_1, \ldots, v_j', \ldots, v_k)$$
$$(v_1, \ldots, cv_j, \ldots, v_k) \sim c(v_1, \ldots, v_k)$$

In other words, in the quotient $v_1 \otimes \cdots \otimes v_k = \prod(V_1, \ldots, v_k)$.

Proposition

$V_1 \otimes \cdots \otimes V_k$ has a basis $\left\{ E_{i_1}^{(1)} \otimes \cdots \otimes E_{i_k}^{(k)} \; : \; 1 \leq i_j \leq n_j \right\}$.

Proposition

There exists a canonical isomorphism $(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3)$ by sending $(v_1 \otimes v_2) \otimes v_3 \mapsto v_1 \otimes (v_2 \otimes v_3)$.

Proposition

$L(V_1, \ldots, V_k; \mathbb{R}) \cong V_1^* \otimes \cdots \otimes V_k^*$.

Proof Sketch

Define $\Phi : V_1^* \times \cdots \times V_k^* \to L(V_1, \ldots, V_k; \mathbb{R})$ by $(\omega^1, \ldots, \omega^k) \mapsto \omega^1 \otimes \cdots \otimes \omega^k$. By multilinear, this induces an isomorphism

$$\Phi : V_1^* \otimes \cdots \otimes V_k^* \cong L(V_1, \ldots, V_k; \mathbb{R})$$

Recall

$V^{**} \cong V$ for finite dimensional vector spaces, so $V_1 \otimes \cdots \otimes V_k = L(V_1^*, \ldots V_k^*; \mathbb{R})$.

Definition: Tensor

A tensor of (k, l)-type is an element in $\underbrace{V \otimes \cdots \otimes V}_{k} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{l}$.
The collection of such elements in $T^{(k,l)}V$. Most of the time we consider $T^{(0,l)}V$.

Examples

A vector in V is a $(1, 0)$-tensor.
A covector in V^* is a $(0, 1)$-tensor.
A linear map $A \in L(V)$ is a $(1, 1)$-tensor.
An inner product is a $(0, 2)$-tensor.

Symmetric Tensor

We say that $\alpha \in T^{(0,l)}V$ is symmetric if $\alpha(\ldots, v_i, \ldots, v_j \ldots) = \alpha(\ldots, v_j, \ldots, v_i, \ldots)$.

Alternating Tensor

We say that $\alpha \in T^{(0,l)}V$ is alternating if $\alpha(\ldots, v_i, \ldots, v_j \ldots) = -\alpha(\ldots, v_j, \ldots, v_i, \ldots)$.