Analysis III

April 2, 2024

No class Thursday, April 04.
Makeup class (tentatively) on Friday, April 12 at 10:30.
Discussion sections on Fridays (tentatively) at 11:40.

Topological Vector Spaces

Definition: Vector Spaces

V over a field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$).

Definition: Topological Spaces

(X, τ) where $\tau \subseteq \mathcal{P}(X)$ satisfying

1. $\varnothing, X \in \tau$
2. $A, B \in \tau \implies A \cap B \in \tau$
3. $A_\omega \in \tau \implies \bigcup_\omega A_\omega \in \tau$

Recall: $A \in \tau \iff A$ open $\iff X \setminus A$ closed.
$A^\circ = \bigcup_{\substack{U \subseteq A \\ U \text{ open}}} U$ the set of interior points of A.
$\overline{A} = \bigcap_{\substack{F \supseteq A \\ F \text{ closed}}}$ the closure of A.
A' limit points of A.
Compact sets.
Locally compact sets.
Recall: X is Hausdorff iff $\forall x, y \in X$, $\exists U, V \in \tau$ such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$.

Definition: Bases for Topological Spaces

Definition: Let (X, τ) be a topological space. $\sigma \subseteq \tau$ is called a base for topology τ if $\forall x \in X$, $\forall U \in \tau$, $x \in U$, $\exists W \in \sigma$ such that $x \in W \subseteq U$.

Proposition

$\sigma \subseteq \tau$ is a base for τ if and only if every $U \in \tau$ is the union of certain sets taken from σ.

$$(*) \quad \tau = \left\{ \bigcup_{\omega \subset \Omega} W_\omega \;:\; \{W_\omega\}_{\omega \in \Omega} \subseteq \sigma,\, \Omega \right\}$$

- Proof
  ($\Longleftarrow$) ✓
  ($\Longrightarrow$) Take $U \in \tau$ and let $x \in U$, $\rightsquigarrow$ find $W_x \in \sigma$, $x \in W_x \subseteq U$.

  $$U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} W_x \subseteq U$$

  Therefore $\bigcup_{x \in U} W_x = U$.

Proposition

If σ is a base for some topology τ on X, then

1. $\forall x \in X$, $\exists W \in \sigma$ such that $x \in W$.

2. $\forall U, V \in \sigma$, $\forall x \in U \cap V$, $\exists W \in \sigma$ such that $x \in W \subseteq U \cap V$.

Conversely, if $\sigma \in \mathcal{P}(X)$ ($\varnothing \notin \sigma$) satisfies (1) and (2), then σ is the base for a topology τ (and τ is given by $(*)$).
Note that $U, V \in \tau \implies U \cap V \in \tau$ (requires (2)).
If $U = \bigcup U_\alpha$ and $V = \bigcup V_\beta$, then $U \cap V = \bigcup_{\alpha,\beta}(U_\alpha \cap V_\beta) = \bigcup_{\alpha,\beta} \bigcup_{x \in U} W_{\alpha,\beta,x}$.

Example: Metric Spaces

(X, d) is a metric space if $d : X \times X \to [0, +\infty)$ satisfies

1. $d(x, y) = 0$ if and only if $x = y$.

2. $d(x, y) = d(y, x)$.

3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

Definition: Epsilon Neighborhoods

$$B_\varepsilon(x) = \{y \in x \,:\, d(x, y) < \varepsilon\}$$

$A \subseteq X$ is open if and only if $\forall x \in A$, $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A$. $x \in B_\varepsilon(x)$.
τ = set of all open sets.

$$\sigma_1 = \{B_\varepsilon(x) \,:\, x \in X, ; \varepsilon > 0\}$$

is a base for the topology τ.

$$\sigma_2 = \left\{B_{1/n}(x) \,:\, x \in X,\ n \in \mathbb{N}\right\}$$

is also a base for τ.

Definition: Direct Product - Product Topology

Let (X_1, τ_1) and (X_2, τ_2) be topological spaces.
Consider $X = X_1 \times X_2$. The product topology τ on X is given by the base

$$\sigma = \{U_1 \times U_2 \,:\, U_1 \in \tau_1,\ U_2 \in \tau_2\}$$

More explicitly,

$$\tau = \left\{ \bigcup_\omega U_{1,\omega} \times U_{2,\omega} \,:\, U_{i,\omega} \in \tau_i \right\}$$

(X_ω, τ_ω) topological spaces ($\omega \in \Omega$)

$$X = \prod_{\omega \in \Omega} = \{(x_\omega)_{\omega \in \Omega} \,:\, x_\omega \in X_\omega\}$$

Formally, $f \cong (x_\omega)_{\omega \in \Omega}$, $x_\omega = f(\omega)$, $f : \Omega \to \bigcup_{\omega \in \Omega} X_\omega$ such that $f(\omega) \in X_\omega$.
[$x \neq \varnothing \Longleftarrow X_\omega \neq \varnothing$ axiom of choice]

$$\sigma = \left\{ \prod_{\omega \in \Omega} U_\omega \,:\, U_\omega \in \tau_\omega \text{ and all but finitely many } U_\omega = X_\omega \right\}$$

Definition: Subspace Topology

Given (X, τ) and $Y \subseteq X$, then (Y, τ_Y) is also a topological space where

$$\tau_Y \{U \cap Y \,:\, U \in \tau\}$$

Definition: Local Bases for Topological Spaces

A collection $\gamma \subseteq \tau$ is called a local base at $x \in X$ if

1. $\forall U \in \tau$, $x \in U$, $\exists W \in \gamma$ such that $x \in W \subseteq U$.

2. $\forall W \in \gamma$, $x \in W$

Example

Let (X, d) be a metric space.

$$\gamma_x = \{B_\varepsilon(x) \,:\, \varepsilon > 0\}$$

is a local base at x. Similarly,

$$\tilde{\gamma}_x = \left\{B_{1/n} \,:\, n \in \mathbb{N}\right\}$$

is a countable local base.

Proposition

If γ_x ($x \in X$) are local bases for τ at X, then

$$\sigma = \bigcup_{x \in X} \gamma_x$$

is a bse for τ.

Proposition

$\{\gamma_x\}_{x \in X}$ are local bases at x for some topology τ if and only if

1. $\forall x \in X$, γ_x is a non-empty collection of subsets containing x.

2. If $U \in \gamma_x$, $V \in \gamma_y$, and $z \in U \cap V$, then $\exists W \in \gamma_z$ such that $z \in W \subseteq U \cap V$.

Definition: Topological Vector Spaces

Suppose V is a vector space over $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ and let τ be a topology on V.
Then V is a topological vector space (TVS) if

1. $\forall\, x \in V$, $\{x\}$ is closed.

2. The functions f, g (i.e. algebraic operations) are continuous.

   $f : V \times V \to V,\ f(x, y) = x + y$
   $g : \mathbb{F} \times V \to V,\ g(\lambda, x) = \lambda \cdot x$

Notation

For $A_1, A_2 \subseteq V$ and $B \subseteq \mathbb{F}$,

$$\begin{aligned} A_1 + A_2 &= \{a_1 + a_2 \,:\, a_1 \in A_1,\ a_2 \in A_2\} \\ a + A_1 &= \{a + \alpha \,:\, \alpha \in A\} \\ B \cdot A &= \{\beta \cdot a \,:\, \beta \in B,\ a \in A\} \\ \alpha \cdot A &= \{\alpha \cdot a \,:\, a \in A\} \end{aligned}$$

Lemma

Let V be a TVS. Then

1. $\forall\, x, y \in V$, $\forall$ open $U_{x+y} \ni x + y$, $\exists$ open $U_x \ni x$, open $U_y \ni y$ such that $U_x + U_y \subseteq U_{x+y}$.

2. $\forall\, x \in V$, $\alpha \in \mathbb{F}$, $\forall$ open $U_{\alpha x} \ni \alpha x$, $\exists$ open $U_x \ni x$, $U_\alpha \ni \alpha$ such that $U_\alpha \cdot U_x \subseteq U_{\alpha x}$.

Proof of 1

Given $x, y \in X$, $x + y \in U_{x+y}$ open.

$$f(x, y) = x + y \in U_{x+y}$$

and $(x, y) \in f^{-1}(U_{x+y})$ open. In the product topology

$$(x, y) \in U_x \times U_y \subseteq f^{-1}(U_{x+y})$$

which implies $x \in U_x$ and $y \in U_y$, both open, and $U_x + U_y \leq U_{x+y}$.