Manifolds III

March 31, 2025

Review

If X, Y are topological spaces and $f, g : X \to Y$ continuous maps, we say f and g are homotopic (written $f \simeq g$) if there is a homotopy $H : X \times I \to Y$ (where $I = [0,1]$) such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$.
We say that f is null-homotopic if it is homotopic to a constant map.

Proposition

Homotopy is an equivalence relation on the collection of continuous maps.

1. $f \simeq f$ by $H(x,t) := f(x)$.
2. $f \overset{\tilde{H}}{\simeq} g \implies g \simeq f$ by defining $\tilde{H}(x,t) := H(x, 1-t)$.
3. $(f \overset{F}{\simeq} g \wedge g \overset{G}{\simeq} h) \implies f \simeq h$ by

$$H(x,t) := \begin{cases} F(x,2t) & 0 \le t \le 1/2 \\ G(x,2t-1) & 1/2 \le t \le 1 \end{cases}.$$

Proposition

For $f_0, f_1 : X \to Y$ and $g_0, g_1 : Y \to Z$, if $f_0 \overset{F}{\simeq} f_1$ and $g_0 \overset{G}{\simeq} g_1$, then $g_0 \circ f_0 \simeq g_1 \circ f_1$.

Proof

Define $H(x,t) := G(F(x,t),t)$ such that $H(x,0) = G(F(x,0),0) = G(f_0(x),0) = g_0 \circ f_0(x)$.
Similarly, $H(x,1) = g_1 \circ f_1(x)$.

Definition: Homotopic Spaces

We say that two spaces X and Y are homotopic to each other ($X \simeq Y$) if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.

Example

$\mathbb{R}^n$ is homotopic to $\{0\}$ (or any single point) by $\iota : 0 \to \mathbb{R}^n$ and $r : \mathbb{R}^n \to 0$. Then $r \circ \iota : 0 \to 0$ is id_0 and $\iota \circ r : \mathbb{R}^n \ni x \mapsto 0 \in \mathbb{R}^n$ is homotopic to $\mathrm{id}_{\mathbb{R}^n}$. In fact, consider $H : \mathbb{R}^n \times I \to \mathbb{R}^n$ where $H(x,t) = tx$, $H(x,1) = x = \mathrm{id}_{\mathbb{R}^n}(x)$ and $H(x,0) = 0$.

Definition: Path

A path in X from p to q is a continuous map $f : I \to X$ such that $f(0) = p$ and $f(1) = q$.

Definition: Path Homotopic

Let $f, g : I \to X$ be two paths in X from p to q.
We say that f and g are path homotopic (write $f \sim g$) if there is a homotopy $H : I \times I \to X$ such that $H(s,0) = f(s)$, $G(s,1) = g(s)$, $H(0,t) = p$ and $H(1,t) = q$.

Proposition

Path homotopy is an equivalence relation on the collection of paths from p to q.
Write $[f]$, the equivalence class of f in the quotient.

Definition: Loop

In the special case that $p = q$, we say that $f : I \to X$ is a loop

Definition: Fundamental Group

Given (X, p), $\pi_1(X, p)$ (the fundamental group of X at the point p) is the set of all loops at p modulo the path homotopy.

$$\{\text{loops at } p\}/\sim$$

Equivalently, $(S^1, 1)$, $\{\text{loops at } p\} = \{\text{continuous maps } f : (S^1, 1) \to (X, p)\}$ with $f(1) = p$. We say this is the homotopy "relative to $1 \in S^1$". We have $H : S^1 \times I \to X$ such that $H(s, 0) = f(s)$, $H(s, 1) = g(s)$ and $H(1, t) = p$.

Definition: Free Homotopy

For two loops $f, g : S^1 \to X$, we say that f and g are free homotopic if $f \simeq g$.

Lemma

When $f : I \to X$ is a path from p to q, if $f \circ \varphi$ is a reparameterization of f then $(f \circ \varphi) \sim f$ where $\varphi : I \to I$ satisfies $\varphi(0) = 0$ and $\varphi(1) = 1$.

Proof

Note that φ is homotopic to the identity map id_I through $H(s, t) = ts + (1 - t)\varphi(s)$ since $H(s, 0) = \varphi(s)$ and $H(s, 1) = s = \mathrm{id}_I(s)$.
Then consider $f \circ H : I \times I \to X$ which is a path homotopy between f and $f \circ \varphi$.

Fundamental Group

Let $f, g : I \to X$ be two paths with $f(1) = g(0)$.
Then we can "compose" (concatenate) f and g together $(f \cdot g) : I \to X$ by

$$(f \cdot g)(s) := \begin{cases} f(2s) & 0 \le s \le 1/2 \\ g(2s - 1 & 1/2 \le s \le 1 \end{cases}.$$

Lemma

If $f_0 \overset{F}{\sim} f_1$, $g_0 \overset{G}{\sim} g_1$ and $f_0(1) = f_1(1) = g_0(0) = g_1(0)$, then $f_0 \cdot g_0 \sim f_1 \cdot g_1$.

Proof

Define

$$H(s, t) := \begin{cases} F(2s, t) & 0 \le s \le 1/2 \\ G(2s - 1, t) & 1/2 \le s \le 1 \end{cases}.$$

Then

$$H(s,0) = \begin{cases} F(2s,0) = f_0(2s) & 0 \le s \le 1/2 \\ G(2s-1,0 = g_0(2s-1) & 1/2 \le s \le 1 \end{cases}.$$

Similarly $H(s,1) = (f_1 \cdot g_1)(s)$, hence $f_0 \cdot g_0 \sim f_1 \cdot g_1$.
With this, we have a well-defined $[f] \cdot [g] := [f \cdot g]$.

Simple Properties

For f from p to q where c_p is the constant map at p,

1. $[c_p] \cdot [f] = [f] = [f] \cdot [c_q]$ since $c_p \cdot f$ is a reparameterization of f.

2. Let $\overline{f}$ be the inverse path of f (i.e. $\overline{f}(s) = f(1-s)$). Then $[f] \cdot [\overline{f}] = [c_p]$ and $[\overline{f}] \cdot [f] = [c_q]$.

$$H(s,t) := \begin{cases} f(2s) & 0 \le s \le t/2 \\ f(t) & t/2 \le s \le 1 - t/2 \\ f(2-2s) & 1 - t/2 \le s \le 2 \end{cases}.$$

1. $([f] \cdot [g]) \cdot [h] = [f] \cdot ([g] \cdot [h])$, since these are reparameterizations of the same path.

Group Structure

$\pi_1(X,p) = \{\text{loops at } p\} / \sim$.
Define $[f] \cdot [g] := [f \cdot g]$.
It has an identity element $[c_p] = e$.
For any $f \in \pi_1(X,p)$, it has an inverse $[\overline{f}]$ such that $[f] \cdot [\overline{f}] = [\overline{f}] \cdot [f] = [c_p]$.
Finally, it is associative by (3) above.

Proposition

Suppose $p, q \in X$ with X path-connected.
Then $\pi_1(X,p)$ is isomorphic to $\pi_1(X,q)$.
Remark: this isomorphism is not canonical.

Proof

We define a path γ from q to p and $\Phi_\gamma : \pi_1(X,p) \to \pi_1(X,q)$ by $[f] \mapsto [\gamma \cdot f \cdot \overline{\gamma}]$.
Φ_γ is a group homomorphism.

$$\begin{aligned} \Phi_\gamma[f] \cdot \Phi_\gamma[g] &= [\gamma \cdot f \cdot \overline{\gamma}] \cdot [\gamma \cdot g \cdot \overline{\gamma}] \\ &= [\gamma \cdot f \cdot \overline{\gamma} \cdot \gamma \cdot g \cdot \overline{\gamma}] \\ &= [\gamma \cdot f] \cdot \overbrace{[\overline{\gamma} \cdot \gamma]}^{=e} \cdot [g \cdot \overline{\gamma}] \\ &= [\gamma \cdot (f \cdot g) \cdot \overline{\gamma}] \\ &= \Phi_\gamma[f \cdot g]. \end{aligned}$$

Φ_γ has an inverse, $\Phi_{\overline{\gamma}} : \pi_1(X,q) \to \pi_1(X,p)$.

$$\Phi_{\overline{\gamma}} \circ \Phi_\gamma[f] = \Phi_{\overline{\gamma}}[\gamma \cdot f \cdot \overline{\gamma}] = [\overline{\gamma} \cdot \gamma \cdot f \cdot \overline{\gamma} \cdot \gamma] = [f].$$

Induced Homomorphism

$\varphi : (X, p) \to (Y, q)$ induces

$$\begin{aligned} \varphi_* : \pi_1(X, p) &\to \pi_1(Y, q) \\ [f] &\mapsto [\varphi \circ f]. \end{aligned}$$

φ_* is a homomorphism.

$$(\varphi_*[f]) \cdot (\varphi_*[g]) = [\varphi \circ f] \cdot [\varphi \circ g] = [(\varphi \circ f) \cdot (\varphi \circ g)] = [\varphi \circ (f \cdot g)] = \varphi_*[f \cdot g].$$

Proposition

If $\varphi, \psi : (X, p) \to (Y, q)$ are homotopic, then $\varphi_* = \psi_* : \pi_1(X, p) \to \pi_1(Y, q)$.

Proof

Let $[f] \in \pi_1(X, p)$, $\varphi_*[f] = [\varphi \circ f]$ and $\psi_*[f] = [\psi \circ f]$ and $H : X \times I \to Y$ a homotopy between φ and ψ.
Then define $\tilde{H} := I \times I \to Y$ by $\tilde{H}(s, t) = H(f(s), t)$ such that

$$\begin{aligned} \tilde{H}(s, 0) &= H(f(s), 0) = \varphi \circ f(s) \\ \tilde{H}(s, 1) &= H(f(s), 1) = \psi \circ f(s). \end{aligned}$$

Corollary

If $X \simeq Y$, then $\pi_1(X) \simeq \pi_1(Y)$.

Examples (*)

$\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(S^n) = 0$ for $n \geq 2$.
For $n \geq 2$, write $S^n = A_+ \cup A_-$ where A_+ and A_- are large balls centered at the north and south pole respectively.
Then A_+ and A_- are both homeomorphic to $\mathbb{R}^n$ and $A_+ \cap A_-$ (their intersection about the equator) is homemorphic to $S^{n-1} \times \mathbb{R}$.
We fix a base point $p \in A_+ \cap A_-$ and let $f : I \to S^n$ be a loop based at p.
There exists a partition of I, $0 = s_0 < s_1 < \cdots < s_k = 1$, such that $f|_{[s_i, s_{i+1}]}$ is contained in A_- or A_+.
Draw a path γ_i from p to $f(s_i)$ such that $\gamma_i \subseteq A_+ \cap A_-$. Let $f_i = f|_{[s_i, s_{i+1}]}$ such that $f = f_0 \cdot f_1 \cdots f_k$. Then this is path homotopic to

$$(f_0 \cdot \overline{\gamma}_1) \cdot (\gamma_1 \cdot f \cdot \overline{\gamma}_2) \cdots (\gamma_{k-1} \cdot f_{k-1} \cdot \overline{\gamma}_k) \cdot (\gamma_k \cdot f_k).$$

Each $\gamma_i \cdot f_i \cdot \overline{\gamma}_i$ is contained in A_- or A_+, hence $\gamma_i \cdot f_i \overline{\gamma}_{i+1} \sim c_p$, $f \simeq c_p$ and $[f] = e$.

April 2, 2025

Correction

For $\varphi, \psi : (X, x_0) \to (Y, y_0)$ where $\varphi \simeq \psi$, we say a homotopy H between φ and ψ is base point preserving if $H(x_0, t) = y_0$ for all $t \in [0, 1]$.

Proposition

If $\varphi \simeq \psi$ through a base point preserving homotopy, then $\varphi_* = \psi_*$, $\pi_1(X, x_0) \to \pi_1(Y, y_0)$.
For $X \simeq Y$, $\varphi : X \to Y$ and $\psi : Y \to X$ where $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$, in general $\psi \circ \varphi(x_0) \neq x_0$ and $\varphi \circ \psi(y_0) \neq y_0$.
Set up: $\varphi_0, \varphi_1 : X \to Y$ with $\varphi_0 \simeq \varphi_1$ through a homotopy H.
Write $\varphi_t = H(\cdot, t) : X \to Y$ and fix a base point $x_0 \in X$ and set $\gamma(t) = \varphi_t(x_0)$ for $t \in [0,1]$.

Proposition 1

$(\varphi_0)_* = \Phi_\gamma \circ (\varphi_1)_* : \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$.

Proof

Let f be a loop at x_0.

IMAGE 1

Let γ_t be $\gamma|_{[0,t]}$ and then, by rescaling the domain $[0,t]$ to $[0,1]$ i.e.

$$\begin{aligned} \gamma_t : [0,1] &\to Y \\ s &\mapsto \gamma(ts). \end{aligned}$$

from $\varphi_0(x_0)$ to $\gamma(t) = \varphi_t(x_0)$. Then $\gamma_t \cdot (\phi_t \circ f) \cdot \overline{\gamma}_t$ is a homotopy between $(\varphi_0 \circ f)$ and $\gamma \cdot (\varphi_1 \circ f) \cdot \overline{\gamma}$. Hence

$$(\varphi_0)_*[f] = [\varphi_0 \circ f] = [\gamma] \cdot [\varphi_1 \circ f] \cdot [\overline{\gamma}] = \Phi_\gamma \circ (\varphi_1)_*[f].$$

Proposition 2

If $X \simeq Y$, then $\pi_1(X) \cong \pi_1(Y)$.

Proof

Since $(\psi \circ \varphi) \simeq \mathrm{id}_X$, by Proposition 1

$$\psi_* \circ \varphi_* = (\psi \circ \varphi)_* = \Phi_\gamma \circ (\mathrm{id}_x)_* = \Phi_\gamma.$$

Hence $\psi_* \circ \varphi_*$ is an isomorphism (as is $\varphi_* \circ \psi_*$). Therefre φ_* and ψ_* are isomorphisms.

Recall: Covering Map

For $X, \tilde{X}$ connected, $\pi : \tilde{X} \to X$ is a covering map if for each $p \in X$ there exists a neighborhood $U \subset X$ such that $\pi^{-1}(U)$ is a disjoint union

$$\pi^{-1}(U) = \dot{\bigcup}_{\alpha \in A} U_\alpha$$

such that $\pi|_{U_\alpha} : U_\alpha \to U$ Is a homeomorphism.

Lifting Properties

A lift is a map $\tilde{f}$ such that $f = \pi \circ \tilde{f}$.

1. Path Lifting: Let $f : I \to X$ be a path from x_0. Then, for any $\tilde{x}_0 \in \pi^{-1}(x_0)$, there is a unique lift $\tilde{f}$ of f with $\tilde{f}(0) = \tilde{x}_0$.
2. Homotopy Lifting: Let $f_0, f_1 : I \to X$ be paths in X with $f_0(0) = f_1(0) = x_0$ and $f_0(1) = f_1(1)$. Suppose H is a path homotopy between f_0 and f_1. Then for any $\tilde{x}_0 \in \pi^{-1}(x_0)$, there is a unique lift $\tilde{H} : I \times I \to \tilde{X}$ of H. In particular, $\tilde{H}$ is a path homotopy between $\tilde{f}_0$ and $\tilde{f}_1$. That is if $H(0,t) = x_0$ then $\tilde{H}(0,t) \in \pi^{-1}(x_0)$ for all t. Hence $\tilde{H}(0,t) = \tilde{x}_0$, $\forall\, t \in [0,1]$. Similarly, $\tilde{H}(1,t)$ is identically constant. In particular, $\tilde{f}_0(1) = \tilde{H}(1,0) = \tilde{H}(1,1) = \tilde{f}_1(1)$.

Fundamental Group of the Circle

$\pi_1(S^1) = \mathbb{Z}$.

Example

$\pi : \mathbb{R} \to S^1$ by $s \mapsto e^{2\pi i \cdot s}$.

Proof

Take as a base point $1 = x_0 \in S^1 \subseteq \mathbb{C}$. For each $n \in \mathbb{Z}$, we define a loop $\omega_n : [0,1] \to S^1$ by $s \mapsto e^{2\pi i \cdot ns}$. Let f be a loop at $x_0 \in S^1$. We can lift f to $\tilde{f} : I \to \mathbb{R}$ at $0 \in \mathbb{R}$. Then $\tilde{f}(1) \in \pi^{-1}(x_0) = \mathbb{Z} \subseteq \mathbb{R}$. This defines a map φ that sends a loop f to $\tilde{f}(1) \in \mathbb{Z}$. This φ induces $\varphi : \pi_1(S^1, x_0) \to \mathbb{Z}$ well-defined. If $f_0, f_1 : I \to S^1$ at x_0 are path homotopic via H, then we may lift H to $\tilde{H} : I \times I \to \mathbb{R}$ which implies $\tilde{f}_0(1) = \tilde{f}_1(1)$.
φ is surjective, since for any $n \in Z$ we may consider the loop ω_n where $\tilde{\omega}_n(1) = n$.
φ is a group homorphism since $\varphi[f \cdot g] = \widetilde{f \cdot g}(1) = \tilde{g} + \tilde{f}(1) = \varphi[f] + \varphi[g]$.
φ is injective, since if $\varphi[f] = 0$ (i.e. $\tilde{f}(0) = 0$) then $\tilde{f}$ is a loop in $\mathbb{R}$ and $\tilde{f}$ is null-homotopic to c_0 by H. Therefore $\pi \circ \tilde{H}$ is a path-homotopy between f and c_{x_0} (i.e. $[f] = e$).

Path-Lifting

For $f : I \to X$, we have a special case where $\operatorname{im} f \subseteq U$ evenly covered. Write $\pi^{-1}(U) = \dot{\bigcup} \tilde{U}_\alpha$ and pick the $\tilde{U}_\alpha$ which contains $\tilde{x}_0$. Since $\pi|_{\tilde{U}_\alpha} : \tilde{U}_\alpha \to U$ is a homemorphism, $\tilde{f} := (\pi|_{\tilde{U}_\alpha})^{-1} \circ f$ is the unique lift of f at $\tilde{x}_0$.
In general, pick a partititon of $I = [0,1]$, $0 = t_0 < t_1 < \cdots < t_m = 1$, such that $\operatorname{im} f|_{[t_i, t_{i+1}]} \subseteq U_i$ evenly covered. We can lift $f|_{[0,t_1]}$ at $\tilde{x}_0$, giving $\tilde{f} : [0,t] \to \tilde{X}$. Next, we lift $f|_{t_1,t_2]}$ at $\tilde{f}(t_1) \in \tilde{X}$. Since the partition is finite, we may repeat the process until f is entirely lifted. This lift is unique.

Homotopy Lifting

For each fixed $(y_0, t_0) \in I \times I$, by continuity, there is a neighborhood $N(y_0) \times (t_0 - \varepsilon, t_0 + \varepsilon)$ such that H sends $N(y_0) \times (t_0 - \varepsilon, t_0 + \varepsilon)$ inside an evenly covered neighborhood. By compactness of $\{y_0\} \times [0,1]$, there is a finite collection of $N_{t_i}(y_0) \times (t_i - \varepsilon_i, t_i + \varepsilon_i)$ such that they cover $\{y_0\} \times I$ and the image of each under H is contained in an evenly covered neighborhood. Set $N = \bigcap_i N_{t_i}(y_0)$, a neighborhood of y_0, and construct a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ such that $H(N \times [t_i, t_{i+1}] \subseteq U_i$ evenly covered. Then we can start with $H|_{N \times [0,t_1]}$ and lift it at $\tilde{x}_0$ by some $(\pi|_{\tilde{U}_\alpha})^{-1}$. Then lift each $H|_{N \times [t_i, t_{i+1}]}$ one by one. Eventually, we have $\tilde{H} : N \times [0,1] \to \tilde{X}$ that lifts $H : N \times [0,1] \to \tilde{X}$ at $\tilde{x}_0$. This lift holds for any $y_0 \in I$ and, if two strips overlap, then the lift must agree there by the uniqueness of path lifting. This assures that $\tilde{H} : I^2 \to \tilde{X}$ is continuous.

Remark

Given a continuous map $F : Y \times I \to X$ and a covering $\pi : \tilde{X} \to X$, suppose that we have a map $\tilde{F} : Y \times \{0\} \to \tilde{X}$ that lifts $F|_{Y\times\{0\}} : Y \times \{0\} \to X$. Then there is a unique lift $\tilde{F} : Y \times I \to \tilde{X}$ of F which extends $\tilde{F} : Y \times \{0\} \to \tilde{X}$.

Theorem: Fundamental Theorem of Algebra

A polynomial $p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ (with $a_i \in \mathbb{C}$) has a root in $\mathbb{C}$.

Proof

Suppose otherwise. Then $p(z) \neq 0$, $\forall z \in \mathbb{C}$. Consider $f_r : [0,1] \to S^1$ ($r \geq 0$) by

$$f_r(s) = \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}.$$

Then $f_0(s) \equiv 1$ is a constant loop at $1 \in \mathbb{C}$, and $f_r \simeq f_0$ for each $r \geq 0$. Consider $R \geq 1$ large such that $R \gg \sum_{i=1}^n |a_i|$. On $\{z : |z| = R\}$, we have

$$|z^n| > \left(\sum_{i=1}^n |a_i|\right) \cdot |z^{n-1}| \geq \sum_{i=1}^n |a_i| \cdot |z^{n-i}| = \left|\sum_{i=1}^n |a_i z^{n-i}\right|.$$

This implies that p does not have any roots on $\{|z| = R\}$. Moreover, for $p_t(z) = z^n + t(a_1 z^{n-1} + \cdots a_{n-1} z + a_n)$ with $0 \leq t \leq 1$, p_t does not have any roots on $\{|z| = R\}$. Consider

$$f_{R,t}(s) = \frac{p_t(Re^{2\pi i s})/p_t(R)}{|p_t(Re^{2\pi i s})/p_t(R)|}.$$

Then

$$f_{R,0}(s) = \frac{(Re^{2\pi i s})^n/R^n}{|(Re^{2\pi i s})^n/R^n|} = (e^{2\pi i s})^n = \omega_n(s).$$

Therefore $f_{R,1}(s) \simeq f_R(s)$ and $f_R \simeq \omega_n$. But since $\omega_n \not\simeq$ constant so this is a contradiction.

April 7, 2025

Definition: Retraction

Let X be a space and $A \subseteq X$ be a subset. We say that a continuous map $r : X \to A$ is a retraction if $r|_A = \mathrm{id}_A$. In particular, becasue $r \circ \iota_A = \mathrm{id_A}$, for $x_0 \in A$

$$r_* \circ (\iota_A)_* : \pi_1(A, x_0) \to \pi_1(A, x_0)$$

is an isomorphism. Hence $r_* : \pi(X, x_0) \to \pi(A, x_0)$ is surjective.

Corollary

There is no retraction $r : D^2 \to S^1 (= \partial D^2)$.

Proof

Suppose there is such a map r, then

$$r_* : \overbrace{\pi_1(D^2, x_0)}^{=0} \to \overbrace{\pi_1(S^1, x_0)}^{=\mathbb{Z}}$$

is surjective which is a contradiction.

Corollary

Every continuous map $h : D^2 \to D^2$ has a fixed point.

Proof

Suppose $\exists h : D^2 \to D^2$ without fixed points.

IMAGE 1

Define $r : D^2 \to D^2$ as the ray pictured from $h(x)$ through x to the boundary. If $x \in \partial D^2$, then by construction $r(x) = x$. Hence $r : D^2 \to S^1$ is a retraction which is a contradiction.

Corollary (Borsuk-Ulam)

Let $f : S^2 \to \mathbb{R}^2$. Then there exists a pair of antipodal points x and $-x$ on S^2 such that $f(x) = f(-x)$. This carries analogously to higher dimensions.

Proof

Suppose that $f(x) \neq f(-x)$ for all $x \in S^2$. We define $g : S^2 \to S^1$ by $g(x) = \frac{f(x)-f(-x)}{||f(x)-f(-x)||}$. On $S^2 \subseteq \mathbb{R}^3$, we consider a loop γ at the equator by $\gamma(s) = (\cos(2\pi s), \sin(2\pi s), 0)$ for $s \in [0,1]$. Because S^2 is simply connected, $g \circ \gamma : [0,1] \to S^1$ is path-homotopic to a constant loop in S^1. On the other hand, we lift $h := g \circ \gamma$ to $\tilde{h} : [0,1] \to \mathbb{R}$ with $\tilde{h}(0) = 0 \in \mathbb{R}$. Note

$$h(s+1/2) = g \circ \gamma(s+1/2) = g(\cos(2\pi s + \pi), \sin(2\pi s + \pi), 0) = g(-\gamma(s)) = -g(\gamma(s)) = -h(s).$$

Hence $\tilde{h}(s+1/2) \in \pi^{-1}(-h(s))$ where $\pi : \mathbb{R} \to S^1$ is the covering map. Since $\pi^{-1}(-h(s)) = \frac{1}{2} + \tilde{h}(s) + \mathbb{Z}$, for each $s \in [0,1/2]$ there is an integer q_s such that $\tilde{h}(s+1/2) = \frac{1}{2} + \tilde{h}(s) + q_s$ and

$$\tilde{h}(s+1/2) - \tilde{h}(s) = \frac{1}{2} + q_s.$$

The left hand side depends continuously on s and, by continuity, q_s is a constant (call it q). This gives

$$\tilde{h}(1) = \tilde{h}(1/2) + \frac{1}{2} + q = \tilde{h}(0) + 1 + 2q = 1 + 2q \neq 0$$

which contradicts the assertion that h is homotopic to a constant loop.

Corollary (Large Fiber Lemma)

If $f : [0,1]^{n+1} \to \mathbb{R}^n$ is a continuous map, then there exist $a, b \in [0,1]^{n+1}$ such that $f(a) = f(b)$ and $|a - b| \geq 1$.
Remark: if $z = f(a) = f(b)$, then the lemma says that $\operatorname{diam} f^{-1}(z) \geq 1$.

Proof

Take the sphere of radius $1/2$ in $[0,1]^{n+1}$, then by Borsuk-Ulam there exist a pair of antipodal points $a, b \in S^1$ such that $f(a) = f(b)$ and $|a-b| \geq 1$.

Proposition

$\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$

Proof

Write $F : \pi_1(X \times Y) \to \pi_1(X) \times \pi_1(Y)$ by $[f] \mapsto ([g],[h])$. Then $f : [0,1] \to X \times Y$ is a loop at (x_0, y_0), $f(s) = (g(s), h(s))$, and $g : [0,1] \to X$ and $h : [0,1] \to Y$ are loops at x_0 and y_0 respectively.

Definition: Wedge Sum

Let X and Y be path-connected topological spaces. Then $X \vee Y = (X \coprod Y)/x_0 \sim y_0$
Let $\{X_\alpha\}$ be a family of such spaces. Then $\bigvee_\alpha X_\alpha = \coprod_\alpha X_\alpha / \sim$.

Sketch

$$\begin{aligned} \pi_1(S^1_-, x_0) \to \pi_1(X, x_0) \quad \text{gen} \mapsto \alpha \\ \pi_1(S^1_+, x_0) \to \pi_1(X, x_0) \quad \text{gen} \mapsto \beta \end{aligned}$$

with $\alpha \neq \beta$, $\alpha\beta \neq \beta\alpha$. Then $\pi_1(X, x_0)$ should be $\langle \alpha, \beta \rangle$.

Definition: Free Product

Let $\{G_\alpha\}_\alpha$ be a family of groups. $*_\alpha G_\alpha = \{g_1 g_2 \cdots g_k \,:\, \text{each } g_i \text{ is a word in some } A_\alpha\}$.

Proposition

If for each α, there is a group homomorphism $\phi_\alpha : G_\alpha \to H$ then $\{\phi_\alpha\}$ induces a group homomorphism $\Phi : *_\alpha G_\alpha \to H$ by $g_1 \cdots g_k \mapsto \phi_{\alpha_1}(g_1) \cdots \phi_{\alpha_k}(g_k)$.

Van-Kapen Theorem

Setup

Let $X = \bigcup_\alpha A_\alpha$, each A_α open and connected where $\{A_\alpha\}$ have a common point x_0. Assume also that each $A_\alpha \cap A_\beta$ is path connected. Then $j_\alpha : A_\alpha \hookrightarrow X$ induces $j_\alpha : \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$. $\{j_\alpha\}_\alpha$ induces $\Phi : *_\alpha \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$ which is surjective by a similar argument as was used above for Example (*) ($S^2 = A_- \cup A_+$) applied to $X = \bigcup_\alpha A_\alpha$. Now, what is the kernel of Φ?

$$\begin{tikzcd} & A_\alpha \arrow[dr, hook] & \\ A_\alpha \cap A_\beta \arrow[ur, hook] \arrow[dr, hook] & & X \\ & A_\beta \arrow[ur, hook] & \end{tikzcd}$$

$$
\begin{tikzcd}
 & \pi_1(A_\alpha) \arrow[dr, hook] & \\
\pi_1(A_\alpha \cap A_\beta) \arrow[ur, hook, "i_{\alpha\beta}"] \arrow[dr, hook, "i_{\beta\alpha}"'] & & \pi_1(X) \\
 & \pi_1(A_\beta) \arrow[ur, hook] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \_\_ \arrow[dr, hook] & \\
w \arrow[ur, hook] \arrow[dr] & & \_\_ \\
 & \_\_ \arrow[ur, hook] &
\end{tikzcd}
$$

Then $i_{\beta\alpha}(w)i_{\alpha\beta}(w)^{-1}$ is NOT id in $*_\alpha \pi_1(A_\alpha)$. But through Φ, it hsould be $\mathrm{id} \in \pi_1(X, x_0)$. Hence every element in $*_\alpha \pi_1(A_\alpha)$ of the form $i_{\beta\alpha}(w)i_{\alpha\beta}(w)^{-1}$ where $w \in \pi_1(A_\alpha \cap A_\beta$ is in the kernel of Φ.

Theorem (Van-Kampen)

If every $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected, $\ker \Phi$ is the normal subgroup N generated by $\{i_{\beta\alpha}(w)i_{\alpha\beta}(w)^{-1} : \alpha, \beta \in A,\ w \in \pi_1(A_\alpha \cap A_\beta)\}$. Hence $\pi_1(X, x_0) \cong (*_\alpha \pi_1(A_\alpha, x_0))/N$.

Remarks

1. In the case that $X = A_0 \cup A_1$ with $A_0 \cap A_1$ path connected, then the intersection condition holds.
2. If $X = A_0 \cup A_1$ and $A_0 \cap A_1$ is simply connected, then $N = \{\mathrm{id}\}$ and $\pi_1(X) = \pi_1(A_0) * \pi_1(A_1)$.
3. If $X = A_0 \cup A_1$ and A_1 is simply connected, then $\pi_1(X) = \pi_1(A_0)/N$ and N is the normal subgroup generated by

$$
i_{01}(w)\overbrace{i_{10}(w)^{-1}}^{\in \pi_1(A_1, x_0)} = i_{01}(w)
$$

   i.e. N is the normal closure of $i_{01}(\pi_1(A_0 \cap A_1))$.

Example

IMAGE 2

For each $\alpha \in \{1, \ldots, 5\}$, let A_α be a small neighborhood of $T \cup e_1$. Every double/triple intersection is a neighborhood of T. Hence it is path continuous and we have that $\pi_1(A_\alpha) = \mathbb{Z}$. Thus $\pi_1(A_\alpha \cap A_\beta) = \mathrm{id}$, and $\pi_1(X) = *_\alpha \pi_1(A_\alpha)/N = *_1^5 \mathbb{Z}$.

Example

IMAGE 3

By Van-Kampen, $\pi_1(X) = \pi_1(A_0)$ modulo the normal closure of $i(\pi_1(A_0 \cap A_1))$. That is

$$
\left\langle a, b \mid aba^{-1}b^{1} \right\rangle = \mathbb{Z}^2.
$$

Remark

In general, orientable M_g is the connected sum of g many toruses.

April 9, 2025

Recall: Van-Kampen Theorem

Write $\pi_1(X) = (\pi_1(A) * \pi_1(B))/N$ where N is the normal closure of $i_{\alpha\beta}(w)i_{\beta\alpha}(w)^{-1}$ for $w \in \pi_1(A \cap B)$, $i_{\alpha\beta} : \pi_1(A \cap B) \to \pi_1(A)$ and $i_{\beta\alpha} : \pi_1(A \cap B) \to \pi_1(B)$.

Example

M_g is the connected sum of g many tori, and $\pi_1(M_g) = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1 b_1]\cdots[a_g b_g]\rangle$.

Example

N_g is the connected sum of g many $\mathbb{RP}^2$ (e.g. N_2 is the Klein bottle). N_g has a polygon-representation by the $2g$-gon with boundary identified through $a_1 a_1 a_2 a_2 \cdots a_g a_g$. Therefore $\pi_1(N_g) = \left\langle a_1 \cdots a_g \mid a_1^2 \cdots a_g^2 \right\rangle$.

Abelianiztion

1. $\mathrm{Ab}(\pi_1(M_g))$ is the free abelian group generated by $\{a_1, b_1, \ldots, a_g, b_g\} = \mathbb{Z}^{2g}$.
2. $\mathrm{Ab}(\pi_1(N_g)) = \mathrm{Ab}(\left\langle a_1 \cdots a_{g-1} a_1 a_2 \cdots a_g \mid a_1^2 \cdots a_g^2 \right\rangle = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$.

Corollary

None of the surfaces in $\{S^2, M_1, \ldots, M_g, \ldots, N_1, \ldots, N_g, \ldots\}$ are homotopic to each other.

Definition: Cell Complex

0-cells are points; 1-cells, e^1, are intervals; 2-cells, e^2, are disks; n-cells, e^n, are $\overline{B}^n$.
A cell complex for space X is a decomposition (assuming finite dimensions) $X = X^0 \cup X^1 \cup \cdots \cup X^n$ where X^0 is the discrete set of points (i.e. 0-cells), X^1 is the space obtained by gluing 1-cells to X^0 ($\varphi_\alpha : \partial e_\alpha^1 \to X^0$), X^2 is the space obtained by gluing 2-cells to X^1 ($\varphi_\alpha : \partial e_\alpha^2 \to X^1$), and in general X^n is obtained by gluing n-cells $\{e_\alpha^n\}_\alpha$ to X^{n-1} by $\varphi_\alpha : \partial e_\alpha^n = S^{n-1} \to X^n$.

Examples

Cell complexes need not be unique. $S^2 = X^1 \cup_\alpha e_+^2 \cup_\alpha e_-^2$ and $S^2 = \{e^0\} \cup_\alpha \{e^2\}$.
$\mathbb{RP}^2 = \{e^1\} \cup_\alpha \{e^2\}$ where φ_α is given by $z \mapsto z^2$.
$\mathbb{T}^2$ is gluing e^2 to $S^1 \vee S^1$.

Theorem (Computing Fundamental Group)

Set up

Let X be a path-connected space, $Y = X \cup_\alpha e_\alpha^2$ (i.e. X is created by gluing 2-cells $\{e_\alpha^2\}_\alpha$ to X via $\phi_\alpha : \partial e_\alpha^2 \to X$). The inclusion $\iota : X \to Y$ induces $\iota_* : \pi_1(X) \to \pi_1(Y)$. Fix a base point $s_0 \in S^1$. For each α we draw a path γ_α from x_0 to $\varphi_\alpha(s_0)$. Then $\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha$ is a loop based at x_0. Thus $\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha$ is null-homotopic in Y (because φ_α is null-homotopic in e_α^2). That is $\iota_*[\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha] = \mathrm{id}$ in $\pi_1(Y)$ and is therefore in the kernel.

Theorem

Let N be the normal subgroup in $\pi_1(X)$ generated by elements of the form $[\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha]$. Then $\pi_1(Y) \cong \pi_1(X)/N$.

IMAGE 1

Example

$\mathbb{RP}^2$ is X^1 with e^2 glued to it by the map $\varphi : z \mapsto z^2$. Then $\pi_1(\mathbb{RP}^2) = \pi_1(S^1)/N = \left\langle \gamma \mid \gamma^2 \right\rangle$ where N is generated by φ. Similarly, the theorem applies to any M_g or N_g.

Definition: Deformation Retraction

For $X \subseteq Z$, $r : Z \to X$ is a retraction if $r|_X = \mathrm{id}_X$ implies $r \circ \iota = \mathrm{id}_X$. If $\iota \circ r : Z \to Z$ is homotopic to id_X, then $r_* : \pi_1(Z) \to \pi_1(X)$ is an isomorphism.

Proof

For each α, we glue a strip S_α along γ_α. We set the base at z_0 above x_0, $Z = Y \cup_\alpha S_\alpha$. Y is a deformation retraction of Z ($\pi_1(Y) = \pi_1(Z)$).

IMAGE 2

Set $A = Z - \bigcup_\alpha \{y_\alpha\}$, where y_α is a point in e^2_α not intersecting S_α. $B = Z - X$. A deformation retracts to X $\pi_1(A) = \pi_1(X)$. B is the union of some S_α (removing r_α) and some e^2_α (removing ∂e^2_α). B is contractible, $\pi_1(B) = \mathrm{id}$ and $A \cap B$ is the union of strips S_α and open disks punctured at y_α. Therefore

$$\pi_1(Y) = \pi_1(Z) = (\pi_1(A) * \pi_1(B))/N = \pi_1(A)/\iota_*(\pi_1(A \cap B)) \cong \pi_1(X)/\iota_*(\pi_1(A \cap B)).$$

Consider the loop $\delta_\alpha \cdot \gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha \cdot \overline{\delta}_\alpha$ where δ_α runs from z_0 to x_0, call this λ_α. It suffices to show that these generate $\pi_1(A \cap B, z_0)$. Cover $A \cap B$ by $A_\alpha = (A \cap B) - \bigcup_{\beta \neq \alpha} e^2_\beta$. Then A_α is a union of strips (with trivia fundamental group) and a single punctured, open disk $e^2_\alpha - \{y_\alpha\}$ and $\pi_1(A_\alpha) = \mathbb{Z} = \langle \lambda_\alpha \rangle$. So $A_\alpha \cap A_\beta$ is the union of strips, equal to $A_\alpha \cap A_\beta \cap A_\gamma$ and simply connected. By Van-Kampen,

$$\pi_1(A \cap B) = (*_\alpha \pi_1(A_\alpha))/N = *_\alpha \pi_1(A_\alpha)$$

is the free group generated by $\{\lambda_\alpha\}_\alpha$. This completes the proof.

Generalization (Theorem: Part 2)

If $Y = X \cup_\alpha e^n_\alpha$ for $n \geq 3$, then $\pi_1(Y) \cong \pi_1(X)$.
This follows from the same argument where instead A_α is the union of strips and a single punctured ball $B^n - \{y_\alpha\} \simeq S^{n-1}$. So $\pi_1(A_\alpha) = \mathrm{id}$, $\pi_1(A \cap B) = \mathrm{id}$, and $\pi_1(X) \cong \pi_1(Y)$.

Theorem: Part 3

Suppose X has a cell complex $X = X^0 \cup X^1 \cup \cdots \cup X^n$. Then $\pi_1(X) \cong \pi_1(X^2)$.
The proof follows directly from part 2.

Corollary

Given any group represented by generators and relations $G = \left\langle g_\alpha \mid r_\beta \right\rangle$, there is a cell complex X_G, of dimension 2, such that $\pi_1(X_G) \cong G$.

Proof

For each g_α, we draw a circle S^1_α. Then $X^1 = \bigvee_\alpha S^1_\alpha$ has fundamental group $*_\alpha \pi_1(S_\alpha) = \langle g_\alpha \rangle_\alpha$. To construct X_G, for each r_β glue a 2-cell e^2_α along r_β (think of r_β as a loop in X^1). Then in $X_G := X^1 \cup_\beta e^2_\beta$ we have $\pi_1(X_G) = \langle g_\alpha \mid r_\beta \rangle$.

April 14, 2025

Recall: Covering Spaces

Let $p : \tilde{X} \to X$, both X and $\tilde{X}$ path-connected.

1. Path-lifting: let $f : I \to X$ starting at $f(0) = x_0$. There is a unique lifting $\tilde{f}$ of f at $\tilde{x}_0 \in p^{-1}(x_0)$.
2. Homotopy-lifting: let $f_0, f_1 : I \to X$ be two paths with $f_0(0) = f_1(0) = x_0$ and $f_0(1) = f_1(1)$. Suppose f_t is a path-homotopy between f_0 and f_1. Then there exists a unique lift $\tilde{f}_t$ between $\tilde{f}_0$ and $\tilde{f}_1$ at $\tilde{x} \in p^{-1}(x)$.

These come from the following: let $f_t : Y \to X$ be a homotopy between f_0 and f_1. Given $\tilde{f}_0 : Y \to \tilde{X}$ that lifts f_0, there exists a unique lifting $\tilde{f}_t$. For path-lifting, we take Y a point; for homotopy-lifting, $Y = [0, 1]$.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ I \arrow[ur, "f"] \arrow[r, "p \circ f"'] & X \end{tikzcd}$$

Proposition 1.31 (in Hatcher)

The covering map $p : \tilde{X} \to X$ induces $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x)$.

1. p_* is injective.
2. $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ are exactly loops at x_0 that lift to loops at $\tilde{x}_0$.

Proof of 1

Suppose $p_*[f] = \mathrm{id} \in \pi_1(X, x_0)$ where $[f] \in \pi_1(\tilde{X}, \tilde{x}_0)$. Then $[p \circ f] = \mathrm{id}$, and $[p \circ f]$ is path-homotopic to the constant loop c_{x_0}. Hence the lifting $\widetilde{p \circ f} = f$ is path-homotopic to a constant loop $c_{\tilde{x}_0}$.

Proof of 2

Let $[f] \in \pi_1(\tilde{X}, \tilde{x}_0)$. $p_*[f] = [p \circ f]$, $p \circ f$ lifts to f at $\tilde{x}_0$ which is a loop at $\tilde{x}_0$.
Let f be a loop at x_0. Suppose f lifts to a loop $\tilde{f}$ at $\tilde{x}_0$ (i.e. $p \circ \tilde{f} = f$). Hence $[f] = [p \circ \tilde{f}] = p_*[\tilde{f}] \in p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

Example

If $p : S^1 \to S^1$ by $z \to z^2$, then $p_*(\pi_1(S^1, 1)) = 2\mathbb{Z} \le \mathbb{Z} = \pi_1(S^1, 1)$.

Remark

If $p : \tilde{X} \to X$ connected, then $p^{-1}(x)$ has the same cardinality for all $x \in X$.

Proof

Fix $x_0 \in X$. Consider $\mathcal{A} = \{x \in X \;:\; p^{-1}(x) \text{ and } p^{-1}(x_0) \text{ have the same cardinality}\} \neq \varnothing$. Then $\mathcal{A}$ is open since for each $x \in \mathcal{A}$, there is a neighborhood U of x such that U is evenly covered by p (i.e. $p^{-1}(U) = \dot{\bigcup}_{\alpha \in I} V_\alpha$ where $V_\alpha \overset{p}{\cong} U$). Then $p^{-1}(x')$ has cardinality $|I|$ for all $x' \in U$. It follows, since $\mathcal{A}^c$ is open, that $\mathcal{A}$ is also closed.

Proposition

The number of sheets is given by $[\pi_1(X, x_0) : p_*(\pi_1(\tilde{X}, \tilde{x}_0))]$.

Proof

Write $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Define $\Phi : \{H\text{-cosets in } \pi_1(X, x_0)\} \to p^{-1}(x_0)$ by $H[g] \mapsto \tilde{g}(1)$ where $\tilde{g}$ is a lift of g at $\tilde{x}_0$.
This map is well defined, since for $[h \cdot g]$ with $h \in H$, $\widetilde{h \cdot g}(1) = \tilde{g}(1)$ (because $\tilde{h}(1) = \tilde{x}_0$).
Φ is surjective. Let $\tilde{x}_1 \in p^{-1}(x_0)$

IMAGE 1

and let $\tilde{g}$ be a path from $\tilde{x}_0$ to $\tilde{x}_1$. Define $g = p \circ \tilde{g}$, a loop at x_0. Then $\Phi(H[g]) = \tilde{g}(1) = \tilde{x}_1$.
Φ is injective. Suppose $\Phi(H[g_1]) = \Phi(H[g_2])$ (i.e. $\tilde{g}_1(1) = \tilde{g}_2(1)$.

IMAGE 2

Consider the loop $g_1\overline{g}_2$ in X at x_0. It lifts to $\tilde{g}_1\overline{\tilde{g}}_2$, which is a loop at $\tilde{x}_0$. This shows that $[g_1\overline{g}_2] \in H$ (i.e. $H[g_1] = H[g_2]$).

Recall (Manifolds 2)

If a smooth manifold M is non-orientable, then there is a double cover (2 sheets) $p : \hat{M} \to M$ ($\hat{M}$ connected). Consequently, $\pi_1(M)$ has a subgroup of index 2.

Definition: Locally Path-Connected

A topological space is called locally path-connected if for each $x \in X$ and every neighborhood $U \ni X$, there is a neighborhood $V \ni x$ such that $V \subseteq U$ and V is path-connected (i.e. $\forall x \in X$, there exists a local basis $\{U_\alpha\}$ at x such that each U_α is path-connected). For example, the Topologist's sine curve with endpoints identified is path-connected but not locally path-connected.

Proposition: Lifting Criterion

Let Y be path-connected and locally path-connected. Given a covering map $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and a map $f : (Y, y_0) \to (X, x_0)$, f has a lift $\tilde{f}$ at $\tilde{x}_0$ ($\tilde{f}(y_0) = \tilde{x}_0$) if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

Proof

$(\Longrightarrow)$

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & X \end{tikzcd} \qquad \begin{tikzcd} & \pi(\tilde{X}) \arrow[d, "p_*"] \\ \pi_1(Y) \arrow[ur, "\tilde{f}_*"] \arrow[r, "f_*"] & \pi_1(X) \end{tikzcd}$$

$$f_*\pi_1(Y) = (p_* \circ \tilde{f}_*)(\pi_1(Y)) \leq p_*\pi_1(\tilde{X}).$$

$(\Longleftarrow)$ Let $y \in Y$, and draw a path γ from y_0 to y.

IMAGE 3

We lift $f \circ \gamma$ to a path in $\tilde{X}$ starting at $\tilde{x}_0$ and define $\tilde{f}(y)$ as the endpoint (i.e. $\tilde{f}(y) = \widetilde{f \circ \gamma}(a)$).
This is well-defined, since $(f \circ \gamma) \cdot (f \circ \overline{\gamma}')$ is a loop at x_0 and $[(f \circ \gamma) \cdot (f \circ \overline{\gamma}')] = f_*[\gamma \cdot \overline{\gamma}'] \in f_*\pi_1(Y, y_0) \leq p_*\pi_1(\tilde{X}, \tilde{x}_0)$. Hence $(f \circ \gamma) \cdot (f \circ \overline{\gamma}')$ lifts to a loop at $\tilde{x}_0$.

IMAGE 4

Therefore $\widetilde{f \circ \gamma}(1) = \widetilde{f \circ \gamma'}(1)$.
$\tilde{f}$ is continuous. Fix $f(y) \in X$ and let U be a neighborhood of $f(y)$ that is evenly covered by p. Choose a path-connected neighborhood V of y such that $f(V) \subseteq U$. We check $\tilde{f}|_V$.

IMAGE 5

Because V is path-connected, we may draw a path η in V from y to y'. Then $\tilde{f}(y') = \widetilde{f \circ \gamma \circ \eta}(1)$, and $\widetilde{\gamma \cdot \eta}$ is first lifting $f \circ \gamma$ at $\tilde{x}_0$ followed by lifting $f \circ \eta$ at $\tilde{\gamma}(1)$. Let $\tilde{U} \subseteq \tilde{X}$ such that $p|_{\tilde{U}} : \tilde{U} \to U$ is a homeomorphism and $\widetilde{f \circ \gamma}(1) \in \tilde{U}$. Then $\widetilde{f \circ \eta}(1) = (p^{-1})|_U \circ f(y')$. Hence $\tilde{f}(y') = f \circ \widetilde{(\gamma \circ \eta)}(1) = \widetilde{f \circ \eta}(1) = (p^{-1})|_U \circ f(y')$ (i.e. $\tilde{f} = (p^{-1})|_U = f$ on V). Hence $\tilde{f}$ is continuous at y.
$\tilde{f}$ is a lift of f. In fact, $(p \circ \tilde{f})(y) = p \circ (\widetilde{f\gamma}(1)) = f(y)$.

Corollary

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

If Y is simply connected, then $f_*\pi_1(Y) \leq p_*\pi_1(\tilde{X})$ always holds (i.e. we can always lift f to $\tilde{f} : Y \to \tilde{X}$ in this case).

Proposition: Unique Lifting

Given $p : \tilde{X} \to X$ and $f : Y \to X$, if two lifts $\tilde{f}_1$ and $\tilde{f}_2$ of f agree at one point, then the agree everywhere on Y.

Proof

Take $\mathcal{A} = \{y \in Y \ : \ \tilde{f}_1(y) = \tilde{f}_2(y)\} \neq \varnothing$. Locally for each $y \in Y$ there exists a neighborhood V of y such that $\tilde{f} = (p^{-1})|_U \circ f$. If $y \in \mathcal{A}$, then $\tilde{f}_1(y) = \tilde{f}_2(y)$. Take a neighborhood U of $f(y)$ that is evenly covered and $\tilde{U}$ of $\tilde{f}_1(y) = \tilde{f}_2(y)$ such that $p|_{\tilde{U}} : \tilde{U} \to U$ is a homeomorphism. Then on V, a path-connected neighborhood such that $f(V) \subseteq U$, $\tilde{f}_i = (p^{-1})|_U \circ f$ (i.e. $\tilde{f}_1 = \tilde{f}_2$ on V). If $y \in \mathcal{A}^c$, $\tilde{f}_1(y) \neq \tilde{f}_2(y)$. Then $\tilde{U}_i \ni \tilde{f}_i(y)$ with $\tilde{U}_1 \cap \tilde{U}_2 = \varnothing$. Then on V, $\tilde{f}_i = (p^{-1})|_{\tilde{U}_i} \circ f$ (ie $\tilde{f}_1$ and $\tilde{f}_2$ never agree on V). Hence $\mathcal{A} = Y$.

Remark

If $p : \tilde{X} \to X$ is a covering map, recall that a covering transformation is a map $f : \tilde{X} \to \tilde{X}$ such that

$$\begin{tikzcd} \tilde{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

commutes. This $f : \tilde{X} \to \tilde{X}$ is a lift of $p : \tilde{X} \to X$. If we fix $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$, the lifting criterion says that $p_*\pi_1(\tilde{X}, \tilde{x}_1) \leq p_*\pi_1(\tilde{X}, \tilde{x}_2)$. In particular, if $\pi_1(\tilde{X})$ is simply connected, then this holds. Hence there is a unique lift of p (i.e. covering transformation) f such that $f(\tilde{x}_1) = \tilde{x}_2$.