

# Manifolds I

September 26, 2024

## Class Organization

1 Takehome Midterm

1 Takehome Final

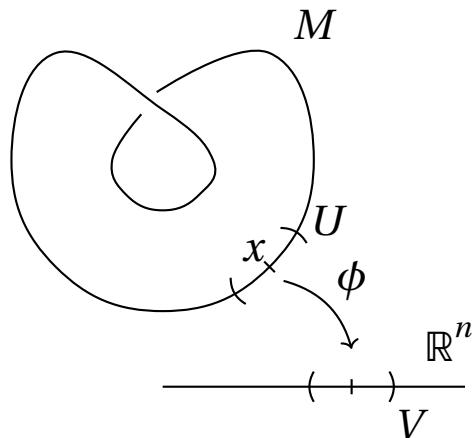
Homeworks assigned, but not graded.

<https://ginzburg.math.ucsc.edu/teaching/208manifolds1-2024/syl.html>

## Definition: Topological Manifolds

For  $M$  a topological space,  $M$  is a topological manifold if  $\forall x \in M, \exists M \ni U \ni x$  and homeomorphism  $\phi: U \rightarrow V \subset \mathbb{R}^n$  for  $V$  open.

To avoid problems (see below), further assume that  $M$  is Hausdorff and second countable.

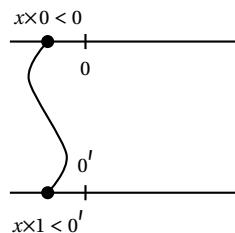


## Exercise

We can require  $V$  to be an open ball.

## Problems

- $M$  need not be Hausdorff.



With  $(\mathbb{R} \times 0 \coprod \mathbb{R} \times 1) / \sim$ .

- $M$  need not be second countable.

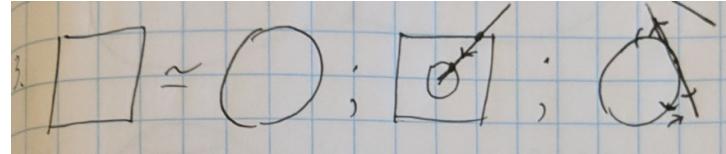
Take  $\coprod_S \mathbb{R}_S$  where  $S$  is an uncountable index.

## Examples

### Example 1

If  $N \underset{\text{homeo}}{\simeq} M$ , this implies  $N$  is a manifold.

### Example 2



### Example 3

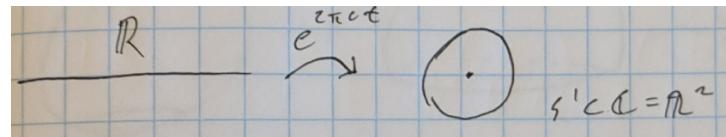
An open subset of a manifold is a manifold.

### Example 4

$M, N$  manifolds implies  $M \times N$  is a manifold.

### Example 5

Take  $\mathbb{R}/\mathbb{Z}$  by the equivalence relation  $t \sim t'$  iff  $t' - t \in \mathbb{Z}$ .



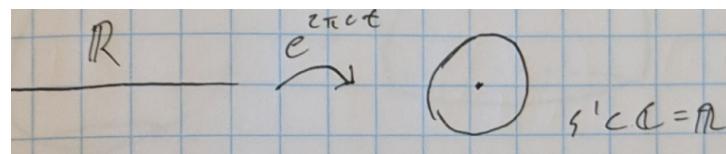
Then  $C^0(S^1)$  relates to periodic functions with period 1.

### Example 6

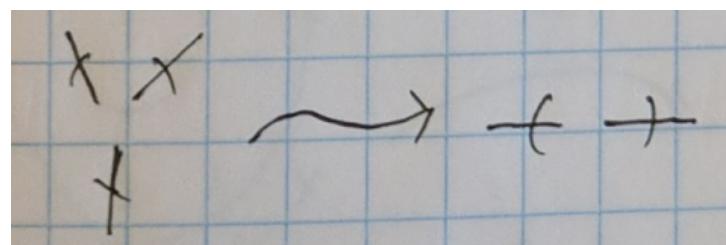
$$\mathbb{T}^n = S^1 \times \cdots \times S^1.$$

### Counterexample 1

$[0, 1]$  is not a manifold.

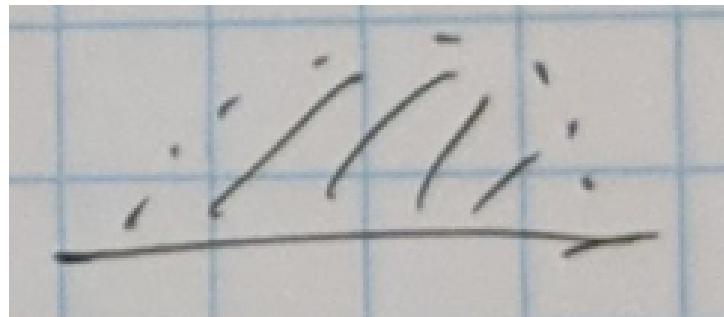


Since 0 must map somewhere in the open interval, its deletion results in a connected space in the former case but a disconnected one in the latter. Similarly, the following breaks into three and two connected components respectively.



## Definition: Manifold with Boundary

There exists a neighborhood  $\forall x \in M$  homeomorphic to either the open ball or the half-closed half-ball.



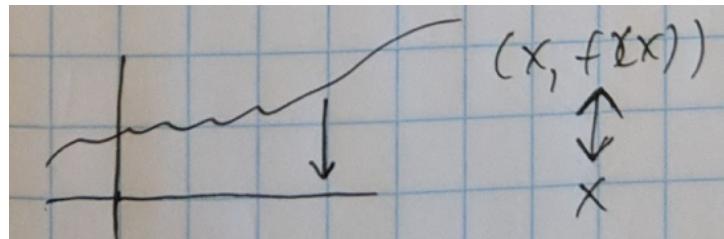
## Exercise

A connected manifold is path-connected.

## Examples

### Example 7

Take  $f : \mathbb{R}^n \xrightarrow{C^0} \mathbb{R}$  with graph  $\Gamma_f = \{(x, f(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^n \times \mathbb{R}$ .

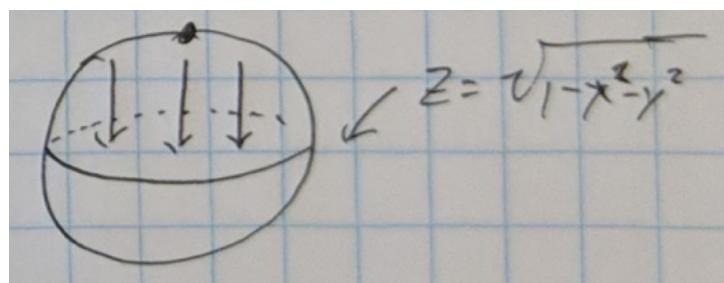


### Example 8

Take  $f : M \rightarrow N$  between manifolds, then  $M \simeq \Gamma_f \subseteq M \times N$ .

### Example 9

$S^n \subset \mathbb{R}^{n+1}$ .



## Definition: Real Projective Spaces

Take  $\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim$  where  $x \sim y \iff x = \lambda y$  for  $\lambda \neq 0$ .  
Informally, the collection of lines through the origin.

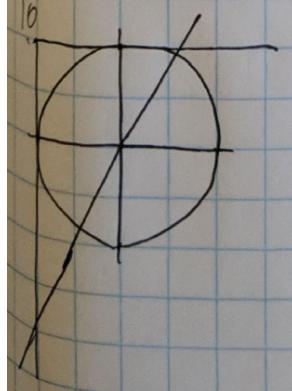
Alternatively,  $\mathbb{RP}^n = S^n / \sim$  where  $x \sim -x$ .

That is, identifying the antipodal points of the unit sphere.

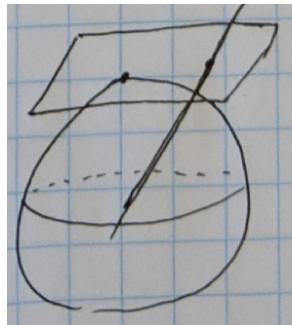
We may also consider  $\mathbb{RP}^n = SO(n+1)/SO(n)$ .

### Claim

$\mathbb{RP}^n$  is a manifold.



$$\mathbb{RP}^1 \setminus \{x\text{-axis}\} \xrightarrow{\text{homeo}} \mathbb{R}.$$



$$\mathbb{RP}^2 \setminus \mathbb{RP}^1 \xrightarrow{\text{homeo}} \mathbb{R}^2$$

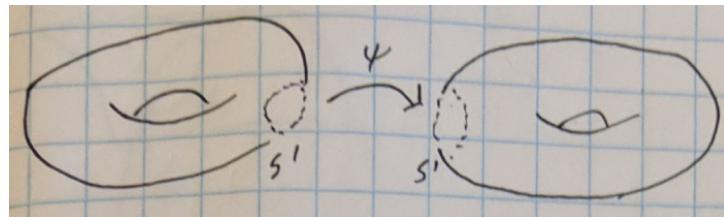
We have that  $\mathbb{RP}^1$  is homeomorphic to the circle, and  $\mathbb{RP}^n = \mathbb{RP}^{n-1} \cup B^n$ .

Take  $x = (x_0, \dots, x_n)$ ,  $y = (y_0, \dots, y_n) = (\lambda x_0, \dots, \lambda x_n)$  and  $[x] = [x_0 : x_1 : \dots : x_n]$ .

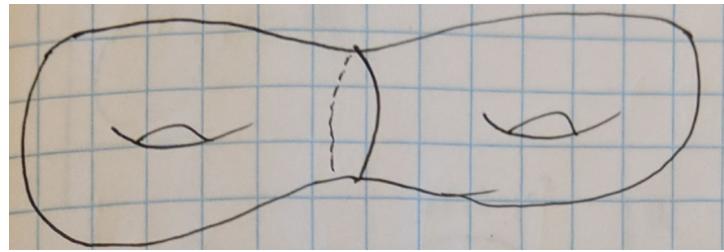
Then for  $U_k \subset \mathbb{RP}^n$  with  $U_k = \{[x] : x_k \neq 0\}$ , we have that  $U_0, \dots, U_n$  covers  $\mathbb{RP}^n$ .

Then define  $U_k \rightarrow \mathbb{R}^n$  by  $[x_0 : \dots : x_n] \mapsto \left( \frac{x_0}{x_k}, \dots, \frac{x_{k-1}}{x_k}, \frac{1}{x_k} \right)$ .

### Connected Sum of Manifolds

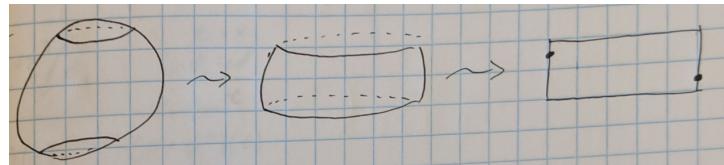


$$M \setminus B^n \coprod N \setminus B^n$$



$M \# N$ .

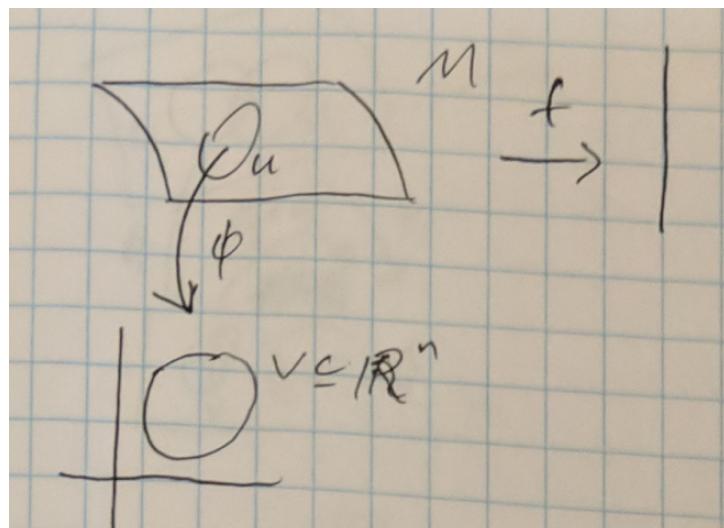
## Möbius Band



October 1, 2024

## A Failed Definition

$$f \in C^{r \geq 1}; f \circ \phi^{-1} : V \xrightarrow{C^r} \mathbb{R}.$$



## Example

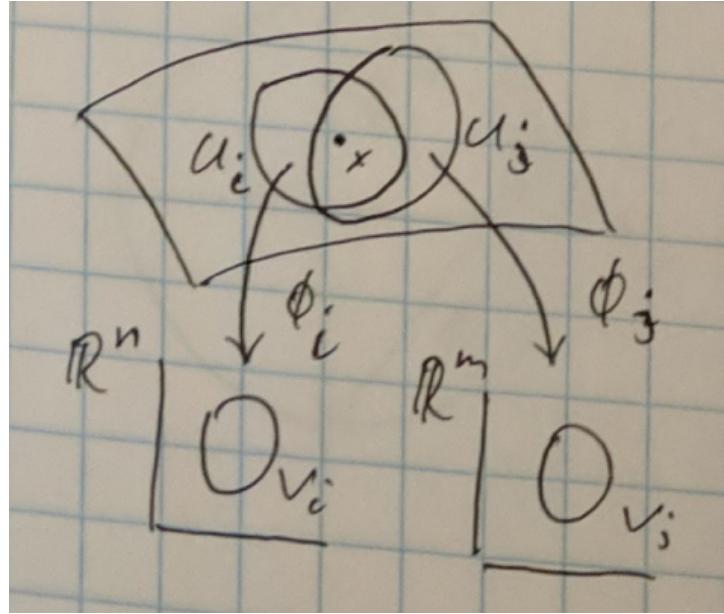
$$\begin{array}{c}
 2. \quad M = \mathbb{R} \xrightarrow{x} \mathbb{R} \\
 \xrightarrow{t = x^3} \phi_2 \quad \xrightarrow{t = x} \phi_1 = id \\
 t \quad \quad \quad t
 \end{array}$$

$f(x) = x^2$   
 $(f \circ \phi_2^{-1})(t) = t^2$   
 $\text{Not } C^1$   
 $\text{cln } C^\infty$

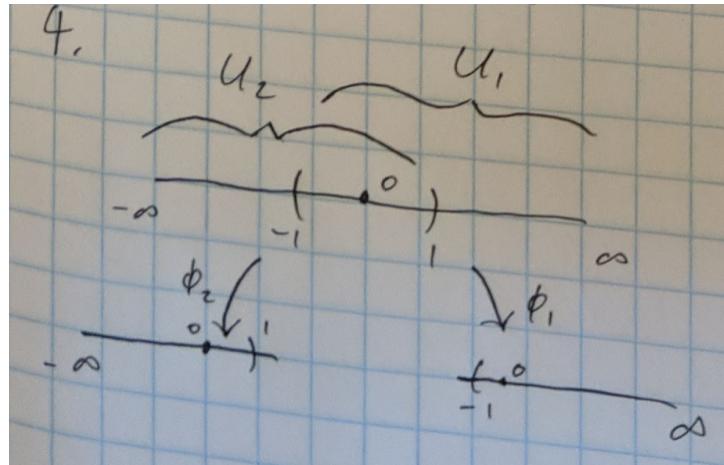
## Definition: Charts

Say there exists a cover  $U_i$  by open sets and  $U_i \xrightarrow{\phi_i} V_i \subseteq \mathbb{R}^n$  fixed.  
Then the pair  $(U_i, \phi_i)$  is a chart.

### What if a point belongs to two charts?



With  $f$  smooth at  $x$ ,  $f \circ \phi_i^{-1}$  smooth at  $\phi_i(x)$  and  $f \circ \phi_j^{-1}$  smooth at  $\phi_j(x)$ .

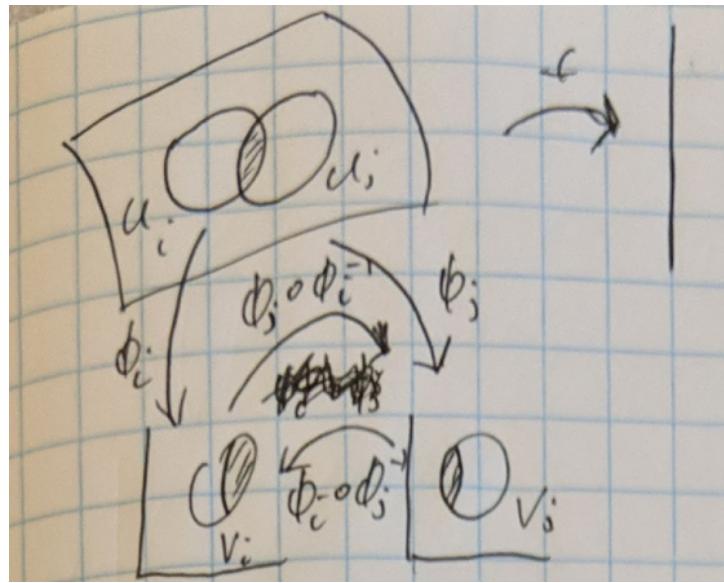


## Notation

The notation  $C^r$  will be used interchangably with the term smooth.

## Definition: Smooth Atlas

Let  $M$  be a topological manifold. A smooth atlas on  $M$  is a cover  $(U_i, \phi_i : U_i \xrightarrow{\sim} V_i \subset \mathbb{R}^n)$  where  $\phi_j \circ \phi_i^{-1}$  and  $\phi_i \circ \phi_j^{-1}$  are smooth for every  $i$  and  $j$ .



Say that the charts are (smooth) compatible.

## Definition: Smooth Function

Say that  $f$  is smooth at  $x \in M$  if there exists a chart  $U_i \ni x$  such that  $f \circ \phi_i$  is smooth at  $\phi_i(x)$ . Equivalently, if for every chart  $U_i \ni x$  we have that  $f \circ \phi_i$  is smooth at  $\phi_i(x)$ .

- Proof

$$f \circ \phi_j^{-1} = (f \circ \phi_i^{-1}) \circ \underbrace{(\phi_i \circ \phi_j^{-1})}_{C^r}$$

## Definition: Compatibility (Equivalence) of Atlases

Atlases  $A_1$  and  $A_2$  are compatible or equivalent if every chart in  $A_1$  is compatible with every chart in  $A_2$ . Equivalently,  $A_1 \cup A_2$  is also an atlas.

- Claim: This is an equivalence relation.

## Example

Consider  $\mathbb{R}$ .

Atlas 1:  $U = \mathbb{R}$  and  $\phi = \text{id}$ .

Atlas 2:  $U_1 = (1, \infty)$ ,  $\phi_1(x) = x^2$ ,  $U_2 = (-\infty, 2)$  and  $\phi_2(x) = x$ .

## Definition: Diffeomorphism

$\mathbb{R}^n \supset V \xrightarrow{F} W \subset \mathbb{R}^n$  is a diffeomorphism if

- $F$  is  $C^r$ ,
- $F$  is invertible, and
- $F^{-1}$  is  $C^r$

## Counterexample

$y = x^3$  is a smooth homeomorphism but not a diffeomorphism.

## Definition: Smooth Structure / Maximal Atlas

Given an atlas, we may take all compatible atlases and define a smooth structure by the union of all such objects (i.e. the maximal atlas).

### Lemma:

Every smooth manifold has a countable, locally finite atlas of precompact charts.

## Examples

- Zero dimensional manifolds (i.e. a point).
- $\mathbb{R}^n$  and open subsets of  $\mathbb{R}^n$ .
- If  $M, N$  are smooth manifolds, then  $M \times N$  is a smooth manifold.

That is, if we have atlases  $(U_i, \phi_i)$  and  $(W_j, \psi_j)$ , we may generate  $(U_i \times W_j, \phi_i \times \psi_j)$ .

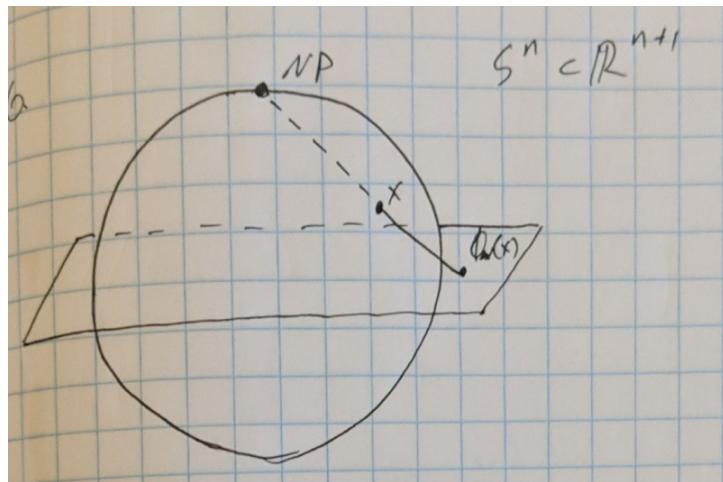
- Take  $F: M \xrightarrow{\text{homeo}} N$  with  $N$  a smooth manifold. Then  $M$  is smooth.

Take an atlas  $A$  on  $N$  and the pullback  $F^{-1}A = \{(F^{-1}(U_i), \phi_i \circ F)\}$ .

- An open subset of a smooth  $M$  is a smooth manifold.
- $\text{GL}(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$ .

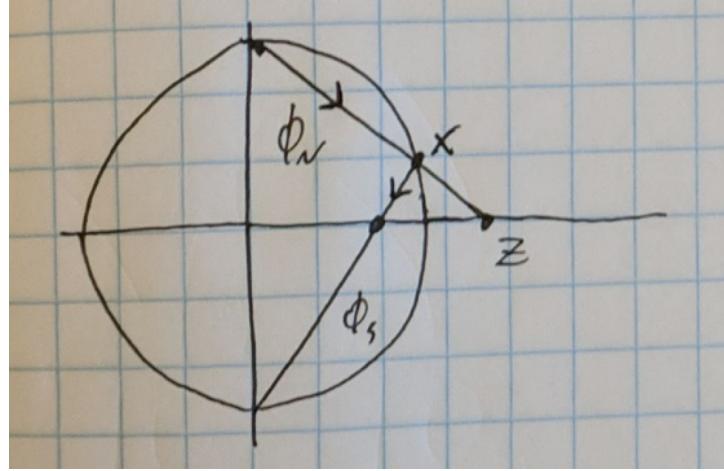
## The n-Sphere

- $S^n$  is a manifold



$$U_N = S^n \setminus NP \xrightarrow{\phi_N} \mathbb{R}^n$$

$$U_S = S^n \setminus SP \xrightarrow{\phi_S} \mathbb{R}^n$$



$$\phi_S \phi_N^{-1}(z) = \frac{z}{|z|^2}.$$

– A different construction for  $S^n$ .

Take hemispheres  $U \xrightarrow{\text{orthogonal projection}} B^n$ .

## Projective Space

$$\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus 0 / \sim \text{ where } x \sim \lambda x \text{ for } \lambda \neq 0.$$

$$[x] = [x_0 : x_1 : \dots : x_n] = [\lambda x_0 : \lambda x_1 : \dots : \lambda x_n].$$

Take  $U_i = \{x_i \neq 0\}$  and open cover, and maps  $U_i \rightarrow \mathbb{R}^n$  given by  $[x_0 : \dots : x_n] \mapsto \left( \frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, 1, \frac{x_i}{x_i}, \dots, \frac{x_n}{x_i} \right)$ . Then for  $j < i$  take

$$\phi_j \phi_i^{-1}(y_1, \dots, y_n) = \left( \frac{y_0}{y_j}, \dots, \frac{y_{j-1}}{y_j}, \frac{y_{j+1}}{y_j}, \dots, \frac{y_{i-1}}{y_1}, 1, \frac{y_i}{y_i}, \dots, \frac{y_n}{y_1} \right)$$

## Definition: Diffeomorphism

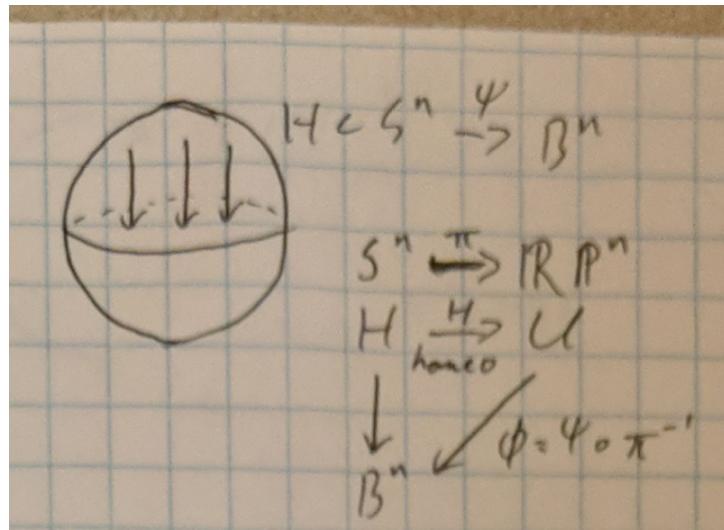
$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ B \subset B_{\max} & & A \supset A_{\max} \end{array}$$

$F$  is a diffeomorphism if  $F$  is a homoeomorphism and  $F^{-1} A_{\max} = B_{\max}$  ( $F^{-1} A \sim B$ ).

**October 3, 2024**

## Recall

$$\mathbb{RP}^n = \begin{cases} \mathbb{R}^{n+1} \setminus 0 / \sim & x \mapsto \lambda x \\ S^n / x \sim -x \end{cases}$$



## Note

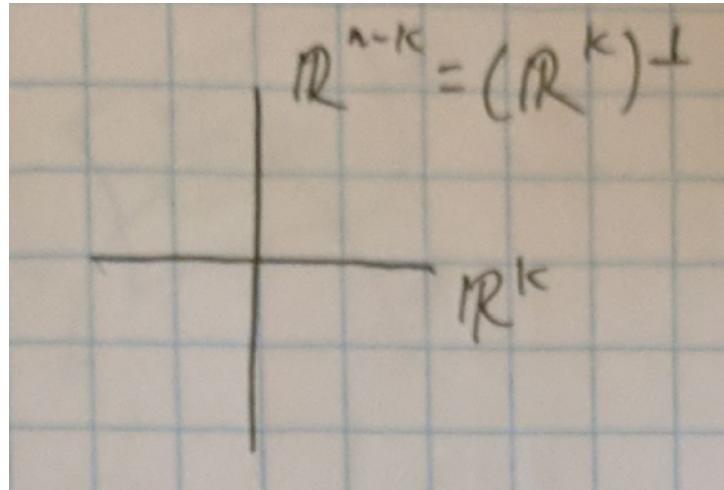
Given a manifold  $M$  and  $A$  a smooth atlas, we generate a continuum of smooth atlases not equivalent to each other. That is, given  $M \xrightarrow[\text{homeo}]{} M$ ,  $F^{-1}A \neq A$ .

## Confer With Groups

$$G \xrightarrow{F} G, a * b = F^{-1}(F(a)F(b)).$$

## Definition: Grassmannians

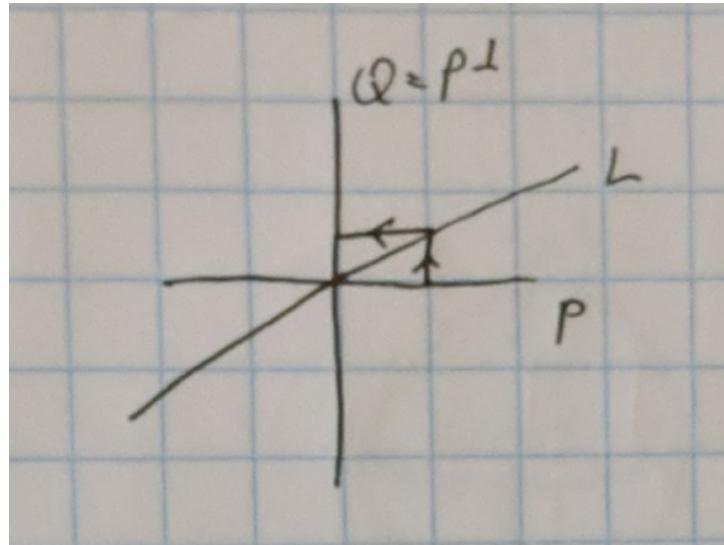
Write  $G_k(n)$ , the collection of all  $k$ -dimensional subspace  $L$  in  $\mathbb{R}^n$ .



Observe that if  $O(i)$  is the collection of orthogonal transformations in dimension  $i$ ,

$$G_k(n) = \frac{O(n)}{O(k) \times O(n-k)}$$

with  $X \sim Y$  when  $Y = XA = X(O(K) \times O(n-k))$ .

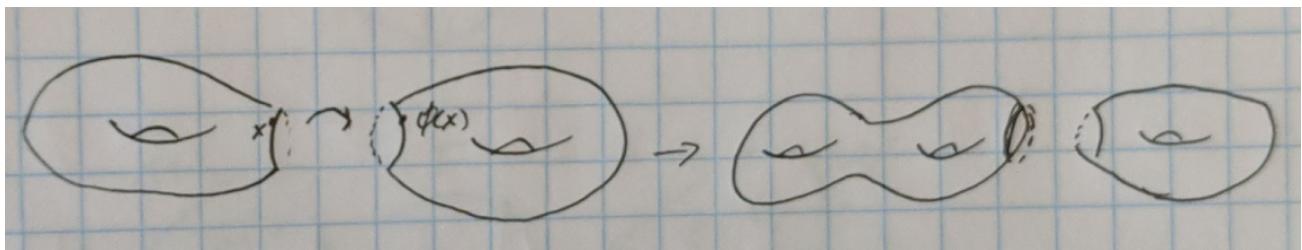


Where  $\dim(L) = k$ ,  $U_p = \{L : L \cap Q = \{0\}\}$ ,  $L = \text{graph}(A : P \rightarrow Q)$ , and we have a homeomorphism

$$U_p \xrightarrow{\phi} \underbrace{\{\text{linear maps } P \rightarrow Q\}}_{\mathbb{R}^{k \times (n-k)}}.$$

## Surfaces

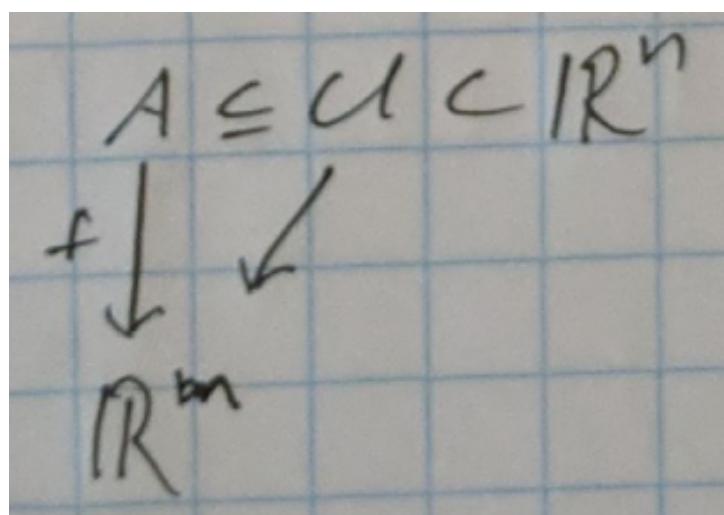
We have explored  $S^2$ ,  $\mathbb{RP}^2$ ,  $\mathbb{T}^2 = S^1 \times S^1$ . We have also connected sums.



## Terminological Remark

Let  $\mathbb{R}^N \ni A \xrightarrow{f} \mathbb{R}^m$ .

Then  $f$  is smooth if it extends to a smooth map



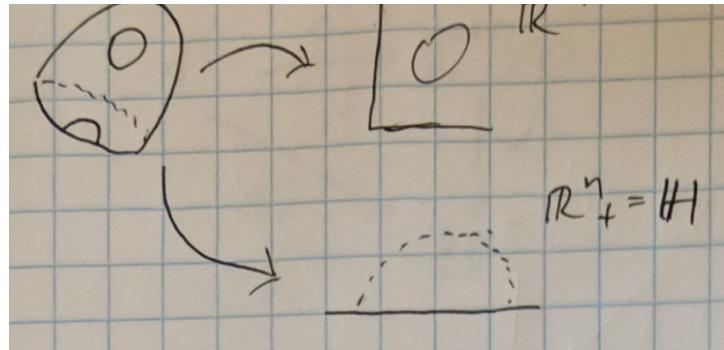
## Exercise

Let  $A = [0, \infty) \subset \mathbb{R}$ .  $f : A \rightarrow \mathbb{R}$  is smooth if and only if it is infinitely differentiable.  
Construct  $(-\varepsilon, \infty)$ .

## Definition: Smooth Manifold with Boundary

A smooth manifold with boundary is a topological space along with an atlas  $\mathcal{A}$  with charts of two types

$$\begin{aligned}\phi : U \rightarrow B^n &\quad (\text{open ball}) \\ \phi : U \rightarrow B^n \cap H\end{aligned}$$



As before,  $\phi_i \circ \phi_j^{-1}$  must be smooth.

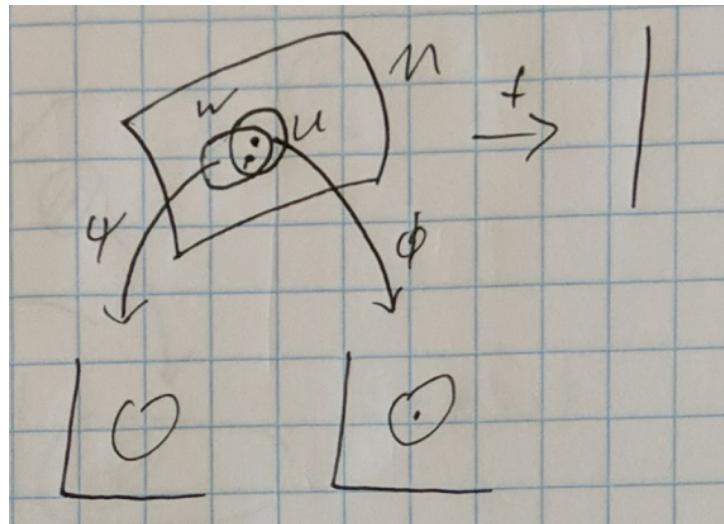
## Examples

- $M \setminus \text{open ball}.$
- The upper half space.

## Definition-Lemma: Smooth Function

A function  $f : M \rightarrow \mathbb{R}$  is smooth at  $p \in M$  if either of the following equivalent conditions is satisfied

1.  $\exists$  a chart  $(U, \phi) \ni p$  such that  $f \circ \phi^{-1}$  is smooth at  $\phi(p)$ .
2.  $\forall$  a chart  $(U, \phi) \ni p$  such that  $f \circ \phi^{-1}$  is smooth at  $\phi(p)$ .



Where  $f \circ \phi^{-1} = f \circ \psi^{-1}(\psi \circ \phi^{-1})$ .

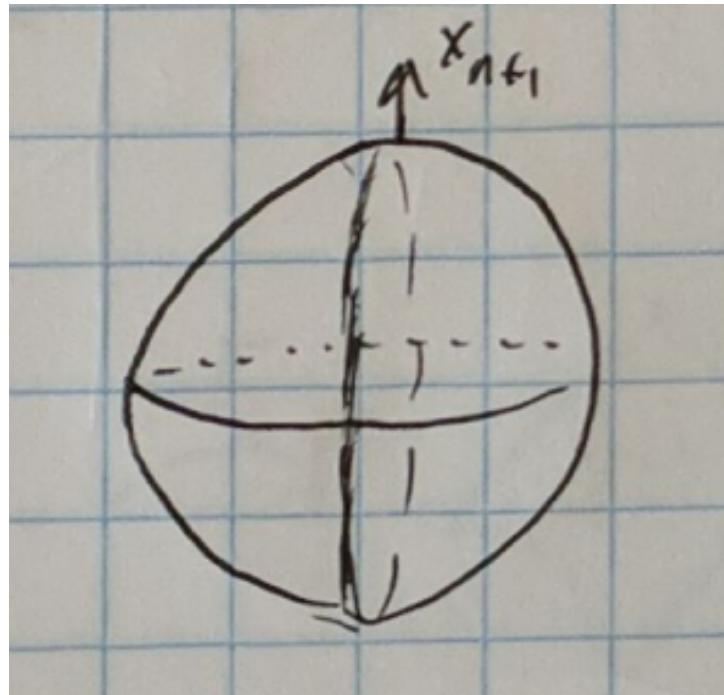
If the above hold for each  $p \in M$ , then  $f$  is smooth.

### Remark

$f$  smooth implies  $f$  is  $C^0$

### Exercise / Sketch

The height function on  $S^n$  is smooth.

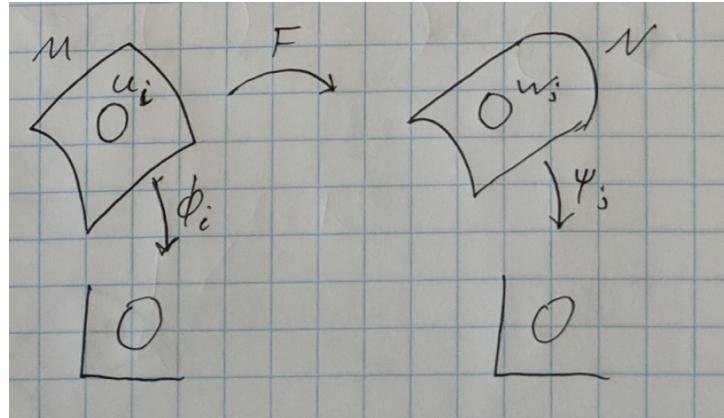


$$\phi : (x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_n).$$

$$f \circ \phi^{-1} = \pm \sqrt{1 - x_1^2 - \dots - x_n^2}.$$

Note that handling the equator requires examining the Eastern and Western hemisphere.  
The stereographic projection leads to a simpler proof.

## Definition: Smooth Function Between Manifolds



$F : M \rightarrow N$  is smooth if  $F$  is  $C^0$  and one of the following equivalent conditions is satisfied

1.  $\exists$  an atlas  $A \subset A_{\max}$  on  $M$  and an atlas  $B \subset B_{\max}$  on  $N$  such that  $\psi_j \circ F \circ \phi_i^{-1}$  is smooth on  $F^{-1}(W_j) \cap U_i$ .
2. The same as a., but for  $A_{\max}$  and  $B_{\max}$ .

Consider as an example  $S^n \rightarrow \mathbb{RP}^n$ .

### Properties of Smooth Maps

$$C^\omega \implies C^\infty \implies C^r \implies C^{r-1} \implies C^1 \implies C^0.$$

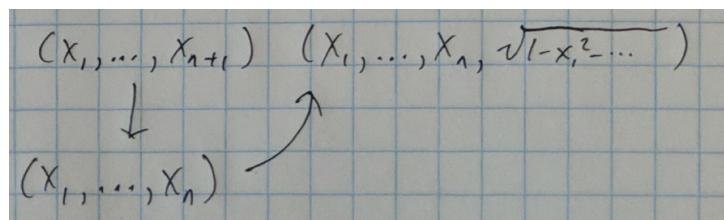
The sum and product of smooth functions is smooth.

### Exercise

The composition of smooth maps is smooth.

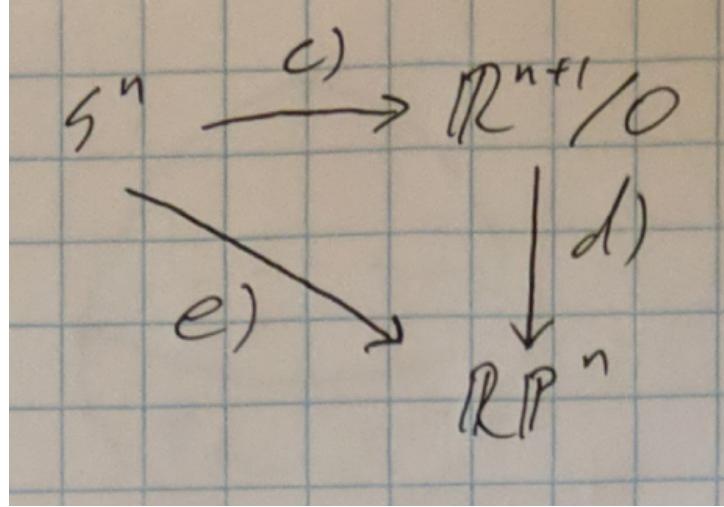
### Examples

1.  $M \times N \xrightarrow{pr} M$  is smooth.
2.  $\underbrace{M \xrightarrow{(F_1, F_2)} N_1 \times N_2}_{\text{smooth}}$  if and only if  $F_1$  and  $F_2$  are smooth.
3.  $S^n \hookrightarrow \mathbb{R}^{n+1}$  is smooth.



1.  $\mathbb{R}^{n+1} \setminus 0 \rightarrow \mathbb{RP}^n$  is smooth with  $[x_0 : \dots : x_n] \mapsto \left( \frac{x_0}{x_i}, \dots, \frac{\hat{x}_i}{x_i}, \dots, \frac{x_n}{x_i} \right)$ .

2.  $S^n \rightarrow \mathbb{RP}^n$ .



## Definition: Diffeomorphism

$$F: M \xrightarrow[A_{\max}]^{\text{diffeo}} N \text{ if } B_{\max}$$

- $F$  is smooth.
- $F$  is invertible.
- $F^{-1}$  is smooth.

## Previous Definition

$$F^{-1}(B_{\max}) = A_{\max}.$$

## Exercise

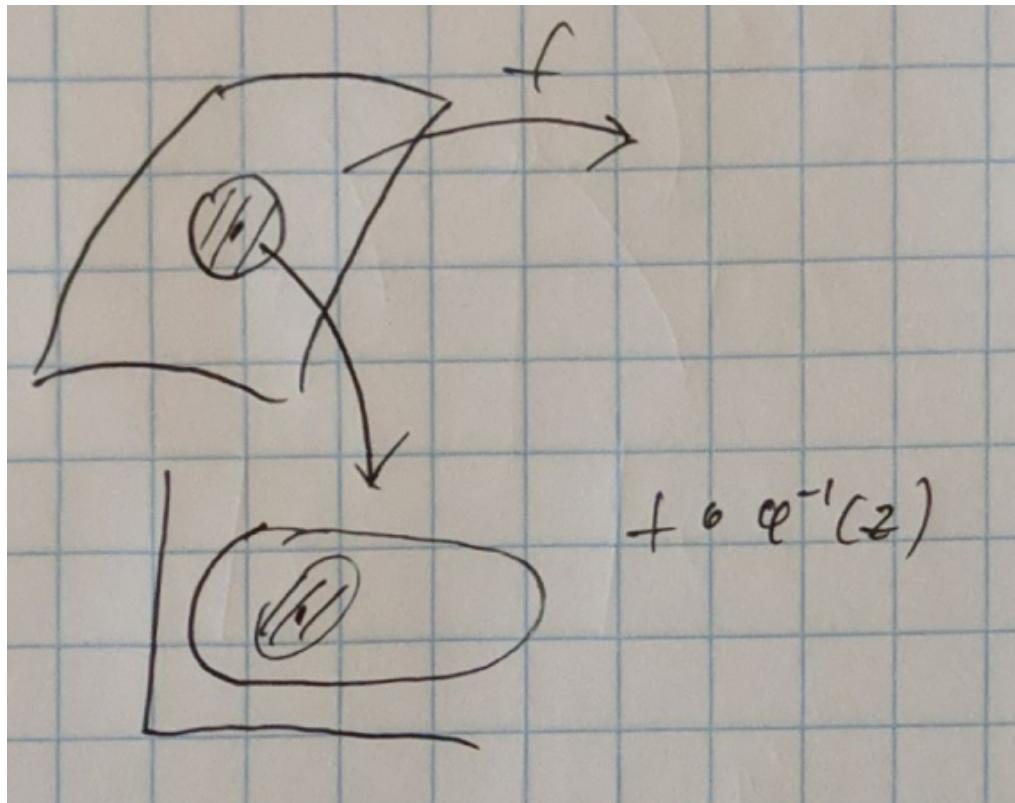
Prove that the definitions are equivalent.

## Examples

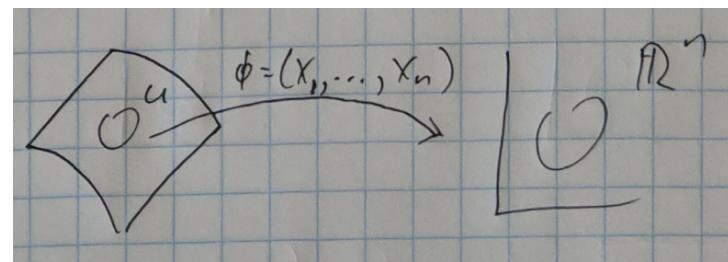
1.  $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \xrightarrow{\text{diffeo}} \mathbb{R}$ .
2.  $x \mapsto x^3$ ,  $\mathbb{R} \rightarrow \mathbb{R}$  is not a diffeomorphism.
3.  $G_k(n) \leftrightarrow G_{n-k}(n)$  with  $P \leftrightarrow P^\perp$ .

## Example 4

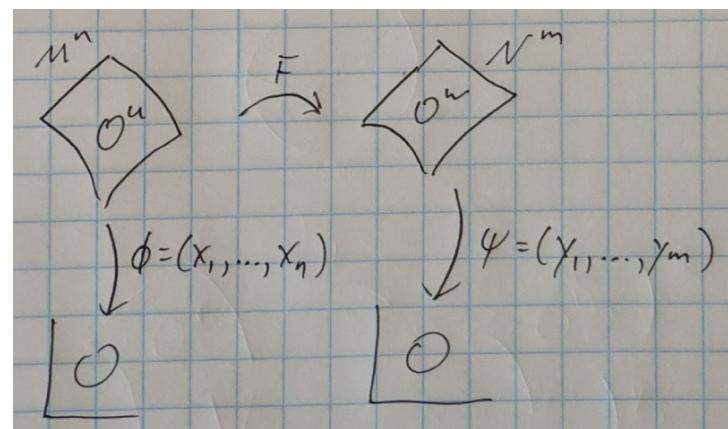
A compact, analytic manifold admits only constant smooth functions by the maximum modulus principle.



### Example 5

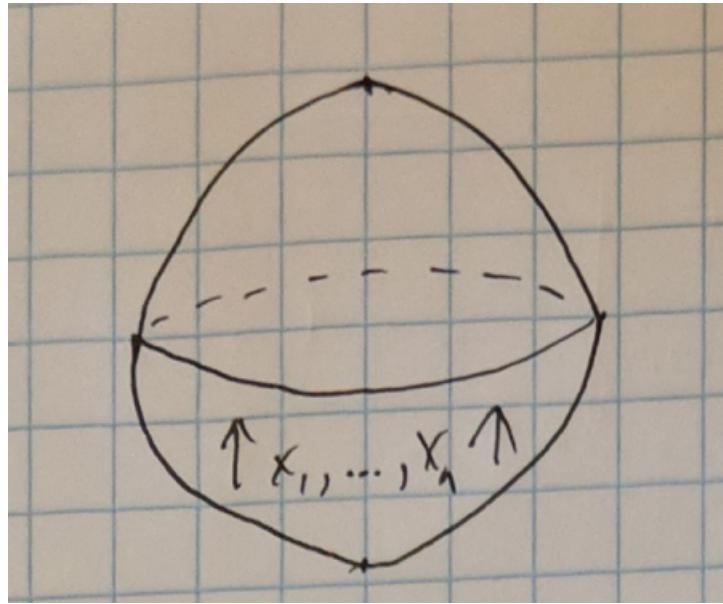


Where  $\phi = (x_1, \dots, x_n)$  and each  $x_i$  is a real-valued function.



$$\psi \circ F \circ \phi^{-1} = (y_1(x_1, \dots, x_n), \dots, y_m(x_1, \dots, x_n)).$$

Then  $S^n \hookrightarrow \mathbb{R}^{n+1}$



where  $y_1 = x_1, \dots, y_n = x_n$ , and  $y_{n+1} = -\sqrt{1 - x_1^2 - \dots - x_n^2}$ .

### Example 6

$$\mathbb{R}^{n+1} \setminus 0 \xrightarrow{F} \mathbb{RP}^n.$$

Need to check that  $\psi_j \circ F$  is smooth.

$$[t_0 : \dots : t_n] \xrightarrow{\psi_0} \left( \frac{t_1}{t_0}, \dots, \frac{t_n}{t_0} \right) \text{ with } U_0 : t_0 \neq 0$$

**October 8, 2024**

## Questions

### Question 1

Given  $M$  smooth and  $x \neq y \in M$ , does there exist  $f \in C^r(M)$  such that  $f(x) = 0$  and  $f(y) = 1$ .

### Question 2

Given  $K \subset U \subset M$  with  $K$  compact and  $U$  open and  $g : K \xrightarrow{C^r} \mathbb{R}$ , does there exist a  $C^r$  extension  $f$  of  $g$  on  $M$  such that  $\text{supp } f \subset U$ .

## Definition: Partitions of Unity

Let  $W_i$  be a locally finite open cover.

A partition of unity subordinated to  $W_i$ , is a collection of functions  $f_i : M \xrightarrow{C^r} \mathbb{R}$  satisfying

- $0 \leq f_i \leq 1$
- $\text{supp } f_i \subset W_i$
- $\sum f_i \equiv 1$

## Definition: Refinement

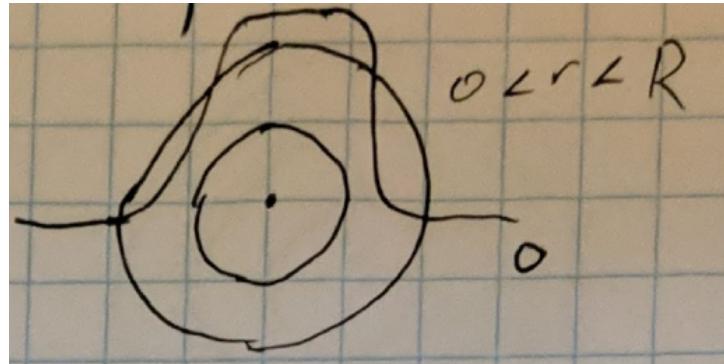
Given covers  $U_j$  and  $W_i$ ,  $U_j$  is a refinement if for each  $j$  we may find  $i$  such that  $U_j \subset W_i$ .

## Theorem

There exists a partition of unity subordinated to  $W_i$ .

### Lemma 1

Take  $B(r) \subset B(R) \subset \mathbb{R}^n$ .



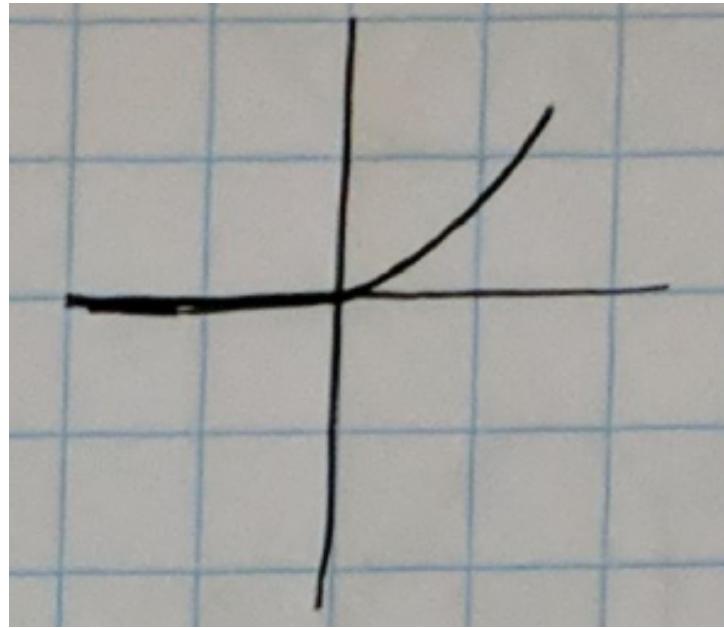
Then there exists  $g \in C^\infty$  such that

- $0 \leq g \leq 1$
- $g|_{\overline{B(r)}} \equiv 1$
- $\text{supp } g \subset B(R)$

## Proof

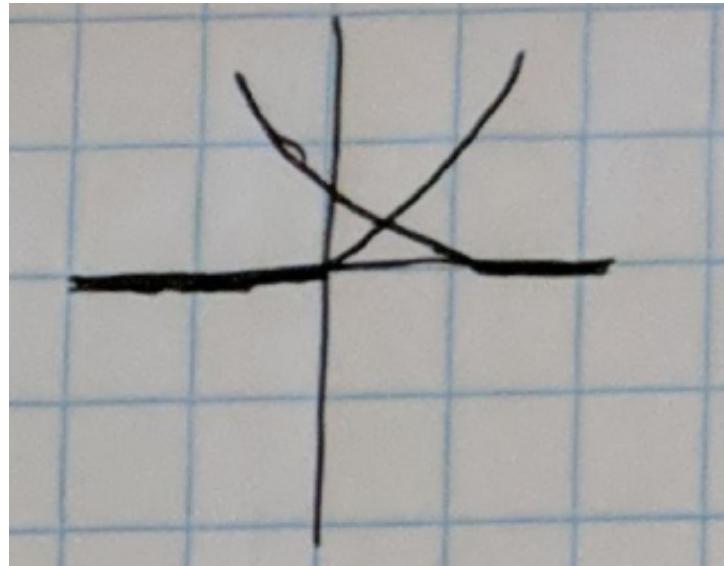
Take

$$h_0(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$



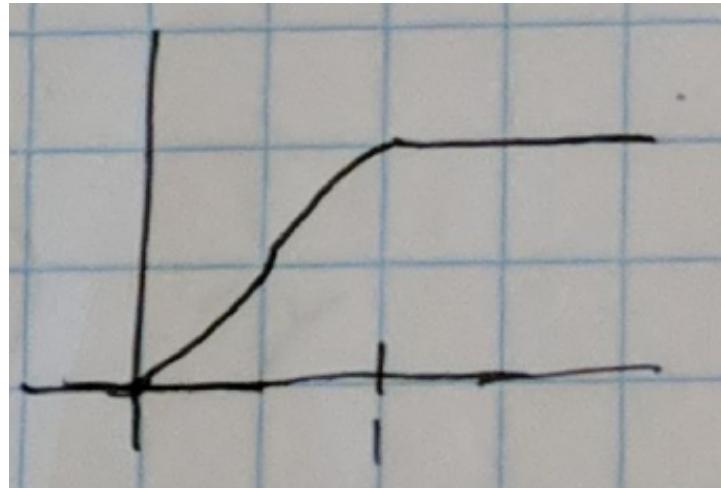
- Exercise: show that  $h_0 \in C^\infty(\mathbb{R})$  (Hint: show that derivatives agree at zero from the right.)

Then take  $h_1(x) = h_0(x) \cdot h_0(1-x)$ .

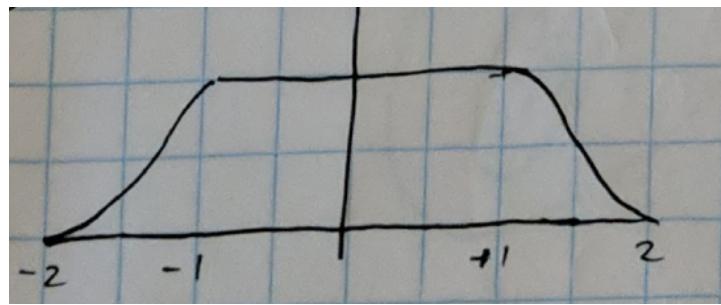


Then define

$$h_2(x) = \frac{1}{\int_{-\infty}^{\infty} h_1(t) dt} \int_{-\infty}^x h_1(t) dt$$



Finally, we define  $h(x) = h_2(x+2) \cdot h_2(2-x)$ .



Then  $g(x) = h(||x||)$  satisfies our requirements.

## Lemma 2

There exists a refinement  $(U_j, \phi_j)$  by coordinate charts such that

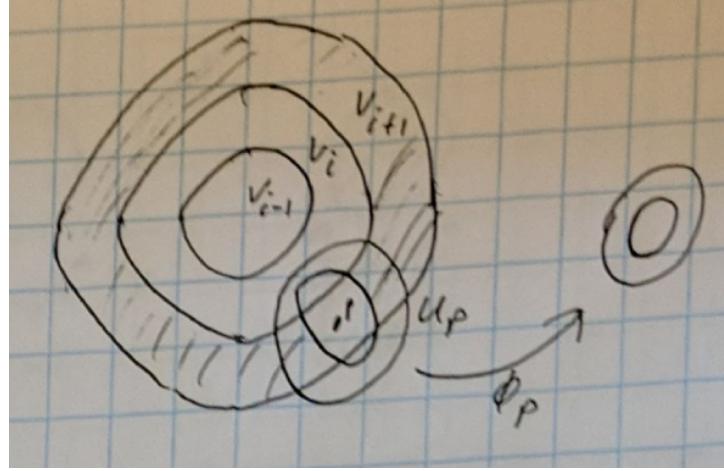
1.  $U_j$  is locally finite.
2.  $\phi_i : U_j \xrightarrow{\text{diffeo}} B^n(2)$ .
3.  $\phi_j^{-1}(B(1))$  is also a cover.

## Proof

There exists a compact exhaustion of  $M$ ,  $C_1 \subset C_2 \subset C_3 \subset \dots$  where  $\bigcup C_i = M$ .

There exists also an open exhaustion by precompact open sets  $\emptyset = V_0 \subset V_1 \subset V_2 \subset \dots$  where  $\overline{V_i} \subset V_{i+1}$ .

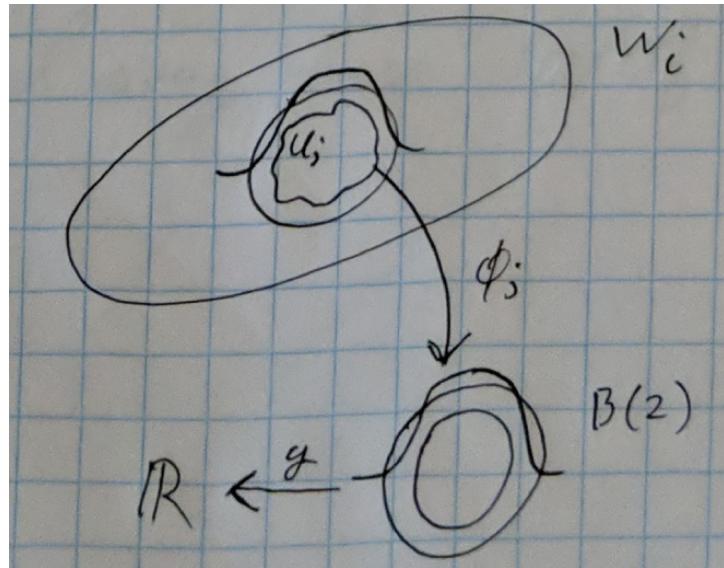
Then define  $V_i := \text{nbhd}(C_i \cup \overline{V}_{i-1})$ .



Take  $A_i = \overline{V}_{i+1} \setminus V_i$  compact,  $p \in A_i$ . Then we have a map  $U_p \xrightarrow{\phi_p} B(2)$ .

- $U_p \subset W_i$  for some  $i$ . (Refinement)
- $U_p \cap V_{i-1} = \emptyset$ . (Locally Finite)
- There exists a finite subcover such that  $\phi_p^{-1}(B(1))$  is also an open cover.

### Proof of Theorem



Set  $g_j = g \circ \phi_j \in C^r$ , extended to 0 outside of  $U_j$ .

- $\text{supp } g_j \subset U_j$
- $\forall p \in M, \exists g_j(p) = 1 \neq 0 \implies \sum g_j > 0$
- $\text{supp } g_j$  is locally finite.

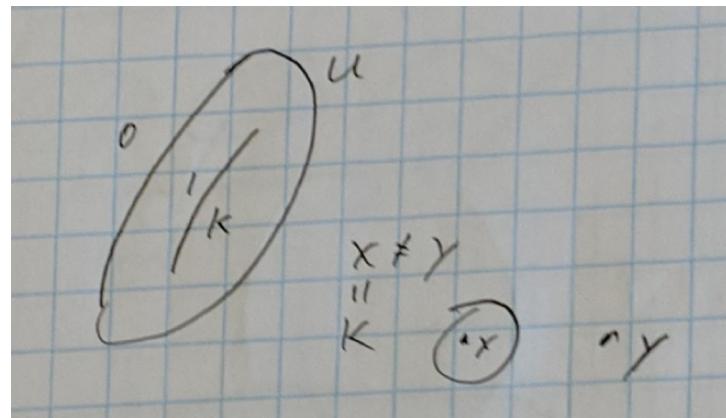
Then we have

$$f_j = \frac{g_j}{\sum g_j}$$

### Corollary 1

For  $K \subset U$ ,  $K$  compact and  $U$  open, there exists  $h \in C^r(M)$  such that

- $h|_K \equiv 1$
- $\text{supp } h \subset U$



### Proof

Take  $U_0 = U$  and  $U_1 = M \setminus K$ .

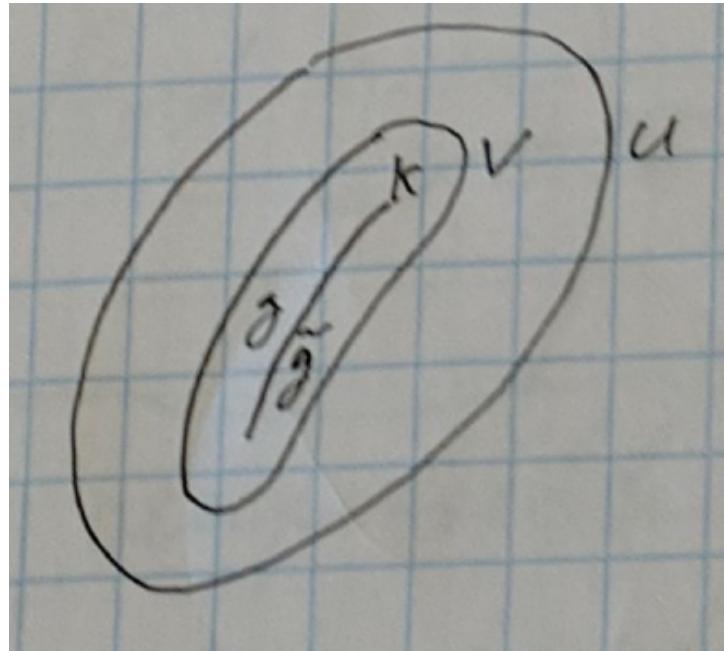
Then there exists a partition of unity  $f_0$  and  $f_1$  where  $f_0 + f_1 = 1$ . Therefore  $f_0$  has support in  $U$  and  $f_1$  has support in  $M \setminus K$  this occurs if and only if  $f_1|_K = 0$ .

### Corollary 2

For  $K \subset U$ ,  $K$  compact and  $U$  open, and  $g : K \xrightarrow{C^r} \mathbb{R}$ , there exists an extension  $f : M \rightarrow \mathbb{R}$  of  $g$ :  $f|_K = g$  such that  $\text{supp } f \subset U$ .

### Proof

For  $g \in C^r(K)$ , there exists a neighborhood  $V$  of  $K$  and a function  $\tilde{g} : V \rightarrow \mathbb{R}$  such that  $\tilde{g}|_K = g$ .



By corollary 1, there exists  $h \in C^r(M)$  such that

- $\text{supp } h \subset V$
- $h|_K = 1$

Therefore  $f = h \cdot \tilde{g}$ .

### Corollary 3

There exists  $f : M \xrightarrow{C^r} \mathbb{R}$  bounded from below and proper.

#### Definition: Proper Function

$f$  is proper if and only if  $f^{-1}(K)$  is compact for  $K$  compact.

#### Proof

See Textbook.

#### Consequence

Take  $\{x : f(x) \leq C_i\} =: E_i$  as  $i \rightarrow \infty$ . Then we get a compact exhaustion  $E_1 \subset E_2 \subset E_3 \subset \dots$   
e.g.  $f(x) = \|x\|^2$ .

**October 10, 2024**

### Algebra = Analysis (Geometry)

Take  $M$  to be either a compact metrizable space ( $A = C^0(M)$ ) or a copact manifold ( $A = C^\infty(M)$ ).

$$M \leftrightarrow A$$

Let  $I \subset A$  be an ideal and take  $V(I) = \{x : f(x) = 0, \forall f^n \in I\}$ .

Take also  $Y \subset M$  closed and consider  $I_Y = \{f : f|_Y = 0\}$  which is also an ideal.

$$\begin{array}{c} Y \rightarrow I_y \\ V(I) \leftarrow I \end{array}$$

Denote  $I_x = \{f : f(x) = 0\}$ .

## Theorem

- $I_x$  is a maximal ideal.
- Every maximal ideal is of this form, and  $x$  is unique.

$$M \leftrightarrow \text{Maximal Ideals of } A$$

## Proof

### Maximal Ideal

Take  $I_x$  and  $g \notin I_x$ .

We want to show that  $g$  with  $I_x$  generates  $A$ .

Then take  $f \in A$  defined as  $f = h + ag$  for some  $h \in I_x$ .

Since  $g \notin I_x$ ,  $g(x) \neq 0$ . Take  $a = \frac{f(x)}{g(x)}$  and define  $h := f - ag$ .

Then  $f(x) - \frac{f(x)}{g(x)}g(x) = 0 \in I_x$ .

### Every Maximal Ideal

Let  $I$  be a proper ideal such that  $V(I) \neq 0$ ,  $\exists x$  such that  $f(x) = 0, \forall f \in I$ .

Maximal  $\implies V(I) = \{x\}$ .

By contradiction, assume not:  $\forall x \in M, \exists f_x \in I, f_x(x) \neq 0$ .

It follows that  $f_x|_{U_x} \neq 0$  where  $U_x \ni x$  is a neighborhood from an open cover.

Then, by compactness, we have a finite subcover

$I \ni f_i = f_{x_i} \neq 0$  on  $U_i$ .

$I \ni \underbrace{\sum f_i^2}_{g>0} > 0$  on  $M$ . But then  $1 = g^{-1}g \in I$ .

### Uniqueness

$$I_{x_1} = I_{x_2} \iff x_1 = x_2$$

( $\iff$ ) is obvious.

( $\implies$ )  $x_1 \neq x_2$  implies that there exists  $f \in A$  such that  $f(x_1) = 1$  and  $f(x_2) = 0$ .

Then  $f \in I_{x_2}$  while  $f \notin I_{x_1}$  implies that  $I_{x_1} \neq I_{x_2}$ .

### Reading the Topology / Smooth Structure

We have a correspondence

$$\text{closed sets} \leftrightarrow \text{ideals}$$

## Algebra Homomorphisms

Take  $\phi : A \rightarrow \mathbb{R}$ . Then  $\ker \phi$  is a maximal ideal, and

$$\begin{aligned} A/I_x &\xrightarrow{\delta_x} \mathbb{R} \\ x &\mapsto \pm f(x) \end{aligned}$$

Then

$$\begin{aligned} M &\leftrightarrow \text{Alg. Hom. } A \rightarrow \mathbb{R} \\ 1 &\mapsto 1 \\ A > 0 &\mapsto \text{pos. \#} \end{aligned}$$

## Counterexample

Take instead  $M = \mathbb{R}$ . Claim: there exist maximal ideals other than  $I_x$ .

### Proof

Take  $J$  to be the ideal of all compactly supported functions such that  $J \subset M$  for some maximal ideal  $M$ . However,  $\forall x \in \mathbb{R}$  there exists a compactly supported function  $f(x) \neq 0$ . So  $M \neq I_x$ .

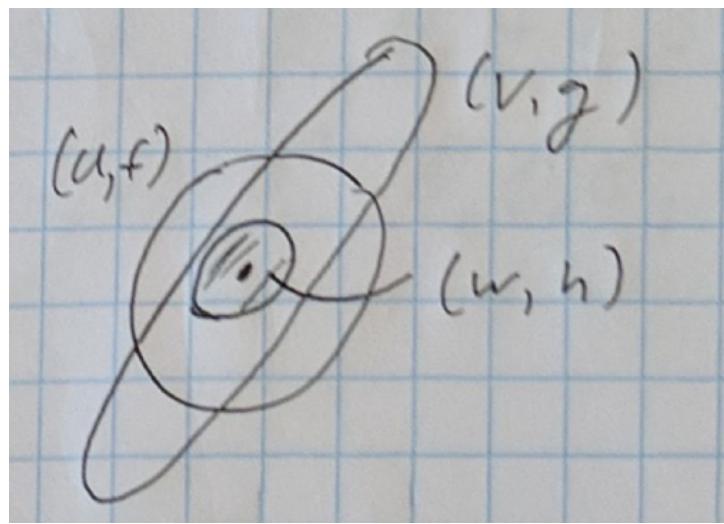
## Functorial Considerations

Take  $F : M \rightarrow N$ ,  $M$  and  $N$  compact. Then

$$\begin{aligned} F : M &\rightarrow N \\ C^{0/\infty}(M) &\xleftarrow{F^A} C^0(N) \\ f \circ F &\leftrightarrow f \end{aligned}$$

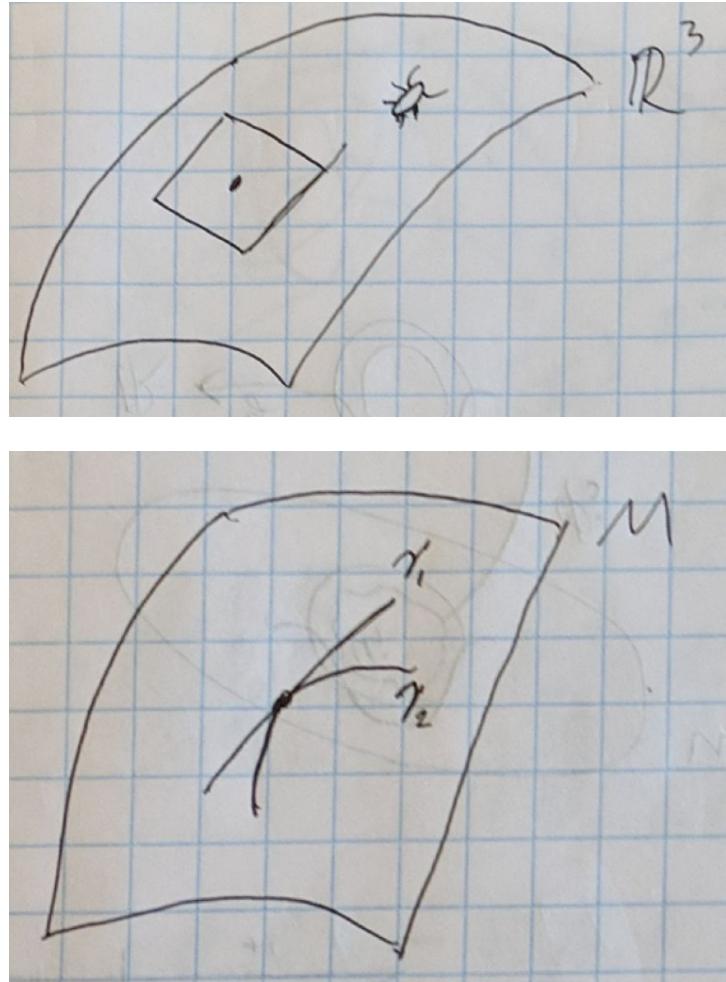
## Definition: Germ

Take  $M \ni P$ .



Where  $(U, f) \sim (V, g)$  if  $f \equiv g$  on some neighborhood of  $p$ .

## Definition: Tangent Spaces



With  $\gamma_1 : (-\varepsilon, \varepsilon) \rightarrow M$ ,  $\gamma_2 : (-\varepsilon, \varepsilon) \rightarrow M$ , and  $\gamma_1 \sim \gamma_2 \iff \gamma'_1(0) = \gamma'_2(0)$ .

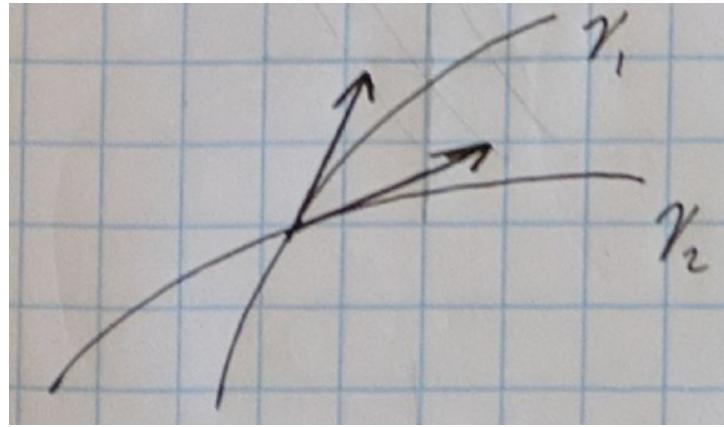
$\gamma(t) = ((x(t), y(t), z(t))$  and  $\gamma'(t) = (x'(t), y'(t), z'(t))$ .

The tangent space to  $M$  at  $p$ , written  $T_p M$ , is the set of equivalence classes.

### Remarks: Tangent Vectors

Take  $V \ni p$  a finite-dimensional vector space ( $= \mathbb{R}^n$ ).

$$\begin{aligned} 0 &\mapsto p \\ \gamma_1, \gamma_2 &: (-\varepsilon, \varepsilon) \rightarrow V \quad (\text{germs}) \\ \gamma'_1 \sim \gamma'_2 &: \gamma'_1(0) = \gamma'_2(0) \end{aligned}$$



Write

$$\gamma'(0) = \lim_{t \rightarrow 0} \frac{\gamma(t) - \gamma(0)}{t}$$

$\gamma = (x_1, \dots, x_n)$  and  $\gamma'(0) = (x'_1(0), \dots, x'_n(0))$ .  
 A tangent vector to  $V$  at  $p$  is an equivalence class  $T_p V$ .  
 Claim:  $T_p V$  is a vector space.

### Operations

Take  $p = 0$ . Write

$$[\gamma_1] + [\gamma_2] = [\gamma_1 + \gamma_2]$$

$$\lambda[\gamma] = [\lambda\gamma]$$

When  $p \neq 0$ , instead

$$[\gamma_1] + [\gamma_2] = [\gamma_1 + \gamma_2 - p]$$

$$\lambda[\gamma] = [\lambda\gamma + (1 - \lambda)p]$$

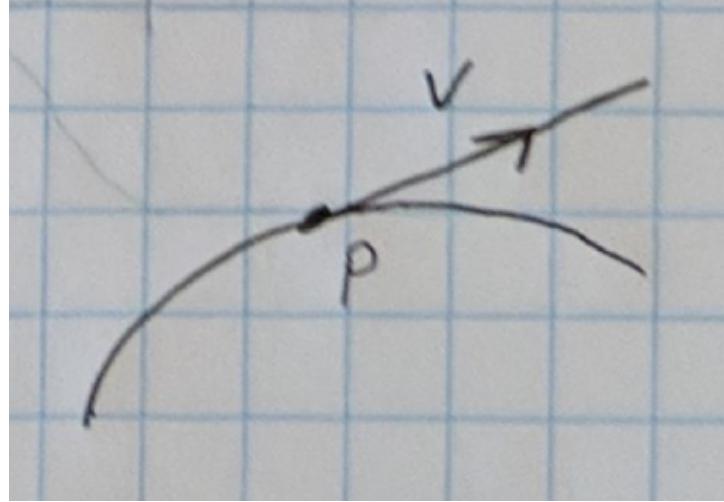
Claim:  $T_p V$  is canonically isomorphic to  $V$

$$[\gamma] \mapsto \gamma'(0)$$

$$[\gamma = (x_1, \dots, x_n)] \mapsto (x'_1(0), \dots, x'_n(0))$$

$$T_p V \rightarrow V$$

$$p + tv \leftrightarrow v$$



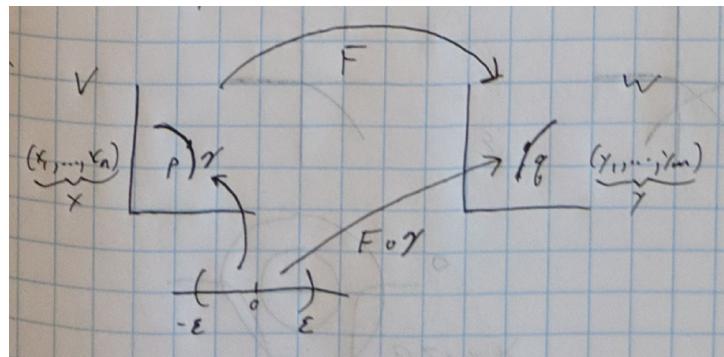
## Proposition

Take

$$\begin{array}{ccc} V & \xrightarrow{C^r} & W \\ \mathbb{R}^n & & \mathbb{R}^m \\ p & \mapsto & q \end{array}$$

Then

$$\begin{aligned} V &\xrightarrow{\sim} W \\ DF_p = F_* : T_p V &\rightarrow T_q W \\ [\gamma] &\mapsto [F \circ \gamma] \end{aligned}$$



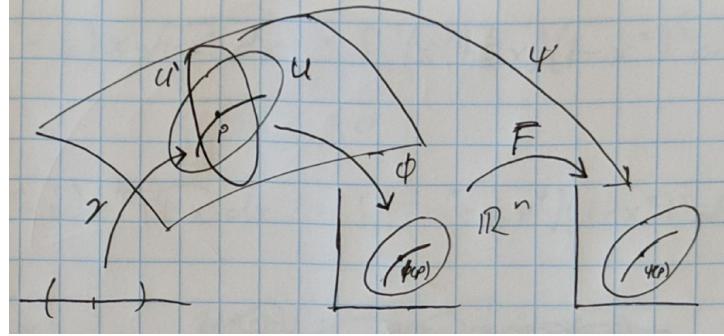
Then  $F$  is well-defined and linear.

$F = (F_1, \dots, F_m)$  and  $F \circ \gamma(F_1(\gamma_1, \dots, \gamma_n), \dots, F_m(\gamma_1, \dots, \gamma_n))$ .

We have that  $[\gamma] = \gamma'(0)$  and  $[F \circ \gamma] = \frac{d}{dt} F \cdot \gamma(t)|_{t=0}$ . By chain rule,

$$\frac{d}{dt}(F \circ \gamma)|_{t=0} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{pmatrix} \begin{pmatrix} \gamma'_1(0) \\ \vdots \\ \gamma'_n(0) \end{pmatrix}$$

## Tangent Space



$\gamma_1 \sim \gamma_2 \iff \phi \circ \gamma_1 \sim \phi \circ \gamma_2$  and  $\gamma_1 \sim \gamma_2 \iff \psi \circ \gamma_1 \sim \psi \circ \gamma_2$ , so

$$(\psi \circ \phi^{-1})(\phi \circ \gamma_1) \sim (\psi \circ \phi^{-1})(\phi \circ \gamma_2)$$

Now, take  $\{[\gamma]\} = T_p M$ . Claim: this is a vector space.

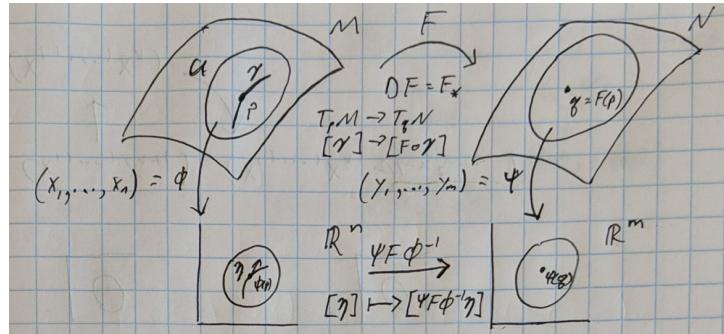
$$\begin{array}{ccc} T_p M & \xrightarrow{D\phi} & T_{\phi(p)} \mathbb{R}^n \\ & \searrow D\psi & \downarrow D(\psi \circ \phi^{-1}) \\ & & T_{\psi(p)} \mathbb{R}^n \end{array}$$

$$[\gamma_1] + [\gamma_2] = [\phi^{-1}(\phi \circ \gamma_1 + \phi \circ \gamma_2)].$$

October 15, 2024

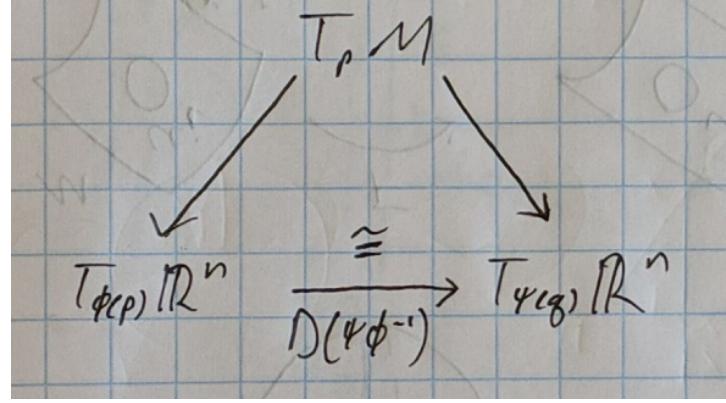
## Recall: Tangent Space by Equivalence Classes

$\gamma : (-\varepsilon, \varepsilon) \rightarrow M, \gamma(0) = p, \gamma_1 \sim \gamma_2 \iff \phi \circ \gamma'_1 \sim \phi \circ \gamma'_2 \iff (\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0).$   
 $T_p M = \{[\gamma]\} \xrightarrow{D\phi=\phi_*} T_{\phi(p)} \mathbb{R}^n = \mathbb{R}^n.$



Then for  $(F_1, \dots, F_m)$ , we have  $D(\psi F \phi^{-1}) : \underbrace{T_{\phi(p)} \mathbb{R}^n}_{\mathbb{R}^n} \rightarrow \underbrace{T_{\psi(q)} \mathbb{R}^m}_{\mathbb{R}^m}$  where

$$D(\psi F \phi^{-1}) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$



## Chain Rule

We have that  $D(FG) = DF \circ DG$  since  $(FG) \circ \gamma = F(G \circ \gamma)$ .

## Example

Take

$$f: \underbrace{M}_{(x_1, \dots, x_n)} \rightarrow \mathbb{R}$$

and  $Df = \left( \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$ .

## Definition: Directional Derivatives

Take  $v = [\gamma] \in T_p M$  and  $f \in C^r(M)$ .

The directional derivative is given by

$$L_v f = \frac{d}{dt} f(\gamma(t))|_{t=0} = \frac{d}{dt} \underbrace{(f \circ \phi^{-1})(\phi \circ \gamma)(t)}_{g \circ \eta}|_{t=0}$$

Then

$$\frac{d}{dt} g(\eta(t))|_{t=0} = \sum \frac{\partial g}{\partial x_i}(\phi) \eta'_i(0)$$

which is determined by  $(\eta'_1(0), \dots, \eta'_n(0)) = \eta'(0)$ .

## Properties

$$L_v : C^\infty(M) \rightarrow \mathbb{R}$$

1. Linear over  $\mathbb{R}$
2.  $L_v(fg) = (L_v f)g(\phi) + f(\phi)(L_v g)$  (product rule)
3. Linear in  $v$ .

## Derivations

The collection  $\mathcal{D}_p = \{C^r(M) \rightarrow \mathbb{R} : (1) \text{ and } (2) \text{ hold}\}$  is called the derivations at  $p$ .

## Algebraic Aside

$\delta_p : A \rightarrow \mathbb{R}$  given by  $f \mapsto f(p)$  yields

$$D(fg) = Df\delta_p(g) + \delta_p(f)Dg$$

## Theorem

$T_p M \rightarrow \mathcal{D}_p$  given by  $v = [\gamma] \mapsto L_v$  is a linear isomorphism.

## Recall: Germ

Take  $(f, U) \ni p$  and  $(g, V) \ni p$ . Then  $(f, U) \sim (g, V)$  if and only iff  $f \equiv g$  on  $W \subset U \cap V$ . The equivalence classes of this relation are germs.

## Hadamard's Lemma

Take  $f \in C^r(\mathbb{R}^n, 0)$  on  $\mathbb{R}^n$  with  $f(0) = 0$ .

There exists  $g_1, \dots, g_n \in C^{r-1}(\mathbb{R}^n, 0)$  such that

$$f(x) = \sum x_i g_i(x)$$

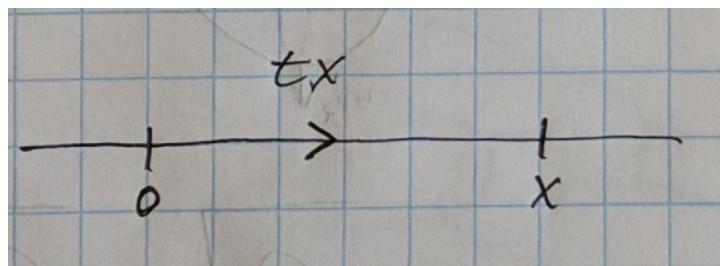
and  $g_i(0) = \frac{\partial f}{\partial x_i}(0)$ .

## Example

For  $n = 1$  and  $f : \mathbb{R} \xrightarrow{C^r} \mathbb{R}$  with  $f(0) = 0$ . Then we have  $f(x) = x \overbrace{g(x)}^{c^{r-1}}$  given by

$$g(x) = \begin{cases} \frac{f(x)}{x} & x \neq 0 \\ f'(0) & x = 0 \end{cases}.$$

## Proof of Example



Take

$$\begin{aligned} \int_0^1 \underbrace{\frac{d}{dt} f(tx) dt}_{f'(tx) \cdot x} &= x \underbrace{\int_0^1 f'(tx) dt}_{g(x)} \\ &= f(1 \cdot x) - f(0 \cdot x) \end{aligned}$$

## Proof of Lemma

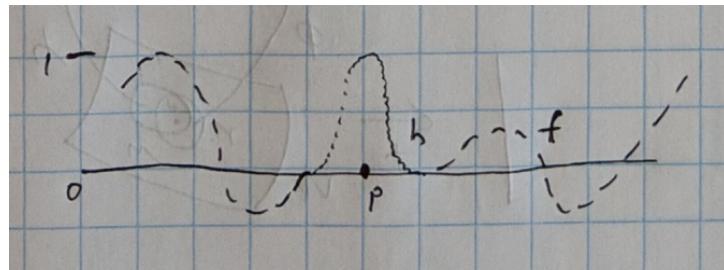
$$\int_0^1 \underbrace{\frac{d}{dt} f(tx) dt}_{\sum \frac{\partial f}{\partial x_i}(tx) \cdot x_i} = \sum x_i \underbrace{\int_0^1 \frac{\partial f}{\partial x_i}(tx) dt}_{g_i}$$

## Lemma

For  $D \in \mathcal{D}_p$ ,  $Df$  depends only on the germ of  $f$ .

## Proof

Need to show that if  $f \equiv 0$  near  $p$ ,  $Df = 0$ .



Where  $\text{supp } h \subset \{x : f(x) = 0\}$ ,  $h(p) = 1$  and, consequently,  $f \circ h = 0$ .

So  $D(0 = f \circ h)$ ,  $D0 = Df \cdot h(p) + f(p)Dh$  and  $0 = Df$ .

## Lemma

$D(k) = 0$  for  $k$  constant.

## Proof

$1 \cdot 1 = 1 \implies D(1) = D(1 \cdot 1) = D(1) \cdot 1 + 1 \cdot D(1) = 2 \cdot D(1)$ , so  $D(1) = 0$ .

Arbitrary constants follow from the fact that  $D$  is linear.

## Proof: One-to-One

Let  $L_v f = L_w f$  for all  $f = x_i$ , then

$$\sum v_i \frac{\partial f}{\partial x_i} = \sum w_i \frac{\partial f}{\partial x_i}$$

for each  $f$  and  $v_i = w_i$ .

## Lemma

$$\text{Der}(C^r(m)@p) = \text{Der}(C^r(M, p))$$

## Proof

Take  $M = \mathbb{R}^n$  and  $p = 0$ .

Given  $D$ , we need  $v = \sum v_i \frac{\partial}{\partial x_i}$  with

$$L_v f = \sum v_i \frac{\partial f}{\partial x_i} = Df$$

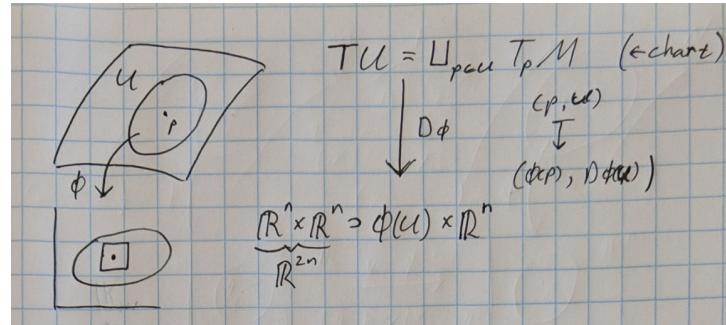
for every  $f$ .

By Hadamard's lemma, write  $f = \sum x_i g_i(x)$ . Then

$$\begin{aligned} Df &= \sum (Dx_i)g_i(0) + \overbrace{x_i(0)}^{=0} Dg_i \\ &= \sum \underbrace{(Dx_i)}_{v_i} \frac{\partial f}{\partial x_i}(0) \end{aligned}$$

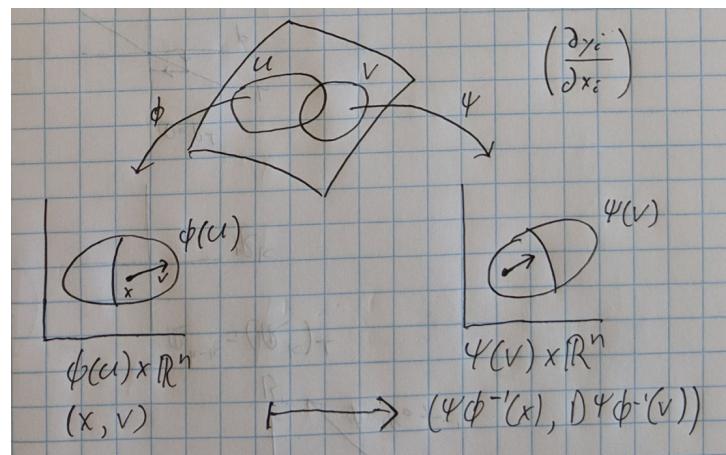
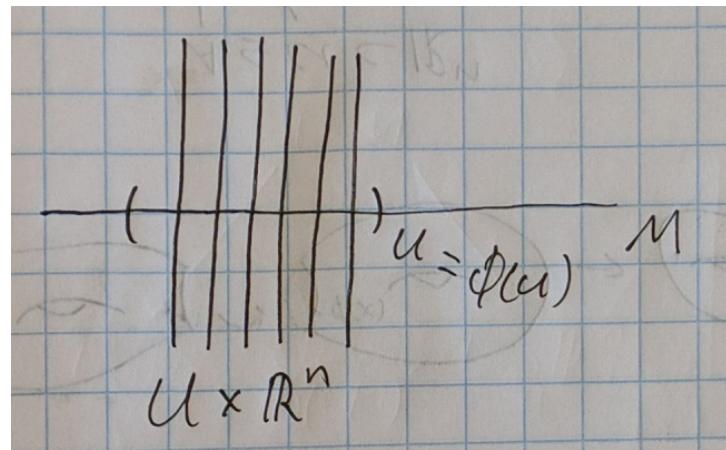
That is, since  $v_i = L_v x_i$ , we have  $v_i = Dx_i$ .

## Definition: Tangent Bundle



For every  $p \in M$ , we have  $T_p M$ .

The tangent bundle  $TM = \coprod_{p \in M} T_p M$ .



When  $M$  is  $C^r$ ,  $TM$  is  $C^{r-1}$ .

Chapter 7 of Lee (Lie Groups) will not be covered in class, but is highly recommended reading.

## Preliminary Definition: Vector Field

Take  $M \xrightarrow{C^\infty} TM$  by  $p \xrightarrow{\nu} \nu(p) \in T_p M$ .

### Space of Vector Fields

Write  $\mathfrak{X}(M)$  to be the collection of all  $C^\infty$  vector fields.

- This is a module over  $C^\infty(M)$ .
- $\mathfrak{X}(M)$  acts on  $C^\infty(M)$  by  $v \mapsto L_v$ .

### Smooth

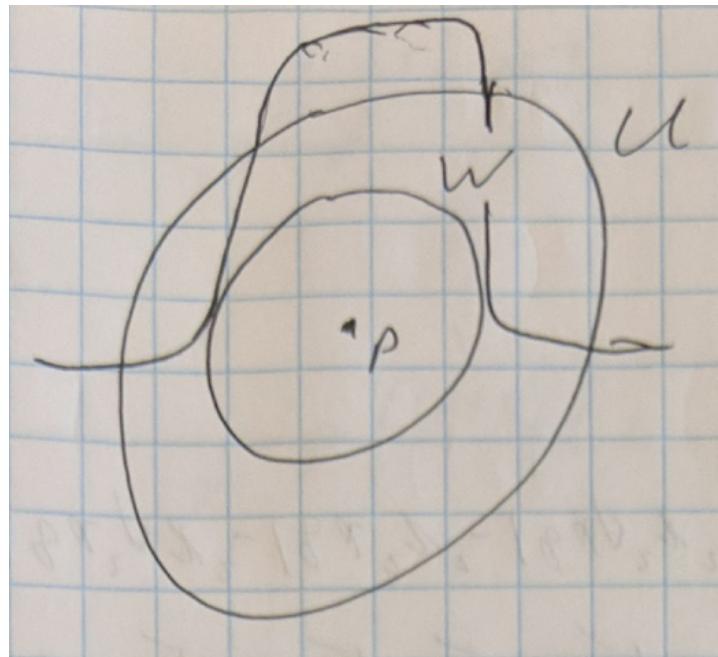
1.  $v$  is  $C^\infty$
2. In local coordinates, for  $p \in U$  and  $(U, \phi) = (x_1, \dots, x_n)$  with  $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$  a basis in  $T_p M$ .

$$v = \sum_{\substack{i \\ \text{functions} \\ \text{on } U}} v_i(x) \frac{\partial}{\partial x_i}$$

3.  $f \in C^\infty(M) \implies L_v f \in C^\infty$

### 2 Implies 3

$$L_v f = \sum v_i \frac{\partial f}{\partial x_i}$$



With  $\phi|_W \equiv 1$ ,  $\text{supp } \phi = U$ ,  $x_i \phi \in C^\infty(M)$ , and  $x_i \phi|_W \equiv W_i$ .

Then  $L_\nu(x_i \phi) \underset{\text{on } W}{=} \nu_i$ .

## Definition: Lie Algebra

Take  $A$  a vector space equipped with (a Lie bracket)  $[\cdot, \cdot] : A \times A \rightarrow A$  such that

- $[a, b] = -[b, a]$  (Skew Symmetric)
- $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$  (Jacobi Identity)

### Example 1

Take  $A$  to be an algebra and define  $[a, b] = ab - ba$ . Then if  $A$  is associative, it satisfies the Jacobi identity.

- $gl(n)$
- $so(n)$  (skew symmetric matrices)
  - $(ab - ba)^T = b^T a^T - a^T b^T = ba - ab = -[a, b]$
- $su(n)$  ( $A^T = A^\dagger$ )

## Theorem:

The space of vector fields  $\mathfrak{X}(M)$  is a Lie algebra:

- $\forall V, W \in \mathfrak{X}(M), \exists! U \in \mathfrak{X}(M)$  such that  $L_U f = L_W L_V f - L_V L_W f$ . Write  $U = [V, W]$ .

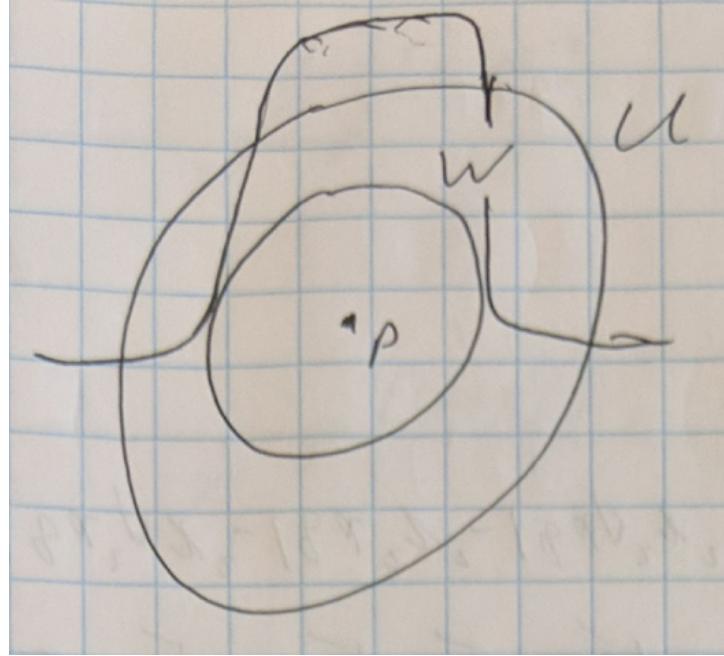
### Lemma

$L_V f = L_W f$ ,  $\forall f$  implies  $V = W$ .

### Proof

$$V = \sum \nu_i \frac{\partial}{\partial x_i} \stackrel{?}{=} \sum w_i \frac{\partial}{\partial x_i} = W$$

Pick  $p \in U$ . We want to find  $\nu_i(p) = w_i(p)$ .



With  $\phi|_W \equiv 1$ ,  $\text{supp } \phi \subset U$  and  $f = x_i \phi \in C^\infty(M)$ .

$$L_V f = L_W f$$

$$\sum v_j \underbrace{\frac{\partial(x_i \phi)}{\partial x_j}}_{\delta_{ij}} = \sum w_j \underbrace{\frac{\partial(x_i \phi)}{\partial x_j}}_{\delta_{ij}}$$

on  $W$ . Therefore  $v_i = w_i$  on  $W$ .

### Variant

For  $W_i$  an open cover,  $L_V f = L_W f$  for all  $f \in C^\infty(W_i)$  implies that  $V = W$ .

### Compute

$$L_V L_W f = L_V \left( \sum w_j \frac{\partial f}{\partial x_j} \right) = \sum v_i \frac{\partial w_j}{\partial x_i} \frac{\partial f}{\partial x_j} + \sum v_i w_j \frac{\partial^2 f}{\partial x_i \partial x_j}$$

$$L_W L_V f = \sum w_i \frac{\partial v_j}{\partial x_i} \frac{\partial f}{\partial x_j} + \sum w_i v_j \frac{\partial^2 f}{\partial x_j \partial x_i}$$

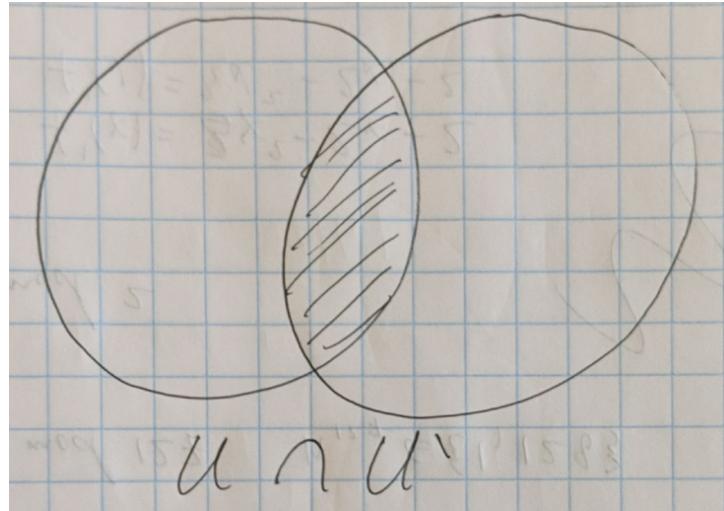
$$L_V L_W f - L_W L_V f = \sum_{i,j} \left( v_i \frac{\partial w_j}{\partial x_i} - w_i \frac{\partial v_j}{\partial x_i} \right) \frac{\partial f}{\partial x_j}$$

$$u = \sum_{i,j} \left( v_i \frac{\partial w_j}{\partial x_i} - w_i \frac{\partial v_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}$$

Therefore  $L_V L_W f - L_W L_V f = L_U f$ .

## Remark

Consider  $U \ni u$  and  $U' \ni u'$



## Properties

- Lie algebra: Skew symmetric and satisfying the Jacobi identity
- Product rule:  $[V, fW] = (L_V f)W + f[V, W]$ .

## Example

Let  $V$  be a finite dimensional vector space (e.g.  $\mathbb{R}^n$ ).

Recall that  $T_p V \cong V$  by  $[p + tv] \mapsto v$ . Then  $TV = V \times V$  ( $p, v$ ).

Take  $A \in \text{End}(V)$  given by

$$v(x) = Ax = \sum v_i \frac{\partial}{\partial x_i} = \begin{pmatrix} & a_{ij} & \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Take also  $w(x) = Bx$ . Then  $[V, W] = -(AB - BA)x = -[A, B]$ .

## Exercise

Given the example with  $W$  constant, determine  $[V, W]$ .

## Theorem (Midterm Problem)

Take  $A = C^\infty(M)$  and the derivations  $D \in \text{Der}(A)$  with  $D : A \xrightarrow{\text{lin.}} A$  over  $\mathbb{R}$  such that  $D(fg) = Df \cdot g + f \cdot Dg$ . There exists a linear isomorphism  $\mathfrak{X}(M) \xrightarrow{\cong} \text{Der}(A)$  given by  $v \mapsto L_v$ .

## Lemma

Take  $D \in \text{Der}(A)$ . Then  $D_p \in \mathcal{D}_p$  where  $D_p f := (Df)(p)$ .

$$D_p(fg) = (D_p f)g(p) + f(p)(D_p g)$$

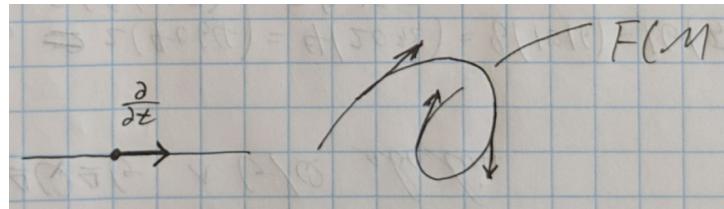
We know from above that  $\mathcal{D}_p \cong T_p m$ . Therefore  $L_v f = Df$ .

## Definition: Push Forward

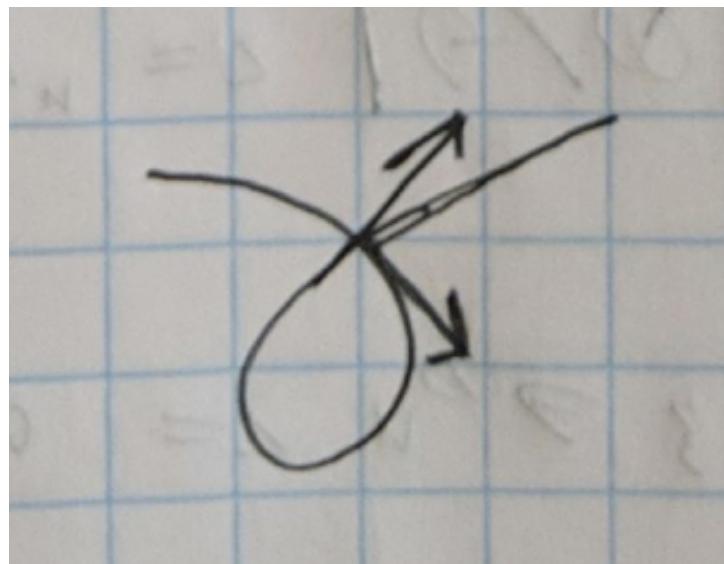
Take  $F : M \rightarrow N$  which gives rise to  $F_* : T_p M \rightarrow T_{F(p)} N$  (equivalent to  $F_* : \mathcal{D}_p \rightarrow \mathcal{D}_{F(p)}$ ) given by  $[\gamma] \mapsto [F \circ \gamma]$ . We see that  $(F_* D)(g) := D(g \circ F)$ .

### You Cannot Push Forward Vector Fields

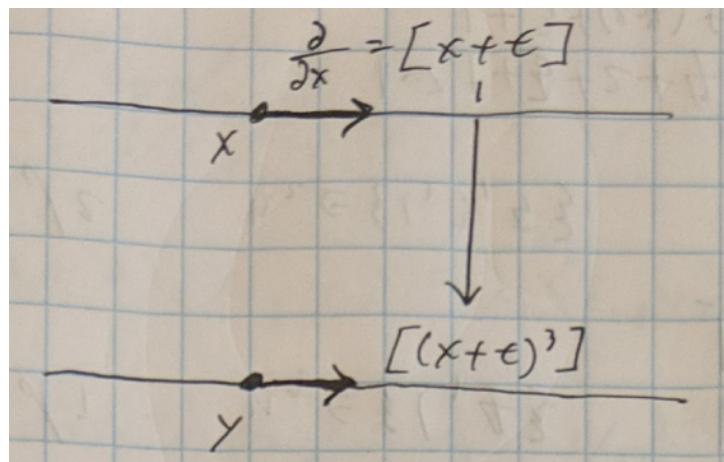
- if  $F$  is not surjective



- if  $F$  is not injective



- it is possible that  $F_* V$  would fail to be smooth. Take  $(F_* v)(y) = 3y^{2/3} \frac{\partial}{\partial y}$



## Remark

Take a diffeomorphism  $F : M \rightarrow N$ .

Then  $F : \mathfrak{X}(M) \rightarrow \mathfrak{X}(N)$  is given by  $F_*[V, W] = [F_*V, F_*W]$ .

**October 22, 2024**

## Dimension of Manifolds

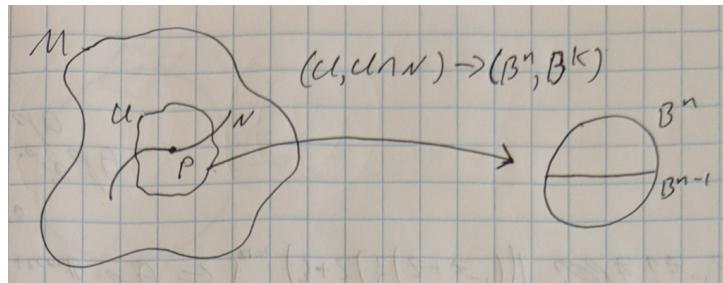
Take connected manifolds  $M^m$  and  $N^n$  with diffeomorphism  $M^m \xrightarrow[G]{F} N^n$ . If  $p \in M^m$  and  $q = F(p)$ , then

$$T_p M \xrightarrow[DG]{DF} T_q N$$

is linear and  $\dim T_p M = \dim T_q N$ .

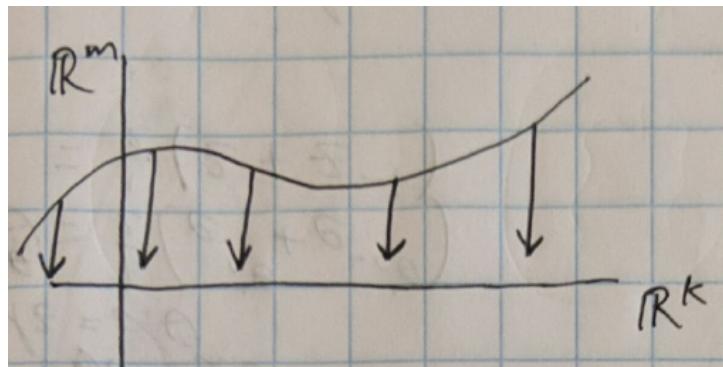
## Definition: Submanifold

$N \subseteq M^n$  is a submanifold if  $\forall p \in N$ , there exists a neighborhood  $U \ni p$  in  $M$  and a diffeomorphism  $U \rightarrow B^n$  which satisfies  $U \cap N \rightarrow B^k = B^n \cap \mathbb{R}^k$ .



## Example 1

Consider  $F : \mathbb{R}^k \rightarrow \mathbb{R}^n$  and its graph  $\text{Graph}(F) = \{(x, F(x))\}$ . This is a submanifold in  $\mathbb{R}^k \times \mathbb{R}^n$ .



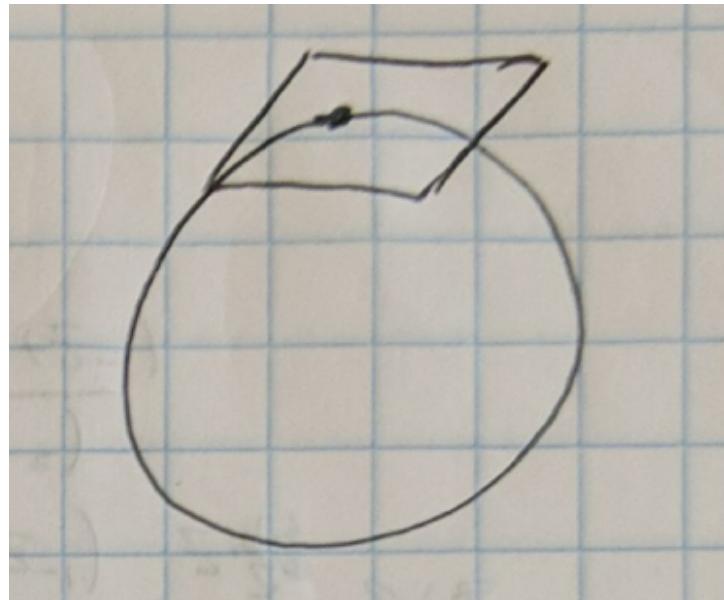
$\mathbb{R}^k \times \mathbb{R}^m \rightarrow \mathbb{R}^k \rightarrow \mathbb{R}^m$  given by  $(x, y) \mapsto (x, y - f(x))$ .

## Example 2

Take  $F : X \rightarrow Y$ ,  $X, Y$  manifolds. Then  $\text{Graph}(F) \subset X \times Y$  is a submanifold.

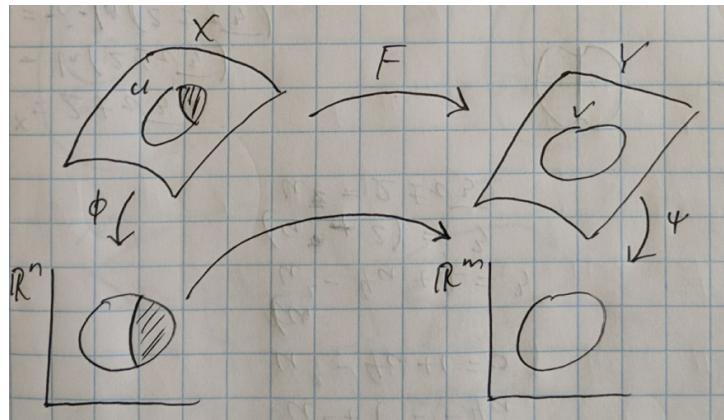
### Example 3

$S^1 \subset \mathbb{R}^2$  or  $S^{n-1} \subset \mathbb{R}^n$



### Example 4

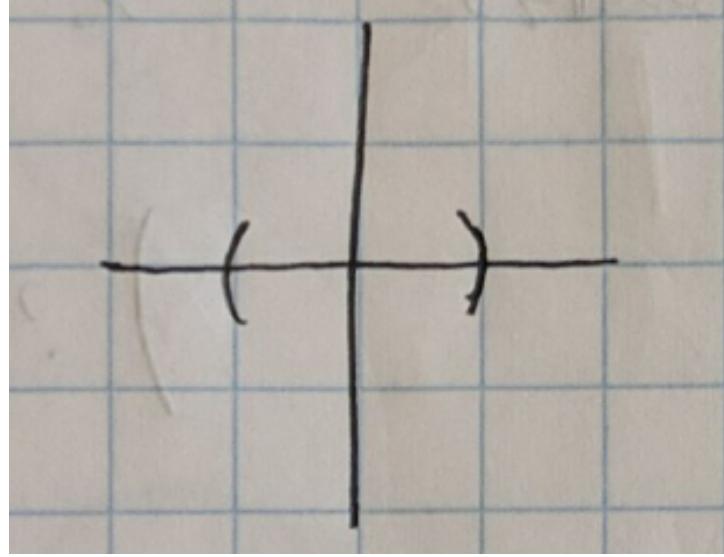
A graph is always a submanifold.



$\text{Graph}(F|_{U \cap F^{-1}(V)}) \subset U \times V$  implies  $\text{Graph}(\psi F p^{-1}|_{\dots}) \subset \phi(U) \times \psi(V)$ .

### Example 5

$(-1, 1) \times 0 \in \mathbb{R}^2$ .



### Example 6

$$\mathbb{RP}^n \subseteq \mathbb{CP}^n.$$

### Definition: Regular Point

Let  $M, N$  be manifolds with  $p \in M$ ,  $q = F(p) \in N$  and take  $F: M \rightarrow N$  with  $DF: T_p M \rightarrow T_q N$ .  $p$  is a regular point if  $DF$  is onto;  $q$  is a regular value if each  $p \in F^{-1}(q)$  is a regular point. Otherwise, they are called critical points or critical values.

### Remark

If  $\dim M < \dim N$ , then all points in  $M$  are critical and all values in  $F(M)$  are critical (while  $N \setminus F(M)$  are regular).

### Remark

For  $f: M \rightarrow \mathbb{R}$  with  $df: T_p M \rightarrow \mathbb{R} \cong T_{f(p)}\mathbb{R}$ . Locally,  $df = \left( \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right) = \nabla f$ . Then  $p$  is regular if and only if  $df \neq 0$ . Equivalently,  $p$  is regular if and only if there exists  $v \in T_p M$  such that  $L_v f \neq 0$ .

### Proof

$$L_v f = df(v) = \sum \frac{\partial f}{\partial x_i} v_i.$$

### Theorem:

Let  $q$  be a regular value. Then  $F^{-1}(q)$  is a submanifold.

### Example 1

$f: \mathbb{R}^n \rightarrow \mathbb{R}$  given by  $f(x) = \sum \lambda_i x_i^2$ ,  $\lambda_i \neq 0$ . Then  $f^{-1}(1)$  is a submanifold.

## Proof

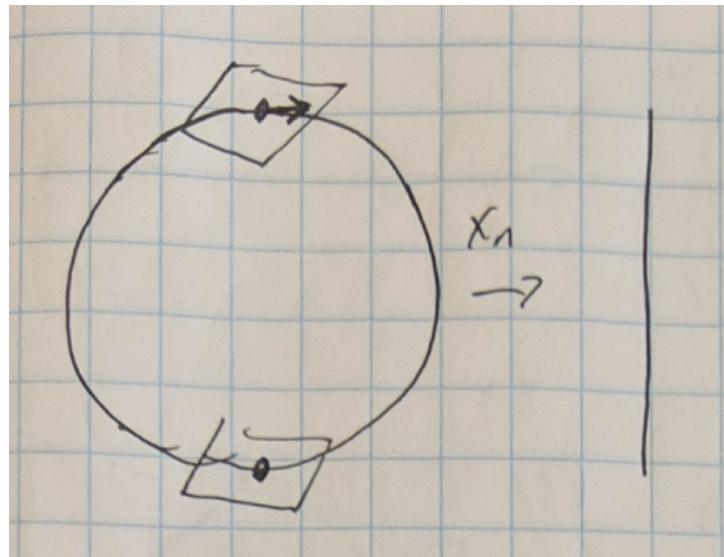
$x$  is a regular point when  $f(x) \neq 0$  ( $\nabla f \neq 0$ ).

$$v = \sum x_i \frac{\partial}{\partial x_i} = x$$

$$L_v f = \sum \underbrace{2 \lambda_i x_i}_{\frac{\partial f}{\partial x_i}} \underbrace{x_i}_{v_i} = 2f(x) \neq 0$$

## Example 2

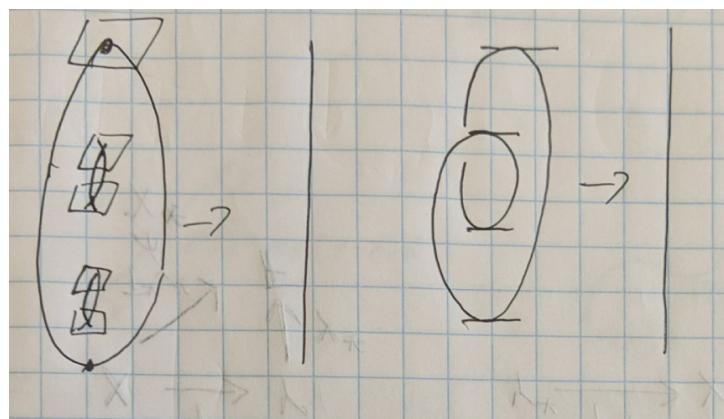
$$S^{n-1} \subset \mathbb{R}$$



$f = x_n : S^{n-1} \rightarrow \mathbb{R}$  the height function, so

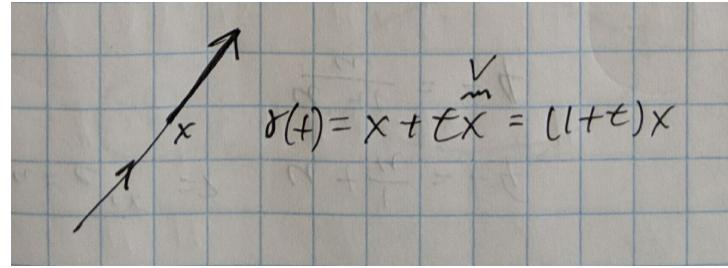
$$L_v f = L_v x_n = v_1 \frac{\partial x_n}{\partial x_1} + \cdots + v_{n-1} \frac{\partial x_n}{\partial x_{n-1}} + \underbrace{v_n \frac{\partial x_n}{\partial x_n}}_{=0}$$

The same follows for all projective maps.



## Theorem:

$$f(\lambda x) = \lambda^k f(x)$$



$$\begin{aligned} L_v f &= \frac{d}{dt} f((1+t)x)|_{t=0} \\ &= \underbrace{\frac{d}{dt} (1+t)^k}_{k} |_{t=0} f(x) \end{aligned}$$

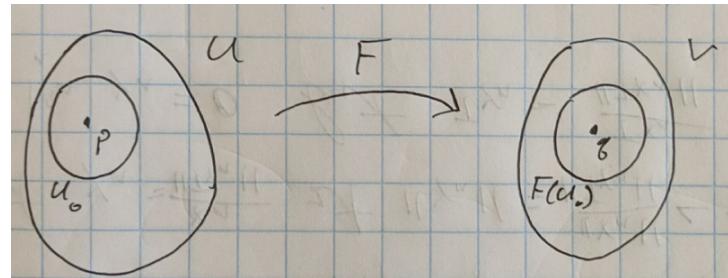
## Theorem: Inverse Function Theorem

Take  $U, V \subseteq \mathbb{R}^n$  open and a map  $F: U \rightarrow V$  with  $p \in U$  and  $q = f(p) \in V$ . Then take

$$\{DF: T_p U \rightarrow T_q V\} = \frac{\partial F_i}{\partial x_j}$$

such that  $\text{Rank } DF = n$  is an isomorphism.

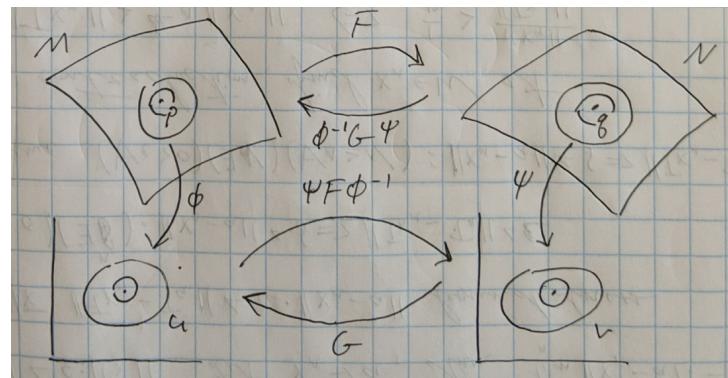
Then there exists  $U_0 \ni p$  such that  $U_0 \xrightarrow{F} F(U_0)$  is a diffeomorphism.



## Corollary

If  $F: M^n \rightarrow N^n$  with  $DF: T_p M \xrightarrow{\cong} T_q N$  ( $\text{Rank } DF = n$ ).

Then there exists a neighborhood  $U_0 \ni p$  such that  $F: U_0 \rightarrow F(U_0)$  is a diffeomorphism.



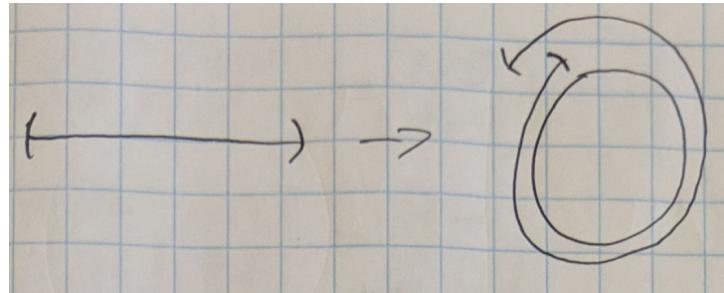
$$D(\psi F \phi^{-1}) = D\psi Df D\phi^{-1}.$$

## Local Diffeomorphism

Diffeomorphisms are local but local diffeomorphisms are not necessarily diffeomorphisms.

### Example 1

Take  $\mathbb{R}$  or any interval longer than 1 and map them to  $S^1$  by  $t \mapsto e^{2\pi i t}$ .



### Example 2

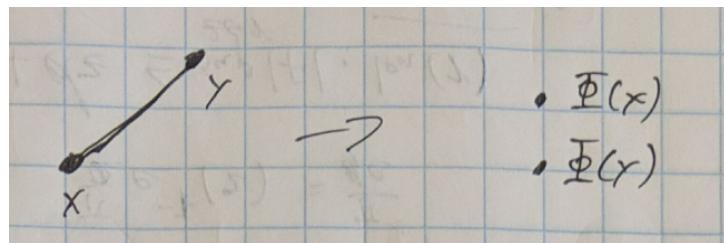
$S^1 \rightarrow S^1$  given by  $z \mapsto z$

### Example 3

$S^n \rightarrow \mathbb{RP}^n$ .

## Contraction Mapping Principle

Take  $\Phi : X \xrightarrow{C^0} X$  with  $X$  complete such that  $d(\Phi(x), \Phi(y)) < c \cdot d(x, y)$  with  $c \in (0, 1)$ . Then there exists a unique fixed point  $\Phi(p) = p$ .



### Example

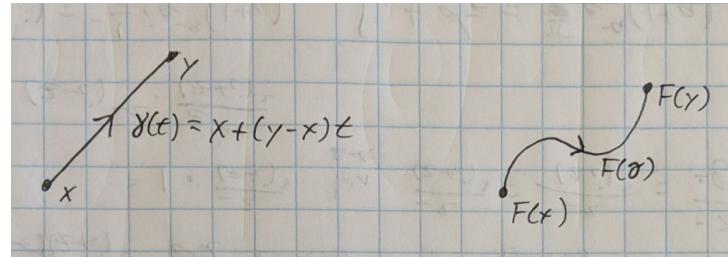
Take  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $\|DF_x\| < c < 1$ .

Recall that the operator norm for an operator  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is given by

$$\|A\|_{\text{op}} = \sup_{v \neq 0} \frac{\|Av\|}{\|v\|} = \sup_{\|v\|=1} \|Av\|$$

Then  $F$  is a contraction by  $c$ .

## Proof



$$d(x, y) = ||y - x|| = \int_0^1 ||\gamma'(t)|| dt$$

Then

$$\begin{aligned} d(F(x), F(y)) &\leq \int_0^1 ||F'(\gamma(t))|| dt \\ &\leq \int_0^1 ||DF_{\gamma(t)} \cdot \gamma'(t)|| dt \\ &\leq \int_0^1 \frac{c}{||DF_{\gamma}(t)||} \cdot ||\gamma'(t)|| dt \\ &\leq c \cdot \int_0^1 ||\gamma'(t)|| dt \\ &\leq c \cdot d(x, y) \end{aligned}$$

## Proof

Write  $\{x_k\} = \{\Phi^k(x)\}$ .

Claim:  $\{x_k\}$  is Cauchy which implies that  $x = \lim_{n \rightarrow \infty} x_n$ . Then

$$\Phi(x) = \lim_{n \rightarrow \infty} \Phi(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = x$$

Uniqueness follows from the observation that for fixed points  $d(x, y) \leq d(x, y)$  implies  $x = y$ .

### Proof that Sequence is Cauchy

For  $n \leq n+k$ ,

$$d(x_n, x_{n+k}) \leq d(\Phi(x_{n-1}, \Phi(x_{n+k-1})) \leq cd(x_{n-1}, x_{n+k-1}) \leq c^n d(x_0, x_k) \leq c^n L \cdot (1 + \dots + c^{k-1}) \leq \frac{L}{1-c} c^n \xrightarrow{n \rightarrow \infty} 0$$

since

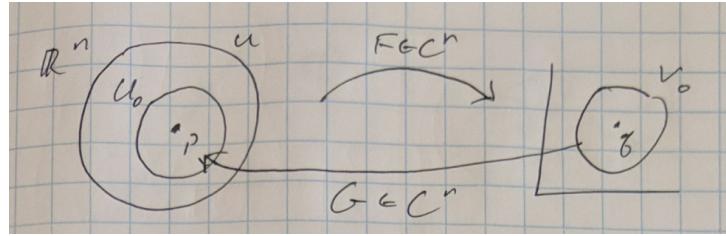
$$d(x_0, x_k) \leq \underbrace{d(x_0, x_1)}_L + \underbrace{d(x_1, x_2)}_{c \cdot L} + \dots + \underbrace{d(x_{k-1}, x_k)}_{c^{k-1} \cdot L}$$

**October 24, 2024**

### Recall: Contraction Mapping Theorem

$\Phi : X \rightarrow X$  complete, then there exists a unique fixed point  $x$  such that  $\Phi(x) = x$ .

## Recall: Inverse Function Theorem



$DF_p : T_p \mathbb{R}^n \rightarrow T_q \mathbb{R}^n$  implies there exist  $U_0$  and  $V_0$  such that

$$F : U_0 \xrightarrow[G]{} V_0$$

is a diffeomorphism with  $F \circ G = I$  and  $G \circ F = I$ .

### Remarks

Assume that  $p = 0 = q$  and that  $DF_0 = I$ .

Then  $F = I + \phi$  and  $D\phi_0 = 0$  (contracting). Look for  $G = I + g$ , then

$$\begin{aligned} F \circ G &= I \\ (I + \phi) \circ (I + g) &= I \\ -\phi \circ (I + g) &= g \end{aligned}$$

which gives us our fixed point. Then  $\Phi : g \mapsto -\phi \circ (I + g)$  with  $g : \overline{B(r)} \rightarrow \mathbb{R}^n$  is a continuous function on a Banach space giving

$$\underbrace{\overline{B(r)}}_{r \text{ adjustable}} \xrightarrow[C^0]{ } \underbrace{B(R)}_{R \text{ fixed}}$$

on a subset with  $\|g\| = \sup_{x \in B(r)} g(x)$  and  $\|g\| < R$ .

### Claim

$\Phi$  is contracting.

The contraction mapping theorem implies there exists a unique fixed point  $\Phi(g) = g$  where  $-\phi \circ (I + g) = g$ .

### Observations

$F$  is  $C^1$  and  $DF$  invertible implies  $G$  is  $C^1$ .

$F \circ G = I$ ,  $F(G(x)) = x$ ,  $(F' \circ G) \circ G' = I$ .

So  $G' = (F' \circ G)^{-1}$ .

### Remark

$$\begin{array}{ccc} \underbrace{U \times \mathbb{R}^n}_{TU} & \xrightarrow{F_*} & \underbrace{\mathbb{R}^n \times \mathbb{R}^n}_{T\mathbb{R}^n} \\ (F, DF) & & \\ & \xleftarrow{G_*} & \\ & (G, DG) & \end{array}$$

Then  $F$  being  $C^2$  implies  $F_*$  is  $C^1$  and invertible, and  $G_*$  is  $C^1$ .

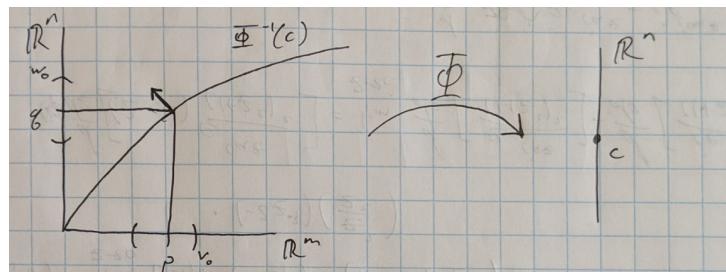
$$F_*(x, v) = (F(x), DF_x v)$$

### Theorem: Implicit Function Theorem

Take

$$(x, y) \in \mathbb{R}^m \times \mathbb{R}^n \supset U \ni (p, q) \xrightarrow[C^r]{\Phi} \mathbb{R}^n \ni c$$

with  $\frac{\partial \Phi}{\partial y}$  an invertible  $n \times n$  matrix.



$$(\nabla \Phi = \left( \frac{\partial \Phi}{\partial x_1}, \dots, \frac{\partial \Phi}{\partial x_n}, \frac{\partial \Phi}{\partial y} \neq 0 \right)).$$

Then there exist  $V_0 \ni p$  and  $W_0 \ni q$  such that in  $V_0 \times W_0$   $\Phi^{-1}(c)$  is the graph of  $g : V_0 \xrightarrow{C^r} W_0$  such that  $\Phi(x, g(x)) = c$ .

### Remark

IFT  $\iff$  IFT.

### Proof that Inverse Implies Implicit

Define  $F : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^m \times \mathbb{R}^n$  by  $(x, y) \mapsto (x, \Phi(x, y))$  (i.e.  $(p, q) \mapsto (p, c)$ ).

$$DF = \begin{matrix} & \mathbb{R}^m & \mathbb{R}^n \\ \mathbb{R}^m & \left[ \begin{array}{c|c} I & * \\ \hline 0 & \frac{\partial F}{\partial y} \end{array} \right] \end{matrix}$$

invertible. Applying the inverse function theorem,  $F \circ G = I$  implies  $G = (\text{id}, g)$  with  $g : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  where  $g(\cdot) = g(\cdot, c)$ . So  $\Phi(x, g(x)) = c$ .

### Theorem: Regular Value Theorem

Take  $F : M^{n+m} \rightarrow N^n \ni c$  with  $c$  a regular value.

Then  $F^{-1}(c)$  is a submanifold.

### Proof

We know that each  $p \in F^{-1}(c)$  is a regular point.

Then in local coordinates  $DF_p : T_p M^{n+m} \rightarrow T_c N^n$  is given by an  $n \times (m+n)$  matrix of rank  $n$ .

$$\begin{matrix} & x & y \\ \mathbb{C} & | & | \frac{\partial F}{\partial y} | \\ & | & | \frac{\partial F}{\partial y} | \end{matrix}$$

then locally  $F^{-1}$  is a graph and therefore a submanifold.

### Definition: Immersion

A map  $F : N^n \rightarrow M^m$  is an immersion if  $DF_p : T_p N \rightarrow T_{f(p)} M$  is one to one at every point ( $m \geq n$ ).

### Definition: Embedding

The map  $F : N^n \rightarrow M^m$  is an embedding if it is an immersion and a homeomorphism on its image.

## Definition: Submersion

The map  $F : N^n \rightarrow M^m$  is a submersion if  $DF_p$  is onto ( $m \leq n$ ).

## Examples

### Example 1

A local diffeomorphism is both an immersion and a submersion.

### Example 2

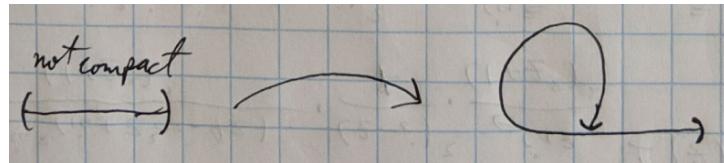
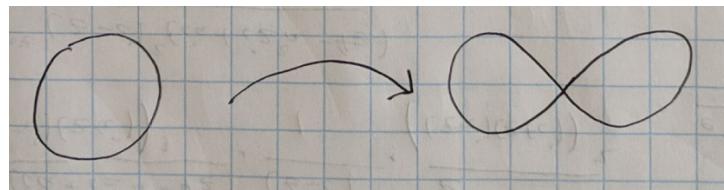
$\gamma : \mathbb{R} \rightarrow M$  is an immersion if and only if  $\gamma'(t) \neq 0$  for all points  $t$ .

### Example 3

For  $N$  compact,  $F$  is an embedding if and only if it is one-to-one and an immersion.

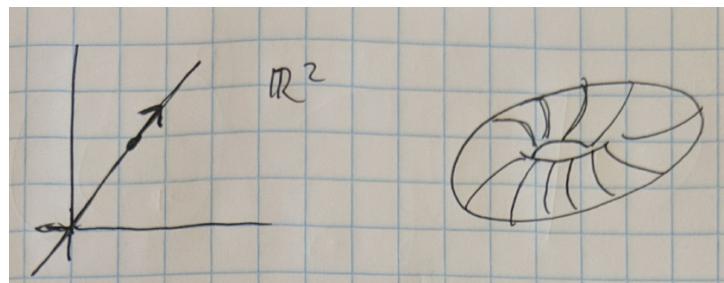
### Counter-example 1

Immersions but not embeddings.



### Counter-example 2

$$\begin{aligned}\mathbb{R} &\rightarrow \mathbb{R} \times \mathbb{R} \rightarrow S^1 \times S^1 = \mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2 \\ t &\mapsto (t, at)\end{aligned}$$



Can be given by composition of  $(e^{2\pi i t}, e^{2\pi i a t})$ .

## Remark

Consider  $A : V \rightarrow V$  with equivalence given by  $BAB^{-1} \sim A$ .

If  $A : V \rightarrow W$  then the equivalence given by  $B_1 A B_2^{-1} \sim A$ .

Consider  $f : \mathbb{R}, 0 \rightarrow \mathbb{R}$  with the assumption that  $f(0) = 0, f'(0) = 1$ . Then there exists a change of coordinates  $x = x(t)$  such that  $f(x(t)) = t$ .

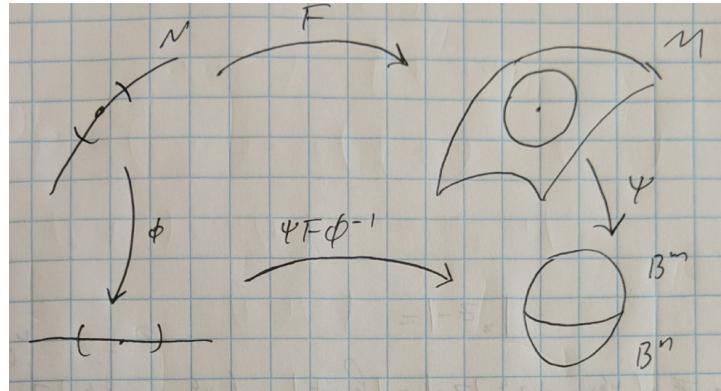
By the inverse function theorem, there exists  $g$  satisfying  $f(g(t)) = t$ .

Consider  $f(z)$  complex analytic with  $f(0) = 0$  and  $f'(0) = 1$ .

## Local Normal

Let  $F : p \in N^n \rightarrow M^m \ni q$  be an immersion.

Then there exist local coordinates  $x_1, \dots, x_n$  near  $p$  and  $y_1, \dots, y_m$  near  $q$  such that  $y_1 = x_1, \dots, y_n = x_n, y_{n+1} = 0, \dots$



**October 29, 2024**

## Recall: Local Normal Form Theorem for Immersions

For  $F : N^n \rightarrow M^m$  an immersion where  $\text{rank}(F) = n \leq m$  and  $p \in N$ , there exists a coordinates  $x_1, \dots, x_n$  near  $p$  and  $y_1, \dots, y_m$  near  $F(p)$  such that

$$\begin{aligned} y_1 &= x_1 \\ &\vdots \\ F: \quad y_n &= x_n \\ y_{n+1} &= 0 \\ &\vdots \\ y_m &= 0 \end{aligned}$$

IMAGE 1

## Proof

$$\begin{aligned} p \in V &\xrightarrow{F} U \ni q \\ p \rightarrow q &\in F(p) \end{aligned}$$

$\text{rank } DF_p = n \leq m$ , for example.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

IMAGE 2

$$\begin{array}{ll} y_1 = z_1 & y_1 = x_1 \\ \vdots & \vdots \\ y_n = z_n & y_n = x_n \\ \text{and} & y_{n+1} = z_{n+1} - F_{n+1}(z_1, \dots, z_n) \\ & \vdots \\ & y_m = z_m - F_m(z_1, \dots, z_m) \end{array}$$

### Theorem: Local Normal Form Theorem for Submersions.

Assume that  $F : N^n \rightarrow M^m$  is a submersion such that  $\text{rank } DF = m \leq n$ .

Then we may take coordinates  $x_1, \dots, x_n$  and  $y_1, \dots, y_m$  such that  $y_1 = x_1, \dots, y_m = x_m$ .  
The procedure is similar to the previous theorem.

### Remark

Immersions look locally like linear subspaces.

Submersions look locally like projections.

### Theorem: Constant Rank Theorem

Assume that  $\text{rank } DF = k$  for a constant  $k$ .

Then for coordinates  $x_1, \dots, x_n$  and  $y_1, \dots, y_m$ , we can have  $y_1 = x_1, \dots, y_k = x_k, y_{k+1} = 0, \dots$

### Homework

Take  $f$  a germ at 0 on  $\mathbb{R}^n$  and take  $df_0 \neq 0$  ( $\nabla f_0 \neq 0$ ).

Then there exists a coordinate system  $y_1, \dots, y_n$  such that  $f(y) = f(0) + y_1$ .

That is, for  $(x_1, \dots, x_n)$  there exists a diffeomorphism  $(f \circ \phi)(x_1, \dots, x_n) = f(0) + x_1$ .

Note that we do not have control over the  $x$  coordinates, so the previous theorems will not work.

### Remark

Take  $K \xrightarrow{G} N \xrightarrow{F} M$ .

If  $G, F$  are submersions, then  $FG$  is a submersion.

If  $G, F$  are immersions, then  $FG$  is an immersion.

### Theorem

Let  $F : N \rightarrow M$  be an embedding.

Then  $F(N)$  is a submanifold. Consequently,  $F : N \rightarrow F(N)$  is a diffeomorphism.

IMAGE 3

If  $N$  is a submanifold of  $M$ , then  $N \hookrightarrow M$  is an embedding.

### Example

Take  $\mathbb{R} \rightarrow \mathbb{R}^2$  by  $t \mapsto (0, t^3)$ .

IMAGE 4

This is not even an immersion.

### Example 2 (Homework)

For  $N$  the graph of  $y = |x|$ , there exists  $F : \mathbb{R} \xrightarrow[C^\infty_{1-1}]{} \mathbb{R}^2$  such that  $N = F(\mathbb{R})$ .

IMAGE 5

### Submanifolds

For any  $q \in X$ , there exists a neighborhood  $U$  and a diffeomorphism  $\phi : (U, U \cap X) \rightarrow (W, W \cap \mathbb{R}^n)$ .

IMAGE 6

### Proof

And embedding is an immersion which is homeomorphic on  $F(N)$

With  $F$  an immersion, we have

IMAGE 7

Since  $F$  is also a homeomorphism, there exists  $U'$  such that  $U' \cap X = F(V)$ .

### Zero Measure Sets

#### Key Point

Zero measure sets are unambiguously defined.

#### Definition: Zero Measure Set

$X \subseteq M$  is a zero measure set if

- For every closed chart  $(U, \phi)$ ,  $\phi(U \cap X) \subset \mathbb{R}^n$  has zero Lebesgue measure.
- There exists a coordinate atlant  $(\{\phi_i\})$  such that  $\phi_i(U_i \cap X)$  has zero measure.

#### Lemma

$\mathbb{R}^n \supset W \xrightarrow[C^1]{F} \mathbb{R}^n$  and  $Y \in W$  where  $m(Y) = 0$ .

Then  $F(Y)$  has zero measure.

## Proof

IMAGE 8

We may cover  $Y$  by a countable collection of open sets  $U$  such that  $\overline{U} \subset W$ .

Then it suffices to show that  $F(Y \cap U)$  has zero measure.

Then  $\|DF\| \leq C$ . Given  $\varepsilon > 0$ , we need a cover  $F(Y)$  by open balls with total volume less than  $\varepsilon$ .

Pick  $p \in Y$  and define  $B_r(p) = \{z \in U : \|z - p\| \leq r\}$ . Then  $F(B_r(p)) \subset B_{Cr}(F(p))$ .

IMAGE 9

For  $p = 0 = F(p)$

IMAGE 10

$$\|F(z)\| \leq \text{length}(t \mapsto F(tz)) = \int_0^1 \left\| \frac{d}{dt} F(tz) \right\| dt \leq \int_0^1 \|DF\| \cdot \|z\| dt \leq Cr$$

Since  $Y$  has zero measure, it can be covered by balls  $B_{r_i}(p_i)$  with total volume less than  $\delta$ .

Therefore  $F(Y)$  is covered by  $B_{Cr_i}(F(p_i))$ . Then the sum is bounded above by  $C^n \delta$ .

Take  $\delta = \frac{\varepsilon}{C^n}$ .

## Definition: Smooth Measure

Let  $\mu$  be a measure on  $M$ .

$\mu$  is smooth if

- for every coordinate chart  $(U, \phi = (x_1, \dots, x_n))$ ,  $\mu = f dx_1 \dots dx_n$  ( $f > 0$ ).
- there exists a coordinate atlas  $(U_i, \phi_i)$  such that the same is true.

## Proposition

A smooth measure exists.

## Proof

Take  $(U_i, \phi_i)$ ,  $dx_1 \dots dx_n = \mu_i$ , and the partition of unity  $h_i$  subordinated to  $U_i$  ( $\text{supp } h_i \subseteq U_i$  and  $\sum h_i \equiv 1$ )  
Then  $h_i \mu_i$  is a measure on  $M$ . Then we may take  $\sum h_i \mu_i$ .

## Proposition

The following three conditions are equivalent.

- $X$  is a zero measure set
- there exists a smooth, positive  $\mu$  such that  $\mu(X) = 0$
- every smooth positive  $\mu$  has  $\mu(X) = 0$ .

**October 31, 2024**

## Theorem: Sard

Let  $F: M^m \xrightarrow{C^r} N^n$  with  $r > \max\{0, m - n\}$ . The set of critical values has zero measure.

### Remarks

There are two cases.  $m \leq n$  permits  $C^1$ . This is the easy case because there are no  $C^1$  space filling curves. If  $m > n$ , then  $r > m - n + 1$ . This case is difficult.

### Corollary

There exist regular values.

The set of regular values is everywhere dense.

### Proof

For  $M = [0, 1]$ ,  $N = \mathbb{R}$ ,  $n = m = 1$ , and  $f: [0, 1] \xrightarrow{C^r} \mathbb{R}$ .

Separate the interval into segments  $I_i$  of length  $1/k$ .

IMAGE 1

Consider all intervals containing critical points.

Then  $f'$  is uniformly continuous. That is for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $|x - y| < \delta \implies |f'(x) - f'(y)| < \varepsilon$ .

Consider the critical values covered by intervals of total length  $< \varepsilon$  and  $f(I_i)$  containing critical points.

Choose  $1/k < \delta$ , then  $|f'|_{I_i}| < \varepsilon$ . So  $|f(I_i)|\varepsilon \cdot 1/k$  and, consequently,

$$\sum_{I_i} |f(I_i)| < k \cdot \frac{k}{\varepsilon} = \varepsilon$$

## Theorem: Brouwer Fixed Point Theorem

A map  $G: \overline{B}^n \xrightarrow{C^0} \overline{B}^n$  has a fixed point  $G(x) = x$ .

### Fact

The one dimensional manifolds are exactly  $\mathbb{R} = (0, 1)$ ,  $S^1$ ,  $[0, 1]$  and  $[0, 1]$ .

### Fact

Consider  $F: M \rightarrow N$  where  $\partial N = \emptyset$  with  $q$  regular for  $F$  and  $F|_{\partial M}$ .

Then  $F^{-1}(q)$  is a manifold with boundary and  $\partial F^{-1}(q) = F^{-1}(q) \cap \partial M$ .

IMAGE 2

### Lemma

Let  $M$  be a compact manifold with boundary. Then no map exists such that  $F: M \xrightarrow{C^\infty} \partial M$  such that  $F|_{\partial M} = \text{id}$ .

## Proof

Write

$$\begin{array}{ccc} F : M & \xrightarrow{\quad} & \partial M \ni q \\ \uparrow & \nearrow id & \\ \partial M & & \end{array}$$

where  $q$  is a regular value.

Then  $F^{-1}(q)$  is a submanifold with dimension  $\dim M - \dim \partial M = n - (n - 1) = 1$ .

Then the only candidate closed manifold with boundary of dimension 1 is  $[0, 1]$ .

## IMAGE 3

But since the restriction is the identity and the boundary must lie in the boundary of  $M$ , we have a contradiction.

## Corollary

There is no function  $F : \overline{B}^n \xrightarrow{C^\infty} S^{n-1}$  such that  $F|_{\partial \overline{B}^n = S^{n-1}} = \text{id}$ .

## Lemma

$G : \overline{B}^n \xrightarrow{C^\infty} \overline{B}^n$  has a fixed point.

## Proof

Assume the contrary, that there exists  $g : \overline{B}^n \xrightarrow{C^\infty} \overline{B}^n$  such that  $g(x) \neq x$ .

## IMAGE 4

Then consider  $x \mapsto G(x)$  with  $G|_{\partial \overline{B}^n} = \text{id}$ . Then, explicitly,

$$g(x) + t \cdot \frac{x - g(x)}{\|x - g(x)\|} = G(x)$$

However, by the previous lemma, this is impossible.

## Fact

Every continuous map  $G : \overline{B}^n \rightarrow \mathbb{R}^k$  can be  $C^0$ -approximated by a smooth map  $P$ .

$$\|G - P\| = \sup_{x \in \overline{B}^n} \|G(x) - P(x)\| < \mu$$

## Proof of Theorem

Assume  $G : \overline{B}^N \rightarrow \overline{B}^n$  with  $G(x) \neq x$  with  $0 < \varepsilon < \min \|G(x) - x\|$ . However  $P : \overline{B}^n \rightarrow \overline{B}^n(1 + \varepsilon)$ . Then for  $F = \frac{1}{1+\varepsilon}P$ ,  $G \approx P \approx F$ .

## Theorem

Let  $M^n$  a smooth manifold. Then there exists an embedding  $M \hookrightarrow \mathbb{R}^N$ .

## Proof

(Assuming that  $M$  is compact)

Pick some finite cover by coordinate charts  $(U_i, \phi_i)$  such that  $V_i \subseteq \overline{V}_i \subseteq U_i$  with  $V_i$  also a finite cover.

IMAGE 5

Let  $f_i$  be such that  $f_i|_{V_i} \equiv 1$  and  $\text{supp } f_i \subset U_i$ . Further, assume that  $f < 1$  outside  $\overline{V}_i$ .

IMAGE 6

Then  $f_i \phi_i : M \rightarrow \mathbb{R}^n$ .

Define  $F : M \rightarrow \mathbb{R}^n$  by  $(f_1 \phi_1, \dots, f_k \phi_k, f_1, \dots, f_k)$  such that  $N = nk + k$ .

Claim:  $F$  is an embedding. That is, it is an immersion which is homeomorphic on its image. Equivalently, it is an immersion and one-to-one.

We need that  $\text{rank } DF = n$ . Take  $x \in V_1$  where  $f_1 \phi_1 = \phi_1$  on  $V_1$ . Then  $D(f_1 \phi_1)_x = (D\phi_1)_x$  which is of rank  $n$ . So  $F$  must be of at least (and, in fact, at most) rank  $n$ . That is,  $F$  is an immersion.

Again, take  $x \in V_1$  and consider  $F(x) = F(y)$ . Then  $f_1(x) = f_1(y)$  and, consequently,  $y \in \overline{V}_1$ .

Therefore  $f_1 \phi_1(x) = f_1 \phi_1(y)$  which implies  $\phi_1(x) = \phi_1(y)$  and  $x = y$ . Therefore  $F$  is one-to-one.

**November 5, 2024**

## Theorem: (Weak) Whitney Embedding Theorem

If  $M^n$  is a smooth manifold, then

- there exists an embedding  $M \hookrightarrow \mathbb{R}^{2n+1}$ .
- there exists an immersion  $M \hookrightarrow \mathbb{R}^{2n}$ .

## Remark: (Strong) Whitney Embedding Theorem

For  $M^n$  smooth and compact,

- there exists an embedding  $M \hookrightarrow \mathbb{R}^{2n}$
- there exists an immersion  $M \hookrightarrow \mathbb{R}^{2n-1}$

This is sharp, however we can do better for some  $M$  (e.g.  $S^n \hookrightarrow \mathbb{R}^{n+1}$ ).

## Tangent Bundle

$$\begin{array}{ccc} TM & & T_p M = \pi^{-1}(p) \\ \downarrow \pi & & \\ U \subset M \ni p & & \end{array}$$

IMAGE 1

Where we have a basis  $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$  in  $T_p M$  for any  $p \in U$ . Then, writing  $v = \sum v_i \frac{\partial}{\partial x_i}$ ,

$$TM|_U = \pi^{-1}(U) \xrightarrow[C^{r-1}]{(\phi, D\phi)} (\underbrace{\phi(p)}_{\phi\pi(p)}, \underbrace{(v_1, \dots, v_n)}_{D\phi(v) \in \mathbb{R}^n})$$

## Properties

$F: M \xrightarrow{C^r} N$  implies  $DF: TM \xrightarrow{C^{r-1}} TN$  by  $v \mapsto DF(v)$  where  $DF(V)$  is  $F_* v$ .

If  $F$  is an embedding (or immersion), then so is  $DF$ .

If  $F$  is a submersion, then so is  $DF$ .

## Examples

$$T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n.$$

$$TV = V \times V.$$

$$TS^1 = S^1 \times \mathbb{R} \text{ by } (p, v) \mapsto (p, a).$$

## IMAGE 2

So any vector field  $v$  may be written as  $v = a \frac{\partial}{\partial \theta}$

If there exist  $v_1, \dots, v_n$  such that  $v_1(p), \dots, v_n(p)$  is a basis, then  $TM = M \times \mathbb{R}^n$ .

$$T\mathbb{T}^n = \mathbb{T}^n \times \mathbb{R}^n.$$

Counterexample:  $TS^2 \not\cong S^2 \times \mathbb{R}^2$ .

## Example

Take an embedding  $M \hookrightarrow \mathbb{R}^k$ . Then  $TM \hookrightarrow T\mathbb{R}^k = \mathbb{R}^k \times \mathbb{R}^k$  is also an embedding.

## IMAGE 3

The map  $(q, v) \mapsto q + v$  is not useful, but  $(q, v) \mapsto v \in \mathbb{R}^k$  is.

We have a Gauss map  $M \xrightarrow{G} \text{Gr}(n, k)$  by  $M \ni q \mapsto T_q M \in \text{Gr}(n, k)$ .

## Example

$$M \text{ a hypersurface } M^n \subset \mathbb{R}^{n+1}$$

## IMAGE 4

$q \mapsto T_q M$  which is " $=$ " to the normal line.

$$M \rightarrow \text{GR}(n+1, n) \xleftrightarrow{\perp} \mathbb{RP}^n$$

$$p \longmapsto v_p$$

$$\begin{array}{ccc} M & \longrightarrow & S^n \subset \mathbb{R}^{n+1} \\ & \searrow & \downarrow \\ & & \mathbb{RP}^n \end{array}$$

## Definition: Riemannian Metric

A Riemannian metric is a smooth map  $g : TM \rightarrow \mathbb{R}$  such that for each  $p \in M$ ,  $g|_{T_p M}$  is a positive-definite quadratic form where  $g(v, v) \geq 0$  and  $g(v, v) = 0$  if and only if  $v = 0$ .

In local coordinates,  $(x_1, \dots, x_n)$  and with  $p \in M$  and  $v \in T_p M$ ,  $g_p(v) = \sum g_{ij}(p)v_i v_j$  where  $g_{ij}(p)$  is symmetric. We have that positive quadratic forms are in one-to-one correspondence with inner products.

$$\langle v, w \rangle = \frac{1}{2}(g(v + w) - g(v) - g(w))$$

In coordinates,

$$\langle v, w \rangle = \sum g_{ij}(p)v_i w_j$$

### Example

Take an immersion  $M \hookrightarrow \mathbb{R}^k$

IMAGE 5

The inner product in  $\mathbb{R}^k$  restricts to a Riemannian metric on  $M$ .

### Theorem

Every manifold admits a Riemannian metric.

#### Proof 1

Since any manifold may be immersed  $M \hookrightarrow \mathbb{R}^k$ , we may define a Riemannian metric.

#### Proof 2

Take a locally finite cover by coordinate charts,  $\{(U_i, \phi_i) = (x_1, \dots, x_n)\}$ .

Within every chart, take a Riemannian metric  $g_i$  such that  $\left\langle \frac{\partial}{\partial x_s}, \frac{\partial}{\partial x_r} \right\rangle = \delta_{sr}$ .

Take a partition of unity  $f_i > 0$  subordinated to  $U_i$  ( $\text{supp } f_i \subset U_i$  and  $\sum f_i \equiv 1$ ). Then

$$g = \sum f_i g_i$$

$$g(v) = \sum f_i(p) g_i(v) > 0$$

with  $v \neq 0$ ,  $g_i > 0$  and  $f_i > 0$  for some  $i$ .

## Definition: Unit Tangent Bundle

Take some manifold  $M$  and a Riemannian metric  $g$ .

Then  $g^{-1}(1) =: STM$  is the unit tangent bundle.

### Claim 1

The unit tangent bundle is a smooth submanifold of codim = 1 and dim =  $2n - 1$ .

## Proof 1

Consider  $\phi_\lambda : TM \rightarrow TM$  by  $v \mapsto \lambda v$  with  $\lambda \neq 0$ .

Since  $g(\phi_\lambda(v)) = \lambda^2 g(v)$ , it follows that (exercise)  $a$  is a regular value of  $g$  if and only if  $\lambda^2 a$  is a regular value. That is, for

$$M \xrightarrow{\phi} M \xrightarrow{f} \mathbb{R}$$

$p$  is a regular value of  $f \circ \phi$  if and only if  $\phi(p)$  is a regular value of  $f$  and  $a$  is a regular value of  $f$  if and only if  $\lambda^2 a$  is a regular value of  $\lambda^2 f$ .

## Proof 2

$(p, v)$  is a regular point if and only if there exists some  $w \in T(TM)$  such that  $L_w g \neq 0$ .

IMAGE 6

For  $w = v$ ,

$$\frac{\partial}{\partial \lambda} g(\lambda v)|_{\lambda=1} = \frac{\partial}{\partial \lambda} \lambda^2 g(v)|_{\lambda=1} = 2g(v) \neq 0$$

## Claim 2

If  $g_0$  and  $g_1$  are Riemannian metrics on  $M$ , then  $STM_{g_0} \cong STM_{g_1}$  are diffeomorphic.

For  $V$ ,  $g_0$  and  $g_1$  with  $S_1$  and  $S_2$ ,

IMAGE 7

The map  $x \mapsto \frac{g_0(x)}{g_1(x)} x$ .

## Exercise

If  $M \hookrightarrow \mathbb{R}^k$  with an induced metric on  $M$ , then  $STM \hookrightarrow \mathbb{R}^k \times S^{k-1}$  by  $(p, v) \mapsto v \in S^{k-1}$  induces  $STM \rightarrow S^{k-1}$ .

IMAGE 8

## Example

$$STS^2 \cong SO(3) \cong \mathbb{RP}^3$$

IMAGE 9

Take  $p \perp v$ ,  $\|p\| = \|v\| = 1$ , we can generate an orthonormal basis  $(p, v, p \times v)$ .

Recall that  $S^n/x \sim -x = \mathbb{RP}^n$ , but that we may also identify  $B^n/x \sim -x$  on  $S^{n-1}$ .

That is,  $\mathbb{RP}^3 = B^3/\sim$  (with radius  $\pi$ ).

IMAGE 10

Take  $A \in SO(3)$ .

**November 7, 2024**

## Theorem: (Weak) Whitney Embedding Theorem

For  $M^n$ ,

1. there exists an embedding  $M \hookrightarrow \mathbb{R}^{2n+1}$ .
2. there exists an immersion  $M \hookrightarrow \mathbb{R}^{2n}$ .

### Proof

(For simplicity, consider  $M$  compact)

$$\begin{array}{ccc} M & \xleftarrow{F} & \mathbb{R}^m \ni v \\ & \searrow F_v & \downarrow \pi_v \\ & & \mathbb{R}^{m-1} \end{array}$$

Where  $\|v\| = 1$ .

Hope: for  $m$  sufficiently large,  $F_v$  is an embedding and immersion.

### What could go wrong?

- $F_v$  not 1-1.

IMAGE 1

$$p - q = t \cdot v, \frac{p - q}{\|p - q\|} = \pm v.$$

So take  $\Phi : M \times M \setminus \Delta \rightarrow S^{m-1}$  (where  $\Delta$  is the diagonal) by  $(p, q) \mapsto \frac{p - q}{\|p - q\|}$ .

We need  $v \notin \Phi(M \times M \setminus \Delta)$ . Since  $2n < m - 1$ , this map has zero measure.

- $F_v$  not an immersion.

IMAGE 2

Take  $\psi : STM \rightarrow S^{m-1}$  by  $(p, w) \mapsto w$ . Then  $v \notin \psi(STM)$ .

Then  $F_v$  is an immersion where  $2n - 1 < m - 1$  ( $2n < m$ ).

Then for immersions, we may continue the process until  $2n = m$ ; for embeddings, only until  $2n + 1 = m$ .

### Remarks

$\Psi : TM \rightarrow \mathbb{R}^k$  by  $(p, w) \mapsto w$ .

$\Phi : (M \times M) \times \mathbb{R} \rightarrow \mathbb{R}^k$  by  $(p, q, t) \mapsto (p - q)t$ .

## Homework

### Problem 1

Take  $L, M \subset \mathbb{R}^k$  submanifolds such that  $\dim L + \dim M < k$ .

Then for almost all  $x$ ,  $(x + L) \cap M = \emptyset$ .

## Problem 2

Take  $M^n \subset \mathbb{R}^k$ ,  $k > 2n$ , and a projection  $\rho_x : p \mapsto \frac{p-x}{\|p-x\|}$ .

IMAGE 3

For almost all  $x \in \mathbb{R}^n$ ,  $\rho_x$  is an immersion.

## Problem 3

Take  $L \subset M \times N \xrightarrow{\pi} N$ .

IMAGE 4

Then  $\pi^{-1}(q) \cap L = L_q$ .

For almost all  $q$ ,  $L_q$  is a smooth submanifold. Hint:  $\pi|_L$ .

## R Smooth Topology

Take  $M$  compact and  $0 \leq r \leq \infty$ .

For  $f : M \rightarrow \mathbb{R}^k$  (with either  $k = 1$  or considered component-wise), define

$$\|f\|_{C^0} = \sup_{x \in M} |f(x)|$$

Then  $C^0(M, \mathbb{R}^k)$  is a Banach space with induced distance  $d(f, g) = \|f - g\|_{C^0}$ .

For  $1 \leq r < \infty$ , take a finite coordinate cover  $(U_\ell, \phi_\ell)$  (which is contained within some coordinate cover  $(W_\ell, \phi_\ell)$ ). For  $r = 1$ , define

$$\|f\|_{C^1} = \|f\|_{C^0} + \sum_{i, \ell} \left| \frac{\partial f}{\partial x_i^{(\ell)}} \right|$$

where  $f = f \circ \phi_\ell$  and we may equivalently take  $\max\{i, \ell\}$  rather than the sum.

Then  $C^1(M, \mathbb{R})$  is a Banach space with well-defined topology. Similarly, for arbitrary  $r$ ,

$$\|f\|_{C^r} = \|f\|_{C^{r-1}} + \sum_{\ell, i_1, \dots, i_r} \left| \frac{\partial^r f}{\partial x_{i_1} \cdots \partial x_{i_r}} \right|$$

with equivalence again to a maximum. Finally, for  $r = \infty$ ,  $C^\infty \subset C^r$  for any finite  $r$ .

The  $C^\infty$  topology is generated by all  $C^r$  topologies.

$$C^0 \supset C^1 \supset C^2 \supset \cdots \supset C^r \supset C^{r+1} \supset \cdots \supset C^\infty$$

Convergence:  $f_i \xrightarrow{C^r} f$  if and only if  $f_i \rightarrow f$ , and so do all partial derivatives up to  $r$ .  $\frac{\partial f_i}{\partial x_c} \rightarrow \frac{\partial f}{\partial x_c}$ .

Note that  $C^\infty(M, \mathbb{R})$  is not a Banach space. It is, however, Fréchet.

## N Smooth Topology

Instead, take  $C^r(M, N)$  with  $0 \leq r \leq \infty$  and  $M$  compact. Define

$$M \xrightarrow{f} N \hookrightarrow \mathbb{R}^k$$

## Theorem

For  $M$  compact,  $r \geq 1$

1. the set of immersion  $M \hookrightarrow \mathbb{R}^k$  is  $C^r$  open.
2. the same is true for embeddings.

## Proof

Let  $F: M \rightarrow \mathbb{R}^k$  be an immersion and  $G C^1$  close to  $F$ .

$$\begin{array}{ccc} T_p(M) & \xrightarrow{DF} & T_{F(p)}\mathbb{R}^k \\ & \searrow DG & \\ & & T_{G(p)}\mathbb{R}^k \end{array} \quad \text{rank} = n$$

## Theorem

For  $M^n \xrightarrow{C^r} \mathbb{R}^k$  with  $M$  compact and  $r \geq 1$ , if  $k \geq 2n$  then immersions form a  $C^r$  open and dense set.  
If  $k \geq 2n + 1$  then the same is true for embeddings.

## Lowering the Dimension

Take  $F: M \rightarrow \mathbb{R}^k$ . We want an embedding/immersion approximation.

We know that there exists an embedding  $M \xrightarrow{G} \mathbb{R}^M$  for sufficiently large  $m$ , so consider  $\tilde{F} := (F, \varepsilon G): M \rightarrow \mathbb{R}^k \times \mathbb{R}^m$ .

Note that  $(F, \varepsilon G) \xrightarrow{C^r} (F, 0)$ .

Take a basis of  $\mathbb{R}^m$ ,  $(e_1, \dots, e_m)$ . We can take vectors close to our basis  $v_1 \approx e_1$ ,  $v_2 \approx e_2$ , etc. We want to construct a series of projections  $\mathbb{R}^k \times \mathbb{R}^m \xrightarrow{\pi_{v_1}} \mathbb{R}^k \times \mathbb{R}^{m-1} \xrightarrow{\pi_{v_2}} \dots \rightarrow \mathbb{R}^k$ .

IMAGE 5

## Review: Ordinary Differential Equations

On  $\mathbb{R}^n$ ,

$$x' = v(x) \text{ on } U \subset \mathbb{R}^n$$

with  $x = (x_1, \dots, x_n)$ . We want a map  $x$  from some open interval to  $U$  satisfying

$$\begin{cases} x'_1(t) = v_1(x(t)) \\ \vdots \\ x'_n(t) = v_n(x(t)) \end{cases}$$

with  $x(0) = p$ . Geometrically

IMAGE 6

where  $x'(t) = v(x(t))$  is an integral curve.

## Theorem

1. Existence: for every  $p \in U$  and  $t_0 \in \mathbb{R}$ , there exists  $\varepsilon > 0$  such that there is an integral curve  $\gamma : (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow U$  (where  $\gamma(t_0) = p$ ).
2. Uniqueness: let  $\gamma : (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow U$  and  $\eta : (t_0 - \delta, t_0 + \delta) \rightarrow U$  be two integral curves through  $p$  at  $t_0$ . Then  $\gamma = \eta$  on the common domain.

We may refine (a) as

1. for every  $p$  and  $t_0$ , there exists  $\varepsilon > 0$  on a neighborhood  $V \ni p$  such that for every  $q \in V$  there exists an integral curve  $\gamma_q : (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow U$  where  $\gamma_q(t_0) = q$ .
2.  $\gamma_q(t)$  is smooth in  $q$  and  $t$ . Namely, on  $V \times (t_0 - \varepsilon, t_0 + \varepsilon)$ .

IMAGE 7

## With Parameters

Consider  $x' = v(x, \lambda)$ . Then we have  $p$  and  $\gamma_{p,\lambda}$  which depends smoothly on  $\lambda$ .

$$x = v(x, \lambda) \quad (p, \lambda) \quad \lambda' = 0$$

For  $x' = v(x, t)$ , given  $p$  and  $t_0$  we have  $\gamma_{p,t_0}$  which depends smoothly on  $p$  and  $t_0$ .

$$x = v(x, t) \quad (p, t_0) \quad t' = 1$$