Random Matrix Theory

April 1, 2025

Preliminaries

Let ξ_{ij}, η_{ij} be normal random variables (i.e. Gaussian, mean 0, variance 1).

e.g. $\mathbb{P}(\xi_{11} < s) = \int_{-\infty}^{s} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$.

$\int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$ is the variance.

$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is the Probability Density Function (PDF).

$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$ is the probability measure on our probability space (i.e. totally finite measure space).
We build matrices

$$\begin{bmatrix} \xi_{11} & \frac{\xi_{12}+i\eta_{12}}{\sqrt{2}} & \frac{\xi_{13}+i\eta_{13}}{\sqrt{2}} & \cdots \\ \frac{\xi_{21}+i\eta_{21}}{\sqrt{2}} & \xi_{22} & \frac{\xi_{22}+i\eta_{22}}{\sqrt{2}} & \\ \frac{\xi_{31}+i\eta_{31}}{\sqrt{2}} & \frac{\xi_{32}+i\eta_{32}}{\sqrt{2}} & \xi_{33} & \\ \vdots & & & \ddots \end{bmatrix}$$

Computing Random Matrices in Matlab

Gassuain, real valued 1x1 matrix.

```
randn
```

Gaussian, real valued 2x2 matrix.

```
randn(2)
```

Gaussian, complex valued 2x2 matrix.

```
randn(2)+sqrt(-1)*randn(2)
```

Gaussian, complex valued, self-adjoint 2x2 matrix.
Note that appending ' to a matrix takes the conjugate transpose, and matlab reserves i for the imaginary unit.

```
m = randn(2)+i*randn(2)
(m+m')/2
```

Producing eigenvalues.

```
m = randn(2)+i*randn(2);
l=(m+m')/2;
eig(l)
```

Running tests to see how many hits we get within the interval $[0, 2]$.

```
edges=[0,2];
H=zeros(1,length(edges)-1);
trials=10;
for j=1:trials
m = randn(2)+i*randn(2);
l=(m+m')/2;
ev=eig(l);
H=H+histcount(ev,edges)
end
```

Homework

Is the PDF of $\frac{a+b}{2}$ the same as $\frac{\xi_{12}}{\sqrt{2}}$ for normal RVs a, b, ξ_{12}?

i.e. $\mathbb{P}\left(\frac{a+b}{2} < s\right) \overset{?}{=} \mathbb{P}\left(\frac{\xi_{12}}{\sqrt{2}} < s\right)$

2x2 Random Matrix

Our matrix L corresponds to eigvenvalues λ_1, λ_2 which are random variables determined by $\{\xi_{ij}, \eta_{ij}\}$.
Then the number of evaulations in the interval B is given by $\sum_{j=1}^{2} \chi_B(\lambda_j)$. We may take the average by

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{j=1}^{2} \chi_B(\lambda_j) \frac{1}{\sqrt{2\pi}} e^{-\xi_{11}^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\xi_{22}^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\xi_{12}^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\eta_{12}^2} \, d\xi_{11} d\xi_{22} d\xi_{12} d\eta_{12}.$$

Expected Evaluations

We have that the expecation of the number of evaluations in the interval (a, b) is given by $\int_a^b G(s) \, ds$ where

$$G(s) = e^{-\frac{s^2}{2}} \sum_{\ell=0}^{2} P_\ell(s)^2$$

and $P_\ell(s)$ is the Hermite polynomial of degree d.

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Differntiability

```
delta = 0.05;
edges=-6:delta:6;
dimensions = 3;
trials = 1000000;

H=zeros(dimensions,trials);

for j=1:trials
m=randn(dimensions)+1i*randn(dimensions);
L=(m+m')/2;
ev=eig(L);
H(:,j) = ev;
end

G = histcounts(H,edges);
plot(edges(1:end-1),G/(trials*delta),'*')
```

Observe that each * in the graph corresponds to the average number of eigenvalues in the interaval (a, b). Therefore, they correspond to $\int_a^b C(\lambda)\, d\lambda$. We may consider the limit of the expectation of hits in each interval

$$\lim_{\Delta \to 0} \frac{\mathbb{E}(\#(a, a+\Delta))}{\Delta}.$$

```
delta = 0.01;
edges=-6:delta:6;
dimensions = 3;
trials = 1000000;

H=zeros(dimensions,trials);

for j=1:trials
m=randn(dimensions)+1i*randn(dimensions);
L=(m+m')/2;
ev=eig(L);
H(:,j) = ev;
end

G = histcounts(H,edges);
plot(edges(1:end-1),G/(trials*delta),'*')
```

As dimension grows large, we observe that the plot tends to a semi-circle with endpoints about $\pm 2\sqrt{\text{dimension}}$. We therefore want a rescaling by $\sqrt{N}$ where $\dim = N$. Then if $G(\alpha) = \frac{d}{d\alpha}\mathbb{E}(\# \text{ of evals in } (a, \alpha))$, we want

$$\int_{-\infty}^{\infty} G(\alpha)\, d\alpha = N.$$

Guess: $G(\alpha) \approx cN^{1/2} \cdot \sqrt{A^2 - \alpha^2/N} \cdot \chi_{(-A\sqrt{N}, A\sqrt{N})}(\alpha)$. We compute

$$\int_{-A\sqrt{N}}^{A\sqrt{N}} cN^{1/2}\sqrt{A^2 - \alpha^2/N}\, d\alpha \overset{\alpha = \sqrt{N}t}{=} cN\int_{-A}^{A} \sqrt{A^2 - t^2}\, dt = \frac{c\pi N A^2}{2}.$$

Choosing $A = 2$ and c such that $\frac{\pi A^2 c}{2} = 1$, we get

$$\int_{-\infty}^{\infty} G(\alpha)\, d\alpha \approx \frac{N^{1/2}}{2\pi}\int_{-\infty}^{\infty} \sqrt{4 - \alpha^2/N}\, d\alpha = N.$$

Number of Eigenvalues in an Interval

Let B be a subset of $\mathbb{R}$ (typically an interval). Write $n(B) = \#\{\text{evaluations in } B\}$, a random variable.
Recall that variance is given by the expectation of the square minus the square of the expectation. That is

$$\mathrm{var}(n(B)) = \mathbb{E}(n(B)^2) - (\mathbb{E}(n(B))^2.$$

Our ultimate goal is to understand PDF and $\mathbb{P}(n(B)) = \ell)$ as (the dimension) $N \to \infty$.

Smallest Scale of Interest

Suppose $B = (0, S)$ and N is large (i.e. $N \to \infty$). How large should we choose s such that $\mathbb{E}(n(B)) = 1$? We compute

$$\int_0^S cN^{1/2}\sqrt{4 - \alpha^2/N}\, d\alpha \overset{\alpha = \sqrt{N}t}{=} \int_0^{\frac{S}{\sqrt{N}}} cN\sqrt{4 - t^2}\, dt \approx cN \cdot 2\frac{S}{\sqrt{N}} = 2cS\sqrt{N}.$$

Sets of size $N^{-1/2}$, the smallest interesting scale, are called the "microscopic scaling regime".

Homework: Largest Scale of Interest

How large should B be to see a fraction of the eigenvalues (on average)? That is, how should we scale a and b such that $\mathbb{E}(n((a, b))) = r \cdot N$ for $0 < r < 1$?

Level Repulsion

```
m=randn(2)+sqrt(-1)*randn(2);
L=(m+m')/2;
ev=eig(L);
subplot(2,1,2),plot(real(ev),imag(ev))
xlim([edges(1),edges(end)])
```

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Macroscopic Scaling Regime for Random Matrices

Suppose $a = \alpha\sqrt{N}$ and $b = \beta\sqrt{N}$ such that $\alpha < \beta$, $-2 < \alpha$ and $\beta < 2$. Then

$$\lim_{n \to \infty} \frac{\mathbb{E}(\# \text{ of evaluations in } (\alpha\sqrt{N}, \beta\sqrt{N}))}{N} = \kappa > 0.$$

Recall that we defined $G(b) = \frac{d}{db}\mathbb{E}(\# \text{ of evaluations in } (a, b))$ and

$$G(b) \approx cN^{1/2}\sqrt{A^2 - x^2/N}\chi_{[-A\sqrt{N}, A\sqrt{n}]}(x).$$

We want that $\int_a^b G(x)\, dx = \kappa N$.

Spacings

Suppose we have eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N = \lambda_{\max}$. We can take the spacing $s_j = \lambda_{j+1} - \lambda_j$.

```
m=randn(2)+sqrt(-1)*randn(2);
L=(m+m')/2;
ev=sort(eig(L));
spacing=diff(ev)
```

0.4839

Summary So Far

Given ξ_{ij} and η_{ij} iid RVs with distribution $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, we have explored

- The behavior of average $n_N(B)$.

- Microscopic, macroscopic (and mesoscopic) scaling.

- That $\lambda_{\max} \sim 2\sqrt{N}$ Tracy-Widom distribution.

- Eigenvalue repulsion.

Induced Distribution

Let M be our matrix built using random variables. Then $M = F\Lambda F^T$ where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \\ \vdots & & \ddots \end{pmatrix}, \quad F = \begin{pmatrix} | & | & & | \\ f_{\lambda_1} & f_{\lambda_2} & \cdots & f_{\lambda_N} \\ | & | & & | \end{pmatrix},$$

and $Mf_{\lambda_j} = \lambda_j f_{\lambda_j}$. What we are interested in is the induced joint PDF on $\{\lambda_1, \ldots, \lambda_N\}$. We may write explicitly

$$\frac{1}{Z^n} e^{-\frac{1}{2}\sum_{j=1}^N \lambda_j^2} \prod_{1 \le j < k \le N} (\lambda_k - \lambda_j)^2.$$

Example

Let $N = 2$ and, suppressing the constant term, write

$$\rho = e^{-\frac{1}{2}(x^2+y^2)}(x-y)^2.$$

Taking partial derivatives, we have that

$$\rho_x = e^{-\frac{1}{2}(x^2+y^2)}(x-y)^2\left(-x+\frac{2}{x-y}\right)$$
$$\rho_y = e^{-\frac{1}{2}(x^2+y^2)}(x-y)^2\left(-x+\frac{2}{y-x}\right)$$

which implies maxima at $x = \pm 1$ and $y = -x$.

Example

If $N = 3$,

$$\rho = e^{-\frac{1}{2}(x^2+y^2+z^2)}(x-y)^2(x-z)^2(y-z)^2.$$

We may visualize the maxima here by level surfaces (homework).

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Recall: Spectral Theorem

Let $M = F\Lambda F^\dagger$ where $F^\dagger F = I = FF^\dagger$

$$\Lambda = \begin{pmatrix} \lambda_N & 0 & \cdots & \\ 0 & \lambda_{N-1} & & \\ \vdots & & \ddots & \\ & & & \lambda_1 \end{pmatrix}, \quad F = \begin{pmatrix} | & | & & | \\ f_{\lambda_1} & f_{\lambda_2} & \cdots & f_{\lambda_N} \\ | & | & & | \end{pmatrix},$$

for $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$.

Deriving the Joint PDF

Let $n = 2$. If

$$F = \begin{pmatrix} | & | \\ V & W \\ | & | \end{pmatrix},$$

then the expectation of eigenvalues may be computed by

$$\mathbb{E}(\mathcal{G}(M)) = \frac{1}{Z_2^4} \int \cdots \int \mathcal{G}(M(\xi_{11}, \xi_{12}, \xi_{22}, \eta_{12})) x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\xi_{11}^2 + \xi_{12}^2 + \xi_{22}^2 + \eta_{12}^2)} \, d\eta_{12} d\xi_{22} d\xi_{12} d\xi_{11}$$

$$= \int \mathcal{G}(M(\lambda_1, \lambda_2, V_1, \phi)) x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\xi_{11}^2 + \xi_{12}^2 + \xi_{22}^2 + \eta_{12}^2)} \, d\eta_{12} d\xi_{22} d\xi_{12} d\xi_{11}.$$

So we need the Jacobian, and therefore a reparameterization using spectral theorem. We want a collection of independent variables which will produce all 2×2 Hermitian matrices. Consider $Mv = \lambda_2 v$ and $||v||^2 = |v_1|^2 + |v_2|^2 = 1$, then multiply by $e^{i\eta}$ such that $v_1 \in \mathbb{R}_+$. Then $v_2 = \sqrt{1 - v_1^2} e^{i\theta}$. That is, $0 \le v_1 \le 1$ and $v_2 = \sqrt{1 - v_1^2}(\cos\theta + i\sin\theta)$.

We want that $|w_1|^2 + |w_2|^2 = 1$ and know that $w \perp v$, so $v_1 w_1 + \overline{v}_2 w_2 = 0$. As before, we can choose w such that $w_2 \in \mathbb{R}_+$. This implies that w_1 and $\overline{v}_2$ have the same argument, and $w_1 = -|w_1|e^{-i\theta}$. Therefore $e^{-i\theta}(-v_1|w_1| + |v_2|w_2) = 0$, and $v_1|w_1| - |v_2|w_2 = 0$. It follows that

$$v_1^2(1 - w_2^2) = w_2^2(1 - v_1^2) \quad \Longleftrightarrow \quad v_1 = w_2.$$

Therefore, the entire system may be parameterized by v_1 and θ. We write

$$F = \begin{pmatrix} v_1 & -\sqrt{1 - v_1^2} e^{-i\theta} \\ \sqrt{1 - v_1^2} e^{i\theta} & v_1 \end{pmatrix}$$

and

$$M = F\Lambda F^\dagger = \begin{pmatrix} v_1 & -\sqrt{1 - v_1^2} e^{-i\theta} \\ \sqrt{1 - v_1^2} e^{i\theta} & v_1 \end{pmatrix} \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix} \begin{pmatrix} v_1 & \sqrt{1 - v_1^2} e^{-i\theta} \\ -\sqrt{1 - v_1^2} e^{i\theta} & v_1 \end{pmatrix}.$$

Therefore

$$M = \begin{pmatrix} \lambda_2 v_1^2 + \lambda_1(1 - v_1^2) & v_1\sqrt{1 - v_1^2} e^{-i\theta}(\lambda_2 - \lambda_1) \\ v_1\sqrt{1 - v_1^2} e^{-i\theta}(\lambda_2 - \lambda_1) & \lambda_2(1 - v_1^2) + \lambda_1 v_1^2 \end{pmatrix}.$$

Recall, we want $\mathcal{G}(M(\xi)) \rightsquigarrow \mathcal{G}(M(\lambda_2, \lambda_1, v_1, \theta))$ and the Jacobian of $M = M(\lambda_2, \lambda_1, v_1, \theta)$. After computation, write

$$|\det J = (\lambda_2 - \lambda_1)^2 \det J' = (\lambda_2 - \lambda_1)^2 Q(v_1, \theta).$$

We integrate

$$\int \cdots \int \mathcal{G}(M(\xi, \eta_{12})) e^{-\frac{1}{2}(\xi_{11}^2 + \xi_{12}^2 + \xi_{22}^2 + \eta_{12}^2)} \frac{1}{(2\pi)^4} \, d\xi_{11} d\xi_{12} d\xi_{22} d\eta_{12}$$

which we may think of uas a function of λ_1 and λ_2 alone. So

$$\frac{1}{(2\pi)^2} \int \cdots \int \mathcal{G}(\lambda_1, \lambda_2) e^{-\frac{1}{2}[M_{11}^2 + M_{22}^2 + 2 \cdot \mathrm{Re}(M_{12})^2 + 2 \cdot \mathrm{Im}(M_{12})^2]} \, d\xi_{11} d\xi_{12} d\xi_{22} d\eta_{12}$$

where we observe that $M_{11}^2 + M_{22}^2 + 2 \cdot \mathrm{Re}(M_{12})^2 + 2 \cdot \mathrm{Im}(M_{12})^2 = \mathrm{Tr}(M^2)$. It follows that we have

$$\frac{1}{(2\pi)^2} \int \cdots \int \mathcal{G}(\lambda_1, \lambda_2) e^{-\frac{1}{2}(\lambda_1^2 + \lambda_2^2)} \, d\xi_{11} d\xi_{12} d\xi_{22} d\eta_{12} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^1 \int \int_{-\infty < \lambda_1 \le \lambda_2 < \infty} \mathcal{G}(\lambda_1, \lambda_2) e^{-\frac{1}{2}(\lambda_1^2 + \lambda_2^2)} (\lambda_2 - \lambda_1)^2 Q(v, \theta) \, d\xi_1$$

$$= \int \int_{-\infty < \lambda_1 \le \lambda_2 < \infty} \mathcal{G}(\lambda_1, \lambda_2) e^{-\frac{1}{2}(\lambda_1^2 + \lambda_2^2)} (\lambda_2 - \lambda_1)^2 \int_0^{2\pi} \int_0^1 \frac{Q(v, \theta)}{(2\pi)^2} \, dv_1 d\theta d$$

$$= c \int \int \mathcal{G}(\lambda_1, \lambda_2) e^{-\frac{1}{2}(\lambda_1^2 + \lambda_2^2)} (\lambda_2 - \lambda_1)^2 \, d\lambda_1 d\lambda_2$$

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Recall: Joint PDF on Evaluation of Eigenvalues

$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$ and PDF $\frac{1}{Z_N} e^{-\frac{1}{2}\sum \lambda_i^2} \prod(\lambda_k - \lambda_j)^2$.
This is the Gaussian Unitary Ensemble.

Hermite Polynomials

Write $p_j = \kappa_j^{(j)} x^j + \kappa_{j-1}^{(j)} x^{j-1} + \cdots + \kappa_0^{(j)}$ where the superscript is usually supressed. Then

$$\int_{-\infty}^{\infty} p_j(x) p_k(x) e^{-\frac{1}{2}x^2} \, dx = \delta_{jk}.$$

Observe that $\{e^{-\frac{1}{4}x^2} p_j(x)\}_{j=0}^{\infty}$ forms a basis for $L^2(\mathbb{R})$. For $f \in L^2$, write the truncation $P^{(N)}(f) = \sum_{\ell=0}^{N-1} \left(\int_{\mathbb{R}} f(y) p_\ell(y) e^{-\frac{1}{4}y^2} \right) e^{-}$
Then

$$P^{(N)} = \int_{\mathbb{R}} \left(e^{-\frac{1}{4}(x^2 + y^2)} \sum_{\ell=0}^{N-1} p_\ell(x) p_\ell(y) \right) f(y) \, dy$$

and we write $K_N(x, y) = e^{-\frac{1}{4}(x^2 + y^2)} \sum_{\ell=0}^{N-1} p_\ell(x) p_\ell(y)$ and $\mathcal{K}_N(f) = \int_{\mathbb{R}} K_n(x, y) f(y) \, dy$.
We have that

$$\frac{1}{Z_N} e^{-\frac{1}{2}\sum \lambda_i^2} \prod(\lambda_k - \lambda_j)^2 = \det \begin{bmatrix} K_N(\lambda_1, \lambda_1) & \cdots & K_N(\lambda_1, \lambda_N) \\ \vdots & \ddots & \vdots \\ K_N(\lambda_N, \lambda_1) & \cdots & K_N(\lambda_N, \lambda_N) \end{bmatrix}.$$

For $N = 2$, we see

$$\frac{1}{Z_N} e^{-\frac{1}{2}\sum \lambda_i^2} \prod(\lambda_k - \lambda_j)^2 = (K_N(\lambda_1, \lambda_1) K_N(\lambda_2, \lambda_2) - K_N(\lambda_1, \lambda_2)^2).$$

Example Computation

Let I be an interval and consider $\mathbb{E}(\#$ of evaluations in $I)$. Then

$$\int_{-\infty < \lambda_1 \leq \lambda_2 < \infty} \left(\sum_{j=1}^{2} \chi_I(\lambda_j)\right) (K_2(\lambda_1, \lambda_1) K_2(\lambda_2, \lambda_2) - K_2(\lambda_1, \lambda_2)^2)\, d\lambda_2 \lambda_2$$

$$= \frac{1}{2!} \int\int_{\mathbb{R}^2} (\chi_I(\lambda_1) + \chi(\lambda_2)) (K_2(\lambda_1, \lambda_1) K_2(\lambda_2, \lambda_2) - K_2(\lambda_1, \lambda_2)^2)\, d\lambda_1 d\lambda_2$$

$$= \frac{1}{2!} \int_I \int_{-\infty}^{\infty} (K_2(\lambda_1, \lambda_1) K_2(\lambda_2, \lambda_2) - K_2(\lambda_1, \lambda_2)^2)\, d\lambda_1 d\lambda_2 + \frac{1}{2!} \int_I \int_{-\infty}^{\infty} (K_2(\lambda_1, \lambda_1) K_2(\lambda_2, \lambda_2) - K_2(\lambda_1, \lambda_2)^2)\, d\lambda_2 d\lambda_1.$$

Observe that $\int_{-\infty}^{\infty} K_2(\lambda_2, \lambda_2)\, d\lambda_2 = \int_{-\infty}^{\infty} e^{-\frac{1}{2}\lambda_2^2}(p_0(\lambda_2)^2 + p_1(\lambda_2)^2\, d\lambda_2 = 2$. We also compute that

$$\int_{-\infty}^{\infty} K_2(\lambda_1, \lambda_2) K_2(\lambda_2, \lambda_1)\, d\lambda_2 = \int_{\mathbb{R}} e^{-\frac{1}{2}(\lambda_1^2 + \lambda_2^2)} \left(\sum_{\ell=0}^{1} p_\ell(\lambda_1) p_\ell(\lambda_2)\right) \left(\sum_{\ell'=0}^{1} p_{\ell'}(\lambda_2) p_{\ell'}(\lambda_1)\right) d\lambda_2$$

$$= \sum_{\ell,\ell'=0}^{1} e^{-\frac{1}{2}\lambda_1^2} p_\ell(\lambda_1) p_{\ell'}(\lambda_1) \int_{\infty}^{\infty} e^{-\frac{1}{2}\lambda_2^2} p_\ell(\lambda_2) p_{\ell'}(\lambda_2)\, d\lambda_2$$

$$= K_2(\lambda_1, \lambda_1).$$

Returning to the first calculation,

$$\frac{1}{2!} \int_I \int_{-\infty}^{\infty} (K_2(\lambda_1, \lambda_1) K_2(\lambda_2, \lambda_2) - K_2(\lambda_1, \lambda_2)^2)\, d\lambda_1 d\lambda_2 + \frac{1}{2!} \int_I \int_{-\infty}^{\infty} (K_2(\lambda_1, \lambda_1) K_2(\lambda_2, \lambda_2) - K_2(\lambda_1, \lambda_2)^2)\, d\lambda_2 d\lambda_1$$

$$= \frac{1}{2!}\left[\int_I (2-1) K_2(\lambda_1, \lambda_1)\, d\lambda_1 + \int_I (2-1) K_2(\lambda_2, \lambda_2)\, d\lambda_2\right]$$

$$= \int_I K_2(\lambda_1, \lambda_1)\, d\lambda_1$$

which is the density function for the average number of evaluations in I. So $K_2(\lambda, \lambda) = \frac{e^{-\frac{1}{2}\lambda^2}}{\sqrt{2\pi}}(1 + \lambda^2)$.

Question:

What is the probability of having zero evaluations in an interval I?
We have an indicator function $(1 - \chi_I(\lambda_1))(1 - \chi_I(\lambda_2))$, so

$$P(\text{no evaluations in } I) = \frac{1}{2} \int_{\mathbb{R}^2} (1 - \chi_I(\lambda_1))(1 - \chi_I(\lambda_2))[K_2(\lambda_1, \lambda_1) K_2(\lambda_2, \lambda_2) - K_2(\lambda_1, \lambda_2)^2]\, d\lambda_1 d\lambda_2$$

$$= \frac{1}{2} \int_{\mathbb{R}^2} (1 - (\chi_I(\lambda_1) + \chi_I(\lambda_2)) + \chi_I(\lambda_1)\chi_I(\lambda_2)) K_2(\lambda_1, \lambda_1) K_2(\lambda_2, \lambda_2) - K_2(\lambda_1, \lambda_2)^2]\, d\lambda_1 d\lambda_2$$

$$= \frac{1}{2}[\int_{\mathbb{R}^2} K_2(\lambda_1, \lambda_1) K_2(\lambda_2, \lambda_2) - K_2(\lambda_1, \lambda_2)^2\, d\lambda_1 d\lambda_2$$

$$- 2\int_I \int_{\mathbb{R}} K_2(\lambda_1, \lambda_1) K_2(\lambda_2, \lambda_2) - K_2(\lambda_1, \lambda_2)^2\, d\lambda_2 d\lambda_1$$

$$+ \int_I \int_I K_2(\lambda_1, \lambda_1) K_2(\lambda_2, \lambda_2) - K_2(\lambda_1, \lambda_2)^2]$$

$$= \frac{1}{2}[4 - 2 - 2\int_I K_2(\lambda_1, \lambda_1)\, d\lambda_1 + \int_I \int_I K_2(\lambda_1, \lambda_1) K_2(\lambda_2, \lambda_2) - K_2(\lambda_1, \lambda_2)^2\, d\lambda_1 \lambda_2]$$

$$= 1 - \int_I K_2(\lambda_1, \lambda_1)\, d\lambda_1 + \int_I \int_I \det(\quad)_{2\times2}\, d^2\lambda$$

$$= \det(1 - \mathcal{K}_2^{(I)})$$

If $I = (0, \infty)$, then the probability is $\frac{\pi - 2}{4\pi}$.

Fredholm Determinant

Write $H_N(I, t) = \det(1 - t\mathcal{K}_N^{(I)})$ where $K_N^{(I)}$ is an integral operator which acts on $L_2(I)$ by

$$\mathcal{K}_N^{(I)}(f) = \int_I K_N(x, y) f(y) \, dy.$$

So the range of $\mathcal{K}_N^{(I)}$ is finite dimensional (i.e. it is a finite rank operator). Then

$$H_N(I, t) = 1 - \int_I K_N(\lambda_1, \lambda_1) \, d\lambda_1 - \frac{t}{2!} \int_I \int_I \det(\quad)_{2\times 2} \, d^2\lambda + \cdots + \frac{(-t)^j}{j!} \overbrace{\int_I \cdots \int_I}^{j} \det(\quad)_{j\times j} \, d^j\lambda + \frac{(-t)^N}{N!} \overbrace{\int_I \cdots \int_I}^{N} \det(\quad)_{N\times N}$$

Then $H_N(I, 1)$ is the probability of no evaluations in I, and $H_N^t(I, 1)$ is negative the probability of exactly one evaluation in I. So

$$H_N^{(j)}(I, 1) = (-1)^j j! P(\text{exactly } j \text{ eigenvalues in } I).$$

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Recall

$$\frac{1}{z_N} e^{-\frac{1}{2} \sum \lambda_j^2} \prod_{j<k} (\lambda_j - \lambda_k)^2 = \frac{1}{N!} \det(K_N(\lambda_j, \lambda_k))_{N\times N}$$

For $n = 2$, we have

$$\mathbb{E}\left(\sum_{j=1}^{2} \chi_B(\lambda_j) \right) = \int_B K_2(\lambda, \lambda) \, d\lambda$$

We also have that

$$\mathbb{E}((1 - \chi_B(\lambda_1))(1 - \chi_B(\lambda_2))) = P(\text{no evaluations})$$

$$= 1 - \lambda_B K_2(\lambda, \lambda) \, d\lambda + \frac{1}{2} \int_B \int_B \det(\quad)_{2\times 2} \, d\lambda$$

where

$$(\lambda_1 - \lambda_2)^2 = \det\begin{pmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{pmatrix} \det\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}$$

$$\overset{q(\lambda_i) = \lambda_i + c_0}{=} \det\begin{pmatrix} 1 & q(\lambda_1) \\ 1 & q(\lambda_2) \end{pmatrix} \det\begin{pmatrix} 1 & 1 \\ q(\lambda_1) & q(\lambda_2) \end{pmatrix}$$

$$= \frac{1}{(\kappa_0^{(0)} \kappa_1^{(1)})^2} \det\begin{pmatrix} \kappa_0^{(0)} & \kappa_1^{(1)} q(\lambda_1) \\ \kappa_0^{(0)} & \kappa_1^{(1)} q(\lambda_2) \end{pmatrix} \det\begin{pmatrix} \kappa_0^{(0)} & \kappa_0^{(0)} \\ \kappa_1^{(1)} q(\lambda_1) & \kappa_1^{(1)} q(\lambda_2) \end{pmatrix}$$

$$= \frac{1}{\prod_0^1 (\kappa_i^{(i)})^2} \det\begin{pmatrix} P_0(\lambda_1) & P_1(\lambda_1) \\ P_0(\lambda_2) & P_1(\lambda_2) \end{pmatrix} \det\begin{pmatrix} P_0(\lambda_1) & P_0(\lambda_2) \\ P_1(\lambda_1) & P_1(\lambda_2) \end{pmatrix}$$

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It follows that

$$e^{-\frac{1}{2}\sum_1^2 \lambda_j^2}(\lambda_2 - \lambda_1)^2 = \prod_{j=0}^{1}(\kappa_j^{(j)})^{-2}\det\begin{pmatrix} P_0(\lambda_1)e^{-\frac{1}{4}\lambda_1^2} & P_1(\lambda_1)e^{-\frac{1}{4}\lambda_1^2} \\ P_0(\lambda_1)e^{-\frac{1}{4}\lambda_2^2} & P_1(\lambda_1)e^{-\frac{1}{4}\lambda_2^2} \end{pmatrix}\det\begin{pmatrix} P_0(\lambda_1)e^{-\frac{1}{4}\lambda_1^2} & P_0(\lambda_1)e^{-\frac{1}{4}\lambda_2^2} \\ P_1(\lambda_1)e^{-\frac{1}{4}\lambda_1^2} & P_1(\lambda_1)e^{-\frac{1}{4}\lambda_2^2} \end{pmatrix}$$

$$= \prod_{j=0}^{1}(\kappa_j^{(j)})^{-2}\det(K_2(\lambda_i, \lambda_j))_{2\times 2}$$

where $K_2(x,y) = e^{-\frac{1}{4}(x^2+y^2)}\sum_{\ell=0}^{1}P_\ell(x)P_\ell(Y)$. So we have

$$\frac{1}{z_N}\prod_0^{1}(\kappa_j^{(j)})^{-2}\det\begin{pmatrix} K_2(\lambda_1, \lambda_1) & K_2(\lambda_1, \lambda_2) \\ K_2(\lambda_2, \lambda_1) & K_2(\lambda_2, \lambda_2) \end{pmatrix}$$

and the fact that

$$\frac{1}{z_N \prod_{j=1}^{2}(\kappa_j^{(j)})}\int_{\mathbb{R}^2}\left[K_2(\lambda_1, \lambda_1)K_2(\lambda_2, \lambda_2) - K_2(\lambda_1, \lambda_2)^2\right]d\lambda_1 d\lambda_2 = 1$$

Observe that (to do: fill in these calculations)

$$\int_{\mathbb{R}^2}K_2(\lambda_1, \lambda_1)K_2(\lambda_2, \lambda_2)\,d\lambda_1 d\lambda_2 = \int_{\mathbb{R}^2}(e^{-\frac{1}{4}\lambda_1^2}(P_0(\lambda_1)P_0(\lambda_1) + P_1(\lambda_1)P_1(\lambda_1))(e^{-\frac{1}{4}\lambda_2^2}(P_0(\lambda_2)P_0(\lambda_2) + P_1(\lambda_2)P_1(\lambda_2))$$

So it must be that

$$\frac{1}{z_N(\kappa_0^{(0)})^2(\kappa_1^{(1)})^2}(2) = 1.$$

We conclude that the original joint PDF can be written as $\frac{1}{2!}\det(K_1(\lambda_i, \lambda_j))_{2\times 2}$.

Vandermonde Determinant

Write

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_N \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_N^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{N-1} & \lambda_2^{N-1} & \cdots & \lambda_N^{N-1} \end{vmatrix} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{N-1} \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{N-1}^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{N-2} & \lambda_2^{N-2} & \cdots & \lambda_{N-1}^{N-2} \end{vmatrix}(\lambda_N - \lambda_1)(\lambda_N - \lambda_2)\cdots(\lambda_N - \lambda_{N-1})$$

$$= \prod_{j<k}(\lambda_k - \lambda_j)$$

and observe that this is zero when $\lambda_i = \lambda_j$. Now write

$$\det\begin{pmatrix} 1 & \cdots & \lambda_1^{N-1} \\ \vdots & & \vdots \\ 1 & \cdots & \lambda_N^{N-1} \end{pmatrix}\det\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ \lambda_1^{N-1} & \cdots & \lambda_N^{N-1} \end{pmatrix}$$

then by using the multilinearity of the dterminant and adding rows we can write

$$\det\begin{pmatrix} 1 & \cdots & \lambda_1^{N-1} \\ \vdots & & \vdots \\ 1 & \cdots & \lambda_N^{N-1} \end{pmatrix} \det\begin{pmatrix} 1 & \cdots & 1 \\ \lambda_1 + c_0 & \cdots & \lambda_N + c_0 \\ \pi_2(\lambda_1) & \cdots & \pi_2(\lambda_N) \\ \vdots & & \vdots \\ \lambda_1^{N-1} & \cdots & \lambda_N^{N-1} \end{pmatrix}$$

So we can write

$$\det\begin{pmatrix} e^{-\frac{1}{4}\lambda_1^2}P_0(\lambda_1) & \cdots & e^{-\frac{1}{4}\lambda_1^2}P_{N-1}(\lambda_1) \\ \vdots & & \vdots \\ e^{-\frac{1}{4}\lambda_N^2}P_0(\lambda_N) & \cdots & e^{-\frac{1}{4}\lambda_1^2}P_{N-1}(\lambda_N) \end{pmatrix} \frac{1}{\prod_{i=0}^{N-1}(\kappa_j^{(j)})^2} \det\begin{pmatrix} P_0(\lambda) & \cdots & P_0\lambda_N) \\ \vdots & & \vdots \\ P_{N-1}(\lambda_1) & \cdots & P_{N-1}(\lambda_N) \end{pmatrix}$$

Examining the (j, k) entry, we have

$$\frac{1}{\prod(\kappa_j^{(j)})^2} e^{-\frac{1}{4}(\lambda_j^2+\lambda_k^2)}(P_0(\lambda_j)P_0(\lambda_k) + P_1(\lambda_j)P_1(\lambda_k) + \cdots + P_{N-1}(\lambda_j)P_{N-1}(\lambda_k)).$$

or

$$\frac{1}{z_N \prod(\kappa_j^{(j)})^2} \det[K_n(\lambda_j, \lambda_k)]_{N\times N}$$

which must integrate across $\mathbb{R}^n$ to exactly 1. From this we conclude that $\dfrac{N!}{z_N \prod(\kappa_j^{(j)})^2} = 1$.