Algebra III

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Chapter 0: Review

Definition: Category

A category $\mathcal{C}$ consists of the following data:

1. A class of objects, Obj$(\mathcal{C})$.
2. For any pair of objects $X, Y \in$ Obj$(\mathcal{C})$, a set of morphisms Mor$_{\mathcal{C}}(X, Y)$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ or $\mathcal{C}(X, Y)$.
3. For any triple of objects $X, Y, Z \in$ Obj$(\mathcal{C})$, a map

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) &\to \mathrm{Hom}_{\mathcal{C}}(Y, Z) \\ (g, f) &\mapsto g \circ f \end{aligned}$$

called compositions subject to the following axioms:

1. Associativity: $f \circ (g \circ h) = (f \circ g) \circ h$ whenever this makes sense.
2. For every object $X \in$ Obj$(\mathcal{C})$, there exists a morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ such that

$$\mathrm{id}_X \circ f = f \quad \text{and} \quad g \circ \mathrm{id}_X = g, \quad \forall f \in \mathrm{Hom}_{\mathcal{C}}(W, X),\ g \in \mathrm{Hom}_{\mathcal{C}}(X, W)$$

Example 1

Let E be a set (or a class).
Define $\mathcal{C}$ by taking Obj$(\mathcal{C}) = E$ and $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \begin{cases} \varnothing & \text{if } x \neq y \\ \{\mathrm{id}_X\} & \text{if } x = y \end{cases}$.

Example 2

Let $\mathcal{C}$ = Set the category of all sets with set functions acting as morphisms.
Let $\mathcal{C}$ = Grp the category of all groups with group homomorphisms acting as morphisms.
Abelian Rings: Ab, Rings: Ring, Commutative Rings: CRing, Vector Spaces over F: Vect$_F$, Topological Spaces: Top, etc.

Example 3

Let G be a group (or more generally a monoid).
Define Obj$(\mathcal{C}) = \{*\}$, $\mathrm{Hom}_{\mathcal{C}}(*, *) = G$ and

$$\mathrm{Hom}_{\mathcal{C}}(*, *) \times \mathrm{Hom}_{\mathcal{C}}(*, *) \to \mathrm{Hom}_{\mathcal{C}}(*, *)$$

the group operator.

Example 4

Let $(E, \leq)$ be a preordered set (i.e. reflexive and transitive).
Define $\mathcal{C}$ by Obj$(\mathcal{C}) = E$,

$$\mathrm{Hom}_{\mathcal{C}}(x, y) = \begin{cases} \varnothing & \text{if } x \nleq y \\ \{f_{xy}\} & \text{if } x \leq y \end{cases}$$

Notation

If $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ we write $X \xrightarrow{f} Y$ in $\mathcal{C}$.

Definition: Isomorphism

A morphism $f : X \to Y$ in $\mathcal{C}$ is an isomorphism if $\exists g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Definition: Endomorphism

A morphism on X with $f : X \to X$.

Definition: Automorphism

An automorphism on X is just an isomorphism $f : X \xrightarrow{\sim} X$ from X to itself.
Note that $\mathrm{Aut}_{\mathcal{C}}(X) \subseteq \mathrm{End}_{\mathcal{C}}(X) = \mathrm{Hom}_{\mathcal{C}}(X, X)$.

Remark:

The collection of all endomorphisms on X form a monoid.
The collection of all automorphisms on X forms a group called the automorphism group of X.

Example 1

Let $\mathcal{C} = \mathrm{Set}$, $X = \{1, \ldots, n\}$. Then $\mathrm{Aut}_{\mathrm{Set}}(\{1, \ldots, n\}) = \mathrm{Perm}(X) = S_n$.

Example 2

Let $\mathcal{C} = \mathrm{Vect}_F$, $X = F^n$. Then $\mathrm{Aut}_{\mathrm{Vect}_F}(F^n) = \mathrm{GL}_n(F)$.

Definition: Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.
A functor $F : \mathcal{C} \to \mathcal{D}$ from $\mathcal{C}$ to $\mathcal{D}$ consists of the following data

1. For each object $X \in \mathrm{Obj}(\mathcal{C})$, a chosen object $F(X) \in \mathrm{Obj}(\mathcal{D})$.

2. For each pair of objects $X, Y \in \mathrm{Obj}(\mathcal{C})$, a function

$$\begin{gathered} \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{D}(F(X), F(Y)) \\ f \mapsto F(f) \end{gathered}$$

such that

1. For any two composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$, we have $F(g \circ f) = F(g) \circ F(f)$.

2. For each object $X \in \mathrm{Obj}(\mathcal{C})$, $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

Example 1

For $\mathcal{D} := \mathcal{C}$, $\mathrm{Id} : \mathcal{C} \to \mathcal{C}$, $X \mapsto X$, $f \mapsto f$.

Example 2: Forgetful Functors

$\mathcal{U} : \mathrm{Grp} \to \mathrm{Set}$ given as $(G, \cdot) \mapsto G$.
$\mathrm{Ring} \to \mathrm{Ab}$ given as $(R, +, \cdot) \mapsto (R, +)$.

Example 3: Tensors

Let R be a commutative ring, $M \in \mathrm{Mod}_R$.
Then $\otimes_R M : \mathrm{Mod}_R \to \mathrm{Mod}_R$ and $\mathrm{Hom}_R(M, -) : \mathrm{Mod}_R \to \mathrm{Mod}_R$.

Definition:

Let X be an object in a category $\mathcal{C}$ and G a group.
An action of G on X is a group homomorphism $G \to \mathrm{Aut}_{\mathcal{C}}(X)$.

Example 1

Let $\mathcal{C} = \mathrm{Set}$.
A G-set is a set $X \in \mathrm{Set}$ equipped wit a group homomorphism

$$G \to \mathrm{Perm}(X) = \mathrm{Aut}_{\mathrm{Set}}(X)$$

Exercise 1

A G-set is the same thing as a functor $G \to \mathrm{Set}$, $* \mapsto X$, $\mathrm{Hom}_{\mathcal{C}}(*, *) \to \mathrm{Hom}_{\mathrm{Set}}(X, X)$ ($G \to \mathrm{Aut}_{\mathrm{Set}}(X)$).

Definition: Adjunctions

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors.
We say that F is left adjoint to G (and that G is right adjoint to F, and that we have a pair of adjoint functors) if for each object $X \in \mathrm{Obj}(\mathcal{C})$ and $Y \in \mathrm{Obj}(\mathcal{D})$, we have a bijection

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \tilde{\to} \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

which is "natural in X and Y":
For any $f : X \to X'$ in $\mathcal{C}$,

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(F(X'), Y) & \tilde{\to} & \mathrm{Hom}_{\mathcal{C}}(X', G(Y)) \\ - \circ F(f) \downarrow & & \downarrow - \circ f \\ \mathrm{Hom}_{\mathcal{D}}(F(X), Y) & \tilde{\to} & \mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \end{array}$$

and for every $g : Y \to Y'$ in $\mathcal{D}$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(F(X), Y) & \tilde{\to} & \mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \\ g \circ - \downarrow & & \downarrow G(g) \circ - \\ \mathrm{Hom}_{\mathcal{D}}(F(X), Y') & \tilde{\to} & \mathrm{Hom}_{\mathcal{C}}(X, G(Y')) \end{array}$$

We write

$$\begin{array}{c} \mathcal{C} \\ F \upharpoonleft\downharpoonright G \\ \mathcal{D} \end{array}$$

Example 1

For $M \in \mathsf{Mod}_R$ we have

$$\begin{array}{c} \mathsf{Mod}_R \\ - \otimes_R M \upharpoonleft\downharpoonright \mathrm{Hom}_R(M, -) \\ \mathsf{Mod}_R \end{array}$$

where

$$\begin{array}{c} \mathrm{Hom}_R(M_1 \otimes M_2, N) \cong \mathrm{Hom}_R(M_1, \mathrm{Hom}_R(M, \mathrm{Hom}_R(M_2, N)) \\ f \mapsto (x \mapsto (y \mapsto f(x \otimes y))) \end{array}$$

Example 2

Let $R \xrightarrow{\phi} S$ be a ring homomorphism.
We can regard an S-module N as an R-module via

$$r \cdot x := \phi(r)x, \quad \forall r \in R, ; x \in N$$

This defines a functor $\mathsf{Mod}_S \to \mathsf{Mod}_R$ called a “restriction of scalars”, which has a left adjoint called “extension of scalars.”

$$\begin{array}{c} \mathsf{Mod}_R \\ S \otimes_R - \upharpoonleft\downharpoonright \\ \mathsf{Mod}_S \end{array}$$

Recall

For commutative ring R, $\rightsquigarrow$ Mod_R.
e.g. $R = F$ a field, $\mathsf{Mod}_R \equiv \mathsf{Vect}_F$; $R = \mathbb{Z}$, $\mathsf{Mod}_R \equiv \mathsf{Ab}$.

Definition: R-Algebra

An R-algebra is an Abelian group $(A, +)$ that has both the structure of

1. an R-module and
2. a ring

which are compatible in that

$$r(ab) = (ra)b = a(rb), \quad \forall r \in R,\ a, b \in A$$

Example 1

The polynomial ring $R[x]$ is an R-algebra.

Example 2

The ring of $n \times n$ matrices $M_n(R)$ is an R-algebra.

Example 3

If $R \xrightarrow{\phi} S$ is a homomorphism of commutative rings, then S is an R-algebra via $r := \phi(r)a$, $\forall r \in R,\ a \in S$.

Example 4

$\mathbb{R} \hookrightarrow \mathbb{C}$. So $\mathbb{C}$ is an $\mathbb{R}$-algebra.
$R \hookrightarrow R[x]$.
More generally, $R[x_1, x_2, \ldots, x_n]$ is an R-algebra.

Commutative R-Algebras

An R-algebra is commutative if it is commutative as a ring.
$\mathrm{CAlg}_R \subset \mathrm{Alg}_R$.

Question: Why are polynomials important?

An algebraic perspective: they are the “free commutative algebras.”

Recall

For R a commutative ring, we have the notion of a free R-module – one that admits a basis.
Categorically, we have an adjunction.

$$\begin{gathered}\mathrm{Set} \\ f \upharpoonleft\downharpoonright \mathcal{U} \\ \mathrm{Mod}_R\end{gathered}$$

The left adjoint of the forgetful functor sends a set I to the free R-module with basis I.

$$F(I) = R^{(I)} = \oplus_{i \in I} R$$

The adjunction says that for any set I and R-module M,

$$\begin{gathered}\mathrm{Hom}_{\mathrm{Mod}_R}(R^{(I)}, M) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Set}}(I, M) \\ \exists!R\text{-linear map}_{\substack{f : R^{(I)} \to M \\ e_i \mapsto x_i}} \leftarrow\!\shortmid \{x_i\}_{i \in I}\end{gathered}$$

Similarly, the forgetful functor $\mathcal{U} : \mathsf{CAlg}_R \to \mathsf{Set}$ has a left adjoint

$$\begin{array}{c} \mathsf{Set} \\ f \upharpoonleft\!\downharpoonright \mathcal{U} \\ \mathsf{CAlg}_R \end{array}$$

which sends a set I to the "free commutative R-algebra on I."
Explicitly, $F(I) = R[\{x_i\}_{i \in I}]$ the polynomial algebra with an indeterminate x_i for each $i \in I$.

Example 1

$I = \{*\} \rightsquigarrow F(\{*\}) = R[x]$.
$I = \{1, \ldots, n\} \rightsquigarrow F(\{1, \ldots, n\}) = R[x_1, \ldots, x_n]$.
$I = \mathbb{N} \rightsquigarrow F(\mathbb{N}) = R[x_1, x_2, \ldots]$.

Adjunction

For any set I and commutative R-algebra $A \in \mathsf{CAlg}_R$, we have a bijection

$$\begin{aligned} \mathrm{Hom}_{\mathsf{CAlg}_R}(R[\{x_i\}_{i \in I}, A) &\cong \mathrm{Hom}_{\mathsf{Set}}(I, A) \\ \exists! R\text{-algebra homomorphism}_{\substack{R[\{x_i\}_{i \in I}] \to A \\ x_i \mapsto a_i}} &\leftarrow\!\shortmid \{a_i\}_{i \in I} \end{aligned}$$

Exmple 1

Let A be a commutative R-algebra.
For any $a \in A$, there exists a unique R-algebra homomorphism $R[x] \to A$ which sends $X \mapsto a$.
Explicitly, $f(x) \mapsto f(a)$.

Corollary

Let $R \xrightarrow{\phi} S$ be a homomorphism of commutative rings.
For any $a \in S$, there is a unique ring $R[x] \xrightarrow{\overline{\phi}} S$ such that $\overline{\phi}|_R = \phi$ and $\overline{\phi}(X) = a$.

Example 1

Let $R \subseteq S$ be a subring.
For each $a \in S$, there is a unique ring homomorphism $R[x] \xrightarrow{\phi} S$ such that $\phi|_R = \mathrm{id}$ and $\phi'(X) = a$.
We call this the "evaluation at a."

$$\begin{aligned} R[x] &\xrightarrow{\mathrm{ev}_a} S \\ f &\mapsto f(a) \end{aligned}$$

Definition: Subalgebra

Let A be a commutative R-algebra, and let $S \subset A$ be a subset.
The subalgebra of A generated by S, denoted $R[S]$, is the intersection of all subalgebras of A which contain S.
Explicitly,

$$R[S] = \{a \in A \,:\, \exists n \geq 1,\ s_1, \ldots, s_n \in S,\ f \in R[x_1, \ldots, x_n],\ a = f(s_1, \ldots, s_n)\}$$

Example 1

Let $A = R[x]$. Then $A \underset{(!)}{=} R[x]$. That is, A is generated by $\{x\}$ as an algebra.
Similarly, $R[x_1, \ldots, x_n]$ is generated as an algebra by $\{x_1, \ldots, x_n\}$.

Example 2

If $R[x]/I$ with $I \subset R[x]$ an ideal, and $x := \overline{X} \in A$, then $A = R[x]$. That is, A is generated by $x = \overline{X}$ as an algebra.
More generally, if $I \subset R[x_1, \ldots, x_n]$ an ideal, then $R[x_1, \ldots, x_n]/I$ is generated by $\{\overline{x}_1, \ldots, \overline{x}_n\}$.

Proposition

If $A \in \mathsf{CAlg}_R$ is a finitely generated, commutative R-algebra, then $A \cong R[x, \ldots, x_n]/I$ for some $n \geq 1$ and ideal $I \subset R[x_1, \ldots, x_n]$.