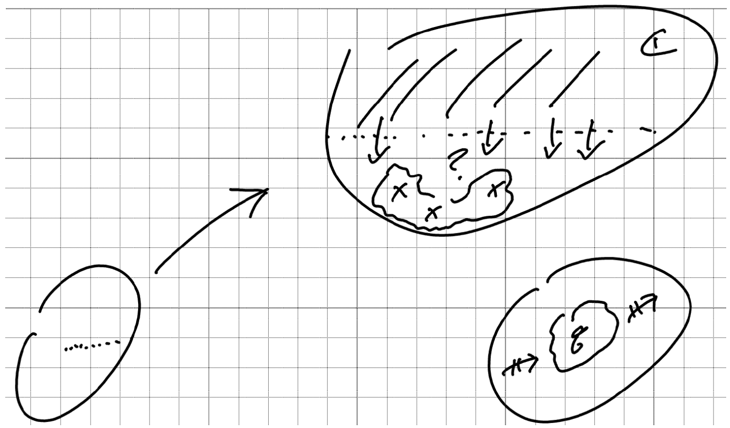
Advanced Analysis

September 25, 2025

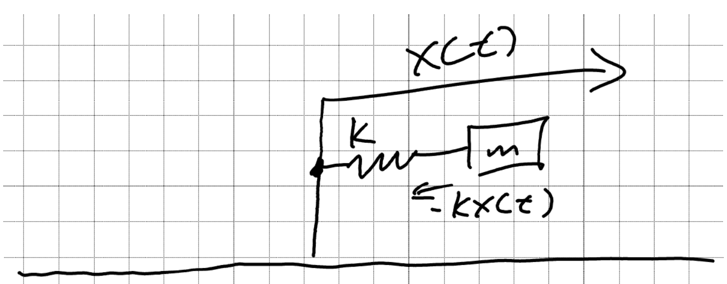
Suppose we have some function of the form $-\Delta + q \in \mathbb{L}(H)$ satisfying $R_A(\lambda)(A - \lambda I)^{-1}$ bounded on $\mathrm{Im}(\lambda) > 0$ and not surjective for $\mathrm{Im}(\lambda) = 0$.



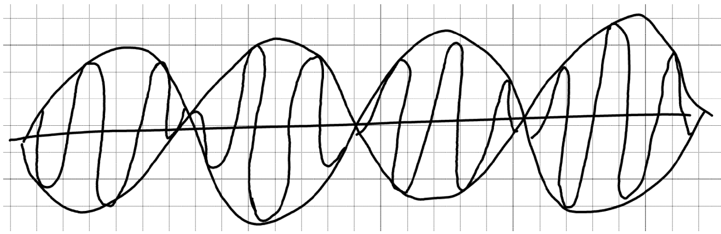
Waves: solutions to $\partial_{tt} u + Au = 0$ on $\mathbb{R}^n$.
Mathematical Tools

- Spectral theory of unbounded operators
- Complex analysis
- Functional analysis
- Microlocal analysis
- Semiclassical analysis

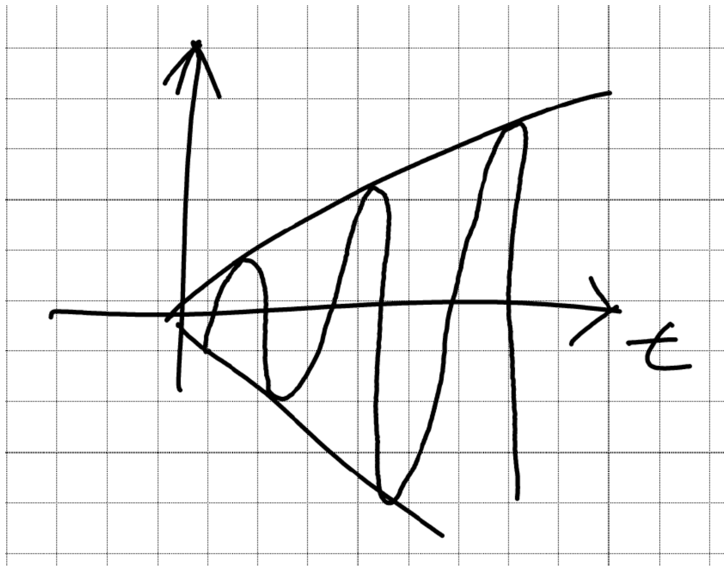
Classical Resonances in ODEs



A harmonic oscillator assuming no friction.
We have an acceleration force, $m\ddot{x}(t) = -kx(t)$ which gives $\ddot{x} + \omega_0^2 x = 0$ with $w_0 = \sqrt{\frac{k}{m}}$ and has solution $x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$.
With forcing, i.e. $m\ddot{x}(t) = -kx(t) + A\sin(\omega t)$, we have $\ddot{x} + \omega_0^2 A' \sin(\omega t)$.
If $|\omega| \neq |\omega_0|$, then $x(t) \sim \mathrm{trig}\left(\left(\frac{\omega - \omega_0}{2}\right) t\right)\left(\left(\frac{\omega - \omega_0}{2}\right) t\right)$ the low and high frequencies respectively.



Beats (non-amplified)
If isntead $|\omega| = |\omega_0|$, then $x(t) \propto \text{trig}(\omega t) t$.



In general, $\dot{x} + Ax = 0$ for $x \in \mathbb{R}^n$, $x(t) = \exp(-tA) + x(0)$.
In the case where A is skew-adjoint, $i.e.$ $\text{sp}(A) \subseteq i\mathbb{R}$, $(x, Ax) = 0 \ \forall x \in \mathbb{R}^n$, then

$$\frac{d}{dt}(x,x) = (\dot{x}, x) + (x, \dot{x}) = (-Ax, x) - (x, Ax) = 0$$

Which implies that $||x(t)||$ is constant and the dynamics are norm perserving.
To generate resonant solutions, if (iw, v) is an eigenpair of A ($\omega \in \mathbb{R}$), consider $\dot{x} + Ax = e^{-i\omega t} v$. As an ansatz, we look for a solution of the form $x(t) = a(t)v$ and the equation becomes $(\dot{a}(t) + i\omega a)v = e^{-i\omega t} v$. Then

$$\begin{aligned} e^{-i\omega t}\frac{d}{dt}(e^{i\omega t} a) &= e^{-i\omega t} \\ \frac{d}{dt}(e^{i\omega t} a) &= 1 \\ a(t) &= te^{-i\omega t}. \end{aligned}$$

Resonances in PDEs

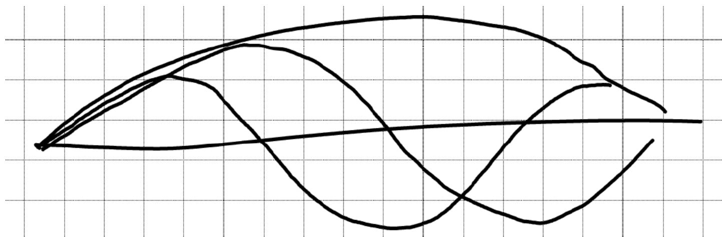
Consider one-dimensional waves on $[0, L]$, $L > 0$.

$$\begin{cases} \partial_{tt} u + \partial_{xx} u = 0 & \\ u|_{t=0} = f & x \in [0, L] \\ \partial_t u|_{t=0} = g & x \in [0, L] \\ u(0,t) = u(L,t) = 0 & \forall t \geq 0 \end{cases}$$

We want to think about this as $\partial_{tt}u = Au = 0$ where A is the Dirichlet Laplacian $Au = -\partial_{xx}u$ with Dirichlet boundary conditions. We then want to find the spectral decomposition of A, $Au - \lambda u = 0 = -\partial_x^2 u - \lambda u$.

$$\begin{aligned}
\lambda = 0. \quad & u(x) = A + Bx \implies A = B = 0 \\
\lambda = -p^2. \quad & u(x) = Ae^{px} + be^{-px} \implies A = B = 0 \\
\lambda = p^2. \quad & u(x) = A\cos(px) + B\sin(px) \implies 0 = u(0) = A \quad 0 = u(L) = B\sin(pl) \implies p = k\pi,\ k \in \mathbb{N}
\end{aligned}$$

Therefore there are infinitely many eigenpairs $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $\phi_n(x) = \sin\left(\frac{k\pi x}{L}\right)$.

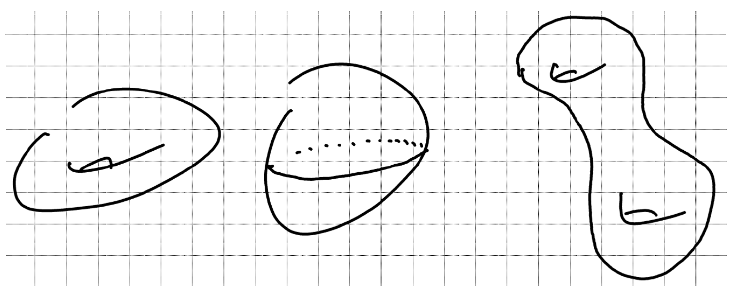


The family $\{\phi_n, n \in \mathbb{N}\}$ is dense in $L^2([0,L])$ where the unbounded operator $(-\partial_x^2)$ with Dirichlet boundary conditions is self-adjoint.

Other Prototypes

(of unbounded self-adjoint operators with discrete spectrum)

- Laplace-Beltrami operators on compact manifolds without boundary.



- On compact domains with boundary there is the Laplacian with Dirichlet boundary conditions.

The (Quantum) Harmonic Oscillator

$H = -\frac{d^2}{dx^2} + x^2$ on $\mathbb{R}$, on $L^2(\mathbb{R})$ with $(f,g) = \int_{\mathbb{R}} f(x)\overline{g(x)}\,dx$.
H acts on the Schwarz space $\mathcal{S}(\mathbb{R}) := \left\{ f \in C^\infty(\mathbb{R}),\ \forall k, \ell \geq 0,\ \sup_{x\in\mathbb{R}} \left| x^k \left(\frac{d}{dx}\right)^\ell f(x) \right| < \infty \right\}$.

- The action of $H : \mathcal{S}(\mathbb{R})$ is continuous.
- H is L^2-symmetric: $\int_{\mathbb{R}} -f''\overline{g} + x^2 f\overline{g}\,dx = (Hf, g) = (f, Hg) = \int_{\mathbb{R}} -\overline{g}'' f + x^2 f\overline{g}\,dx$ (integrating by parts).

We seek eigenvalues $Hu = \lambda u$. If (u,λ) and (v,μ) are eigenpairs, then

$$0 = (Hu, v) - (u, Hv) = (\lambda u, v) - (u, \mu v) = (\lambda - \overline{\mu})(u, v)$$

Where if the difference is nonzero then $(u, v) = 0$.
We can write $H = L^+L^- + I$ where $L^+ = -\frac{d}{dx} + x$ and $L^- = \frac{d}{dx} + x$ and also $[H, L^+] = 2L^+$ and $[H, L^-] = -2L^-$.
Note that H is a non-negative operators

$$(Hf, f) = \int_{\mathbb{R}} ((f')^2 + x^2 f^2)\, dx > 0$$

for $f \neq 0$ and $f \in \mathcal{S}(\mathbb{R})$. Thus $\mathrm{sp}(H) \subseteq (0, \infty)$. If $Hv = \lambda v$, then $H(L^+v) = [H, L^+]v + L^+(Hv) = (\lambda + 2)L^+v$. Similarly $H(L^-v) = (\lambda - 2)L^-v$.
Now we want to solve $L^-\phi_0 = 0$. $\frac{d}{dx}\phi_0 + x\phi_0 = 0$ tells us that $\phi_0(x) = \frac{1}{\sqrt{\pi}} e^{-x^2/2}$ (L^2-normalized). Therefore $H\phi_0 = \phi_0$ and the we have an eigenvalues of one. So we may construct $\phi_n = \frac{(L^+)^n\phi_0}{||(L^+)^n\phi_0||}$ which gives an eigenvector of H with eigenvalues $2n+1$. Note that $||(L^+)^n\phi_0|| = \sqrt{2^n n!}$.
Fact: $\phi_n = p_n(x)e^{-x^2/2}$ where p_n is the Hermite polynomial of degree n.

$$\delta_{nq} = (\phi_n, \phi_q) = \int_{\mathbb{R}} p_n(x)p_q(x)e^{-x^2}\, dx$$

Theorem

$\{\phi_n\}_{n\geq 0}$ is dense in $L^2(\mathbb{R})$ (if $\int_{\mathbb{R}} g\phi_n\, dx = 0$ for all n, then $g = 0$).

Proof (Sketch)

For $g \in L^2$, $\xi \in \mathbb{R}$, $F_g(\xi) = \int_{\mathbb{R}} e^{ix\xi} g(x)\phi_0(x)\, dx = \widehat{g\phi_0}(\xi)$. We observe that

- F_g is real-analytic in ξ.
- $F_g^{(k)}(0) = \int_{\mathbb{R}} (-ix)^k g(x)\phi_0(x)\, dx = 0$ by assumption.

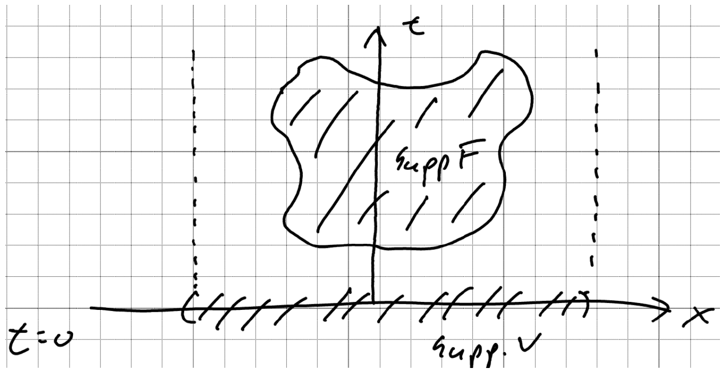
So we have a real-analytic function where all derivatives vanish at a point. So $F_g \equiv 0$, $g\phi_0 = 0$, and $g = 0$.

September 30, 2025

One of the overarching goals is to obtain large time asymptotics of the solution $v(x,t)$ ($x \in \mathbb{R}$, $t > 0$) to

$$\begin{cases} -\partial_{tt}v - P_V v = F(x,t) & \text{on } \mathbb{R}_x \times (0,\infty)_t \\ v(x,0) = \partial_t v(x,0) = 0, & F \in C_C^\infty(R_x \times (0,\infty)_t) \end{cases}$$

where $P_V = D_x^2 + V(x) = -\left(\frac{\partial}{\partial}\right)^2 + V(x)$ and $D_x = \frac{1}{i}\frac{\partial}{\partial x}$. The operator D_x is symmetric and self-adjoint on appropriately chosen domains. For $f(x)$ and $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx$, $\widehat{D_x f} = \xi\hat{f}(\xi)$. $V \in L^\infty_{\text{comp.}}(\mathbb{R})$ (i.e. compactly supported L^∞) is the potential. If $f, g \in \mathcal{S}(\mathbb{R})$, then $(P_V f, g)_{L^2(\mathbb{R})} = (f, P_V g)_{L^2(\mathbb{R})}$.



Another way to look at this assuming v exists, we can consider $u(x,\lambda) := \int_0^\infty e^{it\lambda} v(x,t)\, dt$ (the Fourier-Laplace transform of v) with $\lambda \in \mathbb{C}$, $\mathrm{Im}(\lambda) > 0$. Write $\lambda = \xi + ic,\ c > 0$, such that $u(x, \xi + ic) = \int_0^\infty e^{it\xi} e^{-ct} v(x,t)\, dt = \mathcal{F}_{t \mapsto \xi}(t \mapsto e^{-ct} v(x,t))(x, -\xi)$. Then $u(x,\lambda)$ solves

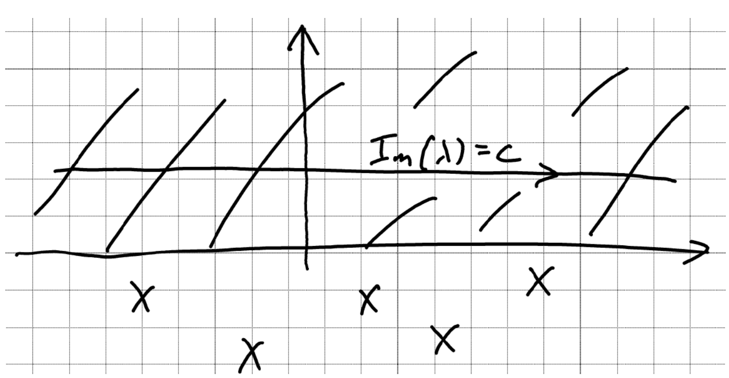
$$\int_0^\infty e^{it\lambda}(-\partial_{tt} v - P_V v)\, dt = \int_0^\infty e^{it\lambda} F(x,t) = \hat{F}(x,\lambda)$$
$$(\lambda^2 - P_V) \underbrace{\int_0^\infty e^{it\lambda} v(x,t)\, dt}_{u(x,\lambda)} = \hat{F}(x,\lambda)$$

which is an entire function in λ.
To Do:

- Study solvability of $(\lambda^2 - P_V)u = \hat{F}(x,\lambda)$.
- Return to v.

For frozen c, we can get $v(x,t)$ back by Fourier inversion.

$$e^{-ct} v(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it\xi} u(x, \xi + ic)\, d\xi$$
$$v(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it(\xi + ic)} u(x, \xi + ic)\, d\xi$$
$$v(x,t) = \frac{1}{2\pi} \int_{\mathrm{Im}(\lambda) = \mathrm{c}} e^{-it\lambda} u(x,\lambda)\, d\lambda$$

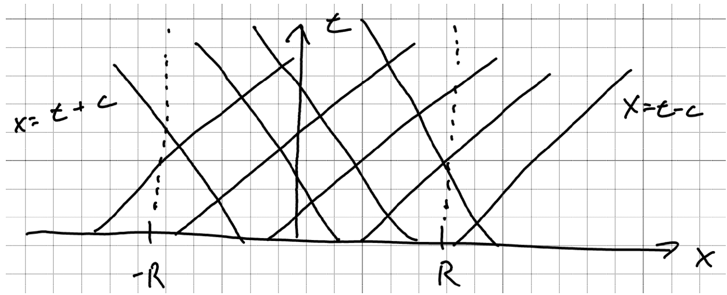


where the spectral problem is invertible.

1D Waves in the Time Domain

Suppose $R > 0$ is such that $\operatorname{supp} V \subset [-R, R]$ and $\operatorname{supp} F \subset [-R, R] \times (0, \infty)$. If $|x| > R$, the PDE looks like $\partial_{tt} v - \partial_{xx} v = 0 = (\partial_t + \partial_x)(\partial_t - \partial_x)v$. Setting $\xi = x + t$ and $\mu = x - t$, then it follows that

$$\partial_\xi \partial_\mu v = 0 \implies v = F(\xi) + G(\mu) = F(x+t) + G(x-t)$$



On $x > R$, we can expect $v(x,t) = F_+(x+t) + G_+(x-t)$; on $x < R$, we expect $v(x,t) = F_-(x+t) + G_-(x-t)$. The terms G_+ and F_- are outgoing whereas the terms F_+ and G_- are incoming and, given that we assumed a source, we expect to be zero.
What does incoming/outgoing look like on the spectral side? $(\lambda^2 - P_v)u = \hat{F}(x, \lambda)$ supported in $|x| \le R$. For $|x| > R$, $(\lambda^2 + \partial_x^2)u = 0$ leads to $u = Ae^{ix\lambda} + Be^{-ix\lambda}$. For $x > R$, $u(x) = a_+ e^{i\lambda|x|} + b_+ e^{-i\lambda|x|}$ for $x < -R$, $u(x) = a_- e^{i\lambda|x|} + b_- e^{-i\lambda|x|}$. u is outgoing if and only if $b_\pm = 0$ and incomming if and only if $a_\pm = 0$.
P_V is an unbounded, symmetric operator on a Hilbert space. For $z \in \mathbb{C}$, $\operatorname{sp}(P_V)$ is the set on the complement of which $(P_V - Z)$ is boundedly invertible. That is, $\forall f$, $\exists! u$ such that $(P_V - z)u = f$ and $||u|| \lesssim ||f|$.

Waves in the Time Domain [Evans, ğ2.4]

Goal: if v solves

$$\begin{aligned} &\partial_{tt} v - \partial_{xx} v = f(x,t) x \in \mathbb{R},\ t > 0,\ f \in C_C^\infty(\mathbb{R} \times (0, \infty)) \\ &v(x,0) = \partial_t v(x,0) = 0 x \in \mathbb{R} \end{aligned}$$

then $v(x,t) = \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(y, t-s)\, dy ds$. We look at

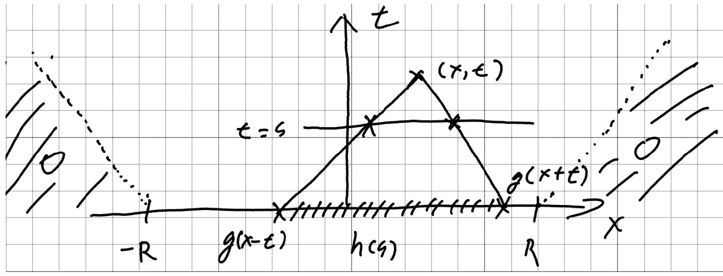
$$\begin{cases} \partial_{tt} v - \partial_{xx} v = 0 \rightsquigarrow v(x,t) = F(x+t) + G(x-t) \\ v(x,0) = g(x),\ \partial_t v(x,0) = h(x) \end{cases}$$

Initial conditions gives us

$$\begin{cases} F(x) + G(x) = g(x) \\ F'(x) - G'(x) = h(x) \end{cases} \qquad \begin{cases} G'(x) = \frac{1}{2}(g'(x) - h(x)) \\ F'(x) = \frac{1}{2}(g'(x) + h(x)) \end{cases}$$

So

$$\begin{aligned} F(x) &= \frac{1}{2}\left(g(x) + \int_0^x h(s)\, ds\right) + C_1 \\ G(x) &= \frac{1}{2}\left(g(x) - \int_0^x h(s)\, ds\right) + C_2 \\ v(x,t) &= \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(s)\, ds + C \end{aligned}$$



This has a finite speed of propogation in the sense that if we suppose $\text{supp}(g,h) \subset [-R,R]$ then $v(x,t) = 0$ whenever $x > R+t$ or $x < -R-t$.

Now we want to go from the homogeneous problem to the inhomogeneous problem. The idea is to think about $v(x,t) = \int_0^t v(x,t;s)\, ds$ where $v(x,t;s)$ solves the homogeneous problem

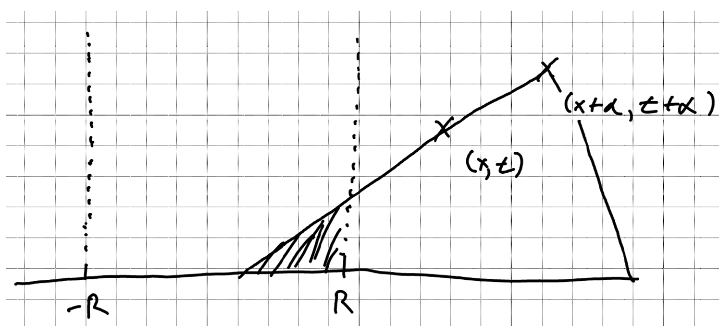
$$\begin{cases} \partial_{tt} v(\cdot,\cdot;s) - \partial_{xx} v(\cdot,\cdot;s) = 0 \\ v(\cdot,s;s) = 0,\ \partial_t v(\cdot,s;s) = f(x,s) \end{cases}$$

Then

$$\partial_{tt} v - \partial_{xx} v = 0 \iff \partial_t \begin{pmatrix} v \\ \partial_t v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \partial_{xx} & 0 \end{pmatrix} \begin{pmatrix} v \\ \partial_t v \end{pmatrix} \quad \text{and} \quad \begin{bmatrix} v \\ \partial_t v \end{bmatrix}_{t=s} = \begin{bmatrix} * \\ * \end{bmatrix}$$

So $v(x,t;s) = \frac{1}{2} \int_{x-(t-s)}^{x+(t-s)} f(y,s)\, ds$ and $v(x,t) = \frac{1}{2} \int_0^t \int_{x-s}^{x+s} f(y,t-s)\, dyds$ follows.

Going back to the original PDE, $(-\partial_{tt} - P_V)v = F$ is equivalent to $(\partial_{tt} - \partial_{xx})v = -(Vv+F)$ which leads to the conclusion that $v(x,t) = -\frac{1}{2} \int_0^t \int_{x-s}^{x+s} (Vv+F)(y,t-s)\, dyds$. For $|x| > R$, v is outgoing.



October 2, 2025

Take come complex vector space and consider the Hilbert space $(\mathcal{H}, (\cdot,\cdot))$ with $(\cdot,\cdot)$ satisfying

$$\begin{cases} (\lambda f, g) = \lambda (f,g) \\ (f, \lambda g) = \overline{\lambda}(f,g) \\ (f,g) = \overline{(g,f)} \\ f \mapsto (f,g) =: ||f||^2 \text{ a norm} \\ (\mathcal{H}, ||\cdot||) \text{ complete with respect to the norm} \end{cases}$$

- Examples
  - $\left(\mathbb{C}^n, (a,b) = \sum_{j=1}^n a_j \overline{b}_j\right)$, $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$

– $L^2(x,\mu)$ (e.g. $[0,1]$ and the Lebesgue measure), $(f,g) = \int_X f\overline{g}\, d\mu$.

Bounded Operators: $T : \mathcal{H} \to \mathcal{H}$ bounded if and only if $\overbrace{\sup_{||x||=1} ||Tx||}^{||T||} < \infty$ satisfying

$$\begin{cases} \mathcal{B}(\mathcal{H}) \text{ the space of bunded operators on } \mathcal{H} \text{ (a complex vector space)} \\ ||\cdot|| \text{ is a norm on } \mathcal{B}(\mathcal{H}) \text{, making it complete} \\ \text{There is a multiplication, } \mathcal{B}(\mathcal{H}) \ni A, B \mapsto AB \text{ and } ||AB|| \le ||A||||B|| \end{cases}$$

Adjoint: if $A \in \mathcal{B}(\mathcal{H})$, $\exists! A^* \in \mathcal{B}(\mathcal{H})$ such that $\forall f, g \in \mathcal{H}$, $(Af, g) = (f, A^* g)$ where A is symmetric/self-adjoint if $A = A^*$. Thes notions are different in the world of unbounded operators.

- Example

  – $\mathcal{H} = \mathbb{C}^n$: $T \in M_n(\mathbb{C})$ symmetric if and only if T is Hermitian. $t_{ij} = \overline{t_{ji}}$.

  – $\mathcal{H} = L^2([0,1])$, $Tf(t) = tf(t)$. $(Tf,g) = \int_0^1 tf(t)\overline{g(t)}\, dt = \int_0^1 f(t)\overline{tg(t)}\, dt = (f, Tg)$.

  – $\mathcal{H} = L^2(\mathbb{R})$ with the Fourier transform. $||f(x)||^2 = c||\hat{f}(\xi)||^2$ (Parseval's Equality).

Finite Dimensional Spectral Theorem

If $A \in M_n(\mathbb{C})$ is Hermitian, there exists an orthonormal basis $(\phi_1, \ldots, \phi_n)$ of $\mathbb{C}^n$ and real eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $A\phi_j = \lambda_j \phi_j$.
Important observation: if A is Hermitian, then λ_j is real for each j, and $\overline{(A\phi_j, \phi_j)} = (\phi_j, A\phi_j) = (A\phi_j, \phi_j) = \lambda_j ||\phi_j||^2$.
So $\lambda_j = \frac{(A\phi_j, \phi_j)}{||\phi_j||^2}$ is real. If $\lambda_j \neq \lambda_k$, then $(\phi_j, \phi_k) = 0$ since $(A\phi_j, \phi_k) - (\phi_j A\phi_k) = (\lambda_j - \overline{\lambda}_k)(\phi_j, \phi_k)$.
Notation: Let $u, v \in \mathbb{C}^n$, denote $u \otimes \overline{v}$ the operator $(u \otimes \overline{v})w = (w, v)u$.
With A as in the theorem, we can write $A = \sum_{j=1}^n \lambda_j \phi_j \otimes \overline{\phi}_j$ $\left(I = \sum_{j=1}^n \phi_j \otimes \overline{\phi}_j\right)$. A second way of writing this is

$$A \underbrace{\begin{bmatrix} \varphi_1 & | & \cdots & | & \varphi_n \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} \varphi_1 & | & \cdots & | & \varphi_n \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}}_{\Lambda}$$

Where $U^* = U^{-1}$ and $A = U\Lambda U^*$. This allows us to construct a functional calculus for A where

$$\begin{cases} A^2 = U\Lambda U^* U\Lambda U^* = U\Lambda^2 U^* \\ A^n = U\Lambda^n U^* \\ p(A) = U \cdot p(\Lambda) \cdot U^*, \ p \text{ a polynomial} \end{cases}$$

Defining $f(A) := U \cdot f(\Lambda) \cdot U^*$, we obtain a Banach algebra homomorphism. Then $f \in C([-||A||, ||A||])$ is also a Banach algebra with sup norm and pointwise multiplication.

$$f(A) := U \cdot \begin{pmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n) \end{pmatrix} \cdot U^*$$

Then we can map $C([-||A||,||A||]) \ni f \mapsto f(A) \in \mathcal{B}(\mathcal{H})$. This is useful for solving ODEs.

- Prototypes
  - Heat equation: $\partial_t u + Au = 0$, $u|_{t=0} = u_0$, $u(t) = e^{-tA}u_0$.
  - Schrödinger equation: $i\partial_t u + Au = 0$, $u|_{t=0} = u_0$, $u(t) = e^{-itA}u_0$.
  - Wave equation: $\partial_{tt}u + Au = 0$, $u|_{t=0} = u_0$, $\partial_t u|_{t=0} = u_1$.

Write $u(t) := \sum_{j=1}^n u_j(t)\phi_j$ with the PDE $\sum_{j=1}^n (u_j'' + \lambda_j u_j)\phi_j = 0$. Then $u_j'' + \lambda_j u_j = 0$, $u_j(0) = u_{j,0}$, and $u_j'(0) = u_{j,1}$. Suppose $\lambda_j > 0$ for all j. Then $u_j(t) = u_{j,0}\cos(\sqrt{\lambda_j}t) + \frac{u_{j,1}}{\sqrt{\lambda_j}}\sin(\sqrt{\lambda_j}t)$. So

$$u(t) = \sum_{j=1}^n \cos(\sqrt{\lambda_j}t)u_{j,0}\phi_j + \frac{1}{\sqrt{\lambda_j}}\sin(\sqrt{\lambda_j}t)u_{j,1}\phi_j$$

Therefore $u = \cos(t\sqrt{A})u_0 + A^{-1/2}\sin(t\sqrt{A})u_1$.

Spectrum of a Bounded Operator

Take $T \in \mathcal{B}(\mathcal{H})$. We say that T is invertible (within $\mathcal{B}(\mathcal{H})$) if and only if $\exists S \in \mathcal{B}(\mathcal{H})$ such that $TS = ST = I$.
Counterexample: take $\mathcal{H} = \ell^2(\mathbb{N}_0) = \{u = (u_n)_{n\geq 0} : \sum |u_n|^2 < \infty\}$ and $Au = \left(\frac{1}{n}u_n\right)_{n\geq 0}$. Then the proxy for $A^{-1}u = (nu_n)_{n\geq 0}$ is not bounded.
Given $T \in \mathcal{B}(\mathcal{H})$, the resolvent set of T is $\rho(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \text{ is invertible}\}$. Invertibility is equivalent to $\forall y \in \mathcal{H}$, $\exists! x$ such that $Tx - \lambda x = y$ with an estimate $||x|| \lesssim ||y||$.
For $\lambda \in \rho(T)$, denote $R(\lambda)$ or $R_T(\lambda) = (T - \lambda I)^{-1} \in \mathcal{B}(\mathcal{H})$ the resolvent of T. Properties of the resolvent set:

1. $\rho(t) \neq 0$ (in fact, if $|\lambda| > ||T||$ then $\lambda \in \rho(T)$.
2. $\rho(t)$ is open.
3. the map $\rho(t) \ni \lambda \mapsto R_T(\lambda) \in \mathcal{B}(\mathcal{H})$ is holomorphic in the sense that $\forall \lambda_0 \in \rho(t)$, $\exists R_T'(\lambda_0) \in \mathcal{B}(\mathcal{H})$ such that $\lim_{\lambda\to\lambda_0}\left|\left|\frac{R_T(\lambda)-R_T(\lambda_0)}{\lambda-\lambda_0} - R_T'(\lambda_0)\right|\right| = 0$.

For a., if $|\lambda| > ||T||$, $Tx - \lambda x = y \iff \left(I - \frac{T}{\lambda}\right)x = -\frac{y}{\lambda} \iff x = \sum_{k=0}^\infty \frac{T^k}{\lambda^{k+1}}y$. Then $R_T(\lambda) = \frac{1}{\lambda}\sum_{k=0}^\infty \left(\frac{T}{\lambda}\right)^k$ and

$$||R_T(\lambda)|| \leq \frac{1}{||\lambda||}\frac{1}{1 - ||T/\lambda||} \leq \frac{1}{|\lambda| - ||T|||}$$

For b., pick $\lambda_0 \in \rho(T)$ and find $r > 0$ such that $|\lambda - \lambda_0| < r \implies \lambda \in \rho(T)$. Then $Tx - \lambda x = y \iff (T - \lambda_0)x - (\lambda - \lambda_0)x = y \iff x - (\lambda - \lambda_0)R_T(\lambda_0)x = R_T(\lambda_0)y$ where if $||(\lambda - \lambda_0)R_T(\lambda_0)|| < 1$ it is boundedly solvable by Neumann series.
For c.,

$$\begin{aligned}
R_T(\lambda) - R_T(\lambda_0) &= (T - \lambda I)^{-1} - (T - \lambda_0 I)^{-1} \\
(T - \lambda I)(R_T(\lambda) - R_T(\lambda_0)) &= I - (T - \lambda_0 I + (\lambda_0 - \lambda)I)(T - \lambda_0 I)^{-1} \\
(T - \lambda I)(R_T(\lambda) - R_T(\lambda_0)) &= I - I + (\lambda - \lambda_0)R_T(\lambda_0) \\
R_T(\lambda) - R_T(\lambda_0) &= (\lambda - \lambda_0)R_T(\lambda)R_T(\lambda_0)
\end{aligned}$$

So $\frac{R_T(\lambda)-R_T(\lambda_0)}{\lambda-\lambda_0} - R_T(\lambda_0)^2 = o(\lambda - \lambda_0)$.
Then we define the spectrum $\sigma(T) := \mathbb{C} \setminus \rho(T)$ which is closed since $\rho(T)$ is open.

Lemma

If $T \in \mathcal{B}(\mathcal{H})$ is self-adjoint, then $\sigma(T) \subseteq [-||T||, ||T||]$.

- Proof

First we know $\sigma(T) \subseteq \{|\lambda| \le ||T||\}$. We want to show that it is real, and that if $\lambda = a+ib$ and $b \neq 0$ then $T-(a+ib)I$ is invertible.
$T-(a+bi)I$ is injective.

$$\begin{aligned} ||(T-(a+ib))x||^2 &= (Tx-(a+ib)x, Tx-(a+ib)x) \\ &= ||Tx||^2 + (a^2+b^2)||x||^2 - (Tx, (a+ib)x) - ((a+ib)x, Tx) \\ &= ||Tx||^2 + (a^2+b^2)||x||^2 - (a-ib)(Tx,x) - (a+ib)(x,Tx) \\ &= ||Tx||^2 + a^2||x||^2 - 2a(x,Tx) + b^2||x||^2 \ge b^2||x||^2 \end{aligned}$$

since $||Tx||^2 + a^2||x||^2 - 2a(x,Tx) \ge 0$by Cauchy-Schwarz. Therefore $T-(a+ib)$ is injective and, by the open mapping theorem, $(T-(a+ib))^* = T-(a-ib)$ is surjective. Similarly for $T-(a-ib)$, and the norm estimate is $||(T-(a+ib))^{-1}|| \le \frac{1}{b}$. Note that $\frac{1}{b} = \frac{1}{\text{dist}(a+ib,\mathbb{R})}$.

Note that the spectrum of T may no longer be made of eigenvalues in the non-finite case. There may exist λ such that $T-\lambda I$ nont injective, $\exists v \neq 0 \; Tv = \lambda v$. Recall the example $Tf(t) = tf(t)$ with $f \in L^2((\cdot,\cdot), dt)$. T is self-adjoint, $||T|| \le 1$, and $(Tf,f) = \int_0^1 t|f(t)|^2 \, dt \ge 0$. So $\sigma(T) \subseteq [0,1]$. For $\lambda \in [0,1]$ is $T-\lambda I$ injective? $Tf = \lambda f \iff tf(t) = \lambda f(t) \iff (t-\lambda)f(t) = 0$ which implies $f \equiv 0$ in $L^2([0,1])$. Is $T-\lambda I$ surjective? $(t-\lambda)f(t) = g(t) \iff f(t) = \frac{g(t)}{t-\lambda}$, so $g(t) \equiv 1 \in L^2([0,1])$ which implies $f(t) = \frac{1}{t-\lambda}$ is not $L^2([0,1])$ and $\sigma(T) = [0,1]$.

October 9, 2025

Spectral Resolution

Take $\mathcal{H}$ a hlibert space, and say that $P \in \mathcal{B}(\mathcal{H})$ is an orthogonal projection if $P^2 = P$ and $P^* = P$. Then let $\mathcal{P}(\mathcal{H}) = \{P \in \mathcal{B}(\mathcal{H}), \; P$ is an orthogonal projection$\}$.

- Examples
  - $\phi \in \mathcal{H}$, $||\phi|| = 1$, $P := \phi \otimes \overline{\phi}$. Then $P\psi = (\psi,\phi)\phi$ and $P^2\psi = P(\psi,\phi)\phi = (\psi,\phi)(\phi,\phi)\phi = P\psi$.
  - $\phi_1, \dots, \phi_n$ an orthonormal family with $P = \sum_{k=1}^n \phi_k \otimes \overline{\phi}_k$.
  - $\mathcal{H} = \ell^2(\mathbb{N})$, $e_j = (0,0,\dots,0,1,0,\dots)$ and $I = \sum_{j=1}^\infty e_j \otimes \overline{e}_j$.
  - $\mathcal{H} = L^2(\mathbb{R})$. Fix I an interval with χ_I the characteristic function for I. Then take $Pf := \chi_I f$.
    - $PPf = \chi_I \chi_I f = \chi_I f = Pf$.
    - $\int_{\mathbb{R}} \chi_I f \overline{g} \, dx = \int_{\mathbb{R}} f \overline{\chi_I g} \, dx$.
    - If I has a nonempty interior, then $\text{Range}(P) = \{f \in L^2(\mathbb{R}), \text{ supp} f \subset I\} \simeq L^2(I)$.

Definition: Spectral Resolution

A spectral resolution is a map $\mathbb{R} \ni \lambda \mapsto E(\lambda) \in \mathcal{P}(\mathcal{H})$ satisfying

1. $\forall f \in \mathcal{H}$, $||E(\lambda)f||$ is increasing.
2. $\exists [a,b]$ such that $E(\lambda) = 0$ if $\lambda < a$ and $E(\lambda) = \mathrm{Id}$ if $\lambda \geq b$.
3. $E(\lambda)$ is right continuous. That is, $\forall f \in \mathcal{H}$, $\lambda \in \mathbb{R}$,

$$\lim_{\substack{\mu \to \lambda \\ \mu > \lambda}} ||E(\mu)f - E(\lambda)f|| = 0$$

Alternatively, we can require $E(\lambda)E(\mu) = E(\min\{\mu, \lambda\})$.

Long story short: the collection of self-adjoint bounded operators is in one-to-one correspondence with the collection of spectral resolutions.

- Examples
  - $A \in M_n(\mathbb{C})$, $A^* = A$, with eigencouples $(\lambda_1, \phi_1), \ldots, (\lambda_n, \phi_n)$ and simple spectrum $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. Define $E(\lambda) := \sum_{j:\lambda_j \leq \lambda} \phi_j \otimes \overline{\phi}_j$.
    * $A = I$ gives $E(\lambda) = 0$ for $\lambda < 1$ and $E(\lambda) = \mathrm{Id}$ for $\lambda \geq 1$.
    * $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ gives $E(\lambda) = \mathrm{Id}_{\lambda \geq 1}\, e_1 \otimes e_1 + \mathrm{Id}_{\lambda \geq 2}\, e_2 \otimes e_2$.
      - If $f = f_1 e_1 + f_2 e_2$, then $||E(\lambda)f||^2 = \mathrm{Id}_{\lambda \geq 1}(\lambda)||f_1||^2 + \mathrm{Id}_{\lambda \geq 2}(\lambda)|f_2|^2$.

Spectral Measures

A spectral resolution gives rise to spectral measures

$$f, g \in \mathcal{H} \quad \lambda \mapsto (E(\lambda)f, g) = F(\lambda) \in \mathbb{C}$$

This defines a Lebesgue-Stieljes measure

$$\mu_F : \mu_F((a,b]) = F(b) - F(a), \quad \forall a, b \in \mathbb{R}$$

We can construct this as follows:

- When $f = g$, $\lambda \mapsto (E(\lambda)f, f) = (E(\lambda)^2 f, f) = ||E(\lambda)f||^2$ (increasing).
- When $f \neq g$,

$$(E(\lambda)f, g) = (E\lambda(f), E(\lambda)g) = \frac{1}{4}\left(||E(\lambda)(f+g)||^2 - ||E(\lambda)(f-g)||^2 - i||E(\lambda)(f+ig)||^2 + i||E(\lambda)(f-ig)||^2\right)$$

$\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ defines a projection-valued measure E_Ω, $\Omega \subset \mathbb{R}$ a Borel set. Start with $E_{(a,b]} = E(b) - E(a)$. We would like for E_Ω to satisfy $E_{\Omega_1}E_{\Omega_2} = E_{\Omega_1 \cap \Omega_2}$, $E_\varnothing = 0$, $E_\mathbb{R} = \mathrm{Id}$.

Theorem: Spectral Theorem

For $A \in \mathcal{B}(\mathcal{H})$ self-adjoint, there exist $a, b \in \mathbb{R}$ and an A-dependent spectral resolution $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ such that

$$A = \int_{a_-}^{b} \lambda \, dE(\lambda)$$

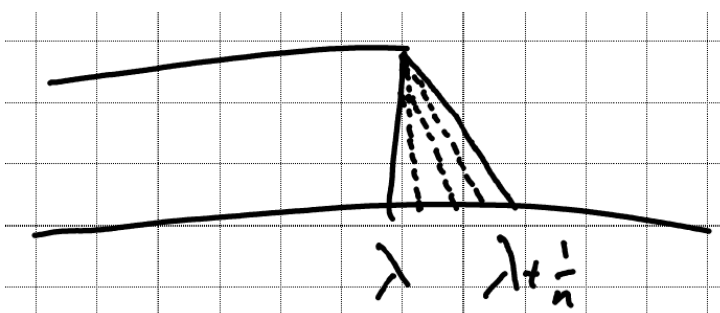
in the sense that $(Af, g) = \int_{a_-}^{b} \lambda \, d(E(\lambda)f, g)$ for all $f, g \in \mathcal{H}$. This is amenable to creating a functional calculus

$$C([-||A||, ||A||]) \to \text{bounded self-adjoint operators that commute with } A$$
$$h \mapsto h(A) := \int_{a_-}^{b} h(\lambda) \, dE(\lambda)$$

The idea is that $E(\lambda) = \chi_{(-\infty, \lambda]}(A)$. Once E is constructed, this leads to E_Ω for Ω Borel. We say that a measure μ is supported in G (Borel) if for every Ω Borel, $\mu(\Omega) = \mu(\Omega \cap G)$. Then $\operatorname{supp} E \subset \sigma(A)$.

Functional Calculus

We want to make sense of $h(A)$ for h in a large enough class. If p is a polynomial, we can make sense of $p(A) = \sum_{k=0}^{n} a_k A^k$ which is self-adjoint and bounded.
For $h \in C(-||A||, ||A||)$, h is uniformly approximated by polynomials. We want to show that p_n is uniformly Cauchy which implies that $p_n(A)$ converges to some $h(A)$.
For $h = \chi_{(-\infty, \lambda]}$, we proceed by approximation by tent functions.



Definition: Positive Operator

If S is a self-adjoint, bounded operator on $\mathcal{H}$, we say that S is positive ($S \geq 0$) if $(Sf, f) \geq 0$, $\forall f \in \mathcal{H}$. For S_1, S_2 self-adjoint and bounded, we say that $S_1 \geq S_2$ if and only if $S_1 - S_2 \geq 0$.
For $T \in \mathcal{B}(\mathcal{H})$, self-adjoint, set $a := \inf_{||f||=1}(Tf, f)$ and $b := \sup_{||f||=1}(Tf, f)$. Then $a\mathrm{Id} \leq T \leq b\mathrm{Id}$.

$$((T - a\mathrm{Id})f, f) = (Tf, f) - a(f, f) = (f, f)\left(\left(T\frac{f}{||f||}, \frac{f}{||f||}\right) - a\right) \geq 0$$

We want to show that if p is a polynomial on $[-||A||, ||A||]$, then $(\inf_{[-||A||,||A||} p)\,\mathrm{Id} \leq p(A) \leq (\sup_{[-||A||,||A||} p)\,\mathrm{Id}$.

Lemma

If T_1 and T_2 are postive and commute, then $T_1 T_2 \geq 0$.

Square Root Lemma

If $A \geq 0$ (i.e. bounded, self-adjoint, and positive), then $\exists! B \geq 0$ such that $B^2 = A$ and B comutes with any operator that commutes with A.

- Proof
  Use the power series of $z \mapsto \sqrt{1-z}$ at $z = 0$.

$$1 + \sum_{k=1}^{\infty} c_k z^k$$

We can find that $c_k < 0$ for all $k \geq 1$ and that the series converges uniformly on $\{|z| \leq 1\}$.
Now let $A \geq 0$ which implies that $0\,\mathrm{Id} \leq I - A \leq 1\,\mathrm{Id}$. Without loss of generality, suppose $\operatorname{supp}||A|| \leq 1$. The idea is to write

$$B = \sqrt{A} = \sqrt{I - (I - A)} = I + \sum_{k=1}^{\infty} c_k (I - A)^k$$

which converges strongly because the series converges uniformly. Then $B^2 = A$. We see that $B \geq 0$ using the fact that $\operatorname{sign}(c_k) < 0$ which implies $\sum_{k \geq 1} c_k \geq -1$. The proof of uniqueness can be found in the text.

Proof of Lemma

Assuming the square root lemma, write $T_2 = B^2$. Then since $[T_1, T_2] = 0$, $[B, T_1] = 0$. Then

$$(T_1 T_2 f, f) = (T_1 B^2 f, f) = (B T_1 B f, f) = (T_1(Bf), Bf) \geq 0$$

Weaker Version

Instead of $(\inf_{[-||A||,||A||} p)\,\mathrm{Id} \leq p(A) \leq (\sup_{[-||A||,||A||} p)\,\mathrm{Id}$, we have that if $\min_{[-||A||,||A||]} p \geq 0$, then $p(A) \geq 0$.

Proof

If $p \geq 0$ on $[-||A||, ||A||]$, we can factor it as a product of positive pieces

$$p(x) = \prod \overbrace{(x - r_j)}^{r_j < -||A||} \overbrace{(s_j - x)}^{s_j \geq ||A||} ((x - a_j)^2 + b_j^2)$$
$$p(A) = \prod \underbrace{(A - r_j)}_{\geq 0} \underbrace{(s_j - A)}_{\geq 0} \underbrace{((A - a_j)^2 b_j^2}_{\geq 0}$$

Using the previous lemma, we have that $P(A) \geq 0$.

Proof

Finally, to show that $(\inf_{[-||A||,||A||} p)\,\mathrm{Id} \leq p(A) \leq (\sup_{[-||A||,||A||} p)\,\mathrm{Id}$, we see that $p - \inf p$ and $\operatorname{supp} f - p$ are positive polynomials and apply the weaker version to them.

Definition

We can define $h(A)$ for $h \in C(-||A||, ||A||)$ by Weierstrass approximation. There exist p_n polynomials such that $\sup_{[-||A||,||A||]} |p_n - h| \underset{n\to\infty}{\longrightarrow} 0$. Then p_n is uniformly Cauchy, so

$$\inf(p_n - p_m)\,\mathrm{Id} \le p_n(A) - p_m(A) \le \sup(p_n - p_m)\,\mathrm{Id}$$

which implies that

$$||p_n(A) - p_{n-1}(A)|| \le \max(\sup(p_n - p_m), -\inf(p_n - p_m)) \underset{n,m\to\infty}{\longrightarrow} 0.$$

So $p_n(A)$ is Cauchy in $(\mathcal{B}(\mathcal{H}), ||\cdot||)$ which means it converges. We call $h(A) = \lim_{n\to\infty} p_n(A)$. We still want to show that $h(A)$ is bounded and self-adjoint.

October 14, 2025

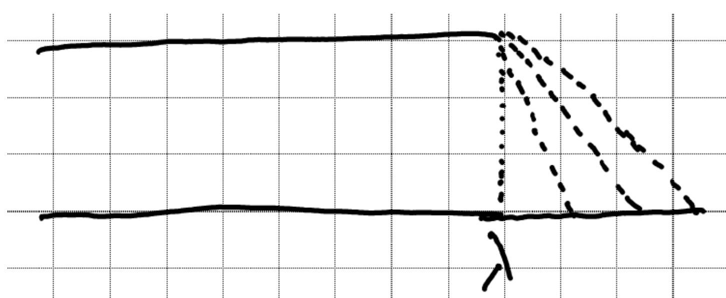
Spectral Theorem for Bounded Self-Adjoint Operators

If $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint such that $a||f||^2 \le (Af, f) \le b||f||^2 \ \forall f \in \mathcal{H}$, then there exists a spectral resolution $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ such that $A = \int_{a^-}^{b} \lambda \, dE(\lambda)$.

Proof (Continued)

We would like that $E(\lambda) = \phi^\lambda(A)$ where $\phi^\lambda := \chi_{(-\infty,\lambda]}$, however ϕ^λ is not continuous so instead we can approximate it as

$$\phi_n^\lambda := \begin{cases} 1, & x \le \lambda \\ \text{linear on } \left[\lambda, \lambda + \frac{1}{n}\right] \\ 0, & x \ge \lambda + \frac{1}{n} \end{cases}$$



To demonstrate this, we need the following proposition.

Proposition

If T_n is a sequence of positive operators and $T_n \ge T_{n+1} \ge 0$, then there exists some $T \ge 0$ such that $T_n f \to Tf$, $\forall f \in \mathcal{H}$.

Proof

Fix $f \in \mathcal{H}$, and consider $(T_n F, f)$ which is decreasing, bounded from below, and therefore converges and is Cauchy. Now as an estimate, we can say that if $S \in \mathcal{B}(\mathcal{H})$ is self-adjoint where $0 \le S \le MI$, then $\forall f \in \mathcal{H}$ we know that

$$||Sf||^2 \le (Sf, f)^{1/2} M^{3/2} ||f||.$$

To see this, we look at $t \mapsto (S(S+tI)f,(S+tI)f) \geq 0$ since $S \geq 0$. Then

$$(S(S+tI)f,(S+tI)f) = (S^2 f, Sf) + 2t||Sf||^2 + t^2(Sf,f)$$

So $\Delta < 0$ if and only if

$$\begin{aligned} ||Sf||^4 - (S^2 f, Sf)(Sf,f) < 0 \\ ||Sf||^4 \leq (Sf,f)(S^2 f, Sf) \\ \leq (Sf,f)\underbrace{||S^2 f||||Sf||}_{M^3||f||^2} \end{aligned}$$

We apply this to $T_n - T_m = S$ where $T_n - T_m \leq T_0$ for all $n \leq m$. Then

$$||(T_n - T_m)f||^2 \leq ((T_n - T_m)f,f)^{1/2}||f||||T_0||^{3/2}$$

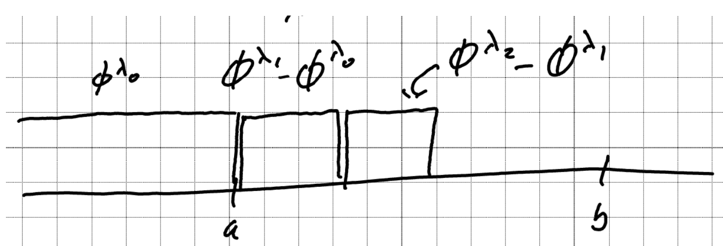
Since $(T_n f, f)$ is Cauchy, $T_n f$ is Cauchy and therefore converges. It remains to check that T is linear, positive, satisfies, $T \leq T_0$, etc.

Spectral Theorem Proof Continued

For each n, $\phi_n^\lambda(A)$ makes sense and is positive since $\phi_n^\lambda(t) \geq 0$. Since $\phi_n^\lambda \geq \phi_{n+1}^\lambda$, $\phi_n^\lambda(A) \geq \phi_{n+1}^\lambda(A)$. Thus, by the preceding proposition, $\phi^\lambda(A) := \lim_{n\to\infty} \phi_n^\lambda(A)$ exists as a bounded, self-adjoint, positive operator.
It remains to check the spectral resolution properties.
The final property to check is that $\forall f \in \mathcal{H}$, $(Af,f) = \int_{a^-}^b \lambda \, d(E(\lambda)f,f)$. The idea is to approximate multiplication by t with piecewise constant functions. We fix a partition $a = \lambda_0 \leq \cdots \leq \lambda_k = b$ such that $\sup(\lambda_{j+1} - \lambda_j) < \delta$. Then

$$t\phi^{\lambda_k}(t) = t = \phi^{\lambda_0}(t) + \sum_{j=1}^k t(\phi^{\lambda_j}(t) - \phi^{\lambda_{j-1}}(t))$$

for all $a \leq t \leq b$.

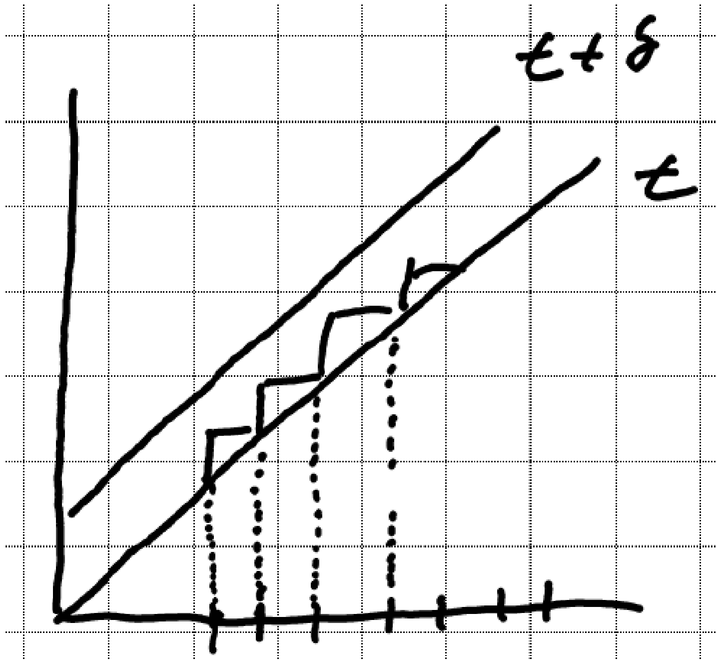


Then

$$\begin{gathered} \lambda^{j-1} \leq t \leq \lambda^j \\ \lambda^{j-1}(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \leq t(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \leq \lambda^j(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \\ \underbrace{\sum_{j=1}^k \lambda^{j-1}(\phi^{\lambda_j} - \phi^{\lambda_{j-1}})}_{\leq \sum_{j=1}^k (\lambda^j - \delta)(\phi^{\lambda_j} - \phi^{\lambda_{j-1}})} \leq t\sum_{j=1}^k (\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \leq \sum_{j=1}^k \lambda^j(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \end{gathered}$$

Adding $\lambda_0\phi^{\lambda_0}$, we see that

$$t \le \lambda_0\phi^{\lambda_0} + \sum_{j=1}^{k} \lambda^j(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \le t + \delta$$



When we apply this to A,

$$A \le \lambda_0 E(\lambda_0) + \sum_{j=1}^{k} \lambda^j(E(\lambda_j) - E(\lambda_{j-1})) \le A + \delta I$$

Finally, we see that
As one refines the partition, $\delta \to 0$ and $(A, f, f) = \int_{a^-}^{b} \lambda \; d(E(\lambda)f, f)$.
Then if $\phi \in C([-||A||, ||A||])$, $\phi(A) = \int_{a^-}^{b} \phi(\lambda) \; dE(\lambda)$.

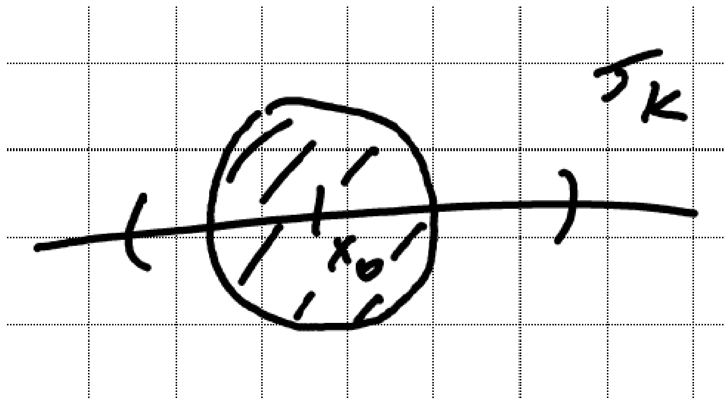
$$\left|(Af, f) - \lambda_0(E(\lambda_0)f, f) - \sum_{j=1}^{k} \lambda^j(E(\lambda_j)f, f) - (E(\lambda_{j-1}f, f))\right| \le \delta||I||^2$$

Functional Calculus

We observe that for $g \in C_C^{\infty}(\mathbb{C})$, $g(w) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{1}{w-z}\partial_{\overline{z}} g(z) \; d^2z$. Then $f(A) := \frac{i}{\pi}\int_{\mathbb{C}}(A-z)^{-1}\partial_{\overline{z}}\tilde{f}(z) \; d^2z$ where $\partial_{\overline{z}}\tilde{f} = O((\mathrm{Im}(z)^n)$ for $n \ge 3$ as $\mathrm{Im}(z) \to 0$.

Proposition

For every $f \in \mathcal{H}$, the Lebesgue-Stieltjesmeasure corresponding to $F(\lambda) = (E(\lambda f, f)$ is supported on $\sigma(A)$.
Since $\sigma(A) \subseteq [a, b]$ is closed, $[a, b] \setminus \sigma(A)$ is open (i.e. $\bigcup_{k\in\mathbb{N}} J_k$ for open intervals J_k). We want to show that $F(\lambda)$ is constant on each J_k (equivalently: $\int_{J_k} dF(\lambda) = 0$.



Fix $J_k \ni x_0$. Then $R_A(x_0) = (A - x_0 I)^{-1}$ exists, and we can pick $\varepsilon > 0$ such that $\forall z \in \overline{B_\varepsilon(x_0)}$, $||R_A(z)|| \leq M$. Then for $z \in \overline{B_\varepsilon(x_0)} + \mathrm{Im}(z) \neq 0$,

$$R_A(z) = (A - zI)^{-1} = \phi_z(A)$$

and $\phi_z(t) = \frac{1}{t-z} \in C([a,b])$. Consider

$$R_A(z)R_A(\overline{z}) = \psi_z(A) = \int \frac{1}{|\lambda - z|^2}\, dE(\lambda)$$

with $\psi_z(t) = \frac{1}{|t-z|^2}$. It follows that for all $z \in \overline{B_\varepsilon(x_0)} \setminus \mathbb{R}$,

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{1}{|\lambda - z|^2}\, dF(\lambda) \leq \int \frac{1}{|\lambda - z|^2}\, dF(\lambda) = (R_A(z)R_A(\overline{z})f, f) \leq M^2||f||^2$$

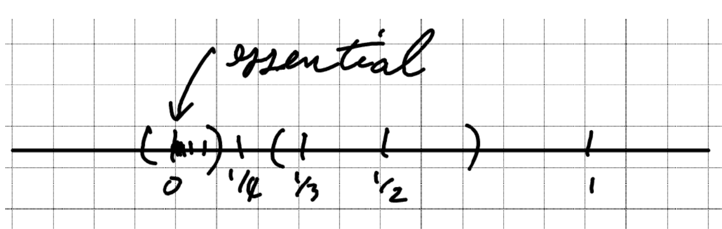
which stays true for all $z \in \overline{B_\varepsilon(x_0)}$. In particular for $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$,

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} dx \int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{1}{|\lambda - z|^2}\, dF(\lambda) \leq 2\varepsilon M^2||f||^2$$

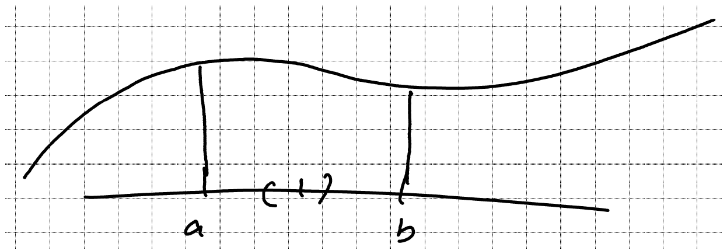
Since Fubini holds, we observe that $\int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{1}{|\lambda - x|^2}\, dx = \infty$, so it must be the case that $\int_{x_0-\varepsilon}^{x_0+\varepsilon} dF(\lambda) = 0$.

Discrete Spectrum vs Essential Spectrum

Recall the spectral measure of A, $E_\Omega(A)$ for $\Omega \subset \mathbb{R}$ Borel where $E_{(a,b]} = E(b) - E(a)$. We say that $\lambda \in \sigma_d(A)$ (the discrete spectrum of A) if there exists $\varepsilon > 0$ such that $\dim(\mathrm{range}(E_{(\lambda-\varepsilon,\lambda+\varepsilon)})) < \infty$. Likewise, $\lambda \in \sigma_{\mathrm{ess}}(A)$ (the essential spectrum) if $\forall \varepsilon > 0$, $\dim(\mathrm{range}(E_{(\lambda-\varepsilon,\lambda+\varepsilon)})) = \infty$.
As an example, take $\mathcal{H} = \ell^2(\mathbb{N})$ with $(Au)_n = \frac{u_n}{n}$ and $A = \sum_{j=1}^{\infty} \frac{1}{j} e_j \otimes e_j$.



Discrete spectra include eigenvalues of finite multiplicity.
Essential spectra include accumulation points of eigenvalues, eigenvalues of infinite multiplicity, absolutely continuous spectrum, s.c. spectrum.
Another example if $Af(t) = tf(t)$ on $L^2([0,1])$. Then $E(\lambda)f(t) = \chi_{(-\infty,\lambda]}f(t)$, and $E_{(a,b]}f(t) = \chi_{(a,b]}f(t)$. $\forall x_0 \in [0,1]$, we have that $\mathrm{range}(\mathrm{E}_{(x_0-\varepsilon,x_0+\varepsilon)} = L^2((x_0 - \varepsilon, x_0 + \varepsilon))$.



October 16, 2025

Compact Operators and Analytic Fredholm Theorem

Definition: Spectral RAdius

For $A \in \mathcal{B}(\mathcal{H})$, we say that the spectral radius of A is $r(a) = \sup_{\lambda \in \sigma(A)} |\lambda| < \infty$

Theorem

1. if $A \in \mathcal{B}(\mathcal{H})$, then $r(A) = \limsup_{n\to\infty} ||A^n||^{1/n}$.
2. If A is, in addition, self-adjoint, then $r(A) = ||A||$.

  As a non-example, consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $\sigma(A) = \{0\}$, but $r(A) = 0 \neq ||A||$.

Proof

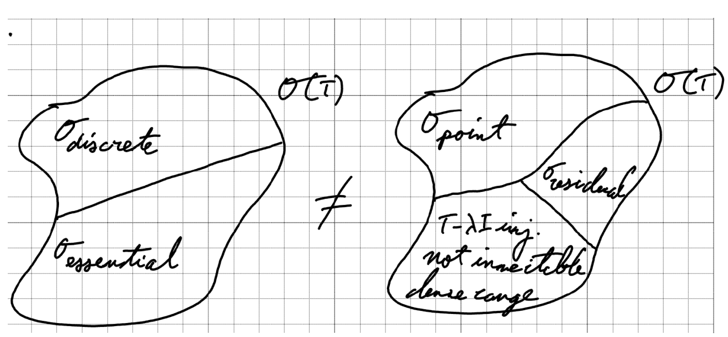
Recall Hadamard's Formula: $\sum_{k=0}^{\infty} a_k z^k$ has radius of convergence R computed by $\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}$. This holds even when a_k are members of a Banach algebra (e.g. $\mathcal{B}(\mathcal{H})$).
Set $z = \frac{1}{\lambda}$, such that $0 < |z| < \frac{1}{r(A)}$ and implies the rexistence of

$$R_A\left(\frac{1}{z}\right) = \left(A - \frac{1}{z}I\right)^{-1} = -(I - zA)^{-1} = -z\sum_{k=0}^{\infty} A^k z^k.$$

So we have that $r(A) = \frac{1}{R} = \limsup_{k\to\infty} ||A^k||^{1/k}$.
Now if A is self-adjoint, $||A^2|| = ||A||^2$ since $||A^2|| \leq ||A||^2$ by submultiplicativity and $||A^2|| \geq \sup_{||x||=1}(x, A^2 x) = \sup_{||x||=1} ||Ax||^2 = ||A||^2$ using Cauchy-Schwarz.
By induction, $||A^{2^k}||^{1/2^k} = ||A||$ which implies that $r(A) = ||A||$.

Another Spectral Decomposition

For $T : \mathcal{H} \to \mathcal{H}$ bounded, we say that λ is in the point spectrum of T if $T - \lambda I$ is not-injective. We say that λ is in the residual spectrum of T if $T - \lambda I$ is injective but does not have dense range.



Self-adjoint operators have no residual spectrum (RS, Thm VI.8).

Definition: Compact Operators

$K \in \mathcal{B}(\mathcal{H})$ is compact if K maps bounded sequences to sequences with a limit point. Equivalently, if $B_{\mathcal{H}} = \{x \in H : ||x|| \leq 1\}$ then K is compact if $K(B_{\mathcal{H}})$ has compact closure (i.e. is precompact).
$\mathcal{K}(\mathcal{H})$, the collection of compact operators on $\mathcal{H}$, is a closed linear subspace of $\mathcal{B}(\mathcal{H})$ since if $T, S \in \mathcal{K}(\mathcal{H})$, then $T + S \in \mathcal{K}(\mathcal{H})$. We have also that $\mathcal{K}(\mathcal{H})$ is a 2-sided ideal of $\mathcal{B}(\mathcal{H})$ since when T is compact and S is bounded, ST and TS are compact.
Examples

- For a finite-rank operator $A : \mathcal{H} \to \mathcal{H}$, $\text{range}(A) < \infty$. The general form of A is $A = \sum_{j=1}^{n} \phi_j \otimes \overline{\psi}_j$ for $\phi_j, \psi_j \in \mathcal{H}$.
- Strong limits of finite-rank operators.
- $(Au)_n = \frac{1}{n} u_n$ on $\ell^2(\mathbb{N})$. Note that $A_n = \sum_{j=1}^{n} \frac{1}{j} e_j \otimes e_j$ shows that $||A - A_n|| \leq \frac{1}{n+1}$.
- The inclusion $h^1 \hookrightarrow \ell^2$ where $h^1 = \{u \in \mathbb{C}^{\mathbb{N}} : \sum_{j=1}^{\infty} j^2 |u_j|^2 < \infty\}$.

Proposition

If $\mathcal{H}$ is separable, all compact operators arise as limits of finite-rank operators.

Proof

We want to show that for $A \in \mathcal{K}(\mathcal{H})$, $\forall \varepsilon > 0$, $\exists A_\varepsilon$ finite-rank such that $||A - A_\varepsilon|| < \varepsilon$.
Let $\varepsilon > 0$ be given. Since $A(B_{\mathcal{H}})$ is precompact, it is totally bounded. That is $\exists y_1, \ldots, y_n \in A(B_{\mathcal{H}})$ such that $\forall x \in A(B_{\mathcal{H}})$, $\min_{1 \leq j \leq n} ||x - y_j|| < \varepsilon$.
Let P_ε be the orthogonal projection onto $\text{span}\{y_1, \ldots, y_n\}$, and set $A_\varepsilon = P_\varepsilon A$. Then for $f \in B_{\mathcal{H}}$ and $x = Af \in A(B_{\mathcal{H}})$,

$$||Af - A_\varepsilon f|| = ||x - P_\varepsilon x|| \leq \min_{1 \leq j \leq n} ||x - y_j|| < \varepsilon$$

So $||A - A_\varepsilon|| < \varepsilon$.
Exercise: confirm whether this argument needs separability.

Theorem: Analytic Fredholm Theorem

Let $D \subset \mathbb{C}$ be open and connected. Let $f : D \to \mathcal{B}(\mathcal{H})$ be an analytic, operator-valued function such that $f(z) \in \mathcal{K}(\mathcal{H})$ for all $z \in D$. Then

1. either $(I - f(z))^{-1}$ exists for no $z \in D$
2. or $(I - f(z))^{-1}$ exists for all $z \in D \setminus S$, where S is a discrete set (finite-rank, no accumulation points) in D.

Then $(I - f(z))^{-1}$ is meromorphic in D, analytic in $D \setminus S$, residues at poles are finite-rank, and if $z \in S$, $f(z)\psi = \psi$ has a nonzero solution for $\psi \in \mathcal{H}$.

Application

If K is compact, consider $f(z) = zK$. At $z = 0$, $I - f(z) = I$ so this is invertible which implies that the theorem holds for $\frac{1}{z} \in \mathbb{C}$ (taking $D = \mathbb{C} \setminus \{0\}$). We have $R_K(\lambda) = -\frac{1}{\lambda}\left(I - f\left(\frac{1}{\lambda}\right)\right)^{-1}$. Note that K is not necessarily self-adjoint. Note that K

is not necessarily self-adjoint.

Proof

We want to prove that either (a) or (b) hold locally for any $z_0 \in D$.
Fix $z_0 \in D$, $r > 0$ such that $||f(z) - f(z_0)|| < \frac{1}{2}$ for $z \in D_r(z_0)$. Choose $F = \sum_{j=1}^{n} \phi_j \otimes \overline{\psi}_j$ a finite-rank operator such that $||f(z_0) - F|| < \frac{1}{2}$. Then $\forall z \in D_r(z_0)$,

$$||f(z) - F|| \leq ||f(z) - f(z_0)|| + ||f(z_0 - F|| < 1$$

which implies that $(I - (f(z) - F))^{-1}$ exists and is holomorphic on $D_r(z_0)$.

Define $g(z) = F(I - f(z) + F)^{-1}$, and observe that $(I - f(z)) = (I - g(z))(I - f(z) + F) = I - f(z) + F - \overbrace{g(z)(I - f(z) + F)}^{F}$.
Write

$$g(z) = \sum_{j=1}^{n} \phi_j \otimes \overline{\psi}_j \cdot (I - f(Z) + F)^{-1} = \sum_{j=1}^{n} \phi_j \otimes \overline{\psi_j(z)}$$

where $\psi_j(z) = ((I - f(z) + F)^{-1})^* \psi_j$ is holomorphic in z. Then $I - f(z)$ is invertible if and only if $I - g(z)$ is invertible.
We claim that this holds if and only if $d(z) \neq 0$ for some holomorphic function d.
When is $I - g(z)$ invertible?
Injectivity: if $g(z)\phi = \phi$, we expect $\phi = \sum_{j=1}^{n} \beta_j \phi_j$. So $g(z)\phi = \phi$ if and only if

$$\sum_{j=1}^{n} \phi_j \left(\sum_{k=1}^{n} \beta_k \phi_k, \psi_j \right) = \sum_{j=1}^{n} \beta_j \phi_j$$

where $\beta_j = \sum_{k=1}^{n} \beta_k (\phi_k, \psi_j(z))$. If $A_{jk}(z) := (\phi_k, \psi_j(z))$, then this has a solution if and only if $\det(I - A(z)) = 0$. Call $d := \det(I - A(z))$. Moreover, if $d(z) \neq 0$ then $I - g(z)$ is invertible. We can solve $(I - g(z))\phi = \psi$ for ϕ given $\psi \in \mathcal{H}$.
So $\phi = \psi + g(z)\phi$ which motivates an ansatz $\phi = \psi + \sum_{j=1}^{n} \beta_j \phi_j$. Then

$$(I - g(z))\phi = (I - g(z))\psi + \sum_{j=1}^{n} (\beta_j - A_{jk}\beta_k)\phi_j = \psi)$$

If and only if

$$\sum_{j=1}^{n} (\beta_j - A_{jk}\beta_k)\phi_j = \sum_{j=1}^{n} (\psi, \psi_j(z))\phi_j$$

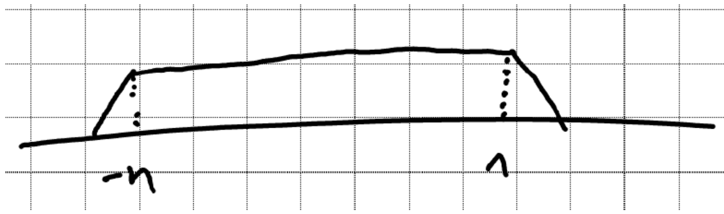
which is boundedly invertible as long as $d(z) \neq 0$.

October 21, 2025

Definition: Unbounded Operator

Let $(\mathcal{H}, (\cdot, \cdot))$ a Hilbert space. An unbounded operator is a linear map $A : \mathcal{D}(A) \to \mathcal{H}$, $\mathcal{D}(A) \subsetneq \mathcal{H}$ ($\mathcal{D}(A)$ the domain of A).
Hypothesis: $\overline{\mathcal{D}(A)} = \mathcal{H}$.
Why “unbounded?” BEcause, generally, $\sup_{f \in \mathcal{D}(A) \setminus \{0\}} \frac{||Af||}{||f||} = \infty$.
Examples

1. $\mathcal{H} = L^2(\mathbb{R})$, $Af(t) = tf(t)$. Then $A\left(\frac{1}{|t|^2}\right) \notin L^2(\mathbb{R})$, and $\mathcal{D}(A) = C_C(\mathbb{R})$. Consider $f_n(t)$ like



1. $\mathcal{H} = L^2(\mathbb{R})$, $Af(x) = D_x f(x) = \frac{1}{i}\frac{\partial}{\partial_x} f(x)$. Then $\mathcal{D}(A) = \mathcal{S}(\mathbb{R})$, and we can see unboundedness from $f(x) = e^{-\frac{x^2}{\sigma}}$ for $\sigma > 0$.

$$\int f^2(x)\,dx = \int e^{-\frac{x^2}{\sigma}}\,dx \overset{y=\frac{x}{\sqrt{(\sigma)}}}{=} \sqrt{\sigma}\int e^{-2y^2}\,dy$$
$$\int (f')^2(x)\,dx = \int \frac{4x^2}{\sigma^2} e^{-\frac{x^2}{\sigma}}\,dx \overset{y=\frac{x}{\sqrt{(\sigma)}}}{=} \frac{1}{\sqrt{\sigma}}\int 4y^2 e^{-2y^2}\,dy$$

So $\frac{||Af||^2}{||f||^2} = \frac{C}{\sigma} \xrightarrow[\sigma\to 0]{} \infty$.

Definition: Extension of an Unbounded Operator

We say that B extends A (and write $A \subseteq B$) if $\mathcal{D}(A) \subset \mathcal{D}(B)$ and $B|_{\mathcal{D}(A)} = A$. We have that $A = B$ when $A \subset B$ and $B \subseteq A$.

Definition: Closed Operator

To recover a notion of continuity, we look at closed operators.
We say that $(A, \mathcal{D}(A))$ is closed if and only if $\forall u_n \in \mathcal{D}(A)$ such that $u_n \to u \in \mathcal{H}$ and $Au_n \to v \in \mathcal{H}$, $u \in \mathcal{D}(A)$ and $v = Au$.
Equivalently, A is closed if the graph of A, $\Gamma(A) = \{(u, Au) \,:\, u \in \mathcal{D}(A)\}$, is closed in $\mathcal{H} \times \mathcal{H}$.
We say that A is closable if there exists some B such that $A \subseteq B$.
Example

- $(P, \mathcal{D}(P), \mathcal{H}) = (D_x, C_C^\infty(\mathbb{R}), L^2(\mathbb{R})$.
  - P is not closed, because there exists $u_n \in C_C^\infty(\mathbb{R})$ such that $u_n \to u \in \mathcal{H}$, $D_x u_n \to v \in \mathcal{H}$, yet $u \notin C_C^\infty(\mathbb{R})$.
  - Take $u \in H^1 \setminus C_C^\infty(\mathbb{R})$, $u_n \in C_C^\infty(\mathbb{R})$ converging to u in H^1.
- Recall that
  - $H^1(\mathbb{R}) = \{f \in L^2(\mathbb{R}) \,:\, f'(x) \in L^2(\mathbb{R})\}$, $||f||_{H^1}^2 = ||f||_{L^2(\mathbb{R})}^2 + ||f'||_{L^2(\mathbb{R})}^2$.
  - $C_C^\infty(\mathbb{R})$ is dense in $H^1(\mathbb{R})$.
  - $H^1(\mathbb{R})$ is complete.

Exercise: prove that if $\mathcal{D}(B) := H^1(\mathbb{R})$ and $Bf = D_x f$, then B is closed and $B|_{C_C^\infty(\mathbb{R})} = P$.

Adjoints of Unbounded Operators

In the bounded case, we have $A \in \mathcal{B}(\mathcal{H})$, $u \in \mathcal{H}$ and define $\ell_u(v) = (u, Av) \in \mathcal{H}'$ ($\mathcal{H}'$ the dual space). By Riesz Representation Theorem, $\exists! w := A^* u$ such that $\forall v \in \mathcal{H}$, $(A^* u, v) = (u, A^* v)$.
In the unbounded case, we need to define A^* and $\mathcal{D}(A^*)$.
For $u \in \mathcal{H}$, set $\ell_u = (u, Av)$, $v \in \mathcal{D}(A)$. If ℓ_u extends to an element of $\mathcal{H}'$ (i.e. if we can prove an estimate $|\ell_u(v)| \le c||v||_{\mathcal{H}}$), then by Riesz Representation Theorem $\exists! w \in \mathcal{H}$, called $A^* u$, such that $\forall v \in \mathcal{D}(A)$, $(A^* u, v) = (u, A^* v)$. Then

$$\mathcal{D}(A^*) = \{u \in \mathcal{H} : \begin{smallmatrix} \mathcal{D}(A) \to \mathbb{C} \\ v \mapsto (u, Av) \end{smallmatrix} \text{ extends to an element of } H'\}.$$

Example

- $(P, \mathcal{D}(P), \mathcal{H}) = (D_x, C_C^\infty(\mathbb{R}), L^2(\mathbb{R})$.
  - Observe that $\forall f, g \in C_C^\infty(\mathbb{R})$, $(D_x f, g) = (f, D_x g)$.
  - Take $f \in \mathcal{H}$ and $g \in C_C^\infty(\mathbb{R})$. Then if $f \in C_C^\infty$ or $f \in H^1(\mathbb{R})$, $g \mapsto (f, D_x g) = (D_x f, g)$ and

    $$|(D_x f, g)| \le ||D_x f||_{L^2(\mathbb{R})} ||g||_{L^2(\mathbb{R})} \le ||f||_{H^1(\mathbb{R})} ||g||_{L^2(\mathbb{R})}$$
  - Then $B(f, g) = D_x f, g) - (f, D_x g)$.
  - So $\mathcal{D}(P^*) \supseteq C_C^\infty(\mathbb{R})$ and, in fact, $\mathcal{D}(P^*) = H^1(\mathbb{R})$.
  - P^* is closed (this is always true of any adjoint).
  - $P \subset P^*$, we say that P is symmetric.

Generally, if $(A, \mathcal{D}(A))$ is an unbounded operator, then $\Gamma(A^*) = J((\Gamma(A))^\perp)$ where $J : \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}$ by $J(u, v) = (-v, u)$. Recall that $(\Gamma(A))^\perp$ is always closed, so A^* is always clsoed.

Definitions:

We say that P is symmetric if $P \subseteq P^*$.
We say that P is self-adjoint if $P = P^*$.
We say that P is essentially self-adjoint if $P^* = \overline{P}$.

Important Questions

1. Given a symmetric operator ($P \subseteq P^*$), to find a self-adjoint opeartor A such that $P \subseteq A = A^* \subseteq P^*$. We call A self-adjoint extension of P. If $P^* = \overline{P}$ (essentially self-adjoint), then it must be the unique extension.
2. How to find or parameterize all self-adjoint extensions? (This problem is hard.)

Example

- $P = (D_x, H_0^1([0, 1]), L^2(0, 1))$ (where elements of $H_0^1([0, 1])$ vanish on the boundary).

– $P^* = (D_x, H^1([0,1])) \underset{\neq}{\supset} P$.

– There exists a "circle" of self-adjoint extensions, parameterized by any point on the unit circle.

General Principle

If $\Omega \subset \mathbb{R}^n$ is open, bounded and withs mooth boundary $\partial\Omega$ ($\mathcal{H} = L^2(\Omega)$), and P is some differential operator (e.g. $P = \Delta$) such that $\forall f, g \in C_C^\infty(\Omega)$. $(Pf, g) = (f, Pg)$. By Green's Identity,,

$$\begin{aligned}\int_\Omega (\Delta f g - f\Delta g)\, dx &= \int_\Omega \nabla((\nabla f)g(-f(\nabla g))\, dx \\ &= \int_{\partial\Omega} (g\nabla f - f\nabla g)\cdot \nu\, ds \\ &= \int_{\partial\Omega} \left(g\frac{\partial f}{\partial \nu} - f\frac{\partial g}{\partial \nu}\right) ds\end{aligned}$$

Examples

- $P = (P = \Delta, C_C^\infty(\Omega), L^2(\Omega))$.
  - P symmetric.
  - $\mathcal{D}(P^*) = \{u \in L^2(\Omega : \Delta u \in L^2\}$.
  - Let $P_{\min} = \overline{(P, C_C^\infty(\Omega)}$ and define $P_{\max} = P^*_{\min}$. Then $\mathcal{D}(P_{\max}) = \{u \in L^2(\Omega) : Pu \in L^2(\Omega)\}$.
  - The self-adjoint extensions lie inbetween.
- $(D_x, H_0^1([0,1]) = \{u \in H^1([0,1]) : u(0) = u(1) = 0\}, L^2([0,1]))$.
  - Observe that the evaluation map $C^\infty([0,1]) \ni f \mapsto f(0)$ extends boundedly to $\tau : H^1([0,1]) \to \mathbb{C}$.
  - For $f, g \in C^\infty([0,1])$,

$$(D_x f, g) - (f, D_x g) = \int_0^1 \frac{1}{i} f'\overline{g} - f\left(\overline{\frac{1}{i}g'}\right) = \frac{1}{i}\int_0^1 (f\overline{g})' = \frac{1}{i}(f(1)\overline{g}(1) - f(0)\overline{g}(0))$$

- Then if $g \in \mathcal{D}(P_{\min})$,

$$\begin{aligned}|(f, D_x g)| &= |(D_x f, g) + \frac{1}{i}(\overbrace{f(1)\overline{g}(1)}^{=0} - \overbrace{f(0)\overline{g}(0)}^{=0})| \\ &\leq ||f||_{H^1}||g||_{L^2}\end{aligned}$$

- Therefore $\mathcal{D}(P^*_{\min}) = \mathcal{D}(P_{\max}) = H^1([0,1])$, and we note that $P^*_{\max} = P_{\min}$.

Now suppose that T is a self-adjoint extension of P ($P_{\min} \subseteq T \subseteq P_{\max}$. If T is self-adjoint, then $\forall f, g \in \mathcal{D}(T) \subseteq H^1([0,1])$, $f(0)\overline{g}(0) - f(1)\overline{g}(1) = 0$.
For $f = g$, $|f(0)|^2 = |f(1)|^2$ for some fixed f, then $\exists \alpha \in S^1$ such that $f(1) = \alpha f(0)$. Then for any other g,

$$0 = f(0)(\overline{g}(0) - \alpha\overline{g}(1) = \alpha f(0)(\overline{\alpha g(0) - g(1)}$$

Therefore $g(1) = \alpha g(0)$ for the same fixed α.

October 23, 2025

Definition: Resolvent Set and Spectrum of and Unbounded Operator

For $(A, \mathcal{D}(A), \mathcal{H})$ unbounded and closed, the resolvent set of A, $\rho(A)$, is the set of $\lambda \in \mathbb{C}$ such that $(A - \lambda) : \mathcal{D}(A) \to \mathcal{H}$ is bijective with bounded inverse $R_A(\lambda) := (A - \lambda)^{-1}$.
The spectrum of A, $\sigma(A)$, is $\sigma(A) = \mathbb{C} \setminus \rho(A)$. If $\rho(A)$ is open, then $\sigma(A)$ is closed.
Example

- Take $P = (D_x, H_0^1([0,1]), L^2(0,1))$ and define two closed extensions P_0 and P_1
  - $\mathcal{D}(P_0) = \{u \in H^1([0,1]) \,:\, u(0) = 0\}$
    * For $\lambda \in \mathbb{C}$, $f \in L^1([0,1])$, we solve $\begin{cases} D_x u + \lambda u = f \\ u(0) = 0 \end{cases}$
    * So $u(x) = ie^{-\lambda x} \int_0^x e^{i\lambda t} f(t)\, dt$ and $\sigma(P_0) = \varnothing$.
  - $\mathcal{D}(P_1) = H^1([0,1])$
    * For $\lambda \in \mathbb{C}$, $\ker(D_x - \lambda) \cap L^2([0,1]) \neq \varnothing$. So $\sigma(P_1) = \mathbb{C}$.

Proposition

If $(A, \mathcal{D}(A))$ is self-adjoint, then $\sigma(A) \subseteq \mathbb{R}$, $\sigma(A) \neq \varnothing$, and we have a resolvent estimate $||R_A(z)|| \leq \frac{1}{|\mathrm{Im}(z)|}$ for $\mathrm{Im}(z) \neq 0$.
The proof of real value and the estimate are the same as in the bounded case.
To show that $\sigma(A) \neq \varnothing$, by contradiction if $\sigma(A) = \varnothing$ then $A^{-1} = R_A(0)$ exists and is bounded. We claim that this requires $\sigma(A^{-1}) = \{0\}$ and pick $\lambda \neq 0$, such that

$$(A^{-1} - \lambda)u = f$$
$$\left(\frac{1}{\lambda} - A\right)u = \frac{1}{\lambda} A f$$

But $\frac{1}{\lambda} \notin \sigma(A)$ implies $u = -R_A\left(\frac{1}{\lambda}\right)\frac{1}{\lambda} A f$. Therefore $R_{A^{-1}}(\lambda) = -\frac{1}{\lambda} A R_A\left(\frac{1}{\lambda}\right)$ and $\lambda \notin \sigma(A^{-1})$. Therefore $A^{-1} = 0$ which contradicts the assumption that $A^{-1}A = \mathrm{Id}_{\mathcal{D}(A)}$.

Generalizing Spectral Resolution

We can extend $\{E_\lambda\}_{a \leq \lambda \leq b}$ to $a = -\infty$ and $b = \infty$ by setting strong limits $E_{-\infty} = 0$ and $E_\infty = I$.
Example

- On $L^2(\mathbb{R})$, $E_\lambda f(t) = \chi_{(-\infty,\lambda)}(t) f(t)$.

Theorem: Spectral Theorem for Unbounded Self-Adjoint Operators

Theorem 8.15 Helffer.
Any self-adjoint operator A on a Hilbert space $\mathcal{H}$ admits a spectral decomposition $\{E_\lambda\}_\lambda$ such that $\forall x, y \in \mathcal{H}$,

$$(Ax, y) = \int_{\mathbb{R}} \lambda \, d(E_\lambda x, y)$$
$$Ax = \int_{\mathbb{R}} \lambda \, d(E_\lambda x)$$

Proof

Largely, we will use the proof for the bounded case.
Suppose A is semibounded from below (i.e. $\exists \mu \in \mathbb{R}$, $\forall x \in \mathcal{D}(A)$, $(Ax, x) \geq \mu||x||^2$) ten $R_A(\lambda)$ exists for $\lambda_0 < \mu$ $(((A-\lambda)x, x) \geq (\mu - \lambda_0)||x||^2$ coercive in the sense that $(\mu - \lambda_0) > 0)$. Then it is also a self-adjoint bounded operator which implies that $R_A(\lambda_0)$ has a spectral representation. Then $A = f(R_A(\lambda_0))$, $f(x) = \lambda_0 + \frac{1}{x}$. We can work with a general case $(A-i)^{-1}$.
Example

- Take $(D_x, H^1(\mathbb{R}), L^2(\mathbb{R}))$, and define the Fourier transform $f(x) \mapsto \mathcal{F}f = \hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx$. Then $\widehat{D_x f}(\xi) = \xi \hat{f}(\xi)$, and we have the following commutative diagram

$$\begin{array}{ccc} H^1(\mathbb{R}) & \xrightarrow{D_x} & L^2(\mathbb{R}) \\ \big\downarrow \mathcal{F} & & \big\downarrow \mathcal{F} \\ H^1(\mathbb{R}) & \xrightarrow[f(t)\mapsto tf(t)]{} & L^2(\mathbb{R}) \end{array}$$

- So $E_\lambda f = \mathcal{F}^{-1}\chi_{(-\infty,\lambda)}\mathcal{F}f$.

- $(D_x^2, H^2(\mathbb{R}), L^2(\mathbb{R}))$. Then $\widehat{D_x^2 f}(\xi) = \xi^2 \hat{f}(\xi)$, $\sigma(D_x^2) = [0, \infty)$ and $D_x^2 f = \mathcal{F}^{-1}\xi^2\mathcal{F}f$.
  - So $E_\lambda f = 0$ for $\lambda < 0$; $E_\lambda f = \mathcal{F}^{-1}\chi_{\xi^2<\lambda}(\xi)\mathcal{F}f$.

- $(D_x^2 + V(x), C_C^\infty(\mathbb{R}), L^2(\mathbb{R}))$, $V \in C(\mathbb{R};\mathbb{R})$ bounded from below. Then $D_x^2 + V$ is semibounded from below and the spectrum depends drastically on V.
  - $V = 0$ gives $\sigma = [0, \infty)$.
  - $V = x^2$ gives $\sigma = \{2n+1 \,:\, n \geq 0\}$
  - $V \in C_C(\mathbb{R})$ gives $\sigma = [0, \infty)$

Proposition

If for some $z_0 \in \rho(A) \cap \mathbb{R}$ $R_A(z_0)$ is compact, then A has purely discrete spectrum $(z_n)_{n\geq 1}$, $|z_n| \to \infty$ as $n \to \infty$.

Proof

$R_A(z_0)$ is compact and self-adjoint, so there exists a basis for $\mathcal{H}$ made of normalized eigenvectors $(\phi_n)_{n\in\mathbb{N}}$ with eigenvalues $|\lambda_n| \searrow 0$.

$$(A - z_0)^{-1}\phi_n = \lambda_n\phi_n$$
$$A\phi_n = \left(z_0 + \frac{1}{\lambda_n}\right)\phi_n$$
$$\mathcal{H} = \left\{u = \sum u_n\phi_n \,:\, \sum |u_n|^2 < \infty\right\}$$
$$\mathcal{D}(A) = \left\{u = \sum u_n\phi_n \,:\, \sum |u_n|^2\left(z_0 + \frac{1}{\lambda_n}\right)^2 < \infty\right\}$$

Applications

- Laplacian (Leplace-Beltrami Operator) on closed Riemann manifolds or Dirichlet Laplace-Beltrami if $\partial M \neq 0$.
- Dirichlet Laplacian ($\Delta = -\sum_{k=1}^n \partial_{x^k}^2$) on bounded domains in $\mathbb{R}^n$ (Ω open in $\mathbb{R}^n$, regular, $\partial\Omega$ smooth).
  - $\Delta_D = (\Delta, H_0^1(\Omega)\cap H^2(\Omega), L^2(\Omega))$ where $H_0^1 = \{u \in H^1(\Omega) \,:\, u|_{\partial\Omega} = 0\}$ ($u|_{\partial\Omega}$ makes sense due to a trace theorem from Evans 5.5) and $H^2(\Omega) = \{u \in L^2(\Omega) \,:\, \partial_i\partial_j u \in L^2,\ \partial_i u \in L^2,\ \forall i,j\}$.
  - $\int_\Omega |\nabla f|^2\, dx = (\Delta f, f) \geq 0$ (Green's Theorem)
  - Claim: $\Delta_D + 1$ has compact inverse.
- Solvability of $\Delta_D + 1$

Our goal is, given $f \in L^2$, to find $u \in ?$ such that $\begin{cases}\Delta u + u = f \\ u|_{\partial\Omega} = 0\end{cases}$. We have a weak formulation, if $v \in H_0^1$, we multiply by v and get an IBP

$$\underbrace{\int \nabla u\cdot\nabla u + \int uv}_{B(u,v)} = \int fv$$

where $\left|\int fv\right| \leq ||f||_{L^2}||v||_{L^2} \leq ||f||_{L^2}(B(v,v))^{1/2}$. By Riesz Representation Theorem, there exists some unique $u \in H_0^1$ such that $\forall v \in H_0^1$, $B(v,v) = \int fv$. Moreover, we can recover H^2 regularity on u. Also, $|B(u,u)| = \left|\int fu\right| \leq ||f||_{L^2}(B(u,u))^{1/2}$ if and only if $||(\Delta_D+1)^{-1}f||_{H_0^1(\Omega)} = ||u||_{H_0^1(\Omega)} \leq ||f||_{L^2}$. Therefore $(\Delta_D+1)^{-1} : L^2(\Omega) \to H_0^1(\Omega)$ is bounded. By Rellich Compactness, $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ compactly. Hence $(\Delta_D+1)^{-1} : L^2 \to L^2$ is compact.

Theorem

On $\mathbb{R}$, $H^1(\mathbb{R})$ does not embed compactly into $L^2(\mathbb{R})$. For example, $f \in H^1$ with $\operatorname{supp} f \subset [0,1]$ defined by $f_n(x) = f(x-n)$.
So if $H \subset L^2(\mathbb{R})$ where $||u||_H^2 = ||u'||_L^2 + ||a(x)^{-1}u||_{L^2}^2$ for $a(x) > 0$ and $\lim_{|x|\to\infty} a(x) = 0$, then H embeds compactly in L^2.
It follows that $D_x^2 + x^2$has compact resolvent and discrete spectrum. $||u||_H^2 = ||d_x u||_{L^2}^2 + ||xu||_{L^2}^2$.

October 30, 2025

Friedrichs Extension

Let $(A, \mathcal{D}(A), \mathcal{H})$ be symmetric, bounded from below in the sense that there exists $\alpha \in \mathbb{R}$ such that $\forall x \in \mathcal{D}(A)$ we have $(Ax, x)_{\mathcal{H}} \geq \alpha ||x||_{\mathcal{H}}^2$.

Theorem:

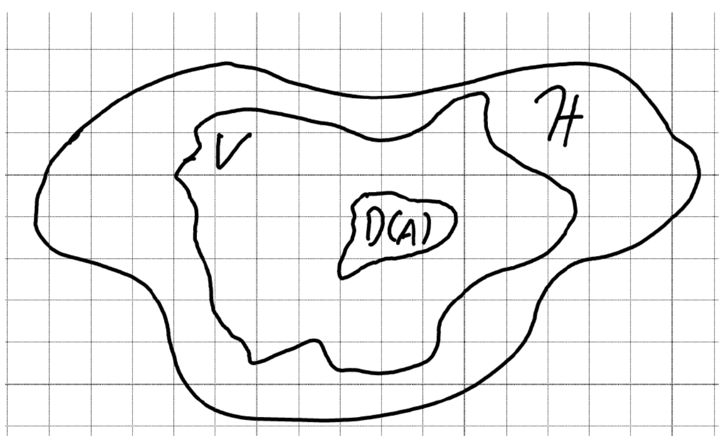
If $(A, \mathcal{D}(A), \mathcal{H})$ is densely defined, symmetric, and bounded below, then A admits a self-adjoint extension $(S, \mathcal{D}(S))$. The procedure for producing this extension is the Friedrichs Extensions.

Proof

Assume that $\alpha = 1$, otherwise consider $A - \lambda I$ for some well-chosen λ. Define the symmetric quadratic form

$$q_A(u, v) = (Au, v), \quad (u, v) \in \mathcal{D}(A) \times \mathcal{D}(A).$$

Note that $q_A(u, u) = (Au, u) \geq ||u||_{\mathcal{H}}^2$, so this form is coercive. Next, we construct $V \subset \mathcal{H}$ by taking $p(u) = (q_a(u, u))^{1/2}$ and letting $u \in \mathcal{D}(A)$ belong to V if $\exists u_n \in \mathcal{H}$ such that $||u_n - u||_{\mathcal{H}} \underset{n\to\infty}{\longrightarrow} 0$ and $\{u_n\}$ is p-cauchy (i.e. $\forall \varepsilon > 0$, $\exists N$, $m, n \geq N \implies p(u_n - u_m) < \varepsilon$).
Then we can define a norm on V: $||u||_V := \lim_{n\to\infty} p(u_n)$ where $u_n \xrightarrow{\mathcal{H}} u$ and u_n is p-Cauchy.
Exercise: show that this definition does not depend on $\{u_n\}$.
We also define $(u, v)_V := \lim_{n\to\infty} q_A(u_n, v_n)$ for u_n and v_n similarly converging and p-cauchy.
So $(V, ||\cdot||_V)$ is complete and $p(u_n) \geq ||u_n||_{\mathcal{H}}$ becomes $||u||_V \geq ||u||_{\mathcal{H}}$ as $n \to \infty$. Hence $V \hookrightarrow \mathcal{H}$ is continuous.



We now construct S starting with a domain $D(S) = \{u \in V : V \ni v \mapsto q_A(u, v)$ satisfies an continuity estimate of the form $\leq C||v||_{\mathcal{H}}\}$. For $u \in \mathcal{D}(S)$, the map $v \mapsto q_A(u, v)$ extends to an element of $\mathcal{H}'$ (bilinear functionals on $\mathcal{H}$). By Riesz-Representation Theorem, there exists some $w := Su$ such that $\forall v \in V$, $q_A(u, v) = (Su, v)_{\mathcal{H}}$.
It remains to show that S is self-adjoint, $\mathcal{D}(S) \supset \mathcal{D}(A)$, $S|_{\mathcal{D}(A)} = A$. Self-adjointness is left as an exercise. If $u \in \mathcal{D}(A)$, then

$$|q_A(u, v)| = |(Au, v)| \leq ||Au||_{\mathcal{H}} ||v||_{\mathcal{H}}$$

so $u \in \mathcal{D}(S)$ and $Su = Au$.

Remark

We could start with a quadratic form $q(u, v)$ which is sesquilinear, coercive and $\mathcal{D}(q) \subset \mathcal{H}$ densely. This construction produces a self-adjoint operator as well.

Example (Dirichlet Laplacian)

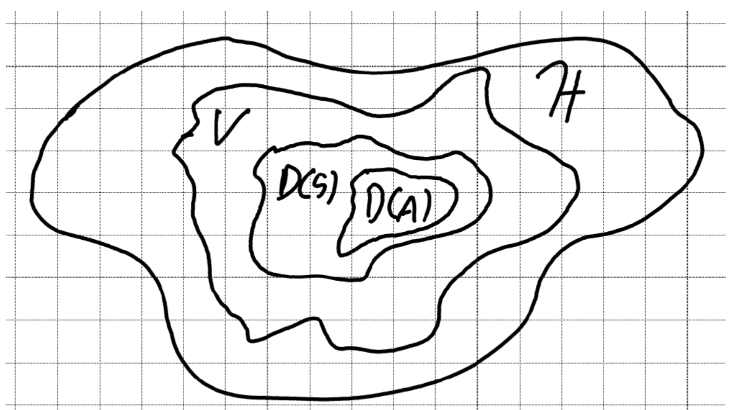
$\Delta = \sum_{j=1}^{n} \partial_{x^j}^2$ with $\Omega = \mathbb{R}^n$ open, bounded, and regular.
$(A = -\Delta, \mathcal{D}(A) = C_C^\infty(\Omega), \mathcal{H} = L^2(\Omega))$. For $u \in \mathcal{D}(A)$,

$$\begin{aligned}(Au, u) &= \int_\Omega -\Delta u \cdot \overline{u}\, dx \\ &= \int_\Omega -\nabla(\nabla u \cdot \overline{u}) + \nabla u \cdot \nabla \overline{u}\, dx \\ &= \overbrace{\int_{\partial\Omega} -\partial_\nu u\, ds}^{=0} + \int_\Omega \nabla u \nabla \overline{u}\, dx \geq 0\end{aligned}$$

So Friedrich's procedure produces a self-adjoint extension

$$q_A(u,v) = \int_\Omega \nabla u \nabla \overline{u}\, dx, \quad u, v \in C_C^\infty(\Omega)$$

which leads to $V = \overline{C_C^\infty(\Omega)}^{H^1(\Omega)} = H_0^1(\Omega)$ (that is the H^1 closure of $C_C^\infty(\Omega)$). What about $\mathcal{D}(S)$?



For $v \in C_C^\infty(\Omega)$, $\int_\Omega \nabla u \cdot \nabla \overline{v}\, dx = \int_\Omega -\Delta u \cdot v \leq C||v||_{L^2}$. Then we have

$$\mathcal{D}(S) = \{u \in H_0^1(\Omega) : \Delta u \in L^2(\Omega)\} = H_0^1(\Omega) \cap H^2(\Omega)$$

Example (Neumann Laplacian)

Start from a quadratic form $q(u,v) = \int_\Omega \nabla u \cdot \nabla \overline{v} + u\overline{v}\, dx$ with $u, v \in C^\infty(\overline{\Omega})$. Then $V = H^1(\Omega)$. We compute $\mathcal{D}(S)$ by first recalling that for $u, v \in C^\infty(\overline{\Omega})$,

$$\int_\Omega -\Delta u \overline{v}\, dx = \int_\Omega \nabla u \cdot \nabla \overline{v}\, dx - \int_{\partial\Omega} \partial_\nu u \overline{v}\, dx$$

We sense that $\Delta u \in L^2(\Omega)$, so we at least need $u \in W(\Omega) = \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\}$. How do we make sense of the Neumann trace on $W(\Omega)$?

- Lemma

- The restriction map $C(\overline{\Omega}) \to C(\partial\Omega)$ extends to a bounded surjective "Dirichlet trace" $\tau_D : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ with bounded right inverse R.

- Example
  - if $\Omega = \mathbb{D}$, $\partial\Omega = S^1$, $f \in C^\infty(S^1) \leftrightarrow f(\theta)\sum_{k\in\mathbb{Z}} f_k e^{ik\theta}$ where $f_k = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-ik\theta}\, d\theta$, we can for each $s \in \mathbb{R}$ define

$$H^S(S^1) = \{f \in \mathfrak{D}'(S^1) \,:\, \sum_{k\in\mathbb{Z}}(1+k^2)^s|f_k|^2 < \infty\} = \mathcal{D}((-\partial_\theta^2)^{s/2})$$

- Fact
  - For $s \geq 0$, $H^{-S}(\partial\Omega) \cong (H^S(\partial\Omega))'$.
  - If $f \in H^{-s}$, $g \in C^\infty(S^1)$,

$$\langle f, g\rangle_{\mathfrak{D}',\mathfrak{D}} = \int(f,g) = c\sum_{k\in\mathbb{Z}} f_k\overline{g_k}\frac{(1+k^2)^{s/2}}{(1+k^2)^{s/2}} \leq c\left(\sum_{k\in\mathbb{Z}}(1+k^2)^{-s}|f_k|^2\right)\left(\sum_{k\in\mathbb{Z}}(1+k^2)^s|g_k|^2\right)$$

Returning to the Neumann Laplacian, the Neumann trace $C^1(\overline{\Omega}) \to C(\partial\Omega)$ by $f \mapsto \partial_\nu f$ extends to a bounded operator $\tau_N : W(\Omega) \to H^{-1/2}(\partial\Omega)$.

- Proof
  If $u \in W(\Omega)$, set

$$\begin{aligned}\Phi_u(v) &:= \int_\Omega -\Delta u\overline{v}\, dx - \int_\Omega \nabla u \cdot \nabla\overline{v}\, dx, \quad v \in H^1(\Omega)\\ &\leq ||\Delta u||_{L^2}||v||_{L^2} + ||\nabla u||_{L^2}||\nabla v||_{L^2}\end{aligned}$$

  Using $R : H^{1/2}(\partial\Omega) \to H^1(\Omega)$, $w \in H^{1/2}(\partial\Omega)$,

$$|\Phi_u(Rw)| \leq (||\Delta u||_{L^2} + ||\nabla u||_{L^2})\,||Rw||_{H^1} \leq (||\Delta u||_{L^2} + ||\nabla u||_{L^2})\,||w||_{H^{1/2}(\Omega)}$$

  which implies that there exists some $h \in H^{-1/2}(\partial\Omega)$ such that $\Phi_u(Rw) = \langle h, w\rangle_{H^{-1/2},H^{1/2}}$.
  We want to extend

$$\int_\Omega -\Delta u\overline{v}\, dx = \int_\Omega \nabla u \cdot \nabla\overline{v}\, dx - \int_{\partial\Omega}\partial_\nu u\overline{v}\, dx$$

  to $u \in W(\Omega$ and $v \in H^1(\Omega)$. First, if $v \in H_0^1(\Omega)$ then

$$\int_\Omega -\Delta u\overline{v}\, dx = \int_\Omega \nabla u \cdot \nabla\overline{v}\, dx$$

  (by passing to the limit with a sequence $v_n \in C_C^\infty(\Omega)$ converging to v in H^1).
  Next, if $v \in H^1$, we write $v = v_0 + R\tau_D v$ with $v_0 \in H_0^1$. Then

$$\int_\Omega -\Delta u\overline{v} - \nabla u \cdot \nabla v\, dx = \overbrace{\int_\Omega -\Delta u\overline{v_0} - \nabla u\cdot\nabla\overline{v_0}\, dx}^{=0} + \Phi_u(R\tau_D v) = \langle\tau_N u, \tau_D v\rangle_{H^{-1/2},H^{1/2}}$$

We conclude that for each $u \in W(\Omega)$, $v \in H^1(\Omega)$

$$\int_\Omega -\Delta u \cdot \overline{v}\, dx = \int_\Omega \nabla u \cdot \nabla \overline{v}\, dx + \langle \tau_N u, \tau_D v\rangle_{H^{-1/2},H^{1/2}}$$

Therefore we have

$$q(u,v) = \int_\Omega (-\Delta u + u)\overline{v}\, dx + \langle \tau_N u, \tau_D v\rangle_{H^{-1/2},H^{1/2}}$$

and $v \mapsto q(u,v)$ is L^2-bounded if and only if $v \mapsto \langle \tau_N u, \tau_D v\rangle_{H^{-1/2},H^{1/2}}$ is L^2-bounded which holds if and only if $\tau_N u = 0$. So

$$\mathcal{D}(S) = \{u \in H^1(\Omega) : -\Delta u \in L^2,\, \tau_N u = 0\} = \{u \in H^2(\Omega) : \tau_N u = 0\}$$

Observation

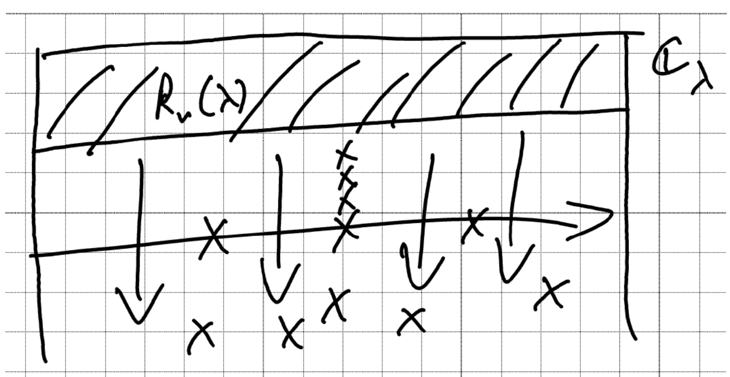
The form domains satisfy $\mathcal{D}(q_D^{-\Delta}) \subseteq \mathcal{D}(q_N^{-\Delta})$.

November 04 and 06, 2025

See class notes on variational principles for eigenvalues and Weyl's criterion for essential spectrum.

November 13, 2025

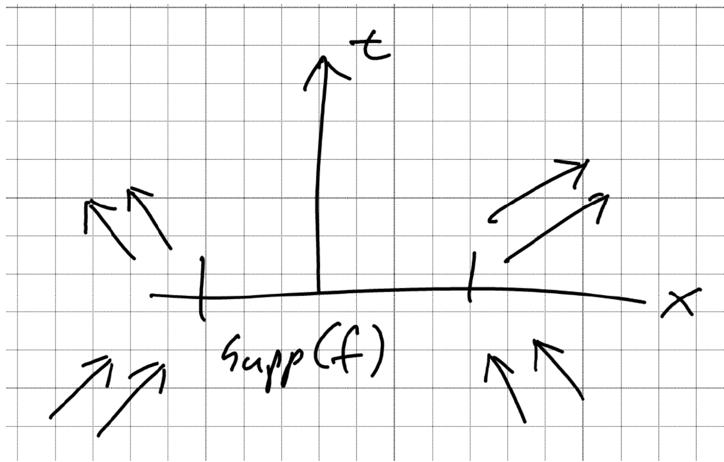
Consider $(D_x^2 - \lambda^2)^{-1} = R_0(\lambda)$, the "outgoing" resolvent and let V be in $C_C^\infty(\mathbb{R},\mathbb{R})$ or $L^\infty_{\text{comp}}(\mathbb{R},\mathbb{R})$. Then we may define $R_V(\lambda) = (D_x^2 + V - \lambda^2)^{-1}$. We want to show that $-R_V(\lambda)$ makes sense for $\operatorname{Im}(\lambda) \gg 0$ as an $L^2 \to L^2$ operator and extends meromorphically to either $\{\operatorname{Im}(\lambda) > 0\}$ as an $L^2 \to L^2$ bounded operator or to $\mathbb{C}$ as a $L^2_{\text{comp}} \to L^2_{\text{loc}}$ opeartor. Recall that $L^2_{\text{comp}} = \{f \in L^2 : \operatorname{supp} f \text{ compact}\}$ and $L^2_{\text{loc}} = \{f : \mathbb{R} \to \mathbb{R} : \forall K \text{ compact}, \int_K |f|^2 < \infty\}$.



We will further see that there are no poles (embedded eigenvalues) on $\mathbb{R} \setminus \{0\}$.

Outgoing/Incoming Solution

We have that $(D_x^2 - \lambda^2)u = 0$ if and only if $u(x) = Ae^{i\lambda x} + Be^{-i\lambda x}$. A solution to $(D_x^2 - \lambda^2)u = f \in C_C^\infty(\mathbb{R})$ is said to be outgoing if $x \gg 0$ implies that $u(x) = A_+ e^{i\lambda x}$ and $x \ll 0$ implies that $u(x) = B_+ e^{-i\lambda x}$.



Likewise, the solution is incomming if $x \gg 0$ implies $u(x) = B_- e^{-i\lambda x}$ and $x \ll 0$ implies $u(x) = A_- e^{i\lambda x}$.
We have seen previously that the D'Alembert Solution to

$$\begin{cases} (\partial_t^2 - \partial_x^2 + V)u = f \\ u|_{t=0} = \partial_t u|_{t=0} = 0 \end{cases}$$

is outgoing.
We will find that an outgoing solution to $(D_x^2 - \lambda^2)u = f$ is $R_0(\lambda)f := u_+ = \frac{i}{2\lambda}\int_{\mathbb{R}} f(y)e^{i\lambda|x-y|}\,dy$. The incoming solution, then, is $u_-(x,\lambda) = -\frac{i}{2\lambda}\int_{\mathbb{R}} f(y)e^{-i\lambda|x-y|}\,dy$.
Note that $u_+(x,\lambda) = u_-(x,-\lambda) = \overline{u_-(x,\lambda)}$. Note also that the outgoing solution is in $L^2(\mathbb{R})$ if and only if $\mathrm{Im}(\lambda) > 0$, and the incoming solution if and only if $\mathrm{Im}(\lambda) < 0$.
The above solution may be derived via Fourier transform. Namely

$$(\xi^2 - \lambda^2)\hat{u}(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\,dx$$

which implies that $\hat{u}(\xi) = \frac{\hat{f}(\xi)}{\xi^2-\lambda^2}$ if and only if $\mathrm{dist}(\lambda^2, [0,\infty)) > 0$. Considering the outgoing solution, this occurs only when $\mathrm{Im}(\lambda) > 0$.
Note, that with respect to Plancherel Theorem giving $||u||_{L^2} = c||\hat{u}(\xi)||_{L^2}$,

$$\frac{1}{c}||u||_{L^2} = ||\hat{u}(\xi)||_{L^2} \leq \frac{1}{\mathrm{dist}(\lambda^2, [0,\infty))}||\hat{f}(\xi)||_{L^2} = \frac{1}{c}||f||_{L^2}$$

This gives continuity estimates for $R_0(\lambda)$. Then $u(x) = h_\lambda(x) * f(x)$ where $h_\lambda(x) = \int_{\mathbb{R}} e^{ix\xi}\frac{d\xi}{\xi^2-\lambda^2}$. We claim, computing by residue theorem, that $h_\lambda(x) = \frac{i}{2\lambda}e^{i\lambda|x|}$.
Exercise: do the residue computation.
So we have $R_0(\lambda) = \frac{i}{2\lambda}\int_{\mathbb{R}} f(y)e^{i\lambda|x-y|}\,dy$ with Schwarz kernel $R_0(x,y,\lambda) = \frac{i}{2\lambda}e^{i\lambda|x-y|}$. When x and y are frozen, this looks mermorphic in λ witha simple pole at $\lambda = 0$ and residue $\frac{i}{2}$. This stil makes sense as a map $C_c^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ or $L^2_{\mathrm{comp}}(\mathbb{R}) \to L^2_{\mathrm{loc}}(\mathbb{R})$.
Alternativey, we can formulate this as $\forall \rho \in C_C^\infty(\mathbb{R})$, $\rho R_0 \rho : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ extends meromorphically to $\mathbb{C}$. Then $\rho R_0(\lambda)\rho$ has Schwarz kernel $\frac{i}{2\lambda}\rho(x)\rho(y)e^{i\lambda|x-y|}$.
Exercise: if $\lambda \neq 0$, this is $L^2 \to L^2$ bounded.
Then $R_0(\lambda) = \frac{P}{\lambda} + Q(\lambda)$ where $Pf(x) = \frac{i}{2}\int_{\mathbb{R}} f(y)\,dy$, $P = \phi \otimes \overline{\phi}$ and $\phi(x) = \frac{e^{i\pi/4}}{\sqrt{2}}$. Here $Q : L^2_{\mathrm{comp}}(\mathbb{R}) \to L^2_{\mathrm{loc}}(\mathbb{R})$ is entire.

Compactly Supported Potential

Take $V \in L^\infty_{\mathrm{comp}}(\mathbb{R})$, where $\sigma_{\mathrm{ess}}(D_x^2 + V) = [0,\infty)$+(possibly negative eigenvalues)+(embedded eigenvalues).

Theorem:

The outgoing resolvent $(D_x^2 + V - \lambda^2)^{-1}$ extends mermorphically to $\mathbb{C}$ with no poles at $\lambda \in \mathbb{R} \setminus \{0\}$.

Proof

Goal: when can we solve $(D_x^2 + V - \lambda^2)u = f$?
We start with the identity $(D_x^2 + V - \lambda^2)R_0(\lambda) = I + VR_0(\lambda)$. For $\mathrm{Im}(\lambda) > 0$, $||VR_0(\lambda)||_{L^2 \to L^2} \le \frac{||V||_\infty}{\mathrm{dist}(\lambda^2, [0,\infty))}$. So if $\mathrm{Im}(\lambda) \gg 1$, $||VR_0(\lambda)||_{L^2 \to L^2} < 1$. Then

$$(D_x^2 + V - \lambda^2)\overbrace{R_0(\lambda)\sum_{p=0}^{\infty}(-VR_0(\lambda))^p}^{R_V(\lambda)} = I.$$

So $R_V(\lambda)$ is an inverse for $D_x^2 + V - \lambda^2$ on $\mathrm{Im} \gg 1$, bounded with operator norm below $||R_0(\lambda)||\frac{1}{1-||VR_0(\lambda)||}$. It produced outgoing solutions due to its factored form.
Now, to extend $R_V(\lambda)$ mermorphically as $L^2_{\mathrm{comp}} \to L^2_{\mathrm{loc}}$, we look at $\rho R_V(\lambda)\rho$ for $\rho \in C_C^\infty(\mathbb{R})$ such that $\rho V = V$. Then for $\mathrm{Im}(\lambda) \gg 0$,

$$R_V(\lambda)\rho = R_0(\lambda)(I + VR_0(\lambda))^{-1}\rho = R_0(\lambda)\rho\sum_{p=0}^{\infty}(VR_0(\lambda)\rho)^p = R_0(\lambda)\rho(I + VR_0(\lambda)\rho)^{-1}$$

Therefore

$$\rho R_V(\lambda)\rho = \rho R_0(\lambda)\rho(I + \rho VR_0(\lambda)\rho)^{-1}$$

We claim that $\lambda \mapsto \rho VR_0(\lambda)\rho$ is a meromorphic on $\mathbb{C} \setminus \{0\}$ family of compact operators (at least on $V \in C_C^\infty(\mathbb{R})$).
To see this, we use $R_0(\lambda) : L^2(\mathbb{R}) \to H^2(\mathbb{R})$, then $\rho VR_0(\lambda)\rho : L^2(\mathbb{R}) \to \{u \in H^2(\mathbb{R}) : \mathrm{supp}\, u \subset \mathrm{supp}\, V\}$ embeds compactly into $L^2(\mathbb{R})$.
By analytic Fredholm theory, since $I + \rho VR_0(\lambda)\rho$ is invertible for $\mathrm{Im}(\lambda) \gg 1$, it is invertible on $\mathbb{C} \setminus \{0\}$ outside of a discrete set with finite-dimensional obstructions at that discrete set.
To upgrade this to a meromorphic extension of $R_V(\lambda) : L^2_{\mathrm{comp}} \to L^2_{\mathrm{loc}}$, pick an increasing sequence of cutoff functions. If ρ_1 is such that $\rho_1\rho = \rho$, then $\rho_1 R_V(\lambda)\rho_1$ extends $\rho R_V(\lambda)\rho$ in the sense that $R_V(\lambda)\rho_1 f = R_V(\lambda)\rho f$ when $\mathrm{supp}\, f \subset \{\rho = 1\}$.
Claim: if $\lambda \in \mathbb{R} \setminus \{0\}$, then λ is not a pole of $R_V(\lambda)$.

- Step 1: if λ is a pole, then there exists an outgoing solution to $(D_x^2 + V - \lambda^2)u = 0$.
- Step 2: if $\lambda \in \mathbb{R} \setminus \{0\}$, and u is an outgoing solution to $(D_x^2 + V - \lambda^2)u = 0$, then u has compact support.
- Step 3: if u is a compactly supported solution of $(D_x^2 + V - \lambda^2)u = 0$, then $u = 0$.

Step 1. If $\lambda = \lambda_0$ is a pole, we may write $R_V(\lambda) = \frac{P_N}{(\lambda-\lambda_0)^N} + \cdots + \frac{P_1}{\lambda - \lambda_0} + Q(\lambda)$ near $\lambda = \lambda_0$ where each term is $L^2_{\mathrm{comp}} \to L^2_{\mathrm{loc}}$. If $\lambda \neq \lambda_0$, then

$$(\lambda - \lambda_0)^N\overbrace{(D_x^2 + V - \lambda^2)R_V(\lambda)}^{I} = (D_x^2 + V - \lambda^2)P_N + (\lambda - \lambda_0)(\cdots)$$

Then as $\lambda \to \lambda_0$, $(D_x^2 + V - \lambda^2)P_n = 0$. So P_N produces outgoing solutions in L^2_{loc}.
Step 2. If $\lambda \in \mathbb{R}$ and u solves $(D_x^2 + V - \lambda^2)u = 0$, then $\overline{u}$ also solves $(D_x^2 + V - \lambda^2)\overline{u} = 0$.
Since u has compact support, write $u(x) = A_+ e^{i\lambda x} + B_- e^{-i\lambda x}$ when $x \gg 0$ and $u(x) = A_- e^{i\lambda x} + B_+ e^{-i\lambda x}$ when $x \ll 0$.
Compute the Wronskian, $W(u, \overline{u}) = u\overline{u}' - u'\overline{u}$ and $\frac{d}{dx}W(u,\overline{u}) = u\overline{u}'' - u''\overline{u} = 0$. We find that this is $|A_+|^2 - |B_-|^2$ for $x \gg 0$ and $|A_-|^2 - |B_+|^2$ for $x \ll 0$.
Therefore $|A_+|^2 + |B_+|^2 = |A_-|^2 + |B_-|^2$. That is, if u is outgoing then $A_- = B_- = 0$ and $A_+ = B_+ = 0$.
Step 3 is insane.

- Lemma

  If we take $u \in L^\infty(\mathbb{R})$, $(D_x^2 + W)u = 0$ where $W \in L^\infty(\mathbb{R})$, then if $u = 0$ on $(-\infty, 0)$ then $u \equiv 0$.

Fix $h > 0$ and let $v = e^{-x/h}u$.

$$\begin{aligned}
||e^{-x/h}(hD_x)^2 e^{x/h} v||_{L^2} &= ||(h^2 D_x^2 - 2ihD_x - 1)v||_{L^2} \\
&= ||(h\xi - i)^2 \hat{v}||_{L^2} \\
&\geq ||\hat{v}||_{L^2} \\
&= ||v||_L^2
\end{aligned}$$

Equivalently

$$||e^{-x/h}u||_{L^2} \leq h^2 ||e^{-x/h} D_x^2 u|| \leq h^2 ||W||_{L^\infty} ||e^{-x/h}u||_{L^2}$$