Random Matrix Theory

April 1, 2025

Preliminaries

Let ξ_{ij}, η_{ij} be normal random variables (i.e. Gaussian, mean 0, variance 1).
e.g. $\mathbb{P}(\xi_{11} < s) = \int_{-\infty}^{s} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$.
$\int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$ is the variance.
$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is the Probability Density Function (PDF).
$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$ is the probability measure on our probability space (i.e. totally finite measure space).
We build matrices

$$\begin{bmatrix} \xi_{11} & \frac{\xi_{12}+i\eta_{12}}{\sqrt{2}} & \frac{\xi_{13}+i\eta_{13}}{\sqrt{2}} & \cdots \\ \frac{\xi_{21}+i\eta_{21}}{\sqrt{2}} & \xi_{22} & \frac{\xi_{22}+i\eta_{22}}{\sqrt{2}} & \\ \frac{\xi_{31}+i\eta_{31}}{\sqrt{2}} & \frac{\xi_{32}+i\eta_{32}}{\sqrt{2}} & \xi_{33} & \\ \vdots & & & \ddots \end{bmatrix}$$

Computing Random Matrices in Matlab

Gassuain, real valued 1x1 matrix.

```
randn
```

Gaussian, real valued 2x2 matrix.

```
randn(2)
```

Gaussian, complex valued 2x2 matrix.

```
randn(2)+sqrt(-1)*randn(2)
```

Gaussian, complex valued, self-adjoint 2x2 matrix.
Note that appending ’ to a matrix takes the conjugate transpose, and matlab reserves i for the imaginary unit.

```
m = randn(2)+i*randn(2)
(m+m')/2
```

Producing eigenvalues.

```
m = randn(2)+i*randn(2);
l=(m+m')/2;
eig(l)
```

Running tests to see how many hits we get within the interval $[0, 2]$.

```
edges=[0,2];
H=zeros(1,length(edges)-1);
trials=10;
for j=1:trials
m = randn(2)+i*randn(2);
l=(m+m')/2;
ev=eig(l);
H=H+histcount(ev,edges)
end
```

Homework

Is the PDF of $\frac{a+b}{2}$ the same as $\frac{\xi_{12}}{\sqrt{2}}$ for normal RVs a, b, ξ_{12}?
i.e. $\mathbb{P}\left(\frac{a+b}{2} < s\right) \stackrel{?}{=} \mathbb{P}\left(\frac{\xi_{12}}{\sqrt{2}} < s\right)$

2x2 Random Matrix

Our matrix L corresponds to eigvenvalues λ_1, λ_2 which are random variables determined by $\{\xi_{ij}, \eta_{ij}\}$. Then the number of evaulations in the interval B is given by $\sum_{j=1}^{2} \chi_B(\lambda_j)$. We may take the average by

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{j=1}^{2} \chi_B(\lambda_j) \frac{1}{\sqrt{2\pi}} e^{-\xi_{11}^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\xi_{22}^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\xi_{12}^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\eta_{12}^2} \, d\xi_{11} d\xi_{22} d\xi_{12} d\eta_{12}.$$

Expected Evaluations

We have that the expecation of the number of evaluations in the interval (a, b) is given by $\int_a^b G(s)\, ds$ where

$$G(s) = e^{-\frac{s^2}{2}} \sum_{\ell=0}^{2} P_\ell(s)^2$$

and $P_\ell(s)$ is the Hermite polynomial of degree d.

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Differntiability

```
delta = 0.05;
edges=-6:delta:6;
dimensions = 3;
trials = 1000000;

H=zeros(dimensions,trials);

for j=1:trials
m=randn(dimensions)+1i*randn(dimensions);
L=(m+m')/2;
ev=eig(L);
H(:,j) = ev;
end

G = histcounts(H,edges);
plot(edges(1:end-1),G/(trials*delta),'*')
```

IMAGE 1

Observe that each * in the graph corresponds to the average number of eigenvalues in the interaval (a,b). Therefore, they correspond to $\int_a^b C(\lambda)\,d\lambda$. We may consider the limit of the expectation of hits in each interval

$$\lim_{\Delta\to 0}\frac{\mathbb{E}(\#(a,a+\Delta))}{\Delta}.$$

```
delta = 0.01;
edges=-6:delta:6;
dimensions = 3;
trials = 1000000;

H=zeros(dimensions,trials);

for j=1:trials
m=randn(dimensions)+1i*randn(dimensions);
L=(m+m')/2;
ev=eig(L);
H(:,j) = ev;
end

G = histcounts(H,edges);
plot(edges(1:end-1),G/(trials*delta),'*')
```

As dimension grows large, we observe that the plot tends to a semi-circle with endpoints about $\pm 2\sqrt{\text{dimension}}$.
We therefore want a rescaling by $\sqrt{N}$ where $\dim = N$. Then if $G(\alpha) = \frac{d}{d\alpha}\mathbb{E}(\#\text{ of evals in }(a,\alpha))$, we want

$$\int_{-\infty}^{\infty} G(\alpha)\,d\alpha = N.$$

Guess: $G(\alpha) \approx cN^{1/2}\cdot\sqrt{A^2-\alpha^2/N}\cdot\chi_{(-A\sqrt{N},A\sqrt{N})}(\alpha)$. We compute

$$\int_{-A\sqrt{N}}^{A\sqrt{N}} cN^{1/2}\sqrt{A^2-\alpha^2/N}\,d\alpha \overset{\alpha=\sqrt{N}t}{=} cN\int_{-A}^{A}\sqrt{A^2-t^2}\,dt = \frac{c\pi NA^2}{2}.$$

Choosing $A=2$ and c such that $\frac{\pi A^2 c}{2}=1$, we get

$$\int_{-\infty}^{\infty} G(\alpha)\,d\alpha \approx \frac{N^{1/2}}{2\pi}\int_{-\infty}^{\infty}\sqrt{4-\alpha^2/N}\,d\alpha = N.$$

Number of Eigenvalues in an Interval

Let B be a subset of $\mathbb{R}$ (typically an interval). Write $n(B) = \#\{\text{evaluations in }B\}$, a random variable.
Recall that variance is given by the expectation of the square minus the square of the expectation. That is

$$\operatorname{var}(n(B)) = \mathbb{E}(n(B)^2) - (\mathbb{E}(n(B))^2.$$

Our ultimate goal is to understand PDF and $\mathbb{P}(n(B)) = \ell)$ as (the dimension) $N\to\infty$.

Smallest Scale of Interest

Suppose $B=(0,S)$ and N is large (i.e. $N\to\infty$). How large should we choose s such that $\mathbb{E}(n(B))=1$? We compute

$$\int_0^S cN^{1/2}\sqrt{4-\alpha^2/N}\,d\alpha \overset{\alpha=\sqrt{N}t}{=} \int_0^{\frac{S}{\sqrt{N}}} cN\sqrt{4-t^2}\,dt \approx cN\cdot 2\frac{S}{\sqrt{N}} = 2cS\sqrt{N}.$$

Sets of size $N^{-1/2}$, the smallest interesting scale, are called the “microscopic scaling regime”.

Homework: Largest Scale of Interest

How large should B be to see a fraction of the eigenvalues (on average)? That is, how should we scale a and b such that $\mathbb{E}(n((a,b)))=r\cdot N$ for $0<r<1$?

Level Repulsion

```
m=randn(2)+sqrt(-1)*randn(2);
L=(m+m')/2;
ev=eig(L);
subplot(2,1,2),plot(real(ev),imag(ev))
xlim([edges(1),edges(end)])
```

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Macroscopic Scaling Regime for Random Matrices

Suppose $a=\alpha\sqrt{N}$ and $b=\beta\sqrt{N}$ such that $\alpha<\beta$, $-2<\alpha$ and $\beta<2$. Then

$$\lim_{n\to\infty}\frac{\mathbb{E}(\#\text{ of evaluations in }(\alpha\sqrt{N},\beta\sqrt{N}))}{N}=\kappa>0.$$

Recall that we defined $G(b)=\frac{d}{db}\mathbb{E}(\#\text{ of evaluations in }(a,b))$ and

$$G(b)\approx cN^{1/2}\sqrt{A^2-x^2/N}\chi_{[-A\sqrt{N},A\sqrt{n}]}(x).$$

We want that $\int_a^b G(x)\,dx=\kappa N$.

Spacings

Suppose we have eigenvalues $\lambda_1\le\lambda_2\le\cdots\le\lambda_N=\lambda_{\max}$. We can take the spacing $s_j=\lambda_{j+1}-\lambda_j$.

```
m=randn(2)+sqrt(-1)*randn(2);
L=(m+m')/2;
ev=sort(eig(L));
spacing=diff(ev)
```

0.4839

Summary So Far

Given ξ_{ij} and η_{ij} iid RVs with distribution $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, we have explored

- The behavior of average $n_N(B)$.
- Microscopic, macroscopic (and mesoscopic) scaling.
- That $\lambda_{\max} \sim 2\sqrt{N}$ Tracy-Widom distribution.
- Eigenvalue repulsion.

Induced Distribution

Let M be our matrix built using random variables. Then $M = F\Lambda F^T$ where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \\ \vdots & & \ddots \end{pmatrix}, \quad F = \begin{pmatrix} | & | & & | \\ f_{\lambda_1} & f_{\lambda_2} & \cdots & f_{\lambda_N} \\ | & | & & | \end{pmatrix},$$

and $Mf_{\lambda_j} = \lambda_j f_{\lambda_j}$. What we are interested in is the induced joint PDF on $\{\lambda_1, \ldots, \lambda_N\}$. We may write explicitly

$$\frac{1}{Z^n} e^{-\frac{1}{2}\sum_{j=1}^N \lambda_j^2} \prod_{1 \le j < k \le N} (\lambda_k - \lambda_j)^2.$$

Example

Let $N = 2$ and, suppressing the constant term, write

$$\rho = e^{-\frac{1}{2}(x^2+y^2)}(x-y)^2.$$

Taking partial derivatives, we have that

$$\rho_x = e^{-\frac{1}{2}(x^2+y^2)}(x-y)^2(-x + \frac{2}{x-y})$$
$$\rho_y = e^{-\frac{1}{2}(x^2+y^2)}(x-y)^2(-x + \frac{2}{y-x})$$

which implies maxima at $x = \pm 1$ and $y = -x$.

Example

If $N = 3$,

$$\rho = e^{-\frac{1}{2}(x^2+y^2+z^2)}(x-y)^2(x-z)^2(y-z)^2.$$

We may visualize the maxima here by level surfaces (homework).

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Recall: Spectral Theorem

Let $M = F\Lambda F^\dagger$ where $F^\dagger F = I = FF^\dagger$

$$\Lambda = \begin{pmatrix} \lambda_N & 0 & \cdots & \\ 0 & \lambda_{N-1} & & \\ \vdots & & \ddots & \\ & & & \lambda_1 \end{pmatrix}, \quad F = \begin{pmatrix} | & | & & | \\ f_{\lambda_1} & f_{\lambda_2} & \cdots & f_{\lambda_N} \\ | & | & & | \end{pmatrix},$$

for $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$.

Deriving the Joint PDF

Let $n = 2$. If

$$F = \begin{pmatrix} | & | \\ V & W \\ | & | \end{pmatrix},$$

then the expectation of eigenvalues may be computed by

$$\begin{aligned}\mathbb{E}(\mathcal{G}(M)) &= \frac{1}{Z_2^4}\int\cdots\int \mathcal{G}(M(\xi_{11},\xi_{12},\xi_{22},\eta_{12}))x\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(\xi_{11}^2+\xi_{12}^2+\xi_{22}^2+\eta_{12}^2)}\,d\eta_{12}d\xi_{22}d\xi_{12}d\xi_{11} \\ &= \int \mathcal{G}(M(\lambda_1,\lambda_2,V_1,\phi))x\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(\xi_{11}^2+\xi_{12}^2+\xi_{22}^2+\eta_{12}^2)}\,d\eta_{12}d\xi_{22}d\xi_{12}d\xi_{11}.\end{aligned}$$

So we need the Jacobian, and therefore a reparameterization using spectral theorem. We want a collection of independent variables which will produce all 2×2 Hermitian matrices. Consider $Mv = \lambda_2 v$ and $||v||^2 = |v_1|^2 + |v_2|^2 = 1$, then multiply by $e^{i\eta}$ such that $v_1 \in \mathbb{R}_+$. Then $v_2 = \sqrt{1-v_1^2}e^{i\theta}$. That is, $0 \le v_1 \le 1$ and $v_2 = \sqrt{1-v_1^2}(\cos\theta + i\sin\theta)$.
We want that $|w_1|^2 + |w_2|^2 = 1$ and know that $w \perp v$, so $v_1 w_1 + \overline{v}_2 w_2 = 0$. As before, we can choose w such that $w_2 \in \mathbb{R}_+$. This implies that w_1 and $\overline{v}_2$ have the same argument, and $w_1 = -|w_1|e^{-i\theta}$. Therefore $e^{-i\theta}(-v_1|w_1| + |v_2|w_2) = 0$, and $v_1|w_1| - |v_2|w_2 = 0$. It follows that

$$v_1^2(1-w_2^2) = w_2^2(1-v_1^2) \quad\iff\quad v_1 = w_2.$$

Therefore, the entire system may be parameterized by v_1 and θ. We write

$$F = \begin{pmatrix} v_1 & -\sqrt{1-v_1^2}e^{-i\theta} \\ \sqrt{1-v_1^2}e^{i\theta} & v_1 \end{pmatrix}$$

and

$$M = F\Lambda F^\dagger = \begin{pmatrix} v_1 & -\sqrt{1-v_1^2}e^{-i\theta} \\ \sqrt{1-v_1^2}e^{i\theta} & v_1 \end{pmatrix}\begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix}\begin{pmatrix} v_1 & \sqrt{1-v_1^2}e^{-i\theta} \\ -\sqrt{1-v_1^2}e^{i\theta} & v_1 \end{pmatrix}.$$

Therefore

$$M = \begin{pmatrix} \lambda_2 v_1^2 + \lambda_1(1-v_1^2) & v_1\sqrt{1-v_1^2}e^{-i\theta}(\lambda_2-\lambda_1) \\ v_1\sqrt{1-v_1^2}e^{-i\theta}(\lambda_2-\lambda_1) & \lambda_2(1-v_1^2)+\lambda_1 v_1^2 \end{pmatrix}.$$

Recall, we want $\mathcal{G}(M(\xi)) \rightsquigarrow \mathcal{G}(M(\lambda_2, \lambda_1, v_1, \theta))$ and the Jacobian of $M = M(\lambda_2, \lambda_1, v_1, \theta)$. After computation, write

$$|\det J = (\lambda_2 - \lambda_1)^2 \det J' = (\lambda_2 - \lambda_1)^2 Q(v_1, \theta).$$

We integrate

$$\int \cdots \int \mathcal{G}(M(\xi, \eta_{12})) e^{-\frac{1}{2}(\xi_{11}^2 + \xi_{12}^2 + \xi_{22}^2 + \eta_{12}^2)} \frac{1}{(2\pi)^4} \, d\xi_{11} d\xi_{12} d\xi_{22} d\eta_{12}$$

which we may think of uas a function of λ_1 and λ_2 alone. So

$$\frac{1}{(2\pi)^2} \int \cdots \int \mathcal{G}(\lambda_1, \lambda_2) e^{-\frac{1}{2}[M_{11}^2 + M_{22}^2 + 2 \cdot \mathrm{Re}(M_{12})^2 + 2 \cdot \mathrm{Im}(M_{12})^2]} \, d\xi_{11} d\xi_{12} d\xi_{22} d\eta_{12}$$

where we observe that $M_{11}^2 + M_{22}^2 + 2 \cdot \mathrm{Re}(M_{12})^2 + 2 \cdot \mathrm{Im}(M_{12})^2 = \mathrm{Tr}(M^2)$. It follows that we have

$$\begin{aligned} \frac{1}{(2\pi)^2} \int \cdots \int \mathcal{G}(\lambda_1, \lambda_2) e^{-\frac{1}{2}(\lambda_1^2 + \lambda_2^2} \, d\xi_{11} d\xi_{12} d\xi_{22} d\eta_{12} &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^1 \int \int_{-\infty < \lambda_1 \le \lambda_2 < \infty} \mathcal{G}(\lambda_1, \lambda_2) e^{-\frac{1}{2}(\lambda_1^2 + \lambda_2^2} (\lambda_2 - \lambda_1)^2 Q(v, \theta) \, d\xi_1 \\ &= \int \int_{-\infty < \lambda_1 \le \lambda_2 < \infty} \mathcal{G}(\lambda_1, \lambda_2) e^{-\frac{1}{2}(\lambda_1^2 + \lambda_2^2)} (\lambda_2 - \lambda_1)^2 \int_0^{2\pi} \int_0^1 \frac{Q(v, \theta)}{(2\pi)^2} \, dv_1 d\theta d \\ &= c \int \int \mathcal{G}(\lambda_1, \lambda_2) e^{-\frac{1}{2}(\lambda_1^2 + \lambda_2^2)} (\lambda_2 - \lambda_1)^2 \, d\lambda_1 d\lambda_2 \end{aligned}$$

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Recall: Joint PDF on Evaluation of Eigenvalues

$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$ and PDF $\frac{1}{Z_N} e^{-\frac{1}{2}\sum \lambda_i^2} \prod (\lambda_k - \lambda_j)^2$.
This is the Gaussian Unitary Ensemble.

Hermite Polynomials

Write $p_j = \kappa_j^{(j)} x^j + \kappa_{j-1}^{(j)} x^{j-1} + \cdots + \kappa_0^{(j)}$ where the superscript is usually supressed. Then

$$\int_{-\infty}^{\infty} p_j(x) p_k(x) e^{-\frac{1}{2}x^2} \, dx = \delta_{jk}.$$

Observe that $\{e^{-\frac{1}{4}x^2} p_j(x)\}_{j=0}^{\infty}$ forms a basis for $L^2(\mathbb{R})$. For$f \in L^2$, write the truncation $P^{(N)}(f) = \sum_{\ell=0}^{N-1} \left(\int_{\mathbb{R}} f(y) p_\ell(y) e^{-\frac{1}{4}y^2} \right) e^{-}$
Then

$$P^{(N)} = \int_{\mathbb{R}} \left(e^{-\frac{1}{4}(x^2 + y^2)} \sum_{\ell=0}^{N-1} p_\ell(x) p_\ell(y) \right) f(y) \, dy$$

and we write $K_N(x, y) = e^{-\frac{1}{4}(x^2 + y^2)} \sum_{\ell=0}^{N-1} p_\ell(x) p_\ell(y)$ and $\mathcal{K}_N(f) = \int_{\mathbb{R}} K_n(x, y) f(y) \, dy$.
We have that

$$\frac{1}{Z_N} e^{-\frac{1}{2}\sum \lambda_i^2} \prod (\lambda_k - \lambda_j)^2 = \det \begin{bmatrix} K_N(\lambda_1, \lambda_1) & \cdots & K_N(\lambda_1, \lambda_N) \\ \vdots & \ddots & \vdots \\ K_N(\lambda_N, \lambda_1) & \cdots & K_N(\lambda_N, \lambda_N) \end{bmatrix}.$$

For $N = 2$, we see

$$\frac{1}{Z_N} e^{-\frac{1}{2}\sum \lambda_i^2} \prod (\lambda_k - \lambda_j)^2 = (K_N(\lambda_1, \lambda_1) K_N(\lambda_2, \lambda_2) - K_N(\lambda_1, \lambda_2)^2).$$

Example Computation

Let I be an interval and consider $\mathbb{E}(\#$ of evaluations in $I)$. Then

$$\begin{aligned}
&\int_{-\infty<\lambda_1\leq\lambda_2<\infty}\left(\sum_{j=1}^{2}\chi_I(\lambda_j)\right)(K_2(\lambda_1,\lambda_1)K_2(\lambda_2,\lambda_2)-K_2(\lambda_1,\lambda_2)^2)\,d\lambda_2\lambda_2\\
&=\frac{1}{2!}\int\int_{\mathbb{R}^2}(\chi_I(\lambda_1)+\chi(\lambda_2))(K_2(\lambda_1,\lambda_1)K_2(\lambda_2,\lambda_2)-K_2(\lambda_1,\lambda_2)^2)\,d\lambda_1 d\lambda_2\\
&=\frac{1}{2!}\int_I\int_{-\infty}^{\infty}(K_2(\lambda_1,\lambda_1)K_2(\lambda_2,\lambda_2)-K_2(\lambda_1,\lambda_2)^2)\,d\lambda_1 d\lambda_2+\frac{1}{2!}\int_I\int_{-\infty}^{\infty}(K_2(\lambda_1,\lambda_1)K_2(\lambda_2,\lambda_2)-K_2(\lambda_1,\lambda_2)^2)\,d\lambda_2 d\lambda_1.
\end{aligned}$$

Observe that $\int_{-\infty}^{\infty}K_2(\lambda_2,\lambda_2)\,d\lambda_2=\int_{-\infty}^{\infty}e^{-\frac{1}{2}\lambda_2^2}(p_0(\lambda_2)^2+p_1(\lambda_2)^2\,d\lambda_2=2$. We also compute that

$$\begin{aligned}
\int_{-\infty}^{\infty}K_2(\lambda_1,\lambda_2)K_2(\lambda_2,\lambda_1)\,d\lambda_2&=\int_{\mathbb{R}}e^{-\frac{1}{2}(\lambda_1^2+\lambda_2^2}\left(\sum_{\ell=0}^{1}p_\ell(\lambda_1)p_\ell(\lambda_2)\right)\left(\sum_{\ell'=0}^{1}p_{\ell'}(\lambda_2)p_{\ell'}(\lambda_1)\right)\,d\lambda_2\\
&=\sum_{\ell,\ell'=0}^{1}e^{-\frac{1}{2}\lambda_1^2}p_\ell(\lambda_1)p_{\ell'}(\lambda_1)\int_{\infty}^{\infty}e^{-\frac{1}{2}\lambda_2^2}p_\ell(\lambda_2)p_{\ell'}(\lambda_2)\,d\lambda_2\\
&=K_2(\lambda_1,\lambda_1).
\end{aligned}$$

Returning to the first calculation,

$$\begin{aligned}
&\frac{1}{2!}\int_I\int_{-\infty}^{\infty}(K_2(\lambda_1,\lambda_1)K_2(\lambda_2,\lambda_2)-K_2(\lambda_1,\lambda_2)^2)\,d\lambda_1 d\lambda_2+\frac{1}{2!}\int_I\int_{-\infty}^{\infty}(K_2(\lambda_1,\lambda_1)K_2(\lambda_2,\lambda_2)-K_2(\lambda_1,\lambda_2)^2)\,d\lambda_2 d\lambda_1\\
&=\frac{1}{2!}\left[\int_I(2-1)K_2(\lambda_1,\lambda_1)\,d\lambda_1+\int_I(2-1)K_2(\lambda_2,\lambda_2)\,d\lambda_2\right]\\
&=\int_I K_2(\lambda_1,\lambda_1)\,d\lambda_1
\end{aligned}$$

which is the density function for the average number of evaluations in I. So $K_2(\lambda,\lambda)=\frac{e^{-\frac{1}{2}\lambda^2}}{\sqrt{2\pi}}(1+\lambda^2)$.

Question:

What is the probability of having zero evaluations in an interval I?
We have an indicator function $(1-\chi_I(\lambda_1))(1-\chi_I(\lambda_2))$, so

$$\begin{aligned}
P(\text{no evaluations in } I)&=\frac{1}{2}\int_{\mathbb{R}^2}(1-\chi_I(\lambda_1))(1-\chi_I(\lambda_2))[K_2(\lambda_1,\lambda_1)K_2(\lambda_2,\lambda_2)-K_2(\lambda_1,\lambda_2)^2]\,d\lambda_1 d\lambda_2\\
&=\frac{1}{2}\int_{\mathbb{R}^2}(1-(\chi_I(\lambda_1)+\chi_I(\lambda_2))+\chi_I(\lambda_1)\chi_I(\lambda_2))K_2(\lambda_1,\lambda_1)K_2(\lambda_2,\lambda_2)-K_2(\lambda_1,\lambda_2)^2]\,d\lambda_1 d\lambda_2\\
&=\frac{1}{2}[\int_{\mathbb{R}^2}K_2(\lambda_1,\lambda_1)K_2(\lambda_2,\lambda_2)-K_2(\lambda_1,\lambda_2)^2\,d\lambda_1 d\lambda_2\\
&\quad -2\int_I\int_{\mathbb{R}}K_2(\lambda_1,\lambda_1)K_2(\lambda_2,\lambda_2)-K_2(\lambda_1,\lambda_2)^2\,d\lambda_2 d\lambda_1\\
&\quad +\int_I\int_I K_2(\lambda_1,\lambda_1)K_2(\lambda_2,\lambda_2)-K_2(\lambda_1,\lambda_2)^2]\\
&=\frac{1}{2}[4-2-2\int_I K_2(\lambda_1,\lambda_1)\,d\lambda_1+\int_I\int_I K_2(\lambda_1,\lambda_1)K_2(\lambda_2,\lambda_2)-K_2(\lambda_1,\lambda_2)^2\,d\lambda_1\lambda_2]\\
&=1-\int_I K_2(\lambda_1,\lambda_1)\,d\lambda_1+\int_I\int_I\det(\quad)_{2\times 2}\,d^2\lambda\\
&=\det(1-\mathcal{K}_2^{(I)})
\end{aligned}$$

If $I=(0,\infty)$, then the probability is $\frac{\pi-2}{4\pi}$.

Fredholm Determinant

Write $H_N(I,t)=\det(1-t\mathcal{K}_N^{(I)})$ where $K_N^{(I)}$ is an integral operator which acts on $L_2(I)$ by

$$\mathcal{K}_N^{(I)}(f)=\int_I K_N(x,y)f(y)\,dy.$$

So the range of $\mathcal{K}_N^{(I)}$ is finite dimensional (i.e. it is a finite rank operator). Then

$$H_N(I,t)=1-\int_I K_N(\lambda_1,\lambda_1)\,d\lambda_1-\frac{t}{2!}\int_I\int_I \det(\quad)_{2\times 2}\,d^2\lambda+\cdots+\frac{(-t)^j}{j!}\overbrace{\int_I\cdots\int_I}^{j}\det(\quad)_{j\times j}\,d^j\lambda+\frac{(-t)^N}{N!}\overbrace{\int_I\cdots\int_I}^{N}\det(\quad)_{N\times N}$$

Then $H_N(I,1)$ is the probability of no evaluations in I, and $H_N'(I,1)$ is negative the probability of exactly one evaluation in I. So

$$H_N^{(j)}(I,1)=(-1)^j j!P(\text{exactly } j \text{ eigenvalues in } I).$$

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Recall

$$\frac{1}{z_N}e^{-\frac{1}{2}\sum\lambda_j^2}\prod_{j<k}(\lambda_j-\lambda_k)^2=\frac{1}{N!}\det(K_N(\lambda_j,\lambda_k))_{N\times N}$$

For $n=2$, we have

$$\mathbb{E}\left(\sum_{j=1}^{2}\chi_B(\lambda_j)\right)=\int_B K_2(\lambda,\lambda)\,d\lambda$$

We also have that

$$\begin{aligned}\mathbb{E}((1-\chi_B(\lambda_1))(1-\chi_B(\lambda_2))&=P(\text{no evaluations})\\&=1-\lambda_B K_2(\lambda,\lambda)\,d\lambda+\frac{1}{2}\int_B\int_B\det(\quad)_{2\times 2}\,d\lambda\end{aligned}$$

where

$$\begin{aligned}(\lambda_1-\lambda_2)^2&=\det\begin{pmatrix}1&\lambda_1\\1&\lambda_2\end{pmatrix}\det\begin{pmatrix}1&1\\\lambda_1&\lambda_2\end{pmatrix}\\&\overset{q(\lambda_i)=\lambda_i+c_0}{=}\det\begin{pmatrix}1&q(\lambda_1)\\1&q(\lambda_2)\end{pmatrix}\det\begin{pmatrix}1&1\\q(\lambda_1)&q(\lambda_2)\end{pmatrix}\\&=\frac{1}{(\kappa_0^{(0)}\kappa_1^{(1)})^2}\det\begin{pmatrix}\kappa_0^{(0)}&\kappa_1^{(1)}q(\lambda_1)\\\kappa_0^{(0)}&\kappa_1^{(1)}q(\lambda_2)\end{pmatrix}\det\begin{pmatrix}\kappa_0^{(0)}&\kappa_0^{(0)}\\\kappa_1^{(1)}q(\lambda_1)&\kappa_1^{(1)}q(\lambda_2)\end{pmatrix}\\&=\frac{1}{\prod_0^1(\kappa_i^{(i)})^2}\det\begin{pmatrix}P_0(\lambda_1)&P_1(\lambda_1)\\P_0(\lambda_2)&P_1(\lambda_2)\end{pmatrix}\det\begin{pmatrix}P_0(\lambda_1)&P_0(\lambda_2)\\P_1(\lambda_1)&P_1(\lambda_2)\end{pmatrix}\end{aligned}$$

It follows that

$$e^{-\frac{1}{2}\sum_1^2 \lambda_j^2}(\lambda_2-\lambda_1)^2 = \prod_{j=0}^{1}(\kappa_j^{(j)})^{-2}\det\begin{pmatrix} P_0(\lambda_1)e^{-\frac{1}{4}\lambda_1^2} & P_1(\lambda_1)e^{-\frac{1}{4}\lambda_1^2} \\ P_0(\lambda_1)e^{-\frac{1}{4}\lambda_2^2} & P_1(\lambda_1)e^{-\frac{1}{4}\lambda_2^2}\end{pmatrix}\det\begin{pmatrix} P_0(\lambda_1)e^{-\frac{1}{4}\lambda_1^2} & P_0(\lambda_1)e^{-\frac{1}{4}\lambda_2^2} \\ P_1(\lambda_1)e^{-\frac{1}{4}\lambda_1^2} & P_1(\lambda_1)e^{-\frac{1}{4}\lambda_2^2}\end{pmatrix}$$
$$= \prod_{j=0}^{1}(\kappa_j^{(j)})^{-2}\det(K_2(\lambda_i,\lambda_j))_{2\times 2}$$

where $K_2(x,y) = e^{-\frac{1}{4}(x^2+y^2)}\sum_{\ell=0}^{1} P_\ell(x)P_\ell(Y)$. So we have

$$\frac{1}{z_N}\prod_0^1(\kappa_j^{(j)})^{-2}\det\begin{pmatrix} K_2(\lambda_1,\lambda_1) & K_2(\lambda_1,\lambda_2) \\ K_2(\lambda_2,\lambda_1) & K_2(\lambda_2,\lambda_2)\end{pmatrix}$$

and the fact that

$$\frac{1}{z_N\prod_{j=1}^{2}(\kappa_j^{(j)})}\int_{\mathbb{R}^2}[K_2(\lambda_1,\lambda_1)K_2(\lambda_2,\lambda_2) - K_2(\lambda_1,\lambda_2)^2]\, d\lambda_1 d\lambda_2 = 1$$

Observe that (to do: fill in these calculations)

$$\int_{\mathbb{R}^2} K_2(\lambda_1,\lambda_1)K_2(\lambda_2,\lambda_2)\, d\lambda_1 d\lambda_2 = \int_{\mathbb{R}^2}(e^{-\frac{1}{4}\lambda_1^2}(P_0(\lambda_1)P_0(\lambda_1) + P_1(\lambda_1)P_1(\lambda_1))(e^{-\frac{1}{4}\lambda_2^2}(P_0(\lambda_2)P_0(\lambda_2) + P_1(\lambda_2)P_1(\lambda_2))$$

So it must be that

$$\frac{1}{z_N(\kappa_0^{(0)})^2(\kappa_1^{(1)})^2}(2) = 1.$$

We conclude that the original joint PDF can be written as $\frac{1}{2!}\det(K_1(\lambda_i,\lambda_j))_{2\times 2}$.

Vandermonde Determinant

Write

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_N \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_N^2 \\ \vdots & & & \vdots \\ \lambda_1^{N-1} & \lambda_2^{N-1} & \dots & \lambda_N^{N-1}\end{vmatrix} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{N-1} \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{N-1}^2 \\ \vdots & & & \vdots \\ \lambda_1^{N-2} & \lambda_2^{N-2} & \dots & \lambda_{N-1}^{N-2}\end{vmatrix}(\lambda_N-\lambda_1)(\lambda_N-\lambda_2)\cdots(\lambda_N-\lambda_{N-1})$$
$$= \prod_{j<k}(\lambda_k - \lambda_j)$$

and observe that this is zero when $\lambda_i = \lambda_j$. Now write

$$\det\begin{pmatrix} 1 & \cdots & \lambda_1^{N-1} \\ \vdots & & \vdots \\ 1 & \dots & \lambda_N^{N-1}\end{pmatrix}\det\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ \lambda_1^{N-1} & \dots & \lambda_N^{N-1}\end{pmatrix}$$

then by using the multilinearity of the dterminant and adding rows we can write

$$\det\begin{pmatrix}1 & \cdots & \lambda_1^{N-1}\\ \vdots & & \vdots\\ 1 & \cdots & \lambda_N^{N-1}\end{pmatrix}\det\begin{pmatrix}1 & \cdots & 1\\ \lambda_1+c_0 & \cdots & \lambda_N+c_0\\ \pi_2(\lambda_1) & \cdots & \pi_2(\lambda_N)\\ \vdots & & \vdots\\ \lambda_1^{N-1} & \cdots & \lambda_N^{N-1}\end{pmatrix}$$

So we can write

$$\det\begin{pmatrix}e^{-\frac{1}{4}\lambda_1^2}P_0(\lambda_1) & \cdots & e^{-\frac{1}{4}\lambda_1^2}P_{N-1}(\lambda_1)\\ \vdots & & \vdots\\ e^{-\frac{1}{4}\lambda_N^2}P_0(\lambda_N) & \cdots & e^{-\frac{1}{4}\lambda_1^2}P_{N-1}(\lambda_N)\end{pmatrix}\frac{1}{\prod_{i=0}^{N-1}(\kappa_j^{(j)})^2}\det\begin{pmatrix}P_0(\lambda) & \cdots & P_0\lambda_N)\\ \vdots & & \vdots\\ P_{N-1}(\lambda_1) & \cdots & P_{N-1}(\lambda_N)\end{pmatrix}$$

Examining the (j,k) entry, we have

$$\frac{1}{\prod(\kappa_j^{(j)})^2}e^{-\frac{1}{4}(\lambda_j^2+\lambda_k^2)}(P_0(\lambda_j)P_0(\lambda_k)+P_1(\lambda_j)P_1(\lambda_k)+\cdots+P_{N-1}(\lambda_j)P_{N-1}(\lambda_k)).$$

or

$$\frac{1}{z_N\prod(\kappa_j^{(j)})^2}\det[K_n(\lambda_j,\lambda_k)]_{N\times N}$$

which must integrate across $\mathbb{R}^n$ to exactly 1. From this we conclude that $\frac{N!}{z_N\prod(\kappa_j^{(j)})^2}=1$.

April 24, 2025

Determinants

$$\begin{aligned}\int_{\mathbb{R}}\det\begin{pmatrix}K_N(\lambda_1,\lambda_1) & K_N(\lambda_1,\lambda_2)\\ K_N(\lambda_2,\lambda_1) & K_N(\lambda_2,\lambda_2)\end{pmatrix}d\lambda_2 &= \int_R K_N(\lambda_1,\lambda_1)K_N(\lambda_2,\lambda_2)-K_N(\lambda_1,\lambda_2)K_N(\lambda_2,\lambda_1)\,d\lambda_2\\ &= K_N(\lambda_1,\lambda_1)\int_{\mathbb{R}}e^{-\frac{1}{2}\lambda_2^2}P_\ell(\lambda_2)^2\,d\lambda_2-0\\ &= NK_N(\lambda_1,\lambda_1)\end{aligned}$$

We have that $\int_{\mathbb{R}} K_N(\lambda,x)K_N(x,\mu)\,dx = K_N(\lambda,\mu)$. Then

$$\begin{aligned}
\int_{\mathbb{R}} \begin{vmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{vmatrix} d\lambda^3 &= \int_{\mathbb{R}} K_{31}\begin{vmatrix} K_{12} & K_{13} \\ K_{22} & K_{23} \end{vmatrix} - K_{32}\begin{vmatrix} K_{11} & K_{13} \\ K_{21} & K_{23} \end{vmatrix} + K_{33}\begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} d\lambda^3 \\
&= \begin{vmatrix} K_{12} & \int_{\mathbb{R}} K(\lambda_1,\lambda_3)K(\lambda_3,\lambda_1)\,d\lambda^3 \\ K_{22} & \int_{\mathbb{R}} K(\lambda_2,\lambda_3)K(\lambda_3,\lambda_1)\,d\lambda^3 \end{vmatrix} \\
&\quad - \begin{vmatrix} K_{11} & \int_{\mathbb{R}} K(\lambda_1,\lambda_3)K(\lambda_3,\lambda_2)\,d\lambda^3 \\ K_{21} & \int_{\mathbb{R}} K(\lambda_2,\lambda_3)K(\lambda_3,\lambda_2)\,d\lambda^3 \end{vmatrix} \\
&\quad + \begin{vmatrix} K_{11} & \int_{\mathbb{R}} K(\lambda_1,\lambda_2)K(\lambda_3,\lambda_3)\,d\lambda^3 \\ K_{21} & \int_{\mathbb{R}} K(\lambda_2,\lambda_2)K(\lambda_3,\lambda_3)\,d\lambda^3 \end{vmatrix} \\
&= \begin{vmatrix} K_{12} & K_{11} \\ K_{22} & K_{21} \end{vmatrix} - \begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} + N\begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} \\
&= -\begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} - \begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} + N\begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} \\
&= (N-2)\det(\quad)_{2\times 2}
\end{aligned}$$

So we have that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} (N-2)\begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} d\lambda_2 d\lambda_1 = (N-2)(N-1)N$$

In general, we see that

$$\int_{\mathbb{R}} \begin{vmatrix} K_{11} & \cdots & K_{1j} \\ \vdots & \ddots & \vdots \\ K_{j1} & \cdots & K_{jj} \end{vmatrix} d\lambda_j = (N-(j-1))\begin{vmatrix} K_{11} & \cdots & K_{1j} \\ \vdots & \ddots & \vdots \\ K_{j1} & \cdots & K_{j-1j-1} \end{vmatrix}$$

or

$$\begin{aligned}
\int_{\mathbb{R}} \begin{vmatrix} K_{11} & \cdots & K_{1j} \\ \vdots & \ddots & \vdots \\ K_{j1} & \cdots & K_{jj} \end{vmatrix} d\lambda_j &= (-1)^{j+1}K_{j1}\begin{vmatrix} K_{12} & \cdots & K_{1j} \\ \vdots & \ddots & \vdots \\ K_{(j-1)2} & \cdots & K_{(j-1)j} \end{vmatrix} + (-1)^{j+2}K_{j2}\begin{vmatrix} K_{11} & \cdots & K_{1j} \\ \vdots & \ddots & \vdots \\ K_{(j-1)1} & \cdots & K_{(j-1)j} \end{vmatrix} + \cdots \\
&\quad \cdots + K_{jj}\begin{vmatrix} K_{11} & \cdots & K_{1(j-1)} \\ \vdots & \ddots & \vdots \\ K_{(j-1)1} & \cdots & K_{(j-1)(j-1)} \end{vmatrix} \\
&= (-1)^{j+1}\begin{vmatrix} K_{12} & \cdots & K_{1j} \\ \vdots & \ddots & \vdots \\ K_{(j-1)2} & \cdots & K_{(j-1)1} \end{vmatrix} + (-1)^{j+2}K_{j2}\begin{vmatrix} K_{11} & \cdots & K_{12} \\ \vdots & \ddots & \vdots \\ K_{(j-1)1} & \cdots & K_{(j-1)2} \end{vmatrix} + \cdots \\
&\quad \cdots + N\begin{vmatrix} K_{11} & \cdots & K_{1(j-1)} \\ \vdots & \ddots & \vdots \\ K_{(j-1)1} & \cdots & K_{(j-1)(j-1)} \end{vmatrix}
\end{aligned}$$

It takes, for example, $j-1$ column moves to convert the leading matrix into the final matrix. So it picks up a leading -1. In fact, we see that each term save the last will be negative. It follows that we have that the integral may be written

$$(N-(j-1))\begin{vmatrix} K_{11} & \cdots & K_{1(j-1)} \\ \vdots & \ddots & \vdots \\ K_{(j-1)1} & \cdots & K_{(j-1)(j-1)} \end{vmatrix}$$

Evaluations

Now consider

$$\begin{aligned}
\mathbb{E}(\#\text{ of evaluations in } B) &= \int_{\mathbb{R}^n} \left(\sum_{j=1}^{N} \chi_B(\lambda_j) \right) \\
&= \sum_{j=1}^{N} \int_{\mathbb{R}^n} \chi_B(\lambda_j) \frac{1}{N!} \det(\quad)_{N\times N}\, d\lambda_N \cdots d\lambda_1
\end{aligned}$$

With a change of variables where $\mu_\ell = \lambda_\ell$ for $\ell \in \{1, j\}$ such that $\mu_j = \lambda_1$ and $\mu_1 = \lambda_j$,

$$\begin{aligned}
\sum_{j=1}^{N} \int_{\mathbb{R}^n} \chi_B(\mu_1) \frac{1}{N!} \det(K_N(\mu_j, \mu_k))_{N\times N}\, d\mu_N \cdots d\mu_1 &= \sum_{j=1}^{N} \int_{\mathbb{R}} \chi_B(\mu_1) \frac{(N-1)!}{N!} K_N(\mu_1, \mu_1)\, d\mu_1 \\
&= \int_B K_N(\mu_1, \mu_1)\, d\mu_1
\end{aligned}$$

Variance

Let $n_N(B) = (\#\text{ of evaluations in } B)$ be a random variable. What is the variance? We have that

$$n_N(B) = \sum_{j=1}^{N} \chi_B(\lambda_j)$$

so

$$\begin{aligned}
\operatorname{var}(n_N(B)) &= \mathbb{E}((n_N(B))^2) - (\mathbb{E}(n_N(B)))^2 \\
&= \int_{\mathbb{R}^n} \left(\sum_{j=1}^{N} \chi_B(\lambda_j) \right)^2 \frac{1}{N!} \det(\quad)_{N\times N}\, d\lambda^N - [\quad]^2 \\
&= \int_{\mathbb{R}^n} \left(\sum_k \sum_j \chi_B(\lambda_j)\chi_B(\lambda_k) \right) \frac{1}{N!} \det(\quad)_{N\times N}\, d\lambda^N - [\quad]^2
\end{aligned}$$

April 29, 2025

Variance

Compute

$$\operatorname{Var}(n_N(B) = \mathbb{E}(n_N(B)^2) - (\mathbb{E}(n_N(B)))^2 = \int_B K_n(\lambda_1, \lambda_1)\, d\lambda_1 - \int_{B\times B} K_N(\lambda_1, \lambda_2)^2\, d\lambda_1 d\lambda_2$$

This follows from $n_N(B) = \sum_{j=1}^{N} \chi_B(\lambda_1)$, so

$$\begin{aligned}
\mathbb{E}\left(\left(\sum_{i=1}^{n} \chi_B(\lambda_j) \right)^2 \right) &= \mathbb{E}\left(\sum_{j=1}^{n} \sum_{k=1}^{n} \chi_B(\lambda_1)\chi_B(\lambda_2) \right) \\
&= \mathbb{E}\left(\sum_{j=1}^{n} \chi_B(\lambda_j) \right) + \mathbb{E}\left(\sum_{j\neq k} \chi_B(\lambda_j)\chi_B(\lambda_k) \right) \\
&= \int_B K_N(\lambda_1, \lambda_1)\, d\lambda_1 + \sum_{j\neq k} \int_{\mathbb{R}^n} \chi_B(\lambda_j)\chi_B(\lambda_k) \frac{1}{N!} \det(K_N(\lambda_m, \lambda_n))_{N\times N}\, d^N\lambda
\end{aligned}$$

Then using the same trick as before such that $\lambda_j = \mu_1$ and $\lambda_k = \mu_2$, we rewrite this

$$\int_B K_N(\lambda_1,\lambda_1)\, d\lambda_1 + \overbrace{\sum_{j\neq k}\int_{\mathbb{R}^n} \chi_B(\mu_1)\chi_B(\mu_2)\frac{1}{N!}\det(K_N(\mu_m,\mu_n)_{N\times N}\, d^N\mu}^{:=I}$$

Then we have

$$\begin{aligned} I &= \sum_{j\neq j}\overbrace{\int\cdots\int}^{N-1} \chi_B(\mu_1)\chi_B(\mu_2)\frac{(1)}{N!}\det(\quad)_{(N-1)\times(N-1)}\, d^{N-1}\mu \\ &= \sum_{j\neq k}\int_B\int_B \frac{(N-2)!}{N!}\det(\quad)_{2\times 2}\, d\mu \\ &= \frac{N!}{N!}\int_B\int_B \begin{vmatrix} K_N(\mu_1,\mu_1) & K_N(\mu_1,\mu_2) \\ K_N(\mu_2,\mu_1) & K_N(\mu_2,\mu_2)\end{vmatrix}\, d^2\mu \end{aligned}$$

Then we have

$$\mathbb{E}(n_N(B)^2) = \int_B K_N(\lambda_1,\lambda_1)\, d\lambda_1 + \int_B\int_B K_N(\lambda_1,\lambda_1)K_N(\lambda_2,\lambda_2) - K_N(\lambda_1,\lambda_2)^2\, d^2\lambda$$

as well as

$$(\mathbb{E}(n_N(B)))^2 = \left(\int_B K_N(\lambda_1,\lambda_1)\, d\lambda_1\right)^2$$

Then, since $\int_B\int_B K_N(\lambda_1,\lambda_1)K_N(\lambda_2,\lambda_2)\, d^2\lambda = \left(\int_B K_N(\lambda_1,\lambda_1)\, d\lambda_1\right)^2$, the terms cancel and we get the expression we want.

Probability of No Evaluations

Now consider $\prod_{j=1}^N (1-\chi_B(\lambda_j))$ which returns 1 if there are no evaluations in B and 0 otherwise. Therefore

$$\int \prod_{j=1}^N (1-\chi_B(\lambda_j))\frac{1}{N!}\det(K_N(\lambda_j,\lambda_k)_{N\times N}\, d^N\lambda$$

is the probabiltiy of having zero eigenvalues in B (i.e. the probability that $n_N(B) = 0$). If we use the case where $B = (a,\infty)$, then this returns the probability that the largest eigenvalue is less than a. Consider

$$\sum_{k=1}^N \chi_B(\lambda_k)\prod_{\substack{j=1 \\ j\neq k}}^N (1-\chi_B(\lambda_k))$$

and suppose we have exactly one eigenvalues (λ_3) in B. This returns 1 when we have exactly one eigvenalue in B and 0 otherwise. So

$$\int \sum \chi_B(\lambda_k)\prod_{j=1}^{N}{}^{(k)}(1-\chi_B(\lambda_1)\frac{1}{N!}\det(\quad)_{N\times N}\, d^N\lambda,$$

where the product skips the k-th term, is the probability $\mathbb{P}\{n_N(B)=1\}$. Now write

$$H(B,t)=\mathbb{E}\left(\prod_{j=1}^{N}(1-t\chi_B(\lambda_j))\right)$$

which gives $H(B,1)=\mathbb{P}\{n_N(B)=1\}$. Then the derivative with respect to t,

$$H^{\prime}(B,t)=\mathbb{E}\left(\sum_{k=1}^{N}(-\chi_B(\lambda_k))\prod_{j=1}^{N}{}^{(k)}(1-t\chi_B(\lambda_j)\right)$$

so $H^{\prime}(B,1)=-\mathbb{P}\{n_N(B)=1\}$, and $H^{\prime\prime}(B,1)=2\mathbb{P}\{n_N(B)=2\}$. It follows that $H^{j}(B,1)=(-1)^j\cdot j!\cdot\mathbb{P}\{n_N=j\}$. Then $n_N(B)$ is the number of evaluations in B, and this process gives us the number statistics. We compute this fact as follows

$$\begin{aligned}
H^{\prime\prime}(B,t)&=\mathbb{E}\left(\sum_{k=1}^{N}\chi_B(\lambda_k)\sum_{\ell=1}^{N}{}^{(k)}\chi_B(\lambda_\ell)\prod_{j=1}^{N}{}^{(k,\ell)}(1-t\chi_B(\lambda_j))\right)\\
&=\mathbb{E}\left(\sum_{k\neq\ell}\chi_B(\lambda_k)\chi_B(\lambda_\ell)\prod_{j=1}^{N}{}^{(k,\ell)}(1-t\chi_B(\lambda_j)\right)\\
&=2!\cdot\mathbb{P}\{n_N(B)=2\}\quad(t=1)
\end{aligned}$$

and

$$\begin{aligned}
H^{j}(B,t)&=\mathbb{E}\left(\sum_{k_1=1}^{N}\sum_{k_2=1}^{N}{}^{(k_1)}\cdots\sum_{k_j=1}^{N}{}^{(k_1,k_2,\ldots,k_{j-1})}\prod_{\nu=1}^{j}\chi_B(\lambda_{k_{i_\nu}})\prod_{j=1}^{N}{}'(1-t\chi_B(\lambda_j)\right)\\
&=(-1)^j\cdot j!\cdot\mathbb{P}\{n_N(B)=j\}\quad(t=1)
\end{aligned}$$

Coming Next

We know that $\mathbb{E}(n_N(B))=\int_B K_N(\lambda,\lambda)\,d\lambda$. We will define an integral operator on functions $f\in L^2(B)$

$$\begin{aligned}
\mathcal{K}_N^{(B)}(f)&=\int_B K_N(x,y)f(y)\,dy\\
&=\int_B e^{-\frac{1}{4}(x^2+y^2)}\sum_{\ell=0}^{N-1}P_\ell(x)P_\ell(y)f(y)\,dy
\end{aligned}$$

We can define the trace of this operator,

$$\mathrm{Tr}(\mathcal{K}_N^{(B)})=\int_B K_N(\lambda,\lambda)\,d\lambda=\mathbb{E}(n_N(B))$$

Then

$$H(B,t)=\det(1-t\mathcal{K}_N^{(B)})$$

May 6, 2025

Expectation

Recall that

$$
\begin{aligned}
\mathbb{E}\left(\prod_{j=1}^{N}(1-t\chi_B(\lambda_i)\right) &= \mathbb{E}\left(1-t\sum_{j=1}^{N}\chi_B(\lambda_j)+t^2\sum_{j<k}\chi_B(\lambda_j)\chi_B(\lambda_k)-t^3\sum_{j<k<m}\chi_B(\lambda_j)\chi_B(\lambda_k)\chi_B(\lambda_m)\right) \\
&= 1-t\int_B K_N(\lambda_1,\lambda_1)\,d\lambda_1+t_2\sum_{j<k}\int\chi_B(\lambda_j)\chi_B(\lambda_k)\frac{1}{N!}\det(\quad)_{N\times N}\,d^N\lambda \\
&= 1-t\int_B K_N(\lambda_1,\lambda_1)\,d\lambda_1+\frac{t^2}{2}\int_B\int_B|\quad|_{2\times 2}\,d^2\lambda-\frac{t^3}{3!}\int_B\int_B\int_B|\quad|_{3\times 3}\,d^3\lambda+\cdots+\int_B\cdots\int_B|\quad|_{N\times N}\,d^N\lambda
\end{aligned}
$$

We claim that

$$
\int_{\mathbb{R}^n}\left(\sum_{k_1<\cdots<k_j}\prod_{\ell=1}^{j}\chi_B(\lambda_{K_\ell})\frac{\det}{N!}(K_N(\lambda_m,\lambda_n)_{N\times N}\right)d^N\lambda=\frac{1}{j!}\int_B\cdots\int_B\det(K_N(\lambda_m,\lambda_n))_{j\times j}\,d\lambda_j\cdots d\lambda_1
$$

Observing that $\binom{N}{j}\cdot\frac{(N-j)!}{N!}=\frac{N!}{j!(N-j)!}=\frac{1}{j!}$, write

$$
\begin{aligned}
\sum_{k_1<\cdots<k_j}\int_{\mathbb{R}^n}\left(\prod_{\ell=1}^{j}\chi_B(\lambda_\ell)\right)\frac{\det}{N!}(\quad)_{N\times N}\,d\lambda_{j+1}\cdots d\lambda_1 &= \sum_{k_1<\cdots<k_j}\int_{\mathbb{R}^{n-1}}\left(\prod\chi_B(\lambda_\ell)\right)\frac{\det}{N!}(\quad)_{(N-1)\times(N-1)}\,d\lambda_{N-1}\cdots d\lambda_1 \\
&= \sum_{k_1<\cdots<k_j}\int_{\mathbb{R}^j}\left(\prod_{\ell=1}^{j}\chi_B(\lambda_\ell)\right)\frac{1\cdot 2\cdot 3\cdot\cdots\cdot(N-j)}{N!}\det(\quad)_{j\times j}\,d^j\lambda \\
&= \frac{1}{j!}\int\cdots\int\det(\quad)_{j\times j}\,d^j\lambda
\end{aligned}
$$

So we have

$$
H(B,t)=1+\sum_{j=1}^{N}\frac{(-1)^j t^j}{j!}\int_B\cdots\int_B\det\begin{pmatrix}K_N(\lambda_1,\lambda_1) & \\ & K_N(\lambda_j,\lambda_j)\end{pmatrix}d^j\lambda=\det(1-t\mathcal{K}_N)
$$

So if A is a symmetric, real $N\times N$ matrix with $||A||<1$,

$$
\det(I-tA)=e^{\log\det(I-ta)}=e^{\log\prod_{j=1}^{N}(1-t\mu_j^A}\overset{?}{=}e^{\sum_{j=1}^{N}\log(1-t\mu_j^A)}\overset{?}{=}e^{\operatorname{Tr}\log(I-tA)}
$$

May 8, 2025

Continuing

We want to show that $\det(I-tA)=e^{\operatorname{Tr}\log(I-tA)}$.
Write

$$
\log(1-x)=\int_0^x-\frac{1}{1-s}\,ds=-\int_0^x\sum_{j=0}^{\infty}s^i\,ds=-\sum_{j=0}^{\infty}\frac{x^{j+1}}{j+1}=-\sum_{j=1}^{\infty}\frac{x^j}{j}.
$$

So $\log(I-tA) := -\sum_{j=1}^{\infty}\frac{(tA)^j}{j}$ converges in the matrix norm (for sufficiently small t). Equivalently, $e^{-\sum_{j=1}^{\infty}\frac{(tA)^j}{j}} = I-tA$. Suppose we have an eigenvalue μ of A, then $\log(1-t\mu)$ is an eigenvalue of $\log(I-tA)$. More commonly, this is presented as $\log\det(I-tA) = \operatorname{Tr}\log(I-tA)$. In the end, we have

$$\det(I-tA) = \exp[\operatorname{Tr}(\log(I-tA))] = \exp\left[-\operatorname{Tr}\sum_{j=1}^{\infty}\frac{(tA)^j}{j}\right].$$

Trace of an Operator

Given $\mathcal{K}_n(f) = \int_B K_N(x,y)f(y)\,dy$, we define $\operatorname{Tr}(\mathcal{K}_n) = \int_B K_N(x,x)\,dx$. We want to consider the trace of $\mathcal{K}_N^j$. Taking $j=2$ as an example,

$$\begin{aligned}
\mathcal{K}_N^2(f) &= \mathcal{K}_N\left[\int_B K_N(x_1,x_2)f(x_2)\,dx_2\right] \\
&= \int_B K_N(x,x_1)\int_B K_N(x_1,x_2)f(x_2)\,dx_2\,dx \\
&= \int_B\left[\int_B K_N(x,x_1)K_N(x_1,x_2)\,dx\right]f(x_2)\,dx_2
\end{aligned}$$

we note that $\int_B K_N(x,x_1)K_n(x_1,x_2)\,dx$ is our new kernel. Then $\operatorname{Tr}(\mathcal{K}_N^2) = \int_B\int_B K_N(x_1,x_2)K_N(x_2,x_1)\,d^2x$. Therefore $\mathcal{K}_N^\ell$ has a kernel given by

$$\overbrace{\int_B\cdots\int_B}^{\ell-1} K_N(x,x_1)\cdots K_N(x_{\ell-1},x_\ell)\,dx_1\cdots dx_{\ell-1}.$$

with trace given by

$$\operatorname{Tr}(\mathcal{K}_N^\ell) = \overbrace{\int_B\cdots\int_B}^{\ell} K_N(x_\ell,x_1)\cdots K_N(x_{\ell-1},X_\ell)\,d^\ell x.$$

Continuing Computation

$$\det(I-tA) = \sum_{k=0}^{\infty}\frac{1}{k!}\left[-\operatorname{Tr}\sum_{j=1}^{\infty}\frac{(tA)^j}{j}\right]^k = 1-\left[\sum_{j=1}^{\infty}\frac{t^j}{j}\operatorname{Tr}(A^j)\right]+\frac{1}{2!}\left[\sum_{j=1}^{\infty}\frac{t^j}{j}\operatorname{Tr}(A^j)\right]^2-\frac{1}{3!}\left[\sum_{j=1}^{\infty}\frac{t^j}{j}\operatorname{Tr}(A^j)\right]^3+\cdots$$

If we are intersetd in $O(t^1)$, we examine the first non-trivial term and have $-\operatorname{Tr}(A)$. For $O(t^2)$ we have $-\frac{1}{2}\operatorname{Tr}(A^2)+\frac{1}{2}(\operatorname{Tr}A)^2$, which we compute as

$$\frac{1}{2}\left[\int_B K_N(\lambda_1,\lambda_1)\,d\lambda_1\int_B K_N(\lambda_2,\lambda_2)\,d\lambda_2 - \int_B\int_B K_N(\lambda_1,\lambda_2),K_N(\lambda_2,\lambda_1)\,d^2\lambda\right]$$

Examining $O(t^3)$, we have $-\frac{1}{3}\operatorname{Tr}(A^3)+\frac{1}{4}(\operatorname{Tr}A)\operatorname{Tr}(A^2)+\frac{1}{4}\operatorname{Tr}(A^2)(\operatorname{Tr}A)-\frac{1}{3!}(\operatorname{Tr}A)^3$ giving

$$-\frac{t^3}{3!}\left[(\operatorname{Tr}A)^3-\frac{3}{2}(\operatorname{Tr}A)\operatorname{Tr}(A^2)-\frac{3}{2}\operatorname{Tr}(A^2)(\operatorname{Tr}A)+2\operatorname{Tr}(A^3)\right]$$

which we compare with

$$-\frac{t^3}{3!}\left[\int_B\int_B\int_B\begin{vmatrix}K_{11} & K_{12} & K_{13}\\ K_{21} & K_{22} & K_{23}\\ K_{31} & K_{32} & K_{33}\end{vmatrix}\,d^3\lambda\right] = -\frac{t^3}{3!}\left[\int_B\int_B\int_B K_{11}(K_{22}K_{33}-K_{23}K_{32})-K_{12}(K_{21}K_{33}-K_{23}K_{31})+K_{13}(K_{21}K_{32}-K_{22}K_{31})\right.$$

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Recall

$H(B,t) = \mathbb{E}\left(\prod_{j=1}^{N}(1-t\chi_B(\lambda_j))\right) \overset{!}{=} \det(1-t\mathcal{K}_N^B)$, the generating function for number statistics.

$$H(B,t) = \mathbb{E}\left(\prod_{j=1}^{N}(1-t\chi_B(\lambda_j))\right) = 1 - t\int_B K_N(\lambda_1,\lambda_1)\,d\lambda_1 + \cdots + \frac{(-1)^j}{j!}t^j \overbrace{\int\cdots\int}^{j} \det[K_N(\lambda_m,\lambda_n)_{j\times j}\,d^j\lambda + \frac{(-1)^N t^N}{N!}\int\cdots\int$$

Observations

We have that $\mathcal{K}_N^B(f) = \int_B K_N(x,y)f(y)\,dy$ where $K_N(xy) = e^{-\frac{x^2+y^2}{4}}\sum_{\ell=0}^{N-1}P_\ell(x)P_\ell(y)$. We know that $\mathcal{K}_N^B$ is bounded and, taking $B=\mathbb{R}$, we see that (1) $||\mathcal{K}_N^B|| < 1$ since $\mathcal{K}_N^{\mathbb{R}}$ is the projection onto the first N orthonormal functions.
Homework: prove this inequality (1).
(2) It follows that $(1-t\mathcal{K}_N)$ is invertible for small t (namely for $t=1$ for $N<\infty$). (3) It is also true that $\mathcal{K}_N^B$ is a finite-rank operator.
Then we may pick an $L^2(B)$ basis such that $1-t\mathcal{K}_N$ is realized as a matrix

$$1 - t\mathcal{K}_N = \begin{bmatrix} I - t\mathfrak{K}_N & 0 \\ 0 & I \end{bmatrix}$$

For example, we can orthonormalize $\{e^{-\frac{x^2}{4}}x^j\}_{j=0}^{N-1}$ and take the first N.
Homework: do this. $\int_B ?\, q_j\, ?$.
Now $\det(1-t\mathcal{K}_N) = e^{\mathrm{Tr}\log(1-t\mathcal{K}_N)}$ where both trace and log are basis agnostic. Then for $\mathcal{K}_N$ of finite rank,

$$\mathrm{Tr}\,\mathcal{K}_N = \sum_{j=1}^{N}(\mathfrak{K}_N)_{jj} = \int K_N(x,x)\,dx$$

It follows that

$$\det(1-t\mathcal{K}_N) = \exp\left(-\mathrm{Tr}\sum_{j=1}^{\infty}\frac{t^j(\mathcal{K}_N)^j}{j}\right) = \exp\left(-\sum_{j=1}^{\infty}\frac{t^j}{j}\int_B\cdots\int_B K_N(\lambda_1,\lambda_2)\cdots K_N(\lambda_j,\lambda_1)\,d^j\lambda\right)$$

Coefficients

So

$$\begin{aligned} 1 - t\int_B K_N(\lambda_1,\lambda_1)\,d\lambda_1 + \cdots + \frac{(-1)^j}{j!}t^j \overbrace{\int\cdots\int}^{j} \det[K_N(\lambda_m,\lambda_n)_{j\times j}\,d^j\lambda + \frac{(-1)^N t^N}{N!}\int\cdots\int \\ = \exp\left(-\sum_{j=1}^{\infty}\frac{t^j}{j}\int_B\cdots\int_B K_N(\lambda_1,\lambda_2)\cdots K_N(\lambda_j,\lambda_1)\,d^j\lambda\right) \end{aligned}$$

Where on the left-hand side we have coefficients

$$\frac{(-1)^n t^n}{n!}\overbrace{\int_B\cdots\int_B}^{n}\det[K_N(\lambda,\lambda)]_{n\times n}\,d^n\lambda$$

and on the right-hand side

$$1-\left(-\operatorname{Tr}\sum_{j=1}^{\infty}\frac{t^j(\mathcal{K}_N)^j}{j}\right)+\frac{1}{2!}(\quad)^2-\frac{1}{3!}(\quad)^3+\cdots+\frac{(-1)^n}{n!}(\quad)^n$$

so

$$-\frac{1}{n}\operatorname{Tr}(\mathcal{K}_N^n)+\frac{1}{2!}\left[\frac{\operatorname{Tr}(\mathcal{K}_N)\operatorname{Tr}(\mathcal{K}_N^{n-1})}{1\cdot(n-1)}+\frac{\operatorname{Tr}(\mathcal{K}_N^2)\operatorname{Tr}(\mathcal{K}_N^{n-2})}{2\cdot(n-2)}+\cdots+\frac{\operatorname{Tr}(\mathcal{K}_N^{n-1})\operatorname{Tr}(\mathcal{K}_N)}{(n-1)\cdot 1}\right]-\cdots$$

Note that the i-th power is constructed by the integer partitions $\mu\vdash n$ for $|\mu|=\ell$. Then the n-th term is

$$t^n\sum_{\mu\vdash n}\frac{(-1)^\ell}{\ell!}\frac{\prod_{k=1}^{\ell}(\operatorname{Tr}(\mathcal{K}_N^{\mu_k}))}{\prod_{k=1}^{\ell}\mu_k}\tilde{C}_n(\mu)$$

where $\tilde{C}_n(\mu)$ is the number of unordered partitions associated to μ. Returning to the left-hand side,

$$\int\cdots\int\sum_{\sigma\in S_n}\operatorname{sign}(\sigma)K_N(\lambda_1,\lambda_{\sigma(1)})K_N(\lambda_2,\lambda_{\sigma(2)})\cdots K_N(\lambda_n,\lambda_{\sigma(n)})\,d^n\lambda$$

When $\sigma=\mathrm{id}$,

$$\int\cdots\int K_N(\lambda_1,\lambda_1)\cdots K_N(\lambda_n,\lambda_n)\,d^n\lambda$$

We can consider the integer partition μ representing the cycle structure of σ. Then if $\tilde{\sigma}$ are permutations with cycles of lengths associated to μ,

$$\sum_{\mu\vdash n}\sum_{\tilde{\sigma}}\operatorname{sign}(\tilde{\sigma})\prod_{k=1}^{\ell}\operatorname{Tr}(\mathcal{K}_N^{\mu_k})=\sum_{\mu\vdash n}\prod_{k=1}^{\ell}\left(\operatorname{Tr}(\mathcal{K}_N^{\mu_k})\right)\sum_{\tilde{\sigma}}\operatorname{sign}(\tilde{\sigma})=\sum_{\mu\vdash n}\prod_{k=1}^{\ell}\left(\operatorname{Tr}(\mathcal{K}_N^{\mu_k})\right)C_n(\mu)\operatorname{sign}(\mu)$$

since $\operatorname{sign}(\tilde{\sigma})=\operatorname{sign}(\tilde{\tilde{\sigma}})$ if their cycle lengths agree. Then

$$\frac{(-1)^n}{n!}\operatorname{sign}(\mu)C_n(\mu)=\frac{(-1)^\ell}{\ell!}\frac{\tilde{C}_n(\mu)}{\prod_{k=1}^{\ell}\mu_k}$$

Returning to the Microscopic Regime

$$\mathbb{P}(\text{no evaluations in }(a,b))=\det(1-\mathcal{K}_N)$$

on $L^2(a,b)$. Suppose $a=\sqrt{N}\alpha$ and $b=\sqrt{N}\left(\alpha+\frac{S}{N}\right)$.
Then $K_N(x,y): x=\alpha\sqrt{N}+\frac{\xi}{\sqrt{N}}, y=\alpha\sqrt{N}+\frac{\eta}{\sqrt{N}}$ for ξ and η in a range between 0 and S. Write

$$\mathcal{K}_N(f)=\int_a^b K_N(x,y)f(y)\,dy=\int_0^S\frac{1}{\sqrt{N}}K_N\left(x=\alpha\sqrt{N}+\frac{\xi}{\sqrt{N}},y=\alpha\sqrt{N}+\frac{\eta}{\sqrt{N}}\right)f\left(\alpha\sqrt{N}+\frac{\eta}{\sqrt{N}}\right)d\eta$$

so that we may define a new kernel

$$\tilde{K}(\xi,\eta) = \frac{1}{\sqrt{N}} K\left(x = \alpha\sqrt{N} + \frac{\xi}{\sqrt{N}}, y = \alpha\sqrt{N} + \frac{\eta}{\sqrt{N}}\right)$$

which acts on functions in $(0, S)$. We can show that $\mathbb{P}(\text{no evaluations in } (a,b)) \stackrel{?}{=} \det(1-\tilde{\mathcal{K}}_N)$. We may write

$$\frac{1}{\sqrt{N}} K\left(x = \alpha\sqrt{N} + \frac{\xi}{\sqrt{N}}, y = \alpha\sqrt{N} + \frac{\eta}{\sqrt{N}}\right) = \square \frac{\sin(\pi(\xi-\eta))}{\pi(\xi-\eta)} + O\left(\frac{1}{n}\right)$$

($\square$ some appropriate constant) and can prove that $\det(1-\tilde{\mathcal{K}}_N) = \det[1-\mathcal{S}]_{L^2(0,S)}$ where $\mathcal{S}$ has the rewritten kernel.