Advanced Analysis

September 25, 2025

Suppose we have some function of the form $-\Delta + q \in \mathbb{L}(H)$ satisfying $R_A(\lambda)(A-\lambda I)^{-1}$ bounded on $\mathrm{Im}(\lambda) > 0$ and not surjective for $\mathrm{Im}(\lambda) = 0$.

IMAGE 1

Waves: solutions to $\partial_{tt}u + Au = 0$ on $\mathbb{R}^n$.
Mathematical Tools

- Spectral theory of unbounded operators
- Complex analysis
- Functional analysis
- Microlocal analysis
- Semiclassical analysis

Classical Resonances in ODEs

IMAGE 2

A harmonic oscillator assuming no friction.
We have an acceleration force, $m\ddot{x}(t) = -kx(t)$ which gives $\ddot{x} + \omega_0^2 x = 0$ with $w_0 = \sqrt{\frac{k}{m}}$ and has solution $x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$.
With forcing, i.e. $m\ddot{x}(t) = -kx(t) + A\sin(\omega t)$, we have $\ddot{x} + \omega_0^2 A' \sin(\omega t)$.
If $|\omega| \neq |\omega_0|$, then $x(t) \sim \mathrm{trig}\left(\left(\frac{\omega-\omega_0}{2}\right)t\right)\left(\left(\frac{\omega-\omega_0}{2}\right)t\right)$ the low and high frequencies respectively.

IMAGE 3

Beats (non-amplified)
If isntead $|\omega| = |\omega_0|$, then $x(t) \propto \mathrm{trig}(\omega t)t$.

IMAGE 4

In general, $\dot{x} + Ax = 0$ for $x \in \mathbb{R}^n$, $x(t) = \exp(-tA) + x(0)$.
In the case where A is skew-adjoint, $i.e.$ $\mathrm{sp}(A) \subseteq i\mathbb{R}$, $(x, Ax) = 0\ \forall x \in \mathbb{R}^n$, then

$$\frac{d}{dt}(x,x) = (\dot{x},x) + (x,\dot{x}) = (-Ax,x) - (x,Ax) = 0$$

Which implies that $||x(t)||$ is constant and the dynamics are norm perserving.
To generate resonant solutions, if (iw, v) is an eigenpair of A ($\omega \in \mathbb{R}$), consider $\dot{x} + Ax = e^{-i\omega t}v$. As an ansatz, we look for a solution of the form $x(t) = a(t)v$ and the equation becomes $(\dot{a}(t) + i\omega a)v = e^{-i\omega t}v$. Then

$$e^{-i\omega t}\frac{d}{dt}(e^{i\omega t}a) = e^{-i\omega t}$$
$$\frac{d}{dt}(e^{i\omega t}a) = 1$$
$$a(t) = te^{-i\omega t}.$$

Resonances in PDEs

Consider one-dimensional waves on $[0,L]$, $L>0$.

$$\begin{cases} \partial_{tt}u + \partial_{xx}u = 0 & \\ u|_{t=0} = f & x\in[0,L] \\ \partial_t u|_{t=0} = g & x\in[0,L] \\ u(0,t) = u(L,t) = 0 & \forall\, t\geq 0 \end{cases}$$

We want to think about this as $\partial_{tt}u = Au = 0$ where A is the Dirichlet Laplacian $Au = -\partial_{xx}u$ with Dirichlet boundary conditions. We then want to find the spectral decomposition of A, $Au - \lambda u = 0 = -\partial_x^2 u - \lambda u$.

$$\begin{aligned} \lambda = 0. &\quad u(x) = A + Bx \implies A = B = 0 \\ \lambda = -p^2. &\quad u(x) = Ae^{px} + be^{-px} \implies A = B = 0 \\ \lambda = p^2. &\quad u(x) = A\cos(px) + B\sin(px) \implies 0 = u(0) = A \quad 0 = u(L) = B\sin(pl) \implies p = k\pi,\ k\in\mathbb{N} \end{aligned}$$

Therefore there are infinitely many eigenpairs $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $\phi_n(x) = \sin\left(\frac{k\pi x}{L}\right)$.

IMAGE 5

The family $\{\phi_n, n\in\mathbb{N}\}$ is dense in $L^2([0,L])$ where the unbounded operator $(-\partial_x^2)$ with Dirichlet boundary conditions is self-adjoint.

Other Prototypes

(of unbounded self-adjoint operators with discrete spectrum)

- Laplace-Beltrami operators on compact manifolds without boundary.

IMAGE 6

- On compact domains with boundary there is the Laplacian with Dirichlet boundary conditions.

The (Quantum) Harmonic Oscillator

$H = -\frac{d^2}{dx^2} + x^2$ on $\mathbb{R}$, on $L^2(\mathbb{R})$ with $(f,g) = \int_{\mathbb{R}} f(x)\overline{g(x)}\,dx$.
H acts on the Schwarz space $\mathcal{S}(\mathbb{R}) := \left\{ f\in C^\infty(\mathbb{R}),\ \forall k,\ell\geq 0,\ \sup_{x\in\mathbb{R}} \left| x^k \left(\frac{d}{dx}\right)^\ell f(x)\right| < \infty \right\}$.

- The action of $H : \mathcal{S}(\mathbb{R})$ is continuous.
- H is L^2-symmetric: $\int_{\mathbb{R}} -f''\overline{g} + x^2 f\overline{g}\,dx = (Hf,g) = (f,Hg) = \int_{\mathbb{R}} -\overline{g}''f + x^2 f\overline{g}\,dx$ (integrating by parts).

We seek eigenvalues $Hu = \lambda u$. If (u,λ) and (v,μ) are eigenpairs, then

$$0 = (Hu,v) - (u,Hv) = (\lambda u, v) - (u,\mu v) = (\lambda - \overline{\mu})(u,v)$$

Where if the difference is nonzero then $(u, v) = 0$.
We can write $H = L^+L^- + I$ where $L^+ = -\frac{d}{dx} + x$ and $L^- = \frac{d}{dx} + x$ and also $[H, L^+] = 2L^+$ and $[H, L^-] = -2L^-$.
Note that H is a non-negative operators

$$(Hf, f) = \int_{\mathbb{R}} ((f')^2 + x^2 f^2)\, dx > 0$$

for $f \neq 0$ and $f \in \mathcal{S}(\mathbb{R})$. Thus $\mathrm{sp}(H) \subseteq (0, \infty)$. If $Hv = \lambda v$, then $H(L^+ v) = [H, L^+]v + L^+(Hv) = (\lambda + 2)L^+ v$. Similarly $H(L^- v) = (\lambda - 2)L^- v$.
Now we want to solve $L^- \phi_0 = 0$. $\frac{d}{dx}\phi_0 + x\phi_0 = 0$ tells us that $\phi_0(x) = \frac{1}{\sqrt{\pi}} e^{-x^2/2}$ (L^2-normalized). Therefore $H\phi_0 = \phi_0$ and the we have an eigenvalues of one. So we may construct $\phi_n = \frac{(L^+)^n \phi_0}{||(L^+)^n \phi_0||}$ which gives an eigenvector of H with eigenvalues $2n + 1$. Note that $||(L^+)^n \phi_0|| = \sqrt{2^n n!}$.
Fact: $\phi_n = p_n(x) e^{-x^2/2}$ where p_n is the Hermite polynomial of degree n.

$$\delta_{nq} = (\phi_n, \phi_q) = \int_{\mathbb{R}} p_n(x) p_q(x) e^{-x^2}\, dx$$

Theorem

$\{\phi_n\}_{n \geq 0}$ is dense in $L^2(\mathbb{R})$ (if $\int_{\mathbb{R}} g\phi_n\, dx = 0$ for all n, then $g = 0$).

Proof (Sketch)

For $g \in L^2$, $\xi \in \mathbb{R}$, $F_g(\xi) = \int_{\mathbb{R}} e^{ix\xi} g(x)\phi_0(x)\, dx = \widehat{g\phi_0}(\xi)$. We observe that

- F_g is real-analytic in ξ.
- $F_g^{(k)}(0) = \int_{\mathbb{R}} (-ix)^k g(x)\phi_0(x)\, dx = 0$ by assumption.

So we have a real-analytic function where all derivatives vanish at a point. So $F_g \equiv 0$, $g\phi_0 = 0$, and $g = 0$.