

# Advanced Analysis

September 25, 2025

Suppose we have some function of the form  $-\Delta + q \in \mathbb{L}(H)$  satisfying  $R_A(\lambda)(A - \lambda I)^{-1}$  bounded on  $\text{Im}(\lambda) > 0$  and not surjective for  $\text{Im}(\lambda) = 0$ .

IMAGE 1

Waves: solutions to  $\partial_{tt}u + Au = 0$  on  $\mathbb{R}^n$ .

Mathematical Tools

- Spectral theory of unbounded operators
- Complex analysis
- Functional analysis
- Microlocal analysis
- Semiclassical analysis

## Classical Resonances in ODEs

IMAGE 2

A harmonic oscillator assuming no friction.

We have an acceleration force,  $m\ddot{x}(t) = -kx(t)$  which gives  $\ddot{x} + \omega_0^2 x = 0$  with  $\omega_0 = \sqrt{\frac{k}{m}}$  and has solution  $x(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t)$ .

With forcing, i.e.  $m\ddot{x}(t) = -kx(t) + A \sin(\omega t)$ , we have  $\ddot{x} + \omega_0^2 x = A' \sin(\omega t)$ .

If  $|\omega| \neq |\omega_0|$ , then  $x(t) \sim \text{trig}\left(\left(\frac{\omega - \omega_0}{2}\right)t\right) \left(\left(\frac{\omega + \omega_0}{2}\right)t\right)$  the low and high frequencies respectively.

IMAGE 3

Beats (non-amplified)

If instead  $|\omega| = |\omega_0|$ , then  $x(t) \propto \text{trig}(\omega t)t$ .

IMAGE 4

In general,  $\dot{x} + Ax = 0$  for  $x \in \mathbb{R}^n$ ,  $x(t) = \exp(-tA) + x(0)$ .

In the case where  $A$  is skew-adjoint, i.e.  $\text{sp}(A) \subseteq i\mathbb{R}$ ,  $(x, Ax) = 0 \forall x \in \mathbb{R}^n$ , then

$$\frac{d}{dt}(x, x) = (\dot{x}, x) + (x, \dot{x}) = (-Ax, x) - (x, Ax) = 0$$

Which implies that  $\|x(t)\|$  is constant and the dynamics are norm perserving.

To generate resonant solutions, if  $(i\omega, v)$  is an eigenpair of  $A$  ( $\omega \in \mathbb{R}$ ), consider  $\dot{x} + Ax = e^{-i\omega t}v$ . As an ansatz, we look for a solution of the form  $x(t) = a(t)v$  and the equation becomes  $(\dot{a}(t) + i\omega a)v = e^{-i\omega t}v$ . Then

$$\begin{aligned} e^{-i\omega t} \frac{d}{dt}(e^{i\omega t} a) &= e^{-i\omega t} \\ \frac{d}{dt}(e^{i\omega t} a) &= 1 \\ a(t) &= te^{-i\omega t}. \end{aligned}$$

## Resonances in PDEs

Consider one-dimensional waves on  $[0, L]$ ,  $L > 0$ .

$$\begin{cases} \partial_{tt}u + \partial_{xx}u = 0 \\ u|_{t=0} = f & x \in [0, L] \\ \partial_t u|_{t=0} = g & x \in [0, L] \\ u(0, t) = u(L, t) = 0 & \forall t \geq 0 \end{cases}$$

We want to think about this as  $\partial_{tt}u = Au = 0$  where  $A$  is the Dirichlet Laplacian  $Au = -\partial_{xx}u$  with Dirichlet boundary conditions. We then want to find the spectral decomposition of  $A$ ,  $Au - \lambda u = 0 = -\partial_x^2 u - \lambda u$ .

$$\begin{aligned} \lambda = 0. \quad u(x) = A + Bx &\implies A = B = 0 \\ \lambda = -p^2. \quad u(x) = Ae^{px} + be^{-px} &\implies A = B = 0 \\ \lambda = p^2. \quad u(x) = A\cos(px) + B\sin(px) &\implies 0 = u(0) = A \quad 0 = u(L) = B\sin(pl) \implies p = k\pi, k \in \mathbb{N} \end{aligned}$$

Therefore there are infinitely many eigenpairs  $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ ,  $\phi_n(x) = \sin\left(\frac{k\pi x}{L}\right)$ .

IMAGE 5

The family  $\{\phi_n, n \in \mathbb{N}\}$  is dense in  $L^2([0, L])$  where the unbounded operator  $(-\partial_x^2)$  with Dirichlet boundary conditions is self-adjoint.

## Other Prototypes

(of unbounded self-adjoint operators with discrete spectrum)

- Laplace-Beltrami operators on compact manifolds without boundary.

IMAGE 6

- On compact domains with boundary there is the Laplacian with Dirichlet boundary conditions.

## The (Quantum) Harmonic Oscillator

$H = -\frac{d^2}{dx^2} + x^2$  on  $\mathbb{R}$ , on  $L^2(\mathbb{R})$  with  $(f, g) = \int_{\mathbb{R}} f(x) \overline{g(x)} dx$ .

$H$  acts on the Schwarz space  $\mathcal{S}(\mathbb{R}) := \left\{ f \in C^\infty(\mathbb{R}), \forall k, \ell \geq 0, \sup_{x \in \mathbb{R}} \left| x^k \left( \frac{d}{dx} \right)^\ell f(x) \right| < \infty \right\}$ .

- The action of  $H : \mathcal{S}(\mathbb{R})$  is continuous.
- $H$  is  $L^2$ -symmetric:  $\int_{\mathbb{R}} -f'' \overline{g} + x^2 f \overline{g} dx = (Hf, g) = (f, Hg) = \int_{\mathbb{R}} -\overline{g}'' f + x^2 f \overline{g} dx$  (integrating by parts).

We seek eigenvalues  $Hu = \lambda u$ . If  $(u, \lambda)$  and  $(v, \mu)$  are eigenpairs, then

$$0 = (Hu, v) - (u, Hv) = (\lambda u, v) - (u, \mu v) = (\lambda - \bar{\mu})(u, v)$$

Where if the difference is nonzero then  $(u, v) = 0$ .

We can write  $H = L^+ L^- + I$  where  $L^+ = -\frac{d}{dx} + x$  and  $L^- = \frac{d}{dx} + x$  and also  $[H, L^+] = 2L^+$  and  $[H, L^-] = -2L^-$ .

Note that  $H$  is a non-negative operators

$$(Hf, f) = \int_{\mathbb{R}} ((f')^2 + x^2 f^2) dx > 0$$

for  $f \neq 0$  and  $f \in \mathcal{S}(\mathbb{R})$ . Thus  $\text{sp}(H) \subseteq (0, \infty)$ . If  $Hv = \lambda v$ , then  $H(L^+ v) = [H, L^+]v + L^+(Hv) = (\lambda + 2)L^+ v$ . Similarly  $H(L^- v) = (\lambda - 2)L^- v$ .

Now we want to solve  $L^- \phi_0 = 0$ .  $\frac{d}{dx} \phi_0 + x \phi_0 = 0$  tells us that  $\phi_0(x) = \frac{1}{\sqrt{\pi}} e^{-x^2/2}$  ( $L^2$ -normalized). Therefore  $H\phi_0 = \phi_0$  and the we have an eigenvalues of one. So we may construct  $\phi_n = \frac{(L^+)^n \phi_0}{|| (L^+)^n \phi_0 ||}$  which gives an eigenvector of  $H$  with eigenvalues  $2n + 1$ . Note that  $|| (L^+)^n \phi_0 || = \sqrt{2^n n!}$ .

Fact:  $\phi_n = p_n(x) e^{-x^2/2}$  where  $p_n$  is the Hermite polynomial of degree  $n$ .

$$\delta_{nq} = (\phi_n, \phi_q) = \int_{\mathbb{R}} p_n(x) p_q(x) e^{-x^2} dx$$

## Theorem

$\{\phi_n\}_{n \geq 0}$  is dense in  $L^2(\mathbb{R})$  (if  $\int_{\mathbb{R}} g \phi_n dx = 0$  for all  $n$ , then  $g = 0$ ).

## Proof (Sketch)

For  $g \in L^2$ ,  $\xi \in \mathbb{R}$ ,  $F_g(\xi) = \int_{\mathbb{R}} e^{ix\xi} g(x) \phi_0(x) dx = \widehat{g\phi_0}(\xi)$ . We observe that

- $F_g$  is real-analytic in  $\xi$ .
- $F_g^{(k)}(0) = \int_{\mathbb{R}} (-ix)^k g(x) \phi_0(x) dx = 0$  by assumption.

So we have a real-analytic function where all derivatives vanish at a point. So  $F_g \equiv 0$ ,  $g\phi_0 = 0$ , and  $g = 0$ .

## September 30, 2025

One of the overarching goals is to obtain large time asymptotics of the solution  $v(x, t)$  ( $x \in \mathbb{R}$ ,  $t > 0$ ) to

$$\begin{cases} -\partial_{tt} v - P_V v = F(x, t) & \text{on } \mathbb{R}_x \times (0, \infty)_t \\ v(x, 0) = \partial_t v(x, 0) = 0, & F \in C_C^\infty(\mathbb{R}_x \times (0, \infty)_t) \end{cases}$$

where  $P_V = D_x^2 + V(x) = -\left(\frac{\partial}{\partial x}\right)^2 + V(x)$  and  $D_x = \frac{1}{i} \frac{\partial}{\partial x}$ . The operator  $D_x$  is symmetric and self-adjoint on appropriately chosen domains. For  $f(x)$  and  $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx$ ,  $\widehat{D_x f} = \xi \hat{f}(\xi)$ .  $V \in L_{\text{comp}}^\infty(\mathbb{R})$  (i.e. compactly supported  $L^\infty$ ) is the potential. If  $f, g \in \mathcal{S}(\mathbb{R})$ , then  $(P_V f, g)_{L^2(\mathbb{R})} = (f, P_V g)_{L^2(\mathbb{R})}$ .

## IMAGE 1

Another way to look at this assuming  $v$  exists, we can consider  $u(x, \lambda) := \int_0^\infty e^{it\lambda} v(x, t) dt$  (the Fourier-Laplace transform of  $v$ ) with  $\lambda \in \mathbb{C}$ ,  $\text{Im}(\lambda) > 0$ . Write  $\lambda = \xi + ic$ ,  $c > 0$ , such that  $u(x, \xi + ic) = \int_0^\infty e^{it\xi} e^{-ct} v(x, t) dt = \mathcal{F}_{t \mapsto \xi}(t \mapsto$

$e^{-ct} v(x, t))(x, -\xi)$ . Then  $u(x, \lambda)$  solves

$$\begin{aligned} \int_0^\infty e^{it\lambda} (-\partial_{tt} v - P_V v) dt &= \int_0^\infty e^{it\lambda} F(x, t) dt = \hat{F}(x, \lambda) \\ (\lambda^2 - P_V) \underbrace{\int_0^\infty e^{it\lambda} v(x, t) dt}_{u(x, \lambda)} &= \hat{F}(x, \lambda) \end{aligned}$$

which is an entire function in  $\lambda$ .

To Do:

- Study solvability of  $(\lambda^2 - P_V)u = \hat{F}(x, \lambda)$ .
- Return to  $v$ .

For frozen  $c$ , we can get  $v(x, t)$  back by Fourier inversion.

$$\begin{aligned} e^{-ct} v(x, t) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it\xi} u(x, \xi + ic) d\xi \\ v(x, t) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it(\xi + ic)} u(x, \xi + ic) d\xi \\ v(x, t) &= \frac{1}{2\pi} \int_{\text{Im}(\lambda)=c} e^{-it\lambda} u(x, \lambda) d\lambda \end{aligned}$$

IMAGE 2

where the spectral problem is invertible.

## 1D Waves in the Time Domain

Suppose  $R > 0$  is such that  $\text{supp } V \subset [-R, R]$  and  $\text{supp } F \subset [-R, R] \times (0, \infty)$ . If  $|x| > R$ , the PDE looks like  $\partial_{tt} v - \partial_{xx} v = 0 = (\partial_t + \partial_x)(\partial_t - \partial_x)v$ . Setting  $\xi = x + t$  and  $\mu = x - t$ , then it follows that

$$\partial_\xi \partial_\mu v = 0 \implies v = F(\xi) + G(\mu) = F(x + t) + G(x - t)$$

IMAGE 3

On  $x > R$ , we can expect  $v(x, t) = F_+(x + t) + G_+(x - t)$ ; on  $x < -R$ , we expect  $v(x, t) = F_-(x + t) + G_-(x - t)$ . The terms  $G_+$  and  $F_-$  are outgoing whereas the terms  $F_+$  and  $G_-$  are incoming and, given that we assumed a source, we expect to be zero.

What does incoming/outgoing look like on the spectral side?  $(\lambda^2 - P_V)u = \hat{F}(x, \lambda)$  supported in  $|x| \leq R$ . For  $|x| > R$ ,  $(\lambda^2 + \partial_x^2)u = 0$  leads to  $u = Ae^{ix\lambda} + Be^{-ix\lambda}$ . For  $x > R$ ,  $u(x) = a_+ e^{i\lambda|x|} + b_+ e^{-i\lambda|x|}$  for  $x < -R$ ,  $u(x) = a_- e^{i\lambda|x|} + b_- e^{-i\lambda|x|}$ .  $u$  is outgoing if and only if  $b_\pm = 0$  and incoming if and only if  $a_\pm = 0$ .

$P_V$  is an unbounded, symmetric operator on a Hilbert space. For  $z \in \mathbb{C}$ ,  $\text{sp}(P_V)$  is the set on the complement of which  $(P_V - Z)$  is boundedly invertible. That is,  $\forall f, \exists ! u$  such that  $(P_V - z)u = f$  and  $\|u\| \lesssim \|f\|$ .

## Waves in the Time Domain [Evans, §2.4]

Goal: if  $v$  solves

$$\begin{aligned}\partial_{tt}v - \partial_{xx}v &= f(x, t) \quad x \in \mathbb{R}, \quad t > 0, \quad f \in C_C^\infty(\mathbb{R} \times (0, \infty)) \\ v(x, 0) &= \partial_t v(x, 0) = 0 \quad x \in \mathbb{R}\end{aligned}$$

then  $v(x, t) = \frac{1}{2} \int_0^t \int_{x-s}^{x+s} f(y, t-s) dy ds$ . We look at

$$\begin{cases} \partial_{tt}v - \partial_{xx}v = 0 \rightsquigarrow v(x, t) = F(x+t) + G(x-t) \\ v(x, 0) = g(x), \quad \partial_t v(x, 0) = h(x) \end{cases}$$

Initial conditions gives us

$$\begin{cases} F(x) + G(x) = g(x) \\ F'(x) - G'(x) = h(x) \end{cases} \quad \begin{cases} G'(x) = \frac{1}{2}(g'(x) - h(x)) \\ F'(x) = \frac{1}{2}(g'(x) + h(x)) \end{cases}$$

So

$$\begin{aligned}F(x) &= \frac{1}{2} \left( g(x) + \int_0^x h(s) ds \right) + C_1 \\ G(x) &= \frac{1}{2} \left( g(x) - \int_0^x h(s) ds \right) + C_2 \\ v(x, t) &= \frac{1}{2} (g(x+t) + g(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} h(s) ds + C\end{aligned}$$

IMAGE 4

This has a finite speed of propagation in the sense that if we suppose  $\text{supp}(g, h) \subset [-R, R]$  then  $v(x, t) = 0$  whenever  $x > R+t$  or  $x < -R-t$ .

Now we want to go from the homogeneous problem to the inhomogeneous problem. The idea is to think about  $v(x, t) = \int_0^t v(x, t; s) ds$  where  $v(x, t; s)$  solves the homogeneous problem

$$\begin{cases} \partial_{tt}v(\cdot, \cdot; s) - \partial_{xx}v(\cdot, \cdot; s) = 0 \\ v(\cdot, s; s) = 0, \quad \partial_t v(\cdot, s; s) = f(x, s) \end{cases}$$

Then

$$\partial_{tt}v - \partial_{xx}v = 0 \iff \partial_t \begin{pmatrix} v \\ \partial_t v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \partial_{xx} & 0 \end{pmatrix} \begin{pmatrix} v \\ \partial_t v \end{pmatrix} \quad \text{and} \quad \begin{bmatrix} v \\ \partial_t v \end{bmatrix}_{t=s} = \begin{bmatrix} * \\ * \end{bmatrix}$$

So  $v(x, t; s) = \frac{1}{2} \int_{x-(t-s)}^{x+(t-s)} f(y, s) dy$  and  $v(x, t) = \frac{1}{2} \int_0^t \int_{x-s}^{x+s} f(y, t-s) dy ds$  follows.

Going back to the original PDE,  $(-\partial_{tt} - P_V)v = F$  is equivalent to  $(\partial_{tt} - \partial_{xx})v = -(Vv + F)$  which leads to the conclusion that  $v(x, t) = -\frac{1}{2} \int_0^t \int_{x-s}^{x+s} (Vv + F)(y, t-s) dy ds$ . For  $|x| > R$ ,  $v$  is outgoing.

IMAGE 5

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Take some complex vector space and consider the Hilbert space  $(\mathcal{H}, (\cdot, \cdot))$  with  $(\cdot, \cdot)$  satisfying

$$\begin{cases} (\lambda f, g) = \lambda(f, g) \\ (f, \lambda g) = \overline{\lambda}(f, g) \\ (f, g) = \overline{(g, f)} \\ f \mapsto (f, g) =: \|f\|^2 \text{ a norm} \\ (\mathcal{H}, \|\cdot\|) \text{ complete with respect to the norm} \end{cases}$$

• Examples

- $(\mathbb{C}^n, (a, b) = \sum_{j=1}^n a_j \overline{b_j})$ ,  $a = (a_1, \dots, a_n)$ ,  $b = (b_1, \dots, b_n)$
- $L^2(x, \mu)$  (e.g.  $[0, 1]$  and the Lebesgue measure),  $(f, g) = \int_X f \overline{g} d\mu$ .

Bounded Operators:  $T : \mathcal{H} \rightarrow \mathcal{H}$  bounded if and only if  $\overbrace{\sup_{\|x\|=1} \|Tx\|}^{\|T\|} < \infty$  satisfying

$$\begin{cases} \mathcal{B}(\mathcal{H}) \text{ the space of bounded operators on } \mathcal{H} \text{ (a complex vector space)} \\ \|\cdot\| \text{ is a norm on } \mathcal{B}(\mathcal{H}), \text{ making it complete} \\ \text{There is a multiplication, } \mathcal{B}(\mathcal{H}) \ni A, B \mapsto AB \text{ and } \|AB\| \leq \|A\| \|B\| \end{cases}$$

Adjoint: if  $A \in \mathcal{B}(\mathcal{H})$ ,  $\exists! A^* \in \mathcal{B}(\mathcal{H})$  such that  $\forall f, g \in \mathcal{H}$ ,  $(Af, g) = (f, A^*g)$  where  $A$  is symmetric/self-adjoint if  $A = A^*$ . These notions are different in the world of unbounded operators.

• Example

- $\mathcal{H} = \mathbb{C}^n$ :  $T \in M_n(\mathbb{C})$  symmetric if and only if  $T$  is Hermitian.  $t_{ij} = \overline{t_{ji}}$ .
- $\mathcal{H} = L^2([0, 1])$ ,  $Tf(t) = tf(t)$ .  $(Tf, g) = \int_0^1 tf(t) \overline{g(t)} dt = \int_0^1 f(t) \overline{tg(t)} dt = (f, Tg)$ .
- $\mathcal{H} = L^2(\mathbb{R})$  with the Fourier transform.  $\|f(x)\|^2 = c \|\hat{f}(\xi)\|^2$  (Parseval's Equality).

## Finite Dimensional Spectral Theorem

If  $A \in M_n(\mathbb{C})$  is Hermitian, there exists an orthonormal basis  $(\phi_1, \dots, \phi_n)$  of  $\mathbb{C}^n$  and real eigenvalues  $\lambda_1, \dots, \lambda_n$  such that  $A\phi_j = \lambda_j\phi_j$ .

Important observation: if  $A$  is Hermitian, then  $\lambda_j$  is real for each  $j$ , and  $\overline{(A\phi_j, \phi_j)} = (\phi_j, A\phi_j) = (A\phi_j, \phi_j) = \lambda_j \|\phi_j\|^2$ .

So  $\lambda_j = \frac{(A\phi_j, \phi_j)}{\|\phi_j\|^2}$  is real. If  $\lambda_j \neq \lambda_k$ , then  $(\phi_j, \phi_k) = 0$  since  $(A\phi_j, \phi_k) - (\phi_j, A\phi_k) = (\lambda_j - \overline{\lambda_k})(\phi_j, \phi_k)$ .

Notation: Let  $u, v \in \mathbb{C}^n$ , denote  $u \otimes \overline{v}$  the operator  $(u \otimes \overline{v})w = (w, v)u$ .

With  $A$  as in the theorem, we can write  $A = \sum_{j=1}^n \lambda_j \phi_j \otimes \overline{\phi_j}$  ( $I = \sum_{j=1}^n \phi_j \otimes \overline{\phi_j}$ ). A second way of writing this is

IMAGE 1

Where  $U^* = U^{-1}$  and  $A = U\Lambda U^*$ . This allows us to construct a functional calculus for  $A$  where

$$\begin{cases} A^2 = U\Lambda U^* U\Lambda U^* = U\Lambda^2 U^* \\ A^n = U\Lambda^n U^* \\ p(A) = U \cdot p(\Lambda) \cdot U^*, \text{ } p \text{ a polynomial} \end{cases}$$

Defining  $f(A) := U \cdot f(\Lambda) \cdot U^*$ , we obtain a Banach algebra homomorphism. Then  $f \in C([-||A||, ||A||])$  is also a Banach algebra with sup norm and pointwise multiplication.

## IMAGE 2

Then we can map  $C([-||A||, ||A||]) \ni f \mapsto f(A) \in \mathcal{B}(\mathcal{H})$ . This is useful for solving ODEs.

### • Prototypes

- Heat equation:  $\partial_t u + Au = 0$ ,  $u|_{t=0} = u_0$ ,  $u(t) = e^{-tA}u_0$ .
- Schrödinger equation:  $i\partial_t u + Au = 0$ ,  $u|_{t=0} = u_0$ ,  $u(t) = e^{-itA}u_0$ .
- Wave equation:  $\partial_{tt} u + Au = 0$ ,  $u|_{t=0} = u_0$ ,  $\partial_t u|_{t=0} = u_1$ .

Write  $u(t) := \sum_{j=1}^n u_j(t)\phi_j$  with the PDE  $\sum_{j=1}^n (u_j'' + \lambda_j u_j)\phi_j = 0$ . Then  $u_j'' + \lambda_j u_j = 0$ ,  $u_j(0) = u_{j,0}$ , and  $u_j'(0) = u_{j,1}$ . Suppose  $\lambda_j > 0$  for all  $j$ . Then  $u_j(t) = u_{j,0} \cos(\sqrt{\lambda_j}t) + \frac{u_{j,1}}{\sqrt{\lambda_j}} \sin(\sqrt{\lambda_j}t)$ . So

$$u(t) = \sum_{j=1}^n \cos(\sqrt{\lambda_j}t) u_{j,0} \phi_j + \frac{1}{\sqrt{\lambda_j}} \sin(\sqrt{\lambda_j}t) u_{j,1} \phi_j$$

Therefore  $u = \cos(t\sqrt{A})u_0 + A^{-1/2} \sin(t\sqrt{A})u_1$ .

## Spectrum of a Bounded Operator

Take  $T \in \mathcal{B}(\mathcal{H})$ . We say that  $T$  is invertible (within  $\mathcal{B}(\mathcal{H})$ ) if and only if  $\exists S \in \mathcal{B}(\mathcal{H})$  such that  $TS = ST = I$ .

Counterexample: take  $\mathcal{H} = \ell^2(\mathbb{N}_0) = \{u = (u_n)_{n \geq 0} : \sum |u_n|^2 < \infty\}$  and  $Au = \left(\frac{1}{n}u_n\right)_{n \geq 0}$ . Then the proxy for  $A^{-1}u = (nu_n)_{n \geq 0}$  is not bounded.

Given  $T \in \mathcal{B}(\mathcal{H})$ , the resolvent set of  $T$  is  $\rho(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \text{ is invertible}\}$ . Invertibility is equivalent to  $\forall y \in \mathcal{H}$ ,  $\exists! x$  such that  $Tx - \lambda x = y$  with an estimate  $||x|| \leq ||y||$ .

For  $\lambda \in \rho(T)$ , denote  $R(\lambda)$  or  $R_T(\lambda) = (T - \lambda I)^{-1} \in \mathcal{B}(\mathcal{H})$  the resolvent of  $T$ . Properties of the resolvent set:

1.  $\rho(T) \neq \emptyset$  (in fact, if  $|\lambda| > ||T||$  then  $\lambda \in \rho(T)$ ).
2.  $\rho(T)$  is open.
3. the map  $\rho(T) \ni \lambda \mapsto R_T(\lambda) \in \mathcal{B}(\mathcal{H})$  is holomorphic in the sense that  $\forall \lambda_0 \in \rho(T)$ ,  $\exists R_T'(\lambda_0) \in \mathcal{B}(\mathcal{H})$  such that  $\lim_{\lambda \rightarrow \lambda_0} \left\| \frac{R_T(\lambda) - R_T(\lambda_0)}{\lambda - \lambda_0} - R_T'(\lambda_0) \right\| = 0$ .

For a., if  $|\lambda| > ||T||$ ,  $Tx - \lambda x = y \iff \left(I - \frac{T}{\lambda}\right)x = -\frac{y}{\lambda} \iff x = \sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}}y$ . Then  $R_T(\lambda) = \frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{T}{\lambda}\right)^k$  and

$$||R_T(\lambda)|| \leq \frac{1}{||\lambda||} \frac{1}{1 - ||T/\lambda||} \leq \frac{1}{||\lambda| - ||T||}$$

For b., pick  $\lambda_0 \in \rho(T)$  and find  $r > 0$  such that  $|\lambda - \lambda_0| < r \implies \lambda \in \rho(T)$ . Then  $Tx - \lambda x = y \iff (T - \lambda_0)x - (\lambda - \lambda_0)x = y \iff x - (\lambda - \lambda_0)R_T(\lambda_0)x = R_T(\lambda_0)y$  where if  $||(\lambda - \lambda_0)R_T(\lambda_0)|| < 1$  it is boundedly solvable by Neumann series.

For c.,

$$\begin{aligned}
R_T(\lambda) - R_T(\lambda_0) &= (T - \lambda I)^{-1} - (T - \lambda_0 I)^{-1} \\
(T - \lambda I)(R_T(\lambda) - R_T(\lambda_0)) &= I - (T - \lambda_0 I + (\lambda_0 - \lambda)I)(T - \lambda_0 I)^{-1} \\
(T - \lambda I)(R_T(\lambda) - R_T(\lambda_0)) &= I - I + (\lambda - \lambda_0)R_T(\lambda_0) \\
R_T(\lambda) - R_T(\lambda_0) &= (\lambda - \lambda_0)R_T(\lambda)R_T(\lambda_0)
\end{aligned}$$

So  $\frac{R_T(\lambda) - R_T(\lambda_0)}{\lambda - \lambda_0} - R_T(\lambda_0)^2 = o(\lambda - \lambda_0)$ .

Then we define the spectrum  $\sigma(T) := \mathbb{C} \setminus \rho(T)$  which is closed since  $\rho(T)$  is open.

### Lemma

If  $T \in \mathcal{B}(\mathcal{H})$  is self-adjoint, then  $\sigma(T) \subseteq [-||T||, ||T||]$ .

• Proof

First we know  $\sigma(T) \subseteq \{|\lambda| \leq ||T||\}$ . We want to show that it is real, and that if  $\lambda = a + ib$  and  $b \neq 0$  then  $T - (a + ib)I$  is invertible.

$T - (a + bi)I$  is injective.

$$\begin{aligned}
||(T - (a + ib))x||^2 &= (Tx - (a + ib)x, Tx - (a + ib)x) \\
&= ||Tx||^2 + (a^2 + b^2)||x||^2 - (Tx, (a + ib)x) - ((a + ib)x, Tx) \\
&= ||Tx||^2 + (a^2 + b^2)||x||^2 - (a - ib)(Tx, x) - (a + ib)(x, Tx) \\
&= ||Tx||^2 + a^2||x||^2 - 2a(x, Tx) + b^2||x||^2 \geq b^2||x||^2
\end{aligned}$$

since  $||Tx||^2 + a^2||x||^2 - 2a(x, Tx) \geq 0$  by Cauchy-Schwarz. Therefore  $T - (a + ib)$  is injective and, by the open mapping theorem,  $(T - (a + ib))^* = T - (a - ib)$  is surjective. Similarly for  $T - (a - ib)$ , and the norm estimate is  $||(T - (a + ib))^{-1}|| \leq \frac{1}{b}$ . Note that  $\frac{1}{b} = \frac{1}{\text{dist}(a + ib, \mathbb{R})}$ .

Note that the spectrum of  $T$  may no longer be made of eigenvalues in the non-finite case. There may exist  $\lambda$  such that  $T - \lambda I$  is not injective,  $\exists v \neq 0$   $Tv = \lambda v$ . Recall the example  $Tf(t) = tf(t)$  with  $f \in L^2((\cdot, \cdot), dt)$ .  $T$  is self-adjoint,  $||T|| \leq 1$ , and  $(Tf, f) = \int_0^1 t|f(t)|^2 dt \geq 0$ . So  $\sigma(T) \subseteq [0, 1]$ . For  $\lambda \in [0, 1]$  is  $T - \lambda I$  injective?  $Tf = \lambda f \iff tf(t) = \lambda f(t) \iff (t - \lambda)f(t) = 0$  which implies  $f \equiv 0$  in  $L^2([0, 1])$ . Is  $T - \lambda I$  surjective?  $(t - \lambda)f(t) = g(t) \iff f(t) = \frac{g(t)}{t - \lambda}$ , so  $g(t) \equiv 1 \in L^2([0, 1])$  which implies  $f(t) = \frac{1}{t - \lambda}$  is not  $L^2([0, 1])$  and  $\sigma(T) = [0, 1]$ .

## October 9, 2025

### Spectral Resolution

Take  $\mathcal{H}$  a Hilbert space, and say that  $P \in \mathcal{B}(\mathcal{H})$  is an orthogonal projection if  $P^2 = P$  and  $P^* = P$ . Then let  $\mathcal{P}(\mathcal{H}) = \{P \in \mathcal{B}(\mathcal{H}), P \text{ is an orthogonal projection}\}$ .

• Examples

$$- \phi \in \mathcal{H}, ||\phi|| = 1, P := \phi \otimes \bar{\phi}. \text{ Then } P\psi = (\psi, \phi)\phi \text{ and } P^2\psi = P(\psi, \phi)\phi = (\psi, \phi)(\phi, \phi)\phi = P\psi.$$



- $\phi_1, \dots, \phi_n$  an orthonormal family with  $P = \sum_{k=1}^n \phi_k \otimes \bar{\phi}_k$ .
- $\mathcal{H} = \ell^2(\mathbb{N})$ ,  $e_j = (0, 0, \dots, 0, 1, 0, \dots)$  and  $I = \sum_{j=1}^{\infty} e_j \otimes \bar{e}_j$ .
- $\mathcal{H} = L^2(\mathbb{R})$ . Fix  $I$  an interval with  $\chi_I$  the characteristic function for  $I$ . Then take  $Pf := \chi_I f$ .
  - \*  $PPf = \chi_I \chi_I f = \chi_I f = Pf$ .
  - \*  $\int_{\mathbb{R}} \chi_I f \bar{g} dx = \int_{\mathbb{R}} f \overline{\chi_I g} dx$ .
  - \* If  $I$  has a nonempty interior, then  $\text{Range}(P) = \{f \in L^2(\mathbb{R}), \text{supp } f \subset I\} \simeq L^2(I)$ .

## Definition: Spectral Resolution

A spectral resolution is a map  $\mathbb{R} \ni \lambda \mapsto E(\lambda) \in \mathcal{P}(\mathcal{H})$  satisfying

1.  $\forall f \in \mathcal{H}$ ,  $\|E(\lambda)f\|$  is increasing.
2.  $\exists [a, b]$  such that  $E(\lambda) = 0$  if  $\lambda < a$  and  $E(\lambda) = \text{Id}$  if  $\lambda \geq b$ .
3.  $E(\lambda)$  is right continuous. That is,  $\forall f \in \mathcal{H}$ ,  $\lambda \in \mathbb{R}$ ,

$$\lim_{\substack{\mu \rightarrow \lambda \\ \mu > \lambda}} \|E(\mu)f - E(\lambda)f\| = 0$$

Alternatively, we can require  $E(\lambda)E(\mu) = E(\min\{\mu, \lambda\})$ .

Long story short: the collection of self-adjoint bounded operators is in one-to-one correspondence with the collection of spectral resolutions.

### • Examples

- $A \in M_n(\mathbb{C})$ ,  $A^* = A$ , with eigencouples  $(\lambda_1, \phi_1), \dots, (\lambda_n, \phi_n)$  and simple spectrum  $\lambda_1 < \lambda_2 < \dots < \lambda_n$ . Define  $E(\lambda) := \sum_{j: \lambda_j \leq \lambda} \phi_j \otimes \bar{\phi}_j$ .
  - \*  $A = I$  gives  $E(\lambda) = 0$  for  $\lambda < 1$  and  $E(\lambda) = \text{Id}$  for  $\lambda \geq 1$ .
  - \*  $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$  gives  $E(\lambda) = \text{Id}_{\lambda \geq 1} e_1 \otimes e_1 + \text{Id}_{\lambda \geq 2} e_2 \otimes e_2$ .
    - If  $f = f_1 e_1 + f_2 e_2$ , then  $\|E(\lambda)f\|^2 = \text{Id}_{\lambda \geq 1}(\lambda) \|f_1\|^2 + \text{Id}_{\lambda \geq 2}(\lambda) \|f_2\|^2$ .

## Spectral Measures

A spectral resolution gives rise to spectral measures

$$f, g \in \mathcal{H} \quad \lambda \mapsto (E(\lambda)f, g) = F(\lambda) \in \mathbb{C}$$

This defines a Lebesgue-Stieljes measure

$$\mu_F : \mu_F((a, b]) = F(b) - F(a), \quad \forall a, b \in \mathbb{R}$$

We can construct this as follows:

- When  $f = g$ ,  $\lambda \mapsto (E(\lambda)f, f) = (E(\lambda)^2 f, f) = \|E(\lambda)f\|^2$  (increasing).
- When  $f \neq g$ ,

$$(E(\lambda)f, g) = (E(\lambda)f, E(\lambda)g) = \frac{1}{4} \left( \|E(\lambda)(f+g)\|^2 - \|E(\lambda)(f-g)\|^2 - i\|E(\lambda)(f+ig)\|^2 + i\|E(\lambda)(f-ig)\|^2 \right)$$

$\{E(\lambda)\}_{\lambda \in \mathbb{R}}$  defines a projection-valued measure  $E_\Omega$ ,  $\Omega \subset \mathbb{R}$  a Borel set. Start with  $E_{(a,b]} = E(b) - E(a)$ . We would like for  $E_\Omega$  to satisfy  $E_{\Omega_1} E_{\Omega_2} = E_{\Omega_1 \cap \Omega_2}$ ,  $E_\emptyset = 0$ ,  $E_{\mathbb{R}} = \text{Id}$ .

### Theorem: Spectral Theorem

For  $A \in \mathcal{B}(\mathcal{H})$  self-adjoint, there exist  $a, b \in \mathbb{R}$  and an  $A$ -dependent spectral resolution  $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$  such that

$$A = \int_{a-}^b \lambda dE(\lambda)$$

in the sense that  $(Af, g) = \int_{a-}^b \lambda d(E(\lambda)f, g)$  for all  $f, g \in \mathcal{H}$ . This is amenable to creating a functional calculus

$C([-||A||, ||A||]) \rightarrow$  bounded self-adjoint operators that commute with  $A$

$$h \mapsto h(A) := \int_{a-}^b h(\lambda) dE(\lambda)$$

The idea is that  $E(\lambda) = \chi_{(-\infty, \lambda]}(A)$ . Once  $E$  is constructed, this leads to  $E_\Omega$  for  $\Omega$  Borel. We say that a measure  $\mu$  is supported in  $G$  (Borel) if for every  $\Omega$  Borel,  $\mu(\Omega) = \mu(\Omega \cap G)$ . Then  $\text{supp } E \subset \sigma(A)$ .

### Functional Calculus

We want to make sense of  $h(A)$  for  $h$  in a large enough class. If  $p$  is a polynomial, we can make sense of  $p(A) = \sum_{k=0}^n a_k A^k$  which is self-adjoint and bounded.

For  $h \in C([-||A||, ||A||])$ ,  $h$  is uniformly approximated by polynomials. We want to show that  $p_n$  is uniformly Cauchy which implies that  $p_n(A)$  converges to some  $h(A)$ .

For  $h = \chi_{(-\infty, \lambda]}$ , we proceed by approximation by tent functions.

IMAGE 1

### Definition: Positive Operator

If  $S$  is a self-adjoint, bounded operator on  $\mathcal{H}$ , we say that  $S$  is positive ( $S \geq 0$ ) if  $(Sf, f) \geq 0$ ,  $\forall f \in \mathcal{H}$ . For  $S_1, S_2$  self-adjoint and bounded, we say that  $S_1 \geq S_2$  if and only if  $S_1 - S_2 \geq 0$ .

For  $T \in \mathcal{B}(\mathcal{H})$ , self-adjoint, set  $a := \inf_{||f||=1} (Tf, f)$  and  $b := \sup_{||f||=1} (Tf, f)$ . Then  $a\text{Id} \leq T \leq b\text{Id}$ .

$$((T - a\text{Id})f, f) = (Tf, f) - a(f, f) = (f, f) \left( \left( T \frac{f}{||f||}, \frac{f}{||f||} \right) - a \right) \geq 0$$

We want to show that if  $p$  is a polynomial on  $[-||A||, ||A||]$ , then  $(\inf_{[-||A||, ||A||]} p) \text{Id} \leq p(A) \leq (\sup_{[-||A||, ||A||]} p) \text{Id}$ .

### Lemma

If  $T_1$  and  $T_2$  are positive and commute, then  $T_1 T_2 \geq 0$ .

## Square Root Lemma

If  $A \geq 0$  (i.e. bounded, self-adjoint, and positive), then  $\exists! B \geq 0$  such that  $B^2 = A$  and  $B$  commutes with any operator that commutes with  $A$ .

- **Proof**

Use the power series of  $z \mapsto \sqrt{1-z}$  at  $z = 0$ .

$$1 + \sum_{k=1}^{\infty} c_k z^k$$

We can find that  $c_k < 0$  for all  $k \geq 1$  and that the series converges uniformly on  $\{|z| \leq 1\}$ .

Now let  $A \geq 0$  which implies that  $0 \text{Id} \leq I - A \leq 1 \text{Id}$ . Without loss of generality, suppose  $\text{supp } \|A\| \leq 1$ . The idea is to write

$$B = \sqrt{A} = \sqrt{I - (I - A)} = I + \sum_{k=1}^{\infty} c_k (I - A)^k$$

which converges strongly because the series converges uniformly. Then  $B^2 = A$ . We see that  $B \geq 0$  using the fact that  $\text{sign}(c_k) < 0$  which implies  $\sum_{k \geq 1} c_k \geq -1$ . The proof of uniqueness can be found in the text.

## Proof of Lemma

Assuming the square root lemma, write  $T_2 = B^2$ . Then since  $[T_1, T_2] = 0$ ,  $[B, T_1] = 0$ . Then

$$(T_1 T_2 f, f) = (T_1 B^2 f, f) = (B T_1 B f, f) = (T_1 (B f), B f) \geq 0$$

## Weaker Version

Instead of  $(\inf_{[-\|A\|, \|A\|]} p) \text{Id} \leq p(A) \leq (\sup_{[-\|A\|, \|A\|]} p) \text{Id}$ , we have that if  $\min_{[-\|A\|, \|A\|]} p \geq 0$ , then  $p(A) \geq 0$ .

## Proof

If  $p \geq 0$  on  $[-\|A\|, \|A\|]$ , we can factor it as a product of positive pieces

$$p(x) = \prod_{\substack{r_j < -\|A\| \\ s_j \geq \|A\|}} \underbrace{(x - r_j)}_{\geq 0} \underbrace{(s_j - x)}_{\geq 0} ((x - a_j)^2 + b_j^2)$$

$$p(A) = \prod_{\substack{r_j < -\|A\| \\ s_j \geq \|A\|}} \underbrace{(A - r_j)}_{\geq 0} \underbrace{(s_j - A)}_{\geq 0} \underbrace{((A - a_j)^2 + b_j^2)}_{\geq 0}$$

Using the previous lemma, we have that  $P(A) \geq 0$ .

## Proof

Finally, to show that  $(\inf_{[-\|A\|, \|A\|]} p) \text{Id} \leq p(A) \leq (\sup_{[-\|A\|, \|A\|]} p) \text{Id}$ , we see that  $p - \inf p$  and  $\text{supp } f - p$  are positive polynomials and apply the weaker version to them.

## Definition

We can define  $h(A)$  for  $h \in C(-||A||, ||A||)$  by Weierstrass approximation. There exist  $p_n$  polynomials such that  $\sup_{[-||A||, ||A||]} |p_n - h| \xrightarrow{n \rightarrow \infty} 0$ . Then  $p_n$  is uniformly Cauchy, so

$$\inf(p_n - p_m) \text{Id} \leq p_n(A) - p_m(A) \leq \sup(p_n - p_m) \text{Id}$$

which implies that

$$||p_n(A) - p_{n-1}(A)|| \leq \max(\sup(p_n - p_m), -\inf(p_n - p_m)) \xrightarrow{n, m \rightarrow \infty} 0.$$

So  $p_n(A)$  is Cauchy in  $(\mathcal{B}(\mathcal{H}), || \cdot ||)$  which means it converges. We call  $h(A) = \lim_{n \rightarrow \infty} p_n(A)$ . We still want to show that  $h(A)$  is bounded and self-adjoint.

## October 14, 2025

### Spectral Theorem for Bounded Self-Adjoint Operators

If  $A \in \mathcal{B}(\mathcal{H})$  is self-adjoint such that  $a||f||^2 \leq (Af, f) \leq b||f||^2 \quad \forall f \in \mathcal{H}$ , then there exists a spectral resolution  $\{E(\lambda)_{\lambda \in \mathbb{R}}\}$  such that  $A = \int_a^b \lambda dE(\lambda)$ .

### Proof (Continued)

We would like that  $E(\lambda) = \phi^\lambda(A)$  where  $\phi^\lambda := \chi_{(-\infty, \lambda]}$ , however  $\phi^\lambda$  is not continuous so instead we can approximate it as

$$\phi_n^\lambda := \begin{cases} 1, & x \leq \lambda \\ \text{linear on } \left[ \lambda, \lambda + \frac{1}{n} \right] \\ 0, & x \geq \lambda + \frac{1}{n} \end{cases}$$

IMAGE 1

To demonstrate this, we need the following proposition.

### Proposition

If  $T_n$  is a sequence of positive operators and  $T_n \geq T_{n+1} \geq 0$ , then there exists some  $T \geq 0$  such that  $T_n f \rightarrow T f, \forall f \in \mathcal{H}$ .

### Proof

Fix  $f \in \mathcal{H}$ , and consider  $(T_n f, f)$  which is decreasing, bounded from below, and therefore converges and is Cauchy. Now as an estimate, we can say that if  $S \in \mathcal{B}(\mathcal{H})$  is self-adjoint where  $0 \leq S \leq MI$ , then  $\forall f \in \mathcal{H}$  we know that

$$||Sf||^2 \leq (Sf, f)^{1/2} M^{3/2} ||f||.$$

To see this, we look at  $t \mapsto (S(S + tI)f, (S + tI)f) \geq 0$  since  $S \geq 0$ . Then

$$(S(S + tI)f, (S + tI)f) = (S^2 f, S f) + 2t||Sf||^2 + t^2(Sf, f)$$

So  $\Delta < 0$  if and only if

$$\begin{aligned} ||Sf||^4 - (S^2 f, Sf)(Sf, f) &< 0 \\ ||Sf||^4 &\leq (Sf, f)(S^2 f, Sf) \\ &\leq (Sf, f) \underbrace{||S^2 f|| ||Sf||}_{M^3 ||f||^2} \end{aligned}$$

We apply this to  $T_n - T_m = S$  where  $T_n - T_m \leq T_0$  for all  $n \leq m$ . Then

$$||(T_n - T_m)f||^2 \leq ((T_n - T_m)f, f)^{1/2} ||f|| ||T_0||^{3/2}$$

Since  $(T_n f, f)$  is Cauchy,  $T_n f$  is Cauchy and therefore converges. It remains to check that  $T$  is linear, positive, satisfies,  $T \leq T_0$ , etc.

### Spectral Theorem Proof Continued

For each  $n$ ,  $\phi_n^\lambda(A)$  makes sense and is positive since  $\phi_n^\lambda(t) \geq 0$ . Since  $\phi_n^\lambda \geq \phi_{n+1}^\lambda$ ,  $\phi_n^\lambda(A) \geq \phi_{n+1}^\lambda(A)$ . Thus, by the preceding proposition,  $\phi^\lambda(A) := \lim_{n \rightarrow \infty} \phi_n^\lambda(A)$  exists as a bounded, self-adjoint, positive operator.

It remains to check the spectral resolution properties.

The final property to check is that  $\forall f \in \mathcal{H}$ ,  $(Af, f) = \int_a^b \lambda d(E(\lambda)f, f)$ . The idea is to approximate multiplication by  $t$  with piecewise constant functions. We fix a partition  $a = \lambda_0 \leq \dots \leq \lambda_k = b$  such that  $\sup(\lambda_{j+1} - \lambda_j) < \delta$ . Then

$$t\phi^{\lambda_k}(t) = t = \phi^{\lambda_0}(t) + \sum_{j=1}^k t(\phi^{\lambda_j}(t) - \phi^{\lambda_{j-1}}(t))$$

for all  $a \leq t \leq b$ .

IMAGE 2

Then

$$\begin{aligned} \lambda^{j-1} &\leq t \leq \lambda^j \\ \lambda^{j-1}(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) &\leq t(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \leq \lambda^j(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \\ \sum_{j=1}^k \lambda^{j-1}(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) &\leq t \sum_{j=1}^k (\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \leq \sum_{j=1}^k \lambda^j(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \\ &\leq \sum_{j=1}^k (\lambda^j - \delta)(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \end{aligned}$$

Adding  $\lambda_0 \phi^{\lambda_0}$ , we see that

$$t \leq \lambda_0 \phi^{\lambda_0} + \sum_{j=1}^k \lambda^j(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \leq t + \delta$$

IMAGE 3

When we apply this to  $A$ ,

$$A \leq \lambda_0 E(\lambda_0) + \sum_{j=1}^k \lambda^j (E(\lambda_j) - E(\lambda_{j-1})) \leq A + \delta I$$

Finally, we see that

As one refines the partition,  $\delta \rightarrow 0$  and  $(A, f, f) = \int_a^b \lambda d(E(\lambda)f, f)$ .

Then if  $\phi \in C([-||A||, ||A||])$ ,  $\phi(A) = \int_a^b \phi(\lambda) dE(\lambda)$ .

$$\left| (Af, f) - \lambda_0(E(\lambda_0)f, f) - \sum_{j=1}^k \lambda^j (E(\lambda_j)f, f) - (E(\lambda_{j-1}f, f)) \right| \leq \delta ||I||^2$$

## Functional Calculus

We observe that for  $g \in C_C^\infty(\mathbb{C})$ ,  $g(w) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{w-z} \partial_{\bar{z}} g(z) d^2 z$ . Then  $f(A) := \frac{i}{\pi} \int_{\mathbb{C}} (A-z)^{-1} \partial_{\bar{z}} \tilde{f}(z) d^2 z$  where  $\partial_{\bar{z}} \tilde{f} = O((\text{Im}(z))^n)$  for  $n \geq 3$  as  $\text{Im}(z) \rightarrow 0$ .

## Proposition

For every  $f \in \mathcal{H}$ , the Lebesgue-Stieltjesmeasure corresponding to  $F(\lambda) = (E(\lambda)f, f)$  is supported on  $\sigma(A)$ .

Since  $\sigma(A) \subseteq [a, b]$  is closed,  $[a, b] \setminus \sigma(A)$  is open (i.e.  $\bigcup_{k \in \mathbb{N}} J_k$  for open intervals  $J_k$ ). We want to show that  $F(\lambda)$  is constant on each  $J_k$  (equivalently:  $\int_{J_k} dF(\lambda) = 0$ ).

IMAGE 4

Fix  $J_k \ni x_0$ . Then  $R_A(x_0) = (A - x_0 I)^{-1}$  exists, and we can pick  $\varepsilon > 0$  such that  $\forall z \in \overline{B_\varepsilon(x_0)}$ ,  $||R_A(z)|| \leq M$ . Then for  $z \in B_\varepsilon(x_0) + \text{Im}(z) \neq 0$ ,

$$R_A(z) = (A - zI)^{-1} = \phi_z(A)$$

and  $\phi_z(t) = \frac{1}{t-z} \in C([a, b])$ . Consider

$$R_A(z)R_A(\bar{z}) = \psi_z(A) = \int \frac{1}{|\lambda - z|^2} dE(\lambda)$$

with  $\psi_z(t) = \frac{1}{|t-z|^2}$ . It follows that for all  $z \in \overline{B_\varepsilon(x_0)} \setminus \mathbb{R}$ ,

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{1}{|\lambda - z|^2} dF(\lambda) \leq \int \frac{1}{|\lambda - z|^2} dF(\lambda) = (R_A(z)R_A(\bar{z})f, f) \leq M^2 ||f||^2$$

which stays true for all  $z \in \overline{B_\varepsilon(x_0)}$ . In particular for  $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ ,

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} dx \int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{1}{|\lambda - z|^2} dF(\lambda) \leq 2\varepsilon M^2 ||f||^2$$

Since Fubini holds, we observe that  $\int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{1}{|\lambda - x|^2} dx = \infty$ , so it must be the case that  $\int_{x_0-\varepsilon}^{x_0+\varepsilon} dF(\lambda) = 0$ .

## Discrete Spectrum vs Essential Spectrum

Recall the spectral measure of  $A$ ,  $E_\Omega(A)$  for  $\Omega \subset \mathbb{R}$  Borel where  $E_{(a,b]} = E(b) - E(a)$ . We say that  $\lambda \in \sigma_d(A)$  (the discrete spectrum of  $A$ ) if there exists  $\varepsilon > 0$  such that  $\dim(\text{range}(E_{(\lambda-\varepsilon, \lambda+\varepsilon)})) < \infty$ . Likewise,  $\lambda \in \sigma_{\text{ess}}(A)$  (the essential spectrum) if  $\forall \varepsilon > 0$ ,  $\dim(\text{range}(E_{(\lambda-\varepsilon, \lambda+\varepsilon)})) = \infty$ .

As an example, take  $\mathcal{H} = \ell^2(\mathbb{N})$  with  $(Au)_n = \frac{u_n}{n}$  and  $A = \sum_{j=1}^{\infty} \frac{1}{j} e_j \otimes e_j$ .

## IMAGE 5

Discrete spectra include eigenvalues of finite multiplicity.

Essential spectra include accumulation points of eigenvalues, eigenvalues of infinite multiplicity, absolutely continuous spectrum, s.c. spectrum.

Another example if  $Af(t) = tf(t)$  on  $L^2([0, 1])$ . Then  $E(\lambda)f(t) = \chi_{(-\infty, \lambda]}f(t)$ , and  $E_{(a, b]}f(t) = \chi_{(a, b]}f(t)$ .  $\forall x_0 \in [0, 1]$ , we have that  $\text{range}(E_{(x_0 - \varepsilon, x_0 + \varepsilon)}) = L^2((x_0 - \varepsilon, x_0 + \varepsilon))$ .

## IMAGE 6

**October 16, 2025**

### Compact Operators and Analytic Fredholm Theorem

#### Definition: Spectral Radius

For  $A \in \mathcal{B}(\mathcal{H})$ , we say that the spectral radius of  $A$  is  $r(A) = \sup_{\lambda \in \sigma(A)} |\lambda| < \infty$

#### Theorem

1. if  $A \in \mathcal{B}(\mathcal{H})$ , then  $r(A) = \limsup_{n \rightarrow \infty} \|A^n\|^{1/n}$ .
2. If  $A$  is, in addition, self-adjoint, then  $r(A) = \|A\|$ .

As a non-example, consider  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ . Then  $\sigma(A) = \{0\}$ , but  $r(A) = 0 \neq \|A\|$ .

#### Proof

Recall Hadamard's Formula:  $\sum_{k=0}^{\infty} a_k z^k$  has radius of convergence  $R$  computed by  $\frac{1}{R} = \limsup_{n \rightarrow \infty} |a_n|^{1/n}$ . This holds even when  $a_k$  are members of a Banach algebra (e.g.  $\mathcal{B}(\mathcal{H})$ ).

Set  $z = \frac{1}{\lambda}$ , such that  $0 < |z| < \frac{1}{r(A)}$  and implies the existence of

$$R_A\left(\frac{1}{z}\right) = \left(A - \frac{1}{z}I\right)^{-1} = -(I - zA)^{-1} = -z \sum_{k=0}^{\infty} A^k z^k.$$

So we have that  $r(A) = \frac{1}{R} = \limsup_{k \rightarrow \infty} \|A^k\|^{1/k}$ .

Now if  $A$  is self-adjoint,  $\|A^2\| = \|A\|^2$  since  $\|A^2\| \leq \|A\|^2$  by submultiplicativity and  $\|A^2\| \geq \sup_{\|x\|=1} (x, A^2 x) = \sup_{\|x\|=1} \|Ax\|^2 = \|A\|^2$  using Cauchy-Schwarz.

By induction,  $\|A^{2^k}\|^{1/2^k} = \|A\|$  which implies that  $r(A) = \|A\|$ .

#### Another Spectral Decomposition

For  $T : \mathcal{H} \rightarrow \mathcal{H}$  bounded, we say that  $\lambda$  is in the point spectrum of  $T$  if  $T - \lambda I$  is not-injective. We say that  $\lambda$  is in the residual spectrum of  $T$  if  $T - \lambda I$  is injective but does not have dense range.

## IMAGE 1

Self-adjoint operators have no residual spectrum (RS, Thm VI.8).

## Definition: Compact Operators

$K \in \mathcal{B}(\mathcal{H})$  is compact if  $K$  maps bounded sequences to sequences with a limit point. Equivalently, if  $B_{\mathcal{H}} = \{x \in \mathcal{H} : \|x\| \leq 1\}$  then  $K$  is compact if  $K(B_{\mathcal{H}})$  has compact closure (i.e. is precompact).

$\mathcal{K}(\mathcal{H})$ , the collection of compact operators on  $\mathcal{H}$ , is a closed linear subspace of  $\mathcal{B}(\mathcal{H})$  since if  $T, S \in \mathcal{K}(\mathcal{H})$ , then  $T + S \in \mathcal{K}(\mathcal{H})$ . We have also that  $\mathcal{K}(\mathcal{H})$  is a 2-sided ideal of  $\mathcal{B}(\mathcal{H})$  since when  $T$  is compact and  $S$  is bounded,  $ST$  and  $TS$  are compact.

Examples

- For a finite-rank operator  $A : \mathcal{H} \rightarrow \mathcal{H}$ ,  $\text{range}(A) < \infty$ . The general form of  $A$  is  $A = \sum_{j=1}^n \phi_j \otimes \bar{\psi}_j$  for  $\phi_j, \psi_j \in \mathcal{H}$ .
- Strong limits of finite-rank operators.
- $(Au)_n = \frac{1}{n}u_n$  on  $\ell^2(\mathbb{N})$ . Note that  $A_n = \sum_{j=1}^n \frac{1}{j}e_j \otimes e_j$  shows that  $\|A - A_n\| \leq \frac{1}{n+1}$ .
- The inclusion  $h^1 \hookrightarrow \ell^2$  where  $h^1 = \{u \in \mathbb{C}^{\mathbb{N}} : \sum_{j=1}^{\infty} j^2 |u_j|^2 < \infty\}$ .

## Proposition

If  $\mathcal{H}$  is separable, all compact operators arise as limits of finite-rank operators.

### Proof

We want to show that for  $A \in \mathcal{K}(\mathcal{H})$ ,  $\forall \varepsilon > 0$ ,  $\exists A_\varepsilon$  finite-rank such that  $\|A - A_\varepsilon\| < \varepsilon$ .

Let  $\varepsilon > 0$  be given. Since  $A(B_{\mathcal{H}})$  is precompact, it is totally bounded. That is  $\exists y_1, \dots, y_n \in A(B_{\mathcal{H}})$  such that  $\forall x \in A(B_{\mathcal{H}})$ ,  $\min_{1 \leq j \leq n} \|x - y_j\| < \varepsilon$ .

Let  $P_\varepsilon$  be the orthogonal projection onto  $\text{span}\{y_1, \dots, y_n\}$ , and set  $A_\varepsilon = P_\varepsilon A$ . Then for  $f \in B_{\mathcal{H}}$  and  $x = Af \in A(B_{\mathcal{H}})$ ,

$$\|Af - A_\varepsilon f\| = \|x - P_\varepsilon x\| \leq \min_{1 \leq j \leq n} \|x - y_j\| < \varepsilon$$

So  $\|A - A_\varepsilon\| < \varepsilon$ .

Exercise: confirm whether this argument needs separability.

## Theorem: Analytic Fredholm Theorem

Let  $D \subset \mathbb{C}$  be open and connected. Let  $f : D \rightarrow \mathcal{B}(\mathcal{H})$  be an analytic, operator-valued function such that  $f(z) \in \mathcal{K}(\mathcal{H})$  for all  $z \in D$ . Then

1. either  $(I - f(z))^{-1}$  exists for no  $z \in D$
2. or  $(I - f(z))^{-1}$  exists for all  $z \in D \setminus S$ , where  $S$  is a discrete set (finite-rank, no accumulation points) in  $D$ .

Then  $(I - f(z))^{-1}$  is meromorphic in  $D$ , analytic in  $D \setminus S$ , residues at poles are finite-rank, and if  $z \in S$ ,  $f(z)\psi = \psi$  has a nonzero solution for  $\psi \in \mathcal{H}$ .

### Application

If  $K$  is compact, consider  $f(z) = zK$ . At  $z = 0$ ,  $I - f(z) = I$  so this is invertible which implies that the theorem holds for  $\frac{1}{z} \in \mathbb{C}$  (taking  $D = \mathbb{C} \setminus \{0\}$ ). We have  $R_K(\lambda) = -\frac{1}{\lambda} \left( I - f\left(\frac{1}{\lambda}\right) \right)^{-1}$ . Note that  $K$  is not necessarily self-adjoint. Note that  $K$  is not necessarily self-adjoint.



## Proof

We want to prove that either (a) or (b) hold locally for any  $z_0 \in D$ .

Fix  $z_0 \in D$ ,  $r > 0$  such that  $\|f(z) - f(z_0)\| < \frac{1}{2}$  for  $z \in D_r(z_0)$ . Choose  $F = \sum_{j=1}^n \phi_j \otimes \bar{\psi}_j$  a finite-rank operator such that  $\|f(z_0) - F\| < \frac{1}{2}$ . Then  $\forall z \in D_r(z_0)$ ,

$$\|f(z) - F\| \leq \|f(z) - f(z_0)\| + \|f(z_0) - F\| < 1$$

which implies that  $(I - (f(z) - F))^{-1}$  exists and is holomorphic on  $D_r(z_0)$ .

Define  $g(z) = F(I - f(z) + F)^{-1}$ , and observe that  $(I - f(z)) = (I - g(z))(I - f(z) + F) = I - f(z) + F - \overbrace{g(z)(I - f(z) + F)}^F$ .  
Write

$$g(z) = \sum_{j=1}^n \phi_j \otimes \bar{\psi}_j \cdot (I - f(z) + F)^{-1} = \sum_{j=1}^n \phi_j \otimes \bar{\psi}_j(z)$$

where  $\psi_j(z) = ((I - f(z) + F)^{-1})^* \psi_j$  is holomorphic in  $z$ . Then  $I - f(z)$  is invertible if and only if  $I - g(z)$  is invertible. We claim that this holds if and only if  $d(z) \neq 0$  for some holomorphic function  $d$ .

When is  $I - g(z)$  invertible?

Injectivity: if  $g(z)\phi = \phi$ , we expect  $\phi = \sum_{j=1}^n \beta_j \phi_j$ . So  $g(z)\phi = \phi$  if and only if

$$\sum_{j=1}^n \phi_j \left( \sum_{k=1}^n \beta_k \phi_k, \psi_j \right) = \sum_{j=1}^n \beta_j \phi_j$$

where  $\beta_j = \sum_{k=1}^n \beta_k (\phi_k, \psi_j(z))$ . If  $A_{jk}(z) := (\phi_k, \psi_j(z))$ , then this has a solution if and only if  $\det(I - A(z)) = 0$ . Call  $d := \det(I - A(z))$ . Moreover, if  $d(z) \neq 0$  then  $I - g(z)$  is invertible. We can solve  $(I - g(z))\phi = \psi$  for  $\phi$  given  $\psi \in \mathcal{H}$ . So  $\phi = \psi + g(z)\phi$  which motivates an ansatz  $\phi = \psi + \sum_{j=1}^n \beta_j \phi_j$ . Then

$$(I - g(z))\phi = (I - g(z))\psi + \sum_{j=1}^n (\beta_j - A_{jk}\beta_k)\phi_j = \psi$$

If and only if

$$\sum_{j=1}^n (\beta_j - A_{jk}\beta_k)\phi_j = \sum_{j=1}^n (\psi, \psi_j(z))\phi_j$$

which is boundedly invertible as long as  $d(z) \neq 0$ .

## October 21, 2025

### Definition: Unbounded Operator

Let  $(\mathcal{H}, (\cdot, \cdot))$  a Hilbert space. An unbounded operator is a linear map  $A : \mathcal{D}(A) \rightarrow \mathcal{H}$ ,  $\mathcal{D}(A) \subsetneq \mathcal{H}$  ( $\mathcal{D}(A)$  the domain of  $A$ ).

Hypothesis:  $\overline{\mathcal{D}(A)} = \mathcal{H}$ .

Why "unbounded?" BEcause, generally,  $\sup_{f \in \mathcal{D}(A) \setminus \{0\}} \frac{\|Af\|}{\|f\|} = \infty$ .

Examples

1.  $\mathcal{H} = L^2(\mathbb{R})$ ,  $Af(t) = tf(t)$ . Then  $A\left(\frac{1}{|t|}\right) \notin L^2(\mathbb{R})$ , and  $\mathcal{D}(A) = C_C(\mathbb{R})$ . Consider  $f_n(t)$  like

IMAGE 1

1.  $\mathcal{H} = L^2(\mathbb{R})$ ,  $Af(x) = D_x f(x) = \frac{1}{i} \frac{\partial}{\partial x} f(x)$ . Then  $\mathcal{D}(A) = \mathcal{S}(\mathbb{R})$ , and we can see unboundedness from  $f(x) = e^{-\frac{x^2}{\sigma}}$  for  $\sigma > 0$ .

$$\begin{aligned} \int f^2(x) dx &= \int e^{-\frac{x^2}{\sigma}} dx \stackrel{y=\frac{x}{\sqrt{\sigma}}}{=} \sqrt{\sigma} \int e^{-y^2} dy \\ \int (f')^2(x) dx &= \int \frac{4x^2}{\sigma^2} e^{-\frac{x^2}{\sigma}} dx \stackrel{y=\frac{x}{\sqrt{\sigma}}}{=} \frac{1}{\sqrt{\sigma}} \int 4y^2 e^{-y^2} dy \end{aligned}$$

So  $\frac{\|Af\|^2}{\|f\|^2} = \frac{C}{\sigma} \xrightarrow{\sigma \rightarrow 0} \infty$ .

### Definition: Extension of an Unbounded Operator

We say that  $B$  extends  $A$  (and write  $A \subseteq B$ ) if  $\mathcal{D}(A) \subset \mathcal{D}(B)$  and  $B|_{\mathcal{D}(A)} = A$ . We have that  $A = B$  when  $A \subset B$  and  $B \subseteq A$ .

### Definition: Closed Operator

To recover a notion of continuity, we look at closed operators.

We say that  $(A, \mathcal{D}(A))$  is closed if and only if  $\forall u_n \in \mathcal{D}(A)$  such that  $u_n \rightarrow u \in \mathcal{H}$  and  $Au_n \rightarrow v \in \mathcal{H}$ ,  $u \in \mathcal{D}(A)$  and  $v = Au$ .

Equivalently,  $A$  is closed if the graph of  $A$ ,  $\Gamma(A) = \{(u, Au) : u \in \mathcal{D}(A)\}$ , is closed in  $\mathcal{H} \times \mathcal{H}$ .

We say that  $A$  is closable if there exists some  $B$  such that  $A \subseteq B$ .

Example

- $(P, \mathcal{D}(P), \mathcal{H}) = (D_x, C_C^\infty(\mathbb{R}), L^2(\mathbb{R}))$ .
  - $P$  is not closed, because there exists  $u_n \in C_C^\infty(\mathbb{R})$  such that  $u_n \rightarrow u \in \mathcal{H}$ ,  $D_x u_n \rightarrow v \in \mathcal{H}$ , yet  $u \notin C_C^\infty(\mathbb{R})$ .
  - Take  $u \in H^1 \setminus C_C^\infty(\mathbb{R})$ ,  $u_n \in C_C^\infty(\mathbb{R})$  converging to  $u$  in  $H^1$ .
- Recall that
  - $H^1(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : f'(x) \in L^2(\mathbb{R})\}$ ,  $\|f\|_{H^1}^2 = \|f\|_{L^2(\mathbb{R})}^2 + \|f'\|_{L^2(\mathbb{R})}^2$ .
  - $C_C^\infty(\mathbb{R})$  is dense in  $H^1(\mathbb{R})$ .
  - $H^1(\mathbb{R})$  is complete.

Exercise: prove that if  $\mathcal{D}(B) := H^1(\mathbb{R})$  and  $Bf = D_x f$ , then  $B$  is closed and  $B|_{C_C^\infty(\mathbb{R})} = P$ .

### Adjoint of Unbounded Operators

In the bounded case, we have  $A \in \mathcal{B}(\mathcal{H})$ ,  $u \in \mathcal{H}$  and define  $\ell_u(v) = (u, Av) \in \mathcal{H}'$  ( $\mathcal{H}'$  the dual space). By Riesz Representation Theorem,  $\exists! w := A^* u$  such that  $\forall v \in \mathcal{H}$ ,  $(A^* u, v) = (u, A v)$ .

In the unbounded case, we need to define  $A^*$  and  $\mathcal{D}(A^*)$ .

For  $u \in \mathcal{H}$ , set  $\ell_u = (u, Av)$ ,  $v \in \mathcal{D}(A)$ . If  $\ell_u$  extends to an element of  $\mathcal{H}'$  (i.e. if we can prove an estimate  $|\ell_u(v)| \leq$

$c||v||_{\mathcal{H}}$ , then by Riesz Representation Theorem  $\exists! w \in \mathcal{H}$ , called  $A^*u$ , such that  $\forall v \in \mathcal{D}(A)$ ,  $(A^*u, v) = (u, A^*v)$ . Then

$$\mathcal{D}(A^*) = \{u \in \mathcal{H} : \begin{matrix} \mathcal{D}(A) \rightarrow \mathbb{C} \\ v \mapsto (u, Av) \end{matrix} \text{ extends to an element of } H^1\}.$$

Example

- $(P, \mathcal{D}(P), \mathcal{H}) = (D_x, C_C^\infty(\mathbb{R}), L^2(\mathbb{R}))$ .

- Observe that  $\forall f, g \in C_C^\infty(\mathbb{R})$ ,  $(D_x f, g) = (f, D_x g)$ .

- Take  $f \in \mathcal{H}$  and  $g \in C_C^\infty(\mathbb{R})$ . Then if  $f \in C_C^\infty$  or  $f \in H^1(\mathbb{R})$ ,  $g \mapsto (f, D_x g) = (D_x f, g)$  and

$$|(D_x f, g)| \leq \|D_x f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})} \leq \|f\|_{H^1(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}$$

- Then  $B(f, g) = D_x f, g) - (f, D_x g)$ .

- So  $\mathcal{D}(P^*) \supseteq C_C^\infty(\mathbb{R})$  and, in fact,  $\mathcal{D}(P^*) = H^1(\mathbb{R})$ .

- $P^*$  is closed (this is always true of any adjoint).

- $P \subset P^*$ , we say that  $P$  is symmetric.

Generally, if  $(A, \mathcal{D}(A))$  is an unbounded operator, then  $\Gamma(A^*) = J((\Gamma(A))^\perp)$  where  $J : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H} \times \mathcal{H}$  by  $J(u, v) = (-v, u)$ . Recall that  $(\Gamma(A))^\perp$  is always closed, so  $A^*$  is always closed.

### Definitions:

We say that  $P$  is symmetric if  $P \subseteq P^*$ .

We say that  $P$  is self-adjoint if  $P = P^*$ .

We say that  $P$  is essentially self-adjoint if  $P^* = \overline{P}$ .

### Important Questions

1. Given a symmetric operator  $(P \subseteq P^*)$ , to find a self-adjoint operator  $A$  such that  $P \subseteq A = A^* \subseteq P^*$ . We call  $A$  self-adjoint extension of  $P$ . If  $P^* = \overline{P}$  (essentially self-adjoint), then it must be the unique extension.
2. How to find or parameterize all self-adjoint extensions? (This problem is hard.)

Example

- $P = (D_x, H_0^1([0, 1]), L^2(0, 1))$  (where elements of  $H_0^1([0, 1])$  vanish on the boundary).

- $P^* = (D_x, H^1([0, 1])) \supsetneq P$ .

- There exists a “circle” of self-adjoint extensions, parameterized by any point on the unit circle.

## General Principle

If  $\Omega \subset \mathbb{R}^n$  is open, bounded and with smooth boundary  $\partial\Omega$  ( $\mathcal{H} = L^2(\Omega)$ ), and  $P$  is some differential operator (e.g.  $P = \Delta$ ) such that  $\forall f, g \in C_C^\infty(\Omega)$ .  $(Pf, g) = (f, Pg)$ . By Green's Identity,,

$$\begin{aligned} \int_{\Omega} (\Delta f g - f \Delta g) dx &= \int_{\Omega} \nabla((\nabla f)g - f(\nabla g)) dx \\ &= \int_{\partial\Omega} (g \nabla f - f \nabla g) \cdot \nu ds \\ &= \int_{\partial\Omega} \left( g \frac{\partial f}{\partial \nu} - f \frac{\partial g}{\partial \nu} \right) ds \end{aligned}$$

## Examples

- $P = (P = \Delta, C_C^\infty(\Omega), L^2(\Omega))$ .
  - $P$  symmetric.
  - $\mathcal{D}(P^*) = \{u \in L^2(\Omega) : \Delta u \in L^2\}$ .
  - Let  $P_{\min} = \overline{(P, C_C^\infty(\Omega))}$  and define  $P_{\max} = P_{\min}^*$ . Then  $\mathcal{D}(P_{\max}) = \{u \in L^2(\Omega) : Pu \in L^2(\Omega)\}$ .
  - The self-adjoint extensions lie inbetween.
- $(D_x, H_0^1([0, 1])) = \{u \in H^1([0, 1]) : u(0) = u(1) = 0\}, L^2([0, 1])$ .
  - Observe that the evaluation map  $C^\infty([0, 1]) \ni f \mapsto f(0)$  extends boundedly to  $\tau : H^1([0, 1]) \rightarrow \mathbb{C}$ .
  - For  $f, g \in C^\infty([0, 1])$ ,

$$(D_x f, g) - (f, D_x g) = \int_0^1 \frac{1}{i} f' \bar{g} - f \left( \frac{1}{i} \bar{g}' \right) = \frac{1}{i} \int_0^1 (f \bar{g})' = \frac{1}{i} (f(1) \bar{g}(1) - f(0) \bar{g}(0))$$

- Then if  $g \in \mathcal{D}(P_{\min})$ ,

$$\begin{aligned} |(f, D_x g)| &= |(D_x f, g) + \frac{1}{i} (\overbrace{f(1) \bar{g}(1)}^{=0} - \overbrace{f(0) \bar{g}(0)}^{=0})| \\ &\leq \|f\|_{H^1} \|g\|_{L^2} \end{aligned}$$

- Therefore  $\mathcal{D}(P_{\min}^*) = \mathcal{D}(P_{\max}) = H^1([0, 1])$ , and we note that  $P_{\max}^* = P_{\min}$ .

Now suppose that  $T$  is a self-adjoint extension of  $P$  ( $P_{\min} \subseteq T \subseteq P_{\max}$ ). If  $T$  is self-adjoint, then  $\forall f, g \in \mathcal{D}(T) \subseteq H^1([0, 1])$ ,  $f(0) \bar{g}(0) - f(1) \bar{g}(1) = 0$ .

For  $f = g$ ,  $|f(0)|^2 = |f(1)|^2$  for some fixed  $f$ , then  $\exists \alpha \in S^1$  such that  $f(1) = \alpha f(0)$ . Then for any other  $g$ ,

$$0 = f(0)(\bar{g}(0) - \alpha \bar{g}(1)) = \alpha f(0)(\overline{\alpha g(0) - g(1)})$$

Therefore  $g(1) = \alpha g(0)$  for the same fixed  $\alpha$ .

October 23, 2025

### Definition: Resolvent Set and Spectrum of and Unbounded Operator

For  $(A, \mathcal{D}(A), \mathcal{H})$  unbounded and closed, the resolvent set of  $A$ ,  $\rho(A)$ , is the set of  $\lambda \in \mathbb{C}$  such that  $(A - \lambda) : \mathcal{D}(A) \rightarrow \mathcal{H}$  is bijective with bounded inverse  $R_A(\lambda) := (A - \lambda)^{-1}$ .

The spectrum of  $A$ ,  $\sigma(A)$ , is  $\sigma(A) = \mathbb{C} \setminus \rho(A)$ . If  $\rho(A)$  is open, then  $\sigma(A)$  is closed.

Example

- Take  $P = (D_x, H_0^1([0, 1]), L^2(0, 1))$  and define two closed extensions  $P_0$  and  $P_1$

$$- \mathcal{D}(P_0) = \{u \in H^1([0, 1]) : u(0) = 0\}$$

$$* \text{ For } \lambda \in \mathbb{C}, f \in L^1([0, 1]), \text{ we solve } \begin{cases} D_x u + \lambda u = f \\ u(0) = 0 \end{cases}$$

$$* \text{ So } u(x) = i e^{-\lambda x} \int_0^x e^{i\lambda t} f(t) dt \text{ and } \sigma(P_0) = \emptyset.$$

$$- \mathcal{D}(P_1) = H^1([0, 1])$$

$$* \text{ For } \lambda \in \mathbb{C}, \ker(D_x - \lambda) \cap L^2([0, 1]) \neq \emptyset. \text{ So } \sigma(P_1) = \mathbb{C}.$$

### Proposition

If  $(A, \mathcal{D}(A))$  is self-adjoint, then  $\sigma(A) \subseteq \mathbb{R}$ ,  $\sigma(A) \neq \emptyset$ , and we have a resolvent estimate  $\|R_A(z)\| \leq \frac{1}{|\operatorname{Im}(z)|}$  for  $\operatorname{Im}(z) \neq 0$ .

The proof of real value and the estimate are the same as in the bounded case.

To show that  $\sigma(A) \neq \emptyset$ , by contradiction if  $\sigma(A) = \emptyset$  then  $A^{-1} = R_A(0)$  exists and is bounded. We claim that this requires  $\sigma(A^{-1}) = \{0\}$  and pick  $\lambda \neq 0$ , such that

$$\begin{aligned} (A^{-1} - \lambda)u &= f \\ \left(\frac{1}{\lambda} - A\right)u &= \frac{1}{\lambda} A f \end{aligned}$$

But  $\frac{1}{\lambda} \notin \sigma(A)$  implies  $u = -R_A\left(\frac{1}{\lambda}\right)\frac{1}{\lambda} A f$ . Therefore  $R_{A^{-1}}(\lambda) = -\frac{1}{\lambda} A R_A\left(\frac{1}{\lambda}\right)$  and  $\lambda \notin \sigma(A^{-1})$ . Therefore  $A^{-1} = 0$  which contradicts the assumption that  $A^{-1} A = \operatorname{Id}_{\mathcal{D}(A)}$ .

### Generalizing Spectral Resolution

We can extend  $\{E_\lambda\}_{a \leq \lambda \leq b}$  to  $a = -\infty$  and  $b = \infty$  by setting strong limits  $E_{-\infty} = 0$  and  $E_\infty = I$ .

Example

- On  $L^2(\mathbb{R})$ ,  $E_\lambda f(t) = \chi_{(-\infty, \lambda)}(t) f(t)$ .

### Theorem: Spectral Theorem for Unbounded Self-Adjoint Operators

Theorem 8.15 Helffer.

Any self-adjoint operator  $A$  on a Hilbert space  $\mathcal{H}$  admits a spectral decomposition  $\{E_\lambda\}_\lambda$  such that  $\forall x, y \in \mathcal{H}$ ,

$$(Ax, y) = \int_{\mathbb{R}} \lambda d(E_{\lambda} x, y)$$

$$Ax = \int_{\mathbb{R}} \lambda d(E_{\lambda} x)$$

### Proof

Largely, we will use the proof for the bounded case.

Suppose  $A$  is semibounded from below (i.e.  $\exists \mu \in \mathbb{R}, \forall x \in \mathcal{D}(A), (Ax, x) \geq \mu \|x\|^2$ ) then  $R_A(\lambda)$  exists for  $\lambda_0 < \mu$  ( $((A - \lambda)x, x) \geq (\mu - \lambda_0) \|x\|^2$  coercive in the sense that  $(\mu - \lambda_0) > 0$ ). Then it is also a self-adjoint bounded operator which implies that  $R_A(\lambda_0)$  has a spectral representation. Then  $A = f(R_A(\lambda_0))$ ,  $f(x) = \lambda_0 + \frac{1}{x}$ . We can work with a general case  $(A - i)^{-1}$ .

Example

- Take  $(D_x, H^1(\mathbb{R}), L^2(\mathbb{R}))$ , and define the Fourier transform  $f(x) \mapsto \mathcal{F}f = \hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx$ . Then  $\widehat{D_x f}(\xi) = \xi \hat{f}(\xi)$ , and we have the following commutative diagram

$$\begin{array}{ccc} H^1(\mathbb{R}) & \xrightarrow{D_x} & L^2(\mathbb{R}) \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\ H^1(\mathbb{R}) & \xrightarrow{f(t) \mapsto t f(t)} & L^2(\mathbb{R}) \end{array}$$

- So  $E_{\lambda} f = \mathcal{F}^{-1} \chi_{(-\infty, \lambda)} \mathcal{F} f$ .
- $(D_x^2, H^2(\mathbb{R}), L^2(\mathbb{R}))$ . Then  $\widehat{D_x^2 f}(\xi) = \xi^2 \hat{f}(\xi)$ ,  $\sigma(D_x^2) = [0, \infty)$  and  $D_x^2 f = \mathcal{F}^{-1} \xi^2 \mathcal{F} f$ .
  - So  $E_{\lambda} f = 0$  for  $\lambda < 0$ ;  $E_{\lambda} f = \mathcal{F}^{-1} \chi_{\xi^2 < \lambda}(\xi) \mathcal{F} f$ .
- $(D_x^2 + V(x), C_c^{\infty}(\mathbb{R}), L^2(\mathbb{R}))$ ,  $V \in C(\mathbb{R}; \mathbb{R})$  bounded from below. Then  $D_x^2 + V$  is semibounded from below and the spectrum depends drastically on  $V$ .
  - $V = 0$  gives  $\sigma = [0, \infty)$ .
  - $V = x^2$  gives  $\sigma = \{2n + 1 : n \geq 0\}$
  - $V \in C_c(\mathbb{R})$  gives  $\sigma = [0, \infty)$

### Proposition

If for some  $z_0 \in \rho(A) \cap \mathbb{R}$   $R_A(z_0)$  is compact, then  $A$  has purely discrete spectrum  $(z_n)_{n \geq 1}$ ,  $|z_n| \rightarrow \infty$  as  $n \rightarrow \infty$ .

### Proof

$R_A(z_0)$  is compact and self-adjoint, so there exists a basis for  $\mathcal{H}$  made of normalized eigenvectors  $(\phi_n)_{n \in \mathbb{N}}$  with eigenvalues  $|\lambda_n| \searrow 0$ .

$$\begin{aligned}
(A - z_0)^{-1} \phi_n &= \lambda_n \phi_n \\
A \phi_n &= \left( z_0 + \frac{1}{\lambda_n} \right) \phi_n \\
\mathcal{H} &= \left\{ u = \sum u_n \phi_n : \sum |u_n|^2 < \infty \right\} \\
\mathcal{D}(A) &= \left\{ u = \sum u_n \phi_n : \sum |u_n|^2 \left( z_0 + \frac{1}{\lambda_n} \right)^2 < \infty \right\}
\end{aligned}$$

## Applications

- Laplacian (Leplace-Beltrami Operator) on closed Riemann manifolds or Dirichlet Laplace-Beltrami if  $\partial M \neq \emptyset$ .
- Dirichlet Laplacian ( $\Delta = -\sum_{k=1}^n \partial_{x^k}^2$ ) on bounded domains in  $\mathbb{R}^n$  ( $\Omega$  open in  $\mathbb{R}^n$ , regular,  $\partial\Omega$  smooth).
  - $\Delta_D = (\Delta, H_0^1(\Omega) \cap H^2(\Omega), L^2(\Omega))$  where  $H_0^1 = \{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\}$  ( $u|_{\partial\Omega}$  makes sense due to a trace theorem from Evans 5.5) and  $H^2(\Omega) = \{u \in L^2(\Omega) : \partial_i \partial_j u \in L^2, \partial_i u \in L^2, \forall i, j\}$ .
  - $\int_{\Omega} |\nabla f|^2 dx = (\Delta f, f) \geq 0$  (Green's Theorem)
  - Claim:  $\Delta_D + 1$  has compact inverse.
- Solvability of  $\Delta_D + 1$

Our goal is, given  $f \in L^2$ , to find  $u \in ?$  such that  $\begin{cases} \Delta u + u = f \\ u|_{\partial\Omega} = 0 \end{cases}$ . We have a weak formulation, if  $v \in H_0^1$ , we multiply by  $v$  and get an IBP

$$\underbrace{\int \nabla u \cdot \nabla u + \int uv}_{B(u,v)} = \int f v$$

where  $|\int f v| \leq \|f\|_{L^2} \|v\|_{L^2} \leq \|f\|_{L^2} (B(v, v))^{1/2}$ . By Riesz Representation Theorem, there exists some unique  $u \in H_0^1$  such that  $\forall v \in H_0^1, B(v, v) = \int f v$ . Moreover, we can recover  $H^2$  regularity on  $u$ . Also,  $|B(u, u)| = |\int f u| \leq \|f\|_{L^2} (B(u, u))^{1/2}$  if and only if  $\|(\Delta_D + 1)^{-1} f\|_{H_0^1(\Omega)} = \|u\|_{H_0^1(\Omega)} \leq \|f\|_{L^2}$ . Therefore  $(\Delta_D + 1)^{-1} : L^2(\Omega) \rightarrow H_0^1(\Omega)$  is bounded. By Rellich Compactness,  $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$  compactly. Hence  $(\Delta_D + 1)^{-1} : L^2 \rightarrow L^2$  is compact.

## Theorem

On  $\mathbb{R}$ ,  $H^1(\mathbb{R})$  does not embed compactly into  $L^2(\mathbb{R})$ . For example,  $f \in H^1$  with  $\text{supp } f \subset [0, 1]$  defined by  $f_n(x) = f(x - n)$ .

So if  $H \subset L^2(\mathbb{R})$  where  $\|u\|_H^2 = \|u'\|_{L^2}^2 + \|a(x)^{-1} u\|_{L^2}^2$  for  $a(x) > 0$  and  $\lim_{|x| \rightarrow \infty} a(x) = 0$ , then  $H$  embeds compactly in  $L^2$ .

It follows that  $D_x^2 + x^2$  has compact resolvent and discrete spectrum.  $\|u\|_H^2 = \|d_x u\|_{L^2}^2 + \|xu\|_{L^2}^2$ .