Analysis I

October 2, 2023

Lecture Notes

Class will not have dedicated lecture notes. Many are available already.
Undergraduate notes are available on Canvas.
Lecture 1 overview available on Canvas (lecture1.pdf).

Tentative Office Hours

Mondays 2-3pm and Tuesday 1-2pm.

Homework

Nominally due at beginning of class; ask for leeway if needed.
First week homework will be review of undergraduate proofs.
First homework due Wednesday, October 11.

Notation

Natural Numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$
Non Negative Integers: $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
Integers: $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$
Rationals: $\mathbb{Q} = \left\{\frac{p}{q},\ p \in \mathbb{Z},\ q \in \mathbb{Z}\right\} = \mathbb{Z} \times \mathbb{N}/\infty$

- Equivalent representation of rationals: $(p_1, q_1) \sim (p_2, q_2)$ iff $p_1 q_2 = p_2 q_1$

Sequence of Rationals: $\{u_n\}_{n \in \mathbb{N}},\ u_n \in \mathbb{Q},\ \forall n.$

Properties of the Rationals

$(\mathbb{Q}, +, \cdot)$ is a (ii) totally ordered (i) field satisfying the (iii) Archimedean property.

(i) Field

1. $+$ is associative: $(a + b) + c = a + (b + c)$

2. $+$ is commutative: $a + b = b + a$

3. $\cdot$ is associative and commutative.

4. $\exists 0 \in \mathbb{Q}$ such that $\forall a \in \mathbb{Q}$, $0 + a = a + 0$

5. $\exists 1 \in \mathbb{Q} \setminus \{0\}$ such that $\forall a \in \mathbb{Q}$, $1 \cdot a = a \cdot 1 = a$

6. $\forall a \in \mathbb{Q} \setminus \{0\}$ $\exists b \in \mathbb{Q}$, $a \cdot b = b \cdot a = 1$

    - $b = a^{-1} = \frac{1}{a}$

(ii) Totally Ordered

$\exists$ a set $\mathbb{Q}_+ \subseteq Q$ of "Positive Numbers" stable under $+$ and $\cdot$ such that $\forall A \in \mathbb{Q}$ either $a > 0$ ($a \in \mathbb{Q}_+$), $-a > 0$ (also $a < 0$) or $a = 0$.

- Ordering: $\forall a, b \in \mathbb{Q}$, $a < b$ if and only if $b - a > -0$.

- Trichotomy: $\forall a, b \in \mathbb{Q}$ either $a < b$, $a > b$, or $a = b$.

- $\max(a, b) = \begin{cases} a & \text{if } a > b \\ b & \text{otherwise} \end{cases}$.

- $|a| = \max(a, -a)$ (helps measure distance in $\mathbb{Q}$).

- $\operatorname{dist}(a, b) := |b - a|$

- Triangle Inequality: $|u \pm v| \leq |u| + |v|$

- Observe also: $||u| - |v|| \leq |u \pm v|$. The triangle inequality may be used to prove this.

- Proof of Triangle Inequality $-|u| \leq u \leq |u|$ and $-|v| \leq v \leq |v|$, therefore $-|u| - |v| \leq u + v \leq |u| + |v|$. Therefore $u + v \leq |u| + |v|$ and $-(u + v) \leq |u| + |v|$ implies $|u + v| \leq |u| + |v|$.

(iii) Archimedian Property:

$\forall \epsilon > 0,\ \exists N,\ \forall n \geq N,\ \frac{1}{n} < \epsilon$.

Bounded Sequence of Rationals

$\{u_n\}_{n \in \mathbb{N}}$ is bounded if $\exists m \in \mathbb{Q}_+$ such that $|u_n| \leq M,\ \forall n$.
$\{u_n\}_{n \in \mathbb{N}}$ converges to $a \in \mathbb{Q}$ ($\lim_{n \to \infty} u_n = a$) if $\forall \epsilon > 0, \exists N, \forall n \geq N, |u_n - a| < \epsilon$.

Famous Limits

Decaying Rational

1. $\lim_{n\to\infty} \frac{1}{n} = 0$

    - $\forall \epsilon \in \mathbb{Q}_+,\ \exists n \in \mathbb{N},\ 0 < \frac{1}{n} < \epsilon$

    - $\forall n \in \mathbb{N},\ \exists n \in \mathbb{N},\ n \geq N$

        - b. and c. are equivalent.

Decaying Exponential Rational

$r \in \mathbb{Q},\ 0 < r < 1,\ \lim_{n\to\infty} r^n = 0.$

- Proof: Write $r = \frac{1}{1+k}$ for some $k > 0$. Then $r^n = \frac{1}{(1+k)^n} \overset{\text{Bernoulli}}{\leq} \frac{1}{1+nk}$.

Geometric

1. $r \in \mathbb{Q},\ 0 < r < 1,\ u_n = 1 + r + \cdots r^n = \frac{1-r^{n+1}}{1-r} \to \frac{1}{1-r}$

Features of Limits

Limits are Unique

If the limit of a sequence exists, it is unique.

Squeezing Lemma

If $\{a_n\}$, $\{b_n\}$ are such that $0 \leq a_n \leq b_n$, and $b_n \to 0$ as $n \to \infty$, then $a_n \to 0$.

Limits Preserve Order

If $a_n \leq b_n$ $\forall n$ and a_n and b_n converge, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

Limit Algebraic Rules

$\lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = \lim_{n\to\infty}(a_n + b_n)$ when a_n and b_n converge.
If $\lim_{n\to\infty} b_n \neq 0$, then $\frac{a_n}{b_n} \to \frac{\lim a_n}{\lim b_n}$.

Peculiarity of the Rationals

$\mathbb{Q}$ lacks completeness.

Examples

Consider $u_1 = 1$ and $u_{n+1} = \frac{1}{2}(u_n + \frac{2}{u_n})$.
Then $u_n \in \mathbb{Q},\ \forall n \in \mathbb{N}$.
It can further be proven, by induction, that $u_n \geq 1,\ \forall n$. $\left(u_{n+1} - 1 = \frac{1}{2}(u_n + \frac{1}{u_n}) - 1 = \frac{1}{2u_n}((u_n - 1)^2 + 1)\right)$.
$\lim_{n\to\infty} u_n^2 = 2$.

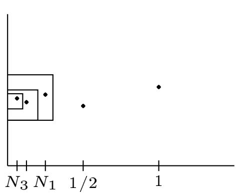
$$\begin{aligned} u_{n+1}^2 - 2 &= \left(\frac{1}{2}(u_n + \frac{2}{u_n})\right)^2 - 2 \\ &= \left(1\frac{1}{2u_n}(u_n^2 + 2)^2 - 4u_n\right) \\ &= 1\frac{4}{u_n^2}\left(u_n^2 - 2)^2\right) \\ &\leq \frac{1}{4}(u_n^2 - 2)^2 \end{aligned}$$

If u_n converged in $\mathbb{Q}$ to L, by algebraic limit rules, $2 = \lim u_n^2 = (\lim u_n)^2 = L^2$, yet $\sqrt{2} \notin \mathbb{Q}$.

Cauchy Criterion

A sequence $\{u_n\}_{n\in\mathbb{N}}$ of rationals is Cauchy if $\forall \epsilon > 0,\ \exists n \in \mathbb{N},\ \forall p, q \geq n,\ |u_p - u_q| < \epsilon$.

Visual Justification



Example 1

The sequence from before is Cauchy.
$|u_p - u_q| = \frac{|u_p^2 - u_q^2|}{|u_p + u_q|} \leq \frac{1}{2}|u_p^2 - u_q^2|$

Example 2

$u_n = \sum_{k=0}^{n} \frac{1}{k!} \in \mathbb{Q}$.

- This is increasing.

- It is bounded above by 3:

$$\begin{aligned} 1 + 1 + \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots + \frac{1}{2\cdots n} &\leq 1 + 1 + \cdots \frac{1}{2^{n-1}} \\ &\leq 1 + \frac{1 - 2^{-n}}{1 - \frac{1}{2}} \\ &\leq 3 \end{aligned}$$

Convergence, Cauchy and Boundedness.

Given a sequence $\{u_n\}_{n\in\mathbb{N}}$,
$\{u_n\}$ converges $\implies$ $\{u_n\}$ is Cauchy $\implies$ $\{u_n\}$ is bounded.
Note that in $\mathbb{Q}$ none of these implications may be reversed.

Construction of the Real Numbers

Short version: If the limit of a sequence isn't in the set, then define the limit as the sequence itself.
Let $C_{\mathbb{Q}} = \{\text{Cauchy sequences of rationals.}\}$.

Two Operations

- Termwise Addition $\{u_n\}_n + \{v_n\}_n := \{u_n + v_n\}_{n\in\mathbb{N}}$

- Termwise Multiplication $\{u_n\}_n \cdot \{v_n\}_n := \{u_n \cdot v_n\}_{n\in\mathbb{N}}$

Closure of Cauchy Sequence

If $\{u_n\}_n,\ \{v_n\}_n \in C_{\mathbb{Q}}$, then $\{u_n\}_n + \{v_n\}_n \in C_n$ and $\{u_n\}_n \cdot \{v_n\}_n \in C_n$.

Example

Infinite decimal expansion.
Fix $N \in \mathbb{Z},\ a_1 \cdots a_n \in \{0, \dots, 9\}$.
Then let $u_n = N + \sum_{k=1}^{n} a_k (10)^{-k}$ (that is the number $N.a_1 a_2 \dots a_n$).
This is always increasing and bounded above by $N + \sum_{k=1}^{n} 9 \cdot (10)^{-k} = N + \frac{9}{10} \cdot \sum_{k=1}^{n} (10)^{-(k+1)} \leq N + 1$.
Hence, it is Cauchy.

Increasing and Bounded Above Implies Cauchy

By contrapositive, increasing and not Cauchy implies not bounded.
By the negation of Cauchy and letting $p \geq q$ without loss of generality, we can force $u_p > u_q + \epsilon$.

Negation of Cauchy

$\exists \epsilon > 0,\ \forall N,\ \exists p, q \geq N,\ |u_p - u_q| > \epsilon$.

Real Numbers as Equivalence Classes of Cauchy Sequences

On $C_{\mathbb{Q}}$ define the relation $\{x_n\}_n \sim \{y_n\}_n$ if and only if $\lim_{n\to\infty} |(x_n - y_n)| = 0$.

Equivalence Relation

Reflexive: $x_n - x_n = 0$
Transitive: Uses algebraic limit rules. $x_n - z_n = x_n - y_n + y_n - z_n$.
Symmetric.

Definition of the Reals

$\mathbb{R} := C_{\mathbb{Q}} / \sim$
Then $x \in \mathbb{R},\ x = [\{x_n\}_n]$.

Addition and Multiplication of Reals

- Addition $x + y := [\{x_n + y_n\}_n]$.

- Multiplication $x \cdot y := [\{x_n \cdot y_n\}_n]$.

Operations Do Not Depend on Choice of Representative

If $\{x_n\}_n \sim \{x_n'\}_n$ and $\{y_n\}_n \sim \{y_n'\}_n$, then $\{x_n\}_n + \{y_n\}_n \sim \{x_n'\}_n + \{y_n'\}_n$.
If $\{x_n\}_n \sim \{x_n'\}_n$ and $\{y_n\}_n \sim \{y_n'\}_n$, then $\{x_n\}_n \cdot \{y_n\}_n \sim \{x_n'\}_n \cdot \{y_n'\}_n$.

The Reals are a Field

There are nine properties to check, eight of which are "obvious":

Commutativity of Addition (and Other "Obvious" Features)

$[\{x_n\}_n] + [\{y_n\}_n] = [\{x_n + y_n\}_n] = [\{y_n + x_n\}] = [\{y_n\}_n] + [\{x_n\}_n]$
That is, the Reals inherit most field features from the Rationals.

- Zero Element $0_{\mathbb{R}} = [\{0_{\mathbb{Q}}\}_n]$

- One Element $1_{\mathbb{R}} = [\{1_{\mathbb{Q}}\}_n]$

Multiplicative Inverses

How to define x^{-1} for $x \in \mathbb{R}$ where $x \neq 0$?

- Idea If $x = [\{x_n\}_n]$ choose $x^{-1} = [\{\frac{1}{x}\}_n]$.
  If $x \in \mathbb{R},\ x \neq 0$ then

  1. $\exists \{x_n\}_n \in C_{\mathbb{Q}}$ representing x with non zero entries.

  2. $\{\frac{1}{x_n}\}_n$ is Cauchy.

  – Proof of 1 Pick any $\{x_n\}_n$ representing x.

    ∗ $x \neq 0$, so NOT ($\lim_{n\to\infty} x_n = 0$: $\exists \epsilon_0 > 0,\ \forall N,\ \exists n \geq N,\ |x_n| > \epsilon_0$.

    ∗ $\{x_n\}$ is Cauchy: $\forall \epsilon > 0, \exists N,\ \forall p, q \geq N,\ |x_p - x_q| < \epsilon$.

Therefore, $\exists N$ such that $\forall p, q \geq N_1,\ |x_p - x_q| < \frac{\epsilon_0}{2}$
And $\exists N_2 \geq N,\ , |x_{N_2} > \epsilon_0$.
For $q \geq N_2$, the Cauchy Criterion states that $|x_q| = |x_q - x_{N_2} + x_{N_2} \geq |x_{N_2}| - |x_{N_2} - x_q| \geq \epsilon_0 - \frac{\epsilon_0}{2} \geq \frac{\epsilon_0}{2}$.
Therefore, the sought sequence is $\{x_{N_2} + k\}_{k \in \mathbb{N}}$.

- Proof of 2 $\left|\frac{1}{x_p} - \frac{1}{x_q}\right| = \frac{|x_p - x_q|}{|x_p||x_q|} \leq \frac{4}{\epsilon_0^2}|x_p - x_q|$.

Order on the Reals

Let $x \neq 0$, $\exists \{x_n\}_{n \in \mathbb{N}}$ be a representation of x and $\epsilon_0 > 0$.
Then for $|x_n| > \epsilon_0,\ \forall n \in \mathbb{N}$, there is a dichotomy:

- Either $\exists N \in \mathbb{N},\ x_n > \epsilon_0,\ \forall n \geq N$ (in which case we write $x > 0$)
- Or $\exists N \in \mathbb{N},\ x_n < -\epsilon_0,\ \forall n \geq N$ (in which case we write $x < 0$

Thus the Reals are totally ordered.

October 4, 2023

Overview

Completeness of $\mathbb{R}$.
Topology of the Real Line.

Non-zero Reals Are Either Positive or Negative

Given $x \in \mathbb{R} \setminus \{0\},\ \exists \delta \in \mathbb{Q}_+$ such that $\forall \{x_n\}_n$ representing x, $\exists N \in \mathbb{N}$ such that $|x_n| > \delta, \forall n \geq N$.
Moreover, one of the following (but not both) holds:

1. $\forall \{x_n\}_n \in x,\ \exists,\ x_n > \delta,\ \forall n \geq N$ (i.e. $x > 0$)
2. $\forall \{x_n\}_n \in x,\ \exists,\ x_n < -\delta,\ \forall n \geq N$ (i.e. $x < 0$)

Recall that $x \in \mathbb{R} \setminus \{0\}$ is an equivalence class of Cauchy sequences.

Total Ordering of the Reals

$x > 0$ produces a total ordering of $\mathbb{R}$ where $x < y$ if and only if $y - x > 0$.
$\rightsquigarrow \max(x, y) = \begin{cases} x & \text{if } x > y \\ y & \text{otherwise} \end{cases}$
$|x| = \max(x, -x)$ (which satisfies the triangle inequality)

Lemma A

Let $x, y \in \mathbb{R}$. If $\{x_n\}_n, \{y_n\}_n$ represent x, y and satisfy $x_n < y_n,\ \exists N \in \mathbb{N},\ \forall n \geq N$, then $x \leq y$.

- Proof By contradiction, suppose $x > y$ and $\exists \{x_n\}_n, \{y_n\}_n$ representing x, y such that $x_n \leq y_n,\ \forall n \geq N_1$.
  Then, by definition, $x - y > 0 \implies \exists \delta > 0,\ \exists N_2,\ x_n - y_n > \delta$ for $n \geq N_2$.
  But $x_n \leq y_n$ contradicts $x_n - y_n > \delta$.

Sequences of Reals

$\{x_n\}_n,\ x_n \in \mathbb{R}$
The definition of bounded, convergent and Cauchy sequences are the same as in $\mathbb{Q}$.

Injection of Rationals

$\iota : \mathbb{Q} \to \mathbb{R}$ such that $r \mapsto [\{u_n = r\}_n]$
This is isometric in the sense that $|\iota(r) - \iota(s)|_{\mathbb{R}} = |r - s|_{\mathbb{Q}}$

Theorem (Completeness 1)

Let $\{x_n\}_n \in C_{\mathbb{Q}}$ and $x = [\{x_n\}_n]$, then $\{\iota(x_n)\}_n$ converges to x.

Proof

What to show: $\forall \epsilon > 0,\ \exists N,\ \forall n \geq N,\ |\iota(x_n) - x| < \epsilon$.
Let $\epsilon \in \mathbb{Q}_+$. By the Cauchy criterion, $\exists N, \forall q, p \geq N,\ |x_p - x_q| < \epsilon$.
This is equivalent to $x_q - \epsilon \leq x_p \leq x_q + \epsilon$ where p is frozen.
Then by Lemma A, $x - \epsilon \leq \iota(x_p) \leq x + \epsilon$.
It follows that $\forall p \geq N,\ |\iota(x_p) - x \leq \epsilon$.

Corollary

$\mathbb{Q} \cong \iota(\mathbb{Q})$ is dense in $\mathbb{R}$. That is, $\forall \epsilon > 0,\ \forall x \in \mathbb{R},\ \exists r \in \mathbb{Q},\ |\iota(r) - x| < \epsilon$.

The Isometric Copy of Rationals

For brevity, the ι notation will be dropped and the $\mathbb{Q}$ will be understood as $\iota(\mathbb{Q})$.

Completeness of the Real Numbers

A sequence of real numbers converges in $\mathbb{R}$ if and only if it is Cauchy.

Proof

($\Longrightarrow$) This is clear.
($\Longleftarrow$) Take a Cauchy sequence of reals $\{x_n\}_n$. Then $\forall \epsilon > 0,\ \exists N,\ \forall p, q \geq\ \ |x_p - x_q| < \epsilon$.
Using the density of $\mathbb{Q}$, $\forall n \in \mathbb{N},\ \exists r_n \in \mathbb{Q}$ such that $|x_n - r_n| < \frac{1}{n}$.

Claim: $\{r_n\}_n$ is Cauchy. Indeed,

$$\begin{aligned} |r_p - r_q| &= |r_p - x_p + x_p - x_q + x_q - r_q| \\ &\leq |r_p - x_p| + |x_p - x_q| + |x_q - r_q| \\ &\leq \frac{1}{p} + |x_p - x_q| + \frac{1}{q} \end{aligned}$$

Take $\epsilon > 0$. $\{x_n\}$ cauchy implies $\exists N_1,\ \forall p,q \geq N, |x_p - x_q| \leq \frac{\epsilon}{3}$ and $\exists N_2,\ \forall p,q \geq N_2, \frac{1}{p} \leq \frac{\epsilon}{3},\ \frac{1}{q} \leq \frac{\epsilon}{3}$ for $p,q \geq \max(N_1,N_2)$ $|r_p - r_q| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$.
Then, for Cauchy $\{r_n\}_n$, call $r = [\{r_n\}_n]$, then $\lim_{n\to\infty} r_n = r$ by the above theorem.
Then my algebraic limit rules, $x_n(x_n - r_n) + r_n$ where $(x_n - r_n) \to 0$ and $r_n \to r$ as $n \to \infty$. So $\{x_n\}$ converges.

Example

Let $x_1 = 1$, $x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n})$.
Then $\{x_n\}_n \in C_{\mathbb{Q}}$, and it converges to $L \in \mathbb{R}$.
By algebraic limit rules, $L^2(\lim x_n)^2 = \lim x_n^2 = 2$.

Subsets of the Reals, Infimum and Supremum

Notation

Subset: $S \subseteq \mathbb{R}$
Inclusion: $x \in S$
Open Interval: $(a,b) = \{x \in \mathbb{R} | a < x < b\}$
Semiclosed Interval: $(a,b] = \{x \in \mathbb{R} | a < x \leq b\}$
Closed Interval: $[a,b] = \{x \in \mathbb{R} | a \leq x \leq b\}$
Unbounded Semiclosed Interval: $(-\infty, a] = \{x \in \mathbb{R} | x \leq a\}$
Unbounded Open: $(-\infty, a) = \{x \in \mathbb{R} | x < a\}$

Suprememum

$S \subseteq \mathbb{R}$ is bouned above (respectively below) if $\exists M \in \mathbb{R},\ \forall x \in S,\ x \leq M$ (respectively $\exists L \in \mathbb{R},\ \forall x \in S,\ L \leq X$)
S ad mits a least upper bound, LUB, suprememum or sup M if

1. $\forall x \in S, x \leq M$

2. $\forall M' \in \mathbb{R}$, upper bound of S, $M \leq M'$

If $\sup S$ exists, it is unique.
If $x \in S$ and x is an upper bound for S, then $x = \sup S$.

Example 1

$\sup(0,1) = \sup[0,1] = 1$

Example 2

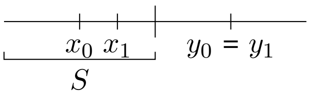
$S = \{x \in \mathbb{Q},\ x^2 < 2\}$ does not have a greatest element in $\mathbb{Q}$, nor a least upper bound in $\mathbb{Q}$.

Theorem (Completness 2)

Every subset $S \subseteq \mathbb{R}$, nonempty and bouned above, has a supremum in $\mathbb{R}$.

Proof

By dichotomy.
$S \neq \varnothing \implies \exists x_0 \in S$ and S bounded above implies $\exists y_0 \in \mathbb{R},\ \forall x \in S,\ x \leq y_0$ (in particular $x_0 \leq y_0$).
If $x_0 = y_0$, done. Otherwise, consider $m_0 = \frac{x_0+y_0}{2}$.



Two options exist: if m_0 is an upper bound for S, set $y_1 = m_0$ and $x_1 = x_0$.
Otherwise, $\exists x_1 \in S$, such that $m_0 < x_1$ so set $y_1 = y_0$.
Repeat this process forever to construct two sequences $x_n,\ y_n$.
$\forall n,\ x_n \in S$, y_n is an upper bound for S.

- $x_n \leq y_n$
- x_n is increasing and bounded above by y_0, so it must be Cauchy and converging to x.
- y_n is decreasing and bounded below by x_0, so it must be Cauchy and converging to y.
- $|x_{n+1} - y_{n+1}| \leq \frac{|x_n - y_n|}{2}$ which implies $|x_n - y_n| \leq \frac{1}{2^n}|x_0 - y_0|$ and $x = y = z$.

Therefore, the process may be understood as $x_0 \leq \cdots \leq x_n \leq x_{n+1} \leq y_{n+1} \leq y_n \leq \cdots \leq y_0$.
There remain two things to check: (1) z is an upper bound for S and (2) z is no larger than any other upper bound for S.

1. Take $x \in S,\ \forall n,\ x \leq y_n \xrightarrow{n\to\infty} x \leq Z$.
2. Take upper bound for S, z'. $x_n \leq z',\ \forall n \xrightarrow{n\to\infty} z \leq z'$.

So $z = \sup S$.

Monotone Convergence Theorem (Completeness 3)

An increasing sequence of reals, $\{x_n\}_n$, that is bounded above, converges to $\sup X = \sup\{x_n | n \in \mathbb{N}\}$.
To prove that this converges, since it is monotone and bounded above it is Cauchy and therefore must be convergent.

Proof

Call x the limit, then $\forall n,\ x_n \leq x$. To see this, suppose $\exists n_0,\ x < x_{n_0}$ then $\forall m \geq m_0,\ x < x_{m_0} \leq x_m \implies |x_m - x| \geq |x_{n_0} - x| > 0,\ \forall m \geq n_0$ is a contradiction.
Let M be an upper bound of X. Then $x_n \leq M,\ \forall n \overset{n\to\infty}{\longrightarrow} x \leq M \implies x = \sup X$.

Theorem (Existence of Roots)

$\forall x \in \mathbb{R}$ where $x > 0,\ p \in \{2, 3,\ \dots\ ,\},\ \exists! y > 0$ such that $y^p = x$.

Proof

Left as an exercise.
Either by dichotomy or consider $S = \{y \in \mathbb{R} | y^p < x\}$, show: $S \neq 0$, bounded above and $(\sup S)^p = x$.
For uniqueness, show $y_1^p = y_2^p = x \iff 0 = y_1^p - y_2^p = (y_1 - y_2)(\dots \neq 0) \implies y_1 = y_2$.

Topological Properties

$S \subseteq \mathbb{R}$ is open if $\forall x \in S,\ \exists a, b \in \mathbb{R},\ x \in (a, b) \subset S$.
x is an accumulation or limit point of S if $\forall \epsilon > 0,\ \exists y \in S,\ 0 < |x - y| < \epsilon$.
$S \subseteq \mathbb{R}$ is closed if it contains all its limit points.
A set may be both open closed, just open, just closed or neither.
Given $S \subseteq \mathbb{R}$, the interior of S is $\bigcup_{S' \text{ open} \subset S} S' = S^{\text{int}} = S^0$.
The closure is $\bigcap_{F \text{ closed} \supseteq S} F = \overline{S} \overset{\text{wts}}{=} S \cup \{\text{limit points of } S\}$.

Example

$\{x\}$ is not open, but, since the limit points of x are $\varnothing$, it is closed.

Propositions

1. Arbitrary unions and finite intersections of open sets are open.

2. S is open if and only the complement $S^c = \mathbb{R} \setminus S$ is closed.

3. Arbitrary intersections and finite unions of closed sets are closed.

Bolzano-Weierstrass Theorem

A bounded sequence in $\mathbb{R}$ ad mits a convergent (Cauchy) subsequence. $\exists M,\ |x_n| \leq M,\ \forall n$

Proof by Dichotomy

Suppose $I_0 = [a, b]$ contains the sequence.
Construct a sequence of intervals by indicators: if $\left[a, \frac{a+b}{2}\right]$ contains infinitely terms of $\{x_n\}_n$, choose n such that $x_{n_1} \in \left[a, \frac{a+b}{2}\right]$ and call $I_1 = \left[a, \frac{a+b}{2}\right]$.

Otherwise, $\left[\frac{a+b}{2}, b\right]$ must contain infinitely many terms. Choose n in a similar fashion as above such that $I_1 = \left[\frac{a+b}{2}, b\right]$.
This process may be repeated to create a sequence of intervals such that $I_k \supseteq I_{k+1} \supseteq I_{k+2}$ and $l(I_k) = \frac{b-a}{2^k}$.
A subsequence $\{u_{n_k}\}_k$ such that $u_{n_k} \in I_l$ for $k \geq l$.

Exercise

Extract a Cauchy criterion out of the above.

October 9, 2023

Overview

- Topology of $\mathbb{R}$ continued.

- Numerical series.

Next Wednesday

- Absolute and Conditional Convergence.

- Rearrangement theorem.

Last Time

Finished with Bolzano-Weierstrass.

Limits

Limit Point

We say $x \in \mathbb{R}$ is a limit point of $\{x_n\}_n$ if a subsequence of $\{x_n\}_n$ converges to x.
Equivalently, $\forall \epsilon > 0,\ \forall n_0 \in \mathbb{N},\ \exists n \geq n_0,\ |x_n - x| < \epsilon$.
That is, the sequence revisits an epsilon neighborhood of x infinitely many times.

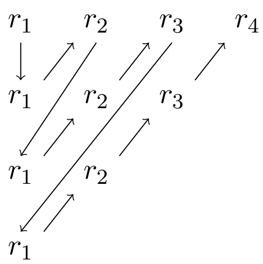
Limit Set

The limit set of $\{x_n\}_n : \mathrm{LS}(\{x_n\}_n)$ = the set of limit points of $\{x_n\}_n$.

- Comments

  - if $\lim_{n\to\infty}\{x_n\} = x$, then $\mathrm{LS}(\{x_n\}_n) = \{x\}$.

  - The limit set can be as big as $\mathbb{R}$!



- What Bolzano-Weierstrass says is that if $\{x_n\}$ is bounded, then $\mathrm{LS}(\{x_n\}) \neq \varnothing$.

- Examples $\mathrm{LS}(\{n\}_n) = \varnothing$.
  $\mathrm{LS}(\{x_n\}_n)$ is closed (good exercise).

Limit Superior

If $\{x_n\}_n \in [a, b]$ is bouneded, $\forall k \in \mathbb{N}$, $\sup\{x_j | j \geq k\}$ exists in $\mathbb{R}$.
Because

$$a \leq \sup\{x_j | j \geq k+1\} = y_{k+1} \leq \sup\{x_j | j \geq k\} = y_k$$

by the Monotone Convergence Theorem, $\{y_k\}_k$ converges. Call its limit $\limsup_n x_n = \inf_n \sup\{x_j | j \geq n\}$.

Limit Inferior

Similarly, define $\lim_n \inf x_n = \sup_n \inf\{x_j | j \geq n\}$.

Limit Superior and Limit Inferior Always Exist

What to show: $\limsup x_n, \liminf x_n \in \mathrm{LS}(\{x_n\})$.
Left as an exercise.

Convergence at the Limit

A bounded sequence $\{x_n\}_n$ converges if and only if $\liminf_n x_n = \limsup_n x_n$.

- Proof Technique Often it is useful to structure a proof such that

$$L < \liminf_n x_n \leq \limsup_n x_n < L$$

Topology of the Reals Continued

Compactness

Let $A \subseteq \mathbb{R}$.
A is (sequentially) compact if every sequence in A has a limit point in A.
A is (Heine-Borel) compact if every open cover of A has a finite subcover.

- Open Cover $\{O_\alpha\}_{\alpha \in I}$, with O_α open, is an open cover of A if $A \subseteq \bigcup_{\alpha \in I} O_\alpha$.

- Finite Subcover $O_1, \ldots, O_n,\ n \in \mathbb{N}$.

Heine-Borel Theorem

Let $A \subseteq \mathbb{R}$.
The following are equivalent

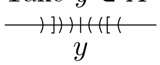
1. A is Heine-Borel compact.

2. A is closed and bounded.

3. A is sequentially compact.

Proof

$(1) \implies (2) \implies (3) \implies (1)$

- Heine-Borel Compact Implies Closed and Bounded Suppose A satisfies the Heine-Borel property.
  Consider $\{(-n, n)\}_{n \in \mathbb{N}}$. Clearly $\bigcup_n (-n, n) = \mathbb{R} \supseteq A$.
  By Heine-Borel, $\exists n_0, \ldots, n_p$ such that $A \supseteq \bigcup_{j=0}^{p} (-n_j,\ n_j) = (-N, N),\ N = \max(n_0, \ldots, n_p)$. So A is bounded.
  A is closed if $y \notin A \implies y$ is not a limit point of A.
  Take $y \in A^c$, then $A \subseteq \mathbb{R} \setminus \{y\} = \bigcup_{n \in \mathbb{N}} (-\infty, y - \frac{1}{n}) \cup (y + \frac{1}{n}, \infty)$.
  
  By the Heine-Borel property,

$$
\begin{aligned}
A &\subseteq \bigcup_{n_0, \ldots, n_p} (-\infty, y - \frac{1}{n}) \cup (y + \frac{1}{n}, \infty) \\
&= (-\infty, y - \frac{1}{N}) \cup (y + \frac{1}{N}, \infty)
\end{aligned}
$$

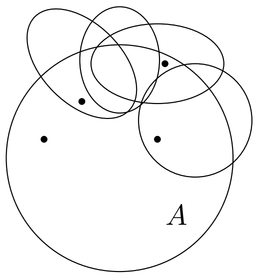
  Which implies $A \cap [y - \frac{1}{N}, y + \frac{1}{N} = \varnothing$ and y is not a limit point of A.
  That is, A contains its limit points.

- Closed and Bounded Implies Sequential Compactness Suppose A is both closed and bounded.
  Let $\{x_n\}_n \in A$. Then $\{x_n\}_n$ is bounded. By Bolzano-Weierstrass, it has a limit point x and a subsequence $\{x_{n_k}\}_k$ converging to x.
  Since A is closed, $\lim_{k \to \infty} x_{n_k} = x \in A$. $\blacksquare$

- Sequential Compactness Implies Heine-Borel Suppose $A \subseteq \mathbb{R}$ is sequentially compact.
  Consider an open cover of A, $\{O_\alpha | \alpha \in I\}$.
  First, turn it into a countable cover:

  - $\forall \alpha \in I,\ O_\alpha \subseteq \left(r_\alpha^1, r_\alpha^2\right),\ r_\alpha^1, r_\alpha^2 \in \mathbb{Q}$

Assume that $\{O_\alpha\}_\alpha$ can be made countable $(O_1, \ldots, O_n)$
By contradiction, suppose $\forall n \in \mathbb{N},\ A \setminus \left(\bigcup_{j=1}^n O_j\right) \neq \varnothing$.
Take $x_n \in A \setminus \left(\bigcup_{j=1}^n O_j\right)$. Since A is sequentially compact, $\exists \{x_{n_k}\}_k$ subsequence of $\{x_n\}_n$ converging to $x \in A$.
Since $A \subset \bigcup_{j\in\mathbb{N}} O_j$, $\exists j_0,\ x \in O_{j_0}$, O_{j_0} is open: $\exists \delta > 0,\ (x-\delta, x+\delta) \subseteq O_{j_0}$.
Then $\exists N,\ k \geq N \implies x_{n_k} \in (x-\delta, x+\delta) \subseteq O_{j_0}$. But if k is such that $n_k > j_0$, we also have $x_{n_k} \notin O_{j_0}$ which is a contradiction!



Structure of Open and Closed Sets

A is open in $\mathbb{R}$ if and only if it can be written as an at most countable, disjoint union of open intervals.

TODO Proof

For $x \in A,\ \exists (a,b)$, such that $x \in (a,b) \subseteq A$.
Let $I_x = \bigcup_{\text{open interval containing} x,\ I\subseteq A} I$. This is the maximal interval containing x in A.
Then, $A \subseteq \bigcup_{x\in A}\{x\} \subseteq \bigcup_{x\in A} I_x \subseteq A$.
That is, $A = \bigcup_{x\in A} I_x \quad (*)$.
Next, if $x, y \in A$, then $\begin{cases} \text{either } I_x = I_y \\ \text{or } I_x \cap I_y = \varnothing \end{cases}$
IMAGE HERE
The union $(*)$, as a disjoint union, is at most countable because each distinct one must contain a distinct rational and $\mathbb{Q}$ is countable.

Long Story Short

The topology of the reals avoids exotic sets. Closed sets, on the other hand, can be quite complicated.

TODO Cantor Set

$C := \bigcap_{k\in\mathbb{N}_0} I_k$. I_{k+1} is obtained by removing the middle open third of each interval making I_k.
IMAGE HERE - CANTOR
$I_0 = [0,1]$. One interval of length 1.
$I_1 = [0,1/3] \cup [2/3,1]$. Two intervals of length 2/3.
$I_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9]$. Four intervals of $(2/3)^2$
I_k is 2^k intervals of length $(2/3)^k$.
$I_{k+1} \subseteq I_k \implies C \subseteq I_k,\ \forall k \implies l(C) \leq l(I_k) = (2/3)^k \implies l(C) = 0$.

TODO Triadic Expansions

Goal:

1. C is perfect (i.e. every point in C is a limit point of C).

2. C contains no open intervals.

Property 2 is easy because $C \subseteq I_k$, which does contain interval of length greater than $(1/3)^k$.

1. C is uncountable.

Every $x \in [0,1]$ can be written in the form $x = \sum_{k=1}^{\infty} \frac{a_k}{3^k},\ a_k \in \{0,1,2\}$.
That is, $x = 0.a_1a_2\ldots$ in base 3. This is not always unique (e.g. $1/3 = 0.100\ldots = 0.022\ldots$).
IMAGE HERE - THIRDS OF INTERVAL
Note that the Cantor set is removing all decimal expansions with leading 1s. That is, $x \in C$ if and only if it has a triadic expansion only made of 0s and 2s.

- Proof of 1 If $x \in C,\ x = \sum_{k\geq 1} \frac{a_k}{3^k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{3^k}$, then $x_n \in C,\ \forall n$ and $x_n = 0.a_1\ldots a_n0000\ldots$ where $a_1, a_n \in \{0,2\}$.
  Unique representation can be maintained by forcing the behavior of the $n+1$th digit.

- Proof of 3 Every point in $[0,1]$ can also be written as $x = \sum_{n=1}^{\infty} = \frac{b_n}{2^n},\ b_n \in \{0,1\}$ (i.e. a binary expansion).
  Then $C \mapsto [0,1]$ gives $x = \sum \frac{a_k}{3^k} \mapsto \sum \frac{b_k}{2^k},\ b_k = \frac{a_k}{2}$ for $a_k \in \{0,2\}$ is a bijection!

October 11, 2023

Overview: Numeric Series

- Series with non-negative terms.

- Series with general terms.

- Convergence criteria.

- Algebraic rules.

- Rearrangements.

General Notation

Sequence $\{x_n\}_{n\geq n_0}$ (often $n_0 \in \{0,1\}$)

Definition: Partial Sum

$S_n = \sum_{k=n_0}^{n} x_k$ ($x_n = S_n - S_{n-1}$)
We say $\sum_n x_n$ converges if $\lim_{n\to\infty} S_n$ exists.
We denote $\sum_{k=n_0}^{\infty} x_k = \lim_{n\to\infty} S_n$

- Example: Geometric Series $\sum_{k=0}^{n} r^k = S_n,\ r \in (0,1)$
  $\frac{1-r^{n-1}}{1-r} \to \frac{1}{1-r}$

- Example: P Series $\sum_{k=1}^{n} \frac{1}{k^p},\ p > 0$

- Example: Exponential $\sum_{k=0}^{n} \frac{1}{k!}$

Series without Non-negative Terms

The series has non-negative terms if $x_n \geq 0,\ \forall n$.

Obvious Algebraic Limit Rules

If $\sum_{n \geq n_0} a_n$ and $\sum_{n \geq n_0} b_n$ converge and $\alpha \in \mathbb{R}$, then $\sum_{n \geq n_0} (a_n + \alpha b_n)$ converges to

$$\sum_{n=n_0}^{\infty} a_n + \alpha \sum_{n=n_0}^{\infty} b_n = \sum_{n=n_0} (a_n + \alpha b_n)$$

- Proof (Sketch) Reason on the partial sums; use algebraic limit rules on sequences.

Proposition

If $\sum_n x_n$ converges in $\mathbb{R}$, then $\lim_{n \to \infty} x_n = 0$.

- Proof $x_n = S_n - S_{n-1} \overset{n\to\infty}{\to} S - S = 0$
  Since $S_n \overset{n\to\infty}{\to} S$ and $S_{n-1} \overset{n\to\infty}{\to} S = \sum_{n=n_0}^{\infty} x_n$.

Series with Non-negative Terms

If $x_n \geq 0,\ \forall n,\ S_n = \sum_{k=n_0}^{n} x_k$ is non-decreasing.
By monotone convergence theorem, S_n is either bouned, and therefore converges, or unbounded from above where

$$\forall m > 0,\ \ \exists n_0 \in \mathbb{N},\ \forall n \geq n_0,\ S_n \geq M$$

This is "diverging to $+\infty$."

Theorem: Convergence Criteria

- Term Test If $0 \leq a_n \leq b_n,\ \forall n \geq n_0$ and $\sum_n b_n$ converges, then $\sum_n a_n$ converges.

  - Proof Suppose $0 \leq a_n \leq b_n$, and $t_n = \sum_{k=n_0}^{n} b_k$ converges and, therefore, is bounded above by $B = \sum_{k=n_0}^{\infty} b_k$.
    Then $\forall n,\ \sum_{k=n_0}^{n} a_k \leq \sum_{k=n_0}^{n} b_k \leq B$.
    Thus, by monotone convergence theorem, $\sum_{k=n_0}^{n} a_k$ converges.

- Ratio Test If $a_n > 0,\ \forall n$ and $\exists n_0 \in \mathbb{R}$ such that $\frac{a_{n+1}}{a_n} \le r < 1,\ \forall n \ge n_0$, then $\sum_n a_n$ converges.

  - Clarification The harmonic series has ratio $\frac{k}{k+1} < 1$ but since $\frac{k}{k+1} \overset{k\to\infty}{\to} 1$, there is no r which satisfies the ratio test.

  - Proof Suppose $a_{n+1} \le r a_n$ for $n \ge n_0$.
    Then $a_{m_0+p} \le a_{m_0+(p-1)} r \le a_{m_0+(p-2)} r^2 \le \cdots \le a_{m_0} r^p$.
    Then for $n \ge n_0$,

$$\sum_{k=n_0}^{n} a_k = \sum_{k=n_0}^{m_0} a_k + \sum_{k=m_0+1}^{n} a_k \le \sum_{k=m_0}^{m_0+(n-m_0)} a_{m_0} r^{n-m_0} \le a_{m_0} \sum_{k=m_0}^{n-m_0} r^{n-m_0} \le \frac{1}{1-r}$$

  - Rate of Convergnce The above proof shows that the ratio test implies a geometric rate of convergence.

- Root Test If $\exists n_0 \in \mathbb{N}$ and $r \in (0,1)$ such that $a_n^{1/n} \le r$, then $\sum_n a_n$ converges.

  - Proof (Sketch) Same story as the ratio test: $a_n^{1/n} \le r \implies a_n \le r^n$.

- Rejection of Ratio/Root If $\exists n_0 \in \mathbb{N}$ such that either $\frac{a_{n+1}}{a_n} \ge 1$ for $n \ge n_0$ or $a_n^{1/n} \ge 1$ for $n \ge n_0$, then $\sum_n a_n$ diverges to $+\infty$.

  - Proof (Sketch) In either case, a_n cannot converge to zero. Therefore the series cannot converge.

Prototype Scales

Geometric Rates

$\sum_{n\ge1} \frac{1}{n^\alpha}$ converges if and only if $\alpha > 1$ (to $\zeta(\alpha)$)
$a_k = \frac{1}{k^\alpha} \rightsquigarrow 2^k a_{2^k} = \frac{2^k}{2^{k\alpha}} = \left(\frac{1}{2^{\alpha-1}}\right)^k \implies t_n = \sum_{k=1}^n 2^k a_{2^k}$ converges if and only if $\frac{1}{2^{\alpha-1}} < 1$ if and only if $\alpha > 1$.

Log Geometric Case

$\sum_{n\ge1} \frac{1}{n(\log(n))^\beta}$ converges if and only if $\beta > 1$.
$a_k = \frac{1}{k(\log(k))^\beta} \rightsquigarrow 2^k a_{2^k} = \frac{2^k}{2^k(\log(2^k)^\beta} = \frac{1}{(\log(2)^\beta k^\beta}$ converges if and only if $\beta > 1$.

Lemma:

Suppose a_n decreases to 0.
Then the sequence $S_n = \sum_{k=1}^n a_k$ converges if and only if $t_n = \sum_{k=1}^n 2^k a_{2^k}$ converges.

$$S_{2^n} = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \cdots$$

- Proof $\quad a_3 + a_3 \le \qquad \le a_2 + a_3$

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \cdots \\ &= a_1 + \sum_{k=1}^{n} \sum_{p=1}^{2^k - 1} a_{2^k + p} \\ &\le a_1 + 2^k a_{2^{k+1}} + \end{aligned}$$

This gives

$$\frac{1}{2}(t_n - a_1) \le S_{2^n} - a_1 \le t_{n-1}$$

Therefore S_{2^n} converges, which implies that t_n converges, and, since S_n is monotone, S_n itself converges.

Series with General Terms

General term is signed.

Trick

Write $a_n = a_n^+ - a_n^-$ and $a_n^\pm = \max(0, \pm a)$. Then

$$S_n = \sum_{k=n_0}^{n} a_k = \left(\sum_{k=n_0}^{n} a_k^+\right) - \left(\sum_{k=n_0}^{n} a_k^-\right)$$

Convergence Outcomes

	$\sum_{k=n_0}^{\infty} a_k^+ < \infty$	$\sum_{k=n_0}^{\infty} a_k^+ = \infty$
$\sum_{k=n_0}^{\infty} a_k^- < \infty$	absolute convergence	$\lim S_n = +\infty$
$\sum_{k=n_0}^{\infty} a_k^+ = \infty$	$\lim S_n = -\infty$	lots of things can happen; divergence, convergence, any limit sequence

If S_n^+ and S_n^- converge, we can return to algebraic limit rules.
S_n converges to $\lim_{n\to\infty} S_n^+ - \lim_{n\to\infty} S_n^-$

Definition: Absolute Convergence

We say $\sum_n a_n$ converges absolutely if and only if $\sum_n |a_n|$ converges.

Note

$|a_n| = a_n^+ + a_n^-$

Proposition: Absolute Convergence Implies Convergence

Proof

Absolute convergence $\implies \sum |a_n|$ converges $\implies \sum a_n^+$ and $\sum a_n^-$ converges $\implies \sum(a_n^+ - a_n^-)$ converges.

Definition: Conditional Convergence

$\sum_n a_n$ converges conditionally if and only if $\sum_n a_n$ converges while $\sum_n |a_n|$ diverges.

Criteria for Convergence

For absolute convergence, run root/ratio/term test on $\sum_n |a_n|$.
Other criteria which might indicate conditional convergence.

Alternating Series Test

If $a_n(-1)^n b_n,\ b_n \geq 0$ decreases to zero, the series is conditionally convergent.

Dirichlet Test

If $a_n = b_n c_n$, where b_n decreases to zero and c_n satisfies $|c_0 + c_1 + \cdots + c_n| \leq C,\ \exists C \in \mathbb{R}, \forall n \in \mathbb{N}$, then $\sum_{n\geq 0} a_n$ converges conditionally.

- Applications $\sum_{n\geq 1} \frac{(-1)^n}{n}$
  $\sum_{n\geq 1} \frac{\cos(n)}{n}$

- Proof Write $C_n = c_0 + c_1 + \cdots + c_n$, such that $|C_n| \leq C,\ \forall n$.
  Then $c_n = C_n - C_{n-1}$, and

$$\sum_{k=0}^{n} b_k c_k = \sum_{k=0}^{n} b_k (C_k - C_{k-1}) = \sum_{k=0}^{n} b_k C_k - \sum_{k=0}^{n} b_k C_{k-1} \overset{l=k-1}{=} \sum_{k=0}^{n} b_k C_k - \sum_{l=0}^{n-1} b_{l+1} C_l = b_n C_n + \sum_{k=0}^{n-1} (b_k - b_{k+1}) C_k$$

  Then, since $b_n C_n \overset{n\to\infty}{\to} 0$, we only need to show that the final term converges absolutely. Consider

$$\sum_{k=0}^{n-1} |b_k - b_{k+1}||C_k| \leq C \sum_{k=0}^{n-1} (b_k - b_{k+1}) = C(b_0 - b_n) \leq C(b_0)$$

  independent of n. Hence, $\sum_{k=0}^{n} b_k c_k$ converges.

Definition: Rearrangement

Take $\sigma : \mathbb{N} \to \mathbb{N}$ a bijection and $\sum_{n\geq 1} a_n$ a series such that $S_n = \sum_{k=1}^{n} a_k$.
Then define a rearranged sum $S_n^{(\sigma)} = \sum_{k=1}^{n} a_{\sigma(k)}$.

Q: When does the rarranged sum converge; to where?

- Theorem: Rearrangement of Absolute Convergence If $\sum a_n$ converges absolutely, then $\forall \sigma,\ \lim_{n\to\infty} S_n^{(\sigma)} = \lim_{n\to\infty} S_n$.

- Theorem: Rearrangement of Conditional Convergence If $\sum a_n$ converges conditionally, then $\forall x \in \mathbb{R},\ \exists \sigma$ such that $\lim_{n\to\infty} S_n^{(\sigma)} = x$.

October 16, 2023

Overview

Sequences and Series of Functions
Things that will be glossed over for time

- Limits
- Continuity
- Differentiability
- Integrabillity

Why care about sequences and series?

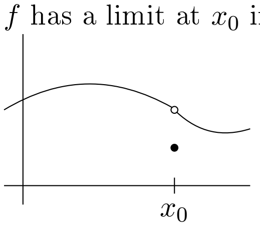
Extending features of functions.
Approximations.

Limits and Continuity

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R},\ x_0 \in I$.

Definition: Limit

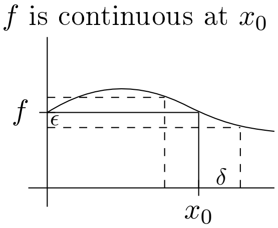
f has a limit at x_0 if $\exists \ell \in \mathbb{R}, \forall \epsilon > 0, \exists \delta > 0, 0 < |x - x_0| < \delta \implies |f(x) - \ell| < \epsilon$



- Equivalently
  For every sequence $\{x_n\}_n$ in I converging to x (but distinct to x), $\lim_{n\to\infty} f(x_n) = \ell$.

Definition: Continuous

f is continuous at x_0 if $\lim_{x\to x_0} f(x) = f(x_0)$.



- Modulus of Continuity $\forall \epsilon > 0, \exists \delta > 0, \forall x \in I, |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$
  Then $\delta(x_0, \epsilon)$ is the modulus of continuity.

Definition: Uniform Continuity on I

f is uniformly continuous on I if $\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in I, |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$.
Where δ is $\delta(\epsilon)$. That is, the modulus of continuity does not depend on the points.

Special Types of Uniform Continuity

Hölder Continuous

f is α-Hölder continuous on I for $\alpha \in (0, i]$, if $\exists c > 0$ such that $\forall x, y \in I, |f(x) - f(y)| \leq c|x - y|^\alpha$
$\alpha = 1$ implies that f is "Lipschitz-continuous"

- Example
  If f' exists and is bounded on $[a, b]$ by M, then by the Mean Value Theorem:
  $|f(x) - f(y)| = |f'(\xi)||x - y| \leq M|x - y|$, where $x \leq \xi \leq y$.

Continuity on Compact Sets

Let $K \subseteq \mathbb{R}$ be a compact set and $f : K \to \mathbb{R}$ be continuous.
Then

1. $f(K)$ is compact. In particular, f is bounded on K.

2. f achieves its extrema on K. (e.g. $\exists M \in K$ such that $f(M) = \sup\{f(x) \mid x \in K\}$.

3. f is uniformly continuous on K.

Note: the proofs of these features are good practice. In particular, proofs that exploit the Heine-Borel property.

Proof 1: Compact

Let y_n be a sequence in $f(K)$.
Then, $\forall n, y_n = f(x_n)$ for $x_n \in K$.
It follows that there exists a subsequence $\{x_{n_k}\}_k$ converging to x in K.
By continuity, $y_{n_k} = f(x_{n_k}) \overset{k\to\infty}{\to} f(x) \in f(k)$.

Proof 2: Achieves Its Extrema

Construct M.
By the suprememum property, $S = \sup\{f(x) \mid x \in \mathbb{R}\}$, $\forall n, \exists x_n \in K$ such that $S - \frac{1}{n} \leq f(x_n) < S$.
Since K is compact, there exists a subsequence $\{x_{n_k}\}_k$ converging to $x \in K$.
Since f is continuous at x, $f(x_{n_k}) \overset{k\to\infty}{\to} f(x)$, and also $S - \frac{1}{n_k} \leq f(x_{n_k} \leq S \overset{k\to\infty}{\to} S = f(x)$.

Proof 3: Uniformly Continuous

Suppose, for sake of contradiction, that $\exists \epsilon > 0, \forall \delta > 0, \exists x_\delta, y_\delta \in K, |x_\delta - y_\delta| < \delta$ and $|f(x_\delta) - f(y_\delta)| \geq \epsilon$.
Letting $\delta = \frac{1}{n}$, we may write $x_n, y_n \in K, |x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \epsilon$.
Since K is compact, there exists a subsequence $\{x_{n_k}\}_k$ which converges to $x \in K$.
Since $|x_{n_k} - y_{n_k}| < \frac{1}{n_k}$, then $\{y_{n_k}\}_k$ also converges to x.
By continuity of f at x, $\lim_{k\to\infty} f(x_{n_k}) - f(y_{n_k}) = 0$. However, this contradicts the established fact that $|f(x_n) - f(y_n)| \geq \epsilon$ for $\epsilon > 0$.

Notation

Let $I \subseteq \mathbb{R}$ be an interval.

Sequence of Functions

$f_n(x), x \in I, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$

Series of Functions

$S_n(x) = \sum_{k=0}^{n} f_k(x)$

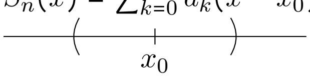
Definition: Pointwise Convergence

A sequence or series of functions converges pointwise on I if and only if $\forall x \in I, \{f_n(x)\}_n$ is convergent.
Call $f(x)$ the limit.

Q: Under what conditions do properties of a sequence (e.g. continuity, differentiability, integrability) propogate to the limit?

Power Series

$\sum_{n\geq 0} a_n(x - x_0)^n$
$S_n(x) = \sum_{k=0}^{n} a_k(x - x_0)^k$



Fourier Series

$S_n = a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx)$ is 2π-periodic.

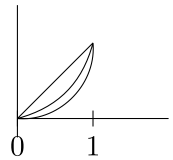
Approximation

For purposes of approximation, it is useful to know if, for example, the integral may be approximated by taking integrals of the partial sums.

Deficiencies of Pointwise Convergence

Example 1

On $[0,1], f_n(x) = x^n \overset{n\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases}$,



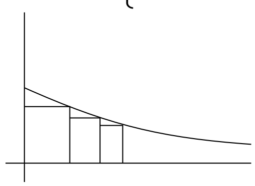
f_n is continuous on $[0,1], \forall n$, but f is not.

- Exercise

  Show that there is no uniform convergence here.
  Hint: negate uniform convergence and prove the negation.

Example 2

$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$ is not Riemann-integrable on $[0,1]$.



If r_n denotes a denumeration of rationals in $[0,1]$, define $f_n(x) = \begin{cases} 1 & \text{if } x \in \{r_0, \dots, r_n \\ 0 & \text{otherwise} \end{cases}$.

So f_n converges pointwise on $\chi_{\mathbb{Q}}$.
Yet, $\forall n, f_n$ is Riemann-integrable and $\int_0^1 f_n(x)\, dx = 0$.

Definition: Uniform Convergence

We say $f_n : D \to \mathbb{R}$ (e.g. D an interval) converges uniformly to f on D (notation $f_n \rightrightarrows f$ on D) if $\forall \epsilon > 0, \exists n \in \mathbb{N}, n \geq \mathbb{N} \implies \begin{cases} |f_n(x) - f(x)| < \epsilon, \forall x \in D \\ \sup_D |f_n - f| < \epsilon \end{cases}$

Compare with Pointwise Convergence

Compare to $f_n \to f$ pointwise on D.
$\forall x \in D, \forall \epsilon > 0, \exists N \in \mathbb{N}, n \geq \mathbb{N} \implies |f_n(x) - f(x) < \epsilon$.
In this case, the behavior is primarily contingent upon the choice of x. That is $N(x,\epsilon)$ is dependent on x.

Theorem: Weierstrass M-Test

Let $f_n : D \to \mathbb{R}$ be bounded by M_n on D.
If $\sum_{n=1}^{\infty} M_n < \infty$, then the series $S_n(x) = \sum_{k=1}^{n} f_k(x)$ converges uniformly to $S(x)$

Proof

$\forall x \in D, |S_n(x) - S(x)| = |\sum_{k=n+1}^{\infty} f_k(x)| \overset{\text{triangle inequality}}{\leq} \sum_{k=n+1}^{\infty} |f_k(x) \leq \sum_{k=n+1}^{\infty} M_k$, where $\sum_{k=n+1}^{\infty} M_k$ is a uniform bound in x.

Let $\epsilon > 0, \exists n, n \geq N \implies \sum_{k=n+1}^{\infty} M_k < \epsilon$.
Then $\forall x \in D, n \geq N, |S_n(x) - S(x)| \leq \sum_{k=n+1}^{\infty} M_k < \epsilon$. ■

Theorem: Continuity and Uniform Limits

Let $f_n D \to \mathbb{R}$ be continuous on D for all n and $f_n f$ on D ($\lim_{n\to\infty} \sup_D |f_n - f| = 0$).
Then f is continuous on D.

Proof

Fix $x \in D$, with x_n converging to x in D.
What To Show: $f(x_n) \xrightarrow{n\to\infty} f(x)$.
Scratch: $f(x_n) - f(x) = (f(x_n) - f_p(x_n)) + (f_p(x_n) - f_p(x)) + (f_p(x) - f(x))$.
Let $\epsilon > 0$ be given.
$f_n \rightrightarrows f : \exists N, n \geq N \implies |f_n(y) - f(y)| < \frac{\epsilon}{3}, \forall y \in D$.
For $p \geq N, |f_p(y) - f(y)| < \frac{\epsilon}{3}, \forall y \in D \implies \forall n \in \mathbb{N}, |f(x_n) - f(x)| \overset{\text{triangle inequality}}{\leq} \frac{2\epsilon}{3} + |f_p(x_n - f_p(x)|$.
With $p = N$, since f_p is continuous at x, $\exists N_1, n \geq N_1 \implies |f_p(x_n) - f_p(x)| < \frac{\epsilon}{3}$.
Hence, for $n \geq N_1, |f(x_n) - f(x)| \leq \epsilon$. ■

Riemann-Integrability

Fix $D = [a, b]$ and $g : [a, b] \to \mathbb{R}$ bounded by $|g(x)| \leq M, \forall x$.

Definition: Subdivision

$\sigma = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b$

Definition: Upper and Lower Riemann Sums

$S^+(g, \sigma) = \sum_{k=1}^{n} (x_k - x_{k-1}) M_k$ is the upper sum.
$S^-(g, \sigma) = \sum_{k=1}^{n} (x_k - x_{k-1}) m_k$ is the lower sum.
Where $M_k = \sup_{[x_{k-1}, x_k]} g$ and $m_k = \inf_{[x_{k-1}, x_k]} g$.
This gives $-M(b - a) \leq S^-(g, \sigma) \leq S^+(g, \sigma) \leq (b - a)M$.
If $\mathfrak{S}[a, b] = \{\text{subdivisions of } [a, b]\}$, then
$I^-(g) = \sup_{\sigma \in \mathfrak{S}[a,b]} S^-(g, \sigma)$ and $I^+(g) = \inf_{\sigma \in \mathfrak{S}[a,b]} S^+(g, \sigma)$.

Definition: Riemann Integrable

g is Riemann integrable if $I^+(g) = I^-(g)$ and we denote $\int_a^b g(t)\, dt = I^+(g)$.

Lemma

g is Riemann integrable if and only if $\forall \epsilon > 0, \exists \sigma \in \mathfrak{S}[a, b]$ such that $S^+(g, \sigma) - S^-(g, \sigma) < \epsilon$.

Properties

1. Continous functions and monotone functions are Riemann Integrable.

2. $f \mapsto \int_a^b f(t)\, dt$ is linear.

3. If f, g are Riemann Integrable and $f(x) \leq g(x), \forall x \in [a, b]$, then $\int_a^b f(t)\, dt \leq \int_a^b g(t)\, dt$.

Theorem:

If $f_n \rightrightarrows f$ on $[a, b]$ and f_n is Riemann Integrable for all n, then f is Riemann Integrable on $[a, b]$ and $\lim_{n\to\infty} \int_a^b f_n(t)\, dt = \int_a^b \lim_{n\to\infty} f_n(t)\, dt = \int_a^b f(t)\, dt$.

Proof

$\forall n, \forall x \in [a, b], f_n(x) - \epsilon \leq f(x) \leq f_n(x) + \epsilon$ where $\epsilon_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|$ (by hypothesis $e_n \xrightarrow{n\to\infty} 0$)
Then, for any $\sigma \in \mathfrak{S}[a, b]$, $S^-(f_n, \sigma) - \epsilon_n(b - a) \leq S^-(f, \sigma) \leq S^+(f, \sigma) \leq S^+(f_n, \sigma) + \epsilon_n(b - a)$.
It follows that $S^+(f, \sigma) - S^-(f, \sigma) \leq S^+(f_n, \sigma) - S^-(f_n, \sigma) + 2\epsilon_n(b - a)$.
Finishing the proof is left as an exercise.

October 18, 2023

Overview

- Sequences/Series

- Power Series

- Exponential and Logarithms

Fundamental Theorems of Calculus

Full proofs in 105A lecture notes.

Differentiation of the Integral

$f : [a, b] \to \mathbb{R}$ continuous.
$\forall x \in [a, b]$, can define $F(X) = \int_a^x f(t)\, dt$.
Then F is continuously differentiable on $[a, b]$
$F'(x) = f(x)$ for $x \in [a, b]$.

Integration of the Derivative

$f \in C^1[a, b]$ with one-sided derivatives at a and b well defined. (e.g. $\frac{f(a+h)-f(a)}{h} \underset{h>0; h\to 0}{\longrightarrow} f'(a)$.
Then $\forall x, y, a \leq x \leq y \leq b$, $f(y) - f(x) = \int_x^y f'(t)\, dt$.

Theorem: Differentiability of Uniform Limits

Let $f_n : (a, b) \to \mathbb{R}$ be a sequence in $C^1[a, b]$, and assume $f_n(x) \to f(x)$ pointwise while $f_n'(x) \rightrightarrows g(x)$ uniformly. Then $f \in C^1(a, b)$ and $f' = g$.

Proof

Fix $a_0 \in (a, b)$.
Then $\forall x \in (a, b)$, by the Fundamental Theorem of Calculus,

$$f_n(x) - f_n(a_0) = \int_{a_0}^{x} f_n'(t)\, dt$$

Observe that $f_n(x) \xrightarrow[n\to\infty]{} f(x)$ and $f_n(a_0) \xrightarrow[n\to\infty]{} f(a_0)$ pointwise, and $\int_{a_0}^{x} f_n'(t)\, dt \to \int_{a_0}^{x} g(t)\, dt$ by the integrability of uniform limits. Then

$$f(x) - f(a_0) = \int_{a_0}^{x} g(t)\, dt,\ \forall x \in (a, b)$$

which implies $f \in C^1$ and $f' = g$. $\blacksquare$

Interesting Applications

$S_n(x) = \sum_{k=0}^{n} f_k(x)$.
Suppose pointwise convergence, that $S_n'(x) = \sum_{k=0}^{n} f_k'(x)$ is continuous, $|f_k'(x)| \leq M_k$ and $\sum_{k=0}^{\infty} M_k < \infty$.
Long story short, this implies

$$\left(\sum_{k=0}^{\infty} f_k(x)\right)' = \sum_{k=0}^{\infty} f_k'(x)$$

Example

$f(x) = \sum_{n=0}^{\infty} \frac{\cos(nx)}{n^3}$
Call $u_n(x) = \frac{\cos(nx)}{n^3}$, then $|u_n(x)| \leq \frac{1}{n^3}$ summable and $|u_n'(x)| = \left|\frac{-\sin(nx)}{n^2}\right| \leq \frac{1}{n^2}$ summable.
This implies $f'(x) = -\sum_{n=0}^{\infty} \frac{\sin(nx)}{n^2}$.
Repetition of this process informs us that $f \in C^2$.

Power Series

$S_n(x) = \sum_{k=1}^{n} a_k(x - x_0)^k$ for, $x_0 \in \mathbb{R}$ fixed, is 'centered at x_0.' Note that each term is $C^{\infty}(\mathbb{R})$.

Example 1

$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ for $|x| < 1$.

Example 2

$\exp(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges $\forall x \in \mathbb{R}$.

- Why?
  Ratio Test.

$$\frac{a_{k+1}}{a_k} = \frac{x^{k+1}}{(k+1)!} \cdot \frac{k!}{x^k} = \frac{x}{k+1}$$

  So $\left|\frac{a_{k+1}}{a_k}\right| \xrightarrow[k\to\infty]{} 0$

Lemma: Radius of Convergence

Suppose a power series $\sum_{n\geq 0} a_n x^n$ converges at $b \in \mathbb{R}$.

1. Converges absolutely $\forall x, |x| < |b|$.

2. $\forall a \in (0, b)$ converges uniformly on $[-a, a]$.

- Proof of 1
  Suppose $\sum_{n\geq 0} a_n b^n$ converges.
  Then $a_n b^n \to 0$.
  Let x such that $|x| < b$, then

$$|a_n x^n| = \left|a_n b^n \left(\frac{x}{b}\right)^n\right| \leq M \left(\frac{|x|}{b}\right)^n$$

  By term test, $\sum_{n=0}^{\infty} |a_n x^n| < \infty \implies \sum a_n x^n$ converges absolutely.

- Proof of 2
  If $|x| \leq a < b$,

$$|a_n x^n| \leq M \left(\frac{|x|}{b}\right)^n \leq M \left(\frac{a}{b}\right)^n$$

  Thus, by M-test for $x \in [-a, a]$, the series converges uniformly on $[-a, a]$.

- Upshot
  The set where a power series converges is an interval centered at x_0.

Theorem: Radius of Convergence

Given a power series, define R to be such that $\frac{1}{R} = \limsup_n |a_n|^{1/n}$. Then

1. $\forall a \in (0, R)$, the series converges uniformly on $[-a, a]$.

2. If $|x| > R$, the series diverges.

Proof

IMAGE HERE - RADIUS OF CONVERGENCE
Fix x. As an exercise, $\limsup_n |a_n x^n|^{1/n} = |x| \cdot \limsup_n |a_n|^{1/n} = \frac{|x|}{R}$.
Recall that $\limsup_n |a_n x^n|^{1/n} = \lim_{n\to\infty} y_n$ where $y_n = \sup_{k>n}\{|a_k x^k|^{1/k}\}$.
If $\frac{|x|}{R} < 1$, then $\exists N_0, n \geq N_0 \implies y_n < \frac{1+\frac{|x|}{R}}{2} < 1$.
This implies $\forall k \geq N_0, |a_k x^k|^{1/k} \leq \frac{1+\frac{|x|}{R}}{2} < 1$ and, by the root test, the series converges.
If $\frac{|x|}{R} > 1$, $\forall n, \sup_{k>n}\{|a_k x^k|^{1/k}\} \geq \frac{|x|}{R}$.
By the properties of the supremum with $\epsilon = \left(\frac{|x|}{R} - 1\right)/2 > 0$,

$$\forall n, \exists k, 1 \leq \frac{\frac{|x|}{R} + 1}{2} \leq y_n - \epsilon \leq |a_k x^k|^{1/k} \leq y_n$$

Therefore $\forall n, \exists k > n, |a_k x^k|^{1/k} \geq 1$. ■

Observation: Behavior at Endpoints

At the endpoints of $(-R, R)$, a series might

Converge Absolutely

e.g. $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$, $R = 1$, $\frac{1}{R} = \limsup_n \left(\frac{1}{n^2}\right)^{1/n} \xrightarrow{n\to\infty} 1$

Converge Conditionally

e.g. $\sum_{k=1}^{\infty} \frac{x^k}{k}$, $R = 1 \longrightarrow \frac{1}{R} = \limsup_n \left(\frac{1}{n}\right)^{1/n} = 1$
Converges conditionally at $x = -1$.

Diverge

e.g. $\sum_{k=0}^{\infty} x^k$, $R = 1$

Theorem: Power Series Differentiation

Let $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ converge on $(x_0 - R, x_0 + R$.
Then $\forall k > 0, f \in C^k(x_0 - R, x_0 + R)$ and $f^{(k)}(x) = \sum_{n=k}^{\infty} a_n n(n-1)\cdots(n-k+1)(x-x_0)^{n-k}$, $\forall x \in (x_0 - R, x_0 + R)$

Exercise

Show that if $a_n \to a > 0$, then $\limsup a_n b_n = a \limsup b_n$.

Proof (by Induction)

Consider the series $S_n(x) = \sum_{n=1}^{\infty} a_n n(x - x_0^{n-1} = \sum_{n=0}^{\infty} a_{n+1}(n+1)(x - x_0)^n$.
Then

$$(x - x_0)\frac{1}{R \text{ of series of derivatives}} = \limsup_{n\to\infty}(a_n n)^{1/n} \limsup_{n\to\infty} a_n^{1/n} n^{1/n} = \limsup a_n^{1/n} = \frac{1}{R}$$

This implies $\sum_{k=0}^{\infty} \frac{d}{dx}(a_k(x - x_0)^k)$ converges uniformly on $[x_0 - a, x_0 + a]$, $\forall a \in (0, R)$.
By the Theorem on Differentiability of Uniform Limits, $f'(x)$ exists and $\forall x \in (x_0 - R, x_0 + R)$

$$f'(x) = \sum_{n=1}^{\infty} a_n n(x - x_0)^{n-1}$$

Repeat to get higher derivatives.

Integration

It is similarly possible to integrate term by term.

Famous Power Series

- $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \ |x| < 1$

- PSE of $\frac{1}{x}$ centerd at $x_0 > 0$

IMAGE HERE - GRAPH

$$\frac{1}{x} = \frac{1}{x - x_0 + x_0} = \frac{1}{x_0} \cdot \frac{1}{1 + \frac{x - x_0}{x_0}} = \frac{1}{x_0}\sum_{k=0}^{\infty}\left(-\frac{x - x_0}{x_0}\right)^k = \sum_{k=0}^{\infty}\frac{(-1)^k}{x_0^{k+1}}(x - x_0)^k \text{ if } |x - x_0| < |x_0|, x \in (0, 2x_0)$$

- $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

- $\exp(0) = 1$

- $\exp'(x) = \sum_{k=1} \frac{kx^{k-1}}{k!} = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} = \exp(x)$

Law of Exponents

$\exp(a)\exp(b) = \exp(a + b), \forall a, b \in \mathbb{R}$

Proof

Special case of the "Cauchy product of convergent series."
If $\sum_{n\geq 0} a_n$ converges absolutely to A and $\sum_{n\geq 0} b_n$ converges to B, then $\sum_{n\geq 0} c_n$ converges to AB, where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$$

- Heuristics

$$\left(\sum_{p=0}^{\infty} a_p x^p\right)\left(\sum_{l=0}^{\infty} b_l x^l\right) = \sum_{p,l\in\mathbb{N}_0^2} a_p b_l x^{p+l}$$

  IMAGE HERE - CIRCLES FROM L TO P

$$\{(p,l) : p+l = n, p, l \in \mathbb{N}_0\} = \{(0,n), (1,n-1), \ldots, (n,0)\}$$

Proof Continued

Aexp$(a) = \sum_{k=0}^{\infty} \frac{a^k}{k!}$ and $\exp(b) = \sum_{l=0}^{\infty} \frac{b^l}{l!}$, thus $\exp(a)\exp(b) = \sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} \frac{(a+b)^n}{n!} = \exp(a+b)$ \) since

$$c_n = \frac{1}{n!}\sum_{n=0}^{n} \frac{a^k}{k!} \cdot \frac{b^{n-k}}{(n-k)!} \text{ and } n! = \frac{1}{n!}(a+b)^n$$

Power Series Expansion of Exponential

Centered at x_0, we have

$$\exp(x) = \exp(x-x_0)\exp(x_0) = \exp(x_0)\sum_{k=0}^{\infty} \frac{(x-x_0)^k}{k!}$$

Observation:

exp is the only $C^1(\mathbb{R})$ solution of $\begin{cases} \exp'(x) = \exp(x) \\ \exp(0) = 1 \end{cases}$

- Proof If f solves the above, then for some constant c

$$\frac{d}{dx}\left(f(x)\exp(-x)\right) = f'(x)\exp(-x) - f(x)\exp(-x) = 0 \underset{\text{MVT}}{\Longrightarrow} f(x)\exp(-x) = c = f(0)\exp(-0) = 1$$

  this implies

$$f(x) = \exp(x) f(x)\exp(-x) = \exp(x)$$

Exponential Features

$$\exp(x) > 0, \forall x \in \mathbb{R} \implies \begin{cases} \text{if } x \geq 0, \exp(x) \geq 1 > 0 \\ \text{if } x < 0, \exp(x) = \frac{1}{\exp(-x)} > 0 \end{cases}$$

Theorem: Exponential and e

$\exp(x) = (\exp(1))^x \forall x \in \mathbb{R}$ and $e = \exp(1)$

Proof

Using law of exponents for

$$x \in \mathbb{N}: \quad \exp(n) = \exp(1 + (n-1) = e \cdot \exp(n-1) = \cdots = e^n \exp(0)$$

$$x = \frac{1}{q}, q \in \mathbb{N}: \quad \left(\exp\left(\frac{1}{q}\right)\right)^q = \exp\left(\frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}\right) = \exp(1) = e$$

$$\therefore \exp\left(\frac{1}{q}\right) = e^{1/q}$$

$$x = \frac{p}{q}, p, q \in \mathbb{N}: \quad \exp\left(\frac{p}{q}\right) = \exp\overbrace{\left(\frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}\right)}^{p} = \left(e^{1/q}\right)^p = e^{p/q}$$

$$x \in -\mathbb{N}, \mathbb{Q} < 0: \quad \text{left as an exercise}$$

Therefore, the functions $x \mapsto \begin{cases} \exp(x) \\ e^x \end{cases}$ are continous on $\mathbb{R}$ and agree on $\mathbb{Q}$. This implies that they must be equal everywhere.