Analysis I

October 2, 2023

Lecture Notes

Class will not have dedicated lecture notes. Many are available already.
Undergraduate notes are available on Canvas.
Lecture 1 overview available on Canvas (lecture1.pdf).

Tentative Office Hours

Mondays 2-3pm and Tuesday 1-2pm.

Homework

Nominally due at beginning of class; ask for leeway if needed.
First week homework will be review of undergraduate proofs.
First homework due Wednesday, October 11.

Notation

Natural Numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$
Non Negative Integers: $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
Integers: $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$
Rationals: $\mathbb{Q} = \left\{\frac{p}{q},\ p \in \mathbb{Z},\ q \in \mathbb{Z}\right\} = \mathbb{Z} \times \mathbb{N}/\infty$

- Equivalent representation of rationals: $(p_1, q_1) \sim (p_2, q_2)$ iff $p_1 q_2 = p_2 q_1$

Sequence of Rationals: $\{u_n\}_{n \in \mathbb{N}},\ u_n \in \mathbb{Q},\ \forall n.$

Properties of the Rationals

$(\mathbb{Q}, +, \cdot)$ is a (ii) totally ordered (i) field satisfying the (iii) Archimedean property.

(i) Field

1. $+$ is associative: $(a + b) + c = a + (b + c)$

2. $+$ is commutative: $a + b = b + a$

3. $\cdot$ is associative and commutative.

4. $\exists 0 \in \mathbb{Q}$ such that $\forall a \in \mathbb{Q}$, $0 + a = a + 0$

5. $\exists 1 \in \mathbb{Q} \setminus \{0\}$ such that $\forall a \in \mathbb{Q}$, $1 \cdot a = a \cdot 1 = a$

6. $\forall a \in \mathbb{Q} \setminus \{0\}$ $\exists b \in \mathbb{Q}$, $a \cdot b = b \cdot a = 1$

    - $b = a^{-1} = \frac{1}{a}$

(ii) Totally Ordered

$\exists$ a set $\mathbb{Q}_+ \subseteq Q$ of "Positive Numbers" stable under $+$ and $\cdot$ such that $\forall A \in \mathbb{Q}$ either $a > 0$ ($a \in \mathbb{Q}_+$), $-a > 0$ (also $a < 0$) or $a = 0$.

- Ordering: $\forall a, b \in \mathbb{Q}$, $a < b$ if and only if $b - a > -0$.

- Trichotomy: $\forall a, b \in \mathbb{Q}$ either $a < b$, $a > b$, or $a = b$.

- $\max(a, b) = \begin{cases} a & \text{if } a > b \\ b & \text{otherwise} \end{cases}$.

- $|a| = \max(a, -a)$ (helps measure distance in $\mathbb{Q}$).

- $\text{dist}(a, b) := |b - a|$

- Triangle Inequality: $|u \pm v| \leq |u| + |v|$

- Observe also: $||u| - |v|| \leq |u \pm v|$. The triangle inequality may be used to prove this.

- Proof of Triangle Inequality $-|u| \leq u \leq |u|$ and $-|v| \leq v \leq |v|$, therefore $-|u| - |v| \leq u + v \leq |u| + |v|$. Therefore $u + v \leq |u| + |v|$ and $-(u + v) \leq |u| + |v|$ implies $|u + v| \leq |u| + |v|$.

(iii) Archimedian Property:

$\forall \epsilon > 0,\ \exists N,\ \forall n \geq N,\ \frac{1}{n} < \epsilon$.

Bounded Sequence of Rationals

$\{u_n\}_{n \in \mathbb{N}}$ is bounded if $\exists m \in \mathbb{Q}_+$ such that $|u_n| \leq M,\ \forall n$.
$\{u_n\}_{n \in \mathbb{N}}$ converges to $a \in \mathbb{Q}$ ($\lim_{n \to \infty} u_n = a$) if $\forall \epsilon > 0, \exists N, \forall n \geq N, |u_n - a| < \epsilon$.

Famous Limits

Decaying Rational

1. $\lim_{n\to\infty} \frac{1}{n} = 0$

   - $\forall \epsilon \in \mathbb{Q}_+,\ \exists n \in \mathbb{N},\ 0 < \frac{1}{n} < \epsilon$

   - $\forall n \in \mathbb{N},\ \exists n \in \mathbb{N},\ n \geq N$

     - b. and c. are equivalent.

Decaying Exponential Rational

$r \in \mathbb{Q},\ 0 < r < 1,\ \lim_{n\to\infty} r^n = 0.$

- Proof: Write $r = \frac{1}{1+k}$ for some $k > 0$. Then $r^n = \frac{1}{(1+k)^n} \overset{\text{Bernoulli}}{\leq} \frac{1}{1+nk}$.

Geometric

1. $r \in \mathbb{Q},\ 0 < r < 1,\ u_n = 1 + r + \cdots r^n = \frac{1-r^{n+1}}{1-r} \to \frac{1}{1-r}$

Features of Limits

Limits are Unique

If the limit of a sequence exists, it is unique.

Squeezing Lemma

If $\{a_n\}$, $\{b_n\}$ are such that $0 \leq a_n \leq b_n$, and $b_n \to 0$ as $n \to \infty$, then $a_n \to 0$.

Limits Preserve Order

If $a_n \leq b_n$ $\forall n$ and a_n and b_n converge, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.

Limit Algebraic Rules

$\lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = \lim_{n\to\infty}(a_n + b_n)$ when a_n and b_n converge.
If $\lim_{n\to\infty} b_n \neq 0$, then $\frac{a_n}{b_n} \to \frac{\lim a_n}{\lim b_n}$.

Peculiarity of the Rationals

$\mathbb{Q}$ lacks completeness.

Examples

Consider $u_1 = 1$ and $u_{n+1} = \frac{1}{2}(u_n + \frac{2}{u_n})$.
Then $u_n \in \mathbb{Q},\ \forall n \in \mathbb{N}$.
It can further be proven, by induction, that $u_n \geq 1,\ \forall n$. $\left(u_{n+1} - 1 = \frac{1}{2}(u_n + \frac{1}{u_n}) - 1 = \frac{1}{2u_n}((u_n - 1)^2 + 1)\right)$.
$\lim_{n\to\infty} u_n^2 = 2$.

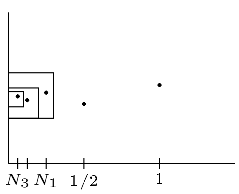
$$\begin{aligned} u_{n+1}^2 - 2 &= \left(\frac{1}{2}(u_n + \frac{2}{u_n})\right)^2 - 2 \\ &= \left(1\frac{1}{2u_n}(u_n^2 + 2)^2 - 4u_n\right) \\ &= 1\frac{4}{u_n^2}\left(u_n^2 - 2)^2\right) \\ &\leq \frac{1}{4}(u_n^2 - 2)^2 \end{aligned}$$

If u_n converged in $\mathbb{Q}$ to L, by algebraic limit rules, $2 = \lim u_n^2 = (\lim u_n)^2 = L^2$, yet $\sqrt{2} \notin \mathbb{Q}$.

Cauchy Criterion

A sequence $\{u_n\}_{n\in\mathbb{N}}$ of rationals is Cauchy if $\forall \epsilon > 0,\ \exists n \in \mathbb{N},\ \forall p, q \geq n,\ |u_p - u_q| < \epsilon$.

Visual Justification



Example 1

The sequence from before is Cauchy.
$|u_p - u_q| = \frac{|u_p^2 - u_q^2|}{|u_p + u_q|} \leq \frac{1}{2}|u_p^2 - u_q^2|$

Example 2

$u_n = \sum_{k=0}^{n} \frac{1}{k!} \in \mathbb{Q}$.

- This is increasing.
- It is bounded above by 3:

$$\begin{aligned} 1 + 1 + \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots + \frac{1}{2\cdots n} &\leq 1 + 1 + \cdots \frac{1}{2^{n-1}} \\ &\leq 1 + \frac{1 - 2^{-n}}{1 - \frac{1}{2}} \\ &\leq 3 \end{aligned}$$

Convergence, Cauchy and Boundedness.

Given a sequence $\{u_n\}_{n\in\mathbb{N}}$,
$\{u_n\}$ converges $\implies$ $\{u_n\}$ is Cauchy $\implies$ $\{u_n\}$ is bounded.
Note that in $\mathbb{Q}$ none of these implications may be reversed.

Construction of the Real Numbers

Short version: If the limit of a sequence isn't in the set, then define the limit as the sequence itself.
Let $C_{\mathbb{Q}} = \{\text{Cauchy sequences of rationals.}\}$.

Two Operations

- Termwise Addition $\{u_n\}_n + \{v_n\}_n := \{u_n + v_n\}_{n\in\mathbb{N}}$

- Termwise Multiplication $\{u_n\}_n \cdot \{v_n\}_n := \{u_n \cdot v_n\}_{n\in\mathbb{N}}$

Closure of Cauchy Sequence

If $\{u_n\}_n,\ \{v_n\}_n \in C_{\mathbb{Q}}$, then $\{u_n\}_n + \{v_n\}_n \in C_n$ and $\{u_n\}_n \cdot \{v_n\}_n \in C_n$.

Example

Infinite decimal expansion.
Fix $N \in \mathbb{Z},\ a_1 \cdots a_n \in \{0, \dots, 9\}$.
Then let $u_n = N + \sum_{k=1}^{n} a_k (10)^{-k}$ (that is the number $N.a_1 a_2 \dots a_n$).
This is always increasing and bounded above by $N + \sum_{k=1}^{n} 9 \cdot (10)^{-k} = N + \frac{9}{10} \cdot \sum_{k=1}^{n} (10)^{-(k+1)} \leq N + 1$.
Hence, it is Cauchy.

Increasing and Bounded Above Implies Cauchy

By contrapositive, increasing and not Cauchy implies not bounded.
By the negation of Cauchy and letting $p \geq q$ without loss of generality, we can force $u_p > u_q + \epsilon$.

Negation of Cauchy

$\exists \epsilon > 0,\ \forall N,\ \exists p, q \geq N,\ |u_p - u_q| > \epsilon$.

Real Numbers as Equivalence Classes of Cauchy Sequences

On $C_{\mathbb{Q}}$ define the relation $\{x_n\}_n \sim \{y_n\}_n$ if and only if $\lim_{n\to\infty} |(x_n - y_n)| = 0$.

Equivalence Relation

Reflexive: $x_n - x_n = 0$
Transitive: Uses algebraic limit rules. $x_n - z_n = x_n - y_n + y_n - z_n$.
Symmetric.

Definition of the Reals

$\mathbb{R} := C_{\mathbb{Q}}/\sim$
Then $x \in \mathbb{R},\ x = [\{x_n\}_n]$.

Addition and Multiplication of Reals

- Addition $x + y := [\{x_n + y_n\}_n]$.

- Multiplication $x \cdot y := [\{x_n \cdot y_n\}_n]$.

Operations Do Not Depend on Choice of Representative

If $\{x_n\}_n \sim \{x_n'\}_n$ and $\{y_n\}_n \sim \{y_n'\}_n$, then $\{x_n\}_n + \{y_n\}_n \sim \{x_n'\}_n + \{y_n'\}_n$.
If $\{x_n\}_n \sim \{x_n'\}_n$ and $\{y_n\}_n \sim \{y_n'\}_n$, then $\{x_n\}_n \cdot \{y_n\}_n \sim \{x_n'\}_n \cdot \{y_n'\}_n$.

The Reals are a Field

There are nine properties to check, eight of which are "obvious":

Commutativity of Addition (and Other "Obvious" Features)

$[\{x_n\}_n] + [\{y_n\}_n] = [\{x_n + y_n\}_n] = [\{y_n + x_n\}] = [\{y_n\}_n] + [\{x_n\}_n]$
That is, the Reals inherit most field features from the Rationals.

- Zero Element $0_{\mathbb{R}} = [\{0_{\mathbb{Q}}\}_n]$

- One Element $1_{\mathbb{R}} = [\{1_{\mathbb{Q}}\}_n]$

Multiplicative Inverses

How to define x^{-1} for $x \in \mathbb{R}$ where $x \neq 0$?

- Idea If $x = [\{x_n\}_n]$ choose $x^{-1} = [\{\frac{1}{x}\}_n]$.
  If $x \in \mathbb{R},\ x \neq 0$ then

  1. $\exists \{x_n\}_n \in C_{\mathbb{Q}}$ representing x with non zero entries.

  2. $\{\frac{1}{x_n}\}_n$ is Cauchy.

  – Proof of 1 Pick any $\{x_n\}_n$ representing x.

    * $x \neq 0$, so NOT ($\lim_{n\to\infty} x_n = 0$: $\exists \epsilon_0 > 0,\ \forall N,\ \exists n \geq N,\ |x_n| > \epsilon_0$.

    * $\{x_n\}$ is Cauchy: $\forall \epsilon > 0, \exists N,\ \forall p, q \geq N,\ |x_p - x_q| < \epsilon$.

Therefore, $\exists N$ such that $\forall p, q \geq N_1,\ |x_p - x_q| < \frac{\epsilon_0}{2}$
And $\exists N_2 \geq N,\ , |x_{N_2} > \epsilon_0$.
For $q \geq N_2$, the Cauchy Criterion states that $|x_q| = |x_q - x_{N_2} + x_{N_2} \geq |x_{N_2}| - |x_{N_2} - x_q| \geq \epsilon_0 - \frac{\epsilon_0}{2} \geq \frac{\epsilon_0}{2}$.
Therefore, the sought sequence is $\{x_{N_2} + k\}_{k \in \mathbb{N}}$.

- Proof of 2 $\left|\frac{1}{x_p} - \frac{1}{x_q}\right| = \frac{|x_p - x_q|}{|x_p||x_q|} \leq \frac{4}{\epsilon_0^2}|x_p - x_q|$.

Order on the Reals

Let $x \neq 0$, $\exists\{x_n\}_{n \in \mathbb{N}}$ be a representation of x and $\epsilon_0 > 0$.
Then for $|x_n| > \epsilon_0,\ \forall n \in \mathbb{N}$, there is a dichotomy:

- Either $\exists N \in \mathbb{N},\ x_n > \epsilon_0,\ \forall n \geq N$ (in which case we write $x > 0$)
- Or $\exists N \in \mathbb{N},\ x_n < -\epsilon_0,\ \forall n \geq N$ (in which case we write $x < 0$

Thus the Reals are totally ordered.

October 4, 2023

Overview

Completeness of $\mathbb{R}$.
Topology of the Real Line.

Non-zero Reals Are Either Positive or Negative

Given $x \in \mathbb{R} \setminus \{0\},\ \exists \delta \in \mathbb{Q}_+$ such that $\forall \{x_n\}_n$ representing x, $\exists N \in \mathbb{N}$ such that $|x_n| > \delta, \forall n \geq N$.
Moreover, one of the following (but not both) holds:

1. $\forall \{x_n\}_n \in x,\ \exists,\ x_n > \delta,\ \forall n \geq N$ (i.e. $x > 0$)
2. $\forall \{x_n\}_n \in x,\ \exists,\ x_n < -\delta,\ \forall n \geq N$ (i.e. $x < 0$)

Recall that $x \in \mathbb{R} \setminus \{0\}$ is an equivalence class of Cauchy sequences.

Total Ordering of the Reals

$x > 0$ produces a total ordering of $\mathbb{R}$ where $x < y$ if and only if $y - x > 0$.
$\rightsquigarrow \max(x, y) = \begin{cases} x & \text{if } x > y \\ y & \text{otherwise} \end{cases}$
$|x| = \max(x, -x)$ (which satisfies the triangle inequality)

Lemma A

Let $x, y \in \mathbb{R}$. If $\{x_n\}_n, \{y_n\}_n$ represent x, y and satisfy $x_n < y_n,\ \exists N \in \mathbb{N},\ \forall n \geq N$, then $x \leq y$.

- Proof By contradiction, suppose $x > y$ and $\exists \{x_n\}_n, \{y_n\}_n$ representing x, y such that $x_n \leq y_n,\ \forall n \geq N_1$.
  Then, by definition, $x - y > 0 \implies \exists \delta > 0,\ \exists N_2,\ x_n - y_n > \delta$ for $n \geq N_2$.
  But $x_n \leq y_n$ contradicts $x_n - y_n > \delta$.

Sequences of Reals

$\{x_n\}_n,\ x_n \in \mathbb{R}$
The definition of bounded, convergent and Cauchy sequences are the same as in $\mathbb{Q}$.

Injection of Rationals

$\iota : \mathbb{Q} \to \mathbb{R}$ such that $r \mapsto [\{u_n = r\}_n]$
This is isometric in the sense that $|\iota(r) - \iota(s)|_{\mathbb{R}} = |r - s|_{\mathbb{Q}}$

Theorem (Completeness 1)

Let $\{x_n\}_n \in C_{\mathbb{Q}}$ and $x = [\{x_n\}_n]$, then $\{\iota(x_n)\}_n$ converges to x.

Proof

What to show: $\forall \epsilon > 0,\ \exists N,\ \forall n \geq N,\ |\iota(x_n) - x| < \epsilon$.
Let $\epsilon \in \mathbb{Q}_+$. By the Cauchy criterion, $\exists N, \forall q, p \geq N,\ |x_p - x_q| < \epsilon$.
This is equivalent to $x_q - \epsilon \leq x_p \leq x_q + \epsilon$ where p is frozen.
Then by Lemma A, $x - \epsilon \leq \iota(x_p) \leq x + \epsilon$.
It follows that $\forall p \geq N,\ |\iota(x_p) - x \leq \epsilon$.

Corollary

$\mathbb{Q} \cong \iota(\mathbb{Q})$ is dense in $\mathbb{R}$. That is, $\forall \epsilon > 0,\ \forall x \in \mathbb{R},\ \exists r \in \mathbb{Q},\ |\iota(r) - x| < \epsilon$.

The Isometric Copy of Rationals

For brevity, the ι notation will be dropped and the $\mathbb{Q}$ will be understood as $\iota(\mathbb{Q})$.

Completeness of the Real Numbers

A sequence of real numbers converges in $\mathbb{R}$ if and only if it is Cauchy.

Proof

($\Longrightarrow$) This is clear.
($\Longleftarrow$) Take a Cauchy sequence of reals $\{x_n\}_n$. Then $\forall \epsilon > 0,\ \exists N,\ \forall p, q \geq\ \ |x_p - x_q| < \epsilon$.
Using the density of $\mathbb{Q}$, $\forall n \in \mathbb{N},\ \exists r_n \in \mathbb{Q}$ such that $|x_n - r_n| < \frac{1}{n}$.

Claim: $\{r_n\}_n$ is Cauchy. Indeed,

$$\begin{aligned}|r_p - r_q| &= |r_p - x_p + x_p - x_q + x_q - r_q| \\ &\leq |r_p - x_p| + |x_p - x_q| + |x_q - r_q| \\ &\leq \frac{1}{p} + |x_p - x_q| + \frac{1}{q}\end{aligned}$$

Take $\epsilon > 0$. $\{x_n\}$ cauchy implies $\exists N_1,\ \forall p,q \geq N, |x_p - x_q| \leq \frac{\epsilon}{3}$ and $\exists N_2,\ \forall p,q \geq N_2, \frac{1}{p} \leq \frac{\epsilon}{3},\ \frac{1}{q} \leq \frac{\epsilon}{3}$ for $p,q \geq \max(N_1,N_2)$ $|r_p - r_q| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$.
Then, for Cauchy $\{r_n\}_n$, call $r = [\{r_n\}_n]$, then $\lim_{n\to\infty} r_n = r$ by the above theorem.
Then my algebraic limit rules, $x_n(x_n - r_n) + r_n$ where $(x_n - r_n) \to 0$ and $r_n \to r$ as $n \to \infty$. So $\{x_n\}$ converges.

Example

Let $x_1 = 1$, $x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n})$.
Then $\{x_n\}_n \in C_{\mathbb{Q}}$, and it converges to $L \in \mathbb{R}$.
By algebraic limit rules, $L^2(\lim x_n)^2 = \lim x_n^2 = 2$.

Subsets of the Reals, Infimum and Supremum

Notation

Subset: $S \subseteq \mathbb{R}$
Inclusion: $x \in S$
Open Interval: $(a,b) = \{x \in \mathbb{R} | a < x < b\}$
Semiclosed Interval: $(a,b] = \{x \in \mathbb{R} | a < x \leq b\}$
Closed Interval: $[a,b] = \{x \in \mathbb{R} | a \leq x \leq b\}$
Unbounded Semiclosed Interval: $(-\infty, a] = \{x \in \mathbb{R} | x \leq a\}$
Unbounded Open: $(-\infty, a) = \{x \in \mathbb{R} | x < a\}$

Suprememum

$S \subseteq \mathbb{R}$ is bouned above (respectively below) if $\exists M \in \mathbb{R},\ \forall x \in S,\ x \leq M$ (respectively $\exists L \in \mathbb{R},\ \forall x \in S,\ L \leq X$)
S ad mits a least upper bound, LUB, suprememum or sup M if

1. $\forall x \in S, x \leq M$

2. $\forall M' \in \mathbb{R}$, upper bound of S, $M \leq M'$

If $\sup S$ exists, it is unique.
If $x \in S$ and x is an upper bound for S, then $x = \sup S$.

Example 1

$\sup(0,1) = \sup[0,1] = 1$

Example 2

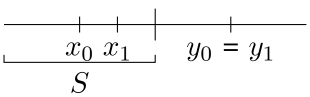
$S = \{x \in \mathbb{Q},\ x^2 < 2\}$ does not have a greatest element in $\mathbb{Q}$, nor a least upper bound in $\mathbb{Q}$.

Theorem (Completness 2)

Every subset $S \subseteq \mathbb{R}$, nonempty and bouned above, has a supremum in $\mathbb{R}$.

Proof

By dichotomy.
$S \neq \varnothing \implies \exists x_0 \in S$ and S bounded above implies $\exists y_0 \in \mathbb{R},\ \forall x \in S,\ x \leq y_0$ (in particular $x_0 \leq y_0$).
If $x_0 = y_0$, done. Otherwise, consider $m_0 = \frac{x_0+y_0}{2}$.



Two options exist: if m_0 is an upper bound for S, set $y_1 = m_0$ and $x_1 = x_0$.
Otherwise, $\exists x_1 \in S$, such that $m_0 < x_1$ so set $y_1 = y_0$.
Repeat this process forever to construct two sequences $x_n,\ y_n$.
$\forall n,\ x_n \in S,\ y_n$ is an upper bound for S.

- $x_n \leq y_n$
- x_n is increasing and bounded above by y_0, so it must be Cauchy and converging to x.
- y_n is decreasing and bounded below by x_0, so it must be Cauchy and converging to y.
- $|x_{n+1} - y_{n+1}| \leq \frac{|x_n - y_n|}{2}$ which implies $|x_n - y_n| \leq \frac{1}{2^n}|x_0 - y_0|$ and $x = y = z$.

Therefore, the process may be understood as $x_0 \leq \cdots \leq x_n \leq x_{n+1} \leq y_{n+1} \leq y_n \leq \cdots \leq y_0$.
There remain two things to check: (1) z is an upper bound for S and (2) z is no larger than any other upper bound for S.

1. Take $x \in S,\ \forall n,\ x \leq y_n \xrightarrow{n\to\infty} x \leq Z$.
2. Take upper bound for S, z'. $x_n \leq z',\ \forall n \xrightarrow{n\to\infty} z \leq z'$.

So $z = \sup S$.

Monotone Convergence Theorem (Completeness 3)

An increasing sequence of reals, $\{x_n\}_n$, that is bounded above, converges to $\sup X = \sup\{x_n | n \in \mathbb{N}\}$.
To prove that this converges, since it is monotone and bounded above it is Cauchy and therefore must be convergent.

Proof

Call x the limit, then $\forall n,\ x_n \leq x$. To see this, suppose $\exists n_0,\ x < x_{n_0}$ then $\forall m \geq m_0,\ x < x_{m_0} \leq x_m \implies |x_m - x| \geq |x_{n_0} - x| > 0,\ \forall m \geq n_0$ is a contradiction.
Let M be an upper bound of X. Then $x_n \leq M,\ \forall n \xrightarrow{n\to\infty} x \leq M \implies x = \sup X$.

Theorem (Existence of Roots)

$\forall x \in \mathbb{R}$ where $x > 0,\ p \in \{2, 3,\ \ldots\ ,\},\ \exists! y > 0$ such that $y^p = x$.

Proof

Left as an exercise.
Either by dichotomy or consider $S = \{y \in \mathbb{R} | y^p < x\}$, show: $S \neq 0$, bounded above and $(\sup S)^p = x$.
For uniqueness, show $y_1^p = y_2^p = x \iff 0 = y_1^p - y_2^p = (y_1 - y_2)(\cdots \neq 0) \implies y_1 = y_2$.

Topological Properties

$S \subseteq \mathbb{R}$ is open if $\forall x \in S,\ \exists a, b \in \mathbb{R},\ x \in (a, b) \subset S$.
x is an accumulation or limit point of S if $\forall \epsilon > 0,\ \exists y \in S,\ |0 < |x - y| < \epsilon$.
$S \subseteq \mathbb{R}$ is closed if it contains all its limit points.
A set may be both open closed, just open, just closed or neither.
Given $S \subseteq \mathbb{R}$, the interior of S is $\bigcup_{S' \text{ open} \subset S} S' = S^{\text{int}} = S^0$.
The closure is $\bigcap_{F \text{ closed} \supseteq S} F = \overline{S} \overset{\text{wts}}{=} S \cup \{\text{limit points of } S\}$.

Example

$\{x\}$ is not open, but, since the limit points of x are $\varnothing$, it is closed.

Propositions

1. Arbitrary unions and finite intersections of open sets are open.

2. S is open if and only the complement $S^c = \mathbb{R} \setminus S$ is closed.

3. Arbitrary intersections and finite unions of closed sets are closed.

Bolzano-Weierstrass Theorem

A bounded sequence in $\mathbb{R}$ ad mits a convergent (Cauchy) subsequence. $\exists M,\ |x_n| \leq M,\ \forall n$

Proof by Dichotomy

Suppose $I_0 = [a, b]$ contains the sequence.
Construct a sequence of intervals by indicators: if $\left[a, \frac{a+b}{2}\right]$ contains infinitely terms of $\{x_n\}_n$, choose n such that $x_{n_1} \in \left[a, \frac{a+b}{2}\right]$ and call $I_1 = \left[a, \frac{a+b}{2}\right]$.

Otherwise, $\left[\frac{a+b}{2}, b\right]$ must contain infinitely many terms. Choose n in a similar fashion as above such that $I_1 = \left[\frac{a+b}{2}, b\right]$.
This process may be repeated to create a sequence of intervals such that $I_k \supseteq I_{k+1} \supseteq I_{k+2}$ and $l(I_k) = \frac{b-a}{2^k}$.
A subsequence $\{u_{n_k}\}_k$ such that $u_{n_k} \in I_l$ for $k \geq l$.

Exercise

Extract a Cauchy criterion out of the above.

October 9, 2023

Overview

- Topology of $\mathbb{R}$ continued.

- Numerical series.

Next Wednesday

- Absolute and Conditional Convergence.

- Rearrangement theorem.

Last Time

Finished with Bolzano-Weierstrass.

Limits

Limit Point

We say $x \in \mathbb{R}$ is a limit point of $\{x_n\}_n$ if a subsequence of $\{x_n\}_n$ converges to x.
Equivalently, $\forall \epsilon > 0,\ \forall n_0 \in \mathbb{N},\ \exists n \geq n_0,\ |x_n - x| < \epsilon$.
That is, the sequence revisits an epsilon neighborhood of x infinitely many times.

Limit Set

The limit set of $\{x_n\}_n : \mathrm{LS}(\{x_n\}_n)$ = the set of limit points of $\{x_n\}_n$.

- **TODO** Comments
  - if $\lim_{n\to\infty}\{x_n\} = x$, then $\mathrm{LS}(\{x_n\}_n) = \{x\}$.

  - The limit set can be as big as $\mathbb{R}$!

IMAGE HERE - ENUMERATE RATIONALS

  - What Bolzano-Weierstrass says is that if $\{x_n\}$ is bounded, then $\text{LS}(\{x_n\}) \neq \varnothing$.

- Examples $\text{LS}(\{n\}_n) = \varnothing$.
  $\text{LS}(\{x_n\}_n)$ is closed (good exercise).

Limit Superior

If $\{x_n\}_n \in [a, b]$ is bouneded, $\forall k \in \mathbb{N}$, $\sup\{x_j | j \geq k\}$ exists in $\mathbb{R}$.
Because

$$a \leq \sup\{x_j | j \geq k+1\} = y_{k+1} \leq \sup\{x_j | j \geq k\} = y_k$$

by the Monotone Convergence Theorem, $\{y_k\}_k$ converges. Call its limit $\limsup_n x_n = \inf_n \sup\{x_j | j \geq n\}$.

Limit Inferior

Similarly, define $\lim_n \inf x_n = \sup_n \inf\{x_j | j \geq n\}$.

Limit Superior and Limit Inferior Always Exist

What to show: $\limsup x_n, \liminf x_n \in \text{LS}(\{x_n\})$.
Left as an exercise.

Convergence at the Limit

A bounded sequence $\{x_n\}_n$ converges if and only if $\liminf_n x_n = \limsup_n x_n$.

- Proof Technique Often it is useful to structure a proof such that

$$L < \liminf_n x_n \leq \limsup_n x_n < L$$

Topology of the Reals Continued

Compactness

Let $A \subseteq \mathbb{R}$.
A is (sequentially) compact if every sequence in A has a limit point in A.
A is (Heine-Borel) compact if every open cover of A has a finite subcover.

- Open Cover $\{O_\alpha\}_{\alpha \in I}$, with O_α open, is an open cover of A if $A \subseteq \bigcup_{\alpha \in I} O_\alpha$.

- Finite Subcover $O_1, \ldots, O_n$, $n \in \mathbb{N}$.

Heine-Borel Theorem

Let $A \subseteq \mathbb{R}$.
The following are equivalent

1. A is Heine-Borel compact.

2. A is closed and bounded.

3. A is sequentially compact.

Proof

(1) $\implies$ (2) $\implies$ (3) $\implies$ (1)

- **TODO** Heine-Borel Compact Implies Closed and Bounded Suppose A satisfies the Heine-Borel property.
  Consider $\{(-n,n)\}_{n\in\mathbb{N}}$. Clearly $\bigcup_n (-n,n) = \mathbb{R} \supseteq A$.
  By Heine-Borel, $\exists n_0, \dots, n_p$ such that $A \supseteq \bigcup_{j=0}^{p} (-n_j,\ n_j) = (-N,N),\ N = \max(n_0,\ \dots,\ n_p)$. So A is bounded.
  A is closed if $y \notin A \implies y$ is not a limit point of A.
  Take $y \in A^c$, then $A \subseteq \mathbb{R} \setminus \{y\} = \bigcup_{n\in\mathbb{N}} (-\infty, y - \frac{1}{n}) \cup (y + \frac{1}{n}, \infty)$.
  IMAGE HERE - OPEN SETS SQUEEZING y
  By the Heine-Borel property,

$$A \subseteq \bigcup_{n_0, \dots, n_p} (-\infty, y - \frac{1}{n}) \cup (y + \frac{1}{n}, \infty)$$
$$= (-\infty, y - \frac{1}{N}) \cup (y + \frac{1}{N}, \infty)$$

  Which implies $A \cap [y - \frac{1}{N}, y + \frac{1}{N} = \varnothing$ and y is not a limit point of A.
  That is, A contains its limit points.

- Closed and Bounded Implies Sequential Compactness Suppose A is both closed and bounded.
  Let $\{x_n\}_n \in A$. Then $\{x_n\}_n$ is bounded. By Bolzano-Weierstrass, it has a limit point x and a subsequence $\{x_{n_k}\}_k$ converging to x.
  Since A is closed, $\lim_{k\to\infty} x_{n_k} = x \in A$. ■

- **TODO** Sequential Compactness Implies Heine-Borel Suppose $A \subseteq \mathbb{R}$ is sequentially compact.
  Consider an open cover of A, $\{O_\alpha | \alpha \in I\}$.
  First, turn it into a countable cover:

  - $\forall \alpha \in I,\ O_\alpha \subseteq \left(r_\alpha^1, r_\alpha^2\right),\ r_\alpha^1, r_\alpha^2 \in \mathbb{Q}$

  Assume that $\{O_\alpha\}_\alpha$ can be made countable $(O_1, \dots, O_n)$
  By contradiction, suppose $\forall n \in \mathbb{N},\ A \setminus \left(\bigcup_{j=1}^{n} O_j\right) \neq \varnothing$.
  Take $x_n \in A \setminus \left(\bigcup_{j=1}^{n} O_j\right)$. Since A is sequentially compact, $\exists \{x_{n_k}\}_k$ subsequence of $\{x_n\}_n$ converging to $x \in A$.

Since $A \subset \bigcup_{j\in\mathbb{N}} O_j,\ \exists j_0,\ x \in O_{j_0}$, O_{j_0} is open: $\exists \delta > 0,\ (x-\delta, x+\delta) \subseteq O_{j_0}$.
Then $\exists N,\ k \geq N \implies x_{n_k} \in (x-\delta, x+\delta) \subseteq O_{j_0}$. But if k is such that $n_k > j_0$, we also have $x_{n_k} \notin O_{j_0}$ which is a contradiction!
IMAGE HERE - COMPACT SET, POINTS AND BALLS

Structure of Open and Closed Sets

A is open in $\mathbb{R}$ if and only if it can be written as an at most countable, disjoint union of open intervals.

Proof

For $x \in A,\ \exists (a,b)$, such that $x \in (a,b) \subseteq A$.
Let $I_x = \bigcup_{\text{open interval containing} x,\ I \subseteq A} I$. This is the maximal interval containing x in A.
Then, $A \subseteq \bigcup_{x\in A}\{x\} \subseteq \bigcup_{x\in A} I_x \subseteq A$.
That is, $A = \bigcup_{x\in A} I_x \quad (*)$.
Next, if $x, y \in A$, then $\begin{cases}\text{either } I_x = I_y \\ \text{or } I_x \cap I_y = \varnothing\end{cases}$
IMAGE HERE
The union $(*)$, as a disjoint union, is at most countable because each distinct one must contain a distinct rational and $\mathbb{Q}$ is countable.

Long Story Short

The topology of the reals avoids exotic sets. Closed sets, on the other hand, can be quite complicated.

Cantor Set

$C := \bigcap_{k\in\mathbb{N}_0} I_k$. I_{k+1} is obtained by removing the middle open third of each interval making I_k.
IMAGE HERE - CANTOR
$I_0 = [0,1]$. One interval of length 1.
$I_1 = [0,1/3] \cup [2/3,1]$. Two intervals of length $2/3$.
$I_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9]$. Four intervals of $(2/3)^2$
I_k is 2^k intervals of length $(2/3)^k$.
$I_{k+1} \subseteq I_k \implies C \subseteq I_k,\ \forall k \implies l(C) \leq l(I_k) = (2/3)^k \implies l(C) = 0$.

Triadic Expansions

Goal:

1. C is perfect (i.e. every point in C is a limit point of C).

2. C contains no open intervals.

Property 2 is easy because $C \subseteq I_k$, which does contain interval of length greater than $(1/3)^k$.

1. C is uncountable.

Every $x \in [0,1]$ can be written in the form $x = \sum_{k=1}^{\infty} \frac{a_k}{3^k},\ a_k \in \{0,1,2\}$.
That is, $x = 0.a_1 a_2 \ldots$ in base 3. This is not always unique (e.g. $1/3 = 0.100\ldots = 0.022\ldots$).
IMAGE HERE - THIRDS OF INTERVAL
Note that the Cantor set is removing all decimal expansions with leading 1s. That is, $x \in C$ if and only if it has a triadic expansion only made of 0s and 2s.

- Proof of 1 If $x \in C,\ x = \sum_{k \geq 1} \frac{a_k}{3^k} = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{a_k}{3^k}$, then $x_n \in C,\ \forall n$ and $x_n = 0.a_1 \ldots a_n 0000\ldots$ where $a_1, a_n \in \{0,2\}$.
  Unique representation can be maintained by forcing the behavior of the $n+1$th digit.

- Proof of 3 Every point in $[0,1]$ can also be written as $x = \sum_{n=1}^{\infty} = \frac{b_n}{2^n},\ b_n \in \{0,1\}$ (i.e. a binary expansion).
  Then $C \mapsto [0,1]$ gives $x = \sum \frac{a_k}{3^k} \mapsto \sum \frac{b_k}{2^k},\ b_k = \frac{a_k}{2}$ for $a_k \in \{0,2\}$ is a bijection!