

# Analysis II

January 9, 2024

## (Real) Analysis

- Calculus
  - Differential
  - Integral (Riemann)
- Functions and Maps
  - Measure Theory
  - (Lebesgue) Integration
- Topology
  - Completeness (as a metric space)
  - Compactness (Bolzano-Weierstrass theorem [real]) (Arzela-Ascoli)
  - Paracompactness / Metrizable / Baire Category Theorem
  - Algebraic / Combinatoric (continuous maps or functions)

### Definition: Cardinality

For sets  $A, B$ ,  $\text{Card}(A) = \text{Card}(B)$  if there exists a one-to-one correspondence  $q : A \leftrightarrow B$ .

Counting, labelling, indexing, etc.

$\text{Card}(A) \leq \text{Card}(B)$  if  $A \subset B$  or there exists a one-to-one mapping  $A \rightarrow B$ .

### Definition: Countable

If  $A \hookrightarrow \mathbb{N}$ , then  $A$  is countable.

### Theorem

The countable union of countable sets is countable.

### Proof

Let  $A_i = \{a_j\}_{j=1}^{\infty}$ ,  $i = 1, 2, \dots$

$$\begin{array}{cccc} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ \vdots & & & \\ a_{k1} & a_{k2} & a_{k3} & \cdots \end{array}$$

Index by diagonalization.

## Theorem

The cartesian product of countable sets is countable.

## Proof

$$X \times Y = \{(x_i, y_j) \mid x_i \in X, y_j \in Y\}$$

$$\begin{array}{cccc}
(x_1, y_1) & (x_1, y_2) & (x_1, y_3) & \cdots \\
(x_2, y_1) & (x_2, y_2) & (x_2, y_3) & \cdots \\
\vdots & & & \\
(x_k, y_1) & (x_k, y_2) & (x_k, y_3) & \cdots
\end{array}$$

## Theorem

$\text{Card}(2^X) > \text{Card}(X)$ , where  $2^X = \{A \subset X\}$  is the power set of  $X$ .

## Proof

For all  $x \in X$ ,  $\{x\} \subset 2^X$ , so  $\text{Card}(X) \leq \text{Card}(2^X)$ .

Assume, for sake of contradiction, that  $\text{Card}(X) = \text{Card}(2^X)$ .

Then, by definition, there exists a one-to-one correspondence  $\phi : X \leftrightarrow 2^X$ .

Set  $A = \{x \in X \mid x \notin \phi(x)\}$ , and let  $a = \phi^{-1}(A)$  (i.e.  $A = \phi(a)$ ).

If  $a \in A$ , then  $a \notin A \subset \phi(a)$ ; but if  $a \notin A$ , then  $a \in A$ , a contradiction.

## Theorem

$$\text{Card}(\mathbb{R}) = \text{Card}(2^{\mathbb{N}}).$$

## Topology of the Real Line

### Completeness (as a metric space)

$$d(a, b) = |a - b|, \quad \forall a, b \in \mathbb{R}.$$

1.  $x_i \rightarrow x$  if  $\forall \varepsilon > 0$ ,  $\exists n \in \mathbb{N}$  such that  $|x_i - x| < \varepsilon$ ,  $\forall i \geq n$ .
2.  $\{x_i\}$  is Cauchy if  $\forall \varepsilon > 0$ ,  $\exists n \in \mathbb{N}$  such that  $|x_i - x_j| < \varepsilon$ ,  $\forall i, j \geq n$ .

### Definition: Open Interval

$(a, b)$  is an open set on the real line.

There exist interior points for any subset  $A$  of real numbers.

$\forall x \in A$ ,  $x$  is interior if  $\exists (a, b)$  such that (1)  $x \in (a, b)$  and (2)  $(a, b) \subset A$ .

- Theorem
  - The union of open sets is open.
  - The intersection of finitely many open sets is open.
  - $\emptyset$  and  $\mathbb{R}$  are open.

### Definition: Limit Point

A limit point  $x \in \mathbb{R}$  of a subset  $A$  is a limit point in  $A$  if for every open neighborhood  $U$  of  $x$ ,  $(U \setminus \{x\}) \cap A \neq \emptyset$ .

### Definition: Closed

$A$  is closed if  $A$  contains all of its limit points.

- Theorem

$A$  is closed if and only if  $A^c = \mathbb{R} \setminus A$  is open.

- Proof

$A$  closed  $\implies A^c$  open.

Otherwise,  $\exists x \in A^c$  such that for every neighborhood  $U$  of  $x$ ,  $(U \setminus \{x\}) \cap A = \emptyset$  which would make it a limit point of  $A$  not in  $A$ . By assumption,  $A$  contains all its limit points so this is a contradiction.

$A^c$  open  $\implies A$  closed.

For any  $x$  a limit point of  $A$ , assume otherwise that  $x \in A^c$ .

Then there exists some neighborhood  $U$  of  $x$  such that  $U \subset A^c$  (since  $A^c$  is open).

It follows that  $(U \setminus \{x\}) \cap A = \emptyset$  and  $x$  is not a limit point of  $A$ , which is a contradiction.

### Definition: Sequential Compactness

$A$  is compact if  $\forall \{x_i\}$ ,  $x_i \in A$  there exists a convergent subsequence  $\{x_{i_k}\}$  and  $x_{i_k} \rightarrow x \in A$ .

- Theorem: Bolzano-Weierstrass

For  $A \subseteq \mathbb{R}$ ,  $A$  is compact if and only if  $A$  is closed and bounded.

- Proof

$A$  compact  $\implies A$  closed and bounded.

Assume that  $A$  is not bounded from above.

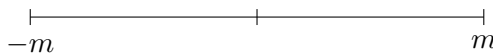
Then there exists a sequence  $\{x_i\}$ ,  $x_i \in A$  where  $x_{i+1} > x_i + 1$  and  $\{x_i\}$  has no convergent subsequences.

Then compactness implies closedness.

$A$  closed and bounded  $\implies A$  (sequentially) compact.

Let any  $\{x_i\}$ ,  $x_i \in A$ .

Claim:  $\forall \{x_i\}$  of reals, if there exists  $m \in \mathbb{R}$  such that  $|x_i| \leq m$ ,  $\forall m$  then there is some convergent subsequence.



Divide and conquer: dividing the interval in half necessitates that at least one half contains infinitely many points. Repeat indefinitely.

- Theorem: Heine-Borel

$A \subseteq \mathbb{R}$  is (sequentially) compact if and only if any open cover has a finite subcover.

- Proof

Heine-Borel Property  $\implies$  closed and bounded.

Assume that  $A$  is unbounded,  $U_n = (-n, n)$  and  $\{U_n\}_{n=1}^{\infty}$  an open cover for  $A \subseteq \mathbb{R}$  has no finite subcover.

Assume  $A$  is not closed, then  $x \in \dot{A}$  (where  $\dot{A}$  is the limit set of  $A$ ) and  $x \notin A$ ,  $U_n \left\{ \left(-\infty, x - \frac{1}{n}\right) \cup \left(x + \frac{1}{n}, +\infty\right) \right\}$ .

Then  $\{U_n\}$  covers  $\mathbb{R} \setminus \{x\} \supset A$  has no finite subcover of  $A$ .

$A$  is bounded and closed  $\implies A$  is Heine-Borel

Divide and conquer: using open sets with respect to open covers.

### Definition: Cantor Set

$C = \{x \in [0, 1] \mid \text{the ternary expansion of } x \text{ has only the digits } \{0, 2\}\}$ .

Equivalently, let  $C_0 = [0, 1]$ ,  $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ ,  $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$ .

Then  $C_n = \bigcup_{k=1}^{2^n} C_n^k$  and  $C = \bigcap_{n=1}^{\infty} C_n$ .

$|C_n| = 2^n \left(\frac{1}{3}\right)^n \rightarrow 0$ .

### Definition: Perfectly Symmetric Sets

Let  $\{\xi_n\}$  where  $\xi_n \in (0, \frac{1}{2})$ .

$E_0 = [0, 1]$ ,  $E_1 = [0, \xi_1] \cup [1 - \xi_1, 1]$ ,  $E_2 = [0, \xi_1 \xi_2] \cup [\xi_1 - \xi_1 \xi_2, \xi_1] \cup [1 - \xi_1, 1 - \xi_1 + \xi_1 \xi_2] \cup [1 - \xi_1 \xi_2, 1]$ .

Then the cantor set is given by  $\xi_n = \frac{1}{3}$ .

$E_n = \bigcup_{k=1}^{2^n} E_n^k$ ,  $|E_n^k| = \xi_1 \xi_2 \cdots \xi_n$ , and  $|E_n| = \sum |E_n^k| = 2^n \xi_1 \xi_2 \cdots \xi_n$ .

Therefore,  $E = \bigcap_{n=1}^{\infty} E_n$  and we define  $|E| = \lim_{n \rightarrow \infty} |E_n| = \lim_{n \rightarrow \infty} (2^n \xi_1 \xi_2 \cdots \xi_n) = \lambda$  where  $\lambda \in [0, 1]$ .

Let

$$2\xi_n = \frac{\left(1 + \frac{\log(\frac{1}{n})}{n-1}\right)^{n-1}}{\left(1 + \frac{\log(\frac{1}{n})}{n}\right)^n} < 1$$

, then

$$2^n \xi_1 \cdots \xi_n = \frac{1}{\left(1 + \frac{\log(\frac{1}{n})}{n}\right)^n} \rightarrow \lambda.$$

### Proof

$\lim_{n \rightarrow \infty} \left( \left(1 + \frac{x}{n}\right)^{n/x} \right)^x = e^x$ , then  $\lim_{y \rightarrow 0} (1 + y)^{1/y} = e$ ,  $\log(1 + y)^{1/y} = \frac{\log(1+y)}{y} \xrightarrow{y \rightarrow 0} 1$ .

Observe that

$$\left( \frac{\log(1+y)}{y} \right)' = \frac{\frac{y}{1+y} - \log(1+y)}{y^2} = \left( 1 + \frac{1}{1+y} - \log(1+y) \right)' = \frac{1}{(1+y)^2} - \frac{1}{1+y} = -\frac{y}{(1+y)^2} < 0$$

### Theorem

Cantor sets and perfect symmetric sets are closed, perfect, uncountable, and nowhere dense.

**January 11, 2024**

### Last Week

Cardinality.

Topology of the reals.

- Cantor (perfect symmetric sets)

$$C_0 = [0, 1]$$

$$C_1 = [0, 1/3] \cup [2/3, 1]$$

$$C_2 = [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 1]$$

$$C_n = \bigcup_{k=1}^{2^n} C_n^k$$

$$|C_n^k| = \left(\frac{1}{3}\right)^n$$

$C = \bigcap_{n=1}^{\infty} C_n$   
 $|C_n| = 2^n \frac{1}{3^n} = \left(\frac{2}{3}\right)^n \implies |C| = \lim_{n \rightarrow \infty} |C_n| = 0$   
 Closed, no interior points and uncountable.

- Perfect Symmetric Sets

$$\begin{aligned}
 \{\xi_k\} &\in (0, \tfrac{1}{2}) \\
 E_0 &= [0, 1] \\
 E_1 &= [0, \xi_1] \cup [1 - \xi_1, 1] \\
 E_2 &= [0, \xi_1 \xi_2] \cup [\xi_1 - \xi_1 \xi_2, \xi_1] \cup [1 - \xi_1, 1 - \xi_1 + \xi_1 \xi_2] \cup [1 - \xi_1 \xi_2, 1] \\
 E_n &= \bigcup_{k=1}^{2^n} E_n^k \\
 |E_n^k| &= \xi_1 \xi_2 \cdots \xi_n \\
 |E_n| &= 2^n \xi_1 \xi_2 \cdots \xi_n \\
 2\xi_n &= \frac{\left(1 + \frac{\log(\frac{1}{n})}{n-1}\right)^{n-1}}{\left(1 + \frac{\log(\frac{1}{n})}{n}\right)^n} < 1 \\
 |E_n| &= \frac{1}{\left(1 + \frac{\log(\frac{1}{n})}{n}\right)^n} \\
 |E| &= \lim_{n \rightarrow \infty} |E_n| = \frac{1}{e^{\log(\frac{1}{n})}} = \lambda, \quad \lambda \in (0, 1)
 \end{aligned}$$

## Volterra's Function

$$\phi(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

IMAGE HERE - graph of phi(x)

$$\phi'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$f(x) = \begin{cases} 0 & x \in E \\ \phi(x-a) & x \in (a, a+y) \\ -\phi(b-x) & x \in (b-y, b) \\ \phi(y) & x \in (a+y, b-y) \end{cases}, \quad (a, b) \in E^c$$

IMAGE HERE - f interval (a,b)

## Propositions

1.  $f'(x) = 0$  for  $x \in E$ .
2.  $f'(x)$  discontinuous on  $E$ .
3.  $f'$  exists on  $[0, 1]$  and is bounded.

Since  $|E| > 0$ ,  $f'(x)$  is not Riemann integrable and, therefore, the fundamental theorem of calculus does not apply.

## Lebesgue Outer Measure

$$|(a, b)| = b - a.$$

Let  $A \subseteq \mathbb{R}$ , then  $m^*(A) = \inf \{ \sum_{n=1}^{\infty} I_n \mid A \subseteq \bigcup_{n=1}^{\infty} I_n \}$

Question:  $m^*(A \cup B) \stackrel{?}{=} m^*(A) + m^*(B)$  for  $A \cap B \neq \emptyset$ ?

### Properties

1.  $A \subseteq B \implies m^*(A) \leq m^*(B)$ .
2.  $m^*(\emptyset) = 0$ .
3. If  $I$  is an interval, then  $m^*(I) = |I|$ .
4. If  $\{A_i\}$  is countable,  $m^*(\bigcup A_i) \leq \sum m^*(A_i)$ .

- Proof of 4

$\forall A_i, \exists \{I_n\}$  open intervals such that  $\sum_n |I_n| < m^*(A_i) + \frac{\varepsilon}{2^i}$ .

Then  $\bigcup_i \bigcup_n I_n^i \supset \bigcup_i A_i$ , and  $\sum_{n,i} |I_n^i| = \sum_i (\sum_n |I_n^i|) \leq \sum_i (m^*(A_i) + \frac{\varepsilon}{2^i})$ .

– Corollary

If  $A$  is countable, then  $m^*(A) = 0$ .

Thus, by contraposition, every interval is uncountable.

### Proposition

For  $A \subseteq \mathbb{R}, \forall \varepsilon > 0, \exists U$  open such that  $A \subseteq U$  and  $m^*(U) \leq m^*(A) + \varepsilon$ .

### Corollary

There exists  $G$  in the intersection of countable open sets such that  $m^*(G) = m^*(A)$  and  $G \supseteq A$ .

### Caratheodory Criteria

If  $\forall E, m^*(E \cap A) + m^*(E \cap A^c) = m^*(E)$ , then  $A$  is Lebesgue measurable.

- Remark:  $m^*(E \cap A) + m^*(E \cap A^c) \leq m^*(E) \leq +\infty$

### Propositions

1. If  $A$  is measurable, then  $A^c$  is measurable.
2.  $m^*(A) = 0$ , then  $A$  is measurable.
3. If  $A, B$  are measurable, then  $A \cup B, A \cap B, A \setminus B$  are measurable.
4. If  $\{A_i\}_{i=1}^k$  are disjoint and measurable, then  $m^*\left(\bigcup_{i=1}^k A_i\right) = \sum_{i=1}^k m^*(A_i)$ .

- Proof of 3

$$\begin{aligned}
m^*(E \cap (A \cup B)) + m^*(E \cap (A \cup B)^c) &= m^*((E \cap A) \cup (E \cap B)) + m^*(E \cap A^c \cap B^c) \\
&= m^*(E \cap A) + m^*((E \cap A^c) \cap B) + m^*((E \cap A^c) \cap B^c) \\
&\leq m^*(E)
\end{aligned}$$

Since  $(A \cap B)^C = A^c \cup B^c$ , this holds from before; similarly,  $A \setminus B = A \cap B^c = A^c \cup B$ .  
If  $A, B$  disjoint, then

$$\begin{aligned}
m^*(A \cup B) &= m^*(E \cap A) + m^*(E \cap A^c) \\
&= m^*(A) + m^*(B)
\end{aligned}$$

## Theorem

If  $\{A_i\}$  is a countable collection of disjoint and measurable sets, then

1.  $\bigcup_i A_i$  is measurable.
2.  $m^*(\bigcup_i A_i) = \sum_i m^*(A_i)$ .

## Proof of 1

Want to show:

$$m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) + m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right) \leq m^*(E)$$

By assumption, since the measure of  $E$  is finite,  $m^*(E \cap \bigcup_{i=1}^{\infty} A_i) < +\infty$ .

Claim:  $\forall \varepsilon > 0, \exists k$  such that

Therefore  $m^*\left(E \cap \bigcup_{i=1}^k A_i\right) \geq m^*(E \cap \bigcup_{i=1}^{\infty} A_i) - \varepsilon$ .

$$m^*(E) \leq m^*\left(E \cap \bigcup_{i=1}^k A_i\right) + \varepsilon + m^*\left(E \cap \left(\bigcup_{i=1}^k A_i\right)^c\right) \leq m^*(E) + \varepsilon$$

## Proof of 2

We have shown  $m^*(\bigcup_i A_i) \leq \sum_{i=1}^{\infty} m^*(A_i)$ .

Assume  $m^*(\bigcup_i A_i) < +\infty$ , then

$$\sum_{i=1}^k m^*(A_i) = m^*\left(\bigcup_{i=1}^k A_i\right) \leq m^*\left(\bigcup_i A_i\right) \implies \sum_{i=1}^{\infty} m^*(A_i) \leq m^*\left(\bigcup_i A_i\right)$$

**January 16, 2024**

Office Hours Tuesday / Thursday 10 AM - 11:30 AM

A note on notation: Latin characters are to be understood as countable indices; greek as possible uncountable.

## Lebesgue Outer Measure

$A \subset \mathbb{R}$

$m^*(A) = \inf \left\{ \sum_{i=1}^{\infty} |I_i| \mid \bigcup_{i=1}^{\infty} I_i \supset A, I_i \text{ open intervals} \right\}$

## Properties

1.  $A \subset B \implies m^*(A) \leq m^*(B)$ .
2.  $m^*(\emptyset) = 0$ .
3.  $m^*(I) = |I|$  for  $I$  an interval.
4. Countable Subadditivity:  $\{A_i\}_{i=1}^\infty \implies m^*(\bigcup_{i=1}^\infty A_i) \leq \sum_{i=1}^\infty m^*(A_i)$ .
5.  $\forall A \subset \mathbb{R}, \forall \varepsilon > 0, \exists$  open neighborhood  $U \supseteq A$  such that  $m^*(U) \leq m^*(A) + \varepsilon$ .
6.  $\exists G \in \bigcap_{n=1}^\infty U_n, U_n$  open,  $U_n \supseteq A \implies G \supseteq A$ , such that  $m^*(G) = m^*(A)$ .

## Measurable (Caratheodory Criterion)

$\forall A \subseteq \mathbb{R}$  is Lebesgue measurable if

$$m^*(A) = m^*(E \cap A) + m^*(E \cap A^c)$$

Essentially,  $m^*(E \cap A) + m^*(E \cap A^c) \leq m^*(E) \leq +\infty$ .

### • Propositions

1.  $A$  measurable  $\implies A^c$  measurable.
2.  $m^*(A) = 0 \implies A$  measurable.
3.  $\{A_i\}_{i=1}^\infty$  countable with  $A_i$  measurable, then
  - (a)  $\bigcap_{i=1}^\infty A_i$  are measurable.
  - (b) Moreover,  $A_i \cap A_j = \emptyset \implies m^*(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty m^*(A_i)$ .
  - (c)  $A, B$  measurable  $\implies A \cup B, A \cap B, A \setminus B$  measurable.
  - (d)  $A \cap B = \emptyset \implies m^*(A \cup B) = m^*(A) + m^*(B)$ .
  - (e)  $\{A_i\}_i^\infty$  with  $A_i$  measurable, then  $\bigcup_{i=1}^\infty A_i$  is measurable and  $A_i \cap A_j = \emptyset \implies m^*(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty m^*(A_i)$ .
- Proof of e  $\forall E \subset \mathbb{R}, m^*(E \cap (\bigcup_{i=1}^\infty A_i)) + m^*(E \cap (\bigcup_{i=1}^\infty A_i)^c)$ .  
 Claim:  $m^*(E \cap (\bigcup_{i=1}^\infty A_i)) = \sum_{i=1}^\infty m^*(E \cap A_i)$  for  $A_i \cap A_j = \emptyset$ .  
 Then,  $\forall \varepsilon > 0, \exists n \in \mathbb{N}$ ,

$$\begin{aligned}
 m^*\left(E \cap \left(\bigcup_{i=1}^\infty A_i\right)\right) &= \sum_{i=1}^\infty m^*(E \cap A_i) \leq \sum_{i=1}^n m^*(E \cap A_i) \\
 \implies m^*\left(E \cap \left(\bigcup_{i=1}^\infty A_i\right)\right) + m^*\left(E \cap \left(\bigcup_{i=1}^\infty A_i\right)^c\right) &\leq m^*\left(E \cap \left(\bigcup_{i=1}^n A_i\right)\right) + m^*\left(E \cap \left(\bigcup_{i=1}^n A_i\right)^c\right) + \varepsilon \leq m^*(E) \\
 &\implies \bigcup_{i=1}^\infty A_i \text{ measurable}
 \end{aligned}$$



Proof of Claim:

Step 1:  $A, B$  measurable and  $A \cap B = \emptyset$ . Since  $A$  is measurable,

$$\begin{aligned} m^*(E \cap (A \cup B)) &= m^*((E \cap (A \cup B)) \cap A) + m^*((E \cap (A \cup B)) \cap A^c) \\ &= m^*(E \cap A) + m^*(E \cap A^c) \end{aligned}$$

For  $\{A_i\}_{i=1}^\infty$ ,  $\bigcup_{i=1}^\infty A_i = \bigcup_{i=1}^\infty A'_i$  with  $A_1 = A'_1$  and  $A'_i = A_i \setminus \bigcup_{k=1}^{i-1} A_k$ ,  $\forall i \geq 2$ .  
Therefore  $A'_i \cap A'_j = \emptyset$  and  $A'_i$  is measurable.

$$\begin{aligned} m^*\left(\bigcup_{i=1}^n A_i\right) &\leq m^*\left(\bigcup_{i=1}^\infty A_i\right) \leq \sum_{i=1}^\infty m^*(A_i) \\ m^*\left(\bigcup_{i=1}^n A_i\right) &= \sum_{i=1}^n m^*(A_i) \leq m^*\left(\bigcup_{k=1}^\infty A_k\right) < +\infty \implies \sum_{i=1}^\infty m^*(A_i) \leq m^*\left(\bigcup_{k=1}^\infty A_k\right) \leq \sum_{i=1}^\infty m^*(A_i) \end{aligned}$$

## Sigma Algebra and Borel Sets

### Definition: Sigma Algebra

Let  $S \subset 2^X$  for some set  $X$ . Then  $S$  is said to be a  $\sigma$ -algebra if

1.  $\emptyset \in S$ .
  2.  $A^c \in S$  if  $A \in S$ .
  3.  $\bigcup_{i=1}^\infty A_i \in S$  if  $A_i \in S$ .
- Equivalently,  $\bigcap_{i=1}^\infty A_i \in S$  if  $A_i \in S$ .

### Theorem:

The collection  $\mathcal{L}$  of all Lebesgue measurable sets is a  $\sigma$ -algebra.

### Definition: Borel Set

Let  $B$  be the  $\sigma$ -algebra generated by open sets of reals (i.e. the smallest  $\sigma$ -algebra containing all open sets of reals). Then  $b \in B$  is called a Borel set.

### Remark

$B$  is generated by  $\{(a, +\infty) \mid a \in \mathbb{R}\}$ .

1.  $(a, +\infty)^c = (-\infty, a]$ .
2.  $\bigcap_{n=1}^\infty (a - \frac{1}{n}, +\infty) = [a, +\infty)$ .
3.  $[a, +\infty)^c = (-\infty, a)$ .
4.  $(-\infty, b) \cap (a, +\infty) = (a, b)$ .
5.  $(-\infty, b] \cap [a, +\infty) = [a, b]$ .

**Theorem:**

Any Borel set is Lebesgue measurable.

**Proof**

It suffices to demonstrate that  $(a, +\infty)$  is measurable  $\forall a \in \mathbb{R}$ .

$\forall E \subset \mathbb{R}$ , we want to show that  $m^*(E \cap (a, +\infty)) + m^*(-\infty, a] \leq m^*(E)$ .

Then,  $\forall \varepsilon > 0$ ,  $\exists \mathcal{C} = \{I_i\}$  with  $I_i$  open intervals such that  $\sum_{I_i \in \mathcal{C}} |I_i| \leq m^*(E) + \varepsilon/2$ . Set

$$\begin{aligned}\mathcal{C}^\ell &= \{I \in \mathcal{C} \mid x < a, \forall x \in I\} \\ \mathcal{C}^r &= \{I \in \mathcal{C} \mid x > a, \forall x \in I\} \\ \mathcal{C}^m &= \{I \in \mathcal{C} \mid a \in I\} = \{I_k\}\end{aligned}$$

Then  $\mathcal{AC} = \mathcal{C}^\ell \cup \mathcal{C}^r \cup \mathcal{C}^m$ .

$\forall I_k \in \mathcal{C}^m = \{I_k\}$ ,  $I_k = (c_k, d_k)$  for some  $c_k, d_k \in \mathbb{R}$ , define

$$\begin{aligned}I_k^\ell &= \left(c_k, a + \frac{\varepsilon}{2^{k+1}}\right) \\ I_k^r &= (a, d_k)\end{aligned}$$

Let  $\mathcal{C}^m = \{I_k^\ell\} \cup \{I_k^r\} = \bar{\mathcal{C}}^{m\ell} \cup \bar{\mathcal{C}}^{mr}$ . Then

$$\begin{aligned}\mathcal{C}^\ell \cup \bar{\mathcal{C}}^{m\ell} &\text{ covers } E \cap (-\infty, k] \\ \mathcal{C}^r \cup \bar{\mathcal{C}}^{mr} &\text{ covers } E \cap (k, +\infty) \\ \mathcal{C}^\ell \cup \mathcal{C}^r \cup \mathcal{C}^m &\text{ covers } E\end{aligned}$$

Observe that

$$|I_k^\ell| + |I_k^r| \leq |I_k| + \frac{\varepsilon}{2^{k+1}}$$

Therefore

$$\begin{aligned}m^*(E \cap (a, +\infty)) &\leq \sum_{I \in \mathcal{C}^r \cup \bar{\mathcal{C}}^{mr}} |I| \\ m^*(E \cap (-\infty, a]) &\leq \sum_{I \in \mathcal{C}^\ell \cup \bar{\mathcal{C}}^{m\ell}} |I|\end{aligned}$$

Therefore

$$\begin{aligned}m^*(E \cap (a, +\infty)) + m^*(E \cap (-\infty, a]) &\leq \sum_{I \in \mathcal{C}^r \cup \bar{\mathcal{C}}^{mr}} |I| + \sum_{I \in \mathcal{C}^\ell \cup \bar{\mathcal{C}}^{m\ell}} |I| \\ &= \sum_{I \in \mathcal{C}^r} |I| + \sum_{I \in \mathcal{C}^\ell} |I| + \sum_k \left(|I_k^\ell| + |I_k^r|\right) \\ &\leq \sum_{I \in \mathcal{C}} |I| + \sum_{k=1}^{+\infty} \frac{\varepsilon}{2^{k+1}} \\ &\leq m^*(E) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &\leq m^*(E) + \varepsilon\end{aligned}$$

## Lebesgue Measurable vs Borel

### Theorem

The following statements are equivalent

1.  $A$  is measurable.
2.  $\forall \varepsilon > 0, \exists U$  open,  $U \supset A$  such that  $m(U \setminus A) < \varepsilon$ .
3.  $\forall \varepsilon > 0, \exists C$  closed,  $C \subset A$  such that  $m(A \setminus C) < \varepsilon$ .
4.  $\forall A \in \mathbb{R}, \exists \bigcap_{n=1}^{\infty} U_n = F \in B, U_n$  open,  $U_n \supset A$  such that  $F \supset A$  and  $m(F \setminus A) = 0$ .
5.  $\exists \{C_n\}, C_n$  closed and  $C_n \subset A$  such that  $G = \bigcup_{n=1}^{\infty} C_n \subset A$  and  $m(A \setminus G) = 0$ .

### Corollary

Every measurable set is the union of a Borel set and a measure zero set.

### Proof 1 Implies 2

Step 1: if  $m(A) < \infty$ , then for  $\varepsilon > 0, \exists U$  open and  $U \supset A$ , then

$$m(U) \leq m(A) + \varepsilon \iff m(U \setminus A) = m(U) - m(A) \leq \varepsilon$$

Step 2: let  $A_n = A \cap (-n, n), n \in \mathbb{N}$ .

Then  $m(A_n) \leq 2n < +\infty$ .

For each  $A_n, \exists U_n$  open with  $U_n \supset A_n$  and  $m(U_n \setminus A_n) < \frac{\varepsilon}{2^n}$

Let  $U = \bigcup_{n=1}^{\infty} U_n$  and  $A = \bigcup_{n=1}^{\infty} A_n$ .

Now verify that

$$m(U \setminus A) = m\left(\bigcup_{n=1}^{\infty} U_n \setminus \bigcup_{n=1}^{\infty} A_n\right) = m\left(\bigcup_{n=1}^{\infty} U_n \setminus A_n\right) \leq \sum_{n=1}^{\infty} m(U_n \setminus A_n) \leq \varepsilon$$

### Proof 2 Implies 3

Write

$$A \setminus C = A \cap C^c = C^c \cap A = C^c \setminus A^c$$

Apply (2).

### Proof 3 Implies 4

$U_n$  comes from 2.

### Proof 4 Implies 5

Follows from 4.

### Proof 5 Implies 1

$A = G \cup (A \setminus G) \implies A$  is measurable.

### Example: Non-measurable Set

Define  $x \sim y$  if  $x - y \in \mathbb{Q}$ ,  $\forall x, y \in \mathbb{R}$ .

Let  $A = \{x \in (0, 1) \mid x \text{ is a representative of each class } \mathbb{R}/\sim\} \subset (0, 1) \subset \mathbb{R}$ .

Claim:  $A$  is not Lebesgue measurable.

Let  $(-1, 1) \supset S = \bigcup_{r \in \mathbb{Q} \cap (0, 1)} (A + r) \supset (0, 1)$ , and observe that  $\mathbb{Q} \cap (0, 1)$  is countable.

So  $(A + r) \cap (A + s) = \emptyset$  for  $s \neq r$ .

Then  $1 < m(S) < 2$ , so  $m(A) = 0$  and  $m(A) > 0$  are both contradictions.

## January 18, 2024

Abstract measure theory.

### Definition: Topological Space

A set  $X$  equipped with a collection of subsets  $\tau \subset 2^X$  where  $\tau$  is a topology if

1.  $\emptyset, X \in \tau$
2. Union of subsets in  $\tau$  remains in  $\tau$ .
3. Intersection of finitely many subsets in  $\tau$  remains in  $\tau$ .

Any subset of  $\tau$  is called an open set of  $X$ .

### Definition: Measure Space

For a set  $X$  with  $\Lambda \subset 2^X$  a  $\sigma$ -algebra such that

1.  $\emptyset \in \Lambda$
2.  $A^c \in \Lambda$  if  $A \in \Lambda$ .
3.  $\bigcup_{i=1}^{\infty} A_i \in \Lambda$  if  $A_i \in \Lambda$ .
4. Remark: Borel Sigma Algebra

The  $\sigma$ -algebra generated by  $\tau$  for a topological space  $(X, \tau)$ .

The measure space  $(X, \Lambda, \mu)$ ,  $\Lambda \subset 2^X$  a  $\sigma$ -algebra equipped with set function  $\mu : \Lambda \rightarrow [0, +\infty]$  such that

1.  $\mu(\emptyset) = 0$
2.  $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$  for  $A_i \in \Lambda$  and  $A_i \cap A_j = \emptyset$  for all  $i \neq j$  (countable additivity).

### Proposition: Monotonicity

$A, B \in \Lambda$ ,  $A \subseteq B \implies \mu(A) \leq \mu(B)$ .

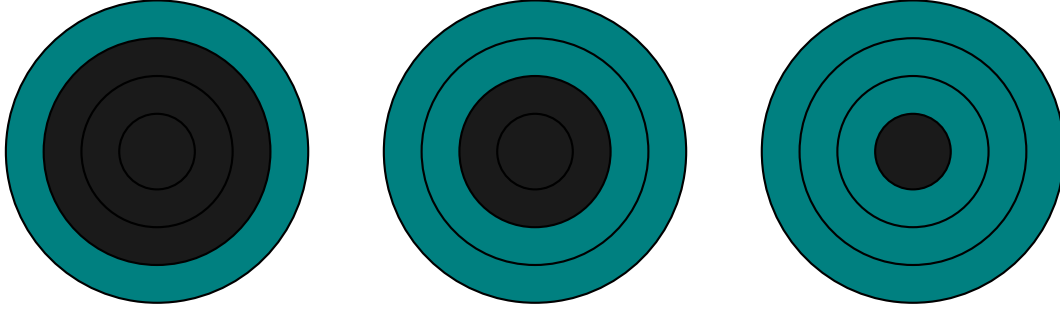
**Proposition: Countable Subadditivity**

$$\mu(\bigcup A_i) \leq \sum \mu(A_i) \text{ if } A_i \in \Lambda$$

**Proposition: Monotone Convergence**

Given  $A_i \subset \Lambda$  such that  $A_i \subset A_{i+1}$  where  $A = \bigcup_{i=1}^{\infty} A_i$ , then  $\mu(A_i) \rightarrow \mu(A)$ .

Similarly, if  $A_i \supset A_{i+1}$  such that  $A = \bigcap_{i=1}^{\infty} A_i$ , then  $\mu(A_i) \rightarrow \mu(A)$  if  $\mu(A_k) < +\infty$  for some  $k = 1, 2, 3, \dots$



$$\text{Given } A'_i = \begin{cases} A_1 & i = 1 \\ A_i \setminus \bigcup_{j=1}^{i-1} A_j & i > 1 \end{cases}, \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} A'_i \text{ and}$$

$$\mu(A) \sum_{i=1}^{\infty} A'_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(A'_i)$$

and

$$\sum_{i=1}^n \mu(A'_i) = \mu(A_1) + (\mu(A_2) - \mu(A_1)) + (\mu(A_3) - \mu(A_2)) + \dots + (\mu(A_n) - \mu(A_{n-1})) = \mu(A_n)$$

Similarly,  $A_1 \setminus A = \bigcup_{i=2}^{\infty} (A_i \setminus A_{i-1})$  where  $\mu(A_1) < +\infty$  gives

$$\mu(A_1) - \mu(A) = \mu(A_1) + \sum_{i=2}^{\infty} (\mu(A_i) - \mu(A_{i-1})) = \lim_{n \rightarrow \infty} \mu(A_n)$$

**Definition: Complete Measure Space**

A measure space  $(X, \Lambda, \mu)$  is complete if  $\forall A \in \Lambda$  with  $\mu(A) = 0$ , then  $\forall B \subset A$  and  $B \in \Lambda$ .

**Example**

The Lebesgue measure space on the reals  $(\mathbb{R}, \mathcal{L}, m)$  is complete.

**Theorem: Completion of a Measure Space**

Given a measure space  $(X, \Lambda, \mu)$ , then there exists  $(X, \overline{\Lambda}, \overline{\mu})$  such that

1.  $\Lambda \subset \overline{\Lambda}$ .
2. If  $A \in \Lambda$ , then  $\overline{\mu}(A) = \mu(A)$ .
3.  $(X, \overline{\Lambda}, \overline{\mu})$  is complete.

### Proof (Construction)

Let  $\bar{\Lambda} = \{A \cup Z \mid A \in \Lambda, \exists D \in \Lambda, m(D) = 0, Z \subset D\}$  and  $\bar{\mu}(A \cup Z) := \mu(A)$ .  
Verify:

1.  $\bar{\Lambda}$  is a  $\sigma$ -Algebra.

(a) If  $A \cup Z \in \bar{\Lambda}$ , then  $(A \cup Z)^c \in \bar{\Lambda}$ .

(b) If  $A_i \cup Z_i \in \bar{\Lambda}$ , then  $\bigcup (A_i \cup Z_i) \in \bar{\Lambda}$ .

2.  $\bar{\mu}$  is a well-defined measure on  $\bar{\Lambda}$ .

3.  $(X, \bar{\Lambda}, \bar{\mu})$  is complete.

• Proof of 1

Given  $A \in \Lambda$  and  $Z \subset D$  where  $\mu(D) = 0$  and  $D \in \Lambda$ , we know  $D^c \subset Z^c$  and  $Z^c = D^c \cup (Z^c \cap D)$ . Therefore

$$(A \cup Z)^c = A^c \cap Z^c = A^c \cap (D^c \cup (Z^c \cap D)) = (A^c \cap D^c) \cup (A^c \cap Z^c \cap D) \in \bar{\Lambda}$$

Since  $A^c \cap D^c \in \Lambda$  and  $A^c \cap Z^c \cap D \in D$

Since  $\bigcup A_i \in \Lambda$  and  $\bigcup Z_i \subset \bigcup D_i$ ,

$$\bigcup_{i=1}^{\infty} (A_i \cup Z_i) = \left( \bigcup_{i=1}^{\infty} A_i \right) \cup \left( \bigcup_{i=1}^{\infty} Z_i \right) \in \bar{\Lambda}$$

• Proof of 2

Given  $A_1 \cup Z_1 = A_2 \cup Z_2$ ,  $A_1 \subset A_2 \cup Z_2 \subset A_2 \cup D_2$  implies  $\mu(A_1) \leq \mu(A_2)$ .

Then,  $\mu(A_2) \leq \mu(A_1) \implies \mu(A_1) = \mu(A_2)$ . So  $\bar{\mu}$  is well defined.

Given  $\{A_i \cup Z_i\}$  with  $(A_i \cup Z_i) \cap (A_j \cup Z_j) = \emptyset$  for all  $i \neq j$ ,

$$\bar{\mu} \left( \bigcup_{i=1}^{\infty} (A_i \cup Z_i) \right) = \bar{\mu} \left( \left( \bigcup_{i=1}^{\infty} A_i \right) \cup \bigcup_{i=1}^{\infty} Z_i \right) = \mu \left( \bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty} \bar{\mu}(A_i \cup Z_i)$$

So  $\bar{\mu}$  is countably additive and therefore a measure.

### Borel Measure and Radon Measure

Given a measure space  $(X, \Lambda, \mu)$  and an underlying topology  $(X, \tau)$ ,

#### Definition: Borel Measure

$\mu$  is a Borel measure if all borel sets  $\tau \subset \Lambda$ .

#### Definition: Locally Finite Measure

$\mu$  is locally finite if  $\forall x \in X, \exists U \subset X$  a neighborhood such that  $\mu(U) < +\infty$ .

**Definition: Borel Regularity**

$\mu$  is Borel regular if  $\forall A \in \Lambda, \exists B$  a Borel set such that  $B \supseteq A$  and  $\mu(B) = \mu(A)$ .

**Definition: Radon Measure**

$\mu$  is a Radon measure if

1. it is a Borel measure.
  2.  $\mu(K) \leq +\infty$  for  $K$  compact.
  3.  $\mu(V) = \sup\{\mu(K) \mid K \subset V, K \text{ compact}\}, V$  open.
  4.  $\mu(A) = \inf\{\mu(V) \mid A \subset V, V \text{ open}\}, \forall A \in \Lambda$ .
- Example 1  
Lebesgue measure.
  - Example 2  
Point charge:  $\mu(\{x\}) = 1$  and  $\mu(A) = 0$  if  $x \notin A$ .

**Theorem:**

Let  $(X, \Lambda, \mu)$  be a Borel regular measure space where the underlying topology  $(X, \tau)$  is a metric space. Then

1. For  $A \in \Lambda$  with  $\mu(A) < +\infty, \forall \varepsilon > 0, \exists C \subseteq A$  closed such that  $\mu(A \setminus C) < \varepsilon$ .
2. For  $A \in \Lambda, \exists \{V_i\}$  open sets such that  $A \subset \bigcup_{i=1}^{\infty} V_i$  and  $\mu(V_i) < +\infty$ . Then  $\forall \varepsilon > 0, \exists U$  open with  $A \subset U$  and  $\mu(U \setminus A) < \varepsilon$ .

**Proof**

Given  $\mu(A) < +\infty, \nu(B) = \mu(B \cap A) < +\infty, \forall B \in \Lambda$  and  $(X, \Lambda, \nu)$ .

Let  $F = \{B \in \Lambda \mid \forall \varepsilon > 0, \exists C \subset B \text{ closed, with } \nu(B \setminus C) < \varepsilon\}$ .

Note that closed sets are in  $F$ .

Claim 1: the Borel  $\sigma$ -algebra is in  $F$ .

Claim 2: if  $A_i \in F, \bigcup A_i, \bigcap A_i \in F$ .

Given claim 2,  $\forall U$  open,  $U^c$  is closed. Then  $U_\varepsilon = \{x \in U \mid \text{dist}(x, U^c) \leq \varepsilon\}$  is closed and, therefore,  $U = \bigcup_{i=1}^{\infty} U_{1/i}$ .

So, given  $A_i \in F, \exists C_i \subset A_i$  closed where  $\nu(A_i \setminus C_i) < \varepsilon/2^{i+1}$ . We want to show that  $\nu(\bigcap A_i \setminus \bigcap C_i) < \varepsilon$ .

Then, for  $x \in \bigcap A_i \setminus \bigcap C_i, x \in A_i$  for all  $i$  and  $x \notin C_{i_0}$  for some  $i_0$ .

Therefore  $x \in A_{i_0}, x \notin C_{i_0}$ , and  $x \in A_{i_0} \setminus C_{i_0}$ . It follows that

$$\begin{aligned} \bigcap_{i=1}^{\infty} A_i \setminus \bigcap_{i=1}^{\infty} C_i &\subset \bigcup_{i=1}^{\infty} (A_i \setminus C_i) \\ \nu\left(\bigcap_{i=1}^{\infty} A_i \setminus \bigcap_{i=1}^{\infty} C_i\right) &\leq \sum_{i=1}^{\infty} \nu(A_i \setminus C_i) < \varepsilon \end{aligned}$$

Therefore

$$\nu \left( \bigcup_{i=1}^{\infty} A_i \setminus \bigcup_{i=1}^n C_i \right) \rightarrow \nu \left( \bigcup_{i=1}^{\infty} A_i \setminus \bigcup_{i=1}^{\infty} C_i \right) \leq \nu \left( \bigcup_{i=1}^{\infty} (A_i \setminus C_i) \right) < \frac{\varepsilon}{2}$$

so  $\exists N \gg 1$  such that  $\nu \left( \bigcup_{i=1}^{\infty} A_i \setminus \bigcup_{i=1}^N C_i \right) < \varepsilon$  with  $\bigcup_{i=1}^N C_i$  closed.

### Restatement

For  $A$  Borel,

$$\varepsilon > \nu(A \setminus C) = \mu((A \setminus C) \cap A) = \mu(A \setminus C)$$

## January 23, 2024

### Review - Abstract Measure

Given  $(X, \Lambda, \mu)$  where  $\Lambda \subseteq 2^X$  is a  $\sigma$ -algebra,  $\mu : \Lambda \rightarrow [0, +\infty]$

1.  $\mu(\emptyset) = 0$ .
2.  $m(\bigcup A_i) = \sum \mu(A_i)$ ,  $A_i \cap A_j = \emptyset$ .

### Properties of a Measure

#### Monotonicity

$$\mu(A) \subseteq \mu(B), A, B \in \Lambda, A \subseteq B$$

#### Countable Subadditivity

$$\mu(\bigcup A_i) \leq \sum \mu(A_i)$$

#### Monotone Convergence

$$\begin{aligned} A_i \subset A_{i+1}, A_i \rightarrow \bigcup A_i &\implies \mu(A) = \mu(\bigcup A_i). \\ A_i \supset A_{i+1}, A_i \rightarrow \bigcap A_i &\implies \mu(A_i) \rightarrow \mu(\bigcap A_i) \text{ if } \mu(A_1) < \infty \end{aligned}$$

- Example  
 $A_n = (n, +\infty)$  gives  $\bigcap A_n = \emptyset$

### Completeness of a Measure

$(X, \Lambda, \mu)$  is complete if  $\forall A \in \Lambda$  with  $\mu(A) = 0$ , then  $\forall B \in \Lambda$  if  $B \subseteq A$ .

#### Theorem:

Given  $(X, \Lambda, \mu)$ , there exists  $(X, \overline{\Lambda}, \overline{\mu})$  such that  $\Lambda \subset \overline{\Lambda}$  and  $\overline{\mu}(A) = \mu(A)$  if  $A \in \Lambda$ .

$$\overline{\Lambda} = \{A \cup Z \mid A \in \Lambda, Z \subset D, D \in \Lambda \text{ with } \mu(D) = 0\}$$



$$\bar{\mu}(A \cup Z) = \mu(A)$$

$(X, \bar{\Lambda}, \bar{\mu})$  is complete.

## Measure Space with Topology

Given a topological space  $(X, \tau)$ , a measure space  $(X, \Lambda, \mu)$

### Definition: Locally Finite

The measure  $\mu$  is locally finite if  $\forall x \in X$ , there exists an open neighborhood  $U$  of  $x$  such that  $U \in \Lambda$  and  $\mu(U) < +\infty$ .

### Definition: Borel Measure

$\mu$  is a Borel measure if the Borel  $\sigma$ -algebra generated by  $\tau$ ,  $\mathcal{B}$ , is a subset of  $\Lambda$ .

### Definition: Borel Regular

$\forall A \in \Lambda$ ,  $\exists B \in \mathcal{B}$  such that  $B \supset A$  and  $\mu(B) = \mu(A)$ .

### Definition: Radon Measure

1. Borel.
2.  $\mu(K) < +\infty$  for  $K$  compact.
3.  $\mu(V) = \sup\{\mu(K) \mid K \text{ compact}, K \subset V\}$ ,  $\forall V$  open.
4.  $\mu(A) = \inf\{\mu(V) \mid V \text{ open}, A \subset V\}$ ,  $\forall A \in \Lambda$ .

### Theorem:

If  $X$  is a metric space equipped with a Borel regular  $(X, \Lambda, \mu)$ , then

1.  $\forall A \in \Lambda$ ,  $\mu(A) < +\infty$ ,  $\forall \varepsilon > 0$ ,  $\exists C$  closed where  $C \subset A$  and  $\mu(C \setminus A) < \varepsilon$ .
2. If  $\exists \{V_i\}$ ,  $V_i$  open and  $\mu(V_i) < +\infty$ , and  $A \in \Lambda$  with  $A \subset \bigcup V_i$ , then  $\exists U$  open such that  $A \subset U$  and  $\mu(U \setminus A) < \varepsilon$ .

### Proof of 1

Define  $\nu(B) = \mu(B \cap A)$  such that  $(X, \Lambda, \nu)$  is a new measure space.

Define  $F = \{B \in \Lambda \mid \forall \varepsilon > 0, \exists C \text{ closed}, C \subset B, \nu(B \setminus C) < \varepsilon\}$ , all closed sets in  $F$ .

Claim 1:  $\bigcap A_i, \bigcup A_i \in F$  if  $A_i \in F$ .

Claim 2:  $U$  is open.

$U = \bigcup U_i$ ,  $U_i = \{x \in U \mid \text{dist}(x, U^c) \leq \frac{1}{i}\}$ , therefore  $\mathcal{B} \subset F$ .

IMAGE HERE - 1

If  $A$  is Borel, then  $\forall \varepsilon > 0$ ,  $\exists C$  closed with  $C \subset A$  and  $\mu(A \setminus C) < \varepsilon$ .

To finish,  $\forall A \in \Lambda$  by Borel Regularity of  $\mu$ ,  $\exists B \in \mathcal{B}$  such that  $B \supset A$  and  $\mu(B) = \mu(A)$ .

Note also that this requires  $\mu(B \setminus A) = 0$  since  $\mu(A) < +\infty$ .

IMAGE HERE - 2

Then  $B \setminus A \in \Lambda$ ,  $\exists D \in \mathcal{B}$  such that  $D \supset B \setminus A$  and  $\mu(D) = \mu(B \setminus A) = 0$ . Then

$$\begin{aligned} B \cap A^c &= B \setminus A \subset D \\ (B \cap A^c)^c &\supset D^c \\ B \cap (B^c \cup A) &\supset D^c \cap B \\ A &\supset B \setminus D \end{aligned}$$

$$A \setminus (B \setminus D) = A \cap (B \cap D^c)^c = A \cap (B^c \cup D) = \overbrace{(A \cap B^c)}^{\emptyset} \cup A \cap D = A \cap D \subset D$$

Therefore  $B \setminus D \subset A$ , and  $\mu(A \setminus (B \setminus D)) = 0$ .

$B \setminus D \in \mathcal{B}$ ,  $\forall \varepsilon > 0$ ,  $\exists C$  closed such that  $C \subset B \setminus D \subset A$ ,  $\mu((B \setminus D) \setminus C) < \varepsilon$ .

This implies that  $\mu(A \setminus C) = \mu(A \setminus (B \setminus D)) + \mu((B \setminus D) \setminus C) < \varepsilon$ .

## Proof of 2

Consider  $V_i \setminus A$  where  $\mu(V_i \setminus A) \leq \mu(V_i) < +\infty$ .

By (1),  $\exists C_i$  closed with  $C_i \subset V_i \setminus A$  and  $\mu((V_i \setminus A) \setminus C_i) < \varepsilon/2^{i+1}$ . Write

$$(V_i \setminus A) \setminus C_i = (V_i \setminus A) \cap C_i^c = V_i \cap A^c \cap C_i^c = (V_i \cap C_i^c) \cap A^c = (V_i \setminus C_i) \setminus A$$

Note that  $V_i \setminus C_i$  is open, since  $C_i$  is closed.

Define  $U = \bigcup (V_i \setminus C_i) \supset A$ . Then,

$$U \setminus A = \left( \bigcup (V_i \setminus C_i) \right) \setminus A = \bigcup ((V_i \setminus C_i) \setminus A)$$

Therefore  $\mu(U \setminus A) \leq \varepsilon_{\frac{\varepsilon}{2^{1+1}}} = \varepsilon$ .

## Remark

$X = \bigcup V_i$ ,  $V_i$  open and  $\mu(V_i) < +\infty$ .

Then  $\forall A \in \Lambda$ ,  $\forall \varepsilon > 0$ ,  $\exists U$  open such that  $U \supset A$  and  $\mu(U \setminus A) < \varepsilon$ .

For  $A^c$ ,  $\exists U \supset A^c$  ( $\implies U^c \subset A$ ),  $\mu(U \setminus A^c) < \varepsilon$ . So

$$U \cap A = U \setminus A^c = A \setminus U^c = A \cap U$$

and  $\mu(A \setminus U^c) < \varepsilon$ ,  $U^c \subset A$  with  $U^c$  closed.

## Corollary

For  $\mathbb{R}^n$ , a measure is Radon if and only if it is locally finite and Borel regular.

- Proof

( $\implies$ )

Let  $B(r, x_0) = \{x \in \mathbb{R}^n \mid |x - x_0| < r\}$  and  $\overline{B(r, x_0)} = \{x \in \mathbb{R}^n \mid |x - x_0| \leq r, \text{ compact}\}$ .

Then  $\mu(B(r, x_0)) \leq \mu(\overline{B(r, x_0)}) < +\infty$ . So  $\mu$  is locally finite.

For  $A \in \Lambda$ , we may assume without loss of generality that  $\mu(A) < +\infty$ .

Then  $\forall i$ ,  $\exists U_i$  open where  $U_i \supset A$  and  $\mu(A) \leq \mu(U_i) \leq \mu(A) + \frac{1}{i} < +\infty$ .

Set  $G = \bigcap U_i \in \mathcal{B}$ , then  $\mu(G) = \mu(A)$ .

( $\longleftarrow$ )

1. Borel regular implies Borel.
2. For  $K$  compact,  $\forall x \in K \ni U_x$  open where  $\mu(U_x) < +\infty$ .

$\{U_\lambda\}_{\lambda \in k}$  is an open cover. Therefore there is a finite subcover  $\{U_{\lambda_i}\}_{i=1}^\lambda$  where

$$\mu(K) \leq \mu\left(\bigcup_{i=1}^k U_{\lambda_i}\right) \leq \sum_{i=1}^k \mu(U_{\lambda_i}) < +\infty$$

3.  $\forall V$  open,  $B(i) = B(i, 0)$ ,  $V \cap B(i)$ ,  $\mu(V \cap B(i)) < +\infty$ ,  $\exists C_i$  closed where  $C_i \subset V \cap B(i)$  so  $C_i$  is bounded and therefore compact.

So  $\mu(C_i) \leq \mu((V \cap B(i)) \setminus C_i) < \frac{1}{i}$  and  $\mu(V \cap B(i)) \leq \mu(C_i) + \frac{1}{i}$ .

Then  $\mu(V) = \lim_{i \rightarrow \infty} \mu(V \cap B(i)) = \lim_{i \rightarrow \infty} \mu(C_i)$ , and  $C_i \subset V \cap B(i) \subset V$  compact.

Therefore  $\mu(V) = \sup\{\mu(K) \mid K \text{ compact}, K \subset V\}$ .

4.  $\forall A \in \Lambda$ ,  $\forall i$ ,  $\exists U_i$  open where  $U_i \supset A$  and  $\mu(U_i \setminus A) < \frac{1}{i}$

This implies that  $\mu(A) \leq \mu(U_i) \leq \mu(A) + \frac{1}{i}$  and therefore  $\mu(A) = \inf\{\mu(U) \mid U \supset A, U \text{ open}\}$ .

## Caratheodory Construction

### Definition: Outer Measure

$\mu^*(A)$ ,  $\forall A \in 2^X$

1.  $\mu^*(\emptyset) = 0$ .
2.  $\mu^*(A) \leq \mu^*(B)$  if  $A \subseteq B$ .
3.  $\mu^*(\bigcup A_i) \leq \sum \mu^*(A_i)$ ,  $\forall A_i \in 2^X$  (countable subadditivity)

Define  $\Lambda = \{A \in 2^X \mid \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c), \forall E \in 2^X\}$ .

Then  $\mu(A) = \mu^*(A)$  if  $A \in \Lambda$ .

$(X, \Lambda, \mu)$  is complete.

**January 25, 2024**

### Theorem: Caratheodory Construction

#### Outer Measure

$u^* : 2^X \rightarrow [0, +\infty]$ .

1.  $\mu^*(\emptyset) = 0$
2. Monotonicity:  $\mu^*(A) \leq \mu^*(B)$ ,  $A \subseteq B$
3. Countable Subadditivity:  $\mu^*(\bigcup_i A_i) \leq \sum_i \mu^*(A_i)$ .

## Caratheodory Criterion

$A \subset X$  is measurable if  $\forall E \in \mathcal{X}$ ,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

## Theorem

The collection  $\Lambda$  of all measurable sets is a  $\sigma$ -algebra.

$(X, \Lambda, \mu)$  is a complete measure space (cf. proof of Lebesgue completeness).

## Hausdorff Measure

$\forall A \subseteq \mathbb{R}^n, \forall s \geq 0, H_s^\delta(A) = \inf \{ \sum_i (d(E_i))^s \mid \bigcup_i E_i \supset A, d(E_i) \leq \delta \}$  where  $d(E_i)$  is the diameter of  $E_i$ .

Notice that  $H_s^{\delta_1}(A) \leq H_s^{\delta_2}(A)$  if  $\delta_2 \leq \delta_1$ .

Let  $H_s^*(A) = \lim_{\delta \rightarrow 0} H_s^\delta(A), \forall A \in 2^{\mathbb{R}^n}$ .

Claim:  $H_s^*$  is an outer measure.

- Verify

1.  $H_s^*(\emptyset) = 0$ .

2.  $H_s^*(A) \leq H_s^*(B), \forall A \subseteq B \subseteq \mathbb{R}^n$ .

3. Given  $A_i \subset \mathbb{R}^n$ ,

$\exists \delta_0 > 0$  such that  $\forall \delta < \delta_0, H_s^*(\bigcup_i A_i) \leq H_s^\delta(\bigcup_i A_i) + \frac{\varepsilon}{2}$ .

Then  $\forall \delta < \delta_0$  fixed,  $\forall A_i, \exists \{E_i^j\}$  such that  $\bigcup_j E_i^j \supset A_i, \sum_j (d(E_i^j))^s \leq H_s^\delta(A_i) + \frac{\varepsilon}{2^{i+1}}$ , and  $d(E_i^j) \leq \delta$ .  
So

$$\begin{aligned} H_s^\delta \left( \bigcup_i A_i \right) &\leq \sum_{i,j} (d(E_i^j))^s \\ &= \sum_i \left( \sum_j (d(E_i^j))^s \right) \\ &= \sum_i \left( H_s^\delta(A_i) + \frac{\varepsilon}{2^{i+1}} \right) \\ &= \sum_i H_s^\delta(A_i) + \frac{\varepsilon}{2} \end{aligned}$$

and

$$H_s^* \left( \bigcup_i A_i \right) \leq \sum_i H_s^\delta(A_i) + \varepsilon \leq \sum_i H_s^*(A_i) + \varepsilon, \quad \forall \varepsilon > 0.$$

Then, since  $H_s^*$  is an outer measure, it is a measure by the Caratheodory construction.

## Definition: Hausdorff Measure

The Hausdorff Measure  $H_s : \Lambda \rightarrow [0, +\infty)$  on a  $\sigma$ -algebra  $\Lambda \subset 2^{\mathbb{R}^n}$ .

## Not Locally Finite

Consider  $B(0, 1) = \{x \mid |x| < 1\}$ .

Then  $H_s(B(0, 1)) = \infty$  for  $s < n$ .

That is, the Hausdorff measure is not locally finite for  $s < n$ .

## Complete

The Hausdorff measure, by the Caratheodory construction, is complete.

## Symmetry

1. Translation Invariance:  $H_s(A + x) = H_s(A)$ .
2. Rotation Invariance:  $H_s(RA) = H_s(A)$ .
3. Scaling:  $H_s(\lambda A) = \lambda^s H_s(A)$ .

## Open Balls Measurable

What about  $B(0, 1) \subset \mathbb{R}^n$ . For  $\delta > 0$ ,

$$H_s^*(E \cap B(0, 1)) + H_s^*(E \cap B(0, 1)^c) \leq H_s^*(E \cap B(0, 1 - \delta)) + H_s^*(E \cap (B(0, 1) \setminus B(0, 1 - \delta))) + H_s^*(E \cap B(0, 1)^c)$$

Want to show that for all  $\varepsilon > 0$ , this is  $\leq H_s^*(E) + \varepsilon$ .

- Lemma 1

$$\begin{aligned} H_s^*(E \cap B(0, 1 - \delta)) + H_s^*(E \cap B(0, 1)^c) &= H_s^*(E \cap (B(0, 1 - \delta) \cup B(0, 1)^c)) \\ &\leq H_s^*(E) \end{aligned}$$

- Lemma 2

$$H_s^*(E \cap (B(0, 1) \setminus B(0, 1 - \delta))) < \varepsilon.$$

- Lemma 1'

If  $A, B \subset \mathbb{R}^n$ ,  $\text{dist}(A, B) > 0$ , then  $H_s^*(A \cup B) = H_s^*(A) + H_s^*(B)$ .

Since  $\{E_i\}$  covering  $A \cup B$ ,  $d(E_i) < \frac{1}{4}\text{dist}(A, B)$  gives

$$\delta < \frac{1}{4}\text{dist}(A, B) \iff \{E_j^A\} \cup \{E_k^B\}$$

if and only if  $\{E_j^A\}$  covers  $A$  and  $\{E_k^B\}$  covers  $B$ . Therefore,

$$\begin{aligned} \sum_i (d(E_i))^s &= \sum_j (d(E_j^A))^s + \sum_k (d(E_k^B))^s \\ \inf \left\{ \sum_i (d(E_i))^s \right\} &= \inf \left\{ \sum_j (d(E_j^A))^s \right\} + \inf \left\{ \sum_k (d(E_k^B))^s \right\} \end{aligned}$$

and  $H_s^\delta(A \cup B) = H_s^\delta(A) + H_s^\delta(B)$ .  
Thus  $H_s^*(A \cup B) = H_s^*(A) + H_s^*(B)$ .

Let  $T_i = E \cap \left( B \left( 0, 1 - \frac{1}{i+1} \right) \right) \setminus B \left( 0, 1 - \frac{1}{i} \right)$ .

IMAGE HERE - 1 CONCENTRIC RINGS

We want to show that  $H_s^*(E \cap (B(0, 1) \setminus B(0, \frac{1}{i}))) < \varepsilon$  for  $i \gg 1$ . Then

$$\begin{aligned} \bigcup_{k=1} T_k &= (B(0, 1) \setminus \{0\}) \cap E \\ \bigcup_{k=i} T_k &= \left( B(0, 1) \setminus B \left( 0, 1 - \frac{1}{i} \right) \right) \cap E \end{aligned}$$

Claim:  $\sum_i H_s^*(T_i) < +\infty$ . It suffices to prove this claim.

$$\begin{aligned} \sum_{i \text{ even}}^{2k} H_s^*(T_i) &= H_s^* \left( \bigcup_{i \text{ even}}^{2k} \right) \leq H_s^*(E) < +\infty \\ \sum_{i \text{ odd}}^{2k+1} H_s^*(T_i) &= H_s^* \left( \bigcup_{i \text{ odd}}^{2k+1} \right) \leq H_s^*(E) < +\infty \end{aligned}$$

Then  $\sum_i^k H_s^*(T_i) \leq \infty$ .

### Borel

Take a countable, dense set  $\{q_i\} \subset \mathbb{R}^n$  and  $\{B(q_i, \frac{1}{k})\}_{i,k}$ .

Claim:  $\forall V \subseteq \mathbb{R}^n$  open, then  $V = \bigcup_l B(q_{i_l}, \frac{1}{k_l})$ .

Then  $\mathcal{B} \subseteq \Lambda$  and the Hausdorff measure is Borel.

### Borel Regular

$\forall A \subset \Lambda$ ,  $\exists B \in \mathcal{B}$  such that  $B \supset A$  and  $H_s(B) = H_s(A)$ .

$\forall \delta = \frac{1}{j}$ ,  $\{E_i^j\}$   $E_i^j$  closed balls with  $d(E_i^j) < \frac{1}{j}$ ,

$$\sum_i (d(E_i))^s \leq H_s^{\frac{1}{j}}(A) + \frac{1}{j}$$

Take  $B = \bigcap_j \left( \bigcup_i E_i^j \right) \in \mathcal{B}$  since  $B = \bigcap_j \bigcup_i E_i^j \supset A$ . Then

$$\begin{aligned} H_s^{\frac{1}{j}}(B) &\leq H_s^{\frac{1}{j}} \left( \bigcup_i E_i^j \right) \\ &\leq \sum_i H_s^{\frac{1}{j}}(E_i^j) \\ &\leq \sum_i \left( d(E_i^j) \right)^s \\ &\leq H_s^{\frac{1}{j}}(A) + \frac{1}{j} \end{aligned}$$

and in the limit as  $j \rightarrow \infty$

$$H_s^*(A) \leq H_s^*(B) \leq H_s^*(A)$$

## Fractional or Hausdorff Dimension

### Theorem:

1.  $H_s^*(A) < +\infty \implies H_t^*(A) = 0, \forall t > s \geq 0.$
2.  $H_t^s > 0 \implies H_s(A) = \infty, \forall 0 \leq s < t$

### Proof

$$\begin{aligned} H_s^\delta(A) &\sim \sum_i (d(E_i))^s \\ &= \sum_i (d(E_i))^t (d(E_i))^{s-t} \end{aligned}$$

So  $s < t$  gives  $\geq \delta^{s-t}$ .

In the other direction, when  $s < t$

$$\begin{aligned} \sum_i (d(E_i))^t &= \sum_i (d(E_i))^s (d(E_i))^{t-s} \\ &\leq \delta^{t-s} \sum_i (d(E_i))^s \end{aligned}$$

### Definition: Hausdorff Dimension

Given  $A \subset \mathbb{R}^n$ ,

$$\begin{aligned} \dim_H(A) &= \sup \{s \mid H_s^*(A) = \infty\} \\ &= \sup \{s \mid H_s^*(A) > 0\} \\ &= \inf \{s \mid H_s^*(A) = 0\} \\ &= \inf \{s \mid H_s^*(A) < +\infty\} \end{aligned}$$

### Example 1

$\mathbb{R}^n$  has  $n$  Hausdorff dimension.

Consider the  $n$ -cube with sides  $d$ ,  $C(d)$ . Then

$$H_s(C(d)) = C(n, s)d^s$$

So  $C(n, s) = C(n, s)2^{nk} \frac{1}{(2^k)^s} = C(n, s)2^{(n-1)k}$ .

If  $s < n$ , this tends to infinity as  $k \rightarrow \infty$ .

Is  $s > n$  it tends to 0.

### Example 2

Cantor set has Hausdorff dimension  $\frac{\log(2)}{\log(3)}$ .

$$\bigcup_{k=1}^{2^n} C_n^k = \frac{\log(2)}{\log(3)}$$

where  $|C_n^k| = \frac{1}{3^n}$ , so  $H_s^\delta(C^n) \sim \frac{2^n}{(3^n)^s} = \left(\frac{2}{3^s}\right)^n$ .

### Example 3

The Koch snowflake has dimension  $\frac{\log(4)}{\log(3)}$ .

**January 30, 2024**

### Lemma:

Given a measure space  $(X, \Lambda, \mu)$  and an extended real-valued function  $f : X \rightarrow [-\infty, +\infty]$ , the following are equivalent

1.  $\forall \alpha \in \mathbb{R}, \{x \in X \mid f(x) > \alpha\} \in \Lambda$ .
2.  $\forall \alpha \in \mathbb{R}, \{x \in X \mid f(x) \geq \alpha\} \in \Lambda$ .
3.  $\forall \alpha \in \mathbb{R}, \{x \in X \mid f(x) < \alpha\} \in \Lambda$ .
4.  $\forall \alpha \in \mathbb{R}, \{x \in X \mid f(x) \leq \alpha\} \in \Lambda$ .
5.  $\forall U \subset \mathbb{R}$  open,  $f^{-1}(U) \in \Lambda$  and  $f^{-1}(\pm\infty) \in \Lambda$ .

### Proof 1 Implies 2

$$\{x \in X \mid f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x \in X \mid f(x) > \alpha - \frac{1}{n}\}.$$

### Proof 2 Implies 3

$$\{x \in X \mid f(x) < \alpha\} = \{x \in X \mid f(x) \geq \alpha\}^c$$

### Proof 3 Implies 4

$$\{x \in X \mid f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty} \{x \in X \mid f(x) < \alpha + \frac{1}{n}\}$$

### Proof 4 Implies 1

$$\{x \in X \mid f(x) > \alpha\} = \{x \in X \mid f(x) \leq \alpha\}^c$$

### Proof of 5

$\forall U \subset \mathbb{R}$  open,  $V = \bigcup_i I_i$  disjoint open intervals.

Therefore  $f^{-1}((a, b)) = \{x \in X \mid f(x) > a\} \cap \{x \in X \mid f(x) < b\}$ .

Similarly,  $f^{-1}(-\infty) = \bigcap_n \{x \in X \mid f(x) < -n\}$  and  $f^{-1}(+\infty) = \bigcap_n \{x \in X \mid f(x) > n\}$ .

### Proof 5 Implies 1

$$\{x \in X \mid f(x) > \alpha\} = f^{-1}((\alpha, +\infty)) \cup f^{-1}(+\infty) \in \Lambda.$$

### Definition: Measurable Function

For a measure space  $(X, \Lambda, \mu)$ , an extended real-valued function  $f : X \rightarrow [-\infty, +\infty]$  is said to be measurable if one or all of (1)-(5) hold.



**Remark:**

If  $(X, \Lambda, \mu)$  is Borel, then continuous functions are always measurable.

**Remark:**

The characteristic function

$$\chi_A = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

is measurable if  $A \in \Lambda$ .

**Definition: Simple Functions**

The function  $\phi$  is simple if

$$\phi(x) = \sum_{i=1}^k \lambda_i \chi_{A_i}, \quad \lambda_i \in \mathbb{R}, A_i \in \Lambda$$

**Proposition:**

Given a measure space  $(X, \Lambda, \mu)$  and measurable, real-valued  $f, g$ ,

- $f \pm g$  is measurable.

$$\{x \in X \mid f(x) + g(x) < \alpha\} = \bigcup_{r \in \mathbb{Q}} (\{x \in X \mid f(x) < r\} \cup \{x \in X \mid g(x) < \alpha - r\}).$$

- $f^2$  is measurable

$$\forall \alpha \geq 0, \{x \in X \mid f^2(x) < \alpha\} = \{x \in X \mid f(x) < \sqrt{\alpha}\} \cap \{x \in X \mid f(x) > -\sqrt{\alpha}\}.$$

- $f \cdot g$  is measurable

$$f(x) \cdot g(x) = \frac{1}{2} ((f+g)^2 - f^2 - g^2).$$

**Definition: Almost Everywhere Equality**

Measurable functions  $f$  and  $g$  on the space  $(X, \Lambda, \mu)$  are the same almost everywhere with respect to  $\mu$  (written  $\mu$ -a.e.) if

$$\mu(\{x \in X \mid f(x) \neq g(x)\}) = 0$$

**Proposition:**

For a complete measure space  $(X, \Lambda, \mu)$ , if  $f$  and  $g$  are equal  $\mu$ -a.e., then  $f$  is measurable if and only if  $g$  is measurable.

**Proof**

$$\begin{aligned} \{x \in X \mid f(x) > \alpha\} &= (\{x \in X \mid f(x) > \alpha\} \cap \{x \in X \mid f(x) = g(x)\}) \cup \{x \in X \mid f(x) > \alpha\} \cap \underbrace{\{x \in X \mid f(x) \neq g(x)\}}_{\mu=0} \\ &= (\{x \in X \mid g(x) > \alpha\} \cap \{x \in X \mid f(x) = g(x)\}) \cup \{x \in X \mid f(x) > \alpha\} \cap \underbrace{\{x \in X \mid f(x) \neq g(x)\}}_{\mu=0} \end{aligned}$$

**Proposition:**

Given  $\{f_k(x)\}$  measurable.

1.  $g_n(x) = \sup\{f_1(x), f_2(x), \dots, f_n(x)\}$  and  $h_n(x) = \inf\{f_1(x), f_2(x), \dots, f_n(x)\}$  measurable.
2.  $g(x) = \sup\{f_n(x)\}$  and  $h(x) = \inf\{f_n(x)\}$  measurable.
3.  $\limsup_{n \rightarrow +\infty} f_n(x) = \inf_n \sup\{f_n(x), f_{n+1}(x), \dots\}$  and  $\liminf_{n \rightarrow +\infty} f_n(x) = \sup_n \inf\{f_n(x), f_{n+1}(x), \dots\}$  measurable.
4.  $f_n(x) \rightarrow f(x)$  pointwise  $\implies f$  measurable.

**Proof of A**

$$\begin{aligned} \{x \in X \mid g_n(x) > \alpha\} &= \bigcup_{k=1}^n \{x \in X \mid f_k(x) > \alpha\} \\ \{x \in X \mid h_n(x) < \alpha\} &= \bigcup_{k=1}^n \{x \in X \mid f_k(x) < \alpha\} \end{aligned}$$

**Proof of B**

$$\begin{aligned} \{x \in X \mid g(x) > \alpha\} &= \bigcup_n \{x \in X \mid f_n(x) > \alpha\} \\ \{x \in X \mid h(x) < \alpha\} &= \bigcup_n \{x \in X \mid f_n(x) < \alpha\} \end{aligned}$$

**Definition: Almost Everywhere Convergence**

For  $f_n(x)$  measurable,  $f_n(x) \rightarrow f(x)$   $\mu$ -a.e. in  $X$  if  $f_n(x) \rightarrow f(x)$  in  $A \subset X$  pointwise where  $\mu(X \setminus A) = 0$ .

**Proposition:**

On a complete measure space  $(X, \Lambda, \mu)$  with  $f_n$  measurable and  $f_n(x) \rightarrow f(x)$   $\mu$ -a.e. in  $X$ ,  $f(x)$  is measurable.

**Proof**

$$\begin{aligned} f_n(x) &\rightarrow f(x) \text{ pointwise in } A \text{ and } \mu(A^c) = 0. \\ \{x \in X \mid f(x) > \alpha\} &= (\{x \in X \mid f(x) > \alpha\} \cap A) \cup (\{x \in X \mid f(x) > \alpha\} \cap A^c). \end{aligned}$$

**Theorem:**

With  $(X, \Lambda, \mu)$  a measure space and  $f$  measurable, there exist simple functions  $\phi_n$  such that

1.  $|\phi_n(x)| \leq |\phi_{n+1}(x)|$ .
2.  $\phi_n(x) \rightarrow f(x)$  pointwise in  $X$ .
3. If  $f$  is bounded, then  $\phi_n(x) \rightrightarrows f(x)$  in  $X$ .

**Proof**

Consider  $(-\infty, -n] \cup (-n, n) \cup [n, +\infty)$ , and define  $N_n = \{x \in X \mid f(x) \leq -n\}$  and  $P_n = \{x \in X \mid f(x) \geq n\}$ . Then  $\bigcap_n (N_n \cup P_n) = \emptyset$ . Define

$$\begin{aligned}
A_{n,k} &= \left\{ x \in X \mid \frac{k-1}{2^n} < f(x) \leq \frac{k}{2^n} \right\}_{k=-1,-2,\dots,-n2^n+1} \\
A_{n,0} &= \left\{ x \in X \mid \frac{-1}{2^n} < f(x) < 0 \right\} \\
A_{n,1} &= \left\{ x \in X \mid 0 < f(x) < \frac{1}{2^n} \right\} \\
A_{n,k} &= \left\{ x \in X \mid \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n} \right\}_{k=2,3,\dots,n2^n}
\end{aligned}$$

and set

$$\phi_n(x) = -n\chi_{N_n} + \sum_{k=0}^{-n2^n+1} \frac{k}{2^n} \chi_{A_{n,k}} + \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \chi_{A_{n,k}} + n\chi_{P_n}$$

Claim:

1.  $\forall x \in X, \phi_n(x) \rightarrow f(x)$ .
2. if  $\exists N \in \mathbb{N}$  such that  $|f(x)| < N \implies \phi_n(x) \rightrightarrows f(x)$  in  $X$ .

**Proof**

$$|\phi_n(x) - f(x)| \leq \frac{1}{2^n}, \forall x \in X \setminus (U_n \cup P_n)$$

Note  $\forall x \in X, \exists m \in \mathbb{N}$  such that  $x \notin N_m \cup P_m$ . So  $|f(x)| < m$ .

Then boundedness implies  $\exists N$  such that  $N_N \cup P_N = \emptyset$ .

Therefore  $\forall x \in X, |\phi_n(x) - f(x)| < \frac{1}{2^n}, \forall n \geq N$ .

**Theorem: Egoroff**

Given a measure space  $(X, \Lambda, \mu)$ ,  $\mu(x) < +\infty$  and  $f_n \rightarrow f$   $\mu$ -a.e. in  $X$ , then  $\forall \delta > 0, \exists A \in \Lambda$  such that  $\mu(X \setminus A) < \delta$  and  $f_n(x) \rightrightarrows f(x)$  in  $A$ .

**Recall: Pointwise Convergence**

$\forall x \in X, f_n(x) \rightarrow f(x)$  if  $\forall \varepsilon > 0, \exists N \in \mathbb{N}$  such that  $|f_n(x) - f(x)| < \varepsilon, \forall n \geq N$ .

$$B_{jN,\varepsilon} = \{x \in X \mid \exists N \in \mathbb{N}, |f_n(x) - f(x)| < \varepsilon, \forall n \geq N\}$$

In negation,  $\exists \varepsilon > 0$  such that  $\forall N \in \mathbb{N}, \exists m \geq N$  such that  $|f_n(x) - f(x)| \geq \varepsilon$ .

$$A_{N,\varepsilon} = B_{N,\varepsilon}^c = \{x \in X \mid \exists m \geq N, |f_n(x) - f(x)| \geq \varepsilon\}$$

$$\text{Then } \{x \in X \mid f_n(x) \rightarrow f(x)\} = \bigcap_{\varepsilon > 0} \bigcup_N B_{N,\varepsilon} = \bigcap_{\varepsilon_i \rightarrow 0} \bigcup_i B_{N_i,\varepsilon_i}$$

$$\text{and } \{x \in X \mid f_n(x) \not\rightarrow f(x)\} = \bigcup_{\varepsilon > 0} \bigcap_N A_{N,\varepsilon} = \bigcup_{\varepsilon_i \rightarrow 0} \bigcap_i A_{N_i,\varepsilon_i} \text{ where } \varepsilon_i = \frac{1}{i}.$$

**February 2, 2024**

**Review: Measurable Function**

An extended, real-valued function  $f : X \rightarrow [-\infty, +\infty]$  is measurable if one or all of the following hold

1.  $\forall \alpha \in \mathbb{R}, \{x \mid f(x) > \alpha\} \in \Lambda$ .

2.  $\forall \alpha \in \mathbb{R}, \{x \mid f(x) \geq \alpha\} \in \Lambda.$
3.  $\forall \alpha \in \mathbb{R}, \{x \mid f(x) < \alpha\} \in \Lambda.$
4.  $\forall \alpha \in \mathbb{R}, \{x \mid f(x) \leq \alpha\} \in \Lambda.$
5.  $\forall V \subseteq \mathbb{R}$  open,  $f^{-1}(U) = \{x \mid f(x) \in V\}$  and  $f^{-1}(-\infty), f^{-1}(+\infty) \in \Lambda.$

## Properties

1. For  $f = g$   $\mu$ -a.e.,  $f$  is measurable if and only if  $g$  is measurable.
2. For  $f, g$  measurable,  $f + g$  and  $f \cdot g$  are measurable.
3. For  $\{f_n\}$  measurable,
  - (a)  $\sup_{n \leq k} \{f_n\}$  and  $\inf_{n \leq k} \{f_n\}$  are measurable.
  - (b)  $\sup_n \{f_n\}$  and  $\inf_n \{f_n\}$  are measurable.
  - (c)  $\limsup_{n \rightarrow \infty} f_n$  and  $\liminf_{n \rightarrow \infty} f_n$  are measurable.
  - (d) if  $f_n \rightarrow f$   $\mu$ -a.e. in  $X$ , then  $f$  is measurable.

## Examples

### Characteristic Functions

$$\chi_A = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}, \quad A \in \Lambda$$

### Simple Functions

$$\sum_{i=1}^k \alpha_i \chi_{A_i}, \quad \alpha_i \in \mathbb{R}, \quad A_i \in \Lambda, \quad A_j \cap A_k = \emptyset$$

### Step Functions

$$\sum_{i=1}^k \alpha_i \chi_{I_i}, \quad I_i \text{ interval}$$

## Theorem:

On a measure space  $(X, \Lambda, \mu)$ , suppose  $f$  is measurable.  
There exists a sequence of simple functions  $\{\phi_n\}$  such that

1.  $\phi_n \rightarrow f$  pointwise.
2.  $\phi_n \rightrightarrows f$  for  $f$  bounded.

**Proof**

Let  $N_n = \{x \mid f(x) \leq -n\}$  and  $A_{n,k} = \{x \mid \frac{k-1}{2^n} < f(x) \leq \frac{k}{2^n}\}$ . Then

$$\begin{aligned} A_{n,0} &= \left\{x \mid -\frac{1}{2^n} < f(x) < 0\right\} \\ A_{n,1} &= \left\{x \mid 0 < f(x) < \frac{1}{2^n}\right\} \\ A_{n,k} &= \left\{x \mid \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n}\right\} \\ P_n &= \{x \mid f(x) \geq n\} \end{aligned}$$

and

$$\phi_n = -n\chi_{N_n} + \sum_{k=-n2^n+1}^D \frac{k}{2^n} \chi_{A_{n,k}} + \sum_1^{n2^n} \chi_{A_{n,k}} + n\chi_{\phi_n}$$

So

$$|\phi_n(x) - f(x)| \leq \frac{1}{2^n}, \quad x \in X \setminus (N_n \cup P_n), \quad \bigcap_n (N_n \cap P_n) = \emptyset$$

**Egoroff Theorem**

Given  $(X, \Lambda, \mu)$  where  $\mu(X) < +\infty$ , if

1.  $f_n(x) \rightarrow f(x)$   $\mu$ -a.e. in  $X$  and
2.  $f_n, f$   $\mu$ -a.e. finite.

Then,  $\forall \delta > 0, \exists A \in \Lambda$  with  $\mu(A) < \delta$  such that  $f_n(x) \rightrightarrows f(x)$  on  $A^c$ .

**Proof**

Define  $D = \{x \mid f_n(x) \rightarrow f(x)\} = X$ .

Then  $\forall \varepsilon > 0, \exists m \in \mathbb{N}$  such that  $|f_n(x) - f(x)| < \varepsilon, \forall n \geq m$ .

Say that the universal quantifier  $\forall$  is equivalent to grand intersection and the existential quantifier  $\exists$  is equivalent to grand union. Then

$$D_{m,\varepsilon} = \{x \mid f_n(x) - f(x) < \varepsilon, \forall n \geq m\}$$

and

$$\bigcap_{\varepsilon > 0} \bigcup_m D_{m,\varepsilon} = X.$$

The negation is

$$D_{n,\varepsilon}^c = \{x \mid \exists n \geq m, |f_n(x) - f(x)| \geq \varepsilon\}$$

Then injection is equivalent to the complement.

Set  $\varepsilon_i = \frac{1}{i}$  such that

$$D = \bigcap_i \bigcup_{m_i} D_{m_i, 1/i}$$

$$\emptyset = D^c = \bigcup_i \bigcap_m D_{m, 1/i}^c$$

So  $\bigcap_m D_{m, 1/i}^c = \emptyset$ ,

$$D_{m, 1/i}^c = A_{m, 1/i} = \left\{ x \mid \exists n \geq m, |f_n(x) - f(x)| \geq \frac{1}{i} \right\}$$

and  $A_{n, 1/i} \supset A_{n+1, 1/i} \supset \dots$ . Therefore

$$\mu(A_{n, 1/i}) \rightarrow \mu\left(\bigcap_m A_{m, 1/i}\right) = 0$$

for  $\mu(X) < +\infty$ .

Thus,  $\forall i, \exists m_i$  such that  $\mu(A_{m_i, 1/i}) < \frac{\delta}{2^{i+1}}$ . It follows that  $A = \bigcup_i (A_{m_i, 1/i})$ ,

$$\mu(A) \leq \sum \mu(A_{m_i, 1/i}) < \delta$$

and

$$x \in A^c = \bigcap_i A_{m_i, 1/i}^c = \bigcap_i D_{m_i, 1/i} = \bigcap_i \left\{ x \mid |f_n(x) - f(x)| < \frac{1}{i}, \forall n \geq m_i \right\}$$

Finally, this implies  $f_n(x) \Rightarrow f(x)$  in  $A^c$ .

### Example

Take  $f_n = \chi_{[n, n+1]}$  on  $\mathbb{R}$ , then  $f_n(x) \rightarrow 0$  in  $\mathbb{R}$  but  $A \subset \mathbb{R}$ ,  $\mu(A) < \frac{1}{2}$ ,  $A^c \cap [n, n+1] \neq \emptyset$ ,  $\forall n$ .

That is,  $\forall n, \exists x \in A^c$  such that  $f_n(x) = 1$  but  $f(x) = 0$ .

Therefore  $f_n(x) \not\Rightarrow f(x)$  on  $\mathbb{R}$ .

### Definition: Essential Bounds

On a measure space  $(X, \Lambda, \mu)$  with  $f$  measurable, define  $\|f\|_\infty = \inf\{M \mid \mu(\{x \mid |f(x)| > M\}) = 0\}$ .

This is the  $L^\infty$ -norm.

### Proposition:

$f_n \Rightarrow f$  on  $A$  where  $\mu(A^c) = 0$  if and only if  $\|f_n - f\|_\infty \rightarrow 0$ .

### Proof

( $\Rightarrow$ )

$\forall \varepsilon > 0, \exists m \in \mathbb{N}$  such that  $|f_n(x) - f(x)| < \frac{\varepsilon}{2}, \forall x \in A$ .

Claim:  $\|f_n(x) - f(x)\|_\infty < \varepsilon, \forall n \geq m$ .

$$\|f_n(x) - f(x)\|_\infty = \inf\{M \mid \mu(\{x \mid |f_n(x) - f(x)| > M\}) = 0\}$$

Where  $\{x \mid |f_n(x) - f(x)| > n\} \subset A^c$  and  $n \geq m$  and  $M \geq \varepsilon/2$ .

( $\Leftarrow$ )

## Recall: Urysohn's Lemma

For  $X$  locally compact and Hausdorff,  $K \subset U$  for  $K$  compact and  $U$  open,  $\exists \phi$  continuous such that  $\phi = \begin{cases} 1 & K \\ 0 & U^c \end{cases}$ .

## Theorem: Vitali-Lusin

On measure space  $(X, \Lambda, \mu)$  with  $X$  locally compact and Hausdorff and  $\mu$  a Radon measure.

For  $f$  measurable,  $\mu$ -a.e. finite and vanishing outside  $A$  where  $\mu(A) < +\infty$ ,

$\forall \varepsilon > 0$ ,  $\exists g$  continuous with compact support such that  $\mu(\{x \mid f(x) \neq g(x)\}) < \varepsilon$ .

## Proof

1.  $\exists C \subset A$  compact with  $\mu(A \setminus C) < \varepsilon$ .
2. For  $A$  compact with  $\mu(A) < +\infty$ ,  $\exists U \supset A$  open neighborhood with compact closure and  $\mu(U \setminus A) < \varepsilon$ .
3.  $\phi_n = -n\chi_{N_n} + \sum_{-n2^n+1}^0 \frac{k}{2^n} \chi_{A_{n,k}} + \sum_1^{n2^n} \frac{k-1}{2^n} \chi_{A_{n,k}} + n\chi_{P_n}$

Since we may minimize  $\mu(N_n \cup P_n) < \varepsilon$ ,

$$\phi_n = \sum_{-n2^n+1}^0 \frac{k}{2^n} \chi_{A_{n,k}} + \sum_1^{n2^n} \frac{k-1}{2^n} \chi_{A_{n,k}}$$

Take  $C_{1,k} \subset A_{1,k}$  compact with  $\mu(C_{1,k}) \geq \mu(A_{1,k}) - 2^{-1}2^{-|k|+1}\varepsilon$ . Write

$$C_1 = \bigcup_k C_{1,k}$$

and inductively define  $C_{n-1,k}$  and  $C_{n-1} = \bigcup_k C_{n-1,k}$  such that  $C_{n,k} \subset A_{n,k} \cap C_{n-1}$  compact and

$$\mu(C_{n,k}) \geq \mu(A_{n,k} \cap C_{n-1}) - 2^{-1}2^{-|k|+1}\varepsilon$$

Define, by Urysohn's Lemma,

$$\tilde{\chi}_{A_{n,k}} := \begin{cases} 1 & C_{n,k} \\ 0 & U^c \cup \bigcup_{l \neq k} C_{n,l} \end{cases}$$

where  $C_n \subset C_{n-1}$ ,  $C = \bigcap C_n$ ,  $C_n = \bigcup_k C_{n,k}$ .

Then define

$$g_n := \sum_{-n2^n+1}^0 \frac{k}{2^n} \tilde{\chi}_{A_{n,k}} + \sum_1^{n2^n} \frac{k-1}{2^n} \tilde{\chi}_{A_{n,k}}$$

Then  $g_n = \phi_n$  on  $C$  for all  $n$ .

Therefore  $g_n = \phi_n \Rightarrow \hat{g} = f$  on  $C$ .

By uniform convergence,  $\hat{g}$  is continuous on  $C$ .

So, again by Urysohn's Lemma,  $g = \phi \hat{g}$  and  $\{x \mid g \neq f\} = U \setminus C$ .

February 8, 2024

## Midterm Review

### Problem 2

Given a finite measure space  $(X, \Lambda, \mu)$ ,  $\mu(X) < +\infty$  and a function  $f$  which is  $\mu$ -a.e. finite. Monotone Convergence Theorem:

1.  $A_1 \subset A_2 \subset \dots$ , then  $\mu(\bigcup_i A_i) = \lim_{i \rightarrow \infty} \mu(A_i)$ .
2.  $A_1 \supset A_2 \supset \dots$ , then  $\mu(\bigcap_i A_i) = \lim_{i \rightarrow \infty} \mu(A_i)$  for  $\mu(A_1) < +\infty$ .

If  $A_k = \{x \mid |f(x)| > k\}$  and

$$F = \bigcap_{k=1}^{\infty} A_k$$

then  $\mu(F) = \lim_{k \rightarrow \infty} \mu(A_k) = 0$  since  $\mu(X) < +\infty$ .

If instead we consider  $A_k^c$ , then

$$\bigcup_k A_k^c = X \setminus F$$

### Problem 3

1. Borel

Given  $(\alpha, +\infty)$ , we want  $\forall E \subset \mathbb{R}$

$$m^*(E \cap (\alpha, +\infty)) + m^*(E \cap (-\infty, \alpha]) \leq m^*(E)$$

$\forall \varepsilon > 0$ ,  $\exists \{I_i\}$  pen intervals

$$\bigcup_i I_i \supset E \quad \sum_i |I_i| \leq m^*(E) + \varepsilon/2$$

Divide  $\{I_i\}$  into 3 groups,

$$\begin{aligned} C^\ell &= \{I \in \{I_i\} \mid I \text{ is to the left of } \alpha\} \\ C^r &= \{I \in \{I_i\} \mid I \text{ is to the right of } \alpha\} \\ C^m &= \{I \in \{I_i\} \mid \alpha \in I\} \end{aligned}$$

Then,  $\forall I_k^m \in C^m = \{I_k^m\}$ , and

$$\begin{aligned} {}^\ell I_k^n &= \left(a_k, \alpha + \frac{2}{2^{k+2}}\right) \\ {}^r I_k^n &= \left(\alpha - \frac{2}{2^{k+2}}, b_k\right) \\ {}^m I_k^n &= (a_k, b_k) \end{aligned}$$



where also

$$\begin{aligned} A_n &\supset (\alpha, +\infty)^c & A_n &= \left(-\infty, \alpha + \frac{1}{2^n}\right) \\ B_n &\supset (\alpha, +\infty) & B_n &= \left(\alpha + \frac{1}{2^n}, +\infty\right) \\ A_n \cap B_n &= \left(\alpha - \frac{1}{2^n}, \alpha + \frac{1}{2^n}\right) \end{aligned}$$

So  ${}^\ell I_k^n \cup {}^r I_k^n = I_k^n$ , and  $|{}^\ell I_k^n| + |{}^r I_k^n| = |I_k^n| + \frac{\varepsilon}{2^{k+1}}$ .  
Finally

$$\begin{aligned} m^*(E \cap (\alpha, +\infty)) + m^*(E \cap (-\infty, \alpha]) &\leq \sum_{I \in C^r} |I| + \sum_k |{}^r I_k^n| + \sum_{I \in C^\ell} |I| + \sum_k |{}^\ell I_k^n| \\ &\leq \sum_{I \in C^r} |I| + \sum_{I \in C^\ell} |I| + \sum_k |I_k^n| + \frac{\varepsilon}{2} \\ &\leq m^*(E) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \end{aligned}$$

2.  $\mu(K) < +\infty$  for  $K \subset \mathbb{R}$  compact.

$K$  is bounded,  $K \subset (-M, M)$  for large  $M$ .  
Therefore  $\mu(K) \leq 2M < +\infty$ .

3.  $\forall U \subset \mathbb{R}$  open, we want to show  $\exists K_n$  compact such that  $K_n \subset U$  and  $\mu(K_n) \rightarrow \mu(U)$ .

Let  $U = \bigcup_i I_i$  a union of countably many disjoint open intervals (e.g.  $I_i = (a_i, b_i)$ ).

Then  $m(U) = \sum_i m(I_i)$ .

Set  $I_i^n = [a_i + \frac{1}{n2^{i+1}}, b_i - \frac{1}{n2^{i+1}}]$ . Then

$$\sum_i^n |I_i^n| \geq \sum_{i=1}^k |I_i| - \frac{1}{n}, \quad \forall k$$

It follows that

$$\sum_{i=1}^k |I_i| \rightarrow \sum_{i=1}^\infty |I_i|, \quad \text{as } k \rightarrow +\infty$$

and

$$\begin{aligned} K_k^n &= \bigcup_{i=1}^k I_i^n \subset U \quad \text{compact} \\ m(U) &\geq m(K_k^n) = \sum_{i=1}^n |I_i^n| \geq \underbrace{\sum_{i=1}^\infty |I_i|}_{m(U)} - \frac{1}{n} \end{aligned}$$

Alternatively, we have the theorem that if  $X$  is a metric space and  $\mu$  is Borel regular on  $(X, \Lambda)$ , then

(a)  $A \in \Lambda$ ,  $\mu(A) < +\infty$ ,  $\forall \varepsilon > 0$ ,  $\exists C$  closed with  $C \subset A$  such that  $\mu(A \setminus C) < \varepsilon$ .

(b)  $\exists \{U_i\}$ ,  $\mu(U_i) < +\infty$ ,  $U_i$  open where  $A \subset \bigcup_i U_i$ ,  $\forall \varepsilon > 0$  there exists  $V$  open such that  $V \supset A$  and  $\mu(V \setminus A) < \varepsilon$ .

With the corollary that for  $\mu$  on  $\mathbb{R}^n$ ,  $\mu$  is Radon if and only if it is locally finite and Borel regular.

4. For  $A \in \Lambda$ ,  $m(A) = \inf\{m(V) \mid V \supset A, V \text{ open}\}$

Recall Borel regularity:  $\forall A \in \Lambda$ , there is some Borel set  $B \supset A$  with  $m(B) = m(A)$ .

We may assume  $m(A) < +\infty$ . Then,  $\forall \varepsilon > 0$ , there is some collection of open intervals  $\{I_i^n\}$  containing  $A$  where

$$\sum_i |I_i^n| \leq m(A) + \varepsilon$$

Set  $\varepsilon = \frac{1}{n}$  and let  $U^n = \bigcup_i I_i^n \supset A$  open. Then

$$m(A) \leq m(U^n) \leq \sum_i |I_i^n| \leq m(A) + \frac{1}{n}$$

If  $B = \bigcap_n U^n$ , then  $\lim_{n \rightarrow \infty} m(U^n) = m(A)$  and  $m(B) = m(A)$ .

#### Problem 4

Given  $f : \mathbb{R} \rightarrow \mathbb{R}$ , continuous outside a measure zero set  $D$ .

That is,  $\bar{f} : \mathbb{R} \setminus D \rightarrow \mathbb{R}$  is continuous.

$\forall V \subset \mathbb{R}$ ,  $f^{-1}(V) = (f^{-1}(V) \cap (\mathbb{R} \setminus D)) \cup (f^{-1}(V) \cap D)$ .

By measure completeness, we are automatically safe on  $f^{-1}(V) \cap D$ .

Claim:  $f^{-1}(V) \cap (\mathbb{R} \setminus D) = \bar{f}^{-1}(V)$ .

Claim:  $\bar{f}^{-1}$  is measurable.

Claim:  $\bar{f}^{-1}(V) = U \cap (\mathbb{R} \setminus D)$  where  $U \subset \mathbb{R}$  open.

Since  $U \cap (\mathbb{R} \setminus D)$  is open in the subspace topology, we are done.

Alternatively (similary to Problem 8 below), for  $D$  such that  $m(D) = 0$ ,  $\forall n$ ,  $\exists U^n$  such that  $m(U^n) \leq 2^{-n}$ ,  $U^n \supset D$  and  $U^n = \bigcup_i (a_i, b_i)$  where  $(a_j, b_j) \cap (a_k, b_k) = \emptyset$  and  $a_i, b_i \in \mathbb{R} \setminus D$ . So

$$f_n = \begin{cases} f(x), & x \in (U^n)^c \\ f(a_i) + \frac{f(b_i) - f(a_i)}{b_i - a_i}(x - a_i), & x \in (a_i, b_i) \subset U^n \end{cases}$$

Then  $\{x \mid f_n(x) \neq f(x)\} \subset U^n$  and  $m(\{x \mid f_n(x) \neq f(x)\}) \leq 2^{-n}$ .

#### Homework 4 Problem 8

Assume  $f(x)$  is decreasing.

1. Discontinuities are limited to jump discontinuities.

2. Discontinuities are countable.

3.  $D = \{x_i\}_i$ ,  $\forall n$  there exists an open cover  $\{I_i^n = (a_i, b_i)\}$  where  $\bigcup_i I_i^n = C^n \supset \{x_i\}_i$  and  $m(C^n) \leq 2^{-n}$ .

Then  $\{x \mid f_n(x) \neq f(x)\} \subset C^n$  and  $\mu(\{x \mid f_n(x) \neq f(x)\}) \leq 2^{-n}$ .

Claim:  $f_n(x) \rightarrow f(x)$  on  $\mathbb{R} \setminus G$  where  $G = \bigcap_n \bigcup_{k=n}^\infty \{x \mid f_k(x) \neq f(x)\}$ .

By monotone convergence,  $\mu(g) = \lim_{n \rightarrow +\infty} \mu(\bigcup_{k=n}^\infty \{x \mid f_n(x) \neq f(x)\}) = \lim_{n \rightarrow +\infty} (\sum_{k=n}^{+\infty} 2^{-k}) = 0$ .

Consider the complement,  $G^c = \bigcap_{n=1}^\infty \bigcap_{k=n}^{+\infty} \{x \mid f_k(x) = f(x)\}$ .

Then  $\forall x \in G^c$ ,  $x \in \bigcap_{k=n_0}^{+\infty} \{x \mid f_k(x) = f(x)\}$ , so  $f_n(x) = f(x) \forall n \geq n_0$ .

## Riemann Integration

Given a function  $f : [a, b] \rightarrow \mathbb{R}$  bounded and  $P$  a partition of  $[a, b]$  where

$$a = x_0 < x_1 < \cdots < x_n = b$$

The Cauchy sum

$$C(P, [a, b]) = \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}), \quad \xi_i \in [x_{i-1}, x_i]$$

alternatively

$$\phi(P, [a, b]) = \sum_i f(\xi_i) \chi_{[x_{i-1}, x_i]}$$

Consider the upper Riemann sum

$$S(P, [a, b]) = \sum_i M_i(x_i - x_{i-1}), \quad M_i = \sup_{[x_{i-1}, x_i]} f(x)$$

and the lower Riemann sum

$$s(P, [a, b]) = \sum_i m_i(x_i - x_{i-1}), \quad m_i = \inf_{[x_{i-1}, x_i]} f(x)$$

then define

$$S = \inf_P S(P, [a, b]) = s = \sup_P s(P, [a, b]) \implies \int_a^b f(x) dx = \lim_{l(P) \rightarrow 0} C(P, [a, b])$$

### Theorem:

$f$  is Riemann integrable on  $[a, b]$  if and only if  $f$  is continuous  $m$ -a.e. (w.r.t Lebesgue measure) on  $[a, b]$ .

### Proof

( $\implies$ ) Let  $f$  be Riemann integrable on  $[a, b]$ .

Define the oscillation

$$\text{Osc}_I(f) = \sup_I f(x) - \inf_I f(x)$$

$$\text{Osc}_x(f) = \lim_{\delta \rightarrow 0} \text{Osc}_{(x-\delta, x+\delta)}(f)$$

and observe that  $f$  is continuous at  $x$  if and only if  $\text{Osc}_x(f) = 0$ .

Let  $D = \{x \mid \text{Osc}_x(f) > 0\}$  and  $D_k = \{x \mid \text{Osc}_x(f) > \frac{1}{k}\}$  such that  $D_k \subset D_{k+1}$  and  $D = \bigcup_k D_k$ .

Therefore  $m(D_k) \rightarrow m(D)$ .

To show that  $m(D) = 0$ , assume otherwise that  $m(D) > 0$ .

Therefore,  $\exists k$  such that  $m(D_k) > d_{k_0}$  for any  $k \geq k_0$ .

Then, for any partition  $P$  we may examine

$$S(P, [a, b]) - s(P, [a, b]) = \sum_{I_i} (M_i - m_i) |I_i|$$

We want to show that this is  $\geq \delta > 0$  for any  $P$ .

## February 13, 2024

### Recall: Riemann Integration

$f(x) \geq 0$  on  $[a, b]$  bounded.

Partition  $P = \{a = x_0 < x_1 < \dots < x_n = b\}$ ,  $[x_{i-1}, x_i]$ .

IMAGE HERE - Riemann Integration

Upper Riemann Sum:  $S_P = \sum_{i=1}^n M_i (x_i - x_{i-1})$  where  $M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$ .

Lower Riemann Sum:  $s_P = \sum_{i=1}^n m_i (x_i - x_{i-1})$  where  $m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$ .

Step Functions:  $\phi_{P,\alpha} = \sum_i \alpha_i \chi_{I_i}$  where  $I_i = [x_{i-1}, x_i]$ .

Set  $S = \inf_P S_P = \inf\{\sum_i \alpha_i |I_i| \mid \phi_{P,\alpha}(x) \geq f(x)\}$

and  $s = \sup_P s_P = \sup\{\sum_i \alpha_i |I_i| \mid \phi_{P,\alpha}(x) \leq f(x)\}$ .

### Definition: Riemann Integrable

The function  $f$  is Riemann integrable if  $S = s$ .

### Remark:

$$S_P - s_P = \sum_{i=1}^n (M_i - m_i)(x_i - x_{i-1}) \rightarrow 0 \text{ as } \ell(P) \rightarrow 0$$

### Remark:

If  $f$  is continuous, then it is Riemann integrable.

### Theorem:

Given  $f : [a, b] \rightarrow \mathbb{R}$  bounded, then  $f$  is Riemann integrable if and only if  $f$  is continuous  $m$ -a.e.

$m(D) = 0$  if and only if  $f$  is Riemann integrable.

### Proof

Recall that  $\text{Osc}_I(f) = \sup_I f(x) - \inf_I f(x)$  and  $\text{Osc}_{x_0}(f) = \lim_{\delta \rightarrow 0} \text{Osc}_{(x_0-\delta, x_0+\delta)}(f)$ .

IMAGE HERE - 2 Oscillation

Write  $D = \{x \in [a, b] \mid f \text{ is not continuous at } x\}$ , and  $D_k = \{x \in [a, b] \mid \text{Osc}_x(f) \geq 1/k\}$  closed (since  $D_k^C$  open).

Then

$$D = \bigcup_k D_k = \{x \in [a, b] \mid \text{Osc}_x(f) > 0\}$$

We have  $m(D_k) \xrightarrow[k \rightarrow \infty]{} m(D)$ .

Then there exists an open cover of  $D_k$ ,  $\{I_i\}$  such that  $m(D_k) + \varepsilon \geq \sum_i |I_i| \geq m(D_k) - \varepsilon$ .

Since  $D_k$  is closed and bounded, it is compact and there exists finite subcover  $\{I_{i_k}\}_{k=1}^\ell \subset \{I_i\}$ .  
 $(\Leftarrow)$  Assume that  $f$  is Riemann integrable and, for sake of contradiction, that  $m(D) > 0$ .

Then  $m(D_k) \geq m > 0, \forall k \geq k_0$ .

Now for any partition  $P = \{x_0, x_1, \dots, x_n\}$ ,

$$\begin{aligned} S_P - s_P &= \sum_{i=1}^n (M_i - m_i)(x_i - x_{i-1}) \\ &\geq \sum_{(x_{i-1}, x_i) \cap D_k \neq \emptyset} (M_i - m_i)(x_i - x_{i-1}) \\ &\geq \frac{1}{k} \sum_{(x_{i-1}, x_i) \cap D_k \neq \emptyset} (x_i - x_{i-1}) \end{aligned}$$

Since  $\bigcup_{(x_{i-1}, x_i) \cap D_k \neq \emptyset} [x_{i-1}, x_i] \supset D_k$ ,

$$\sum_{(x_i, x_{i-1}) \cap D_k \neq \emptyset} (x_i - x_{i-1}) = m \left( \bigcup_{(x_{i-1}, x_i) \cap D_k \neq \emptyset} [x_{i-1}, x_i] \right) \geq m(D_k)$$

we conclude that

$$S_P - s_P \geq \frac{m}{k_0} \geq 0$$

$(\Rightarrow)$  Assume  $m(D) = 0$ .

Then, for any  $k$  satisfying  $\frac{1}{k} < \frac{\varepsilon}{2(b-a)}$ ,  $m(D_k) = 0$  and  $\{I_{i_k}\}_{k=1}^\ell \subset \{I_i\}$  for open intervals  $I_i$ .

We have, also,  $\bigcup_{k=1}^\ell I_{i_k} \supset D_k$  so

$$\sum_{k=1}^\ell |I_{i_k}| \leq \sum_i |I_i| \leq \frac{\varepsilon}{2M}$$

and

$$[a, b] \setminus \bigcup_{k=1}^\ell I_{i_k} \subset D_k^c$$

compact.

Claim: there exists some partition  $P = \{x_i\}_{i=0}^n$  such that  $S_P - s_P < \varepsilon = \frac{1}{k}$ .

Given  $\text{Osc}_x(f) \leq 2M$ ,

$$\begin{aligned} S_P - s_P &= \sum_i (M_i - m_i)(x_i - x_{i-1}) \\ &= \sum_{[x_{i-1}, x_i] \cap D_k = \emptyset} + \sum_{[x_{i-1}, x_i] \cap D_k \neq \emptyset} \\ &\leq \frac{\varepsilon}{2(b-a)}(b-a) + 2M \cdot \frac{\varepsilon}{4M} \end{aligned}$$

## Definition: Lebesgue Integration

Given a measure space  $(X, \Lambda, \mu)$  and simple function  $s = \sum_i \alpha_i \chi_{A_i}$  for  $\alpha_i \in \mathbb{R}$  and  $A_i \in \Lambda$ ,

$$\int_E s \, d\mu = \sum_i \alpha_i \mu(A_i \cap E)$$

Then, for extended real-valued  $f \geq 0$ ,

$$\int_E f \, d\mu = \sup \left\{ \sum_i \alpha_i \mu(A_i \cap E) \mid 0 \leq s(x) \leq f(x) \right\}$$

## Properties

1. For  $0 \leq f \leq g$  on  $E$ ,  $\int_E f \, d\mu \leq \int_E g \, d\mu$ .
  2. For  $A \subset B$  where  $A, B \in \Lambda$ ,  $\int_A f \, d\mu \leq \int_B f \, d\mu$ .
  3. Since  $f \geq 0$ ,  $\forall c \in \mathbb{R}_{\geq 0}$   $\int_E cf \, d\mu = c \int_E f \, d\mu$ .
  4.  $f = 0$   $\mu$ -a.e. if and only if  $\int_X f \, d\mu = 0$ .
  5.  $\int_E f \, d\mu = \int_X f \chi_E \, d\mu$ .
  6. For  $f, g \geq 0$ ,  $\int_E f + g \, d\mu = \int_E f \, d\mu + \int_E g \, d\mu$ .
  7. For  $A, B \in \Lambda$  where  $A \cap B = \emptyset$ ,  $\int_{A \cup B} f \, d\mu = \int_A f \, d\mu + \int_B f \, d\mu$ .
- Proof of 4  
( $\implies$ )  $\sum_i \alpha_i \chi_{A_i} = s(x) = f(x) \implies \alpha_i > 0 \implies \mu(A_i) = 0$ .  
( $\impliedby$ )  $f \geq \alpha > 0$  and  $\mu(A) > 0 \implies f(x) \geq \alpha \chi_A \implies \int_X f \, d\mu \geq \alpha \mu(A) > 0$  a contradiction.
  - Proof of 5  
 $s \chi_E = \sum_i \alpha_i \chi_{A_i \cap E}$ .
  - Proof of 6  
If  $0 \leq s_1 \leq f$  and  $0 \leq s_2 \leq g$ , then  $0 \leq s_1 + s_2 \leq f + g$ .

## Monotone Convergence of Lebesgue Integration

On a measure space  $(X, \Lambda, \mu)$ , let  $f_n \geq 0$  be a sequence of measurable functions which is monotone  $f_i(x) \leq f_{i+1}(x)$  and converging  $f_n(x) \rightarrow f(x)$  for any  $x \in X$ . Then

$$\lim_{n \rightarrow +\infty} \int_X f_n \, d\mu = \int_X f \, d\mu = \int_X \left( \lim_{n \rightarrow +\infty} f_n \right) d\mu$$

## Proof

Observe that  $f_n(x) \leq f(x)$ ,  $\forall x \in X$ , so

$$\int_X f_n \, d\mu \leq \int_X f_{n+1} \, d\mu \leq \int_X f \, d\mu$$

so

$$\lim_{n \rightarrow +\infty} \int_X f_n d\mu \leq \int_X f d\mu$$

We want to show that

$$\lim_{n \rightarrow +\infty} \int_X f_n d\mu \geq \int_X f d\mu$$

Let  $s$  be a simple function satisfying  $0 \leq s(x) \leq f(x)$ , and define

$$E_n = \{x \in X \mid f_n(x) \geq cs(x)\}$$

for some  $c \in (0, 1)$ .

Then  $E_n \subset E_{n+1}$  and  $\bigcup_n E_n = X$ . Consider

$$\int_X f_n d\mu \geq \int_{E_n} f_n d\mu \geq c \int_{E_n} s(x) d\mu = c \sum_i \alpha_i \mu(A_i \cap E_n)$$

For any  $i$ ,  $A_i \cap E_n \rightarrow A_i$ . Therefore  $\mu(A_i \cap E_n) \xrightarrow{n \rightarrow +\infty} \mu(A_i)$ . So

$$\lim_{n \rightarrow +\infty} \int_X f_n d\mu \geq c \sum_i \alpha_i \mu(A_i)$$

for  $0 \leq s = \sum \alpha_i \chi_{A_i} \leq f(x)$ . Since this hold for any  $c$ ,

$$\lim_{n \rightarrow +\infty} \int_X f_n d\mu \geq \int_X f d\mu$$

### Corollary

Given a measurable sequence  $f_n \geq 0$  with  $f(x) = \sum_n f_n(x)$ ,

$$\int_X f d\mu = \sum_n \int_X f_n d\mu$$

and

$$\phi_n(x) = \sum_{k=1}^n f_k(x) \rightarrow f(x)$$

### Definition: Fatou's Lemma

Given a sequence of measurable functions  $f_n \geq 0$ ,

$$\int_X \left( \liminf_{n \rightarrow +\infty} f_n \right) d\mu \leq \liminf_{n \rightarrow +\infty} \int_X f_n d\mu$$

## Proof

Observe that

$$\liminf_{n \rightarrow +\infty} f_n = \lim_{n \rightarrow +\infty} \overbrace{(\inf \{f_n(x), f_{n+1}(x), \dots\})}^{g_n(x)}$$

so, by monotone convergence,

$$\int_X \left( \lim_{n \rightarrow +\infty} g_n(x) \right) d\mu = \lim_{n \rightarrow +\infty} \int_X g_n(x) d\mu$$

and  $g_n(x) \leq f_n(x)$  gives

$$\int_X g_n(x) d\mu \leq \int_X f_n(x) d\mu$$

and implies

$$\lim_{n \rightarrow +\infty} \int_X g_n(x) d\mu \leq \liminf_{n \rightarrow +\infty} \int_X f_n(x) d\mu$$

## Space of Integrable Functions

Write

$$f(x) = f^+(x) - f^-(x)$$

where

$$\begin{aligned} f^+(x) &= \max\{f(x), 0\} \geq 0 \\ f^-(x) &= \min\{-f(x), 0\} \geq 0 \end{aligned}$$

Then for  $\int_X f^+ d\mu$  and  $\int_X f^- d\mu$ ,  $\int_X f d\mu$  is defined when at least one is finite. If both are finite, then

$$L_\mu^1(x) = \int_X |f| d\mu = \int_X f^+ d\mu + \int_X f^- d\mu \leq +\infty$$

## Properties

1. For any  $\alpha, \beta \in \mathbb{R}$ ,

$$\int_X (\alpha f + \beta g) d\mu = \alpha \int_X f d\mu + \beta \int_X g d\mu$$

if  $f, g \in L_\mu^1(x)$ .

2. For  $f \in L_\mu^1(x)$ ,

$$\begin{aligned} \left| \int_X f d\mu \right| &\leq \int_X |f| d\mu \\ \left| \int_X f^+ d\mu - \int_X f^- d\mu \right| &\leq \int_X f^+ d\mu + \int_X f^- d\mu \end{aligned}$$



3. For  $f \leq g$ ,  $f, g \in L^1_\mu(x)$ ,  $\int_X f \, d\mu \leq \int_X g \, d\mu$ .
4.  $\int_{A \cup B} f \, d\mu = \int_A f \, d\mu + \int_B f \, d\mu$ .
5.  $f = 0$   $\mu$ -a.e. if and only if  $\int_X |f| \, d\mu = 0$ .