Manifolds I

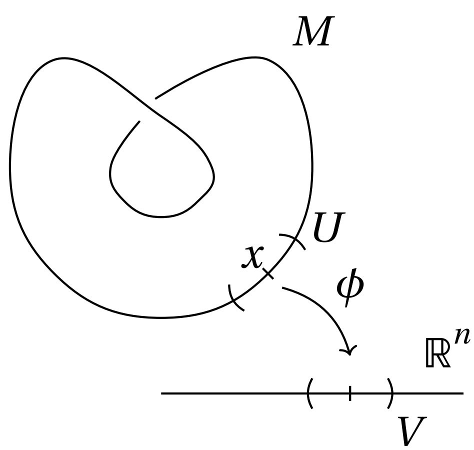
- September 26, 2024

Class Organization

1 Takehome Midterm
1 Takehome Final
Homeworks assigned, but not graded.
`https://ginzburg.math.ucsc.edu/teaching/208manifolds1-2024/syl.html`

Definition: Topological Manifolds

For M a topological space, M is a topological manifold if $\forall x \in M$, $\exists M \supset U \ni x$ and homeomorphism $\phi : U \to V \subset \mathbb{R}^n$ for V open.
To avoid problems (see below), further assume that M is Hausdorff and second countable.

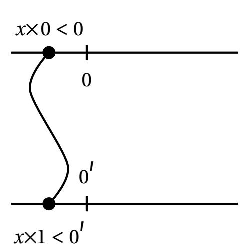


Exercise

We can require V to be an open ball.

Problems

- M need not be Hausdorff.



With $(\mathbb{R} \times 0 \coprod \mathbb{R} \times 1)/\sim$.

- M need not be second countable.

Take $\coprod_S \mathbb{R}_S$ where S is an uncountable index.

Examples

Example 1

If $N \underset{\text{homeo}}{\simeq} M$, this implies N is a manifold.

Example 2

IMAGE 3

Example 3

An open subset of a manifold is a manifold.

Example 4

$M,\ N$ manfiolds implies $M \times N$ is a manifold.

Example 5

Take $\mathbb{R}/\mathbb{Z}$ by the equivalence relation $t \sim t'$ iff $t' - t \in \mathbb{Z}$.

IMAGE 4

Then $C^0(S^1)$ relates to periodic functions with period 1.

Example 6

$\mathbb{T}^n = S^1 \times \cdots \times S^1$.

Counterexample 1

$[0, 1]$ is not a manifold.

IMAGE 5

Since 0 must map somewhere in the open interval, its deletion results in a connected space in the former case but a disconnected one in the latter. Similarly, the following breaks into three and two connected components respectively.

IMAGE 6

Definition: Manifold with Boundary

There exists a neighborhood $\forall x \in M$ homeomorphic to either the open ball or the half-closed half-ball.

IMAGE 7

Exercise

A connected manifold is path-connected.

Examples

Example 7

Take $f : \mathbb{R}^n \overset{C^0}{\to} \mathbb{R}$ with graph $\Gamma_f = \{(x, f(x)) \; : \; x \in \mathbb{R}^n\} \subset \mathbb{R}^n \times \mathbb{R}$.

IMAGE 8

Example 8

Take $f : M \to N$ between manfiolds, then $M \simeq \Gamma_f \subseteq M \times N$.

Example 9

$S^n \subset \mathbb{R}^{n+1}$.

IMAGE 9

Definition: Real Projective Spaces

Take $\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus \{0\}/\sim$ where $x \sim y \iff x = \lambda y$ for $\lambda \neq 0$.
Informally, the collection of lines through the origin.
Alternatively, $\mathbb{RP}^n = S^n/\sim$ where $x \sim -x$.
That is, identifiying the antipodal points of the unit sphere.
We may also consider $\mathbb{RP}^n = SO(n+1)/SO(n)$.

Claim

$\mathbb{RP}^n$ is a manifold.

IMAGE 10

$\mathbb{RP}^1 \setminus \{x\text{-axis}\} \overset{\text{homeo}}{\to} \mathbb{R}$.

IMAGE 11

$\mathbb{RP}^2 \setminus \mathbb{RP}^1 \overset{\text{homeo}}{\to} \mathbb{R}^2$
We have that $\mathbb{RP}^1$ is homeomorphic to the circle, and $\mathbb{RP}^n = \mathbb{RP}^{n-1} \cup B^n$.
Take $x = (x_0, \dots, x_n)$, $y = (y_0, \dots, y_n) = (\lambda x_0, \dots, \lambda x_n)$ and $[x] = [x_0 : x_1 : \dots : x_n]$.
Then for $U_k \subset \mathbb{RP}^n$ with $U_k = \{[x] \; : \; x_k \neq 0\}$, we have that $U_0, \dots, U_n$ covers $\mathbb{RP}^n$.
Then define $U_k \to \mathbb{R}^n$ by $[x_0 : \dots : x_n] \to \left(\frac{x_0}{x_k}, \dots, \frac{x_k}{x_k}, \dots, \frac{x_n}{x_k}\right)$.

Connected Sum of Manfiolds

IMAGE 12

$M \setminus B^n \coprod N \setminus B^n$

IMAGE 13

$M \# N$.

Mobius Band

IMAGE 14