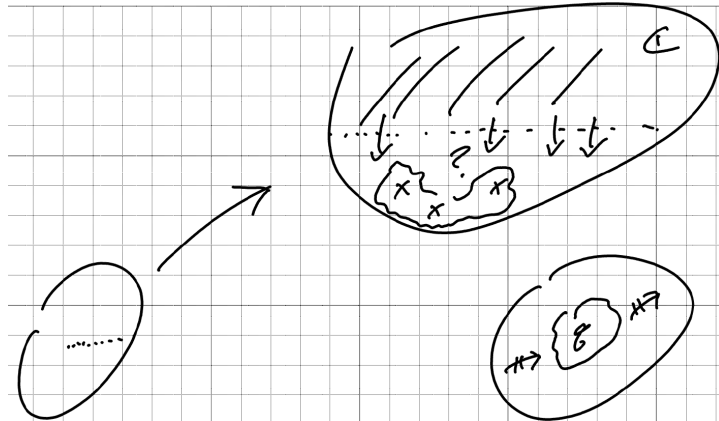


Advanced Analysis

September 25, 2025

Suppose we have some function of the form $-\Delta + q \in \mathbb{L}(H)$ satisfying $R_A(\lambda)(A - \lambda I)^{-1}$ bounded on $\text{Im}(\lambda) > 0$ and not surjective for $\text{Im}(\lambda) = 0$.

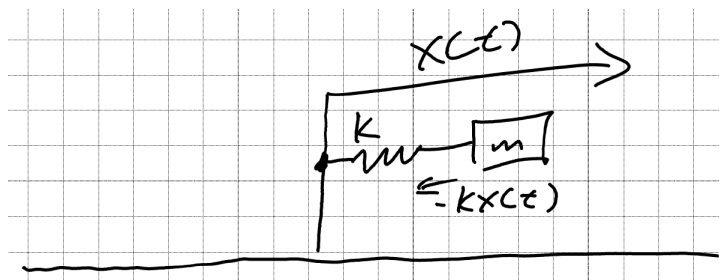


Waves: solutions to $\partial_{tt}u + Au = 0$ on \mathbb{R}^n .

Mathematical Tools

- Spectral theory of unbounded operators
- Complex analysis
- Functional analysis
- Microlocal analysis
- Semiclassical analysis

Classical Resonances in ODEs

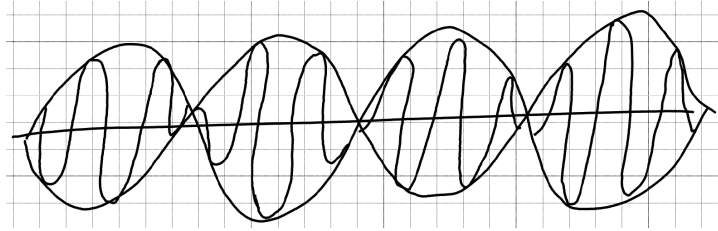


A harmonic oscillator assuming no friction.

We have an acceleration force, $m\ddot{x}(t) = -kx(t)$ which gives $\ddot{x} + \omega_0^2 x = 0$ with $\omega_0 = \sqrt{\frac{k}{m}}$ and has solution $x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$.

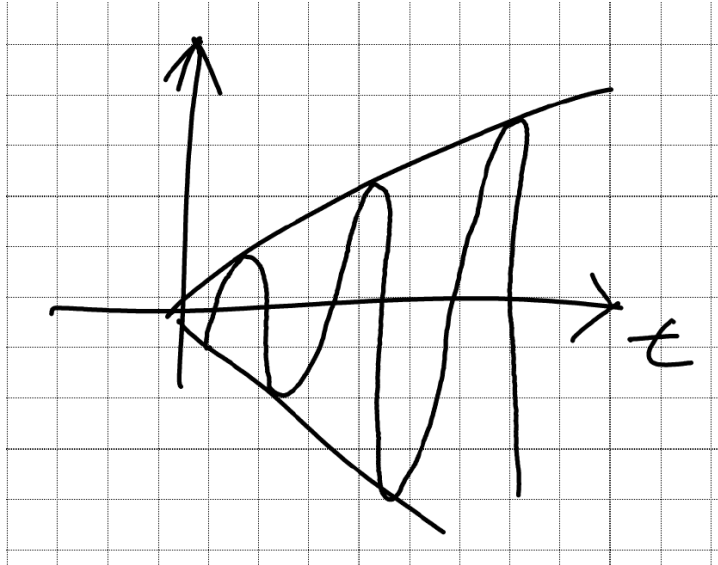
With forcing, i.e. $m\ddot{x}(t) = -kx(t) + A\sin(\omega t)$, we have $\ddot{x} + \omega_0^2 x = A'\sin(\omega t)$.

If $|\omega| \neq |\omega_0|$, then $x(t) \sim \text{trig}\left(\left(\frac{\omega - \omega_0}{2}\right)t\right)\left(\left(\frac{\omega + \omega_0}{2}\right)t\right)$ the low and high frequencies respectively.



Beats (non-amplified)

If instead $|\omega| = |\omega_0|$, then $x(t) \propto \text{trig}(\omega t)t$.



In general, $\dot{x} + Ax = 0$ for $x \in \mathbb{R}^n$, $x(t) = \exp(-tA) + x(0)$.

In the case where A is skew-adjoint, i.e. $\text{sp}(A) \subseteq i\mathbb{R}$, $(x, Ax) = 0 \forall x \in \mathbb{R}^n$, then

$$\frac{d}{dt}(x, x) = (\dot{x}, x) + (x, \dot{x}) = (-Ax, x) - (x, Ax) = 0$$

Which implies that $\|x(t)\|$ is constant and the dynamics are norm preserving.

To generate resonant solutions, if $(i\omega, v)$ is an eigenpair of A ($\omega \in \mathbb{R}$), consider $\dot{x} + Ax = e^{-i\omega t}v$. As an ansatz, we look for a solution of the form $x(t) = a(t)v$ and the equation becomes $(\dot{a}(t) + i\omega a)v = e^{-i\omega t}v$. Then

$$\begin{aligned} e^{-i\omega t} \frac{d}{dt}(e^{i\omega t} a) &= e^{-i\omega t} \\ \frac{d}{dt}(e^{i\omega t} a) &= 1 \\ a(t) &= te^{-i\omega t}. \end{aligned}$$

Resonances in PDEs

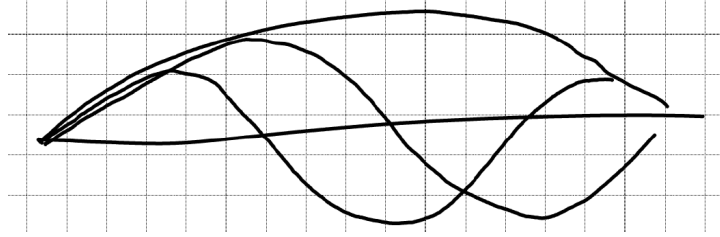
Consider one-dimensional waves on $[0, L]$, $L > 0$.

$$\begin{cases} \partial_{tt}u + \partial_{xx}u = 0 & x \in [0, L] \\ u|_{t=0} = f & x \in [0, L] \\ \partial_t u|_{t=0} = g & x \in [0, L] \\ u(0, t) = u(L, t) = 0 & \forall t \geq 0 \end{cases}$$

We want to think about this as $\partial_{tt}u = Au = 0$ where A is the Dirichlet Laplacian $Au = -\partial_{xx}u$ with Dirichlet boundary conditions. We then want to find the spectral decomposition of A , $Au - \lambda u = 0 = -\partial_x^2 u - \lambda u$.

$$\begin{aligned}\lambda = 0. \quad u(x) &= A + Bx \implies A = B = 0 \\ \lambda = -p^2. \quad u(x) &= Ae^{px} + be^{-px} \implies A = B = 0 \\ \lambda = p^2. \quad u(x) &= A\cos(px) + B\sin(px) \implies 0 = u(0) = A \quad 0 = u(L) = B\sin(pl) \implies p = k\pi, k \in \mathbb{N}\end{aligned}$$

Therefore there are infinitely many eigenpairs $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $\phi_n(x) = \sin\left(\frac{k\pi x}{L}\right)$.

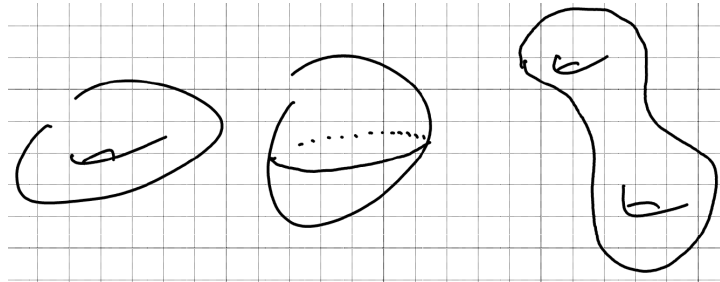


The family $\{\phi_n, n \in \mathbb{N}\}$ is dense in $L^2([0, L])$ where the unbounded operator $(-\partial_x^2)$ with Dirichlet boundary conditions is self-adjoint.

Other Prototypes

(of unbounded self-adjoint operators with discrete spectrum)

- Laplace-Beltrami operators on compact manifolds without boundary.



- On compact domains with boundary there is the Laplacian with Dirichlet boundary conditions.

The (Quantum) Harmonic Oscillator

$H = -\frac{d^2}{dx^2} + x^2$ on \mathbb{R} , on $L^2(\mathbb{R})$ with $(f, g) = \int_{\mathbb{R}} f(x)\overline{g(x)} dx$.

H acts on the Schwarz space $\mathcal{S}(\mathbb{R}) := \left\{ f \in C^\infty(\mathbb{R}), \forall k, \ell \geq 0, \sup_{x \in \mathbb{R}} \left| x^k \left(\frac{d}{dx} \right)^\ell f(x) \right| < \infty \right\}$.

- The action of $H : \mathcal{S}(\mathbb{R})$ is continuous.
- H is L^2 -symmetric: $\int_{\mathbb{R}} -f''\overline{g} + x^2 f\overline{g} dx = (Hf, g) = (f, Hg) = \int_{\mathbb{R}} -\overline{g}'' f + x^2 f\overline{g} dx$ (integrating by parts).

We seek eigenvalues $Hu = \lambda u$. If (u, λ) and (v, μ) are eigenpairs, then

$$0 = (Hu, v) - (u, Hv) = (\lambda u, v) - (u, \mu v) = (\lambda - \bar{\mu})(u, v)$$

Where if the difference is nonzero then $(u, v) = 0$.

We can write $H = L^+ L^- + I$ where $L^+ = -\frac{d}{dx} + x$ and $L^- = \frac{d}{dx} + x$ and also $[H, L^+] = 2L^+$ and $[H, L^-] = -2L^-$.

Note that H is a non-negative operators

$$(Hf, f) = \int_{\mathbb{R}} ((f')^2 + x^2 f^2) dx > 0$$

for $f \neq 0$ and $f \in \mathcal{S}(\mathbb{R})$. Thus $\text{sp}(H) \subseteq (0, \infty)$. If $Hv = \lambda v$, then $H(L^+ v) = [H, L^+]v + L^+(Hv) = (\lambda + 2)L^+ v$. Similarly $H(L^- v) = (\lambda - 2)L^- v$.

Now we want to solve $L^- \phi_0 = 0$. $\frac{d}{dx} \phi_0 + x \phi_0 = 0$ tells us that $\phi_0(x) = \frac{1}{\sqrt{\pi}} e^{-x^2/2}$ (L^2 -normalized). Therefore $H\phi_0 = \phi_0$ and the we have an eigenvalues of one. So we may construct $\phi_n = \frac{(L^+)^n \phi_0}{|| (L^+)^n \phi_0 ||}$ which gives an eigenvector of H with eigenvalues $2n + 1$. Note that $|| (L^+)^n \phi_0 || = \sqrt{2^n n!}$.

Fact: $\phi_n = p_n(x) e^{-x^2/2}$ where p_n is the Hermite polynomial of degree n .

$$\delta_{nq} = (\phi_n, \phi_q) = \int_{\mathbb{R}} p_n(x) p_q(x) e^{-x^2} dx$$

Theorem

$\{\phi_n\}_{n \geq 0}$ is dense in $L^2(\mathbb{R})$ (if $\int_{\mathbb{R}} g \phi_n dx = 0$ for all n , then $g = 0$).

Proof (Sketch)

For $g \in L^2$, $\xi \in \mathbb{R}$, $F_g(\xi) = \int_{\mathbb{R}} e^{ix\xi} g(x) \phi_0(x) dx = \widehat{g\phi_0}(\xi)$. We observe that

- F_g is real-analytic in ξ .
- $F_g^{(k)}(0) = \int_{\mathbb{R}} (-ix)^k g(x) \phi_0(x) dx = 0$ by assumption.

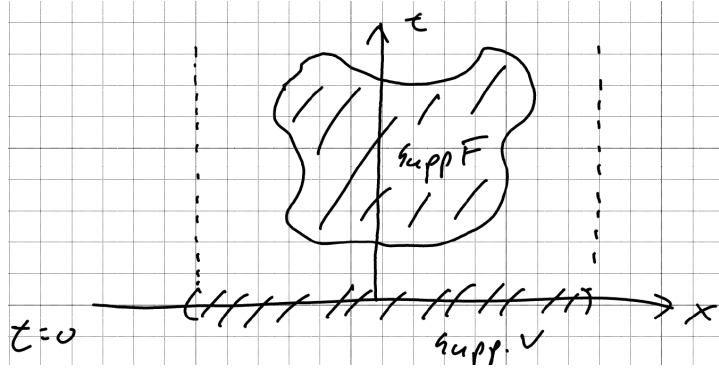
So we have a real-analytic function where all derivatives vanish at a point. So $F_g \equiv 0$, $g\phi_0 = 0$, and $g = 0$.

September 30, 2025

One of the overarching goals is to obtain large time asymptotics of the solution $v(x, t)$ ($x \in \mathbb{R}$, $t > 0$) to

$$\begin{cases} -\partial_{tt} v - P_V v = F(x, t) & \text{on } \mathbb{R}_x \times (0, \infty)_t \\ v(x, 0) = \partial_t v(x, 0) = 0, & F \in C_C^\infty(\mathbb{R}_x \times (0, \infty)_t) \end{cases}$$

where $P_V = D_x^2 + V(x) = -\left(\frac{\partial}{\partial x}\right)^2 + V(x)$ and $D_x = \frac{1}{i} \frac{\partial}{\partial x}$. The operator D_x is symmetric and self-adjoint on appropriately chosen domains. For $f(x)$ and $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx$, $\widehat{D_x f} = \xi \hat{f}(\xi)$. $V \in L_{\text{comp}}^\infty(\mathbb{R})$ (i.e. compactly supported L^∞) is the potential. If $f, g \in \mathcal{S}(\mathbb{R})$, then $(P_V f, g)_{L^2(\mathbb{R})} = (f, P_V g)_{L^2(\mathbb{R})}$.



Another way to look at this assuming v exists, we can consider $u(x, \lambda) := \int_0^\infty e^{it\lambda} v(x, t) dt$ (the Fourier-Laplace transform of v) with $\lambda \in \mathbb{C}$, $\text{Im}(\lambda) > 0$. Write $\lambda = \xi + ic$, $c > 0$, such that $u(x, \xi + ic) = \int_0^\infty e^{it\xi} e^{-ct} v(x, t) dt = \mathcal{F}_{t \rightarrow \xi}(t \mapsto e^{-ct} v(x, t))(x, -\xi)$. Then $u(x, \lambda)$ solves

$$\begin{aligned} \int_0^\infty e^{it\lambda} (-\partial_{tt} v - P_V v) dt &= \int_0^\infty e^{it\lambda} F(x, t) dt = \hat{F}(x, \lambda) \\ (\lambda^2 - P_V) \underbrace{\int_0^\infty e^{it\lambda} v(x, t) dt}_{u(x, \lambda)} &= \hat{F}(x, \lambda) \end{aligned}$$

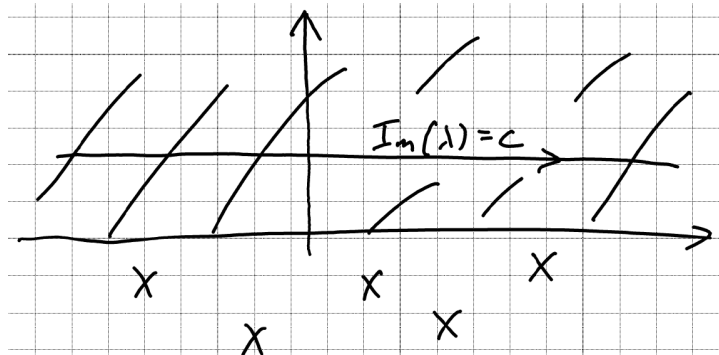
which is an entire function in λ .

To Do:

- Study solvability of $(\lambda^2 - P_V)u = \hat{F}(x, \lambda)$.
- Return to v .

For frozen c , we can get $v(x, t)$ back by Fourier inversion.

$$\begin{aligned} e^{-ct} v(x, t) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it\xi} u(x, \xi + ic) d\xi \\ v(x, t) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-it(\xi + ic)} u(x, \xi + ic) d\xi \\ v(x, t) &= \frac{1}{2\pi} \int_{\text{Im}(\lambda)=c} e^{-it\lambda} u(x, \lambda) d\lambda \end{aligned}$$

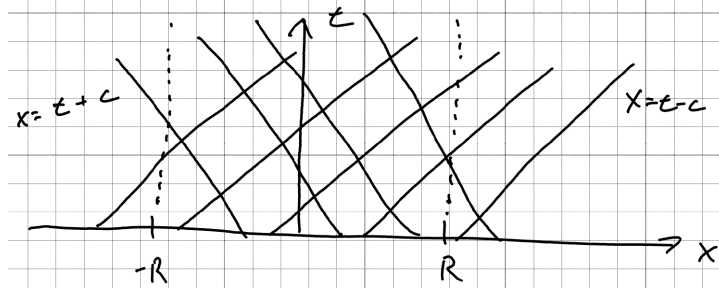


where the spectral problem is invertible.

1D Waves in the Time Domain

Suppose $R > 0$ is such that $\text{supp } V \subset [-R, R]$ and $\text{supp } F \subset [-R, R] \times (0, \infty)$. If $|x| > R$, the PDE looks like $\partial_{tt}v - \partial_{xx}v = 0 = (\partial_t + \partial_x)(\partial_t - \partial_x)v$. Setting $\xi = x + t$ and $\mu = x - t$, then it follows that

$$\partial_{\xi}\partial_{\mu}v = 0 \implies v = F(\xi) + G(\mu) = F(x+t) + G(x-t)$$



On $x > R$, we can expect $v(x, t) = F_+(x+t) + G_+(x-t)$; on $x < -R$, we expect $v(x, t) = F_-(x+t) + G_-(x-t)$. The terms G_+ and F_- are outgoing whereas the terms F_+ and G_- are incoming and, given that we assumed a source, we expect to be zero.

What does incoming/outgoing look like on the spectral side? $(\lambda^2 - P_V)u = \hat{F}(x, \lambda)$ supported in $|x| \leq R$. For $|x| > R$, $(\lambda^2 + \partial_x^2)u = 0$ leads to $u = Ae^{ix\lambda} + Be^{-ix\lambda}$. For $x > R$, $u(x) = a_+e^{i\lambda|x|} + b_+e^{-i\lambda|x|}$ for $x < -R$, $u(x) = a_-e^{i\lambda|x|} + b_-e^{-i\lambda|x|}$. u is outgoing if and only if $b_{\pm} = 0$ and incoming if and only if $a_{\pm} = 0$.

P_V is an unbounded, symmetric operator on a Hilbert space. For $z \in \mathbb{C}$, $\text{sp}(P_V)$ is the set on the complement of which $(P_V - Z)$ is boundedly invertible. That is, $\forall f, \exists! u$ such that $(P_V - z)u = f$ and $\|u\| \lesssim \|f\|$.

Waves in the Time Domain [Evans, §2.4]

Goal: if v solves

$$\begin{aligned} \partial_{tt}v - \partial_{xx}v &= f(x, t) \quad x \in \mathbb{R}, \quad t > 0, \quad f \in C_c^\infty(\mathbb{R} \times (0, \infty)) \\ v(x, 0) &= \partial_t v(x, 0) = 0 \quad x \in \mathbb{R} \end{aligned}$$

then $v(x, t) = \frac{1}{2} \int_0^t \int_{x-s}^{x+s} f(y, t-s) dy ds$. We look at

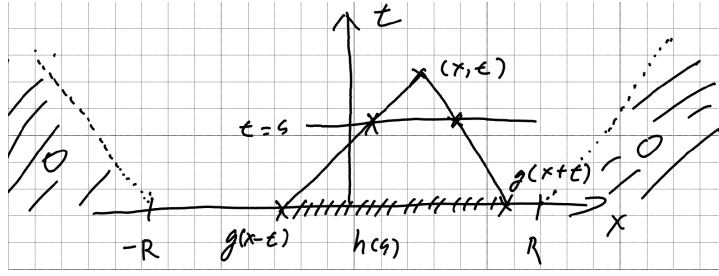
$$\begin{cases} \partial_{tt}v - \partial_{xx}v = 0 \rightsquigarrow v(x, t) = F(x+t) + G(x-t) \\ v(x, 0) = g(x), \quad \partial_t v(x, 0) = h(x) \end{cases}$$

Initial conditions gives us

$$\begin{cases} F(x) + G(x) = g(x) \\ F'(x) - G'(x) = h(x) \end{cases} \quad \begin{cases} G'(x) = \frac{1}{2}(g'(x) - h(x)) \\ F'(x) = \frac{1}{2}(g'(x) + h(x)) \end{cases}$$

So

$$\begin{aligned} F(x) &= \frac{1}{2} \left(g(x) + \int_0^x h(s) ds \right) + C_1 \\ G(x) &= \frac{1}{2} \left(g(x) - \int_0^x h(s) ds \right) + C_2 \\ v(x, t) &= \frac{1}{2} (g(x+t) + g(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} h(s) ds + C \end{aligned}$$



This has a finite speed of propagation in the sense that if we suppose $\text{supp}(g, h) \subset [-R, R]$ then $v(x, t) = 0$ whenever $x > R + t$ or $x < -R - t$.

Now we want to go from the homogeneous problem to the inhomogeneous problem. The idea is to think about $v(x, t) = \int_0^t v(x, t; s) ds$ where $v(x, t; s)$ solves the homogeneous problem

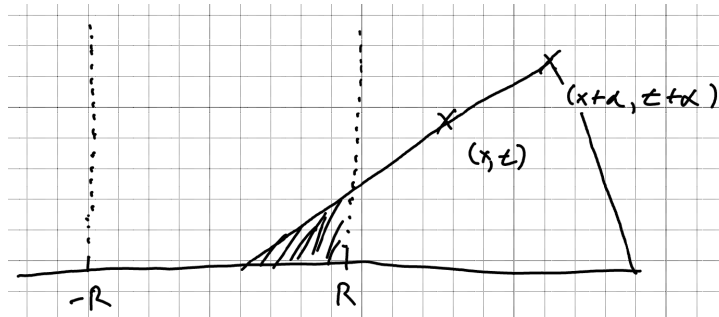
$$\begin{cases} \partial_{tt} v(\cdot, \cdot; s) - \partial_{xx} v(\cdot, \cdot; s) = 0 \\ v(\cdot, s; s) = 0, \partial_t v(\cdot, s; s) = f(x, s) \end{cases}$$

Then

$$\partial_{tt} v - \partial_{xx} v = 0 \iff \partial_t \begin{pmatrix} v \\ \partial_t v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \partial_{xx} & 0 \end{pmatrix} \begin{pmatrix} v \\ \partial_t v \end{pmatrix} \quad \text{and} \quad \begin{bmatrix} v \\ \partial_t v \end{bmatrix}_{t=s} = \begin{bmatrix} * \\ * \end{bmatrix}$$

So $v(x, t; s) = \frac{1}{2} \int_{x-(t-s)}^{x+(t-s)} f(y, s) dy$ and $v(x, t) = \frac{1}{2} \int_0^t \int_{x-s}^{x+s} f(y, t-s) dy ds$ follows.

Going back to the original PDE, $(-\partial_{tt} - P_V)v = F$ is equivalent to $(\partial_{tt} - \partial_{xx})v = -(Vv + F)$ which leads to the conclusion that $v(x, t) = -\frac{1}{2} \int_0^t \int_{x-s}^{x+s} (Vv + F)(y, t-s) dy ds$. For $|x| > R$, v is outgoing.



October 2, 2025

Take some complex vector space and consider the Hilbert space $(\mathcal{H}, (\cdot, \cdot))$ with (\cdot, \cdot) satisfying

$$\begin{cases} (\lambda f, g) = \lambda(f, g) \\ (f, \lambda g) = \overline{\lambda}(f, g) \\ (f, g) = \overline{(g, f)} \\ f \mapsto (f, g) =: ||f||^2 \text{ a norm} \\ (\mathcal{H}, ||\cdot||) \text{ complete with respect to the norm} \end{cases}$$

• Examples

$$- \left(\mathbb{C}^n, (a, b) = \sum_{j=1}^n a_j \overline{b_j} \right), a = (a_1, \dots, a_n), b = (b_1, \dots, b_n)$$

- $L^2(x, \mu)$ (e.g. $[0, 1]$ and the Lebesgue measure), $(f, g) = \int_X f \bar{g} d\mu$.

Bounded Operators: $T : \mathcal{H} \rightarrow \mathcal{H}$ bounded if and only if $\overbrace{\sup_{\|x\|=1} \|Tx\|}^{\|T\|} < \infty$ satisfying

$$\begin{cases} \mathcal{B}(\mathcal{H}) \text{ the space of bounded operators on } \mathcal{H} \text{ (a complex vector space)} \\ \|\cdot\| \text{ is a norm on } \mathcal{B}(\mathcal{H}), \text{ making it complete} \\ \text{There is a multiplication, } \mathcal{B}(\mathcal{H}) \ni A, B \mapsto AB \text{ and } \|AB\| \leq \|A\| \|B\| \end{cases}$$

Adjoint: if $A \in \mathcal{B}(\mathcal{H})$, $\exists! A^* \in \mathcal{B}(\mathcal{H})$ such that $\forall f, g \in \mathcal{H}$, $(Af, g) = (f, A^*g)$ where A is symmetric/self-adjoint if $A = A^*$. These notions are different in the world of unbounded operators.

• Example

- $\mathcal{H} = \mathbb{C}^n$: $T \in M_n(\mathbb{C})$ symmetric if and only if T is Hermitian. $t_{ij} = \overline{t_{ji}}$.
- $\mathcal{H} = L^2([0, 1])$, $Tf(t) = tf(t)$. $(Tf, g) = \int_0^1 tf(t)\overline{g(t)} dt = \int_0^1 f(t)\overline{tg(t)} dt = (f, Tg)$.
- $\mathcal{H} = L^2(\mathbb{R})$ with the Fourier transform. $\|f(x)\|^2 = c\|\hat{f}(\xi)\|^2$ (Parseval's Equality).

Finite Dimensional Spectral Theorem

If $A \in M_n(\mathbb{C})$ is Hermitian, there exists an orthonormal basis (ϕ_1, \dots, ϕ_n) of \mathbb{C}^n and real eigenvalues $\lambda_1, \dots, \lambda_n$ such that $A\phi_j = \lambda_j\phi_j$.

Important observation: if A is Hermitian, then λ_j is real for each j , and $\overline{(A\phi_j, \phi_j)} = (\phi_j, A\phi_j) = (A\phi_j, \phi_j) = \lambda_j\|\phi_j\|^2$.

So $\lambda_j = \frac{(A\phi_j, \phi_j)}{\|\phi_j\|^2}$ is real. If $\lambda_j \neq \lambda_k$, then $(\phi_j, \phi_k) = 0$ since $(A\phi_j, \phi_k) - (\phi_j, A\phi_k) = (\lambda_j - \overline{\lambda_k})(\phi_j, \phi_k)$.

Notation: Let $u, v \in \mathbb{C}^n$, denote $u \otimes \bar{v}$ the operator $(u \otimes \bar{v})w = (w, v)u$.

With A as in the theorem, we can write $A = \sum_{j=1}^n \lambda_j \phi_j \otimes \bar{\phi}_j$ ($I = \sum_{j=1}^n \phi_j \otimes \bar{\phi}_j$). A second way of writing this is

$$A \begin{bmatrix} | & \dots & | \\ \phi_1 & & \phi_n \\ | & \dots & | \end{bmatrix} = \begin{bmatrix} | & \dots & | \\ \phi_1 & & \phi_n \\ | & \dots & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$\underbrace{\hspace{10em}}_U \qquad \underbrace{\hspace{10em}}_U \qquad \underbrace{\hspace{10em}}_\Lambda$

Where $U^* = U^{-1}$ and $A = U\Lambda U^*$. This allows us to construct a functional calculus for A where

$$\begin{cases} A^2 = U\Lambda U^* U\Lambda U^* = U\Lambda^2 U^* \\ A^n = U\Lambda^n U^* \\ p(A) = U \cdot p(\Lambda) \cdot U^*, \quad p \text{ a polynomial} \end{cases}$$

Defining $f(A) := U \cdot f(\Lambda) \cdot U^*$, we obtain a Banach algebra homomorphism. Then $f \in C([- \|A\|, \|A\|])$ is also a Banach algebra with sup norm and pointwise multiplication.

$$f(A) := U \cdot \begin{pmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n) \end{pmatrix} \cdot U^*$$

Then we can map $C([-||A||, ||A||]) \ni f \mapsto f(A) \in \mathcal{B}(\mathcal{H})$. This is useful for solving ODEs.

- Prototypes

- Heat equation: $\partial_t u + Au = 0$, $u|_{t=0} = u_0$, $u(t) = e^{-tA}u_0$.
- Schrödinger equation: $i\partial_t u + Au = 0$, $u|_{t=0} = u_0$, $u(t) = e^{-itA}u_0$.
- Wave equation: $\partial_{tt} u + Au = 0$, $u|_{t=0} = u_0$, $\partial_t u|_{t=0} = u_1$.

Write $u(t) := \sum_{j=1}^n u_j(t)\phi_j$ with the PDE $\sum_{j=1}^n (u_j'' + \lambda_j u_j)\phi_j = 0$. Then $u_j'' + \lambda_j u_j = 0$, $u_j(0) = u_{j,0}$, and $u_j'(0) = u_{j,1}$. Suppose $\lambda_j > 0$ for all j . Then $u_j(t) = u_{j,0} \cos(\sqrt{\lambda_j}t) + \frac{u_{j,1}}{\sqrt{\lambda_j}} \sin(\sqrt{\lambda_j}t)$. So

$$u(t) = \sum_{j=1}^n \cos(\sqrt{\lambda_j}t) u_{j,0} \phi_j + \frac{1}{\sqrt{\lambda_j}} \sin(\sqrt{\lambda_j}t) u_{j,1} \phi_j$$

Therefore $u = \cos(t\sqrt{A})u_0 + A^{-1/2} \sin(t\sqrt{A})u_1$.

Spectrum of a Bounded Operator

Take $T \in \mathcal{B}(\mathcal{H})$. We say that T is invertible (within $\mathcal{B}(\mathcal{H})$) if and only if $\exists S \in \mathcal{B}(\mathcal{H})$ such that $TS = ST = I$.

Counterexample: take $\mathcal{H} = \ell^2(\mathbb{N}_0) = \{u = (u_n)_{n \geq 0} : \sum |u_n|^2 < \infty\}$ and $Au = \left(\frac{1}{n}u_n\right)_{n \geq 0}$. Then the proxy for $A^{-1}u = (nu_n)_{n \geq 0}$ is not bounded.

Given $T \in \mathcal{B}(\mathcal{H})$, the resolvent set of T is $\rho(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \text{ is invertible}\}$. Invertibility is equivalent to $\forall y \in \mathcal{H}$, $\exists! x$ such that $Tx - \lambda x = y$ with an estimate $||x|| \leq ||y||$.

For $\lambda \in \rho(T)$, denote $R(\lambda)$ or $R_T(\lambda) = (T - \lambda I)^{-1} \in \mathcal{B}(\mathcal{H})$ the resolvent of T . Properties of the resolvent set:

1. $\rho(T) \neq \emptyset$ (in fact, if $|\lambda| > ||T||$ then $\lambda \in \rho(T)$).
2. $\rho(T)$ is open.
3. the map $\rho(T) \ni \lambda \mapsto R_T(\lambda) \in \mathcal{B}(\mathcal{H})$ is holomorphic in the sense that $\forall \lambda_0 \in \rho(T)$, $\exists R_T'(\lambda_0) \in \mathcal{B}(\mathcal{H})$ such that $\lim_{\lambda \rightarrow \lambda_0} \left\| \frac{R_T(\lambda) - R_T(\lambda_0)}{\lambda - \lambda_0} - R_T'(\lambda_0) \right\| = 0$.

For a., if $|\lambda| > ||T||$, $Tx - \lambda x = y \iff \left(I - \frac{T}{\lambda}\right)x = -\frac{y}{\lambda} \iff x = \sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}} y$. Then $R_T(\lambda) = \frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{T}{\lambda}\right)^k$ and

$$||R_T(\lambda)|| \leq \frac{1}{||\lambda||} \frac{1}{1 - ||T/\lambda||} \leq \frac{1}{||\lambda| - ||T||}$$

For b., pick $\lambda_0 \in \rho(T)$ and find $r > 0$ such that $|\lambda - \lambda_0| < r \implies \lambda \in \rho(T)$. Then $Tx - \lambda x = y \iff (T - \lambda_0)x - (\lambda - \lambda_0)x = y \iff x - (\lambda - \lambda_0)R_T(\lambda_0)x = R_T(\lambda_0)y$ where if $||(\lambda - \lambda_0)R_T(\lambda_0)|| < 1$ it is boundedly solvable by Neumann series.

For c.,

$$\begin{aligned} R_T(\lambda) - R_T(\lambda_0) &= (T - \lambda I)^{-1} - (T - \lambda_0 I)^{-1} \\ (T - \lambda I)(R_T(\lambda) - R_T(\lambda_0)) &= I - (T - \lambda_0 I + (\lambda_0 - \lambda)I)(T - \lambda_0 I)^{-1} \\ (T - \lambda I)(R_T(\lambda) - R_T(\lambda_0)) &= I - I + (\lambda - \lambda_0)R_T(\lambda_0) \\ R_T(\lambda) - R_T(\lambda_0) &= (\lambda - \lambda_0)R_T(\lambda)R_T(\lambda_0) \end{aligned}$$

So $\frac{R_T(\lambda) - R_T(\lambda_0)}{\lambda - \lambda_0} - R_T(\lambda_0) = o(\lambda - \lambda_0)$.

Then we define the spectrum $\sigma(T) := \mathbb{C} \setminus \rho(T)$ which is closed since $\rho(T)$ is open.

Lemma

If $T \in \mathcal{B}(\mathcal{H})$ is self-adjoint, then $\sigma(T) \subseteq [-||T||, ||T||]$.

• Proof

First we know $\sigma(T) \subseteq \{|\lambda| \leq ||T||\}$. We want to show that it is real, and that if $\lambda = a + ib$ and $b \neq 0$ then $T - (a + ib)I$ is invertible.

$T - (a + bi)I$ is injective.

$$\begin{aligned} ||(T - (a + ib))x||^2 &= (Tx - (a + ib)x, Tx - (a + ib)x) \\ &= ||Tx||^2 + (a^2 + b^2)||x||^2 - (Tx, (a + ib)x) - ((a + ib)x, Tx) \\ &= ||Tx||^2 + (a^2 + b^2)||x||^2 - (a - ib)(Tx, x) - (a + ib)(x, Tx) \\ &= ||Tx||^2 + a^2||x||^2 - 2a(x, Tx) + b^2||x||^2 \geq b^2||x||^2 \end{aligned}$$

since $||Tx||^2 + a^2||x||^2 - 2a(x, Tx) \geq 0$ by Cauchy-Schwarz. Therefore $T - (a + ib)$ is injective and, by the open mapping theorem, $(T - (a + ib))^* = T - (a - ib)$ is surjective. Similarly for $T - (a - ib)$, and the norm estimate is $||(T - (a + ib))^{-1}|| \leq \frac{1}{b}$. Note that $\frac{1}{b} = \frac{1}{\text{dist}(a + ib, \mathbb{R})}$.

Note that the spectrum of T may no longer be made of eigenvalues in the non-finite case. There may exist λ such that $T - \lambda I$ is not injective, $\exists v \neq 0$ $Tv = \lambda v$. Recall the example $Tf(t) = tf(t)$ with $f \in L^2((\cdot, \cdot), dt)$. T is self-adjoint, $||T|| \leq 1$, and $(Tf, f) = \int_0^1 t|f(t)|^2 dt \geq 0$. So $\sigma(T) \subseteq [0, 1]$. For $\lambda \in [0, 1]$ is $T - \lambda I$ injective? $Tf = \lambda f \iff tf(t) = \lambda f(t) \iff (t - \lambda)f(t) = 0$ which implies $f \equiv 0$ in $L^2([0, 1])$. Is $T - \lambda I$ surjective? $(t - \lambda)f(t) = g(t) \iff f(t) = \frac{g(t)}{t - \lambda}$, so $g(t) \equiv 1 \in L^2([0, 1])$ which implies $f(t) = \frac{1}{t - \lambda}$ is not $L^2([0, 1])$ and $\sigma(T) = [0, 1]$.

October 9, 2025

Spectral Resolution

Take \mathcal{H} a Hilbert space, and say that $P \in \mathcal{B}(\mathcal{H})$ is an orthogonal projection if $P^2 = P$ and $P^* = P$. Then let $\mathcal{P}(\mathcal{H}) = \{P \in \mathcal{B}(\mathcal{H}), P \text{ is an orthogonal projection}\}$.

• Examples

- $\phi \in \mathcal{H}$, $||\phi|| = 1$, $P := \phi \otimes \bar{\phi}$. Then $P\psi = (\psi, \phi)\phi$ and $P^2\psi = P(\psi, \phi)\phi = (\psi, \phi)(\phi, \phi)\phi = P\psi$.
- ϕ_1, \dots, ϕ_n an orthonormal family with $P = \sum_{k=1}^n \phi_k \otimes \bar{\phi}_k$.
- $\mathcal{H} = \ell^2(\mathbb{N})$, $e_j = (0, 0, \dots, 0, 1, 0, \dots)$ and $I = \sum_{j=1}^{\infty} e_j \otimes \bar{e}_j$.
- $\mathcal{H} = L^2(\mathbb{R})$. Fix I an interval with χ_I the characteristic function for I . Then take $Pf := \chi_I f$.

$$* PPf = \chi_I \chi_I f = \chi_I f = Pf.$$

$$* \int_{\mathbb{R}} \chi_I f \bar{g} dx = \int_{\mathbb{R}} f \overline{\chi_I g} dx.$$

$$* \text{ If } I \text{ has a nonempty interior, then } \text{Range}(P) = \{f \in L^2(\mathbb{R}), \text{supp } f \subset I\} \simeq L^2(I).$$

Definition: Spectral Resolution

A spectral resolution is a map $\mathbb{R} \ni \lambda \mapsto E(\lambda) \in \mathcal{P}(\mathcal{H})$ satisfying

1. $\forall f \in \mathcal{H}$, $\|E(\lambda)f\|$ is increasing.
2. $\exists [a, b]$ such that $E(\lambda) = 0$ if $\lambda < a$ and $E(\lambda) = \text{Id}$ if $\lambda \geq b$.
3. $E(\lambda)$ is right continuous. That is, $\forall f \in \mathcal{H}$, $\lambda \in \mathbb{R}$,

$$\lim_{\substack{\mu \rightarrow \lambda \\ \mu > \lambda}} \|E(\mu)f - E(\lambda)f\| = 0$$

Alternatively, we can require $E(\lambda)E(\mu) = E(\min\{\mu, \lambda\})$.

Long story short: the collection of self-adjoint bounded operators is in one-to-one correspondence with the collection of spectral resolutions.

• Examples

– $A \in M_n(\mathbb{C})$, $A^* = A$, with eigencouples $(\lambda_1, \phi_1), \dots, (\lambda_n, \phi_n)$ and simple spectrum $\lambda_1 < \lambda_2 < \dots < \lambda_n$. Define $E(\lambda) := \sum_{j: \lambda_j \leq \lambda} \phi_j \otimes \bar{\phi}_j$.

* $A = I$ gives $E(\lambda) = 0$ for $\lambda < 1$ and $E(\lambda) = \text{Id}$ for $\lambda \geq 1$.

* $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ gives $E(\lambda) = \text{Id}_{\lambda \geq 1} e_1 \otimes e_1 + \text{Id}_{\lambda \geq 2} e_2 \otimes e_2$.

• If $f = f_1 e_1 + f_2 e_2$, then $\|E(\lambda)f\|^2 = \text{Id}_{\lambda \geq 1}(\lambda) \|f_1\|^2 + \text{Id}_{\lambda \geq 2}(\lambda) \|f_2\|^2$.

Spectral Measures

A spectral resolution gives rise to spectral measures

$$f, g \in \mathcal{H} \quad \lambda \mapsto (E(\lambda)f, g) = F(\lambda) \in \mathbb{C}$$

This defines a Lebesgue-Stieljes measure

$$\mu_F : \mu_F((a, b]) = F(b) - F(a), \quad \forall a, b \in \mathbb{R}$$

We can construct this as follows:

- When $f = g$, $\lambda \mapsto (E(\lambda)f, f) = (E(\lambda)^2 f, f) = \|E(\lambda)f\|^2$ (increasing).
- When $f \neq g$,

$$(E(\lambda)f, g) = (E(\lambda)(f), E(\lambda)g) = \frac{1}{4} \left(\|E(\lambda)(f+g)\|^2 - \|E(\lambda)(f-g)\|^2 - i\|E(\lambda)(f+ig)\|^2 + i\|E(\lambda)(f-ig)\|^2 \right)$$

$\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ defines a projection-valued measure E_Ω , $\Omega \subset \mathbb{R}$ a Borel set. Start with $E_{(a,b]} = E(b) - E(a)$. We would like for E_Ω to satisfy $E_{\Omega_1} E_{\Omega_2} = E_{\Omega_1 \cap \Omega_2}$, $E_\emptyset = 0$, $E_\mathbb{R} = \text{Id}$.

Theorem: Spectral Theorem

For $A \in \mathcal{B}(\mathcal{H})$ self-adjoint, there exist $a, b \in \mathbb{R}$ and an A -dependent spectral resolution $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ such that

$$A = \int_{a_-}^b \lambda dE(\lambda)$$

in the sense that $(Af, g) = \int_{a_-}^b \lambda d(E(\lambda)f, g)$ for all $f, g \in \mathcal{H}$. This is amenable to creating a functional calculus

$C([-||A||, ||A||]) \rightarrow$ bounded self-adjoint operators that commute with A

$$h \mapsto h(A) := \int_{a_-}^b h(\lambda) dE(\lambda)$$

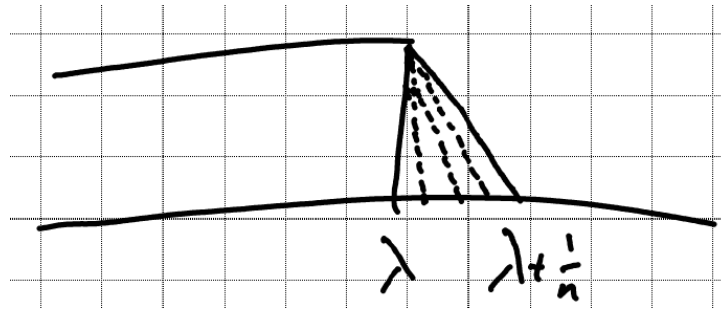
The idea is that $E(\lambda) = \chi_{(-\infty, \lambda]}(A)$. Once E is constructed, this leads to E_Ω for Ω Borel. We say that a measure μ is supported in G (Borel) if for every Ω Borel, $\mu(\Omega) = \mu(\Omega \cap G)$. Then $\text{supp } E \subset \sigma(A)$.

Functional Calculus

We want to make sense of $h(A)$ for h in a large enough class. If p is a polynomial, we can make sense of $p(A) = \sum_{k=0}^n a_k A^k$ which is self-adjoint and bounded.

For $h \in C([-||A||, ||A||])$, h is uniformly approximated by polynomials. We want to show that p_n is uniformly Cauchy which implies that $p_n(A)$ converges to some $h(A)$.

For $h = \chi_{(-\infty, \lambda]}$, we proceed by approximation by tent functions.



Definition: Positive Operator

If S is a self-adjoint, bounded operator on \mathcal{H} , we say that S is positive ($S \geq 0$) if $(Sf, f) \geq 0$, $\forall f \in \mathcal{H}$. For S_1, S_2 self-adjoint and bounded, we say that $S_1 \geq S_2$ if and only if $S_1 - S_2 \geq 0$.

For $T \in \mathcal{B}(\mathcal{H})$, self-adjoint, set $a := \inf_{||f||=1} (Tf, f)$ and $b := \sup_{||f||=1} (Tf, f)$. Then $a\text{Id} \leq T \leq b\text{Id}$.

$$((T - a\text{Id})f, f) = (Tf, f) - a(f, f) = (f, f) \left(\left(T \frac{f}{||f||}, \frac{f}{||f||} \right) - a \right) \geq 0$$

We want to show that if p is a polynomial on $[-||A||, ||A||]$, then $(\inf_{[-||A||, ||A||]} p)\text{Id} \leq p(A) \leq (\sup_{[-||A||, ||A||]} p)\text{Id}$.

Lemma

If T_1 and T_2 are positive and commute, then $T_1 T_2 \geq 0$.

Square Root Lemma

If $A \geq 0$ (i.e. bounded, self-adjoint, and positive), then $\exists! B \geq 0$ such that $B^2 = A$ and B commutes with any operator that commutes with A .

- **Proof**

Use the power series of $z \mapsto \sqrt{1-z}$ at $z = 0$.

$$1 + \sum_{k=1}^{\infty} c_k z^k$$

We can find that $c_k < 0$ for all $k \geq 1$ and that the series converges uniformly on $\{|z| \leq 1\}$.

Now let $A \geq 0$ which implies that $0 \text{Id} \leq I - A \leq 1 \text{Id}$. Without loss of generality, suppose $\text{supp } \|A\| \leq 1$. The idea is to write

$$B = \sqrt{A} = \sqrt{I - (I - A)} = I + \sum_{k=1}^{\infty} c_k (I - A)^k$$

which converges strongly because the series converges uniformly. Then $B^2 = A$. We see that $B \geq 0$ using the fact that $\text{sign}(c_k) < 0$ which implies $\sum_{k \geq 1} c_k \geq -1$. The proof of uniqueness can be found in the text.

Proof of Lemma

Assuming the square root lemma, write $T_2 = B^2$. Then since $[T_1, T_2] = 0$, $[B, T_1] = 0$. Then

$$(T_1 T_2 f, f) = (T_1 B^2 f, f) = (B T_1 B f, f) = (T_1 (B f), B f) \geq 0$$

Weaker Version

Instead of $(\inf_{[-\|A\|, \|A\|]} p) \text{Id} \leq p(A) \leq (\sup_{[-\|A\|, \|A\|]} p) \text{Id}$, we have that if $\min_{[-\|A\|, \|A\|]} p \geq 0$, then $p(A) \geq 0$.

Proof

If $p \geq 0$ on $[-\|A\|, \|A\|]$, we can factor it as a product of positive pieces

$$p(x) = \prod_{\substack{r_j < -\|A\| \\ s_j \geq \|A\|}} \underbrace{(x - r_j)}_{\geq 0} \underbrace{(s_j - x)}_{\geq 0} ((x - a_j)^2 + b_j^2)$$

$$p(A) = \prod_{\substack{r_j < -\|A\| \\ s_j \geq \|A\|}} \underbrace{(A - r_j)}_{\geq 0} \underbrace{(s_j - A)}_{\geq 0} \underbrace{((A - a_j)^2 + b_j^2)}_{\geq 0}$$

Using the previous lemma, we have that $P(A) \geq 0$.

Proof

Finally, to show that $(\inf_{[-\|A\|, \|A\|]} p) \text{Id} \leq p(A) \leq (\sup_{[-\|A\|, \|A\|]} p) \text{Id}$, we see that $p - \inf p$ and $\text{supp } f - p$ are positive polynomials and apply the weaker version to them.

Definition

We can define $h(A)$ for $h \in C(-||A||, ||A||)$ by Weierstrass approximation. There exist p_n polynomials such that $\sup_{[-||A||, ||A||]} |p_n - h| \xrightarrow{n \rightarrow \infty} 0$. Then p_n is uniformly Cauchy, so

$$\inf(p_n - p_m) \text{Id} \leq p_n(A) - p_m(A) \leq \sup(p_n - p_m) \text{Id}$$

which implies that

$$||p_n(A) - p_{n-1}(A)|| \leq \max(\sup(p_n - p_m), -\inf(p_n - p_m)) \xrightarrow{n, m \rightarrow \infty} 0.$$

So $p_n(A)$ is Cauchy in $(\mathcal{B}(\mathcal{H}), ||\cdot||)$ which means it converges. We call $h(A) = \lim_{n \rightarrow \infty} p_n(A)$. We still want to show that $h(A)$ is bounded and self-adjoint.

October 14, 2025

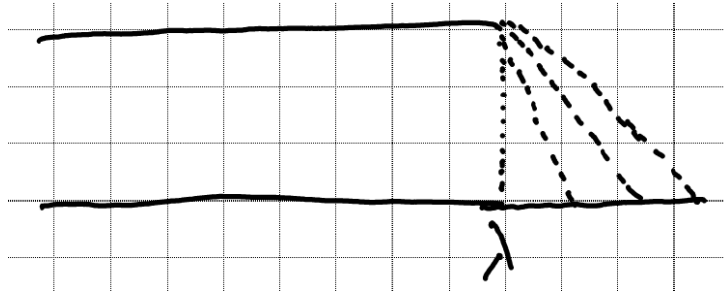
Spectral Theorem for Bounded Self-Adjoint Operators

If $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint such that $a||f||^2 \leq (Af, f) \leq b||f||^2 \quad \forall f \in \mathcal{H}$, then there exists a spectral resolution $\{E(\lambda)\}_{\lambda \in \mathbb{R}}$ such that $A = \int_{a-}^b \lambda dE(\lambda)$.

Proof (Continued)

We would like that $E(\lambda) = \phi^\lambda(A)$ where $\phi^\lambda := \chi_{(-\infty, \lambda]}$, however ϕ^λ is not continuous so instead we can approximate it as

$$\phi_n^\lambda := \begin{cases} 1, & x \leq \lambda \\ \text{linear on } [\lambda, \lambda + \frac{1}{n}] \\ 0, & x \geq \lambda + \frac{1}{n} \end{cases}$$



To demonstrate this, we need the following proposition.

Proposition

If T_n is a sequence of positive operators and $T_n \geq T_{n+1} \geq 0$, then there exists some $T \geq 0$ such that $T_n f \rightarrow T f, \quad \forall f \in \mathcal{H}$.

Proof

Fix $f \in \mathcal{H}$, and consider $(T_n f, f)$ which is decreasing, bounded from below, and therefore converges and is Cauchy. Now as an estimate, we can say that if $S \in \mathcal{B}(\mathcal{H})$ is self-adjoint where $0 \leq S \leq MI$, then $\forall f \in \mathcal{H}$ we know that

$$||Sf||^2 \leq (Sf, f)^{1/2} M^{3/2} ||f||.$$

To see this, we look at $t \mapsto (S(S + tI)f, (S + tI)f) \geq 0$ since $S \geq 0$. Then

$$(S(S + tI)f, (S + tI)f) = (S^2 f, Sf) + 2t||Sf||^2 + t^2(Sf, f)$$

So $\Delta < 0$ if and only if

$$\begin{aligned} ||Sf||^4 - (S^2 f, Sf)(Sf, f) &< 0 \\ ||Sf||^4 &\leq (Sf, f)(S^2 f, Sf) \\ &\leq (Sf, f) \underbrace{||S^2 f||}_{M^3 ||f||^2} ||Sf|| \end{aligned}$$

We apply this to $T_n - T_m = S$ where $T_n - T_m \leq T_0$ for all $n \leq m$. Then

$$||(T_n - T_m)f||^2 \leq ((T_n - T_m)f, f)^{1/2} ||f|| ||T_0||^{3/2}$$

Since $(T_n f, f)$ is Cauchy, $T_n f$ is Cauchy and therefore converges. It remains to check that T is linear, positive, satisfies, $T \leq T_0$, etc.

Spectral Theorem Proof Continued

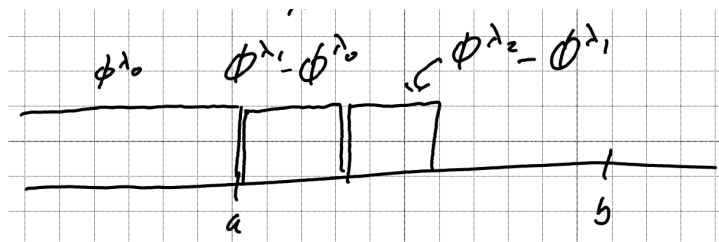
For each n , $\phi_n^\lambda(A)$ makes sense and is positive since $\phi_n^\lambda(t) \geq 0$. Since $\phi_n^\lambda \geq \phi_{n+1}^\lambda$, $\phi_n^\lambda(A) \geq \phi_{n+1}^\lambda(A)$. Thus, by the preceding proposition, $\phi^\lambda(A) := \lim_{n \rightarrow \infty} \phi_n^\lambda(A)$ exists as a bounded, self-adjoint, positive operator.

It remains to check the spectral resolution properties.

The final property to check is that $\forall f \in \mathcal{H}$, $(Af, f) = \int_a^b \lambda d(E(\lambda)f, f)$. The idea is to approximate multiplication by t with piecewise constant functions. We fix a partition $a = \lambda_0 \leq \dots \leq \lambda_k = b$ such that $\sup(\lambda_{j+1} - \lambda_j) < \delta$. Then

$$t\phi^{\lambda_k}(t) = t = \phi^{\lambda_0}(t) + \sum_{j=1}^k t(\phi^{\lambda_j}(t) - \phi^{\lambda_{j-1}}(t))$$

for all $a \leq t \leq b$.

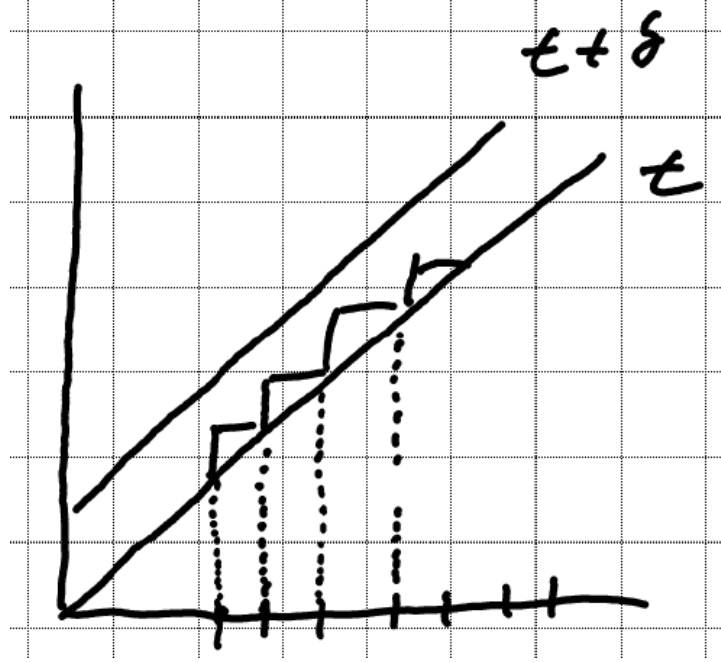


Then

$$\begin{aligned} \lambda^{j-1} &\leq t \leq \lambda^j \\ \lambda^{j-1}(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) &\leq t(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \leq \lambda^j(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \\ \sum_{j=1}^k \lambda^{j-1}(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) &\leq t \sum_{j=1}^k (\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \leq \sum_{j=1}^k \lambda^j(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \\ &\leq \sum_{j=1}^k (\lambda^j - \delta)(\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \end{aligned}$$

Adding $\lambda_0 \phi^{\lambda_0}$, we see that

$$t \leq \lambda_0 \phi^{\lambda_0} + \sum_{j=1}^k \lambda^j (\phi^{\lambda_j} - \phi^{\lambda_{j-1}}) \leq t + \delta$$



When we apply this to A ,

$$A \leq \lambda_0 E(\lambda_0) + \sum_{j=1}^k \lambda^j (E(\lambda_j) - E(\lambda_{j-1})) \leq A + \delta I$$

Finally, we see that

As one refines the partition, $\delta \rightarrow 0$ and $(A, f, f) = \int_{a^-}^b \lambda d(E(\lambda)f, f)$.

Then if $\phi \in C([-||A||, ||A||])$, $\phi(A) = \int_{a^-}^b \phi(\lambda) dE(\lambda)$.

$$\left| (Af, f) - \lambda_0 (E(\lambda_0)f, f) - \sum_{j=1}^k \lambda^j (E(\lambda_j)f, f) - (E(\lambda_{j-1}f, f)) \right| \leq \delta ||I||^2$$

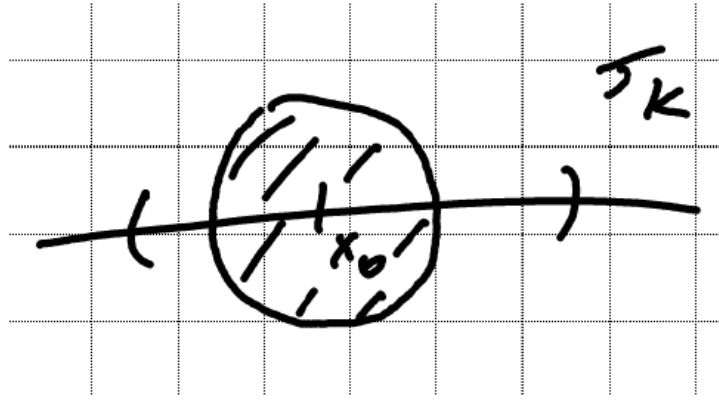
Functional Calculus

We observe that for $g \in C_C^\infty(\mathbb{C})$, $g(w) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{w-z} \partial_{\bar{z}} g(z) d^2 z$. Then $f(A) := \frac{i}{\pi} \int_{\mathbb{C}} (A-z)^{-1} \partial_{\bar{z}} \tilde{f}(z) d^2 z$ where $\partial_{\bar{z}} \tilde{f} = O((\text{Im}(z))^n)$ for $n \geq 3$ as $\text{Im}(z) \rightarrow 0$.

Proposition

For every $f \in \mathcal{H}$, the Lebesgue-Stieltjesmeasure corresponding to $F(\lambda) = (E(\lambda)f, f)$ is supported on $\sigma(A)$.

Since $\sigma(A) \subseteq [a, b]$ is closed, $[a, b] \setminus \sigma(A)$ is open (i.e. $\bigcup_{k \in \mathbb{N}} J_k$ for open intervals J_k). We want to show that $F(\lambda)$ is constant on each J_k (equivalently: $\int_{J_k} dF(\lambda) = 0$).



Fix $J_k \ni x_0$. Then $R_A(x_0) = (A - x_0 I)^{-1}$ exists, and we can pick $\varepsilon > 0$ such that $\forall z \in \overline{B_\varepsilon(x_0)}$, $\|R_A(z)\| \leq M$. Then for $z \in B_\varepsilon(x_0) + i\mathbb{R}$, $\text{Im}(z) \neq 0$,

$$R_A(z) = (A - zI)^{-1} = \phi_z(A)$$

and $\phi_z(t) = \frac{1}{t-z} \in C([a, b])$. Consider

$$R_A(z)R_A(\bar{z}) = \psi_z(A) = \int \frac{1}{|\lambda - z|^2} dE(\lambda)$$

with $\psi_z(t) = \frac{1}{|t-z|^2}$. It follows that for all $z \in \overline{B_\varepsilon(x_0)} \setminus \mathbb{R}$,

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{1}{|\lambda - z|^2} dF(\lambda) \leq \int \frac{1}{|\lambda - z|^2} dF(\lambda) = (R_A(z)R_A(\bar{z})f, f) \leq M^2 \|f\|^2$$

which stays true for all $z \in \overline{B_\varepsilon(x_0)}$. In particular for $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$,

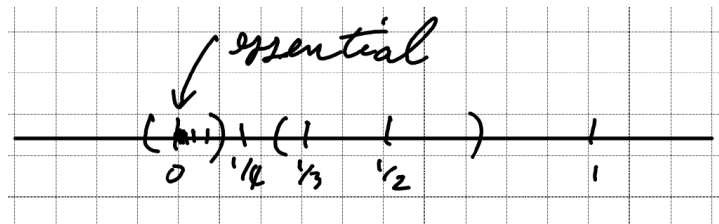
$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} dx \int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{1}{|\lambda - z|^2} dF(\lambda) \leq 2\varepsilon M^2 \|f\|^2$$

Since Fubini holds, we observe that $\int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{1}{|\lambda - x|^2} dx = \infty$, so it must be the case that $\int_{x_0-\varepsilon}^{x_0+\varepsilon} dF(\lambda) = 0$.

Discrete Spectrum vs Essential Spectrum

Recall the spectral measure of A , $E_\Omega(A)$ for $\Omega \subset \mathbb{R}$ Borel where $E_{(a,b]} = E(b) - E(a)$. We say that $\lambda \in \sigma_d(A)$ (the discrete spectrum of A) if there exists $\varepsilon > 0$ such that $\dim(\text{range}(E_{(\lambda-\varepsilon, \lambda+\varepsilon]})) < \infty$. Likewise, $\lambda \in \sigma_{\text{ess}}(A)$ (the essential spectrum) if $\forall \varepsilon > 0$, $\dim(\text{range}(E_{(\lambda-\varepsilon, \lambda+\varepsilon]})) = \infty$.

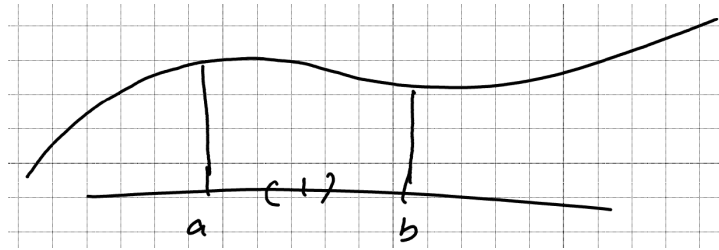
As an example, take $\mathcal{H} = \ell^2(\mathbb{N})$ with $(Au)_n = \frac{u_n}{n}$ and $A = \sum_{j=1}^{\infty} \frac{1}{j} e_j \otimes e_j$.



Discrete spectra include eigenvalues of finite multiplicity.

Essential spectra include accumulation points of eigenvalues, eigenvalues of infinite multiplicity, absolutely continuous spectrum, s.c. spectrum.

Another example if $Af(t) = tf(t)$ on $L^2([0, 1])$. Then $E(\lambda)f(t) = \chi_{(-\infty, \lambda]}f(t)$, and $E_{(a,b]}f(t) = \chi_{(a,b]}f(t)$. $\forall x_0 \in [0, 1]$, we have that $\text{range}(E_{(x_0-\varepsilon, x_0+\varepsilon]}) = L^2((x_0 - \varepsilon, x_0 + \varepsilon))$.



October 16, 2025

Compact Operators and Analytic Fredholm Theorem

Definition: Spectral Radius

For $A \in \mathcal{B}(\mathcal{H})$, we say that the spectral radius of A is $r(A) = \sup_{\lambda \in \sigma(A)} |\lambda| < \infty$

Theorem

1. if $A \in \mathcal{B}(\mathcal{H})$, then $r(A) = \limsup_{n \rightarrow \infty} \|A^n\|^{1/n}$.
2. If A is, in addition, self-adjoint, then $r(A) = \|A\|$.

As a non-example, consider $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $\sigma(A) = \{0\}$, but $r(A) = 0 \neq \|A\|$.

Proof

Recall Hadamard's Formula: $\sum_{k=0}^{\infty} a_k z^k$ has radius of convergence R computed by $\frac{1}{R} = \limsup_{n \rightarrow \infty} |a_n|^{1/n}$. This holds even when a_k are members of a Banach algebra (e.g. $\mathcal{B}(\mathcal{H})$).

Set $z = \frac{1}{\lambda}$, such that $0 < |z| < \frac{1}{r(A)}$ and implies the existence of

$$R_A\left(\frac{1}{z}\right) = \left(A - \frac{1}{z}I\right)^{-1} = -(I - zA)^{-1} = -z \sum_{k=0}^{\infty} A^k z^k.$$

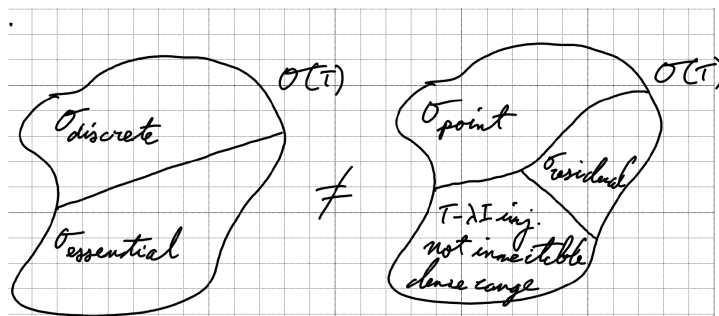
So we have that $r(A) = \frac{1}{R} = \limsup_{k \rightarrow \infty} \|A^k\|^{1/k}$.

Now if A is self-adjoint, $\|A^2\| = \|A\|^2$ since $\|A^2\| \leq \|A\|^2$ by submultiplicativity and $\|A^2\| \geq \sup_{\|x\|=1} (x, A^2 x) = \sup_{\|x\|=1} \|Ax\|^2 = \|A\|^2$ using Cauchy-Schwarz.

By induction, $\|A^{2^k}\|^{1/2^k} = \|A\|$ which implies that $r(A) = \|A\|$.

Another Spectral Decomposition

For $T : \mathcal{H} \rightarrow \mathcal{H}$ bounded, we say that λ is in the point spectrum of T if $T - \lambda I$ is not-injective. We say that λ is in the residual spectrum of T if $T - \lambda I$ is injective but does not have dense range.



Self-adjoint operators have no residual spectrum (RS, Thm VI.8).

Definition: Compact Operators

$K \in \mathcal{B}(\mathcal{H})$ is compact if K maps bounded sequences to sequences with a limit point. Equivalently, if $B_{\mathcal{H}} = \{x \in \mathcal{H} : \|x\| \leq 1\}$ then K is compact if $K(B_{\mathcal{H}})$ has compact closure (i.e. is precompact).

$\mathcal{K}(\mathcal{H})$, the collection of compact operators on \mathcal{H} , is a closed linear subspace of $\mathcal{B}(\mathcal{H})$ since if $T, S \in \mathcal{K}(\mathcal{H})$, then $T + S \in \mathcal{K}(\mathcal{H})$. We have also that $\mathcal{K}(\mathcal{H})$ is a 2-sided ideal of $\mathcal{B}(\mathcal{H})$ since when T is compact and S is bounded, ST and TS are compact.

Examples

- For a finite-rank operator $A : \mathcal{H} \rightarrow \mathcal{H}$, $\text{range}(A) < \infty$. The general form of A is $A = \sum_{j=1}^n \phi_j \otimes \bar{\psi}_j$ for $\phi_j, \psi_j \in \mathcal{H}$.
- Strong limits of finite-rank operators.
- $(Au)_n = \frac{1}{n} u_n$ on $\ell^2(\mathbb{N})$. Note that $A_n = \sum_{j=1}^n \frac{1}{j} e_j \otimes e_j$ shows that $\|A - A_n\| \leq \frac{1}{n+1}$.
- The inclusion $h^1 \hookrightarrow \ell^2$ where $h^1 = \{u \in \mathbb{C}^{\mathbb{N}} : \sum_{j=1}^{\infty} j^2 |u_j|^2 < \infty\}$.

Proposition

If \mathcal{H} is separable, all compact operators arise as limits of finite-rank operators.

Proof

We want to show that for $A \in \mathcal{K}(\mathcal{H})$, $\forall \varepsilon > 0$, $\exists A_\varepsilon$ finite-rank such that $\|A - A_\varepsilon\| < \varepsilon$.

Let $\varepsilon > 0$ be given. Since $A(B_{\mathcal{H}})$ is precompact, it is totally bounded. That is $\exists y_1, \dots, y_n \in A(B_{\mathcal{H}})$ such that $\forall x \in A(B_{\mathcal{H}})$, $\min_{1 \leq j \leq n} \|x - y_j\| < \varepsilon$.

Let P_ε be the orthogonal projection onto $\text{span}\{y_1, \dots, y_n\}$, and set $A_\varepsilon = P_\varepsilon A$. Then for $f \in B_{\mathcal{H}}$ and $x = Af \in A(B_{\mathcal{H}})$,

$$\|Af - A_\varepsilon f\| = \|x - P_\varepsilon x\| \leq \min_{1 \leq j \leq n} \|x - y_j\| < \varepsilon$$

So $\|A - A_\varepsilon\| < \varepsilon$.

Exercise: confirm whether this argument needs separability.

Theorem: Analytic Fredholm Theorem

Let $D \subset \mathbb{C}$ be open and connected. Let $f : D \rightarrow \mathcal{B}(\mathcal{H})$ be an analytic, operator-valued function such that $f(z) \in \mathcal{K}(\mathcal{H})$ for all $z \in D$. Then

1. either $(I - f(z))^{-1}$ exists for no $z \in D$
2. or $(I - f(z))^{-1}$ exists for all $z \in D \setminus S$, where S is a discrete set (finite-rank, no accumulation points) in D .

Then $(I - f(z))^{-1}$ is meromorphic in D , analytic in $D \setminus S$, residues at poles are finite-rank, and if $z \in S$, $f(z)\psi = \psi$ has a nonzero solution for $\psi \in \mathcal{H}$.

Application

If K is compact, consider $f(z) = zK$. At $z = 0$, $I - f(z) = I$ so this is invertible which implies that the theorem holds for $\frac{1}{z} \in \mathbb{C}$ (taking $D = \mathbb{C} \setminus \{0\}$). We have $R_K(\lambda) = -\frac{1}{\lambda} \left(I - f\left(\frac{1}{\lambda}\right) \right)^{-1}$. Note that K is not necessarily self-adjoint. Note that K

is not necessarily self-adjoint.

Proof

We want to prove that either (a) or (b) hold locally for any $z_0 \in D$.

Fix $z_0 \in D$, $r > 0$ such that $\|f(z) - f(z_0)\| < \frac{1}{2}$ for $z \in D_r(z_0)$. Choose $F = \sum_{j=1}^n \phi_j \otimes \bar{\psi}_j$ a finite-rank operator such that $\|f(z_0) - F\| < \frac{1}{2}$. Then $\forall z \in D_r(z_0)$,

$$\|f(z) - F\| \leq \|f(z) - f(z_0)\| + \|f(z_0) - F\| < 1$$

which implies that $(I - (f(z) - F))^{-1}$ exists and is holomorphic on $D_r(z_0)$.

Define $g(z) = F(I - f(z) + F)^{-1}$, and observe that $(I - f(z)) = (I - g(z))(I - f(z) + F) = I - f(z) + \overbrace{F - g(z)(I - f(z) + F)}^F$.
Write

$$g(z) = \sum_{j=1}^n \phi_j \otimes \bar{\psi}_j \cdot (I - f(z) + F)^{-1} = \sum_{j=1}^n \phi_j \otimes \overline{\psi_j(z)}$$

where $\psi_j(z) = ((I - f(z) + F)^{-1})^* \psi_j$ is holomorphic in z . Then $I - f(z)$ is invertible if and only if $I - g(z)$ is invertible.

We claim that this holds if and only if $d(z) \neq 0$ for some holomorphic function d .

When is $I - g(z)$ invertible?

Injectivity: if $g(z)\phi = \phi$, we expect $\phi = \sum_{j=1}^n \beta_j \phi_j$. So $g(z)\phi = \phi$ if and only if

$$\sum_{j=1}^n \phi_j \left(\sum_{k=1}^n \beta_k \phi_k, \psi_j \right) = \sum_{j=1}^n \beta_j \phi_j$$

where $\beta_j = \sum_{k=1}^n \beta_k (\phi_k, \psi_j(z))$. If $A_{jk}(z) := (\phi_k, \psi_j(z))$, then this has a solution if and only if $\det(I - A(z)) = 0$. Call $d := \det(I - A(z))$. Moreover, if $d(z) \neq 0$ then $I - g(z)$ is invertible. We can solve $(I - g(z))\phi = \psi$ for ϕ given $\psi \in \mathcal{H}$. So $\phi = \psi + g(z)\phi$ which motivates an ansatz $\phi = \psi + \sum_{j=1}^n \beta_j \phi_j$. Then

$$(I - g(z))\phi = (I - g(z))\psi + \sum_{j=1}^n (\beta_j - A_{jk}\beta_k)\phi_j = \psi$$

If and only if

$$\sum_{j=1}^n (\beta_j - A_{jk}\beta_k)\phi_j = \sum_{j=1}^n (\psi, \psi_j(z))\phi_j$$

which is boundedly invertible as long as $d(z) \neq 0$.

October 21, 2025

Definition: Unbounded Operator

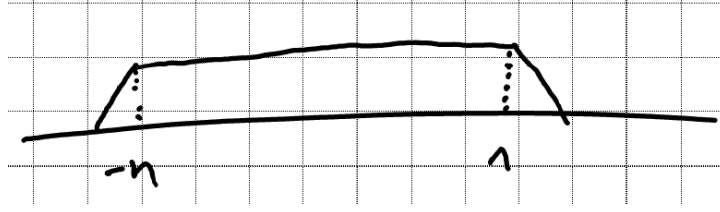
Let $(\mathcal{H}, (\cdot, \cdot))$ a Hilbert space. An unbounded operator is a linear map $A : \mathcal{D}(A) \rightarrow \mathcal{H}$, $\mathcal{D}(A) \subsetneq \mathcal{H}$ ($\mathcal{D}(A)$ the domain of A).

Hypothesis: $\overline{\mathcal{D}(A)} = \mathcal{H}$.

Why "unbounded?" BEcause, generally, $\sup_{f \in \mathcal{D}(A) \setminus \{0\}} \frac{\|Af\|}{\|f\|} = \infty$.

Examples

1. $\mathcal{H} = L^2(\mathbb{R})$, $Af(t) = tf(t)$. Then $A\left(\frac{1}{|t|}\right) \notin L^2(\mathbb{R})$, and $\mathcal{D}(A) = C_C(\mathbb{R})$. Consider $f_n(t)$ like



1. $\mathcal{H} = L^2(\mathbb{R})$, $Af(x) = D_x f(x) = \frac{1}{i} \frac{\partial}{\partial x} f(x)$. Then $\mathcal{D}(A) = \mathcal{S}(\mathbb{R})$, and we can see unboundedness from $f(x) = e^{-\frac{x^2}{\sigma}}$ for $\sigma > 0$.

$$\begin{aligned} \int f^2(x) dx &= \int e^{-\frac{x^2}{\sigma}} dx \stackrel{y=\frac{x}{\sqrt{\sigma}}}{=} \sqrt{\sigma} \int e^{-2y^2} dy \\ \int (f')^2(x) dx &= \int \frac{4x^2}{\sigma^2} e^{-\frac{x^2}{\sigma}} dx \stackrel{y=\frac{x}{\sqrt{\sigma}}}{=} \frac{1}{\sqrt{\sigma}} \int 4y^2 e^{-2y^2} dy \end{aligned}$$

So $\frac{\|Af\|^2}{\|f\|^2} = \frac{C}{\sigma} \xrightarrow{\sigma \rightarrow 0} \infty$.

Definition: Extension of an Unbounded Operator

We say that B extends A (and write $A \subseteq B$) if $\mathcal{D}(A) \subset \mathcal{D}(B)$ and $B|_{\mathcal{D}(A)} = A$. We have that $A = B$ when $A \subset B$ and $B \subseteq A$.

Definition: Closed Operator

To recover a notion of continuity, we look at closed operators.

We say that $(A, \mathcal{D}(A))$ is closed if and only if $\forall u_n \in \mathcal{D}(A)$ such that $u_n \rightarrow u \in \mathcal{H}$ and $Au_n \rightarrow v \in \mathcal{H}$, $u \in \mathcal{D}(A)$ and $v = Au$.

Equivalently, A is closed if the graph of A , $\Gamma(A) = \{(u, Au) : u \in \mathcal{D}(A)\}$, is closed in $\mathcal{H} \times \mathcal{H}$.

We say that A is closable if there exists some B such that $A \subseteq B$.

Example

- $(P, \mathcal{D}(P), \mathcal{H}) = (D_x, C_C^\infty(\mathbb{R}), L^2(\mathbb{R}))$.
 - P is not closed, because there exists $u_n \in C_C^\infty(\mathbb{R})$ such that $u_n \rightarrow u \in \mathcal{H}$, $D_x u_n \rightarrow v \in \mathcal{H}$, yet $u \notin C_C^\infty(\mathbb{R})$.
 - Take $u \in H^1 \setminus C_C^\infty(\mathbb{R})$, $u_n \in C_C^\infty(\mathbb{R})$ converging to u in H^1 .
- Recall that
 - $H^1(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : f'(x) \in L^2(\mathbb{R})\}$, $\|f\|_{H^1}^2 = \|f\|_{L^2(\mathbb{R})}^2 + \|f'\|_{L^2(\mathbb{R})}^2$.
 - $C_C^\infty(\mathbb{R})$ is dense in $H^1(\mathbb{R})$.
 - $H^1(\mathbb{R})$ is complete.

Exercise: prove that if $\mathcal{D}(B) := H^1(\mathbb{R})$ and $Bf = D_x f$, then B is closed and $B|_{C_C^\infty(\mathbb{R})} = P$.

Adjoint of Unbounded Operators

In the bounded case, we have $A \in \mathcal{B}(\mathcal{H})$, $u \in \mathcal{H}$ and define $\ell_u(v) = (u, Av) \in \mathcal{H}'$ (\mathcal{H}' the dual space). By Riesz Representation Theorem, $\exists! w := A^*u$ such that $\forall v \in \mathcal{H}$, $(A^*u, v) = (u, Av)$.

In the unbounded case, we need to define A^* and $\mathcal{D}(A^*)$.

For $u \in \mathcal{H}$, set $\ell_u = (u, Av)$, $v \in \mathcal{D}(A)$. If ℓ_u extends to an element of \mathcal{H}' (i.e. if we can prove an estimate $|\ell_u(v)| \leq c\|v\|_{\mathcal{H}}$), then by Riesz Representation Theorem $\exists! w \in \mathcal{H}$, called A^*u , such that $\forall v \in \mathcal{D}(A)$, $(A^*u, v) = (u, Av)$. Then

$$\mathcal{D}(A^*) = \{u \in \mathcal{H} : \frac{\mathcal{D}(A) \rightarrow \mathbb{C}}{v \mapsto (u, Av)} \text{ extends to an element of } \mathcal{H}'\}.$$

Example

$$\bullet (P, \mathcal{D}(P), \mathcal{H}) = (D_x, C_C^\infty(\mathbb{R}), L^2(\mathbb{R})).$$

– Observe that $\forall f, g \in C_C^\infty(\mathbb{R})$, $(D_x f, g) = (f, D_x g)$.

– Take $f \in \mathcal{H}$ and $g \in C_C^\infty(\mathbb{R})$. Then if $f \in C_C^\infty$ or $f \in H^1(\mathbb{R})$, $g \mapsto (f, D_x g) = (D_x f, g)$ and

$$|(D_x f, g)| \leq \|D_x f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})} \leq \|f\|_{H^1(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}$$

– Then $B(f, g) = D_x f, g - (f, D_x g)$.

– So $\mathcal{D}(P^*) \supseteq C_C^\infty(\mathbb{R})$ and, in fact, $\mathcal{D}(P^*) = H^1(\mathbb{R})$.

– P^* is closed (this is always true of any adjoint).

– $P \subset P^*$, we say that P is symmetric.

Generally, if $(A, \mathcal{D}(A))$ is an unbounded operator, then $\Gamma(A^*) = J((\Gamma(A))^\perp)$ where $J : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H} \times \mathcal{H}$ by $J(u, v) = (-v, u)$. Recall that $(\Gamma(A))^\perp$ is always closed, so A^* is always closed.

Definitions:

We say that P is symmetric if $P \subseteq P^*$.

We say that P is self-adjoint if $P = P^*$.

We say that P is essentially self-adjoint if $P^* = \overline{P}$.

Important Questions

1. Given a symmetric operator $(P \subseteq P^*)$, to find a self-adjoint operator A such that $P \subseteq A = A^* \subseteq P^*$. We call A self-adjoint extension of P . If $P^* = \overline{P}$ (essentially self-adjoint), then it must be the unique extension.
2. How to find or parameterize all self-adjoint extensions? (This problem is hard.)

Example

$$\bullet P = (D_x, H_0^1([0, 1]), L^2(0, 1)) \text{ (where elements of } H_0^1([0, 1]) \text{ vanish on the boundary).}$$

$$- P^* = (D_x, H^1([0, 1])) \supsetneq P.$$

- There exists a “circle” of self-adjoint extensions, parameterized by any point on the unit circle.

General Principle

If $\Omega \subset \mathbb{R}^n$ is open, bounded and with smooth boundary $\partial\Omega$ ($\mathcal{H} = L^2(\Omega)$), and P is some differential operator (e.g. $P = \Delta$) such that $\forall f, g \in C_c^\infty(\Omega)$. $(Pf, g) = (f, Pg)$. By Green's Identity,,

$$\begin{aligned} \int_{\Omega} (\Delta f g - f \Delta g) dx &= \int_{\Omega} \nabla((\nabla f)g - f(\nabla g)) dx \\ &= \int_{\partial\Omega} (g \nabla f - f \nabla g) \cdot \nu ds \\ &= \int_{\partial\Omega} \left(g \frac{\partial f}{\partial \nu} - f \frac{\partial g}{\partial \nu} \right) ds \end{aligned}$$

Examples

- $P = (P = \Delta, C_c^\infty(\Omega), L^2(\Omega))$.

- P symmetric.

- $\mathcal{D}(P^*) = \{u \in L^2(\Omega) : \Delta u \in L^2\}$.

- Let $P_{\min} = \overline{(P, C_c^\infty(\Omega))}$ and define $P_{\max} = P_{\min}^*$. Then $\mathcal{D}(P_{\max}) = \{u \in L^2(\Omega) : Pu \in L^2(\Omega)\}$.

- The self-adjoint extensions lie inbetween.

- $(D_x, H_0^1([0, 1])) = \{u \in H^1([0, 1]) : u(0) = u(1) = 0\}, L^2([0, 1])$.

- Observe that the evaluation map $C^\infty([0, 1]) \ni f \mapsto f(0)$ extends boundedly to $\tau : H^1([0, 1]) \rightarrow \mathbb{C}$.

- For $f, g \in C^\infty([0, 1])$,

$$(D_x f, g) - (f, D_x g) = \int_0^1 \frac{1}{i} f' \bar{g} - f \left(\frac{1}{i} \bar{g}' \right) = \frac{1}{i} \int_0^1 (f \bar{g})' = \frac{1}{i} (f(1) \bar{g}(1) - f(0) \bar{g}(0))$$

- Then if $g \in \mathcal{D}(P_{\min})$,

$$\begin{aligned} |(f, D_x g)| &= |(D_x f, g) + \frac{1}{i} (\overbrace{f(1) \bar{g}(1)}^{=0} - \overbrace{f(0) \bar{g}(0)}^{=0})| \\ &\leq \|f\|_{H^1} \|g\|_{L^2} \end{aligned}$$

- Therefore $\mathcal{D}(P_{\min}^*) = \mathcal{D}(P_{\max}) = H^1([0, 1])$, and we note that $P_{\max}^* = P_{\min}$.

Now suppose that T is a self-adjoint extension of P ($P_{\min} \subseteq T \subseteq P_{\max}$). If T is self-adjoint, then $\forall f, g \in \mathcal{D}(T) \subseteq H^1([0, 1])$, $f(0) \bar{g}(0) - f(1) \bar{g}(1) = 0$.

For $f = g$, $|f(0)|^2 = |f(1)|^2$ for some fixed f , then $\exists \alpha \in S^1$ such that $f(1) = \alpha f(0)$. Then for any other g ,

$$0 = f(0)(\bar{g}(0) - \alpha \bar{g}(1)) = \alpha f(0)(\overline{\alpha g(0) - g(1)})$$

Therefore $g(1) = \alpha g(0)$ for the same fixed α .

October 23, 2025

Definition: Resolvent Set and Spectrum of and Unbounded Operator

For $(A, \mathcal{D}(A), \mathcal{H})$ unbounded and closed, the resolvent set of A , $\rho(A)$, is the set of $\lambda \in \mathbb{C}$ such that $(A - \lambda) : \mathcal{D}(A) \rightarrow \mathcal{H}$ is bijective with bounded inverse $R_A(\lambda) := (A - \lambda)^{-1}$.

The spectrum of A , $\sigma(A)$, is $\sigma(A) = \mathbb{C} \setminus \rho(A)$. If $\rho(A)$ is open, then $\sigma(A)$ is closed.

Example

- Take $P = (D_x, H_0^1([0, 1]), L^2(0, 1))$ and define two closed extensions P_0 and P_1

$$- \mathcal{D}(P_0) = \{u \in H^1([0, 1]) : u(0) = 0\}$$

$$* \text{ For } \lambda \in \mathbb{C}, f \in L^1([0, 1]), \text{ we solve } \begin{cases} D_x u + \lambda u = f \\ u(0) = 0 \end{cases}$$

$$* \text{ So } u(x) = i e^{-\lambda x} \int_0^x e^{i\lambda t} f(t) dt \text{ and } \sigma(P_0) = \emptyset.$$

$$- \mathcal{D}(P_1) = H^1([0, 1])$$

$$* \text{ For } \lambda \in \mathbb{C}, \ker(D_x - \lambda) \cap L^2([0, 1]) \neq \emptyset. \text{ So } \sigma(P_1) = \mathbb{C}.$$

Proposition

If $(A, \mathcal{D}(A))$ is self-adjoint, then $\sigma(A) \subseteq \mathbb{R}$, $\sigma(A) \neq \emptyset$, and we have a resolvent estimate $\|R_A(z)\| \leq \frac{1}{|\operatorname{Im}(z)|}$ for $\operatorname{Im}(z) \neq 0$.

The proof of real value and the estimate are the same as in the bounded case.

To show that $\sigma(A) \neq \emptyset$, by contradiction if $\sigma(A) = \emptyset$ then $A^{-1} = R_A(0)$ exists and is bounded. We claim that this requires $\sigma(A^{-1}) = \{0\}$ and pick $\lambda \neq 0$, such that

$$\begin{aligned} (A^{-1} - \lambda)u &= f \\ \left(\frac{1}{\lambda} - A\right)u &= \frac{1}{\lambda}Af \end{aligned}$$

But $\frac{1}{\lambda} \notin \sigma(A)$ implies $u = -R_A\left(\frac{1}{\lambda}\right)\frac{1}{\lambda}Af$. Therefore $R_{A^{-1}}(\lambda) = -\frac{1}{\lambda}AR_A\left(\frac{1}{\lambda}\right)$ and $\lambda \notin \sigma(A^{-1})$. Therefore $A^{-1} = 0$ which contradicts the assumption that $A^{-1}A = \operatorname{Id}_{\mathcal{D}(A)}$.

Generalizing Spectral Resolution

We can extend $\{E_\lambda\}_{a \leq \lambda \leq b}$ to $a = -\infty$ and $b = \infty$ by setting strong limits $E_{-\infty} = 0$ and $E_\infty = I$.

Example

- On $L^2(\mathbb{R})$, $E_\lambda f(t) = \chi_{(-\infty, \lambda)}(t)f(t)$.

Theorem: Spectral Theorem for Unbounded Self-Adjoint Operators

Theorem 8.15 Helffer.

Any self-adjoint operator A on a Hilbert space \mathcal{H} admits a spectral decomposition $\{E_\lambda\}_\lambda$ such that $\forall x, y \in \mathcal{H}$,

$$(Ax, y) = \int_{\mathbb{R}} \lambda d(E_\lambda x, y)$$

$$Ax = \int_{\mathbb{R}} \lambda d(E_\lambda x)$$

Proof

Largely, we will use the proof for the bounded case.

Suppose A is semibounded from below (i.e. $\exists \mu \in \mathbb{R}, \forall x \in \mathcal{D}(A), (Ax, x) \geq \mu \|x\|^2$) then $R_A(\lambda)$ exists for $\lambda_0 < \mu$ ($((A - \lambda)x, x) \geq (\mu - \lambda_0) \|x\|^2$ coercive in the sense that $(\mu - \lambda_0) > 0$). Then it is also a self-adjoint bounded operator which implies that $R_A(\lambda_0)$ has a spectral representation. Then $A = f(R_A(\lambda_0))$, $f(x) = \lambda_0 + \frac{1}{x}$. We can work with a general case $(A - i)^{-1}$.

Example

- Take $(D_x, H^1(\mathbb{R}), L^2(\mathbb{R}))$, and define the Fourier transform $f(x) \mapsto \mathcal{F}f = \hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx$. Then $\widehat{D_x f}(\xi) = \xi \hat{f}(\xi)$, and we have the following commutative diagram

$$\begin{array}{ccc} H^1(\mathbb{R}) & \xrightarrow{D_x} & L^2(\mathbb{R}) \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\ H^1(\mathbb{R}) & \xrightarrow{f(t) \mapsto t \hat{f}(t)} & L^2(\mathbb{R}) \end{array}$$

- So $E_\lambda f = \mathcal{F}^{-1} \chi_{(-\infty, \lambda)} \mathcal{F} f$.
- $(D_x^2, H^2(\mathbb{R}), L^2(\mathbb{R}))$. Then $\widehat{D_x^2 f}(\xi) = \xi^2 \hat{f}(\xi)$, $\sigma(D_x^2) = [0, \infty)$ and $D_x^2 f = \mathcal{F}^{-1} \xi^2 \mathcal{F} f$.
 - So $E_\lambda f = 0$ for $\lambda < 0$; $E_\lambda f = \mathcal{F}^{-1} \chi_{\xi^2 < \lambda}(\xi) \mathcal{F} f$.
- $(D_x^2 + V(x), C_c^\infty(\mathbb{R}), L^2(\mathbb{R}))$, $V \in C(\mathbb{R}; \mathbb{R})$ bounded from below. Then $D_x^2 + V$ is semibounded from below and the spectrum depends drastically on V .
 - $V = 0$ gives $\sigma = [0, \infty)$.
 - $V = x^2$ gives $\sigma = \{2n + 1 : n \geq 0\}$
 - $V \in C_c(\mathbb{R})$ gives $\sigma = [0, \infty)$

Proposition

If for some $z_0 \in \rho(A) \cap \mathbb{R}$ $R_A(z_0)$ is compact, then A has purely discrete spectrum $(z_n)_{n \geq 1}$, $|z_n| \rightarrow \infty$ as $n \rightarrow \infty$.

Proof

$R_A(z_0)$ is compact and self-adjoint, so there exists a basis for \mathcal{H} made of normalized eigenvectors $(\phi_n)_{n \in \mathbb{N}}$ with eigenvalues $|\lambda_n| \searrow 0$.

$$\begin{aligned} (A - z_0)^{-1} \phi_n &= \lambda_n \phi_n \\ A \phi_n &= \left(z_0 + \frac{1}{\lambda_n} \right) \phi_n \\ \mathcal{H} &= \left\{ u = \sum u_n \phi_n : \sum |u_n|^2 < \infty \right\} \\ \mathcal{D}(A) &= \left\{ u = \sum u_n \phi_n : \sum |u_n|^2 \left(z_0 + \frac{1}{\lambda_n} \right)^2 < \infty \right\} \end{aligned}$$

Applications

- Laplacian (Leplace-Beltrami Operator) on closed Riemann manifolds or Dirichlet Laplace-Beltrami if $\partial M \neq \emptyset$.
- Dirichlet Laplacian ($\Delta = -\sum_{k=1}^n \partial_{x^k}^2$) on bounded domains in \mathbb{R}^n (Ω open in \mathbb{R}^n , regular, $\partial\Omega$ smooth).
 - $\Delta_D = (\Delta, H_0^1(\Omega) \cap H^2(\Omega), L^2(\Omega))$ where $H_0^1 = \{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\}$ ($u|_{\partial\Omega}$ makes sense due to a trace theorem from Evans 5.5) and $H^2(\Omega) = \{u \in L^2(\Omega) : \partial_i \partial_j u \in L^2, \partial_i u \in L^2, \forall i, j\}$.
 - $\int_{\Omega} |\nabla f|^2 dx = (\Delta f, f) \geq 0$ (Green's Theorem)
 - Claim: $\Delta_D + 1$ has compact inverse.
- Solvability of $\Delta_D + 1$

Our goal is, given $f \in L^2$, to find $u \in ?$ such that $\begin{cases} \Delta u + u = f \\ u|_{\partial\Omega} = 0 \end{cases}$. We have a weak formulation, if $v \in H_0^1$, we multiply by v and get an IBP

$$\underbrace{\int \nabla u \cdot \nabla u + \int u v}_{B(u, v)} = \int f v$$

where $|\int f v| \leq \|f\|_{L^2} \|v\|_{L^2} \leq \|f\|_{L^2} (B(v, v))^{1/2}$. By Riesz Representation Theorem, there exists some unique $u \in H_0^1$ such that $\forall v \in H_0^1, B(v, v) = \int f v$. Moreover, we can recover H^2 regularity on u . Also, $|B(u, u)| = |\int f u| \leq \|f\|_{L^2} (B(u, u))^{1/2}$ if and only if $\|(\Delta_D + 1)^{-1} f\|_{H_0^1(\Omega)} = \|u\|_{H_0^1(\Omega)} \leq \|f\|_{L^2}$. Therefore $(\Delta_D + 1)^{-1} : L^2(\Omega) \rightarrow H_0^1(\Omega)$ is bounded. By Rellich Compactness, $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ compactly. Hence $(\Delta_D + 1)^{-1} : L^2 \rightarrow L^2$ is compact.

Theorem

On \mathbb{R} , $H^1(\mathbb{R})$ does not embed compactly into $L^2(\mathbb{R})$. For example, $f \in H^1$ with $\text{supp } f \subset [0, 1]$ defined by $f_n(x) = f(x - n)$.

So if $H \subset L^2(\mathbb{R})$ where $\|u\|_H^2 = \|u'\|_{L^2}^2 + \|a(x)^{-1} u\|_{L^2}^2$ for $a(x) > 0$ and $\lim_{|x| \rightarrow \infty} a(x) = 0$, then H embeds compactly in L^2 .

It follows that $D_x^2 + x^2$ has compact resolvent and discrete spectrum. $\|u\|_H^2 = \|d_x u\|_{L^2}^2 + \|x u\|_{L^2}^2$.

October 30, 2025

Friedrichs Extension

Let $(A, \mathcal{D}(A), \mathcal{H})$ be symmetric, bounded from below in the sense that there exists $\alpha \in \mathbb{R}$ such that $\forall x \in \mathcal{D}(A)$ we have $(Ax, x)_{\mathcal{H}} \geq \alpha \|x\|_{\mathcal{H}}^2$.

Theorem:

If $(A, \mathcal{D}(A), \mathcal{H})$ is densely defined, symmetric, and bounded below, then A admits a self-adjoint extension $(S, \mathcal{D}(S))$. The procedure for producing this extension is the Friedrichs Extensions.

Proof

Assume that $\alpha = 1$, otherwise consider $A - \lambda I$ for some well-chosen λ . Define the symmetric quadratic form

$$q_A(u, v) = (Au, v), \quad (u, v) \in \mathcal{D}(A) \times \mathcal{D}(A).$$

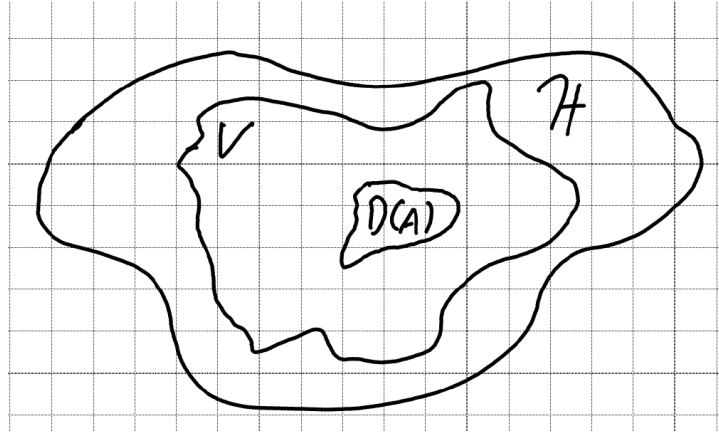
Note that $q_A(u, u) = (Au, u) \geq \|u\|_{\mathcal{H}}^2$, so this form is coercive. Next, we construct $V \subset \mathcal{H}$ by taking $p(u) = (q_A(u, u))^{1/2}$ and letting $u \in \mathcal{D}(A)$ belong to V if $\exists u_n \in \mathcal{H}$ such that $\|u_n - u\|_{\mathcal{H}} \xrightarrow{n \rightarrow \infty} 0$ and $\{u_n\}$ is p -Cauchy (i.e. $\forall \varepsilon > 0, \exists N, m, n \geq N \implies p(u_n - u_m) < \varepsilon$).

Then we can define a norm on V : $\|u\|_V := \lim_{n \rightarrow \infty} p(u_n)$ where $u_n \xrightarrow{\mathcal{H}} u$ and u_n is p -Cauchy.

Exercise: show that this definition does not depend on $\{u_n\}$.

We also define $(u, v)_V := \lim_{n \rightarrow \infty} q_A(u_n, v_n)$ for u_n and v_n similarly converging and p -Cauchy.

So $(V, \|\cdot\|_V)$ is complete and $p(u_n) \geq \|u_n\|_{\mathcal{H}}$ becomes $\|u\|_V \geq \|u\|_{\mathcal{H}}$ as $n \rightarrow \infty$. Hence $V \hookrightarrow \mathcal{H}$ is continuous.



We now construct S starting with a domain $\mathcal{D}(S) = \{u \in V : \exists v \mapsto q_A(u, v) \text{ satisfies a continuity estimate of the form } \leq C\|v\|_{\mathcal{H}}\}$. For $u \in \mathcal{D}(S)$, the map $v \mapsto q_A(u, v)$ extends to an element of \mathcal{H}^1 (bilinear functionals on \mathcal{H}). By Riesz-Representation Theorem, there exists some $w := Su$ such that $\forall v \in V, q_A(u, v) = (Su, v)_{\mathcal{H}}$.

It remains to show that S is self-adjoint, $\mathcal{D}(S) \supset \mathcal{D}(A)$, $S|_{\mathcal{D}(A)} = A$. Self-adjointness is left as an exercise. If $u \in \mathcal{D}(A)$, then

$$|q_A(u, v)| = |(Au, v)| \leq \|Au\|_{\mathcal{H}} \|v\|_{\mathcal{H}}$$

so $u \in \mathcal{D}(S)$ and $Su = Au$.

Remark

We could start with a quadratic form $q(u, v)$ which is sesquilinear, coercive and $\mathcal{D}(q) \subset \mathcal{H}$ densely. This construction produces a self-adjoint operator as well.

Example (Dirichlet Laplacian)

$\Delta = \sum_{j=1}^n \partial_{x_j}^2$ with $\Omega = \mathbb{R}^n$ open, bounded, and regular.

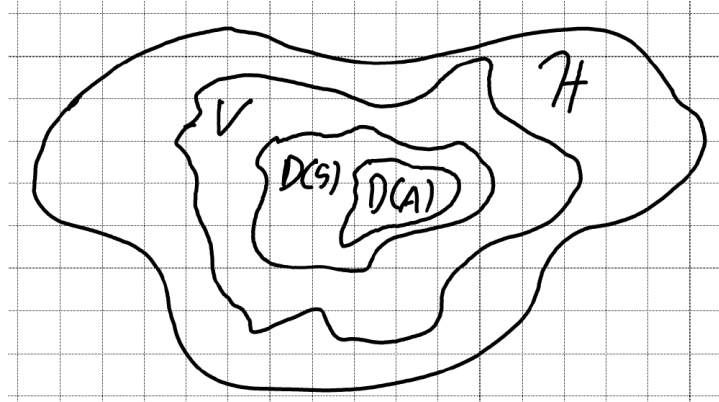
($A = -\Delta, \mathcal{D}(A) = C_C^\infty(\Omega), \mathcal{H} = L^2(\Omega)$). For $u \in \mathcal{D}(A)$,

$$\begin{aligned} (Au, u) &= \int_{\Omega} -\Delta u \cdot \bar{u} \, dx \\ &= \int_{\Omega} -\nabla(\nabla u \cdot \bar{u}) + \nabla u \cdot \nabla \bar{u} \, dx \\ &= \underbrace{\int_{\Omega} -\nabla(\nabla u \cdot \bar{u}) \, dx}_{=0} + \int_{\Omega} \nabla u \cdot \nabla \bar{u} \, dx \geq 0 \end{aligned}$$

So Friedrich's procedure produces a self-adjoint extension

$$q_A(u, v) = \int_{\Omega} \nabla u \cdot \nabla \bar{v} \, dx, \quad u, v \in C_C^\infty(\Omega)$$

which leads to $V = \overline{C_C^\infty(\Omega)}^{H^1(\Omega)} = H_0^1(\Omega)$ (that is the H^1 closure of $C_C^\infty(\Omega)$). What about $\mathcal{D}(S)$?



For $v \in C_C^\infty(\Omega)$, $\int_{\Omega} \nabla u \cdot \nabla \bar{v} \, dx = \int_{\Omega} -\Delta u \cdot v \, dx \leq C \|v\|_{L^2}$. Then we have

$$\mathcal{D}(S) = \{u \in H_0^1(\Omega) : \Delta u \in L^2(\Omega)\} = H_0^1(\Omega) \cap H^2(\Omega)$$

Example (Neumann Laplacian)

Start from a quadratic form $q(u, v) = \int_{\Omega} \nabla u \cdot \nabla \bar{v} + u \bar{v} \, dx$ with $u, v \in C^\infty(\bar{\Omega})$. Then $V = H^1(\Omega)$. We compute $\mathcal{D}(S)$ by first recalling that for $u, v \in C^\infty(\bar{\Omega})$,

$$\int_{\Omega} -\Delta u \bar{v} \, dx = \int_{\Omega} \nabla u \cdot \nabla \bar{v} \, dx - \int_{\partial\Omega} \partial_\nu u \bar{v} \, dx$$

We sense that $\Delta u \in L^2(\Omega)$, so we at least need $u \in W(\Omega) = \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\}$. How do we make sense of the Neumann trace on $W(\Omega)$?

- Lemma

- The restriction map $C(\overline{\Omega}) \rightarrow C(\partial\Omega)$ extends to a bounded surjective “Dirichlet trace” $\tau_D : H^1(\Omega) \rightarrow H^{1/2}(\partial\Omega)$ with bounded right inverse R .

• Example

- if $\Omega = \mathbb{D}$, $\partial\Omega = S^1$, $f \in C^\infty(S^1) \leftrightarrow f(\theta) \sum_{k \in \mathbb{Z}} f_k e^{ik\theta}$ where $f_k = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-ik\theta} d\theta$, we can for each $s \in \mathbb{R}$ define

$$H^s(S^1) = \{f \in \mathcal{D}'(S^1) : \sum_{k \in \mathbb{Z}} (1+k^2)^s |f_k|^2 < \infty\} = \mathcal{D}((- \partial_\theta^2)^{s/2})$$

• Fact

- For $s \geq 0$, $H^{-s}(\partial\Omega) \cong (H^s(\partial\Omega))'$.
- If $f \in H^{-s}$, $g \in C^\infty(S^1)$,

$$\langle f, g \rangle_{\mathcal{D}', \mathcal{D}} = \int (f, g) = c \sum_{k \in \mathbb{Z}} f_k \overline{g_k} \frac{(1+k^2)^{s/2}}{(1+k^2)^{s/2}} \leq c \left(\sum_{k \in \mathbb{Z}} (1+k^2)^{-s} |f_k|^2 \right) \left(\sum_{k \in \mathbb{Z}} (1+k^2)^s |g_k|^2 \right)$$

Returning to the Neumann Laplacian, the Neumann trace $C^1(\overline{\Omega}) \rightarrow C(\partial\Omega)$ by $f \mapsto \partial_\nu f$ extends to a bounded operator $\tau_N : W(\Omega) \rightarrow H^{-1/2}(\partial\Omega)$.

• Proof

If $u \in W(\Omega)$, set

$$\begin{aligned} \Phi_u(v) &:= \int_{\Omega} -\Delta u \bar{v} \, dx - \int_{\Omega} \nabla u \cdot \nabla \bar{v} \, dx, \quad v \in H^1(\Omega) \\ &\leq \|\Delta u\|_{L^2} \|v\|_{L^2} + \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} \end{aligned}$$

Using $R : H^{1/2}(\partial\Omega) \rightarrow H^1(\Omega)$, $w \in H^{1/2}(\partial\Omega)$,

$$|\Phi_u(Rw)| \leq (\|\Delta u\|_{L^2} + \|\nabla u\|_{L^2}) \|Rw\|_{H^1} \leq (\|\Delta u\|_{L^2} + \|\nabla u\|_{L^2}) \|w\|_{H^{1/2}(\Omega)}$$

which implies that there exists some $h \in H^{-1/2}(\partial\Omega)$ such that $\Phi_u(Rw) = \langle h, w \rangle_{H^{-1/2}, H^{1/2}}$.

We want to extend

$$\int_{\Omega} -\Delta u \bar{v} \, dx = \int_{\Omega} \nabla u \cdot \nabla \bar{v} \, dx - \int_{\partial\Omega} \partial_\nu u \bar{v} \, dx$$

to $u \in W(\Omega)$ and $v \in H^1(\Omega)$. First, if $v \in H_0^1(\Omega)$ then

$$\int_{\Omega} -\Delta u \bar{v} \, dx = \int_{\Omega} \nabla u \cdot \nabla \bar{v} \, dx$$

(by passing to the limit with a sequence $v_n \in C_c^\infty(\Omega)$ converging to v in H^1).

Next, if $v \in H^1$, we write $v = v_0 + R\tau_D v$ with $v_0 \in H_0^1$. Then

$$\int_{\Omega} -\Delta u \bar{v} - \nabla u \cdot \nabla \bar{v} \, dx = \overbrace{\int_{\Omega} -\Delta u \bar{v}_0 - \nabla u \cdot \nabla \bar{v}_0 \, dx}^{=0} + \Phi_u(R\tau_D v) = \langle \tau_N u, \tau_D v \rangle_{H^{-1/2}, H^{1/2}}$$

We conclude that for each $u \in W(\Omega)$, $v \in H^1(\Omega)$

$$\int_{\Omega} -\Delta u \cdot \bar{v} \, dx = \int_{\Omega} \nabla u \cdot \nabla \bar{v} \, dx + \langle \tau_N u, \tau_D v \rangle_{H^{-1/2}, H^{1/2}}$$

Therefore we have

$$q(u, v) = \int_{\Omega} (-\Delta u + u) \bar{v} \, dx + \langle \tau_N u, \tau_D v \rangle_{H^{-1/2}, H^{1/2}}$$

and $v \mapsto q(u, v)$ is L^2 -bounded if and only if $v \mapsto \langle \tau_N u, \tau_D v \rangle_{H^{-1/2}, H^{1/2}}$ is L^2 -bounded which holds if and only if $\tau_N u = 0$. So

$$\mathcal{D}(S) = \{u \in H^1(\Omega) : -\Delta u \in L^2, \tau_N u = 0\} = \{u \in H^2(\Omega) : \tau_N u = 0\}$$

Observation

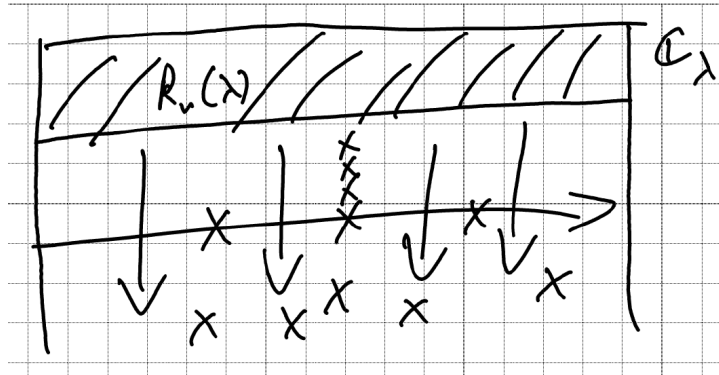
The form domains satisfy $\mathcal{D}(q_D^{-\Delta}) \subseteq \mathcal{D}(q_N^{-\Delta})$.

November 04 and 06, 2025

See class notes on variational principles for eigenvalues and Weyl's criterion for essential spectrum.

November 13, 2025

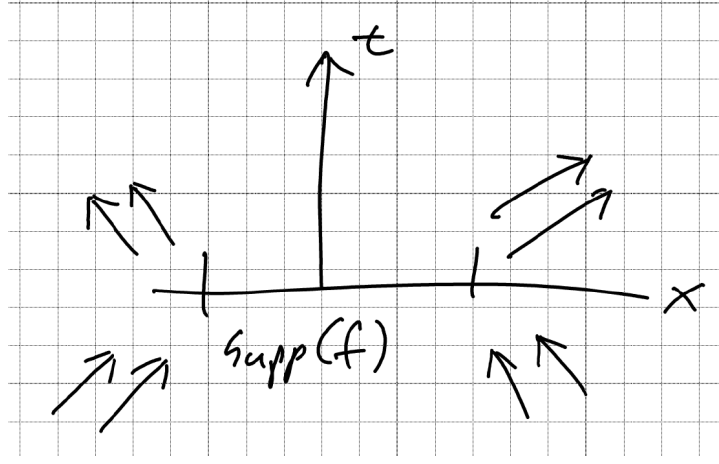
Consider $(D_x^2 - \lambda^2)^{-1} = R_0(\lambda)$, the “outgoing” resolvent and let V be in $C_c^\infty(\mathbb{R}, \mathbb{R})$ or $L_{\text{comp}}^\infty(\mathbb{R}, \mathbb{R})$. Then we may define $R_V(\lambda) = (D_x^2 + V - \lambda^2)^{-1}$. We want to show that $-R_V(\lambda)$ makes sense for $\text{Im}(\lambda) \gg 0$ as an $L^2 \rightarrow L^2$ operator and extends meromorphically to either $\{\text{Im}(\lambda) > 0\}$ as an $L^2 \rightarrow L^2$ bounded operator or to \mathbb{C} as a $L_{\text{comp}}^2 \rightarrow L_{\text{loc}}^2$ operator. Recall that $L_{\text{comp}}^2 = \{f \in L^2 : \text{supp } f \text{ compact}\}$ and $L_{\text{loc}}^2 = \{f : \mathbb{R} \rightarrow \mathbb{R} : \forall K \text{ compact, } \int_K |f|^2 < \infty\}$.



We will further see that there are no poles (embedded eigenvalues) on $\mathbb{R} \setminus \{0\}$.

Outgoing/Incoming Solution

We have that $(D_x^2 - \lambda^2)u = 0$ if and only if $u(x) = Ae^{i\lambda x} + Be^{-i\lambda x}$. A solution to $(D_x^2 - \lambda^2)u = f \in C_c^\infty(\mathbb{R})$ is said to be outgoing if $x \gg 0$ implies that $u(x) = A_+ e^{i\lambda x}$ and $x \ll 0$ implies that $u(x) = B_+ e^{-i\lambda x}$.



Likewise, the solution is incoming if $x \gg 0$ implies $u(x) = B_- e^{-i\lambda x}$ and $x \ll 0$ implies $u(x) = A_- e^{i\lambda x}$. We have seen previously that the D'Alembert Solution to

$$\begin{cases} (\partial_t^2 - \partial_x^2 + V)u = f \\ u|_{t=0} = \partial_t u|_{t=0} = 0 \end{cases}$$

is outgoing.

We will find that an outgoing solution to $(D_x^2 - \lambda^2)u = f$ is $R_0(\lambda)f := u_+ = \frac{i}{2\lambda} \int_{\mathbb{R}} f(y) e^{i\lambda|x-y|} dy$. The incoming solution, then, is $u_-(x, \lambda) = -\frac{i}{2\lambda} \int_{\mathbb{R}} f(y) e^{-i\lambda|x-y|} dy$.

Note that $u_+(x, \lambda) = u_-(x, -\lambda) = \overline{u_-(x, \lambda)}$. Note also that the outgoing solution is in $L^2(\mathbb{R})$ if and only if $\text{Im}(\lambda) > 0$, and the incoming solution if and only if $\text{Im}(\lambda) < 0$.

The above solution may be derived via Fourier transform. Namely

$$(\xi^2 - \lambda^2)\hat{u}(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx$$

which implies that $\hat{u}(\xi) = \frac{\hat{f}(\xi)}{\xi^2 - \lambda^2}$ if and only if $\text{dist}(\lambda^2, [0, \infty)) > 0$. Considering the outgoing solution, this occurs only when $\text{Im}(\lambda) > 0$.

Note, that with respect to Plancherel Theorem giving $\|u\|_{L^2} = c \|\hat{u}(\xi)\|_{L^2}$,

$$\frac{1}{c} \|u\|_{L^2} = \|\hat{u}(\xi)\|_{L^2} \leq \frac{1}{\text{dist}(\lambda^2, [0, \infty))} \|\hat{f}(\xi)\|_{L^2} = \frac{1}{c} \|f\|_{L^2}$$

This gives continuity estimates for $R_0(\lambda)$. Then $u(x) = h_\lambda(x) * f(x)$ where $h_\lambda(x) = \int_{\mathbb{R}} e^{ix\xi} \frac{d\xi}{\xi^2 - \lambda^2}$. We claim, computing by residue theorem, that $h_\lambda(x) = \frac{i}{2\lambda} e^{i\lambda|x|}$.

Exercise: do the residue computation.

So we have $R_0(\lambda) = \frac{i}{2\lambda} \int_{\mathbb{R}} f(y) e^{i\lambda|x-y|} dy$ with Schwarz kernel $R_0(x, y, \lambda) = \frac{i}{2\lambda} e^{i\lambda|x-y|}$. When x and y are frozen, this looks meromorphic in λ with a simple pole at $\lambda = 0$ and residue $\frac{i}{2}$. This still makes sense as a map $C_c^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})$ or $L_{\text{comp}}^2(\mathbb{R}) \rightarrow L_{\text{loc}}^2(\mathbb{R})$.

Alternatively, we can formulate this as $\forall \rho \in C_c^\infty(\mathbb{R})$, $\rho R_0 \rho : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ extends meromorphically to \mathbb{C} . Then $\rho R_0(\lambda) \rho$ has Schwarz kernel $\frac{i}{2\lambda} \rho(x) \rho(y) e^{i\lambda|x-y|}$.

Exercise: if $\lambda \neq 0$, this is $L^2 \rightarrow L^2$ bounded.

Then $R_0(\lambda) = \frac{P}{\lambda} + Q(\lambda)$ where $Pf(x) = \frac{i}{2} \int_{\mathbb{R}} f(y) dy$, $P = \phi \otimes \bar{\phi}$ and $\phi(x) = \frac{e^{i\pi/4}}{\sqrt{2}}$. Here $Q : L_{\text{comp}}^2(\mathbb{R}) \rightarrow L_{\text{loc}}^2(\mathbb{R})$ is entire.

Compactly Supported Potential

Take $V \in L_{\text{comp}}^\infty(\mathbb{R})$, where $\sigma_{\text{ess}}(D_x^2 + V) = [0, \infty) + (\text{possibly negative eigenvalues}) + (\text{embedded eigenvalues})$.

Theorem:

The outgoing resolvent $(D_x^2 + V - \lambda^2)^{-1}$ extends meromorphically to \mathbb{C} with no poles at $\lambda \in \mathbb{R} \setminus \{0\}$.

Proof

Goal: when can we solve $(D_x^2 + V - \lambda^2)u = f$?

We start with the identity $(D_x^2 + V - \lambda^2)R_0(\lambda) = I + VR_0(\lambda)$. For $\text{Im}(\lambda) > 0$, $\|VR_0(\lambda)\|_{L^2 \rightarrow L^2} \leq \frac{\|V\|_\infty}{\text{dist}(\lambda^2, [0, \infty))}$. So if $\text{Im}(\lambda) \gg 1$, $\|VR_0(\lambda)\|_{L^2 \rightarrow L^2} < 1$. Then

$$(D_x^2 + V - \lambda^2)R_0(\lambda) \overbrace{\sum_{p=0}^{\infty} (-VR_0(\lambda))^p}^{R_V(\lambda)} = I.$$

So $R_V(\lambda)$ is an inverse for $D_x^2 + V - \lambda^2$ on $\text{Im} \gg 1$, bounded with operator norm below $\|R_0(\lambda)\| \frac{1}{1 - \|VR_0(\lambda)\|}$. It produced outgoing solutions due to its factored form.

Now, to extend $R_V(\lambda)$ meromorphically as $L_{\text{comp}}^2 \rightarrow L_{\text{loc}}^2$, we look at $\rho R_V(\lambda) \rho$ for $\rho \in C_c^\infty(\mathbb{R})$ such that $\rho V = V$. Then for $\text{Im}(\lambda) \gg 0$,

$$\rho R_V(\lambda) \rho = R_0(\lambda)(I + VR_0(\lambda))^{-1} \rho = R_0(\lambda) \rho \sum_{p=0}^{\infty} (VR_0(\lambda) \rho)^p = R_0(\lambda) \rho (I + VR_0(\lambda) \rho)^{-1}$$

Therefore

$$\rho R_V(\lambda) \rho = \rho R_0(\lambda) \rho (I + \rho VR_0(\lambda) \rho)^{-1}$$

We claim that $\lambda \mapsto \rho VR_0(\lambda) \rho$ is a meromorphic on $\mathbb{C} \setminus \{0\}$ family of compact operators (at least on $V \in C_c^\infty(\mathbb{R})$).

To see this, we use $R_0(\lambda) : L^2(\mathbb{R}) \rightarrow H^2(\mathbb{R})$, then $\rho VR_0(\lambda) \rho : L^2(\mathbb{R}) \rightarrow \{u \in H^2(\mathbb{R}) : \text{supp } u \subset \text{supp } V\}$ embeds compactly into $L^2(\mathbb{R})$.

By analytic Fredholm theory, since $I + \rho VR_0(\lambda) \rho$ is invertible for $\text{Im}(\lambda) \gg 1$, it is invertible on $\mathbb{C} \setminus \{0\}$ outside of a discrete set with finite-dimensional obstructions at that discrete set.

To upgrade this to a meromorphic extension of $R_V(\lambda) : L_{\text{comp}}^2 \rightarrow L_{\text{loc}}^2$, pick an increasing sequence of cutoff functions. If ρ_1 is such that $\rho_1 \rho = \rho$, then $\rho_1 R_V(\lambda) \rho_1$ extends $\rho R_V(\lambda) \rho$ in the sense that $R_V(\lambda) \rho_1 f = R_V(\lambda) \rho f$ when $\text{supp } f \subset \{\rho = 1\}$.

Claim: if $\lambda \in \mathbb{R} \setminus \{0\}$, then λ is not a pole of $R_V(\lambda)$.

- Step 1: if λ is a pole, then there exists an outgoing solution to $(D_x^2 + V - \lambda^2)u = 0$.
- Step 2: if $\lambda \in \mathbb{R} \setminus \{0\}$, and u is an outgoing solution to $(D_x^2 + V - \lambda^2)u = 0$, then u has compact support.
- Step 3: if u is a compactly supported solution of $(D_x^2 + V - \lambda^2)u = 0$, then $u = 0$.

Step 1. If $\lambda = \lambda_0$ is a pole, we may write $R_V(\lambda) = \frac{P_N}{(\lambda - \lambda_0)^N} + \dots + \frac{P_1}{\lambda - \lambda_0} + Q(\lambda)$ near $\lambda = \lambda_0$ where each term is $L_{\text{comp}}^2 \rightarrow L_{\text{loc}}^2$. If $\lambda \neq \lambda_0$, then

$$(\lambda - \lambda_0)^N \overbrace{(D_x^2 + V - \lambda^2)R_V(\lambda)}^I = (D_x^2 + V - \lambda^2)P_N + (\lambda - \lambda_0)(\dots)$$

Then as $\lambda \rightarrow \lambda_0$, $(D_x^2 + V - \lambda^2)P_n = 0$. So P_N produces outgoing solutions in L_{loc}^2 .

Step 2. If $\lambda \in \mathbb{R}$ and u solves $(D_x^2 + V - \lambda^2)u = 0$, then \bar{u} also solves $(D_x^2 + V - \lambda^2)\bar{u} = 0$.

Since u has compact support, write $u(x) = A_+ e^{i\lambda x} + B_- e^{-i\lambda x}$ when $x \gg 0$ and $u(x) = A_- e^{i\lambda x} + B_+ e^{-i\lambda x}$ when $x \ll 0$. Compute the Wronskian, $W(u, \bar{u}) = u\bar{u}' - u'\bar{u}$ and $\frac{d}{dx}W(u, \bar{u}) = u\bar{u}'' - u''\bar{u} = 0$. We find that this is $|A_+|^2 - |B_-|^2$ for $x \gg 0$ and $|A_-|^2 - |B_+|^2$ for $x \ll 0$.

Therefore $|A_+|^2 + |B_+|^2 = |A_-|^2 + |B_-|^2$. That is, if u is outgoing then $A_- = B_- = 0$ and $A_+ = B_+ = 0$.

Step 3 is insane.

- Lemma

If we take $u \in L^\infty(\mathbb{R})$, $(D_x^2 + W)u = 0$ where $W \in L^\infty(\mathbb{R})$, then if $u = 0$ on $(-\infty, 0)$ then $u \equiv 0$.

Fix $h > 0$ and let $v = e^{-x/h}u$.

$$\begin{aligned} \|e^{-x/h}(hD_x)^2 e^{x/h}v\|_{L^2} &= \|(h^2 D_x^2 - 2ihD_x - 1)v\|_{L^2} \\ &= \|(h\xi - i)^2 \hat{v}\|_{L^2} \\ &\geq \|\hat{v}\|_{L^2} \\ &= \|v\|_{L^2}^2 \end{aligned}$$

Equivalently

$$\|e^{-x/h}u\|_{L^2} \leq h^2 \|e^{-x/h}D_x^2 u\| \leq h^2 \|W\|_{L^\infty} \|e^{-x/h}u\|_{L^2}$$

November 25, 2025

Recall

For $V \in C_C^\infty(\mathbb{R})$, $R_V(\lambda)$ is the scattering resolvent ($\text{Im}(\lambda) \gg 0$, $R_V(\lambda) = (D_x^2 + V - \lambda^2)^{-1}$).

This is meromorphic on $\{\text{Im}(\lambda) > 0\}$ as a $L^2 \rightarrow L^2$ bounded family. Similarly, it is meromorphic on $\mathbb{C} \setminus \{0\}$ as a $L_{\text{comp}}^2 \rightarrow L_{\text{loc}}^2$ family.

Then $e_\pm(x, \lambda) = e^{\pm ix\lambda} - R_V(\lambda)(Ve^{\pm \text{id}})$ are two solutions of $(D_x^2 + V - \lambda^2)u = 0$.

IMAGE 1

Then we take $S(\lambda)$, the scattering matrix, defined wherever possible such that

$$S(\lambda) \begin{bmatrix} A_- \\ B_- \end{bmatrix} = \begin{bmatrix} A_+ \\ B_+ \end{bmatrix} \quad S(\lambda) = \begin{bmatrix} T(\lambda) & R_-(\lambda) \\ R_+(\lambda) & T(\lambda) \end{bmatrix}$$

Computing Wronskians gives

$$\begin{aligned} R_-(\lambda) &= -R_+(-\lambda)T(\lambda)/T(-\lambda) \\ S(-\lambda) &= JS(\bar{\lambda})^*J, \quad J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ S(\lambda)JS(-\lambda)J &= I \end{aligned}$$

Last time, we defined $\phi_+(x, \lambda) = \frac{1}{T(-\lambda)}e_+(x, -\lambda)$ and $\phi_-(x, \lambda) = \frac{1}{T(\lambda)}e_-(x, \lambda)$.

IMAGE 2

We have that ehse solve $D_x^2 + V - \lambda = 0$, so $\forall x, \mathbb{R} \ni \lambda \mapsto \phi_{\pm}(x, \lambda)$ is tempered. Therefore, we may take its Fourier transform in λ .

$$A_{\pm}(x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} \phi_{\pm}(x, \lambda) e^{i\lambda y} d\lambda$$

makes sense as a tempered distribution.

Claim

A_{\pm} is a distributional solution of $(D_x^2 + V)A_{\pm}(x, y) = D_y^2 A_{\pm}(x, y)$. That is, if we define $\mathcal{A}_{\pm}f(x) = \int_{\mathbb{R}} A_{\pm}(x, y)f(y) dy$, then $(D_x^2 + V) \circ \mathcal{A}_{\pm} = \mathcal{A}_{\pm} \circ D_x^2$. We compute

$$\begin{aligned} D_y^2 A_{\pm}(x, y) &= \frac{1}{2\pi} \int_{\mathbb{R}} \phi_{\pm}(x, \lambda) \lambda^2 e^{i\lambda y} d\lambda \\ (D_x^2 + V)A_{\pm}(x, y) &= \frac{1}{2\pi} \int_{\mathbb{R}} \lambda^2 \phi_{\pm}(x, \lambda) e^{i\lambda y} d\lambda \end{aligned}$$

In addition, A_{\pm} satisfies $A_{\pm}(x, y) = \delta(x - y)$ for $\pm x \gg 0$ where we can think of the Dirac delta as $\delta(x - y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\lambda(y-x)} d\lambda$. Now, our goal is to get an alternative expression for $S(\lambda)$. We can achieve this by looking at $\phi_{-}(x, \lambda)$.

IMAGE 3

IMAGE 4

We have that $\text{clos}(\text{supp}(V)) \subset [a, b]$. By pictures, for $x \geq b$,

$$\partial_y A_{-}(x, y) = x(y - x) + y(x + y) = \frac{1}{2\pi} \int_{\mathbb{R}} \phi_{-}(x, \lambda) i\lambda e^{i\lambda y} d\lambda$$

for x and y distributions. Via Fourier transform in y , this implies that

$$i\lambda \phi_{-}(x, \lambda) = \int_{\mathbb{R}} e^{-i\lambda y} (X(y - x) + Y(x + y)) dy = e^{-i\lambda x} \hat{X}(\lambda) + e^{i\lambda x} \hat{Y}(\lambda)$$

Exercise: show that $\text{supp}(X) \subset [-2(b - 1), 0]$ and $\text{supp}(Y) \subset [2a, 2b]$.

Copying expressions, we have that

$$\begin{aligned} \frac{1}{T(\lambda)} = \frac{\hat{X}(\lambda)}{i\lambda} &\iff T(\lambda) = \frac{i\lambda}{\hat{X}(\lambda)} \\ \frac{R_{-}(\lambda)}{T(\lambda)} = \frac{\hat{Y}(\lambda)}{i\lambda} &\iff R_{-}(\lambda) = \frac{\hat{Y}(\lambda)}{\hat{X}(\lambda)} \end{aligned}$$

It follows that

$$R_{+}(\lambda) = -R_{-}(-\lambda) \frac{T(\lambda)}{T(-\lambda)} = -\frac{\hat{Y}(\lambda)}{\hat{X}(\lambda)} \cdot \frac{i\lambda}{\hat{X}(\lambda)} \cdot \frac{\hat{X}(-\lambda)}{-i\lambda} = \frac{\hat{Y}(-\lambda)}{\hat{X}(\lambda)}$$

The bottom line, then, is that we may write

$$S(\lambda) = \frac{1}{\hat{X}(\lambda)} \begin{bmatrix} i\lambda & \hat{Y}(\lambda) \\ \hat{Y}(-\lambda) & i\lambda \end{bmatrix}$$

Notation: recall that \mathcal{D}' are distributions, \mathcal{E}' are distributions with compact support, and \mathcal{S}' are tempered distribution. Then, since $X, Y \in \mathcal{E}'(\mathbb{R})$, \hat{X} and \hat{Y} are entire on \mathbb{C} . It follows that $\hat{X}(\lambda) = \int_{-2(b-a)}^0 X(x) e^{-i\lambda x} dx = \langle X, e^{-i\lambda} \rangle$.

If $\hat{X}(\lambda) = 0$, then $i\lambda\phi_-(x, \lambda) = e^{ix\lambda} \hat{Y}(\lambda)$. Then also $\phi_-(x, \lambda) \in L^2(\mathbb{R})$ if $\text{Im}(\lambda) > 0$ which implies an eigenfunction.

Wave Operators

If H is a self-adjoint operator, for $t \in \mathbb{R}$, e^{-itH} makes sense.

$$(e^{-itH})^* = e^{(itH)^*} = e^{itH} = (e^{-itH})^{-1}$$

so e^{-itH} is unitary. For $u \in L^2(\mathbb{R})$, $v(t) := e^{-itH} u$ solves $(i\partial_t - H)v = 0$, $v|_{t=0} = u$.

Given $H_0 = D_x^2$ and $H_V = D_x^2 + V$, $V \in L_{\text{comp}}^\infty(\mathbb{R})$, the idea is that $\forall u \in L^2(\mathbb{R})$ $u \perp \{\text{eigenfunctions of } H\}$. There exists $u_\pm \in L^2(\mathbb{R})$ where $e^{-itH_V} u \approx e^{-itH_0} u_\pm$

IMAGE 5

as $t \rightarrow \pm\infty$.

Theorem

With the above setting, if $u \in L^2(\mathbb{R})$, the following strong limits $W_\pm u = \lim_{t \rightarrow \pm\infty} e^{itH_V} e^{-itH_0} u$.

Also, we have $W_\pm H_0 = H_V W_\pm$ and $||W_\pm u|| = ||u||$. W_\pm are partial isometries intertwining H_0 and H_V .

Then define the scattering operator $\mathcal{S} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ by $\mathcal{S} = W_+^* W_-$.

The connection between scattering/vertex operators comes from defining

$$\begin{aligned} \Phi : L^2(\mathbb{R}) &\rightarrow (L^2([0, \infty]))^2 \\ u(x) &\mapsto \begin{pmatrix} \hat{u}(\lambda) \\ \hat{u}(-\lambda) \end{pmatrix}, \quad \lambda \geq 0 \end{aligned}$$

Theorem

$$\mathcal{S} := \Phi^* \mathcal{S}(\lambda) \Phi = W_+^* W_-.$$

Proof: Existence

For $u \in L^2$, $v(t)u = e^{itH_V} e^{-itH_0} u$.

$$\frac{d}{dt}(v(t)u) = i(e^{itH_V} H_V e^{-itH_0} u - e^{itH_V} H_0 e^{-itH_0} u) = i e^{itH_V} V e^{-itH_0} u$$

So

$$v(t)u = u + i \int_0^t e^{isH_V} V e^{-isH_0} u ds$$

We want to show that

$$\int_{\mathbb{R}} \overbrace{\|e^{isH_v} V e^{-isH_0} u\|_{L^2}}^{\|V e^{-isH_0} u\|_{L^2}} ds < \infty$$

If $u_t(x) = e^{-itH_0} u$, we want to show that $u_t(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(x\xi - t\xi^2)} \hat{u}(\xi) d\xi$. Note that $e^{i(x\xi - t\xi^2)} = \left(\frac{1}{i(x-2t\xi)} \frac{d}{d\xi} \right)^2 e^{i(x\xi - t\xi^2)}$.

Proof of Quantum-Classical

If $v \in C_c^\infty(\mathbb{R})$, define $u := \Phi^* S(\lambda) \Phi v$ and we want to show $u = W_+^* W_- v$ if and only if $W_+ u = W_- v$. Write

$$\begin{aligned} w(x) &= \frac{1}{2\pi} \int_0^\infty (e_+(x, \lambda) \hat{v}(\lambda) + e_-(x, \lambda) \hat{v}(-\lambda)) d\lambda \\ \tilde{w}(x) &= \frac{1}{2\pi} \int_0^\infty (e_+(x, -\lambda) \hat{u}(-\lambda) + e_-(x, -\lambda) \hat{u}(\lambda)) d\lambda \end{aligned}$$

$$\begin{cases} w = W_- v \\ \tilde{w} = W_+ u \\ w = \tilde{w} \end{cases}$$

where the last equality implies that

$$\tilde{w}(x) = \frac{1}{2\pi} \int_0^\infty \overbrace{(e_+(x, -\lambda), e_-(x, -\lambda))}^{(e_+(x, \lambda), e_-(x, \lambda) S(\lambda)^{-1})} \begin{bmatrix} \hat{u}(-\lambda) \\ \hat{u}(\lambda) \end{bmatrix} d\lambda = w(x)$$

So

$$\begin{aligned} e^{-itH_v} w &= \frac{1}{2\pi} \int_0^{+\infty} e^{-it\lambda^2} (e_+(x, \lambda) \hat{v}(\lambda) + e_-(x, \lambda) \hat{v}(-\lambda)) d\lambda \\ &= e^{itH_0} v + \overbrace{\frac{1}{2\pi} \int_0^{+\infty} \left(\underbrace{\overbrace{-R_V(\lambda)(V e^{i\lambda})}_{f_+(x, \lambda)}}_{f_+(x, \lambda)} \hat{v}(\lambda) + \underbrace{\overbrace{-R_V(\lambda) V e^{-i\lambda}}_{f_-(x, \lambda)}}_{f_-(x, \lambda)} \hat{v}(-\lambda) \right) d\lambda}_{:=I} \end{aligned}$$

To show that $\lim_{t \rightarrow \infty} I(t) = 0$, deform $\lambda \in [0, \infty)$ to $\{\mu + i\mu : \mu \geq 0\}$. Then

$$e^{-it\lambda^2} = e^{-it(\mu + i\mu)^2} = e^{2t\mu^2}$$

December 2, 2025

IMAGE 1

Theorem:

Let $V \in L_{\text{comp}}^\infty(\mathbb{R}, \mathbb{R})$, and suppose $w(x, t)$ solves

$$\begin{cases} (D_t^2 - P_V) w(x, t) = 0 & \text{on } \mathbb{R} \times \mathbb{R} \\ w|_{t=0} = w_0 \in H_{\text{comp}}^1(\mathbb{R}) \\ \partial_t w|_{t=0} = w_1 \in L_{\text{comp}}^2(\mathbb{R}) \end{cases}$$

IMAGE 2

Then, for any $A > 0$,

$$w(x, t) = \sum_{\text{Im}(\lambda_j) > -A} \overbrace{\sum_{\ell=0}^{m_R(\lambda_j)-1} t^\ell e^{-i\lambda_j t} f_{j,\ell}(x)}^{-\text{Res}_{\mu=\lambda_j}((iR_V(\mu)w_1 + \mu R_V(\mu)w_0)e^{i\mu t})} + E_A(t)$$

where $m_R(\lambda_j) = \text{rank} \oint_{\lambda} R_V(\xi) d\xi$ and the integral is about a small contour isolating λ . Note that $\sum_{\text{Im}(\lambda_j) > -A}$ is a finite sum.

We also have that $(P_V - \lambda_j^2)f_{j,\ell} = 0$. Then $\forall k > 0$, $\text{supp}(w_j) \subset (-k, k)$, $\exists C_{k,A}, T_{k,A}$ such that

$$||E_A(t)||_{H^2([-k,k])} \leq C_{k,A} e^{-tA} (||w_0||_{H^1} + ||w_1||_{L^2})$$

IMAGE 3

Returning to the Meromorphic Continuation ρb for $\text{Im}(\Lambda) \gg 1$

We have

$$R_V(\lambda) = R_0(\lambda)(I + VR_0(\lambda))^{-1}$$

which is ok via Neumann series, since $||R_0||_{L^2 \rightarrow L^2} \rightarrow 0$ as $\text{Im}(\lambda) \rightarrow \infty$. Fix $\rho \in C_C^\infty(\mathbb{R})$, $\rho V = V$ which implies $(1 - \rho)V = 0$. Then

$$I + VR_0(\lambda) = I + VR_0(\lambda)\rho + VR_0(\lambda)(1 - \rho) = (I + VR_0(\lambda)(1 - \rho))(I + VR_0(\lambda)\rho)$$

Notice also that $(I - VR_0(\lambda)(1 - \rho))(1 + VR_0(\lambda)(1 - \rho)) = I$. Therefore

$$R_V(\lambda) = R_0(\lambda)(I + VR_0(\lambda)\rho)^{-1}(I - VR_0(\lambda)(1 - \rho))$$

and poles of $R_V(\lambda)$ occur when $I + VR_0(\lambda)\rho$ is not invertible. We have an estimate on $VR_0(\lambda)\rho = V\rho R_0(\lambda)\rho$ of

$$||VR_0(\lambda)\rho||_{L^2 \rightarrow L^2} \leq ||V||_\infty ||\rho R_0(\lambda)\rho||_{L^2 \rightarrow L^2}$$

Write $R_0(\lambda)f(x) = \frac{i}{2\lambda} \int_{\mathbb{R}} f(y) e^{i|x-y|\lambda} dy$ and, by Cauchy-Schwarz,

$$|\rho R_0(\lambda)\rho f(x)|^2 = \left| \frac{i}{2\lambda} \rho(x) \int \rho(y) f(y) e^{i|x-y|\lambda} dy \right|^2 \leq \frac{\rho^2(x)}{4|\lambda|^2} ||f||_{L^2}^2 \int \rho^2(y) e^{-2\text{Im}(\lambda)|x-y|} dy$$

where $\text{supp } \rho \subset [-R, R]$ and $\rho \leq 1$. Computing, we may find that

$$||\rho R_0(\lambda)\rho f||_{L^2} \leq \frac{C_1}{|\lambda|} T(\text{Im}(\lambda))_- ||f||_{L^2}$$

where $x_- = \max\{0, -x\}$.

Question:

When is $\|VR_0(\lambda)\rho\|_{L^2 \rightarrow L^2} \leq \frac{1}{2}$?

This happens if and only if $\frac{C}{|\lambda|} e^{T(\operatorname{Im}(\lambda))_-} \leq \frac{1}{2}$, if and only if $x^2 + y^2 = Ce^{\pm Ty}$. If $\operatorname{Im}(\lambda) \geq 0$, then $|\lambda| \geq 2C$; if $\operatorname{Im}(\lambda) < 0$, then $e^{-T\operatorname{Im}(\lambda)} \leq \frac{1}{2C}|\lambda|$ if and only if $\operatorname{Im}(\lambda) \geq -A - \delta \log(1 + |\lambda|)$ with $A, \delta > 0$.

Claim (Exercise)

For every $A' > 0$, the above estimate implies that $\Omega_{A'} = \{\lambda : \operatorname{Im}(\lambda) > -A' - \delta \log(1 + |\lambda|)\}$ has finitely many resonances. Claim: $\Omega_{A'} \cap \Omega_A$ is compact?

Left-Right Symmetry of Resonances

Since P_V is self-adjoint, $(P_V - \bar{z}) = \overline{P_V - z}$, hence it is involutively stable under $z \mapsto \bar{z}$. If $z = \lambda^2$, $\bar{z} = (-\bar{\lambda})^2$ and $R_V(-\bar{\lambda}) = (R_V(\lambda))^*$ is true for $\operatorname{Im} \gg 0$. Hence λ is not a pole if and only if $-\bar{\lambda}$ is not a pole.

Stone's Formula

If $E = E(P)$ is the spectral measure of some self-adjoint operator P , then

$$\frac{1}{2} [E((a, b)) + E([a, b])] = \int_a^b (P - (t + i\epsilon))^{-1} - (P - (t - i\epsilon))^{-1} dt$$

In σ_{ess} , $dE_\lambda = \frac{1}{i\pi} (R_V(\lambda) - R_V(-\lambda)) \lambda d\lambda$.

Expression of the Wave Solution

P_V is self-adjoint, so it admits a functional calculus. So $w = \cos(tP_V)w_0 + \frac{\sin(tP_V)}{P_V}w_1$. Write

$$P_V = \sum_{k=1}^K -\mu_k^2 v_k \otimes v_k + \int_0^\infty z dE_z$$

where we observe that the summands are multiplicity 1 (by ODE theory and outgoing conditions), genuinely L^2 bounded and normalized states. We can restrict to the case where $w_0 = 0$ and $w_1 \in H^2$. In this case

$$w = \frac{\sin(tP_V)}{P_V} w_1 = \sum_{k=1}^K \frac{\sin(t\mu_k)}{\mu_k} (w_1, v_k) v_k + \int_0^\infty \frac{\sin(t\lambda)}{\lambda} dE_\lambda(w_1)$$

Then, applying Stone's Formula,

$$\begin{aligned} \int_0^\infty \frac{\sin(t\lambda)}{\lambda} dE_\lambda(w_1) &= \frac{1}{i\pi} \int_0^\infty \frac{e^{it\lambda} - e^{-it\lambda}}{2i} (R_V(\lambda) - R_V(-\lambda)) w_1 d\lambda \\ &= \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi} \int_{\mathbb{R} \setminus (-\epsilon, \epsilon)} e^{-it\lambda} (R_V(\lambda) - R_V(-\lambda)) w_1 d\lambda \end{aligned}$$

We write $\int_{\mathbb{R} \setminus (-\epsilon, \epsilon)} = \int_{\Sigma_\epsilon} + \int_{\sigma_\epsilon}$ and compute

IMAGE 4

$$\int_{\Sigma_\epsilon} = \frac{1}{2\pi} \lim_{\epsilon \rightarrow 0} \int_{\sigma_\epsilon} e^{it\lambda} \frac{2\pi_0}{\lambda} w_1 d\lambda = -i \frac{(w_1, v_0) v_0}{\pi_0 w_1}$$

Then define $\Gamma = \{\lambda - i(A + \epsilon + \delta \log(1 + |\lambda|)) : \lambda \in \mathbb{R}\}$ and fix $R > 0$.

IMAGE 5

Then $\pi_A(t) := -i \sum_{\lambda_j \in \Omega_A} \text{Res}_{\lambda=\lambda_j} (\rho R_V(\lambda) \rho e^{-i\lambda t})$. By residue theorem, $\rho V(t) \rho = -i m_R(0) \pi_0 \rho + \pi_A(t) + E_{\Gamma_R}(t) + E_{\gamma_R^\pm}(t) + E_{\gamma_R^\infty}(t)$. So

$$E_\gamma(t) := \frac{1}{2\pi} \int_\gamma e^{-it\lambda} \rho (R_V(\lambda) - R_V(-\lambda)) \rho w_1 d\lambda$$

As $R \rightarrow \infty$, $E_{\gamma_R^\pm}(t), E_{\gamma_R^\infty}(t) \rightarrow 0$ and $E_{\Gamma_R}(t) \rightarrow E_\Gamma(t) = O(e^{-at})$.

December 4, 2025

Operators of Interest on Riemannian Manifolds

- Laplace-Beltrami: $\Delta_g \geq 0$ (in the Euclidean case $\Delta_g = -\sum_{k=1}^n \partial_x^2$)
- Shrödinger: $\Delta_g + V$ (where V is the potential)
- Dirac (prototypes of 1st order, elliptic, differential operators)

If M is closed, then Δ_g has one self-adjoint realization with compact resolvent. This implies that $\sigma(\Delta_g) = \{\lambda_0 \leq \lambda_1 \leq \dots \rightarrow \infty\}$.

If $\partial M \neq \emptyset$, then Δ_g^D (Dirichlet boundary conditions) we can define a counting function $N(\lambda) = \#\{j : \lambda_j \leq \lambda\}$.

Trace Class Operators

Let \mathcal{H} separable and Hilbert, and $\mathcal{Q} \in \mathcal{B}(\mathcal{H})$.

We say \mathcal{Q} is trace class if $\exists \{e_n\}_{n \geq 0}$ on orthonormal basis of \mathcal{H} such that

$$\overbrace{\sum_{n=0}^{\infty} \langle |\mathcal{Q}| e_n, e_n \rangle_{\mathcal{H}}}^{\text{call this } \text{tr}(\mathcal{Q})} < \infty, \quad |\mathcal{Q}| = \sqrt{\mathcal{Q}\mathcal{Q}^*}$$

When $\text{tr } \mathcal{Q}$ exists, it does not depend on choice of basis.

Example: Matrices

If $A \in M_n(\mathbb{C})$, $\text{tr}(A) = \sum_j A_{jj} = \sum_{j=1}^n \langle Ae_j, e_j \rangle$.

If A is a self-adjoint, nonnegative, compact operator, then denote $(e_j, \lambda_j)_{j \geq 0}$ a complete eigenset. Then A is traceless if and only if $\sum_{j=0}^{\infty} \lambda_j < \infty$.

If $\sum_{j=0}^{\infty} \lambda_j^2 < \infty$, this is Hilbert-Schmidt; if $\sum_{j=0}^{\infty} \lambda_j^p < \infty$, it is p -Schatten class.

Example

Computing these can extract analytic information.

If $\mathcal{H} = L^2(M, \mu)$ and \mathcal{Q} is self-adjoint, compact, nonnegative, and $(e_j(x), \lambda_j)$ is an eigenset, then

$$\mathcal{Q}f(x) = \int_M \overbrace{\mathcal{K}(x, y)}^{\text{Schwartz kernel}} f(y) d\mu(y) = \sum_{j=0}^{\infty} \lambda_j \left(\overbrace{\int_M f(y) \overline{e_j} d\mu}^{\langle f, e_j \rangle} \right) e_j(x)$$

Recall that $\mathcal{K}(x, y) = \sum_{j=0}^{\infty} \lambda_j e_j(x) \overline{e_j(y)}$, and notice that

$$\sum_{j=0}^{\infty} \lambda_j = \int_M \mathcal{K}(x, y) d\mu(x) = \hbar \mathcal{Q}$$

How to generate trace class operators out of our favorite ones?

Take P self-adjoint, nonnegative with compact resolvent ($\lambda_0 \leq \lambda_1 \leq \dots \rightarrow \infty$). Usually $P = \Delta$.

Example: Heat Trace

$$\begin{cases} \partial_t u + Pu = 0 & \text{(heat equation)} \\ u|_{t=0} = u_0 \end{cases}$$

Set $u(t) = e^{-tP} u_0$ with eigenvalues $e^{-t\lambda_0}, \dots, e^{-t\lambda_n}, \dots$. Then $\text{tr}(e^{-tP}) = \sum_{j=0}^{\infty} e^{-t\lambda_j}$, $t > 0$. This is trace class thanks to Weyl's law ($N(\lambda) \sim C\lambda^{\frac{\dim(M)}{2}}$).

Example: "Half-Wave" Trace

$\frac{1}{i} \partial_t u - Pu = 0$. Any solution solves $\partial_{tt} u = \partial_t (iPu) = iP(\partial_t u) = -P^2 u$ or $\partial_{tt} u + \Delta u = 0$ where $\Delta u = -\sum \partial_{x_j}^2$. Then $u(t) = e^{itP} u_0$ which leads to $\text{tr}(e^{itP}) = \sum_{j=0}^{\infty} e^{it\lambda_j}$ for $t \in \mathbb{R}$.

Counting Functions

Consider $H = -\partial_x^2 + V(x)$ on $[a, b]$.

Then $N(\lambda) \sim \frac{1}{2\pi} \int_a^b \sqrt{\lambda - V(x)} dx$ as $\lambda \rightarrow \infty$. Writing $\sqrt{\lambda - V(x)} = \lambda^{1/2} \left(1 - \frac{V(x)}{\lambda}\right)^{1/2}$, we also see that $N(\lambda) \sim \frac{\lambda^{1/2}}{2\pi} (b-a)$ as λ grows large.

We can consider an "open" analogue $H = -\delta_x^2 + V(x)$ on \mathbb{R} . Taking $R_V(\lambda)$ a meromorphic continuation of the outgoing resolvent for $\lambda \in \mathbb{C}$ and defining $m_R(\lambda) = \text{rank} \int_{\lambda} R_V(\xi) d\xi$. By Proposition 1.9 in the Tang-Zworski notes,

$m_R(\lambda) = \frac{1}{2\pi i} \oint_{\lambda} \frac{\hat{X}'(\xi)}{\hat{X}(\xi)} d\xi$ which is the order of vanishing of \hat{X} at λ .

We have a counting function

$$N(r) = \sum \{m_R(\lambda) : |\lambda| \leq r\} \underset{r \rightarrow \infty}{=} \frac{2|\text{chsupp } V|}{\pi} r(1 + O(1))$$

where $\text{chsupp } V$ is the convex hull of the support of V .

Titchmarsh's Theorem

If $f \in L^1_{\text{comp}}(\mathbb{R})$, then $|\text{chsupp } f| = b - a$. Tang-Zworski upgrades this to $f \in \mathcal{C}^l(\mathbb{R})$.
Then $N_{\hat{f}}(r) := \#\{\text{zeros of } \hat{f} \text{ in } D_R(0) \text{ with multiplicity}\} = \frac{b-a}{\pi}(r + O(r))$.

Poisson Summation Formula

Let $\mathcal{H} = L^2(S^1_T) = \{f : \mathbb{R} \rightarrow \mathbb{C} : T\text{-periodic, } \int_0^T |f(x)|^2 dx < \infty\}$ and $P = D_x = \frac{1}{i}\partial_x$ with eigen decomposition $e_j(x) = \frac{1}{\sqrt{T}}e^{ijwx}$, $w = \frac{2\pi}{T}$, $\lambda_j = jw$ for $j \in \mathbb{Z}$. Then $e^{-itP}f(x) = f(x-t)$.
For any $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\overbrace{\sum_{j \in \mathbb{Z}} \varphi(jw)}^{\text{tr}(\varphi(P))} = T \sum_{k \in \mathbb{Z}} \check{\varphi}(kT)$$

where $\check{\varphi}(\xi) = \frac{1}{2\pi} \int e^{ix\xi} \varphi(x) dx$. Then the left hand side is the spectral story while the right is the Schwarz kernel side.
We can compute

$$\begin{aligned} \varphi(P)f(x) &= \left[\int_{-\infty}^{\infty} e^{-itP} \check{\varphi}(t) dt \right] f(x) \\ &= \int_{-\infty}^{\infty} f(x-t) \check{\varphi}(t) dt \\ &= \sum_{k \in \mathbb{Z}} \int_{kT}^{(k+1)T} f(x-t) \check{\varphi}(t) dt \\ &= \int_0^T f(x-t) \overbrace{\sum_{k \in \mathbb{Z}} \check{\varphi}(t+kT)}^{:=\tilde{\varphi}(t)} dt \\ &= \int_0^T f(s) \overbrace{\tilde{\varphi}(x-s)}^{\mathcal{K}(x,s)} ds \end{aligned}$$

Therefore

$$\overbrace{\sum_{j \in \mathbb{Z}} \varphi(jw)}^{\text{tr}(\varphi(P))} = \int_0^T \mathcal{K}(x, x) dx = T\tilde{\varphi}(0)$$

At the distributional level,

$$\sum_{j \in \mathbb{Z}} \delta_{jw} = \frac{T}{2\pi} \sum_{k \in \mathbb{Z}} e^{ikTx}$$

where δ_{jw} is the Dirac comb.

In 1975 Duistermaat and Guilleman generalized this to manifolds in the study of singular supports and the wave trace.

An Open Analogue

If $V \in L^\infty_{\text{comp}}(\mathbb{R}, \mathbb{R})$, $P_V = -\partial_x^2 + V$, $P_0 = -\partial_x^2$, we have for $f \in \mathcal{S}(\mathbb{R})$ that $f(P_V) - f(P_0)$ is trace class (!) and

$$\text{tr}(f(P_V) - f(P_0)) = \int_0^\infty f(\lambda^2) \frac{\text{tr}(S(\lambda)^{-1} \partial_\lambda S(\lambda))}{2\pi i} d\lambda + \sum_{k=1}^K f(E_k) + \frac{1}{2}(m_R(0) - 1)f(0)$$

where $S(\lambda)$ is the scattering matrix $S(\lambda) = \begin{bmatrix} T(\lambda) & R_-(\lambda) \\ R_+(\lambda) & T(\lambda) \end{bmatrix}$ and we have the Breit-Wigner approximation for $\frac{\text{tr}(S(\lambda)^{-1} \partial_\lambda S(\lambda))}{2\pi i}$ of

$$-\frac{1}{\pi} |\text{chsupp } V| - \frac{1}{\pi} \sum_j \frac{\text{Im } \lambda_j}{|\lambda - \lambda_j|^2}$$

with j the resonances quantity.

Theorem (DZ, 2.21)

Let $f_t(s) \cos(t + \sqrt{s})$. Then for any $\varphi \in C_C^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} \varphi(t) (\cos(t\sqrt{P_V}) - \cos(t\sqrt{P_0})) dt$$

is trace class and

$$2 \text{tr}((\cos(t\sqrt{P_V}) - \cos(t\sqrt{P_0})) = \text{p.v.} \sum_{\lambda \text{ resonances}} m_R(\lambda) e^{-i\lambda|t|} - 2 |\text{chsupp } V| \delta_0 - 1$$

where

$$\left\langle \text{p.v.} \sum_j e^{-i\lambda_j|t|}, \varphi \right\rangle := \lim_{R \rightarrow \infty} \sum_{|\lambda_j| \leq R} \int_{\mathbb{R}} \varphi(t) e^{-i\lambda_j|t|} dt$$

December 11, 2025

Inversion of the Fourier Transform via Riemann-Hilbert Problems

For $q, \frac{dq}{dx} \in L^1(\mathbb{R})$, $\hat{q}(k) := \int_{\mathbb{R}} e^{-ikx} q(x) dx$, we want to prove that $q(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx} \hat{q}(k) dk$. Consider the differential equation

$$\begin{cases} \frac{d\mu}{dx} - ik\mu = q(x) \\ \mu \rightarrow 0, & |x| \rightarrow \infty \end{cases}$$

By integrating factors, $\frac{d}{dx}(e^{-ikx} \mu(x)) = e^{-ikx} q(x)$. So if $\mu(-\infty) = 0$, we get

$$\begin{aligned} e^{-ikx} \mu(x) &= \int_{-\infty}^x e^{-iky} q(y) dy \\ \mu_+(x, k) &= \int_{-\infty}^x e^{ik(x-y)} q(y) dy \end{aligned}$$

IMAGE 1

If instead $\mu(+\infty) = 0$,

$$\mu_-(x, k) = - \int_x^\infty e^{ik(x-y)} q(y) dy$$

It follows that for $k \in \mathbb{R}$

$$\hat{q}(x) = \lim_{x \rightarrow \infty} e^{-ikx} \mu_+(x, k) = \lim_{x \rightarrow -\infty} e^{-ikx} \mu_-(x, k)$$

For $\text{Im}(k) \geq 0$, write $k = \text{Im}(k) + i \text{Im}(k) = k_R + i k_I$ and notice

$$e^{ik(x-y)} = e^{ik_R(x-y)} e^{-k_I(x-y)}$$

where $e^{-k_I(x-y)} \leq 1$ if $x - y \geq 0$ and $k_I \geq 0$. It follows that $\mu_+(x, k)$ makes sense on \mathbb{C}_+ as a holomorphic function on $\{k_I > 0\}$ with continuous limits at $k \in \mathbb{R}$. Likewise, $\mu_-(x, k)$ is the same on \mathbb{C}_- and $\{k_I < 0\}$ respectively.

Notice also that $\hat{q}(k) e^{ikx} = \mu_+(x, k) - \mu_-(x, k)$, $k \in \mathbb{R}$. With x frozen, $\mu(x, k) := \mu_{\pm}(x, k)$ on \mathbb{C}_{\pm}° satisfies

- μ is sectionally analytic on \mathbb{C} with left and right limits as $k \rightarrow \mathbb{R}$.
- $\mu_+(x, k) - \mu_-(x, k) = \hat{q}(k) e^{ikx} := \phi_x(k)$ (prescribed jump condition)
- $\lim_{|k| \rightarrow \infty} \mu_{\pm}(x, k) = 0$

To see the third point,

$$k\mu_+(x, k) = \int_{-\infty}^x \overbrace{ke^{ik(x-y)}}^{\frac{d}{dy} e^{ik(x-y)}} q(y) dy = \overbrace{\left[q(y) e^{ik(x-y)} \right]_{-\infty}^x}^{iq(x)} - \overbrace{\int_{-\infty}^x e^{ik(x-y)} q'(y) dy}^{\text{bounded on } \mathbb{C}}$$

So $\mu_+(x, k) \leq \frac{iq(x)+C}{k}$. We find that the solution to the Riemann-Hilbert Problem is unique and given by

$$\mu(k) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\phi(\ell)}{\ell - k} d\ell = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\hat{\mu}(\ell) e^{i\ell x}}{\ell - k} d\ell, \quad k \in \mathbb{C} \setminus \mathbb{R}$$

Then

$$k\mu(k) = \frac{1}{2\pi i} \int_{\mathbb{R}} \hat{q}(\ell) e^{i\ell x} \frac{1}{\frac{\ell}{k} - 1} d\ell \rightarrow -\frac{1}{2\pi i} \int_{\mathbb{R}} \hat{q}(\ell) e^{i\ell x} d\ell$$

Inverse Scattering

$Lu = -\frac{d^2 u}{dx^2} - V(x)u$ with $V \in C_C^{\infty}(\mathbb{R})$, $\sigma(L) = \sigma_{\text{ess}}(L) = [0 + \infty)$, and $\lambda = s^2$.

Define four solutions to $Lu = s^2 u$ for $s \in \mathbb{R}$,

$$\begin{array}{llll} \psi_p(x; s) = e^{isx}, & x \gg 0 & \begin{array}{c} * \\ \rightarrow \leftarrow \end{array} & \begin{array}{c} * \\ | \\ V \\ | \\ \begin{array}{c} 1 \\ \rightarrow \leftarrow \end{array} \end{array} \\ \phi_p(x; s) = e^{isx}, & x \ll 0 & \begin{array}{c} 1 \\ \rightarrow \leftarrow \end{array} & \begin{array}{c} * \\ | \\ V \\ | \\ \begin{array}{c} * \\ \rightarrow \leftarrow \end{array} \end{array} \\ \psi_m(x; s) = e^{-isx}, & x \gg 0 & \begin{array}{c} * \\ \rightarrow \leftarrow \end{array} & \begin{array}{c} * \\ | \\ V \\ | \\ \begin{array}{c} 0 \\ \rightarrow \leftarrow \end{array} \end{array} \\ \phi_m(x; s) = e^{-isx}, & x \gg 0 & \begin{array}{c} 0 \\ \rightarrow \leftarrow \end{array} & \begin{array}{c} * \\ | \\ V \\ | \\ \begin{array}{c} * \\ \rightarrow \leftarrow \end{array} \end{array} \end{array}$$

Since they are four solutions of a second order ODE, they must be related.

$$\begin{bmatrix} \psi_p(x; s) & \psi_p'(x; s) \\ \psi_m(x; s) & \psi_m'(x; s) \end{bmatrix} = S(s) \begin{bmatrix} \phi_p(x; s) & \phi_p'(x; s) \\ \phi_m(x; s) & \phi_m'(x; s) \end{bmatrix}$$

where

$$\begin{aligned} S(s) &= \begin{bmatrix} \psi_p(x; s) & \psi_p'(x; s) \\ \psi_m(x; s) & \psi_m'(x; s) \end{bmatrix} \frac{1}{W(\phi_p(x; s), \phi_m(x; s))} \begin{bmatrix} \phi_m'(x; s) & -\phi_p'(x; s) \\ -\phi_m(x; s) & \phi_p(x; s) \end{bmatrix} \\ &= -\frac{1}{2is} \begin{bmatrix} W(\psi_p(x; s), \phi_m(x; s)) & W(\phi_p(x; s), \psi_p(x; s)) \\ W(\psi_m(x; s), \phi_m(x; s)) & W(\phi_p(x; s), \psi_m(x; s)) \end{bmatrix} \\ &= \begin{bmatrix} A(s) & B(s) \\ b(s) & a(s) \end{bmatrix} \end{aligned}$$

Note also that $\psi_p = \overline{\psi_m}$, $\phi_p = \overline{\phi_m}$, so $a(s) = \overline{A(s)}$ and $b(s) = \overline{B(s)}$. We have a reflection coefficient $\rho(s) := \frac{b(s)}{a(s)}$. We observe that ψ_p and ϕ_m can be analytically continued to \mathbb{C}_- while ϕ_p and ψ_m extend to \mathbb{C}_+ .

• Example

$$\overbrace{\phi_m(x; z) e^{izx}}^{\Phi_1^-(x; z)} = 1 - \frac{1}{2iz} \int_{-\infty}^x (1 - e^{2iz(x-\tau)}) v(z) \overbrace{\phi_m(\tau; z) e^{iz\tau}}^{\Phi_1^-(\tau; z)} d\tau$$

which is still solvable for $z \in \mathbb{C}_+$.

So $[\psi_m \ \phi_p] = [\phi_m \ \psi_p] \tilde{G}(s)$ where

$$\tilde{G}(s) = \begin{bmatrix} a(s) - \frac{B(s)b(s)}{A(s)} & -\frac{B(s)}{A(s)} \\ \frac{b(s)}{A(s)} & \frac{1}{A(s)} \end{bmatrix}$$

but we have a problem: $\tilde{G}(s)$ has monodromy (i.e. $\det \tilde{G}(s)$ has nonzero winding number). We renormalize

$$\begin{aligned} \begin{bmatrix} \psi_m & \phi_p \end{bmatrix} \begin{bmatrix} e^{izx} & 0 \\ 0 & e^{-izx} \end{bmatrix} &= \begin{bmatrix} 1 & 1 \end{bmatrix} + \mathcal{O}\left(\frac{1}{z}\right), \quad |z| \rightarrow \infty \\ \begin{bmatrix} \phi_m & \psi_p \end{bmatrix} \begin{bmatrix} e^{izx} & 0 \\ 0 & e^{-izx} \end{bmatrix} &= \begin{bmatrix} 1 & 1 \end{bmatrix} + \mathcal{O}\left(\frac{1}{z}\right), \quad |z| \rightarrow \infty \end{aligned}$$

and instead look at

$$\begin{bmatrix} \frac{\psi_m}{a(s)} & \phi_p \end{bmatrix} = \begin{bmatrix} \phi_m & \frac{\psi_p}{A(s)} \end{bmatrix} G(s)$$

for $s \in \mathbb{R}$ where

$$G(s) = \begin{bmatrix} 1 - \rho(s)\overline{\rho}(s) & -\overline{\rho}(s) \\ \rho(s) & 1 \end{bmatrix}$$

and $\det G(s) = 1$. Now we define

$$\Phi(z) := \begin{cases} \Phi^- & := \begin{bmatrix} \frac{\psi_m}{a} & \phi_p \end{bmatrix} \begin{bmatrix} e^{izx} & 0 \\ 0 & e^{-izx} \end{bmatrix}, & \mathbb{C}_+ \\ \Phi^+ & := \begin{bmatrix} \phi_m & \frac{\psi_p}{A} \end{bmatrix} \begin{bmatrix} e^{izx} & 0 \\ 0 & e^{-izx} \end{bmatrix}, & \mathbb{C}_- \end{cases}$$

Normalizing $\Phi(\infty) = \begin{bmatrix} 1 & 1 \end{bmatrix}$, we have a jump

$$\Phi^+(s) = \Phi^-(s) \overbrace{\begin{bmatrix} e^{-isx} & 0 \\ 0 & e^{isx} \end{bmatrix} G(s) \begin{bmatrix} e^{isx} & 0 \\ 0 & e^{-isx} \end{bmatrix}}^{\det=1}$$

Riemann-Hilbert theory tells us that there exists a unique solution Φ to this problem and gives us a way to compute it

$$\partial_x \Phi^-(z)_1 = e^{-2izx} \int_{-\infty}^x e^{-2iz\tau} V(\tau) \Phi_1^-(\tau; z) d\tau \sim \frac{\overbrace{\Phi_1^-(x; z)}^1 V(x)}{2iz}$$