Analysis II

January 9, 2024

(Real) Analysis

- Calculus
  - Differential
  - Integral (Riemann)
- Functions and Maps
  - Measure Theory
  - (Lebesgue) Integration
- Topology
  - Completeness (as a metric space)
  - Compactness (Bolzano-Weierstrass theorem [real]) (Arzela-Ascoli)
  - Paracompactness / Metrizable / Baire Category Theorem
  - Algebraic / Combinatoric (continuous maps or functions)

Definition: Cardinality

For sets A, B, $\mathrm{Card}(A) = \mathrm{Card}(B)$ if there exists a one-to-one correspondence $q : A \leftrightarrow B$.
Counting, labelling, indexing, etc.
$\mathrm{Card}(A) \leq \mathrm{Card}(B)$ if $A \subset B$ or there exists a one-to-one mapping $A \to B$.

Definition: Countable

If $A \hookrightarrow \mathbb{N}$, then A is countable.

Theorem

The countable union of countable sets is countable.

Proof

Let $A_i = \{a_j\}_{j=1}^{\infty}$, $i = 1, 2, \ldots$.

$$\begin{matrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ \vdots & & & \\ a_{k1} & a_{k2} & a_{k3} & \cdots \end{matrix}$$

Index by diagonalization.

Theorem

The cartesian product of countable sets is countable.

Proof

$X \times Y = \{(x_i, y_j \mid x_i \in X, y_j \in Y\}$

$$\begin{matrix} (x_1, y_1) & (x_1, y_2) & (x_1, y_3) & \cdots \\ (x_2, y_1) & (x_2, y_2) & (x_2, y_3) & \cdots \\ \vdots & & & \\ (x_k, y_1) & (x_k, y_2) & (x_k, y_3) & \cdots \end{matrix}$$

Theorem

$\mathrm{Card}\left(2^X\right) > \mathrm{Card}(X)$, where $2^X = \{A \subset X\}$ is the power set of X.

Proof

For all $x \in X$, $\{x\} \subset 2^X$, so $\mathrm{Card}(X) \leq \mathrm{Card}\left(2^X\right)$.
Assume, for sake of contradiction, that $\mathrm{Card}(X) = \mathrm{Card}\left(2^X\right)$.
Then, by definition, there exists a one-to-one correspondence $\phi : X \leftrightarrow 2^X$.
Set $A = \{x \subset X \mid x \notin \phi(x)\}$, and let $a = \phi^{-1}(A)$ (i.e. $A = \phi(a)$).
If $a \in A$, then $a \notin A \subset \phi(a)$; but if $a \notin A$, then $a \in A$, a contradiction.

Theorem

$\mathrm{Card}(\mathbb{R}) = \mathrm{Card}\left(2^{\mathbb{N}}\right)$.

Topology of the Real Line

Completeness (as a metric space)

$d(a, b) = |a - b|, \quad \forall a, b \in \mathbb{R}$.

1. $x_i \to x$ if $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ such that $|x_i - x| < \varepsilon$, $\forall i \geq n$.
2. $\{x_i\}$ is Cauchy if $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ such that $|x_i - x_j| < \varepsilon$, $\forall i, j \geq n$.

Definition: Open Inteval

(a, b) is an open set on the real line.
There exist interior points for any subset A of real numbers.
$\forall x \in A$, x is interior if $\exists (a, b)$ such that (1) $x \in (a, b)$ and (2) $(a, b) \subset A$.

- Theorem
  The union of open sets is open.
  The intersection of finitely many open sets is open.
  $\varnothing$ and $\mathbb{R}$ are open.

Definition: Limit Point

A limit point $x \in \mathbb{R}$ of a subset A is a limit point in A if for every open neighborhood U of X, $(U \setminus \{x\}) \cap A \neq \varnothing$.

Definition: Closed

A is closed if A contains all of its limit points.

- Theorem
  A is closed if and only if $A^c = \mathbb{R} \setminus A$ is open.

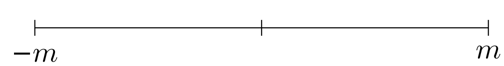
  - Proof
    A closed $\implies$ A^c open.
    Otherwise, $\exists x \in A^c$ such that for every neighborhood U of X, $(U \setminus \{x\}) \cap A = \varnothing$ which would make it a limit point of A not in A. By assumption, A contains all its limit points so this is a contradiction.
    A^c open $\implies$ A closed.
    For any x a limit point of A, assume otherwise that $x \in A^c$.
    Then there exists some neighborhood U of x such that $U \subset A^c$ (since A^c is open).
    It follows that $(U \setminus \{x\}) \cap A = \varnothing$ and x is not a limit point of A, which is a contradiction.

Definition: Sequential Compactness

A is compact if $\forall \{x_i\}$, $x_i \in A$ there exists a convergent subsequence $\{x_{i_k}\}$ and $x_{i_k} \to x \in A$.

- Theorem: Bolzano-Weierstrass
  For $A \subseteq \mathbb{R}$, A is compact if and only if A is closed and bounded.

  - Proof
    A compact $\implies$ A closed and bounded.
    Assume that A is not bounded from abvove.
    Then there exists a sequence $\{x_i\}$, $x_i \in A$ where $x_{i+1} > x_i + 1$ and $\{x_i\}$ has no convergent subsequences.
    Then compactness implies closedness.
    A closed and bounded $\implies$ A (sequentially) compact.
    Let any $\{x_i\}$, $x_i \in A$.
    Claim: $\forall \{x_i\}$ of reals, if there exists $m \in \mathbb{R}$ such that $|x_i| \leq m$, $\forall m$ then there is some convergent subsequence.

    Divide and conquer: dividing the interval in half necessitates that at least one half contains infinitely many points. Repeat indefinitely.

- Theorem: Heine-Borel)
  $A \subseteq \mathbb{R}$ is (sequentially) compact if and only if any open cover has a finite subcover.

  - Proof
    Heine-Borel Property $\implies$ closed and bounded.
    Assume that A is unbounded, $U_n = (-n, n)$ and $\{U_n\}_{n=1}^{\infty}$ an open cover for $A \subseteq \mathbb{R}$ has no finite subcover.
    Assume A is not closed, then $x \in \dot{A}$ (where $\dot{A}$ is the limit set of A) and $x \notin A$, $U_n \left\{\left(-\infty, x - \frac{1}{n}\right) \cup \left(x + \frac{1}{n}, +\infty\right)\right\}$.
    Then $\{U_n\}$ covers $\mathbb{R} \setminus \{x\} \supset A$ has no finite subcover of A.
    A is bounded and closed $\implies$ A is Heine-Borel

Divide and conquer: using open sets with respect to open covers.

Definition: Cantor Set

$C = \{x \in [0,1] \mid$ the ternary expansion of x has only the digits $\{0,2\}\}$.
Equivalenetly, let $C_0 = [0,1]$, $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$.
Then $C_n = \bigcup_{k=1}^{2^n} C_n^k$ and $C = \bigcap_{n=1}^{\infty} C_n$.
$|C_n| = 2^n \left(\frac{1}{3}\right)^n \to 0$.

Definition: Perfectly Symmetric Sets

Let $\{\xi_n\}$ where $\xi_n \in \left(0, \frac{1}{2}\right)$.
$E_0 = [0,1]$, $E_1 = [0, \xi_1] \cup [1 - \xi_1, 1]$, $E_2 = [0, \xi_1\xi_2] \cup [\xi_1 - \xi_1\xi_2, \xi_1] \cup [1 - \xi_1, 1 - \xi_1 + \xi_1\xi_2] \cup [1 - \xi_1\xi_2, 1]$.
Then the cantor set is given by $\xi_n = \frac{1}{3}$.
$E_n = \bigcup_{k=1}^{2^n} E_n^k$, $|E_n^k| = \xi_1\xi_2 \cdots \xi_n$, and $|E_n| = \sum |E_n^k| = 2^n \xi_1\xi_2 \cdots \xi_n$.
Therefore, $E = \bigcap_{n=1}^{\infty} E_n$ and we define $|E| = \lim_{n\to\infty} |E_n| = \lim_{n\to\infty} \left(2^n \xi_1\xi_2 \cdots \xi_n\right) = \lambda$ where $\lambda \in [0,1)$.
Let

$$2\xi_n = \frac{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n-1}\right)^{n-1}}{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n}\right)^{n}} < 1$$

, then

$$2^n \xi_1 \cdots \xi_n = \frac{1}{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n}\right)^{n}} \to \lambda.$$

Proof

$\lim_{n\to\infty} \left(\left(1 + \frac{x}{n}\right)^{n/x}\right)^x = e^x$, then $\lim_{y\to 0} (1+y)^{1/y} = e$, $\log(1+y)^{1/y} = \frac{\log(1+y)}{y} \underset{y\to 0}{\to} 1$.
Observe that

$$\left(\frac{\log(1+y)}{y}\right)' = \frac{\frac{y}{1+y} - \log(1+y)}{y^2} = \left(1 + \frac{1}{1+y} - \log(1+y)\right)' = \frac{1}{(1+y)^2} - \frac{1}{1+y} = -\frac{y}{(1+y)^2} < 0$$

Theorem

Cantor sets and perfect symmetric sets are closed, perfect, uncountable, and nowhere dense.

January 11, 2024

Last Week

Cardinality.
Topology of the reals.

- Cantor (perfect symmetric sets)

$C_0 = [0,1]$
$C_1 = [0,1/3] \cup [2/3,1]$
$C_2 = [0,1/9] \cup [2/9,3/9] \cup [6/9,7/9] \cup [8/9,1]$
$C_n = \bigcup_{n=1}^{2^n} C_n^k$
$|C_n^k| = \left(\frac{1}{3}\right)^n$
$C = \bigcap_{n=1}^{\infty} C_n$
$|C_n| = 2^n \frac{1}{3^n} = \left(\frac{2}{3}\right)^n \implies |C| = \lim_{n\to\infty} |C_n| = 0$
Closed, no interior points and uncountable.

- Perfect Symmetric Sets

  $\{\xi_k\} \in \left(0, \frac{1}{2}\right)$
  $E_0 = [0,1]$
  $E_1 = [0,\xi_1] \cup [1-\xi_1, 1]$
  $E_2 = [0,\xi_1\xi_2] \cup [\xi_1 - \xi_1\xi_2, \xi_1] \cup [1-\xi_1, 1-\xi_1+\xi_1\xi_2] \cup [1-\xi_1\xi_2, 1]$
  $E_n = \bigcup_{n=1}^{2^n} E_n^k$
  $|E_n^k| \xi_1\xi_2 \cdots \xi_n$
  $|E_n| = 2^n \xi_1\xi_2\cdots\xi_n$
  $2\xi_n = \dfrac{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n-1}\right)^{n-1}}{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n}\right)^{n}} < 1$
  $|E_n| = \dfrac{1}{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n}\right)^{n}}$
  $|E| = \lim_{n\to\infty} |E_n| = \frac{1}{e^{\log\left(\frac{1}{n}\right)}} = \lambda, \quad \lambda \in (0,1)$

Volterra's Function

$$\phi(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

IMAGE HERE - graph of phi(x)

$$\phi'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$f(x) = \begin{cases} 0 & x \in E \\ \phi(x-a) & x \in (a, a+y) \\ -\phi(b-x) & x \in (b-y, b) \\ \phi(y) & x \in (a+y, b-y) \end{cases}, \quad (a,b) \in E^c$$

IMAGE HERE - f interval (a,b)

Propositions

1. $f'(x) = 0$ for $x \in E$.

2. $f'(x)$ discontinuous on E.

3. f' exists on $[0,1]$ and is bounded.

Since $|E| > 0$, $f'(x)$ is not Riemann integrable and, therefore, the fundamental theorem of calculus does not apply.

Lebesgue Outer Measure

$|(a,b)| = b - a$.
Let $A \subseteq \mathbb{R}$, then $m^*(A) = \inf \left\{ \sum_{n=1}^{\infty} I_n \mid A \subseteq \bigcup_{n=1}^{\infty} \right\}$
Question: $m^*(A \cup B) \stackrel{?}{=} m^*(A) + m^*(B)$ for $A \cap B \neq \varnothing$?

Properties

1. $A \subseteq B \implies m^*(A) \leq m^*(B)$.

2. $m^*(\varnothing) = 0$.

3. If I is an interval, then $m^*(I) = |I|$.

4. If $\{A_i\}$ is countable, $m^*\left(\bigcup A_i\right) \leq \sum m^*(A_i)$.

- Proof of 4
  $\forall A_i$, $\exists \{I_n\}$ open intervals such that $\sum_n |I_n| < m^*(A_i) + \frac{\varepsilon}{2^i}$.
  Then $\bigcup_i \bigcup_n I_n^i \supset \bigcup_i A_i$, and $\sum_{n,i} |I_n^i| = \sum_i \left(\sum_n |I_n^i|\right) \leq \sum_i \left(m^*(A_i) + \frac{\varepsilon}{2^i}\right)$.

  - Corollary

    If A is countable, then $m^*(A) = 0$.
    Thus, by contraposition, every interval is uncountable.

Proposition

For $A \subseteq \mathbb{R}$, $\forall \varepsilon > 0$, $\exists U$ open such that $A \subseteq U$ and $m^*(U) \leq m^*(A) + \varepsilon$.

Corollary

There exists G in the intersection of countable open sets such that $m^*(G) = m^*(A)$ and $G \supseteq A$.

Caratheodory Criteria

If $\forall E$, $m^*(E \cap A) + m^*(E \cap A^c) = m^*(E)$, then A is Lebesgue measurable.

- Remark: $m^*(E \cap A) + m^*(E \cap A^c) \leq m^*(E) \leq +\infty$

Propositions

1. If A is measurable, then A^c is measurable.

2. $m^*(A) = 0$, then A is measurable.

3. If $A,\ B$ are measurable, then $A \cup B,\ A \cap B,\ A \setminus B$ are measurable.

4. If $\{A_i\}_{i=1}^k$ are disjoint and measurable, then $m^*\left(\bigcup_{i=1}^k A_i\right) = \sum_{i=1}^k m^*(A_i)$.

- Proof of 3

$$\begin{aligned} m^*(E \cap (A \cup B)) + m^*(E \cap (A \cup B)^c) &= m^*((E \cap A) \cup (E \cap B)) + m^*(E \cap A^c \cap B^c) \\ &= m^*(E \cap A) + m^*((E \cap A^c) \cap B) + m^*((E \cap A^c) + B^c) \\ &\le m^*(E) \end{aligned}$$

Since o$(A \cap B)^C = A^c \cup B^c$, this holds from before; similarly, $A \setminus B = A \cap B^c = A^c \cup B$.
If $A,\ B$ disjoint, then

$$\begin{aligned} m^*(A \cup B) &= m^*(E \cap A) + m^*(E \cap A^c) \\ &= m^*(A) + m^*(B) \end{aligned}$$

Theorem

If $\{A_i\}$ is a countable collection of disjoint and measurable sets, then

1. $\bigcup_i A_i$ is measurable.

2. $m^*\left(\bigcup_i A_i\right) = \sum_i m^*(A_i)$.

Proof of 1

Want to show:

$$m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) + m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right) \le m^*(E)$$

By assumption, since the measure of E is finite, $m^*\left(E \cap \bigcup_{i=1}^{\infty} A_i\right) < +\infty$.
Claim: $\forall \varepsilon > 0,\ \exists k$ such that
Therefore $m^*\left(E \cap \bigcup_{i=1}^{k} A_i\right) \ge m^*\left(E \cap \bigcup_{i=1}^{\infty} A_i\right) - \varepsilon$.

$$m^*(E) \le m^*\left(E \cap \bigcup_{i=1}^{k} A_i\right) + \varepsilon + m^*\left(E \cap \left(\bigcup_{i=1}^{k} A_i\right)^c\right) \le m^*(E) + \varepsilon$$

Proof of 2

We have shown $m^*\left(\bigcup_i A_i\right) \le \sum_{i=1}^{\infty} m^*(A_i)$.
Assume $m^*\left(\bigcup_i A_i\right) < +\infty$, then

$$\sum_{i=1}^{k} m^*(A_i) = m^*\left(\bigcup_{i=1}^{k} A_i\right) \le m^*\left(\bigcup_{i}^{\infty} A_i\right) \implies \sum_{i=1}^{\infty} m^*(A_i) \le m^*\left(\bigcup_{i}^{\infty} A_i\right)$$

January 16, 2024

Office Hours Tuesday / Thursday 10 AM - 11:30 AM
A note on notation: Latin characters are to be understood as countable indecies; greek as possible uncountable.

Lebesgue Outer Measure

$A \subset \mathbb{R}$
$m^*(A) = \inf\left\{\sum_{i=1}^{\infty} |I_i| \mid \bigcup_{i=1}^{\infty} I_i \supset A,\ I_i \text{ open intervals}\right\}$

Properties

1. $A \subset B \implies m^*(A) \le m^*(B)$.
2. $m^*(\varnothing) = 0$.
3. $m^*(I) = |I|$ for I an interval.
4. Countable Subadditivity: $\{A_i\}_{i=1}^{\infty} \implies m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} m(A_i)$.
5. $\forall A \subset \mathbb{R},\ \forall \varepsilon > 0,\ \exists$ open neighborhood $U \supseteq A$ such that $m^*(U) \le m^*(A) + \varepsilon$.
6. $\exists G \in \bigcap_{n=1}^{\infty} U_n$, U_n open, $U_n \supseteq A \implies G \supseteq A$, such that $m^*(G) = m^*(A)$.

Measurable (Caratheodory Criterion)

$\forall A \subseteq \mathbb{R}$ is Lebesgue measurable if

$$m^*(A) = m^*(E \cap A) + m^*(E \cap A^c)$$

Essentially, $m^*(E \cap A) + m^*(E \cap A^c) \le m^*(E) \le +\infty$.

- Propositions
  1. A measurable $\implies A^c$ measurable.
  2. $m^*(A) = 0 \implies A$ measurable.
  3. $\{A_i\}_{i=1}^{\infty}$ countable with A_i measurable, then
     (a) $\bigcap_{i=1}^{\infty} A_i$ are measurable.
     (b) Moreover, $A_i \cap A_j = \varnothing \implies m^*\left(\bigcup_{i=1}^{\infty}(A_i) = \sum_{i=1}^{\infty} m^* A_i\right)$.
     (c) A, B measurable $\implies A \cup B$, $A \cap B$, $A \setminus B$ measurable.
     (d) $A \cap B = \varnothing \implies m^*(A \cup B) = m^*(A) + m^*(B)$.
     (e) $\{A_i\}_i^{\infty}$ with A_i measurable, then $\bigcup_{i=1}^{\infty} A_i$ is measurable and $A_i \cap A_j \varnothing \implies m^*\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m^*(A_i)$.

  – Proof of e $\forall E \subset \mathbb{R}$, $m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) + m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right)$.

Claim: $m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) = \sum_{i=1}^{\infty} m^*(E \cap A_I)$ for $A_i \cap A_j = \varnothing$.
Then, $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$,

$$m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) = \sum_{i=1}^{\infty} m^*(E \cap A_i) \le \sum_{i=1}^{n} m^*(E \cap A_i) + \varepsilon$$
$$\implies m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) + m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right) \le m^*\left(E \cap \left(\bigcup_{i=1}^{n} A_i\right)\right) + m^*\left(E \cap \left(\bigcup_{i=1}^{n} A_i\right)^c\right) + \varepsilon \le m^*(E) + \varepsilon$$
$$\implies \bigcup_{i=1}^{\infty} A_i \text{ measurable}$$

Proof of Claim:
Step 1: A, B measurable and $A \cap B = \varnothing$. Since A is measurable,

$$\begin{aligned} m^*(E \cap (A \cup B)) &= m^*((E \cap (A \cup B)) \cap A) + m^*((E \cap (A \cup B)) \cap A^c) \\ &= m^*(E \cap A) + m^*(E \cap A^c) \end{aligned}$$

For $\{A_i\}_{i=1}^{\infty}$, $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} A_i'$ with $A_1 = A_1'$ and $A_i' = A_i \setminus \bigcup_{k=1}^{i-1} A_k$, $\forall i \ge 2$.
Therefore $A_i' \cap A_j' = \varnothing$ and A_i' is measurable.

$$m^*\left(\bigcup_{i=1}^{n} A_i\right) \le m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} m^*(A_i)$$
$$m^*\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} m^*(A_i) \le m^*\left(\bigcup_{k=1}^{\infty} A_k\right) < +\infty \implies \sum_{i=1}^{\infty} m^*(A_i) \le m^*\left(\bigcup_{k=1}^{\infty} A_k\right) \le \sum_{i=1}^{\infty} m^*(A_i)$$

Sigma Algebra and Borel Sets

Definition: Sigma Algebra

Let $S \subset 2^X$ for some set X. Then S is said to be a σ-algebra if

1. $\varnothing \in S$.
2. $A^c \in S$ if A^c.
3. $\bigcup_{i=1}^{\infty} A_i \in S$ if $A_i \in S$.
   - Equivalently, $\bigcap_{i=1}^{\infty} A_i \in S$ if $A_i \in S$.

Theorem:

The collection $\mathcal{L}$ of all Lebesgue measurable sets is a σ-algebra.

Definition: Borel Set

Let B be the σ-algebra generated by open sets of reals (i.e. the smallet σ-algebra containing all open sets of reals).
Then $b \in B$ is called a Borel set.

Remark

B is generated by $\{(a, +\infty) \mid a \in \mathbb{R}\}$.

1. $(a, +\infty)^c = (-\infty, a]$.
2. $\bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, +\infty\right) = [a, +\infty)$.
3. $[a, +\infty)^c = (-\infty, a)$.
4. $(-\infty, b) \cap (a, +\infty) = (a, b)$.
5. $(-\infty, b] \cap [a, +\infty) = [a, b]$.

Theorem:

Any Borel set is Lebesgue measurable.

Proof

It suffices to demonstrate that $(a, +\infty)$ is measurable $\forall a \in \mathbb{R}$.
$\forall E \subset \mathbb{R}$, we want to show that $m^*(E \cap (a, +\infty)) + m^*(-\infty, a]) \leq m^*(E)$.
Then, $\forall \varepsilon > 0$, $\exists \mathcal{C} = \{I_i\}$ with I_i open intervals such that $\sum_{I_i \in \mathcal{C}} |I_i| \leq m^*(E) + \varepsilon/2$. Set

$$\begin{aligned}
\mathcal{C}^\ell &= \{I \in \mathcal{C} \mid x < a, \forall x \in I\} \\
\mathcal{C}^r &= \{I \in \mathcal{C} \mid x > a, \forall x \in I\} \\
\mathcal{C}^m &= \{I \in \mathcal{C} \mid a \in I\} = \{I_k\}
\end{aligned}$$

Then A$\mathcal{C} = \mathcal{C}^\ell \cup \mathcal{C}^r \cup \mathcal{C}^m$.
$\forall I_k \in \mathcal{C}^m = \{I_k\}$, $I_k = (c_k, d_k)$ for some $c_k, d_k \in \mathbb{R}$, define

$$\begin{aligned}
I_k^\ell &= \left(c_k, a + \frac{\varepsilon}{2^{k+1}}\right) \\
I_k^r &= (a, d_k)
\end{aligned}$$

Let $\mathcal{C}^m = \{I_k^\ell\} \cup \{I_k^r\} = \overline{\mathcal{C}}^{m\ell} \cup \overline{\mathcal{C}}^{mr}$. Then

$$\begin{aligned}
&\mathcal{C}^\ell \cup \overline{\mathcal{C}}^{m\ell} \text{ covers } E \cap (-\infty, k] \\
&\mathcal{C}^r \cup \overline{\mathcal{C}}^{mr} \text{ covers } E \cap (k, +\infty) \\
&\mathcal{C}^\ell \cup \mathcal{C}^r \cup \mathcal{C}^m \text{ covers } E
\end{aligned}$$

Observe that

$$|I_k^\ell| + |I_k^r| \leq |I_k| + \frac{\varepsilon}{2^{k+1}}$$

Therefore

$$m^*(E \cap (a, +\infty)) \le \sum_{I \in \mathcal{C}^R + \overline{\mathcal{C}}^{mr}} |I|$$
$$m^*(E \cap [-\infty, a]) \le \sum_{I \in \mathcal{C}^\ell + \overline{\mathcal{C}}^{m\ell}} |I|$$

Therefore

$$\begin{aligned} m^*(E \cap (a, +\infty)) + m^*(E \cap (-\infty, a]) &\le \sum_{I \in \mathcal{C}^r \cup \overline{\mathcal{C}}^{mr}} |I| + \sum_{I \in \mathcal{C}^\ell |I| \cup \overline{\mathcal{C}}^{m\ell}} |I| \\ &= \sum_{I \in \mathcal{C}^r} |I| + \sum_{I \in \mathcal{C}^\ell} |I| + \sum_k \left(|I_k^\ell| + |I_k^r| \right) \\ &\le \sum_{I \in \mathcal{C}} |I| + \sum_{k=1}^{+\infty} \frac{\varepsilon}{2^{k+1}} \\ &\le m^*(E) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &\le m^*(E) + \varepsilon \end{aligned}$$

Lebesgue Measurable vs Borel

Theorem

The following statements are equivalent

1. A is measurable.
2. $\forall \varepsilon > 0$, $\exists U$ open, $U \supset A$ such that $m(U \setminus A) < \varepsilon$.
3. $\forall \varepsilon > 0$, $\exists C$ closed, $C \subset A$ such that $m(A \setminus C) < \varepsilon$.
4. $\forall A \in \mathbb{R}$, $\exists \bigcap_{n=1}^{\infty} U_n = F \in B$, U_n open, $U_n \supset A$ such that $F \supset A$ and $m(F \setminus A) = 0$.
5. $\exists \{C_n\}$, C_n closed and $C_n \subset A$ such that $G = \bigcup_{n=1}^{\infty} C_n \subset A$ and $m(A \setminus G) = 0$.

Corollary

Every measurable set is the union of a Borel set and a measure zero set.

Proof 1 Implies 2

Step 1: if $m(A) < \infty$, then for $\varepsilon > 0$, $\exists U$ open and $U \supset A$, then

$$m(U) \le m(A) + \varepsilon \iff m(U \setminus A) = m(U) - m(A) \le \varepsilon$$

Step 2: let $A_n = A \cap (-n, n)$, $n \in \mathbb{N}$.
Then $m(A_n) \le 2n < +\infty$.
For ech A_n, $\exists U_n$ open with $U_n \supset A_n$ and $m(U_n \setminus A_n) < \frac{\varepsilon}{2^n}$
Let $U = \bigcup_{n=1}^{\infty} U_n$ and $A = \bigcup_{n=1}^{\infty} A_n$.
Now verify that

$$m(U \setminus A) = m\left(\bigcup_{n=1}^{\infty} U_n \setminus \bigcup_{n=1}^{\infty} A_n\right) = m\left(\bigcup_{n=1}^{\infty} U_n \setminus A_n\right) \leq \sum_{n=1}^{\infty} m(U_n \setminus A_n) \leq \varepsilon$$

Proof 2 Implies 3

Write

$$A \setminus C = A \cap C^c = C^c \cap A = C^c \setminus A^c$$

Apply (2).

Proof 3 Implies 4

U_n comes from 2.

Proof 4 Implies 5

Follows from 4.

Proof 5 Implies 1

$A = G \cup (A \setminus G) \implies A$ is measurable.

Example: Non-measurable Set

Define $x \sim y$ if $x - y \in \mathbb{Q}$, $\forall x, y \in \mathbb{R}$.
Let $A = \{x \in (0,1) \mid x \text{ is a representative of each class } \mathbb{R}/\sim\} \subset (0,1) \subset \mathbb{R}$.
Claim: A is not Lebesgue measurable.
Let $(-1,1) \supset S = \bigcup_{r \in \mathbb{Q} \cap (0,1)} (A + r) \supset (0,1)$, and observe that $\mathbb{Q} \cap (0,1)$ is countable.
So $(A + r) \cap (A + s) = \varnothing$ for $s \neq r$.
Then $1 < m(S) < 2$, so $m(A) = 0$ and $m(A) > 0$ are both contradictions.