Advanced Analysis

September 25, 2025

Suppose we have some function of the form $-\Delta + q \in \mathbb{L}(H)$ satisfying $R_A(\lambda)(A - \lambda I)^{-1}$ bounded on $\mathrm{Im}(\lambda) > 0$ and not surjective for $\mathrm{Im}(\lambda) = 0$.

IMAGE 1

Waves: solutions to $\partial_{tt} u + Au = 0$ on $\mathbb{R}^n$.
Mathematical Tools

- Spectral theory of unbounded operators
- Complex analysis
- Functional analysis
- Microlocal analysis
- Semiclassical analysis

Classical Resonances in ODEs

IMAGE 2

A harmonic oscillator assuming no friction.
We have an acceleration force, $m\ddot{x}(t) = -kx(t)$ which gives $\ddot{x} + \omega_0^2 x = 0$ with $w_0 = \sqrt{\frac{k}{m}}$ and has solution $x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$.
With forcing, i.e. $m\ddot{x}(t) = -kx(t) + A\sin(\omega t)$, we have $\ddot{x} + \omega_0^2 A' \sin(\omega t)$.
If $|\omega| \neq |\omega_0|$, then $x(t) \sim \mathrm{trig}\left(\left(\frac{\omega - \omega_0}{2}\right)t\right)\left(\left(\frac{\omega - \omega_0}{2}\right)t\right)$ the low and high frequencies respectively.

IMAGE 3

Beats (non-amplified)
If isntead $|\omega| = |\omega_0|$, then $x(t) \propto \mathrm{trig}(\omega t)t$.

IMAGE 4

In general, $\dot{x} + Ax = 0$ for $x \in \mathbb{R}^n$, $x(t) = \exp(-tA) + x(0)$.
In the case where A is skew-adjoint, $i.e.$ $\mathrm{sp}(A) \subseteq i\mathbb{R}$, $(x, Ax) = 0$ $\forall x \in \mathbb{R}^n$, then

$$\frac{d}{dt}(x, x) = (\dot{x}, x) + (x, \dot{x}) = (-Ax, x) - (x, Ax) = 0$$

Which implies that $||x(t)||$ is constant and the dynamics are norm perserving.
To generate resonant solutions, if (iw, v) is an eigenpair of A ($\omega \in \mathbb{R}$), consider $\dot{x} + Ax = e^{-i\omega t} v$. As an ansatz, we look for a solution of the form $x(t) = a(t)v$ and the equation becomes $(\dot{a}(t) + i\omega a)v = e^{-i\omega t} v$. Then

$$e^{-i\omega t} \frac{d}{dt}(e^{i\omega t} a) = e^{-i\omega t}$$
$$\frac{d}{dt}(e^{i\omega t} a) = 1$$
$$a(t) = te^{-i\omega t}.$$

Resonances in PDEs

Consider one-dimensional waves on $[0, L]$, $L > 0$.

$$\begin{cases} \partial_{tt} u + \partial_{xx} u = 0 & \\ u|_{t=0} = f & x \in [0, L] \\ \partial_t u|_{t=0} = g & x \in [0, L] \\ u(0, t) = u(L, t) = 0 & \forall t \geq 0 \end{cases}$$

We want to think about this as $\partial_{tt} u = Au = 0$ where A is the Dirichlet Laplacian $Au = -\partial_{xx} u$ with Dirichlet boundary conditions. We then want to find the spectral decomposition of A, $Au - \lambda u = 0 = -\partial_x^2 u - \lambda u$.

$$\begin{aligned} \lambda = 0. &\quad u(x) = A + Bx \implies A = B = 0 \\ \lambda = -p^2. &\quad u(x) = Ae^{px} + be^{-px} \implies A = B = 0 \\ \lambda = p^2. &\quad u(x) = A\cos(px) + B\sin(px) \implies 0 = u(0) = A \quad 0 = u(L) = B\sin(pl) \implies p = k\pi,\ k \in \mathbb{N} \end{aligned}$$

Therefore there are infinitely many eigenpairs $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $\phi_n(x) = \sin\left(\frac{k\pi x}{L}\right)$.

IMAGE 5

The family $\{\phi_n, n \in \mathbb{N}\}$ is dense in $L^2([0, L])$ where the unbounded operator $(-\partial_x^2)$ with Dirichlet boundary conditions is self-adjoint.

Other Prototypes

(of unbounded self-adjoint operators with discrete spectrum)

- Laplace-Beltrami operators on compact manifolds without boundary.

IMAGE 6

- On compact domains with boundary there is the Laplacian with Dirichlet boundary conditions.

The (Quantum) Harmonic Oscillator

$H = -\frac{d^2}{dx^2} + x^2$ on $\mathbb{R}$, on $L^2(\mathbb{R})$ with $(f, g) = \int_{\mathbb{R}} f(x)\overline{g(x)}\, dx$.
H acts on the Schwarz space $\mathcal{S}(\mathbb{R}) := \left\{ f \in C^\infty(\mathbb{R}),\ \forall k, \ell \geq 0,\ \sup_{x\in\mathbb{R}} \left| x^k \left(\frac{d}{dx}\right)^\ell f(x) \right| < \infty \right\}$.

- The action of $H : \mathcal{S}(\mathbb{R})$ is continuous.
- H is L^2-symmetric: $\int_{\mathbb{R}} -f''\overline{g} + x^2 f\overline{g}\, dx = (Hf, g) = (f, Hg) = \int_{\mathbb{R}} -\overline{g}'' f + x^2 f\overline{g}\, dx$ (integrating by parts).

We seek eigenvalues $Hu = \lambda u$. If (u, λ) and (v, μ) are eigenpairs, then

$$0 = (Hu, v) - (u, Hv) = (\lambda u, v) - (u, \mu v) = (\lambda - \overline{\mu})(u, v)$$

Where if the difference is nonzero then $(u, v) = 0$.
We can write $H = L^+L^- + I$ where $L^+ = -\frac{d}{dx} + x$ and $L^- = \frac{d}{dx} + x$ and also $[H, L^+] = 2L^+$ and $[H, L^-] = -2L^-$.
Note that H is a non-negative operators

$$(Hf, f) = \int_{\mathbb{R}} ((f')^2 + x^2 f^2)\, dx > 0$$

for $f \neq 0$ and $f \in \mathcal{S}(\mathbb{R})$. Thus $\mathrm{sp}(H) \subseteq (0, \infty)$. If $Hv = \lambda v$, then $H(L^+ v) = [H, L^+]v + L^+(Hv) = (\lambda + 2)L^+ v$. Similarly $H(L^- v) = (\lambda - 2)L^- v$.
Now we want to solve $L^- \phi_0 = 0$. $\frac{d}{dx}\phi_0 + x\phi_0 = 0$ tells us that $\phi_0(x) = \frac{1}{\sqrt{\pi}} e^{-x^2/2}$ (L^2-normalized). Therefore $H\phi_0 = \phi_0$ and the we have an eigenvalues of one. So we may construct $\phi_n = \frac{(L^+)^n \phi_0}{||(L^+)^n \phi_0||}$ which gives an eigenvector of H with eigenvalues $2n + 1$. Note that $||(L^+)^n \phi_0|| = \sqrt{2^n n!}$.
Fact: $\phi_n = p_n(x) e^{-x^2/2}$ where p_n is the Hermite polynomial of degree n.

$$\delta_{nq} = (\phi_n, \phi_q) = \int_{\mathbb{R}} p_n(x) p_q(x) e^{-x^2}\, dx$$

Theorem

$\{\phi_n\}_{n \geq 0}$ is dense in $L^2(\mathbb{R})$ (if $\int_{\mathbb{R}} g\phi_n\, dx = 0$ for all n, then $g = 0$).

Proof (Sketch)

For $g \in L^2$, $\xi \in \mathbb{R}$, $F_g(\xi) = \int_{\mathbb{R}} e^{ix\xi} g(x)\phi_0(x)\, dx = \widehat{g\phi_0}(\xi)$. We observe that

- F_g is real-analytic in ξ.
- $F_g^{(k)}(0) = \int_{\mathbb{R}} (-ix)^k g(x)\phi_0(x)\, dx = 0$ by assumption.

So we have a real-analytic function where all derivatives vanish at a point. So $F_g \equiv 0$, $g\phi_0 = 0$, and $g = 0$.

September 30, 2025

One of the overarching goals is to obtain large time asymptotics of the solution $v(x, t)$ ($x \in \mathbb{R}$, $t > 0$) to

$$\begin{cases} -\partial_{tt} v - P_V v = F(x, t) & \text{on } \mathbb{R}_x \times (0, \infty)_t \\ v(x, 0) = \partial_t v(x, 0) = 0, & F \in C_C^\infty(R_x \times (0, \infty)_t) \end{cases}$$

where $P_V = D_x^2 + V(x) = -\left(\frac{\partial}{\partial}\right)^2 + V(x)$ and $D_x = \frac{1}{i}\frac{\partial}{\partial x}$. The operator D_x is symmetric and self-adjoint on appropriately chosen domains. For $f(x)$ and $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx$, $\widehat{D_x f} = \xi \hat{f}(\xi)$. $V \in L^\infty_{\text{comp.}}(\mathbb{R})$ (i.e. compactly supported L^∞) is the potential. If $f, g \in \mathcal{S}(\mathbb{R})$, then $(P_V f, g)_{L^2(\mathbb{R})} = (f, P_V g)_{L^2(\mathbb{R})}$.

IMAGE 1

Another way to look at this assuming v exists, we can consider $u(x, \lambda) := \int_0^\infty e^{it\lambda} v(x, t)\, dt$ (the Fourier-Laplace transform of v) with $\lambda \in \mathbb{C}$, $\mathrm{Im}(\lambda) > 0$. Write $\lambda = \xi + ic$, $c > 0$, such that $u(x, \xi + ic) = \int_0^\infty e^{it\xi} e^{-ct} v(x, t)\, dt = \mathcal{F}_{t \mapsto \xi}(t \mapsto$

$e^{-ct}v(x,t))(x,-\xi)$. Then $u(x,\lambda)$ solves

$$\int_0^\infty e^{it\lambda}(-\partial_{tt}v - P_V v)\,dt = \int_0^\infty e^{it\lambda}F(x,t) = \hat{F}(x,\lambda)$$
$$(\lambda^2 - P_V)\underbrace{\int_0^\infty e^{it\lambda}v(x,t)\,dt}_{u(x,\lambda)} = \hat{F}(x,\lambda)$$

which is an entire function in λ.
To Do:

- Study solvability of $(\lambda^2 - P_V)u = \hat{F}(x,\lambda)$.
- Return to v.

For frozen c, we can get $v(x,t)$ back by Fourier inversion.

$$e^{-ct}v(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-it\xi}u(x,\xi+ic)\,d\xi$$
$$v(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-it(\xi+ic)}u(x,\xi+ic)\,d\xi$$
$$v(x,t) = \frac{1}{2\pi}\int_{\mathrm{Im}(\lambda)=\mathrm{c}} e^{-it\lambda}u(x,\lambda)\,d\lambda$$

IMAGE 2

where the spectral problem is invertible.

1D Waves in the Time Domain

Suppose $R>0$ is such that $\operatorname{supp} V \subset [-R,R]$ and $\operatorname{supp} F \subset [-R,R]\times(0,\infty)$. If $|x|>R$, the PDE looks like $\partial_{tt}v - \partial_{xx}v = 0 = (\partial_t+\partial_x)(\partial_t-\partial_x)v$. Setting $\xi = x+t$ and $\mu = x-t$, then it follows that

$$\partial_\xi\partial_\mu v = 0 \implies v = F(\xi)+G(\mu) = F(x+t)+G(x-t)$$

IMAGE 3

On $x>R$, we can expect $v(x,t) = F_+(x+t)+G_+(x-t)$; on $x<R$, we expect $v(x,t) = F_-(x+t)+G_-(x-t)$. The terms G_+ and F_- are outgoing whereas the terms F_+ and G_- are incoming and, given that we assumed a source, we expect to be zero.
What does incoming/outgoing look like on the spectral side? $(\lambda^2 - P_v)u = \hat{F}(x,\lambda)$ supported in $|x|\le R$. For $|x|>R$, $(\lambda^2+\partial_x^2)u = 0$ leads to $u = Ae^{ix\lambda}+Be^{-ix\lambda}$. For $x>R$, $u(x) = a_+e^{i\lambda|x|}+b_+e^{-i\lambda|x|}$ for $x<-R$, $u(x) = a_-e^{i\lambda|x|}+b_-e^{-i\lambda|x|}$. u is outgoing if and only if $b_\pm = 0$ and incomming if and only if $a_\pm = 0$.
P_V is an unbounded, symmetric operator on a Hilbert space. For $z\in\mathbb{C}$, $\mathrm{sp}(P_V)$ is the set on the complement of which $(P_V - Z)$ is boundedly invertible. That is, $\forall f$, $\exists! u$ such that $(P_V - z)u = f$ and $||u|| \lesssim ||f|$.

Waves in the Time Domain [Evans, ğ2.4]

Goal: if v solves

$$\begin{aligned} \partial_{tt}v - \partial_{xx}v &= f(x,t) x \in \mathbb{R},\ t>0,\ f \in C_C^\infty(\mathbb{R}\times(0,\infty)) \\ v(x,0) = \partial_t v(x,0) &= 0 x \in \mathbb{R} \end{aligned}$$

then $v(x,t) = \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(y,t-s)\,dyds$. We look at

$$\begin{cases} \partial_{tt}v - \partial_{xx}v = 0 \rightsquigarrow v(x,t) = F(x+t) + G(x-t) \\ v(x,0) = g(x),\ \partial_t v(x,0) = h(x) \end{cases}$$

Initial conditions gives us

$$\begin{cases} F(x) + G(x) = g(x) \\ F'(x) - G'(x) = h(x) \end{cases} \qquad \begin{cases} G'(x) = \frac{1}{2}(g'(x) - h(x)) \\ F'(x) = \frac{1}{2}(g'(x) + h(x)) \end{cases}$$

So

$$\begin{aligned} F(x) &= \frac{1}{2}\left(g(x) + \int_0^x h(s)\,ds\right) + C_1 \\ G(x) &= \frac{1}{2}\left(g(x) - \int_0^x h(s)\,ds\right) + C_2 \\ v(x,t) &= \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(s)\,ds + C \end{aligned}$$

IMAGE 4

This has a finite speed of propogation in the sense that if we suppose $\operatorname{supp}(g,h) \subset [-R,R]$ then $v(x,t) = 0$ whenever $x > R+t$ or $x < -R-t$.

Now we want to go from the homogeneous problem to the inhomogeneous problem. The idea is to think about $v(x,t) = \int_0^t v(x,t;s)\,ds$ where $v(x,t;s)$ solves the homogeneous problem

$$\begin{cases} \partial_{tt}v(\cdot,\cdot;s) - \partial_{xx}v(\cdot,\cdot;s) = 0 \\ v(\cdot,s;s) = 0,\ \partial_t v(\cdot,s;s) = f(x,s) \end{cases}$$

Then

$$\partial_{tt}v - \partial_{xx}v = 0 \iff \partial_t \begin{pmatrix} v \\ \partial_t v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \partial_{xx} & 0 \end{pmatrix}\begin{pmatrix} v \\ \partial_t v \end{pmatrix} \quad \text{and} \quad \begin{bmatrix} v \\ \partial_t v \end{bmatrix}_{t=s} = \begin{bmatrix} * \\ * \end{bmatrix}$$

So $v(x,t;s) = \frac{1}{2}\int_{x-(t-s)}^{x+(t-s)} f(y,s)\,ds$ and $v(x,t) = \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(y,t-s)\,dyds$ follows.

Going back to the original PDE, $(-\partial_{tt} - P_V)v = F$ is equivalent to $(\partial_{tt} - \partial_{xx})v = -(Vv+F)$ which leads to the conclusion that $v(x,t) = -\frac{1}{2}\int_0^t \int_{x-s}^{x+s} (Vv+F)(y,t-s)\,dyds$. For $|x| > R,\ v$ is outgoing.

IMAGE 5