

# Partial Differential Equations I

January 8, 2024

## Homework

Assigned exercises and concept maps. Graded by completion.

## Presentations

Assigned topics; responsible for giving a class.

## Definition: Partial Differential Equation(s) (PDE)

An identity relating an unknown function, its partial derivatives and its variables.

$$F(D^k u, \dots, D^2 u, Du, u, x) = 0, \quad x \in U \subseteq \mathbb{R}^n$$

where  $U$  is an open subset of  $\mathbb{R}^n$ ,  $u : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $Du = (\partial_{x_1} u, \dots, \partial_{x_n} u)$ .

Then  $F : \mathbb{R}^k \times \dots \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ , where  $F$  is given.

$x = (x_1, \dots, x_n)$  is (are) the independent variable(s).

$u$  is the unknown function or dependent variable.

$k$  is the order of the PDE.

## Goal

Given a PDE, we consider

- Existence
- Uniqueness
- Stability

## Recall: Multiindex Notation

$\alpha = (\alpha_1, \dots, \alpha_n)$  a vector such that  $\alpha_i \in \mathbb{Z}_{\geq 0}$ .

Then we say that  $\alpha$  is a multiindex with order  $|\alpha| = \alpha_1 + \dots + \alpha_n$ .

## Notation

$u : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\alpha = (\alpha_1, \dots, \alpha_n)$ .

$u^\alpha := D^\alpha u = \partial_{x_n}^{\alpha_n} \dots \partial_{x_1}^{\alpha_1} u$ , where  $\partial^0 u = u$ .

## Definition: Linear Partial Differential Equation

A linear PDE of order  $k$  is of the form

$$(*) \sum_{|\alpha|=k} a_\alpha(x) D^\alpha u = f(x)$$

## Remark

This means that  $F$  is multilinear in the first  $n^k + n^{k-1} + \dots$  variables.

## Definition: Homogeneous Linear Partial Differential Equation

A linear given by  $(*)$  is homogeneous if  $f(x) \equiv 0$ .

Otherwise, it is non-homogeneous.

## Example 1: Linear Transport Equation

$$\nabla u \cdot (1, b) = u_t + b \cdot Du = f(t, x)$$

This is a linear PDE of order 1 on  $\mathbb{R} \times \mathbb{R}^n \equiv \mathbb{R}^{n+1}$  where  $(t, x)$  are independent variables and  $u$  is dependent. Here,  $x$  is the spatial variable while  $t$  is time and  $Du$  represents the gradient.

$\nabla u = (\partial_t u, \nabla u)$ ,  $b \cdot Du = \sum_{i=1}^n b_i \partial_{x_i} u$ ,  $(b_1, \dots, b_n) \in \mathbb{R}^n$  is fixed.

## Example 2: Laplace Equation

$$\Delta u := \sum_{i=1}^n \partial_{x_i}^2 u = 0$$

This is a linear, homogeneous PDE of order 2.

## Example 3: Poisson Equation

$$-\Delta u := f(u)$$

This is a nonlinear PDE of order 2.

Consider  $f(u) = u^2$ .

## Example 4: Heat Equation (Diffusion Equation)

$$u_t - \Delta u = 0$$

This is a linear, homogeneous PDE of order 2.

## Example 5: Wave Equation

$$u_{tt} - \Delta u = 0$$

This is a linear, homogeneous PDE of order 2.

## Transport Equation

$u : \mathbb{R}^n(0, \infty) \rightarrow \mathbb{R}$  given by

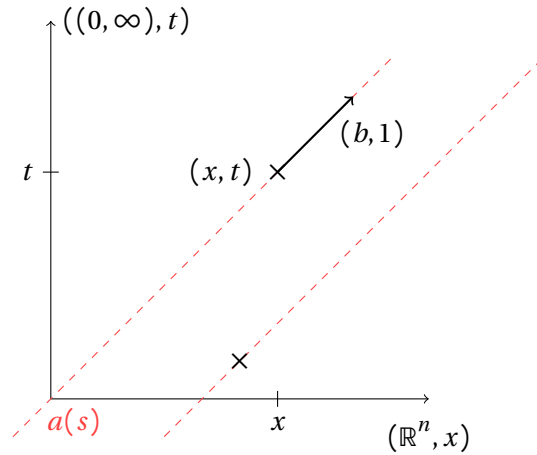
$$u_t + b \cdot Du = 0, \quad b \in \mathbb{R}^n$$

In order to get a solution, first assume that there exists a “nice” (e.g. smooth,  $C^1$ , differentiable, etc.) solution.

## Step 1

Notice that the PDE is equivalent to

$$\nabla u \cdot (b, 1) = 0$$



## Step 2

Consider a curve on  $\mathbb{R}^{n+1}$  with velocity  $(1, b)$  which passes through  $(x, t)$ . i.e.

$$\alpha(s) = (x + sb, t + s)$$

Notice  $\alpha'(s) = (b, 1)$ .

Then, let us study  $u$  along the curve  $\alpha(s)$ .

$$z(s) := u(\alpha(s))$$

Taking the derivative with respect to  $s$ ,

$$z'(s) := \frac{d}{ds}(u \circ \alpha(s)) = \nabla u|_{\alpha(s)} \cdot \alpha'(s) = \nabla u|_{\alpha(s)} \cdot (b, 1) = 0$$

That is  $z'(s) = 0$ ,  $z(s)$  is constant, and  $u$  along  $\alpha(s)$  is constant.

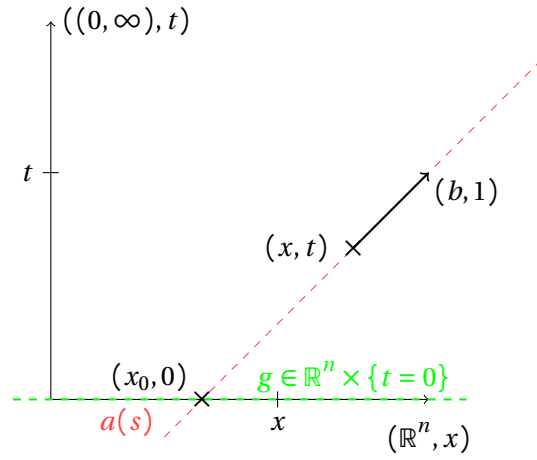
## Conclusion

If we know some value of  $u$  along  $\alpha(s)$ , then we know all values along  $\alpha(s)$ .

If we have some value of  $u$  along every  $\alpha(s)$ , then we know  $u$  on  $\mathbb{R}^n \times (0, \infty)$ .

## Transport Equation - Homogeneous Initial Value Problem

$$(*) \begin{cases} \nabla u \cdot (b, 1) = 0, & \mathbb{R}^n \times (0, \infty) \\ u = g, & \mathbb{R}^n \times \{t = 0\} \end{cases}$$



Here,  $g: \mathbb{R}^n \rightarrow \mathbb{R}$  is given.

Consider  $(x, t)$ ; we want to find  $(x_0, 0)$ .

We know  $\alpha(s) = (x + sb, t + s) = (x_0, 0)$ , therefore

$$\begin{cases} x + sb = x_0 & (1) \\ t + s = 0 \implies s = -t & (2) \end{cases}$$

Then, by replacing (2) in (1),

$$x_0 = x - tb$$

Then from the conclusion

$$u(x, t) = u(x_0, 0) = g(x_0) = g(x - tb)$$

Therefore,  $u(x, t) := g(x - tb)$  (♥).

### Remark

1. If there exists a regular (differentiable or  $C^1$ ) solution  $u$  for  $*$ , then  $u$  should look like ♥.
2. If  $g$  is (differentiable or  $C^1$ ), then  $u$  defined by ♥ is a (differentiable or  $C^1$ ) solution for my problem.

### Homework

Show that ♥ satisfies  $*$ .

### Transport Equation - Non-homogeneous Initial Value Problem

$$(*) \begin{cases} \nabla u \cdot (b, 1) = f(x, t), & \mathbb{R}^n \times (0, \infty) \\ u = g, & \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Where  $g: \mathbb{R}^n \rightarrow \mathbb{R}$  and  $f: \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}$  are given.

## Solution

Notice that the PDE is equivalent to

$$\nabla u \cdot (b, 1) = f(x, t)$$

Define the “characteristic curve”

$$\alpha(s) = (x + sb, t + s)$$

and

$$z(s) := u(\alpha(s))$$

Taking  $\frac{d}{ds}$ ,

$$z'(s) = \nabla u|_{\alpha(s)} \cdot (b, 1) = f(\alpha(s)) \implies z'(s) = f(x + sb, t + s) \quad (c)$$

Notice that  $c$  is an ordinary differential equation. Integrating from  $-t$  to 0.

$$\begin{aligned} \int_{-t}^0 z'(s) ds &= \int_{-t}^0 f(x + sb, t + s) ds \\ z(0) - z(-t) &= \int_{-t}^0 f(x + sb, t + s) ds \end{aligned}$$

Notice that  $z(0) = u(x, t)$  and  $z(-t) = u(\alpha(-t)) = u(x - tb, 0)$ .

$$u(x, t) = u(x - tb, 0) + \int_{-t}^0 f(x + sb, t + s) ds$$

Then

$$\begin{aligned} u(x, t) &= g(x - tb) + \int_{-t}^0 f(x + sb, t + s) ds \\ &\stackrel{\bar{s}=s+t}{=} g(x - tb) + \int_0^t f(x + (\bar{s} - t)b, \bar{s}) d\bar{s} \\ &= g(x - tb) + \int_0^t f(x + (s - t)b, s) ds \end{aligned}$$

## Remark: Method of Characteristics

Try to vert the PDE into an ODE and solve using characteristic curves.

**January 10, 2024**

## Definition: Harmonic Function

If  $u \in C^2$  such that  $\Delta u = 0$ , then  $u$  is a harmonic function.

## Laplace Equation

Consider  $u : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  with  $U$  open, then the homogeneous (Laplace) form is given by

$$\Delta u = 0$$

and the non-homogeneous (Poisson) form is

$$-\Delta u = f$$

where  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is given.

### Remark / Exercise

The Laplace equation is invariant under translation and rotation.

That is, if  $\Delta u(x) = 0$  and  $v(x) = u(x - y)$ , then  $\Delta v = 0$ .

Similarly, if  $w(x) = u(O(x))$  then  $\Delta w = 0$  where  $O$  is an orthogonal matrix.

### Fundamental Solution of the Laplace Equation

Remark: since the Laplace equation is invariant under rotation, we can consider a function in terms of the radius  $v(x) = v(|x|)$ .

Recall  $r = |x| = (x_1^2 + \dots + x_n^2)^{1/2}$ .

Because of this remark, assume that  $u(x) = v(|x|) = v(r(x))$  (\*) where  $v : (0, \infty) \rightarrow \mathbb{R}$ .

Therefore, we need

$$\frac{\partial r}{\partial x_i} = \frac{1}{2} \cdot \frac{2x_i}{(x_1^2 + \dots + x_n^2)^{1/2}} = \frac{x_i}{r}$$

Replace (\*) in the PDE

$$\frac{\partial u}{\partial x_i} = \frac{\partial}{\partial x_i} v(r(x)) = v'(r(x)) \cdot \frac{\partial r}{\partial x_i} = v'(r(x)) \cdot \frac{x_i}{r}$$

and

$$\begin{aligned} \frac{\partial^2 u}{\partial x_i^2} &= \frac{\partial}{\partial x_i} \left( v'(r(x)) \cdot \frac{x_i}{r} \right) \\ &= \frac{\partial}{\partial x_i} (v'(r(x))) \cdot \frac{x_i}{r} + v'(r(x)) \cdot \frac{\partial}{\partial x_i} \left( \frac{x_i}{r} \right) \\ &= v''(r(x)) \cdot \frac{x_i^2}{r^2} + v'(r(x)) \left[ \frac{1}{r} + x_i \frac{\partial}{\partial x_i} \left( \frac{1}{r} \right) \right] \\ &= v'' \frac{x_i^2}{r^2} + v' \left[ \frac{1}{r} - \frac{x_i^2}{r^3} \right] \end{aligned}$$

Then, summing across  $i$ ,

$$\Delta u = v'' + v' \left[ \frac{n}{r} + \frac{1}{r} \right] = 0$$

Then the PDE is equivalent to

$$v''(r) + \frac{v'}{r}(n-1) = 0 \quad (\square)$$

We need to find a solution for  $\square$ .

$$v''(r) = \frac{(1-n)v'}{r}$$

Assume, without loss of generality, that  $v' \neq 0$  such that

$$\frac{v''(r)}{v'(r)} = \frac{1-n}{r} \implies (\log(|v'|))' = \frac{1-n}{r}$$

Then, integrating,

$$\log(|v'|) = (1-n)\log(r) + C = \log(r^{1-n}) + C$$

such that

$$|v'| = Cr^{1-n} \implies v' = Cr^{1-n} \implies v(r) = Cr^{1-n+1} + D = Cr^{2-n} + D$$

### Definition: Fundamental Solution of the Laplace Equation

The function  $\Phi$  given by

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log|x|, & n=2 \\ \frac{1}{n(n-2)\alpha(n)} \cdot \frac{1}{|x|^{n-2}}, & n \geq 3 \end{cases}$$

where  $\alpha(n)$  is the volume of the unit ball in  $\mathbb{R}^n$  is called the fundamental solution.

### Remark

$\Phi$  solves the Laplace equation away from 0.

### Lemma: Estimates for the Fundamental Solution

- First Estimate  
 $|D\Phi(x)| \leq \frac{C}{|x|^{n-1}}, \text{ for } x \neq 0.$

$$\frac{\partial \Phi}{\partial x_i} = C \frac{\partial}{\partial x_i} (|x|^{2-n}) = \frac{C(2-n)}{1-n} |x|^{2-n-1} \frac{\partial |x|}{\partial x_i} = |x|^{1-n} \cdot \frac{x_i}{|x|} = Cx_i |x|^{-n}$$

Therefore

$$|D\Phi(x)| \leq C|x||x|^{-n} \implies |D\Phi(x)| \leq C|X|^{1-n}$$

- Exercise  
 Compute for  $n=2$ .

- Second Estimate  
 $|D^2\Phi(x)| \leq \frac{C}{|x|^n}$ , for  $x \neq 0$ .

$$\begin{aligned}
\frac{\partial^2}{\partial x_j \partial x_i} \Phi &= C \frac{\partial}{\partial x_j} (x_i |x|^{-n}) \\
&= C \left[ \delta_{ij} |x|^{-n} + x_i \frac{\partial}{\partial x_j} |x|^{-n} \right] \\
&= C \left[ \delta_{ij} |x|^{-n} + (-n) \cdot \frac{x_i |x|^{-n-1} x_j}{|x|} \right] \\
&= C \left[ \frac{\delta_{ij} |x|}{|x|^n} + \frac{C x_i x_j}{|x|^{n+1}} \right]
\end{aligned}$$

Then

$$\left| \frac{\partial \Phi}{\partial x_i \partial x_j} \right| \leq \frac{C}{|x|^n} + \frac{C |x_i| |x_j|}{|x|^{n+2}} \leq \frac{2C}{|x|^n} = \frac{C}{|x|^n}$$

Then, we are done since

$$|D^2\Phi(x)| = \sqrt{\sum_j \left( \frac{\partial \Phi}{\partial x_i \partial x_j} \right)^2}$$

## Poisson Equation

### Motivation

Suppose we have  $\Phi(x)$ , the fundamental solution.

Then for an arbitrary, fixed element  $y \in \mathbb{R}^n$ , then we have  $x \rightarrow \Phi(x-y)$  harmonic for  $x \neq y$ .

Consider  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $y \rightarrow f(y)$  then  $x \rightarrow f(y)\Phi(x-y)$  is similarly harmonic for  $x \neq y$ .

Now, if given  $\{y_1, \dots, y_m\}$  where  $y_i \in \mathbb{R}^n$ , then  $x \rightarrow \sum_{i=1}^m f(y_i)\Phi(x-y_i)$  is harmonic  $\forall x \neq \{y_1, \dots, y_m\}$ .

Then, what happens if we consider

$$u(x) := \int_{\mathbb{R}^n} f(y)\Phi(x-y) dy \quad (\square_3)$$

Is  $u$  harmonic? No, since  $\Delta\Phi(x-y)$  is not summable in  $\mathbb{R}^n$  we may not pass the limit into the integral.

(to be covered later) However, since  $\Delta\Phi(x-y)$  acts as  $\delta_{xy}$  in distribution, this may solve the Poisson equation.

### Remark / Exercise

Assume that  $f \in C_c^2(\mathbb{R}^n)$  (i.e  $f$  is twice continuously differentiable with compact support on  $\mathbb{R}^n$ ).

The function  $\Phi$  is integrable near the singularity on compact sets.

Prove using spherical coordinates.

Therefore,  $u$  defined by  $\square_3$  is well defined

$$|u| = \left| \int_{\mathbb{R}^n} f(y)\Phi(x-y) dy \right| = \left| \int_K \Phi(x-y) dy \right| < \infty$$



## Theorem: Solving the Poisson Equation

If  $f \in C_c^2(\mathbb{R}^n)$  and  $u$  is defined by  $\square_3$ , then

1.  $u \in C^2(\mathbb{R}^n)$
2.  $-\Delta u = f$ , in  $\mathbb{R}^n$

• Proof of 1

Since  $\Phi$  presents a problem at  $x = y$  but  $f$  is well behaved, we will change variables such that  $\bar{y} = x - y$ ,  $y = x - \bar{y}$ , and  $\frac{dy}{d\bar{y}}(-1)I_{m \times m}$  and then redefine  $\bar{y} = y$ .

$$u(x) = \int_{\mathbb{R}^n} f(y)\Phi(x-y) dy = \int_{\mathbb{R}^n} f(x-\bar{y})\Phi(\bar{y}) d\bar{y} = \int_{\mathbb{R}^n} f(x-y)\Phi(y) dy$$

In short, we have sent the problem from  $\Phi$  to  $f$ .

Now, let us consider  $e_i = (0, \dots, 1, \dots, 0)$ .

Then for  $h > 0$ ,

$$\frac{u(x + he_i) - u(x)}{h} = \frac{1}{h} \int_{\mathbb{R}^n} \Phi(y) [f(x + he_i - y) - f(x - y)] dy$$

Now, the limit as  $h \rightarrow 0$

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{u(x + he_i) - u(x)}{h} &= \lim_{h \rightarrow 0} \int_{\mathbb{R}^n} \overbrace{\Phi(y) \left[ \frac{f(x + he_i - y) - f(x - y)}{h} \right]}^{H(h,y)} dy \\ &= \int_{\mathbb{R}^n} \Phi(y) \cdot \frac{\partial f(x - y)}{\partial x_i} dy \end{aligned}$$

To justify passing the limit into the integral, take an arbitrary sequence  $h_m \xrightarrow{m \rightarrow 0} 0$  and consider

$$f_m(y) := H(h_m, y)$$

We want to consider

$$\begin{aligned} |H(h_m, y)| &\leq \Phi(y) \left[ \frac{f(x + h_m e_i - y) - f(x - y)}{h} \right] \\ &\leq \Phi(y) f'(c) \end{aligned}$$

Where  $c$  is along the curve between  $f(x + h_m e_i - y)$  and  $f(x - y)$  and chosen by mean value theorem.

– Exercise

$$|H(h_m, y)| \leq \Phi(y) \|f'\|_{L^\infty} \chi_{B(x,R)}(y)$$

Note that

$$C \int_{\mathbb{R}^n} |\Phi(y)| \chi_{B(x,R)}(y) dy = \int_{B(x,R)} |\Phi(y)| dy < \infty$$

Using the fact that a continuous function is uniformly continuous on a compact set, show that  $u \in C^2(\mathbb{R}^n)$ .

## Dominated Convergence Theorem

If  $f_m(x)$  such that  $f_m(x) \xrightarrow[\text{pointwise}]{m \rightarrow \infty} f(x)$ , and  $|f_m(x)| \leq g(x)$  for  $g \in L^1$ , then  $f$  is integrable and

$$\lim_{m \rightarrow \infty} \int f_m(x) dx = \int f(x) dx$$

January 17, 2024

## Recall: Averages

$f : \{1, \dots, n\} \rightarrow \mathbb{R}$   
 $i \rightarrow a(i)$

The average is given as  $\frac{a(1) + \dots + a(n)}{n}$ .

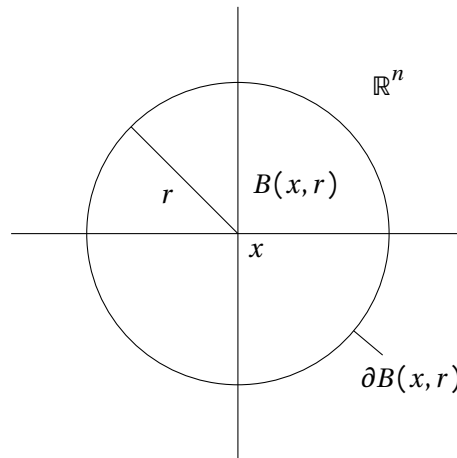
Then for  $f : \Omega \rightarrow \mathbb{R}$ , the average is given as

$$\frac{1}{|\Omega|} \int f(y) dy := \int_{\Omega} f d\mu$$

In our case,  $f : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$

$$\int_{B(x,n)} f d\mu \equiv \frac{1}{|B(x,n)|} \int_{B(x,n)} f d\mu$$

$$\int_{\partial B(x,n)} f d\mu = \frac{1}{|\partial B(x,n)|} \int_{\partial B(x,n)} f d\mu$$

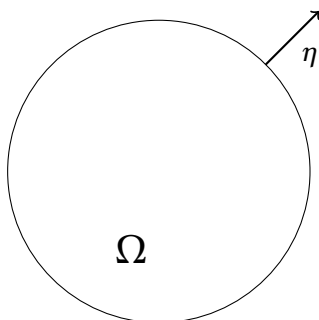


## Theorem: Lebesgue Differentiation

$$u(x) = \lim_{r \rightarrow 0} \int_{B(x,n)} u d\mu = \lim_{r \rightarrow 0} \int_{\partial B(x,n)} u d\mu$$

## Integration by Parts

$$\int_{\Omega} u \Delta v = - \int_{\Omega} \nabla u \cdot \nabla v + \int_{\partial \Omega} u \cdot \frac{\partial v}{\partial \eta}$$



## Recall: Poisson's PDE

$$f \in C_c^2(\mathbb{R}^n), u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) dy.$$

$$\Phi(x) = \begin{cases} \frac{1}{n(n-2)|\alpha(n)|} \frac{1}{|x|^{(n-2)}} \end{cases}$$

$$u(x) = \int_{\mathbb{R}^n} f(x-y) \Phi(y) dy$$

## Part A

$$u \in C^2(\mathbb{R}^n)$$

Then

$$\frac{\partial u}{\partial x_1} = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_1}(x-y) \Phi(y) dy$$

$$\frac{\partial^2 u}{\partial x_1 \partial x_T} = \int_{\mathbb{R}^n} \frac{\partial^2 f}{\partial x_1 \partial x_T}(x-y) \Phi(y) dy$$

Notice that if we prove that

$$g(x) = \int_{\mathbb{R}^n} h(x-y) \Phi(y) dy$$

– where  $h$  is continuous with compact support – is continuous then we are done.

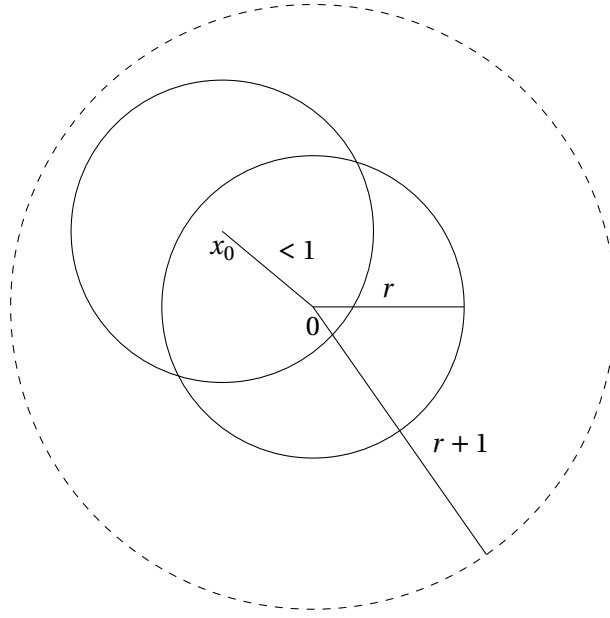
Let us prove that  $g$  is continuous.

Let  $\varepsilon > 0$ ,

$$|g(x) - g(x_0)| \leq \int_{\mathbb{R}^n} \Phi(y) |h(x-y) - h(x_0-y)| dy$$

Without loss of generality,  $h$  has compact support on  $B(0, r)$  for some radius  $r$ .

Therefore  $h(x, y)$  has compact support on  $B(x, r)$  and  $h(x_0, y)$  has compact support on  $B(x_0, r)$ .



Consider  $|x - x_0| < 1$ , then  $|h(x - y) - h(x_0 - y)|$  has compact support on  $B(x_0, r + 1)$ . Then

$$|g(x) - g(x_0)| \leq \int_{B(x_0, r+1)} \Phi(y) |h(x - y) - h(x_0 - y)| dy$$

Since  $h$  is continuous on a compact domain, it is uniformly continuous.

Therefore  $\exists \delta > 0$  such that  $|w - z| < \delta \implies |h(w) - h(z)| < \epsilon$ .

Set  $w = x - y$  and  $z = x_0 - y$  such that  $|w - z| = |x - x_0| < \delta$ , then  $|h(x - y) - h(x_0 - y)| < \epsilon$ . Thus,

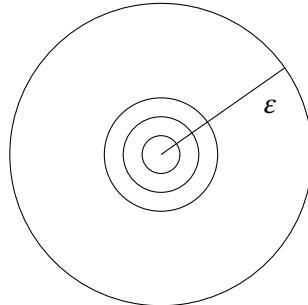
$$|g(x) - g(x_0)| \leq \epsilon \int_{B(x_0, r+1)} \Phi(y) dy$$

## Part B

$$-\Delta u = f$$

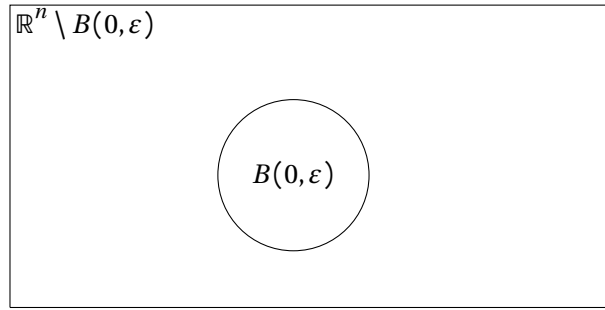
Letting  $\epsilon > 0$  and taking the Laplacian of both sides,

$$\begin{aligned} \Delta_x u(x) &= \int_{\mathbb{R}^n} \Delta_x f(x - y) \Phi(y) dy \\ &= \overbrace{\int_{B(0, \epsilon)} \Delta_x f(x - y) \Phi(y) dy}^{I_\epsilon} + \overbrace{\int_{\mathbb{R}^n \setminus B(0, \epsilon)} \Delta_x f(x - y) \Phi(y) dy}^{J_\epsilon} \end{aligned}$$



Then

$$\begin{aligned}
|I_\varepsilon| &\leq \int_{B(0,\varepsilon)} |\Delta_x f(x-y)| \Phi(y) dy \\
&\leq \| |\nabla^2 f| \|_{L^\infty} \int_{B(0,\varepsilon)} \Phi(y) dy \\
&\leq c \int_0^\varepsilon \int_{\partial B(0,r)} \Phi(y) dS(y) dr \\
&\leq c \int_0^\varepsilon \int_{\partial B(0,r)} \frac{1}{|y|^{n-2}} dS(y) dr \\
&= c \int_0^\varepsilon \int_{\partial B(0,r)} \frac{1}{r^{n-2}} dS(y) dr \\
&= c \int_0^\varepsilon \frac{1}{r^{n-2}} \int_{\partial B(0,r)} dS(y) dr \\
&\leq c \int_0^\varepsilon \frac{r^{n-1}}{r^{n-2}} dr \\
&= c \int_0^\varepsilon r dr = c\varepsilon^2
\end{aligned}$$



As an exercise, attempt the same argument with  $n = 2$ .

Therefore,

$$\Delta_x u = I_\varepsilon + J_\varepsilon$$

and  $\lim_{\varepsilon \rightarrow 0} I_\varepsilon = 0$ .

Now, we need to control  $J_\varepsilon$ .

$$J_\varepsilon = \int_{\mathbb{R}^n} \Delta_x f(x-y) \Phi(y) dy$$

$$\Delta_x f(x-y) = \sum \frac{\partial^2 f}{\partial x_i^2} f(x-y)$$

$$\frac{\partial f}{\partial x}(x-y) = \nabla f|_{z=(x-y)} \cdot e_i = \frac{\partial f}{\partial z_i} |_{z=(x-y)}$$

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{\partial^2 f}{\partial z_i^2} |_{z=(x-y)}$$

$$\begin{aligned}
\Delta_y f(x-y) &= \sum \frac{\partial^2 f}{\partial y_i^2}(x-y) \\
\frac{\partial f}{\partial y_i}(x-y) &= \nabla f|_{z=(x-y)} \cdot -e_i = -\frac{\partial f}{\partial z_i}|_{z=(x-y)} \\
\frac{\partial^2 f}{\partial y_i^2} &= \frac{\partial^2}{\partial z_i^2}|_{z=x-y}
\end{aligned}$$

So

$$\begin{aligned}
J_\varepsilon &= \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Delta_y f(x-y) \Phi(y) dy \\
&= \overbrace{- \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \nabla f(x-y) \nabla \Phi(y) dy}^{K_\varepsilon} + \overbrace{\int_{\partial(\mathbb{R}^n \setminus B(0,\varepsilon))} \frac{\partial f}{\partial \eta} \Phi(y) dS(y)}^{L_\varepsilon}
\end{aligned}$$

and

$$\Delta_x u = I_\varepsilon + K_\varepsilon + L_\varepsilon$$

To control  $L_\varepsilon$ , since

$$\begin{aligned}
|L_\varepsilon| &\leq \int_{\partial B(0,\varepsilon)} \frac{|\partial f|}{\partial \eta} \Phi(y) dy \\
&\leq \int_{\partial B(0,\varepsilon)} |\nabla f| \Phi(y) dy \\
&\leq \|\nabla f\|_{L^\infty} \int_{\partial B(0,\varepsilon)} \Phi(y) dy \\
&\leq c \int_{\partial B(0,\varepsilon)} \frac{1}{|y|^{n-2}} dy \\
&= \frac{c}{\varepsilon^{n-2}} \int_{\partial B(0,\varepsilon)} dy \\
&\leq \frac{c\varepsilon^{n-1}}{\varepsilon^{n-2}} \\
&= c\varepsilon
\end{aligned}$$

and  $K_\varepsilon$ , since  $\frac{\partial \Phi}{\partial \eta} = \nabla \phi \cdot \eta = \frac{-y}{n\alpha(n)|y|^n} \cdot \eta = \frac{|y|^2}{n\alpha(n)|y|^n|y|} = \frac{1}{n\alpha(n)|y|^{n-1}}$

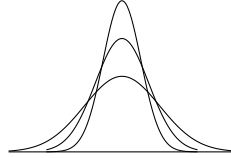
$$\begin{aligned}
|K_\varepsilon| &= - \int_{\mathbb{R}^n \setminus B(0, \varepsilon)} \nabla_y f(x-y) \nabla_y \Phi(y) dy \\
&= \overbrace{\int_{\mathbb{R}^n \setminus B(0, \varepsilon)} f(x-y) \Delta_y \Phi(y) dy}^0 - \int_{\partial(\mathbb{R}^n \setminus B(0, \varepsilon))} f(x-y) \frac{\partial \Phi}{\partial \eta} \\
&= - \int_{\partial B(0, \varepsilon)} f(x-y) \frac{\partial \Phi}{\partial \eta} \\
&= - \int_{\partial B(0, \varepsilon)} f(x-y) \frac{1}{n\alpha(n)|y|^{n-1}} dS(y) \\
&= - \frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B(0, \varepsilon)} f(x-y) dS(y) \\
&= - \underbrace{\frac{1}{n\alpha(n)\varepsilon^{n-1}}}_{\text{volume}} \int_{\partial B(0, \varepsilon)} f(z) dS(z) \\
&= \frac{1}{|\partial B(0, \varepsilon)|} \int_{\partial B(0, \varepsilon)} f(z) dz \\
&= - \oint_{\partial B(x, \varepsilon)} f(z) dz
\end{aligned}$$

## Laplacian as a Distribution

$$-\Delta \Phi(y) = \delta(y)$$

Define the Dirac delta “function” as

$$\delta(y) = \begin{cases} \infty & y = 0 \\ 0 & y \neq 0 \end{cases}$$



such that  $\int_{\mathbb{R}^n} \delta = 1$ .

Translate the Dirac delta as

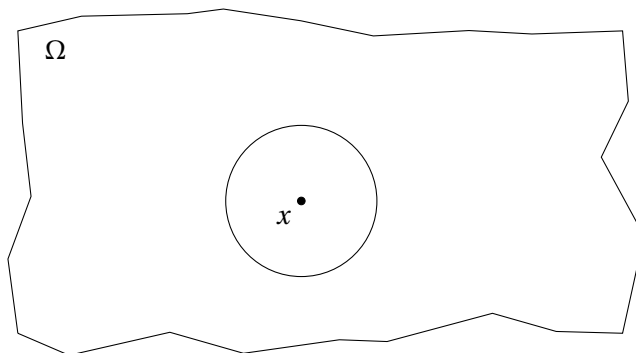
$$\delta_x = \begin{cases} \infty & y = x \\ 0 & y \neq x \end{cases}$$

and recall that

$$\begin{aligned}
 \Delta u(x) &= \Delta \left( \int_{\mathbb{R}^n} \Phi(x-y) f(y) dy \right) \\
 &= \int_{\mathbb{R}^n} \overbrace{\Delta \Phi(x-y)}^{-\delta_x(y)} f(y) dy \\
 &= - \int_{\mathbb{R}^n} \delta_x(y) f(y) dy \\
 &= - \int_{\mathbb{R}^n} \delta_x(y) f(x) dy \\
 &= -f(x) \overbrace{\int_{\mathbb{R}^n} \delta_x(y) dy}^1 \\
 &= -f(x)
 \end{aligned}$$

## Harmonic Functions

Suppose  $u$  is harmonic



$u : \Omega \rightarrow \mathbb{R}^n$  harmonic.

## Mean-value Formulas

Let  $U$  be an open set in  $\mathbb{R}^n$ ,  $u : U \rightarrow \mathbb{R}$  such that  $\Delta u = 0$  in  $U$ . Then

$$\begin{aligned}
 u(x) &= \oint_{\partial B(0,r)} -u(y) dS(y) \\
 &= \oint_{B(x,r)} u(y) dy
 \end{aligned}$$

where  $B(x, r) \subseteq U$ .

IMAGE HERE

## Proof

Consider  $\phi(r) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y) dS(y)$ .

If  $\phi'(r) = 0$ , when we are done since that would make  $\phi$  constant and  $\phi(r) = \lim_{s \rightarrow 0} \phi(s) = u(x)$ . Then



$$\begin{aligned}
\phi(r) &= \frac{1}{|\partial B(x, r)|} \int_{\partial B(x, r)} u(y) dS(y) \\
&\stackrel{y=x+rz}{=} \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(0,1)} u(x+rz)r^{n-1} dS(z) \\
&= \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} u(x+rz) dS(z)
\end{aligned}$$

Therefore

$$\begin{aligned}
\phi'(r) &= \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} \nabla u(x+rz) \cdot z dS(z) \\
&\stackrel{y=x+rz}{=} \frac{1}{|\partial B(0, r)|} \int_{\partial B(x, r)} \nabla u(y) \cdot \frac{y-x}{r} dS(y) \\
&= \frac{1}{|\partial B(0, r)|} \int_{\partial B(x, r)} \nabla u(y) \cdot \eta dS(y) \\
&= \frac{1}{|\partial B(0, r)|} \int_{\partial B(x, r)} \frac{\partial y}{\partial \eta} dS(y) \\
&= \frac{1}{|\partial B(0, r)|} \int_{B(x, r)} \Delta u \\
&= 0
\end{aligned}$$

**January 22, 2024**

### Mean Value Formula

For  $U \subseteq \mathbb{R}^n$ ,  $U$  open with  $u : U \rightarrow \mathbb{R}$  such that  $u \in C^2(U)$ ,  $\Delta u = 0$ , we have

$$u(x) \stackrel{(a)}{=} \oint_{\partial B(x, r)} u \stackrel{(b)}{=} \oint_{B(x, r)} u$$

for all  $B(x, r) \subseteq U$ .

Recall that (a) was proven above by setting  $\phi(r) = \oint_{\partial B(x, r)} u(y) dS(y)$  and showing  $\phi'(r) = 0$ .

For (b), we again apply spherical coordinates such that

$$\begin{aligned}
\int_{B(x, r)} u(y) dy &= \int_0^r \int_{\partial B(x, s)} u(y) dS(y) ds \\
&= \int_0^r |\partial B(x, s)| \overbrace{\oint_{\partial B(x, s)} u(y) dS(y)}^{u(x)} ds \\
&= u(x) \int_0^r |\partial B(x, s)| ds \\
&= u(x) \int_0^r n\alpha(n)S^{n-1} ds \\
&= \frac{u(x)n\alpha(n)S^n}{n} \Big|_0^r \\
&= u(x) \overbrace{\alpha(n)r^n}^{|B(x, r)|}
\end{aligned}$$

## Converse

Recall that

$$\phi'(r) = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u(y) dy$$

Suppose then that we do not know that  $\Delta u = 0$  but we have

$$u(x) = \oint_{\partial B(x,r)} u, \quad \forall B(x,r) \subseteq U$$

Then, necessarily,  $\Delta u = 0$  in  $U$ .

- Proof

Suppose, for sake of contradiction, that  $\Delta u \neq 0$ . Then, without loss of generality, there exists  $y \in U$  such that  $\Delta u(x) > 0$  for  $x \in B(y,n) \subseteq U$ .

IMAGE HERE

$$\phi(r) = \oint_{\partial B(x,r)} u(x) dS(x)$$

and

$$\phi'(r) = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(y,r)} \Delta u(x) dS(x) > 0$$

which contradicts  $\phi'(x) = 0$ .

## Strong Maximum Principle

Let  $U \subseteq \mathbb{R}^n$  be a bounded open set,  $u \in C^2(U) \cap C(\overline{U})$ ,  $\Delta u = 0$  on  $U$ . Then

1.  $\max_{\overline{U}}(u) = \max_{\partial U}(u)$ .
2. If  $U$  is connected and  $u$  has its maximum in an interior point, then  $u$  is constant on  $\overline{U}$ .

IMAGE HERE - 2

### Proof of A

Since  $\partial U \subseteq \overline{U}$ ,  $\max_{\partial U}(u) \leq \max_{\overline{U}}(u)$ .

Let  $x_0 \in \overline{U}$  such that  $u(x_0) = \max_{\overline{U}}(u)$ .

IMAGE HERE - 4

So either  $x_0 \in \partial U$  or  $x_0 \in U$ .

Let  $U'$  be the connected component which contains  $x_0$ . Then  $x_0 \in U'$ , so by part (b)  $u$  is constant on  $\overline{U'}$ . So

$$\max_{\overline{U}}(u) = u(x_0) = \max_{\partial U'}(u) \leq \max_{\partial U}(u)$$

## Proof of B

Then there exists  $x_0 \in U$  such that  $\max_{\overline{U}}(u) = u(x_0) = M$ .

Let us define  $\Omega = \{y \in U : u(y) = M\}$ . Then

1.  $\Omega \neq \emptyset, B \setminus x_0 \in \Omega$ .

2.  $\Omega$  open set.

IMAGE HERE - 3

1.  $\Omega$  is closed, since  $\Omega = u^{-1}(\{M\})$ .

It follows that  $\Omega = U$  and, therefore,  $u(y) = M, \forall y \in U$ .

- Proof of 2

Let  $y \in \Omega, y \in U, u(y) = M$ . Then there exists  $B(y, r) \subseteq U$ , and

$$M = u(y) = \oint_{B(y,r)} u(x) dS(x) \leq M$$

Then

$$\oint_{B(y,r)} u(x) dx = M$$

so  $u(x) = M, \forall x \in B(y, r)$  and, therefore  $B(y, r) \subseteq \Omega$  and  $\Omega$  is open.

## Remark: Boundedness Is Important

1. Consider  $f(x) = x$  on  $\mathbb{R}_{\geq 0}$ .

2. IMAGE HERE - 5

## Remark: Strong Minimum Principle Is Equivalent

### Consequences

1. Positivity of harmonic functions.

2. Uniqueness of the Poisson problem.

## Corollary: Positivity of Harmonic Functions

Suppose that  $U$  is connected and  $u : U \rightarrow \mathbb{R}, u \in C^2(U) \cap C(\overline{U})$  solves the following problem

$$\begin{cases} \Delta u = 0 & \text{on } U \\ u = g & \text{on } \partial U \end{cases}$$

If  $g \geq 0$  on  $\partial U$ , then  $u$  is positive on  $U$  as long as  $g$  is positive in some point.

### Proof

Assume  $\exists x_0 \in \partial U$  where  $x_0$  is the minimum. Then  $u(x_0) = \min_{\overline{U}}(u)$  and,  $\forall x \in U$ ,

$$0 \leq u(x_0) = \min_{\overline{U}}(u) \leq u(x)$$

so  $u$  is non-negative. If  $u(x) = 0$ , then  $u(x_0) = 0$  and the minimum is achieved in the interior. That would mean  $u(x) = 0$ ,  $\forall x \in \overline{U} \supseteq \partial U$  and  $g(x) = 0$ ,  $\forall x \in \partial U$  which would be a contradiction.

### Theorem: Uniqueness of the Poisson Problem

Suppose  $U \subseteq \mathbb{R}^n$  is open, connected and bounded. Then, there exists at most one solution to

$$(*) \begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

where  $u \in C^2(U) \cap C(\overline{U})$ .

### Proof

Let  $u_1$  and  $u_2$  be two solutions of  $*$ .

Consider  $w = u_1 - u_2$  and observe that

$$\Delta w = \Delta u_1 - \Delta u_2 = -f + f = 0, \quad \text{in } U$$

and  $w|_{\partial U} = g - g = 0$  on  $\partial U$ . Therefore

$$\begin{cases} \Delta w = 0 & \text{in } U \\ w = 0 & \text{on } \partial U \end{cases}$$

By applying the minimum and maximum principles,

$$w(x_0) = \min_{\overline{U}}(w) \leq w(x) \leq \max_{\overline{U}}(w) = w(x)$$

so  $w(x) = 0$ ,  $\forall x \in \overline{U}$  and therefore  $u_1 = u_2$ .

### Example

Let's consider  $f : \mathbb{C} \rightarrow \mathbb{C}$  analytic (i.e.  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  for  $a_n, z \in \mathbb{C}$ ). Then

$$f(z) = u(z) + v(z)$$

If  $\mathbb{C} \cong \mathbb{R}^2$ ,

$$f(x + iy) = u(x, y) + v(x, y)$$

for  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $v : \mathbb{R}^2 \rightarrow \mathbb{R}$ .

Claim:  $u$  and  $v$  are Harmonic.

$$u(x, y) + v(x, y) = \sum_{n=0}^{\infty} a_n (x + iy)^n$$

and

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \sum_{n=1}^{\infty} a_n n |x + iy|^{n-1}$$

$$\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} = \sum_{n=1}^{\infty} a_n n |x + iy|^{n-1} i$$

So

$$i \left( \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) = \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

### Recall: Convolution and Smoothing

Let  $U \subseteq \mathbb{R}^n$  be an open set.

For  $\varepsilon > 0$ , define  $U_\varepsilon = \{x \in U : d(x, \partial U) > \varepsilon\}$ .

IMAGE HERE - 6

Define

$$\eta(x) = \begin{cases} ce^{\left(\frac{1}{|x|^2-1}\right)} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

with  $c$  such that  $\int_{\mathbb{R}^n} \eta(x) dx = 1$ ,  $\eta \in C^\infty(\mathbb{R}^n)$

IMAGE HERE - 7

Note that  $\text{supp}(\eta) = B(0, 1)$  and take

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right), \quad \eta_\varepsilon \in C^\infty(\mathbb{R}^n)$$

Then

$$\int_{\mathbb{R}^n} \eta_\varepsilon(x) dx = 1$$

and  $\text{supp}(\eta_\varepsilon) \subseteq B(0, \varepsilon)$ .

If  $f$  is locally integrable on  $U$ , define its mollification

$$f^\varepsilon(x) = \int_U \eta_\varepsilon(x-y) f(y) dy \quad \forall x \in U_\varepsilon$$

**January 24, 2024**

### Recall: Mollifiers

Define

$$\eta(x) = \begin{cases} ce^{\left(\frac{1}{|x|^2-1}\right)} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

where  $\eta \in C^\infty(\mathbb{R}^n)$ ,  $\int_{\mathbb{R}^n} \eta(x) = 1$  and  $\text{supp}(\eta) \subseteq B(0, 1)$ .

Then for  $\varepsilon > 0$ ,  $\eta_\varepsilon(x) = \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right)$  where  $\eta_\varepsilon \in C^\infty(\mathbb{R}^n)$ .

So  $\int_{\mathbb{R}^n} \eta_\varepsilon(x) = 1$  and  $\text{supp}(\eta_\varepsilon) \subseteq B(0, \varepsilon)$

Given  $f$  locally summable;  $f : U \rightarrow \mathbb{R}$ ,

$$\begin{aligned} f^\varepsilon(x) &:= \int_U \eta_\varepsilon(x-y) f(y) dy \quad x \in U_\varepsilon \\ &= \int_{B(x, \varepsilon)} \eta_\varepsilon(x-y) f(y) dy \quad x \in U_\varepsilon \end{aligned}$$

### Properties

1.  $f^\varepsilon \in C^\infty(U_\varepsilon)$ .
2.  $f^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} f$  a.e.
3. If  $f$  continuous, then  $f^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} f$  uniformly on compact sets of  $U$ .

### Theorem 6:

Let  $u \in C(U)$  with  $U \in \mathbb{R}^n$  open and such that  $u$  satisfies the mean-value property (i.e.  $u(x) = \int_{\partial B(x, r)} u(y) dS(y)$ ,  $\forall B(x, r) \subseteq U$ ), then  $u \in C^\infty$ .

### Corollary

If  $u \in C^2(U)$  is harmonic, then  $u \in C^\infty(U)$ .

### Proof

Let us take  $x_0 \in U$

IMAGE HERE - 1

Notice, that if we prove that  $u = u_\varepsilon$  on  $U_\varepsilon$  then we are done.

Let  $x \in U_\varepsilon$ , and noticing that  $\eta(x) = \eta(|x|)$ ,

$$\begin{aligned}
u_\varepsilon(x) &= \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y) u(y) dy \\
&= \frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} \eta \frac{|x-y|}{\varepsilon} u(y) dy \\
&\stackrel{\text{spherical}}{=} \frac{1}{\varepsilon^n} \int_0^\varepsilon \int_{\partial B(x,r)} \eta \frac{\overbrace{|x-y|}^r}{\varepsilon} u(y) dS(y) dr \\
&= \frac{1}{\varepsilon^n} \int_0^\varepsilon \eta \frac{r}{\varepsilon} \int_{\partial B(x,r)} u(y) dS(y) dr \\
&= \frac{1}{\varepsilon^n} \int_0^\varepsilon \eta \frac{r}{\varepsilon} \underbrace{|\partial B(x,r)|}_{|\partial B(0,r)|} u(x) dr \\
&= \frac{u(x)}{\varepsilon^n} \int_0^\varepsilon \eta \frac{r}{\varepsilon} \int_{\partial B(0,r)} dS(y) dr \\
&= u(x) \int_0^\varepsilon \frac{1}{\varepsilon^n} \eta \frac{r}{\varepsilon} dS(y) dr \\
&= u(x) \overbrace{\int_{B(0,\varepsilon)} \eta_\varepsilon(y) dy}^1 = u(x)
\end{aligned}$$

### Theorem 7: Local Estimates of Harmonic Functions

Suppose  $u \in C^2(U)$  a harmonic function.

Then  $|D^\alpha u(x_0)| \leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0,r))}$ ,  $B(x_0,r) \subseteq U$ , where  $\alpha$  is a multiindex of order  $k$ ,  $C_0 = \frac{1}{\alpha(n)}$  and  $C_k = \frac{(2^{n+1}nk)^k}{\alpha(n)}$  for  $k = 1, 2, \dots$

We may take  $\alpha$  since, by previous theorem,  $u \in C^\infty(U)$ .

#### Proof

By induction.

Consider  $k = 0$ .

$$\begin{aligned}
u(x_0) &= \int_{B(x_0,r)} u(y) dy \\
&= \frac{1}{\alpha(n)r^n} \int_{B(x_0,r)} u(y) dy \\
|u(x_0)| &\leq \frac{1}{\alpha(n)r^n} \int_{B(x_0,r)} |u(y)| dy \\
&= \frac{C_0}{r^n} \|u\|_{L^1(B(x_0,r))}
\end{aligned}$$

For  $k = 1$ , if  $|\alpha| = k = 1$  then  $D^\alpha u(x_0) = \frac{\partial u}{\partial x_i}(x)$  for  $i = 1, 2, \dots$

Notice that  $\frac{\partial u}{\partial x_i}$  is also harmonic.

$$\begin{aligned}
\Delta \frac{\partial u}{\partial x_i} &= \sum_{t=1}^n \frac{\partial^2}{\partial x_t^2} \frac{\partial u}{\partial x_i} \\
&= \frac{\partial}{\partial x_i} \underbrace{\sum_{t=1}^n \frac{\partial^2 u}{\partial x_t^2}}_0
\end{aligned}$$

Applying the mean-value formula to  $\frac{\partial u}{\partial x_i}(x_0)$  in  $B(x, r/2)$ .

IMAGE HERE - 2

$$\begin{aligned}\frac{\partial u}{\partial x_i}(x_0) &= \oint_{B(x_0, r/2)} \frac{\partial u}{\partial x_i}(y) dy \\ &= \frac{2^n}{\alpha(n)r^n} \oint_{B(x_0, r/2)} \frac{\partial u}{\partial x_i}(y) dy\end{aligned}$$

Recall  $\int_{\Omega} w \Delta v = - \int_{\Omega} \nabla w \cdot \nabla v + \int_{\Omega} w \frac{\partial v}{\partial \eta}$ .

$$\begin{aligned}&\stackrel{e_i = \nabla y_i}{=} \frac{2^n}{\alpha(n)r^n} \int_{B(x_0, r/2)} \underbrace{\nabla u(y)}_w \cdot \underbrace{\nabla y_i}_v dy \\ &\stackrel{IBP}{=} \frac{2^n}{\alpha(n)r^n} \left[ - \int_{B(x_0, r/2)} u(y) \Delta y_i dy + \int_{\partial B(x_0, r/2)} u(y) \frac{\partial y_i}{\partial \eta} \right]\end{aligned}$$

Note that

$$\frac{\partial y_i}{\partial \eta} = \nabla y_i \cdot \eta = e_i \cdot \eta = \eta_i$$

and

$$\left| \frac{\partial y_i}{\partial \eta} \right| = |\eta_i| \leq |\eta| = 1$$

So,

$$\begin{aligned}\left| \frac{\partial u}{\partial x_i} x_0 \right| &\leq \frac{2^n}{\alpha(n)r^n} \int_{\partial B(x_0, r/2)} |u(y)| dS(y) \\ &= \frac{2^n n \alpha(n) \left(\frac{r}{2}\right)^{n-1}}{\alpha(n)r^n} \|u\|_{L^\infty(\partial B(x_0, r/2))} \\ &= \frac{2n}{r} \underbrace{\|u\|_{L^\infty(\partial B(x_0, r/2))}}_*\end{aligned}$$

Let's analyze \*.

Let  $x \in \partial B(x_0, r/2)$ , then  $B(x, r/2) \subseteq B(x_0, r)$ .

IMAGE HERE - 3

Then we may apply  $k = 0$ .

$$\begin{aligned}|u(x)| &\leq \frac{C_0}{\left(\frac{r}{2}\right)^n} \|u\|_{L^1(B(x, r/2))} \\ &\leq \frac{C_0}{\left(\frac{r}{2}\right)^n} \|u\|_{L^1(B(x_0, r))}\end{aligned}$$

Then

$$\begin{aligned}\left| \frac{\partial u}{\partial x_i}(x_0) \right| &\leq \frac{2n}{r} \frac{C_0}{\left(\frac{r}{2}\right)^n} \|u\|_{L^1(B(x_0, r))} \\ &= \frac{2^{n+1}n}{r^{n+1}\alpha(n)} \|u\|_{L^1(B(x_0, r))}\end{aligned}$$

HOMEWORK: Induct for arbitrary  $k$ .



## Theorem 8: Liouville's Theorem

Suppose  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  is harmonic and bounded. Then  $u$  is constant.

### Proof

$$|D^\alpha u(x)| = \sqrt{\sum_{i=1}^n \left[ \frac{\partial u}{\partial x_i} \right]^2} \leq \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|$$

Let  $r > 0$ ,  $B(x, r) \subseteq \mathbb{R}^n$ . Then, using estimates

$$\left| \frac{\partial u}{\partial x_i}(x) \right| \leq \frac{C_1}{r^{n+1}} \|u\|_{L^1(B(x, r))}$$

Therefore,

$$\begin{aligned} |D^\alpha u(x)| &\leq \frac{nC_1}{r^{n+1}} \|u\|_{L^1(B(x, r))} \\ &= \frac{nC_1}{r^{n+1}} \int_{B(x, r)} |u(y)| \, dy \\ &\leq \frac{nC_1}{r^{n+1}} \|u\|_{L^\infty(B(x, r))} \alpha(n) r^n \\ &= \frac{C \|u\|_{L^\infty(B(x, r))}}{r} \end{aligned}$$

and

$$\begin{aligned} \left| \frac{\partial u}{\partial x_i}(x) \right| &\leq \frac{C \|u\|_{L^\infty(B(x, r))}}{r} \\ \left| \frac{\partial u}{\partial x_i}(x) \right| &\leq C \|u\|_{L^\infty(B(x, r))} \lim_{r \rightarrow \infty} \frac{1}{r} \implies \frac{\partial u}{\partial x_i}(x) = 0 \implies u = Ck \end{aligned}$$

## Theorem: Representation Formula

Recall:  $f \in C_c^2(\mathbb{R}^n)$ ,  $(*) -\Delta u = f$  in  $\mathbb{R}^n$ , we saw that

$$\tilde{u}(x) : \int_{\mathbb{R}^n} \Phi(x-y) f(y) \, dy$$

solves  $*$ .

Let us consider  $u \in C^2(\mathbb{R}^n)$  solving  $-\Delta u = f$  for  $n \geq 3$  where  $f \in C_c^2(\mathbb{R}^n)$  and  $u$  is bounded.

Then  $u(x) = \tilde{u}(x) + C = \int_{\mathbb{R}^n} \Phi(x-y) f(y) \, dy + C$ .

### Proof

Notice that if  $\tilde{u}$  is bounded, then we are done. Because we may consider  $w = u - \tilde{u}$  on  $\mathbb{R}^n$  where

$$\Delta w = \Delta u - \Delta \tilde{u} = -f - (-f) = 0$$

Therefore  $w$  is bounded and, by Liouville's Theorem,  $w = C$  and  $u - \tilde{u} = C \iff u = \tilde{u} + C$ .

$$\begin{aligned} |\tilde{u}(x)| &\leq \int_{B(0, k)} \Phi(x-y) f(y) \, dy \\ &\leq \|f\|_{L^\infty(\mathbb{R}^n)} \int_{B(0, k)} \Phi(x-y) \, dy \end{aligned}$$

If this is less than some  $C$  which does not depend on  $x$ , we are done.

Since  $\Phi(x) \rightarrow 0$  for  $|x| \rightarrow \infty$ , for any  $C_1 \exists \alpha$  such that if  $|x| > \alpha$  then  $|\Phi(x)| < C_1$ .

IMAGE HERE - 4

$\text{dist}(x, B(0, k)) = \text{dist}(x, y_0)$  where  $y_0 \in \overline{B(0, k)}$ .

IMAGE HERE - 5

There are two cases.

- Case 1

$$\text{dist}(x, B(0, k)) \leq \alpha$$

$$B(x, k) \subseteq B(0, \alpha + Ck)$$

Let  $y \in B(x, k)$ , then  $|y - x| < k$  so  $|x - y_0| < \alpha$ .

Therefore  $|y - y_0| \leq k + \alpha \implies |y| \leq k + \alpha + |y_0| = 2k + \alpha \implies y \in B(0, 2k + \alpha)$ . Then

$$\|f\|_{L^\infty(\mathbb{R}^n)} \int_{B(x, k)} \Phi(y) dy \leq \|f\|_{L^\infty(\mathbb{R}^n)} \int_{B(0, \alpha + 2k)} \Phi(y) dy$$

HOMEWORK - Consider the other case.

**January 29, 2024**

## Recall: Representation Formula

For  $n \geq 3$ .

$$\tilde{u}(x) : \int_{\mathbb{R}^n} \Phi(x - y) f(y) dy$$

It is sufficient to show that  $\tilde{u}$  is bounded. Then

$$|\tilde{u}| \leq C \int_{B(0, k)} \Phi(x - y) dy$$

$\forall C_1, \exists \alpha$  such that  $|z| \geq \alpha \implies |\Phi(z)| \leq C_1$ .

## Case 2

For  $\text{dist}(x, B(0, k)) \geq \alpha$ ,  $\text{dist}(x, y) \geq \alpha$ ,  $\forall y \in B(0, k)$ . Then

$$\begin{aligned} |x - y| &\geq \alpha \\ \frac{1}{|x - y|} &\leq \frac{1}{\alpha} \\ \frac{1}{|x - y|^{n-2}} &\leq \frac{1}{\alpha^{n-1}} \end{aligned}$$

and

$$|\tilde{u}(x)| \leq C \int_{B(0, k)} \frac{1}{|x - y|^{n-2}} dy \leq \frac{C}{\alpha^{n-2}} \int_{B(0, k)} dy$$

## Theorem 10: Harmonic Implies Analytic

Let  $U \subseteq \mathbb{R}^n$  open,  $u \in C^2(U)$  harmonic. Then  $u$  is analytic in  $U$ .

## Proof

Let  $x_0 \in U$ . We want to prove that the power series converges to  $u(x)$  for  $x$  in a neighborhood around  $x_0$ .

Let  $r = \text{dist}\left(x_0, \frac{\partial U}{4}\right)$ ,  $M = \frac{1}{\alpha(n)r^n} \|u\|_{L^1(B(x_0, r))} \subset U$ .

IMAGE HERE - 1

We want to analyze  $x \in B(x_0, r)$ .

Notice that  $B(x, r) \subset B(x_0, 2r)$ , and  $z \in B(x, r)$  gives  $|z - x| < r$  so

$$|z - x_0| \leq \underbrace{|z - x|}_{\leq r} + \underbrace{|x - x_0|}_{\leq r} \leq 2r$$

Applying estimates on  $B(x, r)$ ,  $|\alpha| = k$ ,

$$\begin{aligned} |D^\alpha u(x)| &\leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x, r))} \\ &\leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0, 2r))} \end{aligned}$$

and

$$\sup_{x \in B(x_0, r)} |D^\alpha u(x)| \leq \frac{(2^{n+1}nk)^k}{\alpha(n)r^{n+k}} \|u\|_{L^1(B(x_0, 2r))}$$

Notice, by Stirling's approximation or Taylor expansion,  $\frac{k^k}{k!} < e^k$ ,  $\forall k \geq 1$ . So

$$|\alpha|^{|\alpha|} < e^{|\alpha|} |\alpha|!$$

and

$$n^k = \underbrace{(1 + \dots + 1)}_{n\text{-times}}^k = \sum_{|\beta|=k} \frac{|\beta|!}{\beta!} \geq \frac{|\alpha|!}{\alpha!}$$

where  $|\alpha|! \leq \alpha! n^k$ ,  $\beta = (\beta_1, \dots, \beta_n)$  and  $\beta! := \beta_1! \beta_2! \dots \beta_n!$ . Therefore

$$|\alpha|^{|\alpha|} \leq e^{|\alpha|} |\alpha|! \leq e^{|\alpha|} \alpha! n^k$$

and finally

$$(*) \quad k^k \leq e^k \alpha! n^k$$

Applying  $*$  to the above inequality,

$$\begin{aligned} \sup_{x \in B(x_0, r)} |D^\alpha u(x)| &\leq \frac{(2^{n+1}n)^k e^k \alpha! n^k}{\alpha(n)r^n r^k} \|u\|_{L^1(B(x_0, 2r))} \\ &\leq \left( \frac{2^{n+1}n^2 e}{r} \right)^k \cdot \alpha! M \end{aligned}$$

Let us analyze the Taylor expansion

$$\sum_{k=0}^N \sum_{|\alpha|=k} \frac{D^\alpha u(x_0)}{\alpha!} (x - x_0)^\alpha$$

Where  $\alpha = (\alpha_1, \dots, \alpha_n)$ ,  $y \in \mathbb{R}^n$  and  $y^\alpha = y_1^{\alpha_1} \dots y_n^{\alpha_n}$ .

Pick  $|x - x_0| \leq \frac{r}{2^{n+2}n^3e}$ . We want to prove that the remainder  $R_N(x) \xrightarrow{N \rightarrow \infty} 0$ .

$$\begin{aligned} R_N(x) &= u(x) - \sum_{k=0}^{N-1} \sum_{|\alpha|=k} \frac{D^\alpha u(x_0)}{\alpha!} (x - x_0)^\alpha \\ &= \sum_{|\alpha|=N} \frac{D^\alpha u(x_0 + t(x - x_0))(x - x_0)^\alpha}{\alpha!}, \quad \text{for some } |t| \leq 1 \end{aligned}$$

Using the remainder of the Taylor expansion with  $g(t) = u(x_0 + t(x - x_0))$  for  $g : I \rightarrow \mathbb{R}$ .

Homework: show this around  $t = 0$  at  $t = 1$ .

Note that  $u(x_0 + t(x - x_0))$  describes a straight line with  $t = 0 \implies u(x_0)$  and  $t = 1 \implies u(x)$ .

Notice also that  $x_0 + t(x - x_0) \in B(x_0, r)$ . Then, considering the supremum of the remainder,

$$|R_N(x)| \leq \sum_{|\alpha|=N} \left( \frac{2^{n+1}n^2e}{r} \right)^N \cdot M \alpha! \cdot \frac{|(x - x_0)^\alpha|}{\alpha!}$$

Remark: for  $\alpha = (\alpha_1, \dots, \alpha_n)$  and  $y = (y_1, \dots, y_n)$ ,

$$\begin{aligned} |y^\alpha| &= |y_1^{\alpha_1} \dots y_n^{\alpha_n}| \leq |y_1|^{\alpha_1} \dots |y_n|^{\alpha_n} \\ &\leq |y|^{\alpha_1} \dots |y|^{\alpha_n} \\ &= |y|^{\alpha_1 + \alpha_2 + \dots + \alpha_n} \\ &= |y|^\alpha \end{aligned}$$

Therefore

$$\begin{aligned} |R_N(x)| &\leq \sum_{|\alpha|=N} \left( \frac{2^{n+1}n^2e}{r} \right)^N \cdot M |x - x_0|^N \\ &\leq M \cdot \sum_{|\alpha|=N} \left( \frac{2^{n+1}n^2e}{r} \right)^N \left( \frac{r}{2^{n+2}n^3e} \right)^N \\ &= M \cdot \sum_{|\alpha|=N} \left( \frac{1}{2n} \right)^N \\ &\leq M \left( \frac{1}{2n} \right)^N \sum_{|\alpha|=N} 1 \\ &\leq M \left( \frac{1}{2n} \right)^N n^N \\ &= M \left( \frac{1}{2} \right)^N \end{aligned}$$

Note that  $\sum_{|\alpha|=N} 1 \leq n^N$  since

$$\alpha = (\alpha_1, \dots, \alpha_n) = (\alpha_{1_N}, \dots, \alpha_{i_N}) = n^N$$

### Theorem 11: Harnack's Inequality

Define  $V \subset\subset U$  as “ $V$  totally contained in  $U$ ” meaning  $\overline{V}$  compact and  $V \subseteq \overline{V} \subseteq U$ .

IMAGE HERE - 2

Let  $U$  open and  $u \in C^2(U)$  harmonic and non-negative.

Then for each connected open set  $V \subset\subset U$

$$\sup_V u \leq C \inf_V u$$

for some  $C$  that depends on  $V$ .

#### Remark

Then

$$\frac{1}{C} u(y) \leq u(x) \leq C u(y), \quad \forall x, y \in V$$

Since

$$u(x) \leq \sup_V u \leq C \inf_V u \leq C u(y)$$

and

$$\frac{1}{C} u(y) \leq \frac{1}{C} \sup_V u \leq \inf_V u \leq u(x).$$

#### Proof

Take  $r = \frac{\text{dist}(V, \partial U)}{4} > 0$ .

- Case 1

Let us suppose that  $x, y \in V$  such that  $|x - y| < r$ .

IMAGE HERE - 3

Notice  $B(x, 2r) \subseteq U$ . Applying mean-value formulas,

$$u(x) = \oint_{B(x, 2r)} u = \frac{1}{\alpha(n)(2r)^n} \int_{B(x, 2r)} u$$

But notice that  $B(y, r) \subseteq B(x, 2r)$ , so

$$u(x) \geq \frac{1}{\alpha(n)2^n r^n} \int_{B(y, r)} u = \frac{1}{2^n} \oint_{B(y, r)} u = \frac{1}{2^n} u(y)$$

That is, if  $x, y \in V$  such that  $|x - y| < r$ , then  $u(x) \geq \frac{1}{2^n} u(y)$  and, mutatis mutandis,  $u(y) \geq \frac{1}{2^n} u(x)$ .

- Case 2

Let us cover  $\overline{V}$  by an open covering of balls  $\{B_i\}_{i=1}^N$  such that the radius of each ball is  $\frac{r}{2}$  and  $B_i \cap B_{i-1} \neq \emptyset$ .

IMAGE HERE - 4

Then  $u(x) \geq \frac{1}{2^n} u(z) \frac{1}{2^n} u(y)$ , so  $u(x) \geq \frac{1}{2^{2n}} u(y)$ .

In the same way,  $u(y) \geq \frac{1}{2^{2n}} u(x)$ .

IMAGE HERE - 5

For three balls,  $u(x) \geq \frac{1}{2^{3n}} u(y)$  and  $u(y) \geq \frac{1}{2^{3n}} u(x)$ .

Since we have a finite covering of  $N$  balls, the same strategy gives

$$u(x) \geq \frac{1}{2^{Nn}} u(y)$$

$$u(y) \geq \frac{1}{2^{Nn}} u(x)$$

and

$$\frac{1}{2^{Nn}} \leq u(x)$$

Taking the supremum  $y \in V$  ;

$$\sup_{y \in V} u(y) \leq 2^{Nn} u(x)$$

taking the infimum  $x \in V$

$$\inf_{x \in V} u(y)$$

## Recap: Laplace Equation

- Fundamental Solution
  - Poisson Equation in  $\mathbb{R}^n$
- Mean-value Formulas
- Properties
  - Strong Maximum / Minimum Principles
    - \* Uniqueness of the Poisson Equation on Bounded Domains
  - Regularity
  - Derivative Estimates
  - Liouville's Theorem
    - \* Representation Formula
      - Uniqueness of the Poisson Equation up to a Constant on  $\mathbb{R}^n$  for Bounded Functions
  - Analyticity
  - Harnack's Inequality

## Green's Functions

For  $U$  open and bounded,  $\partial U \in C^1$ .

Goal: We want to solve  $-\Delta u = f$  on  $U$  and  $u = g$  on  $\partial U$ .

### Recall: Green's Formula

$$\int_{\Omega} u \Delta v - v \Delta u = \int_{\partial \Omega} u \frac{\partial v}{\partial \eta} - v \frac{\partial u}{\partial \eta}$$

### Obtaining Green's Formula

Let  $x \in U$  and consider  $u(y)$ ,  $\Phi(y-x)$  as functions of  $y$ .

Let  $\varepsilon > 0$  and consider  $V_{\varepsilon} = U \setminus B_{\varepsilon}(x)$ . Applying Green's formula;  $\Omega = V_{\varepsilon}$ ,

$$\int_{V_{\varepsilon}} \underbrace{u(y) \Delta_y \Phi(y-x) - \Phi(y-x) \Delta_y u}_{=0} = \int_{\partial V_{\varepsilon}} u(y) \frac{\partial \Phi(y-x)}{\partial \eta} - \Phi(y-x) \frac{\partial u(y)}{\partial \eta}$$

IMAGE HERE - 6

**January 31, 2024**

## Green's Functions

Goal is to solve for  $U \subseteq \mathbb{R}^n$  open and bounded,

$$\begin{cases} -\Delta u = f, & \text{in } U \\ u = g, & \text{on } \partial U \end{cases}$$

by obtaining Green's function.

Let  $x \in U$  and assume  $u \in C^2(U)$ , and consider  $u(y)$  and  $\Phi(y-x)$ .

Recall Green's formula  $\int_{\Omega} u \Delta v - v \Delta u = \int_{\partial \Omega} u \frac{\partial v}{\partial \eta} - v \frac{\partial u}{\partial \eta}$ .

Then, let  $\varepsilon > 0$  and define  $V_{\varepsilon} = U \setminus B(x, \varepsilon)$ .

IMAGE HERE - 1

By applying Green's Formula,

$$\int_{V_{\varepsilon}} u(y) \underbrace{\Delta \Phi(y-x)}_0 - \Phi(y-x) \Delta u(y) = \int_{\partial V_{\varepsilon}} \underbrace{u \frac{\partial \Phi(y-x)}{\partial \eta}}_{\square_1} - \underbrace{\Phi(y-x) \frac{\partial u}{\partial \eta}}_{\square_2}$$

Notice that  $\partial V_{\varepsilon} = \partial U \cup \partial B(x, \varepsilon)$ .

Let us analyze  $\square$  along  $\partial B(x, \varepsilon)$

For  $\square_2$  along  $\partial B(x, \varepsilon)$ ,

$$\begin{aligned} \left| \int_{\partial B(x, \varepsilon)} \Phi(y-x) \frac{\partial u(y)}{\partial \eta} \right| &\leq \sup_{\bar{U}} |\nabla U| \int_{\partial B(x, \varepsilon)} \Phi(y-x) dS(y) \\ &= \frac{C}{\varepsilon^{n-2}} \int_{\partial B(x, \varepsilon)} dS(y) \\ &= \frac{C \varepsilon^{n-1}}{\varepsilon^{-2}} \\ &= c \varepsilon \end{aligned}$$

Then  $\lim_{\varepsilon \rightarrow 0} \int_{\partial B(x, \varepsilon)} \square_2 = 0$ .

Now, for  $\square_1$  along  $\partial B(x, \varepsilon)$  and recalling  $\nabla \Phi(z) = \frac{-z}{n\alpha(n)|z|^n}$  while  $\eta(z) = \frac{-z}{|z|}$  such that

$$\frac{\partial \Phi}{\partial \eta}(z) = \nabla \Phi \cdot \eta = \frac{|z|^2}{n\alpha(n)|z|^{n+1}} = \frac{1}{n\alpha(n)|z|^{n-1}}$$

we have

$$\begin{aligned} \int_{\partial B(x, \varepsilon)} u(y) \frac{\partial \Phi(y-x)}{\partial \eta} dS(y) & \stackrel{z=y-x}{=} \int_{\partial U(0, \varepsilon)} u(z+x) \frac{\partial \Phi(z)}{\partial \eta} |z| ds(z) \\ & = \frac{1}{n\alpha(n)} \int_{\partial B(0, \varepsilon)} \frac{u(z+x)}{|z|^{n-1}} dS(z) \\ & = \frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B(0, \varepsilon)} u(z+x) dS(z) \\ & \stackrel{y=z+x}{=} \frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B(x, \varepsilon)} u(y) dS(y) \\ & = \oint_{\partial B(x, \varepsilon)} u(y) dS(y) \end{aligned}$$

Then  $\lim_{\varepsilon \rightarrow 0} \int_{\partial B(x, \varepsilon)} \square_1 = u(x)$ . It follows, then, that

$$\int_U -\Phi(y-x) \Delta u(y) = \int_{\partial U} \overbrace{u \frac{\partial \Phi(y-x)}{\partial \eta} - \Phi(y-x) \frac{\partial u}{\partial \eta}}^{\square} + u(x)$$

$\square_1 \qquad \qquad \square_2$

That is

$$u(x) \stackrel{\square_4}{=} - \int_U \Phi(y-x) \Delta u + \int_{\partial U} \Phi(y-x) \frac{\partial u}{\partial \eta} - u \frac{\partial \Phi(y-x)}{\partial \eta}$$

Notice that we have  $-\Delta u = f$  in  $U$  and  $u = g$  on  $\partial U$ , but we will need  $\frac{\partial u}{\partial \eta} |_{\partial U}$ .

### Definition: Corrector Function

Given a domain  $U \subseteq \mathbb{R}^n$  open and bounded with  $x \in U$ , define the function  $\phi^x(y)$  that satisfies the following

$$\begin{cases} \Delta \phi^x(y) = 0, & \text{in } U \\ \phi^x(y) = \Phi(y-x), & \text{on } y \in \partial U \end{cases}$$

Note that we do not know that such a function exists.

### Green's Function Continued

Suppose that we have  $\phi^x(y)$ . Then, applying green's formula for  $u(y)$  and  $\phi^x(y)$ ,

$$\int_U u \underbrace{\Delta \phi^x(y)}_0 - \phi^x(y) \Delta u = \int_{\partial U} u \frac{\partial \phi^x(y)}{\partial \eta} - \underbrace{\phi^x(y) \frac{\partial u}{\partial \eta}}_{\Phi(y-x) \frac{\partial u}{\partial \eta}}$$



Then

$$\int_{\partial U} \Phi(y-x) \frac{\partial u}{\partial \eta} \Big|_{\square_3} = \int_{\partial U} u \frac{\partial \phi^x(y)}{\partial \eta} + \int_U \phi^x(y) \Delta u$$

Replacing  $\square_3$  in  $\square_4$ ,

$$u(x) = - \int_U \Phi(y-x) \Delta u + \int_{\partial U} u \frac{\partial \phi^x(y)}{\partial \eta} + \int_U \phi^x(y) \Delta u - \int_{\partial U} u \frac{\partial \Phi(y-x)}{\partial \eta}$$

and, therefore,

$$u(x) = - \int_U \Delta u [\Phi(y-x) - \phi^x(y)] - \int_{\partial U} u \frac{\partial}{\partial \eta} [\Phi(y-x) - \phi^x(y)]$$

### Definition: Green's Function

Given a domain  $U \subseteq \mathbb{R}^n$ , the Green's function for  $x \in U$  is defined by

$$G(x, y) := \Phi(y-x) - \phi^x(y)$$

### Theorem: Representation Formula

Suppose  $U \subseteq \mathbb{R}^n$ , and  $u \in C^2(U)$  that solves

$$\begin{cases} -\Delta u = f, & \text{in } U \\ u = g, & \text{on } \partial U \end{cases}$$

Then,

$$u(x) = \int_U f G(x, y) - \int_{\partial U} g \frac{\partial G(x, y)}{\partial \eta}$$

### Interpretation of the Green's Functions

$$\Delta_y G(x, y) = \Delta_y \Phi(y-x) - \underbrace{\Delta_y \phi^x(y)}_0 = \delta^x(y)$$

and

$$G(x, y) = \Phi(y-x) - \phi^x(y) = 0, \quad y \in \partial U$$

That is, it is the Dirac delta on the interior which vanishes at the boundary.

### Theorem: Symmetry of the Green's Function

For all  $x, y \in U$ ,  $x \neq y$ , we want to show that  $G(x, y) = G(y, x)$ .

## Proof

Let  $x, y \in U$ ,  $x \neq y$ .

Define  $V(z) := G(x, z)$  and  $W(z) := G(y, z)$ .

Notice that  $\Delta_z V = 0$  for  $z \neq x$  and  $\Delta_z W = 0$  for  $z \neq y$  and  $V(z) = W(z) = 0$  for  $z \in \partial U$ .

IMAGE HERE - 2

Then, let us consider  $\varepsilon > 0$  and

$$\Omega_\varepsilon := U \setminus \left[ B(x, \varepsilon) \sqcup B(y, \varepsilon) \right]$$

Then

$$\begin{aligned} 0 &= \int_{\Omega_\varepsilon} \underbrace{W \Delta V}_0 - \underbrace{V \Delta W}_0 = \int_{\partial \Omega_\varepsilon} W \frac{\partial V}{\partial \eta} - V \frac{\partial W}{\partial \eta} \\ &= \int_{\partial U} W \frac{\partial V}{\partial \eta} - V \frac{\partial W}{\partial \eta} + \int_{\partial B(x, \varepsilon)} W \frac{\partial V}{\partial \eta} - V \frac{\partial W}{\partial \eta} + \int_{\partial B(y, \varepsilon)} W \frac{\partial V}{\partial \eta} - V \frac{\partial W}{\partial \eta} \end{aligned}$$

It follows that

$$\underbrace{\int_{\partial B(x, \varepsilon)} \overbrace{W \frac{\partial V}{\partial \eta} - V \frac{\partial W}{\partial \eta}}^{(a)}}_{\heartsuit_1} = \underbrace{\int_{\partial B(y, \varepsilon)} \overbrace{V \frac{\partial W}{\partial \eta} - W \frac{\partial V}{\partial \eta}}^{(b)}}_{\heartsuit_2}$$

Let us analyze (b), fixing  $\varepsilon_0 > 0$  such that  $\varepsilon < \varepsilon_0$

$$\begin{aligned} \left| \int_{\partial B(x, \varepsilon)} V \frac{\partial W}{\partial \eta} \right| &\leq \sup_{z \in \partial B(x, \varepsilon)} |V(z)| \int_{\partial B(x, \varepsilon)} \left| \frac{\partial W}{\partial \eta}(z) \right| dS(z) \\ &\leq \sup_{z \in \partial B(x, \varepsilon_0)} |\nabla W(z)| \int_{\partial B(x, \varepsilon)} dS(z) \\ &\leq C\varepsilon^{n-1} \sup_{z \in \partial B(x, \varepsilon)} |V(z)| \\ &\leq C\varepsilon^{n-1} \left( \frac{C}{\varepsilon^{n-2} + C} \right) \\ &= C\varepsilon + C\varepsilon^{n-1} \end{aligned}$$

Since, given  $z \in \partial B(x, \varepsilon)$ ,

$$V(z) = G(x, z) = \Phi(z - x) - \phi^x(z)$$

we have

$$\begin{aligned} |V(z)| &\leq |\Phi(z - x)| + |\phi^x(z)| \\ &\leq \frac{C}{\varepsilon^{n-2}} + \sup_{z \in B(x, \varepsilon_0)} |\phi^x(z)| \end{aligned}$$

Thus, we have  $\lim_{\varepsilon \rightarrow 0} \int_{\partial B(x, \varepsilon)} V \frac{\partial W}{\partial \eta} = 0$ .

Let us analyze (a),

$$\begin{aligned}
\int_{\partial B(x, \varepsilon)} W(z) \frac{\partial V}{\partial \eta}(z) dS(z) &= \int_{\partial B(x, \varepsilon)} W(z) \left[ \frac{\Phi(z-x)}{\partial \eta} - \frac{\partial \phi^x(z)}{\partial \eta} \right] dS(z) \\
&= \int_{\partial B(x, \varepsilon)} \overbrace{W(z) \frac{\partial \Phi(z-x)}{\partial \eta}}^{(e)} - \overbrace{W(z) \frac{\partial \phi^x(z)}{\partial \eta}}^{(h)} dS(z)
\end{aligned}$$

Analyzing (h),

$$\begin{aligned}
\left| \int_{\partial B(x, \varepsilon)} W(z) \frac{\partial \phi^x(z)}{\partial \eta} \right| &\leq \sup_{\partial B(x, \varepsilon_0)} |\nabla \phi^x(z)| |W(z)| \int_{\partial B(x, \varepsilon)} dS(z) \\
&= C\varepsilon^{n-1}
\end{aligned}$$

Then  $\lim_{\varepsilon \rightarrow 0} h = 0$  and

$$\lim_{\varepsilon \rightarrow 0} \int_{\partial B(x, \varepsilon)} W(z) \frac{\partial \Phi(z-x)}{\partial \eta} = W(x)$$

So  $\lim_{\varepsilon \rightarrow 0} (a) = W(x)$ . Then

$$\lim_{\varepsilon \rightarrow 0} \int_{\partial B(x, \varepsilon)} W \frac{\partial V}{\partial \eta} - V \frac{\partial W}{\partial \eta} = W(x)$$

Applying the same process,

$$\lim_{\varepsilon \rightarrow 0} \int_{\partial B(y, \varepsilon)} V \frac{\partial W}{\partial \eta} - W \frac{\partial V}{\partial \eta} = V(y)$$

Therefore  $W(x) = V(y)$  and  $G(y, x) = G(x, y)$ .

### Definition: Half Space

Define the half space  $\mathbb{R}_+^n = \{(x_1, \dots, x_n) : x_n > 0\}$ .

IMAGE HERE - 3

### Definition: Reflection

For a  $x = (x_1, \dots, x_n) \in \mathbb{R}_+^n$ , define its reflection  $\tilde{x} = (x_1, \dots, -x_n)$ .

### Green's Function in the Half Space

We want to find  $\phi^x(y)$  that solves

$$(*) \begin{cases} \Delta \phi^x(y) = 0, & \text{in } \mathbb{R}_+^n \\ \phi^x(y) = \Phi(y-x), & y \in \partial \mathbb{R}_+^n \end{cases}$$

Let us consider  $\phi^x(y) := \Phi(y - \tilde{x})$ ,  $x, y \in \mathbb{R}_+^n$ . Then  $\phi^x(y)$  satisfies  $*$ .

Then we can see that  $\Delta \phi^x(y) = 0$ .

Let  $y \in \partial \mathbb{R}_+^n$  such that  $y = (y_1, \dots, y_{n-1}, 0)$ . So

$$\begin{aligned}\phi^x(y) &= \Phi(y - \tilde{x}) \\ &= \Phi(|y - \tilde{x}|) \\ &= \Phi\left(\sqrt{(y_1 - x_1)^2 + \dots + (y_{n-1} - x_{n-1})^2 + (0 - x_n)^2}\right) \\ &= \Phi(|y - x|^2) \\ &= \Phi(y - x)\end{aligned}$$

**February 5, 2024**

**Recall: Green's Function**

$$G(x, y) = \Phi(y - x) - \phi^x(y).$$

For  $U \subset \mathbb{R}_+^n$ , when

$$G(x, y) = \Phi(y - x) - \Phi(y - \tilde{x})$$

we proved that if  $u \in C^2(\overline{U})$ ,

$$\begin{cases} -\Delta u = f, & U \\ u = g, & \partial U \end{cases}$$

then

$$u(x) = \int_U f G(x, y) dy - \int_{\partial U} g \frac{\partial G(x, y)}{\partial \eta} dS(y)$$

Let us consider

$$\begin{cases} \Delta u = 0, & \mathbb{R}_+^n \\ u = g, & \partial \mathbb{R}_+^n \end{cases}$$

such that

$$u(x) = - \int_{\partial \mathbb{R}_+^n} g \frac{\partial G(x, y)}{\partial \eta} dS(y)$$

Let us compute  $\frac{\partial G}{\partial \eta}$ .

IMAGE HERE - 1 UPPER HALF SPACE WITH NORMAL VECOTR ETA

Recall

$$\begin{aligned}\nabla \Phi(z) &= \frac{-z}{n\alpha(n)|z|^n} \\ \frac{\partial \Phi(z)}{\partial z_n} &= \frac{-z_n}{n\alpha(n)|z|^n}\end{aligned}$$

so, since  $y - \tilde{x}_n = y_n + x_n$ ,

$$\begin{aligned}\frac{\partial G}{\partial \eta} &= \nabla G(x, y) \cdot \eta \\ &= -\frac{\partial G(x, y)}{\partial y_{n+1}} \\ &= -\frac{\partial}{\partial y_{n+1}} (\Phi(y - x) - \Phi(y - \tilde{x})) \\ &= -\left[ \frac{-(y_n - x_n)}{n\alpha(n)|y - x|^n} - \frac{-(y_n + x_n)}{n\alpha(n)|x - \tilde{x}|^n} \right]\end{aligned}$$

But recall that if  $y \in \partial\mathbb{R}_+^n$ ,  $|y - x| = |y - \tilde{x}|$ . Then  $y \in \partial\mathbb{R}_+^n$ ,

$$\frac{\partial G(x, y)}{\partial \eta} = -\frac{1}{n\alpha(n)|y - x|^n} [-y_n + x_n + y_n + x_n] = -\frac{2x_n}{n\alpha(n)|y - x|^n}$$

Then

$$u(x) = \int_{\partial\mathbb{R}_+^n} \frac{g(y)2x_n}{n\alpha(n)|y - x|^n} dS(y)$$

**Definition: Poisson Kernel**

$$K(x, y) = \frac{2x_n}{n\alpha(n)|y - x|^n} = \frac{\partial G}{\partial y_n}$$

is called the Poisson Kernel and

$$u(x) = \int_{\partial\mathbb{R}_+^n} g(y)K(x, y) dS(y)$$

is called the Poisson Formula.

Notice (HW):  $\int_{\partial\mathbb{R}_+^n} K(x, y) dy = 1$ ,  $\forall x \in \mathbb{R}_+^n$  (hint: apply spherical coordinates).

**Theorem 14:**

Define

$$(*) \quad u(x) = \int_{\partial\mathbb{R}_+^n} K(x, y)g(y) dS(y)$$

Suppose that  $g \in C^\infty(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$ .

Then

1.  $u \in C^\infty(\mathbb{R}_+^n) \cap L^\infty(\mathbb{R}_+^n)$ .
2.  $\Delta u = 0$ ,  $\mathbb{R}_+^n$ .
3.  $\lim_{x \rightarrow x^0} u(x) = g(x^0)$ ,  $x \in \mathbb{R}_+^n$ ,  $x^0 \in \partial\mathbb{R}_+^n$ .

## Proof

We know  $G(x, y)$  satisfies

$$\Delta_y G(x, y) = \delta^x(y).$$

Notice that  $y \rightarrow G(x, y)$  is harmonic for  $x \neq y$ .

Recall that  $G(x, y) = G(y, x)$ , so  $x \rightarrow G(x, y)$  is harmonic for  $x \neq y$ .

Then  $x \rightarrow \frac{\partial G(x, y)}{\partial y_n}$  is harmonic ( $*_2$ ) for  $x \neq y$  and for  $y \in \partial \mathbb{R}_+^n$ .

Homework: compute this directly.

Noticing that  $K$  is smooth when  $x \neq y$ , then

$$\frac{\partial u}{\partial x_i} = \int_{\partial \mathbb{R}_+^n} \frac{\partial}{\partial x_i} K(x, y) g(y) dS(y)$$

Homework: justify putting the limit inside the integral.

Homework: prove that  $\frac{\partial u}{\partial x_i}$  is continuous.

By repeatedly taking derivatives, we see  $u \in C^\infty(\mathbb{R}_+^n)$ .

Moreover,

$$\Delta_x u = \int_{\partial \mathbb{R}_+^n} \underbrace{\Delta_x K(x, y)}_{=0} g(y) dS(y) = 0$$

by  $*_2$ . Then

$$|u(x)| \leq \int_{\partial \mathbb{R}_+^n} |K(x, y)| |g(y)| dS(y) \leq \|g\|_{L^\infty(\mathbb{R}^{n-1})} \underbrace{\int_{\partial \mathbb{R}_+^n} K(x, y) dS(y)}_{=1} < \infty$$

For part c, consider  $x^0 \in \partial \mathbb{R}_+^n$  and  $\varepsilon > 0$ .

Since  $g \in C^\infty(\mathbb{R}^{n-1})$ , let  $\delta > 0$  such that  $|y - x^0| < \delta \implies |g(y) - g(x^0)| < \varepsilon$  for  $y \in \partial \mathbb{R}_+^n$ .

IMAGE HERE - 2 DELTA BALL AROUND  $x^0$  HALF DELTA BALL WITH  $x$  INSIDE

Now, let us consider  $|x - x^0| < \frac{\delta}{2}$ .

$$\begin{aligned} |u(x) - g(x^0)| &= \left| \int_{\partial \mathbb{R}_+^n} K(x, y) g(y) - K(x, y) g(x^0) dS(y) \right| \\ &\leq \int_{\partial \mathbb{R}_+^n} K(x, y) |g(y) - g(x^0)| dS(y) \\ &= \underbrace{\int_{\partial \mathbb{R}_+^n \cap B(x^0, \delta)} K(x, y) |g(y) - g(x^0)| dS(y)}_I + \underbrace{\int_{\partial \mathbb{R}_+^n \cap B^c(x^0, \delta)} K(x, y) |g(y) - g(x^0)| dS(y)}_{II} \end{aligned}$$

Then

$$I \leq \varepsilon \int_{\partial \mathbb{R}_+^n \cap B(x^0, \delta)} K(x, y) dS(y) \leq \varepsilon$$

Now, we want to control  $II$

$$\begin{aligned} \int_{\partial \mathbb{R}_+^n \cap B^c(x^0, \delta)} K(x, y) |g(y) - g(x^0)| dS(y) &\leq C \|g\|_{L^\infty} \int_{\partial \mathbb{R}_+^n \cap B^c(x^0, \delta)} K(x, y) dS(y) \\ &= \frac{2C}{n\alpha(n)} \int_{\partial \mathbb{R}_+^n \cap B^c(x^0, \delta)} \frac{x_n}{|x - y|^n} dS(y) \end{aligned}$$

We want to control  $|x^0 - y|$  with something related to  $|x - y|$ .  
 We know  $|y - x^0| > \delta$  and we will consider  $|x - x^0| < \frac{\delta}{2}$ . So

$$|y - x^0| \leq |y - x| + |x - x^0| \leq |y - x| + \frac{\delta}{2} \leq |y - x| + \frac{|y - x^0|}{2}$$

So  $\frac{|y - x^0|}{2} \leq |y - x|$  implies that  $\frac{1}{|y - x|^n} \leq \frac{2^n}{|y - x^0|^n}$ . Therefore

$$\begin{aligned} \int_{\partial \mathbb{R}_+^n \cap B^c(x^0, \delta)} K(x, y) |g(y) - g(x^0)| dS(y) &\leq C x_n \int_{\partial \mathbb{R}_+^n \cap B^c(x^0, \delta)} \frac{1}{|y - x^0|^n} dS(y) \\ &= \int_{\delta}^{\infty} \int_{\partial B^{n-1}(x^0, r)} \frac{1}{r^n} dS(y) dr \\ &= C \int_{\delta}^{\infty} \frac{1}{r^n} r^{n-2} dr \\ &= C \int_{\delta}^{\infty} \frac{1}{r^2} dr \\ &= C \left( \frac{1}{r} \right) \Big|_{\delta}^{\infty} \\ &= \frac{C}{\delta} \end{aligned}$$

Then  $II \leq \frac{C x_n}{\delta}$ . Now let us consider  $|x - x^0| < \frac{\delta}{J}$  where  $\frac{1}{J} < \varepsilon$ . Then

$$II \leq \frac{C |x - x^0|}{\delta} \leq C \frac{\delta}{\delta J} \leq C \varepsilon$$

## Energy Methods: Uniqueness

Consider the boundary value problem

$$(*) \quad \begin{cases} -\Delta u = f, & U, f \in C(U) \\ u = g, & \partial U, g \in C(\partial U) \end{cases}$$

with  $U$  open and bounded in  $\mathbb{R}^n$ ,  $u \in C^2(\overline{U})$  and  $\partial U \in C^1$ .

### Theorem 16: Uniqueness

There exists at most one solution  $u \in C^2(\overline{U})$  for  $*$ .

#### Proof

Let us suppose that  $\tilde{u}$  is another solution.

Then  $w := u - \tilde{u}$  solves

$$\begin{cases} \Delta w = 0, & U, w \in C^2(\overline{U}) \\ w = 0, & \partial U \end{cases}$$

where

$$0 = \int_U w \Delta w = - \int_U |\nabla w|^2 + \int_{\partial U} w \frac{\partial w}{\partial \eta}$$

so

$$0 = \int_U |\nabla w|^2 \implies \nabla w = 0 \implies w = 0 \implies u = \tilde{u}$$

### Definition: Energy Functional

Let us consider

$$A = \{w \in C^2(\overline{U}) : W|_{\partial U} = g\}$$

for  $g \in C(\partial U)$  and  $f \in C(U)$ .

Define the energy functional  $I : A \rightarrow \mathbb{R}$  given by  $I(w) := \int_U \frac{|\nabla w|^2}{2} - fw$ .

### Energy Methods: Dirichlet Principle

Calculus of variations applied to the Laplace equation.

#### Theorem:

Suppose  $u \in C^2(\overline{U})$  is a solution to the problem

$$\square \quad \begin{cases} -\Delta u = f, & U \\ u = g, & \partial U \end{cases}$$

Then,

$$(*) \quad I(u) = \min_{w \in A} \{I(w)\}$$

Moreover, if  $u \in A$  such that  $*$  happens, then  $u$  satisfies  $\square$ .

#### Proof

( $\implies$ ) For  $w \in A$ ,

$$\begin{aligned} 0 &= \int_U \underbrace{(-\Delta u - f)}_{=0} (u - w) \\ &= \int_U -\Delta u (u - w) - \int_U f(u - w) \\ &= \int_U \nabla(u - w) \cdot \nabla u - \underbrace{\int_{\partial U} (u - w) \cdot \frac{\partial u}{\partial \eta}}_{=0} - \int_U f(u - w) \\ &= \int_U |\nabla u|^2 - \int_U \nabla w \cdot \nabla u - \int_U fu + \int_U fw \end{aligned}$$

Notice that, since  $|a - b|^2 \geq 0$  implies  $\frac{a^2 + b^2}{2} \geq ab$ ,

$$\int_U \nabla w \cdot \nabla u \leq \int_U |\nabla w| |\nabla u| \leq \frac{1}{2} \int_U |\nabla w|^2 + \frac{1}{2} \int_U |\nabla u|^2$$



Therefore

$$\begin{aligned} \int_U |\nabla u|^2 - \int_U f u &= \int_U \nabla w \cdot \nabla u - \int_U f w \\ &\leq \int_U \frac{|\nabla w|^2}{2} + \int_U \frac{|\nabla u|^2}{2} - \int_U f w \\ \int_U \frac{|\nabla u|^2}{2} - f u &\leq \int_U \frac{|\nabla w|^2}{2} - f w \end{aligned}$$

Then

$$I(u) \leq I(w), \quad \forall w \in A$$

$B/u \in A$ .

**February 7, 2024**

**Recall: Energy Functional**

For  $U \subseteq \mathbb{R}^n$  bounded,  $g \in C(\partial U)$ ,  $f \in C(\overline{U})$

$$A = \{w \in C^2(\overline{U}) : w|_{\partial U} = g\}$$

we have

$$I(w) := \int_U \frac{1}{2} |\nabla w|^2 - f w$$

**Theorem:**

Suppose  $u \in A$  such that  $I(u) = \min\{I(w) : w \in A\}$ . Then  $u$  satisfies

$$\begin{cases} -\Delta u = f, & \text{in } U \\ u = g, & \text{on } \partial U \end{cases}$$

**Proof**

Consider  $v \in C_c^\infty(U)$ .

Define  $i : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\tau \mapsto I(\tau) := I(u + \tau v)$ .

Notice that  $u + \tau v$  is a perturbation of  $u$  and, since  $u + \tau v \in C^2(\overline{U})$  while  $u + \tau v|_{\partial U} = u|_{\partial U} = g$ ,  $u + \tau v \in A$ . Then

$$i(0) = I(u) \leq I(u + \tau v) = i(\tau)$$

so  $i$  has a minimum point at  $\tau = 0$ . Compute

$$\begin{aligned} i(\tau) &= I(u + \tau v) \\ &= \int_U \frac{|\nabla(u + \tau v)|^2}{2} - f(u + \tau v) \\ &= \int_U \frac{|\nabla u|^2}{2} + \tau \langle \nabla u, \nabla v \rangle + \frac{\tau^2 |\nabla v|^2}{2} - \int_U f u - \tau \int_U f v \\ &= \int_U \frac{|\nabla u|^2}{2} + \tau \int_U \langle \nabla u, \nabla v \rangle + \frac{\tau^2}{2} \int_U |\nabla v|^2 - \int_U f u - \tau \int_U f v \end{aligned}$$

So  $i$  is a polynomial in  $\tau$ , and

$$i'(0) = i'(\tau)_{\tau=0} = \left( \int_U \langle \nabla u, \nabla v \rangle + \tau \int_U |\nabla v|^2 - \int_U f v \right)_{\tau=0}$$

So

$$\begin{aligned} 0 &= i'(0) \\ &= \int_U \langle \nabla u, \nabla v \rangle - \int_U f v \\ &= \int_U -\Delta u \cdot v + \underbrace{\int_U \frac{\partial u}{\partial \eta} \cdot v}_{=0} - \int_U f v \\ &= \int_U \underbrace{(-\Delta u - f)}_{=0} v \end{aligned}$$

Since  $0 = \int g v$ ,  $\forall v \in C_c^\infty(U)$  requires  $g \equiv 0$ .  
Then  $-\Delta u - f = 0$ .

## Heat Equation (Diffusion Equation)

The equations

$$(*) \begin{cases} u_t - \Delta u = 0, & \text{homogeneous case} \\ u_t - \Delta u = f, & \text{non-homogeneous case} \end{cases}$$

(note that  $\Delta u = \Delta_x u$ )

subject to some boundary and initial conditions  $t \geq 0$  time and  $x \in \mathbb{R}^n$ , space variable,  $x \in U$  and open set of  $\mathbb{R}^n$ .

$u : U \times (0, \infty) \rightarrow \mathbb{R}$  defined as  $(x, t) \mapsto u(x, t)$  with  $u$  unknown.

IMAGE HERE - 1

## Motivation: Fundamental Solution of the Heat Equation

We would like to have the following:

If  $u$  solves

$$\begin{cases} u_t - \Delta u = 0, & \mathbb{R}^n \times (0, \infty) \\ u(x, 0) = g(x), & \mathbb{R}^n \times \{0\} \end{cases}$$

then

$$u(x, t) = \int_{\mathbb{R}^n} G(x - y, t) g(y) dy$$

How do we get  $G$ ?

Let us suppose that  $u(\tilde{x}, \tilde{t})$  solves

$$\begin{cases} u_{\tilde{t}} - \Delta_{\tilde{x}} u = 0 \\ u(\tilde{x}, 0) = g(\tilde{x}) \end{cases}$$

We would like to have invariance under dilation.

$$v(x, t) := u(\lambda x, \lambda^2 t)$$

Such that

$$\begin{aligned} v_t &= \nabla U|_{(\lambda x, \lambda^2 t)} - \frac{\partial}{\partial t} \left[ \frac{\lambda x}{\lambda^2 t} \right] \\ &= \lambda^2 u_{\tilde{t}}(\lambda x, \lambda^2 t) \\ v_{x_i} &= \lambda u_{\tilde{x}_i}(\lambda x, \lambda^2 t) \\ v_{x_i x_i} &= \lambda^2 u_{\tilde{x}_i \tilde{x}_i}(\lambda x, \lambda^2 t) \end{aligned}$$

Therefore

$$v_t - \Delta_x v = \lambda^2 u_{\tilde{t}} - \lambda^2 \Delta_{\tilde{x}} u = \lambda^2 \underbrace{(u_{\tilde{t}} - \Delta_{\tilde{x}} u)}_{=0} = 0$$

with

$$v(x, 0) = u(\lambda x, 0) = g(\lambda x)$$

Then, applying the motivation,

$$v(x, t) = \int_{\mathbb{R}^n} G(x - y, t) g(\lambda y) dy = \int_{z=\lambda y} \int_{\mathbb{R}^n} G\left(x - \frac{z}{\lambda}, t\right) g(z) \frac{dz}{\lambda^n}$$

On the other hand,

$$v(x, t) = u(\lambda x, \lambda^2 t) = \int_{\mathbb{R}^n} G(\lambda x - z, \lambda^2 t) g(z) dz$$

It follows that

$$\begin{aligned} \frac{1}{\lambda^n} G\left(\overbrace{x - \frac{z}{\lambda}}^w, t\right) &= G(\lambda x - z, \lambda^2 t) \\ \frac{1}{\lambda^n} G(w, t) &= G(\lambda w, \lambda^2 t) \end{aligned}$$

If  $\lambda^2 t = 1$ , then

$$G(w, t) = \frac{1}{t^{n/2}} G\left(\frac{1}{\sqrt{t}} w, 1\right)$$

If we call  $G\left(\frac{w}{\sqrt{t}}, 1\right) = v\left(\frac{w}{\sqrt{t}}\right)$ , then we are looking at  $G(w, t) = \frac{1}{t^{n/2}} v\left(\frac{w}{t^{1/2}}\right)$ .  
So, we have motivation to define

$$u(x, t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right)$$

for  $\alpha, \beta$  appropriate and  $v(y) : \mathbb{R}^n \rightarrow \mathbb{R}$ .

## Obtaining a Fundamental Solution to the Heat Equation

Let us compute  $u_t$  and  $\Delta_x u$ .

$$\begin{aligned}
 u_t &= \frac{\partial}{\partial t} \left( \frac{1}{t^\alpha} v \left( \frac{x}{t^\beta} \right) \right) \\
 &= \frac{(-\alpha)}{t^{\alpha+1}} v \left( \frac{x}{t^\beta} \right) + \frac{1}{t^\alpha} \frac{\partial}{\partial t} \left( v \left( \frac{x}{t^\beta} \right) \right) \\
 &= \frac{(-\alpha)}{t^{\alpha+1}} v \left( \frac{x}{t^\beta} \right) + \frac{1}{t^\alpha} \cdot \nabla v \Big|_{\frac{x}{t^\beta}} \cdot \frac{\partial}{\partial t} \left( \frac{x}{t^\beta} \right) \\
 u_t &= \frac{(-\alpha)}{t^{\alpha+1}} v \left( \frac{x}{t^\beta} \right) + \frac{(-\beta)}{t^\alpha t^{\beta+1}} \nabla v \Big|_{\frac{x}{t^\beta}} \cdot x \quad \square_1
 \end{aligned}$$

and

$$\begin{aligned}
 \frac{\partial u}{\partial x_i} &= \frac{1}{t^\alpha} \frac{\partial}{\partial x_i} \left( v \left( \frac{x}{t^\beta} \right) \right) \\
 &= \frac{1}{t^\alpha} \nabla v \Big|_{\frac{x}{t^\beta}} \cdot \frac{\partial}{\partial x_i} \left( \frac{x}{t^\beta} \right) \\
 &= \frac{1}{t^{\alpha+\beta}} \frac{\partial v}{\partial x_i} \Big|_{\frac{x}{t^\beta}}
 \end{aligned}$$

while

$$\frac{\partial^2 u}{\partial x_i \partial x_i} = \frac{1}{t^{\alpha+2\beta}} \frac{\partial^2 v}{\partial x_i \partial x_i} \Big|_{\frac{x}{t^\beta}} \quad \square_2$$

Then, replacing  $\square_1$  and  $\square_2$  in  $*$ ,

$$-\frac{\alpha}{t^{\alpha+1}} v \left( \frac{x}{t^\beta} \right) - \frac{\beta}{t^{\alpha+\beta+1}} \nabla v \Big|_{\frac{x}{t^\beta}} \cdot x - \frac{1}{t^{\alpha+2\beta}} \Delta v \Big|_{\frac{x}{t^\beta}} \stackrel{?}{=} 0$$

Set  $y := \frac{x}{t^\beta}$

$$-\frac{\alpha}{t^{\alpha+1}} v(y) - \frac{\beta}{t^{\alpha+1}} \nabla v(y) \cdot y - \frac{1}{t^{\alpha+2\beta}} \Delta v(y) = 0$$

Multiplying through by  $-t^{\alpha+1}$ ,

$$\alpha v(y) + \beta \nabla v(y) \cdot y + \frac{1}{t^{2\beta-1}} \Delta v(y) = 0$$

Let us assume that  $2\beta - 1 = 0$  such that  $\beta = \frac{1}{2}$ , giving

$$\alpha v(y) + \frac{1}{2} \nabla v(y) \cdot y + \Delta v(y) = 0$$

Since the Laplacian is rotationally invariant, assume  $v(y) = w(|y|)$  for  $w : \mathbb{R}^+ \rightarrow \mathbb{R}$ .

Recall that  $\frac{\partial}{\partial y_i} |y| = \frac{\partial}{\partial y_i} \left( \sqrt{y_1^2 + \cdots + y_n^2} \right) = \frac{y_i}{|y|}$ . Now

$$\frac{\partial}{\partial y_i} v(y) = \frac{\partial}{\partial y_i} (w(|y|)) = w'(|y|) \cdot \frac{\partial}{\partial y_i} (|y|) = w'(|y|) \cdot \frac{y_i}{|y|}$$

$$\begin{aligned}
\frac{\partial^2 v(y)}{\partial y_i y_i} &= \frac{\partial}{\partial y_i} \left( w'(|y|) \right) \frac{y_i}{|y|} + w'(|y|) \cdot \frac{\partial}{\partial y_i} \left( \frac{y_i}{|y|} \right) \\
&= w''(|y|) \cdot \frac{y_i^2}{|y|^2} + w'(|y|) \left[ \frac{1}{|y|} + y_i \frac{\partial}{\partial y_i} \left( \frac{1}{|y|} \right) \right] \\
&= w''(|y|) \frac{y_i^2}{|y|^2} + w'(|y|) \left[ \frac{1}{|y|} - \frac{y_i^2}{|y|^3} \right]
\end{aligned}$$

Replacing in the PDE of  $v$ ,

$$\begin{aligned}
0 &= \alpha w(|y|) + \frac{1}{2} \frac{w'(|y|)y}{|y|} \cdot y + \sum_{i=1}^n w''(|y|) \frac{y_i^2}{|y|^2} + w'(|y|) \left[ \frac{1}{|y|} - \frac{y_i^2}{|y|^3} \right] \\
&= \alpha w(|y|) + \frac{1}{2} w'(|y|)|y| + w''(|y|) + w'(|y|) \left[ \frac{n}{|y|} - \frac{1}{|y|} \right]
\end{aligned}$$

If  $|y| = r$

$$0 = \alpha w(r) + \frac{1}{2} w'(r)r + w''(r) + w'(r) \frac{n-1}{r}$$

Take  $\alpha = \frac{n}{2}$  and multiply through by  $r^{n-1}$ ,

$$\begin{aligned}
0 &= \frac{nr^{n-1}}{2} w(r) + \frac{r^n}{2} w'(r) + w''(r)r^{n-1} + w'(r)(n-1)r^{n-2} \\
&= \frac{1}{2} [w(r)r^n]' + [w'(r)r^{n-1}]'
\end{aligned}$$

Then by the fundamental theorem of calculus,  $w'(r)r^{n-1} + \frac{w(r)r^n}{2} = C$ .

We would like  $w, w' \xrightarrow[r \rightarrow \infty]{} 0$ . Then  $C = 0$ , so

$$w'(r)r^{n-1} = -\frac{w(r)r^n}{2}$$

Which gives

$$w' = \frac{-wr}{2} \iff \frac{w'}{w} = -\frac{r}{2} \iff (\ln(w))' = \frac{-r}{2} \iff \ln(w) = -\frac{r^2}{4} + d$$

and, finally,

$$w(r) = be^{-\frac{r^2}{4}}$$

Then define

$$\begin{aligned}
u(x, t) &:= \frac{1}{t^{n/2}} v\left(\frac{x}{t^{1/2}}\right) \\
&= \frac{1}{t^{n/2}} w\left(\left|\frac{x}{t^{1/2}}\right|\right) \\
&= \frac{b}{t^{n/2}} e^{-\frac{1}{4}\left|\frac{x}{t^{1/2}}\right|^2} \\
&= \frac{b}{t^{n/2}} e^{-\frac{1}{4t}|x|^2}
\end{aligned}$$

Where  $b$  is chosen such that the expression integrates to 1.

## Definition: Fundamental Solution of the Heat Equation

The fundamental solution for the heat equation is given by

$$\begin{cases} \Phi(x, t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, & x \in \mathbb{R}^n, t > 0 \\ 0, & x \in \mathbb{R}^n, t < 0 \end{cases}$$

where we have chosen  $b = \frac{1}{(4\pi)^{n/2}}$ .

IMAGE HERE - 2

Notice that these match in the limit away from the origin ( $\lim_{(x,t) \rightarrow (x_0,0)} \Phi(x, t) = 0$ ).

Remark:  $\Phi(x, t)$  has a unique singularity at  $(0, 0)$ .

## February 12, 2024

### Recall: Heat Equation

$$\Phi(x, t) = \begin{cases} \frac{b}{(t)^{n/2}} e^{-\frac{|x|^2}{4t}}; & t > 0, x \in \mathbb{R}^n \\ 0; & t < 0 \end{cases}$$

Remark:  $\Phi$  is radial such that  $\Phi(x, t) = \Phi(|x|, t)$ .

### Lemma:

For each  $t > 0$ ,

$$\int_{\mathbb{R}^n} \Phi(x, t) dx = 1$$

### Proof

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi(x, t) dx &= \frac{b}{t^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}} dx \\ &= \frac{b}{t^{n/2}} \int_{\mathbb{R}^n} e^{-\left|\frac{x}{2\sqrt{t}}\right|^2} \\ &= \frac{b}{t^{n/2}} \int_{\mathbb{R}^n} e^{-|z|^2} (2\sqrt{t})^n dz \\ &= b 2^n \int_{\mathbb{R}^n} e^{-|z|^2} dz \\ &= 2^n b \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-z_1^2 - \cdots - z_n^2} dz_1 \cdots dz_n \\ &= 2^n b \left[ \int_{-\infty}^{\infty} e^{-x} dx \right]^n \end{aligned}$$

We need

$$\begin{aligned}
A &= \int_{-\infty}^{\infty} e^{-x^2} dx \\
A^2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} dx dy \\
&= \int_{\mathbb{R}^n} e^{-|z|^2} dz \\
&= \int_0^{\infty} \int_{\partial B_r^2} e^{-r^2} dS(z) dr \\
&= 2\pi \int_0^{\infty} e^{-r^2} r dr \\
&= \pi \int_0^{\infty} e^{-s} ds \\
&= -\pi(e^{-s})|_0^{\infty} = -\pi(0-1) = \pi
\end{aligned}$$

Therefore  $A^2 = \pi$  and  $A = \sqrt{\pi}$ . So, picking  $b = \frac{1}{(4\pi)^{n/2}}$ ,

$$\int_{\mathbb{R}^n} \Phi(x, t) dx = b 2^n A^n = b 2^n \pi^{n/2} = 1$$

**Remark:**

$\Phi$  solves the Heat Equation, except at the point  $(x, t) = (0, 0)$ .

**Remark:**

$\Phi$  is infinitely differentiable on  $\mathbb{R}^n \times (\delta, \infty)$ ,  $\forall \delta > 0$ .

**Cauchy Problem (Initial Value Problem)**

$$\begin{cases} u_t - \Delta_x u = 0, & \mathbb{R}^n \times (0, \infty) \\ u(x, 0) = g(x), & x \in \mathbb{R}^n \end{cases}$$

Recall  $y \in \mathbb{R}^n$ ,

$$(x, t) \rightarrow \Phi(x - y)$$

solves the heat equation except at  $(y, 0)$ .

Define,  $x \in \mathbb{R}^n$ ,  $t > 0$ ,

$$(*) \quad u(x, t) = \int_{\mathbb{R}^n} \Phi(x - y, t) g(y) dy = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y) dt$$

**Theorem (#?): Solution to the Cauchy Problem**

Assume  $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ . Then  $u$  defined by  $*$  satisfies

1.  $u \in C^\infty(\mathbb{R}^n \times (0, \infty))$ .

$$2. \quad u_t(x, t) - \Delta_x u(x, t) = 0, (x, t) \in \mathbb{R}^n \times (0, \infty).$$

$$3. \quad \lim_{\substack{(x,t) \rightarrow (x_0,0) \\ x \in \mathbb{R}^n, t > 0}} u(x, t) = g(x_0), x_0 \in \mathbb{R}^n.$$

### Proof

Homework: justify putting the limit inside to prove (1).

For (2), observe that

$$u_t - \Delta_x u(x, t) = \int_{\mathbb{R}^n} \underbrace{[\Phi_t(x-y, t) - \Delta_x \Phi(x-y, t)]}_{=0} g(y) dy$$

For (3), let  $\varepsilon > 0$ . Let  $\delta > 0$  such that  $|x - x_0| < \delta \implies |g(x) - g(x_0)| < \varepsilon$  (since  $g$  continuous). Then

$$\begin{aligned} |u(x, t) - g(x_0)| &= \left| \int_{\mathbb{R}^n} \Phi(x-y, t) g(y) - g(x_0) \underbrace{\int_{\mathbb{R}^n} \Phi(x-y, t) dy}_{=1} \right| \\ &\leq \int_{\mathbb{R}^n} \Phi(x-y, t) |g(y) - g(x_0)| dy \\ &= \underbrace{\int_{B(x_0, \delta)} \Phi(x-y, t) |g(y) - g(x_0)| dy}_I + \underbrace{\int_{\mathbb{R}^n \setminus B(x_0, \delta)} \Phi(x-y, t) |g(y) - g(x_0)| dy}_J \end{aligned}$$

Bounding  $I$ ,  $|y - x_0| < \delta \implies |g(y) - g(x_0)| < \varepsilon$  gives

$$I \leq \varepsilon \underbrace{\int_{B(x_0, \delta)} \Phi(x-y, t) dy}_{\leq 1} \leq \varepsilon$$

Bounding  $J$ , assume  $|x - x_0| < \frac{\delta}{2}$ . Then

$$|J| \leq \|g\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n \setminus B(x_0, \delta)} \Phi(x-y, t) dy$$

Now we want to compare  $|x - y|$  with  $|x_0 - y|$ . Then, for  $|x - x_0| < \frac{\delta}{2}$  and  $|y - x_0| > \delta$ ,

$$|y - x_0| < |y - x| + |x - x_0| < |y - x| + \frac{\delta}{2} < |y - x| + \frac{|y - x_0|}{2}$$

so  $\frac{|y - x_0|}{2} < |y - x|$ . It follows that

$$\begin{aligned} \frac{|y - x_0|^2}{4} &\leq |y - x|^2 \\ -\frac{|y - x|^2}{4t} &\leq -\frac{|y - x_0|^2}{16t} \\ e^{-\frac{|y - x|^2}{4t}} &\leq e^{-\frac{|y - x_0|^2}{16t}} \end{aligned}$$



Then

$$\begin{aligned} |J| &\leq 2 \|g\|_{L^\infty} \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n \setminus B(x_0, \delta)} e^{-\frac{|y-x_0|^2}{16t}} dy \\ &= \frac{C}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n \setminus B(x_0, \delta)} e^{-\frac{1}{16} \left| \frac{y-x_0}{\sqrt{t}} \right|^2} dy \end{aligned}$$

Letting  $z = \frac{y-x_0}{\sqrt{t}}$  such that  $\sqrt{t} dz = dy$ ,

$$\begin{aligned} |J| &\leq \frac{C}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n \setminus B(0, \delta/\sqrt{t})} e^{-\frac{|z|^2}{16}} \underbrace{(\sqrt{t})^n}_{dy} dz \\ &= \frac{C}{(4\pi)^{n/2}} \int_{\mathbb{R}^n \setminus B(0, \delta/\sqrt{t})} e^{-\frac{|z|^2}{16}} dz \end{aligned}$$

Let  $\delta_2 > 0$  such that  $\delta_2 = \max\{\frac{\delta}{2}, \delta^3\}$ .

If  $|(x, t) - (x_0, 0)| < \delta_2$ ,

$$\begin{aligned} t &< \delta_2 < \delta^3 \\ \sqrt{t} &< \delta^{3/2} \\ \frac{1}{\delta^{3/2}} &< \frac{1}{\sqrt{t}} \\ \frac{1}{\delta^{1/2}} &< \frac{\delta}{\sqrt{t}} \end{aligned}$$

so

$$B(0, 1/\delta^{1/2}) \subseteq B(0, \delta/\sqrt{t}) \quad \text{and} \quad \mathbb{R}^n \setminus B(0, \delta/\sqrt{t}) \subseteq \mathbb{R}^n \setminus B(0, 1/\delta^{1/2})$$

Therefore,

$$|u| \leq C \int_{\mathbb{R}^n \setminus B(0, 1/\sqrt{\delta})} e^{-\frac{|z|^2}{16}} dz \rightarrow 0$$

### Intepretation of Fundamental Solution for the Heat Equation

$$\begin{cases} \Phi_t - \Delta_x \Phi(x, t) = 0, & x \in \mathbb{R}^n, t > 0 \\ \Phi(x, 0) = \delta_0(x), & x \in \mathbb{R}^n \end{cases}$$

Then

$$u(x, t) = \int_{\mathbb{R}^n} \Phi(x - y, t) g(y) dy$$

if  $t = 0$ ,

$$\begin{aligned}
 u(x, 0) &= \int_{\mathbb{R}^n} \Phi(x - y, 0) g(y) dy \\
 &= \int_{\mathbb{R}^n} \underbrace{\delta^x(y)}_{y=x} g(y) dy \\
 &= \int_{\mathbb{R}^n} \delta^x(y) g(x) dy \\
 &= g(x) \underbrace{\int_{\mathbb{R}^n} \delta^x(y) dy}_{=1} = g(x)
 \end{aligned}$$

### Remark: Infinite Propagation Speed

Let  $g \in C(\mathbb{R}^n \cap L^\infty(\mathbb{R}^n))$ ,  $g \geq 0$ ,  $g \neq 0$ . Then

$$u(x, t) \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y) dy > 0, \quad \forall x \in \mathbb{R}^n, \forall t > 0$$

IMAGE HERE - 1

That is, the heat equation forces infinite propagation speed for disturbances.

### Non-Homogeneous Heat Problem

$$(*_2) \quad \begin{cases} u_t - \Delta_x u = f, & f : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R} \\ u(x, 0) = 0, & x \in \mathbb{R}^n \end{cases}$$

### Motivation

Let  $y \in \mathbb{R}^n$ ,  $s > 0$ . Then  $(x, t) \rightarrow \Phi(x - y, t - s)$  solves the heat equation except at  $x = y$  and  $t = s$ . That is, it satisfies the equation on  $\mathbb{R}^n \times (s, \infty)$ .

Then for  $s$  fixed, define

$$(\square) \quad u(x, t; s) := \int_{\mathbb{R}^n} \Phi(x - y, t - s) f(y; s) dy$$

which solves

$$\begin{cases} u_t(x, t; s) - \Delta_x u(x, t; s) = 0, & \mathbb{R}^n \times (s, \infty) \\ u(x, s; s) = f(x; s), & \mathbb{R}^n \times \{s\} \end{cases}$$

which is the IVP with  $t = 0 \iff t = s$  and  $g(y) \iff f(y; s)$ .

### Definition: Duhamel's Principle

If we integrate  $\square$  from 0 to  $t$ ,

$$u(x, t) := \int_0^t u(x, t; s) ds$$

Let us consider,

$$(\square_2) \quad u(x, t) := \int_0^t \int_{\mathbb{R}^n} \Phi(x - y, t - s) f(y, s) dy ds$$

as a candidate solution for  $\ast_2$ .

### Theorem: Solution to the Non-Homogeneous Heat Equation

Suppose  $f \in C_c^2(\mathbb{R}^n \times (0, \infty))$  with compact support.

If we define  $u$  by  $\square_2$ , then

1.  $u \in C_c^2(\mathbb{R}^n \times (0, \infty))$ .
2.  $u_t(x, t) - \Delta_x u(x, t) = f(x, t); x \in \mathbb{R}^n, t > 0$ .
3.  $\lim_{\substack{(x, t) \rightarrow (x_0, 0) \\ x \in \mathbb{R}^n, t > 0}} u(x, t) = 0, \forall x_0 \in \mathbb{R}^n$ .

**February 14, 2024**

### Recall: Non-Homogeneous Heat Equation

Given

$$\begin{cases} u_t - \Delta_x u = f(x, t), & f : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R} \\ u(x, 0) = 0 \end{cases}$$

we have a candidate solution from Duhamel's Principle.

$$\begin{aligned} (*) \quad u(x, t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(x - y, t - s) f(y, s) dy ds \\ &= \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi(t-s))^{n/2}} e^{-\frac{|x-y|^2}{4(t-s)}} f(y, s) dy ds \end{aligned}$$

Note that unlike the homogeneous case, the integral approaches the singularity at  $(0, 0)$  and we cannot pass a limit inside.

### Theorem: Differentiation Under Moving Regions

Take  $\Omega(t) \subseteq \mathbb{R}^n$  a nice region with nice boundaries ( $\partial\Omega(t) \in C^1$  and  $t \in \mathbb{R}$ ) and  $F(z, t)$  smooth.

$$\frac{d}{dt} \left( \int_{\Omega(t)} F(x, t) dz \right) = \int_{\partial\Omega(t)} F v \eta ds(z) + \int_{\partial\Omega(t)} F_t dz$$

where  $v$  is the velocity vector on  $\partial\Omega(t)$  and  $\eta$  is the unit outer normal.

### Theorem:

Suppose  $f \in C_1^2(\mathbb{R}^n \times (0, \infty))$  with compact support.

Then, if  $u$  is defined by  $\ast$ ,

1.  $u \in C_1^2(\mathbb{R}^n \times (0, \infty))$ .
2.  $u_t - \Delta_x u = f(x, t); x \in \mathbb{R}^n, t > 0$
3.  $\lim_{(x,t) \rightarrow (x_0,0)} u(x, t) = 0$  for  $x \in \mathbb{R}^n, t > 0, \forall x_0 \in \mathbb{R}^n$ .

### Proof of 1

Since  $\Phi$  has a singularity at  $(0,0)$ , we cannot differentiate under the integral sign. Define  $\bar{y} = x - y$  and  $\bar{s} = t - s$ , then  $\frac{d\bar{s}}{ds} = -1$ ,  $-d\bar{s} = ds$ , and  $\frac{d\bar{y}}{dy} = (-1)$ . So

$$u(x, t) = - \int_t^0 \int_{\mathbb{R}^n} \Phi(\bar{y}, \bar{s}) f(x - \bar{y}, t - \bar{s}) d\bar{y} d\bar{s}$$

Then, rewrite

$$u(x, t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y, s) f(x - y, t - s) dy ds$$

We may now justify passing the derivative of the space variable inside

$$\frac{\partial u}{\partial x_i} = \int_0^t \int_{\mathbb{R}^n} \Phi(y, s) \frac{\partial}{\partial x_i} f(x - y, t - s) dy ds$$

In the same way, justifying putting the limit inside, we have  $\frac{\partial u}{\partial x_i}$  is continuous.

Now, apply the Differentiation Theorem for Moving Regions (above) where  $\Omega(t) = \mathbb{R}^n \times [0, t]$ .

Define  $F(y, s, t) := \Phi(y, s) f(x - y, t - s)$ .

IMAGE HERE - 1

Then,

$$\begin{aligned} \frac{\partial}{\partial t} u(x, t) &= \int_{\partial\Omega(t)} F(\vec{y}, \vec{s}, t) \nu \eta dS(y, s) + \int_0^t \int_{\mathbb{R}^n} \partial_t F(y, s, t) dy ds \\ &= \int_{\mathbb{R}^n \times \{t\}} F(\vec{y}, \vec{s}, t) dS(y, s) + \int_0^t \int_{\mathbb{R}^n} \partial_t F(y, s, t) dy ds \\ &= \int_{\mathbb{R}^n} F(y, t, t) dy + \int_0^t \int_{\mathbb{R}^n} \partial_t F(y, s, t) dy ds \end{aligned}$$

Therefore

$$\frac{\partial u}{\partial t}(x, t) = \int_{\mathbb{R}^n} \Phi(y, t) f(x - y, 0) dy + \int_0^t \int_{\mathbb{R}^n} \Phi(y, s) \partial_t f(x - y, t - s) dy ds$$

Homework: Prove that  $\frac{\partial u}{\partial t}$  is continuous to complete the proof.

### Proof of 2

$$u_t - \Delta_x u = \overbrace{\int_{\mathbb{R}^n} \Phi(y, t) f(x - y, 0) dy}^K + \int_0^t \int_{\mathbb{R}^n} \Phi(y, s) [f_t(x - y, t - s) - \Delta_x f(x - y, t - s)] dy ds$$

Since  $\Phi$  has a singularity, let  $\varepsilon > 0$  and isolate

$$u_t - \Delta_x u = K + \underbrace{\int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y, s) [f_t(x-y, t-s) - \Delta_x f(x-y, t-s)] dy ds}_{J_\varepsilon} + \underbrace{\int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y, s) [f_t(x-y, t-s) - \Delta_x f(x-y, t-s)] dy ds}_{I_\varepsilon}$$

Controlling  $J_\varepsilon$ ,

$$|J_\varepsilon| \leq (||f_t||_{L^\infty} + ||\nabla_x f||_{L^\infty}) \int_0^\varepsilon \overbrace{\int_{\mathbb{R}^n} \Phi(y, s) dy ds}^1 \leq C\varepsilon$$

So  $J_\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

Controlling  $I_\varepsilon$ , using symmetry of  $t$  and  $s$  and  $x$  and  $y$ ,

$$I_\varepsilon = - \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y, s) \partial_s f(x-y, t-s) dy ds - \int_\varepsilon^t \Phi(y, s) \Delta_y f(x-y, t-s) dy ds$$

Recall that

$$\int_U u_{x_i} v = - \int_U u v_{x_i} + \int_{\partial U} u v \eta^i$$

where  $\eta^{-i}$  is the  $i$ th component of  $\eta$ . and

$$\int_\Omega u \Delta v - v \Delta u = \int_{\partial \Omega} u \frac{\partial v}{\partial \eta} - v \frac{\partial u}{\partial \eta}$$

So, integrating by parts,

$$I_\varepsilon = - \left[ - \int_\varepsilon^t \int_{\mathbb{R}^n} \partial_s \Phi(y, s) f(x-y, t-s) dy ds + \int \int_{\partial(\mathbb{R}^n \times [\varepsilon, t])} \Phi(y, s) f(x-y, t-s) \eta^{n+1} dy ds \right] - \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y, s) \Delta_y f(x-y, t-s) dy ds$$

Since  $\eta^{n+1} = 1$  and  $f$  has compact support, this gives

$$I_\varepsilon = \int_0^t \int_{\mathbb{R}^n} \partial_s \Phi(y, s) f(x-y, t-s) dy ds - K + \int_{\mathbb{R}^n} \Phi(y, \varepsilon) f(x-y, t-\varepsilon) dy - \int_\varepsilon^t \Delta_y \Phi(y, s) f(x-y, t-s) dy ds$$

Notice that the first and last summands solve the heat equation on  $\mathbb{R}^n \times [\varepsilon, t]$ . So

$$I_\varepsilon = -K + \int_{\mathbb{R}^n} \Phi(y, \varepsilon) f(x-y, t-\varepsilon) dy$$

Therefore

$$u_t - \Delta_x u = \lim_{\varepsilon \rightarrow 0} K + 0 - K + \int_{\mathbb{R}^n} \Phi(y, \varepsilon) f(x - y, t - \varepsilon) dy$$

Homework: prove that we may pass the limit inside.

$$\begin{aligned} u_t - \Delta_x u &= \int_{\mathbb{R}^n} \Phi(y, 0) f(x - y, t) dy \\ &= \int_{\mathbb{R}^n} \delta^0(y) f(x - y, t) dy \\ &= \int_{\mathbb{R}^n} \delta^0(y) f(x, t) dy \\ &= f(x, t) \int_{\mathbb{R}^n} \delta^0(y) dy \\ &= f(x, t) \end{aligned}$$

### Proof of 3

Write

$$|u(x, t)| \leq \|f\|_{L^\infty} \int_0^t \overbrace{\int_{\mathbb{R}^n} \Phi(y, s) dy}^{=1} ds \leq ct$$

### General Solution to the Heat Equation

If  $f \in C_1^2(\mathbb{R}^n \times (0, \infty))$  and  $g \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ , then

$$u(x, t) := \int_0^t \int_{\mathbb{R}^n} \Phi(x - y, t - s) f(y, s) dy ds + \int_{\mathbb{R}^n} \Phi(x - y, t) g(y) dy$$

is a solution for

$$\begin{cases} u_t - \Delta_x u = f(x, t) \\ u(x, 0) = g(x) \end{cases}$$

### Mean-Value Formulas for the Heat Equation

#### Definition: Parabolic Cylinder

Let  $U \subseteq \mathbb{R}^n$  be an open set and  $T > 0$ .

The parabolic cylinder  $U_T$  is given by

$$U_T := U \times (0, T]$$

and the parabolic boundary is

$$\Gamma_T = \overline{U_T} - U_T$$

IMAGE HERE - 2

## Motivation for Mean-Formulas

In the harmonic case,

$$\Phi(x) = \frac{c}{|x|^{n-2}}; \quad n \geq 3$$

for  $x$  fixed and  $r$  fixed

$$\begin{aligned} \phi : \mathbb{R}^n &\rightarrow \mathbb{R} \\ y &\rightarrow \Phi(x - y) \end{aligned}$$

Then the balls  $B(x, r)$  are the level surface of  $\phi$ . See that

$$\begin{aligned} \phi^{-1}(c_0) &= \{y \in \mathbb{R}^n : \Phi(x - y) = c_0\} \\ &= \{y \in \mathbb{R}^n : \frac{C}{|x - y|^{n-2}} = c_0\} \\ &= \{y \in \mathbb{R}^n : |x - y|^{n-2} = \sqrt[n-2]{\frac{C}{c_0}}\} \\ &= \partial B\left(x, \sqrt[n-2]{\frac{C}{c_0}}\right) \end{aligned}$$

Then to get the mean-value formula, it is worth it to pay attention to the level surface of the fundamental solution of the heat equation.

**February 26, 2024**

### Recall: Mean-Value Formula for Heat Equation

$$\Phi(x, t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & , x \in \mathbb{R}^n, t > 0 \\ 0 & , x \in \mathbb{R}^n, t < 0 \end{cases}$$

For  $U \subset \mathbb{R}^n$  open and bounded,  $T > 0$ ,  $U_t = U \times (0, T]$ ,  $\Gamma_T = \overline{U}_T - U_T$ .

### Definition: Heat Balls

Let  $x \in \mathbb{R}^n$ ,  $t \in \mathbb{R}$ ,  $r \in \mathbb{R}_+$ . Define the heat ball  $E$  as

$$E(x, t; r) = \{(y, s) \in \mathbb{R}^{n+1} : s \leq t, \Phi(x - y, t - s) \geq 1/r^n\}$$

### Remark 1

$$\frac{1}{4r^n} \int_{E(x, t; r)} \frac{|x - y|^2}{|t - s|^2} dy ds = 1$$

Do as homework.

**Remark 2**

$\partial E(x, t; n)$ ,  $\Phi$  is constant.

**Theorem: Mean-Value Formulas**

Let  $u \in C_1^2(U_T)$  solves the heat equation. Then

$$u(x, t) = \frac{1}{4r^n} \int \int_{E(x, t; r)} u(y, s) \frac{|x - y|^2}{|t - s|^2} dy ds$$

for all  $E(x, t; r) \subseteq U_T$ .

**Proof**

Define

$$\phi(r) := \frac{1}{4r^n} \int \int_{E(x, t; r)} u(y, s) \frac{|x - y|^2}{|t - s|^2} dy ds$$

We want to prove  $\phi$  constant with  $\phi' = 0$ .

Without loss of generality, set  $x = 0$ ,  $t = 0$  such that  $E(r) := E(0, 0, r)$ . Then

$$\phi(r) = \frac{1}{4r^n} \int \int_{E(r)} u(y, s) \frac{|y|^2}{s^2} dy ds$$

Rescaling by  $y = r\bar{y}$  and  $s = r^2\bar{s}$ ,

$$\begin{aligned} \phi(r) &= \frac{1}{4r^n} \int \int_{E(1)} u(r\bar{y}, r^2\bar{s}) \frac{r^2|\bar{y}|^2}{r^4\bar{s}^2} r^n r^2 d\bar{y} d\bar{s} \\ &= \frac{1}{4} \int \int_{E(1)} \frac{|\bar{y}|^2}{\bar{s}^2} d\bar{y} d\bar{s} \end{aligned}$$

Where we have  $E(1)$  because  $(y, s) \in E(r) = E(0, 0, r)$ ,  $s \leq 0$ ,  $\frac{1}{(4\pi(-s))^{n/2}} e^{\frac{-|y|^2}{4(-s)}} \geq \frac{1}{r^n}$ .

So  $r^2\bar{s} \leq 0$  and  $\frac{1}{4\pi(-r^2\bar{s})^{n/2}} e^{\frac{-|r\bar{y}|^2}{4(-r^2\bar{s})}} \geq \frac{1}{r^n}$ .

Therefore

$$\bar{s} \leq 0 \quad \text{and} \quad \frac{1}{4\pi(-\bar{s})^{n/2}} e^{\frac{-|\bar{y}|^2}{4(-\bar{s})}} \geq 1$$



Reindexing  $\bar{y} = y$  and  $\bar{s} = s$ ,

$$\begin{aligned}
4\phi'(r) &= \int \int_{E(1)} \left[ Du|_{(ry, r^2s)} \cdot \begin{pmatrix} y \\ 2rs \end{pmatrix} \right] \frac{|y|^2}{s^2} dy ds \\
&= \int \int_{E(1)} \left[ \sum_{i=1}^n \frac{\partial u}{\partial y_i} |_{(ry, r^2s)} y_i + \frac{\partial u}{\partial s} 2rs \right] \frac{|y|^2}{s^2} dy ds \\
&= \int \int_{E(1)} \frac{|y|^2}{s^2} \sum_{i=1}^n \frac{\partial u}{\partial y_i} |_{(ry, r^2s)} y_i dy ds + 2 \int \int_{E(1)} \frac{\partial u}{\partial s} r \frac{|y|^2}{s} dy ds
\end{aligned}$$

Then, again applying the change of variables,

$$\begin{aligned}
4\phi'(r) &= \int \int_{E(r)} \frac{|\bar{y}|^2 r^4}{r^2 \bar{s}^2} \sum_{i=1}^n \frac{\partial u}{\partial y_i} \frac{\bar{y}_i}{r} \frac{d\bar{y}}{r^n} \frac{d\bar{s}}{r^2} + 2 \int \int_{E(r)} \frac{\partial u}{\partial s} r \frac{|\bar{y}|^2}{r^2} \frac{r^2}{\bar{s}} \frac{d\bar{y}}{r^n} \frac{d\bar{s}}{r^2} \\
&= \underbrace{\frac{|y|^2}{r^{n+1}} \int \int_{E(r)} \frac{1}{s^2} \sum_{i=1}^n \frac{\partial u}{\partial y_i} y_i dy ds}_A + \underbrace{\frac{2}{r^{n+1}} \int \int_{E(r)} \frac{\partial u}{\partial s} \frac{|y|^2}{s} dy ds}_B
\end{aligned}$$

We want to analyze  $B$ .

Let us introduce the notation

$$\psi(y, s) = \frac{-n}{2} \log(-4\pi s) + \frac{|y|^2}{4s} + n \log(n)$$

• Lemma 1

$$\psi(y, s) = 0, (y, s) \in \partial E(r).$$

– Proof

$$\text{If } (y, s) \in \partial E(r), \Phi(-y, -s) = \frac{1}{r^n}, \frac{1}{(4\pi(-s))^{n/2}} e^{\frac{-|-y|^2}{4(-s)}} = \frac{1}{r^n}. \text{ Therefore}$$

$$r^n = (4\pi(-s))^{n/2} e^{\frac{-|-y|^2}{4(-s)}} = e^{\log((4\pi(-s))^{n/2} e^{\frac{-|-y|^2}{4(-s)}})}$$

So

$$n \log(r) = \frac{n}{2} \log(4\pi(-s)) - \frac{-|-y|^2}{4s}$$

• Lemma 2

$$\frac{\partial \psi}{\partial y_i} = \frac{2y_i}{4s} = \frac{y_i}{2s}.$$

– Proof

$$4 \sum_i \frac{\partial \psi}{\partial y_i} y_i = \frac{2|y|^2}{s}$$

- Analyzing B

Then, integrating by parts,

$$\begin{aligned}
 B &= \frac{4}{r^{n+1}} \int \int_{E(r)} \frac{\partial u}{\partial s} \sum_i \frac{\partial \psi}{\partial y_i} y_i dy ds \\
 &= \frac{4}{r^{n+1}} \sum_i \left[ \int \int_{E(r)} \frac{\partial}{\partial y_i} \left( \frac{\partial u}{\partial s} y_i \right) \psi dy ds + \overbrace{\int \int_{\partial E(r)} \frac{\partial u}{\partial s} y_i \psi \eta^i}^{=0} \right] \\
 &= \frac{4}{r^{n+1}} \sum_i \int \int_{E(r)} \psi \left[ \frac{\partial u}{\partial s} + y_i \frac{\partial^2 u}{\partial y_i \partial s} \right] dy ds
 \end{aligned}$$

Then, again integrating by parts,

$$\begin{aligned}
 B &= \underbrace{-\frac{4}{r^{n+1}} \sum_i \int_{E(r)} \psi \frac{\partial u}{\partial s} dy ds}_{C} - \frac{4}{r^{n+1}} \sum_i \int_{E(r)} \psi y_i \frac{\partial^2 u}{\partial y_i \partial s} dy ds \\
 &= C - \frac{4}{r^{n+1}} \sum_i \left[ - \int_{E(r)} y_i \frac{\partial \psi}{\partial s} \frac{\partial u}{\partial y_i} dy ds + \overbrace{\int_{\partial E(r)} \psi y_i \frac{\partial u}{\partial y_i} \eta^s}^{=0} \right] \\
 &= C - \frac{4}{r^{n+1}} \sum_i \int_{E(r)} y_i \frac{\partial u}{\partial y_i} \left[ \frac{n}{2s} + \frac{|y|^2}{4s^2} \right] dy ds
 \end{aligned}$$

Since  $-\int_{E(r)} \sum_i y_i \frac{\partial u}{\partial y_i} \frac{|y|^2}{4s^2} = -A$ , we have

$$B = -\frac{4n}{r^{n+1}} \int_{E(r)} \psi \frac{\partial u}{\partial s} dy ds - \frac{4}{r^{n+1}} \sum_i \int_{E(r)} y_i \frac{\partial u}{\partial y_i} \frac{n}{2s} dy ds - A$$

So, since  $u$  solves the heat equation, we have  $\Delta u = \frac{\partial u}{\partial s}$  and may integrate by parts

$$\begin{aligned}
 4\phi'(r) &= -\frac{4n}{r^{n+1}} \int_{E(r)} \psi \frac{\partial u}{\partial s} dy ds - \frac{4}{r^{n+1}} \sum_i \int_{E(r)} y_i \frac{\partial u}{\partial y_i} \frac{n}{2s} dy ds \\
 &= -\frac{4n}{r^{n+1}} \left[ \int_{E(r)} \nabla \psi \cdot \nabla u dy ds + \overbrace{\int_{\partial E(r)} \psi \frac{\partial u}{\partial \eta}}^{=0} \right] - \frac{4}{r^{n+1}} \sum_i \int_{E(r)} y_i \frac{\partial u}{\partial y_i} \frac{n}{2s} dy ds \\
 &= 0
 \end{aligned}$$

Then we have

$\phi'(r) = 0$  and  $\phi(r)$  constant. We know

$$\begin{aligned}
\phi(r) &= \lim_{t \rightarrow 0} \phi(t) \\
&= \lim_{t \rightarrow 0} \frac{1}{4} \int \int_{E(1)} u(ty, ts^2) \frac{|y|^2}{s^2} dy ds \\
&= \frac{1}{4} \int_{E(1)} u(0,0) \frac{|y|^2}{s^2} dy ds \\
&= u(0,0) \overbrace{\frac{1}{4} \int_{E(1)} \frac{|y|^2}{|s|^2} dy ds}^{=1} \\
&= u(0,0)
\end{aligned}$$

### Theorem: Strong Maximum Principle for Heat Equation

Let  $U$  be bounded,  $u \in C_1^2(U_T) \cap C(\overline{U}_T)$  that satisfies the heat equation.

1.  $\max_{\overline{U}_T} u = \max_{\Gamma_T} u$ .
  2. If  $U$  is connected and  $(x_0, t_0) \in U_T$  such that  $u(x_0, t_0) = \max_{\overline{U}_T} u$ , then  $u$  is constant in  $\overline{U}_{t_0}$ .
- IMAGE HERE - 1 CYLINDER to  $U_{t_0}$

### Proof of 2

Let  $(x_0, t_0) \in U_T$  such that  $u(x_0, t_0) = \max_{\overline{U}_T} u := M$ .

Pick  $r$  small enough such that  $E(x_0, t_0; r) \subseteq U_T$ .

IMAGE HERE - 2 BALL IN CYLINDER

Then, applying the mean-value formula,

$$\begin{aligned}
M &= \frac{1}{4r^n} \int_{E(x_0, t_0; r)} u(y, s) \frac{|x_0 - y|^2}{|t_0 - s|^2} dy ds \\
&\leq \frac{M}{4r^n} \int_{E(x_0, t_0; r)} \frac{|x_0 - y|^2}{|t_0 - s|} dy ds \\
&= M
\end{aligned}$$

Therefore  $u(y, s) = M, \forall (y, s) \in E(x_0, t_0; r)$ .

#### • Part A

Let  $(y_0, s_0) \in U_T$  such that we may connect  $(x_0, t_0)$  and  $(y_0, s_0)$  with a line  $L$  where  $L \subseteq U_T$ .

Then  $u = M$  on  $L$ .