Partial Differential Equations I

January 8, 2024

Homework

Assigned exercises and concept maps. Graded by completion.

Presentations

Assigned topics; responsible for giving a class.

Definition: Partial Differential Equation(s) (PDE)

An identity relating an uknown function, its partial derivatives and its variables.

$$F(D^k u, \dots, D^2 u, Du, u, x) = 0, \quad x \in U \subseteq \mathbb{R}^n$$

where U is an open subset of $\mathbb{R}^n$, $u : U \subset \mathbb{R}^n \to \mathbb{R}$, $Du = (\partial_{x_1} u_1, \dots, \partial_{x_n} u)$.
Then $F : \mathbb{R}^{n^k} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, where F is given.
$x = (x_1 \dots, x_n)$ is (are) the independent variable(s).
u is the unknown function or dependent variable.
k is the order of the PDE.

Goal

Given a PDE, we consider

- Existence
- Uniqueness
- Stability

Recall: Multiindex Notation

$\alpha = (\alpha_1, \dots, \alpha_n)$ a vector such that $\alpha_i \in \mathbb{Z}_{\geq 0}$.
Then we say that α is a multiindex with order $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Notation

$u : U \subseteq \mathbb{R}^n \to \mathbb{R}$, $\alpha = (\alpha_1, \dots, \alpha_n)$.
$u^\alpha := D^\alpha u = \partial_{x_n}^{\alpha_n} \cdots \partial_{x_1}^{\alpha_1} u$, where $\partial^0 u = u$.

Definition: Linear Partial Differential Equation

A linear PDE of order k is of the form

$$(*) \sum_{|\alpha| = k} a_\alpha(x) D^\alpha u = f(x)$$

Remark

This means that F is multilinear in the first $n^k + n^{k-1} + \cdots$ variables.

Definition: Homogeneous Linear Partial Differential Equation

A linear given by $(*)$ is homogeneous if $f(x) \equiv 0$.
Otherwise, it is non-homogeneous.

Example 1: Linear Transport Equation

$$\nabla u \cdot (1, b) = u_t + b \cdot Du = f(t, x)$$

This is a linear PDE of order 1 on $\mathbb{R} \times \mathbb{R}^n \equiv \mathbb{R}^{n+1}$ where (t, x) are independent variables and u is dependent.
Here, x is the spatial variable while t is time and Du represents the gradient.
$\nabla u = (\partial_t u, \nabla u)$, $b \cdot Du = \sum_{i=1}^n b_i \partial_{x_i} u$, $(b_1, \ldots, b_n) \in \mathbb{R}^n$ is fixed.

Example 2: Laplace Equation

$$\Delta u := \sum_{i=1}^{n} \partial_{x_i} u = 0$$

This is a linear, homogeneous PDE of order 2.

Example 3: Poisson Equation

$-\Delta u := f(u)$
This is a nonlinear PDE of order 2.
Consider $f(u) = u^2$.

Example 4: Heat Equation (Diffusion Equation)

$$u_t - \Delta u = 0$$

This is a linear, homogeneous PDE of order 2.

Example 5: Wave Equation

$$u_{tt} - \Delta u = 0$$

This is a linear, homogeneous PDE of order 2.

Transport Equation

$u : \mathbb{R}^n (0, \infty) \to \mathbb{R}$ given by

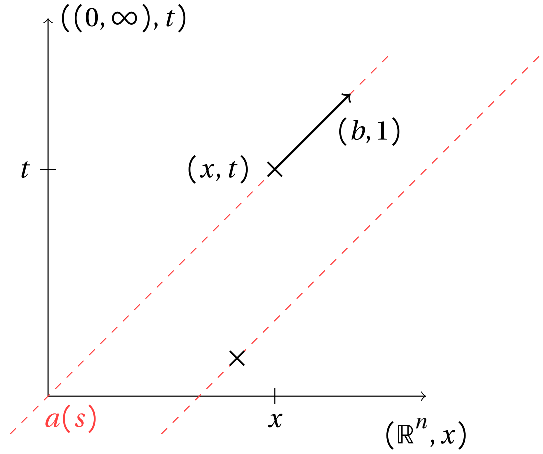
$$u_t + b \cdot Du = 0, \quad b \in \mathbb{R}^n$$

In order to get a solution, first assume that ther exists a “nice” (e.g. smooth, C^1, differentiable, etc.) solution.

Step 1

Notice that the PDE is equivalent to

$$\nabla u \cdot (b, 1) = 0$$



Step 2

Consider a curve on $\mathbb{R}^{n+1}$ with velocity $(1, b)$ which passes through (x, t). i.e.

$$\alpha(s) = (x + sb, t + s)$$

Notice $\alpha'(s) = (b, 1)$.
Then, let us study u along the curve $\alpha(s)$.

$$z(s) := u(\alpha(s))$$

Taking the derivative with respect to s,

$$z'(s) := \frac{d}{ds}(u \circ \alpha(s)) = \nabla u|_{\alpha(s)} \cdot \alpha'(s) = \nabla u|_{\alpha(s)} \cdot (b, 1) = 0$$

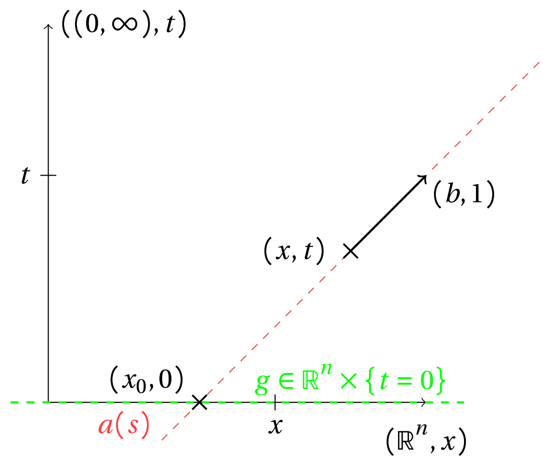
That is $z'(s) = 0$, $z(s)$ is constant, and u along $\alpha(s)$ is constant.

Conclusion

If we know some value of u along $\alpha(s)$, then we know all values along $\alpha(s)$.
If we have some value of u along every $\alpha(s)$, then we know u on $\mathbb{R}^n \times (0, \infty)$.

Transport Equation - Homogeneous Initial Value Problem

$$(*) \begin{cases} \nabla u \cdot (b, 1) = 0, & \mathbb{R}^n \times (0, \infty) \\ u = g, & \mathbb{R}^n \times \{t = 0\} \end{cases}$$



Here, $g : \mathbb{R}^n \to \mathbb{R}$ is given.
Consider (x,t); we want to find $(x_0,0)$.
We know $\alpha(s) = (x+sb, t+s) = (x_0,0)$, therefore

$$\begin{cases} x + sb = x_0 & (1) \\ t + s = 0 \implies s = -t & (2) \end{cases}$$

Then, by replacing (2) in (1),

$$x_0 = x - tb$$

Then from the conclusion

$$u(x,t) = u(x_0,0) = g(x_0) = g(x-tb)$$

Therfore, $u(x,t) := g(x-tb)$ $(\heartsuit)$.

Remark

1. If there exists a regular (differentiable or C^1) solution u for $*$, then u should look like $\heartsuit$.
2. If g is (differentiable or C^1), then u defined by $\heartsuit$ is a (differentiable or C^1) solution for my problem.

Homework

Show that $\heartsuit$ satisfies $*$.

Transport Equation - Non-homogeneous Initial Value Problem

$$(*) \begin{cases} \nabla u \cdot (b,1) = f(x,t), & \mathbb{R}^n \times (0,\infty) \\ u = g, \quad \mathbb{R}^n \times \{t=0\} \end{cases}$$

Where $g : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ are given.

Solution

Notice that the PDE is equivalent to

$$\nabla u \cdot (b,1) = f(x,t)$$

Define the "characteristic curve"

$$\alpha(s) = (x + sb, t + s)$$

and

$$z(s) := u(\alpha(s))$$

Taking $\frac{d}{ds}$,

$$z'(s) = \nabla u|_{\alpha(s)} \cdot (b,1) = f(\alpha(s)) \implies z'(s) = f(x + sb, t + s) \; (c)$$

Notice that c is an ordinary differential equation. Integrating from $-t$ to 0.

$$\begin{aligned} \int_{-t}^{0} z'(s)\, ds &= \int_{-t}^{0} f(x + sb, t + s)\, ds \\ z(0) - z(-t) &= \int_{-t}^{0} f(x + sb, t + s)\, ds \end{aligned}$$

Notice that $z(0) = u(x,t)$ and $z(-t) = u(\alpha(-t)) = u(x - tb, 0)$.

$$u(x,t) = u(x - tb, 0) + \int_{-t}^{0} f(x + sb, t + s)\, ds$$

Then

$$\begin{aligned} u(x,t) &= g(x - tb) + \int_{-t}^{0} f(x + sb, t + s)\, ds \\ &\underset{\bar{s}=s+t}{=} g(x - tb) + \int_{0}^{t} f(x + (\bar{s} - t)b, \bar{s})\, d\bar{s} \\ &= g(x - tb) + \int_{0}^{t} f(x + (s - t)b, s)\, ds \end{aligned}$$

Remark: Method of Characteristics

Try to vert the PDE into an ODE and solve using characteristic curves.

January 10, 2024

Definition: Harmonic Function

If $u \in C^2$ such that $\Delta u = 0$, then u is a harmonic function.

Laplace Equation

Consider $u : U \subseteq \mathbb{R}^n \to \mathbb{R}$ with U open, then the homogeneous (Laplace) form is given by

$$\Delta u = 0$$

and the non-homogeneous (Poisson) form is

$$-\Delta u = f$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is given.

Remark / Exercise

The Laplace equation is invariant under translation and rotation.
That is, if $\Delta u(x) = 0$ and $v(x) = u(x - y)$, then $\Delta v = 0$.
Similarly, if $w(x) = u(O(x))$ then $\Delta w = 0$ where O is an orthogonal matrix.

Fundamental Solution of the Laplace Equation

Remark: since the Laplace equation is invariant under rotation, we can consider a function in terms of the radius $v(x) = v(|x|)$.
Recall $r = |x| = (x_1^2 + \cdots + x_n^2)^{1/2}$.
Because of this remark, assume that $u(x) = v(|x|) = v(r(x)) \quad (*)$ where $v : (0, \infty) \to \mathbb{R}$.
Therefore, we need

$$\frac{\partial r}{\partial x_i} = \frac{1}{2} \cdot \frac{2x_i}{(x_1^2 + \cdots + x_n^2)^{1/2}} = \frac{x_i}{r}$$

Replace $(*)$ in the PDE

$$\frac{\partial u}{\partial x_i} = \frac{\partial}{\partial x_i} v(r(x)) = v'(r(x)) \cdot \frac{\partial r}{\partial x_i} = v'(r(x)) \cdot \frac{x_i}{r}$$

and

$$\begin{aligned}
\frac{\partial^2 u}{\partial x_i^2} = \frac{\partial}{\partial x_i} \left(v'(r(x)) \cdot \frac{x_i}{r} \right) \\
&= \frac{\partial}{\partial x_i} (v'(r(x))) \cdot \frac{x_i}{r} + v'(r(x)) \cdot \frac{\partial}{\partial x_i} \left(\frac{x_i}{r} \right) \\
&= v''(r(x)) \cdot \frac{x_i^2}{r^2} + v'(r(x)) \left[\frac{1}{r} + x_i \frac{\partial}{\partial x_i} (r) \right] \\
&= v'' \frac{x_i^2}{r^2} + v' \left[\frac{1}{r} - \frac{x_i^2}{r^3} \right]
\end{aligned}$$

Thenm, summing across i,

$$\Delta u = v'' + v' \left[\frac{n}{r} + \frac{1}{r} \right] = 0$$

Then the PDE is equivalent to

$$v''(r) + \frac{v'}{r}(n-1) = 0 \; (\square)$$

We need to find a solution for $\square$.

$$v''(r) = \frac{(1-n)v'}{r}$$

Assume, without loss of generality, that $v' \neq 0$ such that

$$\frac{v''(r)}{v'(r)} = \frac{1-n}{r} \implies (\log(|v'|))' = \frac{1-n}{r}$$

Then, integrating,

$$\log(|v'|) = (1-n)\log(r) + C = \log\left(r^{1-n}\right) + C$$

such that

$$|v'| = Cr^{1-n} \implies v' = Cr^{1-n} \implies v(r) = Cr^{1-n+1} + D = Cr^{2-n} + D$$

Definition: Fundamental Solution of the Laplace Equation

The function Φ given by

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x|, & n = 2 \\ \frac{1}{n(n-2)\alpha(n)} \cdot \frac{1}{|x|^{n-2}}, & n \geq 3 \end{cases}$$

where $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$ is called the fundamental solution.

Remark

Φ solves the Laplace equation away from 0.

Lemma: Estimates for the Fundamental Solution

- First Estimate
  $|D\Phi(x)| \leq \frac{C}{|x|^{n-1}}$, for $x \neq 0$.

  $$\frac{\partial \Phi}{\partial x_i} = C\frac{\partial}{\partial x_i}\left(|x|^{2-n}\right) = \frac{C(2-n)}{1-n}|x|^{2-n-1}\frac{\partial|x|}{\partial x_i} = |x|^{1-n} \cdot \frac{x_i}{|x|} = Cx_i|x|^{-n}$$

  Therefore

  $$|D\Phi(x)| \leq C|x||x|^{-n} \implies |D\Phi(x)| \leq C|X|^{1-n}$$

  - Exercise
    Compute for $n = 2$.

- Second Estimate
  $|D^2\Phi(x)| \leq \frac{C}{|x|^n}$, for $x \neq 0$.

$$
\begin{aligned}
\frac{\partial^2}{\partial x_J \partial x_i}\Phi &= C\frac{\partial}{\partial x_J}\left(x_i|x|^{-n}\right) \\
&= C\left[\delta_{iJ}|x|^{-n} + x_i\frac{\partial}{\partial x_J}|x|^{-n}\right] \\
&= C\left[\delta_{iJ}|x|^{-n} + (-n)\cdot\frac{x_i|x|^{-n-1}x_J}{|x}\right] \\
&= C\left[\frac{\delta_{iJ}|x|}{|x|^n} + \frac{Cx_ix_J}{|x|^{n+1}}\right]
\end{aligned}
$$

Then

$$
\left|\frac{\partial\Phi}{\partial x_i\partial x_J}\right| \leq \frac{C}{|x|^n} + \frac{C|x_i||x_J|}{|x|^{n+2}} \leq \frac{2C}{|x|^n} = \frac{C}{|x|^n}
$$

Then, we are done since

$$
|D^2\Phi(x)| = \sqrt{\sum_J\left(\frac{\partial\Phi}{\partial x_i\partial x_J}\right)}
$$

Poisson Equation

Motivation

Suppose we have $\Phi(x)$, the fundamental solution.
Then for an arbitrary, fixed element $y \in \mathbb{R}^n$, then we have $x \to \Phi(x-y)$ harmonic for $x \neq y$.
Consider $f : \mathbb{R}^n \to \mathbb{R}$ such that $y \to f(y)$ then $x \to f(y)\Phi(x-y)$ is similarly harmonic for $x \neq y$.
Now, if given $\{y_1, \ldots, y_m\}$ where $y_i \in \mathbb{R}^n$, then $x \to \sum_{i=1}^m f(y_i)\Phi(x-y_i)$ is harmonic $\forall x \neq \{y_1, \ldots, y_m\}$.
Then, what happens if we consider

$$
u(x) := \int_{\mathbb{R}^n} f(y)\Phi(x-y)\,dy \quad (\square_3)
$$

Is u harmonic? No, since $\Delta\Phi(x-y)$ is not summable in $\mathbb{R}^n$ we may not pass the limit into the integral.
(to be covered later) However, since $\Delta\Phi(x-y)$ acts as δ_{xy} in distribution, this may solve the Poisson equation.

Remark / Exercise

Assume that $f \in C_C^2(\mathbb{R}^n)$ (i.e f is twice continuously differentiable with compact support on $\mathbb{R}^n$).
The function Φ is integrable near the singularity on compact sets.
Prove using spherical coordinates.
Therefore, u defined by $\square_3$ is well defined

$$
|u| = \left|\int_{\mathbb{R}^n} f(y)\Phi(x-y)\,dy\right| = \left|\int_K \Phi(x-y)\,dy\right| < \infty
$$

Theorem: Solving the Poisson Equation

If $f \in C_C^2(\mathbb{R}^n)$ and u is defined by $\square_3$, then

1. $u \in C^2(\mathbb{R}^n)$
2. $-\Delta u = f$, in $\mathbb{R}^n$

- Proof of 1
  Since Φ presents a problem at $x = y$ but f is well behaved, we will change variables such that $\overline{y} = x - y$, $y = x - \overline{y}$, and $\frac{dy}{d\overline{y}}(-1)I_{m\times m}$ and then redefine $\overline{y} = y$.

$$u(x) = \int_{\mathbb{R}^n} f(y)\Phi(x-y)\,dy = \int_{\mathbb{R}^n} f(x-\overline{y})\Phi(\overline{y})\,d\overline{y} = \int_{\mathbb{R}^n} f(x-y)\Phi(y)\,dy$$

  In short, we have sent the problem from Φ to f.
  Now, let us consider $e_i = (0,\ldots,1,\ldots,0)$.
  Then for $h > 0$,

$$\frac{u(x+he_i)-u(x)}{h} = \frac{1}{h}\int_{\mathbb{R}^n} \Phi(y)\left[f(x+he_i-y)-f(x-y)\right]\,dy$$

  Now, the limit as $h \to 0$

$$\begin{aligned}\lim_{h\to 0}\frac{u(x+he_i)-u(x)}{h} &= \lim_{h\to 0}\int_{\mathbb{R}^n} \overbrace{\Phi(y)\left[\frac{f(x+he_i-y)-f(x-y)}{h}\right]}^{H(h,y)}\,dy \\ &= \int_{\mathbb{R}^n}\Phi(y)\cdot\frac{\partial f(x-y)}{\partial x_i}\,dy\end{aligned}$$

  To justify passing the limit into the integral, take an arbitrary sequence $h_m \xrightarrow{m\to 0} 0$ and consider

$$f_m(y) := H(h_m, y)$$

  We want to consider

$$\begin{aligned}|H(h_m,y)| &\leq \Phi(y)\left[\frac{f(x+h_m e_i-y)-f(x-y)}{h}\right] \\ &\leq \Phi(y)f'(c)\end{aligned}$$

  Where c is along the curve between $f(x+h_m e_i - y)$ and $f(x-y)$ and chosen by mean value theorem.

  - Exercise

$$|H(h_m,y)| \leq \Phi(y)||f'||_{L^\infty}\chi_{B(x,R)}(y)$$

  Note that

$$C\int_{\mathbb{R}^n}|\Phi(y)|\chi_{B(x,R)}(y)\,dy = \int_{B(x,R)}|\Phi(y)|\,dy < \infty$$

- Exercise

Using the fact that a continuous function is uniformly continuous on a compact set, show that $u \in C^2(\mathbb{R}^n)$.

Dominated Convergence Theorem

If $f_m(x)$ such that $f_m(x) \underset{\text{pointwise}}{\overset{m\to\infty}{\longrightarrow}} f(x)$, and $|f_m(x)| \le g(x)$ for $g \in L^1$, then f is integrable and

$$\lim_{m\to\infty} \int f_m(x)\,dx = \int f(x)\,dx$$

January 17, 2024

Recall: Averages

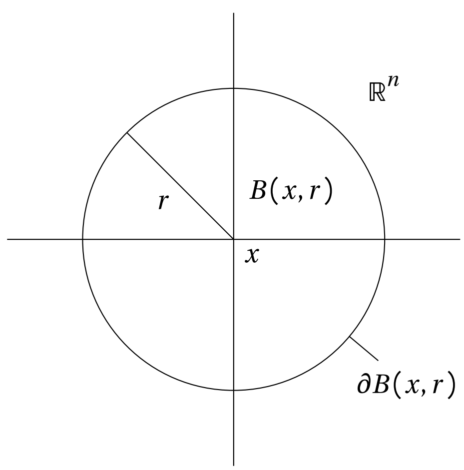
$f : \{1, \ldots, n\} \to \mathbb{R}$
$i \to a(i)$
The average is given as $\frac{a(i)+\cdots+a(n)}{n}$.
Then for $f : \Omega \to \mathbb{R}$, the average is given as

$$\frac{1}{|\Omega|}\int f(y)\,dy := ⨍_\Omega f\,d\mu$$

In our case, $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$

$$⨍_{B(x,n)} f\,d\mu \equiv \frac{1}{|B(x,n)|} ⨍_{B(x,n)} f\,d\mu$$

$$⨍_{\partial B(x,n)} f\,d\mu = \frac{1}{|\partial B(x,n)} ⨍_{\partial B(x,b)} f\,d\mu$$

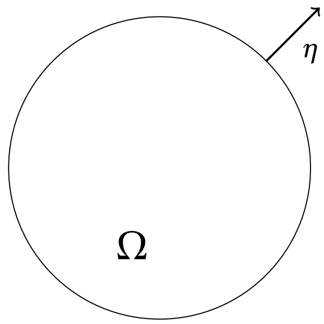


Theorem: Lebesgue Differentiation

$$u|x| = \lim_{r\to 0} ⨍_{B(x,n)} u\,d\mu = \lim_{r\to 0} ⨍_{\partial B(x,n)} u\,d\mu$$

Integration by Parts

$$\int_{\Omega} u\Delta v = -\int_{\Omega} \nabla u \cdot \nabla v + \int_{\partial\Omega} u \cdot \frac{\partial v}{\partial \eta}$$



Recall: Poisson's PDE

$f \in C_c^2|\mathbb{R}^n|$, $u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\, dy$.

$$\Phi(x) = \left\{ \frac{1}{n(n-2)|\alpha(n)|} \frac{1}{|X|(n-2)} \right.$$

$$u(x)\text{“} = \text{“} \int_{\mathbb{R}^n} f(x-y)\Phi(y)\, dy$$

Part A

$u \in C^2(\mathbb{R}^n)$
Then

$$\frac{\partial u}{\partial x_1} = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_1}(x-y)\Phi(y)\, dy$$

$$\frac{\partial^2 u}{\partial x_1 x_T} = \int_{\mathbb{R}^n} \frac{\partial^2 f}{\partial x_1 x_T}(x-y)\Phi(y)\, dy$$

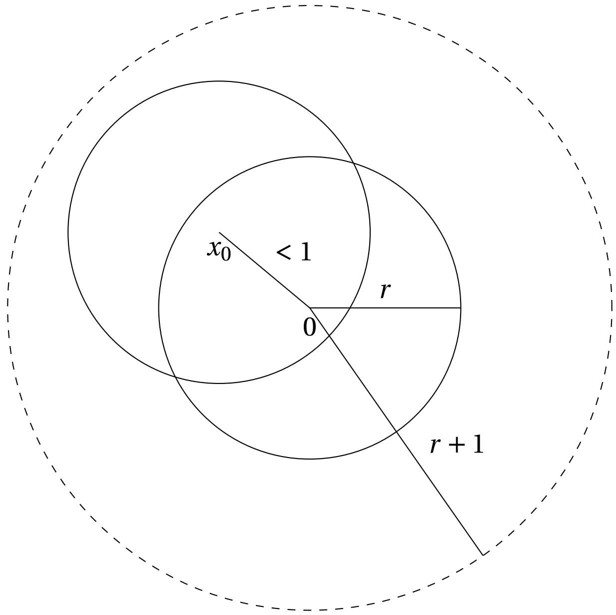
Notice that if we prove that

$$g(x) = \int_{\mathbb{R}^n} h(x-y)\Phi(y)\, dy$$

– where h is continuous with compact support – is continuous then we are done.
Let us prove that g is continuous.
Let $\varepsilon > 0$,

$$|g(x) - g(x_0)| \le \int_{\mathbb{R}^n} \Phi(y)|h(x-y) - h(x_0-y)|\, dy$$

Without loss of generality, h has compact support on $B(0,r)$ for some radius r.
Therefore $h(x,y)$ has compact support on $B(x,r)$ and $h(x_0,y)$ has compact support on $B(x_0,r)$.



Consider $|x - x_0| < 1$, then $|h(x-y) - h(x_0 - y)|$ has compact support on $B(x_0, r+1)$. Then

$$|g(x) - g(x_0)| \leq \int_{B(x_0, r+1)} \Phi(y) |h(x-y) - h(x_0-y)| \, dy$$

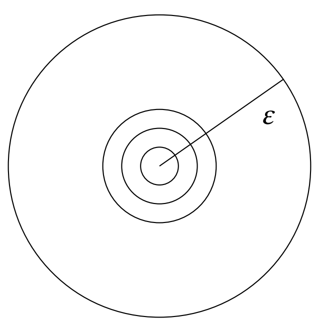
Since h is continuous on a compact domain, it is uniformly continuous.
Therefore $\exists \delta > 0$ such that $|w - z| < \delta \implies |h(w) - h(z)| < \epsilon$.
Set $w = x - y$ and $z = x_0 - y$ such that $|w - z| = |x - x_0| < \delta$, then $|h(x-y) - h(x_0 - y)| < \epsilon$. Thus,

$$|g(x) - g(x_0)| \leq \varepsilon \int_{B(x_0, r+1)} \Phi(y) \, dy$$

Part B

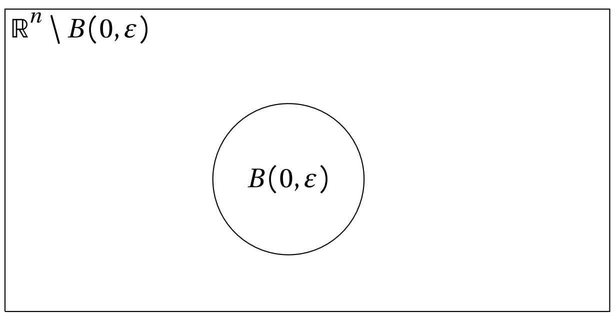
$-\Delta u = f$
Letting $\varepsilon > 0$ and taking the Laplacian of both sides,

$$\begin{aligned}
\Delta_x u(x) &= \int_{\mathbb{R}^n} \Delta_x f(x-y) \Phi(y) \, dy \\
&= \overbrace{\int_{B(0,\varepsilon)} \Delta_x f(x-y) \Phi(y) \, dy}^{I_\varepsilon} + \overbrace{\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Delta_x f(x-y) \Phi(y) \, dy}^{J_\varepsilon}
\end{aligned}$$



Then

$$
\begin{aligned}
|I_\varepsilon| &\leq \int_{B(0,\varepsilon)} |\Delta_x f(x-y)| \Phi(y)\, dy \\
&\leq ||\overset{2}{\nabla} f||_{L^\infty} \int_{B(0,\varepsilon)} \Phi(y)\, dy \\
&\leq c \int_0^\varepsilon \int_{\partial B(0,r)} \Phi(y)\, dS(y)\, dr \\
&\leq c \int_0^\varepsilon \int_{\partial B(0,r)} \frac{1}{|y|^{n-2}}\, dS(y)\, dr \\
&= c \int_0^\varepsilon \int_{\partial B(0,r)} \frac{1}{r^{n-2}}\, dS(y)\, dr \\
&= c \int_0^\varepsilon \frac{1}{r^{n-2}} \int_{\partial B(0,r)} dS(y)\, dr \\
&\leq c \int_0^\varepsilon \frac{r^{n-1}}{r^{n-2}}\, dr \\
&c \int_0^\varepsilon r\, dr = c\varepsilon^2
\end{aligned}
$$



As an exercise, attempt the same argument with $n = 2$.
Therefore,

$$\Delta_x u = I_\varepsilon + J_\varepsilon$$

and $\lim_{\varepsilon \to 0} I_\varepsilon = 0$.
Now, we need to control J_ε.

$$J_\varepsilon = \int_{\mathbb{R}^n} \Delta_x f(x-y) \Phi(y)\, dy$$

$$\Delta_x f(x-y) = \sum \frac{\partial^2 f}{\partial x^2} f(x-y)$$

$$
\begin{aligned}
\frac{\partial f}{\partial x}(x-y) = \nabla f|_{z=(x-y)} \cdot e_i &= \frac{\partial f}{\partial z_i} | z = (x-y) \\
\frac{\partial^2 f}{\partial x_i^2} &= \frac{\partial^2 f}{\partial z_i^2}|_{z=(x-y)}
\end{aligned}
$$

$$\Delta_y f(x-y) = \sum \frac{\partial f^2}{\partial y_i}(x-y)$$
$$\frac{\partial f}{\partial y_i}(x-y) = \nabla f|_{z=(x-y)} \cdot -e_i = -\frac{\partial f}{\partial z_i}|_{z=(x-y)}$$
$$\frac{\partial^2 f}{\partial y_i^2} = \frac{\partial^2}{\partial y_i^2}|_{z=x-y}$$

So

$$\begin{aligned}
J_\varepsilon &= \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Delta_y f(x-y)\Phi(y)\, dy \\
&= \overbrace{-\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \nabla_y f(x-y) \nabla \Phi(y)\, dy}^{K_\varepsilon} + \overbrace{\int_{\partial(\mathbb{R}^n \setminus B(0,\varepsilon))} \frac{\partial f}{\partial \eta} \Phi(y)\, dS(y)}^{L_\varepsilon}
\end{aligned}$$

and

$$\Delta_x u = I_\varepsilon + K_\varepsilon + L_\varepsilon$$

To control L_ε, since

$$\begin{aligned}
|L_\varepsilon| &\leq \int_{\partial B(0,\varepsilon)} \frac{|\partial f|}{\partial \eta} \Phi(y)\, dy \\
&\leq \int_{\partial B(0,\varepsilon)} |\nabla f| \Phi(y)\, dy \\
&\leq ||\nabla f||_{L^\infty} \int_{\partial B(0,\varepsilon)} \Phi(y)\, dy \\
&\leq c \int_{\partial B(0,\varepsilon)} \frac{1}{|y|^{n-2}}\, dy \\
&= \frac{c}{\varepsilon^{n-2}} \int_{\partial B(0,\varepsilon)} dy \\
&\leq \frac{c\varepsilon^{n-1}}{\varepsilon^{n-2}} \\
&= c\varepsilon
\end{aligned}$$

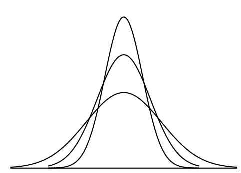
and K_ε, since $\frac{\partial \Phi}{\partial \eta} = \nabla \phi \cdot \eta = \frac{-y}{n\alpha(n)|y|^n} \cdot \eta = \frac{|y|^2}{n\alpha(n)|y|^n|y|} = \frac{1}{n\alpha(n)|y|^{n-1}}$

$$
\begin{aligned}
|K_\varepsilon| &= -\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \nabla_y f(x-y) \nabla_y \Phi(y)\, dy \\
&= \overbrace{\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} f(x-y) \Delta_y \Phi(y)\, dy}^{0} - \int_{\partial(\mathbb{R}^n \setminus B(0,\varepsilon))} f(x-y) \frac{\partial \Phi}{\partial \eta} \\
&= -\int_{\partial B(0,\varepsilon)} f(x-y) \frac{\partial \Phi}{\partial \eta} \\
&= -\int_{\partial B(0,\varepsilon)} f(x-y) \frac{1}{n\alpha(n)|y|^{n-1}}\, dS(y) \\
&= -\frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\delta B(0,\varepsilon)} f(x-y)\, dS(y) \\
&= \overset{\text{volume}}{\underbrace{-\frac{1}{n\alpha(n)\varepsilon^{n-1}}}} \int_{\partial B(0,\varepsilon)} f(z)\, dS(z) \\
&= \frac{1}{|\partial B(0,\varepsilon)|} \int_{\partial B(0,\varepsilon)} f(z)\, dz \\
&= -\fint_{\partial B(x,\varepsilon)} f(z)\, dz
\end{aligned}
$$

Laplacian as a Distribution

$-\Delta \Phi(y) = \delta(y)$
Define the Dirac delta "function" as

$$
\delta(y) = \begin{cases} \infty & y = 0 \\ 0 & y \neq 0 \end{cases}
$$



such that $\int_{\mathbb{R}^n} \delta = 1$.
Translate the Dirac delta as

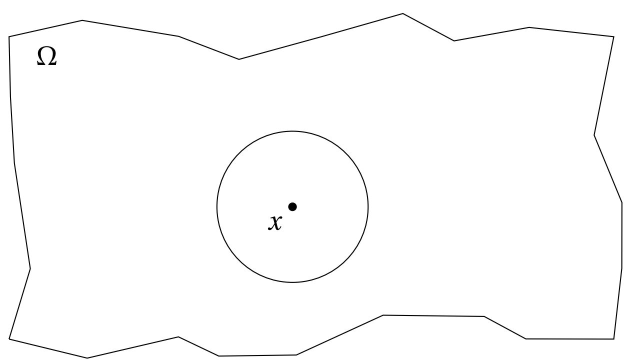
$$
\delta_x = \begin{cases} \infty & y = x \\ 0 & y \neq x \end{cases}
$$

and recall that

$$
\begin{aligned}
\Delta u(x) &= \Delta\left(\int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy\right) \\
&= \int_{\mathbb{R}^n} \overbrace{\Delta\Phi(x-y)}^{-\delta_x(y)} f(y)\, dy \\
&= -\int_{\mathbb{R}^n} \delta_x(y) f(y)\, dy \\
&= -\int_{\mathbb{R}^n} \delta_x(y) f(x)\, dy \\
&= -f(x) \overbrace{\int_{\mathbb{R}^n} \delta_x(y)\, dy}^{1} \\
&= -f(x)
\end{aligned}
$$

Harmonic Functions

Suppose u is harmonic



$u : \Omega \to \mathbb{R}^n$ harmonic.

Mean-value Formulas

Let U be an open set in $\mathbb{R}^n$, $u : U \to \mathbb{R}$ such that $\Delta u = 0$ in U. Then

$$
\begin{aligned}
u(x) &= \fint_{\partial B(0,r)} -u(y)\, dS(y) \\
&= \fint_{B(x,r)} u(y)\, dy
\end{aligned}
$$

where $B(x,r) \subseteq U$.
IMAGE HERE

Proof

Consider $\phi(r) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y)\, dS(y)$.
If $\phi'(r) = 0$, when we are done since that would make ϕ constant and $\phi(r) = \lim_{s\to 0} \phi(s) = u(x)$. Then

$$
\begin{aligned}
\phi(r) &= \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y)\, dS(y) \\
&\underset{y=x+rz}{=} \frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B(0,1)} u(x+rz)r^{n-1}\, dS(z) \\
&= \frac{1}{n\alpha(n)}\int_{\partial B(0,1)} u(x+rz)\, dS(z)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\phi'(r) &= \frac{1}{n\alpha(n)}\int_{\partial B(0,1)} \nabla u(x+rz)\cdot z\, dS(z) \\
&\underset{y=x+rz}{=} \frac{1}{|\partial B(0,r)|}\int_{\partial B(x,r)} \nabla u(y)\cdot \frac{y-x}{r}\, dS(y) \\
&= \frac{1}{|\partial B(0,r)|}\int_{\partial B(x,r)} \nabla u(y)\cdot \eta\, dS(y) \\
&= \frac{1}{|\partial B(0,r)|}\int_{\partial B(x,r)} \frac{\partial y}{\partial \eta}\, dS(y) \\
&= \frac{1}{|\partial B(0,r)|}\int_{B(x,r)} \Delta u \\
&= 0
\end{aligned}
$$

January 22, 2024

Mean Value Formula

For $U \subseteq \mathbb{R}^n$, U open with $u : U \to \mathbb{R}$ such that $u \in C^2(U)$, $\Delta u = 0$, we have

$$
u(x) \underbrace{=}_{\text{(a)}} ⨍_{\partial B(x,r)} u \underbrace{=}_{\text{(b)}} ⨍_{B(x,r)} u
$$

for all $B(x,r) \subseteq U$.
Recall that (a) was proven above by setting $\phi(r) = ⨍_{\partial B(r)} u(y)\, dS(y)$ and showing $\phi'(r) = 0$.
For (b), we again apply spherical coordinates such that

$$
\begin{aligned}
\int_{B(x,r)} u(y)\, dy &= \int_0^r \int_{\partial B(x,s)} u(y)\, dS(y) ds \\
&= \int_0^r |\partial B(x,s)| \overbrace{⨍_{\partial B(x,s)} u(y)\, dS(y)}^{u(x)} ds \\
&= u(x)\int_0^r |\partial B(x,s)|\, ds \\
&= u(x)\int_0^r n\alpha(n)S^{n-1}\, ds \\
&= \frac{u(x)n\alpha(n)S^n}{n}\Big|_0^r \\
&= u(x)\overbrace{\alpha(n)r^n}^{|B(x,r)|}
\end{aligned}
$$

Converse

Recall that

$$\phi'(r) = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u(y)\, dy$$

Suppose then that we do not know that $\Delta u = 0$ but we have

$$u(x) = \fint_{\partial B(x,r)} u, \quad \forall B(x,r) \subseteq U$$

Then, necessarily, $\Delta u = 0$ in U.

- Proof
  Suppose, for sake of contradiction, that $\Delta u \neq 0$. Then, without loss of generality, there exists $y \in U$ such that $\Delta u(x) > 0$ for $x \in B(y,n) \subseteq U$.
  IMAGE HERE

  $$\phi(r) = \fint_{\partial B(x,r)} u(x)\, dS(x)$$

  and

  $$\phi'(r) = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(y,r)} \Delta u(x)\, dS(x) > 0$$

  which contradicts $\phi'(x) = 0$.

Strong Maximum Principle

Let $U \subseteq \mathbb{R}^n$ be a bounded open set, $u \in C^2(U) \cap C(\overline{U})$, $\Delta u = 0$ on U. Then

1. $\max_{\overline{U}}(u) = \max_{\partial U}(u)$.
2. If U is connected and u has its maximum in an interior point, then u is constant on $\overline{U}$.

IMAGE HERE - 2

Proof of A

Since $\partial U \subseteq \overline{U}$, $\max_{\partial U}(u) \leq \max_{\overline{U}}(u)$.
Let $x_0 \in \overline{U}$ such that $u(x_0) = \max_{\overline{U}}(u)$.
IMAGE HERE - 4
So either $x_0 \in \partial U$ or $x_0 \in U$.
Let U' be the connected component which contains x_0. Then $x_0 \in U'$, so by part (b) u is constant on $\overline{U'}$. So

$$\max_{\overline{U}}(u) = u(x_0) = \max_{\partial U'}(u) \leq \max_{\partial U}(u)$$

Proof of B

Then there exists $x_0 \in U$ such that $\max_{\overline{U}}(u) = u(x_0) = M$.
Let us define $\Omega = \{y \in U \,:\, u(y) = M\}$. Then

1. $\Omega \neq \varnothing$, $B \setminus x_0 \in \Omega$.

2. Ω open set.

IMAGE HERE - 3

1. Ω is closed, since $\Omega = u^{-1}(\{M\})$.

It follows that $\Omega = U$ and, therefore, $u(y) = M$, $\forall\, y \in U$.

- Proof of 2
  Let $y \in \Omega$, $y \in U$, $u(y) = M$. Then there exists $B(y, r) \subseteq U$, and

$$M = u(y) = ⨍_{B(y,r)} u(x)\, dS(x) \leq M$$

  Then

$$⨍_{B(y,r)} u(x)\, dx = M$$

  so $u(x) = M$, $\forall\, x \in B(y, r)$ and, therefore $B(y, r) \subseteq \Omega$ and Ω is open.

Remark: Boundedness Is Important

1. Consider $f(x) = x$ on $\mathbb{R}_{\geq 0}$.

2. IMAGE HERE - 5

Remark: Strong Minimum Principle Is Equivalent

Consequences

1. Positivity of harmonic functions.

2. Uniqueness of the Poisson problem.

Corollary: Positivity of Harmonic Functions

Suppose that U is connected and $u : U \to \mathbb{R}$, $u \in C^2(U) \cap C(\overline{U})$ solves the following problem

$$\begin{cases} \Delta u = 0 & \text{on } U \\ u = g & \text{on } \partial U \end{cases}$$

If $g \geq 0$ on ∂U, then u is positive on U as long as g is positive in some point.

Proof

Assume $\exists x_0 \in \partial U$ where x_0 is the minimum. Then $u(x_0) = \min_{\overline{U}}(u)$ and, $\forall x \in U$,

$$0 \leq u(x_0) = \min_{\overline{U}}(u) \leq u(x)$$

so u is non-negative. If $u(x) = 0$, then $u(x_0) = 0$ and the minimum is achieved in the interior. That would mean $u(x) = 0$, $\forall x \in \overline{U} \supseteq \partial U$ and $g(x) = 0$, $\forall x \in \partial U$ which would be a contradiction.

Theorem: Uniqueness of the Poisson Problem

Suppose $U \subseteq \mathbb{R}^n$ is open, connected and bounded. Then, there exists at most one solution to

$$(*) \begin{cases} -\Delta u = f & \text{in } U \\ \quad u = g & \text{on } \partial U \end{cases}$$

where $u \in C^2(U) \cap C(\overline{U})$.

Proof

Let u_1 and u_2 be two solutions of $*$.
Consider $w = u_1 - u_2$ and observe that

$$\Delta w = \Delta u_1 - \Delta u_2 = -f + f = 0, \quad \text{in } U$$

and $w|_{\partial U} = g - g = 0$ on ∂U. Therefore

$$\begin{cases} \Delta w = 0 & \text{in } U \\ \quad w = 0 & \text{on } \partial U \end{cases}$$

By applying the minimum and maximum principles,

$$w(x_0) = \min_{\overline{U}}(w) \leq w(x) \leq \max_{U}(w) = w(x)$$

so $w(x) = 0$, $\forall x \in \overline{U}$ and therefore $u_1 = u_2$.

Example

Let's consider $f : \mathbb{C} \to \mathbb{C}$ analytic (i.e. $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $a_n, z \in \mathbb{C}$). Then

$$f(z) = u(z) + v(z)$$

If $\mathbb{C} \cong \mathbb{R}^2$,

$$f(x + y) = u(x, y) + v(x, y)$$

for $u : \mathbb{R}^2 \to \mathbb{R}$ and $v : \mathbb{R}^2 \to \mathbb{R}$.
Claim: u and v are Harmonic.

$$u(x, y) + v(x, y) = \sum_{n=0}^{\infty} a_n (x + iy)^n$$

and

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \sum_{n=1}^{\infty} a_n n|x+iy|^{n-1}$$
$$\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = \sum_{n=1}^{\infty} a_n n|x+iy|^{n-1} i$$

So

$$i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

Recall: Convolution and Smoothing

Let $U \subseteq \mathbb{R}^n$ be an open set.
For $\varepsilon > 0$, define $U_\varepsilon = \{x \in U \,:\, d(x, \partial U) > \varepsilon\}$.
IMAGE HERE - 6
Define

$$\eta(x) \begin{cases} ce\left(\frac{1}{|x|^2-1}\right) & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

with c such that $\int_{\mathbb{R}^n} \eta(x)\, dx = 1$, $\eta \in C^\infty(\mathbb{R}^n)$
IMAGE HERE - 7
Note that supp$(\eta) = B(0,1)$ and take

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right), \quad \eta_\varepsilon \in C^\infty(\mathbb{R}^n)$$

Then

$$\int_{\mathbb{R}^n} \eta_\varepsilon(x)\, dx = 1$$

and supp$(\eta_\varepsilon) \subseteq B(0,\varepsilon)$.
If f is locally integrable on U, define its mollification

$$f^\varepsilon = (x) = \int_U \eta_\varepsilon(x-y) f(y)\, dy \quad \forall x \in U_\varepsilon$$

January 24, 2024

Recall: Mollifiers

Define

$$\eta(x) = \begin{cases} ce\left(\frac{1}{|x|^2-1}\right) & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

where $\eta \in C^{\infty}(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \eta(x) = 1$ and supp$(\eta) \subseteq B(0,1)$.
Then for $\varepsilon > 0$, $\eta_\varepsilon(x) = \frac{1}{\varepsilon}\left(\frac{x}{\varepsilon}\right)$ where $\eta_\varepsilon \in C^{\infty}(\mathbb{R}^n)$.
So $\int_{\mathbb{R}^n} \eta_\varepsilon(x) = 1$ and supp$(\eta_\varepsilon) \subseteq B(0,\varepsilon)$
Given f locally summable; $f : U \to \mathbb{R}$,

$$
\begin{aligned}
f^{\varepsilon}(x) &:= \int_U \eta_\varepsilon(x-y) f(y)\, dy \quad x \in U_\varepsilon \\
&= \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y) f(y)\, dy \quad x \in U_\varepsilon
\end{aligned}
$$

Properties

1. $f^{\varepsilon} \in C^{\infty}(U_\varepsilon)$.
2. $f^{\varepsilon} \underset{\varepsilon \to 0}{\longrightarrow} f$ a.e.
3. If f continuous, then $f^{\varepsilon} \underset{\varepsilon \to 0}{\longrightarrow} f$ uniformly on compact sets of U.

Theorem 6:

Let $u \in C(U)$ with $U \in \mathbb{R}^n$ open and such that u satisfies the mean-value property (i.e. $u(x) = ⨍_{\partial B(x,r)} u(y)\, dS(y)$, $\forall B(x,r) \subseteq U$), then $u \in C^{\infty}$.

Corollary

If $u \in C^2(U)$ is harmonic, then $u \in C^{\infty}(U)$.

Proof

Let us take $x_0 \in U$
IMAGE HERE - 1
Notice, that if we prove that $u = u_\varepsilon$ on U_ε then we are done.
Let $x \in U_\varepsilon$, and noticing that $\eta(x) = \eta(|x|)$,

$$
\begin{aligned}
u_\varepsilon(x) &= \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)u(y)\,dy \\
&= \frac{1}{\varepsilon^n}\int_{B(x,\varepsilon)} \eta\frac{|x-y|}{\varepsilon}u(y)\,dy \\
&\underset{\text{spherical}}{=} \frac{1}{\varepsilon^n}\int_0^\varepsilon\int_{\partial B(x,r)} \eta\frac{\overbrace{|x-y|}^{r}}{\varepsilon}u(y)\,dS(y)dr \\
&= \frac{1}{\varepsilon^n}\int_0^\varepsilon \eta\frac{r}{\varepsilon}\int_{\partial B(x,r)} u(y)\,dS(y)dr \\
&= \frac{1}{\varepsilon n}\int_0^\varepsilon \eta\frac{r}{\varepsilon}\underbrace{|\partial B(x,r)|}_{|\partial B(0,r)|}u(x)\,dr \\
&= \frac{u(x)}{\varepsilon^n}\int_0^r \eta\frac{r}{\varepsilon}\int_{\partial B(0,r)\,dS(y)dr} \\
&= u(x)\int_0^\varepsilon \frac{1}{\varepsilon^n}\eta\frac{r}{\varepsilon}\,dS(y)dr \\
&= u(x)\overbrace{\int_{B(0,\varepsilon)} \eta_\varepsilon(y)\,dy}^{1} = u(x)
\end{aligned}
$$

Theorem 7: Local Estimates of Harmonic Functions

Suppose $u \in C^2(U)$ a harmonic function.
Then $|D^\alpha u(x_0)| \leq \frac{C_k}{r^{n+k}}||u||_{L^1(B(x_0,r))}$, $B(x_0,r) \subseteq U$, where α is a multiindex of order k, $C_0 = \frac{1}{\alpha(n)}$ and $C_k = \frac{(2^{n+1}nk)^k}{\alpha(n)}$ for $k = 1, 2, \ldots$.
We may take α since, by previous theorem, $u \in C^\infty(U)$.

Proof

By induction.
Consider $k = 0$.

$$
\begin{aligned}
u(x_0) &= ⨍_{B(x_0,r)} u(y)\,dy \\
&= \frac{1}{\alpha(n)r^n}\int_{B(x_0,r)} u(y)\,dy \\
|u(x_0)| &\leq \frac{1}{\alpha(n)r^n}\int_{B(x_0,r)} |u(y)|\,dy \\
&= \frac{C_0}{r^n}||u||_{L^1(B(x_0,r))}
\end{aligned}
$$

For $k = 1$, if $|\alpha| = k = 1$ then $D^\alpha u(X_0) = \frac{\partial u}{\partial x_i}(x)$ for $i = 1, 2, \ldots$.
Notice that $\frac{\partial u}{\partial x_i}$ is also harmonic.

$$
\begin{aligned}
\Delta\frac{\partial u}{\partial x_i} &= \sum_{t=1}^n \frac{\partial^2}{\partial x_t^2}\frac{\partial u}{\partial x_i} \\
&= \frac{\partial}{\partial x_i}\underbrace{\sum_{t=1}^\infty \frac{\partial^2 u}{\partial x_t^2}}_{0}
\end{aligned}
$$

Applying the mean-value formula to $\frac{\partial u}{\partial x_i}(x_0)$ in $B(x, r/2)$.
IMAGE HERE - 2

$$
\begin{aligned}
\frac{\partial u}{\partial x_i}(x_0) &= \fint_{B(x_0,r/2)} \frac{\partial u}{\partial x_i}(y)\, dy \\
&= \frac{2^n}{\alpha(n) r^n}{}_{B(x_0,r/2)} \frac{\partial u}{\partial x_i}(y)\, dy
\end{aligned}
$$

Recall $\int_\Omega w \Delta v = -\int_\Omega \nabla w \cdot \nabla v + \int_\Omega w \frac{\partial v}{\partial \eta}$.

$$
\begin{aligned}
&\underset{e_i = \nabla y_i}{=} \frac{2^n}{\alpha(n) r^n} \int_{B(x_0,r/2)} \nabla \underbrace{u(y)}_{w} \cdot \nabla \underbrace{y_i}_{v}\, dy \\
&\underset{IBP}{=} \frac{2^n}{\alpha(n) r^n} \left[-\int_{B(x_0,r/2)} u(y) \Delta y_i\, dy + \int_{\partial B(x_0,r/2)} u(y) \frac{\partial y_i}{\partial \eta} \right]
\end{aligned}
$$

Note that

$$
\frac{\partial y_i}{\partial \eta} = \nabla y_i \cdot \eta = e_i \cdot \eta = \eta_i
$$

and

$$
\left| \frac{\partial y_i}{\partial \eta} \right| = |\eta_i| \le |\eta| = 1
$$

So,

$$
\begin{aligned}
\left| \frac{\partial u}{\partial x_i} x_0 \right| &\le \frac{2^n}{\alpha(n) r^n} \int_{\partial B(x_0,r/2)} |u(y)|\, dS(y) \\
&= \frac{2^n n \alpha(n) \left(\frac{r}{2}\right)^{n-1}}{\alpha(n) r^n} ||u||_{L^\infty(\partial B(x_0,r/2))} \\
&= \frac{2n}{r} \underbrace{||u||_{L^\infty(\partial B(x_0,r/2)}}_{*}
\end{aligned}
$$

Let’s analyze $*$.
Let $x \in \partial B(x_0, r/2)$, then $B(x, r/2) \subseteq B(x_0, r)$.
IMAGE HERE - 3
Then we may apply $k = 0$.

$$
\begin{aligned}
|u(x)| &\le \frac{C_0}{\left(\frac{r}{2}\right)^n} ||u||_{L^1(B(x,r/2))} \\
&\le \frac{C_0}{\left(\frac{r}{2}\right)^n} ||u||_{L^1(B(x_0,r))}
\end{aligned}
$$

Then

$$
\begin{aligned}
\left| \frac{\partial u}{\partial x_i}(x_0) \right| &\le \frac{2n}{r} \frac{C_0}{\left(\frac{r}{2}\right)^n} ||u||_{L^1(B(x_0,r))} \\
&= \frac{2^{n+1} n}{r^{n+1} \alpha(n)} ||u||_{L^1(B(x_0,r))}
\end{aligned}
$$

HOMEWORK: Induct for arbitrary k.

Theorem 8: Liouville's Theorem

Suppose $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic and bounded. Then u is constant.

Proof

$$|D^\alpha u(x)| = \sqrt{\sum_{i=1}^{n}\left[\frac{\partial u}{\partial x_i}\right]^2} \leq \sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|$$

Let $r > 0$, $B(x,r) \subseteq \mathbb{R}^n$. Then, using estimates

$$\left|\frac{\partial u}{\partial x_i}(x)\right| \leq \frac{C_1}{r^{n+1}}||u||_{L^1(B(x,r))}$$

Therefore,

$$\begin{aligned}|D^\alpha u(x)| &\leq \frac{nC_1}{r^{n+1}}||u||_{L^1(B(x,r))} \\ &= \frac{nC_1}{r^{n+1}}\int_{B(x,r)}|u(y)|\,dy \\ &\leq \frac{nC_1}{r^{n+1}}||u||_{L^\infty(B(x,r))}\alpha(n)r^n \\ &= \frac{C||u||_{L^\infty(B(x,r))}}{r}\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{\partial u}{\partial x_i}(x)\right| \leq \frac{C||u||_{L^\infty(B(x,r))}}{r} \\ &\left|\frac{\partial u}{\partial x_i}(x)\right| \leq C||u||_{L^\infty(B(x,r))}\lim_{r\to\infty}\frac{1}{r} \implies \frac{\partial u}{\partial x_i}(x) = 0 \implies u = Ck\end{aligned}$$

Theorem: Representation Formula

Recall: $f \in C_c^2(\mathbb{R}^n)$, $(*) - \Delta u = f$ in $\mathbb{R}^n$, we saw that

$$\tilde{u}(x) : \int_{\mathbb{R}^n}\Phi(x-y)f(y)\,dy$$

solves $*$.
Let us consider $u \in C^2(\mathbb{R}^n)$ solving $-\Delta u = f$ for $n \geq 3$ where $f \in C_c^2(\mathbb{R}^n)$ and u is bounded.
Then $u(x) = \tilde{u}(x) + C = \int_{\mathbb{R}^n}\Phi(x-y)f(y)\,dy + C$.

Proof

Notice that if $\tilde{u}$ is bounded, then we are done. Because we may consider $w = u - \tilde{u}$ on $\mathbb{R}^n$ where

$$\Delta w = \Delta u - \Delta\tilde{u} = -f - (-f) = 0$$

Therefore w is bounded and, by Liouville's Theorem, $w = C$ and $u - \tilde{u} = c \iff u = \tilde{u} + C$.

$$\begin{aligned}|\tilde{u}(x)| &\leq \int_{B(0,k)}\Phi(x-y)f(y)\,dy \\ &\leq ||f||_{L^\infty(\mathbb{R}^n)}\int_{B(0,k)}\Phi(x-y)\,dy\end{aligned}$$

If this is less than some C which does not depend on x, we are done.
Since $\Phi(x) \to 0$ for $|x| \to \infty$, for any C_1 $\exists \alpha$ such that if $|x| > \alpha$ then $|\Phi(x)| < C_1$.
IMAGE HERE - 4
dist$(x, B(0,k))$ = dist(x, y_0) where $y_0 \in \overline{B(0,k)}$.
IMAGE HERE - 5
There are two cases.

- Case 1
  dist$(x, B(0,k)) \leq \alpha$
  $B(x,k) \subseteq B(0, \alpha + Ck)$
  Let $y \in B(x,k)$, then $|y - x| < k$ so $|x - y_0| < \alpha$.
  Therefore $|y - y_0| \leq k + \alpha \implies |y| \leq k + \alpha + |y_0| = 2k + \alpha \implies y \in B(0, 2k + \alpha)$. Then

$$||f||_{L^\infty(\mathbb{R}^n)} \int_{B(x,k)} \Phi(y)\, dy \leq ||f||_{L^\infty(\mathbb{R}^n)} \int_{B(0,\alpha+2k)} \Phi(y)\, dy$$

  HOMEWORK - Consider the other case.

January 29, 2024

Recall: Representation Formula

For $n \geq 3$.

$$\tilde{u}(x) : \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$$

It is sufficient to show that $\tilde{u}$ is bounded. Then

$$|\tilde{u}| \leq C \int_{B(0,k)} \Phi(x-y)\, dy$$

$\forall C_1$, $\exists \alpha$ such that $|z| \geq \alpha \implies |\Phi(z)| \leq C_1$.

Case 2

For dist$(x, B(0,k)) \geq \alpha$, dist$(x,y) \geq \alpha$, $\forall y \in B(0,k)$. Then

$$|x-y| \geq \alpha$$
$$\frac{1}{|x-y|} \leq \frac{1}{\alpha}$$
$$\frac{1}{|x-y|^{n-2}} \leq \frac{1}{\alpha^{n-1}}$$

and

$$|\tilde{u}(x)| \leq C \int_{B(0,k)} \frac{1}{|x-y|^{n-2}}\, dy \leq \frac{C}{\alpha^{n-2}} \int_{B(0,k)} dy$$

Theorem 10: Harmonic Implies Analytic

Let $U \subseteq \mathbb{R}^n$ open, $u \in C^2(U)$ harmonic. Then u is analytic in U.

Proof

Let $x_0 \in U$. We want to prove that the power series converges to $u(x)$ for x in a neighborhood around x_0.
Let $r = \text{dist}\left(x_0, \frac{\partial U}{4}\right)$, $M = \frac{1}{\alpha(n) r^n} ||u||_{L^1(B(x_0,r)} \subset U$.
IMAGE HERE - 1
We want to analyze $x \in B(x_0, r)$.
Notice that $B(x,r) \leq B(x_0, 2r)$, and $z \in B(x,r)$ gives $|z - x| < r$ so

$$|z - x_0| \leq \underbrace{|z - x|}_{\leq r} + \underbrace{|x - x_0|}_{\leq r} \leq 2r$$

Applying estimates on $B(x,r)$, $|\alpha| = k$,

$$\begin{aligned} |D^\alpha u(x)| &\leq \frac{C_k}{r^{n+k}} ||u||_{L^1(B(x,r))} \\ &\leq \frac{C_k}{r^{n+k}} ||u||_{L^1(B(x_0,2r))} \end{aligned}$$

and

$$\sup_{x \in B(x_0,r)} |D^\alpha u(x)| \leq \frac{\left(2^{n+1} nk\right)^k}{\alpha(n) r^{n+k}} ||u||_{L^1(B(x_0,2r))}$$

Notice, by Stirling's approximation or Taylor expansion, $\frac{k^k}{k!} < e^k$, $\forall\, k \geq 1$. So

$$|\alpha|^{|\alpha|} < e^{|\alpha|} |\alpha|!$$

and

$$n^k = (\underbrace{\overset{k}{1 + \cdots + 1}}_{n-\text{times}}) = \sum_{|\beta| = k} \frac{|\beta|!}{\beta!} \geq \frac{|\alpha|!}{\alpha!}$$

where $|\alpha|! \leq \alpha! n^k$, $\beta = (\beta_1, \ldots, \beta_2)$ and $\beta! := \beta_1! \beta_2! \cdots \beta_n!$. Therefore

$$|\alpha|^{|\alpha|} \leq e^{|\alpha|} |\alpha|! \leq e^{|\alpha|} \alpha! n^k$$

and finally

$$(*) \quad k^k \leq e^k \alpha! n^k$$

Applying $*$ to the above inequality,

$$\begin{aligned} \sup_{X \in B(x_0,r)} |D^\alpha u(x)| &\leq \frac{\left(2^{n+1} n\right)^k e^k \alpha! n^k}{\alpha(n) r^n r^k} ||u||_{L^1(B(x_0,2r))} \\ &\leq \left(\frac{2^{n+1} n^2 e}{r}\right)^k \cdot \alpha! M \end{aligned}$$

Let us analyze the Taylor expansion

$$\sum_{k=0}^{N}\sum_{|\alpha|=k}\frac{D^{\alpha}u(x_0)}{\alpha!}(x-x^0)^{\alpha}$$

Where $\alpha=(\alpha_1,\dots,\alpha_n)$, $y\in\mathbb{R}^n$ and $y^{\alpha}=y_1^{\alpha_1}\cdots y_n^{\alpha_n}$.
Pick $|x-x_0|\le\frac{r}{2^{n+2}n^3e}$. We want to prove that the remainder $R_N(x)\underset{N\to\infty}{\longrightarrow}0$.

$$\begin{aligned}R_N(x)&=u(x)-\sum_{k=0}^{N-1}\sum_{|\alpha|=k}\frac{D^{\alpha}u(x_0)}{\alpha!}(x-x_0)^{\alpha}\\&=\sum_{|\alpha|=N}\frac{D^{\alpha}u(x_0+t(x-x_0))(x-x_0)^{\alpha}}{\alpha!},\quad\text{for some }|t|\le 1\end{aligned}$$

Using the remainder of the Taylor expansion with $g(t)=u(x_0+t(x-x_0))$ for $g:I\to\mathbb{R}$.
Homework: show this around $t=0$ at $t=1$.
Note that $u(x_0+t(x-x_0))$ describes a straight long with $t=0\implies u(x_0)$ and $t=1\implies u(x)$.
Notice also that $x_0+t(x-x_0)\in B(x_0,r)$. Then, considering the superemum of the remainder,

$$|R_n(x)|\le\sum_{|\alpha|=N}\left(\frac{2^{n+1}n^2e}{r}\right)^N\cdot M\alpha!\cdot\frac{|(x-x_0)^{\alpha}|}{\alpha!}$$

Remark: for $\alpha=(\alpha_1,\dots,\alpha_n)$ and $y=(y_1,\dots,y_n)$,

$$\begin{aligned}|y^{\alpha}|=|y_1^{\alpha_1}\cdots y_n^{\alpha_n}|&\le|y_1|^{\alpha_1}\cdots|y_n|^{\alpha_n}\\&\le|y|^{\alpha_1}\cdots|y|^{\alpha_n}\\&=|y|^{\alpha_1+\alpha_2+\cdots+\alpha_n}\\&=|y|^{\alpha}\end{aligned}$$

Therefore

$$\begin{aligned}|R_n(x)|&\le\sum_{|\alpha|=N}\left(\frac{2^{n+1}n^2e}{r}\right)^N\cdot M|x-x_0|^N\\&\le M\cdot\sum_{|\alpha|=N}\left(\frac{2^{n+1}n^2e}{r}\right)^N\left(\frac{r}{2^{n+2}n^3e}\right)^N\\&=M\cdot\sum_{|\alpha|=N}\left(\frac{1}{2n}\right)^N\\&\le M\left(\frac{1}{2n}\right)^N\sum_{|\alpha|=N}\\&\le M\left(\frac{1}{2n}\right)^N n^N\\&=M\left(\frac{1}{2}\right)^N\end{aligned}$$

Note that $\sum_{|\alpha|=N}(1)\le n^N$ since

$$\alpha=(\alpha_1,\dots,\alpha_n)=(\alpha_{1_N},\dots,\alpha_{i_N})=n^N$$

Theorem 11: Harnack's Inequality

Define $V \subset\subset U$ as "V totally contained in U" meaning $\overline{V}$ compact and $V \subseteq \overline{V} \subseteq U$.
IAMGE HERE - 2
Let U open and $u \in C^2(U)$ harmonic and non-negative.
Then for each connected open set $V \subset\subset U$

$$\sup_V u \leq C \inf_V u$$

for some C that depends on V.

Remark

Then

$$\frac{1}{C} u(y) \leq u(x) \leq C u(y), \quad \forall x, y \in V$$

Since

$$u(x) \leq \sup_V u \leq C \inf_V u \leq C u(y)$$

and

$$\frac{1}{C} u(y) \leq \frac{1}{C} \sup_V u \leq \inf_V u \leq u(x).$$

Proof

Take $r = \frac{\text{dist}(v, \partial U)}{4} > 0$.

- Case 1
  Let us suppose that $x, y \in V$ such that $|x - y| < r$.
  IMAGE HERE - 3
  Notice $B(x, 2r) \subseteq U$. Applying mean-value formulas,

$$u(x) = ⨍_{B(x,2r)} u = \frac{1}{\alpha(n)(2r)^n} \int_{B(x,2r)} u$$

  But notice that $B(y, r) \subseteq B(x, 2r)$, so

$$u(x) \geq \frac{1}{\alpha(n) 2^n r^n} \int_{B(y,r)} u = \frac{1}{2^n} ⨍_{B(y,r)} u = \frac{1}{2^n} u(y)$$

  That is, if $x, y \in V$ such that $|x - y| < r$, then $u(x) \geq \frac{1}{2^n} u(y)$ and, mutatis mutandis, $u(y) \geq \frac{1}{2^n} u(x)$.

- Case 2
  Let us cover $\overline{V}$ by an open covering of balls $\{B_i\}_{i=1}^N$ such that the radius of each ball is $\frac{r}{2}$ and $B_i \cap B_{i-1} \neq \emptyset$.
  IMAGE HERE - 4
  Then $u(x) \geq \frac{1}{2^n} u(z) \frac{1}{2^n 2^n} u(y)$, so $u(x) \geq \frac{1}{2^{2n}} u(y)$.

In the same way, $u(y) \geq \frac{1}{2^{2n}} u(x)$.
IMAGE HERE - 5
For three balls, $u(x) \geq \frac{1}{2^{3n}} u(y)$ and $u(y) \geq \frac{1}{2^{3n}} u(x)$.
Since we have a finite covering of N balls, the same strategy gives

$$u(x) \geq \frac{1}{2^{Nn}} u(y) \qquad\qquad u(y) \geq \frac{1}{2^{Nn}} u(x)$$

and

$$\frac{1}{2^{Nn}} \leq u(x)$$

Taking the supremum $y \in V$;

$$\sup_{y \in V} u(y) \leq 2^{Nn} u(x)$$

taking the infemum $x \in V$

$$\inf_{x \in V} u(y)$$

Recap: Laplace Equation

- Fundemental Solution
  - Poisson Equation in $\mathbb{R}^n$
- Mean-value Formulas
- Properties
  - Strong Maximum / Minimum Principles
    - Uniqueness of the Poisson Equation on Bounded Domains
  - Regularity
  - Derivative Estimates
  - Liouville's Theorem
    - Representation Formula
      - Uniqueness of the Poisson Equation up to a Constant on $\mathbb{R}^n$ for Bounded Functions
  - Analyticity
  - Harnack's Inequality

Green's Functions

For U open and bounded, $\partial U \in C^1$.
Goal: We want to solve $-\Delta u = f$ on U and $u = g$ on ∂U.

Recall: Green's Formula

$$\int_{\Omega} u\Delta v - v\Delta u = \int_{\partial\Omega} u\frac{\partial v}{\partial \eta} - v\frac{\partial u}{\partial \eta}$$

Obtaining Green's Formula

Let $x \in U$ and consider $u(y)$, $\Phi(y-x)$ as functions of y.
Let $\varepsilon > 0$ and consider $V_\varepsilon = U \setminus B_\varepsilon(x)$. Applying Green's formula; $\Omega = V_\varepsilon$,

$$\int_{V_\varepsilon} \underbrace{u(y)\Delta_y \Phi(y-x)}_{=0} - \Phi(y-x)\Delta_y u = \int_{\partial V_\varepsilon} u(y)\frac{\partial \Phi(y-x)}{\partial \eta} - \Phi(y-x)\frac{\partial u(y)}{\partial \eta}$$

IMAGE HERE - 6

January 31, 2024

Green's Functions

Goal is to solve for $U \subseteq \mathbb{R}^n$ open and bounded,

$$\begin{cases} -\Delta u = f, & \text{in } U \\ u = g, & \text{on } \partial U \end{cases}$$

by obtaining Green's function.
Let $x \in U$ and assume $u \in C^2(U)$, and consider $u(y)$ and $\Phi(y-x)$.
Recall Green's formula $\int_\Omega u\Delta v - v\Delta u = \int_{\partial\Omega} u\frac{\partial u}{\partial \eta} - v\frac{\partial v}{\partial \eta}$.
Then, let $\varepsilon > 0$ and define $V_\varepsilon U \setminus B(x,\varepsilon)$.
IMAGE HERE - 1
By applying Green's Formula,

$$\int_{V_\varepsilon} u(y)\underbrace{\Delta\Phi(y-x)}_{0} - \Phi(y-x)\Delta u(y) = \int_{\partial V_\varepsilon} \overbrace{\underbrace{u\frac{\partial \Phi(y-x)}{\partial \eta}}_{\square_1} - \underbrace{\Phi(y-x)\frac{\partial u}{\partial \eta}}_{\square_2}}^{\square}$$

Notice that $\partial V_\varepsilon = \partial U \cup \partial B(x,\varepsilon)$.
Let us analyze $\square$ along $\partial B(x,\varepsilon)$
For $\square_2$ along $\partial B(x,\varepsilon)$,

$$\begin{aligned} \left| \int_{\partial B(x,\varepsilon)} \Phi(y-x)\frac{\partial u(y)}{\partial \eta} \right| &\leq \sup_{\overline{U}} |\nabla U| \int_{\partial B(x,\varepsilon)} \Phi(y-x)\, dS(y) \\ &= \frac{C}{\varepsilon^{n-2}} \int_{\partial B(x,\varepsilon)} dS(y) \\ &= \frac{C\varepsilon^{n-1}}{\varepsilon^{-2}} \\ &= c\varepsilon \end{aligned}$$

Then $\lim_{\varepsilon\to 0}\int_{\partial B(x,\varepsilon)} \square_2 = 0$.
Now, for $\square_1$ along $\partial B(x,\varepsilon)$ and recalling $\nabla\Phi(z) = \frac{-z}{n\alpha(n)|z|^n}$ while $\eta(z) = \frac{-z}{|z|}$ such that

$$\frac{\partial\Phi}{\partial\eta}(z) = \nabla\Phi\cdot\eta = \frac{|z|^2}{n\alpha(n)|z|^{n+1}} = \frac{1}{n\alpha(n)|z|^{n-1}}$$

we have

$$\begin{aligned}
\int_{\partial B(x,\varepsilon)} u(y)\frac{\partial\Phi(y-x)}{\partial\eta}\,dS(y) &\underset{z=y-x}{=} \int_{\partial U(0,\varepsilon)} u(z+x)\frac{\partial\Phi(z)}{\partial\eta}|z|\,ds(z)\\
&= \frac{1}{n\alpha(n)}\int_{\partial B(0,\varepsilon)} \frac{u(z+x)}{|z|^{n-1}}\,dS(z)\\
&= \frac{1}{n\alpha(n)\varepsilon^{n-1}}\int_{\partial B(0,\varepsilon)} u(z+x)\,dS(z)\\
&\underset{y=z+x}{=} \frac{1}{n\alpha(n)\varepsilon^{n-1}}\int_{\partial B(x,\varepsilon)} u(y)\,dS(y)\\
&= \fint_{\partial B(x,\varepsilon)} u(y)\,dS(y)
\end{aligned}$$

Then $\lim_{\varepsilon\to 0}\int_{\partial B(x,\varepsilon)} \square_1 = u(x)$. It follows, then, that

$$\int_U -\Phi(y-x)\Delta u(y) = \int_{\partial U} \overbrace{\underbrace{u\frac{\partial\Phi(y-x)}{\partial\eta}}_{\square_1} - \underbrace{\Phi(y-x)\frac{\partial u}{\partial\eta}}_{\square_2}}^{\square} + u(x)$$

That is

$$u(x) \underset{\square_4}{=} -\int_U \Phi(y-x)\Delta u + \int_{\partial u} \Phi(y-x)\frac{\partial u}{\partial\eta} - u\frac{\partial\Phi(y-x)}{\partial\eta}$$

Notice that we have $-\Delta u = f$ in U and $u = g$ on ∂U, but we will need $\frac{\partial u}{\partial\eta}\mid_{\partial U}$.

Definition: Corrector Function

Given a domain $U\subseteq\mathbb{R}^n$ open and bounded with $x\in U$, define the function $\phi^x(y)$ that satisfies the following

$$\begin{cases} \Delta\phi^x(y) = 0, & \text{in } U\\ \phi^x(y) = \Phi(y-x), & \text{on } y\in\partial U\end{cases}$$

Note that we do not know that such a function exists.

Green's Function Continued

Suppose that we have $\phi^x(y)$. Then, applying green's formula for $u(y)$ and $\phi^x(y)$,

$$\int_U u\Delta\underbrace{\phi^x(y)}_{0} - \phi^x(y)\Delta u = \int_{\partial U} u\frac{\partial\phi^x(y)}{\partial\eta} - \underbrace{\phi^x(y)\frac{\partial u}{\partial\eta}}_{\Phi(y-x)\frac{\partial u}{\partial\eta}}$$

Then

$$\int_{\partial U} \Phi(y-x)\frac{\partial u}{\partial \eta} \underset{\square_3}{=} \int_{\partial U} u\frac{\partial \phi^x(y)}{\partial \eta} + \int_U \phi^x(y)\Delta u$$

Replacing $\square_3$ in $\square_4$,

$$u(x) = -\int_U \Phi(y-x)\Delta u + \int_{\partial U} u\frac{\partial \phi^x(y)}{\partial \eta} + \int_U \phi^x(y)\Delta u - \int_{\partial U} u\frac{\partial \Phi(y-x)}{\partial \eta}$$

and, therefore,

$$u(x) = -\int_U \Delta u\left[\Phi(y-x) - \phi^x(y)\right] - \int_{\partial U} u\frac{\partial}{\partial \eta}\left[\Phi(y-x) - \phi^x(y)\right]$$

Definition: Green's Function

Given a domain $U \subseteq \mathbb{R}^n$, the Green's function for $x \in U$ is defined by

$$G(x,y) := \Phi(y-x) - \phi^x(y)$$

Theorem: Representation Formula

Suppose $U \subseteq \mathbb{R}^n$, and $u \in C^2(U)$ that solves

$$\begin{cases} -\Delta u = f, & \text{in } U \\ \quad u = g, & \text{on } \partial U \end{cases}$$

Then,

$$u(x) = \int_U fG(x,y) - \int_{\partial U} g\frac{\partial G(x,y)}{\partial \eta}$$

Interpretation of the Green's Functions

$$\Delta_y G(x,y) = \Delta_y \Phi(y-x) - \underbrace{\Delta_y \phi^x(y)}_{0} = \delta^x(y)$$

and

$$G(x,y) = \Phi(y-x) - \phi^x(y) = 0, \quad y \in \partial U$$

That is, it is the Dirac delta on the interior which vanishes at the boundary.

Theorem: Symmetry of the Green's Function

For all $x, y \in U$, $x \neq y$, we want to show that $G(x,y) = G(y,x)$.

Proof

Let $x, y \in U$, $x \neq y$.
Define $V(z) := G(x,z)$ and $W(z) := G(y,z)$.
Notice that $\Delta_z V = 0$ for $z \neq x$ and $\Delta_z W = 0$ for $w \neq y$ and $V(z) = W(z) = 0$ for $z \in \partial U$.
IMAGE HERE - 2
Then, let us consider $\varepsilon > 0$ and

$$\Omega_\varepsilon := U \setminus \left[B(x,\varepsilon) \coprod B(y,\varepsilon) \right]$$

Then

$$\begin{aligned} 0 = \int_{\Omega_\varepsilon} W \underbrace{\Delta V}_{0} - V \underbrace{\Delta W}_{0} &= \int_{\partial \Omega_\varepsilon} W \frac{\partial V}{\partial \eta} - V \frac{\partial W}{\partial \eta} \\ &= \int_{\partial U} W \frac{\partial V}{\partial \eta} - V \frac{\partial W}{\partial \eta} + \int_{\partial B(x,\varepsilon)} W \frac{\partial V}{\partial \eta} - V \frac{\partial W}{\partial \eta} + \int_{\partial B(y,\varepsilon)} W \frac{\partial V}{\partial \eta} - V \frac{\partial W}{\partial \eta} \end{aligned}$$

It follows that

$$\underbrace{\int_{\partial B(x,\varepsilon)} \overbrace{W \frac{\partial V}{\partial \eta}}^{(a)} - \overbrace{V \frac{\partial W}{\partial \eta}}^{(b)}}_{\heartsuit_1} = \underbrace{\int_{\partial B(y,\varepsilon)} V \frac{\partial W}{\partial \eta} - W \frac{\partial V}{\partial \eta}}_{\heartsuit_2}$$

Let us analyze (b), fixing $\varepsilon_0 > 0$ such that $\varepsilon < \varepsilon_0$

$$\begin{aligned} \left| \int_{B(x,\varepsilon)} V \frac{\partial W}{\partial \eta} \right| &\leq \sup_{z \in \partial B(x,\varepsilon)} |V(z)| \int_{B(x,\varepsilon)} \left| \frac{\partial W}{\partial \eta}(z) \right| \, dS(z) \\ &\leq \sup_{z \in \partial B(x,\varepsilon_0)} |\nabla W(z)| \int_{\partial B(x,\varepsilon)} dS(z) \\ &\leq C \varepsilon^{n-1} \sup_{z \in \partial B(x,\varepsilon)} |V(z)| \\ &\leq C \varepsilon^{n-1} \left(\frac{C}{\varepsilon^{n-2} + C} \right) \\ &= C\varepsilon + C\varepsilon^{n-1} \end{aligned}$$

Since, given $z \in \partial B(x,\varepsilon)$,

$$V(z) = G(x,z) = \Phi(z-x) - \phi^x(z)$$

we have

$$\begin{aligned} |V(z)| &\leq |\Phi(z-x)| + |\phi^x(z)| \\ &\leq \frac{C}{\varepsilon^{n-2}} + \sup_{z \in B(x,\varepsilon_0)} |\phi^x(z)| \end{aligned}$$

Thus, we have $\lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} V \frac{\partial W}{\partial \eta} = 0$.
Let us analyze (a),

$$\int_{\partial B(x,\varepsilon)} W(z)\frac{\partial V}{\partial \eta}(z)\,dS(z) = \int_{\partial B(x,\varepsilon)} W(z)\left[\frac{\Phi(z-x)}{\partial \eta} - \frac{\partial \phi^x(z)}{\partial \eta}\right] dS(z)$$

$$= \int_{\partial B(x,\varepsilon)} \overbrace{W(z)\frac{\partial \Phi(z-x)}{\partial \eta}}^{(e)} - \overbrace{W(z)\frac{\partial \phi^x(z)}{\partial \eta}}^{(h)}\,dS(z)$$

Analyzing (h),

$$\left|\int_{\partial B(x,\varepsilon)} W(z)\frac{\partial \phi^x(z)}{\partial \eta}\right| \le \sup_{\partial B(x,\varepsilon_0)} |\nabla \phi^x(z)||W(z)| \int_{\partial B(x,\varepsilon)} dS(z)$$
$$= C\varepsilon^{n-1}$$

Then $\lim_{\varepsilon\to 0} h = 0$ and

$$\lim_{\varepsilon\to 0}\int_{\partial B(x,\varepsilon)} W(z)\frac{\partial \Phi(z-x)}{\partial \eta} = W(x)$$

So $\lim_{\varepsilon\to 0}(a) = W(x)$. Then

$$\lim_{\varepsilon\to 0}\int_{\partial B(x,\varepsilon)} W\frac{\partial V}{\partial \eta} - V\frac{\partial W}{\partial \eta} = W(x)$$

Applying the same process,

$$\lim_{\varepsilon\to 0}\int_{\partial B(y,\varepsilon)} V\frac{\partial W}{\partial \eta} - W\frac{\partial V}{\partial \eta} = V(y)$$

Therefore $W(x) = V(y)$ and $G(y,x) = G(x,y)$.

Definition: Half Space

Define the half space $\mathbb{R}^n_+ = \{(x_1,\ldots,x_n) : x_n > 0$.
IMAGE HERE - 3

Definition: Reflection

For a $x = (x_1,\ldots,x_n) \in \mathbb{R}^n_+$, define its reflection $\tilde{x} = (x_1,\ldots,-x_n)$.

Green's Function in the Half Space

We want to find $\phi^x(y)$ that solves

$$(*)\begin{cases} \Delta\phi^x(y) = 0, & \text{in } \mathbb{R}^n_+ \\ \phi^x(y) = \Phi(y-x), & y \in \partial\mathbb{R}^n_+ \end{cases}$$

Let us consider $\phi^x(y) := \Phi(y-\tilde{x})$, $x, y \in \mathbb{R}^n_+$. Then $\phi^x(y)$ satisfies $*$.
Then we can see that $\Delta\phi^x(y) = 0$.

Let $y \in \partial\mathbb{R}^n_+$ such that $y = (y_1, \ldots, y_{n-1}, 0)$. So

$$
\begin{aligned}
\phi^x(y) &= \Phi(y - \tilde{x}) \\
&= \Phi(|y - \tilde{x}|) \\
&= \Phi\left(\sqrt{(y_1 - x_1)^2 + \cdots + (y_{n-1} - x_{n-1})^2 + (0 + x^n)^2}\right) \\
&= \Phi\left(|y - x|^2\right) \\
&= \Phi(y - x)
\end{aligned}
$$

February 5, 2024

Recall: Green's Function

$G(x, y) = \Phi(y - x) - \phi^x(y)$.
For $U \subset \mathbb{R}^n_+$, when

$$G(x, y) = \Phi(y - x) - \Phi(y - \tilde{x})$$

we proved that if $u \in C^2(\overline{U})$,

$$\begin{cases} -\Delta u = f, & U \\ u = g, & \partial U \end{cases}$$

then

$$u(x) = \int_U f G(x, y)\, dy - \int_{\partial U} g \frac{\partial G(x, y)}{\partial \eta}\, dS(y)$$

Let us consider

$$\begin{cases} \Delta u = 0, & \mathbb{R}^n_+ \\ u = g, & \partial\mathbb{R}^n_+ \end{cases}$$

such that

$$u(x) = -\int_{\partial\mathbb{R}^n_+} g \frac{\partial G(x, y)}{\partial \eta}\, dS(y)$$

Let us compute $\frac{\partial G}{\partial \eta}$.
IMAGE HERE - 1 UPPER HALF SPACE WITH NORMAL VECOTR ETA
Recall

$$
\begin{aligned}
\nabla \Phi(z) &= \frac{-z}{n\alpha(n)|z|^n} \\
\frac{\partial \Phi(z)}{\partial z_n} &= \frac{-z_n}{n\alpha(n)|z|^n}
\end{aligned}
$$

so, since $y - \tilde{x}_n = y_n + x_n$,

$$\begin{aligned}\frac{\partial G}{\partial \eta} &= \nabla G(x,y)\cdot\eta \\ &= -\frac{\partial G(x,y)}{\partial y_{n+1}} \\ &= -\frac{\partial}{\partial y_{n+1}}\left(\Phi(y-x)-\Phi(y-\tilde{x})\right) \\ &= -\left[\frac{-(y_n - x_n)}{n\alpha(n)|y-x|^n} - \frac{-(y_n+x_n)}{n\alpha(n)|x-\tilde{x}|^n}\right]\end{aligned}$$

But recall that if $y \in \partial\mathbb{R}^n_+$, $|y-x| = |y-\tilde{x}|$. Then $y \in \partial\mathbb{R}^n_+$,

$$\frac{\partial G(x,y)}{\partial \eta} = -\frac{1}{n\alpha(n)|y-x|^n}\left[-y_n + x_n + y_n + x_n\right] = -\frac{2x_n}{n\alpha(n)|y-x|^n}$$

Then

$$u(x) = \int_{\partial\mathbb{R}^n_+} \frac{g(y)2x_n}{n\alpha(n)|y-x|^n}\, dS(y)$$

Definition: Poisson Kernel

$$K(x,y) = \frac{2x_n}{n\alpha(n)|y-x|^n} = \frac{\partial G}{\partial y_n}$$

is called the Poisson Kernel and

$$u(x)\int_{\partial\mathbb{R}^n_+} g(x)K(x,y)\, dS(y)$$

is called the Poisson Formula.
Notice (HW): $\int_{\partial\mathbb{R}^n_+} K(x,y)\, dy = 1,\ \forall x \in \mathbb{R}^n_+$ (hint: apply spherical coordinates).

Theorem 14:

Define

$$(*)\quad u(x) = \int_{\partial\mathbb{R}^n_+} K(x,y)g(y)\, dS(y)$$

Suppose that $g \in C^\infty(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$.
Then

1. $u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$.
2. $\Delta u = 0$, $\mathbb{R}^n_+$.
3. $\lim_{x\to x^0} u(x) = g(x^0)$, $x \in \mathbb{R}^n_+$, $x^0 \in \partial\mathbb{R}^n_+$.

Proof

We know $G(x, y)$ satisfies

$$\Delta_y G(x, y) = \delta^x(y).$$

Notice that $y \to G(x, y)$ is harmonic for $x \neq y$.
Recall that $G(x, y) = G(y, x)$, so $x \to G(x, y)$ is harmonic for $x \neq y$.
Then $x \to \frac{\partial G(x,y)}{\partial y_n}$ is harmonic $(*_2)$ for $x \neq y$ and for $y \in \partial\mathbb{R}^n_+$.
Homework: compute this directly.
Noticing that K is smooth when $x \neq y$, then

$$\frac{\partial u}{\partial x_i} = \int_{\partial\mathbb{R}^n_+} \frac{\partial}{\partial x_i} K(x, y) g(y)\, dS(y)$$

Homework: justify puting the limit inside the integral.
Homework: prove that $\frac{\partial u}{\partial x_i}$ is continuous.
By repeatedly taking derivaties, we see $u \in C^\infty(\mathbb{R}^n_+)$.
Moreover,

$$\Delta_x u = \int_{\partial\mathbb{R}^n} \underbrace{\Delta_x K(x, y)}_{=0} g(y)\, dS(y) = 0$$

by $*_2$. Then

$$|u(x)| \leq \int_{\partial\mathbb{R}^n_+} |K(x, y)||g(y)|\, dS(y) \leq ||g||_{L^\infty(\mathbb{R}^{n-1}} \underbrace{\int_{\partial\mathbb{R}^n_+} K(x, y)\, dS(y)}_{=1} < \infty$$

For part c, consider $x^0 \in \partial\mathbb{R}^n_+$ and $\varepsilon > 0$.
Since $g \in C^\infty(\mathbb{R}^{n-1})$, let $\delta > 0$ such that $|y - x^0| < \delta \implies |g(y) - g(x^0)| < \varepsilon$ for $y \in \partial\mathbb{R}^n_+$.
IMAGE HERE - 2 DELTA BALL AROUND X0 HALF DELTA BALL WITH X INSIDE
Now, let us consider $|x - x_0| < \frac{\delta}{2}$.

$$\begin{aligned}
|u(x) - g(x^0)| &= \left| \int_{\partial\mathbb{R}^n_+} K(x, y)g(y) - K(x, y)g(x^0)\, dS(y) \right| \\
&\leq \int_{\partial\mathbb{R}^n_+} K(x, y) \left| g(y) - g(x^0) \right|\, dS(y) \\
&= \underbrace{\int_{\partial\mathbb{R}^n_+ \cap B(x^0,\delta)} K(x, y)|g(y) - g(x^0)|\, dS(y)}_{I} + \underbrace{\int_{\partial\mathbb{R}^n_+ \cap B^c(x^0,\delta)} K(x, y)|g(y) - g(x^0)|\, dS(y)}_{II}
\end{aligned}$$

Then

$$I \leq \varepsilon \int_{\partial\mathbb{R}^n_+ \cap B(x^0,\delta)} K(x, y) \leq \varepsilon$$

Now, we want to control II

$$\begin{aligned}
\int_{\partial\mathbb{R}^n_+ \cap B^c(x^0,\delta)} K(x, y)|g(y) - g(x^0)|\, dS(y) &\leq C||g||_{L^\infty} \int_{\partial\mathbb{R}^n_+ \cap B^c(x^0,\delta)} \\
&= \frac{2C}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+ \cap B^c(x^0,\delta)} \frac{x_n}{|x - y|^n}\, dS(y)
\end{aligned}$$

We want to control $|x^0 - y|$ with something related to $|x - y|$.
We know $|y - x^0| > \delta$ and we will consider $|x - x^0| < \frac{\delta}{2}$. So

$$|y - x^0| \leq |y - x| + |x - x^0| \leq |y - x| + \frac{\delta}{2} \leq |y - x| + \frac{|y - x^0|}{2}$$

So $\frac{|y-x_0|}{2} \leq |y - x|$ implies that $\frac{1}{|y-x|^n} \leq \frac{2^n}{|y-x_0|^n}$. Therefore

$$\begin{aligned}
\int_{\partial\mathbb{R}^n_+ \cap B^c(x^0,\delta)} K(x,y)|g(y) - g(x^0)|\, dS(y) &\leq Cx_n \int_{\partial\mathbb{R}^n_+ \cap B^c(x^0,\delta)} \frac{1}{|y - x^0|^n}\, dS(y) \\
&= \int_\delta^\infty \int_{\partial B^{n-1}(x^0,r)} \frac{1}{r^n}\, dS(y) dr \\
&= C \int_\delta^\infty \frac{1}{r^n} r^{n-2}\, dr \\
&= C \int_\delta^\infty \frac{1}{r^2}\, dr \\
&= C(\frac{1}{r})|_\delta^\infty \\
&= \frac{C}{\delta}
\end{aligned}$$

Then $II \leq \frac{Cx_n}{\delta}$. Now let us consider $|x - x^0| < \frac{\delta}{J}$ where $\frac{1}{J} < \varepsilon$. Then

$$II \leq \frac{C|x - x^0|}{\delta} \leq C\frac{\delta}{\delta J} \leq C\varepsilon$$

Energy Methods: Uniqueness

Consider the boundary value problem

$$(*) \quad \begin{cases} -\Delta u = f, & U,\ f \in C(U) \\ \quad u = g, & \partial U,\ g \in C(\partial U) \end{cases}$$

with U open and bounded in $\mathbb{R}^n$, $u \in C^2(\overline{U})$ and $\partial U \in C^1$.

Theorem 16: Uniqness

There exists at most one solution $u \in C^2(\overline{U})$ for $*$.

Proof

Let us suppose that $\tilde{u}$ is another solution.
Then $w := u - \tilde{u}$ solves

$$\begin{cases} \Delta w = 0, & U,\ w \in C^2(\overline{U}) \\ w = 0,^{\partial U} \end{cases}$$

where

$$0 = \int_U w\Delta w = -\int_U |\nabla W|^2 + \int_{\partial U} w\frac{\partial w}{\partial \eta}$$

so

$$0 = \int_U |\nabla w|^2 \implies \nabla w = 0 \implies w = 0 \implies u = \tilde{u}$$

Definition: Energy Functional

Let us consider

$$A = \left\{ w \in C^2(\overline{U}) \,:\, W|_{\partial U} = g \right\}$$

for $g \in C(\partial U)$ and $f \in C(U)$.
Define the energy functional $I : A \to \mathbb{R}$ given by $I(w) := \int_U \frac{|\nabla w|^2}{2} - f w$.

Energy Methods: Dirichlet Principle

Calculus of variations applied to the Laplace equation.

Theorem:

Suppose $u \in C^2(\overline{U})$ is a solution to the problem

$$\square \quad \begin{cases} -\Delta u = f, & U \\ \quad u = g, & \partial U \end{cases}$$

Then,

$$(*) \quad I(u) = \min_{w \in A}\{I(w)\}$$

Moreover, if $u \in A$ such that $*$ happens, then u satisfies $\square$.

Proof

($\implies$) For $w \in A$,

$$\begin{aligned}
0 &= \int_U \underbrace{(-\Delta u - f)}_{=0}(u - w) \\
&= \int_U -\Delta u(u - w) - \int_U f(u - w) \\
&= \int_U \nabla(u - w) \cdot \nabla u - \underbrace{\int_{\partial U} (u - w) \cdot \frac{\partial u}{\partial \eta}}_{=0} - \int_U f(u - w) \\
&= \int_U |\nabla u|^2 - \int_U \nabla w \cdot \nabla u - \int_U f u + \int_U f w
\end{aligned}$$

Notice that, since $|a - b|^2 \geq 0$ implies $\frac{a^2 + b^2}{2} \geq ab$,

$$\int_U \nabla w \cdot \nabla u \leq \int_U |\nabla w||\nabla u| \leq \frac{1}{2}\int_U |\nabla w|^2 + \frac{1}{2}\int_U |\nabla u|^2$$

Therefore

$$\begin{aligned}\int_U |\nabla u|^2 - \int_U fu &= \int_U \nabla w \cdot \nabla u - \int_U fw \\ &\le \int_U \frac{|\nabla w|^2}{2} + \int_U \frac{|\nabla u|^2}{2} - \int_U fw \\ \int_U \frac{|\nabla u|^2}{2} - fu &\le \int_U \frac{|\nabla w|^2}{2} - fw\end{aligned}$$

Then

$$I(u) \le I(w), \quad \forall w \in A$$

$B/u \in A$.

February 7, 2024

Recall: Energy Functional

For $U \subseteq \mathbb{R}^n$ bounded, $g \in C(\partial U)$, $f \in C(\overline{U})$

$$A = \left\{ w \in C^2(\overline{U}) \,:\, w|_{\partial U} = g \right\}$$

we have

$$I(w) := \int_U \frac{1}{2} |\nabla w|^2 - fw$$

Theorem:

Suppose $u \in A$ such that $I(u) = \min\{I(w) \,:\, w \in A\}$. Then u satisfies

$$\begin{cases} -\Delta u = f, & \text{in } U \\ \quad u = g, & \text{on } \partial U \end{cases}$$

Proof

Consider $v \in C_c^\infty(U)$.
Define $i : \mathbb{R} \to \mathbb{R}$ such that $\tau \mapsto I(\tau) := I(u + \tau v)$.
Notice that $u + \tau v$ is a perturbation of u and, since $u + \tau v \in C^2(\overline{U})$ while $u + \tau v|_{\partial U} = u|_{\partial U} = g$, $u + \tau v \in A$. Then

$$i(0) = I(u) \le I(u + \tau v) = i(\tau)$$

so i has a minimum point at $\tau = 0$. Compute

$$\begin{aligned} i(\tau) &= I(u + \tau v) \\ &= \int_U \frac{|\nabla(u + \tau v)|^2}{2} - f(u + \tau v) \\ &= \int_U \frac{|\nabla u|^2}{2} + \tau \langle \nabla u, \nabla v \rangle + \frac{\tau^2 |\nabla v|^2}{2} - \int_U fu - \tau \int_U fv \\ &= \int_U \frac{|\nabla u|^2}{2} + \tau \int_U \langle \nabla u, \nabla v \rangle + \frac{\tau^2}{2} \int_U |\nabla v|^2 - \int_U fu - \tau \int_U fv \end{aligned}$$

So i is a polynomial in τ, and

$$i'(0) = i'(\tau)_{\tau=0} = \left(\int_U \langle \nabla u, \nabla v \rangle + \tau \int_U |\nabla v|^2 - \int_U f v \right)_{\tau=0}$$

So

$$\begin{aligned} 0 &= i'(0) \\ &= \int_U \langle \nabla u, \nabla v \rangle - \int_U f v \\ &= \int_U -\Delta u \cdot v + \underbrace{\int_{\partial U} \frac{\partial u}{\partial \eta} \cdot v}_{=0} - \int_U f v \\ &= \int_U \underbrace{(-\Delta u - f)}_{=0} v \end{aligned}$$

Since $0 = \int g v$, $\forall v \in C_c^{\infty}(U)$ requires $g \equiv 0$.
Then $-\Delta u - f = 0$.

Heat Equation (Diffusion Equation)

The equations

$$(*) \begin{cases} u_t - \Delta u = 0, & \text{homogeneous case} \\ u_t - \Delta u = f, & \text{non-homogeneous case} \end{cases}$$

(note that $\Delta u = \Delta_x u$)
subject to some boundary and initial conditions $t \geq 0$ time and $x \in \mathbb{R}^n$, space variable, $x \in U$ and opsen set of $\mathbb{R}^n$.
$u : U \times (0, \infty) \to \mathbb{R}$ defined as $(x, t) \mapsto u(x, t)$ with u unknown.
IMAGE HERE - 1

Motivation: Fundamental Solution of the Heat Equation

We would like to have the following:
If u solves

$$\begin{cases} u_t - \Delta u = 0, & \mathbb{R}^n \times (0, \infty) \\ u(x, 0) = g(x), & \mathbb{R}^n \times \{0\} \end{cases}$$

then

$$u(x, t) = \int_{\mathbb{R}^n} G(x - y, t) g(y)\, dy$$

How do we get G?
Let us suppose that $u(\tilde{x}, \tilde{t})$ solves

$$\begin{cases} u_{\tilde{t}} - \Delta_{\tilde{x}} u = 0 \\ u(\tilde{x}, 0) = g(\tilde{x}) \end{cases}$$

We would like to have invariance under dilation.

$$v(x,t) := u(\lambda x, \lambda^2 t)$$

Such that

$$\begin{aligned} v_t &= \nabla U\big|_{(\lambda x, \lambda^2 t)} - \frac{\partial}{\partial t}\begin{bmatrix} \lambda x \\ \lambda^2 t \end{bmatrix} \\ &= \lambda^2 u_{\tilde{t}}(\lambda x, \lambda^2 t) \\ v_{x_i} &= \lambda u_{\tilde{x}_i}(\lambda x, \lambda^2 t) \\ v_{x_i x_i} &= \lambda^2 u_{\tilde{x}_i \tilde{x}_i}(\lambda x, \lambda^2 t) \end{aligned}$$

Therefore

$$v_t - \Delta_x v = \lambda^2 u_{\tilde{t}} - \lambda^2 \Delta_{\tilde{x}} u = \lambda^2 (\underbrace{u_{\tilde{t}} - \Delta_{\tilde{x}} u}_{=0}) = 0$$

with

$$v(x,0) = u(\lambda x, 0) = g(\lambda x)$$

Then, applying the motivation,

$$v(x,t) = \int_{\mathbb{R}^n} G(x-y,t) g(\lambda y)\, dy \underset{z=\lambda y}{=} \int_{\mathbb{R}^n} G\left(x - \frac{z}{\lambda}, t\right) g(z) \frac{dz}{\lambda^n}$$

On the other hand,

$$v(x,t) = u(\lambda x, \lambda^2 t) = \int_{\mathbb{R}^n} G(\lambda x - z, \lambda^2 t) g(z)\, dz$$

It follows that

$$\begin{aligned} \frac{1}{\lambda^n} G\left(\overbrace{x - \frac{z}{\lambda}}^{w}, t\right) &= G(\lambda x - z, \lambda^2 t) \\ \frac{1}{\lambda^n} G(w,t) &= G(\lambda w, \lambda^2 t) \end{aligned}$$

If $\lambda^2 t = 1$, then

$$G(w,t) = \frac{1}{t^{n/2}} G\left(\frac{1}{\sqrt{t}} w, 1\right)$$

If we call $G\left(\frac{w}{\sqrt{t}}, 1\right) = v\left(\frac{w}{\sqrt{t}}\right)$, then we are looking at $G(w,t) = \frac{1}{t^{n/2}} v\left(\frac{w}{t^{1/2}}\right)$.
So, we have motivation to define

$$u(x,t) = \frac{1}{t^{\alpha}} v\left(\frac{x}{t^{\beta}}\right)$$

for α, β appropriate and $v(y) : \mathbb{R}^n \to \mathbb{R}$.

Obtaining a Fundamental Solution to the Heat Equation

Let us compute u_t and $\Delta_x u$.

$$\begin{aligned} u_t &= \frac{\partial}{\partial t}\left(\frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right)\right) \\ &= \frac{(-\alpha)}{t^{\alpha+1}} v\left(\frac{x}{t^\beta}\right) + \frac{1}{t^\alpha}\frac{\partial}{\partial t}\left(v\left(\frac{x}{t^\beta}\right)\right) \\ &= \frac{(-\alpha)}{t^{\alpha+1}} v\left(\frac{x}{t^\beta}\right) + \frac{1}{t^\alpha}\cdot \nabla v|_{\frac{x}{t^\beta}} \cdot \frac{\partial}{\partial t}\left(\frac{x}{t^\beta}\right) \\ u_t &= \frac{(-\alpha)}{t^{\alpha+1}} v\left(\frac{x}{t^\beta}\right) + \frac{(-\beta)}{t^\alpha t^{\beta+1}} \nabla v|_{\frac{x}{t^\beta}} \cdot x \quad \square_1 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial u}{\partial x_i} &= \frac{1}{t^\alpha}\frac{\partial}{\partial x_i}\left(v\left(\frac{x}{t^\beta}\right)\right) \\ &= \frac{1}{t^\alpha}\nabla v|_{\frac{x}{t^\beta}} \cdot \frac{\partial}{\partial x_i}\left(\frac{x}{t^\beta}\right) \\ &= \frac{1}{t^{\alpha+\beta}}\frac{\partial v}{\partial x_i}|_{\frac{x}{t^\beta}} \end{aligned}$$

while

$$\frac{\partial^2 u}{\partial x_i x_i} = \frac{1}{t^{\alpha+2\beta}}\frac{\partial^2 v}{\partial x_i x_i}|_{\frac{x}{t^\beta}} \quad \square_2$$

Then, replacing $\square_1$ and $\square_2$ in $*$,

$$-\frac{\alpha}{t^{\alpha+1}} v\left(\frac{x}{t^\beta}\right) - \frac{\beta}{t^{\alpha+\beta+1}}\nabla v|_{\frac{x}{t^\beta}} \cdot x - \frac{1}{t^{\alpha+2\beta}}\Delta v|_{\frac{x}{t^\beta}} \stackrel{?}{=} 0$$

Set $y := \frac{x}{t^\beta}$

$$-\frac{\alpha}{t^{\alpha+1}} v(y) - \frac{\beta}{t^{\alpha+1}}\nabla v(y)\cdot y - \frac{1}{t^{\alpha+2\beta}}\Delta v(y) = 0$$

Multiplying through by $-t^{\alpha+1}$,

$$\alpha v(y) + \beta \nabla v(y) \cdot y + \frac{1}{t^{2\beta-1}}\Delta v(y) = 0$$

Let us assume that $2\beta - 1 = 0$ such that $\beta = \frac{1}{2}$, giving

$$\alpha v(y) + \frac{1}{2}\nabla v(y)\cdot y + \Delta v(y) = 0$$

Since the Laplacian is rotationally invariant, assume $v(y) = w(|y|)$ for $w : \mathbb{R}^+ \to \mathbb{R}$.
Recall that $\frac{\partial}{\partial y_i}|y| = \frac{\partial}{\partial y_i}\left(\sqrt{y_1^2 + \cdots + y_n^2}\right) = \frac{y_i}{|y|}$. Now

$$\frac{\partial}{\partial y_i} v(y) = \frac{\partial}{\partial y_i}(w(|y|)) = w'(|y|)\cdot \frac{\partial}{\partial y_i}(|y|) = w'(|y|)\cdot \frac{y_i}{|y|}$$

$$
\begin{aligned}
\frac{\partial^2 v(y)}{\partial y_i y_i} &= \frac{\partial}{\partial y_i}\left(w'(|y|)\right)\frac{y_i}{|y|} + w'(|y|)\cdot\frac{\partial}{\partial y_i}\left(\frac{y_i}{|y|}\right) \\
&= w''(|y|)\cdot\frac{y_i^2}{|y|^2} + w'(|y|)\left[\frac{1}{|y|} + y_i\frac{\partial}{\partial y_i}\left(\frac{1}{|y|}\right)\right] \\
&= w''(|y|)\frac{y_i^2}{|y|^2} + w'(|y|)\left[\frac{1}{|y|} - \frac{y_i^2}{|y|^3}\right]
\end{aligned}
$$

Replacing in the PDE of v,

$$
\begin{aligned}
0 &= \alpha w(|y|) + \frac{1}{2}\frac{w'(|y|)y}{|y|}\cdot y + \sum_{i=1}^{n} w''(|y|)\frac{y_i^2}{|y|^2} + w'(|y|)\left[\frac{1}{|y|} - \frac{y_i^2}{|y|^3}\right] \\
&= \alpha w(|y|) + \frac{1}{2}w'(|y|)|y| + w''(|y|) + w'(|y|)\left[\frac{n}{|y|} - \frac{1}{|y|}\right]
\end{aligned}
$$

If $|y| = r$

$$
0 = \alpha w(r) + \frac{1}{2}w'(r)r + w''(r) + w'(r)\frac{n-1}{r}
$$

Take $\alpha = \frac{n}{2}$ and multiply through by r^{n-1},

$$
\begin{aligned}
0 &= \frac{nr^{n-1}}{2}w(r) + \frac{r^n}{2}w'(r) + w''(r)r^{n-1} + w'(r)(n-1)r^{n-2} \\
&= \frac{1}{2}\left[w(r)r^n\right]' + \left[w'(r)r^{n-1}\right]'
\end{aligned}
$$

Then by the fundamental theorem of calculus, $w'(r)r^{n-1} + \frac{w(r)r^n}{2} = C$.
We would like $w, w' \xrightarrow[r\to\infty]{} 0$. Then $C = 0$, so

$$
w'(r)r^{n-1} = -\frac{w(r)r^n}{2}
$$

Which gives

$$
w' = \frac{-wr}{2} \iff \frac{w'}{w} = -\frac{r}{2} \iff (\ln(w))' = \frac{-r}{2} \iff \ln(w) = -\frac{r^2}{4} + d
$$

and, finally,

$$
w(r) = be^{-\frac{r^2}{4}}
$$

Then define

$$
\begin{aligned}
u(x,t) &:= \frac{1}{t^{n/2}}v\left(\frac{x}{t^{1/2}}\right) \\
&= \frac{1}{t^{n/2}}w\left(\left|\frac{x}{t^{1/2}}\right|\right) \\
&= \frac{b}{t^{n/2}}e^{-\frac{1}{4}\left|\frac{x}{t^{1/2}}\right|^2} \\
&= \frac{b}{t^{n/2}}e^{-\frac{1}{4t}|x|^2}
\end{aligned}
$$

Where b is chosen such that the expression integrates to 1.

Definition: Fundamental Solution of the Heat Equation

The fundamental solution for the heat equation is given by

$$\begin{cases} \Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, & x \in \mathbb{R}^n,\ t > 0 \\ \qquad\qquad 0, & x \in \mathbb{R}^n,\ t < 0 \end{cases}$$

where we have chosen $b = \frac{1}{(4\pi)^{n/2}}$.
IMAGE HERE - 2
Notice that these match in the limit away from the origin ($\lim_{(x,t)\to(x_0,0)} \Phi(x,t) = 0$).
Remark: $\Phi(x,t)$ has a unique singularity at $(0,0)$.

February 12, 2024

Recall: Heat Equation

$$\Phi(x,t) = \begin{cases} \frac{b}{(t)^{n/2}} e^{\frac{-|x|^2}{4t}}; & t > 0, x \in \mathbb{R}^n \\ 0; & t < 0 \end{cases}$$

Remark: Φ is radial such that $\Phi(x,t) = \Phi(|x|,t)$.

Lemma:

For each $t > 0$,

$$\int_{\mathbb{R}^n} \Phi(x,t)\, dx = 1$$

Proof

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi(x,t)\, dx &= \frac{b}{t^{n/2}} \int_{\mathbb{R}^n} e^{\frac{-|x|^2}{4t}}\, dx \\ &= \frac{b}{t^{n/2}} \int_{\mathbb{R}^n} e^{-\left|\frac{x}{2\sqrt{t}}\right|^2} \\ &\underset{z=\frac{x}{2\sqrt{t}}}{=} \frac{b}{t^{n/2}} \int_{\mathbb{R}^n} e^{-|z|^2} (2\sqrt{t})^n\, dz \\ &= b2^n \int_{\mathbb{R}^n} e^{-|z|^2}\, dz \\ &= 2^n b \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-z_1^2-\cdots-z_n^2}\, dz_1 \cdots dz_n \\ &= 2^n b \left[\int_{-\infty}^{\infty} e^{-x}\, dx \right]^n \end{aligned}$$

We need

$$
\begin{aligned}
A &= \int_{-\infty}^{\infty} e^{-x^2}\,dx \\
A^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dxdy \\
&= \int_{\mathbb{R}^n} e^{-|z|^2}\,dz \\
&= \int_0^{\infty}\int_{\partial B_r^2} e^{-r^2}\,dS(z)dr \\
&= 2\pi \int_0^{\infty} e^{-r^2} r\,dr \\
&\underset{s=r^2}{=} \pi \int_0^{\infty} e^{-s}\,ds \\
&= -\pi(e^{-s})\big|_0^{\infty} = -\pi(0-1) = \pi
\end{aligned}
$$

Therefore $A^2 = \pi$ and $A = \sqrt{\pi}$. So, picking $b = \frac{1}{(4\pi)^{n/2}}$,

$$
\int_{\mathbb{R}^n} \Phi(x,t)\,dx = b2^n A^n = b2^n \pi^{n/2} = 1
$$

Remark:

Φ solves the Heat Equation, except at the point $(x,t) = (0,0)$.

Remark:

Φ is infinitely differentiable on $\mathbb{R}^n \times (\delta, \infty)$, $\forall \delta > 0$.

Cauchy Problem (Initial Value Problem)

$$
\begin{cases}
u_t - \Delta_x u = 0, & \mathbb{R}^n \times (0,\infty) \\
u(x,0) = g(x), & x \in \mathbb{R}^n
\end{cases}
$$

Recall $y \in \mathbb{R}^n$,

$$
(x,t) \to \Phi(x-y)
$$

solves the heat equation except at $(y,0)$.
Define, $x \in \mathbb{R}^n$, $t > 0$,

$$
(*) \quad u(x,t) = \int_{R^n} \Phi(x-y,t)g(y)\,dy = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{\frac{-|x-y|^2}{4t}} g(y)\,dt
$$

Theorem (#?): Solution to the Cauchy Problem

Assume $g \in C(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$. Then u defined by $*$ satisfies

1. $u \in C^{\infty}(\mathbb{R}^n \times (0,\infty))$.

2. $u_t(x,t) - \Delta_x(x,t) = 0$, $(x,t) \in \mathbb{R}^n \times (0,\infty)$.

3. $\lim_{\substack{(x,t)\to(x_0,0) \\ x\in\mathbb{R}^n,\, t>0}} u(x,t) = g(x_0)$, $x_0 \in \mathbb{R}^n$.

Proof

Homework: justify putting the limit inside to prove (1).
For (2), observe that

$$u_t - \Delta_x u(x,t) = \int_{\mathbb{R}^n} \underbrace{[\Phi_t(x-y,t) - \Delta_x \Phi(x-y,t)]}_{=0} g(y)\, dy$$

For (3), let $\varepsilon > 0$. Let $\delta > 0$ such that $|x - x_0| < \delta \implies |g(x) - g(x_0)| < \varepsilon$ (since g continuous). Then

$$\begin{aligned}
|u(x,t) - g(x_0)| &= \left| \int_{\mathbb{R}^n} \Phi(x-y,t) g(y) - g(x_0) \underbrace{\int_{\mathbb{R}^n} \Phi(x-y,t)\, dy}_{=1} \right| \\
&\leq \int_{\mathbb{R}^n} \Phi(x-y,t) |g(y) - g(x_0)|\, dy \\
&= \underbrace{\int_{B(x_0,\delta)} \Phi(x-y,t)|g(y) - g(x_0)|\, dy}_{I} + \underbrace{\int_{\mathbb{R}^n \setminus B(x_0,\delta)} \Phi(x-y,t)|g(y)-g(x_0)|\, dy}_{J}
\end{aligned}$$

Bounding I, $|y - x_0| < \delta \implies |g(y) - g(x_0)| < \varepsilon$ gives

$$I \leq \varepsilon \underbrace{\int_{B(x_0,\delta)} \Phi(x-y,t)\, dy}_{\leq 1} \leq \varepsilon$$

Bounding J, assume $|x - x_0| < \frac{\delta}{2}$. Then

$$|J| \leq ||g||_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n \setminus B(x_0,\delta)} \Phi(x-y,t)\, dy$$

Now we want to compare $|x - y|$ with $|x_0 - y|$. Then, for $|x - x_0| < \frac{\delta}{2}$ and $|y - x_0| > \delta$,

$$|y - x_0| < |y - x| + |x - x_0| < |y - x| + \frac{\delta}{2} < |y - x| + \frac{|y - x_0|}{2}$$

so $\frac{|y-x_0|}{2} < |y - x|$. It follows that

$$\begin{aligned}
\frac{|y-x_0|^2}{4} &\leq |y-x|^2 \\
-\frac{|y-x|^2}{4t} &\leq -\frac{|y-x_0|^2}{16t} \\
e^{-\frac{|y-x|^2}{4t}} &\leq e^{-\frac{|y-x_0|^2}{16t}}
\end{aligned}$$

Then

$$
\begin{aligned}
|J| &\le 2||g||_{L^\infty} \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n \setminus B(x_0,\delta)} e^{-\frac{|y-x_0|^2}{16t}}\, dy \\
&= \frac{C}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n \setminus B(x_0,\delta)} e^{-\frac{1}{16}\left|\frac{y-x_0}{\sqrt{t}}\right|^2}\, dy
\end{aligned}
$$

Letting $z = \frac{y-x_0}{\sqrt{t}}$ such that $\sqrt{t}\, dz = dy$,

$$
\begin{aligned}
|J| &\le \frac{C}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n \setminus B(0,\delta/\sqrt{t})} e^{-\frac{|z|^2}{16}} \underbrace{(\sqrt{t})^n\, dz}_{dy} \\
&= \frac{C}{(4\pi)^{n/2}} \int_{\mathbb{R}^n \setminus B(0,\delta/\sqrt{t})} e^{-\frac{|z|^2}{16}}\, dz
\end{aligned}
$$

Let $\delta_2 > 0$ such that $\delta_2 = \max\left\{\frac{\delta}{2}, \delta^3\right\}$.
If $|(x,t) - (x_0, 0) < \delta_2$,

$$
\begin{aligned}
t < \delta_2 &< \delta^3 \\
\sqrt{t} &< \delta^{3/2} \\
\frac{1}{\delta^{3/2}} &< \frac{1}{\sqrt{t}} \\
\frac{1}{\delta^{1/2}} &< \frac{\delta}{\sqrt{t}}
\end{aligned}
$$

so

$$
B(0, 1/\delta^{1/2}) \subseteq B(0, \delta/\sqrt{t}) \quad \text{and} \quad \mathbb{R}^n \setminus B(0,\delta/\sqrt{t}) \subseteq \mathbb{R}^n \setminus B(0, 1/\delta^{1/2})
$$

Therefore,

$$
|u| \le C \int_{\mathbb{R}^n \setminus B(0,1/\sqrt{\delta})} e^{-\frac{|z|^2}{16}}\, dz \to 0
$$

Intepretation of Fundamental Solution for the Heat Equation

$$
\begin{cases}
\Phi_t - \Delta_x \Phi(x,t) = 0, & x \in \mathbb{R}^n,\ t > 0 \\
\Phi(x,0) = \delta_0(x), & x \in \mathbb{R}^n
\end{cases}
$$

Then

$$
u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy
$$

if $t = 0$,

$$\begin{aligned} u(x,0) &= \int_{\mathbb{R}^n} \Phi(x-y,0)g(y)\,dy \\ &= \int_{\mathbb{R}^n} \underbrace{\delta^x(y)g(y)}_{y=x}\,dy \\ &= \int_{\mathbb{R}^n} \delta^x(y)g(x)\,dy \\ &= g(x)\underbrace{\int_{\mathbb{R}^n} \delta^x(y)\,dy}_{=1} = g(x) \end{aligned}$$

Remark: Infinite Propagation Speed

Let $g \in C(\mathbb{R}^n \cap L^\infty(\mathbb{R}^n))$, $g \geq 0$, $g \neq 0$. Then

$$u(x,t)\frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy > 0, \quad \forall x \in \mathbb{R}^n,\ \forall t > 0$$

IMAGE HERE - 1
That is, the heat equation forces infinite propagation speed for disturbances.

Non-Homogeneous Heat Problem

$$(*_2) \quad \begin{cases} u_t - \Delta_x u = f, & f : \mathbb{R}^n \times (0,\infty) \to \mathbb{R} \\ u(x,0) = 0, & x \in \mathbb{R}^n \end{cases}$$

Motivation

Let $y \in \mathbb{R}^n$, $s > 0$. Then $(x,t) \to \Phi(x-y,t-s)$ solves the heat equation except at $x = y$ and $t = s$.
That is, it satisfies the equation on $\mathbb{R}^n \times (s,\infty)$.
Then for s fixed, define

$$(\square) \quad u(x,t;s) := \int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y;s)\,dy$$

which solves

$$\begin{cases} u_t(x,t;s) - \Delta_x u(x,t;s) = 0, & \mathbb{R}^n \times (s,\infty) \\ u(x,s;s) = f(x;s), & \mathbb{R}^n \times \{s\} \end{cases}$$

which is the IVP with $t = 0 \iff t = s$ and $g(y) \iff f(y;s)$.

Definition: Duhamel's Principle

If we integrate $\square$ from 0 to t,

$$u(x,t) := \int_0^t u(x,t;s)\,ds$$

Let us consider,

$$(\square_2) \quad u(x,t) := \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y;s)\, dyds$$

as a candidate solution for $*_2$.

Theorem: Solution to the Non-Homogeneous Heat Equation

Suppose $f \in C_c^2((\mathbb{R}^n \times (0,\infty))$ with compact support.
If we define u by $\square_2$, then

1. $u \in C_c^2(\mathbb{R}^n \times (0,\infty))$.
2. $u_t(x,t) - \Delta_x u(x,t) = f(x,t)$; $x \in \mathbb{R}^n$, $t > 0$.
3. $\lim_{\substack{(x,t)\to(x_0,0) \\ x\in\mathbb{R}^n,\ t>0}} u(x,t) = 0$, $\forall x_0 \in \mathbb{R}^n$.

February 14, 2024

Recall: Non-Homogeneous Heat Equation

Given

$$\begin{cases} u_f - \Delta_x u = f(x,t), & f : \mathbb{R}^n \times (0,\infty) \to \mathbb{R} \\ \qquad u(x,0) = 0 \end{cases}$$

we have a candidate solution from Duhamel's Principle.

$$\begin{aligned} (*) \quad u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\, dyds \\ &= \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi(t-s))^{n/2}} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\, dyds \end{aligned}$$

Note that unlike the homogeneous case, the integral approaches the singularity at $(0,0)$ and we cannot pass a limit inside.

Theorem: Differentiation Under Moving Regions

Take $\Omega(t) \subseteq \mathbb{R}^n$ a nice region with nice boundaries ($\partial\Omega(t) \in C^1$ and $t \in \mathbb{R}$) and $F(z,t)$ smooth.

$$\frac{d}{dt}\left(\int_{\Omega(t)} F(x,t)\, dz \right) = \int_{\partial\Omega(t)} F v \eta\, ds(z) + \int_{\partial\Omega(t)} F_t\, dz$$

where v is the velocity vector on $\partial\Omega(t)$ and η is the unit outer normal.

Theorem:

Supopse $f \in C_1^2(\mathbb{R}^n \times (0,\infty))$ with compact support.
Then, if u is defined by $*$,

1. $u \in C_1^2(\mathbb{R}^n \times (0,\infty))$.

2. $u_t - \Delta_x u = f(x,t)$; $x \in \mathbb{R}^n$, $t > 0$

3. $\lim_{(x,t)\to(x_0,0)} u(x,t) = 0$ for $x \in \mathbb{R}^n$, $t > 0$, $\forall x_0 \in \mathbb{R}^n$.

Proof of 1

Since Φ has a singularity at $(0,0)$, we cannot differentiate under the integral sign.
Define $\overline{y} = x - y$ and $\overline{s} = t - s$, then $\frac{d\overline{s}}{ds} = -1$, $-d\overline{s} = ds$, and $\frac{d\overline{y}}{dy} = (-1)$. So

$$u(x,t) = -\int_t^0 \int_{\mathbb{R}^n} \Phi(\overline{y},\overline{s}) f(x-\overline{y}, t-\overline{s})\, d\overline{y} d\overline{s}$$

Then, rewrite

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) f(x-y,t-s)\, dyds$$

We may now justify passing the derivative of the space variable inside

$$\frac{\partial u}{\partial x_i} = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) \frac{\partial}{\partial x_i} f(x-y,t-s)\, dyds$$

In the same way, justifying putting the limit inside, we have $\frac{\partial u}{\partial x_i}$ is continuous.
Now, apply the Differentiation Theorem for Moving Regions (above) where $\Omega(t) = \mathbb{R}^n \times [0,t]$.
Define $F(y,s,t) := \Phi(y,s) f(x-y,t-s)$.
IMAGE HERE - 1
Then,

$$\begin{aligned}
\frac{\partial}{\partial t} u(x,t) &= \int_{\partial\Omega(t)} F(\overbrace{y,s}^{z},t) v\eta\, dS(y,s) + \int_0^t \int_{\mathbb{R}^n} \partial_t F(y,s,t)\, dyds \\
&= \int_{\mathbb{R}^n \times \{t\}} F(\overbrace{y,s}^{z},t)\, dS(y,s) + \int_0^t \int_{\mathbb{R}^n} \partial_t F(y,s,t)\, dyds \\
&= \int_{\mathbb{R}^n} F(y,t,t)\, dy + \int_0^t \int_{\mathbb{R}^n} \partial_t F(y,s,t)\, dyds
\end{aligned}$$

Therefore

$$\frac{\partial u}{\partial t}(x,t) = \int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy + \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) \partial_t f(x-y,t-s)\, dyds$$

Homework: Prove that $\frac{\partial u}{\partial t}$ is continuous to complete the proof.

Proof of 2

$$u_t - \Delta_x u = \overbrace{\int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\, dy}^{K} + \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)\, [f(_t(x-y,t-s) - \Delta_x f(x-y,t-s)]\, dyds$$

Since Φ has a signularity, let $\varepsilon > 0$ and isolate

$$u_t - \Delta_x u = K + \underbrace{\int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,s)\left[f_t(x-y,t-s) - \Delta_x f(x-y,t-s)\right]\,dyds}_{J_\varepsilon}$$
$$+ \underbrace{\int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s)\left[f_t(x-y,t-s) - \Delta_x f(x-y,t-s)\right]\,dyds}_{I_\varepsilon}$$

Controlling J_ε,

$$\begin{aligned}|J_\varepsilon| &\le (||f_t||_{L^\infty} + ||\nabla_x f||_{L^\infty}) \int_0^\varepsilon \overbrace{\int_{\mathbb{R}^n} \Phi(y,s)\,dy}^{1}ds \\ &\le C\varepsilon\end{aligned}$$

So $J_\varepsilon \to 0$ as $\varepsilon \to 0$.
Controlling I_ε, using symmetry of t and s and x and y,

$$I_\varepsilon = -\int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s)\partial_s f(x-y,t-s)\,dyds - \int_\varepsilon^t \Phi(y,s)\Delta_y f(x-y,t-s)\,dyds$$

Recall that

$$\int_U u_{x_i} v = -\int_U u v_{x_i} + \int_{\partial U} u v \eta^i$$

where η^{-i} is the ith component of η. and

$$\int_\Omega u\Delta v - v\Delta u = \int_{\partial\Omega} u\frac{\partial v}{\partial \eta} - v\frac{\partial u}{\partial \eta}$$

So, integrating by parts,

$$\begin{aligned}I_\varepsilon = &-\left[-\int_\varepsilon^t \int_{\mathbb{R}^n} \partial_s\Phi(y,s) f(x-y,t-s)\,dyds + \int\int_{\partial(\mathbb{R}^n\times[\varepsilon,t])} \Phi(y,s) f(x-y,t-s)\eta^{n+1}\,dyds\right] \\ &-\int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s)\Delta_y f(x-y,t-s)\,dyds\end{aligned}$$

Since $\eta^{n+1} = 1$ and f has compact support, this gives

$$\begin{aligned}I_\varepsilon = &\int_0^t \int_{\mathbb{R}^n} \partial_s\Phi(y,s) f(x-y,t-s)\,dyds - K + \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\,dy \\ &- \int_\varepsilon^t \Delta_y \phi(y,s) f(x-y,t-s)\,dyds\end{aligned}$$

Notice that the first and last summands solve the heat equation on $\mathbb{R}^n \times [\varepsilon, t]$. So

$$I_\varepsilon = -K + \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\,dy$$

Therefore

$$u_t - \Delta_x u = \lim_{\varepsilon \to 0} K + 0 - K + \int_{\mathbb{R}^n} \Phi(y, \varepsilon) f(x-y, t-\varepsilon)\, dy$$

Homework: prove that we may pass the limit inside.

$$\begin{aligned} u_t - \Delta_x u &= \int_{\mathbb{R}^n} \Phi(y,0) f(x-y,t)\, dy \\ &= \int_{\mathbb{R}^n} \delta^0(y) f(x-y,t)\, dy \\ &= \int_{\mathbb{R}^n} \delta^0(y) f(x,t)\, dy \\ &= f(x,t) \int_{\mathbb{R}^n} \delta^0(y)\, dy \\ &= f(x,t) \end{aligned}$$

Proof of 3

Write

$$|u(x,t)| \le ||f||_{L^\infty} \int_0^t \overbrace{\int_{\mathbb{R}^n} \Phi(y,s)\, dy}^{=1} ds \le ct$$

General Solution to the Heat Equation

If $f \in C_1^2(\mathbb{R}^n \times (0,\infty))$ and $g \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, then

$$u(x,t) := \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\, dyds + \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy$$

is a solution for

$$\begin{cases} u_t - \Delta_x u = f(x,t) \\ u(x,0) = g(x) \end{cases}$$

Mean-Value Formulas for the Heat Equation

Definition: Parabolic Cylinder

Let $U \subseteq \mathbb{R}^n$ be an open set and $T > 0$.
The parabolic cylinder U_T is given by

$$U_T := U \times (0, T]$$

and the parabolic boundary is

$$\Gamma_T = \overline{U}_T - U_T$$

IMAGE HERE - 2

Motivation for Mean-Formulas

In the harmonic case,

$$\Phi(x) = \frac{c1}{|x|^{n-2}}; \quad n \geq 3$$

for x fixed and r fixed

$$\begin{aligned}\phi : \mathbb{R}^n &\to \mathbb{R}\\ y &\to \Phi(x-y)\end{aligned}$$

Then the balls $B(x, r)$ are the level surface of ϕ. See that

$$\begin{aligned}\phi^{-1}(c_0) &= \{y \in \mathbb{R}^n \, : \, \Phi(x-y) = c_0\}\\ &= \{y \in \mathbb{R}^n \, : \, \frac{C}{|x-y|^{n-2}} = c_0\}\\ &= \{y \in \mathbb{R}^n \, : \, |x-y|^{n-2} = \sqrt[n-2]{\frac{c}{c_0}}\}\\ &= \partial B\left(x, \sqrt[n-2]{\frac{c}{c_0}}\right)\end{aligned}$$

Then to get the mean-value formula, it is worth it to pay attention to the level surface of the fundemental solution of the heat equation.

February 26, 2024

Recall: Mean-Value Formula for Heat Equation

$$\Phi(x, t) = \begin{cases}\frac{1}{(4\pi t)^{n/2}} e^{\frac{|x|^2}{4t}} & , \, x \in \mathbb{R}^n, t > 0\\ 0 & , \, x \in \mathbb{R}^n, t < 0\end{cases}$$

For $U \subset \mathbb{R}^n$ open and bounded, $T > 0$, $U_t = U \times (0, T]$, $\Gamma_T = \overline{U}_T - U_T$.

Definition: Heat Balls

Let $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $r \in \mathbb{R}_+$. Defint the heat ball E as

$$E(x, t; r) = \{(y, s) \in \mathbb{R}^{n+1} \, : \, s \leq t, \Phi(x-y, t-s) \geq 1/r^n\}$$

Remark 1

$$\frac{1}{4r^n} \int_{E(x,t;r)} \frac{|x-y|^2}{|t-s|^2} \, dyds = 1$$

Do as homework.

Remark 2

$\partial E(x,t;n)$, Φ is constant.

Theorem: Mean-Value Formulas

Let $u \in C_1^2(U_T)$ solves the heat equation. Then

$$u(x,t) = \frac{1}{4r^n} \iint\limits_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{|t-s|^2}\, dyds$$

for all $E(x,t;r) \subseteq U_T$.

Proof

Define

$$\phi(r) := \frac{1}{4r^n} \iint\limits_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{|t-s|^2}\, dyds$$

We want to prove ϕ constant with $\phi' = 0$.
Without loss of generality, set $x = 0,\ t = 0$ such that $E(r) := E(0,0,r)$. Then

$$\phi(r) = \frac{1}{4r^n} \iint\limits_{E(r)} u(y,s) \frac{|y|^2}{s^2}\, dyds$$

Rescaling by $y = r\overline{y}$ and $s = r^2\overline{s}$,

$$\begin{aligned}\phi(r) &= \frac{1}{4r^n} \iint\limits_{E(1)} u(r\overline{y}, r^2\overline{s}) \frac{r^2|\overline{y}|^2}{r^4\overline{s}^2} r^n r^2\, d\overline{y}d\overline{s} \\ &= \frac{1}{4} \iint\limits_{E(1)} \frac{|\overline{y}|^2}{\overline{s}^2}\, d\overline{y}d\overline{s}\end{aligned}$$

Where we have $E(1)$ because $(y,s) \in E(r) = E(0,0,r),\ s \le 0,\ \frac{1}{(4\pi(-s))^{n/2}} e^{\frac{-|-y|^2}{4(-s)}} \ge \frac{1}{r^n}$.
So $r^2\overline{s} \le 0$ and $\frac{1}{4\pi(-r^2\overline{s})^{n/2}} e^{\frac{-|-r\overline{y}|^2}{4(-r^2\overline{s}}} \ge \frac{1}{r^n}$.
Therefore

$$\overline{s} \le 0 \quad \text{and} \quad \frac{1}{4\pi(-\overline{s}))^{n/2}} e^{\frac{-|-\overline{y}|^2}{4(-\overline{s})}} \ge 1$$

Reindexing $\overline{y} = y$ and $\overline{s} = s$,

$$4\phi'(r) = \iint\limits_{E(1)} \left[Du|_{(ry,r^2s)} \cdot \begin{pmatrix} y \\ 2rs \end{pmatrix} \right] \frac{|y|^2}{s^2}\, dyds$$

$$= \iint\limits_{E(1)} \left[\sum_{i=1}^{n} \frac{\partial u}{\partial y_i}|_{(ry,r^2s)} y_i + \frac{\partial u}{\partial s} 2rs \right] \frac{|y|^2}{s^2}\, dyds$$

$$= \iint\limits_{E(1)} \frac{|y|^2}{s^2} \sum_{i=1}^{n} \frac{\partial u}{\partial y_i}|_{(ry,r^2s)} y_i\, dyds + 2 \iint\limits_{E(1)} \frac{\partial u}{\partial s} r \frac{|y|^2}{s}\, dyds$$

Then, again applying the change of variables,

$$4\phi'(r) = \iint\limits_{E(r)} \frac{|\overline{y}|^2 r^4}{r^2 \overline{s}^2} \sum_{i=1}^{n} \frac{\partial u}{\partial y_i} \frac{\overline{y}_i}{r} \frac{d\overline{y}}{r^n} \frac{d\overline{s}}{r^2} + 2 \iint\limits_{E(r)} \frac{\partial u}{\partial s} \frac{r|\overline{y}|^2}{r^2} \frac{r^2}{\overline{s}} \frac{d\overline{y}}{r^n} \frac{d\overline{s}}{r^2}$$

$$= \underbrace{\frac{|y|^2}{r^{n+1}} \iint\limits_{E(r)} \frac{1}{s^2} \sum_{i=1}^{n} \frac{\partial u}{\partial y_i} y_i\, dyds}_{A} + \underbrace{\frac{2}{r^{n+1}} \iint\limits_{E(r)} \frac{\partial u}{\partial s} \frac{|y|^2}{s}\, dyds}_{B}$$

We want to analyze B.
Let us introduce the notation

$$\psi(y,s) = \frac{-n}{2} \log(-4\pi s) + \frac{|y|^2}{4s} + n\log(n)$$

- Lemma 1
  $\psi(y,s) = 0$, $(y,s) \in \partial E(r)$.

  - Proof

  If $(y,s) \in \partial E(r)$, $\Phi(-y,-s) = \frac{1}{r^n}$, $\frac{1}{(4\pi(-s))^{n/2}} e^{\frac{-|-y|^2}{4(-s)}} = \frac{1}{r^n}$. Therefore

$$r^n = (4\pi(-s))^{n/2} e^{\frac{|-y|^2}{4(-s)}} = e^{\log((4\pi(-s))^{n/2} e^{\frac{|-y|^2}{4(-s)}})}$$

  So

$$n\log(r) = \frac{n}{2}\log(4\pi(-s)) - \frac{-|-y|^2}{4s}$$

- Lemma 2

$$\frac{\partial\psi}{\partial y_i} = \frac{2y_i}{4s} = \frac{y_i}{2s}.$$

  - Proof

$$4\sum_i \frac{\partial\psi}{\partial y_i} y_i = \frac{2|y|^2}{s}$$

- Analyzing B
  Then, integrating by parts,

$$B = \frac{4}{r^{n+1}} \int\int_{E(r)} \frac{\partial u}{\partial s} \sum_i \frac{\partial \psi}{\partial y_i} y_i \, dyds$$
$$= \frac{4}{r^{n+1}} \sum_i \left[\int\int_{E(r)} \frac{\partial}{\partial y_i}\left(\frac{\partial u}{\partial s} y_i\right) \psi \, dyds + \overbrace{\int\int_{\partial E(r)} \frac{\partial u}{\partial s} y_i \psi \eta^i}^{=0} \right]$$
$$= \frac{4}{r^{n+1}} \sum_i \int_{E(r)} \int \psi \left[\frac{\partial u}{\partial s} + y_i \frac{\partial^2 u}{\partial y_i \partial s} \right] dyds$$

Then, again integrating by parts,

$$B = \underbrace{-\frac{4}{r^{n+1}} \sum_i \int_{E(r)} \psi \frac{\partial u}{\partial s} \, dyds}_{C} - \frac{4}{r^{n+1}} \sum_i \int_{E(r)} \psi y_i \frac{\partial^2 u}{\partial y_i \partial s} \, dyds$$
$$= C - \frac{4}{r^{n+1}} \sum_i \left[-\int_{E(r)} y_i \frac{\partial \psi}{\partial s} \frac{\partial u}{\partial y_i} \, dys + \overbrace{\int_{\partial E(r)} \psi y_i \frac{\partial u}{\partial y_i} \eta^s}^{=0} \right]$$
$$= C - \frac{4}{r^{n+1}} \sum_i \int_{E(r)} y_i \frac{\partial u}{\partial y_i} \left[\frac{n}{2s} + \frac{|y|^2}{4s^2} \right] dyds$$

Since $-\int_{E(r)} \sum_i y_i \frac{\partial u}{\partial y_i} \frac{|y|^2}{4s^2} = -A$, we have

$$B = -\frac{4n}{r^{n+1}} \int_{E(r)} \psi \frac{\partial u}{\partial s} \, dyds - \frac{4}{r^{n+1}} \sum_i \int_{E(r)} y_i \frac{\partial u}{\partial y_i} \frac{n}{2s} \, dyds - A$$

So, since u solves the heat equation, we have $\Delta u = \frac{\partial u}{\partial s}$ and may integrate by parts

$$4\phi'(r) = -\frac{4n}{r^{n+1}} \int_{E(r)} \psi \frac{\partial u}{\partial s} \, dyds - \frac{4}{r^{n+1}} \sum_i \int_{E(r)} y_i \frac{\partial u}{\partial y_i} \frac{n}{2s} \, dyds$$
$$= -\frac{4n}{r^{n+1}} \left[\int_{E(r)} \nabla \psi \cdot \nabla u \, dyds + \overbrace{\int_{\partial E(r)} \psi \frac{\partial u}{\partial \eta}}^{=0} \right] - \frac{4}{r^{n+1}} \sum_i \int_{E(r)} y_i \frac{\partial u}{\partial y_i} \frac{n}{2s} \, dyds$$
$$= 0$$

Then we have
$)\phi'(r) = 0$ and $\phi(r)$ constant. We know

$$
\begin{aligned}
\phi(r) &= \lim_{t\to 0}\phi(t) \\
&= \lim_{t\to 0}\frac{1}{4}\int\int_{E(1)} u(ty, ts^2)\frac{|y|^2}{s^2}\,dyds \\
&= \frac{1}{4}\int_{E(1)} u(0,0)\frac{|y|^2}{s^2}\,dyds \\
&= u(0,0)\overbrace{\frac{1}{4}\int_{E(1)}\frac{|y|^2}{|s|^2}\,dyds}^{=1} \\
&= u(0,0)
\end{aligned}
$$

Theorem: Strong Maximum Principle for Heat Equation

Let U be bounded, $u \in C_1^2(U_T) \cap C(\overline{U}_T)$ that satisfies the heat equation.

1. $\max_{\overline{U}_T} u = \max_{\Gamma_T} u$.
2. If U is connected and $(x_0, t_0) \in U_T$ such that $u(x_0, t_0) = \max_{\overline{U}_T} u$, then u is constant in $\overline{U}_{t_0}$.
   IMAGE HERE - 1 CYLINDER to Ut0

Proof of 2

Let $(x_0, t_0) \in U_T$ such that $u(x_0, t_0) = \max_{\overline{U}_T} u := M$.
Pick r small enough such that $E(x_0, t_0; r) \subseteq U_T$.
IMAGE HERE - 2 BALL IN CYLINDER
Then, applying the mean-value formula,

$$
\begin{aligned}
M &= \frac{1}{4r^n}\int_{E(x_0,t_0;r)} u(y,s)\frac{|x_0 - y|^2}{|t_0 - s|^2}\,dyds \\
&\leq \frac{M}{4r^n}\int_{E(x_0,t_0;r)} \frac{|x_0 - y|^2}{|t_0 - s|}\,dyds \\
&= M
\end{aligned}
$$

Therefore $u(y,s) = M$, $\forall (y,s) \in E(x_0, t_0; r)$.

- Part A
  Let $(y_0, s_0) \in U_T$ such that we may connect (x_0, t_0) and (y_0, s_0) with a line L where $L \subseteq U_T$.
  Then $u = M$ on L.

February 28, 2024

Recall: Strong Maximum Principle

Let $u \in C_1^2(U_T) \cap C(\overline{U}_T)$, where $U_T = U \times [0, T]$, solve the heat equation. Then

1. $\max_{\overline{U}_T} u = \max_{\Gamma_T} u$

2. If U is connected and if $\exists(x_0, t_0) \in U_T$ such that $\max_{\overline{U}_T} u = u(x_0, t_0)$, then u constant on $\overline{U}_{t_0}$

Proof of 2

Let $(x_0, t_0) \in U_T$, $M = \max_{\overline{U}_T} = u(x_0, t_0)$.
Using mean-value formula, we proved $\exists r > 0$ such that $u = M$ is constant on $E(x_0, t_0; r)$.

- Part A
  Let (y_0, s_0), $s_0 < t_0$, such that (y_0, s_0) and (x_0, t_0) are connected by a line $L \subseteq U_T$.
  So $\Omega = \{s \geq s_0 \,:\, u(x, t) = M, \forall(x, t) \in L, s \leq t \leq t_0\}$ is nonempty since $t_0 \in \Omega$
  We know $\inf(\Omega)$ exists and, since u is continuous, $\min(\Omega)$ exists.
  Set $r_0 := \min\{\Omega\}$. From the construction, $s_0 \leq r_0$.
  We want to show that $s_0 = r_0$.
  Suppose $s_0 < r_0$. Then $\exists z_0 \in U$ such that $M = u(z_0, r_0) \in L \subset U_T$.
  IMAGE HERE - 1
  Applying the argument from the beginning, $\exists r$ such that $u = M$ on $E(z_0, r_0; r)$.
  But $E(z_0, r_0; r)$ contains points on $L \cap \{r_0 - \sigma \leq t \leq r_0\}$, for some $\sigma > 0$.
  This implies that $r_0 - \sigma \in \Omega$ which contradicts the assumption that r_0 was the minimum of Ω.
  Therefore, $u(y_0, s_0) = M = \max_{\overline{U}_T} u$.

- Part B
  Let $x \in U,\ t < t_0$.
  Since U is connected, there exists a finite set of points $x_0, \ldots, x_m = x$ such that the line connected x_i with x_{i-1} is contained in U.
  Then we may define a finite set of times, $t_0 > t_1 > \cdots > t_m = t$ such that the straight line L_i connecting (x_i, t_i) and (x_{i-1}, t_{i-1}) is totally contained in U_T.
  Then, applying Part A on each L_i, we have $u(x, t) = M$.

Proof of 1

Trivially, $\max_{\Gamma_T} u \leq \max_{\overline{U}_T} u$.
Assume that U is connected, and let $(x_0, t_0) \in \overline{U}_T$ be such that $u(x_0, t_0) = \max_{\overline{U}_T} u$.
If $(x_0, t_0) \in \Gamma_T$, then $\max_{\overline{U}_T} = u(x_0, t_0) \leq \max_{\Gamma_t} u$.
If $(x_0, t_0) \in U_T$, then, using 2, $u = \max_{\overline{U}_T} u$ is constant on $\overline{U}_{t_0}$.
Then we may pick $(x_1, t_0) \in \overline{U}_{t_0}$ and $x_1 \in \partial U$ such that

$$M = u(x_0, t_0) = u(x_1, t_0) \leq \max_{\Gamma_{t_0}} u \leq \max_{\Gamma_T} u$$

If U is not connected, we may take $U = \bigcup_{i \in \Lambda} U_i$,

$$\max_{\Gamma_T} u = \max_{i \in \Lambda}\{\max_{\Gamma_T^i} u\} = \max_{i \in \Lambda}\{\max_{\overline{U}} u\} = \max_{\overline{U}} u$$

Remark: Strong Minimum

Given that strong maximum principle, we have also the strong minimum principle.

Remark: Infinite Propagation Speed for Disturbances on Bounded Domains

Let U be bounded and connected, and $u \in C_1^2(U_T) \cap C(\overline{U}_T)$ which solves

$$\begin{cases} u_t - \Delta u = 0 \\ \qquad u \geq 0 \quad \text{on } \partial U \times [0, T] \\ \qquad u = g \quad \text{on } U \times \{0\} \end{cases}$$

If g is postive, where $g(x) \geq 0$, $\forall x$ and $\exists x_1$ for $g(x_1) > 0$, then $u(x,t) > 0$, $\forall (x,t) \in U_T$.

Proof

By the strong minimum principle,

$$u(x,t) \geq \min_{\overline{U}_T} u = \min_{\Gamma_T} u \geq 0$$

If $u(x,t) = 0 = \min_{\overline{U}_T} u$, then u is constant on $\overline{U}_t$ which contradicts the assumption that g is positive.

Theorem 5: Uniqguness on Bounded Domains

Let $g \in C(\Gamma_T)$ and $f \in C(U_T)$ with U bounded and connected.
Then there exists at most one solution $u \in C_1^2(U_T) \cap C(\overline{U}_T)$ satisfying

$$(*) \begin{cases} u_t - \Delta u = f & U_T \\ u = g & \Gamma_T \end{cases}$$

Proof

Suppose that $u, \tilde{u}$ solve $*$. Then

$$(u - \tilde{u})_t - \Delta(u - \tilde{u}) = (u_t - \Delta u) - (\tilde{u}_t - \Delta \tilde{u}) = f - f = 0$$

Then $u - \tilde{u} \equiv 0$ on Γ_T. Applying the strong maximum and minimum principles to extend to $\overline{U}_T$, we have

$$u - \tilde{u} \equiv 0 \iff u = \tilde{u}$$

Theorem 6: Strong Maximum (Supremum) Principle for Unbounded Domains

Let $u \in C_1^2(\mathbb{R}^n \times [0, T]) \cap C(\overline{\mathbb{R}^n \times (0, T]})$ satisfy

$$\begin{cases} u_t - \Delta u = 0, & \mathbb{R}^n \times (0, T] \\ \qquad u = g, & \mathbb{R}^n \times \{0\} \end{cases}$$

with $|u(x,t)| \leq Ae^{a|x|^2}$ for some $A, a \geq 0$.
Then, $\sup_{\overline{\mathbb{R}^n \times (0,T]}} u = \sup g$.

Proof

Trivially, $\sup g \leq \sup_{\overline{\mathbb{R}^n \times [0,T)}} u$.

- Part 1
  Assume $4aT < 1$, then for some $\varepsilon > 0$ $4a(T+\varepsilon) < 1$.
  For $y \in \mathbb{R}^n$, $\mu > 0$,

$$v(x,t) := u(x,t) - \frac{\mu e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}}}{(T+\varepsilon-t)^{n/2}}; \quad x \in \mathbb{R}^n, t > 0$$

Notice that $v_t - \Delta v = 0$.
IMAGE HERE - 2
Then let $r > 0$ and let us consider $U = B_r(y)$ bounded.
Then we may apply the strong maximum principle for bounded domains to the function v.

$$U_T = B_r(y) \times (0,T]$$
$$\Gamma_T = (\partial B_r(y) \times (0,T]) \cup (B_r(y) \times \{0\})$$

Then $\max_{\Gamma_T} v = \max_{\overline{U_T}} u$. We need to analyze v on Γ_T.
Consider $B_r(y) \times \{0\}$ and $v(x,0)$ where $x \in B_r(y)$.

$$u(x,0) - \frac{\mu e^{\frac{|x-y|^2}{4(T+\varepsilon}}}{(T+\varepsilon)^{n/2}} \leq u(x,0) = g(x)$$

Let $|x-y| = r$,

$$v(x,t) = u(x,t) - \frac{\mu e^{\frac{r^2}{4(T+\varepsilon-t)}}}{(t+\varepsilon-t^{n/2}}$$
$$\leq Ae^{a(|y|+r)^2} - \frac{\mu e^{\frac{r^2}{4(T+\varepsilon-t)}}}{(T+\varepsilon-t)^{n/2}}$$

We know $T-\varepsilon-t \leq T+\varepsilon$, so

$$(T+\varepsilon-t)^{n/2} \leq (T+\varepsilon)^{n/2}$$
$$-\frac{\mu}{(T+\varepsilon-t)^{n/2}} \leq -\frac{\mu}{(T+\varepsilon)^{n/2}} \leq 0$$

and

$$\frac{4(T+\varepsilon-t)}{r^2} \leq \frac{4(T+\varepsilon)}{r^2}$$
$$e^{\frac{r^2}{4(T+\varepsilon-t)}} \geq e^{\frac{r^2}{4(T+\varepsilon)}}$$

Therefore

$$\frac{-\mu e^{\frac{r^2}{4(T+\varepsilon-t)}}}{(T+\varepsilon-t)^{n/2}} \leq \frac{-\mu e^{\frac{r^2}{4(T+\varepsilon)}}}{(T+\varepsilon)^{n/2}}$$

and

$$v(x,t) \leq Ae^{a(|y|+r)^2} - \frac{\mu e^{\frac{r^2}{4(T+\varepsilon)}}}{(T+\varepsilon)^{n/2}}$$

Then for $a < \frac{1}{4(T+\varepsilon)}$, there exists γ such that $a+\gamma = \frac{1}{4(T+\varepsilon)}$. So

$$v(x,t) \leq Ae^{a(|y|+r)^2} - \mu e^{r^2(a+\gamma)}(4(a+\gamma))^{n/2}$$

If $\sup g = \infty$, we are done. Otherwise, we claim that $\exists r$ big enough such that $v(x,t) \leq \sup g$.
Idea: we want $r^2(a+\gamma) >> a(|y|+r)^2$, for r big enough. Write

$$(a+\gamma) > a\left(\frac{|y|}{r}+1\right)^2 \geq a\left(\frac{|y|}{r}+1\right)$$

and

$$\gamma > \frac{a|y|}{r}$$

March 4, 2024

Notation

The disjoint union between A and B is denoted $A \uplus B$.
The interior of U is denoted $\mathring{U}$.

Recall: Strong Maximum Principle of the Cauchy Problem

Let $u \in C_1^2(\mathbb{R}^n \times (0,t]) \cap C(\mathbb{R}^n \times [0,t])$ satisfy

$$\begin{cases} u_t - \Delta u = 0 & \mathbb{R}^n \times (0,t] \\ u = g & \mathbb{R}^n \times \{0\} \end{cases}$$

with $u(x,t) \leq Ae^{a|x|^2}$, $A, a > 0$ constants. Then

$$\frac{\sup}{\mathbb{R}^n \times (0,T]} = \sup g$$

Proof

Trivially, $\sup g \leq \sup_{\mathbb{R}^n \times [0,T]} u$.

- Part 1
  Let us assume $4aT < 1$ and, for ε small enough, $4a(T+\varepsilon) < 1$.
  Then for $y \in \mathbb{R}^n$, $\mu > 0$, define

$$v(x,t) := u(x,t) - \frac{\mu e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}}}{(T+\varepsilon-t)^{n/2}}; \quad x \in \mathbb{R}^n, t > 0$$

Notice that v satisfies $v_t - \Delta v = 0$.
For $r > 0$, define $U = B_r(y)$. Consider U_{T_i} and appy the maximum principle for bounded domains

$$\max_{\overline{U}_T} v = \max_{\Gamma_T} v$$

- Part 2
  Analyzing v on Γ_T. Note that

$$\Gamma_T = (\partial B(y,r) \times [0,T]) \cup \underbrace{(B(y,r) \times \{0\})}_{v(x,0) \leq g(x)}$$

If $|x-y| = r$, we proved that for r big enough such that $v(x,t) \leq \sup_{\mathbb{R}^n} g$,

$$v(y,t) \leq \max_{\overline{U}_T} v = \max_{\Gamma_T} v \leq \sup_{\mathbb{R}^n} g, \quad \forall\, t \in [0,T]$$

Then if $\mu \to 0$,

$$v(y,t) \leq \sup_{\mathbb{R}^n} g, \quad \forall\, t \in [0,T]$$

Therefore,

$$\sup_{\overline{U}_T} u(y,t) \leq \sup_{\mathbb{R}^n} g$$

That is, if $T < \frac{1}{4a}$, the maximum is achieved at $T = 0$.

- Part 3
  If $4aT \geq 1$, we will divide $[0,T]$ into subintervals such that each subinterval has length smaller than $\frac{1}{4a}$. Then

$$\begin{aligned}
\sup_{\mathbb{R}^n \times [0,T]} u &= \sup\{ \sup_{\mathbb{R}^n \times [0,T_1]} u, \sup_{\mathbb{R}^n \times [T_1,T_2]} u, \dots, \sup_{\mathbb{R}^n \times [T_{n-1},T_n]} u\} \\
&= \sup\{ \sup_{\mathbb{R}^n \times \{0\}} u, \sup_{\mathbb{R}^n \times \{T_1\}} u, \dots, \sup_{\mathbb{R}^n \times \{T_{n-1}\}} u\} \\
&\leq \sup\{ \sup_{x \in \mathbb{R}^n} g, \sup_{\mathbb{R}^n \times [0,T_1]} u, \dots, \sup_{\mathbb{R}^n \times [T_{n-2},T_{n-1}]} u\} \\
&\leq \sup g
\end{aligned}$$

Theorem: Smoothness of the Heat Equation

Let $u \in C_1^2(U_T)$ satisfy the heat equation. Then $u \in C^\infty(U_T)$ ($u \in C^\infty(\mathring{U}_T)$).

Proof: Step 1

IMAGE HERE - 2
Take

$$c(x,t;r) = \{(y,s) : |x-y| \leq r, t - r^2 \leq s \leq t\}$$

for $(x_0, t_0) \in \overset{\circ}{U}_T$. Then

$$C := C(x_0, t_0; r)$$
$$C' := C\left(x_0, t_0; \frac{3}{4} r\right)$$
$$C'' := C\left(x_0, t_0 \frac{r}{2}\right)$$

IMAGE HERE - 3
Let $\zeta \in C^{\infty}$ be a cutoff function such that

$$\begin{cases} 0 \le \zeta \le 1, & C \\ \zeta = 1, & C' \\ \zeta = 0 & \text{near parabolic boundary of } C \end{cases}$$

IMAGE HERE - 4
We may extend $\zeta = 0$ outside of C.
Remark: ζ_t, $\nabla \zeta$, $\Delta \zeta$, $\mathbb{R}^{n+1}$ vanishes outside C.

Proof: Step 2

Suppose $w \in C^{\infty}(U_T)$ and define

$$v(x,t) := w(x,t)\zeta(x,t), \quad x \in \mathbb{R}^n, 0 \le t \le t_0$$

We have

$$v_t = w_t \zeta + w \zeta_t$$
$$\frac{\partial v}{\partial x_i} = w_{x_i} \zeta + w \zeta_i$$
$$\frac{\partial v^2}{\partial^2 x_i} = w_{x_i x_i} \zeta + w_{x_i} \zeta_{x_i} + w_{x_i} \zeta_{x_i} + w \zeta_{x_i x_i}$$

and

$$\Delta v = \zeta \Delta w + 2\langle \nabla w, \nabla \zeta \rangle + w \Delta \zeta$$

So define

$$w_t \zeta + w \zeta_t - \zeta \Delta w - 2\langle \nabla w, \nabla \zeta \rangle - w \Delta \zeta := \tilde{f}$$

such that

$$(*) \quad \begin{cases} v_t - \Delta v = \tilde{f} \\ v(x,0) = 0, \quad \mathbb{R}^n \times \{0\} \end{cases}$$

Notice that $\tilde{f}$ has compact support on $\mathbb{R}^n \times [0, t_0]$. Then by Theorem 2 (existence), we have

$$\tilde{v}(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) \tilde{f}(y,s) \, dy ds$$

also solves $(*)$.
Claim: $|v|, |\tilde{v}| \leq A$ for some constant A.

$$|v(x,t)| \leq |w(x,t)||\zeta(x,t)| \leq |w(x,t)|\chi_C(x,t) \leq A'$$

$$|\tilde{v}(x,t)| \leq \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)|\tilde{f}(y,s)|\,dyds \leq \tilde{A}\int_0^1 \Phi(x-y,t-s)\,dyds \leq \tilde{A}t_0 \leq A''$$

Set $A = \max\{A', A''\}$. Then $A \leq Ae^{|x|^2}$ and, trivially, v and $\tilde{v}$ satisfy the growth control.
By the strong maximum principle, we have uniqueness of solutions and conclude $v = \tilde{v}$. So

$$v(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)\tilde{f}(y,s)\,dyds$$

Proof: Step 3

For $(x,t) \in C'' \subset C'$ given $w \in C^\infty(U_T)$ solves the heat equation on C, $\zeta = 1$ while $\zeta, \zeta_t, \Delta\zeta$ have support in C. Therefore

$$\begin{aligned}
w(x,t) &= v(x,t) \\
&= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)\tilde{f}(y,s)\,dyds \\
&= \int_C \Phi(x-y,t-s)[w_t\zeta + w\zeta_t - \zeta\Delta w - 2\langle\nabla w, \nabla\zeta\rangle - w\Delta\zeta]\,dyds
\end{aligned}$$

If W solves the heat equation, $w_t\zeta - \zeta\Delta w = \zeta(w_t - \Delta w) = 0$. So for $(x,t) \in C''$.

$$w(x,t) = \int_C \Phi(x-y,t-s)[w\zeta_t - 2\langle\nabla w, \nabla\zeta\rangle - w\Delta\zeta]\,dyds$$

Notice that we do not have problems around the singularity (x,t), because $w\zeta_t - 2\langle\nabla w, \nabla\zeta\rangle - w\Delta\zeta$ vanishes around (x,t) since $\zeta = 1$ on C'.
Let us analyze

$$\begin{aligned}
\int_C \Phi(x-y,t-s)\langle\nabla w, \nabla\zeta\rangle\,dyds &= \sum_{i=1}^n \int_{t-r^2}^t \int_{B(x_0,r)} \Phi\frac{\partial\zeta}{\partial y_i}\frac{\partial w}{\partial y_i}\,dyds \\
&\underset{\text{IBP}}{=} \sum_{i=1}^n \int_{t-r^2}^t \left[-\int_{B(x_0,r)} \frac{\partial}{\partial y_i}\left(\Phi\frac{\partial\zeta}{\partial y_i}\right) w\,dy + \overbrace{\int_{\partial B(x_0,r)} \Phi\frac{\partial\zeta}{\partial y_i} w\eta^i\,dy}^{=0}\right] ds
\end{aligned}$$

Where the latter term is zero since $\zeta = 0$ near the parabolic boundary.

$$\begin{aligned}
\int_C \Phi(x-y,t-s)\langle\nabla w, \nabla\zeta\rangle\,dyds &= \sum_{i=1}^n \int_{t-r^2}^t \int_{B(x_0,r)} w\left[\frac{\partial\Phi}{\partial y_i}\frac{\partial\zeta}{\partial y_i} - \Phi\frac{\partial^2\zeta}{\partial y_i^2}\right] dyds \\
&= \int_C w\langle\nabla\Phi, \nabla\zeta\rangle - w\Phi\Delta\zeta\,dyds
\end{aligned}$$

So

$$\begin{aligned}
w(x,t) &= \int_C \Phi w\zeta_t - \phi w\Delta\zeta - 2w\langle\nabla\Phi, \nabla\zeta\rangle + 2w\Phi\Delta\zeta\,dyds \\
&= \int_C \Phi w\zeta_t + \phi w\Delta\zeta - 2w\langle\nabla\Phi, \nabla\zeta\rangle\,dyds
\end{aligned}$$

Proof: Step 4

Then, define

$$u^{\varepsilon} := \eta_{\varepsilon} * u, \quad (U_T)_{\varepsilon}$$

the convolution on $\mathbb{R}^{n+1}$.
IMAGE HERE - 5
We know u^{ε} is smooth. Moreover, by properties of convolution, u^{ε} satisfies the heat equation. Applying Step 3 to u^{ε},

$$u^{\varepsilon}(x,t) = \int_C u^{\varepsilon}(y,s) \overbrace{\left[\Phi\zeta_t + \Phi\Delta\zeta - 2\langle\nabla\Phi, \nabla\zeta\rangle\right]}^{K(x,t,y,s)} \, dyds$$

When $\varepsilon \to 0$,

$$u(x,t) = \int_C u(y,s) K(x,t,y,s) \, dyds$$

To be continued.