Algebra I

September 28, 2023

Grade Weights

50% Homework + 50% Final
Participation matters for pass/fail.

Office Hours

Tuesday / Thursday 11:25 - 12:00
Or by appointment (jusuh@ucsc.edu)

Recommended Text

Abstract Algebra (3e) - Dummit and Foote
Finite Groups: An Introduction (2nd revised) - Jean-Pierre Serre
Robert Boltje's Lecture Notes - (`https://boltje.math.ucsc.edu/courses/f17/f17m200notes.pdf`)

Definition: Binary Operation

Let S be a set. A binary operation on S is a function $f : S \times S \to S$. We will almost never use f for the binary operation ($f(s,t)$).
The usual notation for binary operations is $s * t$.

Example

1. $S = \mathbb{R}^3$, define $f : S \times S \to S$ as $(\vec{x}, \vec{y}) \rightsquigarrow \vec{x} + \vec{y}$.

2. $S = \mathbb{R}^3$, define $S \times S \xrightarrow{f} S$ as $(\vec{x}, \vec{y}) \rightsquigarrow \vec{x} + \vec{y}$.

    - Note that $(\vec{x}, \vec{y}) \rightsquigarrow \vec{x} \cdot \vec{y}$ is not a binary operation.

3. $S = \mathbb{Z}$ as $(m, n) \mapsto m \cdot n$.

4. $S = \mathbb{R}^3$ as $(\vec{x}, \vec{y}) \rightsquigarrow \frac{\vec{x}+\vec{y}}{2}$

5. Let $n \geq 1$ be an interger and $S = M_{n\times n}(\mathbb{R}) = \{n \times n \text{ real matricies}\}$. Then $(A, B) \rightsquigarrow AB$.

Observations

Examples 1,3,5 are associative; examples 2,4 are not.
Examples 1-4 are commutative; example 5 commutes only when $n = 1$.
$\vec{0}$ for example 1, 1 for example 3, and I_n for example 5.

Q: What is a Group?

A group is a set equipped with a binary operation which satisfies three axioms.
Let $*$ be a binary operation on a set S.

1. Say $*$ is associative if $\forall a, b, c \in S$, $(a * b) * c = a * (b * c)$.

2. Say $*$ is commutative if $\forall a, b \in S$, $a * b = b * a$.

3. An element $e \in S$ is a neutral element (with respect to $*$) if $\forall a \in S$, $a * e = a = e * a$.

    - If there exists a neutral element, then it is unique.

4. Suppose $(S, *)$ has a neutral element e. Let $a \in S$. Then $b \in S$ is called an inverse of a (with respect to $*$) if $a * b = e = b * a$.

Definition: Group

A group is a set G equipped with a binary operation $*$ such that

1. $*$ is associative.

2. $*$ has a neutral element e.

3. Every $g \in G$ has an inverse.

If, in addition, $*$ is commutative, we say $(G, *)$ is an abelian or commutative group.

Examples

$(\mathbb{R}^3, +)$ is a commutative group.
$(\mathbb{R}^3, \times)$ has no neutral element.
$(\mathbb{Z}, \cdot)$ has no inverse (except ± 1).
$(\mathbb{R}^3, \text{mid})$ is not associative. (the midpoint)
$(M_{n\times n}(\mathbb{R}), \cdot)$ has no inverse of $0_{n\times n}$.
For $n \geq 1$, $(\mathbb{R}^n, +)$ and $(\mathbb{C}^n, +)$ are abelian groups.

Proof that the Neutral Element is unique.

Let e, e' be neutral eements. Then $e' = e * e' = e$. ■

Proof that the Inverse is unique.

Left to the reader.

Definition: Subgroup

Let G be a group, and let H be a subset of G. We say that H is a subgroup of G if

1. $\forall h_1, h_2 \in H$, $h_1 * h_2 \in H$.
2. $e \in H$.
3. $\forall h \in H$, $h^{-1} \in H$.

Examples

$\mathbb{Z}^n \subseteq \mathbb{R}^n$ is a subgroup $(* = +)$.
$G = \{A \in M_{n\times n} : \det(A) \neq 0\}$. Then $(G, \cdot)$ is a group.

- This is the General Linear Group on $\mathbb{R}$: $\mathrm{GL}_n(\mathbb{R})$.
- Recall $A^{-1} = \frac{1}{\det(A)}\left((-1)^{itj}\det(M_{\alpha_i})\right)$.

Definition: General Linear Subgroups

$S = \{A \in \mathrm{GL}_n(\mathbb{R}) : a_{ij} \in \mathbb{Z},\ \forall 1 \leq i, j \leq n\}$.
S is closed under $\cdot$ and $I_n \in S$, but for example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \frac{1}{1\cdot 4 - 2\cdot 3}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

so S is not a subgroup.
However, $T = \{A \in S : \det(A) = \pm 1\} \subseteq \mathrm{GL}_n(\mathbb{R})$.

- Note that if $AA' = I_n$ then $\det(A)\det(A') = 1$.

Definition: Additive Groups

For groups like $\mathbb{Z}^n$, $\mathbb{R}^n$ and $\mathbb{C}^n$, we will use $+$ for the binary operation and say that they are additive groups.
The Neutral Element is denoted as 0.
The inverse is denoted as $-g$.
For $m \geq 1$ and $g \in G$, $mg = g + \overset{m}{\cdots} + g$ and $(-m)g = -(mg)$.

Definition: Multiplicative Groups

For groups like $\mathrm{GL}_n(\mathbb{C})$ or $\mathrm{GL}_n(\mathbb{Z})$, we say that the group is multiplicative.
Denote the neutral element as 1.
Denote the inverse of g as g^{-1}.
For $m \geq 1$, $g^m = g \overset{m}{\cdots} g$.
$g^0 = 1$.
$g^{-m} = (g^m)^{-1}$.

Definition: Group Element Order

Let G be a group, $g \in G$, and $m \geq 1$.
Say g has order m if $g^m = 1$ and $g^k \neq 1, \ \forall k$ such that $1 \leq k \leq m$.
An element has infinite order if $g^m \neq 1, \ \forall m \in \mathbb{Z}^+$.

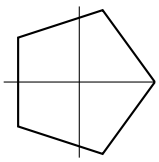
Examples

In D_{10}, I_2 has order 1, rotations have order 5 and reflections have order 2.

Groups from Geometry

Pentagon

Consider the regular pentagon P.



$H = \{T \in \mathrm{GL}_2(\mathbb{R}) : T(P) = P\}$.
This is the symmetry group of P or D_{10} (sometimes D_5)
$H \leq \mathrm{GL}_2(\mathbb{R})$.

- Proof of closure.

  Suppose $T_1, T_2 \in H$. Then $T_1(P) = P$, $T_2(P) = P$ and $(T_1 \circ T_2)(P) = T_1(T_2(P)) = T_1(P) = P$.
  Therefore H is closed under $\circ$.

- Proof of identity.

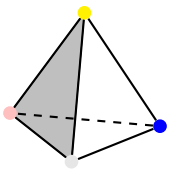
  $\mathrm{Id}_{\mathrm{GL}_2} = I_2$ does satisfy $I_2(P)$.

- Proof of inverse.

  If $T \in H$ (i.e. $T \in \mathrm{GL}_2(\mathbb{R})$ and $T(P) = P$, apply T^{-1} and get $T^{-1}(T(P)) = T^{-1}(P)$. Therefore $P = T^{-1}(P)$.

Tetrahedron

Let X be the regular tetrahedron and $A = \{\text{rotational symmetries of } X\}$.



Then A contains

- The identity: 1.
- $2 \cdot 4 = 8$ rotations by 120°.
- 3 rotations of 180°.

So we have a bijection $r : \{B, P, W, Y\} \to \{B, P, W, Y\}$ where

B $\longrightarrow$ B
P $\nearrow$ P
W $\searrow$ W
Y $\nearrow$ Y

Definition: Symmetric Group

Let S be a set (e.g. $E = \{B, P, W, Y\}$). The Symmetric Group $\mathrm{Sym}(E)$ is the set of bijections $f : E \to E$ equipped with the binary operation $\circ$ (composition).

October 3, 2023

Homework

First homework should be released this Thursday, October 5th.
Next lecture will be on group actions.

Propositions: Symmetric Group

Let X be a set.
When $|X| = n$ denote the elements $\{1, 2, \ldots, n\}$.
$\mathrm{Sym}(X) = \{f : X \to X | f \text{ is bijective}\}$.
With $\circ$ (composition of functions) as a binary operation, $\mathrm{Sym}(X)$ is a group.

Symmetric Group Order

If $|X| = n$, then $|\mathrm{Sym}(X)| = n!$

- Proof

  Let $X = \{1, 2, \ldots, n\}$. A bijection f consists of $f(1), f(2), \ldots, f(n)$.
  For $f(1)$, we have n choices; for $f(2)$ we have $n - 1$ choices. This continues until only 1 choice remains for $f(n)$.
  Therefore the choices are $(n)(n-1)\cdots(1) = n!$

Example

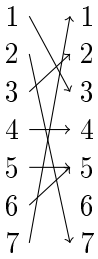
For the symmetric group on four letters $\{a, b, c, d\}$, $|\mathrm{Sym}(4)| = 4! = 24$

Definition: Cycles

Let $x = \{1, \ldots, n\}$, $m \geq 1$ be an integer and $a_1, a_2, \ldots, a_m$ distinct elements in X.
Then the m-cycle denoted by $(a_1\ a_2 \cdots a_m)$ is the element of $\mathrm{Sym}(X)$ which maps a_1 to a_2, a_2 to a_3,..., a_{m-1} to a_m, and a_m to a_1.

Example

Let $n = 7$ and $m = 4$. Then $(2\ 7\ 1\ 3)$ is a bijection.



Degenerate Case

$m = 1$ gives Id_X.

First Non-Degenerate Case

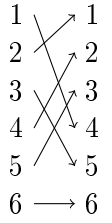
A transposition is, by definition a 2-cycle: $(a_1\ a_2)$.

Proposition: Symmetric Group as Cycle Composition

Every elenent in $\mathrm{Sym}(X)$ is the product (using $\circ$) of m-cycles, where m can vary.

- Proof

  Consider $\mathrm{Sym}(6)$.

  This gives a bijection $\pi = (1\ 4\ 2)(3\ 5)(6)$ which is the composition of cycles.
  We say that this π has cycle type $3 + 2 + 1$.

- Cycle Type

  If instead $\pi = (1\ 4\ 2)(3\ 5\ 6)$ then the cycle type is given as $3 + 3$.

Finite Symmetric Groups

For $n = 2$, $\mathrm{Sym}(X) = \{\mathrm{Id}, (1\ 2)\}$.
This gives cylce types $1 + 1$ and 2.
For $n = 3$, $\mathrm{Sym}(X) = \{\mathrm{Id}, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$.
This gives cycle types $1 + 1$, $2 + 1$ and 3.

Symmetric Group for Tetrahedron

For $n = 4$ let $X = \{B, P, W, Y\}$.
Partitions of $n = 4$ are

$4 = 4$	6	$(B\ P\ W\ Y)$	$\cdots$					sign = -1
$= 3 + 1$	$4 \cdot 2 = 8$	$(P\ W\ Y)$	$\cdots$					sign = $+1$
$= 2 + 2$	$\frac{\binom{4}{2}}{2} = 3$	$(B\ P)(W\ Y)$	$(B\ W)(P\ Y)$	$(B\ Y)(P\ W)$				sign = $+1$
$= 2 + 1 + 1$	$\binom{4}{2} = 6$	$(B\ P)$	$(B\ W)$	$(B\ Y)$	$(P\ W)$	$(P\ Y)$	$(W\ Y)$	sign = -1
$= 1 + 1 + 1 + 1$	1	Id_X						sign = $+1$

Rotation Group for Tetrahedron

$$\begin{aligned} A &= \{\text{Rotational Symmetries}\} \\ &= \{\mathrm{Id}_X, 8\ 3\text{-cycles}, 3 \text{ of type } 2{+}2\} \end{aligned}$$

Note, from the sign, that $A \leq \mathrm{Sym}(4)$.

Symmetries Not in Rotation

Why, for example, is $(B\ P)$ not in the rotation group?
If it were, it should be possible to swap vertices and then undo the switch with only rotation.
However, the two tetrahedra are mirror images across a plane.
Observe that the right hand rule with respect to P, W and Y will give opposite, orthogonal vectors.

Rotation as a Subgroup of Symmetry

Q: Is A a subgroup of $\mathrm{Sym}(4)$?
Following the definition, it would be necessary to veryify

- $\mathrm{Id} \in A$
- A is closed under inverse.
- A is closed under composition.

Group Homomorphism

Let G and H be groups (whose binary operations are denoted by $g_1 \cdot g_2$).
A (group) homomorphism from G to H is a function $\phi : G \to H$ such that

- $\phi(g_1 \underset{G}{\cdot} g_2) = \phi(g_1) \underset{H}{\cdot} \phi(g_2)$

Properties of Group Homomorphism

1. $\phi(1_G) = 1_H$

2. $\phi(g^{-1}) = [\phi(g)]^{-1}, \ \forall g \in G$

- Proof

  By definition, $\phi(1_G \cdot 1_G) = \phi(1_G) \cdot \phi(1_G)$.
  Letting $e = \phi(1_G)$, we get $e = e \cdot e$.
  By multiplying both sides by e^{-1}, we get $1_H = e$.
  Part two is left as an exercise.

Example 1

Let $n \geq 1$ and $G = \mathrm{GL}_n(\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) | \det(A) \neq 0\}$.
In particular, when $n = 1$, $\mathrm{GL}_1(\mathbb{R}) = \mathbb{R}^* = \{r \in \mathbb{R} | r \neq 0\}$ (with multiplication as the binary operation).
Then $\det : G \to H$ is a group homomorphism.
That is $\det(AB) = \det(A)\det(B)$ (as learned in MATH 21).

Example 2

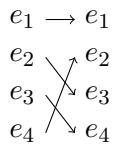
Let $n \geq 1$, $G = \mathrm{Sym}(n)$, $H = \mathrm{GL}_n(\mathbb{R})$.
Construct a group homomorphism $\rho : G \to H$.
Recall that a linear transformation $A \in H$ is completely determined by $Ae_1, \ Ae_2, \ \ldots \ , Ae_n$
where $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$.
For $\pi \in G = \mathrm{Sym}(n)$, $\rho(\pi)$ is the linear transformation that maps e_i to e_j whenever π maps i to j.
This is a surjective linear transformation on a vector space and, therefore, invertible.

- Example

  For $n = 4$ and $\pi = (2\ 3\ 4)$

  $1 \longrightarrow 1$
  $2 \searrow 2$
  $3 \searrow 3$
  $4 \nearrow 4$

  $\rho(\pi)$



Therefore

$$\rho(\pi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- Is this a group homomorphism?
  Let $\pi_1, \pi_2 \in G$ be arbitrary elements.
  Need to show: $\rho(\pi_1 \circ \pi_2) = \rho(\pi_1) \circ \rho(\pi_2)$.
  Both sides are linear transformations and, hence, determined by their actions on e_i for $i = 1, \ldots, n$.

$$\begin{aligned} \rho(\pi_1 \circ \pi_2)e_i &= e_{\pi(i)} \\ &= e_{\pi_1(\pi_2(i))} \\ \rho(\pi_1)(\rho(\pi_2)e_i) &= \rho(\pi_1)(e_{\pi_2(i)}) \end{aligned}$$

Composition of Group Homomorphisms

Let G, H and K be groups and $G \xrightarrow{\phi} H$ and $H \xrightarrow{\psi} K$ be homomorphisms.
Then the composite $\psi \circ \phi : G \to K$ is a group homomorphism.

Proof

Let $g_1, g_2 \in G$ be arbitrary.

$$\begin{aligned} (\psi \circ \phi)(g_1 g_2) &= \psi(\phi(g_1 g_2)) && \text{by definition of } \circ \\ &= \psi(\phi(g_1 \phi(g_2) && \text{since } \phi \text{ is a group homomorphism} \\ &= \psi(\phi(g_1))\psi(\phi(g_2)) && \text{since } \psi \text{ is a group homomorphism} \\ &= (\psi \circ \phi)(g_1) \circ (\psi \circ \phi)(g_2) && \text{by definition of } \circ \end{aligned}$$

Definition: Sign Homomorphism

Let $n \geq 1$ and $G = \mathrm{Sym}(n)$.
Ths sign homomorphism is the composition sign: $G \xrightarrow{\rho} \mathrm{GL}_n(\mathbb{R}) \xrightarrow{\det} \mathbb{R}^*$

Sign of Symmetric Group

$\mathrm{sign}(\mathrm{sym}(n)) \subseteq \{1, -1\} \leq \mathbb{R}^*$

- Lemma

  Let $a_1, \ldots, a_m$ be distinct numbers between 1 and n.
  Then $(a_1 \cdots a_m)$ is equal to $(a_1 \cdots a_{m-1})(a_{m-1}\ a_m)$.
  This will be proven on homework.

- Corollary

  Any m cycle is the composition of $m-1$ transpositions.
  Namely, $(a_1, \ldots, a_m) = (a_1\ a_2)(a_2\ a_3)\cdots(a_{m-1}\ a_m)$.
  Easily check: $\text{sign}((a_i\ a_{i+1})) = -1$.
  Now any $g \in \text{Sym}(n)$ allows a cycle decomposition.

Definition: Kernel of a Homomorphism

Let $G \xrightarrow{\phi} H$ be a group homomorphism.
The kernel of ϕ is $\ker(\phi) := \{g \in G | \phi(g) = 1_H\}$.

The Kernel is a Subgroup

Let $g_1, g_2 \in \ker(\phi)$. Then

$$\begin{aligned} \phi(g_1 g_2) &= \phi(g_1)\phi(g_2) && \phi \text{ is a homomorphism} \\ &= 1_H 1_H && g_1, g_2 \in \ker(\phi) \\ &= 1_H && g_1, g_2 \in \ker(\phi) \end{aligned}$$

Similarly, $1_G \in \ker(\phi)$ and $g^{-1} \in \ker(\phi)$ if $g \in \ker(\phi)$. ■

Definition: Alternating Group

Let X be a set, $|X| = n \leq \infty$.
The alternating group on X is the $\text{Alt}(X) = \ker(\text{sign} : \text{Sym}(X) \to \{\pm 1\})$.

October 5, 2023

Definition: Group Action

Let G be a group and X a set.
A (left) action of G on X is a function $\alpha : G \times X \to X$ which satisfies two conditions:

1. $\alpha(1_G, x) = x$ for all $x \in X$.

2. $\alpha(g_1, \alpha(g_2, x)) = \alpha(g_1 g_2, x)$ for all $g_1, g_2 \in G$ and $x \in X$.

Notation

Write $\alpha(g, x) = g * x = g \cdot x = gx$.

Example A

Let X be any set, and let $G = \mathrm{Sym}(X) = \{f : X \to X \text{ bijections}\}$ where the group operation $\circ$ is the composition of functions.
Then G acts (on the left) on X by $f * x \underset{\text{def}}{=} f(x)$.
Then the features

1. $\mathrm{Id}_X(x) = x, \ \forall x \in X$

2. $g_1 * (g_2 * x) = (g_1 \circ g_2) * x, \ \forall g_1, g_2 \in G, \ \forall x \in X$

    - Or $g_1(g_2(x)) = (g_1 \circ g_2)(x)$

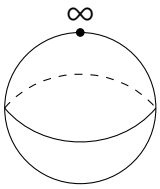
are satisfied.

Example B

Let $G = \mathrm{Sym}(\{B, P, W, Y\})$ which acts on $X = \{B, P, W, Y\}$.
If $H \leq G$, then H acts on X as well, define $h * x = \dot{h} * x$ (where $\dot{h}$ is regarded as in the alternating group of G).
In particular, $\mathrm{Alt}(\{B, P, W, Y\})$ acts on X by rotations.

Example C*

This example is not required for this class.
From complex Analysis we have the Riemann sphere $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$.



Let $G = \mathrm{SL}_2(\mathbb{C})$.
Define G-action on $X = \mathbb{P}^1(\mathbb{C})$ by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} z := \frac{\alpha z + \beta}{\gamma z + \delta} \qquad (\infty \text{ if } \gamma z + \delta = 0)$$

This is called the Möbius group action on $\mathbb{P}^1(\mathbb{C})$.
Exercise: show that 1. and 2. are satisfied.

Definitions

Let G act on X. (Say X is a (left) G-set)

Stabilizer

Let $x \in X$. The stabilizer of x in G is $\text{Stab}_G(x) = \{g \in G | g * x = x\} \subseteq G$.

- Example 1

  Let G be any group and X a G-set.
  Then for any $x \in X$, $\text{Stab}_G(x) \leq G$.

  - Proof

    1. $1_G \in \text{Stab}_G(x)$ since, by definition, $1_G * x = x$.
       Therefore the identity is present.

    2. If $g_1, g_2 \in \text{Stab}_G(x)$ are such that $g_1 * x = x$ and $g_2 * x = x$, then $(g_1 g_2) * x \underset{\text{2nd Axiom}}{=} g_1 * (g_2 * x) = g_1 * x = x$.
       Therefore the stabilizer is closed under composition.

    3. Say $g \in \text{Stab}_G(x)$ and $g * x = x$. Apply g^{-1} to both sdies to get

$$x \underset{\text{1st Axiom}}{=} 1_G * x = (g^{-1} g) * x \underset{\text{2nd Axiom}}{=} g^{-1} * (g * x) = g^{-1} * x$$

    Therefore the stabilizer is closed under inverse.

- Example 2

  Let $G = \text{Alt}(\{B,\ P,\ W,\ Y\})$ and consider $H = \text{Stab}_G(W) = \{\text{Id}, (B\ P\ Y), (B\ Y\ P)$.
  Fact: H does not act transitively on X, since W is fixed and no element $g \in H$ satisfies $g * W = B$.

Orbit

Let $x \in X$. The G-orbit of x in X is $G \cdot x = \{g * x | g \in G\} \subseteq X$.
Let G act on X and $x, y \in X$. Either $G \cdot x = G \cdot Y$ or $G \cdot x \cap G \cdot y = \varnothing$.
So X is the disjoint union of G-orbits.
e.g. $\{B,\ P,\ W,\ Y\} = \{W\} \coprod \{B,\ P,\ Y\}$ gives the $\text{Stab}_G(W)$-orbits.

- Example 1

  When $G = \text{Alt}(X)$, for $X = \{B,\ P,\ W,\ Y\}$, there is only one orbit since $\forall x \in X$, $G \cdot x = X$.

- Example 2

  When $G = \text{Stab}_G(W)$, for $X = \{B,\ P,\ W,\ Y\}$, then $G \cdot W = \{W\}$ while

$$\begin{aligned} G \cdot B &= \{\text{Id}(B),\ (B\ P\ Y)(B),\ (B\ Y\ P)(B)\} = \{B,\ P,\ Y\} \\ &= G \cdot P = \{\text{Id}(P),\ (B\ P\ Y)(P),\ (B\ Y\ P)(P)\} = \{P,\ Y,\ B\} \\ &= G \cdot Y \end{aligned}$$

Transitivity

Say G acts transitively on X (or the action is transitive) if, for any pair $x, y \in X$, there exists $g \in G$ (depending on x and y) such that $g * x = y$.

- Example

  $G = \mathrm{Alt}(\{B,\ P,\ W,\ Y\}) \circlearrowright \{B,\ P,\ W,\ Y\}$ is transitive.

  - Proof

    Let $x, y \in X$ be arbitrary.
    If $x = y$, then take $g = \mathrm{Id}_X$ and we have $g * x = y$.
    Suppose $x \neq y$, then write $X = \{x,\ y,\ z,\ w\}$ and take $g = (x\ y)(z\ w)$. We have $g * x = y$.
    e.g. $x = P$, $y = Y$, $z = B$ and $w = W$ gives $g = (P\ Y)(B\ W)$.

- Exercise *

  This exercise is not required for the course.
  Prove that $\mathrm{SL}_2(\mathbb{C})$ acts transitively on $\mathbb{P}^1(\mathbb{C})$.
  Say $\mathbb{P}^1(\mathbb{C})$ is a homogeneous space under $\mathrm{SL}_2(\mathbb{C})$.

Proposition: Group Action Gives Group Homomorphisms

($\longrightarrow$) Let G act on X. Then

1. For any $g \in G$, the function $\pi_g : X \to X$ defined by $\pi_g(x) = g * x$ is a bijection of X, hence $\pi_G \in \mathrm{Sym}(X)$.

2. The function $G \xrightarrow{\phi} \mathrm{Sym}(X)$ given by $\phi(g) = \pi_g$ is a group homomorphism.

Proof of 1

Need to show that π_g is injective and surjective.
(Inj) Let $x, y \in X$ and assume $\pi_g(x) = \pi_g(y)$ (i.e. $g * x = g * y$).
Apply $g^{-1}*$ on both sides, such that $x = g^{-1} * (g * x) = g^{-1} * (g * y) = y$.
(Sur) Let $x \in X$ be arbitrary. Need to find $y \in X$ such that $\pi_g(y) = x$.
Take $y = g^{-1} * x$, and $\pi_g(y) = g * (g^{-1} * x) = x$.

Proof of 2

Need to show that $\forall g_1, g_2 \in G$, $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$.
$\phi(g_1 g_2) \in \mathrm{Sym}(X)$ is characterized by $[\phi(g_1 g_2)](x) = \pi_{g_1 g_2}(x) = (g_1 g_2) * x$.
On the other hand, $\phi(g_1)\phi(g_2) \in \mathrm{Sym}(X)$ is characterized by $[\phi(g_1)\phi(g_2)](x) = \phi(g_1)[\phi(g_2)(x)] = g_1 * (g_2 * x)$.
By the second group action axiom, these must be the same.

Proposition: Group Homomorphism Admits Group Action

($\longleftarrow$) Let $G \xrightarrow{\rho} \mathrm{Sym}(X)$ be a group homomorphism.
Then, by letting $g * x = \rho(g)(x) \in X$ we get a left G-action on X.

Proof

1. $1_G * x = \rho(1_G)(x) = \mathrm{Id}_X(x) = x$.

2. Let $g_1, g_2 \in G$ and $x \in X$. Then $(g_1 g_2) * x = [\rho(g_1 g_2)](x) = [\rho(g_1) \circ \rho(g_2)](x) = \rho(g_1)[\rho(g_2)(x)] = g_1 * (g_2 * x)$.

Definition: Right Group Actions

Let G be a group and X be a set. A right G-action on X is a function $\beta : X \times G \to X$ such that

1. $\beta(x, 1_G) = x, \ \forall x \in X$.

2. $\beta(x, g_1 g_2) = \beta(\beta(x, g_1), g_2), \ \forall g_1, g_2 \in G, \ \forall x \in X$.

Notation

$\beta(x, g) = x * g = x \cdot g = xg$

Remark

If $\alpha : G \times X \to X$ is a left action, we get a right action $\beta : X \times G \to X$ by $\beta(x, g) = \alpha(g^{-1}, x)$ and vice versa.
That is $x * g = g^{-1} * x$.
Proof recommended as an exercise.

Analogues

Stability, orbit and transitivty all have analogues which can be demonstrated by converting to left actions.

Definition: Cosets

Let $H \leq G$, and let $X = G$.
We have left action $H \times X \to X$ and $h * x = hx$ (taken in G).
As well as right action $X \times H \to X$ where $x * h = xh$.
A (left) H-coset is an orbit xH for some $x \in X$.
A (right) H-coset is an orbit Hx for some $x \in X$.

Example

Let $G = \mathrm{Alt}(4)$, $H = \mathrm{Stab}_G(W) = \{\mathrm{Id}, \ (B\ P\ Y), \ (B\ Y\ P)\}$.

1. Take any $x \in H$, $xH = H$.

2. Take $x = (B\ P)(W\ Y)$, and $xH = \{(B\ P)(W\ Y), (B\ P)(W\ Y)(B\ P\ Y) = (P\ W\ Y), \ (B\ P)(W\ Y)(B\ Y\ P) = (B\ W\ Y)\}$.

3. There are two more; what are they?

October 10, 2023

Cosets Revisited

Let G be a group, $H \leq G$. Then a (left) H-coset in G is a set of the form

$$gH = \{gh | h \in H\}$$

, where $g \in G$

Coset Space

G/H is the set of H-cosets.

- Example

  For $G = \mathrm{Alt}(4)$, given $C_1 = H = \mathrm{Stab}_G(B) = \{1, (P\ W\ Y), (P\ Y\ W)\}$, we have
  $C_2 = (B\ P\ W)H = \{(B\ P\ W), (B\ P\ )(W\ Y), (B\ P\ Y)\}$
  $(B\ P\ W) \circ (P\ W\ Y) = (B\ P)(W\ Y)$

  P ⟵ B ⟵ B
  B ⟵ W ⟵ P
  Y ⟵ Y ⟵ W
  W ⟵ P ⟵ Y

  $(B\ P\ W) \circ (P\ Y\ W) = (B\ P\ Y)$

  P ⟵ B ⟵ B
  Y ⟵ Y ⟵ P
  W ⟵ P ⟵ W
  B ⟵ W ⟵ Y

  $C_3 = (B\ W\ P)H = \{(B\ W\ P), (B\ W\ Y), (B\ W)(P\ Y)\}$
  $C_4 = (B\ Y\ P)H = \{(B\ Y\ P), (B\ Y\ )(P\ W), (B\ Y\ W)\}$
  Then $G/H = \{C_1, C_2, C_3, C_4\}$.

  - Q: What do the 3 elements in C3 have in common in geometric terms?
    C_3 sends B to W.
    Similarly, the cosets send B to all other vertices (including to itself).

Definition: Transporter

Let G be a group and X a G-set.
For two points, $x, y \in X$, the transporter $\mathrm{Trsp}_G(x, y) = \{g \in G | gx = y\}$.

Example

$G/H = \{\mathrm{Trsp}_G(B,B), \mathrm{Trsp}_G(B,P), \mathrm{Trsp}_G(B,W), \mathrm{Trsp}_G(B,Y)\}$

Note

When $x = y$, we recover $\mathrm{Trsp}_G(x,x) = \mathrm{Stab}_G(x)$.
For general G and H, there may not be a nice geometric action associated with it.
But G/H is still a G-set since $g'(gH) = (g'g)H$.

Proposition (B)

Let $H \leq G$ be a subgroup and let $g \in G$.
Then the map $H \xrightarrow{f} gH$ defined by $h \mapsto f(h) = gh$ is a bijection.

Proof

(Surjective) Any element x in gH is, by definition, of the form gh for some $h \in H$. So $x = f(h)$.
(Injective) Say $h_1, h_2 \in H$ satisfy $f(h_1) = f(h_2)$. That is $gh_1 = gh_2$. Multiplying g^{-1} on the left, we get $h_1 = h_2$.

Proposition (C)

Let G act on X, $x \in X$, and $g \in G$.
Take $y := gx$ and $H = \mathrm{Stab}_G(x)$. Then $gH = \mathrm{Trsp}_G(x,y)$.

Proof

($\subseteq$) Let $gh \in gH$ be arbitrary. Then

$$(gh) * x \underset{\text{Axiom 2}}{=} g * (h * x) \underset{h \in \mathrm{Stab}_G(x)}{=} g * x \underset{y = g*x}{=} y$$

Therefore $gh \in \mathrm{Trsp}_G(x,y)$.
($\supseteq$) Suppose $g' \in \mathrm{Trsp}_G(x,y)$. Consider $g^{-1}g'$. Then

$$(g^{-1}g') * x = g^{-1} * (g' * x) \underset{g' \in \mathrm{Trsp}_G(x,y)}{=} g^{-1}(y) = x$$

Therefore $(g^{-1}g') \in \mathrm{Stab}_G(x)$. Setting $g^{-1}g' := h$, so $g' = gh \in gH$.

Theorem: Orbit-Stabilizer Theorem

Let G act transitively on a set X (so that there is only one orbit in X, namely X itself).
If $|G| < \infty$, then for any $x \in X$ we have

$$|X| \cdot |\mathrm{Stab}_G(x)| = |G|$$

Proof

Let us count $|G|$ by partitioning G into transporters.

$$G = \bigsqcup_{y \in X} \mathrm{Trsp}_G(x, y)$$

Therefore

$$|G| = \sum_{y \in X} |\mathrm{Trsp}_G(x, y)| \underset{B+C}{=} \sum_{y \in X} |\mathrm{Stab}_G(x)| = |X||\mathrm{Stab}_G(x)| \quad \blacksquare$$

Theorem: Lagrange's Theorem

If G is a finite group and $H \leq G$, then $|G| = |H| \cdot |G/H|$.

Proof (Sketch)

Apply the Orbit-Stabilizer Theorem to $X = G/H$.
This action is transitive as $g(1H) = gH$.
Note $gH = H \iff g \in H$ and $g1 \in H$.
Therefore $\mathrm{Stab}_G(1H) = \{g \in G \mid g(1H) = 1H\} = H$.

Corollary

If $H \leq G$ and $|G| < \infty$, then $|H| \mid |G|$.
The converse is not true. No subgroup of order 6 in Alt(4) (where $|\mathrm{Alt}(4)| = 12$).

Definition: Conjugate

Let G be a group, $H \leq G$, $g \in G$.

1. For $x \in G$ the g-conjugate of x is $gxg^{-1} = {}^g x$.

2. The g-conjugate of H is $gHg^{-1} = {}^g H = \{gxg^{-1} \mid x \in H\}$.

Example

Let $G = \mathrm{Alt}(4)$ and $H = \mathrm{Stab}_G(B) = \{1, (P\ W\ Y), (P\ Y\ W)\}$. Then, for $g = (B\ Y\ P)$

$$gHg^{-1} = \{1, (B\ W\ P), (B\ P\ W)\} = \mathrm{Stab}_G(Y)$$

$(B\ Y\ P)1(B\ P\ Y) = 1$
$(B\ Y\ P)(P\ W\ Y)(B\ P\ Y) = (B\ W\ P)$

W ← W ← P ← B
B ← P ← Y ← P
P ← Y ← W ← W
Y ← B ← B ← Y

- Note: Shortcut
  $(gxg^{-1})^{-1} = (g^{-1})^{-1}x^{-1}g^{-1} = gx^{-1}g^{-1}$.
  Applying this to $g = (B\ Y\ P)$ with $x = (P\ W\ Y)$
  Therefore, from the previous calculation, $gx^{-1}g^{-1} = (gxg^{-1})^{-1} = (B\ P\ W)$.

Proposition: Geometric Meaning of Conjugate

Let G act on a set X, $x \in X$, $g \in G$, and define $y := g * x$.
Then for $H = \text{Stab}_G(x)$, we have

$$gHg^{-1} = \text{Stab}_G(y)$$

That is, the conjugate of a a stabilizer is a stabilizer.

Proof

$(\subseteq)$ Let $ghg^{-1} \in gHg^{-1}$ be arbitrary.
Then .

$$(ghg^{-1}) * y = g * (h * (g^{-1} * y))g * (h * x) = g * x = y$$

Therefore $ghg^{-1} \subseteq \text{Stab}_G(y)$.
$(\supseteq)$ Let $g' \in \text{Stab}_G(y)$ be arbitrary.
Consider $g^{-1}g'g$. Then

$$(g^{-1}g'g) * x = g^{-1} * (g' * (g * x)) = g^{-1} * (g' * y) = g^{-1} * y = x$$

Therefore $h := g^{-1}g'g \in H$. Then by multiplying g on the left and g^{-1} on the right, we get

$$g' = ghg^{-1} \in gHg^{-1}$$

Orbit-Stablizer Theorem and Lagrange

1. If G acts transitively on X, then all the stabilizers have the same cardinality because they are all conjugates. So the Orbit-Stabilizer Theorem is consistent.

2. If $X = G/H$, then $\text{Stab}_G(1H) = H$. What about $\text{Stab}_G(gH) = gHg^{-1}$?

October 12, 2023

Recall: Conjugate

$h \in G,\ H \leq G, g \in G$
The conjugates $ghg^{-1} = {}^g h,\ gHg^{-1} = \{ghg^{-1} \mid h \in H\}$.

Meaning

If G acts on X, $x \in X$, $g \in G$, $g * x =: y$, $H = \mathrm{Stab}_G(x)$, then $gHg^{-1} = \mathrm{Stab}_G(y)$.

- Example

  $G = \mathrm{Alt}(4)$, $x = B$, $H = \{1, (P\ W\ Y), (P\ Y\ W)\}$, $g = (B\ P)(Y\ W)$
  gives $gHg^{-1} = \mathrm{Stab}_G(P) = \{1, (B\ W\ Y), (B\ Y\ W)\}$.

Definition: Cyclic Subgroup

The cyclic subgroup generated by g is given as $\langle g \rangle = \{g^n \mid n \in \mathbb{Z}\} = \{\ldots, g^{-2}, g^{-1}, 1, g, g^1, g^2, \ldots\}$.

Definition: Normal Subgroup

Let $H \leq G$.
Say H is normal in G and write $H \trianglelefteq G$ if for all $g \in G$, $gHg^{-1} = H$.
Equivalently, $gH = Hg$ for all $g \in G$.

Trivial Example

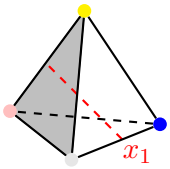
If $H = \{1\}$ ir $H = G$, then $H \trianglelefteq G$.

Example 1: Non-Normal Subgroup

$G = \mathrm{Alt}(4)$, $H = \mathrm{Stab}_G(B)$ is not normal.

Example 2: Alternating Group

Consider $G = \mathrm{Alt}(4)$, $X = \{x_1, x_2, x_3\}, H = \mathrm{Stab}_G(x_1)$.



Then we have the following facts.

- First: Transitivity

  G acts transitively.

- Second: Order of H

  By the Orbit-Stabilizer Theorem,

$$|H||X| = |G| \implies |H| \cdot 3 = 12 \implies |H| = 4$$

- Third: Description of G

  We know

$$G = \left\{1, \binom{8}{3\text{-cycles}}, \binom{3}{2+2\text{-cycles}}\right\}$$

- Q: Can a 3-cycle belong to H?

  A: No. Lagrange's Theorem.
  Suppose a 3-cycle $g \in H$. Then the cyclic subgroup $\langle g \rangle = \{1, g, g^2\}$, but $\langle g \rangle \leq H$ and $3 \nmid 4$.

- Fourth: Description of H

  By Lagrange's Theorem, $H \subseteq \left\{1, \binom{3}{2+2\text{-cycles}}\right\} \implies H = \left\{1, \binom{3}{2+2\text{-cycles}}\right\}$.

- Fifth: H is a Normal Subgroup

  Run the argument with x_1 replaced with x_2, then $\mathrm{Stab}_G(x_2) = H = \mathrm{Stab}_G(x_3)$.
  Therefore, $H \trianglelefteq G$.

Example 3: Non-Normal

From Homework 1, $\mathbb{P}_1(1)$, $H = \{A \in \mathrm{SL}_2\mathbb{C} \mid A^\dagger A = I_2\}$ is not normal.
$\mathbb{P}_1(2)$, $S = \{A = A^\dagger\} \not\leq G$.

Example 4: SL As Kernel

$\mathrm{SL}_n(\mathbb{R}) = \ker(\det \mid \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^x) \trianglelefteq \mathrm{GL}_n(\mathbb{R})$

Example 5: Alternating Group As Kernel

$\mathrm{Alt}(n) = \ker(\mathrm{sign} \mid \mathrm{Sym}(n) \to \{\pm 1\}) \trianglelefteq \mathrm{Sym}(n)$

- Proof

  Let $g \in G$ and $k \in K$ be arbitrary.
  We need to show that $gkg^{-1} \in K$.
  $\phi(gkg^{-1}) \underset{\phi \text{ homomorphism}}{=} \phi(g)\phi(k)\phi(g^{-1}) \underset{k \in K}{=} \phi(g)[\phi(g)]^{-1} = 1_H$.

Exercise 1

$\mathrm{Stab}_{\mathrm{Alt}(4)}(1 \text{ of the 6 edges}) \overset{?}{\trianglelefteq} \mathrm{Alt}(4)$.

Exercise 2

Can you have a subset closed under conjugation?
On Homework 2 we will examine conjugate classes.

Theorem: Quotient Group

Let $N \trianglelefteq G$. Then on G/N, we have a structure of a group, the quotient group, with the binary operation $g_1N * g_2N := (g_1g_2)N$.
The identity element $1N = N$.
The inverse $(gN)^{-1} = g^{-1}N$.

Proof

The main difficulty is in demonstrating that $*$ is well-defined.
That is, if g_1', g_2' are other elements such that (1) $g_1'N = g_1N$ and (2) $g_2' = g_2N$, then we need to show that $(g_1g_2)N = (g_1'g_2')N$.
But (1) means that $g_1' = g_1n_1$ for some $n_1 \in N$, while (2) similarly means $g_2' = g_2n_2$ for $n_2 \in N$.
It follows that $g_1'g_2' = g_1n_1g_2n_2$.
Recall that $N \trianglelefteq G$ implies $Ng_2 = g_2N$, so $n_1g_2 = g_2n_3$ for some $n_3 \in N$.
Therefore $g_1n_1g_2n_2 = g_1g_2n_3n_2 \in N$.
Finally, multiplying N on the right side gives $(g_1g_2)N = (g_1'g_2')N$.

- Associativity

  Proof of associativity is left as an exercise.

Remark

If $H \leq G$ and $g_1H * g_2H \underset{\text{def}}{=} g_1g_2H$ defines (well) a group, then $H \trianglelefteq G$.

Proposition: Kernal Is Normal

Let $G \xrightarrow{\phi} H$ be a group homomorphism, then $K := \ker(\phi) = \{g \in G \mid \phi(g) = 1_H\}$ is a normal subgroup of G.

Proposition: Abelian Subgroups are Normal

If G is abelian, and $H \leq G$, then $H \trianglelefteq G$.

Proof

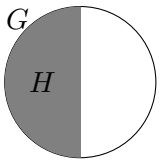
$ghg^{-1} = h$

Proposition: Subgroup of Index 2

Let $H \leq G$ of index 2 (i.e. $[G : H] = |G/H| = 2$), then $H \trianglelefteq G$.

Proof

Let $g \in G$ be arbitrary.
Need to show that o$gHg^{-1} = H$ or, equivalently, $gH = Hg$.
If $g \in H$, there is nothing to prove. So assume $g \notin H$.
But both gH and Hg are the (set) complement of H in G.



- Example

  $\mathrm{Alt}(n) \trianglelefteq \mathrm{Sym}(n)$
  If $n = 1$, there is nothing to prove.
  If $n \geq 2$, then $|\mathrm{Alt}(n)| = \frac{n!}{2} = \frac{|\mathrm{Sym}(n)|}{2}$.

Definition: Simple Group

A group G is simple if G has exactly 2 normal subgroups, namely $\{1\} \neq G$.

Non-Example

$G = \mathrm{Alt}(4) \trianglerighteq H = \mathrm{Stab}_G(x_1),\ H \neq \{1\}$ or G
So G is not simple.

Proposition: Group of Prime Order

Let p be a prime number.
Any group G of order p is both cyclic and simple.
$\mathbb{Z}/p\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{p-1}\}$ with $+$.

- Proof

  Suppose $g \in G$ is not the identity.
  Form $\langle g \rangle = \{g^n \mid n \in \mathbb{Z}\} =: H \leq G$.
  By Lagrange's Theorem, $|H||G| = p$. Since the only positive divisors of p are 1 and p, $\langle g \rangle = H = G$ for any $g \neq 1_G$.
  If $H \leq G$ is any normal subgroup, $|H| \mid |G|$ so $H = \{1\}$ or $H = G$.

Remarks*

For all $n \geq 5$, $\mathrm{Alt}(n)$ is a simple group.
This may be asked as a homework exercise.
If $p \geq 5$ is a prime, then $\mathrm{SL}_2(\mathbb{F}_p)/\{\pm I_2\}$ is a simple group.
Relatedly, if $n \geq 4$ then $\mathrm{SL}_n(\mathbb{F}_p)/Z(\mathrm{SL}_n(\mathbb{F}_p))$ is simple.
The point is that there are infinitely many finite simple groups.
Circa 1980, classification of finite simple groups was announced.
Requires $\sim 10^4$ pages (not everyone has been convinced).

Definition: Center

Let G be a group.
The center $Z(G) = \{z \in G \mid zg = gz,\ \forall g \in G\}$ is a normal subgroup ($gzg^{-1} = z, \forall g, z$).

More Group Action Terminology

Let G act on a set X.
Recall that this corresponds to a group homomorphism $G \xrightarrow{\rho} \text{Sym}(x)$.

Definition: Faithful

Say the action is faifthful if $\ker(\rho) = \{1_G\}$.
$\forall g \in G : \ gx = x, \ \forall x \in X \implies g = 1_G$

Definition: Free

Say the action is free if $\forall g \in G, \ \forall x \in X, \ gx = x \implies g = 1_G$.
$\text{Stab}_G(x) = \{1_G\}, \ \forall x \in X$

Example 1

Let G act on $X = G$ by left multiplication. Then this is free.

Example 2

$G = \text{Alt}(4)$
$X_1 = \{B, P, W, Y\}$ is faifthful but not free.
$X_2 = \{6 \text{ edges}\}$ is faifthful but not free.
$X_3 = \{3 \text{ strings}\}$ is not faithful.

October 17, 2023

Recall: Normal Subgroup

A subgroup $N \leq G$ is normal & $N \trianglelefteq G$, if $gNg^{-1} = N$ for all $g \in G$.

Notation

$G \xrightarrow{\pi} G/N$
$g \mapsto gN$

Recall: Quotient Group

We have constructed, for $N \trianglelefteq G$, the quotient group G/N, with $g_1N * g_2N = g_1g_2N$.

Definition: Group Homomorphisms

For groups G, K, $\text{Hom}(G, K) = \{G \to K \mid \text{group homomorphisms}\}$.

Theorem: Universal Mapping Property of the Quotient Group

Let H be any group.

Part 1

For any group homomorphism $\overline{f} : G/N \to H$, by composing, we get a homomorphism

$$\overline{f} \circ \pi = f : G \xrightarrow{\pi} G/N \xrightarrow{\overline{f}} H$$

such that $f(N) = \{1_H\}$.

- Proof
  Let $\overline{f} \in \mathrm{Hom}(G/N, H)$ and $f := \overline{f} \circ \pi \in \mathrm{Hom}(G, H)$.
  Then

$$f(n) = \overline{f}(\pi(n)) = \overline{f}(nN) = \overline{f}(N) \underset{N \in G/N \text{ is the identity}}{=} 1_H$$

Part 2

Conversely, for any group homomorphism $f : G \to H$ such that $f(N) = \{1_H\}$ we get a unique homomorphism $\overline{f} : G/N \to H$ such that $f = \overline{f} \circ \pi$.

- Proof
  - Well Defined
    Let $f \in \mathrm{Hom}(G, H)$ satisfy $f(N) = \{1_H\}$.
    Define $\overline{f}(gN) = f(g)$ for every $g \in G$.
    Need to Show: If $g_1 N = g_2 N$, then $f(g_1) = f(g_2)$.
    But $g_1 N = g_2 N$ implies, since $g_1 1_G \in g_1 N$, that $g_1 = g_2 n$ for some $n \in N$.
    Since f is a group homomorphism, we have

$$f(g_1) = f(g_2)f(n) \underset{f(N)=\{1_H\}}{=} f(g_2)1_H = f(g_2)$$

  - Homomorphism
    $\overline{f}$ is a homomorphism since for any $g_1 N_1, g_2 N_2 \in G/N$, we have

$$\overline{f}(g_1 N * g_2 N) \underset{\text{definition of } *}{=} \overline{f}(g_1 g_2 N) \underset{\text{definition of } \overline{f}}{=} f(g_1 g_2) \underset{f \text{ is homomorphism}}{=} f(g_1)f(g_2) \underset{\text{definition of } \overline{f}}{=} \overline{f}(g_1 N)\overline{f}(g_2 N$$

  - Uniqueness
    Suppose $l \in \mathrm{Hom}(G/N, H)$ is any element satisfying $f = l \circ \pi$.
    Then for any $g \in G$, we have

$$\overline{f}(gN) = f(g) = l(\pi(g)) = l(gN)$$

    Therefore $\overline{f} = l$.

Rephrase: Universal Mapping Property of Quotient Group

$\mathrm{Hom}(G/N, H) \underset{2}{\overset{1}{\rightleftarrows}} \{f \in \mathrm{Hom}(G,H) \mid f(N) = \{1_H\}\}$
Equivalently, $f(N) = \{1_H\} \iff N \leq \ker(f)$.
Note: $f(N) = \{1_H\} \iff N \subseteq f^{-1}(\{1_H\}) = \ker(f)$

Loosely: Universal Mapping Property of Quotient Group

Giving a homomorphism $G/N \to H$ is the same as giving a homomorphism $G \to H$ that kills N.

Definition: Group Generators

Let G be a group and let $S \subseteq G$.

Definition: Word

A word in S of length $l \geq 1$ is an expression $x_1 x_2 \cdots x_l$ where $x_i \in \{s, s^{-1}\}$ for some $s \in S$ and $i = 1, \ldots, l$.
The word of length zero is, by convention, 1_G.

Definition: Generated Subgrouo

The subgroup generated by S, $\langle S \rangle$ is the subset of G consisting of all the words in S of all possible lengths.
Fact: $\langle S \rangle \leq G$.

- Example

  $S = \{A, C, I, T, L\}$.
  One word in S of length 3 is $CA^{-1}T$.
  Inverse: $(CA^{-1}T) = T^{-1}AC^{-1}$.
  Composition: $CA^{-1}T * T^{-1}A^{-1}IL = CA^{-1}TT^{-1}A^{-1}IL = CA^{-2}IL$. (Note that, strictly, $A^{-2} \notin S$ but rather $A^{-2} \equiv A^{-1}A^{-1}$).

Common Usage

We usually do not use the bare definition to describe what $\langle S \rangle$.

- Example

  Let $G = \mathrm{Sym}(n)$ and let $S = \{\text{transpositions } (a\ b) \mid 1 \leq a < b \leq n\}$.
  Then $\langle S \rangle = G$.

  - Proof

    Step 1: Any $\pi \in G$ has a cycle decomposition. So $\pi = \gamma_1 \gamma_2 \cdots \gamma_k$, where γ_i is an l_i-cycle for some $l_i \geq 1$.
    Step 2: If γ is an l-cycle, $l \geq 2$, say $\gamma = (a_1\ a_2 \cdots a_l)$, then $\gamma \overset{\text{HW1}}{=} (a_1\ a_2)(a_2\ a_3)\cdots(a_{l-1}\ a_l)$.

Definition: Commutator

A commutator in G is an element of the form $[g,h] := ghg^{-1}h^{-1}$ for some $g, h \in G$.

Definition: Commutator Subgroup

The commutator subgroup (or derived subgroup) $D(G)$ (sometimes $[G,G]$) is the subgroup of G generated by all the commutators.
That is, a typical element in $D(G)$ is $C_1C_2\cdots C_l$ where $C_i = [g_i, h_i]$ for some $g_i, h_i \in G, \forall i$.
Note that $[g,h]^{-1} = (ghg^{-1}h^{-1})^{-1} = hgh^{-1}g^{-1} = [h,g]$.

Proposition: Commutator Subgroup Is Normal

$D(G) \trianglelefteq G$.

Lemma:

Let $g_0 \in G$ be fixed.
The map $G \to G$ defined as $g \mapsto g_0 g g_0^{-1}$ – called the inner homomorphism and denoted $\text{Int}(g_0)$ – is a group homormorphism.

- Proof

$$\text{Int}(g_0)(g_1g_2) = g_0g_1g_2g_0^{-1} = g_0g_1g_0^{-1}g_0g_2g_0^{1} = [\text{Int}(g_0)(g_1)][\text{Int}(g_0)(g_2)]$$

Proof

Let $g_0 \in G$ and $C_1\cdots C_l \in D(G)$ be arbitrary.

$$g_0(C_1\cdots C_l)g_0^{-1} = \text{Int}(g_0)(C_1\cdots C_l) = \underset{\text{lemma}}{=} [\text{Int}(g_0)(C_1)]\cdots[\text{Int}(g_0)(C_l)]$$

It suffices to prove that $\text{Int}(g_0)(C)$ is a commutator for any commutator C.

$$\text{Int}(g_0)(C) = \text{Int}(g_0)(ghg^{-1}h^{-1}) = [\text{Int}(g_0)(g), \text{Int}(g_0)(h)] \quad \blacksquare$$

Definition: Abelianization

The quotient group $G/D(G)$ is called the Abelianization G^{ab} of G.

Property: Commutativity

G^{ab} is commutative.

- Proof

  Let $D = D(G)$ and let $gD, hD \in G^{\text{ab}}$ be arbitrary.
  Then

$$[gD, hD] = (gD)(hD)(gD)^{-1}(hD)^{-1} = ghg^{-1}h^{-1}D = D = 1_{G^{\text{ab}}}$$

  Therefore, by multiplying on the right by $(hD)(gD)$, we get

$$(gD)(hD) = (hD)(gD)$$

  So G^{ab} is Abelian.

Note: Abelian Groups

G is Abelian if and only if $D(G) = \{1_G\}$.

Proposition: Minimal Normal Subgroup with Commutative Quotient

$D(G)$ is the smallest normal subgroup $N \trianglelefteq G$ such that G/N is commutative.

Proof

Suppose $N \trianglelefteq G$ has Abelian G/N.
Need to Show: $D(G) \leq N$.
So let $g, h \in G$ be arbitrary and let $C = ghg^{-1}h^{-1}$.
Then in G/N, $[gN, hN] = 1_{G/N} = N$.
Therefore $[g, h] \in N$ and $[G, G] \leq N$. ■

Remark*: Homology in Algebraic Topology

Let X be a connected manifold, say
IMAGE HERE - DOUBLE TORUS WITH CHUNK BEING REMOVED
In Algebraic Topology, we study loops up to homotopy $G = \pi_1(X, x_0)$.
Then $G^{\text{ab}} \underset{Hurewicz}{=} H_1(X; \mathbb{Z})$ is a homology.

Theorem: Universal Mapping Property of Abelianization

Let G be any group and H be any Abeian group.
Then we have a bijection $\text{Hom}(G, H) \rightleftarrows \text{Hom}(G^{\text{ab}}, H)$.

Proof

Since H is Abelian, any homomorphism $f : G \to H$ will satisfy $f([G, G]) = \{1_H\}$.
So use UMP for quotient G/N. ■

October 19, 2023

Review: Commutator

Let G be a group.
A commutator is an element of the form $[g, h] = ghg^{-1}h^{-1}$.
The derived (or commutator) subgroup $DG = [G, G]$ is the subgroup generated by all the commutators.

Properties of Commutators

$DG \trianglelefteq G$
$G^{\text{ab}} := G/DG$ is abelian.
G^{ab} satisfies the universal mapping property. That is, for all abelian H.

$$\text{Hom}(G, H) \leftrightarrows \text{Hom}(G^{\text{ab}}, H)$$

Example: G Abelian

If G is abelian, then $D(G) = \{1_g\}$ and $G \tilde{\to} G^{\text{ab}} = G/\{1_G\}$ (usually written $G = G^{\text{ab}}$).
Conversely, $DG = \{1_G\}$ implies G is abelian.

Example: G Simple and Non-abelian

If G is simple and non-abelian, then $DG = G$.

Definition: Perfect Group

A group G is called perfect if $D(G) = G$.
cf. Poincaré Conjecture (Perelmon circa 2002: Every closed 3-manifold M with $\pi_1(M) = \{1\}$ is homeomorphic to S^3.

Example: G Symmetric Group 3

Let $G = \text{Sym}(3)$.
By Lagrange, if $H \leq G$, then $|H| \mid |G| = 3! = 6$. Consider

$$|H| = 1, 2, 3, \text{ or } 6$$

Then
IMAGE HERE - LINES BETWEEN GENERATORS, GROUP AND TRIVIAL GROUP
Which one is DG?
It cannot be $\{1\}$ since $G \neq G^{\text{ab}}$.
It cannot be $\langle\tau\rangle, \tau^2 = 1$, since they are not normal.
$G/\langle(1\ 2\ 3)\rangle$ is a group of order 2 and, therefore, abelian.

Recall

DG is the smallest normal subgroup N of G such that G/N is abelian.

Therefore, $DG = D(\text{Sym}(3)) = \langle(1\ 2\ 3)\rangle$.

Note

For $G = \text{Sym}(3)$,

$$G \underset{\neq}{\geq} D(G) = \langle(1\ 2\ 3)\rangle \underset{\neq}{\geq} D(D(G)) = \{1\}$$

Definition: Solvable Group

Suppose G is a group such that

$$D(D(D(\cdots D(G)\cdots) = \{1\}$$

Then G is solvable (or soluble).

Definition: Conjugacy Class

The conjugacy class of $\in G$ is the set of elements in G that are conjugate to $\overline{x}$.
Let G act on (the set) $G = X$ by conjugation: $g * x = gxg^{-1}$.
Then the conjugacy class of x is the G-orbit of $x \in X$.

Definition: Centralizer

The centralizer $x \in G$ is the stabilizer of x in this conjugation cation.

$$C_G(x) = \{g \in G \mid gxg^{-1} = x\} \iff gx = xg$$

Example: G Alternating Group 4

Let $G = \text{Alt}(4)$.
Then, by Lagrange, $H \leq G \implies |H| \mid |G| = 12$.
So $|H| = 1, 2, 3, 4, 6,$ or 12
IMAGE HERE - SUBGROUP DIAGRAM

Proposition: No Order 6 Subgroup

No subgroup H of order 6 in AAlt(4).

- Proof

  Suppose $H \leq G$ has order six.
  Then $[G : H] = \frac{12}{6} = 2$ and, therefore, $H \trianglelefteq G$.
  So H must be a union of certain conjugacy classes.
  From Homework 1, G acts transitively on {6 edges}, so Stab($\overline{BP}$), Stab($\overline{BW}$),and Stab($\overline{BY}$) are conjugate to each other (HW2 P1(b)).
  Therefore the 3 elements of order 2, $T = \{(B\ P)(W\ Y), (B\ W)(P\ Y), (B\ Y)(P\ W)\}$ form a conjugacy class in G
  It follows that $G = \{1\} \coprod T \coprod \{8 \text{ elments of order } 3\}$.
  By the following lemma, this resolves to

$$G = \{1\} \bigsqcup T \bigsqcup \{p\}^{\#} \bigsqcup \{p^2\}^{\#}$$

- Lemma (Key)

  Every element p of order 3 in $G = \text{Alt}(4)$ has conjugacy class of size 4.

  - Proof

    $|(\text{conjugacy class of } p| \geq 4$ since G acts transitively on $\{B, P, W, Y\}$ and $\exists p \in \text{Stab}(P)$ such that, without loss of generality, $gpg^{-1} = \text{Stab}(B)$.
    Since $p = 1$ implies $gpg^{-1} = 1$, each g sends non-trivial p to either $(P\ W\ Y)$ or $(P\ Y\ W)$ in Stab(B).
    Then, also

$$|(\text{conjugacy class of } p| = |\text{orbit of } p \text{ under the conjugation action}| = |G|/|C_G(p)| \leq 12/3 = 4$$

    Therefore no normal group of order 6 is in $G = \text{Alt}(4)$.

Example Continued: 6 Alternating Group 4

$\{1\}$ is non-abelian.
Stab($\overline{BP}$) inhabits a conjugate class.
Because $|G/\text{Stab(String)}| = 3$ it is cyclic and $DG = \{1\} \coprod T$
Observe that DG is abelian.

Proof

What to Show: For $t = (B\ P)(W\ Y)$ and every $t \in T$, $C_G(t) \supseteq DG$.
Then

$$|C_G(T)| \underset{OST}{=} \frac{|G|}{|T|} = \frac{12}{3} = 4$$

Therefore, $C_G(t) = DG$.

Remark: Enumerability of Subgroups

There are very few groups G such that we can enumerate all the subgroups of G.

Theorem: Product of Integers is a Subgroup

Let $G = (\mathbb{Z}, +)$.

1. For any $a \in \mathbb{Z}$, the subset $a\mathbb{Z} = \{an \mid n \in \mathbb{Z}\}$ is a subgroup of G.

2. Conversely, any subgroup $H \leq \mathbb{Z}$ is of the form $H = a\mathbb{Z}$ for some $a \in G$.

Proof of 1

Need to Show: $a\mathbb{Z}$ contains 0, is closed under $+$ and is closed under (additive) inverse.
But, $0 = a0$, and $an + am = a(n + m)$ and $-(an) = a(-n)$, for $n, m \in \mathbb{Z}$.

Proof of 2

Suppose $H \leq (\mathbb{Z}, +)$.
Since $+$ is commutative, all subgroups will be normal.
If $H = \{0\}$, then $h = 0\mathbb{Z}$, and we're done.
If not, say $H \ni x \neq 0$, then $h \ni -x$.
Therefore $S = \{x \in H \mid x > 0\}$ must be non-empty.
Let a be the smallest element in S.
Then we claim $H = a\mathbb{Z}$.

- Proof of the Claim

  ($\supseteq$) Since $H \ni a$ and $H \leq G$.
  ($\subseteq$) Let $x \in H$ be arbitrary. Apply Division Algorithm and get

$$x = aq + r$$

where $q, r \in \mathbb{Z}$ and $0 \leq r \leq a$.
If $r > 0$, then

$$r = x - aq \in H$$

so $r \in S$.
However, since $r < a$, this contradicts the very choice of a as the smallest element.
So $r = 0$.
But then $x = aq \in a\mathbb{Z}$.
Therefore $H \subseteq a\mathbb{Z}$. ■

Homework 2 Question

If $H \leq G$, then $gHg^{-1} \leq G$.
Consider the set X_G of all the subgroups H of G.
The group G acts on X_G by conjugation: $g * H := gHg^{-1}$.

Definition: Normalizer

The normalizer of H in G is $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$.

Note

$H \leq N_G(H) \leq G$.