

Manifolds I

September 26, 2024

Class Organization

1 Takehome Midterm

1 Takehome Final

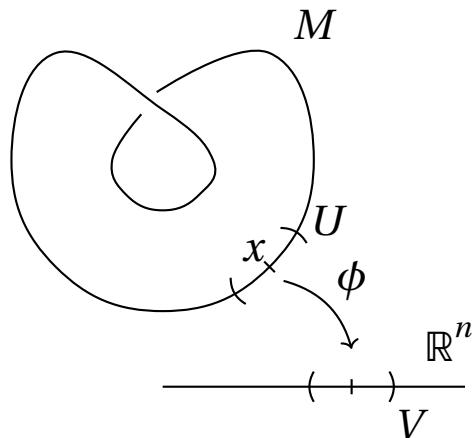
Homeworks assigned, but not graded.

<https://ginzburg.math.ucsc.edu/teaching/208manifolds1-2024/syl.html>

Definition: Topological Manifolds

For M a topological space, M is a topological manifold if $\forall x \in M, \exists M \ni U \ni x$ and homeomorphism $\phi: U \rightarrow V \subset \mathbb{R}^n$ for V open.

To avoid problems (see below), further assume that M is Hausdorff and second countable.

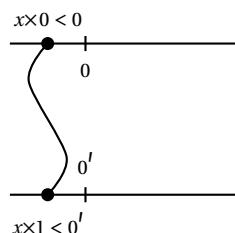


Exercise

We can require V to be an open ball.

Problems

- M need not be Hausdorff.



With $(\mathbb{R} \times 0 \coprod \mathbb{R} \times 1) / \sim$.

- M need not be second countable.

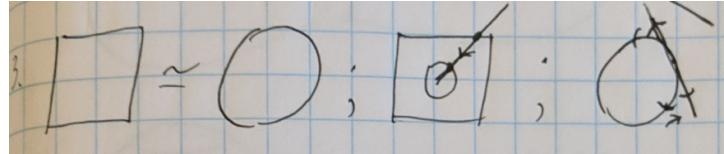
Take $\coprod_S \mathbb{R}_S$ where S is an uncountable index.

Examples

Example 1

If $N \underset{\text{homeo}}{\simeq} M$, this implies N is a manifold.

Example 2



Example 3

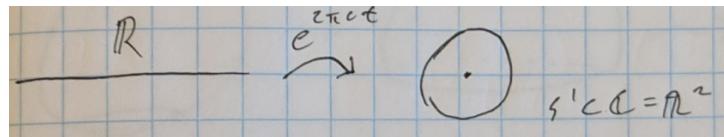
An open subset of a manifold is a manifold.

Example 4

M, N manifolds implies $M \times N$ is a manifold.

Example 5

Take \mathbb{R}/\mathbb{Z} by the equivalence relation $t \sim t'$ iff $t' - t \in \mathbb{Z}$.



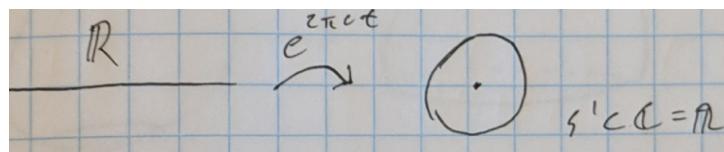
Then $C^0(S^1)$ relates to periodic functions with period 1.

Example 6

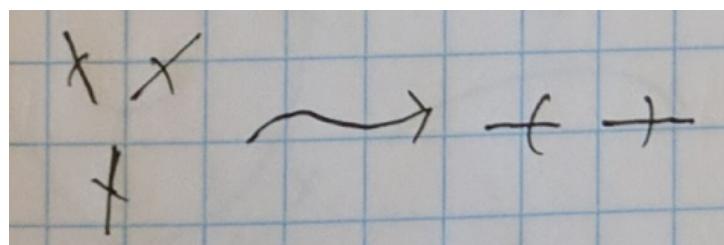
$$\mathbb{T}^n = S^1 \times \cdots \times S^1.$$

Counterexample 1

$[0, 1]$ is not a manifold.

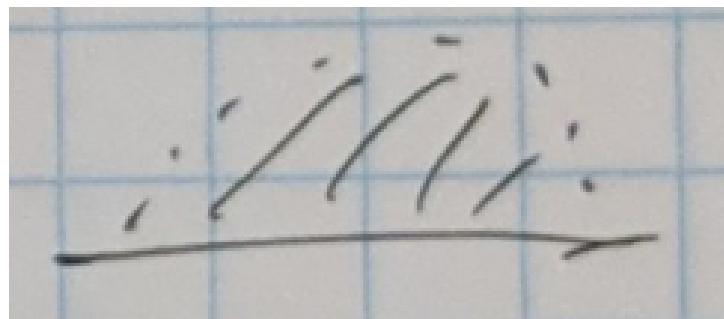


Since 0 must map somewhere in the open interval, its deletion results in a connected space in the former case but a disconnected one in the latter. Similarly, the following breaks into three and two connected components respectively.



Definition: Manifold with Boundary

There exists a neighborhood $\forall x \in M$ homeomorphic to either the open ball or the half-closed half-ball.



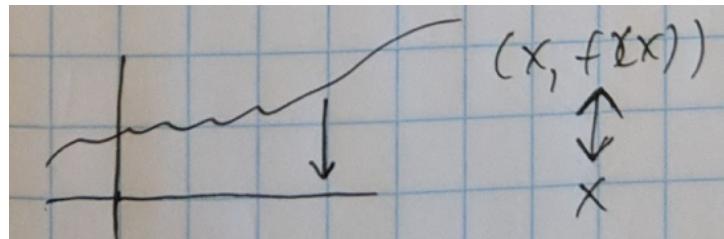
Exercise

A connected manifold is path-connected.

Examples

Example 7

Take $f : \mathbb{R}^n \xrightarrow{C^0} \mathbb{R}$ with graph $\Gamma_f = \{(x, f(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^n \times \mathbb{R}$.

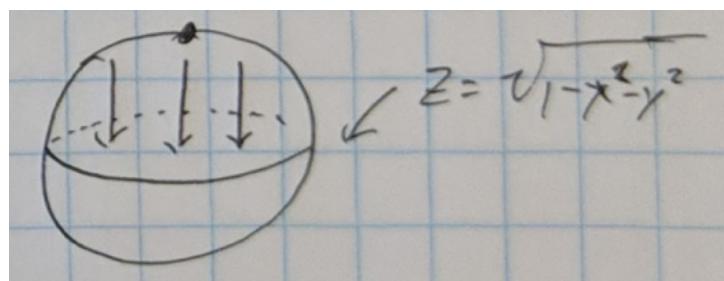


Example 8

Take $f : M \rightarrow N$ between manifolds, then $M \simeq \Gamma_f \subseteq M \times N$.

Example 9

$S^n \subset \mathbb{R}^{n+1}$.



Definition: Real Projective Spaces

Take $\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim$ where $x \sim y \iff x = \lambda y$ for $\lambda \neq 0$.
Informally, the collection of lines through the origin.

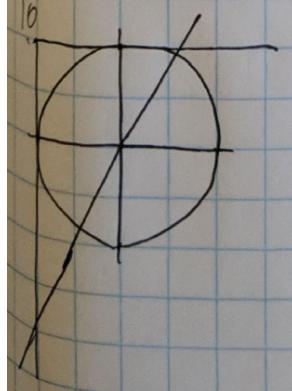
Alternatively, $\mathbb{RP}^n = S^n / \sim$ where $x \sim -x$.

That is, identifying the antipodal points of the unit sphere.

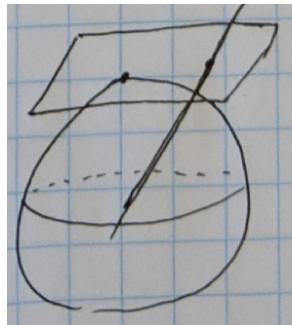
We may also consider $\mathbb{RP}^n = SO(n+1)/SO(n)$.

Claim

\mathbb{RP}^n is a manifold.



$$\mathbb{RP}^1 \setminus \{x\text{-axis}\} \xrightarrow{\text{homeo}} \mathbb{R}.$$



$$\mathbb{RP}^2 \setminus \mathbb{RP}^1 \xrightarrow{\text{homeo}} \mathbb{R}^2$$

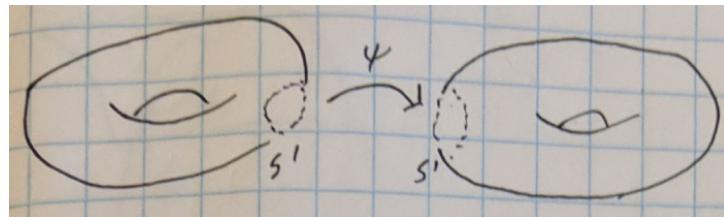
We have that \mathbb{RP}^1 is homeomorphic to the circle, and $\mathbb{RP}^n = \mathbb{RP}^{n-1} \cup B^n$.

Take $x = (x_0, \dots, x_n)$, $y = (y_0, \dots, y_n) = (\lambda x_0, \dots, \lambda x_n)$ and $[x] = [x_0 : x_1 : \dots : x_n]$.

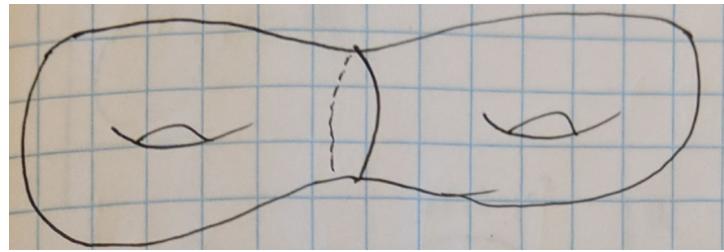
Then for $U_k \subset \mathbb{RP}^n$ with $U_k = \{[x] : x_k \neq 0\}$, we have that U_0, \dots, U_n covers \mathbb{RP}^n .

Then define $U_k \rightarrow \mathbb{R}^n$ by $[x_0 : \dots : x_n] \mapsto \left(\frac{x_0}{x_k}, \dots, \frac{x_{k-1}}{x_k}, \frac{x_{k+1}}{x_k}, \dots, \frac{x_n}{x_k} \right)$.

Connected Sum of Manifolds

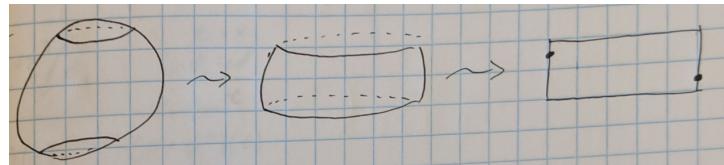


$$M \setminus B^n \coprod N \setminus B^n$$



$M \# N$.

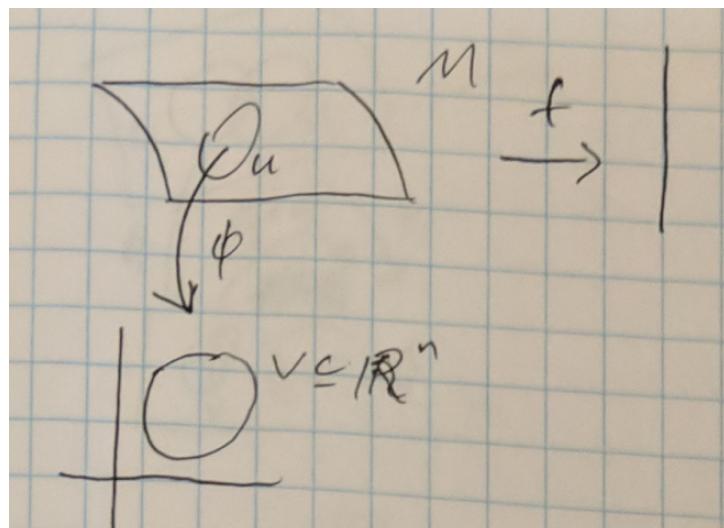
Möbius Band



October 1, 2024

A Failed Definition

$$f \in C^{r \geq 1}; f \circ \phi^{-1} : V \xrightarrow{C^r} \mathbb{R}.$$



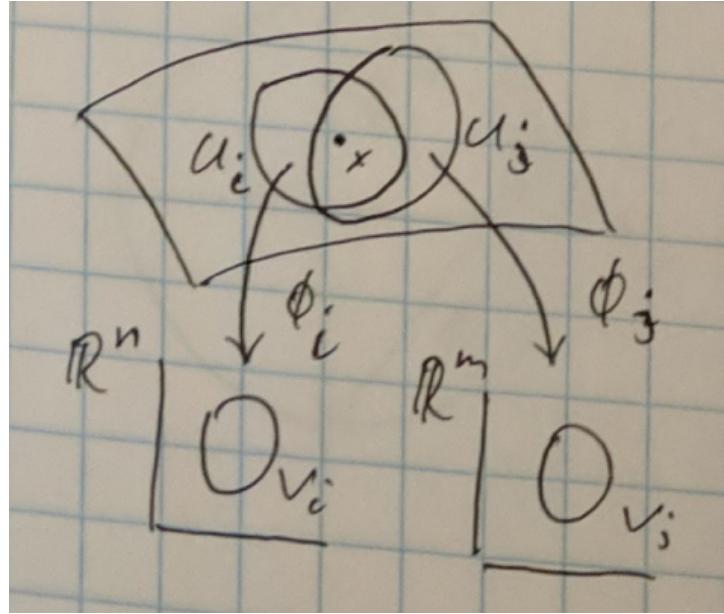
Example

$$\begin{array}{c}
 2. \quad \frac{M = \mathbb{R}}{t = x^3} \quad \frac{x}{\phi_2} \quad \frac{f(x) = x^2}{t = x} \quad \frac{| \mathbb{R}}{\phi_1 = id} \\
 \frac{t}{\epsilon} \quad \frac{\epsilon}{\epsilon} \quad \frac{\epsilon}{\epsilon} \\
 (f \circ \phi_2^{-1})(\epsilon) = \epsilon^2 \\
 \text{Not } C^1 \\
 \text{in } C^\infty
 \end{array}$$

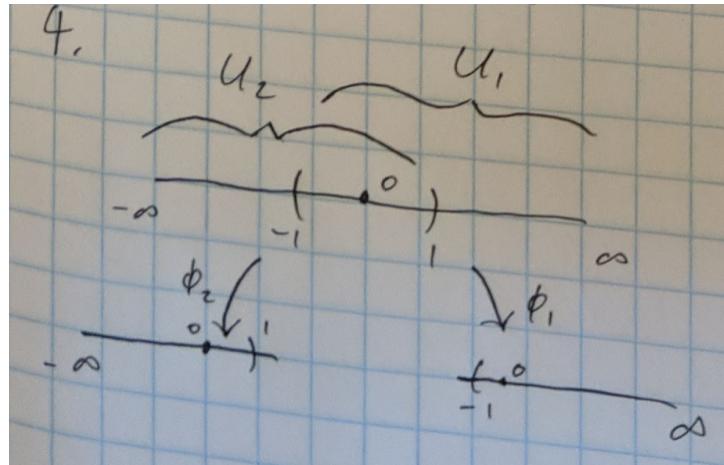
Definition: Charts

Say there exists a cover U_i by open sets and $U_i \xrightarrow{\phi_i} V_i \subseteq \mathbb{R}^n$ fixed.
Then the pair (U_i, ϕ_i) is a chart.

What if a point belongs to two charts?



With f smooth at x , $f \circ \phi_i^{-1}$ smooth at $\phi_i(x)$ and $f \circ \phi_j^{-1}$ smooth at $\phi_j(x)$.

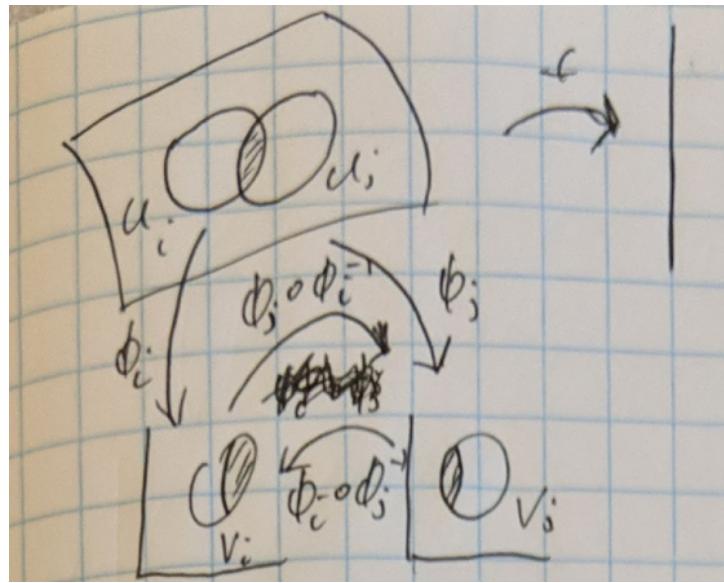


Notation

The notation C^r will be used interchangably with the term smooth.

Definition: Smooth Atlas

Let M be a topological manifold. A smooth atlas on M is a cover $(U_i, \phi_i : U_i \xrightarrow{\sim} V_i \subset \mathbb{R}^n)$ where $\phi_j \circ \phi_i^{-1}$ and $\phi_i \circ \phi_j^{-1}$ are smooth for every i and j .



Say that the charts are (smooth) compatible.

Definition: Smooth Function

Say that f is smooth at $x \in M$ if there exists a chart $U_i \ni x$ such that $f \circ \phi_i$ is smooth at $\phi_i(x)$. Equivalently, if for every chart $U_i \ni x$ we have that $f \circ \phi_i$ is smooth at $\phi_i(x)$.

- Proof

$$f \circ \phi_j^{-1} = (f \circ \phi_i^{-1}) \circ \underbrace{(\phi_i \circ \phi_j^{-1})}_{C^r}$$

Definition: Compatibility (Equivalence) of Atlases

Atlases A_1 and A_2 are compatible or equivalent if every chart in A_1 is compatible with every chart in A_2 . Equivalently, $A_1 \cup A_2$ is also an atlas.

- Claim: This is an equivalence relation.

Example

Consider \mathbb{R} .

Atlas 1: $U = \mathbb{R}$ and $\phi = \text{id}$.

Atlas 2: $U_1 = (1, \infty)$, $\phi_1(x) = x^2$, $U_2 = (-\infty, 2)$ and $\phi_2(x) = x$.

Definition: Diffeomorphism

$\mathbb{R}^n \supset V \xrightarrow{F} W \subset \mathbb{R}^n$ is a diffeomorphism if

- F is C^r ,
- F is invertible, and
- F^{-1} is C^r

Counterexample

$y = x^3$ is a smooth homeomorphism but not a diffeomorphism.

Definition: Smooth Structure / Maximal Atlas

Given an atlas, we may take all compatible atlases and define a smooth structure by the union of all such objects (i.e. the maximal atlas).

Lemma:

Every smooth manifold has a countable, locally finite atlas of precompact charts.

Examples

- Zero dimensional manifolds (i.e. a point).
- \mathbb{R}^n and open subsets of \mathbb{R}^n .
- If M, N are smooth manifolds, then $M \times N$ is a smooth manifold.

That is, if we have atlases (U_i, ϕ_i) and (W_j, ψ_j) , we may generate $(U_i \times W_j, \phi_i \times \psi_j)$.

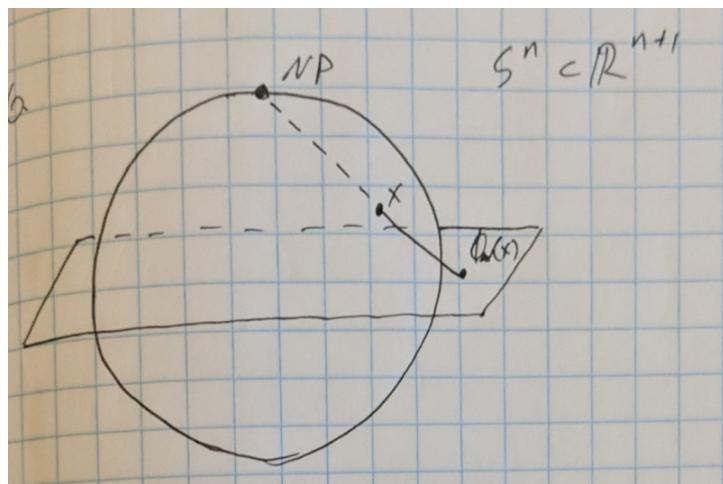
- Take $F: M \xrightarrow{\text{homeo}} N$ with N a smooth manifold. Then M is smooth.

Take an atlas A on N and the pullback $F^{-1}A = \{(F^{-1}(U_i), \phi_i \circ F)\}$.

- An open subset of a smooth M is a smooth manifold.
- $\text{GL}(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$.

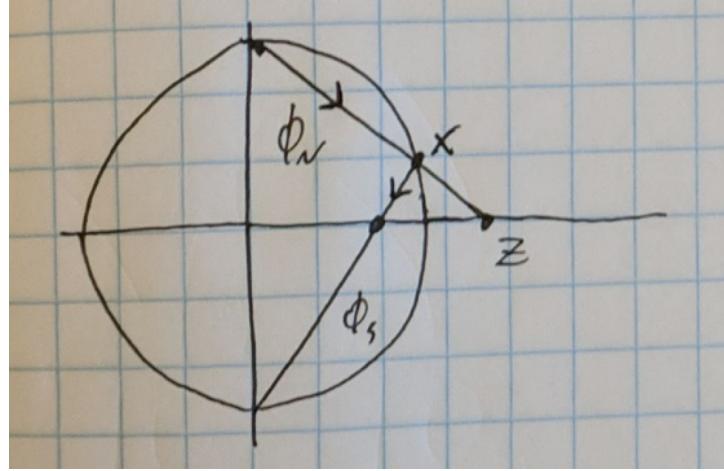
The n-Sphere

- S^n is a manifold



$$U_N = S^n \setminus NP \xrightarrow{\phi_N} \mathbb{R}^n$$

$$U_S = S^n \setminus SP \xrightarrow{\phi_S} \mathbb{R}^n$$



$$\phi_S \phi_N^{-1}(z) = \frac{z}{|z|^2}.$$

– A different construction for S^n .

Take hemispheres $U \xrightarrow{\text{orthogonal projection}} B^n$.

Projective Space

$$\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus 0 / \sim \text{ where } x \sim \lambda x \text{ for } \lambda \neq 0.$$

$$[x] = [x_0 : x_1 : \dots : x_n] = [\lambda x_0 : \lambda x_1 : \dots : \lambda x_n].$$

Take $U_i = \{x_i \neq 0\}$ and open cover, and maps $U_i \rightarrow \mathbb{R}^n$ given by $[x_0 : \dots : x_n] \mapsto \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, 1, \frac{x_i}{x_i}, \dots, \frac{x_n}{x_i} \right)$. Then for $j < i$ take

$$\phi_j \phi_i^{-1}(y_1, \dots, y_n) = \left(\frac{y_0}{y_j}, \dots, \frac{y_{j-1}}{y_j}, \frac{y_{j+1}}{y_j}, \dots, \frac{y_{i-1}}{y_1}, 1, \frac{y_i}{y_i}, \dots, \frac{y_n}{y_1} \right)$$

Definition: Diffeomorphism

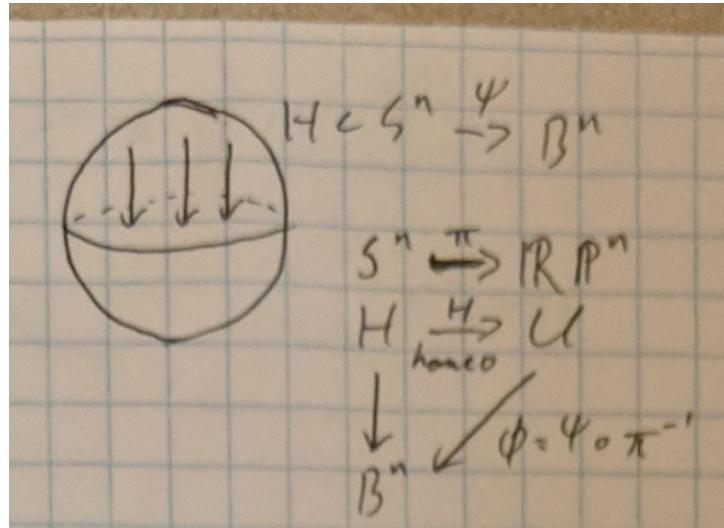
$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ B \subset B_{\max} & & A \supset A_{\max} \end{array}$$

F is a diffeomorphism if F is a homoeomorphism and $F^{-1} A_{\max} = B_{\max}$ ($F^{-1} A \sim B$).

October 3, 2024

Recall

$$\mathbb{RP}^n = \begin{cases} \mathbb{R}^{n+1} \setminus 0 / \sim & x \mapsto \lambda x \\ S^n / x \sim -x \end{cases}$$



Note

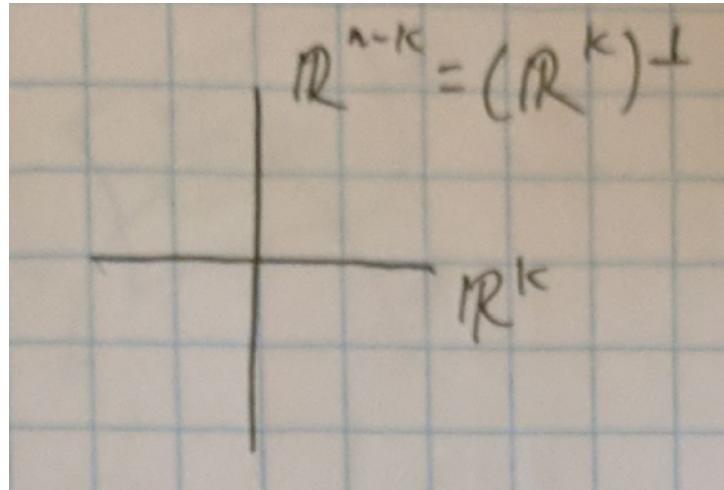
Given a manifold M and A a smooth atlas, we generate a continuum of smooth atlases not equivalent to each other. That is, given $M \xrightarrow[\text{homeo}]{} M$, $F^{-1}A \neq A$.

Confer With Groups

$$G \xrightarrow{F} G, a * b = F^{-1}(F(a)F(b)).$$

Definition: Grassmannians

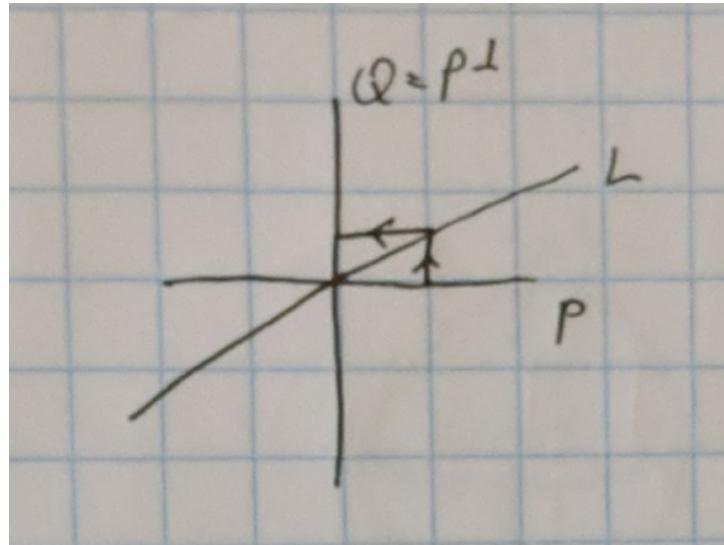
Write $G_k(n)$, the collection of all k -dimensional subspace L in \mathbb{R}^n .



Observe that if $O(i)$ is the collection of orthogonal transformations in dimension i ,

$$G_k(n) = \frac{O(n)}{O(k) \times O(n-k)}$$

with $X \sim Y$ when $Y = XA = X(O(K) \times O(n-k))$.

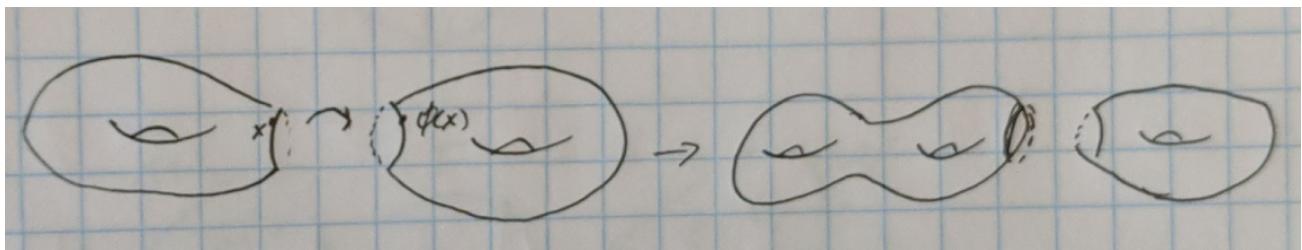


Where $\dim(L) = k$, $U_p = \{L : L \cap Q = \{0\}\}$, $L = \text{graph}(A : P \rightarrow Q)$, and we have a homeomorphism

$$U_p \xrightarrow{\phi} \underbrace{\{\text{linear maps } P \rightarrow Q\}}_{\mathbb{R}^{k \times (n-k)}}.$$

Surfaces

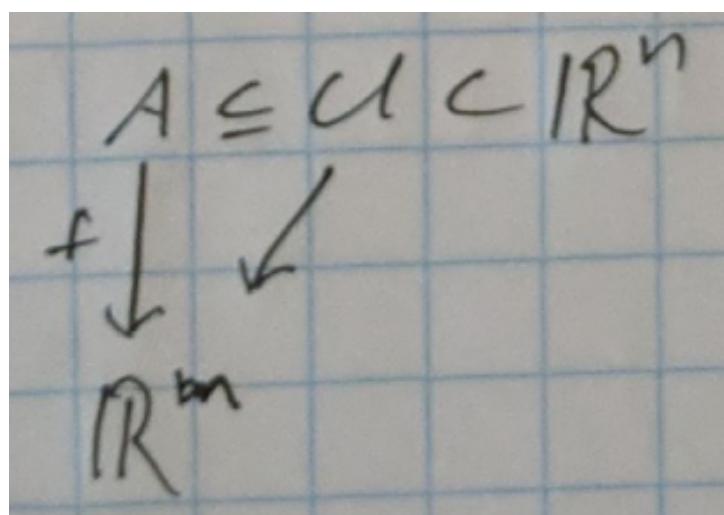
We have explored S^2 , \mathbb{RP}^2 , $\mathbb{T}^2 = S^1 \times S^1$. We have also connected sums.



Terminological Remark

Let $\mathbb{R}^N \ni A \xrightarrow{f} \mathbb{R}^m$.

Then f is smooth if it extends to a smooth map



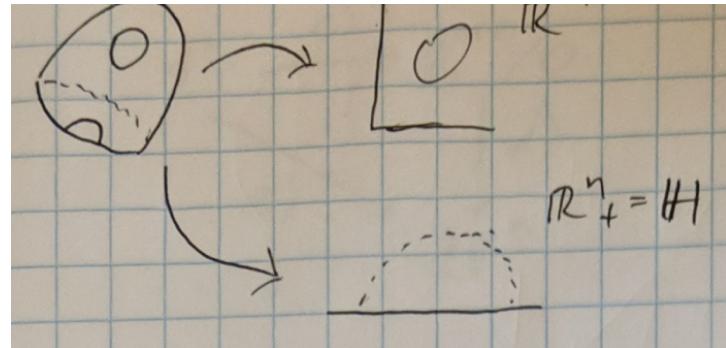
Exercise

Let $A = [0, \infty) \subset \mathbb{R}$. $f : A \rightarrow \mathbb{R}$ is smooth if and only if it is infinitely differentiable.
Construct $(-\varepsilon, \infty)$.

Definition: Smooth Manifold with Boundary

A smooth manifold with boundary is a topological space along with an atlas \mathcal{A} with charts of two types

$$\begin{aligned}\phi : U \rightarrow B^n &\quad (\text{open ball}) \\ \phi : U \rightarrow B^n \cap H\end{aligned}$$



As before, $\phi_i \circ \phi_j^{-1}$ must be smooth.

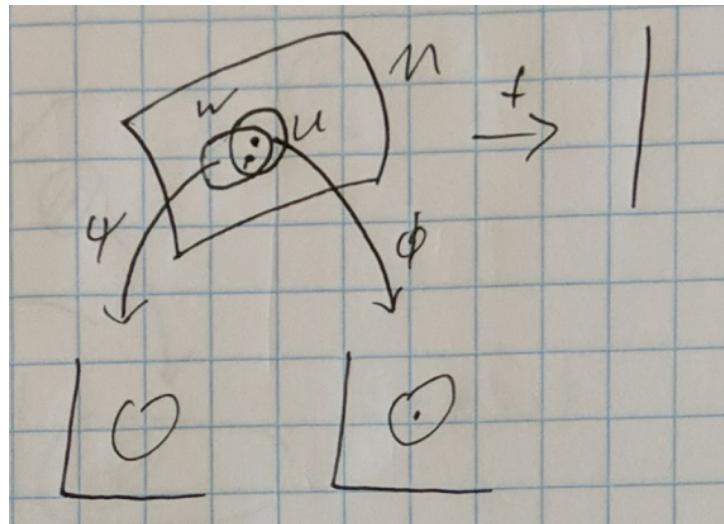
Examples

- $M \setminus \text{open ball}.$
- The upper half space.

Definition-Lemma: Smooth Function

A function $f : M \rightarrow \mathbb{R}$ is smooth at $p \in M$ if either of the following equivalent conditions is satisfied

1. \exists a chart $(U, \phi) \ni p$ such that $f \circ \phi^{-1}$ is smooth at $\phi(p)$.
2. \forall a chart $(U, \phi) \ni p$ such that $f \circ \phi^{-1}$ is smooth at $\phi(p)$.



Where $f \circ \phi^{-1} = f \circ \psi^{-1}(\psi \circ \phi^{-1})$.

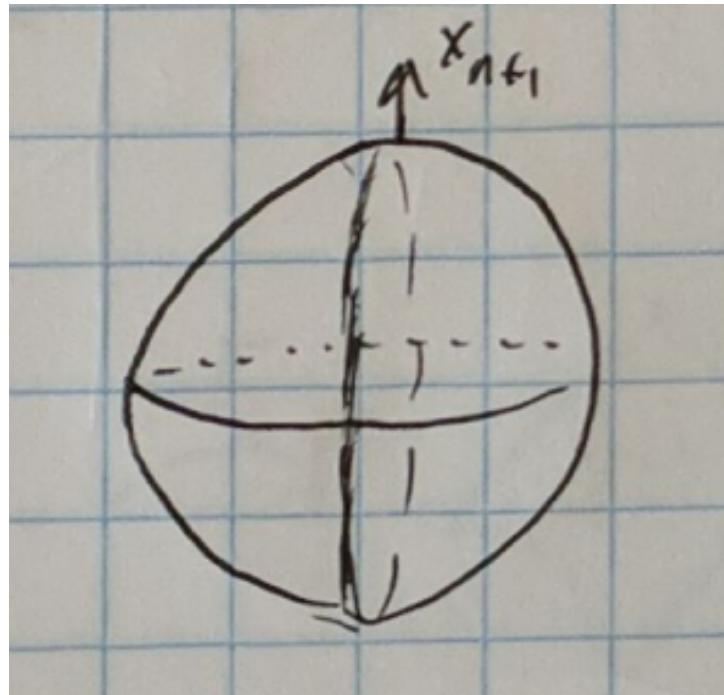
If the above hold for each $p \in M$, then f is smooth.

Remark

f smooth implies f is C^0

Exercise / Sketch

The height function on S^n is smooth.

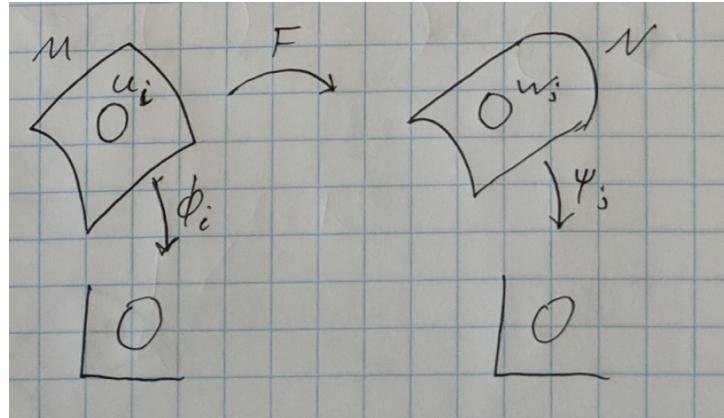


$$\phi : (x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_n).$$

$$f \circ \phi^{-1} = \pm \sqrt{1 - x_1^2 - \dots - x_n^2}.$$

Note that handling the equator requires examining the Eastern and Western hemisphere.
The stereographic projection leads to a simpler proof.

Definition: Smooth Function Between Manifolds



$F : M \rightarrow N$ is smooth if F is C^0 and one of the following equivalent conditions is satisfied

1. \exists an atlas $A \subset A_{\max}$ on M and an atlas $B \subset B_{\max}$ on N such that $\psi_j \circ F \circ \phi_i^{-1}$ is smooth on $F^{-1}(W_j) \cap U_i$.
2. The same as a., but for A_{\max} and B_{\max} .

Consider as an example $S^n \rightarrow \mathbb{RP}^n$.

Properties of Smooth Maps

$$C^\omega \implies C^\infty \implies C^r \implies C^{r-1} \implies C^1 \implies C^0.$$

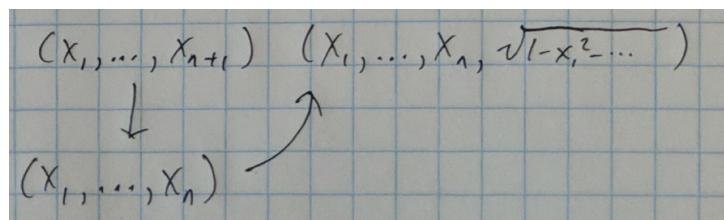
The sum and product of smooth functions is smooth.

Exercise

The composition of smooth maps is smooth.

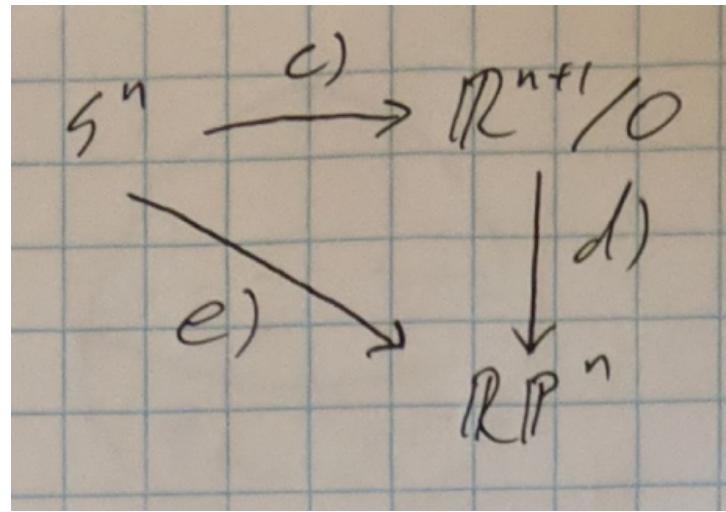
Examples

1. $M \times N \xrightarrow{pr} M$ is smooth.
2. $\underbrace{M \xrightarrow{(F_1, F_2)} N_1 \times N_2}_{\text{smooth}}$ if and only if F_1 and F_2 are smooth.
3. $S^n \hookrightarrow \mathbb{R}^{n+1}$ is smooth.



1. $\mathbb{R}^{n+1} \setminus 0 \rightarrow \mathbb{RP}^n$ is smooth with $[x_0 : \dots : x_n] \mapsto \left(\frac{x_0}{x_i}, \dots, \frac{\hat{x}_i}{x_i}, \dots, \frac{x_n}{x_i} \right)$.

2. $S^n \rightarrow \mathbb{RP}^n$.



Definition: Diffeomorphism

$$F: M \xrightarrow[A_{\max}]^{\text{diffeo}} N \text{ if } B_{\max}$$

- F is smooth.
- F is invertible.
- F^{-1} is smooth.

Previous Definition

$$F^{-1}(B_{\max}) = A_{\max}.$$

Exercise

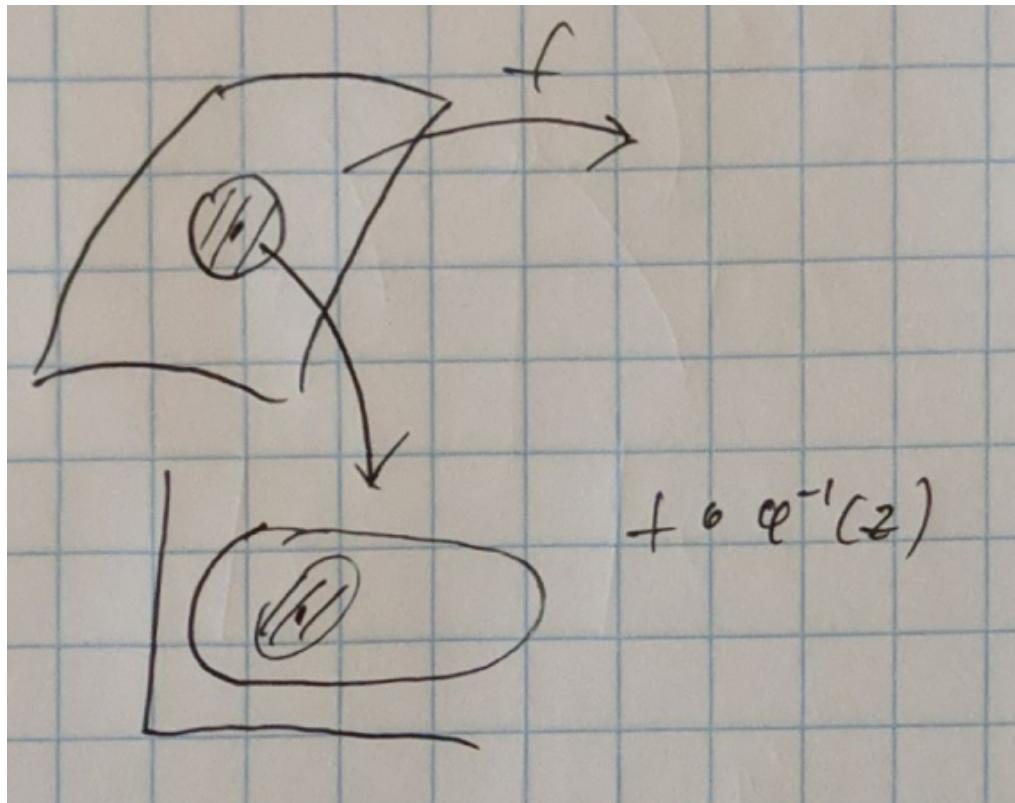
Prove that the definitions are equivalent.

Examples

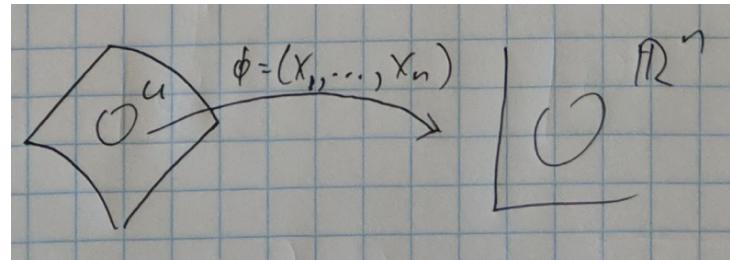
1. $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \xrightarrow{\text{diffeo}} \mathbb{R}$.
2. $x \mapsto x^3$, $\mathbb{R} \rightarrow \mathbb{R}$ is not a diffeomorphism.
3. $G_k(n) \leftrightarrow G_{n-k}(n)$ with $P \leftrightarrow P^\perp$.

Example 4

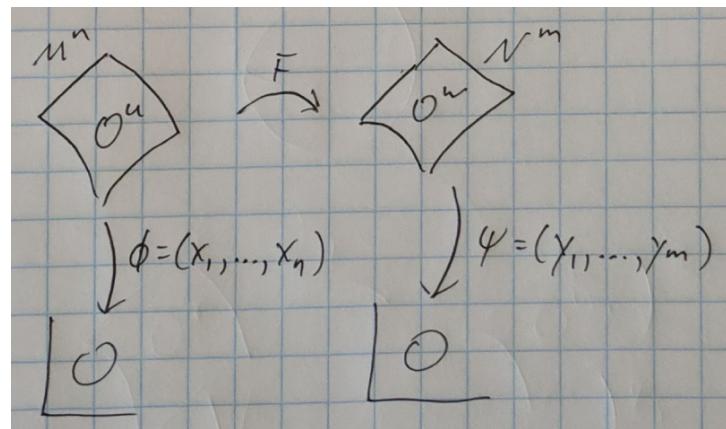
A compact, analytic manifold admits only constant smooth functions by the maximum modulus principle.



Example 5

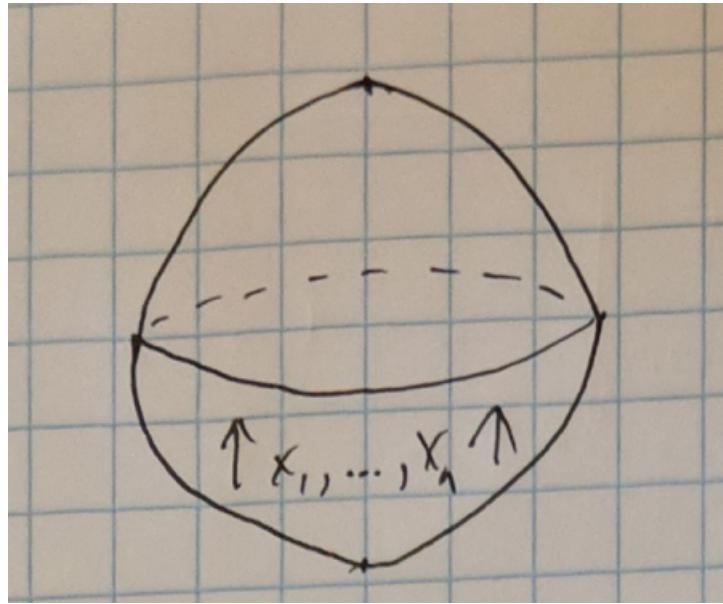


Where $\phi = (x_1, \dots, x_n)$ and each x_i is a real-valued function.



$$\psi \circ F \circ \phi^{-1} = (y_1(x_1, \dots, x_n), \dots, y_m(x_1, \dots, x_n)).$$

Then $S^n \hookrightarrow \mathbb{R}^{n+1}$



where $y_1 = x_1, \dots, y_n = x_n$, and $y_{n+1} = -\sqrt{1 - x_1^2 - \dots - x_n^2}$.

Example 6

$$\mathbb{R}^{n+1} \setminus 0 \xrightarrow{F} \mathbb{RP}^n.$$

Need to check that $\psi_j \circ F$ is smooth.

$$[t_0 : \dots : t_n] \xrightarrow{\psi_0} \left(\frac{t_1}{t_0}, \dots, \frac{t_n}{t_0} \right) \text{ with } U_0 : t_0 \neq 0$$

October 8, 2024

Questions

Question 1

Given M smooth and $x \neq y \in M$, does there exist $f \in C^r(M)$ such that $f(x) = 0$ and $f(y) = 1$.

Question 2

Given $K \subset U \subset M$ with K compact and U open and $g : K \xrightarrow{C^r} \mathbb{R}$, does there exist a C^r extension f of g on M such that $\text{supp } f \subset U$.

Definition: Partitions of Unity

Let W_i be a locally finite open cover.

A partition of unity subordinated to W_i , is a collection of functions $f_i : M \xrightarrow{C^r} \mathbb{R}$ satisfying

- $0 \leq f_i \leq 1$
- $\text{supp } f_i \subset W_i$
- $\sum f_i \equiv 1$

Definition: Refinement

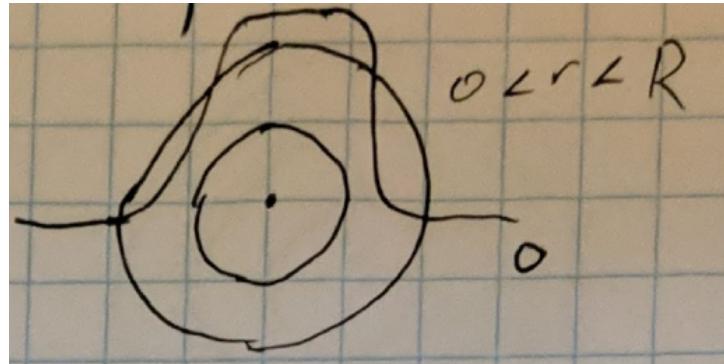
Given covers U_j and W_i , U_j is a refinement if for each j we may find i such that $U_j \subset W_i$.

Theorem

There exists a partition of unity subordinated to W_i .

Lemma 1

Take $B(r) \subset B(R) \subset \mathbb{R}^n$.



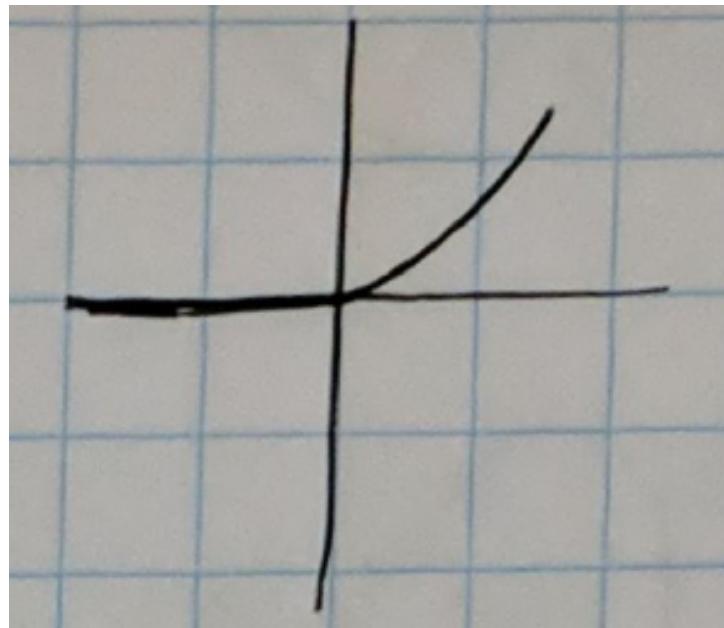
Then there exists $g \in C^\infty$ such that

- $0 \leq g \leq 1$
- $g|_{\overline{B(r)}} \equiv 1$
- $\text{supp } g \subset B(R)$

Proof

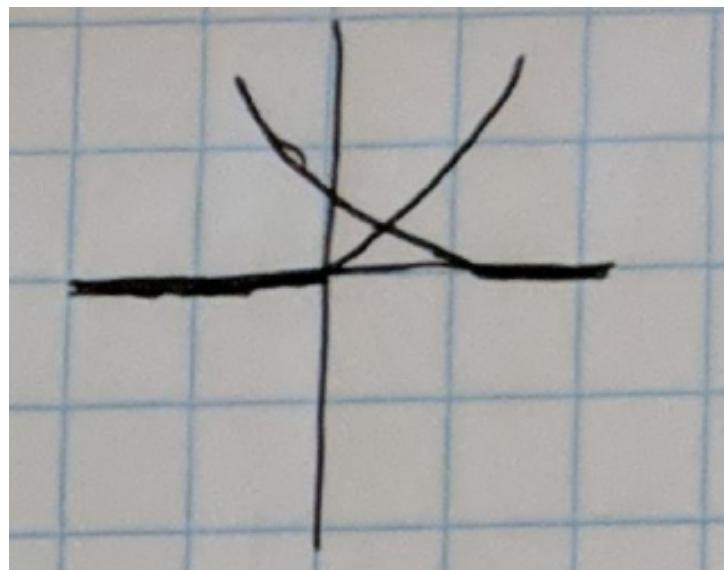
Take

$$h_0(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$



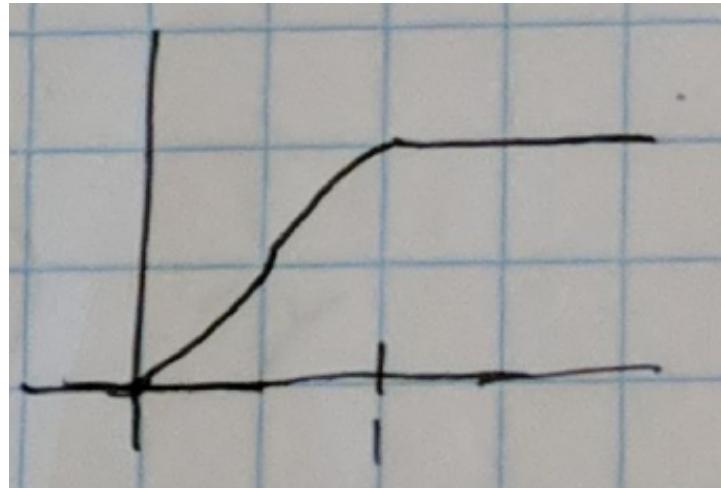
- Exercise: show that $h_0 \in C^\infty(\mathbb{R})$ (Hint: show that derivatives agree at zero from the right.)

Then take $h_1(x) = h_0(x) \cdot h_0(1-x)$.

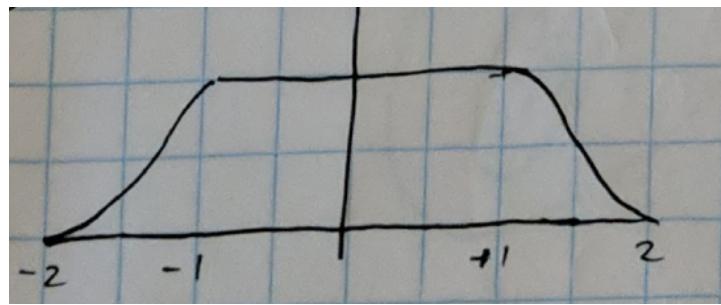


Then define

$$h_2(x) = \frac{1}{\int_{-\infty}^{\infty} h_1(t) dt} \int_{-\infty}^x h_1(t) dt$$



Finally, we define $h(x) = h_2(x+2) \cdot h_2(2-x)$.



Then $g(x) = h(||x||)$ satisfies our requirements.

Lemma 2

There exists a refinement (U_j, ϕ_j) by coordinate charts such that

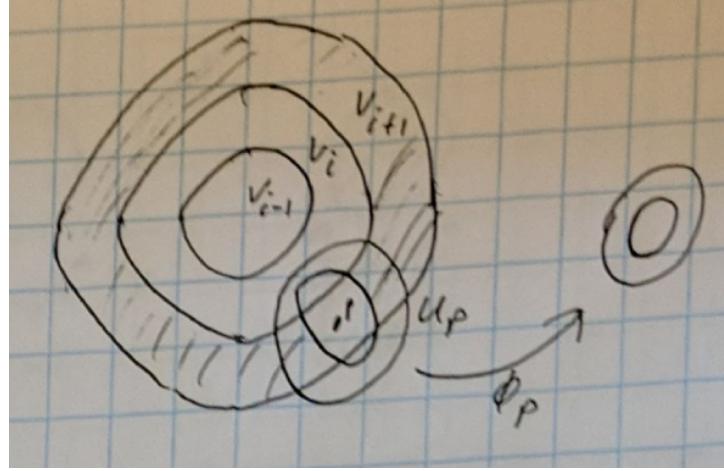
1. U_j is locally finite.
2. $\phi_i : U_j \xrightarrow{\text{diffeo}} B^n(2)$.
3. $\phi_j^{-1}(B(1))$ is also a cover.

Proof

There exists a compact exhaustion of M , $C_1 \subset C_2 \subset C_3 \subset \dots$ where $\bigcup C_i = M$.

There exists also an open exhaustion by precompact open sets $\emptyset = V_0 \subset V_1 \subset V_2 \subset \dots$ where $\overline{V_i} \subset V_{i+1}$.

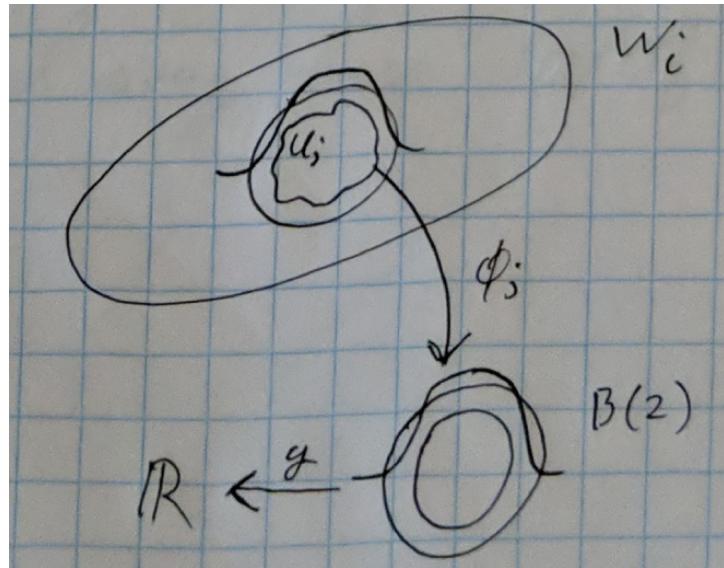
Then define $V_i := \text{nbhd}(C_i \cup \overline{V}_{i-1})$.



Take $A_i = \overline{V}_{i+1} \setminus V_i$ compact, $p \in A_i$. Then we have a map $U_p \xrightarrow{\phi_p} B(2)$.

- $U_p \subset W_i$ for some i . (Refinement)
- $U_p \cap V_{i-1} = \emptyset$. (Locally Finite)
- There exists a finite subcover such that $\phi_p^{-1}(B(1))$ is also an open cover.

Proof of Theorem



Set $g_j = g \circ \phi_j \in C^r$, extended to 0 outside of U_j .

- $\text{supp } g_j \subset U_j$
- $\forall p \in M, \exists g_j(p) = 1 \neq 0 \implies \sum g_j > 0$
- $\text{supp } g_j$ is locally finite.

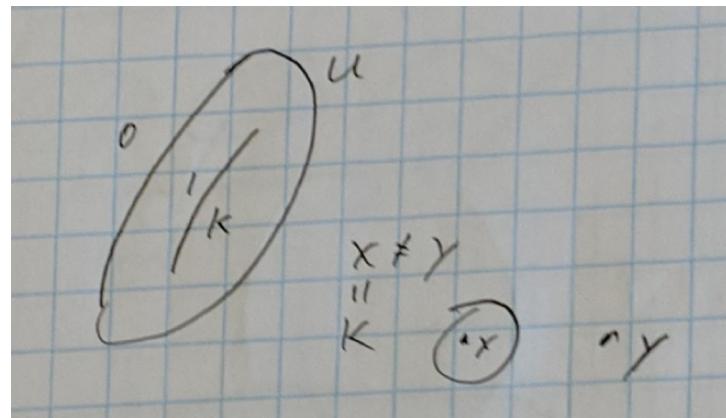
Then we have

$$f_j = \frac{g_j}{\sum g_j}$$

Corollary 1

For $K \subset U$, K compact and U open, there exists $h \in C^r(M)$ such that

- $h|_K \equiv 1$
- $\text{supp } h \subset U$



Proof

Take $U_0 = U$ and $U_1 = M \setminus K$.

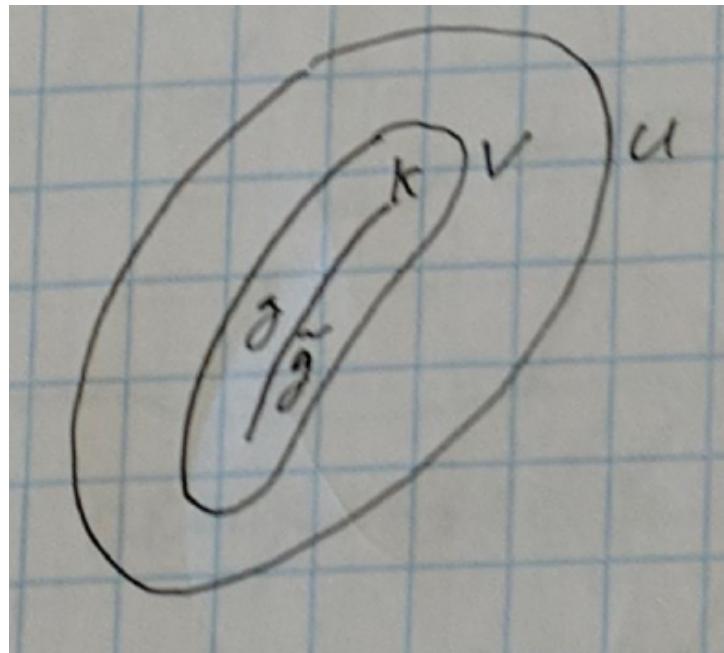
Then there exists a partition of unity f_0 and f_1 where $f_0 + f_1 = 1$. Therefore f_0 has support in U and f_1 has support in $M \setminus K$ this occurs if and only if $f_1|_K = 0$.

Corollary 2

For $K \subset U$, K compact and U open, and $g : K \xrightarrow{C^r} \mathbb{R}$, there exists an extension $f : M \rightarrow \mathbb{R}$ of g : $f|_K = g$ such that $\text{supp } f \subset U$.

Proof

For $g \in C^r(K)$, there exists a neighborhood V of K and a function $\tilde{g} : V \rightarrow \mathbb{R}$ such that $\tilde{g}|_K = g$.



By corollary 1, there exists $h \in C^r(M)$ such that

- $\text{supp } h \subset V$
- $h|_K = 1$

Therefore $f = h \cdot \tilde{g}$.

Corollary 3

There exists $f : M \xrightarrow{C^r} \mathbb{R}$ bounded from below and proper.

Definition: Proper Function

f is proper if and only if $f^{-1}(K)$ is compact for K compact.

Proof

See Textbook.

Consequence

Take $\{x : f(x) \leq C_i\} =: E_i$ as $i \rightarrow \infty$. Then we get a compact exhaustion $E_1 \subset E_2 \subset E_3 \subset \dots$
e.g. $f(x) = \|x\|^2$.

October 10, 2024

Algebra = Analysis (Geometry)

Take M to be either a compact metrizable space ($A = C^0(M)$) or a copact manifold ($A = C^\infty(M)$).

$$M \leftrightarrow A$$

Let $I \subset A$ be an ideal and take $V(I) = \{x : f(x) = 0, \forall f^n \in I\}$.

Take also $Y \subset M$ closed and consider $I_Y = \{f : f|_Y = 0\}$ which is also an ideal.

$$\begin{array}{c} Y \rightarrow I_y \\ V(I) \leftarrow I \end{array}$$

Denote $I_x = \{f : f(x) = 0\}$.

Theorem

- I_x is a maximal ideal.
- Every maximal ideal is of this form, and x is unique.

$$M \leftrightarrow \text{Maximal Ideals of } A$$

Proof

Maximal Ideal

Take I_x and $g \notin I_x$.

We want to show that g with I_x generates A .

Then take $f \in A$ defined as $f = h + ag$ for some $h \in I_x$.

Since $g \notin I_x$, $g(x) \neq 0$. Take $a = \frac{f(x)}{g(x)}$ and define $h := f - ag$.

Then $f(x) - \frac{f(x)}{g(x)}g(x) = 0 \in I_x$.

Every Maximal Ideal

Let I be a proper ideal such that $V(I) \neq 0$, $\exists x$ such that $f(x) = 0, \forall f \in I$.

Maximal $\implies V(I) = \{x\}$.

By contradiction, assume not: $\forall x \in M, \exists f_x \in I, f_x(x) \neq 0$.

It follows that $f_x|_{U_x} \neq 0$ where $U_x \ni x$ is a neighborhood from an open cover.

Then, by compactness, we have a finite subcover

$I \ni f_i = f_{x_i} \neq 0$ on U_i .

$I \ni \underbrace{\sum f_i^2}_{g>0} > 0$ on M . But then $1 = g^{-1}g \in I$.

Uniqueness

$$I_{x_1} = I_{x_2} \iff x_1 = x_2$$

(\iff) is obvious.

(\implies) $x_1 \neq x_2$ implies that there exists $f \in A$ such that $f(x_1) = 1$ and $f(x_2) = 0$.

Then $f \in I_{x_2}$ while $f \notin I_{x_1}$ implies that $I_{x_1} \neq I_{x_2}$.

Reading the Topology / Smooth Structure

We have a correspondence

$$\text{closed sets} \leftrightarrow \text{ideals}$$

Algebra Homomorphisms

Take $\phi : A \rightarrow \mathbb{R}$. Then $\ker \phi$ is a maximal ideal, and

$$\begin{aligned} A/I_x &\xrightarrow{\delta_x} \mathbb{R} \\ x &\mapsto \pm f(x) \end{aligned}$$

Then

$$\begin{aligned} M &\leftrightarrow \text{Alg. Hom. } A \rightarrow \mathbb{R} \\ 1 &\mapsto 1 \\ A > 0 &\mapsto \text{pos. \#} \end{aligned}$$

Counterexample

Take instead $M = \mathbb{R}$. Claim: there exist maximal ideals other than I_x .

Proof

Take J to be the ideal of all compactly supported functions such that $J \subset M$ for some maximal ideal M . However, $\forall x \in \mathbb{R}$ there exists a compactly supported function $f(x) \neq 0$. So $M \neq I_x$.

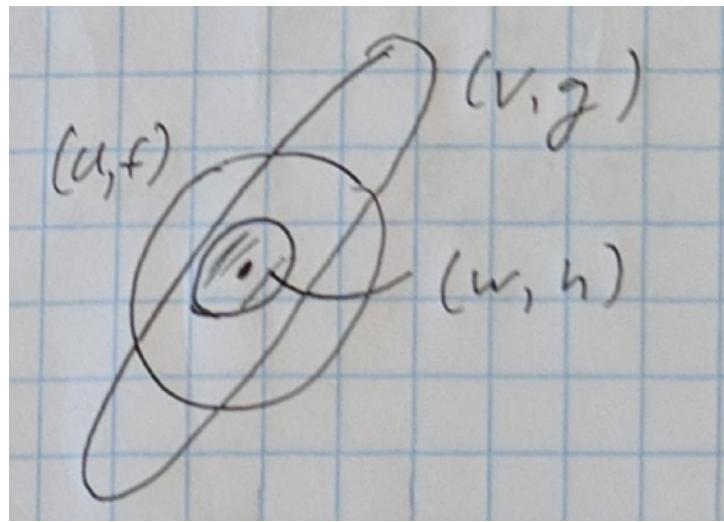
Functorial Considerations

Take $F : M \rightarrow N$, M and N compact. Then

$$\begin{aligned} F : M &\rightarrow N \\ C^{0/\infty}(M) &\xleftarrow{F^A} C^0(N) \\ f \circ F &\leftrightarrow f \end{aligned}$$

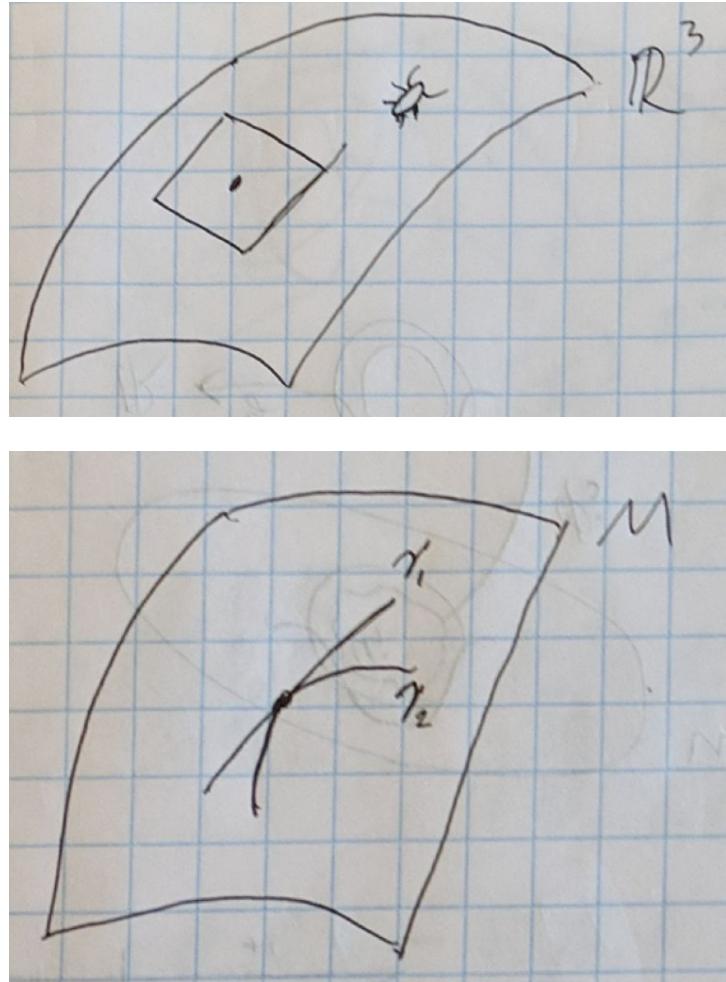
Definition: Germ

Take $M \ni P$.



Where $(U, f) \sim (V, g)$ if $f \equiv g$ on some neighborhood of p .

Definition: Tangent Spaces



With $\gamma_1 : (-\varepsilon, \varepsilon) \rightarrow M$, $\gamma_2 : (-\varepsilon, \varepsilon) \rightarrow M$, and $\gamma_1 \sim \gamma_2 \iff \gamma'_1(0) = \gamma'_2(0)$.

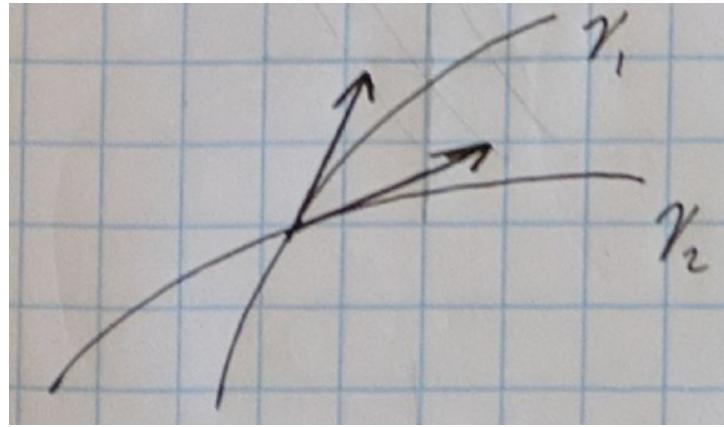
$\gamma(t) = ((x(t), y(t), z(t))$ and $\gamma'(t) = (x'(t), y'(t), z'(t))$.

The tangent space to M at p , written $T_p M$, is the set of equivalence classes.

Remarks: Tangent Vectors

Take $V \ni p$ a finite-dimensional vector space ($= \mathbb{R}^n$).

$$\begin{aligned} 0 &\mapsto p \\ \gamma_1, \gamma_2 &: (-\varepsilon, \varepsilon) \rightarrow V \quad (\text{germs}) \\ \gamma'_1 \sim \gamma'_2 &: \gamma'_1(0) = \gamma'_2(0) \end{aligned}$$



Write

$$\gamma'(0) = \lim_{t \rightarrow 0} \frac{\gamma(t) - \gamma(0)}{t}$$

$\gamma = (x_1, \dots, x_n)$ and $\gamma'(0) = (x'_1(0), \dots, x'_n(0))$.
 A tangent vector to V at p is an equivalence class $T_p V$.
 Claim: $T_p V$ is a vector space.

Operations

Take $p = 0$. Write

$$[\gamma_1] + [\gamma_2] = [\gamma_1 + \gamma_2]$$

$$\lambda[\gamma] = [\lambda\gamma]$$

When $p \neq 0$, instead

$$[\gamma_1] + [\gamma_2] = [\gamma_1 + \gamma_2 - p]$$

$$\lambda[\gamma] = [\lambda\gamma + (1 - \lambda)p]$$

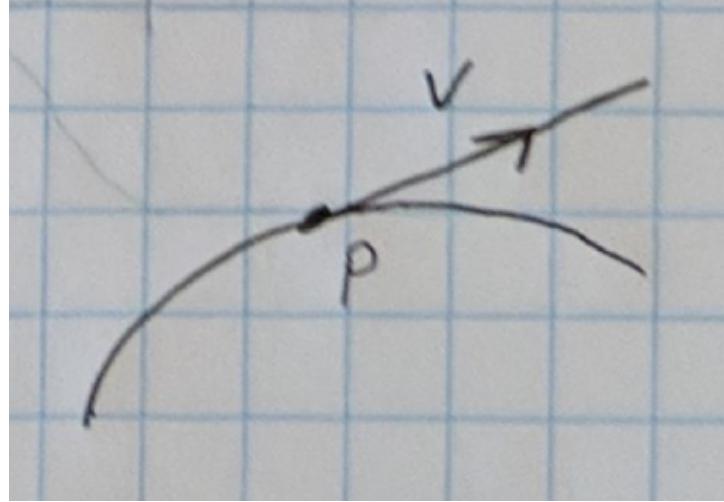
Claim: $T_p V$ is canonically isomorphic to V

$$[\gamma] \mapsto \gamma'(0)$$

$$[\gamma = (x_1, \dots, x_n)] \mapsto (x'_1(0), \dots, x'_n(0))$$

$$T_p V \rightarrow V$$

$$p + tv \leftrightarrow v$$



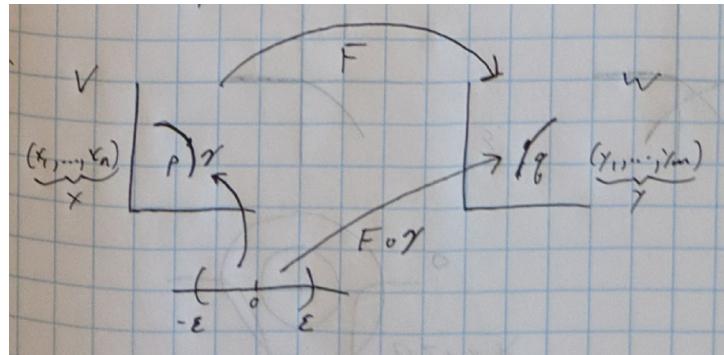
Proposition

Take

$$\begin{array}{ccc} V & \xrightarrow{C^r} & W \\ \mathbb{R}^n & & \mathbb{R}^m \\ p & \mapsto & q \end{array}$$

Then

$$\begin{aligned} V &\xrightarrow{\sim} W \\ DF_p = F_* : T_p V &\rightarrow T_q W \\ [\gamma] &\mapsto [F \circ \gamma] \end{aligned}$$



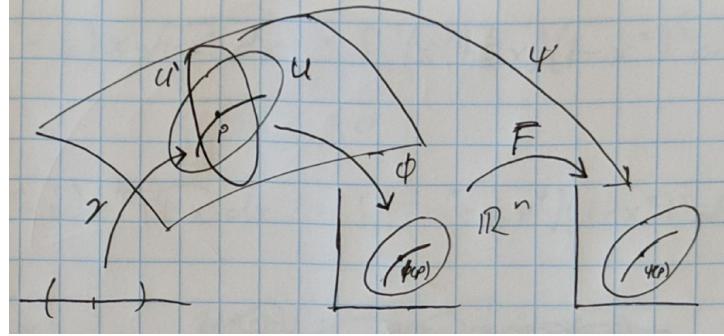
Then F is well-defined and linear.

$F = (F_1, \dots, F_m)$ and $F \circ \gamma(F_1(\gamma_1, \dots, \gamma_n), \dots, F_m(\gamma_1, \dots, \gamma_n))$.

We have that $[\gamma] = \gamma'(0)$ and $[F \circ \gamma] = \frac{d}{dt} F \cdot \gamma(t)|_{t=0}$. By chain rule,

$$\frac{d}{dt}(F \circ \gamma)|_{t=0} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{pmatrix} \begin{pmatrix} \gamma'_1(0) \\ \vdots \\ \gamma'_n(0) \end{pmatrix}$$

Tangent Space



$\gamma_1 \sim \gamma_2 \iff \phi \circ \gamma_1 \sim \phi \circ \gamma_2$ and $\gamma_1 \sim \gamma_2 \iff \psi \circ \gamma_1 \sim \psi \circ \gamma_2$, so

$$(\psi \circ \phi^{-1})(\phi \circ \gamma_1) \sim (\psi \circ \phi^{-1})(\phi \circ \gamma_2)$$

Now, take $\{[\gamma]\} = T_p M$. Claim: this is a vector space.

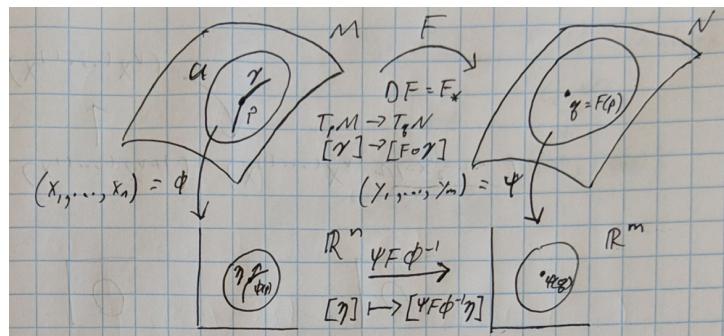
$$\begin{array}{ccc} T_p M & \xrightarrow{D\phi} & T_{\phi(p)} \mathbb{R}^n \\ & \searrow D\psi & \downarrow D(\psi \circ \phi^{-1}) \\ & & T_{\psi(p)} \mathbb{R}^n \end{array}$$

$$[\gamma_1] + [\gamma_2] = [\phi^{-1}(\phi \circ \gamma_1 + \phi \circ \gamma_2)].$$

October 15, 2024

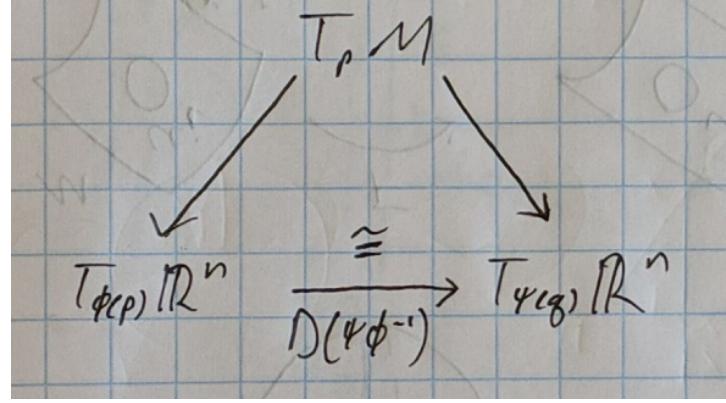
Recall: Tangent Space by Equivalence Classes

$\gamma : (-\varepsilon, \varepsilon) \rightarrow M, \gamma(0) = p, \gamma_1 \sim \gamma_2 \iff \phi \circ \gamma'_1 \sim \phi \circ \gamma'_2 \iff (\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0).$
 $T_p M = \{[\gamma]\} \xrightarrow{D\phi=\phi_*} T_{\phi(p)} \mathbb{R}^n = \mathbb{R}^n.$



Then for (F_1, \dots, F_m) , we have $D(\psi F \phi^{-1}) : \underbrace{T_{\phi(p)} \mathbb{R}^n}_{\mathbb{R}^n} \rightarrow \underbrace{T_{\psi(q)} \mathbb{R}^m}_{\mathbb{R}^m}$ where

$$D(\psi F \phi^{-1}) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$



Chain Rule

We have that $D(FG) = DF \circ DG$ since $(FG) \circ \gamma = F(G \circ \gamma)$.

Example

Take

$$f: \underbrace{M}_{(x_1, \dots, x_n)} \rightarrow \mathbb{R}$$

and $Df = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$.

Definition: Directional Derivatives

Take $v = [\gamma] \in T_p M$ and $f \in C^r(M)$.

The directional derivative is given by

$$L_v f = \frac{d}{dt} f(\gamma(t))|_{t=0} = \frac{d}{dt} \underbrace{(f \circ \phi^{-1})(\phi \circ \gamma)(t)}_{g \circ \eta}|_{t=0}$$

Then

$$\frac{d}{dt} g(\eta(t))|_{t=0} = \sum \frac{\partial g}{\partial x_i}(\phi) \eta'_i(0)$$

which is determined by $(\eta'_1(0), \dots, \eta'_n(0)) = \eta'(0)$.

Properties

$$L_v : C^\infty(M) \rightarrow \mathbb{R}$$

1. Linear over \mathbb{R}
2. $L_v(fg) = (L_v f)g(\phi) + f(\phi)(L_v g)$ (product rule)
3. Linear in v .

Derivations

The collection $\mathcal{D}_p = \{C^r(M) \rightarrow \mathbb{R} : (1) \text{ and } (2) \text{ hold}\}$ is called the derivations at p .

Algebraic Aside

$\delta_p : A \rightarrow \mathbb{R}$ given by $f \mapsto f(p)$ yields

$$D(fg) = Df\delta_p(g) + \delta_p(f)Dg$$

Theorem

$T_p M \rightarrow \mathcal{D}_p$ given by $v = [\gamma] \mapsto L_v$ is a linear isomorphism.

Recall: Germ

Take $(f, U) \ni p$ and $(g, V) \ni p$. Then $(f, U) \sim (g, V)$ if and only iff $f \equiv g$ on $W \subset U \cap V$. The equivalence classes of this relation are germs.

Hadamard's Lemma

Take $f \in C^r(\mathbb{R}^n, 0)$ on \mathbb{R}^n with $f(0) = 0$.

There exists $g_1, \dots, g_n \in C^{r-1}(\mathbb{R}^n, 0)$ such that

$$f(x) = \sum x_i g_i(x)$$

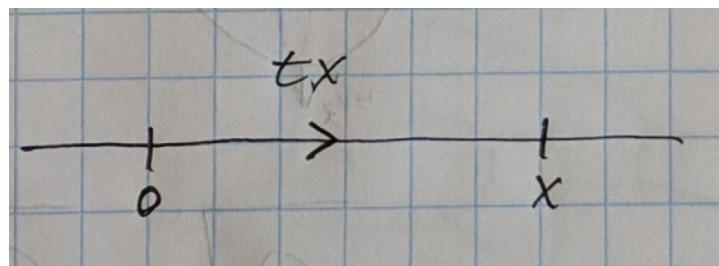
and $g_i(0) = \frac{\partial f}{\partial x_i}(0)$.

Example

For $n = 1$ and $f : \mathbb{R} \xrightarrow{C^r} \mathbb{R}$ with $f(0) = 0$. Then we have $f(x) = x \overbrace{g(x)}^{c^{r-1}}$ given by

$$g(x) = \begin{cases} \frac{f(x)}{x} & x \neq 0 \\ f'(0) & x = 0 \end{cases}.$$

Proof of Example



Take

$$\begin{aligned} \int_0^1 \underbrace{\frac{d}{dt} f(tx) dt}_{f'(tx) \cdot x} &= x \underbrace{\int_0^1 f'(tx) dt}_{g(x)} \\ &= f(1 \cdot x) - f(0 \cdot x) \end{aligned}$$

Proof of Lemma

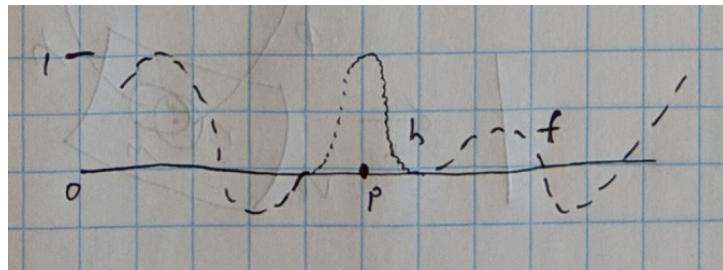
$$\int_0^1 \underbrace{\frac{d}{dt} f(tx) dt}_{\sum \frac{\partial f}{\partial x_i}(tx) \cdot x_i} = \sum x_i \underbrace{\int_0^1 \frac{\partial f}{\partial x_i}(tx) dt}_{g_i}$$

Lemma

For $D \in \mathcal{D}_p$, Df depends only on the germ of f .

Proof

Need to show that if $f \equiv 0$ near p , $Df = 0$.



Where $\text{supp } h \subset \{x : f(x) = 0\}$, $h(p) = 1$ and, consequently, $f \circ h = 0$.

So $D(0 = f \circ h)$, $D0 = Df \cdot h(p) + f(p)Dh$ and $0 = Df$.

Lemma

$D(k) = 0$ for k constant.

Proof

$1 \cdot 1 = 1 \implies D(1) = D(1 \cdot 1) = D(1) \cdot 1 + 1 \cdot D(1) = 2 \cdot D(1)$, so $D(1) = 0$.

Arbitrary constants follow from the fact that D is linear.

Proof: One-to-One

Let $L_v f = L_w f$ for all $f = x_i$, then

$$\sum v_i \frac{\partial f}{\partial x_i} = \sum w_i \frac{\partial f}{\partial x_i}$$

for each f and $v_i = w_i$.

Lemma

$$\text{Der}(C^r(m)@p) = \text{Der}(C^r(M, p))$$

Proof

Take $M = \mathbb{R}^n$ and $p = 0$.

Given D , we need $v = \sum v_i \frac{\partial}{\partial x_i}$ with

$$L_v f = \sum v_i \frac{\partial f}{\partial x_i} = Df$$

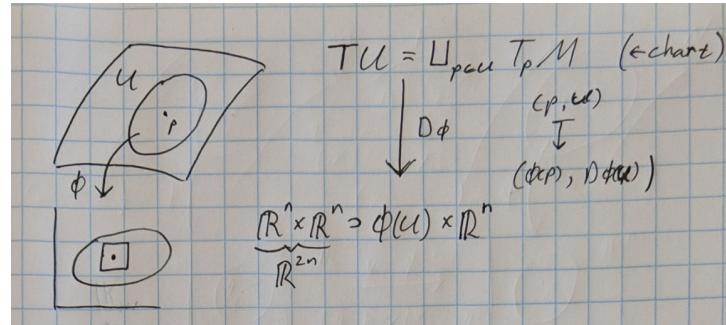
for every f .

By Hadamard's lemma, write $f = \sum x_i g_i(x)$. Then

$$\begin{aligned} Df &= \sum (Dx_i)g_i(0) + \overbrace{x_i(0)}^{=0} Dg_i \\ &= \sum \underbrace{(Dx_i)}_{v_i} \frac{\partial f}{\partial x_i}(0) \end{aligned}$$

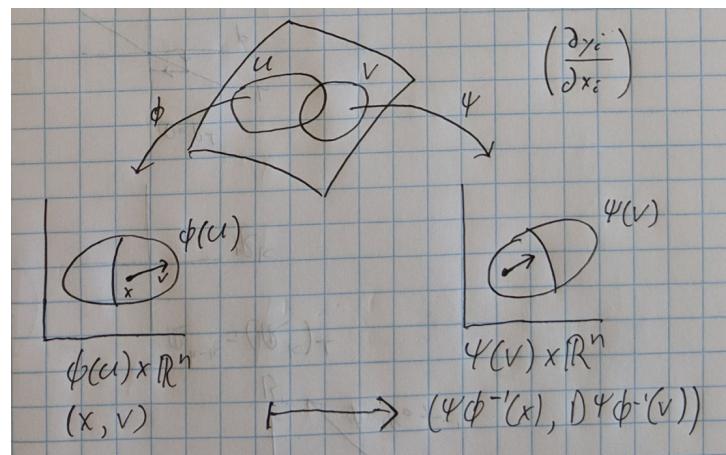
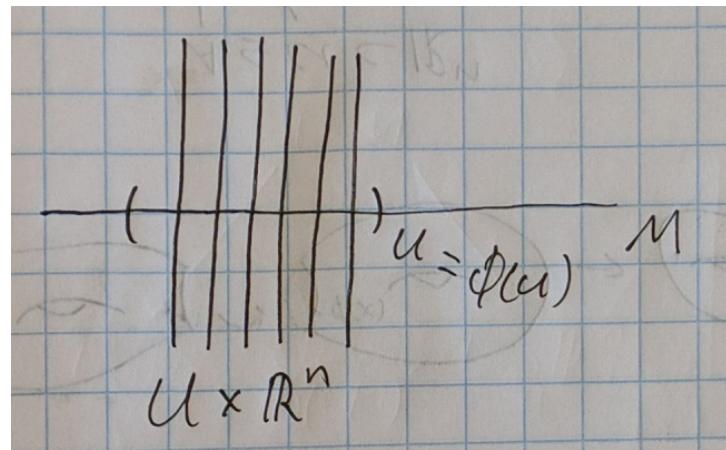
That is, since $v_i = L_v x_i$, we have $v_i = Dx_i$.

Definition: Tangent Bundle



For every $p \in M$, we have $T_p M$.

The tangent bundle $TM = \bigsqcup_{p \in M} T_p M$.



When M is C^r , TM is C^{r-1} .

Chapter 7 of Lee (Lie Groups) will not be covered in class, but is highly recommended reading.

Preliminary Definition: Vector Field

Take $M \xrightarrow{C^\infty} TM$ by $p \xrightarrow{\nu} \nu(p) \in T_p M$.

Space of Vector Fields

Write $\mathfrak{X}(M)$ to be the collection of all C^∞ vector fields.

- This is a module over $C^\infty(M)$.
- $\mathfrak{X}(M)$ acts on $C^\infty(M)$ by $v \mapsto L_v$.

Smooth

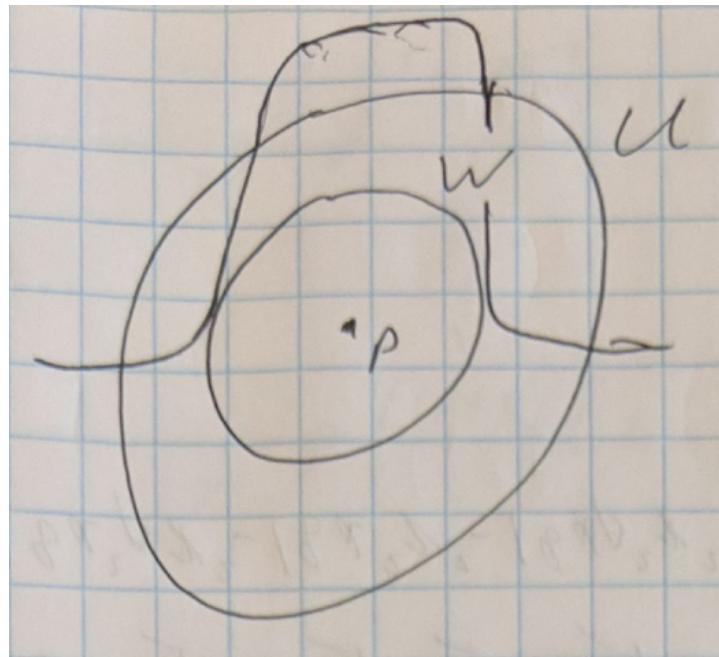
1. v is C^∞
2. In local coordinates, for $p \in U$ and $(U, \phi) = (x_1, \dots, x_n)$ with $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ a basis in $T_p M$.

$$v = \sum_{\substack{i \\ \text{functions} \\ \text{on } U}} v_i(x) \frac{\partial}{\partial x_i}$$

3. $f \in C^\infty(M) \implies L_v f \in C^\infty$

2 Implies 3

$$L_v f = \sum v_i \frac{\partial f}{\partial x_i}$$



With $\phi|_W \equiv 1$, $\text{supp } \phi = U$, $x_i \phi \in C^\infty(M)$, and $x_i \phi|_W \equiv W_i$.

Then $L_\nu(x_i \phi) \underset{\text{on } W}{=} \nu_i$.

Definition: Lie Algebra

Take A a vector space equipped with (a Lie bracket) $[\cdot, \cdot] : A \times A \rightarrow A$ such that

- $[a, b] = -[b, a]$ (Skew Symmetric)
- $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$ (Jacobi Identity)

Example 1

Take A to be an algebra and define $[a, b] = ab - ba$. Then if A is associative, it satisfies the Jacobi identity.

- $gl(n)$
- $so(n)$ (skew symmetric matrices)
 - $(ab - ba)^T = b^T a^T - a^T b^T = ba - ab = -[a, b]$
- $su(n)$ ($A^T = A^\dagger$)

Theorem:

The space of vector fields $\mathfrak{X}(M)$ is a Lie algebra:

- $\forall V, W \in \mathfrak{X}(M), \exists! U \in \mathfrak{X}(M)$ such that $L_U f = L_W L_V f - L_V L_W f$. Write $U = [V, W]$.

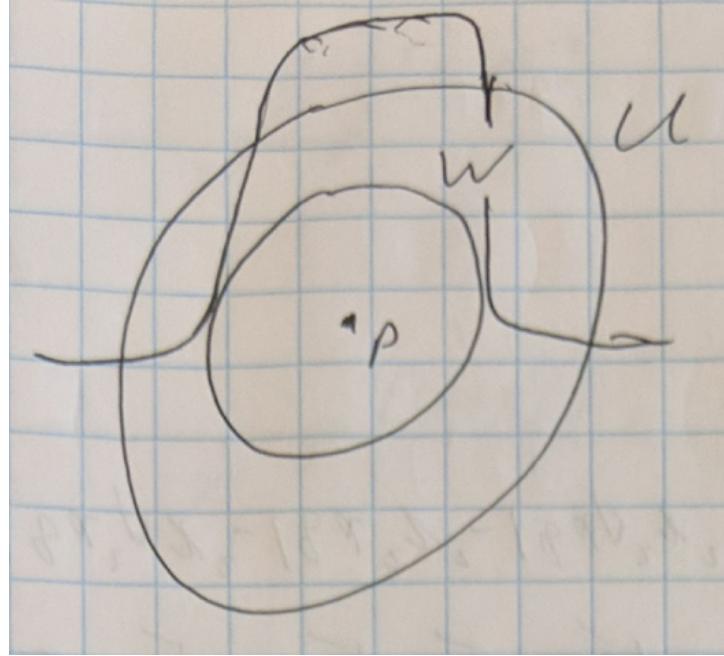
Lemma

$L_V f = L_W f$, $\forall f$ implies $V = W$.

Proof

$$V = \sum \nu_i \frac{\partial}{\partial x_i} \stackrel{?}{=} \sum w_i \frac{\partial}{\partial x_i} = W$$

Pick $p \in U$. We want to find $\nu_i(p) = w_i(p)$.



With $\phi|_W \equiv 1$, $\text{supp } \phi \subset U$ and $f = x_i \phi \in C^\infty(M)$.

$$L_V f = L_W f$$

$$\sum v_j \underbrace{\frac{\partial(x_i \phi)}{\partial x_j}}_{\delta_{ij}} = \sum w_j \underbrace{\frac{\partial(x_i \phi)}{\partial x_j}}_{\delta_{ij}}$$

on W . Therefore $v_i = w_i$ on W .

Variant

For W_i an open cover, $L_V f = L_W f$ for all $f \in C^\infty(W_i)$ implies that $V = W$.

Compute

$$L_V L_W f = L_V \left(\sum w_j \frac{\partial f}{\partial x_j} \right) = \sum v_i \frac{\partial w_j}{\partial x_i} \frac{\partial f}{\partial x_j} + \sum v_i w_j \frac{\partial^2 f}{\partial x_i \partial x_j}$$

$$L_W L_V f = \sum w_i \frac{\partial v_j}{\partial x_i} \frac{\partial f}{\partial x_j} + \sum w_i v_j \frac{\partial^2 f}{\partial x_j \partial x_i}$$

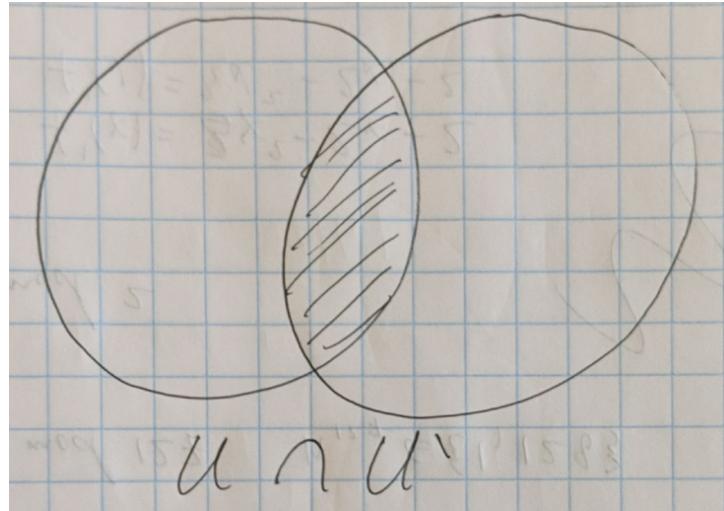
$$L_V L_W f - L_W L_V f = \sum_{i,j} \left(v_i \frac{\partial w_j}{\partial x_i} - w_i \frac{\partial v_j}{\partial x_i} \right) \frac{\partial f}{\partial x_j}$$

$$u = \sum_{i,j} \left(v_i \frac{\partial w_j}{\partial x_i} - w_i \frac{\partial v_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}$$

Therefore $L_V L_W f - L_W L_V f = L_U f$.

Remark

Consider $U \ni u$ and $U' \ni u'$



Properties

- Lie algebra: Skew symmetric and satisfying the Jacobi identity
- Product rule: $[V, fW] = (L_V f)W + f[V, W]$.

Example

Let V be a finite dimensional vector space (e.g. \mathbb{R}^n).

Recall that $T_p V \cong V$ by $[p + tv] \mapsto v$. Then $TV = V \times V$ (p, v).

Take $A \in \text{End}(V)$ given by

$$v(x) = Ax = \sum v_i \frac{\partial}{\partial x_i} = \begin{pmatrix} & a_{ij} & \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Take also $w(x) = Bx$. Then $[V, W] = -(AB - BA)x = -[A, B]$.

Exercise

Given the example with W constant, determine $[V, W]$.

Theorem (Midterm Problem)

Take $A = C^\infty(M)$ and the derivations $D \in \text{Der}(A)$ with $D : A \xrightarrow{\text{lin.}} A$ over \mathbb{R} such that $D(fg) = Df \cdot g + f \cdot Dg$. There exists a linear isomorphism $\mathfrak{X}(M) \xrightarrow{\cong} \text{Der}(A)$ given by $v \mapsto L_v$.

Lemma

Take $D \in \text{Der}(A)$. Then $D_p \in \mathcal{D}_p$ where $D_p f := (Df)(p)$.

$$D_p(fg) = (D_p f)g(p) + f(p)(D_p g)$$

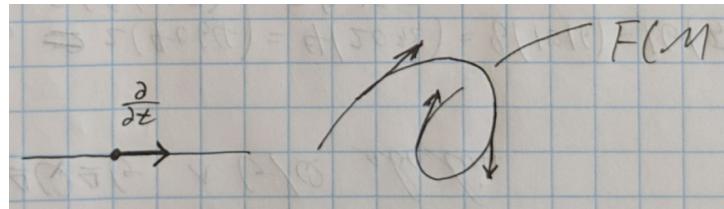
We know from above that $\mathcal{D}_p \cong T_p m$. Therefore $L_v f = Df$.

Definition: Push Forward

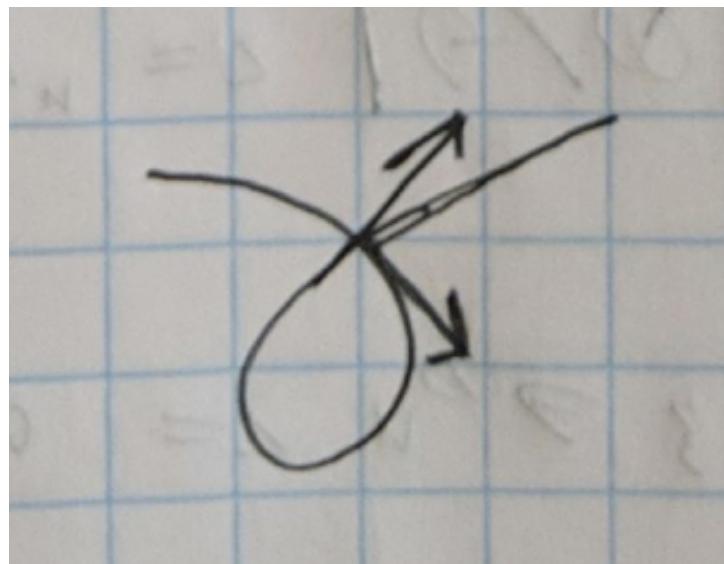
Take $F : M \rightarrow N$ which gives rise to $F_* : T_p M \rightarrow T_{F(p)} N$ (equivalent to $F_* : \mathcal{D}_p \rightarrow \mathcal{D}_{F(p)}$) given by $[\gamma] \mapsto [F \circ \gamma]$. We see that $(F_* D)(g) := D(g \circ F)$.

You Cannot Push Forward Vector Fields

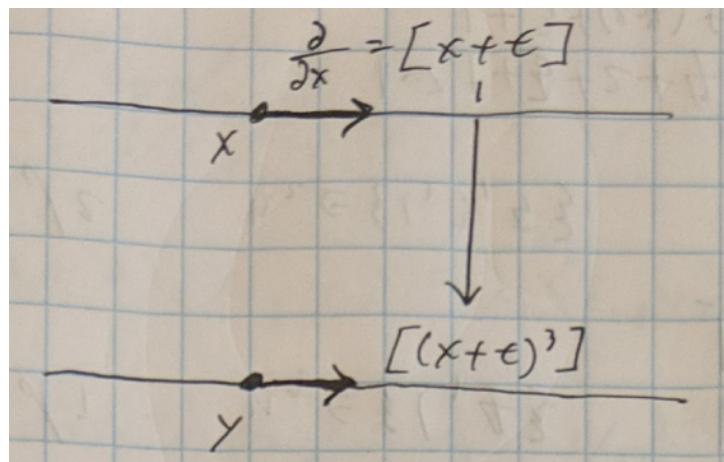
- if F is not surjective



- if F is not injective



- it is possible that $F_* V$ would fail to be smooth. Take $(F_* v)(y) = 3y^{2/3} \frac{\partial}{\partial y}$



Remark

Take a diffeomorphism $F : M \rightarrow N$.

Then $F : \mathfrak{X}(M) \rightarrow \mathfrak{X}(N)$ is given by $F_*[V, W] = [F_*V, F_*W]$.

October 22, 2024

Dimension of Manifolds

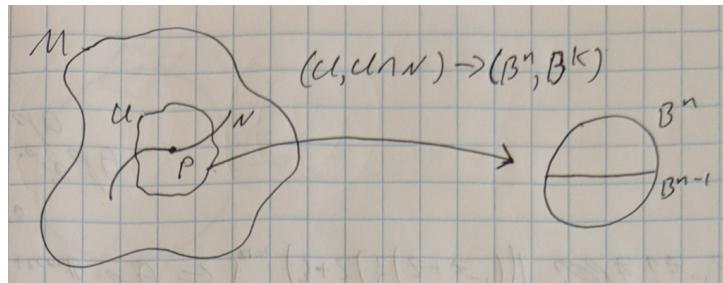
Take connected manifolds M^m and N^n with diffeomorphism $M^m \xrightarrow[G]{F} N^n$. If $p \in M^m$ and $q = F(p)$, then

$$T_p M \xrightarrow[DG]{DF} T_q N$$

is linear and $\dim T_p M = \dim T_q N$.

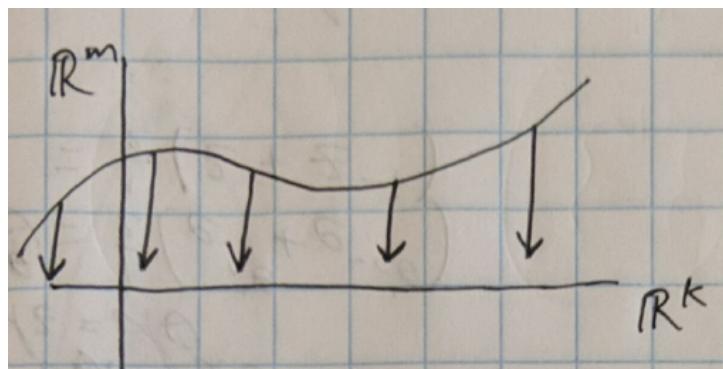
Definition: Submanifold

$N \subseteq M^n$ is a submanifold if $\forall p \in N$, there exists a neighborhood $U \ni p$ in M and a diffeomorphism $U \rightarrow B^n$ which satisfies $U \cap N \rightarrow B^k = B^n \cap \mathbb{R}^k$.



Example 1

Consider $F : \mathbb{R}^k \rightarrow \mathbb{R}^n$ and its graph $\text{Graph}(F) = \{(x, F(x))\}$. This is a submanifold in $\mathbb{R}^k \times \mathbb{R}^n$.



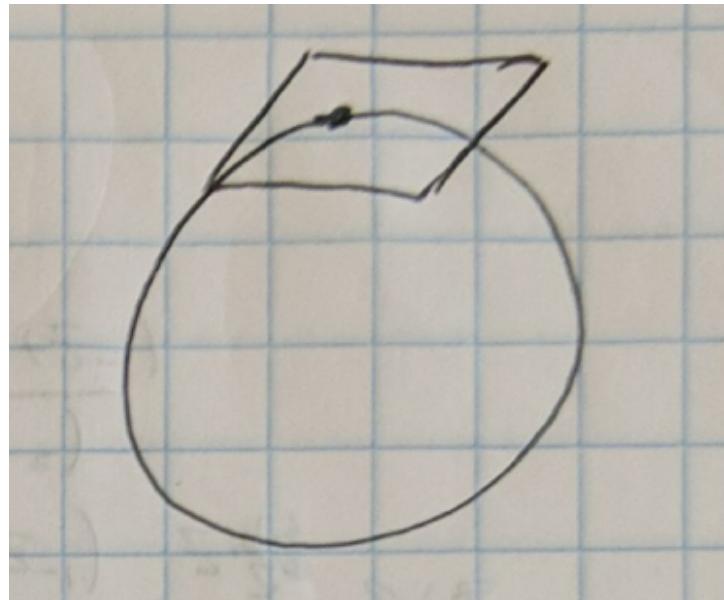
$\mathbb{R}^k \times \mathbb{R}^m \rightarrow \mathbb{R}^k \rightarrow \mathbb{R}^m$ given by $(x, y) \mapsto (x, y - f(x))$.

Example 2

Take $F : X \rightarrow Y$, X, Y manifolds. Then $\text{Graph}(F) \subset X \times Y$ is a submanifold.

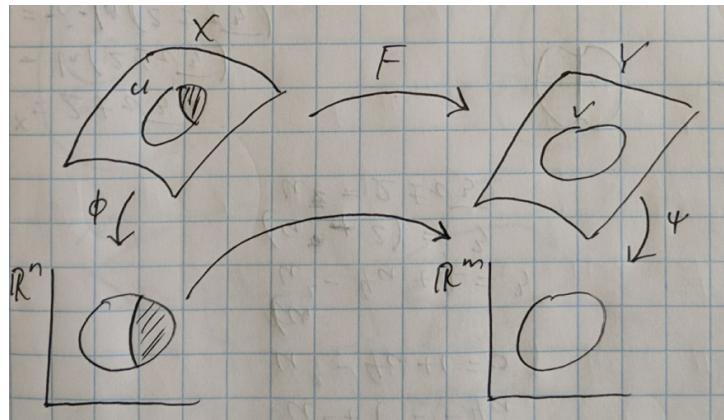
Example 3

$S^1 \subset \mathbb{R}^2$ or $S^{n-1} \subset \mathbb{R}^n$



Example 4

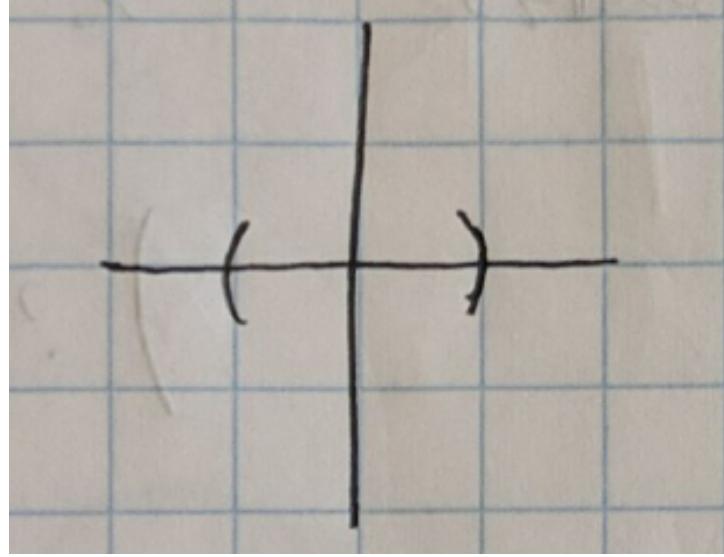
A graph is always a submanifold.



$\text{Graph}(F|_{U \cap F^{-1}(V)}) \subset U \times V$ implies $\text{Graph}(\psi F p^{-1}|_{\dots}) \subset \phi(U) \times \psi(V)$.

Example 5

$(-1, 1) \times 0 \in \mathbb{R}^2$.



Example 6

$$\mathbb{RP}^n \subseteq \mathbb{CP}^n.$$

Definition: Regular Point

Let M, N be manifolds with $p \in M$, $q = F(p) \in N$ and take $F: M \rightarrow N$ with $DF: T_p M \rightarrow T_q N$. p is a regular point if DF is onto; q is a regular value if each $p \in F^{-1}(q)$ is a regular point. Otherwise, they are called critical points or critical values.

Remark

If $\dim M < \dim N$, then all points in M are critical and all values in $F(M)$ are critical (while $N \setminus F(M)$ are regular).

Remark

For $f: M \rightarrow \mathbb{R}$ with $df: T_p M \rightarrow \mathbb{R} \cong T_{f(p)}\mathbb{R}$. Locally, $df = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right) = \nabla f$. Then p is regular if and only if $df \neq 0$. Equivalently, p is regular if and only if there exists $v \in T_p M$ such that $L_v f \neq 0$.

Proof

$$L_v f = df(v) = \sum \frac{\partial f}{\partial x_i} v_i.$$

Theorem:

Let q be a regular value. Then $F^{-1}(q)$ is a submanifold.

Example 1

$f: \mathbb{R}^n \rightarrow \mathbb{R}$ given by $f(x) = \sum \lambda_i x_i^2$, $\lambda_i \neq 0$. Then $f^{-1}(1)$ is a submanifold.

Proof

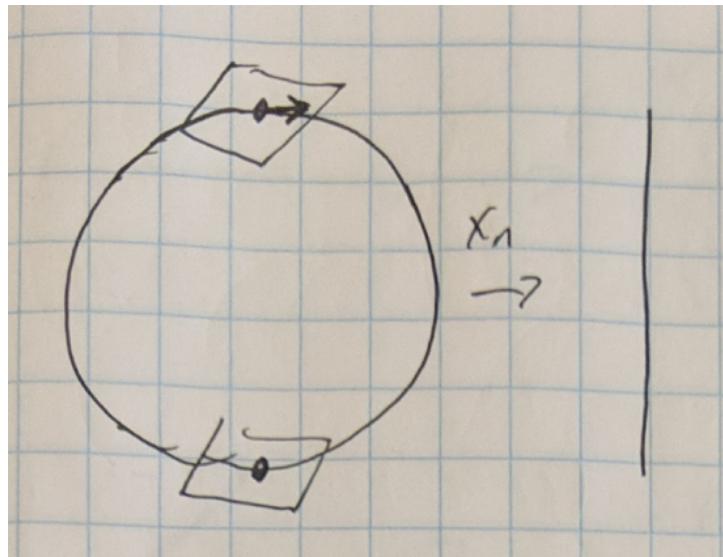
x is a regular point when $f(x) \neq 0$ ($\nabla f \neq 0$).

$$v = \sum x_i \frac{\partial}{\partial x_i} = x$$

$$L_v f = \sum \underbrace{2 \lambda_i x_i}_{\frac{\partial f}{\partial x_i}} \underbrace{x_i}_{v_i} = 2f(x) \neq 0$$

Example 2

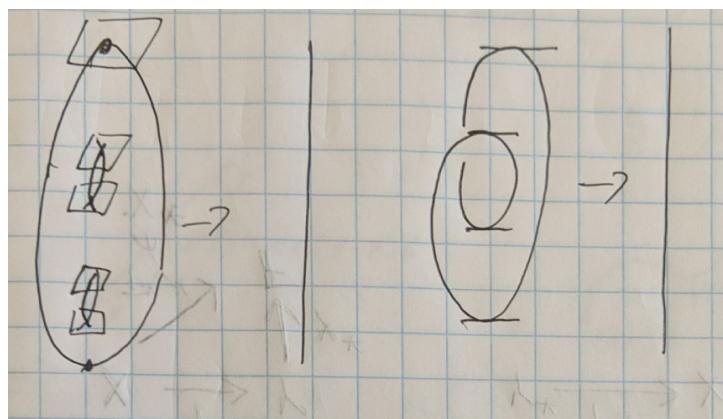
$$S^{n-1} \subset \mathbb{R}$$



$f = x_n : S^{n-1} \rightarrow \mathbb{R}$ the height function, so

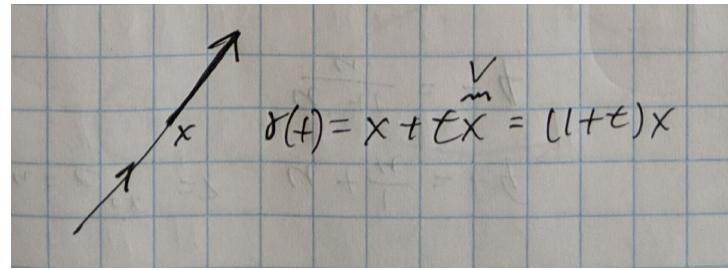
$$L_v f = L_v x_n = v_1 \frac{\partial x_n}{\partial x_1} + \cdots + v_{n-1} \frac{\partial x_n}{\partial x_{n-1}} + \underbrace{v_n \frac{\partial x_n}{\partial x_n}}_{=0}$$

The same follows for all projective maps.



Theorem:

$$f(\lambda x) = \lambda^k f(x)$$



$$\begin{aligned} L_v f &= \frac{d}{dt} f((1+t)x)|_{t=0} \\ &= \underbrace{\frac{d}{dt} (1+t)^k}_{k} |_{t=0} f(x) \end{aligned}$$

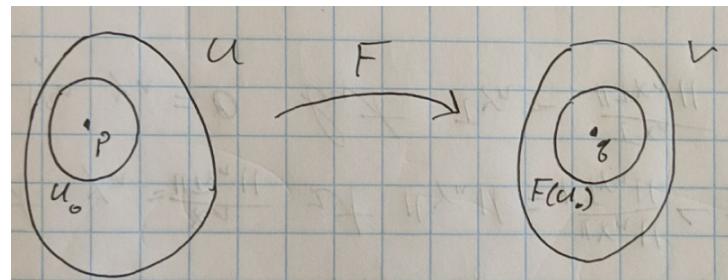
Theorem: Inverse Function Theorem

Take $U, V \subseteq \mathbb{R}^n$ open and a map $F: U \rightarrow V$ with $p \in U$ and $q = f(p) \in V$. Then take

$$\{DF: T_p U \rightarrow T_q V\} = \frac{\partial F_i}{\partial x_j}$$

such that $\text{Rank } DF = n$ is an isomorphism.

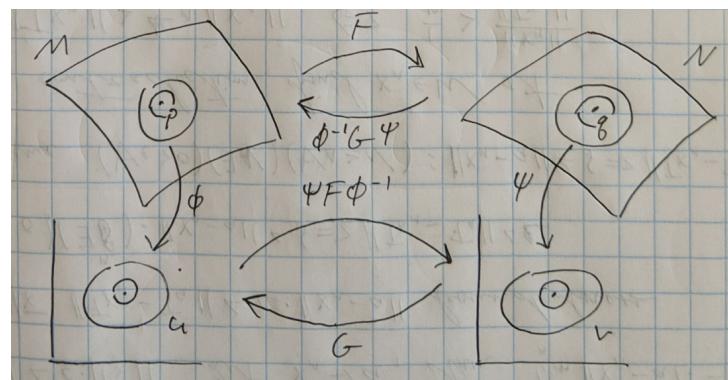
Then there exists $U_0 \ni p$ such that $U_0 \xrightarrow{F} F(U_0)$ is a diffeomorphism.



Corollary

If $F: M^n \rightarrow N^n$ with $DF: T_p M \xrightarrow{\cong} T_q N$ ($\text{Rank } DF = n$).

Then there exists a neighborhood $U_0 \ni p$ such that $F: U_0 \rightarrow F(U_0)$ is a diffeomorphism.



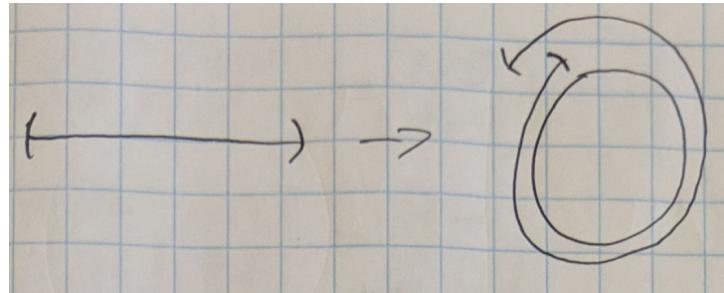
$$D(\psi F \phi^{-1}) = D\psi Df D\phi^{-1}.$$

Local Diffeomorphism

Diffeomorphisms are local but local diffeomorphisms are not necessarily diffeomorphisms.

Example 1

Take \mathbb{R} or any interval longer than 1 and map them to S^1 by $t \mapsto e^{2\pi i t}$.



Example 2

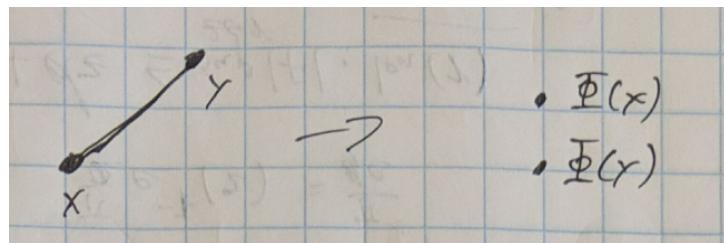
$S^1 \rightarrow S^1$ given by $z \mapsto z$

Example 3

$S^n \rightarrow \mathbb{RP}^n$.

Contraction Mapping Principle

Take $\Phi : X \xrightarrow{C^0} X$ with X complete such that $d(\Phi(x), \Phi(y)) < c \cdot d(x, y)$ with $c \in (0, 1)$. Then there exists a unique fixed point $\Phi(p) = p$.



Example

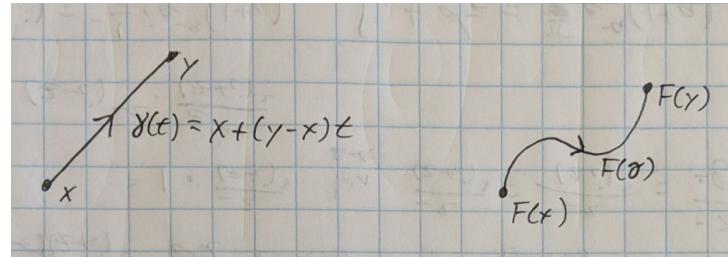
Take $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\|DF_x\| < c < 1$.

Recall that the operator norm for an operator $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is given by

$$\|A\|_{\text{op}} = \sup_{v \neq 0} \frac{\|Av\|}{\|v\|} = \sup_{\|v\|=1} \|Av\|$$

Then F is a contraction by c .

Proof



$$d(x, y) = ||y - x|| = \int_0^1 ||\gamma'(t)|| dt$$

Then

$$\begin{aligned} d(F(x), F(y)) &\leq \int_0^1 ||F'(\gamma(t))|| dt \\ &\leq \int_0^1 ||DF_{\gamma(t)} \cdot \gamma'(t)|| dt \\ &\leq \int_0^1 \frac{c}{||DF_{\gamma}(t)||} \cdot ||\gamma'(t)|| dt \\ &\leq c \cdot \int_0^1 ||\gamma'(t)|| dt \\ &\leq c \cdot d(x, y) \end{aligned}$$

Proof

Write $\{x_k\} = \{\Phi^k(x)\}$.

Claim: $\{x_k\}$ is Cauchy which implies that $x = \lim_{n \rightarrow \infty} x_n$. Then

$$\Phi(x) = \lim_{n \rightarrow \infty} \Phi(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = x$$

Uniqueness follows from the observation that for fixed points $d(x, y) \leq d(x, y)$ implies $x = y$.

Proof that Sequence is Cauchy

For $n \leq n+k$,

$$d(x_n, x_{n+k}) \leq d(\Phi(x_{n-1}, \Phi(x_{n+k-1})) \leq cd(x_{n-1}, x_{n+k-1}) \leq c^n d(x_0, x_k) \leq c^n L \cdot (1 + \dots + c^{k-1}) \leq \frac{L}{1-c} c^n \xrightarrow{n \rightarrow \infty} 0$$

since

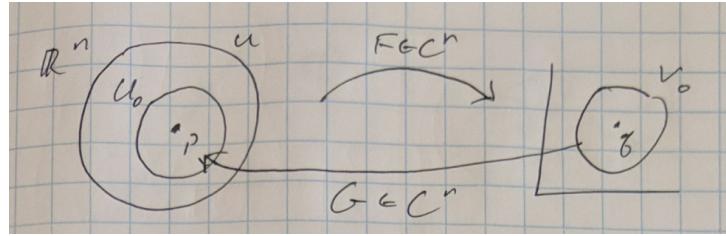
$$d(x_0, x_k) \leq \underbrace{d(x_0, x_1)}_L + \underbrace{d(x_1, x_2)}_{c \cdot L} + \dots + \underbrace{d(x_{k-1}, x_k)}_{c^{k-1} \cdot L}$$

October 24, 2024

Recall: Contraction Mapping Theorem

$\Phi : X \rightarrow X$ complete, then there exists a unique fixed point x such that $\Phi(x) = x$.

Recall: Inverse Function Theorem



$DF_P : T_p \mathbb{R}^n \rightarrow T_q \mathbb{R}^n$ implies there exist U_0 and V_0 such that

$$F : U_0 \xrightarrow[G]{} V_0$$

is a diffeomorphism with $F \circ G = I$ and $G \circ F = I$.

Remarks

Assume that $p = 0 = q$ and that $DF_0 = I$.

Then $F = I + \phi$ and $D\phi_0 = 0$ (contracting). Look for $G = I + g$, then

$$\begin{aligned} F \circ G &= I \\ (I + \phi) \circ (I + g) &= I \\ -\phi \circ (I + g) &= g \end{aligned}$$

which gives us our fixed point. Then $\Phi : g \mapsto -\phi \circ (I + g)$ with $g : \overline{B(r)} \rightarrow \mathbb{R}^n$ is a continuous function on a Banach space giving

$$\underbrace{\overline{B(r)}}_{r \text{ adjustable}} \xrightarrow[C^0]{ } \underbrace{B(R)}_{R \text{ fixed}}$$

on a subset with $\|g\| = \sup_{x \in B(r)} g(x)$ and $\|g\| < R$.

Claim

Φ is contracting.

The contraction mapping theorem implies there exists a unique fixed point $\Phi(g) = g$ where $-\phi \circ (I + g) = g$.

Observations

F is C^1 and DF invertible implies G is C^1 .

$F \circ G = I$, $F(G(x)) = x$, $(F' \circ G) \circ G' = I$.

So $G' = (F' \circ G)^{-1}$.

Remark

$$\begin{array}{ccc} \underbrace{U \times \mathbb{R}^n}_{TU} & \xrightarrow{F_*} & \underbrace{\mathbb{R}^n \times \mathbb{R}^n}_{T\mathbb{R}^n} \\ (F, DF) & & \\ & \xleftarrow{G_*} & \\ & (G, DG) & \end{array}$$

Then F being C^2 implies F_* is C^1 and invertible, and G_* is C^1 .

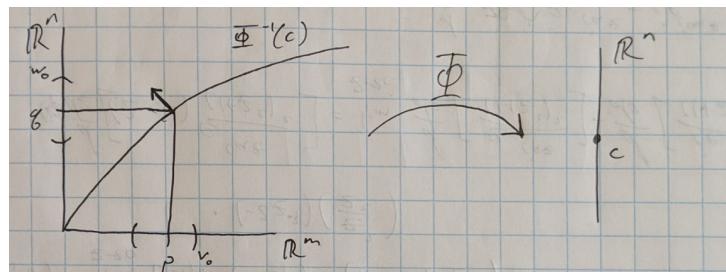
$$F_*(x, v) = (F(x), DF_x v)$$

Theorem: Implicit Function Theorem

Take

$$(x, y) \in \mathbb{R}^m \times \mathbb{R}^n \supset U \ni (p, q) \xrightarrow[C^r]{\Phi} \mathbb{R}^n \ni c$$

with $\frac{\partial \Phi}{\partial y}$ an invertible $n \times n$ matrix.



$$(\nabla \Phi = \left(\frac{\partial \Phi}{\partial x_1}, \dots, \frac{\partial \Phi}{\partial x_n}, \frac{\partial \Phi}{\partial y} \neq 0 \right)).$$

Then there exist $V_0 \ni p$ and $W_0 \ni q$ such that in $V_0 \times W_0$ $\Phi^{-1}(c)$ is the graph of $g : V_0 \xrightarrow{C^r} W_0$ such that $\Phi(x, g(x)) = c$.

Remark

IFT \iff IFT.

Proof that Inverse Implies Implicit

Define $F : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^m \times \mathbb{R}^n$ by $(x, y) \mapsto (x, \Phi(x, y))$ (i.e. $(p, q) \mapsto (p, c)$).

$$DF = \begin{matrix} & \mathbb{R}^m & \mathbb{R}^n \\ \mathbb{R}^m & \left[\begin{array}{c|c} I & * \\ \hline 0 & \frac{\partial F}{\partial y} \end{array} \right] \end{matrix}$$

invertible. Applying the inverse function theorem, $F \circ G = I$ implies $G = (\text{id}, g)$ with $g : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ where $g(\cdot) = g(\cdot, c)$. So $\Phi(x, g(x)) = c$.

Theorem: Regular Value Theorem

Take $F : M^{n+m} \rightarrow N^n \ni c$ with c a regular value.

Then $F^{-1}(c)$ is a submanifold.

Proof

We know that each $p \in F^{-1}(c)$ is a regular point.

Then in local coordinates $DF_p : T_p M^{n+m} \rightarrow T_c N^n$ is given by an $n \times (m+n)$ matrix of rank n .

$$\begin{matrix} & x & y \\ \mathbb{C} & | & | \frac{\partial F}{\partial y} | \\ & | & | \frac{\partial F}{\partial y} | \end{matrix}$$

then locally F^{-1} is a graph and therefore a submanifold.

Definition: Immersion

A map $F : N^n \rightarrow M^m$ is an immersion if $DF_p : T_p N \rightarrow T_{f(p)} M$ is one to one at every point ($m \geq n$).

Definition: Embedding

The map $F : N^n \rightarrow M^m$ is an embedding if it is an immersion and a homeomorphism on its image.

Definition: Submersion

The map $F : N^n \rightarrow M^m$ is a submersion if DF_p is onto ($m \leq n$).

Examples

Example 1

A local diffeomorphism is both an immersion and a submersion.

Example 2

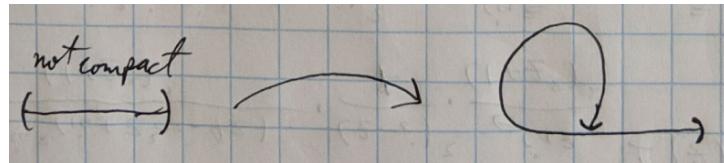
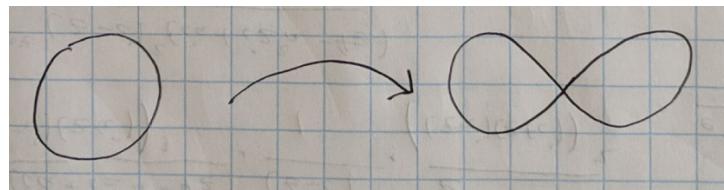
$\gamma : \mathbb{R} \rightarrow M$ is an immersion if and only if $\gamma'(t) \neq 0$ for all points t .

Example 3

For N compact, F is an embedding if and only if it is one-to-one and an immersion.

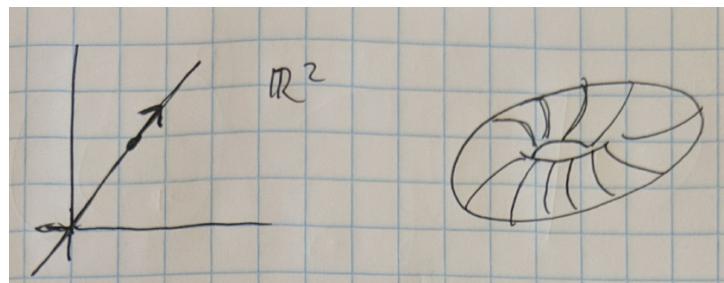
Counter-example 1

Immersions but not embeddings.



Counter-example 2

$$\begin{aligned}\mathbb{R} &\rightarrow \mathbb{R} \times \mathbb{R} \rightarrow S^1 \times S^1 = \mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2 \\ t &\mapsto (t, at)\end{aligned}$$



Can be given by composition of $(e^{2\pi i t}, e^{2\pi i a t})$.

Remark

Consider $A : V \rightarrow V$ with equivalence given by $BAB^{-1} \sim A$.

If $A : V \rightarrow W$ then the equivalence given by $B_1 A B_2^{-1} \sim A$.

Consider $f : \mathbb{R}, 0 \rightarrow \mathbb{R}$ with the assumption that $f(0) = 0, f'(0) = 1$. Then there exists a change of coordinates $x = x(t)$ such that $f(x(t)) = t$.

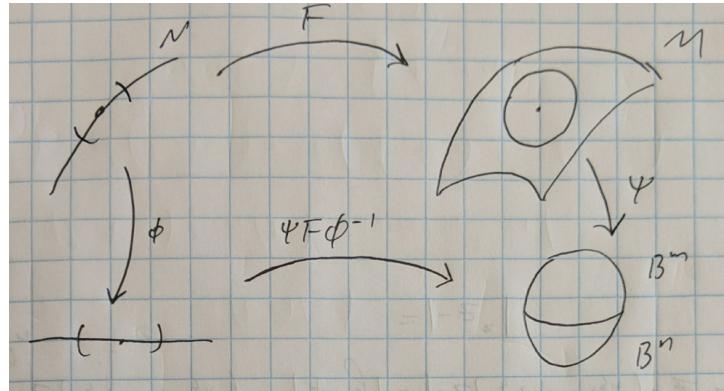
By the inverse function theorem, there exists g satisfying $f(g(t)) = t$.

Consider $f(z)$ complex analytic with $f(0) = 0$ and $f'(0) = 1$.

Local Normal

Let $F : p \in N^n \rightarrow M^m \ni q$ be an immersion.

Then there exist local coordinates x_1, \dots, x_n near p and y_1, \dots, y_m near q such that $y_1 = x_1, \dots, y_n = x_n, y_{n+1} = 0, \dots$



October 29, 2024

Recall: Local Normal Form Theorem for Immersions

For $F : N^n \rightarrow M^m$ an immersion where $\text{rank}(F) = n \leq m$ and $p \in N$, there exists a coordinates x_1, \dots, x_n near p and y_1, \dots, y_m near $F(p)$ such that

$$\begin{aligned} y_1 &= x_1 \\ &\vdots \\ F: \quad y_n &= x_n \\ y_{n+1} &= 0 \\ &\vdots \\ y_m &= 0 \end{aligned}$$

IMAGE 1

Proof

$$\begin{aligned} p \in V &\xrightarrow{F} U \ni q \\ p \rightarrow q &\in F(p) \end{aligned}$$

$\text{rank } DF_p = n \leq m$, for example.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

IMAGE 2

$$\begin{array}{ll} y_1 = z_1 & y_1 = x_1 \\ \vdots & \vdots \\ y_n = z_n & y_n = x_n \\ \text{and} & y_{n+1} = z_{n+1} - F_{n+1}(z_1, \dots, z_n) \\ & \vdots \\ & y_m = z_m - F_m(z_1, \dots, z_m) \end{array}$$

Theorem: Local Normal Form Theorem for Submersions.

Assume that $F : N^n \rightarrow M^m$ is a submersion such that $\text{rank } DF = m \leq n$.

Then we may take coordinates x_1, \dots, x_n and y_1, \dots, y_m such that $y_1 = x_1, \dots, y_m = x_m$.
The procedure is similar to the previous theorem.

Remark

Immersions look locally like linear subspaces.

Submersions look locally like projections.

Theorem: Constant Rank Theorem

Assume that $\text{rank } DF = k$ for a constant k .

Then for coordinates x_1, \dots, x_n and y_1, \dots, y_m , we can have $y_1 = x_1, \dots, y_k = x_k, y_{k+1} = 0, \dots$

Homework

Take f a germ at 0 on \mathbb{R}^n and take $df_0 \neq 0$ ($\nabla f_0 \neq 0$).

Then there exists a coordinate system y_1, \dots, y_n such that $f(y) = f(0) + y_1$.

That is, for (x_1, \dots, x_n) there exists a diffeomorphism $(f \circ \phi)(x_1, \dots, x_n) = f(0) + x_1$.

Note that we do not have control over the x coordinates, so the previous theorems will not work.

Remark

Take $K \xrightarrow{G} N \xrightarrow{F} M$.

If G, F are submersions, then FG is a submersion.

If G, F are immersions, then FG is an immersion.

Theorem

Let $F : N \rightarrow M$ be an embedding.

Then $F(N)$ is a submanifold. Consequently, $F : N \rightarrow F(N)$ is a diffeomorphism.

IMAGE 3

If N is a submanifold of M , then $N \hookrightarrow M$ is an embedding.

Example

Take $\mathbb{R} \rightarrow \mathbb{R}^2$ by $t \mapsto (0, t^3)$.

IMAGE 4

This is not even an immersion.

Example 2 (Homework)

For N the graph of $y = |x|$, there exists $F : \mathbb{R} \xrightarrow[C^\infty_{1-1}]{} \mathbb{R}^2$ such that $N = F(\mathbb{R})$.

IMAGE 5

Submanifolds

For any $q \in X$, there exists a neighborhood U and a diffeomorphism $\phi : (U, U \cap X) \rightarrow (W, W \cap \mathbb{R}^n)$.

IMAGE 6

Proof

And embedding is an immersion which is homeomorphic on $F(N)$

With F an immersion, we have

IMAGE 7

Since F is also a homeomorphism, there exists U' such that $U' \cap X = F(V)$.

Zero Measure Sets

Key Point

Zero measure sets are unambiguously defined.

Definition: Zero Measure Set

$X \subseteq M$ is a zero measure set if

- For every closed chart (U, ϕ) , $\phi(U \cap X) \subset \mathbb{R}^n$ has zero Lebesgue measure.
- There exists a coordinate atlant $(\{\phi_i\})$ such that $\phi_i(U_i \cap X)$ has zero measure.

Lemma

$\mathbb{R}^n \supset W \xrightarrow[C^1]{F} \mathbb{R}^n$ and $Y \in W$ where $m(Y) = 0$.

Then $F(Y)$ has zero measure.

Proof

IMAGE 8

We may cover Y by a countable collection of open sets U such that $\overline{U} \subset W$.

Then it suffices to show that $F(Y \cap U)$ has zero measure.

Then $\|DF\| \leq C$. Given $\varepsilon > 0$, we need a cover $F(Y)$ by open balls with total volume less than ε .

Pick $p \in Y$ and define $B_r(p) = \{z \in U : \|z - p\| \leq r\}$. Then $F(B_r(p)) \subset B_{Cr}(F(p))$.

IMAGE 9

For $p = 0 = F(p)$

IMAGE 10

$$\|F(z)\| \leq \text{length}(t \mapsto F(tz)) = \int_0^1 \left\| \frac{d}{dt} F(tz) \right\| dt \leq \int_0^1 \|DF\| \cdot \|z\| dt \leq Cr$$

Since Y has zero measure, it can be covered by balls $B_{r_i}(p_i)$ with total volume less than δ .

Therefore $F(Y)$ is covered by $B_{Cr_i}(F(p_i))$. Then the sum is bounded above by $C^n \delta$.

Take $\delta = \frac{\varepsilon}{C^n}$.

Definition: Smooth Measure

Let μ be a measure on M .

μ is smooth if

- for every coordinate chart $(U, \phi = (x_1, \dots, x_n))$, $\mu = f dx_1 \dots dx_n$ ($f > 0$).
- there exists a coordinate atlas (U_i, ϕ_i) such that the same is true.

Proposition

A smooth measure exists.

Proof

Take (U_i, ϕ_i) , $dx_1 \dots dx_n = \mu_i$, and the partition of unity h_i subordinated to U_i ($\text{supp } h_i \subseteq U_i$ and $\sum h_i \equiv 1$)
Then $h_i \mu_i$ is a measure on M . Then we may take $\sum h_i \mu_i$.

Proposition

The following three conditions are equivalent.

- X is a zero measure set
- there exists a smooth, positive μ such that $\mu(X) = 0$
- every smooth positive μ has $\mu(X) = 0$.

October 31, 2024

Theorem: Sard

Let $F: M^m \xrightarrow{C^r} N^n$ with $r > \max\{0, m - n\}$. The set of critical values has zero measure.

Remarks

There are two cases. $m \leq n$ permits C^1 . This is the easy case because there are no C^1 space filling curves. If $m > n$, then $r > m - n + 1$. This case is difficult.

Corollary

There exist regular values.

The set of regular values is everywhere dense.

Proof

For $M = [0, 1]$, $N = \mathbb{R}$, $n = m = 1$, and $f: [0, 1] \xrightarrow{C^r} \mathbb{R}$.

Separate the interval into segments I_i of length $1/k$.

IMAGE 1

Consider all intervals containing critical points.

Then f' is uniformly continuous. That is for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta \implies |f'(x) - f'(y)| < \varepsilon$.

Consider the critical values covered by intervals of total length $< \varepsilon$ and $f(I_i)$ containing critical points.

Choose $1/k < \delta$, then $|f'|_{I_i}| < \varepsilon$. So $|f(I_i)|\varepsilon \cdot 1/k$ and, consequently,

$$\sum_{I_i} |f(I_i)| < k \cdot \frac{k}{\varepsilon} = \varepsilon$$

Theorem: Brouwer Fixed Point Theorem

A map $G: \overline{B}^n \xrightarrow{C^0} \overline{B}^n$ has a fixed point $G(x) = x$.

Fact

The one dimensional manifolds are exactly $\mathbb{R} = (0, 1)$, S^1 , $[0, 1]$ and $[0, 1]$.

Fact

Consider $F: M \rightarrow N$ where $\partial N = \emptyset$ with q regular for F and $F|_{\partial M}$.

Then $F^{-1}(q)$ is a manifold with boundary and $\partial F^{-1}(q) = F^{-1}(q) \cap \partial M$.

IMAGE 2

Lemma

Let M be a compact manifold with boundary. Then no map exists such that $F: M \xrightarrow{C^\infty} \partial M$ such that $F|_{\partial M} = \text{id}$.

Proof

Write

$$\begin{array}{ccc} F : M & \xrightarrow{\quad} & \partial M \ni q \\ \uparrow & \nearrow id & \\ \partial M & & \end{array}$$

where q is a regular value.

Then $F^{-1}(q)$ is a submanifold with dimension $\dim M - \dim \partial M = n - (n - 1) = 1$.

Then the only candidate closed manifold with boundary of dimension 1 is $[0, 1]$.

IMAGE 3

But since the restriction is the identity and the boundary must lie in the boundary of M , we have a contradiction.

Corollary

There is no function $F : \overline{B}^n \xrightarrow{C^\infty} S^{n-1}$ such that $F|_{\partial \overline{B}^n = S^{n-1}} = \text{id}$.

Lemma

$G : \overline{B}^n \xrightarrow{C^\infty} \overline{B}^n$ has a fixed point.

Proof

Assume the contrary, that there exists $g : \overline{B}^n \xrightarrow{C^\infty} \overline{B}^n$ such that $g(x) \neq x$.

IMAGE 4

Then consider $x \mapsto G(x)$ with $G|_{\partial \overline{B}^n} = \text{id}$. Then, explicitly,

$$g(x) + t \cdot \frac{x - g(x)}{\|x - g(x)\|} = G(x)$$

However, by the previous lemma, this is impossible.

Fact

Every continuous map $G : \overline{B}^n \rightarrow \mathbb{R}^k$ can be C^0 -approximated by a smooth map P .

$$\|G - P\| = \sup_{x \in \overline{B}^n} \|G(x) - P(x)\| < \mu$$

Proof of Theorem

Assume $G : \overline{B}^N \rightarrow \overline{B}^n$ with $G(x) \neq x$ with $0 < \varepsilon < \min \|G(x) - x\|$. However $P : \overline{B}^n \rightarrow \overline{B}^n(1 + \varepsilon)$. Then for $F = \frac{1}{1+\varepsilon}P$, $G \approx P \approx F$.

Theorem

Let M^n a smooth manifold. Then there exists an embedding $M \hookrightarrow \mathbb{R}^N$.

Proof

(Assuming that M is compact)

Pick some finite cover by coordinate charts (U_i, ϕ_i) such that $V_i \subseteq \overline{V}_i \subseteq U_i$ with V_i also a finite cover.

IMAGE 5

Let f_i be such that $f_i|_{V_i} \equiv 1$ and $\text{supp } f_i \subset U_i$. Further, assume that $f < 1$ outside \overline{V}_i .

IMAGE 6

Then $f_i \phi_i : M \rightarrow \mathbb{R}^n$.

Define $F : M \rightarrow \mathbb{R}^n$ by $(f_1 \phi_1, \dots, f_k \phi_k, f_1, \dots, f_k)$ such that $N = nk + k$.

Claim: F is an embedding. That is, it is an immersion which is homeomorphic on its image. Equivalently, it is an immersion and one-to-one.

We need that $\text{rank } DF = n$. Take $x \in V_1$ where $f_1 \phi_1 = \phi_1$ on V_1 . Then $D(f_1 \phi_1)_x = (D\phi_1)_x$ which is of rank n . So F must be of at least (and, in fact, at most) rank n . That is, F is an immersion.

Again, take $x \in V_1$ and consider $F(x) = F(y)$. Then $f_1(x) = f_1(y)$ and, consequently, $y \in \overline{V}_1$.

Therefore $f_1 \phi_1(x) = f_1 \phi_1(y)$ which implies $\phi_1(x) = \phi_1(y)$ and $x = y$. Therefore F is one-to-one.

November 5, 2024

Theorem: (Weak) Whitney Embedding Theorem

If M^n is a smooth manifold, then

- there exists an embedding $M \hookrightarrow \mathbb{R}^{2n+1}$.
- there exists an immersion $M \hookrightarrow \mathbb{R}^{2n}$.

Remark: (Strong) Whitney Embedding Theorem

For M^n smooth and compact,

- there exists an embedding $M \hookrightarrow \mathbb{R}^{2n}$
- there exists an immersion $M \hookrightarrow \mathbb{R}^{2n-1}$

This is sharp, however we can do better for some M (e.g. $S^n \hookrightarrow \mathbb{R}^{n+1}$).

Tangent Bundle

$$\begin{array}{ccc} TM & & T_p M = \pi^{-1}(p) \\ \downarrow \pi & & \\ U \subset M \ni p & & \end{array}$$

IMAGE 1

Where we have a basis $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ in $T_p M$ for any $p \in U$. Then, writing $v = \sum v_i \frac{\partial}{\partial x_i}$,

$$TM|_U = \pi^{-1}(U) \xrightarrow[C^{r-1}]{(\phi, D\phi)} (\underbrace{\phi(p)}_{\phi\pi(p)}, \underbrace{(v_1, \dots, v_n)}_{D\phi(v) \in \mathbb{R}^n})$$

Properties

$F: M \xrightarrow{C^r} N$ implies $DF: TM \xrightarrow{C^{r-1}} TN$ by $v \mapsto DF(v)$ where $DF(V)$ is $F_* v$.

If F is an embedding (or immersion), then so is DF .

If F is a submersion, then so is DF .

Examples

$$T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n.$$

$$TV = V \times V.$$

$$TS^1 = S^1 \times \mathbb{R} \text{ by } (p, v) \mapsto (p, a).$$

IMAGE 2

So any vector field v may be written as $v = a \frac{\partial}{\partial \theta}$

If there exist v_1, \dots, v_n such that $v_1(p), \dots, v_n(p)$ is a basis, then $TM = M \times \mathbb{R}^n$.

$$T\mathbb{T}^n = \mathbb{T}^n \times \mathbb{R}^n.$$

Counterexample: $TS^2 \not\cong S^2 \times \mathbb{R}^2$.

Example

Take an embedding $M \hookrightarrow \mathbb{R}^k$. Then $TM \hookrightarrow T\mathbb{R}^k = \mathbb{R}^k \times \mathbb{R}^k$ is also an embedding.

IMAGE 3

The map $(q, v) \mapsto q + v$ is not useful, but $(q, v) \mapsto v \in \mathbb{R}^k$ is.

We have a Gauss map $M \xrightarrow{G} \text{Gr}(n, k)$ by $M \ni q \mapsto T_q M \in \text{Gr}(n, k)$.

Example

$$M \text{ a hypersurface } M^n \subset \mathbb{R}^{n+1}$$

IMAGE 4

$q \mapsto T_q M$ which is " $=$ " to the normal line.

$$M \rightarrow \text{GR}(n+1, n) \xleftrightarrow{\perp} \mathbb{RP}^n$$

$$p \longmapsto v_p$$

$$\begin{array}{ccc} M & \longrightarrow & S^n \subset \mathbb{R}^{n+1} \\ & \searrow & \downarrow \\ & & \mathbb{RP}^n \end{array}$$

Definition: Riemannian Metric

A Riemannian metric is a smooth map $g : TM \rightarrow \mathbb{R}$ such that for each $p \in M$, $g|_{T_p M}$ is a positive-definite quadratic form where $g(v, v) \geq 0$ and $g(v, v) = 0$ if and only if $v = 0$.

In local coordinates, (x_1, \dots, x_n) and with $p \in M$ and $v \in T_p M$, $g_p(v) = \sum g_{ij}(p)v_i v_j$ where $g_{ij}(p)$ is symmetric. We have that positive quadratic forms are in one-to-one correspondence with inner products.

$$\langle v, w \rangle = \frac{1}{2}(g(v+w) - g(v) - g(w))$$

In coordinates,

$$\langle v, w \rangle = \sum g_{ij}(p)v_i w_j$$

Example

Take an immersion $M \hookrightarrow \mathbb{R}^k$

IMAGE 5

The inner product in \mathbb{R}^k restricts to a Riemannian metric on M .

Theorem

Every manifold admits a Riemannian metric.

Proof 1

Since any manifold may be immersed $M \hookrightarrow \mathbb{R}^k$, we may define a Riemannian metric.

Proof 2

Take a locally finite cover by coordinate charts, $\{(U_i, \phi_i) = (x_1, \dots, x_n)\}$.

Within every chart, take a Riemannian metric g_i such that $\left\langle \frac{\partial}{\partial x_s}, \frac{\partial}{\partial x_r} \right\rangle = \delta_{sr}$.

Take a partition of unity $f_i > 0$ subordinated to U_i ($\text{supp } f_i \subset U_i$ and $\sum f_i \equiv 1$). Then

$$g = \sum f_i g_i$$

$$g(v) = \sum f_i(p) g_i(v) > 0$$

with $v \neq 0$, $g_i > 0$ and $f_i > 0$ for some i .

Definition: Unit Tangent Bundle

Take some manifold M and a Riemannian metric g .

Then $g^{-1}(1) =: STM$ is the unit tangent bundle.

Claim 1

The unit tangent bundle is a smooth submanifold of codim = 1 and dim = $2n - 1$.

Proof 1

Consider $\phi_\lambda : TM \rightarrow TM$ by $v \mapsto \lambda v$ with $\lambda \neq 0$.

Since $g(\phi_\lambda(v)) = \lambda^2 g(v)$, it follows that (exercise) a is a regular value of g if and only if $\lambda^2 a$ is a regular value. That is, for

$$M \xrightarrow{\phi} M \xrightarrow{f} \mathbb{R}$$

p is a regular value of $f \circ \phi$ if and only if $\phi(p)$ is a regular value of f and a is a regular value of f if and only if $\lambda^2 a$ is a regular value of $\lambda^2 f$.

Proof 2

(p, v) is a regular point if and only if there exists some $w \in T(TM)$ such that $L_w g \neq 0$.

IMAGE 6

For $w = v$,

$$\frac{\partial}{\partial \lambda} g(\lambda v)|_{\lambda=1} = \frac{\partial}{\partial \lambda} \lambda^2 g(v)|_{\lambda=1} = 2g(v) \neq 0$$

Claim 2

If g_0 and g_1 are Riemannian metrics on M , then $STM_{g_0} \cong STM_{g_1}$ are diffeomorphic.

For V , g_0 and g_1 with S_1 and S_2 ,

IMAGE 7

The map $x \mapsto \frac{g_0(x)}{g_1(x)} x$.

Exercise

If $M \hookrightarrow \mathbb{R}^k$ with an induced metric on M , then $STM \hookrightarrow \mathbb{R}^k \times S^{k-1}$ by $(p, v) \mapsto v \in S^{k-1}$ induces $STM \rightarrow S^{k-1}$.

IMAGE 8

Example

$$STS^2 \cong SO(3) \cong \mathbb{RP}^3$$

IMAGE 9

Take $p \perp v$, $\|p\| = \|v\| = 1$, we can generate an orthonormal basis $(p, v, p \times v)$.

Recall that $S^n/x \sim -x = \mathbb{RP}^n$, but that we may also identify $B^n/x \sim -x$ on S^{n-1} .

That is, $\mathbb{RP}^3 = B^3/\sim$ (with radius π).

IMAGE 10

Take $A \in SO(3)$.

November 7, 2024

Theorem: (Weak) Whitney Embedding Theorem

For M^n ,

1. there exists an embedding $M \hookrightarrow \mathbb{R}^{2n+1}$.
2. there exists an immersion $M \hookrightarrow \mathbb{R}^{2n}$.

Proof

(For simplicity, consider M compact)

$$\begin{array}{ccc} M & \xleftarrow{F} & \mathbb{R}^m \ni v \\ & \searrow F_v & \downarrow \pi_v \\ & & \mathbb{R}^{m-1} \end{array}$$

Where $\|v\| = 1$.

Hope: for m sufficiently large, F_v is an embedding and immersion.

What could go wrong?

- F_v not 1-1.

IMAGE 1

$$p - q = t \cdot v, \frac{p - q}{\|p - q\|} = \pm v.$$

So take $\Phi : M \times M \setminus \Delta \rightarrow S^{m-1}$ (where Δ is the diagonal) by $(p, q) \mapsto \frac{p - q}{\|p - q\|}$.

We need $v \notin \Phi(M \times M \setminus \Delta)$. Since $2n < m - 1$, this map has zero measure.

- F_v not an immersion.

IMAGE 2

Take $\psi : STM \rightarrow S^{m-1}$ by $(p, w) \mapsto w$. Then $v \notin \psi(STM)$.

Then F_v is an immersion where $2n - 1 < m - 1$ ($2n < m$).

Then for immersions, we may continue the process until $2n = m$; for embeddings, only until $2n + 1 = m$.

Remarks

$\Psi : TM \rightarrow \mathbb{R}^k$ by $(p, w) \mapsto w$.

$\Phi : (M \times M) \times \mathbb{R} \rightarrow \mathbb{R}^k$ by $(p, q, t) \mapsto (p - q)t$.

Homework

Problem 1

Take $L, M \subset \mathbb{R}^k$ submanifolds such that $\dim L + \dim M < k$.

Then for almost all x , $(x + L) \cap M = \emptyset$.

Problem 2

Take $M^n \subset \mathbb{R}^k$, $k > 2n$, and a projection $\rho_x : p \mapsto \frac{p-x}{\|p-x\|}$.

IMAGE 3

For almost all $x \in \mathbb{R}^n$, ρ_x is an immersion.

Problem 3

Take $L \subset M \times N \xrightarrow{\pi} N$.

IMAGE 4

Then $\pi^{-1}(q) \cap L = L_q$.

For almost all q , L_q is a smooth submanifold. Hint: $\pi|_L$.

R Smooth Topology

Take M compact and $0 \leq r \leq \infty$.

For $f : M \rightarrow \mathbb{R}^k$ (with either $k = 1$ or considered component-wise), define

$$\|f\|_{C^0} = \sup_{x \in M} |f(x)|$$

Then $C^0(M, \mathbb{R}^k)$ is a Banach space with induced distance $d(f, g) = \|f - g\|_{C^0}$.

For $1 \leq r < \infty$, take a finite coordinate cover (U_ℓ, ϕ_ℓ) (which is contained within some coordinate cover (W_ℓ, ϕ_ℓ)). For $r = 1$, define

$$\|f\|_{C^1} = \|f\|_{C^0} + \sum_{i, \ell} \left| \frac{\partial f}{\partial x_i^{(\ell)}} \right|$$

where $f = f \circ \phi_\ell$ and we may equivalently take $\max\{i, \ell\}$ rather than the sum.

Then $C^1(M, \mathbb{R})$ is a Banach space with well-defined topology. Similarly, for arbitrary r ,

$$\|f\|_{C^r} = \|f\|_{C^{r-1}} + \sum_{\ell, i_1, \dots, i_r} \left| \frac{\partial^r f}{\partial x_{i_1} \cdots \partial x_{i_r}} \right|$$

with equivalence again to a maximum. Finally, for $r = \infty$, $C^\infty \subset C^r$ for any finite r .

The C^∞ topology is generated by all C^r topologies.

$$C^0 \supset C^1 \supset C^2 \supset \cdots \supset C^r \supset C^{r+1} \supset \cdots \supset C^\infty$$

Convergence: $f_i \xrightarrow{C^r} f$ if and only if $f_i \rightarrow f$, and so do all partial derivatives up to r . $\frac{\partial f_i}{\partial x_c} \rightarrow \frac{\partial f}{\partial x_c}$.

Note that $C^\infty(M, \mathbb{R})$ is not a Banach space. It is, however, Fréchet.

N Smooth Topology

Instead, take $C^r(M, N)$ with $0 \leq r \leq \infty$ and M compact. Define

$$M \xrightarrow{f} N \hookrightarrow \mathbb{R}^k$$

Theorem

For M compact, $r \geq 1$

1. the set of immersion $M \hookrightarrow \mathbb{R}^k$ is C^r open.
2. the same is true for embeddings.

Proof

Let $F: M \rightarrow \mathbb{R}^k$ be an immersion and $G C^1$ close to F .

$$\begin{array}{ccc} T_p(M) & \xrightarrow{DF} & T_{F(p)}\mathbb{R}^k \\ & \searrow DG & \\ & & T_{G(p)}\mathbb{R}^k \end{array} \quad \text{rank} = n$$

Theorem

For $M^n \xrightarrow{C^r} \mathbb{R}^k$ with M compact and $r \geq 1$, if $k \geq 2n$ then immersions form a C^r open and dense set.
If $k \geq 2n + 1$ then the same is true for embeddings.

Lowering the Dimension

Take $F: M \rightarrow \mathbb{R}^k$. We want an embedding/immersion approximation.

We know that there exists an embedding $M \xrightarrow{G} \mathbb{R}^M$ for sufficiently large m , so consider $\tilde{F} := (F, \varepsilon G): M \rightarrow \mathbb{R}^k \times \mathbb{R}^m$.

Note that $(F, \varepsilon G) \xrightarrow{C^r} (F, 0)$.

Take a basis of \mathbb{R}^m , (e_1, \dots, e_m) . We can take vectors close to our basis $v_1 \approx e_1, v_2 \approx e_2$, etc. We want to construct a series of projections $\mathbb{R}^k \times \mathbb{R}^m \xrightarrow{\pi_{v_1}} \mathbb{R}^k \times \mathbb{R}^{m-1} \xrightarrow{\pi_{v_2}} \dots \rightarrow \mathbb{R}^k$.

IMAGE 5

Review: Ordinary Differential Equations

On \mathbb{R}^n ,

$$x' = v(x) \text{ on } U \subset \mathbb{R}^n$$

with $x = (x_1, \dots, x_n)$. We want a map x from some open interval to U satisfying

$$\begin{cases} x'_1(t) = v_1(x(t)) \\ \vdots \\ x'_n(t) = v_n(x(t)) \end{cases}$$

with $x(0) = p$. Geometrically

IMAGE 6

where $x'(t) = v(x(t))$ is an integral curve.

Theorem

1. Existence: for every $p \in U$ and $t_0 \in \mathbb{R}$, there exists $\varepsilon > 0$ such that there is an integral curve $\gamma : (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow U$ (where $\gamma(t_0) = p$).
2. Uniqueness: let $\gamma : (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow U$ and $\eta : (t_0 - \delta, t_0 + \delta) \rightarrow U$ be two integral curves through p at t_0 . Then $\gamma = \eta$ on the common domain.

We may refine (a) as

1. for every p and t_0 , there exists $\varepsilon > 0$ on a neighborhood $V \ni p$ such that for every $q \in V$ there exists an integral curve $\gamma_q : (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow U$ where $\gamma_q(t_0) = q$.
2. $\gamma_q(t)$ is smooth in q and t . Namely, on $V \times (t_0 - \varepsilon, t_0 + \varepsilon)$.

IMAGE 7

With Parameters

Consider $x' = v(x, \lambda)$. Then we have p and $\gamma_{p,\lambda}$ which depends smoothly on λ .

$$x = v(x, \lambda) \quad (p, \lambda) \quad \lambda' = 0$$

For $x' = v(x, t)$, given p and t_0 we have γ_{p,t_0} which depends smoothly on p and t_0 .

$$x = v(x, t) \quad (p, t_0) \quad t' = 1$$

November 12, 2024

Recall: Solutions to ODEs

Given $x' = v(x)$ on U .

For each $p \in U$, there exists $\varepsilon > 0$ such that $\gamma_p : (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow U$ where $\gamma_p(t_0) = p$.

Two solutions with the same initial conditions agree on their common domain.

Proof

Choose $p = 0$ and $t_0 = 0$, and write $\gamma'(t) = v(\gamma(t))$ with $\gamma(0) = 0$ for $t \in (-\varepsilon, \varepsilon)$. Then,

$$\gamma(t) = \int_0^t v(\gamma(\tau)) d\tau$$

Take $B = C^0([-\varepsilon, \varepsilon], \mathbb{R}^n)$ with $\gamma(0) = 0$, and a map $\Phi : B \rightarrow B$ given by

$$\Phi(\gamma) = \int_0^t v(\gamma(\tau)) d\tau$$

Claim: Φ is a contraction provided that ε is small. Then $\gamma = \Phi(\gamma)$ has a unique solution by the contraction mapping theorem.

Uniqueness

Fix t_0 and p . Then there exists a maximal integral curve. That is, given $\gamma_p : (a, b) \rightarrow \mathbb{R}^n$ and $\tilde{\gamma}_p : (a', b') \rightarrow \mathbb{R}^n$, $\gamma_p = \tilde{\gamma}_p$ on $(a', b') \cap (a, b)$ and they are defined on $(a, b) \cup (a', b')$.

Examples

On \mathbb{R}^n

- $v(p) = 0$ implies $\gamma_p(f) \equiv p$
- $v(x) = Ax$ gives $\gamma_p(t) = e^{At}p$.
- $v(x) = 1$ on $U = (0, 1)$ with $\gamma_p(t) = p + t$

IMAGE 1

- $t_0 = 0$, $x' = \lambda x^2$, $\frac{dx}{dt} = \lambda x^2$, $\frac{dx}{\lambda x^2} = dt$, and $x = \gamma_p(t)$, gives

$$\int_p^x \frac{dy}{\lambda y^2} = \int_0^t dt$$

and

$$\frac{2}{\lambda y} \Big|_p^x = t$$

implies

$$\frac{-1}{\lambda x(t)} + \frac{1}{\lambda p} = t \iff x(t) = \frac{p}{1 - \lambda t p}$$

IMAGE 2

So the max interval is $(-\infty, \frac{1}{\lambda p})$.

Definition: Integral Curve

Let M be a manifold and v a vector field. $\gamma : (a, b) \rightarrow M$ is an integral curve if $\gamma'(t) = v(\gamma(t))$.

IMAGE 3

$\tau \rightarrow \gamma(t + \tau)$.

IMAGE 4

$$(\phi \circ \gamma)' = (\phi_* v)(\phi(\gamma(t))).$$

Existence

For v a vector field on M , and for each $p \in M$ and t_0 , there exists $\varepsilon > 0$ and an integral curve $\gamma_p : (t_0 - \varepsilon, t_0 + \varepsilon) \rightarrow M$ with $\gamma_p(t_0) = p$.

Definition: Complete Vector Field

v is complete if every maximal integral curve has \mathbb{R} as a domain.

Examples

$v(x) = x^2 \frac{\partial}{\partial x}$ on \mathbb{R} is not complete.

$v(x) = \frac{\partial}{\partial x}$ on $(0, 1)$ is not complete.

$v(x) = Ax$ on \mathbb{R}^n is complete.

If $\|v(x)\| < a|x| + b$ on \mathbb{R}^n , $v(x)$ will be complete.

Definition: Support of a Vector Field

$$\text{supp } v = \overline{\{x : v(x) \neq 0\}}$$

Theorem

Suppose v is compactly supported. Then v is complete.

Corollary

If M is closed, then every vector field is complete.

Phase Portraits

Take $v = -x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$.

IMAGE 5

Consider $x' = -x$, then $x(t) = e^{-t} p$. Then it takes infinite time to travel from p to 0.

IMAGE 6

Definition: Flow

A flow is a 1-parameter subgroup in $\text{Diff}(M)$, the group of diffeomorphisms on M . That is, $\phi^t \in \text{Diff}(M)$, $t \in \mathbb{R}$ such that

- $\phi^0 = \text{id}$
- $\phi^{t+\tau} = \phi^t \phi^\tau$

Note that this implies $(\phi^t)^{-1} = \phi^{-t}$.

Said differently, $\mathbb{R} \rightarrow \text{Diff}(M)$ by $t \mapsto \phi^t$ is a group homomorphism.

Example: $\phi^t = e^{At} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a one parameter subgroup $\text{GL}(n) \subset \text{Diff}(\mathbb{R}^n)$.

$F : M \times \mathbb{R} \rightarrow M$ given by $(x, t) \mapsto \phi^t(x)$ gives $F(x, 0) = x$ and $F(F(x, t), \tau) = F(x, t + \tau)$.

Definition: Local Flow

Take $\phi^t(x)$ defined only on $t \in (-\varepsilon(x), \varepsilon(x))$.

For a neighborhood $U \subset M \times \{0\} \subset M \times \mathbb{R}$, $F : U \rightarrow M$.

IMAGE 7

Theorem

There is a 1-1 correspondence between local flows and vector fields; similarly, there is a correspondence between flows and complete vector fields.

Explicitly, $\phi^t \mapsto v(x) = \frac{d}{dt}\phi^t(x)|_{t=0}$ and, conversely, $v \mapsto \gamma'_p(t) =: \phi^t(p)$.

Proof

First, we need that $\phi^t : p \mapsto \gamma_p(t)$ is a flow.

Flow lines are integral curves of the respective vector field.

$$\underbrace{\phi^{t+\tau}(p)}_{=\gamma_p(t+\tau)} = \underbrace{\phi^t(\phi^\tau(p))}_{=\gamma_{\gamma_p(\tau)}(t)}$$

So $t \mapsto \gamma_p(t + \tau)$ and $0 \mapsto \gamma_p(\tau)$ while $t \mapsto \gamma_{\gamma_p(\tau)}(t)$ while $0 \mapsto \gamma_p(\tau)$.

IMAGE 8

Therefore, by uniqueness, these are the same.

Now we need $v \rightsquigarrow \phi \rightsquigarrow w$ where $w = v$. Compute

$$w(p) = \frac{d}{dt} \underbrace{\phi^t(p)}_{\gamma_p(t)} \Big|_{t=0} = \frac{d}{dt} \gamma_p(t) \Big|_{t=0} = v(p)$$

where $\gamma'_p(t) = v(\gamma_p(t))$ and $\gamma_p(0) = p$.

Finally, we want $\phi \rightsquigarrow v \rightsquigarrow \psi$ where $\psi = \phi$. We have

$$\frac{d}{dt} \psi^t(p) = v(\psi^t(p)) \quad \text{and} \quad v(q) = \frac{d}{dh} \phi^h(q) \Big|_{h=0}$$

and we need $\frac{d}{dt} \phi^t(p) = v(\phi^t(p))$. Compute

$$\frac{d}{dt} \phi^t(p) = \frac{d}{dh} \psi^{t+h}(p) \Big|_{h=0} = \frac{d}{dh} \underbrace{\phi^h(\phi^t(p))}_{q} \Big|_{h=0} = v(\phi^t(p))$$

Examples

Take \mathbb{R}^n and $v = v_0$ a constant.

IMAGE 9

Solutions have the form $\phi^t(p) = p + v_0 t$.

Take instead $v(x) = Ax$, $\phi^t(p) = e^{At}p$ and $e^{A(t+\tau)} = e^{At}e^{A\tau}$

Remark

In general, $e^{A+B} \neq e^A e^B$ but they are equal when $[A, B] = AB - BA = 0$.

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Proposition

Let v be a vector field and $v(p) \neq 0$, then there exist x_1, \dots, x_n near p such that $v(x) = \frac{\partial}{\partial x_n}$ and $\phi^t(x) = (x_1, \dots, x_{n-1}, x_n + t)$.

IMAGE 1

Proof

Take $y_1, \dots, y_{n-1}, y_n, p = 0$, and $v(p) = \frac{\partial}{\partial y_n}$.

IMAGE 2

Then there exists time τ such that $y_n(\phi^{-\tau}(q)) = 0$ where $\tau(q)$ depends on q .

Define a map $\pi : q \mapsto \phi^{-\tau}(q)$. Then $q = \phi^{\tau(q)}(\pi(q))$. Then

$$\begin{aligned} x_1 &= y_1 \circ \pi \\ &\vdots \\ x_{n-1} &= y_{n-1} \circ \pi \\ x_n &= \tau(x) \end{aligned}$$

Lie Brackets Via Flows

Take two vector fields v, w and consider the flow ϕ^t of v

IMAGE 3

Then we may take $D\phi^t W_{\phi^{-t}(p)} \in T_p M$ and define $-\frac{d}{dt} D\phi^t W_{\phi^{-t}(p)} \Big|_{t=0} = L_v w(p)$, the Lie derivative.
Compare with

IMAGE 4

where $L_v f(p) = -\frac{d}{dt} f(\phi^{-t}(p)) \Big|_{t=0}$ which depends only on $v(p)$.

Remark

If M is compact, then ϕ^t is defined for all times.

Consider $\mathfrak{X}(M)$ of C^∞ vector spaces which is topologically Fréchet (complete, locally convex, with an invariant metric). Then

$$D\phi^{-t}(w) = w_t \in \mathfrak{X}(M) \quad \text{and} \quad L_v w = -\frac{d}{dt} w_t \Big|_{t=0}$$

Example 1

Take v a constant vector flow on \mathbb{R}^n with $\phi^t(x) = x + tv$ and $D\phi^t = \text{id}$. Then for $w = \sum w_i \frac{\partial}{\partial x_i}$,

$$L_v w = -\frac{\partial}{\partial t} \sum w_i(x - tv) \frac{\partial}{\partial x_i} \Big|_{t=0} = \sum L_w w_i(x) \frac{\partial}{\partial x_i}$$

Example 2

Let $w = Ax$ with v constant. Then

$$L_v w = -\frac{d}{dt} A(x - vt) \Big|_{t=0} = Av$$

Example 3

Let $v = Ax$ with w constant. Then $\phi^t(x)e^{At}x$ and $D\phi^t = e^{At}$ and

$$L_v w = -\frac{d}{dt} e^{At} w \Big|_{t=0} = -Aw$$

Note that examples 2 and 3 demonstrate the skew symmetry of the Lie bracket.

Example 4

Let $v = Ax$ and $w = Bx$. Then $\phi^t(x) = e^{At}x$, $D\phi^t = e^{At}$, and

$$L_v w = -\frac{d}{dt} \underbrace{e^{At}}_{D\phi^t} \underbrace{Be^{-At}}_{w_{\phi^{-t}(x)}} \Big|_{t=0} = -(AB - BA) = -[A, B]$$

Theorem

$$L_v w = [v, w].$$

Proof

Let $K = \{p : v(p) = 0\}$ and $U = M \setminus K$.

Then $M = U \coprod \partial K \coprod \text{int}(K)$. Since U and the interior of K are together dense, and the flow for $\text{int}(K)$ is $\phi^t = \text{id}$, we need only examine U .

Then with the local normal form near p , write

$$w = \sum w_i(x) \frac{\partial}{\partial x_i}, \quad L_v w = \sum \frac{\partial w_i}{\partial x_n} \frac{\partial}{\partial x_i} \quad \text{and} \quad [v, w] = \sum \frac{\partial w_i}{\partial x_n} \frac{\partial}{\partial x_n}$$

Remark

Consider flows $v \sim \phi^t$ and $w \sim \psi^t$.

The bracket $[v, w]$ measures to what extent ϕ^t and ψ^s do not commute.

Example

$[A, B]$ measures to what extent e^{At} and e^{Bs} do not commute.

Theorem

$$[v, w] = 0 \text{ if and only if } \phi^t \psi^s = \psi^s \phi^t.$$

Observation 1

Write $D\phi^t = \phi_*^t$ and observe that $\phi_*^t v = v$ while $\psi_*^s w = w$.

IMAGE 5

Proof

Consider $F : M \rightarrow M$. Recall that $v \mapsto F_* v$ by $[\eta] \mapsto [F \circ \eta]$. Compute

$$\eta(\tau) = \gamma_p(\tau) = \phi^\tau(p)$$

so $\frac{d}{dt}\phi^t(p) = v$ and $\phi^t \eta(\tau) = \phi^t(\phi^\tau(p))\phi^{t+\tau}(p)$. Therefore

$$\frac{d}{dt}\phi^{t+\tau}(p) = \frac{d}{dt}\phi^\tau(\phi^t(p)) = v(\phi^t(p))$$

Observation 2

Next, observe that for $F : M \rightarrow M$, $F\phi^t = \phi^t F$ if and only if $F_* v = v$.

Proof

Recall that $v_p = [t \mapsto \phi^t(p)]$.

If $F\phi^t = \phi^t F$, then $F\phi^t(p) = \phi^t F(p)$.

IMAGE 6

If instead, $F_* v = v$, then $F \circ \phi^t \circ F^{-1} = \phi^t$. Left as an exercise.

Proof

(\Leftarrow) Set $\psi^s := F$ such that, by previous observation, $\psi_*^s v = v \in \mathfrak{X}(M)$. Write

$$-[w, v] = -L_w v = \frac{d}{ds}\psi_*^s v = 0$$

(\Rightarrow) If $[v, w] = 0$, then $\psi_*^s v = v$ for $F := \psi^s$. By the second observation, $\psi^s \phi^t = \phi^t \psi^s$. To see that ψ_*^s is constant, compute

$$\frac{d}{ds}\psi_*^s v = \frac{d}{dh}\psi_*^{s+h} v \Big|_{h=0} = \frac{d}{dh}\psi_*^s \psi_*^h v \Big|_{h=0} = \psi_*^s \frac{d}{dh}\psi_*^h v \Big|_{h=0} = -\psi_*^s [w, v] = 0$$

Theorem

IMAGE 7

If $c(t) = \psi^{-t} \phi^{-t} \psi^t \phi^t(p)$, then $c'(t) = 0$ and $c''(0) = 2[v, w](p)$.

Fact

If $c : (-\varepsilon, \varepsilon) \rightarrow M$, $c(0) = p$, and $c'(0) = 0$, then $c''(0) \in T_p M$ is well-defined.