Manifolds I

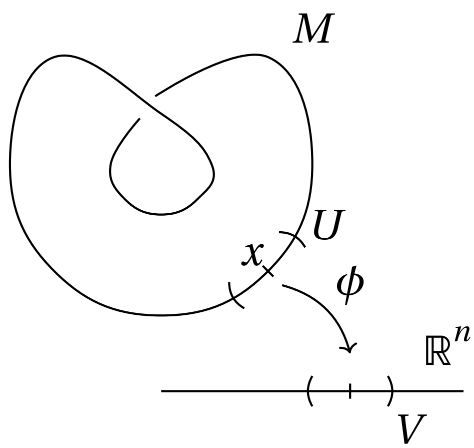
September 26, 2024

Class Organization

1 Takehome Midterm
1 Takehome Final
Homeworks assigned, but not graded.
`https://ginzburg.math.ucsc.edu/teaching/208manifolds1-2024/syl.html`

Definition: Topological Manifolds

For M a topological space, M is a topological manifold if $\forall x \in M$, $\exists M \supset U \ni x$ and homeomorphism $\phi : U \to V \subset \mathbb{R}^n$ for V open.
To avoid problems (see below), further assume that M is Hausdorff and second countable.

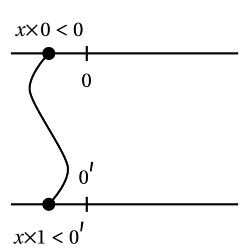


Exercise

We can require V to be an open ball.

Problems

- M need not be Hausdorff.



With $(\mathbb{R} \times 0 \coprod \mathbb{R} \times 1) / \sim$.

- M need not be second countable.

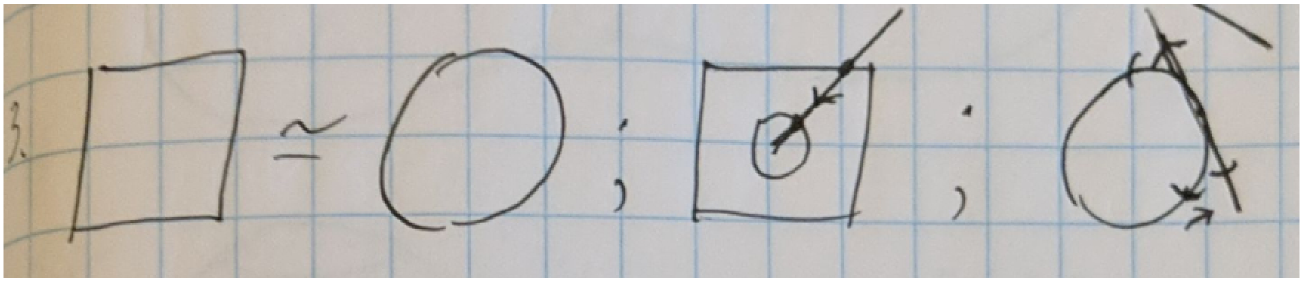
Take $\coprod_S \mathbb{R}_S$ where S is an uncountable index.

Examples

Example 1

If $N \underset{\text{homeo}}{\simeq} M$, this implies N is a manifold.

Example 2



Example 3

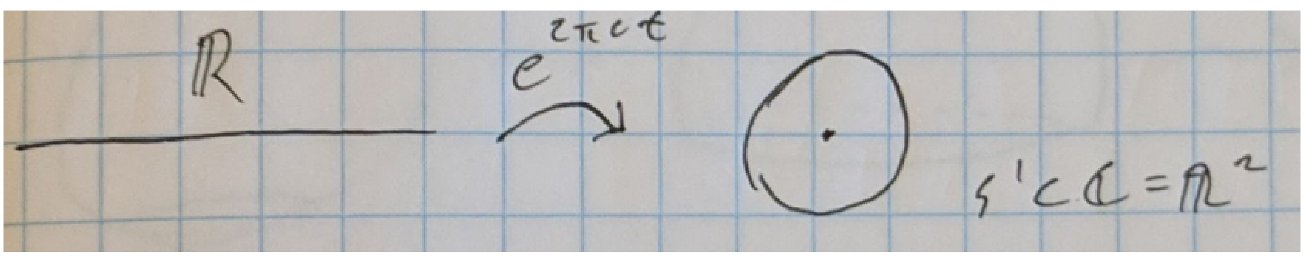
An open subset of a manifold is a manifold.

Example 4

$M,\ N$ manfiolds implies $M \times N$ is a manifold.

Example 5

Take $\mathbb{R}/\mathbb{Z}$ by the equivalence relation $t \sim t'$ iff $t' - t \in \mathbb{Z}$.



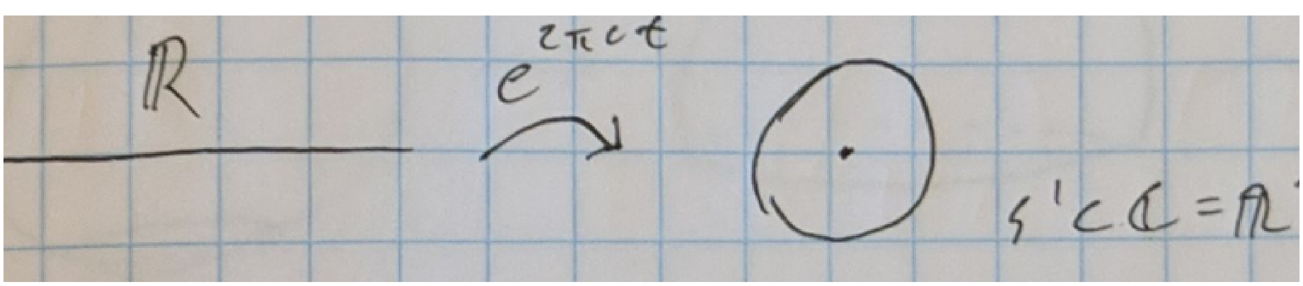
Then $C^0(S^1)$ relates to periodic functions with period 1.

Example 6

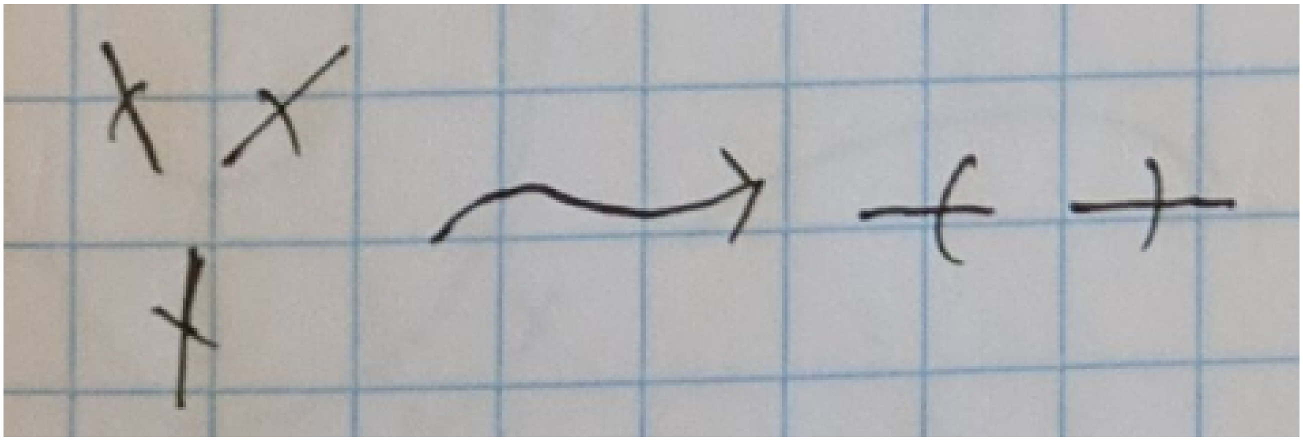
$\mathbb{T}^n = S^1 \times \cdots \times S^1$.

Counterexample 1

$[0,1]$ is not a manifold.

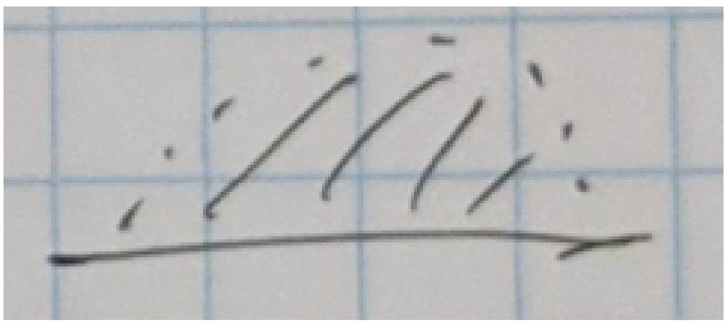


Since 0 must map somewhere in the open interval, its deletion results in a connected space in the former case but a disconnected one in the latter. Similarly, the following breaks into three and two connected components respectively.



Definition: Manifold with Boundary

There exists a neighborhood $\forall x \in M$ homeomorphic to either the open ball or the half-closed half-ball.



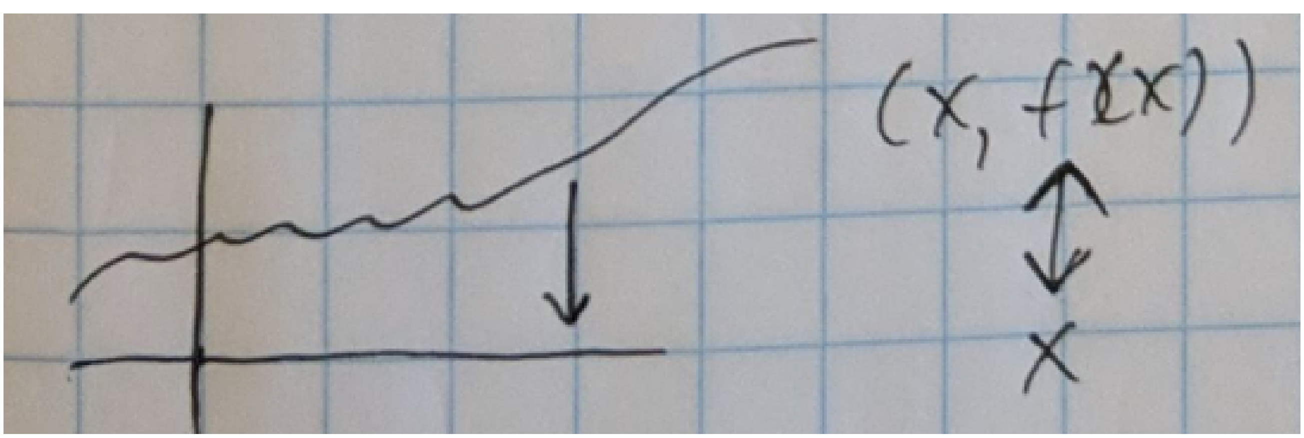
Exercise

A connected manifold is path-connected.

Examples

Example 7

Take $f : \mathbb{R}^n \xrightarrow{C^0} \mathbb{R}$ with graph $\Gamma_f = \{(x, f(x)) \,:\, x \in \mathbb{R}^n\} \subset \mathbb{R}^n \times \mathbb{R}$.

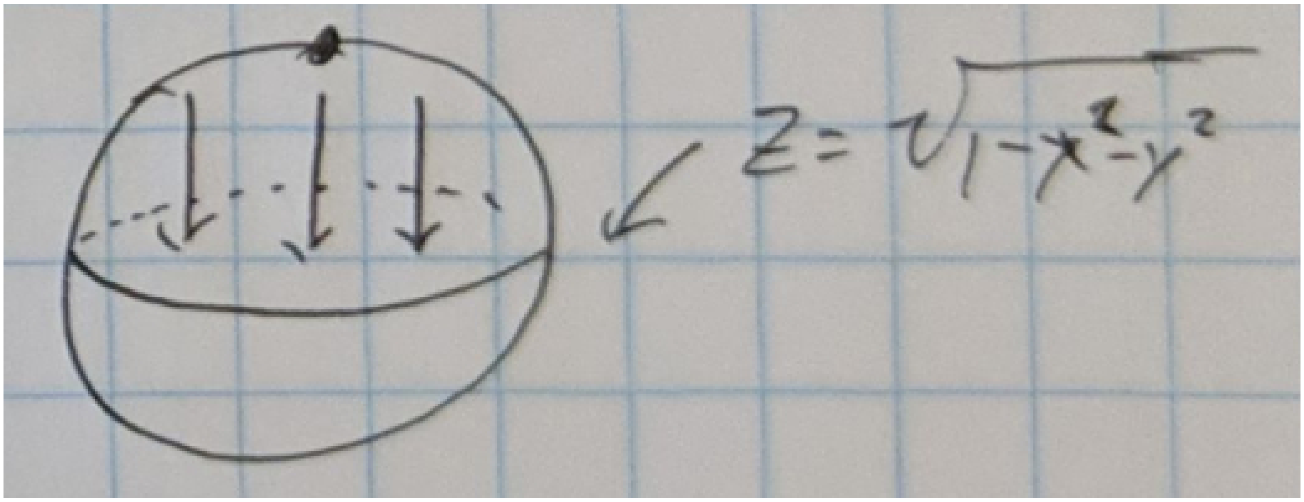


Example 8

Take $f : M \to N$ between manfiolds, then $M \simeq \Gamma_f \subseteq M \times N$.

Example 9

$S^n \subset \mathbb{R}^{n+1}$.

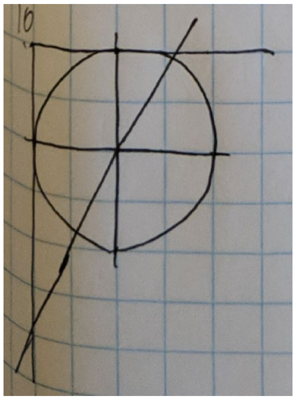


Definition: Real Projective Spaces

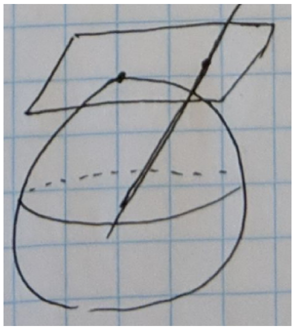
Take $\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim$ where $x \sim y \iff x = \lambda y$ for $\lambda \neq 0$.
Informally, the collection of lines through the origin.
Alternatively, $\mathbb{RP}^n = S^n / \sim$ where $x \sim -x$.
That is, identifiying the antipodal points of the unit sphere.
We may also consider $\mathbb{RP}^n = SO(n+1)/SO(n)$.

Claim

$\mathbb{RP}^n$ is a manifold.

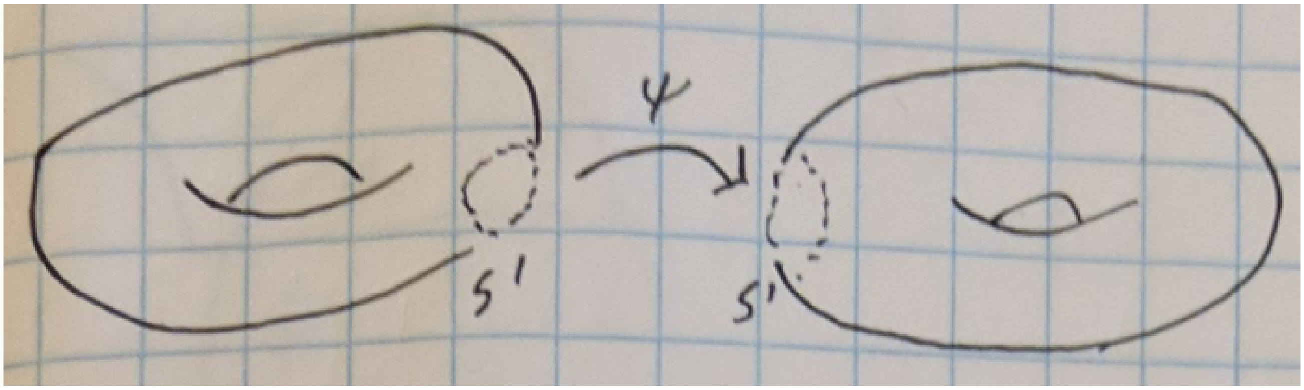


$\mathbb{RP}^1 \setminus \{x\text{-axis}\} \overset{\text{homeo}}{\to} \mathbb{R}$.

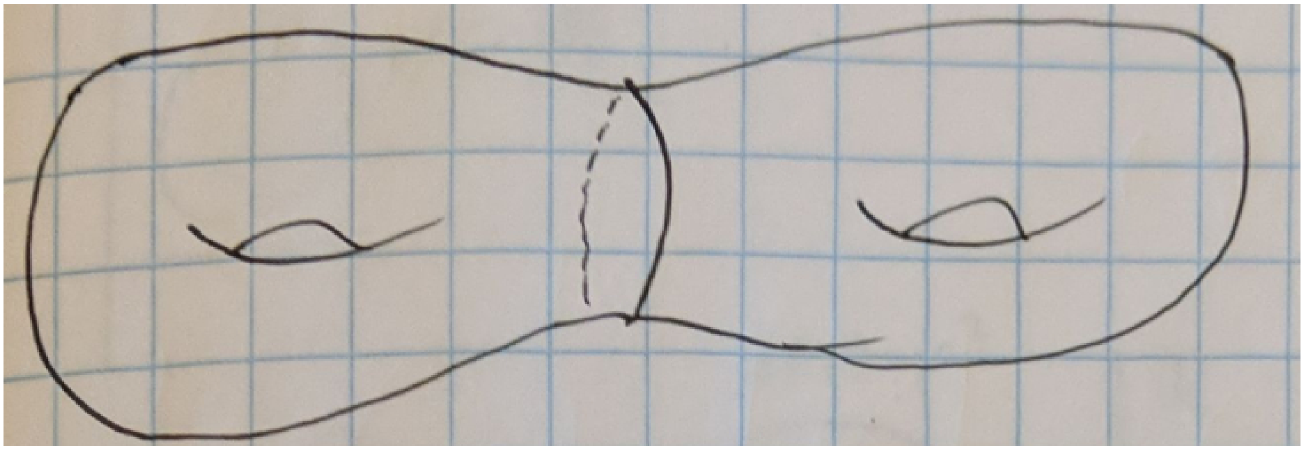


$\mathbb{RP}^2 \setminus \mathbb{RP}^1 \overset{\text{homeo}}{\to} \mathbb{R}^2$
We have that $\mathbb{RP}^1$ is homeomorphic to the circle, and $\mathbb{RP}^n = \mathbb{RP}^{n-1} \cup B^n$.
Take $x = (x_0, \ldots, x_n)$, $y = (y_0, \ldots, y_n) = (\lambda x_0, \ldots, \lambda x_n)$ and $[x] = [x_0 : x_1 : \cdots : x_n]$.
Then for $U_k \subset \mathbb{RP}^n$ with $U_k = \{[x] : x_k \neq 0\}$, we have that $U_0, \ldots, U_n$ covers $\mathbb{RP}^n$.
Then define $U_k \to \mathbb{R}^n$ by $[x_0 : \cdots : x_n] \to \left(\frac{x_0}{x_k}, \ldots, \frac{x_k}{x_k}, \ldots, \frac{x_n}{x_k}\right)$.

Connected Sum of Manfiolds

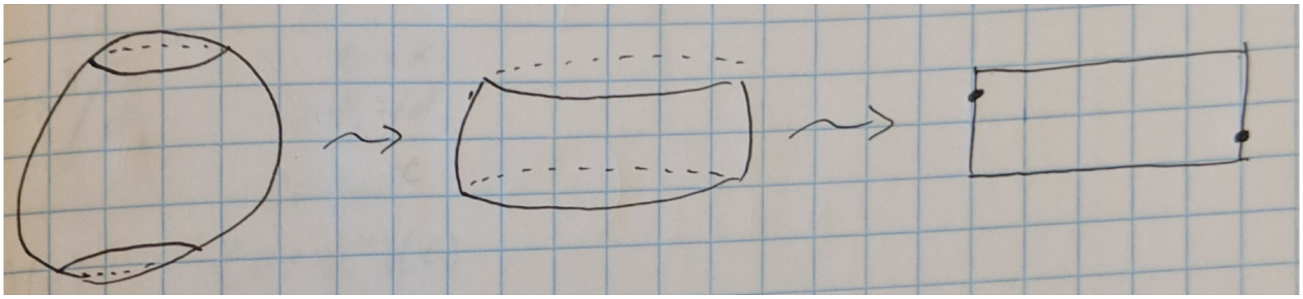


$M \setminus B^n \coprod N \setminus B^n$



$M\#N$.

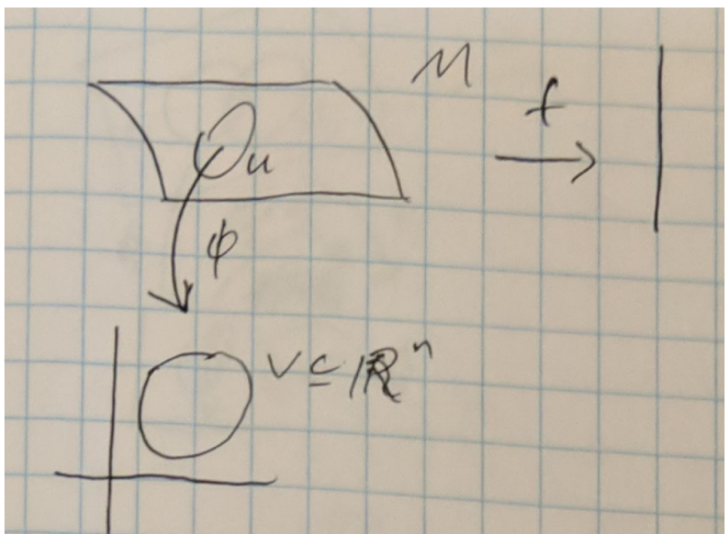
Mobius Band



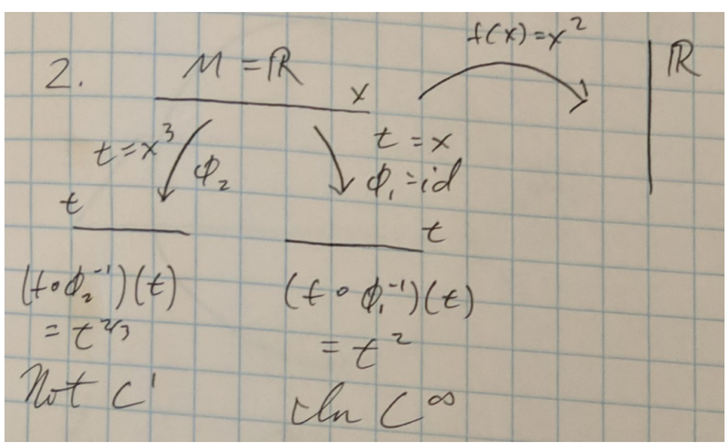
October 1, 2024

A Failed Definition

$f \in C^{r \geq 1}$; $f \circ \phi^{-1} : V \xrightarrow{C^r} \mathbb{R}$.



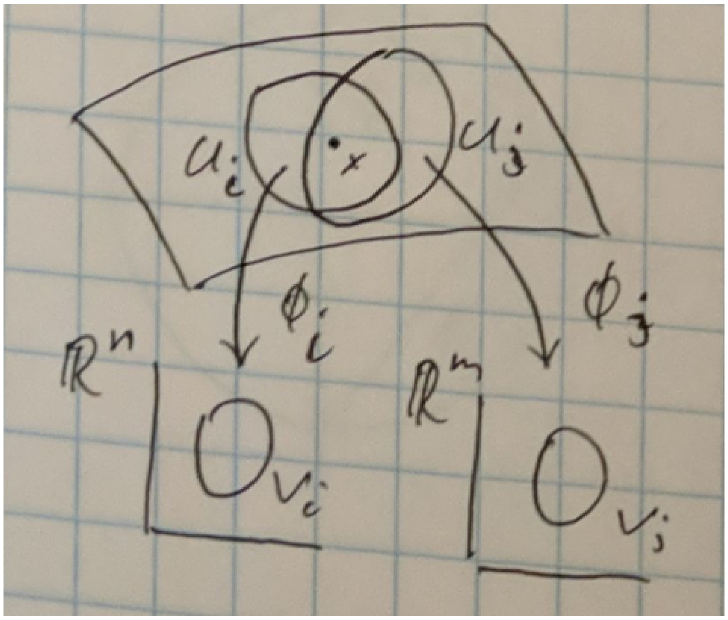
Example



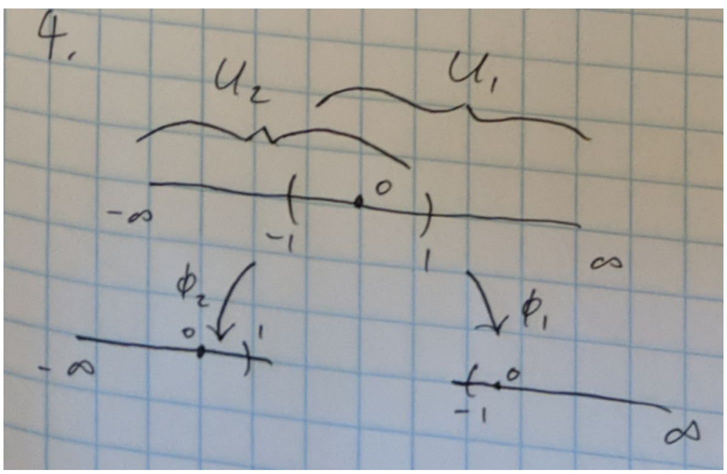
Definition: Charts

Say there exists a cover U_i by open sets and $U_i \xrightarrow{\phi_i} V_i \subseteq \mathbb{R}^n$ fixed.
Then the pair (U_i, ϕ_i) is a chart.

What if a point belongs to two charts?



With f smooth at x, $f \circ \phi_i^{-1}$ smooth at $\phi_i(x)$ and $f \circ \phi_j^{-1}$ smooth at $\phi_j(x)$.

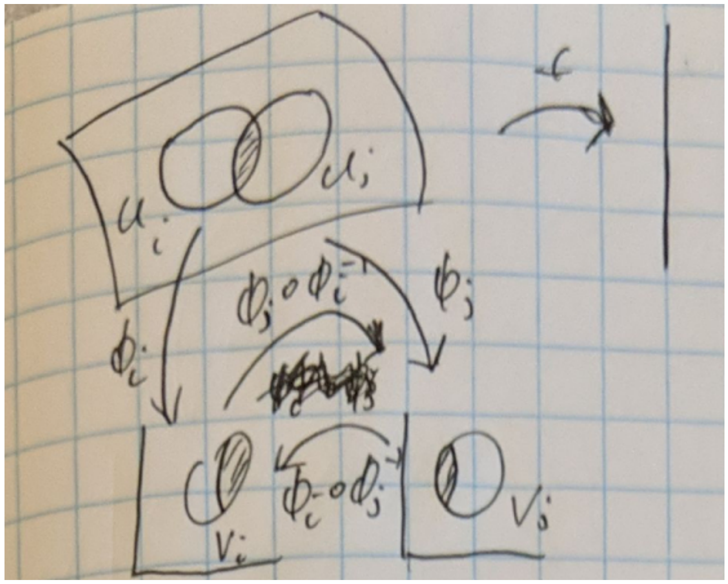


Notation

The notation C^r will be used interchangably with the term smooth.

Definition: Smooth Atlas

Let M be a topological manifold. A smooth atlas on M is a cover $(U_i, \phi_i : U_i \xrightarrow{\sim} V_i \subset \mathbb{R}^n)$ where $\phi_j \circ \phi_i^{-1}$ and $\phi_i \circ \phi_j^{-1}$ are smooth for every i and j.



Say that the charts are (smooth) compatible.

Definition: Smooth Function

Say that f is smooth at $x \in M$ if there exists a chart $U_i \ni x$ such that $f \circ \phi_i$ is smooth at $\phi_i(x)$.
Equivalently, if for every chart $U_i \ni x$ we have that $f \circ \phi_i$ is smooth at $\phi_i(x)$.

- Proof

$$f \circ \phi_j^{-1} = (f \circ \phi_i^{-1}) \circ \underbrace{(\phi_i \circ \phi_j^{-1})}_{C^r}$$

Definition: Compatibility (Equivalence) of Atlases

Atlases A_1 and A_2 are compatible or equivalent if every chart in A_1 is compatible with every chart in A_2.
Equivalently, $A_1 \cup A_2$ is also an atlas.

- Claim: This is an equivalence relation.

Example

Consider $\mathbb{R}$.
Atlas 1: $U = \mathbb{R}$ and $\phi = \mathrm{id}$.
Atlas 2: $U_1 = (1, \infty)$, $\phi_1 = (x) = x^2$, $U_2 = (-\infty, 2)$ and $\phi_2(x) = x$.

Definition: Diffeomorphism

$\mathbb{R}^n \supset V \xrightarrow{F} W \subset \mathbb{R}^n$ is a diffeomorphism if

- F is C^r,
- F is invertible, and
- F^{-1} is C^r

Counterexample

$y = x^3$ is a smooth homemorphism but not a diffeomorphism.

Definition: Smooth Structure / Maximal Atlas

Given an atlas, we may take all compatible atlases and define a smooth structure by the union of all such objects (i.e. the maximal atlas).

Lemma:

Every smooth manifold has a countable, locally finite atlas of precompact charts.

Examples

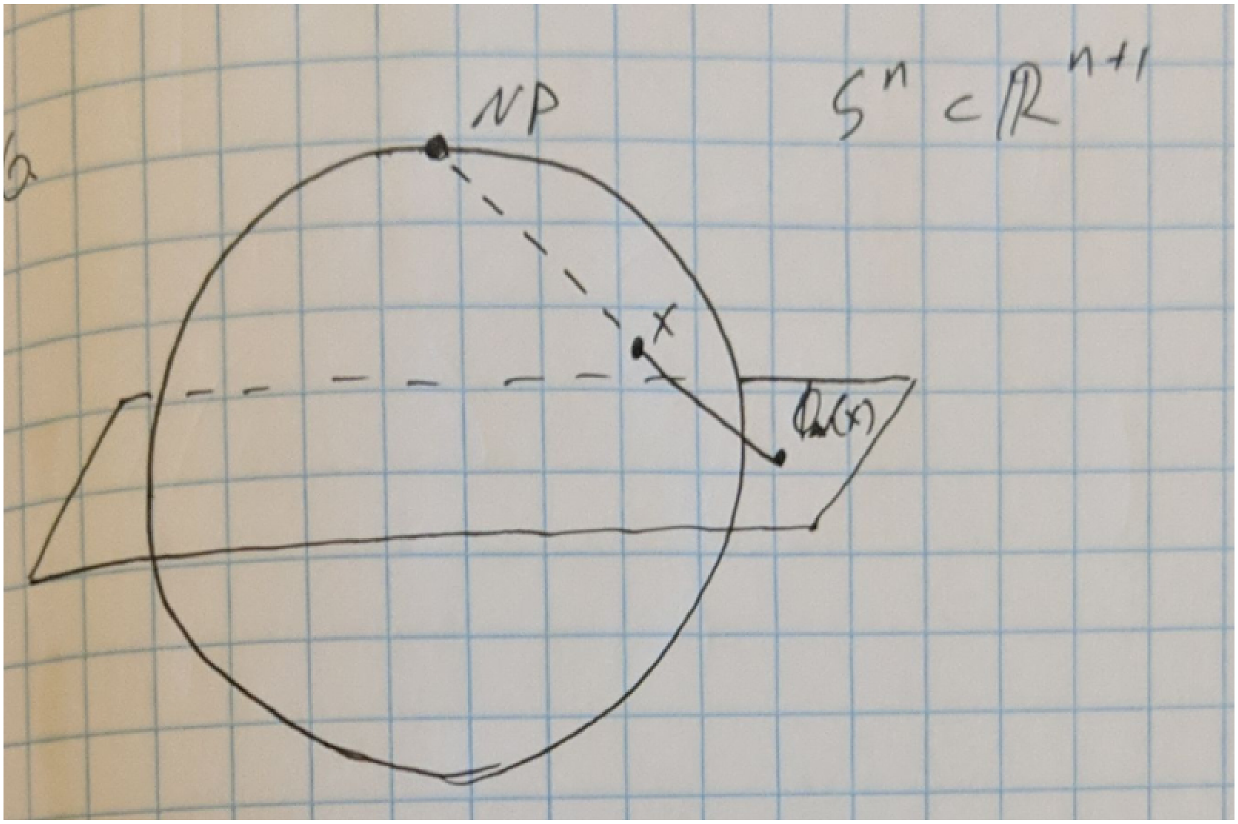
- Zero dimensional manfiolds (i.e. a point).
- $\mathbb{R}^n$ and open subsets of $\mathbb{R}^n$.
- If $M,\ N$ are smooth manfiolds, then $M \times N$ is a a smooth manfiold.

  That is, if we have atlases (U_i, ϕ_i) and (W_j, ψ_j), we may generate $(U_i \times W_j, \phi_i \times \psi_j)$.
- Take $F : M \overset{\text{homeo}}{\to} N$ with N a smooth manfiold. Then M is smooth.

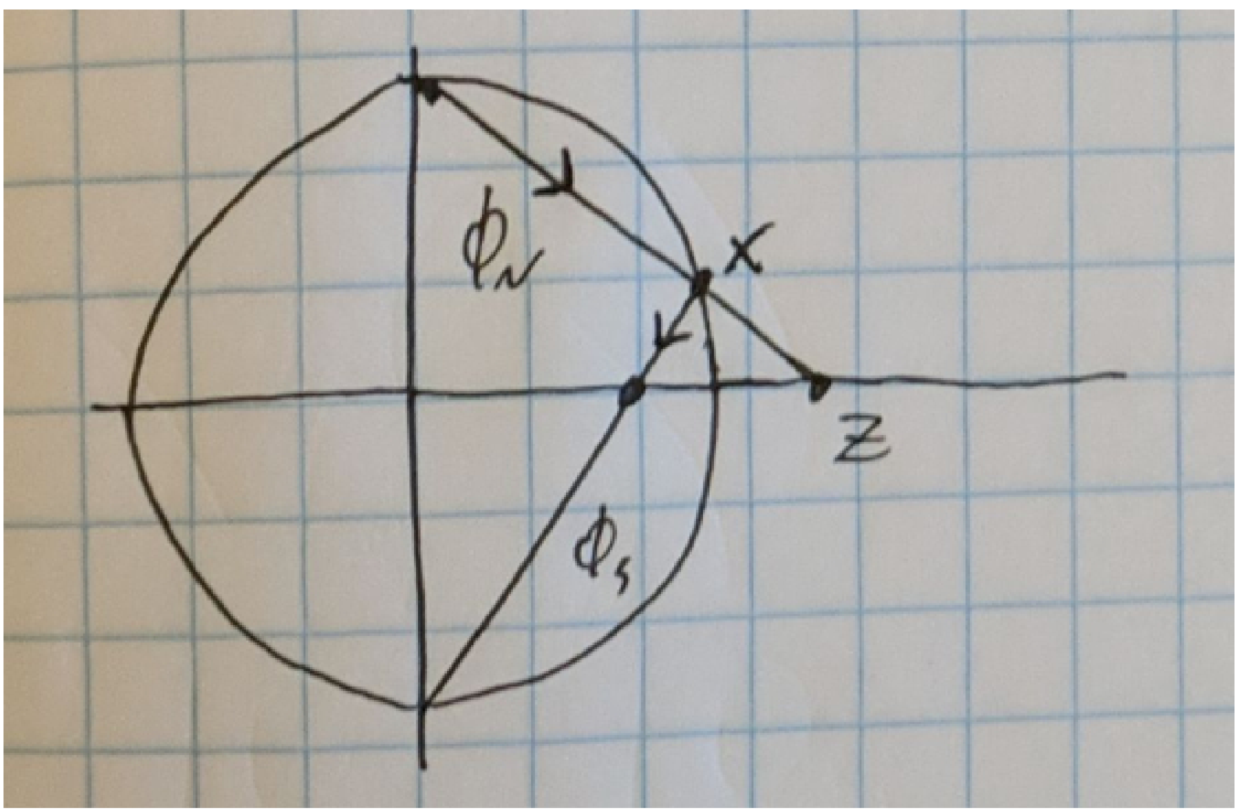
  Take an atlas A on N and the pullback $F^{-1}A = \{(F^{-1}(U_i), \phi_i \circ F)\}$.
- An open subset of a smooth M is a smooth manifold.
- $\mathrm{GL}(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$.

The n-Sphere

- S^n is a manifold



$$U_N = S^n \setminus NP \xrightarrow{\phi_N} \mathbb{R}^n$$
$$U_S = S^n \setminus SP \xrightarrow[\phi_S]{} \mathbb{R}^n$$



$\phi_S\phi_N^{-1}(z) = \frac{z}{|z|^2}$.

– A different construction for S^n.

  Take hemispheres $U \overset{\text{orthogonal projection}}{\longrightarrow} B^n$.

Projective Space

$\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus 0 / \sim$ where $x \sim \lambda x$ for $\lambda \neq 0$.
$[x] = [x_0 : x_2 : \cdots : x_n] = [\lambda x_0 : \lambda x_2 : \cdots : \lambda x_n]$.
Take $U_i = \{x_i \neq 0\}$ and open cover, and maps $U_i \to \mathbb{R}^n$ given by $[x_0 : \cdots : x_n] \mapsto \left(\frac{x_0}{x_i}, \ldots, \frac{x_i}{x_i}, \ldots, \frac{x_n}{x_i}\right)$. Then for $j < i$ take

$$\phi_j \phi_i^{-1}(y_1, \ldots, y_n) = \left(\frac{y_0}{y_j}, \ldots, \frac{y_{j-1}}{y_j}, \frac{y_{i+1}}{y_j}, \ldots, \frac{y_{i-1}}{y_1}, \frac{1}{y_i}, \frac{y_i}{y_i}, \ldots, \frac{y_n}{y_1}\right)$$

Definition: Diffeomorphism

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ B \subset B_{\text{max}} & & A \supset A_{\text{max}} \end{array}$$

F is a diffeomorphism if F is a homoeomorphism and $F^{-1} A_{\text{max}} = B_{\text{max}}$ ($F^{-1} A \sim B$).