Analysis II

January 9, 2024

(Real) Analysis

- Calculus
  - Differential
  - Integral (Riemann)
- Functions and Maps
  - Measure Theory
  - (Lebesgue) Integration
- Topology
  - Completeness (as a metric space)
  - Compactness (Bolzano-Weierstrass theorem [real]) (Arzela-Ascoli)
  - Paracompactness / Metrizable / Baire Category Theorem
  - Algebraic / Combinatoric (continuous maps or functions)

Definition: Cardinality

For sets A, B, $\text{Card}(A) = \text{Card}(B)$ if there exists a one-to-one correspondence $q : A \leftrightarrow B$.
Counting, labelling, indexing, etc.
$\text{Card}(A) \leq \text{Card}(B)$ if $A \subset B$ or there exists a one-to-one mapping $A \to B$.

Definition: Countable

If $A \hookrightarrow \mathbb{N}$, then A is countable.

Theorem

The countable union of countable sets is countable.

Proof

Let $A_i = \{a_j\}_{j=1}^{\infty}$, $i = 1, 2, \ldots$.

$$\begin{matrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ \vdots & & & \\ a_{k1} & a_{k2} & a_{k3} & \cdots \end{matrix}$$

Index by diagonalization.

Theorem

The cartesian product of countable sets is countable.

Proof

$X \times Y = \{(x_i, y_j \mid x_i \in X, y_j \in Y\}$

$$\begin{array}{cccc} (x_1, y_1) & (x_1, y_2) & (x_1, y_3) & \cdots \\ (x_2, y_1) & (x_2, y_2) & (x_2, y_3) & \cdots \\ \vdots & & & \\ (x_k, y_1) & (x_k, y_2) & (x_k, y_3) & \cdots \end{array}$$

Theorem

$\mathrm{Card}\left(2^X\right) > \mathrm{Card}(X)$, where $2^X = \{A \subset X\}$ is the power set of X.

Proof

For all $x \in X$, $\{x\} \subset 2^X$, so $\mathrm{Card}(X) \leq \mathrm{Card}\left(2^X\right)$.
Assume, for sake of contradiction, that $\mathrm{Card}(X) = \mathrm{Card}\left(2^X\right)$.
Then, by definition, there exists a one-to-one correspondence $\phi : X \leftrightarrow 2^X$.
Set $A = \{x \subset X \mid x \notin \phi(x)\}$, and let $a = \phi^{-1}(A)$ (i.e. $A = \phi(a)$).
If $a \in A$, then $a \notin A \subset \phi(a)$; but if $a \notin A$, then $a \in A$, a contradiction.

Theorem

$\mathrm{Card}(\mathbb{R}) = \mathrm{Card}\left(2^{\mathbb{N}}\right)$.

Topology of the Real Line

Completeness (as a metric space)

$d(a, b) = |a - b|, \quad \forall\, a, b \in \mathbb{R}$.

1. $x_i \to x$ if $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ such that $|x_i - x| < \varepsilon$, $\forall i \geq n$.

2. $\{x_i\}$ is Cauchy if $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ such that $|x_i - x_j| < \varepsilon$, $\forall i, j \geq n$.

Definition: Open Inteval

(a, b) is an open set on the real line.
There exist interior points for any subset A of real numbers.
$\forall x \in A$, x is interior if $\exists (a, b)$ such that (1) $x \in (a, b)$ and (2) $(a, b) \subset A$.

- Theorem
  The union of open sets is open.
  The intersection of finitely many open sets is open.
  $\varnothing$ and $\mathbb{R}$ are open.

Definition: Limit Point

A limit point $x \in \mathbb{R}$ of a subset A is a limit point in A if for every open neighborhood U of X, $(U \setminus \{x\}) \cap A \neq \varnothing$.

Definition: Closed

A is closed if A contains all of its limit points.

- Theorem
  A is closed if and only if $A^c = \mathbb{R} \setminus A$ is open.

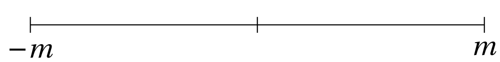
  - Proof
    A closed $\implies$ A^c open.
    Otherwise, $\exists x \in A^c$ such that for every neighborhood U of X, $(U \setminus \{x\}) \cap A = \varnothing$ which would make it a limit point of A not in A. By assumption, A contains all its limit points so this is a contradiction.
    A^c open $\implies$ A closed.
    For any x a limit point of A, assume otherwise that $x \in A^c$.
    Then there exists some neighborhood U of x such that $U \subset A^c$ (since A^c is open).
    It follows that $(U \setminus \{x\}) \cap A = \varnothing$ and x is not a limit point of A, which is a contradiction.

Definition: Sequential Compactness

A is compact if $\forall \{x_i\}$, $x_i \in A$ there exists a convergent subsequence $\{x_{i_k}\}$ and $x_{i_k} \to x \in A$.

- Theorem: Bolzano-Weierstrass
  For $A \subseteq \mathbb{R}$, A is compact if and only if A is closed and bounded.

  - Proof
    A compact $\implies$ A closed and bounded.
    Assume that A is not bounded from abvove.
    Then there exists a sequence $\{x_i\}$, $x_i \in A$ where $x_{i+1} > x_i + 1$ and $\{x_i\}$ has no convergent subsequences.
    Then compactness implies closedness.
    A closed and bounded $\implies$ A (sequentially) compact.
    Let any $\{x_i\}$, $x_i \in A$.
    Claim: $\forall \{x_i\}$ of reals, if there exists $m \in \mathbb{R}$ such that $|x_i| \leq m$, $\forall m$ then there is some convergent subsequence.

    Divide and conquer: dividing the interval in half necessitates that at least one half contains infinitely many points. Repeat indefinitely.

- Theorem: Heine-Borel)
  $A \subseteq \mathbb{R}$ is (sequentially) compact if and only if any open cover has a finite subcover.

  - Proof
    Heine-Borel Property $\implies$ closed and bounded.
    Assume that A is unbounded, $U_n = (-n, n)$ and $\{U_n\}_{n=1}^{\infty}$ an open cover for $A \subseteq \mathbb{R}$ has no finite subcover.
    Assume A is not closed, then $x \in \dot{A}$ (where $\dot{A}$ is the limit set of A) and $x \notin A$, $U_n \left\{\left(-\infty, x - \frac{1}{n}\right) \cup \left(x + \frac{1}{n}, +\infty\right)\right\}$.
    Then $\{U_n\}$ covers $\mathbb{R} \setminus \{x\} \supset A$ has no finite subcover of A.
    A is bounded and closed $\implies$ A is Heine-Borel

Divide and conquer: using open sets with respect to open covers.

Definition: Cantor Set

$C = \{x \in [0,1] \mid \text{the ternary expansion of } x \text{ has only the digits } \{0,2\}\}$.
Equivalenetly, let $C_0 = [0,1]$, $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$.
Then $C_n = \bigcup_{k=1}^{2^n} C_n^k$ and $C = \bigcap_{n=1}^{\infty} C_n$.
$|C_n| = 2^n \left(\frac{1}{3}\right)^n \to 0$.

Definition: Perfectly Symmetric Sets

Let $\{\xi_n\}$ where $\xi_n \in \left(0, \frac{1}{2}\right)$.
$E_0 = [0,1]$, $E_1 = [0, \xi_1] \cup [1 - \xi_1, 1]$, $E_2 = [0, \xi_1\xi_2] \cup [\xi_1 - \xi_1\xi_2, \xi_1] \cup [1 - \xi_1, 1 - \xi_1 + \xi_1\xi_2] \cup [1 - \xi_1\xi_2, 1]$.
Then the cantor set is given by $\xi_n = \frac{1}{3}$.
$E_n = \bigcup_{k=1}^{2^n} E_n^k$, $|E_n^k| = \xi_1\xi_2\cdots\xi_n$, and $|E_n| = \sum |E_n^k| = 2^n \xi_1\xi_2\cdots\xi_n$.
Therefore, $E = \bigcap_{n=1}^{\infty} E_n$ and we define $|E| = \lim_{n\to\infty} |E_n| = \lim_{n\to\infty} \left(2^n \xi_1\xi_2\cdots\xi_n\right) = \lambda$ where $\lambda \in [0,1)$.
Let

$$2\xi_n = \frac{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n-1}\right)^{n-1}}{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n}\right)^{n}} < 1$$

, then

$$2^n \xi_1 \cdots \xi_n = \frac{1}{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n}\right)^{n}} \to \lambda.$$

Proof

$\lim_{n\to\infty} \left(\left(1 + \frac{x}{n}\right)^{n/x}\right)^x = e^x$, then $\lim_{y\to 0} (1+y)^{1/y} = e$, $\log(1+y)^{1/y} = \frac{\log(1+y)}{y} \underset{y\to 0}{\to} 1$.
Observe that

$$\left(\frac{\log(1+y)}{y}\right)' = \frac{\frac{y}{1+y} - \log(1+y)}{y^2} = \left(1 + \frac{1}{1+y} - \log(1+y)\right)' = \frac{1}{(1+y)^2} - \frac{1}{1+y} = -\frac{y}{(1+y)^2} < 0$$

Theorem

Cantor sets and perfect symmetric sets are closed, perfect, uncountable, and nowhere dense.

January 11, 2024

Last Week

Cardinality.
Topology of the reals.

- Cantor (perfect symmetric sets)

$C_0 = [0,1]$
$C_1 = [0,1/3] \cup [2/3,1]$
$C_2 = [0,1/9] \cup [2/9,3/9] \cup [6/9,7/9] \cup [8/9,1]$
$C_n = \bigcup_{n=1}^{2^n} C_n^k$
$|C_n^k| = \left(\frac{1}{3}\right)^n$
$C = \bigcap_{n=1}^{\infty} C_n$
$|C_n| = 2^n \frac{1}{3^n} = \left(\frac{2}{3}\right)^n \implies |C| = \lim_{n\to\infty} |C_n| = 0$
Closed, no interior points and uncountable.

- Perfect Symmetric Sets

$\{\xi_k\} \in \left(0, \frac{1}{2}\right)$
$E_0 = [0,1]$
$E_1 = [0,\xi_1] \cup [1-\xi_1, 1]$
$E_2 = [0,\xi_1\xi_2] \cup [\xi_1 - \xi_1\xi_2, \xi_1] \cup [1-\xi_1, 1-\xi_1+\xi_1\xi_2] \cup [1-\xi_1\xi_2, 1]$
$E_n = \bigcup_{n=1}^{2^n} E_n^k$
$|E_n^k| \xi_1\xi_2 \cdots \xi_n$
$|E_n| = 2^n \xi_1\xi_2\cdots\xi_n$
$2\xi_n = \dfrac{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n-1}\right)^{n-1}}{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n}\right)^{n}} < 1$
$|E_n| = \dfrac{1}{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n}\right)^{n}}$
$|E| = \lim_{n\to\infty} |E_n| = \frac{1}{e^{\log\left(\frac{1}{n}\right)}} = \lambda, \quad \lambda \in (0,1)$

Volterra's Function

$$\phi(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

IMAGE HERE - graph of phi(x)

$$\phi'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$f(x) = \begin{cases} 0 & x \in E \\ \phi(x-a) & x \in (a, a+y) \\ -\phi(b-x) & x \in (b-y, b) \\ \phi(y) & x \in (a+y, b-y) \end{cases} \quad , \quad (a,b) \in E^c$$

IMAGE HERE - f interval (a,b)

Propositions

1. $f'(x) = 0$ for $x \in E$.

2. $f'(x)$ discontinuous on E.

3. f' exists on $[0,1]$ and is bounded.

Since $|E| > 0$, $f'(x)$ is not Riemann integrable and, therefore, the fundamental theorem of calculus does not apply.

Lebesgue Outer Measure

$|(a,b)| = b - a$.
Let $A \subseteq \mathbb{R}$, then $m^*(A) = \inf\{\sum_{n=1}^{\infty} I_n \mid A \subseteq \bigcup_{n=1}^{\infty}\}$
Question: $m^*(A \cup B) \stackrel{?}{=} m^*(A) + m^*(B)$ for $A \cap B \neq \varnothing$?

Properties

1. $A \subseteq B \implies m^*(A) \leq m^*(B)$.

2. $m^*(\varnothing) = 0$.

3. If I is an interval, then $m^*(I) = |I|$.

4. If $\{A_i\}$ is countable, $m^*(\bigcup A_i) \leq \sum m^*(A_i)$.

- Proof of 4
  $\forall A_i$, $\exists\{I_n\}$ open intervals such that $\sum_n |I_n| < m^*(A_i) + \frac{\varepsilon}{2^i}$.
  Then $\bigcup_i \bigcup_n I_n^i \supset \bigcup_i A_i$, and $\sum_{n,i} |I_n^i| = \sum_i \left(\sum_n |I_n^i|\right) \leq \sum_i \left(m^*(A_i) + \frac{\varepsilon}{2^i}\right)$.

  - Corollary

    If A is countable, then $m^*(A) = 0$.
    Thus, by contraposition, every interval is uncountable.

Proposition

For $A \subseteq \mathbb{R}$, $\forall \varepsilon > 0$, $\exists U$ open such that $A \subseteq U$ and $m^*(U) \leq m^*(A) + \varepsilon$.

Corollary

There exists G in the intersection of countable open sets such that $m^*(G) = m^*(A)$ and $G \supseteq A$.

Caratheodory Criteria

If $\forall E$, $m^*(E \cap A) + m^*(E \cap A^c) = m^*(E)$, then A is Lebesgue measurable.

- Remark: $m^*(E \cap A) + m^*(E \cap A^c) \leq m^*(E) \leq +\infty$

Propositions

1. If A is measurable, then A^c is measurable.

2. $m^*(A) = 0$, then A is measurable.

3. If $A,\ B$ are measurable, then $A \cup B,\ A \cap B,\ A \setminus B$ are measurable.

4. If $\{A_i\}_{i=1}^k$ are disjoint and measurable, then $m^*\left(\bigcup_{i=1}^k A_i\right) = \sum_{i=1}^k m^*(A_i)$.

- Proof of 3

$$
\begin{aligned}
m^*(E \cap (A \cup B)) + m^*(E \cap (A \cup B)^c) &= m^*((E \cap A) \cup (E \cap B)) + m^*(E \cap A^c \cap B^c) \\
&= m^*(E \cap A) + m^*((E \cap A^c) \cap B) + m^*((E \cap A^c) + B^c) \\
&\le m^*(E)
\end{aligned}
$$

Since o$(A \cap B)^C = A^c \cup B^c$, this holds from before; similarly, $A \setminus B = A \cap B^c = A^c \cup B$.
If $A,\ B$ disjoint, then

$$
\begin{aligned}
m^*(A \cup B) &= m^*(E \cap A) + m^*(E \cap A^c) \\
&= m^*(A) + m^*(B)
\end{aligned}
$$

Theorem

If $\{A_i\}$ is a countable collection of disjoint and measurable sets, then

1. $\bigcup_i A_i$ is measurable.

2. $m^*\left(\bigcup_i A_i\right) = \sum_i m^*(A_i)$.

Proof of 1

Want to show:

$$
m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) + m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right) \le m^*(E)
$$

By assumption, since the measure of E is finite, $m^*\left(E \cap \bigcup_{i=1}^{\infty} A_i\right) < +\infty$.
Claim: $\forall \varepsilon > 0,\ \exists k$ such that
Therefore $m^*\left(E \cap \bigcup_{i=1}^{k} A_i\right) \ge m^*\left(E \cap \bigcup_{i=1}^{\infty} A_i\right) - \varepsilon$.

$$
m^*(E) \le m^*\left(E \cap \bigcup_{i=1}^{k} A_i\right) + \varepsilon + m^*\left(E \cap \left(\bigcup_{i=1}^{k} A_i\right)^c\right) \le m^*(E) + \varepsilon
$$

Proof of 2

We have shown $m^*\left(\bigcup_i A_i\right) \le \sum_{i=1}^{\infty} m^*(A_i)$.
Assume $m^*\left(\bigcup_i A_i\right) < +\infty$, then

$$
\sum_{i=1}^{k} m^*(A_i) = m^*\left(\bigcup_{i=1}^{k} A_i\right) \le m^*\left(\bigcup_{i}^{\infty} A_i\right) \implies \sum_{i=1}^{\infty} m^*(A_i) \le m^*\left(\bigcup_{i}^{\infty} A_i\right)
$$

January 16, 2024

Office Hours Tuesday / Thursday 10 AM - 11:30 AM
A note on notation: Latin characters are to be understood as countable indecies; greek as possible uncountable.

Lebesgue Outer Measure

$A \subset \mathbb{R}$
$m^*(A) = \inf\{\sum_{i=1}^\infty |I_i| \mid \bigcup_{i=1}^\infty I_i \supset A,\ I_i \text{ open intervals}\}$

Properties

1. $A \subset B \implies m^*(A) \le m^*(B)$.
2. $m^*(\varnothing) = 0$.
3. $m^*(I) = |I|$ for I an interval.
4. Countable Subadditivity: $\{A_i\}_{i=1}^\infty \implies m^*\left(\bigcup_{i=1}^\infty A_i\right) \le \sum_{i=1}^\infty m(A_i)$.
5. $\forall A \subset \mathbb{R},\ \forall \varepsilon > 0,\ \exists$ open neighborhood $U \supseteq A$ such that $m^*(U) \le m^*(A) + \varepsilon$.
6. $\exists G \in \bigcap_{n=1}^\infty U_n$, U_n open, $U_n \supseteq A \implies G \supseteq A$, such that $m^*(G) = m^*(A)$.

Measurable (Caratheodory Criterion)

$\forall A \subseteq \mathbb{R}$ is Lebesgue measurable if

$$m^*(A) = m^*(E \cap A) + m^*(E \cap A^c)$$

Essentially, $m^*(E \cap A) + m^*(E \cap A^c) \le m^*(E) \le +\infty$.

- Propositions
  1. A measurable $\implies A^c$ measurable.
  2. $m^*(A) = 0 \implies A$ measurable.
  3. $\{A_i\}_{i=1}^\infty$ countable with A_i measurable, then
     (a) $\bigcap_{i=1}^\infty A_i$ are measurable.
     (b) Moreover, $A_i \cap A_j = \varnothing \implies m^*\left(\bigcup_{i=1}^\infty (A_i) = \sum_{i=1}^\infty m^* A_i\right)$.
     (c) A, B measurable $\implies A \cup B$, $A \cap B$, $A \setminus B$ measurable.
     (d) $A \cap B = \varnothing \implies m^*(A \cup B) = m^*(A) + m^*(B)$.
     (e) $\{A_i\}_i^\infty$ with A_i measurable, then $\bigcup_{i=1}^\infty A_i$ is measurable and $A_i \cap A_j \varnothing \implies m^*\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty m^*(A_i)$.
  - Proof of e $\forall E \subset \mathbb{R}$, $m^*\left(E \cap \left(\bigcup_{i=1}^\infty A_i\right)\right) + m^*\left(E \cap \left(\bigcup_{i=1}^\infty A_i\right)^c\right)$.
    Claim: $m^*\left(E \cap \left(\bigcup_{i=1}^\infty A_i\right)\right) = \sum_{i=1}^\infty m^*(E \cap A_I)$ for $A_i \cap A_j = \varnothing$.

Then, $\forall \varepsilon > 0, \exists n \in \mathbb{N}$,

$$m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) = \sum_{i=1}^{\infty} m^*(E \cap A_i) \leq \sum_{i=1}^{n} m^*(E \cap A_i) + \varepsilon$$
$$\implies m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) + m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right) \leq m^*\left(E \cap \left(\bigcup_{i=1}^{n} A_i\right)\right) + m^*\left(E \cap \left(\bigcup_{i=1}^{n} A_i\right)^c\right) + \varepsilon \leq m^*(E) + \varepsilon$$
$$\implies \bigcup_{i=1}^{\infty} A_i \text{ measurable}$$

Proof of Claim:
Step 1: A, B measurable and $A \cap B = \varnothing$. Since A is measurable,

$$\begin{aligned} m^*(E \cap (A \cup B)) &= m^*((E \cap (A \cup B)) \cap A) + m^*((E \cap (A \cup B)) \cap A^c) \\ &= m^*(E \cap A) + m^*(E \cap A^c) \end{aligned}$$

For $\{A_i\}_{i=1}^{\infty}$, $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} A_i'$ with $A_1 = A_1'$ and $A_i' = A_i \setminus \bigcup_{k=1}^{i-1} A_k$, $\forall i \geq 2$.
Therefore $A_i' \cap A_j' = \varnothing$ and A_i' is measurable.

$$m^*\left(\bigcup_{i=1}^{n} A_i\right) \leq m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} m^*(A_i)$$
$$m^*\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} m^*(A_i) \leq m^*\left(\bigcup_{k=1}^{\infty} A_k\right) < +\infty \implies \sum_{i=1}^{\infty} m^*(A_i) \leq m^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{i=1}^{\infty} m^*(A_i)$$

Sigma Algebra and Borel Sets

Definition: Sigma Algebra

Let $S \subset 2^X$ for some set X. Then S is said to be a σ-algebra if

1. $\varnothing \in S$.
2. $A^c \in S$ if A^c.
3. $\bigcup_{i=1}^{\infty} A_i \in S$ if $A_i \in S$.
    - Equivalently, $\bigcap_{i=1}^{\infty} A_i \in S$ if $A_i \in S$.

Theorem:

The collection $\mathcal{L}$ of all Lebesgue measurable sets is a σ-algebra.

Definition: Borel Set

Let B be the σ-algebra generated by open sets of reals (i.e. the smallet σ-algebra containing all open sets of reals).
Then $b \in B$ is called a Borel set.

Remark

B is generated by $\{(a,+\infty) \mid a \in \mathbb{R}\}$.

1. $(a,+\infty)^c = (-\infty, a]$.
2. $\bigcap_{n=1}^{\infty}\left(a - \frac{1}{n}, +\infty\right) = [a,+\infty)$.
3. $[a,+\infty)^c = (-\infty, a)$.
4. $(-\infty, b) \cap (a,+\infty) = (a,b)$.
5. $(-\infty, b] \cap [a,+\infty) = [a,b]$.

Theorem:

Any Borel set is Lebesgue measurable.

Proof

It suffices to demonstrate that $(a,+\infty)$ is measurable $\forall\, a \in \mathbb{R}$.
$\forall\, E \subset \mathbb{R}$, we want to show that $m^*(E \cap (a,+\infty)) + m^*(-\infty, a]) \leq m^*(E)$.
Then, $\forall \varepsilon > 0$, $\exists \mathcal{C} = \{I_i\}$ with I_i open intervals such that $\sum_{I_i \in \mathcal{C}} |I_i| \leq m^*(E) + \varepsilon/2$. Set

$$
\begin{aligned}
\mathcal{C}^{\ell} &= \{I \in \mathcal{C} \mid x < a, \forall\, x \in I\} \\
\mathcal{C}^{r} &= \{I \in \mathcal{C} \mid x > a, \forall\, x \in I\} \\
\mathcal{C}^{m} &= \{I \in \mathcal{C} \mid a \in I\} = \{I_k\}
\end{aligned}
$$

Then A$\mathcal{C} = \mathcal{C}^{\ell} \cup \mathcal{C}^{r} \cup \mathcal{C}^{m}$.
$\forall\, I_k \in \mathcal{C}^m = \{I_k\}$, $I_k = (c_k, d_k)$ for some $c_k, d_k \in \mathbb{R}$, define

$$
\begin{aligned}
I_k^{\ell} &= \left(c_k, a + \frac{\varepsilon}{2^{k+1}}\right) \\
I_k^{r} &= (a, d_k)
\end{aligned}
$$

Let $\mathcal{C}^m = \{I_k^{\ell}\} \cup \{I_k^{r}\} = \overline{\mathcal{C}}^{m\ell} \cup \overline{\mathcal{C}}^{mr}$. Then

$$
\begin{aligned}
&\mathcal{C}^{\ell} \cup \overline{\mathcal{C}}^{m\ell} \text{ covers } E \cap (-\infty, k] \\
&\mathcal{C}^{r} \cup \overline{\mathcal{C}}^{mr} \text{ covers } E \cap (k, +\infty) \\
&\mathcal{C}^{\ell} \cup \mathcal{C}^{r} \cup \mathcal{C}^{m} \text{ covers } E
\end{aligned}
$$

Observe that

$$
|I_k^{\ell}| + |I_k^{r}| \leq |I_k| + \frac{\varepsilon}{2^{k+1}}
$$

Therefore

$$\begin{aligned}
m^*(E \cap (a, +\infty)) &\le \sum_{I \in \mathcal{C}^R + \overline{\mathcal{C}}^{mr}} |I| \\
m^*(E \cap [-\infty, a]) &\le \sum_{I \in \mathcal{C}^\ell + \overline{\mathcal{C}}^{m\ell}} |I|
\end{aligned}$$

Therefore

$$\begin{aligned}
m^*(E \cap (a, +\infty)) + m^*(E \cap (-\infty, a]) &\le \sum_{I \in \mathcal{C}^r \cup \overline{\mathcal{C}}^{mr}} |I| + \sum_{I \in \mathcal{C}^\ell |I| \cup \overline{\mathcal{C}}^{m\ell}} |I| \\
&= \sum_{I \in \mathcal{C}^r} |I| + \sum_{I \in \mathcal{C}^\ell} |I| + \sum_k \left(|I_k^\ell| + |I_k^r| \right) \\
&\le \sum_{I \in \mathcal{C}} |I| + \sum_{k=1}^{+\infty} \frac{\varepsilon}{2^{k+1}} \\
&\le m^*(E) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&\le m^*(E) + \varepsilon
\end{aligned}$$

Lebesgue Measurable vs Borel

Theorem

The following statements are equivalent

1. A is measurable.
2. $\forall \varepsilon > 0$, $\exists U$ open, $U \supset A$ such that $m(U \setminus A) < \varepsilon$.
3. $\forall \varepsilon > 0$, $\exists C$ closed, $C \subset A$ such that $m(A \setminus C) < \varepsilon$.
4. $\forall A \in \mathbb{R}$, $\exists \bigcap_{n=1}^{\infty} U_n = F \in B$, U_n open, $U_n \supset A$ such that $F \supset A$ and $m(F \setminus A) = 0$.
5. $\exists \{C_n\}$, C_n closed and $C_n \subset A$ such that $G = \bigcup_{n=1}^{\infty} C_n \subset A$ and $m(A \setminus G) = 0$.

Corollary

Every measurable set is the union of a Borel set and a measure zero set.

Proof 1 Implies 2

Step 1: if $m(A) < \infty$, then for $\varepsilon > 0$, $\exists U$ open and $U \supset A$, then

$$m(U) \le m(A) + \varepsilon \iff m(U \setminus A) = m(U) - m(A) \le \varepsilon$$

Step 2: let $A_n = A \cap (-n, n)$, $n \in \mathbb{N}$.
Then $m(A_n) \le 2n < +\infty$.
For ech A_n, $\exists U_n$ open with $U_n \supset A_n$ and $m(U_n \setminus A_n) < \frac{\varepsilon}{2^n}$
Let $U = \bigcup_{n=1}^{\infty} U_n$ and $A = \bigcup_{n=1}^{\infty} A_n$.

Now verify that

$$m(U \setminus A) = m\left(\bigcup_{n=1}^{\infty} U_n \setminus \bigcup_{n=1}^{\infty} A_n\right) = m\left(\bigcup_{n=1}^{\infty} U_n \setminus A_n\right) \le \sum_{n=1}^{\infty} m(U_n \setminus A_n) \le \varepsilon$$

Proof 2 Implies 3

Write

$$A \setminus C = A \cap C^c = C^c \cap A = C^c \setminus A^c$$

Apply (2).

Proof 3 Implies 4

U_n comes from 2.

Proof 4 Implies 5

Follows from 4.

Proof 5 Implies 1

$A = G \cup (A \setminus G) \implies A$ is measurable.

Example: Non-measurable Set

Define $x \sim y$ if $x - y \in \mathbb{Q}$, $\forall x, y \in \mathbb{R}$.
Let $A = \{x \in (0,1) \mid x \text{ is a representative of each class } \mathbb{R}/\sim\} \subset (0,1) \subset \mathbb{R}$.
Claim: A is not Lebesgue measurable.
Let $(-1,1) \supset S = \bigcup_{r \in \mathbb{Q} \cap (0,1)} (A + r) \supset (0,1)$, and observe that $\mathbb{Q} \cap (0,1)$ is countable.
So $(A + r) \cap (A + s) = \varnothing$ for $s \neq r$.
Then $1 < m(S) < 2$, so $m(A) = 0$ and $m(A) > 0$ are both contradictions.

January 18, 2024

Abstract measure theory.

Definition: Topological Space

A set X equipped with a collection of subsets $\tau \subset 2^X$ where τ is a topology if

1. $\varnothing, X \in \tau$
2. Union of subsets in τ remains in τ.
3. Intersection of finitely many subsets in τ remains in τ.

Any subset of τ is called an open set of X.

Definition: Measure Space

For a set X with $\Lambda \subset 2^X$ a σ-algebra such that

1. $\varnothing \in \Lambda$
2. $A^c \in \Lambda$ if $A \in \Lambda$.
3. $\bigcup_{i=1}^{\infty} A_i \in \Lambda$ if $A_i \in \Lambda$.
4. Remark: Borel Sigma Algebra

The σ-algebra generated by τ for a topological space (X, τ).
The measure space (X, Λ, μ), $\Lambda \subset 2^X$ a σ-algebra equipped with set function $\mu : \Lambda \to [0, +\infty]$ such that

1. $\mu(\varnothing) = 0$
2. $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m(A_i)$ for $A_i \in \Lambda$ and $A_i \cap A_j = \varnothing$ for all $i \neq j$ (countable additivity).

Proposition: Monotonicity

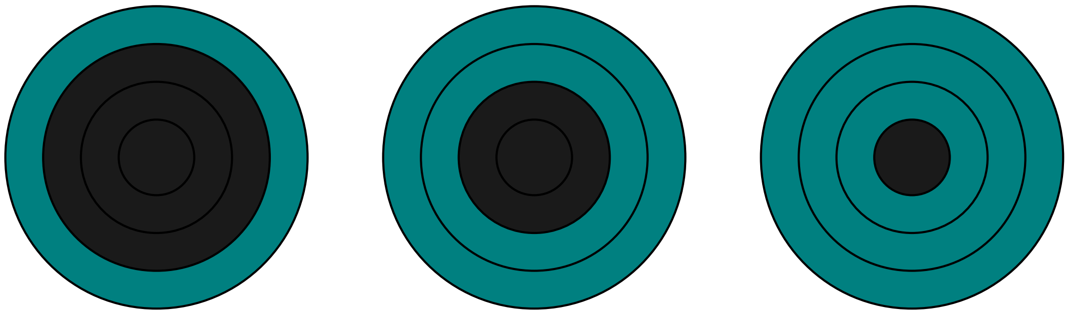
$A, B \in \Lambda,\ A \subseteq B \implies \mu(A) \leq \mu(B)$.

Proposition: Countable Subadditivity

$\mu(\bigcup A_i) \leq \sum \mu(A_i)$ if $A_i \in \Lambda$

Proposition: Monotone Convergence

Given $A_i \subset \Lambda$ such that $A_i \subset A_{i+1}$ where $A = \bigcup_{i=1}^{\infty} A_i$, then $\mu(A_i) \to \mu(A)$.
Similarly, if $A_i \supset A_{i+1}$ such that $A = \bigcap_{i=1}^{\infty} A_i$, then $\mu(A_i) \to \mu(A)$ if $\mu(A_k) < +\infty$ for some $k = 1, 2, 3, \ldots$.



Given $A_i' = \begin{cases} A_1 & i = 1 \\ A_i \setminus \bigcup_{j=1}^{i-1} A_j & i > 1 \end{cases}$, $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} A_i'$ and

$$\mu(A) \sum_{i=1}^{\infty} A_i' = \lim_{n\to\infty} \sum_{i=1}^{\infty} \mu(A_i')$$

and

$$\sum_{i=1}^{n} \mu(A_i') = \mu(A_1) + (\mu(A_2) - \mu(A_1)) + (\mu(A_3) - \mu(A_2)) + \cdots + (\mu(A_n) - \mu(A_{n-1})) = \mu(A_n)$$

Similarly, $A_1 \setminus A = \bigcup_{i=2}^{\infty} (A_i \setminus A_{i-1})$ where $\mu(A_1) < +\infty$ gives

$$\mu(A_1) - \mu(A) = \mu(A_1) + \sum_{i=2}^{\infty} (\mu(A_i) - \mu(A_{i-1})) = \lim_{n\to\infty} \mu(A_n)$$

Definition: Complete Measure Space

A measure space (X, Λ, μ) is complete if $\forall A \in \Lambda$ with $\mu(A) = 0$, then $\forall B \subset A$ and $B \in \Lambda$.

Example

The Lebesgue measure space on the reals $(\mathbb{R}, \mathcal{L}, m)$ is complete.

Theorem: Completion of a Measure Space

Given a measure space (X, Λ, μ), then there exists $(X, \overline{\Lambda}, \overline{\mu})$ such that

1. $\Lambda \subset \overline{\Lambda}$.
2. If $A \in \Lambda$, then $\overline{\mu}(A) = \mu(A)$.
3. $(X, \overline{\Lambda}, \overline{\mu})$ is complete.

Proof (Construction)

Let $\overline{\Lambda} = \{A \cup Z \mid A \in \Lambda, \exists D \in \Lambda, m(D) = 0, Z \subset D\}$ and $\overline{\mu}(A \cup Z) := \mu(A)$.
Verify:

1. $\overline{\Lambda}$ is a σ-Algebra.
   (a) If $A \cup Z \in \overline{\Lambda}$, then $(A \cup Z)^c \in \overline{\Lambda}$.
   (b) If $A_i \cup Z_i \in \overline{\Lambda}$, then $\bigcup (A_i \cup Z_i) \in \overline{\Lambda}$.
2. $\overline{\mu}$ is a well-defined measure on $\overline{\Lambda}$.
3. $(X, \overline{\Lambda}, \overline{\mu})$ is complete.

- Proof of 1
  Given $A \in \Lambda$ and $Z \subset D$ where $\mu(D) = 0$ and $D \in \Lambda$, we know $D^c \subset Z^c$ and $Z^c = D^c \cup (Z^c \cap D)$. Therefore

$$(A \cup Z)^C = A^c \cap Z^c = A^c \cap (D^c \cup (Z^c \cap D)) = (A^c \cap D^c) \cup (A^c \cap Z^c \cap D) \in \overline{\Lambda}$$

  Since $A^c \cap D^c \in \Lambda$ and $A^c \cap Z^c \cap D \in D$
  Since $\bigcup A_i \in \Lambda$ and $\bigcup Z_i \subset \bigcup D_i$,

$$\bigcup_{i=1}^{\infty} (A_i \cup Z_i) = \left(\bigcup_{i=1}^{\infty} A_i \right) \cup \left(\bigcup_{i=1}^{\infty} Z_i \right) \in \overline{\Lambda}$$

- Proof of 2
  Given $A_1 \cup Z_1 = A_2 \cup Z_2$, $A_1 \subset A_2 \cup Z_2 \subset A_2 \cup D_2$ implies $\mu(A_1) \leq \mu(A_2)$.
  Then, $\mu(A_2) \leq \mu(A_1) \implies \mu(A_1) = \mu(A_2)$. So $\overline{\mu}$ is well defined.
  Given $\{A_i \cup Z_i\}$ with $(A_i \cup Z_i) \cap (A_j \cup Z_j) = \varnothing$ for all $i \neq j$,

$$\overline{\mu}\left(\bigcup_{i=1}^{\infty}(A_i \cup Z_i)\right) = \overline{\mu}\left(\left(\bigcup_{i=1}^{\infty} A_i\right) \cup \bigcup_{i=1}^{\infty} Z_i\right) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty} \overline{\mu}(A_i \cup Z_i)$$

  So $\overline{\mu}$ is countably additive and therefore a measure.

Borel Measure and Radon Measure

Given a measure space (X, Λ, μ) and an underlying topology (X, τ),

Definition: Borel Measure

μ is a Borel measure if all borel sets $\tau \subset \Lambda$.

Definition: Locally Finite Measure

μ is locally finite if $\forall x \in X$, $\exists U \subset X$ a neighborhood such that $\mu(U) < +\infty$.

Definition: Borel Regularity

μ is Borel regular if $\forall A \in \Lambda$, $\exists B$ a Borel set such that $B \supseteq A$ and $\mu(B) = \mu(A)$.

Definition: Radon Measure

μ is a Radon measure if

1. it is a Borel measure.

2. $\mu(K) \leq +\infty$ for K compact.

3. $\mu(V) = \sup\{\mu(K) \mid K \subset V, K \text{ compact}\}$, V open.

4. $\mu(A) = \inf\{\mu(V) \mid A \subset V, V \text{ open}\}$, $\forall A \in \Lambda$.

- Example 1
  Lebesgue measure.

- Example 2
  Point charge: $\mu(\{x\}) = 1$ and $\mu(A) = 0$ if $x \notin A$.

Theorem:

Let (X, Λ, μ) be a Borel regular measure space where the underlying topology (X, τ) is a metric space. Then

1. For $A \in \Lambda$ with $\mu(A) < +\infty$, $\forall \varepsilon > 0$, $\exists C \subseteq A$ closed such that $\mu(A \setminus C) < \varepsilon$.

2. For $A \in \Lambda$, $\exists \{V_i\}$ open sets such that $A \subset \bigcup_{i=1}^{\infty} V_i$ and $\mu(V_i) < +\infty$. Then $\forall \varepsilon > 0$, $\exists U$ open with $A \subset U$ and $\mu(U \setminus A) < \varepsilon$.

Proof

Given $\mu(A) < +\infty$, $\nu(B) = \mu(B \cap A) < +\infty$, $\forall B \in \Lambda$ and (X, Λ, ν).
Let $F = \{B \in \Lambda \mid \forall \varepsilon > 0, \exists C \subset B$ closed, with $\nu(B \setminus C) < \varepsilon\}$.
Note that closed sets are in F.
Claim 1: the Borel σ-algebra is in F.
Claim 2: if $A_i \in F$, $\bigcup A_i, \bigcap A_i \in F$.
Given claim 2, $\forall U$ open, U^c is closed. Then $U_\varepsilon = \{x \in U \mid \text{dist}(x, U^c) \leq \varepsilon\}$ is closed and, therefore, $U = \bigcup_{i=1}^{\infty} U_{1/i}$.
So, given $A_i \in F$, $\exists C_i \subset A_i$ closed where $\nu(A_i \setminus C_i) < \varepsilon/2^{i+1}$. We want to show that $\nu\left(\bigcap A_i \setminus \bigcap C_i\right) < \varepsilon$.
Then, for $x \in \bigcap A_i \setminus \bigcap C_i$, $x \in A_i$ for all i and $x \notin C_{i_0}$ for some i_0.
Therefore $x \in A_{i_0}$, $x \notin C_{i_0}$, and $x \in A_{i_0} \setminus C_{i_0}$. It follows that

$$\bigcap_{i=1}^{\infty} A_i \setminus \bigcap_{i=1}^{\infty} C_i \subset \bigcup_{i=1}^{\infty} (A_i \setminus C_i)$$
$$\nu\left(\bigcap_{i=1}^{\infty} A_i \setminus \bigcap_{i=1}^{\infty} C_i\right) \leq \sum_{i=1}^{\infty} \nu(A_i \setminus C_i) < \varepsilon$$

Therefore

$$\nu\left(\bigcup_{i=1}^{\infty} A_i \setminus \bigcup_{i=1}^{n} C_i\right) \to \nu\left(\bigcup_{i=1}^{\infty} A_i \setminus \bigcup_{i=1}^{\infty} C_i\right) \leq \nu\left(\bigcup_{i=1}^{\infty} (A_i \setminus C_i\right) < \frac{\varepsilon}{2}$$

so $\exists N >> 1$ such that $\nu\left(\bigcup_{i=1}^{\infty} \setminus \bigcup_{i=1}^{N} C_i < \varepsilon\right)$ with $\bigcup_{i=1}^{N} C_i$ closed.

Restatement

For A Borel,

$$\varepsilon > \nu(A \setminus C) = \mu((A \setminus C) \cap A) = \mu(A \setminus C)$$

January 23, 2024

Review - Abstract Measure

Given (X, Λ, μ) where $\Lambda \subseteq 2^X$ is a σ-algebra, $\mu : \Lambda \to [0, +\infty]$

1. $\mu(\varnothing) = 0$.

2. $m\left(\bigcup A_i\right) = \sum \mu(A_i)$, $A_i \cap A_j = \varnothing$.

Properties of a Measure

Monotonicity

$\mu(A) \subseteq \mu(B)$, $A, B \in \Lambda$, $A \subseteq B$

Countable Subadditivity

$\mu\left(\bigcup A_i\right) \leq \sum \mu(A_i)$

Monotone Convergence

$A_i \subset A_{i+1}$, $A_i \to \bigcup A_i \implies \mu(A) = \mu(\bigcup A_i)$.
$A_i \supset A_{i+1}$, $A_i \to \bigcap A_i \implies \mu(A_i) \to \mu(\bigcap A_i)$ if $\mu(A_1) < \infty$

- Example
  $A_n = (n, +\infty)$ gives $\bigcap A_n = \varnothing$

Completeness of a Measure

(X, Λ, μ) is complete if $\forall A \in \Lambda$ with $\mu(A) = 0$, then $\forall B \in \Lambda$ if $B \subseteq A$.

Theorem:

Given (X, Λ, μ), there exists $(X, \overline{\Lambda}, \overline{\mu})$ such that $\Lambda \subset \overline{\Lambda}$ and $\overline{\mu}(A) = \mu(A)$ if $A \in \Lambda$.

$$\overline{\Lambda} = \{A \cup Z \mid A \in \Lambda, Z \subset D, D \in \Lambda \text{ with } \mu(D) = 0\}$$

$$\overline{\mu}(A \cup Z) = \mu(A)$$

$(X, \overline{\Lambda}, \overline{\mu})$ is complete.

Measure Space with Topology

Given a topological space (X, τ), a measure space (X, Λ, μ)

Definition: Locally Finite

The measure μ is locally finite if $\forall x \in X$, there exists an open neighborhood U of x such that $U \in \Lambda$ and $\mu(U) < +\infty$.

Definition: Borel Measure

μ is a Borel measure if the Borel σ-algebra generated by τ, $\mathcal{B}$, is a subset of Λ.

Definition: Borel Regular

$\forall A \in \Lambda$, $\exists B \in \mathcal{B}$ such that $B \supset A$ and $\mu(B) = \mu(A)$.

Definition: Radon Measure

1. Borel.
2. $\mu(K) < +\infty$ for K compact.
3. $\mu(V) = \sup\{\mu(K) \mid K \text{ compact}, K \subset V\}$, $\forall V$ open.
4. $\mu(A) = \inf\{\mu(V) \mid V \text{ open}, A \subset V\}$, $\forall A \in \Lambda$.

Theorem:

If X is a metric space equipped with a Borel regular (X, Λ, μ), then

1. $\forall A \in \Lambda$, $\mu(A) < +\infty$, $\forall \varepsilon > 0$, $\exists C$ closed where $C \subset A$ and $\mu(C \setminus A) < \varepsilon$.

2. If $\exists \{V_i\}$, V_i open and $\mu(V_i) < +\infty$, and $A \in \Lambda$ with $A \subset \bigcup V_i$, then $\exists U$ open such that $A \subset U$ and $\mu(U \setminus A) < \varepsilon$.

Proof of 1

Define $\nu(B) = \mu(B \cap A)$ such that (X, Λ, ν) is a new measure space.
Define $F = \{B \in \Lambda \mid \forall \varepsilon > 0, \exists C \text{ closed}, C \subset B, \nu(B \setminus C) < \varepsilon\}$, all closed sets in F.
Claim 1: $\bigcap A_i, \bigcap A_i \in F$ if $A_i \in F$.
Claim 2: U is open.
$U = \bigcup U_i$, $U_i = \left\{x \in U \mid \text{dist}(x, U^c) \leq \frac{1}{i}\right\}$, therefore $\mathcal{B} \subset F$.
IMAGE HERE - 1
If A is Borel, then $\forall \varepsilon > 0$, $\exists C$ closed with $C \subset A$ and $\mu(A \setminus C) < \varepsilon$.
To finish, $\forall A \subset \Lambda$ by Borel Regularity of μ, $\exists B \in \mathcal{B}$ such that $B \supset A$ and $\mu(B) = \mu(A)$.
Note also that this requires $\mu(B \setminus A) = 0$ since $\mu(A) < +\infty$.
IMAGE HERE - 2
Then $B \setminus A \in \Lambda$, $\exists D \in \mathcal{B}$ such that $D \supset B \setminus A$ and $\mu(D) = \mu(B \setminus A) = 0$. Then

$$\begin{aligned} B \cap A^c = B \setminus A \subset D \\ (B \cap A^c)^c \supset D^c \\ B \cap (B^c \cup A) \supset D^c \cap B \\ A \supset B \setminus D \end{aligned}$$

$$A \setminus (B \setminus D) = A \cap (B \cap D^c)^c = A \cap (B^c \cup D = \overbrace{(A \cap B^c)}^{\varnothing} \cup A \cap D = A \cap D \subset D$$

Therefore $B \setminus D \subset A$, and $\mu(A \setminus (B \setminus D)) = 0$.
$B \setminus D \in \mathcal{B}$, $\forall \varepsilon > 0$, $\exists C$ closed such that $C \subset B \setminus D \subset A$, $\mu((B \setminus D) \setminus C) < \varepsilon$.
This implies that $\mu(A \setminus C) = \mu(A \setminus (B \setminus D)) + \mu((B \setminus D) \setminus C) < \varepsilon$.

Proof of 2

Consider $V_i \setminus A$ where $\mu(V_i \setminus A) \leq \mu(V_i) < +\infty$.
By (1), $\exists C_i$ closed with $C_i \subset V_i \setminus A$ and $\mu((V_i \setminus A) \setminus C_i) < \varepsilon / 2^{i+1}$. Write

$$(V_i \setminus A) \setminus C_i = (V_i \setminus A) \cap C_i^c = V_i \cap A^c \cap C_i^c = (V_i \cap C_i^c) \cap A^c = (V_i \setminus C_i) \setminus A$$

Note that $V_i \setminus C_i$ is open, since C_i is closed.
Define $U = \bigcup (V_i \setminus C_i) \supset A$. Then,

$$U \setminus A = \left(\bigcup (V_i \setminus C_i)\right) \setminus A = \bigcup ((V_i \setminus C_i) \setminus A)$$

Therefore $\mu(U \setminus A) \leq \varepsilon \frac{\varepsilon}{2^{1+1}} = \varepsilon$.

Remark

$X = \bigcup V_i$, V_i open and $\mu(V_i) < +\infty$.
Then $\forall A \in \Lambda$, $\forall \varepsilon > 0$, $\exists U$ open such that $U \supset A$ and $\mu(U \setminus A) < \varepsilon$.
For A^c, $\exists U \supset A^c$ ($\implies U^c \subset A$), $\mu(U \setminus A^c) < \varepsilon$. So

$$U \cap A = U \setminus A^c = A \setminus U^c = A \cap U$$

and $\mu(A \setminus U^c) < \varepsilon$, $U^c \subset A$ with U^c closed.

Corollary

For $\mathbb{R}^n$, a measure is Radon if and only if it is locally finite and Borel regular.

- Proof
  ($\Longrightarrow$)
  Let $B(r, x_0) = \{x \in \mathbb{R}^n \mid |x - x_0| < r\}$ and $\overline{B(r, x_0)} = \{x \in \mathbb{R}^n \mid |x - x_0| \leq r, \text{ compact}\}$.
  Then $\mu(B(r, x_0)) \leq \mu(\overline{B(r, x_0)}) < +\infty$. So μ is locally finite.
  For $A \in \Lambda$, we may assume without loss of generality that $\mu(A) < +\infty$.
  Then $\forall i$, $\exists U_i$ open where $U_i \supset A$ and $\mu(A) \leq \mu(U_i) \leq \mu(A) + \frac{1}{i} < +\infty$.
  Set $G = \bigcap U_i \in \mathcal{B}$, then $\mu(G) = \mu(A)$.
  ($\Longleftarrow$)

  1. Borel regular implies Borel.

  2. For K compact, $\forall x \in K \ni U_x$ open where $\mu(U_x) < +\infty$.

     $\{U_\lambda\}_{\lambda \in k}$ is an open cover. Therefore there is a finite subcover $\{U_{\lambda_i}\}_{i=1}^{\lambda}$ where

     $$\mu(K) \leq \mu\left(\bigcup_{i=1}^{k} U_{\lambda_i}\right) \leq \sum_{i=1}^{k} \mu\left(U_{x_i}\right) < +\infty$$

  3. $\forall V$ open, $B(i) = B(i, 0)$, $V \cap B(i)$, $\mu(V \cap B(i)) < +\infty$, $\exists C_i$ closed where $C_i \subset V_{\cap B(i)}$ so C_i is bounded and therefore compact.

     So $\mu(C_i) \leq \mu((V \cap B(i)) \setminus C_i) < \frac{1}{i}$ and $\mu(V \cap B(i)) \leq \mu(C_i) + \frac{1}{i}$.
     Then $\mu(V) = \lim_{i \to \infty} \mu(V \cap B(i)) = \lim_{i \to \infty} \mu(C_i)$, and $C_i \subset V \cap B(i) \subset V$ compact.
     Therefore $\mu(V) = \sup\{\mu(K) \mid K \text{ compact}, K \subset V\}$.

  4. $\forall A \in \Lambda$, $\forall i$, $\exists U_i$ open where $U_i \supset A$ and $\mu(U_i \setminus A) < \frac{1}{i}$

     This implies that $\mu(A) \leq \mu(U_i) \leq \mu(A) + \frac{1}{i}$ and therefore $\mu(A) = \inf\{\mu(U) \mid U \supset A, U \text{ open}\}$.

Caratheodory Construction

Definition: Outer Measure

$\mu^*(A)$, $\forall A \in 2^X$

1. $\mu^*(\emptyset) = 0$.

2. $\mu^*(A) \leq \mu^*(B)$ if $A \subseteq B$.

3. $\mu^*(\bigcup A_i) \leq \sum \mu^*(A_i)$, $\forall A_i \in 2^X$ (countable subadditivity)

Define $\Lambda = \{A \in 2^x \mid \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c), \forall E \in 2^X\}$.
Then $\mu(A) = \mu^*(A)$ if $A \in \Lambda$.
(X, Λ, μ) is complete.

January 25, 2024

Theorem: Caratheodory Construction

Outer Measure

$u^* : 2^X \to [0, +\infty]$.

1. $\mu^*(\varnothing) = 0$
2. Monotonicity: $\mu^*(A) \le \mu^*(B)$, $A \subseteq B$
3. Countable Subadditivity: $\mu^*\left(\bigcup_i A_i\right) \le \sum_i \mu^*(A_i)$.

Caratheodory Criterion

$A \subset X$ is measurable if $\forall E \in X$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

Theorem

The collection Λ of all measurable sets is a σ-algebra.
(X, Λ, μ) is a complete measure space (cf. proof of Lebesgue completeness).

Hausdorff Measure

$\forall A \subseteq \mathbb{R}^n$, $\forall s \ge 0$, $H_s^\delta(A) = \inf\left\{\sum_i (d(E_i))^s \mid \bigcup_i E_i \supset A,\ d(E_i) \le \delta\right\}$ where $d(E_i)$ is the diameter of E_i.
Notice that $H_s^{\delta_1}(A) \le H_s^{\delta_2}(A)$ if $\delta_2 \le \delta_1$.
Let $H_s^*(A) = \lim_{\delta \to 0} H_s^\delta(A)$, $\forall A \in 2^{\mathbb{R}^n}$.
Claim: H_s^* is an outer measure.

- Verify

  1. $H_s^*(\varnothing) = 0$.
  2. $H_s^*(A) \le H_s^*(B)$, $\forall A \subseteq B \subseteq \mathbb{R}^n$.
  3. Given $A_i \subset \mathbb{R}^N$,

     $\exists \delta_0 > 0$ such that $\forall \delta < \delta_0$, $H_s^*\left(\bigcup_i A_i\right) \le H_s^\delta\left(\bigcup_i A_i\right) + \frac{\varepsilon}{2}$.
     Then $\forall \delta < \delta_0$ fixed, $\forall A_i$, $\exists \{E_i^j\}$ such that $\bigcup_j E_i^j \supset A_i$, $\sum_j (d(E_i^j))^s \le H_s^\delta(A_i) + \frac{\varepsilon}{2^{i+1}}$, and $d(E_j^j) \le \delta$. So

$$
\begin{aligned}
H_s^\delta\left(\bigcup_i A_i\right) &\leq \sum_{i,j} (d(E_i^j))^s \\
&= \sum_i \left(\sum_j (d(E_i^j)^s\right) \\
&= \sum_i \left(H_s^\delta(A_i) + \frac{\varepsilon}{2^{i+1}}\right) \\
&= \sum_i H_s^\delta(A_i) + \frac{\varepsilon}{2}
\end{aligned}
$$

and

$$
H_s^*\left(\bigcup_i A_i\right) \leq \sum_i H_s^\delta(A_i) + \varepsilon \leq \sum_i H_s^*(A_i) + \varepsilon, \quad \forall \varepsilon > 0.
$$

Then, since H_s^* is an outer measure, it is a measure by the Caratheodory construction.

Definition: Hausdorff Measure

The Hausdroff Measure $H_s : \Lambda \to [0, +\infty)$ on a σ-algebra $\Lambda \subset 2^{\mathbb{R}^n}$.

Not Locally Finite

Consider $B(0,1) = \{x \mid |x| < 1\}$.
Then $H_s(B(0,1)) = \infty$ for $s < n$.
That is, the Hausdorff measure is not locally finite for $s < n$.

Complete

The Hausdorff measure, by the Caratheodory construction, is complete.

Symmetry

1. Translation Invariance: $H_s(A + x) = H_s(A)$.
2. Rotation Invariance: $H_s(RA) = H_s(A)$.
3. Scaling: $H_s(\lambda A) = \lambda^s H_s(A)$.

Open Balls Measurable

What about $B(0,1) \subset \mathbb{R}^n$. For $\delta > 0$,

$$
H_s^*(E \cap B(0,1)) + H_s^*(E \cap B(0,1)^c) \leq H_s^*(E \cap B(0,1-\delta)) + H_s^*(E \cap (B(0,1) \setminus B(0,1-\delta))) + H_s^*(E \cap B(0,1)^c)
$$

Want to show that for all $\varepsilon > 0$, this is $\leq H_s^*(E) + \varepsilon$.

- Lemma 1

$$
\begin{aligned}
H_s^*(E\cap B(0,1-\delta)) + H_s^*(E\cap B(0,1)^c) &= H_s^*(E\cap (B(0,1-\delta)\cup B(0,1)^c))\\
&\le H_s^*(E)
\end{aligned}
$$

- Lemma 2

$$H_s^*(E\cap (B(0,1)\setminus B(0,1-\delta)) < \varepsilon.$$

- Lemma 1'

  If $A, B\subset \mathbb{R}^n$, $\operatorname{dist}(A,B)>0$, then $H_s^*(A\cup B) = H_s^*(A)+H_s^*(B)$.
  Since $\{E_i\}$ covering $A\cup B$, $d(E_i) < \frac{1}{4}\operatorname{dist}(A,B)$ gives

  $$\delta < \frac{1}{4}\operatorname{dist}(A,B) \iff \{E_j^A\}\cup\{E_k^B\}$$

  if and only if $\{E_j^A\}$ covers A and $\{E_k^B\}$ covers B. Therefore,

  $$
  \begin{aligned}
  \sum_i (d(E_i))^s &= \sum_j (d(E_j^A))^s + \sum_k (d(E_k^B))^s\\
  \inf\left\{\sum_i (d(E_i))^s\right\} &= \inf\left\{\sum_j (d(E_j^A))^s\right\} + \inf\left\{\sum_k (d(E_k^B))^s\right\}
  \end{aligned}
  $$

  and $H_s^\delta(A\cup B) = H_s^\delta(A) + H_s^\delta(B)$.
  Thus $H_s^*(A\cup B) = H_s^*(A)+H_s^*(B)$.

Let $T_i = E\cap \left(B\left(0,1-\frac{1}{i+1}\right)\right)\setminus B\left(0,1-\frac{1}{i}\right)$.
IMAGE HERE - 1 CONCENTRIC RINGS
We want to show that $H_s^*\left(E\cap\left(B(0,1)\setminus B\left(0,\frac{1}{i}\right)\right)\right) < \varepsilon$ for $i >> 1$. Then

$$
\begin{aligned}
\bigcup_{k=1} T_k &= (B(0,1)\setminus\{0\})\cap E\\
\bigcup_{k=i} T_k &= \left(B(0,1)\setminus B\left(0,1-\frac{1}{i}\right)\right)\cap E
\end{aligned}
$$

Claim: $\sum_i H_s^*(T_i) < +\infty$. It suffices to prove this claim.

$$\sum_{i \text{ even}}^{2k} H_s^*(T_i) = H_s^*\left(\bigcup_{i \text{ even}}^{2k}\right) \le H_s^*(E) < +\infty$$

$$\sum_{i \text{ odd}}^{2k+1} H_s^*(T_i) = H_s^*\left(\bigcup_{i \text{ odd}}^{2k+1}\right) \le H_s^*(E) < +\infty$$

Then $\sum_i^k H_s^*(T_i) <= \infty$.

Borel

Take a countable, dense set $\{q_i\} \subset \mathbb{R}^n$ and $\left\{B\left(q_i, \frac{1}{k}\right)\right\}_{i,k}$.
Claim: $\forall V \subseteq \mathbb{R}^n$ open, then $V = \bigcup_l B\left(q_{i_l}, \frac{1}{k_l}\right)$.
Then $\mathcal{B} \subseteq \Lambda$ and the Hausdorff measure is Borel.

Borel Regular

$\forall A \subset \Lambda$, $\exists B \in \mathcal{B}$ such that $B \supset A$ and $H_s(B) = H_s(A)$.
$\forall \delta = \frac{1}{j}$, $\{E_i^j\}$ E_i^j closed balls with $d(E_i^j) < \frac{1}{j}$,

$$\sum_i (d(E_i))^s \leq H_s^{\frac{1}{j}}(A) + \frac{1}{j}$$

Take $B = \bigcap_j \left(\bigcup_i E_i^j\right) \in \mathcal{B}$ since $B = \bigcap_j \bigcup_i E_i^j \supset A$. Then

$$\begin{aligned} H_s^{\frac{i}{j}}(B) &\leq H_s^{\frac{1}{j}}\left(\bigcup_i E_i^j\right) \\ &\leq \sum_i H_s^{\frac{1}{j}}\left(E_i^j\right) \\ &\leq \sum_i \left(d(E_i^j)\right)^s \\ &\leq H_s^{\frac{1}{j}}(A) + \frac{1}{j} \end{aligned}$$

and in the limit as $j \to \infty$

$$H_s^*(A) \leq H_s^*(B) \leq H_s^*(A)$$

Fractional or Hausdorff Dimension

Theorem:

1. $H_s^*(A) < +\infty \implies H_t^*(A) = 0$, $\forall t > s \geq 0$.
2. $H_t^s > 0 \implies H_s(A) = \infty$, $\forall 0 \leq s < t$

Proof

$$\begin{aligned} H_s^\delta(A) &\sim \sum_i (d(E_i))^s \\ &= \sum_i (d(E_i))^t (d(E_i))^{s-t} \end{aligned}$$

So $s < t$ gives $\geq \delta^{s-t}$.
In the other direction, when $s < t$

$$\begin{aligned} \sum_i (d(E_i))^t &= \sum (d(E_i))^s (d(E_i))^{t-s} \\ &\leq \delta^{t-s} \sum_i (d(E_i))^s \end{aligned}$$

Definition: Hausdorff Dimension

Given $A \subset \mathbb{R}^n$,

$$\begin{aligned}\dim_H(A) &= \sup\{s \mid H_s^*(A) = \infty\} \\ &= \sup\{s \mid H_s^*(A) > 0\} \\ &= \inf\{s \mid H_s^*(A) = 0\} \\ &= \inf\{s \mid H_s^*(A) < +\infty\}\end{aligned}$$

Example 1

$\mathbb{R}^n$ has n Hausdorff dimension.
Consider the n-cube with sides d, $C(d)$. Then

$$H_s(C(d)) = C(n,s)d^s$$

So $C(n,s) = C(n,s)2^{nk}\frac{1}{(2^k)^s} = C(n,s)2^{(n-1)k}$.
If $s < n$, this tends to infinity as $k \to \infty$.
Is $s > n$ it tends to 0.

Example 2

Cantor set has Hausdorff dimension $\frac{\log(2)}{\log(3)}$.

$$\bigcup_{k=1}^{2^n} C_n^k = \frac{\log(2)}{\log(3)}$$

where $|C_n^k| = \frac{1}{3^n}$, so $H_s^\delta(C^n) \sim \frac{2^n}{(3^n)^s} = \left(\frac{2}{3^s}\right)^n$.

Example 3

The Koch snowflake has dimension $\frac{\log(4)}{\log(3)}$.

January 30, 2024

Lemma:

Given a measure space (X, Λ, μ) and an extended real-valued function $f : X \to [-\infty, +\infty]$, the following are equivalent

1. $\forall \alpha \in \mathbb{R}, \{x \in X \mid f(x) > \alpha\} \in \Lambda$.
2. $\forall \alpha \in \mathbb{R}, \{x \in X \mid f(x) \geq \alpha\} \in \Lambda$.
3. $\forall \alpha \in \mathbb{R}, \{x \in X \mid f(x) < \alpha\} \in \Lambda$.
4. $\forall \alpha \in \mathbb{R}, \{x \in X \mid f(x) \leq \alpha\} \in \Lambda$.
5. $\forall U \subset \mathbb{R}$ open, $f^{-1}(U) \in \Lambda$ and $f^{-1}(\pm\infty) \in \Lambda$.

Proof 1 Implies 2

$\{x \in X \mid f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \left\{x \in X \mid f(x) > \alpha - \frac{1}{n}\right\}$.

Proof 2 Implies 3

$\{x \in X \mid f(x) < \alpha\} = \{x \in X \mid f(x) \geq \alpha\}^c$

Proof 3 Implies 4

$\{x \in X \mid f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty} \left\{x \in X \mid f(x) < \alpha + \frac{1}{n}\right\}$

Proof 4 Implies 1

$\{x \in X \mid f(x) > \alpha\} = \{x \in X \mid f(x) \leq \alpha\}^c$

Proof of 5

$\forall U \subset \mathbb{R}$ open, $V = \bigcup_i I_i$ disjoint open intervals.
Therefore $f^{-1}((a,b)) = \{x \in X \mid f(x) > a\} \cap \{x \in X \mid f(x) < b\}$.
Similarly, $f^{-1}(-\infty) = \bigcap_n \{x \in X \mid f(x) < -n\}$ and $f^{-1}(\infty) = \bigcap_n \{x \in X \mid f(x) > n\}$.

Proof 5 Implies 1

$\{x \in X \mid f(x) > \alpha\} = f^{-1}((\alpha, +\infty)) \cup f^{-1}(+\infty) \in \Lambda$.

Definition: Measurable Function

For a measure space (X, Λ, μ), an extended real-valued function $f : X \to [-\infty, +\infty]$ is said to be measurable if one or all of (1)-(5) hold.

Remark:

If (X, Λ, μ) is Borel, then continuous functions are always measurable.

Remark:

The characteristic function

$$\chi_A = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

is measurable if $A \in \Lambda$.

Definition: Simple Functions

The function ϕ is simple if

$$\phi(x) = \sum_{i=1}^{k} \lambda_i \chi_{A_i}, \quad \lambda_I \in \mathbb{R},\ A_i \in \Lambda$$

Proposition:

Given a measure space (X, Λ, μ) and measurable, real-valued f, g,

- $f \pm g$ is measruable.

  $\{x \in X \mid f(x) + g(x) < \alpha\} = \bigcup_{r\in\mathbb{Q}}(\{x \in X \mid f(x) < r\} \cup \{x \in X \mid g(x) < \alpha - r\})$.

- f^2 is measurable

  $\forall \alpha \geq 0, \{x \in X \mid f^2(x) < \alpha\} = \{x \in x \mid f(x) < \sqrt{\alpha}\} \cap \{x \in X \mid f(x) > -\sqrt{\alpha}\}$.

- $f \cdot g$ is measurable

  $f(x) \cdot g(x) = \frac{1}{2}\left((f+g)^2 - f^2 - g^2\right)$.

Definition: Almost Everywhere Equality

Measurable functions f and g on the space (X, Λ, μ) are the same almost everywhere with respect to μ (written μ-a.e.) if

$$\mu(\{x \in X \mid f(x) \neq g(x)\}) = 0$$

Propositon:

For a complete measure space (X, Λ, μ), if f and g are equal μ-a.e., then f is measurable if and only if g is measurable.

Proof

$$\begin{aligned}\{x \in X \mid f(x) > \alpha\} &= (\{x \in X \mid f(x) > \alpha\} \cap \{x \in X \mid f(x) = g(x)\}) \cup \{x \in X \mid f(x) > \alpha\} \cap \{x \in X \mid f(x) \neq g(x)\} \\ &= (\{x \in X \mid g(x) > \alpha\} \cap \{x \in X \mid f(x) = g(x)\}) \cup \{x \in X \mid f(x) > \alpha\} \cap \underbrace{\{x \in X \mid f(x) \neq g(x)\}}_{\mu=0}\end{aligned}$$

Proppsotion:

Given $\{f_k(x)\}$ measurable.

1. $g_n(x) = \sup\{f_1(x), f_2(x), \ldots, f_n(x)\}$ and $h_n(x) = \inf\{f_1(x), f_2(x), \ldots, f_n(x)\}$ measurable.

2. $g(x) = \sup\{f_n(x)\}$ and $h(x) = \inf\{f_n(x)\}$ measurable.

3. $\limsup_{n\to+\infty} f_n(x) = \inf_n \sup\{f_n(x), f_{n+1}(x), \ldots\}$ and $\liminf_{n\to+\infty} f_n(x) = \sup_n \inf\{f_n(x), f_{n+1}(x), \ldots\}$ measurable.

4. $f_n(x) \to f(x)$ pointwise $\implies f$ measurable.

Proof of A

$\{x \in X \mid g_n(x) > \alpha\} = \bigcup_{k=1}^{n}\{x \in X \mid f_k(x) > \alpha\}$
$\{x \in X \mid h_n(x) < \alpha\} = \bigcup_{k=1}^{n}\{x \in X \mid f_k(x) < \alpha\}$

Proof of B

$\{x \in X \mid g(x) > \alpha\} = \bigcup_n\{x \in X \mid f_n(x) > \alpha\}$
$\{x \in X \mid h(x) < \alpha\} = \bigcup_n\{x \in X \mid f_n(x) < \alpha\}$

Definition: Almost Everywhere Convergence

For $f_n(x)$ measurable, $f_n(x) \to f(x)$ μ-a.e. in X if $f_n(x) \to f(x)$ in $A \subset X$ pointwise where $\mu(X \setminus A) = 0$.

Proposition:

On a complete measure space (X, Λ, μ) with f_n measurable and $f_n(x) \to f(x)$ μ-a.e. in X, $f(x)$ is measurable.

Proof

$f_n(x) \to f(x)$ pointwise in A and $\mu(A^c) = 0$.
$\{x \in X \mid f(x) > \alpha\} = (\{x \in X \mid f(x) > \alpha\} \cap A) \cup (\{x \in X \mid f(x) > \alpha\} \cap A^c)$.

Theorem:

With (X, Λ, μ) a measure space and f measurable, there exist simple functions ϕ_n such that

1. $|\phi_n(x)| \leq |\phi_{n+1}(x)|$.
2. $\phi_n(x) \to f(x)$ pointwise in X.
3. If f is bounded, then $\phi_n(x) \rightrightarrows f(x)$ in X.

Proof

Consider $(-\infty, -n] \cup (-n, n) \cup [n, +\infty)$, and define $N_n = \{x \in X \mid f(x) \leq -n\}$ and $P_n = \{x \in X \mid f(x) \geq n\}$.
Then $\bigcap_n (N_n \cup P_n) = \varnothing$.
Define

$$A_{n,k} = \left\{x \in X \mid \frac{k-1}{2^n} < f(x) \leq \frac{k}{2^n}\right\}_{k=-1,-2,\ldots,-n2^n+1}$$
$$A_{n,0} = \left\{x \in X \mid \frac{-1}{2^n} < f(x) < 0\right\}$$
$$A_{n,1} = \left\{x \in X \mid 0 < f(x) < \frac{1}{2^n}\right\}$$
$$A_{n,k} = \left\{x \in X \mid \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n}\right\}_{k=2,3,\ldots,n2^n}$$

and set

$$\phi_n(x) = -n\chi_{N_n} + \sum_{k=0}^{-n2^n+1} \frac{k}{2^n}\chi_{A_{n,k}} + \sum_{k=1}^{n2^n} \frac{k-1}{2^n}\chi_{A_{n,k}} + n\chi_{P_n}$$

Claim:

1. $\forall x \in X, \phi_n(x) \to f(x)$.
2. if $\exists N \in \mathbb{N}$ such that $|f(x)| < N \implies \phi_n(x) \rightrightarrows f(x)$ in X.

Proof

$|\phi_n(x) - f(x)| \le \frac{1}{2^n}$, $\forall x \in X \setminus (U_n \cup P_n)$
Note $\forall x \in X$, $\exists m \in \mathbb{N}$ such that $x \notin N_m \cup P_m$. So $|f(x)| < m$.
Then boundedness implies $\exists N$ such that $N_N \cup P_N = \varnothing$.
Therefore $\forall x \in X$, $|\phi_n(x) - f(x)| < \frac{1}{2^n}$, $\forall n \ge N$.

Theorem: Egoroff

Given a measure space (X, Λ, μ), $\mu(x) < +\infty$ and $f_n \to f$ μ-a.e. in X, then $\forall \delta > 0$, $\exists A \in \Lambda$ such that $\mu(X \setminus A) < \delta$ and $f_n(x) \rightrightarrows f(x)$ in A.

Recall: Pointwise Convergence

$\forall x \in X$, $f_n(x) \to f(x)$ if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$, $\forall n \ge N$.
$Bjj_{N,\varepsilon} = \{x \in X \mid \exists N \in \mathbb{N},\ |f_n(x) - f(x)| < \varepsilon,\ \forall n \ge N\}$
In negation, $\exists \varepsilon > 0$ such that $\forall N \in \mathbb{N}$, $\exists m \ge N$ such that $|f_n(x) - f(x)| \ge \varepsilon$.
$A_{N,\varepsilon} = B^c_{N,\varepsilon} = \{x \in X \mid \exists m \ge N,\ |f_n(x) - f(x)| \ge \varepsilon\}$
Then $\{x \in X \mid f_n(x) \to f(x)\} = \bigcap_{\varepsilon>0} \bigcup_N B_{N,\varepsilon} = \bigcap_{\varepsilon_i \to 0} \bigcup_i B_{N_i,\varepsilon_i}$
and $\{x \in X \mid f_n(x) \not\to f(x)\} = \bigcup_{\varepsilon>0} \bigcap_N A_{N,\varepsilon} = \bigcup_{\varepsilon_i \to 0} \bigcap_i A_{N_i,\varepsilon_i}$ where $\varepsilon_i = \frac{1}{i}$.

February 2, 2024

Review: Measurable Function

An extended, real-valued function $f : X \to [-\infty, +\infty]$ is measurable if one or all of the following hold

1. $\forall \alpha \in \mathbb{R}$, $\{x \mid f(x) > \alpha\} \in \Lambda$.

2. $\forall \alpha \in \mathbb{R}$, $\{x \mid f(x) \ge \alpha\} \in \Lambda$.

3. $\forall \alpha \in \mathbb{R}$, $\{x \mid f(x) < \alpha\} \in \Lambda$.

4. $\forall \alpha \in \mathbb{R}$, $\{x \mid f(x) \le \alpha\} \in \Lambda$.

5. $\forall V \subseteq \mathbb{R}$ open, $f^{-1}(U) = \{x \mid f(x) \in V\}$ and $f^{-1}(-\infty), f^{-1}(+\infty) \in \Lambda$.

Properties

1. For $f = g$ μ-a.e., f is measurable if and only if g is measurable.

2. For $f,\ g$ measurable, $f + g$ and $f \cdot g$ are measurable.

3. For $\{f_n\}$ measurable,

   (a) $\sup_{n \le k}\{f_n\}$ and $\inf_{n \le k}\{f_n\}$ are measurable.

   (b) $\sup_n\{f_n\}$ and $\inf_n\{f_n\}$ are measurable.

   (c) $\limsup_{n\to\infty} f_n$ and $\liminf_{n\to\infty} f_n$ are measurable.

   (d) if $f_n \to f$ μ-a.e. in X, then f is measurable.

Examples

Characteristic Functions

$$\chi_A = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}, \quad A \in \Lambda$$

Simple Functions

$$\sum_{i=1}^{k} \alpha_i \chi_{A_i}, \quad \alpha_i \in \mathbb{R},\ A_i \in \Lambda,\ A_j \cap A_k = \varnothing$$

Step Functions

$$\sum_{i=1}^{k} \alpha_i \chi_{I_i}, \quad I_i \text{ interval}$$

Theorem:

On a measure space (X, Λ, μ), suppose f is measurable.
There exists a sequence of simple functions $\{\phi_n\}$ such that

1. $\phi_n \to f$ pointwise.
2. $\phi_n \rightrightarrows f$ for f bounded.

Proof

Let $N_n = \{x \mid f(x) \le -n\}$ and $A_{n,k} = \left\{x \mid \frac{k-1}{2^n} < f(x) \le \frac{k}{2^n}\right\}$. Then

$$\begin{aligned} A_{n,0} &= \left\{x \mid -\frac{1}{2^n} < f(x) < 0\right\} \\ A_{n,1} &= \left\{x \mid 0 < f(x) < \frac{1}{2n}\right\} \\ A_{n,k} &= \left\{x \mid \frac{k-1}{2^n} \le f(x) < \frac{k}{2^n}\right\} \\ P_n &= \{x \mid f(x) \ge n\} \end{aligned}$$

and

$$\phi_n = -n\chi_{N_n} + \sum_{k=-n2^n+1}^{D} \frac{k}{2^n}\chi_{A_{n,k}} + \sum_{1}^{n2^n} \chi_{A_{n,k}} + n\chi_{\phi_n}$$

So

$$|\phi_n(x) - f(x)| \le \frac{1}{2^n}, \quad x \in X \setminus (N_n \cup P_n), \quad \bigcap_n (N_n \cap P_p) = \varnothing$$

Egoroff Theorem

Given (X, Λ, μ) where $\mu(X) < +\infty$, if

1. $f_n(x) \to f(x)$ μ-a.e. in X and
2. f_n, f μ-a.e. finite.

Then, $\forall \delta > 0$, $\exists A \in \Lambda$ with $\mu(A) < \delta$ such that $f_n(x) \rightrightarrows f(x)$ on A^c.

Proof

Define $D = \{x \mid f_n(x) \to f(x)\} = X$.
Then $\forall \varepsilon > 0$, $\exists m \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$, $\forall n \geq m$.
Say that the universal quantifier $\forall$ is equivalent to grand intersection and the existential quantifier $\exists$ is equivalent to grand union. Then

$$D_{m,\varepsilon} = \{x \mid f_n(x) - f(x) < \varepsilon, \ \forall n \geq m\}$$

and

$$\bigcap_{\varepsilon > 0} \bigcup_m D_{m,\varepsilon} = X.$$

The negation is

$$D^c_{n,\varepsilon} = \{x \mid \exists n \geq m, \ |f_n(x) - f(x)| \geq \varepsilon\}$$

Then injection is equivalent to the complement.
Set $\varepsilon_i = \frac{1}{i}$ such that

$$D = \bigcap_i \bigcup_{m_i} D_{m_i, 1/i}$$
$$\varnothing = D^c = \bigcup_i \bigcap_m D^c_{m,1/i}$$

So $\bigcap_m D^c_{m,1/i} = \varnothing$,

$$D^c_{m,1/i} = A_{m,1/i} = \left\{x \mid \exists n \geq m, |f_n(x) - f(x)| \geq \frac{1}{i}\right\}$$

and $A_{n,1/i} \supset A_{n+1,1/i} \supset \cdots$. Therefore

$$\mu(A_{n,1/i}) \to \mu\left(\bigcap_m A_{m,1/i}\right) = 0$$

for $\mu(X) < +\infty$.
Thus, $\forall i$, $\exists m_i$ such that $\mu(A_{m_i,1/i}) < \frac{\delta}{2^{i+1}}$. It follows that $A = \bigcup_i (A_{m_i,1/i})$,

$$\mu(A) \leq \sum \mu(A_{m_i,1/i}) < \delta$$

and

$$x \in A^c = \bigcap_i A^c_{m_i,1/i} = \bigcap_i D_{m_i,1/i} = \bigcap_i \left\{x \mid |f_n(x) - f(x)| < \frac{1}{i}, \ \forall n \geq m_i\right\}$$

Finally, this implies $f_n(x) \rightrightarrows f(x)$ in A^c.

Example

Take $f_n = \chi_{[n,n+1]}$ on $\mathbb{R}$, then $f_n(x) \to 0$ in $\mathbb{R}$ but $A \subset \mathbb{R}$, $\mu(A) < \frac{1}{2}$, $A^c \cap [n,n+1] \neq \varnothing$, $\forall n$.
That is, $\forall n$, $\exists x \in A^c$ such that $f_n(x) = 1$ but $f(x) = 0$.
Therefore $f_n(x) \not\rightrightarrows f(x)$ on $\mathbb{R}$.

Definition: Essential Bounds

On a measure space (X, Λ, μ) with f measurable, define $||f||_\infty = \inf\{M \mid \mu(\{x \mid |f(x)| > M\}) = 0\}$.
This is the L^∞-norm.

Proposition:

$f_n \rightrightarrows f$ on A where $\mu(A^c) = 0$ if and only if $||f_n - f||_\infty \to 0$.

Proof

$(\Longrightarrow)$
$\forall \varepsilon > 0$, $\exists m \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$, $\forall x \in A$.
Claim: $||f_n(x) - f(x)||)\infty < \varepsilon$, $\forall n \geq m$.

$$||f_n(x) - f(x)||_\infty = \inf\{M \mid \mu(\{x \mid |f_n(x) - f(x)| > M\}) = 0\}$$

Where $\{x \mid |f_n(x) - f(x)| > n\} \subset A^c$ and $n \geq m$ and $M \geq \varepsilon/2$.
$(\Longleftarrow)$

Recall: Urysohn's Lemma

For X locally compact and Hausdorff, $K \subset U$ for K compact and U open, $\exists \phi$ continuous such that $\phi = \begin{cases} 1 & K \\ 0 & U^c \end{cases}$.

Theorem: Vitali-Lusin

On measure space (X, Λ, μ) with X locally compact and Hausdorff and μ a Radon measure.
For f measurable, μ-a.e. finite and vanishing outside A where $\mu(A) < +\infty$,
$\forall \varepsilon > 0$, $\exists g$ continuous with compact support such that $\mu(\{x \mid f(x) \neq g(x)\}) < \varepsilon$.

Proof

1. $\exists C \subset A$ compact with $\mu(A \setminus C) < \varepsilon$.

2. For A compact with $\mu(A) < +\infty$, $\exists U \supset A$ open neighborhood with compact closure and $\mu(U \setminus A) < \varepsilon$.

3. $\phi_n = -n\chi_{N_n} + \sum_{-n2^n+1}^{0} \frac{k}{2^n}\chi_{A_{n,k}} + \sum_1^{n2^n} \frac{k-1}{2^n}\chi_{A_{n,k}} + n\chi_{P_n}$

Since we may minimize $\mu(N_n \cup P_n) < \varepsilon$,

$$\phi_n = \sum_{-n2^n+1}^{0} \frac{k}{2^n} \chi_{A_{n,k}} + \sum_{1}^{n2^n} \frac{k-1}{2^n} \chi_{A_{n,k}}$$

Take $C_{1,k} \subset A_{1,k}$ compact with $\mu(C_{1,k}) \geq \mu(A_{1,k}) - 2^{-1}2^{-|k|+1}\varepsilon$. Write

$$C_1 = \bigcup_k C_{1,k}$$

and inductively define $C_{n-1,k}$ and $C_{n-1} = \bigcup_k C_{n-1,k}$ such that $C_{n,k} \subset A_{n,k} \cap C_{n-1}$ compact and

$$\mu(C_{n,k}) \geq \mu(A_{n,k} \cap C_{n-1}) - 2^{-1}2^{-|k|+1}\varepsilon$$

Define, by Urysohn's Lemma,

$$\tilde{\chi}_{A_{n,k}} := \begin{cases} 1 & C_{n,k} \\ 0 & U^c \cup \bigcup_{l \neq k} C_{n,l} \end{cases}$$

where $C_n \subset C_{n-1}$, $C = \bigcap C_n$, $C_n = \bigcup_k C_{n,k}$.
Then define

$$g_n := \sum_{-n2^n+1}^{0} \frac{k}{2^n} \tilde{\chi}_{A_{n,k}} + \sum_{1}^{n2^n} \frac{k-1}{2^n} \tilde{\chi}_{A_{n,k}}$$

Then $g_n = \phi_n$ on C for all n.
Therefore $g_n = \phi_n \rightrightarrows \hat{g} = f$ on C.
By uniform convergence, $\hat{g}$ is continuous on C.
So, again by Urysohn's Lemma, $g = \phi\hat{g}$ and $\{x \mid g \neq f\} = U \setminus C$.

February 8, 2024

Midterm Review

Problem 2

Given a finite measure space (X, Λ, μ), $\mu(X) < +\infty$ and a function f which is μ-a.e. finite.
Monotone Convergence Theorem:

1. $A_1 \subset A_2 \subset \cdots$, then $\mu\left(\bigcup_i A_i\right) = \lim_{i\to\infty} \mu(A_i)$.
2. $A_1 \supset A_2 \supset \cdots$, then $\mu\left(\bigcap_i A_i\right) \lim_{i\to\infty} \mu(A_i)$ for $\mu(A_1) < +\infty$.

If $A_k = \{x \mid |f(x)| > k\}$ and

$$F = \bigcap_{k=1}^{\infty} A_k$$

then $\mu(F) = \lim_{k\to\infty} \mu(A_k) = 0$ since $\mu(X) < +\infty$.
If instead we consider A_k^c, then

$$\bigcup_k A_k^c = X \setminus F$$

Problem 3

1. Borel

   Given $(\alpha, +\infty)$, we want $\forall E \subset \mathbb{R}$

   $$m^*(E \cap (\alpha, +\infty)) + m^*(E \cap (-\infty, \alpha]) \le m^*(E)$$

   $\forall \varepsilon > 0$, $\exists \{I_i\}$ pen intervals

   $$\bigcup_i I_i \supset E \quad \sum_i |I_i| \le m^*(E) + \varepsilon/2$$

   Divide $\{I_i\}$ into 3 groups,

   $$\begin{aligned} C^\ell &= \{I \in \{I_i\} \mid I \text{ is to the left of } \alpha\} \\ C^r &= \{I \in \{I_i\} \mid I \text{ is to the right of } \alpha\} \\ C^m &= \{I \in \{I_i\} \mid \alpha \in I\} \end{aligned}$$

   Then, $\forall I_k^m \in C^m = \{I_k^m\}$, and

   $$\begin{aligned} {}^\ell I_k^n &= \left(a_k, \alpha + \frac{2}{2^{k+2}}\right) \\ {}^r I_k^n &= \left(\alpha - \frac{2}{2^{k+2}}, b_k\right) \\ {}^m I_k^n &= (a_k, b_k) \end{aligned}$$

   where also

   $$\begin{aligned} A_n \supset (\alpha, +\infty)^c \quad A_n &= \left(-\infty, \alpha + \frac{1}{2^n}\right) \\ B_n \supset (\alpha, +\infty) \quad B_n &= \left(\alpha + \frac{1}{2^n}, +\infty\right) \\ A_n \cap B_n &= \left(\alpha - \frac{1}{2^n}, \alpha + \frac{1}{2^n}\right) \end{aligned}$$

   So ${}^\ell I_k^n \cup {}^r I_k^n = I_k^n$, and $|{}^\ell I_k^n| + |{}^r I_k^n| = |I_k^n| + \frac{\varepsilon}{2^{k+1}}$.
   Finally

   $$\begin{aligned} m^*(E \cap (\alpha, +\infty)) + m^*(E \cap (-\infty, \alpha]) &\le \sum_{I \in C^r} |I| + \sum_k |{}^r I_k^n| + \sum_{I \in C^\ell} |I| + \sum_k |{}^\ell I_k^n| \\ &\le \sum_{I \in C^r} |I| + \sum_{I \in C^\ell} |I| + \sum_k |I_k^n| + \frac{\varepsilon}{2} \\ &\le m^*(E) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \end{aligned}$$

2. $\mu(K) < +\infty$ for $K \subset \mathbb{R}$ compact.

   K is bounded, $k \subset (-M, M)$ for large M.
   Therefore $\mu(K) \le 2M < +\infty$.

3. $\forall U \subset \mathbb{R}$ open, we want to show $\exists K_n$ compact such that $K_n \subset U$ and $\mu(K_n) \to \mu(U)$.

   Let $U = \bigcup_i I_i$ a union of countably many disjoint open intervals (e.g. $I_i = (a_i, b_i)$).
   Then $m(U) = \sum_i m(I_i)$.
   Set $I_i^n = \left[a_i + \frac{1}{n2^{i+1}}, b_i - \frac{1}{n2^{i+1}}\right]$. Then

$$\sum_{i}^{k} |I_i^n| \geq \sum_{i=1}^{k} |I_i| - \frac{1}{n}, \quad \forall k$$

   It follows that

$$\sum_{i=1}^{k} |I_i| \to \sum_{i=1}^{\infty} |I_i|, \quad \text{as } k \to +\infty$$

   and

$$K_k^n = \bigcup_{i=1}^{k} U_i^n \subset U \quad \text{compact}$$

$$m(U) \geq m(K_k^n) = \sum_{i=1}^{n} |I_i^n| \geq \underbrace{\sum_{i=1}^{\infty} |I_i|}_{m(U)} - \frac{1}{n}$$

   Alternatively, we have the theorem that if X is a metric space and μ is Borel regular on (X, Λ), then

   (a) $A \in \Lambda$, $\mu(A) < +\infty$, $\forall \varepsilon > 0$, $\exists C$ closed with $C \subset A$ such that $\mu(A \setminus C) < \varepsilon$.

   (b) $\exists \{U_i\}$, $\mu(U_i) < +\infty$, U_i open where $A \subset \bigcup_i U_i$, $\forall \varepsilon > 0$ there exists V open such that $V \supset A$ and $\mu(V \setminus A)\varepsilon$.

   With the corollary that for μ on $\mathbb{R}^n$, μ is Radon if and only if it is locally finite and Borel regular.

4. For $A \in \Lambda$, $m(A) = \inf\{m(V) \mid V \supset A,\ V \text{ open}\}$

   Recall Borel regularity: $\forall A \in \Lambda$, there is some Borel set $B \supset A$ with $m(B) = m(A)$.
   We may assume $m(A) < +\infty$. Then, $\forall \varepsilon > 0$, there is some collection of open intervals $\{I_i^n\}$ containing A where

$$\sum_i |I_i^n| \leq m(A) + \varepsilon$$

   Set $\varepsilon = \frac{1}{n}$ and let $U^n = \bigcup_i I_i^n \supset A$ open. Then

$$m(A) \leq m(U^n) \leq \sum_i |I_i^n| \leq m(A) + \frac{1}{n}$$

   If $B = \bigcap_n U_n$, then $\lim_{m\to\infty} m(U^n) = m(A)$ and $m(B) = m(A)$.

Problem 4

Given $f : \mathbb{R} \to \mathbb{R}$, continuous outisde a measure zero set D.
That is, $\overline{f} : \mathbb{R} \setminus D \to \mathbb{R}$ is continuous.
$\forall V \subset \mathbb{R}$, $f^{-1}(V) = (f^{-1})V \cap (\mathbb{R} \setminus D)) \cup (f^{-1}(V) \cap D)$.
By measure completeness, we are automatically safe on $f^{-1}(V) \cap D$.
Claim: $f^{-1}(V) \cap (\mathbb{R} \setminus D) = \overline{f}^{-1}(V)$.
Claim: $\overline{f}^{-1}$ is measurable.
Claim: $\overline{f}^{-1}(V) = U \cap (\mathbb{R} \setminus D)$ where $U \subset \mathbb{R}$ open.
Since $U \cap (\mathbb{R} \setminus D)$ is open in the subspace topology, we are done.
Alternatively (similary to Probelm 8 below), for D such that $m(D) = 0$, $\forall n$, $\exists U^n$ such that $m(U^n) \leq 2^{-n}$, $U^n \supset D$ and $U^n = \bigcup_i (a_i, b_i)$ where $(a_j, b_j) \cap (a_k, b_k) = \emptyset$ and $a_i, b_i \in \mathbb{R} \setminus D$. So

$$f_n = \begin{cases} f(x), & x \in (U^n)^c \\ f(a_i) + \frac{f(b_i) - f(a_i)}{b_i - a_i}(x - a_i), & x \in (a_i, b_i) \subset U^n \end{cases}$$

Then $\{x \mid f_n(x) \neq f(x)\} \subset U^n$ and $m(\{x \mid f_n(x) \neq f(x)\}) \leq 2^{-n}$.

Homework 4 Problem 8

Assume $f(x)$ is decreasing.

1. Discontinuities are limited to jump discontinuities.
2. Discontinuities are countable.
3. $D = \{x_i\}_i$, $\forall n$ there exists an open cover $\{I_i^n = (a_i, b_i)\}$ where $\bigcup_i I_i^n = C^n \supset \{x_i\}_i$ and $m(C^n) \leq 2^{-n}$.

Then $\{x \mid f_n(x) \neq f(x)\} \subset C^n\}$ and $\mu(\{x \mid f_n(x) \neq f(x)\}) \leq 2^{-n}$.
Claim: $f_n(x) \to f(x)$ on $\mathbb{R} \setminus G$ where $G = \bigcap_n^\infty \bigcup_{k=n}^\infty \{x \mid f_k(x) \neq f(x)\}$.
By monotone convergence, $\mu(g) = \lim_{n\to+\infty} \mu\left(\bigcup_{k=n}^\infty \{x \mid f_n(x) \neq f(x)\}\right) = \lim_{n\to+\infty}\left(\sum_{k=n}^{+\infty} 2^{ik}\right) = 0$.
Consider the complement, $G^c = \bigcap_{n=1}^\infty \bigcap_{k=n}^{+\infty} \{x \mid f_k(x) \neq f(x)\}$.
Then $\forall x \in G^c$, $x \in \bigcap_{k=n_0}^{+\infty} \{x \mid f_k(x) = f(x)$, so $f_n(x) = f(x)$ $\forall n \geq n_0$.

Riemann Integration

Given a function $f : [a, b] \to \mathbb{R}$ bounded and P a partiation of $[a, b]$ where

$$a = x_0 < x_1 < \cdots < x_n = b$$

The Cauchy sum

$$C(P, [a, b]) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i+1}), \quad \xi_i \in [x_i, x_{i+1})$$

alternatively

$$\phi(P, [a, b]) = \sum_i f(\xi_i)\chi_{[x_i, x_i+1)}$$

Consider the upper Riemann sum

$$S(P,[a,b]) = \sum_i M_i(x_i, x_{i+1}), \quad M_i = \sup_{[x_i,x_{i+1}]} f(x)$$

and the lower Riemann sum

$$s(P,[a,b]) = \sum_i m_i(x_i, x_{i+1}), \quad m_i = \inf_{[x_i,x_{i+1}]} f(x)$$

then define

$$S = \inf_P S(P,[a,b]) = s = \sup_P s(P,[a,b]) \implies \int_a^b f(x)\, dx = \lim_{l(P)\to 0} C(P,[a,b])$$

Theorem:

f is Riemann integrable on $[a,b]$ if and only if f is continuous m-a.e. (w.r.t Lebesgue measure) on $[a,b]$.

Proof

($\implies$) Let f be Riemann integrable on $[a,b]$.
Define the oscillation

$$\begin{aligned}\mathrm{Osc}_I(f) &= \sup_I f(x) - \inf_I f(x)\\ \mathrm{Osc}_x(f) &= \lim_{\delta\to 0} \mathrm{Osc}_{(x-\delta,x+\delta)}(f)\end{aligned}$$

and observe that f is continuous at x if and only if $\mathrm{Osc}_x(f) = 0$.
Let $D = \{x \mid \mathrm{Osc}_x(f) > 0\}$ and $D_k = \left\{x \mid \mathrm{Osc}_x(f) > \frac{1}{k}\right\}$ such that $D_k \subset D_{k+1}$ and $D = \bigcup_k D_k$.
Therefore $m(D_k) \to m(D)$.
To show that $m(D) = 0$, assume otherwise that $m(D) > 0$.
Therefore, $\exists k$ such that $m(D_k) > d_{k_0}$ for any $k \geq k_0$.
Then, for any partition P we may examine

$$S(P,[a,b]) - s(P,[a,b]) = \sum_{I_i} (M_i - m_i)|I_i|$$

We want to show that this is $\geq \delta > 0$ for any P.

February 13, 2024

Recall: Riemann Integration

$f(x) \geq 0$ on $[a,b]$ bounded.
Partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, $[x_{i-1}, x_i]$.
IMAGE HERE - Riemann Integration
Upper Riemann Sum: $S_P = \sum_{i=1}^n M_i(x_i - x_{i-1})$ where $M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$.
Lower Riemann Sum: $s_P = \sum_{i=1}^n m_i(x_i - x_{i-1})$ where $m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$.
Step Functions: $\phi_{P,\alpha} = \sum_i \alpha_i \chi_{I_i}$ where $I_i = [x_{i-1}, x_i]$.
Set $S = \inf_P S_P = \inf\left\{\sum_i \alpha_i |I_i| \mid \phi_{P,\alpha}(x) \geq f(x)\right\}$
and $s = \sup_P s_P = \sup\left\{\sum_i \alpha_i |I_i| \mid \phi_{P,\alpha}(x) \leq f(x)\right\}$.

Definition: Riemann Integrable

The function f is Riemann integrable if $S = s$.

Remark:

$$S_P - s_P = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \to 0 \text{ as } \ell(P) \to 0$$

Remark:

If f is continuous, then it is Riemann integrable.

Theorem:

Given $f : [a, b] \to \mathbb{R}$ bounded, then f is Riemann integrable if and only if f is continuous m-a.e.
$m(D) = 0$ if and only if f is Riemann integrable.

Proof

Recall that $\text{Osc}_I(f) = \sup_I f(x) - \inf_I f(x)$ and $\text{Osc}_{x_0}(f) = \lim_{\delta\to 0} \text{Osc}_{(x_0-\delta, x_0+\delta)}(f)$.
IMAGE HERE - 2 Oscillation
Write $D = \{x \in [a, b] \mid f \text{ is not continuous at } x\}$, and $D_k\{x \in [a, b] \mid \text{Osc}_x(f) \geq 1/k\}$ closed (since D_k^C open). Then

$$D = \bigcup_k D_k = \{x \in [a, b] \mid \text{Osc}_x(f) > 0\}$$

We have $m(D_k) \underset{k\to\infty}{\longrightarrow} m(D)$.
Then there exists an open cover of D_k, $\{I_i\}$ such that $m(D_k) + \varepsilon \geq \sum_i |I_i| \geq m(D_k) - \varepsilon$.
Since D_k is closed and bounded, it is compact and there exists finite subcover $\{I_{i_k}\}_{k=1}^{\ell} \subset \{I_i\}$.
$(\Longleftarrow)$ Assume that f is Riemann integrable and, for sake of contradiction, that $m(D) > 0$.
Then $m(D_k) \geq m > 0$, $\forall\, k \geq k_0$.
Now for any partition $P = \{x_0, x_1, \ldots, x_n\}$,

$$\begin{aligned}
S_P - s_P &= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \\
&\geq \sum_{(x_{i-1},x_i)\cap D_k \neq \varnothing} (M_i - m_i)(x_i - x_{i-1}) \\
&\geq \frac{1}{k} \sum_{(x_{i-1},x_i)\cap D_k \neq \varnothing} (x_i - x_{i-1})
\end{aligned}$$

Since $\bigcup_{(x_{i-1},x_i)\cap D_k \neq \varnothing} [x_{i-1}, x_i] \supset D_k$,

$$\sum_{(x_i,x_{i-1})\cap D_k \neq \varnothing} (x_i - x_{i-1}) = m\left(\bigcup_{(x_{i-1},x_i)\cap D_k \neq 0} [x_{i-1}, x_i] \right) \geq m(D_k)$$

we conclude that

$$S_P - s_P \geq \frac{m}{k_0} \geq 0$$

($\Longrightarrow$) Assume $m(D) = 0$.
Then, for any k satisfying $\frac{1}{k} < \frac{\varepsilon}{2(b-a)}$, $m(D_k) = 0$ and $\{I_{i_k}\}_{k=1}^{\ell} \subset \{I_i\}$ for open intervals I_i.
We have, also, $\bigcup_{k=1}^{\ell} I_{i_k} \supset D_k$ so

$$\sum_{k=1}^{\ell} |I_{i_k}| \leq \sum_i |I_i| \leq \frac{\varepsilon}{2M}$$

and

$$[a,b] \setminus \bigcup_{k=1}^{\ell} I_{i_k} \subset D_k^c$$

compact.
Claim: there exists some partition $P = \{x_i\}_{i=0}^{n}$ such that $S_P - s_P < \varepsilon = \frac{1}{k}$.
Given $\mathrm{Osc}_x(f) \leq 2M$,

$$\begin{aligned} S_P - s_P &= \sum_i (M_i - m_i)(x_i - x_{i-1}) \\ &= \sum_{[x_{i-1},x_i]\cap D_k = \varnothing} + \sum_{[x_{i-1},x_i]\cap D_k \neq \varnothing} \\ &\leq \frac{\varepsilon}{2(b-a)}(b-a) + 2M \cdot \frac{\varepsilon}{4M} \end{aligned}$$

Definition: Lebesgue Integration

Given a measure space (X, Λ, μ) and simple function $s = \sum_i \alpha_i \chi_{A_i}$ for $\alpha_i \in \mathbb{R}$ and $A_i \in \Lambda$,

$$\int_E s \, d\mu = \sum_i \alpha_i \mu(A_i \cap E)$$

Then, for extended real-valued $f \geq 0$,

$$\int_E f \, d\mu = \sup \left\{ \sum_i \alpha_i \mu(A \cap E) \mid 0 \leq s(x) \leq f(x) \right\}$$

Properties

1. For $0 \leq f \leq g$ on E, $\int_E f \, d\mu \leq \int_E g \, d\mu$.
2. For $A \subset B$ where $A, B \in \Lambda$, $\int_A f \, d\mu \leq \int_B f \, d\mu$.
3. Since $f \geq 0$, $\forall c \in \mathbb{R}_{\geq 0}$ $\int_E cf \, d\mu = c \int_E f \, d\mu$.
4. $f = 0$ μ-a.e. if and only if $\int_X f \, d\mu = 0$.
5. $\int_E f \, d\mu = \int_X f \chi_E \, d\mu$.
6. For $f, g \geq 0$, $\int_E f + g \, d\mu = \int_E f \, d\mu + \int_E g \, d\mu$.
7. For $A, B \in \Lambda$ where $A \cap B = \varnothing$, $\int_{A\cup B} f \, d\mu = \int_A f \, d\mu + \int_B f \, d\mu$.

- Proof of 4
  $(\Longrightarrow)\ \sum_i \alpha_i \chi_{A_i} = s(x) = f(x) \implies \alpha_i > 0 \implies \mu(A_i) = 0$.
  $(\Longleftarrow)\ f \geq \alpha > 0$ and $\mu(A) > 0 \implies f(x) \geq \alpha \chi_A \implies \int_X f\, d\mu \geq \alpha_{\mu(A)} > 0$ a contradiction.

- Proof of 5
  $s\chi_E = \sum_i \alpha_i \chi_{A_i \cap E}$.

- Proof of 6
  If $0 \leq s_1 \leq f$ and $0 \leq s_2 \leq g$, then $0 \leq s_1 + s_2 \leq f + g$.

Monotone Convergence of Lebesgue Integration

On a measure space (X, Λ, μ), let $f_n \geq 0$ be a sequence of measurable functions which is monotone $f_i(x) \leq f_{i+1}(x)$ and converging $f_n(x) \to f(x)$ for any $x \in X$. Then

$$\lim_{n\to+\infty} \int_X f_n\, d\mu = \int_X f\, d\mu = \int_X \left(\lim_{n\to+\infty} f_n\right) d\mu$$

Proof

Observe that $f_n(x) \leq f(x)$, $\forall x \in X$, so

$$\int_X f_n\, d\mu \leq \int_X f_{n+1}\, d\mu \leq \int_X f\, d\mu$$

so

$$\lim_{n\to+\infty} \int_X f_n\, d\mu \leq \int_X f\, d\mu$$

We want to show that

$$\lim_{n\to+\infty} \int_X f_n\, d\mu \geq \int_X f\, d\mu$$

Let s be a simple function satisfying $0 \leq s(x) \leq f(x)$, and define

$$E_n = \{x \in X \mid f_n(x) \geq cs(x)\}$$

for some $c \in (0, 1)$.
Then $E_n \subset E_{n+1}$ and $\bigcup_n E_n = X$. Consider

$$\int_X f_n\, d\mu \geq \int_{E_n} f_n\, d\mu \geq c \int_{E_n} s(x)\, d\mu = c \sum_i \alpha_i \mu(A_i \cap E_n)$$

For any i, $A_i \cap E_n \to A_i$. Therefore $\mu(A_i \cap E_n) \underset{n\to+\infty}{\longrightarrow} \mu(A_i)$. So

$$\lim_{n\to+\infty} \int_X f_n\, d\mu \geq c \sum_i \alpha_i \mu(A_i)$$

for $0 \leq s = \sum \alpha_i \chi_{A_i} \leq f(x)$. Since this hold for any c,

$$\lim_{n\to+\infty} \int_X f_n\, d\mu \geq \int_X f\, d\mu$$

Corollary

Given a measurable sequence $f_n \geq 0$ with $f(x) = \sum_n f_n(x)$,

$$\int_X f \, d\mu = \sum_n \int_X f_n \, d\mu$$

and

$$\phi_n(x) = \sum_{k=1}^{n} f_k(x) \to f(x)$$

Definition: Fatou's Lemma

Given a sequence of measurable functions $f_n \geq 0$,

$$\int_X \left(\liminf_{n \to +\infty} f_n \right) \, d\mu \leq \liminf_{n \to +\infty} \int_X f_n \, d\mu$$

Proof

Observe that

$$\liminf_{n \to +\infty} f_n = \lim_{n \to +\infty} \overbrace{\left(\inf\{f_n(x), f_{n+1}(x), \ldots\}\right)}^{g_n(x)}$$

so, by monotone convergence,

$$\int_X \left(\lim_{n \to +\infty} g_n(x) \right) \, d\mu = \lim_{n \to +\infty} \int_X g_n(x) \, d\mu$$

and $g_n(x) \leq f_n(x)$ gives

$$\int_X g_n(x) \, d\mu \leq \int_X f_n(x) \, d\mu$$

and implies

$$\lim_{n \to +\infty} \int_X g_n(x) \, d\mu \leq \liminf_{n \to +\infty} \int_X f_n(x) \, d\mu$$

Space of Integrable Functions

Write

$$f(x) = f^+(x) - f^-(x)$$

where

$$f^+(x) = \max\{f(x), 0\} \geq 0$$
$$f^+(x) = \min\{-f(x), 0\} \geq 0$$

Then for $\int_X f^+ \, d\mu$ and $\int_X f^- \, d\mu$, $\int_X f \, d\mu$ is defined when at least one is finite.
If both are finite, then

$$L^1_\mu(x) = \int_X |f| \, d\mu = \int_X f^+ \, d\mu + \int_X f^- \, d\mu \leq +\infty$$

Properties

1. For any $\alpha, \beta \in \mathbb{R}$,

$$\int_X (\alpha f + \beta g)\, d\mu = \alpha \int_X f\, d\mu + \beta \int_X g\, d\mu$$

   if $f, g \in L^1_\mu(x)$.

2. For $f \in L^1_\mu(x)$,

$$\left| \int_X f\, d\mu \right| \leq \int_X |f|\, d\mu$$
$$\left| \int_X f^+\, d\mu - \int_x f^{-1}\, d\mu \right| \leq \int_X f^+\, d\mu + \int_X f^-\, d\mu$$

3. For $f \leq g$, $f, g \in L^1_\mu(x)$, $\int_X f\, d\mu \leq \int_X g\, d\mu$.

4. $\int_{A \cup B} f\, d\mu = \int_A f\, d\mu + \int_B f\, d\mu$.

5. $f = 0$ μ-a.e. if and only if $\int_X |f|\, d\mu = 0$.