

Analysis II

January 9, 2024

(Real) Analysis

- Calculus
 - Differential
 - Integral (Riemann)
- Functions and Maps
 - Measure Theory
 - (Lebesgue) Integration
- Topology
 - Completeness (as a metric space)
 - Compactness (Bolzano-Weierstrass theorem [real]) (Arzela-Ascoli)
 - Paracompactness / Metrizable / Baire Category Theorem
 - Algebraic / Combinatoric (continuous maps or functions)

Definition: Cardinality

For sets A, B , $\text{Card}(A) = \text{Card}(B)$ if there exists a one-to-one correspondence $q : A \leftrightarrow B$.

Counting, labelling, indexing, etc.

$\text{Card}(A) \leq \text{Card}(B)$ if $A \subset B$ or there exists a one-to-one mapping $A \rightarrow B$.

Definition: Countable

If $A \hookrightarrow \mathbb{N}$, then A is countable.

Theorem

The countable union of countable sets is countable.

Proof

Let $A_i = \{a_j\}_{j=1}^{\infty}$, $i = 1, 2, \dots$

$$\begin{array}{cccc} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ \vdots & & & \\ a_{k1} & a_{k2} & a_{k3} & \cdots \end{array}$$

Index by diagonalization.

Theorem

The cartesian product of countable sets is countable.

Proof

$$X \times Y = \{(x_i, y_j) \mid x_i \in X, y_j \in Y\}$$

$$\begin{array}{cccc}
(x_1, y_1) & (x_1, y_2) & (x_1, y_3) & \cdots \\
(x_2, y_1) & (x_2, y_2) & (x_2, y_3) & \cdots \\
\vdots & & & \\
(x_k, y_1) & (x_k, y_2) & (x_k, y_3) & \cdots
\end{array}$$

Theorem

$\text{Card}(2^X) > \text{Card}(X)$, where $2^X = \{A \subset X\}$ is the power set of X .

Proof

For all $x \in X$, $\{x\} \subset 2^X$, so $\text{Card}(X) \leq \text{Card}(2^X)$.

Assume, for sake of contradiction, that $\text{Card}(X) = \text{Card}(2^X)$.

Then, by definition, there exists a one-to-one correspondence $\phi : X \leftrightarrow 2^X$.

Set $A = \{x \in X \mid x \notin \phi(x)\}$, and let $a = \phi^{-1}(A)$ (i.e. $A = \phi(a)$).

If $a \in A$, then $a \notin A \subset \phi(a)$; but if $a \notin A$, then $a \in A$, a contradiction.

Theorem

$$\text{Card}(\mathbb{R}) = \text{Card}(2^{\mathbb{N}}).$$

Topology of the Real Line

Completeness (as a metric space)

$$d(a, b) = |a - b|, \quad \forall a, b \in \mathbb{R}.$$

1. $x_i \rightarrow x$ if $\forall \varepsilon > 0, \exists n \in \mathbb{N}$ such that $|x_i - x| < \varepsilon, \forall i \geq n$.
2. $\{x_i\}$ is Cauchy if $\forall \varepsilon > 0, \exists n \in \mathbb{N}$ such that $|x_i - x_j| < \varepsilon, \forall i, j \geq n$.

Definition: Open Interval

(a, b) is an open set on the real line.

There exist interior points for any subset A of real numbers.

$\forall x \in A$, x is interior if $\exists (a, b)$ such that (1) $x \in (a, b)$ and (2) $(a, b) \subset A$.

- Theorem

The union of open sets is open.

The intersection of finitely many open sets is open.

\emptyset and \mathbb{R} are open.

Definition: Limit Point

A limit point $x \in \mathbb{R}$ of a subset A is a limit point in A if for every open neighborhood U of x , $(U \setminus \{x\}) \cap A \neq \emptyset$.

Definition: Closed

A is closed if A contains all of its limit points.

- Theorem

A is closed if and only if $A^c = \mathbb{R} \setminus A$ is open.

- Proof

A closed $\implies A^c$ open.

Otherwise, $\exists x \in A^c$ such that for every neighborhood U of x , $(U \setminus \{x\}) \cap A \neq \emptyset$ which would make it a limit point of A not in A . By assumption, A contains all its limit points so this is a contradiction.

A^c open $\implies A$ closed.

For any x a limit point of A , assume otherwise that $x \in A^c$.

Then there exists some neighborhood U of x such that $U \subset A^c$ (since A^c is open).

It follows that $(U \setminus \{x\}) \cap A = \emptyset$ and x is not a limit point of A , which is a contradiction.

Definition: Sequential Compactness

A is compact if $\forall \{x_i\}$, $x_i \in A$ there exists a convergent subsequence $\{x_{i_k}\}$ and $x_{i_k} \rightarrow x \in A$.

- Theorem: Bolzano-Weierstrass

For $A \subseteq \mathbb{R}$, A is compact if and only if A is closed and bounded.

- Proof

A compact $\implies A$ closed and bounded.

Assume that A is not bounded from above.

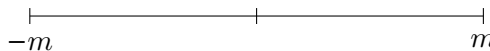
Then there exists a sequence $\{x_i\}$, $x_i \in A$ where $x_{i+1} > x_i + 1$ and $\{x_i\}$ has no convergent subsequences.

Then compactness implies closedness.

A closed and bounded $\implies A$ (sequentially) compact.

Let any $\{x_i\}$, $x_i \in A$.

Claim: $\forall \{x_i\}$ of reals, if there exists $m \in \mathbb{R}$ such that $|x_i| \leq m$, $\forall m$ then there is some convergent subsequence.



Divide and conquer: dividing the interval in half necessitates that at least one half contains infinitely many points. Repeat indefinitely.

- Theorem: Heine-Borel

$A \subseteq \mathbb{R}$ is (sequentially) compact if and only if any open cover has a finite subcover.

- Proof

Heine-Borel Property \implies closed and bounded.

Assume that A is unbounded, $U_n = (-n, n)$ and $\{U_n\}_{n=1}^{\infty}$ an open cover for $A \subseteq \mathbb{R}$ has no finite subcover.

Assume A is not closed, then $x \in \dot{A}$ (where \dot{A} is the limit set of A) and $x \notin A$, $U_n \left\{ \left(-\infty, x - \frac{1}{n} \right) \cup \left(x + \frac{1}{n}, +\infty \right) \right\}$.

Then $\{U_n\}$ covers $\mathbb{R} \setminus \{x\} \supset A$ has no finite subcover of A .

A is bounded and closed $\implies A$ is Heine-Borel

Divide and conquer: using open sets with respect to open covers.

Definition: Cantor Set

$C = \{x \in [0, 1] \mid \text{the ternary expansion of } x \text{ has only the digits } \{0, 2\}\}$.

Equivalently, let $C_0 = [0, 1]$, $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$.

Then $C_n = \bigcup_{k=1}^{2^n} C_n^k$ and $C = \bigcap_{n=1}^{\infty} C_n$.

$|C_n| = 2^n \left(\frac{1}{3}\right)^n \rightarrow 0$.

Definition: Perfectly Symmetric Sets

Let $\{\xi_n\}$ where $\xi_n \in \left(0, \frac{1}{2}\right)$.

$E_0 = [0, 1]$, $E_1 = [0, \xi_1] \cup [1 - \xi_1, 1]$, $E_2 = [0, \xi_1 \xi_2] \cup [\xi_1 - \xi_1 \xi_2, \xi_1] \cup [1 - \xi_1, 1 - \xi_1 + \xi_1 \xi_2] \cup [1 - \xi_1 \xi_2, 1]$.

Then the cantor set is given by $\xi_n = \frac{1}{3}$.

$E_n = \bigcup_{k=1}^{2^n} E_n^k$, $|E_n^k| = \xi_1 \xi_2 \cdots \xi_n$, and $|E_n| = \sum |E_n^k| = 2^n \xi_1 \xi_2 \cdots \xi_n$.

Therefore, $E = \bigcap_{n=1}^{\infty} E_n$ and we define $|E| = \lim_{n \rightarrow \infty} |E_n| = \lim_{n \rightarrow \infty} (2^n \xi_1 \xi_2 \cdots \xi_n) = \lambda$ where $\lambda \in [0, 1)$.

Let

$$2\xi_n = \frac{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n-1}\right)^{n-1}}{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n}\right)^n} < 1$$

, then

$$2^n \xi_1 \cdots \xi_n = \frac{1}{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n}\right)^n} \rightarrow \lambda.$$

Proof

$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^{n/x} = e^x$, then $\lim_{y \rightarrow 0} (1 + y)^{1/y} = e$, $\log(1 + y)^{1/y} = \frac{\log(1+y)}{y} \xrightarrow{y \rightarrow 0} 1$.

Observe that

$$\left(\frac{\log(1 + y)}{y}\right)' = \frac{\frac{y}{1+y} - \log(1 + y)}{y^2} = \left(1 + \frac{1}{1 + y} - \log(1 + y)\right)' = \frac{1}{(1 + y)^2} - \frac{1}{1 + y} = -\frac{y}{(1 + y)^2} < 0$$

Theorem

Cantor sets and perfect symmetric sets are closed, perfect, uncountable, and nowhere dense.

January 11, 2024

Last Week

Cardinality.

Topology of the reals.

- Cantor (perfect symmetric sets)

$$\begin{aligned}
C_0 &= [0, 1] \\
C_1 &= [0, 1/3] \cup [2/3, 1] \\
C_2 &= [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 1] \\
C_n &= \bigcup_{k=1}^{2^n} C_n^k \\
|C_n^k| &= \left(\frac{1}{3}\right)^n \\
C &= \bigcap_{n=1}^{\infty} C_n \\
|C_n| &= 2^n \frac{1}{3^n} = \left(\frac{2}{3}\right)^n \implies |C| = \lim_{n \rightarrow \infty} |C_n| = 0 \\
&\text{Closed, no interior points and uncountable.}
\end{aligned}$$

- Perfect Symmetric Sets

$$\begin{aligned}
\{\xi_k\} &\in \left(0, \frac{1}{2}\right) \\
E_0 &= [0, 1] \\
E_1 &= [0, \xi_1] \cup [1 - \xi_1, 1] \\
E_2 &= [0, \xi_1 \xi_2] \cup [\xi_1 - \xi_1 \xi_2, \xi_1] \cup [1 - \xi_1, 1 - \xi_1 + \xi_1 \xi_2] \cup [1 - \xi_1 \xi_2, 1] \\
E_n &= \bigcup_{k=1}^{2^n} E_n^k \\
|E_n^k| &= \xi_1 \xi_2 \cdots \xi_n \\
|E_n| &= 2^n \xi_1 \xi_2 \cdots \xi_n \\
&= \frac{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n-1}\right)^{n-1}}{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n}\right)^n} < 1 \\
|E_n| &= \frac{1}{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n}\right)^n} \\
|E| &= \lim_{n \rightarrow \infty} |E_n| = \frac{1}{e^{\log\left(\frac{1}{n}\right)}} = \lambda, \quad \lambda \in (0, 1)
\end{aligned}$$

Volterra's Function

$$\phi(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

IMAGE HERE - graph of phi(x)

$$\phi'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$f(x) = \begin{cases} 0 & x \in E \\ \phi(x-a) & x \in (a, a+y) \\ -\phi(b-x) & x \in (b-y, b) \\ \phi(y) & x \in (a+y, b-y) \end{cases}, \quad (a, b) \in E^c$$

IMAGE HERE - f interval (a,b)

Propositions

1. $f'(x) = 0$ for $x \in E$.

2. $f'(x)$ discontinuous on E .
3. f' exists on $[0, 1]$ and is bounded.

Since $|E| > 0$, $f'(x)$ is not Riemann integrable and, therefore, the fundamental theorem of calculus does not apply.

Lebesgue Outer Measure

$$|(a, b)| = b - a.$$

Let $A \subseteq \mathbb{R}$, then $m^*(A) = \inf \left\{ \sum_{n=1}^{\infty} |I_n| \mid A \subseteq \bigcup_{n=1}^{\infty} I_n \right\}$

Question: $m^*(A \cup B) \stackrel{?}{=} m^*(A) + m^*(B)$ for $A \cap B \neq \emptyset$?

Properties

1. $A \subseteq B \implies m^*(A) \leq m^*(B)$.
2. $m^*(\emptyset) = 0$.
3. If I is an interval, then $m^*(I) = |I|$.
4. If $\{A_i\}$ is countable, $m^*\left(\bigcup A_i\right) \leq \sum m^*(A_i)$.

• Proof of 4

$\forall A_i, \exists \{I_n\}$ open intervals such that $\sum_n |I_n| < m^*(A_i) + \frac{\varepsilon}{2^i}$.

Then $\bigcup_i \bigcup_n I_n^i \supset \bigcup_i A_i$, and $\sum_{n,i} |I_n^i| = \sum_i \left(\sum_n |I_n^i| \right) \leq \sum_i \left(m^*(A_i) + \frac{\varepsilon}{2^i} \right)$.

– Corollary

If A is countable, then $m^*(A) = 0$.

Thus, by contraposition, every interval is uncountable.

Proposition

For $A \subseteq \mathbb{R}$, $\forall \varepsilon > 0$, $\exists U$ open such that $A \subseteq U$ and $m^*(U) \leq m^*(A) + \varepsilon$.

Corollary

There exists G in the intersection of countable open sets such that $m^*(G) = m^*(A)$ and $G \supseteq A$.

Caratheodory Criteria

If $\forall E, m^*(E \cap A) + m^*(E \cap A^c) = m^*(E)$, then A is Lebesgue measurable.

- Remark: $m^*(E \cap A) + m^*(E \cap A^c) \leq m^*(E) \leq +\infty$

Propositions

1. If A is measurable, then A^c is measurable.

2. $m^*(A) = 0$, then A is measurable.
3. If A, B are measurable, then $A \cup B, A \cap B, A \setminus B$ are measurable.
4. If $\{A_i\}_{i=1}^k$ are disjoint and measurable, then $m^*\left(\bigcup_{i=1}^k A_i\right) = \sum_{i=1}^k m^*(A_i)$.

• Proof of 3

$$\begin{aligned}
m^*(E \cap (A \cup B)) + m^*(E \cap (A \cup B)^c) &= m^*((E \cap A) \cup (E \cap B)) + m^*(E \cap A^c \cap B^c) \\
&= m^*(E \cap A) + m^*((E \cap A^c) \cap B) + m^*((E \cap A^c) \cap B^c) \\
&\leq m^*(E)
\end{aligned}$$

Since $(A \cap B)^C = A^c \cup B^c$, this holds from before; similarly, $A \setminus B = A \cap B^c = A^c \cup B$.
If A, B disjoint, then

$$\begin{aligned}
m^*(A \cup B) &= m^*(E \cap A) + m^*(E \cap A^c) \\
&= m^*(A) + m^*(B)
\end{aligned}$$

Theorem

If $\{A_i\}$ is a countable collection of disjoint and measurable sets, then

1. $\bigcup_i A_i$ is measurable.
2. $m^*\left(\bigcup_i A_i\right) = \sum_i m^*(A_i)$.

Proof of 1

Want to show:

$$m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) + m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right) \leq m^*(E)$$

By assumption, since the measure of E is finite, $m^*(E \cap \bigcup_{i=1}^{\infty} A_i) < +\infty$.

Claim: $\forall \varepsilon > 0, \exists k$ such that

Therefore $m^*\left(E \cap \bigcup_{i=1}^k A_i\right) \geq m^*(E \cap \bigcup_{i=1}^{\infty} A_i) - \varepsilon$.

$$m^*(E) \leq m^*\left(E \cap \bigcup_{i=1}^k A_i\right) + \varepsilon + m^*\left(E \cap \left(\bigcup_{i=1}^k A_i\right)^c\right) \leq m^*(E) + \varepsilon$$

Proof of 2

We have shown $m^*\left(\bigcup_i A_i\right) \leq \sum_{i=1}^{\infty} m^*(A_i)$.

Assume $m^*\left(\bigcup_i A_i\right) < +\infty$, then

$$\sum_{i=1}^k m^*(A_i) = m^*\left(\bigcup_{i=1}^k A_i\right) \leq m^*\left(\bigcup_i A_i\right) \implies \sum_{i=1}^{\infty} m^*(A_i) \leq m^*\left(\bigcup_i A_i\right)$$