Manifolds III

March 31, 2025

Review

If X, Y are topological spaces and $f, g : X \to Y$ continuous maps, we say f and g are homotopic (written $f \simeq g$) if there is a homotopy $H : X \times I \to Y$ (where $I = [0,1]$) such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$.
We say that f is null-homotopic if it is homotopic to a constant map.

Proposition

Homotopy is an equivalence relation on the collection of continuous maps.

1. $f \simeq f$ by $H(x,t) := f(x)$.
2. $f \overset{\tilde{H}}{\simeq} g \implies g \simeq f$ by defining $\tilde{H}(x,t) := H(x, 1-t)$.
3. $(f \overset{F}{\simeq} g \wedge g \overset{G}{\simeq} h) \implies f \simeq h$ by

$$H(x,t) := \begin{cases} F(x,2t) & 0 \leq t \leq 1/2 \\ G(x,2t-1) & 1/2 \leq t \leq 1 \end{cases}.$$

Proposition

For $f_0, f_1 : X \to Y$ and $g_0, g_1 : Y \to Z$, if $f_0 \overset{F}{\simeq} f_1$ and $g_0 \overset{G}{\simeq} g_1$, then $g_0 \circ f_0 \simeq g_1 \circ f_1$.

Proof

Define $H(x,t) := G(F(x,t),t)$ such that $H(x,0) = G(F(x,0),0) = G(f_0(x),0) = g_0 \circ f_0(x)$.
Similarly, $H(x,1) = g_1 \circ f_1(x)$.

Definition: Homotopic Spaces

We say that two spaces X and Y are homotopic to each other ($X \simeq Y$) if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.

Example

$\mathbb{R}^n$ is homotopic to $\{0\}$ (or any single point) by $\iota : 0 \to \mathbb{R}^n$ and $r : \mathbb{R}^n \to 0$. Then $r \circ \iota : 0 \to 0$ is id_0 and $\iota \circ r : \mathbb{R}^n \ni x \mapsto 0 \in \mathbb{R}^n$ is homotopic to $\mathrm{id}_{\mathbb{R}^n}$. In fact, consider $H : \mathbb{R}^n \times I \to \mathbb{R}^n$ where $H(x,t) = tx$, $H(x,1) = x = \mathrm{id}_{\mathbb{R}^n}(x)$ and $H(x,0) = 0$.

Definition: Path

A path in X from p to q is a continuous map $f : I \to X$ such that $f(0) = p$ and $f(1) = q$.

Definition: Path Homotopic

Let $f, g : I \to X$ be two paths in X from p to q.
We say that f and g are path homotopic (write $f \sim g$) if there is a homotopy $H : I \times I \to X$ such that $H(s,0) = f(s)$, $G(s,1) = g(s)$, $H(0,t) = p$ and $H(1,t) = q$.

Proposition

Path homotopy is an equivalence relation on the collection of paths from p to q.
Write $[f]$, the equivalence class of f in the quotient.

Definition: Loop

In the special case that $p = q$, we say that $f : I \to X$ is a loop

Definition: Fundamental Group

Given (X, p), $\pi_1(X, p)$ (the fundamental group of X at the point p) is the set of all loops at p modulo the path homotopy.

$$\{\text{loops at } p\}/\sim$$

Equivalently, $(S^1, 1)$, $\{\text{loops at } p\} = \{\text{continuous maps } f : (S^1, 1) \to (X, p)\}$ with $f(1) = p$. We say this is the homotopy "relative to $1 \in S^1$". We have $H : S^1 \times I \to X$ such that $H(s, 0) = f(s)$, $H(s, 1) = g(s)$ and $H(1, t) = p$.

Definition: Free Homotopy

For two loops $f, g : S^1 \to X$, we say that f and g are free homotopic if $f \simeq g$.

Lemma

When $f : I \to X$ is a path from p to q, if $f \circ \varphi$ is a reparameterization of f then $(f \circ \varphi) \sim f$ where $\varphi : I \to I$ satisfies $\varphi(0) = 0$ and $\varphi(1) = 1$.

Proof

Note that φ is homotopic to the identity map id_I through $H(s, t) = ts + (1 - t)\varphi(s)$ since $H(s, 0) = \varphi(s)$ and $H(s, 1) = s = \mathrm{id}_I(s)$.
Then consider $f \circ H : I \times I \to X$ which is a path homotopy between f and $f \circ \varphi$.

Fundamental Group

Let $f, g : I \to X$ be two paths with $f(1) = g(0)$.
Then we can "compose" (concatenate) f and g together $(f \cdot g) : I \to X$ by

$$(f \cdot g)(s) := \begin{cases} f(2s) & 0 \le s \le 1/2 \\ g(2s - 1 & 1/2 \le s \le 1 \end{cases}.$$

Lemma

If $f_0 \overset{F}{\sim} f_1$, $g_0 \overset{G}{\sim} g_1$ and $f_0(1) = f_1(1) = g_0(0) = g_1(0)$, then $f_0 \cdot g_0 \sim f_1 \cdot g_1$.

Proof

Define

$$H(s, t) := \begin{cases} F(2s, t) & 0 \le s \le 1/2 \\ G(2s - 1, t) & 1/2 \le s \le 1 \end{cases}.$$

Then

$$H(s,0)=\begin{cases}F(2s,0)=f_0(2s) & 0\le s\le 1/2\\ G(2s-1,0=g_0(2s-1) & 1/2\le s\le 1\end{cases}.$$

Similarly $H(s,1)=(f_1\cdot g_1)(s)$, hence $f_0\cdot g_0\sim f_1\cdot g_1$.
With this, we have a well-defined $[f]\cdot[g]:=[f\cdot g]$.

Simple Properties

For f from p to q where c_p is the constant map at p,

1. $[c_p]\cdot[f]=[f]=[f]\cdot[c_q]$ since $c_p\cdot f$ is a reparameterization of f.

2. Let $\overline{f}$ be the inverse path of f (i.e. $\overline{f}(s)=f(1-s)$). Then $[f]\cdot[\overline{f}]=[c_p]$ and $[\overline{f}]\cdot[f]=[c_q]$.

$$H(s,t):=\begin{cases}f(2s) & 0\le s\le t/2\\ f(t) & t/2\le s\le 1-t/2\\ f(2-2s) & 1-t/2\le s\le 2\end{cases}.$$

1. $([f]\cdot[g])\cdot[h]=[f]\cdot([g]\cdot[h])$, since these are reparameterizations of the same path.

Group Structure

$\pi_1(X,p)=\{\text{loops at } p\}/\sim$.
Define $[f]\cdot[g]:=[f\cdot g]$.
It has an identity element $[c_p]=e$.
For any $f\in\pi_1(X,p)$, it has an inverse $[\overline{f}]$ such that $[f]\cdot[\overline{f}]=[\overline{f}]\cdot[f]=[c_p]$.
Finally, it is associative by (3) above.

Proposition

Suppose $p,q\in X$ with X path-connected.
Then $\pi_1(X,p)$ is isomorphic to $\pi_1(X,q)$.
Remark: this isomorphism is not canonical.

Proof

We define a path γ from q to p and $\Phi_\gamma:\pi_1(X,p)\to\pi_1(X,q)$ by $[f]\mapsto[\gamma\cdot f\cdot\overline{\gamma}]$.
Φ_γ is a group homomorphism.

$$\begin{aligned}\Phi_\gamma[f]\cdot\Phi_\gamma[g]&=[\gamma\cdot f\cdot\overline{\gamma}]\cdot[\gamma\cdot g\cdot\overline{\gamma}]\\ &=[\gamma\cdot f\cdot\overline{\gamma}\cdot\gamma\cdot g\cdot\overline{\gamma}]\\ &=[\gamma\cdot f]\cdot\overbrace{[\overline{\gamma}\cdot\gamma]}^{=e}\cdot[g\cdot\overline{\gamma}]\\ &=[\gamma\cdot(f\cdot g)\cdot\overline{\gamma}]\\ &=\Phi_\gamma[f\cdot g].\end{aligned}$$

Φ_γ has an inverse, $\Phi_{\overline{\gamma}}:\pi_1(X,q)\to\pi_1(X,p)$.

$$\Phi_{\overline{\gamma}}\circ\Phi_\gamma[f]=\Phi_{\overline{\gamma}}[\gamma\cdot f\cdot\overline{\gamma}]=[\overline{\gamma}\cdot\gamma\cdot f\cdot\overline{\gamma}\cdot\gamma]=[f].$$

Induced Homomorphism

$\varphi : (X,p) \to (Y,q)$ induces

$$\begin{aligned}\varphi_* : \pi_1(X,p) &\to \pi_1(Y,q)\\ [f] &\mapsto [\varphi \circ f].\end{aligned}$$

φ_* is a homomorphism.

$$(\varphi_*[f]) \cdot (\varphi_*[g]) = [\varphi \circ f] \cdot [\varphi \circ g] = [(\varphi \circ f) \cdot (\varphi \circ g)] = [\varphi \circ (f \cdot g)] = \varphi_*[f \cdot g].$$

Proposition

If $\varphi, \psi : (X,p) \to (Y,q)$ are homotopic, then $\varphi_* = \psi_* : \pi_1(X,p) \to \pi_1(Y,q)$.

Proof

Let $[f] \in \pi_1(X,p)$, $\varphi_*[f] = [\varphi \circ f]$ and $\psi_*[f] = [\psi \circ f]$ and $H : X \times I \to Y$ a homotopy between φ and ψ.
Then define $\tilde{H} := I \times I \to Y$ by $\tilde{H}(s,t) = H(f(s),t)$ such that

$$\begin{aligned}\tilde{H}(s,0) &= H(f(s),0) = \varphi \circ f(s)\\ \tilde{H}(s,1) &= H(f(s),1) = \psi \circ f(s).\end{aligned}$$

Corollary

If $X \simeq Y$, then $\pi_1(X) \simeq \pi_1(Y)$.

Examples (*)

$\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(S^n) = 0$ for $n \geq 2$.
For $n \geq 2$, write $S^n = A_+ \cup A_-$ where A_+ and A_- are large balls centered at the north and south pole respectively.
Then A_+ and A_- are both homeomorphic to $\mathbb{R}^n$ and $A_+ \cap A_-$ (their intersection about the equator) is homemorphic to $S^{n-1} \times \mathbb{R}$.
We fix a base point $p \in A_+ \cap A_-$ and let $f : I \to S^n$ be a loop based at p.
There exists a partition of I, $0 = s_0 < s_1 < \cdots < s_k = 1$, such that $f|_{[s_i, s_{i+1}]}$ is contained in A_- or A_+.
Draw a path γ_i from p to $f(s_i)$ such that $\gamma_i \subseteq A_+ \cap A_-$. Let $f_i = f|_{[s_i, s_{i+1}]}$ such that $f = f_0 \cdot f_1 \cdots f_k$. Then this is path homotopic to

$$(f_0 \cdot \overline{\gamma}_1) \cdot (\gamma_1 \cdot f \cdot \overline{\gamma}_2) \cdots (\gamma_{k-1} \cdot f_{k-1} \cdot \overline{\gamma}_k) \cdot (\gamma_k \cdot f_k).$$

Each $\gamma_i \cdot f_i \cdot \overline{\gamma}_i$ is contained in A_- or A_+, hence $\gamma_i \cdot f_i \overline{\gamma}_{i+1} \sim c_p$, $f \simeq c_p$ and $[f] = e$.

April 2, 2025

Correction

For $\varphi, \psi : (X, x_0) \to (Y, y_0)$ where $\varphi \simeq \psi$, we say a homotopy H between φ and ψ is base point preserving if $H(x_0, t) = y_0$ for all $t \in [0,1]$.

Proposition

If $\varphi \simeq \psi$ through a base point preserving homotopy, then $\varphi_* = \psi_*$, $\pi_1(X, x_0) \to \pi_1(Y, y_0)$.
For $X \simeq Y$, $\varphi : X \to Y$ and $\psi : Y \to X$ where $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$, in general $\psi \circ \varphi(x_0) \neq x_0$ and $\varphi \circ \psi(y_0) \neq y_0$.
Set up: $\varphi_0, \varphi_1 : X \to Y$ with $\varphi_0 \simeq \varphi_1$ through a homotopy H.
Write $\varphi_t = H(\cdot, t) : X \to Y$ and fix a base point $x_0 \in X$ and set $\gamma(t) = \varphi_t(x_0)$ for $t \in [0,1]$.

Proposition 1

$(\varphi_0)_* = \Phi_\gamma \circ (\varphi_1)_* : \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$.

Proof

Let f be a loop at x_0.

IMAGE 1

Let γ_t be $\gamma|_{[0,t]}$ and then, by rescaling the domain $[0,t]$ to $[0,1]$ i.e.

$$\begin{aligned} \gamma_t : [0,1] &\to Y \\ s &\mapsto \gamma(ts). \end{aligned}$$

from $\varphi_0(x_0)$ to $\gamma(t) = \varphi_t(x_0)$. Then $\gamma_t \cdot (\phi_t \circ f) \cdot \overline{\gamma}_t$ is a homotopy between $(\varphi_0 \circ f)$ and $\gamma \cdot (\varphi_1 \circ f) \cdot \overline{\gamma}$. Hence

$$(\varphi_0)_*[f] = [\varphi_0 \circ f] = [\gamma] \cdot [\varphi_1 \circ f] \cdot [\overline{\gamma}] = \Phi_\gamma \circ (\varphi_1)_*[f].$$

Proposition 2

If $X \simeq Y$, then $\pi_1(X) \cong \pi_1(Y)$.

Proof

Since $(\psi \circ \varphi) \simeq \mathrm{id}_X$, by Proposition 1

$$\psi_* \circ \varphi_* = (\psi \circ \varphi)_* = \Phi_\gamma \circ (\mathrm{id}_x)_* = \Phi_\gamma.$$

Hence $\psi_* \circ \varphi_*$ is an isomorphism (as is $\varphi_* \circ \psi_*$). Therefre φ_* and ψ_* are isomorphisms.

Recall: Covering Map

For $X, \tilde{X}$ connected, $\pi : \tilde{X} \to X$ is a covering map if for each $p \in X$ there exists a neighborhood $U \subset X$ such that $\pi^{-1}(U)$ is a disjoint union

$$\pi^{-1}(U) = \dot{\bigcup}_{\alpha \in A} U_\alpha$$

such that $\pi|_{U_\alpha} : U_\alpha \to U$ Is a homeomorphism.

Lifting Properties

A lift is a map $\tilde{f}$ such that $f = \pi \circ \tilde{f}$.

1. Path Lifting: Let $f : I \to X$ be a path from x_0. Then, for any $\tilde{x}_0 \in \pi^{-1}(x_0)$, there is a unique lift $\tilde{f}$ of f with $\tilde{f}(0) = \tilde{x}_0$.
2. Homotopy Lifting: Let $f_0, f_1 : I \to X$ be paths in X with $f_0(0) = f_1(0) = x_0$ and $f_0(1) = f_1(1)$. Suppose H is a path homotopy between f_0 and f_1. Then for any $\tilde{x}_0 \in \pi^{-1}(x_0)$, there is a unique lift $\tilde{H} : I \times I \to \tilde{X}$ of H. In particular, $\tilde{H}$ is a path homotopy between $\tilde{f}_0$ and $\tilde{f}_1$. That is if $H(0,t) = x_0$ then $\tilde{H}(0,t) \in \pi^{-1}(x_0)$ for all t. Hence $\tilde{H}(0,t) = \tilde{x}_0$, $\forall\, t \in [0,1]$. Similarly, $\tilde{H}(1,t)$ is identically constant. In particular, $\tilde{f}_0(1) = \tilde{H}(1,0) = \tilde{H}(1,1) = \tilde{f}_1(1)$.

Fundamental Group of the Circle

$\pi_1(S^1) = \mathbb{Z}$.

Example

$\pi : \mathbb{R} \to S^1$ by $s \mapsto e^{2\pi i \cdot s}$.

Proof

Take as a base point $1 = x_0 \in S^1 \subseteq \mathbb{C}$. For each $n \in \mathbb{Z}$, we define a loop $\omega_n : [0,1] \to S^1$ by $s \mapsto e^{2\pi i \cdot ns}$. Let f be a loop at $x_0 \in S^1$. We can lift f to $\tilde{f} : I \to \mathbb{R}$ at $0 \in \mathbb{R}$. Then $\tilde{f}(1) \in \pi^{-1}(x_0) = \mathbb{Z} \subseteq \mathbb{R}$. This defines a map φ that sends a loop f to $\tilde{f}(1) \in \mathbb{Z}$. This φ induces $\varphi : \pi_1(S^1, x_0) \to \mathbb{Z}$ well-defined. If $f_0, f_1 : I \to S^1$ at x_0 are path homotopic via H, then we may lift H to $\tilde{H} : I \times I \to \mathbb{R}$ which implies $\tilde{f}_0(1) = \tilde{f}_1(1)$.
φ is surjective, since for any $n \in Z$ we may consider the loop ω_n where $\tilde{\omega}_n(1) = n$.
φ is a group homorphism since $\varphi[f \cdot g] = \widetilde{f \cdot g}(1) = \tilde{g} + \tilde{f}(1) = \varphi[f] + \varphi[g]$.
φ is injective, since if $\varphi[f] = 0$ (i.e. $\tilde{f}(0) = 0$) then $\tilde{f}$ is a loop in $\mathbb{R}$ and $\tilde{f}$ is null-homotopic to c_0 by H. Therefore $\pi \circ \tilde{H}$ is a path-homotopy between f and c_{x_0} (i.e. $[f] = e$).

Path-Lifting

For $f : I \to X$, we have a special case where $\operatorname{im} f \subseteq U$ evenly covered. Write $\pi^{-1}(U) = \dot{\bigcup} \tilde{U}_\alpha$ and pick the $\tilde{U}_\alpha$ which contains $\tilde{x}_0$. Since $\pi|_{\tilde{U}_\alpha} : \tilde{U}_\alpha \to U$ is a homemorphism, $\tilde{f} := (\pi|_{\tilde{U}_\alpha})^{-1} \circ f$ is the unique lift of f at $\tilde{x}_0$.
In general, pick a partititon of $I = [0,1]$, $0 = t_0 < t_1 < \cdots < t_m = 1$, such that $\operatorname{im} f|_{[t_i, t_{i+1}]} \subseteq U_i$ evenly covered. We can lift $f|_{[0,t_1]}$ at $\tilde{x}_0$, giving $\tilde{f} : [0,t] \to \tilde{X}$. Next, we lift $f|_{t_1, t_2]}$ at $\tilde{f}(t_1) \in \tilde{X}$. Since the partition is finite, we may repeat the process until f is entirely lifted. This lift is unique.

Homotopy Lifting

For each fixed $(y_0, t_0) \in I \times I$, by continuity, there is a neighborhood $N(y_0) \times (t_0 - \varepsilon, t_0 + \varepsilon)$ such that H sends $N(y_0) \times (t_0 - \varepsilon, t_0 + \varepsilon)$ inside an evenly covered neighborhood. By compactness of $\{y_0\} \times [0,1]$, there is a finite collection of $N_{t_i}(y_0) \times (t_i - \varepsilon_i, t_i + \varepsilon_i)$ such that they cover $\{y_0\} \times I$ and the image of each under H is contained in an evenly covered neighborhood. Set $N = \bigcap_i N_{t_i}(y_0)$, a neighborhood of y_0, and construct a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ such that $H(N \times [t_i, t_{i+1}] \subseteq U_i$ evenly covered. Then we can start with $H|_{N \times [0,t_1]}$ and lift it at $\tilde{x}_0$ by some $(\pi|_{\tilde{U}_\alpha})^{-1}$. Then lift each $H|_{N \times [t_i, t_{i+1}]}$ one by one. Eventually, we have $\tilde{H} : N \times [0,1] \to \tilde{X}$ that lifts $H : N \times [0,1] \to \tilde{X}$ at $\tilde{x}_0$. This lift holds for any $y_0 \in I$ and, if two strips overlap, then the lift must agree there by the uniqueness of path lifting. This assures that $\tilde{H} : I^2 \to \tilde{X}$ is continuous.

Remark

Given a continuous map $F : Y \times I \to X$ and a covering $\pi : \tilde{X} \to X$, suppose that we have a map $\tilde{F} : Y \times \{0\} \to \tilde{X}$ that lifts $F|_{Y\times\{0\}} : Y \times \{0\} \to X$. Then there is a unique lift $\tilde{F} : Y \times I \to \tilde{X}$ of F which extends $\tilde{F} : Y \times \{0\} \to \tilde{X}$.

Theorem: Fundamental Theorem of Algebra

A polynomial $p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ (with $a_i \in \mathbb{C}$) has a root in $\mathbb{C}$.

Proof

Suppose otherwise. Then $p(z) \neq 0$, $\forall z \in \mathbb{C}$. Consider $f_r : [0,1] \to S^1$ $(r \geq 0)$ by

$$f_r(s) = \frac{p(re^{2\pi is})/p(r)}{|p(re^{2\pi is})/p(r)|}.$$

Then $f_0(s) \equiv 1$ is a constant loop at $1 \in \mathbb{C}$, and $f_r \simeq f_0$ for each $r \geq 0$. Consider $R \geq 1$ large such that $R \gg \sum_{i=1}^n |a_i|$. On $\{z : |z| = R\}$, we have

$$|z^n| > \left(\sum_{i=1}^n |a_i|\right) \cdot |z^{n-1}| \geq \sum_{i=1}^n |a_i| \cdot |z^{n-i}| = \left|\sum_{i=1}^n |a_i z^{n-i}\right|.$$

This implies that p does not have any roots on $\{|z| = R\}$. Moreover, for $p_t(z) = z^n + t(a_1 z^{n-1} + \cdots a_{n-1} z + a_n)$ with $0 \leq t \leq 1$, p_t does not have any roots on $\{|z| = R\}$. Consider

$$f_{R,t}(s) = \frac{p_t(Re^{2\pi is})/p_t(R)}{|p_t(Re^{2\pi is})/p_t(R)|}.$$

Then

$$f_{R,0}(s) = \frac{(Re^{2\pi is})^n/R^n}{|(Re^{2\pi is})^n/R^n|} = (e^{2\pi is})^n = \omega_n(s).$$

Therefore $f_{R,1}(s) \simeq f_R(s)$ and $f_R \simeq \omega_n$. But since $\omega_n \not\simeq$ constant so this is a contradiction.

April 7, 2025

Definition: Retraction

Let X be a space and $A \subseteq X$ be a subset. We say that a continuous map $r : X \to A$ is a retraction if $r|_A = \mathrm{id}_A$. In particular, becasue $r \circ \iota_A = \mathrm{id_A}$, for $x_0 \in A$

$$r_* \circ (\iota_A)_* : \pi_1(A, x_0) \to \pi_1(A, x_0)$$

is an isomorphism. Hence $r_* : \pi(X, x_0) \to \pi(A, x_0)$ is surjective.

Corollary

There is no retraction $r : D^2 \to S^1 (= \partial D^2)$.

Proof

Suppose there is such a map r, then

$$r_* : \overbrace{\pi_1(D^2, x_0)}^{=0} \to \overbrace{\pi_1(S^1, x_0)}^{=\mathbb{Z}}$$

is surjective which is a contradiction.

Corollary

Every continuous map $h : D^2 \to D^2$ has a fixed point.

Proof

Suppose $\exists h : D^2 \to D^2$ without fixed points.

IMAGE 1

Define $r : D^2 \to D^2$ as the ray pictured from $h(x)$ through x to the boundary. If $x \in \partial D^2$, then by construction $r(x) = x$. Hence $r : D^2 \to S^1$ is a retraction which is a contradiction.

Corollary (Borsuk-Ulam)

Let $f : S^2 \to \mathbb{R}^2$. Then there exists a pair of antipodal points x and $-x$ on S^2 such that $f(x) = f(-x)$. This carries analogously to higher dimensions.

Proof

Suppose that $f(x) \neq f(-x)$ for all $x \in S^2$. We define $g : S^2 \to S^1$ by $g(x) = \frac{f(x)-f(-x)}{||f(x)-f(-x)||}$. On $S^2 \subseteq \mathbb{R}^3$, we consider a loop γ at the equator by $\gamma(s) = (\cos(2\pi s), \sin(2\pi s), 0)$ for $s \in [0,1]$. Because S^2 is simply connected, $g \circ \gamma : [0,1] \to S^1$ is path-homotopic to a constant loop in S^1. On the other hand, we lift $h := g \circ \gamma$ to $\tilde{h} : [0,1] \to \mathbb{R}$ with $\tilde{h}(0) = 0 \in \mathbb{R}$. Note

$$h(s+1/2) = g \circ \gamma(s+1/2) = g(\cos(2\pi s + \pi), \sin(2\pi s + \pi), 0) = g(-\gamma(s)) = -g(\gamma(s)) = -h(s).$$

Hence $\tilde{h}(s+1/2) \in \pi^{-1}(-h(s))$ where $\pi : \mathbb{R} \to S^1$ is the covering map. Since $\pi^{-1}(-h(s)) = \frac{1}{2} + \tilde{h}(s) + \mathbb{Z}$, for each $s \in [0,1/2]$ there is an integer q_s such that $\tilde{h}(s+1/2) = \frac{1}{2} + \tilde{h}(s) + q_s$ and

$$\tilde{h}(s+1/2) - \tilde{h}(s) = \frac{1}{2} + q_s.$$

The left hand side depends continuously on s and, by continuity, q_s is a constant (call it q). This gives

$$\tilde{h}(1) = \tilde{h}(1/2) + \frac{1}{2} + q = \tilde{h}(0) + 1 + 2q = 1 + 2q \neq 0$$

which contradicts the assertion that h is homotopic to a constant loop.

Corollary (Large Fiber Lemma)

If $f : [0,1]^{n+1} \to \mathbb{R}^n$ is a continuous map, then there exist $a, b \in [0,1]^{n+1}$ such that $f(a) = f(b)$ and $|a - b| \geq 1$.
Remark: if $z = f(a) = f(b)$, then the lemma says that $\operatorname{diam} f^{-1}(z) \geq 1$.

Proof

Take the sphere of radius $1/2$ in $[0,1]^{n+1}$, then by Borsuk-Ulam there exist a pair of antipodal points $a, b \in S^1$ such that $f(a) = f(b)$ and $|a - b| \geq 1$.

Proposition

$\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$

Proof

Write $F : \pi_1(X \times Y) \to \pi_1(X) \times \pi_1(Y)$ by $[f] \mapsto ([g],[h])$. Then $f : [0,1] \to X \times Y$ is a loop at (x_0, y_0), $f(s) = (g(s), h(s))$, and $g : [0,1] \to X$ and $h : [0,1] \to Y$ are loops at x_0 and y_0 respectively.

Definition: Wedge Sum

Let X and Y be path-connected topological spaces. Then $X \vee Y = (X \coprod Y)/x_0 \sim y_0$
Let $\{X_\alpha\}$ be a family of such spaces. Then $\bigvee_\alpha X_\alpha = \coprod_\alpha X_\alpha / \sim$.

Sketch

$$\begin{aligned} \pi_1(S^1_-, x_0) \to \pi_1(X, x_0) &\quad \text{gen} \mapsto \alpha \\ \pi_1(S^1_+, x_0) \to \pi_1(X, x_0) &\quad \text{gen} \mapsto \beta \end{aligned}$$

with $\alpha \neq \beta$, $\alpha\beta \neq \beta\alpha$. Then $\pi_1(X, x_0)$ should be $\langle \alpha, \beta \rangle$.

Definition: Free Product

Let $\{G_\alpha\}_\alpha$ be a family of groups. $*_\alpha G_\alpha = \{g_1 g_2 \cdots g_k \,:\, \text{each } g_i \text{ is a word in some } A_\alpha\}$.

Proposition

If for each α, there is a group homomorphism $\phi_\alpha : G_\alpha \to H$ then $\{\phi_\alpha\}$ induces a group homomorphism $\Phi : *_\alpha G_\alpha \to H$ by $g_1 \cdots g_k \mapsto \phi_{\alpha_1}(g_1) \cdots \phi_{\alpha_k}(g_k)$.

Van-Kapen Theorem

Setup

Let $X = \bigcup_\alpha A_\alpha$, each A_α open and connected where $\{A_\alpha\}$ have a common point x_0. Assume also that each $A_\alpha \cap A_\beta$ is path connected. Then $j_\alpha : A_\alpha \hookrightarrow X$ induces $j_\alpha : \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$. $\{j_\alpha\}_\alpha$ induces $\Phi : *_\alpha \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$ which is surjective by a similar argument as was used above for Example (*) ($S^2 = A_- \cup A_+$) applied to $X = \bigcup_\alpha A_\alpha$. Now, what is the kernel of Φ?

$$\begin{tikzcd} & A_\alpha \arrow[dr, hook] & \\ A_\alpha \cap A_\beta \arrow[ur, hook] \arrow[dr, hook] & & X \\ & A_\beta \arrow[ur, hook] & \end{tikzcd}$$

$$\begin{tikzcd}
 & \pi_1(A_\alpha) \arrow[dr, hook] & \\
\pi_1(A_\alpha \cap A_\beta) \arrow[ur, hook, "i_{\alpha\beta}"] \arrow[dr, hook, "i_{\beta\alpha}"'] & & \pi_1(X) \\
 & \pi_1(A_\beta) \arrow[ur, hook] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \underline{\quad} \arrow[dr, hook] & \\
w \arrow[ur, hook] \arrow[dr, hook] & & \underline{\quad} \\
 & \underline{\quad} \arrow[ur, hook] &
\end{tikzcd}$$

Then $i_{\beta\alpha}(w)i_{\alpha\beta}(w)^{-1}$ is NOT id in $*_\alpha \pi_1(A_\alpha)$. But through Φ, it hsould be $\mathrm{id} \in \pi_1(X, x_0)$. Hence every element in $*_\alpha \pi_1(A_\alpha)$ of the form $i_{\beta\alpha}(w)i_{\alpha\beta}(w)^{-1}$ where $w \in \pi_1(A_\alpha \cap A_\beta$ is in the kernel of Φ.

Theorem (Van-Kampen)

If every $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected, $\ker \Phi$ is the normal subgroup N generated by $\{i_{\beta\alpha}(w)i_{\alpha\beta}(w)^{-1} : \alpha, \beta \in A,\ w \in \pi_1(A_\alpha \cap A_\beta)\}$. Hence $\pi_1(X, x_0) \cong (*_\alpha \pi_1(A_\alpha, x_0))/N$.

Remarks

1. In the case that $X = A_0 \cup A_1$ with $A_0 \cap A_1$ path connected, then the intersection condition holds.

2. If $X = A_0 \cup A_1$ and $A_0 \cap A_1$ is simply connected, then $N = \{\mathrm{id}\}$ and $\pi_1(X) = \pi_1(A_0) * \pi_1(A_1)$.

3. If $X = A_0 \cup A_1$ and A_1 is simply connected, then $\pi_1(X) = \pi_1(A_0)/N$ and N is the normal subgroup generated by

$$i_{01}(w)\overbrace{i_{10}(w)^{-1}}^{\in \pi_1(A_1, x_0)} = i_{01}(w)$$

i.e. N is the normal closure of $i_{01}(\pi_1(A_0 \cap A_1))$.

Example

IMAGE 2

For each $\alpha \in \{1, \ldots, 5\}$, let A_α be a small neighborhood of $T \cup e_1$. Every double/triple intersection is a neighborhood of T. Hence it is path continuous and we have that $\pi_1(A_\alpha) = \mathbb{Z}$. Thus $\pi_1(A_\alpha \cap A_\beta) = \mathrm{id}$, and $\pi_1(X) = *_\alpha \pi_1(A_\alpha)/N = *_1^5 \mathbb{Z}$.

Example

IMAGE 3

By Van-Kampen, $\pi_1(X) = \pi_1(A_0)$ modulo the normal closure of $i(\pi_1(A_0 \cap A_1))$. That is

$$\left\langle a, b \mid aba^{-1}b^{1} \right\rangle = \mathbb{Z}^2.$$

Remark

In general, orientable M_g is the connected sum of g many toruses.

April 9, 2025

Recall: Van-Kampen Theorem

Write $\pi_1(X) = (\pi_1(A) * \pi_1(B))/N$ where N is the normal closure of $i_{\alpha\beta}(w)i_{\beta\alpha}(w)^{-1}$ for $w \in \pi_1(A \cap B)$, $i_{\alpha\beta} : \pi_1(A \cap B) \to \pi_1(A)$ and $i_{\beta\alpha} : \pi_1(A \cap B) \to \pi_1(B)$.

Example

M_g is the connected sum of g many tori, and $\pi_1(M_g) = \langle a_1, b_1, \dots, a_g, b_g \mid [a_1 b_1] \cdots [a_g b_g] \rangle$.

Example

N_g is the connected sum of g many $\mathbb{RP}^2$ (e.g. N_2 is the Klein bottle). N_g has a polygon-representation by the $2g$-gon with boundary identified through $a_1 a_1 a_2 a_2 \cdots a_g a_g$. Therefore $\pi_1(N_g) = \left\langle a_1 \cdots a_g \mid a_1^2 \cdots a_g^2 \right\rangle$.

Abelianiztion

1. $\mathrm{Ab}(\pi_1(M_g))$ is the free abelian group generated by $\{a_1, b_1, \dots, a_g, b_g\} = \mathbb{Z}^{2g}$.
2. $\mathrm{Ab}(\pi_1(N_g)) = \mathrm{Ab}(\left\langle a_1 \cdots a_{g-1} a_1 a_2 \cdots a_g \mid a_1^2 \cdots a_g^2 \right\rangle = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$.

Corollary

None of the surfaces in $\{S^2, M_1, \dots, M_g, \dots, N_1, \dots, N_g, \dots\}$ are homotopic to each other.

Definition: Cell Complex

0-cells are points; 1-cells, e^1, are intervals; 2-cells, e^2, are disks; n-cells, e^n, are $\overline{B}^n$.
A cell complex for space X is a decomposition (assuming finite dimensions) $X = X^0 \cup X^1 \cup \cdots \cup X^n$ where X^0 is the discrete set of points (i.e. 0-cells), X^1 is the space obtained by gluing 1-cells to X^0 ($\varphi_\alpha : \partial e_\alpha^1 \to X^0$), X^2 is the space obtained by gluing 2-cells to X^1 ($\varphi_\alpha : \partial e_\alpha^2 \to X^1$), and in general X^n is obtained by gluing n-cells $\{e_\alpha^n\}_\alpha$ to X^{n-1} by $\varphi_\alpha : \partial e_\alpha^n = S^{n-1} \to X^n$.

Examples

Cell complexes need not be unique. $S^2 = X^1 \cup_\alpha e_+^2 \cup_\alpha e_-^2$ and $S^2 = \{e^0\} \cup_\alpha \{e^2\}$.
$\mathbb{RP}^2 = \{e^1\} \cup_\alpha \{e^2\}$ where φ_α is given by $z \mapsto z^2$.
$\mathbb{T}^2$ is gluing e^2 to $S^1 \vee S^1$.

Theorem (Computing Fundamental Group)

Set up

Let X be a path-connected space, $Y = X \cup_\alpha e_\alpha^2$ (i.e. X is created by gluing 2-cells $\{e_\alpha^2\}_\alpha$ to X via $\phi_\alpha : \partial e_\alpha^2 \to X$). The inclusion $\iota : X \to Y$ induces $\iota_* : \pi_1(X) \to \pi_1(Y)$. Fix a base point $s_0 \in S^1$. For each α we draw a path γ_α from x_0 to $\varphi_\alpha(s_0)$. Then $\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha$ is a loop based at x_0. Thus $\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha$ is null-homotopic in Y (because φ_α is null-homotopic in e_α^2). That is $\iota_*[\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha] = \mathrm{id}$ in $\pi_1(Y)$ and is therefore in the kernel.

Theorem

Let N be the normal subgroup in $\pi_1(X)$ generated by elements of the form $[\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha]$. Then $\pi_1(Y) \cong \pi_1(X)/N$.

IMAGE 1

Example

$\mathbb{RP}^2$ is X^1 with e^2 glued to it by the map $\varphi : z \mapsto z^2$. Then $\pi_1(\mathbb{RP}^2) = \pi_1(S^1)/N = \langle \gamma \mid \gamma^2 \rangle$ where N is generated by φ. Similarly, the theorem applies to any M_g or N_g.

Definition: Deformation Retraction

For $X \subseteq Z$, $r : Z \to X$ is a retraction if $r|_X = \mathrm{id}_X$ implies $r \circ \iota = \mathrm{id}_X$. If $\iota \circ r : Z \to Z$ is homotopic to id_X, then $r_* : \pi_1(Z) \to \pi_1(X)$ is an isomorphism.

Proof

For each α, we glue a strip S_α along γ_α. We set the base at z_0 above x_0, $Z = Y \cup_\alpha S_\alpha$. Y is a deformation retraction of Z ($\pi_1(Y) = \pi_1(Z)$).

IMAGE 2

Set $A = Z - \bigcup_\alpha \{y_\alpha\}$, where y_α is a point in e^2_α not intersecting S_α. $B = Z - X$. A deformation retracts to X $\pi_1(A) = \pi_1(X)$. B is the union of some S_α (removing r_α) and some e^2_α (removing ∂e^2_α). B is contractible, $\pi_1(B) = \mathrm{id}$ and $A \cap B$ is the union of strips S_α and open disks punctured at y_α. Therefore

$$\pi_1(Y) = \pi_1(Z) = (\pi_1(A) * \pi_1(B))/N = \pi_1(A)/\iota_*(\pi_1(A \cap B)) \cong \pi_1(X)/\iota_*(\pi_1(A \cap B)).$$

Consider the loop $\delta_\alpha \cdot \gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma}_\alpha \cdot \overline{\delta}_\alpha$ where δ_α runs from z_0 to x_0, call this λ_α. It suffices to show that these generate $\pi_1(A \cap B, z_0)$. Cover $A \cap B$ by $A_\alpha = (A \cap B) - \bigcup_{\beta \neq \alpha} e^2_\beta$. Then A_α is a union of strips (with trivia fundamental group) and a single punctured, open disk $e^2_\alpha - \{y_\alpha\}$ and $\pi_1(A_\alpha) = \mathbb{Z} = \langle \lambda_\alpha \rangle$. So $A_\alpha \cap A_\beta$ is the union of strips, equal to $A_\alpha \cap A_\beta \cap A_\gamma$ and simply connected. By Van-Kampen,

$$\pi_1(A \cap B) = (*_\alpha \pi_1(A_\alpha))/N = *_\alpha \pi_1(A_\alpha)$$

is the free group generated by $\{\lambda_\alpha\}_\alpha$. This completes the proof.

Generalization (Theorem: Part 2)

If $Y = X \cup_\alpha e^n_\alpha$ for $n \geq 3$, then $\pi_1(Y) \cong \pi_1(X)$.
This follows from the same argument where instead A_α is the union of strips and a single punctured ball $B^n - \{y_\alpha\} \simeq S^{n-1}$. So $\pi_1(A_\alpha) = \mathrm{id}$, $\pi_1(A \cap B) = \mathrm{id}$, and $\pi_1(X) \cong \pi_1(Y)$.

Theorem: Part 3

Suppose X has a cell complex $X = X^0 \cup X^1 \cup \cdots \cup X^n$. Then $\pi_1(X) \cong \pi_1(X^2)$.
The proof follows directly from part 2.

Corollary

Given any group represented by generators and relations $G = \langle g_\alpha \mid r_\beta \rangle$, there is a cell complex X_G, of dimension 2, such that $\pi_1(X_G) \cong G$.

Proof

For each g_α, we draw a circle S^1_α. Then $X^1 = \bigvee_\alpha S^1_\alpha$ has fundamental group $*_\alpha \pi_1(S_\alpha) = \langle g_\alpha \rangle_\alpha$. To construct X_G, for each r_β glue a 2-cell e^2_α along r_β (think of r_β as a loop in X^1). Then in $X_G := X^1 \cup_\beta e^2_\beta$ we have $\pi_1(X_G) = \langle g_\alpha \mid r_\beta \rangle$.

April 14, 2025

Recall: Covering Spaces

Let $p : \tilde{X} \to X$, both X and $\tilde{X}$ path-connected.

1. Path-lifting: let $f : I \to X$ starting at $f(0) = x_0$. There is a unique lifting $\tilde{f}$ of f at $\tilde{x}_0 \in p^{-1}(x_0)$.

2. Homotopy-lifting: let $f_0, f_1 : I \to X$ be two paths with $f_0(0) = f_1(0) = x_0$ and $f_0(1) = f_1(1)$. Suppose f_t is a path-homotopy between f_0 and f_1. Then there exists a unique lift $\tilde{f}_t$ between $\tilde{f}_0$ and $\tilde{f}_1$ at $\tilde{x} \in p^{-1}(x)$.

These come from the following: let $f_t : Y \to X$ be a homotopy between f_0 and f_1. Given $\tilde{f}_0 : Y \to \tilde{X}$ that lifts f_0, there exists a unique lifting $\tilde{f}_t$. For path-lifting, we take Y a point; for homotopy-lifting, $Y = [0,1]$.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ I \arrow[ur, "f"] \arrow[r, "p \circ f"'] & X \end{tikzcd}$$

Proposition 1.31 (in Hatcher)

The covering map $p : \tilde{X} \to X$ induces $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x)$.

1. p_* is injective.

2. $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ are exactly loops at x_0 that lift to loops at $\tilde{x}_0$.

Proof of 1

Suppose $p_*[f] = \mathrm{id} \in \pi_1(X, x_0)$ where $[f] \in \pi_1(\tilde{X}, \tilde{x}_0)$. Then $[p \circ f] = \mathrm{id}$, and $[p \circ f]$ is path-homotopic to the constant loop c_{x_0}. Hence the lifting $\widetilde{p \circ f} = f$ is path-homotopic to a constant loop $c_{\tilde{x}_0}$.

Proof of 2

Let $[f] \in \pi_1(\tilde{X}, \tilde{x}_0)$. $p_*[f] = [p \circ f]$, $p \circ f$ lifts to f at $\tilde{x}_0$ which is a loop at $\tilde{x}_0$.
Let f be a loop at x_0. Suppose f lifts to a loop $\tilde{f}$ at $\tilde{x}_0$ (i.e. $p \circ \tilde{f} = f$). Hence $[f] = [p \circ \tilde{f}] = p_*[\tilde{f}] \in p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

Example

If $p : S^1 \to S^1$ by $z \to z^2$, then $p_*(\pi_1(S^1, 1)) = 2\mathbb{Z} \leq \mathbb{Z} = \pi_1(S^1, 1)$.

Remark

If $p : \tilde{X} \to X$ connected, then $p^{-1}(x)$ has the same cardinality for all $x \in X$.

Proof

Fix $x_0 \in X$. Consider $\mathcal{A} = \{x \in X \ : \ p^{-1}(x)$ and $p^{-1}(x_0)$ have the same cardinality$\} \neq \varnothing$. Then $\mathcal{A}$ is open since for each $x \in \mathcal{A}$, there is a neighborhood U of x such that U is evenly covered by p (i.e. $p^{-1}(U) = \dot{\bigcup}_{\alpha \in I} V_\alpha$ where $V_\alpha \overset{p}{\cong} U$). Then $p^{-1}(x')$ has cardinality $|I|$ for all $x' \in U$. It follows, since $\mathcal{A}^c$ is open, that $\mathcal{A}$ is also closed.

Proposition

The number of sheets is given by $[\pi_1(X, x_0) : p_*(\pi_1(\tilde{X}, \tilde{x}_0)]$.

Proof

Write $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Define $\Phi : \{H\text{-cosets in } \pi_1(X, x_0)\} \to p^{-1}(x_0)$ by $H[g] \mapsto \tilde{g}(1)$ where $\tilde{g}$ is a lift of g at $\tilde{x}_0$.
This map is well defined, since for $[h \cdot g]$ with $h \in H$, $\widetilde{h \cdot g}(1) = \tilde{g}(1)$ (because $\tilde{h}(1) = \tilde{x}_0$).
Φ is surjective. Let $\tilde{x}_1 \in p^{-1}(x_0)$

IMAGE 1

and let $\tilde{g}$ be a path from $\tilde{x}_0$ to $\tilde{x}_1$. Define $g = p \circ \tilde{g}$, a loop at x_0. Then $\Phi(H[g]) = \tilde{g}(1) = \tilde{x}_1$.
Φ is injective. Suppose $\Phi(H[g_1]) = \Phi(H[g_2])$ (i.e. $\tilde{g}_1(1) = \tilde{g}_2(1)$.

IMAGE 2

Consider the loop $g_1 \overline{g}_2$ in X at x_0. It lifts to $\tilde{g}_1 \overline{\tilde{g}}_2$, which is a loop at $\tilde{x}_0$. This shows that $[g_1 \overline{g}_2] \in H$ (i.e. $H[g_1] = H[g_2]$).

Recall (Manifolds 2)

If a smooth manifold M is non-orientable, then there is a double cover (2 sheets) $p : \hat{M} \to M$ ($\hat{M}$ connected). Consequently, $\pi_1(M)$ has a subgroup of index 2.

Definition: Locally Path-Connected

A topological space is called locally path-connected if for each $x \in X$ and every neighborhood $U \ni X$, there is a neighborhood $V \ni x$ such that $V \subseteq U$ and V is path-connected (i.e. $\forall x \in X$, there exists a local basis $\{U_\alpha\}$ at x such that each U_α is path-connected). For example, the Topologist's sine curve with endpoints identified is path-connected but not locally path-connected.

Proposition: Lifting Criterion

Let Y be path-connected and locally path-connected. Given a covering map $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and a map $f : (Y, y_0) \to (X, x_0)$, f has a lift $\tilde{f}$ at $\tilde{x}_0$ ($\tilde{f}(y_0) = \tilde{x}_0$) if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

Proof

$(\Longrightarrow)$

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & X \end{tikzcd} \qquad \begin{tikzcd} & \pi(\tilde{X}) \arrow[d, "p_*"] \\ \pi_1(Y) \arrow[ur, "\tilde{f}_*"] \arrow[r, "f_*"] & \pi_1(X) \end{tikzcd}$$

$$f_* \pi_1(Y) = (p_* \circ \tilde{f}_*)(\pi_1(Y)) \leq p_* \pi_1(\tilde{X}).$$

$(\Longleftarrow)$ Let $y \in Y$, and draw a path γ from y_0 to y.

IMAGE 3

We lift $f \circ \gamma$ to a path in $\tilde{X}$ starting at $\tilde{x}_0$ and define $\tilde{f}(y)$ as the endpoint (i.e. $\tilde{f}(y) = \widetilde{f \circ \gamma}(a)$).
This is well-defined, since $(f \circ \gamma) \cdot (f \circ \overline{\gamma}')$ is a loop at x_0 and $[(f \circ \gamma) \cdot (f \circ \overline{\gamma}')] = f_*[\gamma \cdot \overline{\gamma}'] \in f_* \pi_1(Y, y_0) \leq p_* \pi_1(\tilde{X}, \tilde{x}_0)$.
Hence $(f \circ \gamma) \cdot (f \circ \overline{\gamma}')$ lifts to a loop at $\tilde{x}_0$.

IMAGE 4

Therefore $\widetilde{f \circ \gamma}(1) = \widetilde{f \circ \gamma'}(1)$.
$\tilde{f}$ is continuous. Fix $f(y) \in X$ and let U be a neighborhood of $f(y)$ that is evenly covered by p. Choose a path-connected neighborhood V of y such that $f(V) \subseteq U$. We check $\tilde{f}|_V$.

IMAGE 5

Because V is path-connected, we may draw a path η in V from y to y'. Then $\tilde{f}(y') = \widetilde{f \circ \gamma \circ \eta}(1)$, and $\widetilde{\gamma \cdot \eta}$ is first lifting $f \circ \gamma$ at $\tilde{x}_0$ followed by lifting $f \circ \eta$ at $\tilde{\gamma}(1)$. Let $\tilde{U} \subseteq \tilde{X}$ such that $p|_{\tilde{U}} : \tilde{U} \to U$ is a homeomorphism and $\widetilde{f \circ \gamma}(1) \in \tilde{U}$.
Then $\widetilde{f \circ \eta}(1) = (p^{-1})|_U \circ f(y')$. Hence $\tilde{f}(y') = f \circ \widetilde{(\gamma \circ \eta)}(1) = \widetilde{f \circ \eta}(1) = (p^{-1})|_U \circ f(y')$ (i.e. $\tilde{f} = (p^{-1})|_U = f$ on V). Hence $\tilde{f}$ is continuous at y.
$\tilde{f}$ is a lift of f. In fact, $(p \circ \tilde{f})(y) = p \circ (\widetilde{f\gamma}(1)) = f(y)$.

Corollary

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

If Y is simply connected, then $f_* \pi_1(Y) \leq p_* \pi_1(\tilde{X})$ always holds (i.e. we can always lift f to $\tilde{f} : Y \to \tilde{X}$ in this case).

Proposition: Unique Lifting

Given $p : \tilde{X} \to X$ and $f : Y \to X$, if two lifts $\tilde{f}_1$ and $\tilde{f}_2$ of f agree at one point, then the agree everywhere on Y.

Proof

Take $\mathcal{A} = \{y \in Y \ : \ \tilde{f}_1(y) = \tilde{f}_2(y)\} \neq \emptyset$. Locally for each $y \in Y$ there exists a neighborhood V of y such that $\tilde{f} = (p^{-1})|_U \circ f$. If $y \in \mathcal{A}$, then $\tilde{f}_1(y) = \tilde{f}_2(y)$. Take a neighborhood U of $f(y)$ that is evenly covered and $\tilde{U}$ of $\tilde{f}_1(y) = \tilde{f}_2(y)$ such that $p|_{\tilde{U}} : \tilde{U} \to U$ is a homeomorphism. Then on V, a path-connected neighborhood such that $f(V) \subseteq U$, $\tilde{f}_i = (p^{-1})|_U \circ f$ (i.e. $\tilde{f}_1 = \tilde{f}_2$ on V). If $y \in \mathcal{A}^c$, $\tilde{f}_1(y) \neq \tilde{f}_2(y)$. Then $\tilde{U}_i \ni \tilde{f}_i(y)$ with $\tilde{U}_1 \cap \tilde{U}_2 = \emptyset$. Then on V, $\tilde{f}_i = (p^{-1})|_{\tilde{U}_i} \circ f$ (ie $\tilde{f}_1$ and $\tilde{f}_2$ never agree on V). Hence $\mathcal{A} = Y$.

Remark

If $p : \tilde{X} \to X$ is a covering map, recall that a covering transformation is a map $f : \tilde{X} \to \tilde{X}$ such that

$$\begin{tikzcd} \tilde{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

commutes. This $f : \tilde{X} \to \tilde{X}$ is a lift of $p : \tilde{X} \to X$. If we fix $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$, the lifting criterion says that $p_* \pi_1(\tilde{X}, \tilde{x}_1) \leq p_* \pi_1(\tilde{X}, \tilde{x}_2)$. In particular, if $\pi_1(\tilde{X})$ is, then this holds. Hence there is a unique lift of p (i.e. covering transformation) f such that $f(\tilde{x}_1) = \tilde{x}_2$.

April 16, 2025

Question

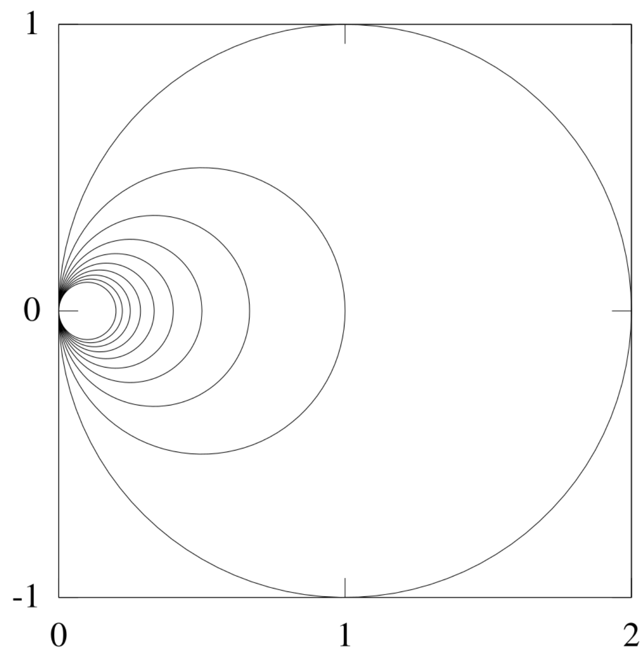
Given X path-connected and locally path-connected, when does X admit a simply connected covering space $p : \tilde{X} \to X$?

Definition: Semi-locally Simply Connected

We say that X is semi-locally simply connected if for any $x \in X$ there exists a neighborhood U such that every loop in U is null-homotopic in X. That is $\mathrm{Im}(\pi_1(U) \to \pi_1(X))$ is trivial.

Non-example

The Hawaiian earing in $\mathbb{R}^2$.



Example

The cones over the Hawaiian earing.

IMAGE 1

In fact, this is simply connected.

Example

The double Hawaiian earing with cones.

IMAGE 2

Theorem

X has a simply connected covering space (i.e. a universal covering) if and only if X is semi-locally simply connected.

Proof

($\Longrightarrow$) Let $x \in X$ and pick a neighborhood U of x that is evenly covered by p. Let f be a loop at x in U. f lifts to $\tilde{f}$ at $\tilde{x}_0$, which is a loop. Retract $\tilde{f}$ to $c_{\tilde{x}_0}$ by a path-homotopy H. Then $p \circ H$ shows that f is null-homotopic in X.
($\Longleftarrow$) We construct $\tilde{X}$ as follows: fix $x_0 \in X$ and set $\tilde{X} = \{[\gamma] \text{ path homotopies } : \gamma \text{ is a path starting at } x_0\}$. Let $\mathcal{U} = \{U : \mathrm{Im}(\pi_1(U) \to \pi_1(X)) \text{ is trivial}\}$. By assumption $\mathcal{U}$ is a basis for X. For each $u \in \mathcal{U}$ and each γ from x_0 to a point in U, we define $U_{[\gamma]} = \{\gamma \cdot \eta] : \eta \text{ starting at } \gamma(1) \text{ stays in } U\}$. Then $p : \tilde{X} \to X$ by $[\gamma] \to \gamma(1)$.
We need to check that $\{U_{[\gamma]} : U \in \mathcal{U}, \gamma \text{ a path from } x_0 \text{ to a point in } U\}$ generates a topology on $\tilde{X}$.
We need also to check that $p : U_{[\gamma]} \to U$ is bijective. It is clearly surjective, and if $p[\gamma \cdot \eta] = p[\gamma \cdot \delta]$ with η, δ paths starting at $\gamma(1)$ and staying in U. Then $\eta(1) = \delta(1)$ and, since η, δ share the same endpoints and they stay in $U_{[\gamma]}$, then $[\eta] = [\delta]$. Hence $[\gamma \cdot \eta] = [\gamma \cdot \delta]$ and p is injective.
Further, we need to check that $p : U_{[\gamma]} \to U$ is a homemorphism and that $p^{-1}(U) = \dot{\bigcup}_{[\gamma]} U_{[\gamma]}$. Hence p is a covering map.
Finally, we need to check that $\tilde{X}$ is simply connected. Recall that $p : \tilde{X} \to X$ induces an injective homomorphism $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$. It suffices to show that $p_* \pi_1(\tilde{X}, \tilde{x}_0) = \mathrm{id}$. We se t$\tilde{x}_0 = [C_{x_0}] \in \tilde{X}$. Recall also that elements in $p_* \pi_1(\tilde{X}, \tilde{x}_0)$ are exactly the loops in X at x_0 such that they lift to loops at $\tilde{x}_0$. Suppose $[\gamma] \in p_* \pi_1(\tilde{X}, \tilde{x}_0)$. Then γ lifts to a loop $\tilde{\gamma}$ at $\tilde{x}_0 = [C_{x_0}]$. For $t \in [0,1]$, consider the path γ_t which follows γ on $[0,t]$ then stays stationary at $\gamma(t)$ for the remaining time. Then $t \mapsto [\gamma_t]$ is a path on $\tilde{X}$, $p([\gamma_t]) = \gamma_t(1) = \gamma(t)$, and $t \mapsto [\gamma_t]$ is a lift of γ at $\tilde{x}_0 = [C_{x_0}]$. Then $t \mapsto [\gamma_t]$ is a loop (i.e. $[\gamma] = [\gamma_1] = \tilde{x}_0 = [C_{x_0}]$) and γ is null-homotopic. This shows that $p_* \pi_1(\tilde{X}, \tilde{x}_0) = \mathrm{id}$ (i.e. $\tilde{X}$ is simply connected).

Group Actions on Fibers (Monodromy Action)

Given $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ a covering map, $\pi_1(X, x_0)$ acts on p^{-1} as follows: $p^{-1}(x_0) \times \pi_1(X, x_0) \to p^{-1}(x_0)$ by $(e, [f]) \mapsto \tilde{f}_e(1)$ where $\tilde{f}_e$ is the (unique) lift of f at $e \in p^{-1}(x_0)$. This is a right $\underline{\pi_1(X, x)}$ action.
We want to check that $(e \cdot [f]) \cdot [g] = e \cdot [f \cdot g]$. We have that $e \cdot [f \cdot g] = \widetilde{(f \cdot g)}_e(1)$, but $\widetilde{(f \cdot g)}_e$ is the lift of f at e followed by the lift of g at the endpoint of $\tilde{f}_e$, call it $\tilde{f}_e(1) = z$. Then $\widetilde{(f \cdot g)}_e(1) = \tilde{g}_z(1) = z \cdot [g] = (e \cdot [f]) \cdot [g]$.
This action is transitive. Given e and e', draw a path connecting them $\tilde{g}$. Under the map p, we have that $p \circ \tilde{g} = g$ which is a loop at x_0. Then $e \cdot [g] = \tilde{g}(1) = e'$.
Recall: Given a right G-set S, $G_S = \{g \in G : s \cdot g = s\}$ is the isotropy subgroup at $s \in S$.
Given $e \in p^{-1}(x_0)$, the isotropy subgroup at e is all the loops such that their lfts at e are loops (i.e. the isotropy subgroup at e is precisely $p_* \pi_1(\tilde{X}, e)$).
Recall: $G \cdot S = G / G_s$. Here, this tells us that $p^{-1}(x_0) = \pi_1(X, x_0) / p_* \pi_1(\tilde{X}, e)$. This recovers the fact that the number of sheets is equal to the index of $\mathrm{im}(p_*)$.
In particular, if $\tilde{X}$ is simply connected, then

- $\pi_1(X, x_0)$ acts freely on $p^{-1}(x_0)$ and
- the number of sheets equals the cardinality of $\pi_1(X, x_0)$.

Definition: Universal Cover

A covering space $p : \tilde{X} \to X$ is called universal if it has the universal property (i.e. for any covering space $q : Y \to X$, there is a covering map $\tilde{p} : \tilde{X} \to Y$ such that the associated diagram commutes).

$$\begin{tikzcd} \tilde{X} \arrow[r, dashed, "\tilde{p}"] \arrow[d, "p"'] & Y \arrow[dl, "q"] \\ X & \end{tikzcd}$$

Definition: Covering Homomorphism

Let $p_1 : X_1 \to X$ and $p_2 : X_2 \to X$ be two covering spaces. A covering homomorphism is a map $\varphi : X_1 \to X_2$ such that the associated diagram commutes

$$\begin{tikzcd} X_1 \arrow[rr, "\varphi"] \arrow[dr, "p_1"'] & & X_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

By definition, φ is a lift of p_1.

Proposition

1. A covering homomorphism φ is uniquely determined by its value at one point.

2. For each $x \in X$, $\varphi|_{p_1^{-1}(x)} : p_1^{-1}(x) \to p_2^{-1}(x)$ is $\pi_1(X, x_0)$-equivariant.

3. A covering homomorphism $\varphi : X_1 \to X_2$ is a covering map. Assuming this, the universal cover is unique.

Recall: if S_1, S_2 are right G-sets, a G-equivariant map $\varphi : S_1 \to S_2$ is a map such that the associated diagram commutes

$$\begin{tikzcd} S_1 \arrow[r, "\varphi"] \arrow[d, "\cdot g"] & S_2 \arrow[d, "\cdot g"] \\ S_1 \arrow[r, "\varphi"] & S_2 \end{tikzcd}$$

Proof of 2

Let $e \in p_1^{-1}(x)$. We need to show taht $\varphi(e)\cdot g = \varphi(e\cdot g)$. We have that $g \in \pi_1(X, x_0)$ is represented by a loop f at x_0. So $e\cdot g = e\cdot[f] = \tilde{f}_e(1) \in X_1$, and $\varphi(e\cdot g) = \varphi(\tilde{f}_e(1))$. On the left hand side, we have that $\varphi(e)\cdot g = f_{\varphi(e)}(1) \in X_2$. We need to verify that $\varphi(\tilde{f}_e) = \tilde{f}_{\varphi(e)}$ which are both lifts of f at $\varphi(e)$. But since the diagram commutes, $p_2(\varphi \circ \tilde{f}_e) = p_1 \circ \tilde{f}_e = f$.

Uniqueness in 3

Suppose we have

$$\begin{tikzcd} X_1 \arrow[rr, shift left, leftarrow, "\psi"] \arrow[rr, shift right, "\varphi"'] \arrow[dr, "p_1"'] & & X_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

with $\varphi(e_1) = e_2$ and $\psi(e_2) = e_1$. Then $\psi \circ \varphi(e_1) = e_1$. Hence $\psi \circ \varphi = \mathrm{id}$ and, similarly, $\psi \circ \varphi = \mathrm{id}$. Hence φ is a bijection and a homemorphism.

Proof of 3

φ is surjective. Given any $e' \in X_2$, set $x_0 = p_2(e)$ and let $e \in p_1^{-1}(x_0)$ so $\varphi(e) \in p_2^{-1}(x_0)$. Since $\pi_1(X, x_0)$ acts transitively on $p_2^{-1}(x_0)$, there exists $g \in \pi_1(X, x_0)$ such that $e' = \varphi(e) \cdot g = \varphi(e \cdot g)$
φ is a covering map. Let V be a neighborhood of $x_0 \in X$ such that V is evenly covered by both p_1 and p_2. Let U be a component in $p_2^{-1}(V)$ that contains e_2. Then $p_1^{-1}(X) = \dot{\bigcup} U_\alpha$. U as a component in $p_2^{-1}(V)$ is both open and closed.

Hence $\varphi^{-1}(U)$ is open and closed in $p_1^{-1}(V) = \dot{\bigcup} U_\alpha$. It follows that $\varphi^{-1}(U)$ is the disjoint union of several components of $\{U_\alpha\}_\alpha$, and each component is homemorphic to V and consequently homeomorphic to U. This shows that φ is a covering map.

April 21, 2025

Recall: Lifting Criterion

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & (X, x_0) \end{tikzcd}$$

There exists a lift $\tilde{f}$ of f at $\tilde{x}_0$ if and only if $f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)$.

If $(\tilde{X}, \tilde{x}_0) \xrightarrow{p} (Y, y_0)$, $\pi_1(X, x_0)$ acts transitively on $p^{-1}(x_0)$ by path lifting (a right action where $e \in p^{-1}(x_0)$ by $e \cdot [\gamma] = \tilde{\gamma}_e(1)$). The isotropy subgroup at e is $p_*\pi_1(\tilde{X}, e)$.

Covering Transformations

$$\begin{tikzcd} \tilde{X} \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

Write $\mathrm{Aut}(\tilde{X} \xrightarrow{p} X)$ for the covering group $\{\varphi : \tilde{X} \to \tilde{X} \text{ covering transformations}\}$.

1. $\varphi : \tilde{X} \to \tilde{X}$ is uniquely determined by its value at one point.

2. Given $e_1, e_2 \in p^{-1}(x)$, there is $\varphi \in \mathrm{Aut}(\tilde{X} \xrightarrow{p} X)$ if and only if $p_*\pi_1(\tilde{X}, e_1) = p_*\pi_1(\tilde{X}, e_2)$. In fact, for $\varphi \in \mathrm{Aut}(\tilde{X} \xrightarrow{p} X)$ with $p_*\pi_1(\tilde{X}, e_1) \subseteq p_*\pi_1(\tilde{X}, e_2)$.

3. $\varphi|_{p^{-1}(x)} : p^{-1}(x) \to p^{-1}(x)$ is $\pi_1(X, x)$-equivariant (i.e. $\varphi(e) \cdot \gamma = \varphi(e \cdot \gamma)$.

Example

Given $p : \mathbb{R} \to S^1$, what is $\mathrm{Aut}(\mathbb{R} \xrightarrow{p} S^1)$?
$1 \in S^1$, $p^{-1}(1) = \mathbb{Z} \subseteq \mathbb{R}$, $\forall \varphi \in \mathrm{Aut}(\mathbb{R} \xrightarrow{p} S^1)$, $\varphi(0) = k \in \mathbb{Z}$. Then $\varphi(x) = x + k$. In fact, the map $x \mapsto x + k$ is a covering transofmratio that agrees with φ at $0 \in \mathbb{R}$. Hence they agree everywhere (i.e. $\varphi(x) = x + k$ for all x).

Example

Given $p : S^2 \to \mathbb{RP}^2$, then $\mathrm{Aut}(S^2 \xrightarrow{p} \mathbb{RP}^2) = \{\mathrm{id}, A\}$ with A the antipodal map.

Proposition: Normal Covering

Let $\tilde{X} \xrightarrow{p} X$ be a covering map. The following are equivalent

1. There exists $x \in X$ such that $p_*\pi_1(\tilde{X}, e)$ is normal for one (thus for all) $e \in p^{-1}(x)$.

2. For every $x \in X$ and each $e \in p^{-1}(x)$, $p_*\pi_1(\tilde{X}, e)$ is normal.

3. $\mathrm{Aut}(\tilde{X} \xrightarrow{p} X)$ acts transitively on some (thus all) fiber $p^{-1}(x)$.

If any of these hold, we say that $p : \tilde{X} \to X$ is a normal covering.

Proof

Suppose $e, e' \in p^{-1}(x)$ with $p_*\pi_1(\tilde{X}, e)$ and $p_*\pi_1(\tilde{X}, e')$. These are the isotropy subgroups at e and e' respectively. We know also $\pi_1(X, x)$ acts transitively on $p^{-1}(x)$.
Fact: If S is a right G-set, then $G_s = \{h \in G : s \cdot h = s\}$ and $G_{sg} = \{h \in G : s \cdot g \cdot h = s \cdot g\} = \{h \in S : s \cdot g \cdot h \cdot g^{-1} = s\}$. So $g \cdot G_{sg} \cdot g^{-1} \in G_s$ which implies that $G_{sg} = g^{-1} \cdot G_s \cdot g$. So if G_s is normal then so is G_{sg}.

IMAGE 1

$$\begin{tikzcd} \pi_1(\tilde{X}, e_0) \arrow[r, "\Phi_{\tilde{h}}"] \arrow[d, "p_*"] & \pi_1(\tilde{X}, e) \arrow[d, "p_*"] \\ \pi_1(X, x_0) \arrow[r, "\Phi_h"] & \pi_1(X, x) \end{tikzcd}$$

commutes. Hence Φ_h maps $p_*\pi_1(\tilde{X}, e_0)$ to $p_*\pi_1(\tilde{X}, e)$, and $\Phi_h : \pi_1(X, x_0) \xrightarrow{\sim} \pi_1(X, x)$ preserves normal subgroups.

(3) implies (1)

Finally, for every $e_1, e_2 \in p^{-1}(x)$, there exists $\varphi \in \mathrm{Aut}(\tilde{X} \xrightarrow{p} X)$ such that $\varphi(e_1) = e_2$. This holds if and only if $p_*\pi_1(\tilde{X}, e_1) = p_*\pi_1(\tilde{X}, e_2)$ for every $e_1, e_2 \in p^{-1}(x)$. That is, $e_2 = e_1 \cdot \gamma$ for some $\gamma \in \pi_1(X, x)$ and $H = \gamma^{-1} H \gamma$ for every $\gamma \in \pi_1(X, x)$. So H is normal.

Remark

The (simply connected) universal cover is always normal because $\{\mathrm{id}\}$ is normal in $\pi_1(X, x)$.

Theorem

Let $p : \tilde{X} \to X$ be a covering map with $x \in X$ and $e \in p^{-1}(x)$. Then $\mathrm{Aut}(\tilde{X} \xrightarrow{p} X) \cong \frac{N_G(H)}{H}$ where $G = \pi_1(X, x)$, $H = p_*\pi_1(\tilde{X}, e)$, and $N_G(H) = \{g \in G : g^{-1} H g = H\}$.

Special Case 1

If $p : \tilde{X} \to X$ is a normal covering, then H is normal in G. Then also $\mathrm{Aut}(\tilde{X} \xrightarrow{p} X) \cong G/H$.

Special Case 2

If $p : \tilde{X} \to X$ is the (simply connected) universal covering, then $\mathrm{Aut}(\tilde{X} \xrightarrow{p} X) \cong \pi_1(X, x)$.

Proof

Let S be a right G-set with transitive action and $\mathrm{Aut}_G(S)\{\varphi : S \to S$ G-equivariant bijections$\}$. Fix $s \in S$. Then $\mathrm{Aut}_G(S) \cong \frac{N_G(H)}{H}$ where $h = G_s$.
Define $\Phi : N_G(H) \to \mathrm{Aut}_G(S)$ by $\gamma \mapsto \Phi(\gamma) = \varphi_\gamma$ with $\varphi_\gamma : S \to S$ defined by

$$G_{s \cdot \gamma} = \gamma^{-1} H \gamma = H = G_s.$$

Then there exists a unique $\varphi_\gamma \in \mathrm{Aut}_G(S)$ such that $\varphi_\gamma(s) = s \cdot \gamma$.

- Lemma

  For each $s' \in S$, $s' = s \cdot \gamma'$ for some $\gamma' \in G$. Then $\varphi_\gamma(s') = \varphi_\gamma(s \cdot \gamma') = \varphi_\gamma(s) \cdot \gamma' = s \cdot \gamma\gamma'$. This is well defined. If $s' = s \cdot \gamma''$, then $s = s(\gamma \cdot \gamma'' \cdot (\gamma')^{-1} \cdot \gamma^{-1})$ which implies that $\gamma \cdot \gamma''(\gamma')^{-1} \cdot \gamma^{-1} \in G_s$ and $\gamma'' \cdot (\gamma')^{-1} \in G_s$.

Φ is a group homomorphism since

$$\varphi_{\gamma_1} \circ \varphi_{\gamma_2}(s) = \varphi_{\gamma_1}(s \cdot \gamma_2) = \varphi_{\gamma_1}(s) \cdot \gamma_2 = s \cdot \gamma_1 \cdot \gamma_2.$$

Φ is surjective since letting $\varphi \in \mathrm{Aut}_G(S)$, it maps s to some $\varphi(s) = s' = s \cdot \gamma$ and hence $\varphi = \varphi_\gamma$.
If $\varphi_\gamma = \mathrm{id}$, then $\varphi_\gamma(s) = s$ and $\gamma \in G_s = H$. So Φ induces $\frac{N_G(H)}{H} \cong \mathrm{Aut}_G(S)$.
Take $G = \pi_1(X, x)$ and $\mathrm{Aut}(\tilde{X} \xrightarrow{p} X) \to \mathrm{Aut}_G(p^{-1}(x)) \cong \frac{N_G(H)}{H}$ by $\varphi \mapsto \varphi|_{p^{-1}(x)}$ where H is the isotropy subgroup of the $\pi_1(X, x)$ action at e $(p_*\pi_1(\tilde{X}, e))$. Then $\varphi \mapsto \varphi|_{p^{-1}(x)}$ is injective because it is uniquely determined by its value at one point.
$\varphi \mapsto \varphi|_{p^{-1}(x)}$ is surjective. Letting $\eta \in \mathrm{Aut}_g(p^{-1}(x))$ and $e_1 \in p^{-1}(x)$, we set $e_2 = \eta(e_1)$ and see that $p_*\pi_1(\tilde{X}, e_1) = G_{e_1} = G_{e_2} = p_*\pi_1(\tilde{X}, e_2)$. By the lifting criterion, there exists $\varphi \in \mathrm{Aut}(\tilde{X} \xrightarrow{p} X)$ such that $\varphi(e_1) = e_2$. Then $\varphi|_{p^{-1}(x)} = \eta$ since both are in $\mathrm{Aut}_G(p^{-1}(x))$ and they agree at one point (hence everywhere). Thus we conclude that the map is a bijection and

$$\mathrm{Aut}(\tilde{X} \xrightarrow{p} X) \cong \mathrm{Aut}_G(p^{-1}(x)) \cong \frac{N_G(H)}{H}.$$

Definition: Covering Space Action

Let X be connected and locally path connected with a group action Γ acting by homeomorphism. The quotient map $p : X \to X/\Gamma$ will be a covering map if we impose $(*)$ for all $x \in X$, there exists a neighborhood U of x such that $U \cap (g \cdot U) = \varnothing$ for each $g \in \Gamma - \{\mathrm{id}\}$. In particular, G acts freely on X. We say that a Γ-action on X is a covering space action if $(*)$ if fulfilled.

Counter-example

Consider an $\mathbb{R}$ action on $\mathbb{R}^2$ by translation. Then $U \cap (g \cdot U) \neq \varnothing$.

IMAGE 2

Remark

Assuming $(*)$, $\{g \cdot U : g \in \Gamma\}$ is a disjoint family of open sets.

Example

Take a $\mathbb{Z}$-action by $\mathbb{R}^2$ given by $\gamma(x, y) = (x + 1, -y)$.

IMAGE 3

Example

S^2 with $\mathbb{Z}_2$-action $(\{\mathrm{id}, A\})$.

Theorem

If Γ acts on X as a covering space action, then $q : X \to X/\Gamma$ is a normal covering map.

Proof

Let $\overline{x} \in X/\Gamma$ and pick $x \in q^{-1}(\overline{x})$. By $(*)$, we have a neighborhood U such that $\{g \cdot U \, : \, g \in \Gamma\}$ is a disjoint collection. Let $V = q(U)$, an open neighborhood of $\overline{x}$ in X/Γ. Then $q^{-1}(V) = \{g \cdot U \, : \, g \in \Gamma\}$. Moreover, $g \cdot U \to V$ is a homeomorphism. If there exist $x', g'x' \in g \cdot U$, then $x' = h_1 \cdot u_1$ and $g' \cdot x' = h_2 \cdot u_2$. So $h_1^{-1}x' \in U$ and $h_2^{-1}g' \cdot x' \in U$ but this holds only for the identity map. So the covering map is injective.

Classifications of Covering Spaces

Take X path-connected, locally path-connected and semi-locally simply path connected (guaranteeing a simply connected universal cover). Then there is a $1-1$ correspondence between

$$\left\{\begin{matrix}\text{isomorphism classes of covering} \\ \text{spaces } p{:}\hat{X}\to X\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{conjugacy classes of} \\ \text{subgroups in } \pi_1(X,x)\end{matrix}\right\}$$
$$(p : \hat{X} \to X) \leftrightarrows [p_* \pi_1(\hat{X}, \hat{e})]$$

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Recall: Theorem

For X path-connected, locally path-connected and semi-locally simply path connected, Γ acts on X as a covering group action (i.e. $\forall x \in X$, there exists a neighborhood U of x such that $U \cap (g \cdot U) = \varnothing$ for all $g \in \Gamma \setminus \{e\}$).
Then $p : X \to X/\Gamma$ is a normal covering map. Moreover $\mathrm{Aut}(X \xrightarrow{p} X/\Gamma) = \Gamma$.

Proof

$(\supseteq)$ this follows from

$$\begin{tikzcd} X \arrow[rr, "g\cdot"] \arrow[dr, "p"'] & & X \arrow[dl, "p"] \\ & X/\Gamma & \end{tikzcd}$$

$(\subseteq)$ Let $\varphi \in \mathrm{Aut}(X \xrightarrow{p} X/\Gamma)$. That is

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & X \arrow[dl, "p"] \\ & X/\Gamma & \end{tikzcd}$$

commutes with φ a homeomorphism. Now let $x \in p^{-1}(\overline{x})$ where $\overline{x} \in X/\Gamma$, and let $x' = \varphi(x)$. Then $p(x) = \overline{x} = p(x')$, hence $x, x' \in p^{-1}(\overline{x})$. Hence there is $g \in \Gamma$ such that $gx = x'$. So we have

$$\varphi : X \to X \varphi(x) = x'$$
$$g : X \to X g(x) = x'$$

so φ is equivalent to an action by g.

Theorem

Take X path-connected, locally path-connected and semi-locally simply path connected (guaranteeing a simply connected universal cover). Then there is a $1-1$ correspondence between

$$\left\{\begin{matrix}\text{isomorphism classes of}\\ \text{covering maps } p:\hat{X}\to X\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{conjugacy classes of}\\ \text{subgroups in } \pi_1(X,x_0)\end{matrix}\right\}$$

→ Assign a subgroup $H = p_*(\hat{X}, \hat{e})$ for $\hat{e} \in p^{-1}(x_0)$.
← Given a conjugacy class of subgroups, pick a subgroup H in the class.

$$H \leq \pi_1(X, x_0) \cong \mathrm{Aut}(\tilde{X} \xrightarrow{p} X)$$

Hence H acts naturally on $\tilde{X}$ as covering transformations. Consider $q : \tilde{X} \to \tilde{X}/H =: \hat{X}$, a normal covering map.

$$\begin{tikzcd}
\tilde{X} \arrow[dd, "p"'] \arrow[dr, "/H"] & \\
& \hat{X} \arrow[dl, dashed] \\
X &
\end{tikzcd}$$

Since $\tilde{X}/\pi_1(X, x_0) = x$, we have an induced map $\hat{p} : \hat{X} \to X$. We need to show that $\hat{p} : \hat{X} \to X$ is a covering map with $\hat{p}_*\pi_1(\hat{X}, \hat{e}) = H$ for some $\hat{e} \in \hat{p}^{-1}(x)$. Let U be a neighborhood of x such that $p^{-1}(U) = \dot{\bigcup}_\alpha \tilde{U}_\alpha$. Then $\{\tilde{U}_\alpha\}$ is a collect iof disjoint open sets and identical to $\{g \cdot \tilde{U} \,:\, g \in \pi_1(X, x)\}$ where $\tilde{U}$ is a component of $p^{-1}(U)$. The H-action permutes the copies in $\{g \cdot \tilde{U}\} = \{\tilde{U}_\alpha\}$. Hence $q|_{\tilde{U}_\alpha} : \tilde{U}_\alpha \to \hat{X}$ is a homeomorphism. Let $\hat{U}$ be a component in $\hat{p}^{-1}(U)$. Then $q^{-1}(\hat{p}^{-1}(U)) = p^{-1}(U) = \dot{\bigcup}_\alpha \tilde{U}_\alpha$ where $q^{-1}(\hat{U})$ is a union of components in $\dot{\bigcup}_\alpha \tilde{U}_\alpha$. Hence $\hat{U}$ is homeomorphic to U, and $\hat{p}^{-1}(U)$ is a union of components that are homemorphic to U.
Lastly, we show that $\hat{p}_*\pi_1(\hat{X}, \hat{e}_0) = H$. This is the isotropy subgroup of $\pi_1(X, x_0)$-actions at $\hat{e}_0$. $q|_{p^{-1}(x_0)} : p^{-1}(x_0) \to \hat{p}^{-1}(x_0)$ is $\pi_1(X, x_0)$-equivariant (i.e. $q(e \cdot \gamma) = q(e) \cdot \gamma$, $q(e) = \hat{e}$ for $e \in \tilde{X}$). Hence γ fixes $q(e) = \hat{e}$ if and only if $q(e \cdot \gamma) = q(e)$, if and only if $e \cdot \gamma$ and e are in the same H-orbit, if and only if $\gamma \in H$.

Example 1

$X = S^1$ with $\pi_1(S^1) = \mathbb{Z}$.
$\mathbb{Z}$ has subgroups $\mathbb{Z}$, $2\mathbb{Z}$, $3\mathbb{Z}$, $\ldots$, $k\mathbb{Z}$, $\ldots$ where $k\mathbb{Z}$ corresponds to the covering map $p_k : z \mapsto z^k$.

Example 2

X the Mobius strip with $\pi_1(X) = \mathbb{Z}$ with $\pi_1(X) = \langle\gamma\rangle$ and $\gamma(x, y) = (x + 1, -y)$.
Take $H = 2\mathbb{Z} = \langle 2\gamma\rangle \leq \mathbb{Z}$. Then $2\gamma(x, y) = (x + 2, y)$ and $\mathbb{R}^2/H$ is the cylinder while the cylinder modulo $\mathbb{Z}_2$ is the mobius strip.

Example 3

The Klein bottle, $K = \mathbb{R}^2/\Gamma$ with Γ genereted by $g(x, y) = (x + 1, -y)$ and $h(x, y) = (x, y + 1)$.
So $\pi_1(K) = \langle g, h\rangle$. $g^2(x, y) = (x + 2, y)$ commutes with h, so $\mathbb{Z}^2 \cong \langle g^2, h\rangle \leq \pi_1(K)$ and $\mathbb{R}^2/\langle g^2, h\rangle = \mathbb{T}^2$ covers K.

Simplexes

IMAGE 1

The standard n-simplex is

$$\Delta^n = \left\{(t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \,:\, \sum_{i=0}^{n} t_i = 1,\ t_i \geq 0, \forall i\right\}$$

$$\Delta^1 = \left\{(t_0, t_1) \in \mathbb{R}^2 \,:\, t_0 + t_1 = 1,\ t_0, t_1 \geq 0\right\}$$

IMAGE 2

$$\Delta^2 = \left\{(t_0, t_1, t_2) \in \mathbb{R}^3 \ : \ t_0 + t_1 + t_2 = 1,\ t_0, t_1, t_2 \geq 0\right\}$$

IMAGE 3

Δ^n has $(n+1)$-many faces ($(n+1)$-simplex) where the ith face is $\Delta^{n-1} \to \Delta^n$ by $(t_0, \ldots, t_{n-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$.
Let X be a topological space. A Δ-complex structure on X is a family of maps $\sigma_\alpha : \Delta^n \to X$ (n may depend on α) such that

1. $\sigma_\alpha|_{\mathring{\Delta}^n} : \mathring{\Delta}^n \to X$ is injective and each point is in the image of at most one of $\sigma_\alpha|_{\circ\Delta^n}$.
2. $\sigma_\alpha|_{\text{a face of } \Delta^N}$ is some $\sigma_\beta : \Delta^{n-1} \to X$ in the family.
3. $A \subseteq X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for all α.

σ_β is $\Delta^{n-1} \overset{i\text{th face}}{\to} \Delta^n \overset{\sigma}{\to} X$.

Example

S^1 is the following iwht 1-simplex

IMAGE 4

Then the "body" of $\Delta^1 \overset{\sigma}{\to} X$ is

IMAGE 5

with $\sigma \circ \delta_0 : \Delta^0 \to X$ and $\sigma \circ \delta_1 : \Delta^0 \to X$. The boundary $\partial\sigma = \sum_{i=0}^n (-1)^i \sigma \circ \delta_i$. They define $\delta : C_n(X) \to C_{n-1}(X)$. For this example, we have $\partial\sigma = \sigma \circ \delta_0 + (-1)\sigma \circ \delta_1 = 0$.
The ith face is $\delta_i : \Delta^{n-1} \to \Delta^n$ by $(t_0, \ldots, t_{n-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$.
In Hatcher's notation, the boundary is $\partial\sigma = \sum_{i=0}^n (-1)^i \sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}$ where we shoild think of $[v_0, \ldots, \hat{v}_i, \ldots, v_n]$ as the ith face. So $\sigma : \Delta^n = [v_0, \ldots, v_n] \to X$. Now we have

$$\cdots \overset{\partial}{\to} C_n(X) \overset{\partial}{\to} C_{n-1}(X) \cdots$$

where $\partial^2 = 0$.

Proof

$$\begin{aligned}
\partial(\partial\sigma) &= \partial\left(\sum_{i=0}^n (-1)^i \sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}\right) \\
&= \sum_{i=0}^n (-1)^i \partial(\sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}) \\
&= \sum_{j<i} (-1)^i (-1)^j \sigma_{[v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots, v_n]} + \sum_{j>i} (-1)^i (-1)^{j-1} \sigma_{[v_0, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_n]} \\
&= 0
\end{aligned}$$

Homoology Associated to the Delta Complex

We have $\ker\partial \supseteq \operatorname{im}\partial$ where $\ker\partial$ are the n-cycles and $\operatorname{im}\partial$ are the n-bodies, and

$$H_n^{\delta}(X) = \frac{\{n\text{-cycles}\}}{\{n\text{-bodies}\}} = \frac{\ker\partial}{\operatorname{im}\partial}$$

Example

For the circle, $C_1(X) = \mathbb{Z} = \langle\sigma\rangle$ and $C_0(X) = \mathbb{Z} = \langle v\rangle$. Therefore

$$\overbrace{C_2(X)}^{=0} \to \overbrace{C_1(X)}^{=\mathbb{Z}} \xrightarrow{0} \overbrace{C_0(X)}^{=\mathbb{Z}} \to 0$$

Then $H_1^{\Delta}(X) = \frac{\ker\partial}{\operatorname{im}\partial} = \mathbb{Z}/\{0\} = \mathbb{Z}$ and $H_0^{\Delta}(X) = \frac{\ker\partial}{\operatorname{im}\partial} = \mathbb{Z}$.

An Aside

IMAGE 7

$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

Example

For the torus, draw

IMAGE 6

So $C_0(X) = \langle v\rangle = \mathbb{Z}$, $C_1(X) = \langle a, b, c\rangle = \mathbb{Z}^3$ and $C_2(X) = \langle U, L\rangle = \mathbb{Z}^2$. Then also $\partial U = a + b - c$ and $\partial L = a + b - c$, so $\partial(U - L) = 0$ and $\ker\partial_2 = \langle U - L\rangle \cong \mathbb{Z}$. That is $H_2^{\Delta}(X) = \frac{\ker\partial}{\operatorname{im}\partial} \cong \mathbb{Z}$. Now $\partial a = 0 = \partial b = \partial c$, so $\ker\partial_1 = \langle a, b, c\rangle$ and $H_1^{\Delta}(X) = \frac{\ker\partial_1}{\operatorname{im}\partial_2} = \frac{\langle a,b,a+b-c\rangle}{\langle a+b-c\rangle} \cong \mathbb{Z}^2$. Finally we have that $H_0^{\Delta} = \frac{\ker\partial_0}{\operatorname{im}\partial_1} = \frac{\langle v\rangle}{\{0\}} \cong \mathbb{Z}$.

Example

For $\mathbb{RP}^2$, draw

IMAGE 8

A$C_0(X) = \langle v, w\rangle \cong \mathbb{Z}^2$, $C_1(X) = \langle a, b, c\rangle \cong \mathbb{Z}^3$, and $C_2(X) = \langle U, L\rangle \cong \mathbb{Z}^2$. Then $\partial U = a + b + c$ while $\partial L = a + b - c$, so $\ker\partial_2 = \{0\}$ and $H_2^{\Delta}\frac{\ker\partial_2}{\operatorname{im}\partial_3} = \{0\}$. $\partial_1(a) = w - v$, $\partial_1(b) = v - w$ and $\partial_1(c) = 0$, so $\ker\partial_1 = \langle c, a - b\rangle$ and

$$H_1^{\Delta}(X) = \frac{\ker\partial_1}{\operatorname{im}\partial_2} = \frac{\langle c, a+b\rangle}{\langle a+b+c, a+b-c\rangle} = \langle a+b+c, c\rangle/\langle a+b+c, 2c\rangle \cong \langle c\rangle/\langle 2c\rangle \cong \mathbb{Z}^2.$$

April 28th, 2025

Recall:

For X with a Δ-complex structure, we have $H_n^{\Delta}(X)$.

Definition: Singular Simplex

A singular n-simplex is a continuous map $\sigma : \Delta^n \to X$.
The singular chain $C_n(X)$ is the free Abelian group generated by singular n-simplecies. Write

$$C_n(X) = \left\{ \sum n_i \sigma_i \ : \ |\sum n_i \sigma_i| < \infty,\ n_i \in \mathbb{Z},\ \sigma_i : \Delta^n \to X \right\}$$

$$\cdots \xrightarrow{\partial} C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial} \cdots$$

$$\partial(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}$$

While $\partial^2 = 0$ and $H_n(X) = \ker \partial_n / \operatorname{Im} \partial_{n+1}$ is the singular homology.

Proposition

If $X = \coprod_\alpha X_\alpha$ with X_α connected components of X, then $H_n(X) \cong \oplus_\alpha H_n(X_\alpha)$.

Proof

$\sigma : \Delta^n \to x$, $\operatorname{im}\sigma \subseteq X_\alpha$ for some α. So $C_n(X) = \oplus_\alpha C_n(X_\alpha)$ and $\partial : C_n(X) \to C_{n-1}(X)$ maps $C_n(X_\alpha)$ to $C_{n-1}(X_\alpha)$. Therefore $\ker \partial_n = \oplus_\alpha \ker(\partial|_{C_n(X_\alpha)})$ and $\operatorname{im} \partial_{n+1} = \oplus_\alpha \operatorname{im}(\partial|_{C_{n+1}(X_\alpha)})$. Then $H_n(X) \cong \oplus_\alpha \ker(\partial|_{C_n(X_\alpha)}) / \oplus_\alpha \operatorname{im}(\partial|_{C_{n+1}(X_\alpha)}) \cong \oplus_\alpha H_n(X_\alpha)$.

Proposition

Let X be a point. Then

$$H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \geq 1 \end{cases}$$

Proof

For each n, $C_n(X)$ is generated by a single element $\sigma_n : \Delta^n \to p$ so $C_n(X) \cong \mathbb{Z}$. Then

$$\partial \sigma_n = \sum_{i=0}^{n} (-1)^i \sigma_n|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]} = \sum_{i=0} (-1)^i \sigma_{n-1} = \begin{cases} 0 & n \text{ odd} \\ \sigma_{n-1} & n \text{ even} \end{cases}$$

$$\cdots \longrightarrow C_{n+1}(X) \longrightarrow C_n(X) \longrightarrow C_{n-1}(X) \longrightarrow \cdots$$

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \cdots$$ We see that

$$\partial_n = \begin{cases} \cong & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Therefore $\ker / \operatorname{im} = 0$ or $\ker / im = \mathbb{Z}/\mathbb{Z} = 0$. Because

$$C_1(X) \xrightarrow{0} C_0(X) \xrightarrow{0} 0$$ we have thata $H_0(X) = \ker / \operatorname{im} = \mathbb{Z}/\{0\} = \mathbb{Z}$.

Proposition

If X is path connected, then $H_0(X) \cong \mathbb{Z}$.

Proof

Define a map $\epsilon : C_0(X) \to Z$ by $\sum n_i\sigma_i \mapsto \sum n_i$ given that $\sigma_i : \{v\} \to X$. Then ϵ is surjective. Also,

$$H_0(X) = \ker \partial_0 / \operatorname{im} \partial_1 = C_0(X)/\operatorname{im}\partial_1 = C_0(X)/\ker\epsilon \cong \operatorname{im}\epsilon \cong \mathbb{Z}$$

We claim that $\ker\epsilon = \operatorname{im}\delta$.
($\supseteq$) Let $\sigma : \Delta^1 \to X$, $\epsilon(\delta_1(\sigma)) = \epsilon(v_1 - v_0) = 1 - 1 = 0$.
($\subseteq$) Let $\sum n_i\sigma_i \in C_0(X)$ such that $0 = \epsilon(\sum n_i\sigma_i) = \sum n_i$. We fix a point $x_0 \in X$. Because X is path-connected, we can draw paths τ_i from x_0 to σ_i. Consider $\sum n_i\tau_i \in C_1(X)$, then

$$\partial(\sum n_i\tau_i) = \sum n_i(\partial\tau_i) = n_i(\sigma_i - x_0) = \sum n_i\sigma_i - \overbrace{\sum n_i}^{=0} x_0 = \sum n_i\sigma_i$$

Reduced Homology

$$\cdots \longrightarrow C_n(X) \longrightarrow \cdots \longrightarrow C_1(X) \longrightarrow C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

Usually written as $\tilde{H}_n(X)$, and $\tilde{H}_n(X) = H_n(X)$ if $n \geq 1$. We have that $\tilde{H}_0(X) = \ker\epsilon/\operatorname{im}\partial_1$ and $\epsilon|_{\operatorname{im}\partial_1} = 0$ so ϵ induces a map $\tilde{\epsilon} H_0(X) \hookrightarrow \mathbb{Z}$. Then $\ker\tilde{\epsilon} = \tilde{H}_0(X)$. It follows that

$$0 \longrightarrow \tilde{H}_0(X) \longrightarrow H_0(X) \longrightarrow \mathbb{Z} \longrightarrow 0$$

is a split exact sequence since $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$. In particualr, $\tilde{H}(\text{pt}) = \{0\}$.

Remark

$$\pi_1/[\pi_1, \pi_1] \cong H_1$$

Homotopy Invariance

Suppose we have $f : X \to Y$ continuous. It induces $f_\sharp : C_n(X) \to C_n(Y)$ by $\sigma \mapsto f \circ \sigma$. $f_\sharp$ is called a chain map and the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] & C_n(X) \arrow[r, "\partial"] \arrow[d, "f_\sharp"] & C_{n-1} \arrow[r, "\partial"] \arrow[d, "f_\sharp"] & \cdots \\
\cdots \arrow[r] & C_n(X) \arrow[r, "\partial"] & C_{n-1} \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

Let $\sigma \in C_n(X)$ and

$$f_\sharp(\partial\sigma) = f_\sharp\left(\sum_{i=0}^{n}(-1)^i \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}\right) = \sum_{i=0}^{n}(-1)^i (f \circ \sigma)|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]} = \partial(f_\sharp\sigma)$$

Then $f_\sharp$ maps cycles to cycles ($\partial c = 0$, $\partial(f_\sharp c) = f_\sharp(\sigma c) = 0$) and boundaries to boundaries ($f_\sharp(\partial c) = \partial(f_\sharp c)$). So $f_\sharp$ induces $f_* : H_n(X) \to H_n(Y)$.

Theorem

If $f, g : X \to Y$ are homotopic, then $f_* = g_* : H_n(X) \to H_n(Y)$ for all n.

Corollary

If $X \simeq Y$ are homotopic, then $H_n(X) \cong H_n(Y)$. $g \circ f \simeq \mathrm{id}_X$, $f \circ g \simeq \mathrm{id}_Y$,

$$g_* \circ f_* = (g \circ f)_* = (\mathrm{id}_X)_* = \mathrm{id}$$

and similarly $g_* \circ f_* = \mathrm{id}$. So f_* and g_* are isomorphisms.

Definition

Let $f, g : C.(X) \to C.(Y)$ be two chain maps. We say that f and g are chain homotopic if there is a map $p : C_n(X) \to C_{n+1}(Y)$ such that $\partial P + P\partial = g - f$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_n(X) \arrow[r, "\partial"] \arrow[d, "f,g"] \arrow[dl, "P"'] & C_{n-1} \arrow[r, "\partial"] \arrow[d, "f,g"] \arrow[dl, "P"'] & \cdots \\
\cdots \arrow[r] & C_n(X) \arrow[r, "\partial"] & C_{n-1} \arrow[r, "\partial"] & \cdots
\end{tikzcd}$$

Theorem

If $f \simeq g$ are homotopic, then

1. $f_\sharp$ and $g_\sharp$ are chain homotopic,
2. $f_* = g_*$ on homoology
3. For any n-cycle, $c \in C_n(X)$, $g(c) - f(c) = \partial P(c) + \overbrace{P(\partial c)}^{=0}$. Hence $g_*[c] = f_*[c]$.

Proof

Consider $\Delta^n \times I$, and set $\Delta^n \times \{0\} = [v_0, \ldots, v_n]$ and $\Delta^n \times \{1\} = [w_0, \ldots, w_n]$. Then the following are all n-simplicies

$$\begin{aligned}
[v_0, v_1, \ldots, v_{n-1}, v_n] \\
[v_0, v_1, \ldots, v_{n-1}, w_n] \\
[v_0, v_1, \ldots, w_{n-1}, w_n] \\
\vdots \\
[v_0, w_1, \ldots, w_{n-1}, w_n] \\
[w_0, w_1, \ldots, w_{n-1}, w_n]
\end{aligned}$$

They divide $\Delta^n \times I$ into $(n+1)$-simplicies, $\{[v_0, \ldots, v_i, w_i, \ldots, w_n] : i = 0, \ldots, n\}$. Now let $F : X \times I \to Y$ be a homotopy between f and g. Consider

$$\begin{tikzcd}
\Delta^n \times I \arrow[r, "\sigma \times \mathrm{id}"] & X \times I \arrow[r, "F"] & Y
\end{tikzcd}$$

and define $P : C_N(X) \to C_{n+1}(Y)$ by $\sigma \mapsto \sum_{i=0}^{n} (-1)^i F \circ (\sigma \times \mathrm{id})|_{[v_0, \ldots, v_i, w_i, \ldots, w_n]}$. We need to check that $\partial P + P\partial = g_\sharp - f_\sharp$.

Short Exact Sequences of Chain Complexes Induce Long Exact Sequences of Homology Groups

Applications

1. Rleative homology group.
2. Meyer-Vietoris sequence.

F

Suppose we have sequences

$$\begin{tikzcd}
0 \arrow[d] & & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
A_* \arrow[r] \arrow[d, "i"] & \cdots \arrow[r] & & A_{n+1} \arrow[r, "\partial"] \arrow[d] & A_n \arrow[r, "\partial"] \arrow[d] & A_{n-1} \arrow[r] \arrow[d] & \cdots \\
B_* \arrow[r] \arrow[d, "j"] & \cdots \arrow[r] & & B_{n+1} \arrow[r, "\partial"] \arrow[d] & B_n \arrow[r, "\partial"] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & \cdots \\
C_* \arrow[r] \arrow[d] & \cdots \arrow[r] & & C_{n+1} \arrow[r, "\partial"] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & \cdots \\
0 & & & 0 & 0 & 0
\end{tikzcd}$$

So H induces a long exact sequence

$$\xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C)$$

$$\xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} \cdots$$

where $\partial : H_n(C) \to H_{n-1}(A)$ by $[c] \mapsto [a]$, our connecting homomorphism, for $c \in C_n$. Then we have that the following commutes

$$\begin{tikzcd}
 & a \arrow[r, mapsto, "\partial"] \arrow[d, mapsto, "i"] & {} \arrow[d, "i"] \\
b \arrow[r, mapsto] \arrow[d, mapsto, "j"] & \partial b \arrow[r, mapsto] \arrow[d, mapsto, "j"] & 0 \\
c \arrow[r, mapsto] & 0 &
\end{tikzcd}$$

So a is a cycle. We need to show that $\partial a = 0$. Note that $i(\partial a) = \partial(ia) = \partial(\partial b) = 0$. Because i is injective, $\partial a = 0$. ∂ is well defined since

- choice of a: i is injective
- choice of b: suppose $b' \in B_n$ such tha t $j(b') = j(b) = c$. Then $b - b'$ satisfies $j(b - b') = 0$ and $b - b' \in \ker j = \operatorname{im} i$ (i.e. there exists $a' \in A_n$ such that $i(a') = b - b'$, so $b' = b + i(a')$. Then

$$\begin{tikzcd} a' \arrow[r, mapsto] \arrow[d, mapsto] & \partial a' \\ b-b' \arrow[d, mapsto] & \\ 0 & \end{tikzcd}$$

So $a+\partial a'$ satisfies

$$i(a+\partial a') = i(a) + i(\partial a') = \partial b + \partial(i a') = \partial b'$$

and

$$\begin{tikzcd} & a+\partial a' \arrow[d, mapsto] \\ b' \arrow[r, mapsto] \arrow[d, mapsto] & \partial b' \\ c & \end{tikzcd}$$

since $[a+\partial a'] = [a]$.

- We need to check choice of c, but we will skip this.
- We need to check that δ is a homomorphism, which follows from the definitions.
- Finally, check that the induced long sequence is exact. We will check only exactness about $H_n(C)$ (i.e. $\operatorname{im} j_* = \ker\delta$).

  $\operatorname{im} j_* \subseteq \ker\delta$: $\delta(j_*[b]) = 0$ because

$$\begin{tikzcd} & 0 \arrow[d, mapsto] \\ b \arrow[r, mapsto, "\partial"] \arrow[d, mapsto, "j"] & 0 \\ j(b) & \end{tikzcd}$$

$\ker\delta \subseteq \operatorname{im} j_*$: Suppose $[c] \in H_n(C)$ such that $\partial[c] = 0$, then

$$\begin{tikzcd} a' \arrow[r, mapsto, "\partial"] & a = \partial a' \arrow[d, mapsto] \\ b \arrow[r, mapsto] \arrow[d, mapsto, "j"] & \partial b \\ c & \end{tikzcd}$$

Consider $b - i(a')$, then $j(b-i(a')) = j(b) - \overbrace{j \circ i(a')}^{=0} = j(b) = c$. So $[c] = j_*[b-i(a')] \in \operatorname{im} j_*$. This is a cycle, since $\partial(b) - \partial(i(a')) = \partial b - i(\partial a') = \partial b - \partial b = 0$.