

# Random Matrix Theory

April 1, 2025

## Preliminaries

Let  $\xi_{ij}, \eta_{ij}$  be normal random variables (i.e. Gaussian, mean 0, variance 1).

e.g.  $\mathbb{P}(\xi_{11} < s) = \int_{-\infty}^s \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ .

$\int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$  is the variance.

$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$  is the Probability Density Function (PDF).

$\frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$  is the probability measure on our probability space (i.e. totally finite measure space).

We build matrices

$$\begin{bmatrix} \xi_{11} & \frac{\xi_{12} + i\eta_{12}}{\sqrt{2}} & \frac{\xi_{13} + i\eta_{13}}{\sqrt{2}} & \dots \\ \frac{\xi_{21} + i\eta_{21}}{\sqrt{2}} & \xi_{22} & \frac{\xi_{22} + i\eta_{22}}{\sqrt{2}} & \\ \frac{\xi_{31} + i\eta_{31}}{\sqrt{2}} & \frac{\xi_{32} + i\eta_{32}}{\sqrt{2}} & \xi_{33} & \\ \vdots & & & \ddots \end{bmatrix}$$

## Computing Random Matrices in Matlab

Gaussian, real valued 1x1 matrix.

```
randn
```

Gaussian, real valued 2x2 matrix.

```
randn(2)
```

Gaussian, complex valued 2x2 matrix.

```
randn(2)+sqrt(-1)*randn(2)
```

Gaussian, complex valued, self-adjoint 2x2 matrix.

Note that appending ' to a matrix takes the conjugate transpose, and matlab reserves i for the imaginary unit.

```
m = randn(2)+i*randn(2);  
(m+m')/2
```

Producing eigenvalues.

```
m = randn(2)+i*randn(2);  
l=(m+m')/2;  
eig(l)
```

Running tests to see how many hits we get within the interval  $[0, 2]$ .

```

edges=[0,2];
H=zeros(1,length(edges)-1);
trials=10;
for j=1:trials
m = randn(2)+i*randn(2);
l=(m+m')/2;
ev=eig(l);
H=H+histcount(ev,edges)
end

```

## Homework

Is the PDF of  $\frac{a+b}{2}$  the same as  $\frac{\xi_{12}}{\sqrt{2}}$  for normal RVs  $a, b, \xi_{12}$ ?

i.e.  $\mathbb{P}\left(\frac{a+b}{2} < s\right) \stackrel{?}{=} \mathbb{P}\left(\frac{\xi_{12}}{\sqrt{2}} < s\right)$

## 2x2 Random Matrix

Our matrix  $L$  corresponds to eigenvalues  $\lambda_1, \lambda_2$  which are random variables determined by  $\{\xi_{ij}, \eta_{ij}\}$ . Then the number of evaluations in the interval  $B$  is given by  $\sum_{j=1}^2 \chi_B(\lambda_j)$ . We may take the average by

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \sum_{j=1}^2 \chi_B(\lambda_j) \frac{1}{\sqrt{2\pi}} e^{-\xi_{11}^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\xi_{22}^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\xi_{12}^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\eta_{12}^2} d\xi_{11} d\xi_{22} d\xi_{12} d\eta_{12}.$$

## Expected Evaluations

We have that the expectation of the number of evaluations in the interval  $(a, b)$  is given by  $\int_a^b G(s) ds$  where

$$G(s) = e^{-\frac{s^2}{2}} \sum_{\ell=0}^2 P_{\ell}(s)^2$$

and  $P_{\ell}(s)$  is the Hermite polynomial of degree  $d$ .

## April 3, 2025

### Differentiability

```

delta = 0.05;
edges=-6:delta:6;
dimensions = 3;
trials = 1000000;

H=zeros(dimensions,trials);

for j=1:trials
m=randn(dimensions)+1i*randn(dimensions);
L=(m+m')/2;
ev=eig(L);
H(:,j) = ev;
end

G = histcounts(H,edges);
plot(edges(1:end-1),G/(trials*delta),'*')

```

## IMAGE 1

Observe that each \* in the graph corresponds to the average number of eigenvalues in the interval  $(a, b)$ . Therefore, they correspond to  $\int_a^b C(\lambda) d\lambda$ . We may consider the limit of the expectation of hits in each interval

$$\lim_{\Delta \rightarrow 0} \frac{\mathbb{E}(\#(a, a + \Delta))}{\Delta}.$$

```
delta = 0.01;
edges=-6:delta:6;
dimensions = 3;
trials = 1000000;

H=zeros(dimensions,trials);

for j=1:trials
m=randn(dimensions)+1i*randn(dimensions);
L=(m+m')/2;
ev=eig(L);
H(:,j) = ev;
end

G = histcounts(H,edges);
plot(edges(1:end-1),G/(trials*delta),'*')
```

As dimension grows large, we observe that the plot tends to a semi-circle with endpoints about  $\pm 2\sqrt{\text{dimension}}$ . We therefore want a rescaling by  $\sqrt{N}$  where  $\text{dim} = N$ . Then if  $G(\alpha) = \frac{d}{d\alpha} \mathbb{E}(\# \text{ of evals in } (a, \alpha))$ , we want

$$\int_{-\infty}^{\infty} G(\alpha) d\alpha = N.$$

Guess:  $G(\alpha) \approx cN^{1/2} \cdot \sqrt{A^2 - \alpha^2/N} \cdot \chi_{(-A\sqrt{N}, A\sqrt{N})}(\alpha)$ . We compute

$$\int_{-A\sqrt{N}}^{A\sqrt{N}} cN^{1/2} \sqrt{A^2 - \alpha^2/N} d\alpha \stackrel{\alpha=\sqrt{N}t}{=} cN \int_{-A}^A \sqrt{A^2 - t^2} dt = \frac{c\pi NA^2}{2}.$$

Choosing  $A = 2$  and  $c$  such that  $\frac{\pi A^2 c}{2} = 1$ , we get

$$\int_{-\infty}^{\infty} G(\alpha) d\alpha \approx \frac{N^{1/2}}{2\pi} \int_{-\infty}^{\infty} \sqrt{4 - \alpha^2/N} d\alpha = N.$$

### Number of Eigenvalues in an Interval

Let  $B$  be a subset of  $\mathbb{R}$  (typically an interval). Write  $n(B) = \#\{\text{evaluations in } B\}$ , a random variable. Recall that variance is given by the expectation of the square minus the square of the expectation. That is

$$\text{var}(n(B)) = \mathbb{E}(n(B)^2) - (\mathbb{E}(n(B)))^2.$$

Our ultimate goal is to understand PDF and  $\mathbb{P}(n(B)) = \ell$  as (the dimension)  $N \rightarrow \infty$ .

## Smallest Scale of Interest

Suppose  $B = (0, S)$  and  $N$  is large (i.e.  $N \rightarrow \infty$ ). How large should we choose  $s$  such that  $\mathbb{E}(n(B)) = 1$ ? We compute

$$\int_0^S cN^{1/2} \sqrt{4 - \alpha^2/N} d\alpha \stackrel{\alpha = \sqrt{N}t}{=} \int_0^{\frac{S}{\sqrt{N}}} cN \sqrt{4 - t^2} dt \approx cN \cdot 2 \frac{S}{\sqrt{N}} = 2cS\sqrt{N}.$$

Sets of size  $N^{-1/2}$ , the smallest interesting scale, are called the “microscopic scaling regime”.

## Homework: Largest Scale of Interest

How large should  $B$  be to see a fraction of the eigenvalues (on average)? That is, how should we scale  $a$  and  $b$  such that  $\mathbb{E}(n((a, b))) = r \cdot N$  for  $0 < r < 1$ ?

## Level Repulsion

```
m=randn(2)+sqrt(-1)*randn(2);
L=(m+m')/2;
ev=eig(L);
subplot(2,1,2),plot(real(ev),imag(ev))
xlim([edges(1),edges(end)])
```

**April 8, 2025**

## Macroscopic Scaling Regime for Random Matrices

Suppose  $a = \alpha\sqrt{N}$  and  $b = \beta\sqrt{N}$  such that  $\alpha < \beta$ ,  $-2 < \alpha$  and  $\beta < 2$ . Then

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}(\# \text{ of evaluations in } (\alpha\sqrt{N}, \beta\sqrt{N}))}{N} = \kappa > 0.$$

Recall that we defined  $G(b) = \frac{d}{db} \mathbb{E}(\# \text{ of evaluations in } (a, b))$  and

$$G(b) \approx cN^{1/2} \sqrt{A^2 - x^2/N} \chi_{[-A\sqrt{N}, A\sqrt{N}]}(x).$$

We want that  $\int_a^b G(x) dx = \kappa N$ .

## Spacings

Suppose we have eigenvalues  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N = \lambda_{\max}$ . We can take the spacing  $s_j = \lambda_{j+1} - \lambda_j$ .

```
m=randn(2)+sqrt(-1)*randn(2);
L=(m+m')/2;
ev=sort(eig(L));
spacing=diff(ev)
```

0.4839

## Summary So Far

Given  $\xi_{ij}$  and  $\eta_{ij}$  iid RVs with distribution  $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ , we have explored

- The behavior of average  $n_N(B)$ .
- Microscopic, macroscopic (and mesoscopic) scaling.
- That  $\lambda_{\max} \sim 2\sqrt{N}$  Tracy-Widom distribution.
- Eigenvalue repulsion.

## Induced Distribution

Let  $M$  be our matrix built using random variables. Then  $M = F\Lambda F^T$  where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \\ \vdots & & \ddots \end{pmatrix}, \quad F = \begin{pmatrix} | & | & \cdots & | \\ f_{\lambda_1} & f_{\lambda_2} & \cdots & f_{\lambda_N} \\ | & | & & | \end{pmatrix},$$

and  $Mf_{\lambda_j} = \lambda_j f_{\lambda_j}$ . What we are interested in is the induced joint PDF on  $\{\lambda_1, \dots, \lambda_N\}$ . We may write explicitly

$$\frac{1}{Z^n} e^{-\frac{1}{2} \sum_{j=1}^N \lambda_j^2} \prod_{1 \leq j < k \leq N} (\lambda_k - \lambda_j)^2.$$

### Example

Let  $N = 2$  and, suppressing the constant term, write

$$\rho = e^{-\frac{1}{2}(x^2+y^2)}(x-y)^2.$$

Taking partial derivatives, we have that

$$\begin{aligned} \rho_x &= e^{-\frac{1}{2}(x^2+y^2)}(x-y)^2(-x + \frac{2}{x-y}) \\ \rho_y &= e^{-\frac{1}{2}(x^2+y^2)}(x-y)^2(-x + \frac{2}{y-x}) \end{aligned}$$

which implies maxima at  $x = \pm 1$  and  $y = -x$ .

### Example

If  $N = 3$ ,

$$\rho = e^{-\frac{1}{2}(x^2+y^2+z^2)}(x-y)^2(x-z)^2(y-z)^2.$$

We may visualize the maxima here by level surfaces (homework).

**April 15, 2025**

## Recall: Spectral Theorem

Let  $M = F\Lambda F^\dagger$  where  $F^\dagger F = I = FF^\dagger$

$$\Lambda = \begin{pmatrix} \lambda_N & 0 & \cdots & \\ 0 & \lambda_{N-1} & & \\ \vdots & & \ddots & \\ & & & \lambda_1 \end{pmatrix}, \quad F = \begin{pmatrix} | & | & \cdots & | \\ f_{\lambda_1} & f_{\lambda_2} & \cdots & f_{\lambda_N} \\ | & | & & | \end{pmatrix},$$

for  $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ .

## Deriving the Joint PDF

Let  $n = 2$ . If

$$F = \begin{pmatrix} | & | \\ V & W \\ | & | \end{pmatrix},$$

then the expectation of eigenvalues may be computed by

$$\begin{aligned} \mathbb{E}(\mathcal{G}(M)) &= \frac{1}{Z_2^4} \int \cdots \int \mathcal{G}(M(\xi_{11}, \xi_{12}, \xi_{22}, \eta_{12})) x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\xi_{11}^2 + \xi_{12}^2 + \xi_{22}^2 + \eta_{12}^2)} d\eta_{12} d\xi_{22} d\xi_{12} d\xi_{11} \\ &= \int \mathcal{G}(M(\lambda_1, \lambda_2, V_1, \phi)) x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\xi_{11}^2 + \xi_{12}^2 + \xi_{22}^2 + \eta_{12}^2)} d\eta_{12} d\xi_{22} d\xi_{12} d\xi_{11}. \end{aligned}$$

So we need the Jacobian, and therefore a reparameterization using spectral theorem. We want a collection of independent variables which will produce all  $2 \times 2$  Hermitian matrices. Consider  $Mv = \lambda_2 v$  and  $||v||^2 = |v_1|^2 + |v_2|^2 = 1$ , then multiply by  $e^{i\eta}$  such that  $v_1 \in \mathbb{R}_+$ . Then  $v_2 = \sqrt{1 - v_1^2} e^{i\theta}$ . That is,  $0 \leq v_1 \leq 1$  and  $v_2 = \sqrt{1 - v_1^2}(\cos \theta + i \sin \theta)$ .

We want that  $|w_1|^2 + |w_2|^2 = 1$  and know that  $w \perp v$ , so  $v_1 w_1 + \bar{v}_2 w_2 = 0$ . As before, we can choose  $w$  such that  $w_2 \in \mathbb{R}_+$ . This implies that  $w_1$  and  $\bar{v}_2$  have the same argument, and  $w_1 = -|w_1| e^{-i\theta}$ . Therefore  $e^{-i\theta}(-v_1 |w_1| + |v_2| w_2) = 0$ , and  $v_1 |w_1| - |v_2| w_2 = 0$ . It follows that

$$v_1^2(1 - w_2^2) = w_2^2(1 - v_1^2) \iff v_1 = w_2.$$

Therefore, the entire system may be parameterized by  $v_1$  and  $\theta$ . We write

$$F = \begin{pmatrix} v_1 & -\sqrt{1 - v_1^2} e^{-i\theta} \\ \sqrt{1 - v_1^2} e^{i\theta} & v_1 \end{pmatrix}$$

and

$$M = F\Lambda F^\dagger = \begin{pmatrix} v_1 & -\sqrt{1 - v_1^2} e^{-i\theta} \\ \sqrt{1 - v_1^2} e^{i\theta} & v_1 \end{pmatrix} \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix} \begin{pmatrix} v_1 & \sqrt{1 - v_1^2} e^{-i\theta} \\ -\sqrt{1 - v_1^2} e^{i\theta} & v_1 \end{pmatrix}.$$

Therefore

$$M = \begin{pmatrix} \lambda_2 v_1^2 + \lambda_1(1 - v_1^2) & v_1 \sqrt{1 - v_1^2} e^{-i\theta} (\lambda_2 - \lambda_1) \\ v_1 \sqrt{1 - v_1^2} e^{-i\theta} (\lambda_2 - \lambda_1) & \lambda_2(1 - v_1^2) + \lambda_1 v_1^2 \end{pmatrix}.$$

Recall, we want  $\mathcal{G}(M(\xi)) \rightsquigarrow \mathcal{G}(M(\lambda_2, \lambda_1, v_1, \theta))$  and the Jacobian of  $M = M(\lambda_2, \lambda_1, v_1, \theta)$ . After computation, write

$$|\det J| = (\lambda_2 - \lambda_1)^2 \det J' = (\lambda_2 - \lambda_1)^2 Q(v_1, \theta).$$

We integrate

$$\int \cdots \int \mathcal{G}(M(\xi, \eta_{12})) e^{-\frac{1}{2}(\xi_{11}^2 + \xi_{12}^2 + \xi_{22}^2 + \eta_{12}^2)} \frac{1}{(2\pi)^4} d\xi_{11} d\xi_{12} d\xi_{22} d\eta_{12}$$

which we may think of as a function of  $\lambda_1$  and  $\lambda_2$  alone. So

$$\frac{1}{(2\pi)^2} \int \cdots \int \mathcal{G}(\lambda_1, \lambda_2) e^{-\frac{1}{2}[M_{11}^2 + M_{22}^2 + 2 \cdot \operatorname{Re}(M_{12})^2 + 2 \cdot \operatorname{Im}(M_{12})^2]} d\xi_{11} d\xi_{12} d\xi_{22} d\eta_{12}$$

where we observe that  $M_{11}^2 + M_{22}^2 + 2 \cdot \operatorname{Re}(M_{12})^2 + 2 \cdot \operatorname{Im}(M_{12})^2 = \operatorname{Tr}(M^2)$ . It follows that we have

$$\begin{aligned} \frac{1}{(2\pi)^2} \int \cdots \int \mathcal{G}(\lambda_1, \lambda_2) e^{-\frac{1}{2}(\lambda_1^2 + \lambda_2^2)} d\xi_{11} d\xi_{12} d\xi_{22} d\eta_{12} &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^1 \int \int_{-\infty < \lambda_1 \leq \lambda_2 < \infty} \mathcal{G}(\lambda_1, \lambda_2) e^{-\frac{1}{2}(\lambda_1^2 + \lambda_2^2)} (\lambda_2 - \lambda_1)^2 Q(v, \theta) d\xi_{11} d\xi_{12} d\xi_{22} d\eta_{12} \\ &= \int \int_{-\infty < \lambda_1 \leq \lambda_2 < \infty} \mathcal{G}(\lambda_1, \lambda_2) e^{-\frac{1}{2}(\lambda_1^2 + \lambda_2^2)} (\lambda_2 - \lambda_1)^2 \int_0^{2\pi} \int_0^1 \frac{Q(v, \theta)}{(2\pi)^2} dv_1 d\theta d\lambda_1 d\lambda_2 \\ &= c \int \int \mathcal{G}(\lambda_1, \lambda_2) e^{-\frac{1}{2}(\lambda_1^2 + \lambda_2^2)} (\lambda_2 - \lambda_1)^2 d\lambda_1 d\lambda_2 \end{aligned}$$

**April 17, 2025**

## Recall: Joint PDF on Evaluation of Eigenvalues

$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$  and PDF  $\frac{1}{Z_N} e^{-\frac{1}{2} \sum \lambda_i^2} \prod (\lambda_k - \lambda_j)^2$ .

This is the Gaussian Unitary Ensemble.

## Hermite Polynomials

Write  $p_j = \kappa_j^{(j)} x^j + \kappa_{j-1}^{(j)} x^{j-1} + \cdots + \kappa_0^{(j)}$  where the superscript is usually suppressed. Then

$$\int_{-\infty}^{\infty} p_j(x) p_k(x) e^{-\frac{1}{2}x^2} dx = \delta_{jk}.$$

Observe that  $\{e^{-\frac{1}{4}x^2} p_j(x)\}_{j=0}^{\infty}$  forms a basis for  $L^2(\mathbb{R})$ . For  $f \in L^2$ , write the truncation  $P^{(N)}(f) = \sum_{\ell=0}^{N-1} \left( \int_{\mathbb{R}} f(y) p_{\ell}(y) e^{-\frac{1}{4}y^2} dy \right) e^{-\frac{1}{4}x^2} p_{\ell}(x)$ .

Then

$$P^{(N)} = \int_{\mathbb{R}} \left( e^{-\frac{1}{4}(x^2+y^2)} \sum_{\ell=0}^{N-1} p_{\ell}(x) p_{\ell}(y) \right) f(y) dy$$

and we write  $K_N(x, y) = e^{-\frac{1}{4}(x^2+y^2)} \sum_{\ell=0}^{N-1} p_{\ell}(x) p_{\ell}(y)$  and  $\mathcal{K}_N(f) = \int_{\mathbb{R}} K_N(x, y) f(y) dy$ .

We have that

$$\frac{1}{Z_N} e^{-\frac{1}{2} \sum \lambda_i^2} \prod (\lambda_k - \lambda_j)^2 = \det \begin{bmatrix} K_N(\lambda_1, \lambda_1) & \cdots & K_N(\lambda_1, \lambda_N) \\ \vdots & \ddots & \vdots \\ K_N(\lambda_N, \lambda_1) & \cdots & K_N(\lambda_N, \lambda_N) \end{bmatrix}.$$

For  $N = 2$ , we see

$$\frac{1}{Z_N} e^{-\frac{1}{2} \sum \lambda_i^2} \prod (\lambda_k - \lambda_j)^2 = (K_N(\lambda_1, \lambda_1) K_N(\lambda_2, \lambda_2) - K_N(\lambda_1, \lambda_2)^2).$$

### Example Computation

Let  $I$  be an interval and consider  $\mathbb{E}(\# \text{ of evaluations in } I)$ . Then

$$\begin{aligned} & \int_{-\infty < \lambda_1 \leq \lambda_2 < \infty} \left( \sum_{j=1}^2 \chi_I(\lambda_j) \right) (K_2(\lambda_1, \lambda_1)K_2(\lambda_2, \lambda_2) - K_2(\lambda_1, \lambda_2)^2) d\lambda_2 d\lambda_1 \\ &= \frac{1}{2!} \int \int_{\mathbb{R}^2} (\chi_I(\lambda_1) + \chi_I(\lambda_2)) (K_2(\lambda_1, \lambda_1)K_2(\lambda_2, \lambda_2) - K_2(\lambda_1, \lambda_2)^2) d\lambda_1 d\lambda_2 \\ &= \frac{1}{2!} \int_I \int_{-\infty}^{\infty} (K_2(\lambda_1, \lambda_1)K_2(\lambda_2, \lambda_2) - K_2(\lambda_1, \lambda_2)^2) d\lambda_1 d\lambda_2 + \frac{1}{2!} \int_I \int_{-\infty}^{\infty} (K_2(\lambda_1, \lambda_1)K_2(\lambda_2, \lambda_2) - K_2(\lambda_1, \lambda_2)^2) d\lambda_2 d\lambda_1. \end{aligned}$$

Observe that  $\int_{-\infty}^{\infty} K_2(\lambda_2, \lambda_2) d\lambda_2 = \int_{-\infty}^{\infty} e^{-\frac{1}{2}\lambda_2^2} (p_0(\lambda_2)^2 + p_1(\lambda_2)^2) d\lambda_2 = 2$ . We also compute that

$$\begin{aligned} \int_{-\infty}^{\infty} K_2(\lambda_1, \lambda_2)K_2(\lambda_2, \lambda_1) d\lambda_2 &= \int_{\mathbb{R}} e^{-\frac{1}{2}(\lambda_1^2 + \lambda_2^2)} \left( \sum_{\ell=0}^1 p_{\ell}(\lambda_1)p_{\ell}(\lambda_2) \right) \left( \sum_{\ell'=0}^1 p_{\ell'}(\lambda_2)p_{\ell'}(\lambda_1) \right) d\lambda_2 \\ &= \sum_{\ell, \ell'=0}^1 e^{-\frac{1}{2}\lambda_1^2} p_{\ell}(\lambda_1)p_{\ell'}(\lambda_1) \int_{-\infty}^{\infty} e^{-\frac{1}{2}\lambda_2^2} p_{\ell}(\lambda_2)p_{\ell'}(\lambda_2) d\lambda_2 \\ &= K_2(\lambda_1, \lambda_1). \end{aligned}$$

Returning to the first calculation,

$$\begin{aligned} & \frac{1}{2!} \int_I \int_{-\infty}^{\infty} (K_2(\lambda_1, \lambda_1)K_2(\lambda_2, \lambda_2) - K_2(\lambda_1, \lambda_2)^2) d\lambda_1 d\lambda_2 + \frac{1}{2!} \int_I \int_{-\infty}^{\infty} (K_2(\lambda_1, \lambda_1)K_2(\lambda_2, \lambda_2) - K_2(\lambda_1, \lambda_2)^2) d\lambda_2 d\lambda_1 \\ &= \frac{1}{2!} \left[ \int_I (2-1)K_2(\lambda_1, \lambda_1) d\lambda_1 + \int_I (2-1)K_2(\lambda_2, \lambda_2) d\lambda_2 \right] \\ &= \int_I K_2(\lambda_1, \lambda_1) d\lambda_1 \end{aligned}$$

which is the density function for the average number of evaluations in  $I$ . So  $K_2(\lambda, \lambda) = \frac{e^{-\frac{1}{2}\lambda^2}}{\sqrt{2\pi}}(1 + \lambda^2)$ .

### Question:

What is the probability of having zero evaluations in an interval  $I$ ?

We have an indicator function  $(1 - \chi_I(\lambda_1))(1 - \chi_I(\lambda_2))$ , so

$$\begin{aligned} P(\text{no evaluations in } I) &= \frac{1}{2} \int_{\mathbb{R}^2} (1 - \chi_I(\lambda_1))(1 - \chi_I(\lambda_2)) [K_2(\lambda_1, \lambda_1)K_2(\lambda_2, \lambda_2) - K_2(\lambda_1, \lambda_2)^2] d\lambda_1 d\lambda_2 \\ &= \frac{1}{2} \int_{\mathbb{R}^2} (1 - (\chi_I(\lambda_1) + \chi_I(\lambda_2)) + \chi_I(\lambda_1)\chi_I(\lambda_2)) [K_2(\lambda_1, \lambda_1)K_2(\lambda_2, \lambda_2) - K_2(\lambda_1, \lambda_2)^2] d\lambda_1 d\lambda_2 \\ &= \frac{1}{2} \left[ \int_{\mathbb{R}^2} K_2(\lambda_1, \lambda_1)K_2(\lambda_2, \lambda_2) - K_2(\lambda_1, \lambda_2)^2 d\lambda_1 d\lambda_2 \right. \\ &\quad - 2 \int_I \int_{\mathbb{R}} K_2(\lambda_1, \lambda_1)K_2(\lambda_2, \lambda_2) - K_2(\lambda_1, \lambda_2)^2 d\lambda_2 d\lambda_1 \\ &\quad \left. + \int_I \int_I K_2(\lambda_1, \lambda_1)K_2(\lambda_2, \lambda_2) - K_2(\lambda_1, \lambda_2)^2 d\lambda_1 d\lambda_2 \right] \\ &= \frac{1}{2} \left[ 4 - 2 - 2 \int_I K_2(\lambda_1, \lambda_1) d\lambda_1 + \int_I \int_I K_2(\lambda_1, \lambda_1)K_2(\lambda_2, \lambda_2) - K_2(\lambda_1, \lambda_2)^2 d\lambda_1 d\lambda_2 \right] \\ &= 1 - \int_I K_2(\lambda_1, \lambda_1) d\lambda_1 + \int_I \int_I \det \begin{pmatrix} & \\ & \end{pmatrix}_{2 \times 2} d^2 \lambda \\ &= \det(1 - \mathcal{K}_2^{(I)}) \end{aligned}$$



If  $I = (0, \infty)$ , then the probability is  $\frac{\pi-2}{4\pi}$ .

## Fredholm Determinant

Write  $H_N(I, t) = \det(1 - t\mathcal{K}_N^{(I)})$  where  $\mathcal{K}_N^{(I)}$  is an integral operator which acts on  $L_2(I)$  by

$$\mathcal{K}_N^{(I)}(f) = \int_I K_N(x, y) f(y) dy.$$

So the range of  $\mathcal{K}_N^{(I)}$  is finite dimensional (i.e. it is a finite rank operator). Then

$$H_N(I, t) = 1 - \int_I K_N(\lambda_1, \lambda_1) d\lambda_1 - \frac{t}{2!} \int_I \int_I \det \begin{pmatrix} & \\ & \end{pmatrix}_{2 \times 2} d^2 \lambda + \cdots + \frac{(-t)^j}{j!} \overbrace{\int_I \cdots \int_I}^j \det \begin{pmatrix} & \\ & \end{pmatrix}_{j \times j} d^j \lambda + \frac{(-t)^N}{N!} \overbrace{\int_I \cdots \int_I}^N \det \begin{pmatrix} & \\ & \end{pmatrix}_{N \times N} d^N \lambda$$

Then  $H_N(I, 1)$  is the probability of no evaluations in  $I$ , and  $H'_N(I, 1)$  is negative the probability of exactly one evaluation in  $I$ . So

$$H_N^{(j)}(I, 1) = (-1)^j j! P(\text{exactly } j \text{ eigenvalues in } I).$$

**April 22, 2025**

## Recall

$$\frac{1}{z_N} e^{-\frac{1}{2} \sum \lambda_j^2} \prod_{j < k} (\lambda_j - \lambda_k)^2 = \frac{1}{N!} \det(K_N(\lambda_j, \lambda_k))_{N \times N}$$

For  $n = 2$ , we have

$$\mathbb{E} \left( \sum_{j=1}^2 \chi_B(\lambda_j) \right) = \int_B K_2(\lambda, \lambda) d\lambda$$

We also have that

$$\begin{aligned} \mathbb{E}((1 - \chi_B(\lambda_1))(1 - \chi_B(\lambda_2))) &= P(\text{no evaluations}) \\ &= 1 - \lambda_B K_2(\lambda, \lambda) d\lambda + \frac{1}{2} \int_B \int_B \det \begin{pmatrix} & \\ & \end{pmatrix}_{2 \times 2} d\lambda \end{aligned}$$

where

$$\begin{aligned} (\lambda_1 - \lambda_2)^2 &= \det \begin{pmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{pmatrix} \det \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \\ &\stackrel{q(\lambda_i) = \lambda_i + c_0}{=} \det \begin{pmatrix} 1 & q(\lambda_1) \\ 1 & q(\lambda_2) \end{pmatrix} \det \begin{pmatrix} 1 & 1 \\ q(\lambda_1) & q(\lambda_2) \end{pmatrix} \\ &= \frac{1}{(\kappa_0^{(0)} \kappa_1^{(1)})^2} \det \begin{pmatrix} \kappa_0^{(0)} & \kappa_1^{(1)} q(\lambda_1) \\ \kappa_0^{(0)} & \kappa_1^{(1)} q(\lambda_2) \end{pmatrix} \det \begin{pmatrix} \kappa_0^{(0)} & \kappa_0^{(0)} \\ \kappa_1^{(1)} q(\lambda_1) & \kappa_1^{(1)} q(\lambda_2) \end{pmatrix} \\ &= \frac{1}{\prod_0^1 (\kappa_i^{(i)})^2} \det \begin{pmatrix} P_0(\lambda_1) & P_1(\lambda_1) \\ P_0(\lambda_2) & P_1(\lambda_2) \end{pmatrix} \det \begin{pmatrix} P_0(\lambda_1) & P_0(\lambda_2) \\ P_1(\lambda_1) & P_1(\lambda_2) \end{pmatrix} \end{aligned}$$

It follows that

$$\begin{aligned} e^{-\frac{1}{2} \sum_1 \lambda_j^2} (\lambda_2 - \lambda_1)^2 &= \prod_{j=0}^1 (\kappa_j^{(j)})^{-2} \det \begin{pmatrix} P_0(\lambda_1) e^{-\frac{1}{4} \lambda_1^2} & P_1(\lambda_1) e^{-\frac{1}{4} \lambda_1^2} \\ P_0(\lambda_1) e^{-\frac{1}{4} \lambda_2^2} & P_1(\lambda_1) e^{-\frac{1}{4} \lambda_2^2} \end{pmatrix} \det \begin{pmatrix} P_0(\lambda_1) e^{-\frac{1}{4} \lambda_1^2} & P_0(\lambda_1) e^{-\frac{1}{4} \lambda_2^2} \\ P_1(\lambda_1) e^{-\frac{1}{4} \lambda_1^2} & P_1(\lambda_1) e^{-\frac{1}{4} \lambda_2^2} \end{pmatrix} \\ &= \prod_{j=0}^1 (\kappa_j^{(j)})^{-2} \det(K_2(\lambda_i, \lambda_j))_{2 \times 2} \end{aligned}$$

where  $K_2(x, y) = e^{-\frac{1}{4}(x^2+y^2)} \sum_{\ell=0}^1 P_\ell(x) P_\ell(y)$ . So we have

$$\frac{1}{z_N} \prod_0^1 (\kappa_j^{(j)})^{-2} \det \begin{pmatrix} K_2(\lambda_1, \lambda_1) & K_2(\lambda_1, \lambda_2) \\ K_2(\lambda_2, \lambda_1) & K_2(\lambda_2, \lambda_2) \end{pmatrix}$$

and the fact that

$$\frac{1}{z_N \prod_{j=1}^2 (\kappa_j^{(j)})} \int_{\mathbb{R}^2} [K_2(\lambda_1, \lambda_1) K_2(\lambda_2, \lambda_2) - K_2(\lambda_1, \lambda_2)^2] d\lambda_1 d\lambda_2 = 1$$

Observe that (to do: fill in these calculations)

$$\int_{\mathbb{R}^2} K_2(\lambda_1, \lambda_1) K_2(\lambda_2, \lambda_2) d\lambda_1 d\lambda_2 = \int_{\mathbb{R}^2} (e^{-\frac{1}{4} \lambda_1^2} (P_0(\lambda_1) P_0(\lambda_1) + P_1(\lambda_1) P_1(\lambda_1))) (e^{-\frac{1}{4} \lambda_2^2} (P_0(\lambda_2) P_0(\lambda_2) + P_1(\lambda_2) P_1(\lambda_2)))$$

So it must be that

$$\frac{1}{z_N (\kappa_0^{(0)})^2 (\kappa_1^{(1)})^2} (2) = 1.$$

We conclude that the original joint PDF can be written as  $\frac{1}{2!} \det(K_1(\lambda_i, \lambda_j))_{2 \times 2}$ .

## Vandermonde Determinant

Write

$$\begin{aligned} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_N \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{N-1} & \lambda_2^{N-1} & \cdots & \lambda_N^{N-1} \end{vmatrix} &= \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{N-1} \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{N-1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{N-2} & \lambda_2^{N-2} & \cdots & \lambda_{N-1}^{N-2} \end{vmatrix} (\lambda_N - \lambda_1)(\lambda_N - \lambda_2) \cdots (\lambda_N - \lambda_{N-1}) \\ &= \prod_{j < k} (\lambda_k - \lambda_j) \end{aligned}$$

and observe that this is zero when  $\lambda_i = \lambda_j$ . Now write

$$\det \begin{pmatrix} 1 & \cdots & \lambda_1^{N-1} \\ \vdots & & \vdots \\ 1 & \cdots & \lambda_N^{N-1} \end{pmatrix} \det \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ \lambda_1^{N-1} & \cdots & \lambda_N^{N-1} \end{pmatrix}$$

then by using the multilinearity of the determinant and adding rows we can write

$$\det \begin{pmatrix} 1 & \cdots & \lambda_1^{N-1} \\ \vdots & & \vdots \\ 1 & \cdots & \lambda_N^{N-1} \end{pmatrix} \det \begin{pmatrix} 1 & \cdots & 1 \\ \lambda_1 + c_0 & \cdots & \lambda_N + c_0 \\ \pi_2(\lambda_1) & \cdots & \pi_2(\lambda_N) \\ \vdots & & \vdots \\ \lambda_1^{N-1} & \cdots & \lambda_N^{N-1} \end{pmatrix}$$

So we can write

$$\det \begin{pmatrix} e^{-\frac{1}{4}\lambda_1^2} P_0(\lambda_1) & \cdots & e^{-\frac{1}{4}\lambda_1^2} P_{N-1}(\lambda_1) \\ \vdots & & \vdots \\ e^{-\frac{1}{4}\lambda_N^2} P_0(\lambda_N) & \cdots & e^{-\frac{1}{4}\lambda_N^2} P_{N-1}(\lambda_N) \end{pmatrix} \frac{1}{\prod_{i=0}^{N-1} (\kappa_j^{(j)})^2} \det \begin{pmatrix} P_0(\lambda) & \cdots & P_0(\lambda_N) \\ \vdots & & \vdots \\ P_{N-1}(\lambda_1) & \cdots & P_{N-1}(\lambda_N) \end{pmatrix}$$

Examining the  $(j, k)$  entry, we have

$$\frac{1}{\prod (\kappa_j^{(j)})^2} e^{-\frac{1}{4}(\lambda_j^2 + \lambda_k^2)} (P_0(\lambda_j)P_0(\lambda_k) + P_1(\lambda_j)P_1(\lambda_k) + \cdots + P_{N-1}(\lambda_j)P_{N-1}(\lambda_k)).$$

or

$$\frac{1}{z_N \prod (\kappa_j^{(j)})^2} \det[K_n(\lambda_j, \lambda_k)]_{N \times N}$$

which must integrate across  $\mathbb{R}^n$  to exactly 1. From this we conclude that  $\frac{N!}{z_N \prod (\kappa_j^{(j)})^2} = 1$ .

**April 24, 2025**

**Determinants**

$$\begin{aligned} \int_{\mathbb{R}} \det \begin{pmatrix} K_N(\lambda_1, \lambda_1) & K_N(\lambda_1, \lambda_2) \\ K_N(\lambda_2, \lambda_1) & K_N(\lambda_2, \lambda_2) \end{pmatrix} d\lambda_2 &= \int_{\mathbb{R}} K_N(\lambda_1, \lambda_1)K_N(\lambda_2, \lambda_2) - K_N(\lambda_1, \lambda_2)K_N(\lambda_2, \lambda_1) d\lambda_2 \\ &= K_N(\lambda_1, \lambda_1) \int_{\mathbb{R}} e^{-\frac{1}{2}\lambda_2^2} P_\ell(\lambda_2)^2 d\lambda_2 - 0 \\ &= NK_N(\lambda_1, \lambda_1) \end{aligned}$$

We have that  $\int_{\mathbb{R}} K_N(\lambda, x) K_N(x, \mu) dx = K_N(\lambda, \mu)$ . Then

$$\begin{aligned}
\int_{\mathbb{R}} \begin{vmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{vmatrix} d\lambda^3 &= \int_{\mathbb{R}} K_{31} \begin{vmatrix} K_{12} & K_{13} \\ K_{22} & K_{23} \end{vmatrix} - K_{32} \begin{vmatrix} K_{11} & K_{13} \\ K_{21} & K_{23} \end{vmatrix} + K_{33} \begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} d\lambda^3 \\
&= \begin{vmatrix} K_{12} & \int_{\mathbb{R}} K(\lambda_1, \lambda_3) K(\lambda_3, \lambda_1) d\lambda^3 \\ K_{22} & \int_{\mathbb{R}} K(\lambda_2, \lambda_3) K(\lambda_3, \lambda_1) d\lambda^3 \end{vmatrix} \\
&\quad - \begin{vmatrix} K_{11} & \int_{\mathbb{R}} K(\lambda_1, \lambda_3) K(\lambda_3, \lambda_2) d\lambda^3 \\ K_{21} & \int_{\mathbb{R}} K(\lambda_2, \lambda_3) K(\lambda_3, \lambda_2) d\lambda^3 \end{vmatrix} \\
&\quad + \begin{vmatrix} K_{11} & \int_{\mathbb{R}} K(\lambda_1, \lambda_2) K(\lambda_3, \lambda_3) d\lambda^3 \\ K_{21} & \int_{\mathbb{R}} K(\lambda_2, \lambda_2) K(\lambda_3, \lambda_3) d\lambda^3 \end{vmatrix} \\
&= \begin{vmatrix} K_{12} & K_{11} \\ K_{22} & K_{21} \end{vmatrix} - \begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} + N \begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} \\
&= - \begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} - \begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} + N \begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} \\
&= (N-2) \det \begin{pmatrix} & \\ & \end{pmatrix}_{2 \times 2}
\end{aligned}$$

So we have that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} (N-2) \begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} d\lambda_2 d\lambda_1 = (N-2)(N-1)N$$

In general, we see that

$$\int_{\mathbb{R}} \begin{vmatrix} K_{11} & \cdots & K_{1j} \\ \vdots & \ddots & \vdots \\ K_{j1} & \cdots & K_{jj} \end{vmatrix} d\lambda_j = (N-(j-1)) \begin{vmatrix} K_{11} & \cdots & K_{1j} \\ \vdots & \ddots & \vdots \\ K_{j1} & \cdots & K_{j-1j-1} \end{vmatrix}$$

or

$$\begin{aligned}
\int_{\mathbb{R}} \begin{vmatrix} K_{11} & \cdots & K_{1j} \\ \vdots & \ddots & \vdots \\ K_{j1} & \cdots & K_{jj} \end{vmatrix} d\lambda_j &= (-1)^{j+1} K_{j1} \begin{vmatrix} K_{12} & \cdots & K_{1j} \\ \vdots & \ddots & \vdots \\ K_{(j-1)2} & \cdots & K_{(j-1)j} \end{vmatrix} + (-1)^{j+2} K_{j2} \begin{vmatrix} K_{11} & \cdots & K_{1j} \\ \vdots & \ddots & \vdots \\ K_{(j-1)1} & \cdots & K_{(j-1)j} \end{vmatrix} + \cdots \\
&\quad \cdots + K_{jj} \begin{vmatrix} K_{11} & \cdots & K_{1(j-1)} \\ \vdots & \ddots & \vdots \\ K_{(j-1)1} & \cdots & K_{(j-1)(j-1)} \end{vmatrix} \\
&= (-1)^{j+1} \begin{vmatrix} K_{12} & \cdots & K_{1j} \\ \vdots & \ddots & \vdots \\ K_{(j-1)2} & \cdots & K_{(j-1)1} \end{vmatrix} + (-1)^{j+2} K_{j2} \begin{vmatrix} K_{11} & \cdots & K_{12} \\ \vdots & \ddots & \vdots \\ K_{(j-1)1} & \cdots & K_{(j-1)2} \end{vmatrix} + \cdots \\
&\quad \cdots + N \begin{vmatrix} K_{11} & \cdots & K_{1(j-1)} \\ \vdots & \ddots & \vdots \\ K_{(j-1)1} & \cdots & K_{(j-1)(j-1)} \end{vmatrix}
\end{aligned}$$

It takes, for example,  $j-1$  column moves to convert the leading matrix into the final matrix. So it picks up a leading  $-1$ . In fact, we see that each term save the last will be negative. It follows that we have that the integral may be written

$$(N-(j-1)) \begin{vmatrix} K_{11} & \cdots & K_{1(j-1)} \\ \vdots & \ddots & \vdots \\ K_{(j-1)1} & \cdots & K_{(j-1)(j-1)} \end{vmatrix}$$

## Evaluations

Now consider

$$\begin{aligned}\mathbb{E}(\# \text{ of evaluations in } B) &= \int_{\mathbb{R}^n} \left( \sum_{j=1}^N \chi_B(\lambda_j) \right) \\ &= \sum_{j=1}^N \int_{\mathbb{R}^n} \chi_B(\lambda_j) \frac{1}{N!} \det( \quad )_{N \times N} d\lambda_N \cdots d\lambda_1\end{aligned}$$

With a change of variables where  $\mu_\ell = \lambda_\ell$  for  $\ell \in \{1, j\}$  such that  $\mu_j = \lambda_1$  and  $\mu_1 = \lambda_j$ ,

$$\begin{aligned}\sum_{j=1}^N \int_{\mathbb{R}^n} \chi_B(\mu_1) \frac{1}{N!} \det(K_N(\mu_j, \mu_k))_{N \times N} d\mu_N \cdots d\mu_1 &= \sum_{j=1}^N \int_{\mathbb{R}} \chi_B(\mu_1) \frac{(N-1)!}{N!} K_N(\mu_1, \mu_1) d\mu_1 \\ &= \int_B K_N(\mu_1, \mu_1) d\mu_1\end{aligned}$$

## Variance

Let  $n_N(B) = (\# \text{ of evaluations in } B)$  be a random variable. What is the variance? We have that

$$n_N(B) = \sum_{j=1}^N \chi_B(\lambda_j)$$

so

$$\begin{aligned}\text{var}(n_N(B)) &= \mathbb{E}((n_N(B))^2) - (\mathbb{E}(n_N(B)))^2 \\ &= \int_{\mathbb{R}^n} \left( \sum_{j=1}^N \chi_B(\lambda_j) \right)^2 \frac{1}{N!} \det( \quad )_{N \times N} d\lambda^N - [ \quad ]^2 \\ &= \int_{\mathbb{R}^n} \left( \sum_k \sum_j \chi_B(\lambda_j) \chi_B(\lambda_k) \right) \frac{1}{N!} \det( \quad )_{N \times N} d\lambda^N - [ \quad ]^2\end{aligned}$$

**April 29, 2025**

## Variance

Compute

$$\text{Var}(n_N(B)) = \mathbb{E}(n_N(B)^2) - (\mathbb{E}(n_N(B)))^2 = \int_B K_n(\lambda_1, \lambda_1) d\lambda_1 - \int_{B \times B} K_N(\lambda_1, \lambda_2)^2 d\lambda_1 d\lambda_2$$

This follows from  $n_N(B) = \sum_{j=1}^N \chi_B(\lambda_j)$ , so

$$\begin{aligned}\mathbb{E}\left(\left(\sum_{i=1}^n \chi_B(\lambda_j)\right)^2\right) &= \mathbb{E}\left(\sum_{j=1}^n \sum_{k=1}^n \chi_B(\lambda_1) \chi_B(\lambda_2)\right) \\ &= \mathbb{E}\left(\sum_{j=1}^n \chi_B(\lambda_j)\right) + \mathbb{E}\left(\sum_{j \neq k} \chi_B(\lambda_j) \chi_B(\lambda_k)\right) \\ &= \int_B K_N(\lambda_1, \lambda_1) d\lambda_1 + \sum_{j \neq k} \int_{\mathbb{R}^n} \chi_B(\lambda_j) \chi_B(\lambda_k) \frac{1}{N!} \det(K_N(\lambda_m, \lambda_n))_{N \times N} d^N \lambda\end{aligned}$$

Then using the same trick as before such that  $\lambda_j = \mu_1$  and  $\lambda_k = \mu_2$ , we rewrite this

$$\int_B K_N(\lambda_1, \lambda_1) d\lambda_1 + \overbrace{\sum_{j \neq k} \int_{\mathbb{R}^n} \chi_B(\mu_1) \chi_B(\mu_2) \frac{1}{N!} \det(K_N(\mu_m, \mu_n)_{N \times N}) d^N \mu}^{:=I}$$

Then we have

$$\begin{aligned} I &= \sum_{j \neq j} \int \cdots \int \chi_B(\mu_1) \chi_B(\mu_2) \frac{(1)}{N!} \det(\quad)_{(N-1) \times (N-1)} d^{N-1} \mu \\ &= \sum_{j \neq k} \int_B \int_B \frac{(N-2)!}{N!} \det(\quad)_{2 \times 2} d\mu \\ &= \frac{N!}{N!} \int_B \int_B \begin{vmatrix} K_N(\mu_1, \mu_1) & K_N(\mu_1, \mu_2) \\ K_N(\mu_2, \mu_1) & K_N(\mu_2, \mu_2) \end{vmatrix} d^2 \mu \end{aligned}$$

Then we have

$$\mathbb{E}(n_N(B)^2) = \int_B K_N(\lambda_1, \lambda_1) d\lambda_1 + \int_B \int_B K_N(\lambda_1, \lambda_1) K_N(\lambda_2, \lambda_2) - K_N(\lambda_1, \lambda_2)^2 d^2 \lambda$$

as well as

$$(\mathbb{E}(n_N(B)))^2 = \left( \int_B K_N(\lambda_1, \lambda_1) d\lambda_1 \right)^2$$

Then, since  $\int_B \int_B K_N(\lambda_1, \lambda_1) K_N(\lambda_2, \lambda_2) d^2 \lambda = \left( \int_B K_N(\lambda_1, \lambda_1) d\lambda_1 \right)^2$ , the terms cancel and we get the expression we want.

## Probability of No Evaluations

Now consider  $\prod_{j=1}^N (1 - \chi_B(\lambda_j))$  which returns 1 if there are no evaluations in  $B$  and 0 otherwise. Therefore

$$\int \prod_{j=1}^N (1 - \chi_B(\lambda_j)) \frac{1}{N!} \det(K_N(\lambda_j, \lambda_k)_{N \times N}) d^N \lambda$$

is the probability of having zero eigenvalues in  $B$  (i.e. the probability that  $n_N(B) = 0$ ). If we use the case where  $B = (a, \infty)$ , then this returns the probability that the largest eigenvalue is less than  $a$ . Consider

$$\sum_{k=1}^N \chi_B(\lambda_k) \prod_{\substack{j=1 \\ j \neq k}}^N (1 - \chi_B(\lambda_k))$$

and suppose we have exactly one eigenvalues ( $\lambda_3$ ) in  $B$ . This returns 1 when we have exactly one eigenvalue in  $B$  and 0 otherwise. So

$$\int \sum \chi_B(\lambda_k) \prod_{j=1}^N (1 - \chi_B(\lambda_1)) \frac{1}{N!} \det(\quad)_{N \times N} d^N \lambda,$$

where the product skips the  $k$ -th term, is the probability  $\mathbb{P}\{n_N(B) = 1\}$ . Now write

$$H(B, t) = \mathbb{E} \left( \prod_{j=1}^N (1 - t\chi_B(\lambda_j)) \right)$$

which gives  $H(B, 1) = \mathbb{P}\{n_N(B) = 1\}$ . Then the derivative with respect to  $t$ ,

$$H'(B, t) = \mathbb{E} \left( \sum_{k=1}^N (-\chi_B(\lambda_k)) \prod_{j=1}^{(k)} (1 - t\chi_B(\lambda_j)) \right)$$

so  $H'(B, 1) = -\mathbb{P}\{n_N(B) = 1\}$ , and  $H''(B, 1) = 2\mathbb{P}\{n_N(B) = 2\}$ . It follows that  $H^j(B, 1) = (-1)^j \cdot j! \cdot \mathbb{P}\{n_N = j\}$ . Then  $n_N(B)$  is the number of evaluations in  $B$ , and this process gives us the number statistics. We compute this fact as follows

$$\begin{aligned} H''(B, t) &= \mathbb{E} \left( \sum_{k=1}^N \chi_B(\lambda_k) \sum_{\ell=1}^N {}^{(k)}\chi_B(\lambda_\ell) \prod_{j=1}^N {}^{(k, \ell)}(1 - t\chi_B(\lambda_j)) \right) \\ &= \mathbb{E} \left( \sum_{k \neq \ell} \chi_B(\lambda_k) \chi_B(\lambda_\ell) \prod_{j=1}^N {}^{(k, \ell)}(1 - t\chi_B(\lambda_j)) \right) \\ &= 2! \cdot \mathbb{P}\{n_N(B) = 2\} \quad (t = 1) \end{aligned}$$

and

$$\begin{aligned} H^j(B, t) &= \mathbb{E} \left( \sum_{k_1=1}^N \sum_{k_2=1}^N {}^{(k_1)} \dots \sum_{k_j=1}^N {}^{(k_1, k_2, \dots, k_{j-1})} \prod_{v=1}^j \chi_B(\lambda_{k_{i_v}}) \prod_{j=1}^N {}^{(k_1, \dots, k_j)}(1 - t\chi_B(\lambda_j)) \right) \\ &= (-1)^j \cdot j! \cdot \mathbb{P}\{n_N(B) = j\} \quad (t = 1) \end{aligned}$$

## Coming Next

We know that  $\mathbb{E}(n_N(B)) = \int_B K_N(\lambda, \lambda) d\lambda$ . We will define an integral operator on functions  $f \in L^2(B)$

$$\begin{aligned} \mathcal{K}_N^{(B)}(f) &= \int_B K_N(x, y) f(y) dy \\ &= \int_B e^{-\frac{1}{4}(x^2 + y^2)} \sum_{\ell=0}^{N-1} P_\ell(x) P_\ell(y) f(y) dy \end{aligned}$$

We can define the trace of this operator,

$$\text{Tr}(\mathcal{K}_N^{(B)}) = \int_B K_N(\lambda, \lambda) d\lambda = \mathbb{E}(n_N(B))$$

Then

$$H(B, t) = \det(1 - t\mathcal{K}_N^{(B)})$$

May 6, 2025

## Expectation

Recall that

$$\begin{aligned}\mathbb{E}\left(\prod_{j=1}^N (1 - t\chi_B(\lambda_j))\right) &= \mathbb{E}\left(1 - t \sum_{j=1}^N \chi_B(\lambda_j) + t^2 \sum_{j < k} \chi_B(\lambda_j) \chi_B(\lambda_k) - t^3 \sum_{j < k < m} \chi_B(\lambda_j) \chi_B(\lambda_k) \chi_B(\lambda_m)\right) \\ &= 1 - t \int_B K_N(\lambda_1, \lambda_1) d\lambda_1 + t_2 \sum_{j < k} \int \chi_B(\lambda_j) \chi_B(\lambda_k) \frac{1}{N!} \det(\quad)_{N \times N} d^N \lambda \\ &= 1 - t \int_B K_N(\lambda_1, \lambda_1) d\lambda_1 + \frac{t^2}{2} \int_B \int_B | \quad |_{2 \times 2} d^2 \lambda - \frac{t^3}{3!} \int_B \int_B \int_B | \quad |_{3 \times 3} d^3 \lambda + \cdots + \int_B \cdots \int_B | \quad |_{N \times N} d^N \lambda\end{aligned}$$

We claim that

$$\int_{\mathbb{R}^n} \left( \sum_{k_1 < \cdots < k_j} \prod_{\ell=1}^j \chi_B(\lambda_{k_\ell}) \frac{\det}{N!}(K_N(\lambda_m, \lambda_n)_{N \times N}) \right) d^N \lambda = \frac{1}{j!} \int_B \cdots \int_B \det(K_N(\lambda_m, \lambda_n))_{j \times j} d\lambda_j \cdots d\lambda_1$$

Observing that  $\binom{N}{j} \cdot \frac{(N-j)!}{N!} = \frac{N!}{j!(N-j)!} = \frac{1}{j!}$ , write

$$\begin{aligned}\sum_{k_1 < \cdots < k_j} \int_{\mathbb{R}^n} \left( \prod_{\ell=1}^j \chi_B(\lambda_{k_\ell}) \right) \frac{\det}{N!}(\quad)_{N \times N} d\lambda_{j+1} \cdots d\lambda_1 &= \sum_{k_1 < \cdots < k_j} \int_{\mathbb{R}^{n-1}} \left( \prod \chi_B(\lambda_\ell) \right) \frac{\det}{N!}(\quad)_{(N-1) \times (N-1)} d\lambda_{N-1} \cdots d\lambda_1 \\ &= \sum_{k_1 < \cdots < k_j} \int_{\mathbb{R}^j} \left( \prod_{\ell=1}^j \chi_B(\lambda_\ell) \right) \frac{1 \cdot 2 \cdot 3 \cdots (N-j)}{N!} \det(\quad)_{j \times j} d^j \lambda \\ &= \frac{1}{j!} \int \cdots \int \det(\quad)_{j \times j} d^j \lambda\end{aligned}$$

So we have

$$H(B, t) = 1 + \sum_{j=1}^N \frac{(-1)^j t^j}{j!} \int_B \cdots \int_B \det \begin{pmatrix} K_N(\lambda_1, \lambda_1) & & \\ & \ddots & \\ & & K_N(\lambda_j, \lambda_j) \end{pmatrix} d^j \lambda = \det(1 - tK_N)$$

So if  $A$  is a symmetric, real  $N \times N$  matrix with  $\|A\| < 1$ ,

$$\det(I - tA) = e^{\log \det(I - tA)} = e^{\log \prod_{j=1}^N (1 - t\mu_j^A)} \stackrel{?}{=} e^{\sum_{j=1}^N \log(1 - t\mu_j^A)} \stackrel{?}{=} e^{\text{Tr} \log(I - tA)}$$

May 8, 2025

## Continuing

We want to show that  $\det(I - tA) = e^{\text{Tr} \log(I - tA)}$ .

Write

$$\log(1 - x) = \int_0^x -\frac{1}{1-s} ds = - \int_0^x \sum_{j=0}^{\infty} s^j ds = - \sum_{j=0}^{\infty} \frac{x^{j+1}}{j+1} = - \sum_{j=1}^{\infty} \frac{x^j}{j}.$$



So  $\log(I - tA) := -\sum_{j=1}^{\infty} \frac{(tA)^j}{j}$  converges in the matrix norm (for sufficiently small  $t$ ). Equivalently,  $e^{-\sum_{j=1}^{\infty} \frac{(tA)^j}{j}} = I - tA$ . Suppose we have an eigenvalue  $\mu$  of  $A$ , then  $\log(1 - t\mu)$  is an eigenvalue of  $\log(I - tA)$ . More commonly, this is presented as  $\log \det(I - tA) = \text{Tr} \log(I - tA)$ . In the end, we have

$$\det(I - tA) = \exp[\text{Tr}(\log(I - tA))] = \exp\left[-\text{Tr} \sum_{j=1}^{\infty} \frac{(tA)^j}{j}\right].$$

## Trace of an Operator

Given  $\mathcal{K}_n(f) = \int_B K_N(x, y) f(y) dy$ , we define  $\text{Tr}(\mathcal{K}_n) = \int_B K_N(x, x) dx$ . We want to consider the trace of  $\mathcal{K}_N^j$ . Taking  $j = 2$  as an example,

$$\begin{aligned} \mathcal{K}_N^2(f) &= \mathcal{K}_N \left[ \int_B K_N(x_1, x_2) f(x_2) dx_2 \right] \\ &= \int_B K_N(x, x_1) \int_B K_N(x_1, x_2) f(x_2) dx_2 dx \\ &= \int_B \left[ \int_B K_N(x, x_1) K_N(x_1, x_2) dx \right] f(x_2) dx_2 \end{aligned}$$

we note that  $\int_B K_N(x, x_1) K_N(x_1, x_2) dx$  is our new kernel. Then  $\text{Tr}(\mathcal{K}_N^2) = \int_B \int_B K_N(x_1, x_2) K_N(x_2, x_1) d^2 x$ . Therefore  $\mathcal{K}_N^\ell$  has a kernel given by

$$\overbrace{\int_B \cdots \int_B}^{\ell-1} K_N(x, x_1) \cdots K_N(x_{\ell-1}, x_\ell) dx_1 \cdots dx_{\ell-1}.$$

with trace given by

$$\text{Tr}(\mathcal{K}_N^\ell) = \overbrace{\int_B \cdots \int_B}^{\ell} K_N(x_\ell, x_1) \cdots K_N(x_{\ell-1}, x_\ell) d^\ell x.$$

## Continuing Computation

$$\det(I - tA) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[ -\text{Tr} \sum_{j=1}^{\infty} \frac{(tA)^j}{j} \right]^k = 1 - \left[ \sum_{j=1}^{\infty} \frac{t^j}{j} \text{Tr}(A^j) \right] + \frac{1}{2!} \left[ \sum_{j=1}^{\infty} \frac{t^j}{j} \text{Tr}(A^j) \right]^2 - \frac{1}{3!} \left[ \sum_{j=1}^{\infty} \frac{t^j}{j} \text{Tr}(A^j) \right]^3 + \cdots$$

If we are interested in  $O(t^1)$ , we examine the first non-trivial term and have  $-\text{Tr}(A)$ . For  $O(t^2)$  we have  $-\frac{1}{2} \text{Tr}(A^2) + \frac{1}{2} (\text{Tr} A)^2$ , which we compute as

$$\frac{1}{2} \left[ \int_B K_N(\lambda_1, \lambda_1) d\lambda_1 \int_B K_N(\lambda_2, \lambda_2) d\lambda_2 - \int_B \int_B K_N(\lambda_1, \lambda_2) K_N(\lambda_2, \lambda_1) d^2 \lambda \right]$$

Examining  $O(t^3)$ , we have  $-\frac{1}{3} \text{Tr}(A^3) + \frac{1}{4} (\text{Tr} A) \text{Tr}(A^2) + \frac{1}{4} \text{Tr}(A^2)(\text{Tr} A) - \frac{1}{3!} (\text{Tr} A)^3$  giving

$$-\frac{t^3}{3!} \left[ (\text{Tr} A)^3 - \frac{3}{2} (\text{Tr} A) \text{Tr}(A^2) - \frac{3}{2} \text{Tr}(A^2)(\text{Tr} A) + 2 \text{Tr}(A^3) \right]$$

which we compare with

$$-\frac{t^3}{3!} \left[ \int_B \int_B \int_B \begin{vmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{vmatrix} d^3 \lambda \right] = -\frac{t^3}{3!} \left[ \int_B \int_B \int_B K_{11}(K_{22}K_{33} - K_{23}K_{32}) - K_{12}(K_{21}K_{33} - K_{23}K_{31}) + K_{13}(K_{21}K_{32} - K_{22}K_{31}) \right]$$