

September 30, 2024

Chapter 1: Banach Algebras

1.1: Definitions and Basic Properties

Definition: Banach Space

A Banach space X (over \mathbb{C}) is a normed vector space with algebraic operations

$$\begin{aligned}(x, y) &\mapsto x + y && \text{addition} \\ (\lambda, y) &\mapsto \lambda y && \text{scalar multiplication}\end{aligned}$$

and a norm

$$x \mapsto \|x\|$$

which is complete (i.e. every Cauchy sequence converges).

Definition: (Complex) Banach Algebra

A (complex) Banach algebra B is a Banach space in which there is multiplication

$$B \times B \ni (x, y) \mapsto xy \in B$$

such that

1. $x(yz) = (xy)z$
2. $(x + y)z = xz + yz$ and $x(y + z) = xy + xz$
3. $\lambda(xy) = (\lambda x)y = x(\lambda y)$
4. $\|xy\| \leq \|x\| \cdot \|y\|$

Definition: Unital Banach Algebra

B is called a unital Banach algebra if $\exists e \in B$ such that

$$xe = ex = x \quad \text{and} \quad \|e\| = 1.$$

If e exists, it is unique.

1.2: Examples

Example 1

If X is a Banach space, then $B = \mathcal{L}(X)$ (the set of all bounded linear operators $A : X \rightarrow X$) equipped with algebraic operations

$$(A+B)x = Ax + Bx$$

$$(\lambda A)x = \lambda(Ax)$$

$$(AB)x = A(Bx)$$

and the operator norm

$$\|A\|_{\mathcal{L}(X)} = \sup_{x \neq 0} \frac{\|Ax\|_X}{\|x\|_X}.$$

$B = \mathcal{L}(X)$ is complete because X is complete.

The unit element is given by $I_X x = x$.

Example 2

If $X = \mathbb{C}^n$, then $B = \mathcal{L}(\mathbb{C}^n) \cong \mathbb{C}^{n \times n}$.

$$A = (a_{ij})_{i,j=1}^n$$

$$Ax = y$$

$$\sum_{j=1}^n a_{ij} x_j = y_i.$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

The norm in \mathbb{C}^n leads to a norm in $\mathbb{C}^{n \times n}$

$$\|(x_i)\| = \left(\sum |x_i|^2 \right)^{1/2}$$

$$\|(x_i)\| = \sum |x_i|$$

$$\|(x_i)\| = \max |x_i|$$

$$\|A\| =$$

$$\|A\| = \max_j \sum_i |a_{ij}|$$

$$\|A\| = \max_i \sum_j |a_{ij}|$$

All norms are equivalent.

Example 3

Take $B = C(K)$ with K a compact Hausdorff space, $f : K \rightarrow \mathbb{C}$ continuous and $\|f\| = \max_{t \in K} |f(t)|$.

Example 4

Take $B = A(K)$, $K \subseteq \mathbb{C}$ compact with $\text{int}(K) \neq \emptyset$, $f : K \rightarrow \mathbb{C}$ continuous where f is holomorphic on $\text{int}(K)$ and

$$\|f\| = \max_{t \in K} |f(t)| = \max_{t \in K \setminus \text{int}(K)} |f(t)|$$

e.g. $K = \overline{\mathbb{D}} = \{t \in \mathbb{C} : |t| \leq 1\}$. Then $A(K) \subseteq C(K)$.

Example 5

Take $B = \ell^\infty(\mathbb{N})$ or $B = L^\infty(S, \sigma, \mu)$ with (S, σ, μ) a measure space, $f : S \rightarrow \mathbb{C}$ essentially bounded functions and

$$||f|| = \text{ess sup}_{t \in S} |f(t)| = \inf_{\substack{N \subseteq S \\ \mu(N) = 0}} \left(\sup_{t \in S \setminus N} |f(t)| \right)$$

Example 6

Take $B = \ell^1(\mathbb{Z})$ or $B = L^1(\mathbb{R}^d)$ with $||\{x_n\}|| = \sum |x_n|$ and $||f|| = \int_{\mathbb{R}^d} |f(t)| dt$ respectively. Multiplication is given by the convolution. e.g.

$$fg = (f * g)(x) = \int_{\mathbb{R}^d} f(x-t)g(t) dt$$

$\ell^1(\mathbb{Z})$ is unital, but $L^1(\mathbb{R}^d)$ is non-unital (since the unit of convolution is the Dirac delta; see Example 7).

Example 7

Take $B = M(\mathbb{R}^d)$ the complex measures on \mathbb{R}^d with bounded variation. Then multiplications is given as

$$(\mu * \nu)(A) = \int_{\mathbb{R}^d} \mu(A-x) d\nu(x)$$

and norm

$$||\mu|| = \sup_{\substack{\mathbb{R}^d = \bigcup_{i=1}^n A_i \\ \text{disjoint}}} \sum_{i=1}^n |\mu(A_i)| < +\infty.$$

Then, $f dm = d\mu$ gives $L^1(\mathbb{R}^d) \rightarrow M(\mathbb{R}^d)$.

Example 8

Take $B = C^{n \times n}[K]$ with K compcat and Hausdorff, continuous functions $f : K \rightarrow \mathbb{C}^{n \times n}$ and norm

$$||f||_B = \max_{t \in K} ||f(t)||_{C^{n \times n}}.$$

Then $B \cong (C(K))^{n \times n}$ the $n \times n$ matrices with entries from $C(K)$.

1.3: Remarks

- If B does not have a unit element, consider $B_1 = B \times \mathbb{C}$ with operations

$$\begin{aligned} (b_1, \lambda_1) + (b_2, \lambda_2) &= (b_1 + b_2, \lambda_1 + \lambda_2) \\ \alpha(b, \lambda) &= (\alpha b, \alpha \lambda) \\ (b_1, \lambda_1)(b_2, \lambda_2) &= b_1 b_2 + \lambda_1 b_2 + \lambda_2 b_1, \lambda_1 \lambda_2 \end{aligned}$$

and norm

$$||(b, \lambda)|| = ||b|| + |\lambda|.$$

Then B_1 is a unital Banach algebra with $e = (0, 1)$. One writes $(b, \lambda) = (b, 0) + \lambda(0, 1) = b + \lambda \cdot e$. In some sense, $B \subseteq B_1$ where $b \in B \mapsto (b, 0) \in B_1$.

1.4: Definitions

Definition: Commutative Banach Algebra

B is called commutative if $xy = yx$.

Definition: Banach Subalgebra

A subset B_0 of a B -algebra is called a subalgebra if it is closed with respect to the algebraic operations

$$x, y \in B_0, \lambda \in \mathbb{C} \leadsto x + y, xy, \lambda x \in B$$

Definition: Closed Subalgebra

B_0 is a closed subalgebra or Banach subalgebra if it is norm-closed.

- Proposition: B_0 is a Banach algebra.

Definition: Generated Subalgebra

Let $M \neq \emptyset$ be a subset of a Banach algebra B .

The Banach subalgebra generated by M is the smallest closed subalgebra containing M .

$$\text{alg } M = (\text{clos alg}_B M)$$

- Remark

$\text{alg } M$ is the intersection of all closed subalgebras containing M .

$\text{alg } M = \text{clos} \left\{ \sum_{i=1}^N \lambda_i a_1^{(i)} a_2^{(i)} \cdots a_{n_i}^{(i)} \right\}$ is the norm-closure of finite linear combinations of finite products of $a_j^{(i)} \in M$.

1.5: Examples

Example 1

Take B unital, $b \in B$. Then

$$\text{alg}\{e, b\} = \text{clos}_B \left\{ \sum_{i=0}^N \lambda_i b^i : \lambda_i \in \mathbb{C}, N \in \mathbb{N} \right\}$$

where $b^0 = e$.

1.6 Definitions

Definition: Banach Algebra Homomorphism

A Banach algebra homomorphism is a map $\phi : B_1 \rightarrow B_2$ between Banach algebras B_1 and B_2 such that

- ϕ is linear
- ϕ is bounded (continuous)
- ϕ is multiplicative

$$\phi(b_1 b_2) = \phi(b_1) \cdot \phi(b_2)$$

- ϕ is unital if both B_1, B_2 have units and $\phi(e_{B_1}) = e_{B_2}$.

Definition: Banach Algebra Isomorphism

A Banach algebra homomorphism which is bijective is called a Banach algebra isomorphism.

Then $\phi^{-1} : B_2 \rightarrow B_1$ is an isomorphism as well.

Definition: Banach Algebra Isometry

ϕ is an isometry if $||\phi(x)|| = ||x||$.

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Recall

Given $M \subseteq \mathcal{L}(X)$ with X a Banach space (and $\mathcal{L}(X)$ itself a Banach algebra), we may construct $B = \text{alg}_{\mathcal{L}(X)} M$.

1.7 Proposition

Let B be a unital Banach algebra. Then the map

$$\phi : B \ni x \rightarrow L_x \in \mathcal{L}(B)$$

is an isometric isomorphism onto a closed subalgebra of $\mathcal{L}(B)$ where

$$L_x : B \ni z \mapsto xz \in B$$

is the left-representation of x .

Proof

L_x is in $\mathcal{L}(B)$ since $L_x z = xz$

- is linear in z and
- $||L_x z|| = ||xz|| \leq ||x|| \cdot ||z||$ implies $||L_x|| \leq ||x||$ (i.e. L_x is a bounded).

The map $\phi : x \mapsto L_x$ is linear

$$L_{x_1+x_2}z = (x_1+x_2)z = x_1z + x_2z = L_{x_1}z + L_{x_2}z = (L_{x_1} + L_{x_2})z$$

ϕ is multiplicative

$$L_{x_1x_2}z = (x_1x_2)z = x_1(x_2z) = L_{x_1}(L_{x_2}z)$$

From the above, we conclude that ϕ is a homomorphism.

To show that ϕ is an isometry,

$$\|L_x\| = \sup_{z \neq 0} \frac{\|L_x z\|}{\|z\|} \geq \frac{\|L_x e\|}{\|e\|} = \frac{\|x\|}{1} = \|x\|.$$

Then also ϕ is injective and $\text{im } \phi$ is closed. Since $\text{im } \phi$ is a Banach algebra, it is therefore a closed subalgebra.

1.7 Remark: Right-Regular Representation

Every unital Banach algebra is isometrically isomorphic to a Banach algebra of operators.

Right-regular representation:

$$R_x = z \mapsto zx$$

Chapter 2: Group of Invertible Elements in a Banach Algebra

2.1 Definition: Invertible Element

Let B be a unital Banach algebra. An element $x \in B$ (in B) if there exists $y \in B$ such that $xy = yx = e$.

Note that $y = x^{-1}$ is uniquely determined.

Write GB for the set of all invertible elements of B .

Remark

GB is a (multiplicative group).

- $x, y \in GB \implies xy \in GB$ and $(xy)^{-1} = y^{-1}x^{-1}$,
- $x \in GB \implies x^{-1} \in GB$ and $(x^{-1})^{-1} = x$, and
- $e \in GB$.

2.2 Lemma

If $x \in B$ and $\|x\| < 1$, then $e - x \in GB$.

Proof

Take the Neumann series

$$e + x + x^2 + x^3 + \dots$$

which converges to some $s \in B$

$$s_n = e + x + \cdots + x^n$$

where s_n are Cauchy:

$$||s_{n+k} - s_n|| = ||x^{n+1} + \cdots + x^{n+k}|| \leq ||x||^{n+1} + ||x||^{n+2} + \cdots = \frac{||x||^{n+1}}{1 - ||x||}.$$

So $s_n \rightarrow S$,

$$(e - x)s_n = s_n(e - x)e - x^{n+1}.$$

Taking $n \rightarrow \infty$

$$(e - x)s = s(e - x) = e.$$

2.3 Proposition

The group GB is open in B and the map $\Lambda : GB \ni x \mapsto x^{-1} \in GB$ is continuous (in the norm).

Proof

Take $x \in GB$ and consider $y \in B$ with $||y|| < \frac{1}{||x^{-1}||} = \varepsilon$.

Then $x + y \in B_\varepsilon(x)$ is invertible,

$$x + y = x(e + x^{-1}y),$$

and

$$||x^{-1}y|| \leq ||x^{-1}|| \cdot ||y|| < 1.$$

Therefore GB is open, since $B_\varepsilon(X) \subseteq GB$. The inverse

$$(x + y)^{-1} = (e + x^{-1}y)^{-1}x^{-1} = \sum_{n=0}^{\infty} (-x^{-1}y)^n x^{-1} = x^{-1} + \sum_{n=1}^{\infty} (-x^{-1}y)^n x^{-1}$$

so

$$||(x + y)^{-1} - x^{-1}|| \leq \sum_{n=1}^{\infty} ||x^{-1}||^{n+1} ||y||^n = \frac{||x^{-1}||^2 ||y||}{1 - ||x^{-1}|| \cdot ||y||}.$$

This converges to zero as $||y|| \rightarrow 0$.

2.4 Examples

Example 1

$B = C(K)$, K compact Hausdorff, $f : K \rightarrow \mathbb{C}$ continuous.

$GB = \{f \in C(K) : f(t) \neq 0, \forall t \in K\}$.

Example 2

$$B = \mathbb{C}^{n \times n}.$$

$$GB = \{A \in \mathbb{C}^{n \times n} : \det A \neq 0\}.$$

2.5 Definition:

Let $G_0 B$ stand for the connected component of GB containing e .

Remarks

- the ε -neighborhoods $B_\varepsilon(x) \subseteq B$ are (path-)connected.

$$B_\varepsilon(x) = \{y \in B : \|x - y\| < \varepsilon\}$$

For $y_1, y_2 \in B_\varepsilon(x)$, there is a continuous path

$$\sigma : [0, 1] \ni \lambda \mapsto y_1 \lambda + y_2(1 - \lambda) \in B_\varepsilon(x)$$

- Because GB is open and $B_\varepsilon(x)$ is path-connected, GB is locally (path-)connected (i.e. every $x \in GB$ has a (path-)connected open neighborhood in GB).
- In this context, connectedness and path-connectedness are equivalent. Therefore the components of GB are the path-components of GB .
- GB is the union of disjoint (path-)components where each component is both open and closed in GB .
- $x, y \in GB$ belong to the same path-component if there exists a continuous path $\gamma : [0, 1] \rightarrow GB$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Here, $x \sim y$ is an equivalence relation.
- $G_0 B = \{x \in GB : \exists \text{ a path in } GB \text{ connecting } e \text{ and } x\}.$

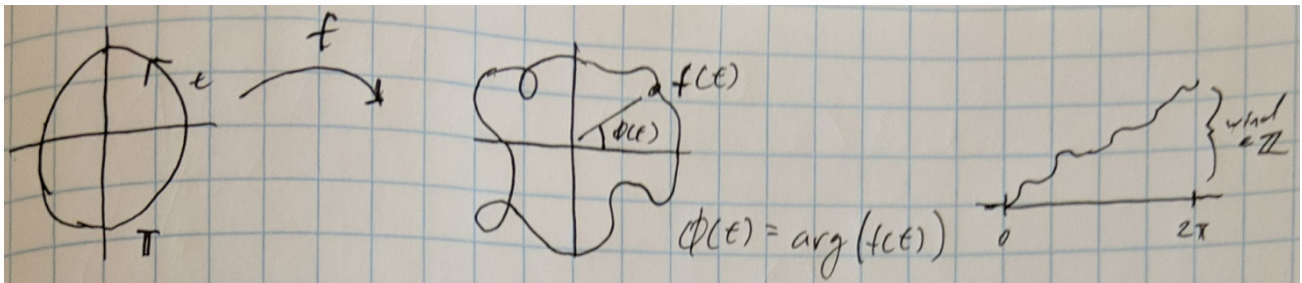
2.6 Examples

Example 1

Take $B = C(\mathbb{T})$ with $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and continuous functions $f : \mathbb{T} \rightarrow \mathbb{C}$.

GB is the non-vanishing continuous functions $f : \mathbb{T} \rightarrow \mathbb{C}$ ($f(t) \neq 0, \forall t \in \mathbb{T}$).

For $f \in GB$ one can define a winding number.



We have $\frac{1}{2\pi} \arg f(e^{ix})$ a continuous function with

$$\text{wind}(t) = \left[\frac{1}{2\pi} \arg f(e^{ix}) \right]_{x=0}^{2\pi} = \phi(2\pi) - \phi(0)$$

and $\text{wind}(t) \in \mathbb{Z}$.

The map $GB \ni f \mapsto \text{wind}(t) \in \mathbb{Z}$ is continuous, hence locally constant (i.e. constant on each connected component).

Therefore $G_0C(\mathbb{T}) \subseteq \{f \in GC(\mathbb{T}) : \text{wind}(f) = 0\}$. In fact, we will see that we have equality.

That is, f can be contracted (in GB) to the constant function $e(t) = 1$.

2.7 Proposition

G_0B is a normal subgroup of GB .

Proof

- G_0B is a group.

For any $x, y \in G_0B$, there exist paths $\gamma_1 : [0, 1] \rightarrow GB$ and $\gamma_2 : [0, 1] \rightarrow GB$ with $\gamma_1(0) = \gamma_2(0) = e$, $\gamma_1(1) = x$ and $\gamma_2(1) = y$.

Define $\gamma(t) = \gamma_1(t)\gamma_2(t)$ a path in GB such that $\gamma(0) = e$ and $\gamma(1) = xy$. Then $xy \in G_0B$.

Following from Lemma 2.2, $\hat{\gamma} = (\gamma_1(t))^{-1}$ is a continuous path with $\hat{\gamma}_1(0) = e$, $\hat{\gamma}_1(1) = x^{-1}$ and $x^{-1} \in G_0B$.

- G_0B is a normal subgroup of GB .

For every $y \in GB$, $yG_0By^{-1} \subseteq G_0B$ if and only if $yG_0B = G_0By$.

Take $x \in G_0B$ with path γ , then

$$\delta(t) = y\gamma(t)y^{-1}, \quad \delta(0) = yey^{-1} = e, \quad \text{and} \quad \delta(1) = yxy^{-1} \in G_0B.$$

2.8 Definition: Abstract Index Group

The quotient group GB/G_0B is called the abstract index group of B .

Remark

GB/G_0B is in 1-to-1 correspondence with the set of connected components of GB .

Indeed, the (path-)connected components of GB are given by $yG_0B = G_0By$ (for $y \in GB$).

$$y_1G_0B = y_2G_0B \iff y_2^{-1}y_1G_0B = G_0B \iff y_2^{-1}y_1 \in G_0B \iff [y_2] = [y_1] \text{ in } GB/G_0B.$$

2.9 Definition: Exponential Map

For $x \in B$, we define the exponential map $B \ni x \mapsto \exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

2.10 Lemma

The exponential map $B \ni x \mapsto \exp(x) \in GB$ is well-defined and continuous.

For $xy = yx$, we have $\exp(x+y) = \exp(x)\exp(y)$.

In particular, $(\exp(x))^{-1} = \exp(-x)$.

Proof

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergent.

$$\sum_{n=0}^{\infty} \frac{||x||^n}{n!} < +\infty.$$

It follows that $s_n = \sum_{k=0}^n \frac{x^k}{k!}$ is a Cauchy sequence and therefore converges. Continuity left as an exercise. Need to show:

$$\left\| \sum \frac{x^n}{n!} - \sum \frac{y^n}{n!} \right\| \leq \|x - y\| \cdot M_{x,y}$$

The fact that $\exp(x + y) = \exp(x) \exp(y)$ follows from multiplying terms and the binomial formula.

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Recall

GB $e + x$.

G_0B connected component of GB containing e .

GB/G_0B is the abstract index group.

$B = C(\mathbb{T}) \leadsto f \in GC(\mathbb{T}) \leadsto \text{ind}(f)$.

Definition: Exponential Map

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \in GB$$

Lemma:

For $y \in B$, $\|y\| < 1$, there exists $x \in B$ such that $\exp(x) = e + y$.

Proof

Define

$$\log(e + y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \dots \in B.$$

This converges absolutely ($\|y\| < 1$), therefore it converges in B by completeness.

Identities

$$\exp(\log(e + y)) = \sum_{n=0}^{\infty} \frac{\left(\sum_k \frac{y^k}{k} (-1)^{k-1} \right)^n}{n!} = e + y$$

Proof

G_0B is equal to the set of all finite products of exponentials of elements in B .

$$G_0B = \bigcup_{n=0}^{\infty} \Gamma_n = \bigcup_{n=0}^{\infty} \{ \exp(a_1) \exp(a_2) \cdots \exp(a_n) \in B \}$$

Proof

Call $\Gamma = \bigcup_{n=0}^{\infty} \Gamma^n$.

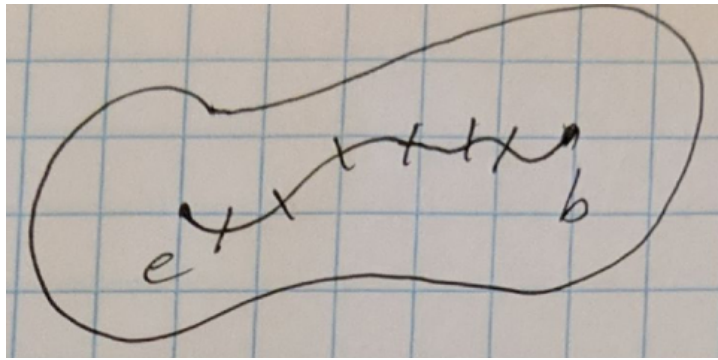
Then observe that each Γ_n is path-connected and contains e .

For $b = \exp(a_1) \cdots \exp(a_n) \in \Gamma_n$, define a path

- $\sigma : [0, 1] \rightarrow \Gamma_n$
- $\sigma(t) = \exp(ta_1) \cdots \exp(ta_n)$ is continuous with $\sigma(0) = e$ and $\sigma(1) = b$.

Therefore, Γ is path-connected and contains e . It follows that $\Gamma \subseteq G_0B$.

To prove that $G_0B \subseteq \Gamma$, take $b \in G_0B$ and show that there exists a path in GB $\gamma : [0, 1] \rightarrow GB$ continuous with $\gamma(0) = e$ and $\gamma(1) = b$.



We have that $(\gamma(t))^{-1}$ is continuous and bounded in the norm. Then $\gamma(t)$ is uniformly continuous.

$$\|\gamma^{-1}(t)\| \leq M.$$

$(\exists N) : |t - s| \leq \frac{1}{N} \implies \|\gamma(t) - \gamma(s)\| \leq \frac{1}{M} \cdot \frac{1}{2}$. Write

$$b = \gamma(1) \cdot \gamma^{-1}(0) = \gamma(1) \gamma^{-1}\left(\frac{N-1}{N}\right) \gamma\left(\frac{N-1}{N}\right) \gamma^{-1}\left(\frac{N-2}{N}\right) \cdots \gamma\left(\frac{1}{N}\right) \gamma^{-1}\left(\frac{1}{N}\right) \gamma(0) = \prod_{k=1}^N \gamma^{-1}\left(\frac{k}{N}\right) \gamma\left(\frac{k-1}{N}\right).$$

Therefore, with $s_k = \gamma^{-1}\left(\frac{k}{N}\right) \gamma\left(\frac{k-1}{N}\right)$, $b = \prod_{k=1}^N \exp(\log(s_k))$.

$$\|s_k - e\| \leq \|\gamma^{-1}\left(\frac{k}{N}\right)\| \cdot \|\gamma\left(\frac{k-1}{N}\right) - \gamma\left(\frac{k}{N}\right)\| \leq M \cdot \frac{1}{2M} \leq \frac{1}{2}.$$

Corollary

If B is commutative, $G_0B = \{\exp(a) : a \in B\}$.

Remark

Special case: $B = C(K)$ (K compact Hausdorff space).

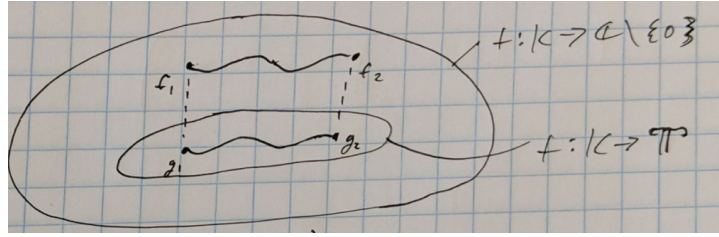
$$G_0B = \{\exp(a) : a \in C(K)\}.$$

GB/G_0B is an equivalence class of functions $f : K \rightarrow \mathbb{C} \setminus \{0\}$ with respect to path-connectedness.

That is, $f_1 \sim f_2$ if and only if there exists continuous $F(t, x) : [0, 1] \times K \rightarrow \mathbb{C} \setminus \{0\}$ with $F(0, x) = f_1(x)$ and $F(1, x) = f_2(x)$.

These are the homotopy classes of continuous functions $f : K \rightarrow \mathbb{C} \setminus \{0\}$.

This corresponds to homotopy classes of continuous functions $f : K \rightarrow \mathbb{T}$ (with $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$) called the 1st co-homotopy group of K $\pi^1(K)$.



$f: K \rightarrow \mathbb{C} \setminus \{0\}$ and $\frac{f}{|f|}: K \rightarrow \mathbb{C} \setminus \{0\}$ are path-connected by $\sigma(s) = \frac{f}{|f|^s}$, $s \in [0, 1]$.

$f_1 \sim f_2$ in $K \rightarrow \mathbb{C} \setminus \{0\}$ implies that $\frac{f_1}{\|f_1\|} \sim \frac{f_2}{\|f_2\|}$ in $K \rightarrow \mathbb{T}$ by $F(s, x)$ and $\frac{F(s, x)}{|F(s, x)|}$.

We conclude that $\pi^1(K) \cong GC(K)/G_0C(K)$.

Example

Let $B = C(\mathbb{T})$.

$$G_0B = \{\exp(a) : a \in C(\mathbb{T}) = \{f \in GC(\mathbb{T}) : \text{wind}(f) = 0\}\}$$

For $f \in GC(\mathbb{T})$, $\text{wind}(f) = 0$ implies that $f = \exp(a)$ has a logarithm.

This implies that $f \in G_0B$ which itself implies that $\text{wind}(f) = 0$, since $\text{wind}(f)$ is continuous on $GC(\mathbb{T})$ and therefore constant on the component.

Therefore, $GB/G_0B \cong \mathbb{Z}$ via the winding number.

For connected components of GB , define $\chi_n(t) = t^n$, $|t| = 1$, where $\text{wind}(\chi_n) = n$.

Remark: Closed Subalgebras and Invertibility

Let A be a closed subalgebra of B (both being unital, $e \in A$, $e \in B$).

Obviously, if $a \in A$ is invertible in A (i.e. $a^{-1} \in A$) then a is invertible in B . Then $GA \subseteq GB \cap A \subseteq GB$.

Example

Take $B = C(\mathbb{T})$ and $A = \{f \in C(\mathbb{T}) : f_n = 0, \forall n < 0\} = C_+(\mathbb{T})$ where $f_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{ix}) e^{-inx} dx$ is the n th Fourier coefficient.

Formally: $f(t) \cong \sum_{n=-\infty}^{\infty} f_n t^n$ in $B = C(\mathbb{T})$, $|t| = 1$.

$f \in A$: $f(t) = \sum_{n=0}^{\infty} f_n t^n$, $|t| = 1$ has an analytic extension into the unit disk $|t| < 1$.

More precisely, $\phi: A(\overline{\mathbb{D}}) \rightarrow C_+(\mathbb{T}) \subseteq C(\mathbb{T})$ by $f \mapsto f|_{\mathbb{T}}$.

Where $A(\overline{\mathbb{D}}) = \{f \in \overline{D} \rightarrow \mathbb{C} \text{ continuous, holomorphic on } \mathbb{D}\}$ and $\mathbb{D} = \{t \in \mathbb{C} : |t| \leq 1\}$.

Then, for $f \in A(\overline{\mathbb{D}})$ with $n \in \{-1, -2, -3, \dots\}$,

$$f_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{ix}) e^{-inx} dx = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(z)}{z^{n+1}} dz = \lim_{r \rightarrow 1^-} \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(rz)}{z^{n+1}} dz = 0$$

- In fact, ϕ is an isometry.

$$\|f\|_{A(\overline{\mathbb{D}})} = \sup_{|z| \leq 1} |f(z)| = \max_{|z|=1} |f(z)| = \|f|_{\mathbb{T}}\|_{C(\mathbb{T})}$$

By maximum modulus principle of holomorphic functions, since ϕ is not constant.

- ϕ is linear and multiplicative.

- $C_+(\mathbb{T})$ is a closed subset of $C(\mathbb{T})$.

$$\Lambda_n : C(\mathbb{T}) \ni f \mapsto f_n \in \mathbb{C}$$

is a continuous linear functional.

$$C_+(\mathbb{T}) = \bigcap_{n=0} \ker \Lambda_n$$

- Less trivially, ϕ is surjective and $C_+(\mathbb{T})$ is an algebra.

Example

$\chi_1(t) = t$ is invertible in $C(\mathbb{T}) = B$.
 $\chi_1^{-1}(t) = \frac{1}{t} = x_{-1}(t) \notin C_+(\mathbb{T})$ while $\chi_1(t) \in C_+(\mathbb{T})$.
Therefore $GA \subseteq GB \cap A$ may not be equal.

Definition: Boundary

The boundary of a subset U of a topological space X is $\partial U = \overline{U} \setminus \text{int}(U)$.

Remark

For $U \subseteq X$, $X = \text{int}(U) \cup \partial U \cup \text{int}(X \setminus U)$ a union of disjoint sets.

Lemma:

1. if $a \in \partial GA$, then $a \notin GA$ and there exists a sequence $a_n \in GA$ such that $a_n \rightarrow a$.
2. if $a \in \partial a$ and $a_n \in GA$ such that $a_n \rightarrow a$, then $\|a_n^{-1}\| \rightarrow +\infty$.

Proof of 1

$a \in GA$ would imply $a \in \text{int}(GA)$ and not a boundary point.

Proof of 2

Otherwise, there would exist a bounded subsequence $\|a_{n_i}^{-1}\| \leq M$.

$$\|a_{n_i}^{-1} - a_{n_j}^{-1}\| \leq \|a_{n_i}^{-1}\| \cdot \|a_{n_j} - a_{n_i}\| \cdot \|a_{n_j}^{-1}\| \leq M^2 \|a_{n_i} - a_{n_j}\|$$

Since a_n converges, $\{a_n\}$ is Cauchy which implies $a_{n_i}^{-1}$ is Cauchy.

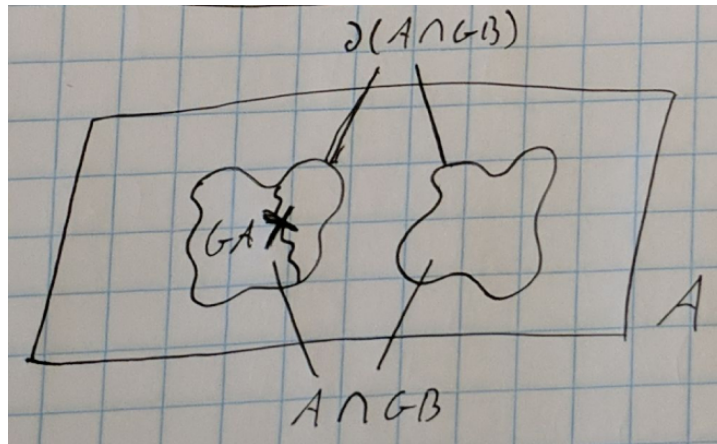
Then $a_{n_i}^{-1} \rightarrow b \in A$. $e = a_{n_i} a_{n_i}^{-1} \rightarrow ab$ implies $a^{-1} = b$ and $a \in GA$. However $a \notin GA$.

Proposition

Let A be a closed subalgebra of B ($e \in A$, $e \in B$). Then $\partial GA \subseteq \partial(A \cap GB)$ (both boundaries are considered in A).

Remark

Both GA and $A \cap GB$ are open subsets of A .



Proof

Take $a \in \partial GA$ and suppose $a \notin \partial(A \cap GB)$.

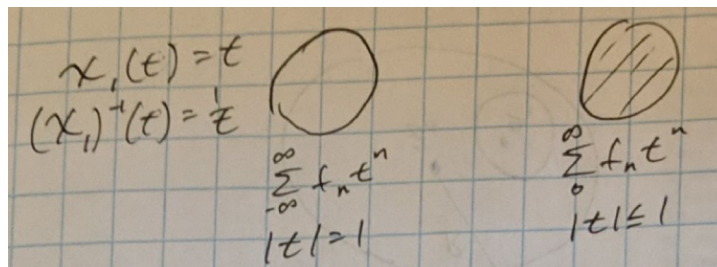
Take $a \in \partial GA$: $a_n \in GA$, $a \notin GA$, $a_n \rightarrow a$, $\|a_n^{-1}\| \rightarrow +\infty$.

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Recall

$A \subseteq B$, $GA \subseteq A \cap GB$.

If $A = C_+(\mathbb{T}) \cong A(\overline{\mathbb{D}})$ and $B = C(\mathbb{T})$.



Recall: Theorem

For GA , $A \cap GB$ open sets in A , $U \subseteq X$, $\partial U = \overline{U} \setminus \text{int } U$, we have that $\partial GA \subseteq \partial(A \cap GB)$.

Proof

Take $a \in \partial GA$, $a_n \rightarrow a$, $a \notin GA$, $a \in A$.

Since $a_n \in GA$, $\|a_n^{-1}\| \rightarrow +\infty$.

However, $a \notin GB$ otherwise $a \in GB$, $a_n \rightarrow a$ implies $a_n^{-1} \rightarrow a^{-1}$ (in GB) and, consequently, $\sup \|a_n^{-1}\| < +\infty$, a contradiction.

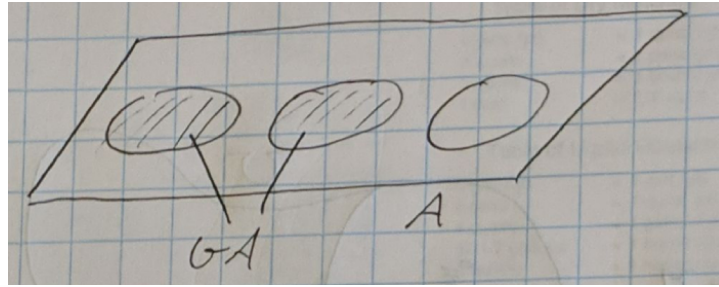
Therefore $a \notin A \cap GB$ and, consequently, $a \in \partial(A \cap GB) = \overline{(A \cap GB)} \setminus (A \cap GB)$.

Theorem

Let A be a closed subalgebra of B .

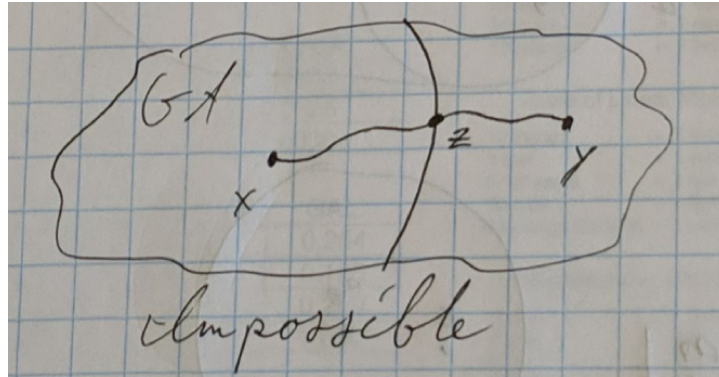
GA is equal to the union of some components of $A \cap GB$.

Proof



Let U be a component of $A \cap GB$.

We want to show that either $U \cap GA \neq \emptyset$ or $U \subseteq GA$.



The above cannot occur since, by path-connectedness, for $x, y \in U$, $x \in GA$, $y \notin GA$, there would need to be some $z \in \partial GA$ with $z \notin A \cap GB$ a contradiction.

Alternatively, take $A \cap GB$ open in A .

Then $A \cap GB \cap \partial(A \cap GB) = \emptyset$ and $(A \cap GB) \cap \partial GA = \emptyset$ by the previous theorem.

Write $A = GA \cup \partial GA \cup \text{int}(A \setminus GA)$. Then

$$A \cap GB = GA \cup \emptyset \cup \text{int}(A \setminus GA) \cap (A \cap GB)$$

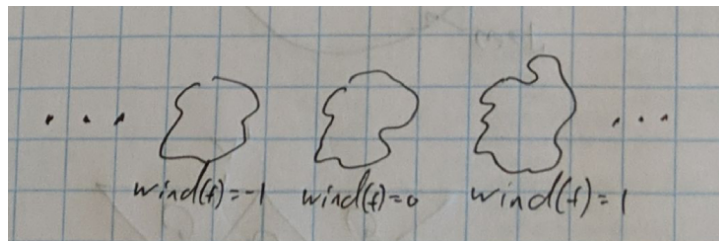
and $U = (GA \cap U) \cup \text{int}(A \setminus GB) \cap U$ where $(GA \cap U) \cap \text{int}(A \setminus GA) = \emptyset$ and open in U .

Therefore either $GA \cap U = \emptyset$ or $GA \cap U = U$ which implies that $U \subseteq GA$.

Example

Take $B(\mathbb{T})$ and $A = C_+(\mathbb{T}) \cong A(\overline{D})$.

Then $GB = \{f : \mathbb{T} \rightarrow \mathbb{C} : f(t) \neq 0\}$.



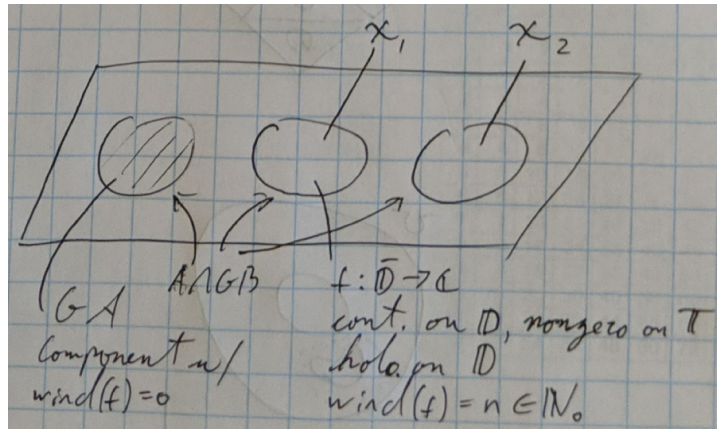
Then take

$$A \cap GB = \{f : \mathbb{T} \rightarrow \mathbb{C} \text{ continuous, } f(t) \neq 0, |t| = 1 \text{ with analytic continuation into } |t| < 1\}$$

such that $f \in A \cap GB$ which implies $\text{wind}(f) \in \{0, 1, 2, 3, \dots\}$ gives the number of zeroes of f inside \mathbb{D} .

$$\begin{aligned}\text{wind}(f) &= \frac{1}{2\pi i} \left[\log f(e^{ix}) \right]_{x=0}^{2\pi} \\ &= \frac{1}{2\pi i} \lim_{r \rightarrow 1^-} \left[\log f(re^{ix}) \right]_{x=0}^{2\pi} \\ &= \frac{1}{2\pi i} \lim_{r \rightarrow 1^-} \int_0^{2\pi} \frac{f'(re^{ix})}{f(re^{ix})} ire^{ix} dx \\ &= \frac{1}{2\pi i} \lim_{r \rightarrow 0} \int_{|z|=r} \frac{f'(z)}{f(z)} dz\end{aligned}$$

Which gives the number of zeros of $f(z)$ inside $|z| < 1$



Chapter 3: Holomorphic Vector-Valued Functions

Goal

Define the notion of holomorphic/analytic functions $f : \Omega \rightarrow X$ where $\Omega \subset \mathbb{C}$ open and X a (complex) Banach space.

Summary

- Basically all classical results remain true.
- There is a strong and a weak version of holomorphy, but they are equivalent.

Theorem

For a function $f : \Omega \rightarrow X$, $\Omega \subseteq \mathbb{C}$ open and X Banach, the following are equivalent

1. f is differentiable at every $z_0 \in \Omega$, i.e. there exists $f'(z_0) \in X$ such that

$$\lim_{z \rightarrow z_0} \left\| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right\|_X = 0$$

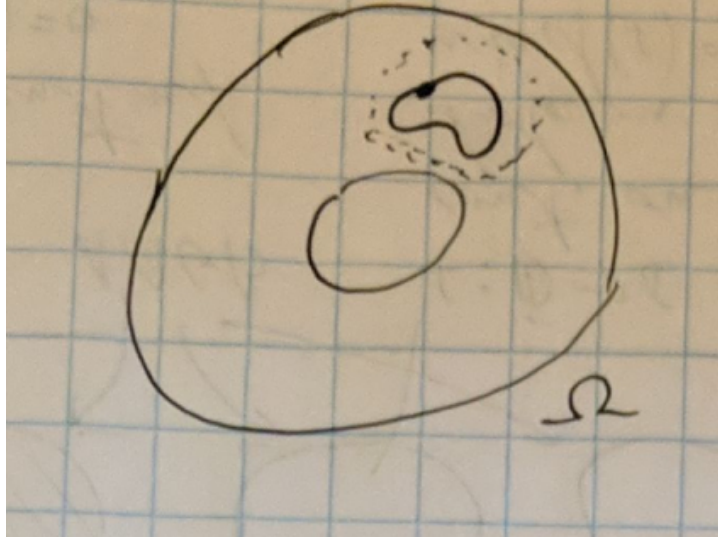
2. f is analytic at each point $z_0 \in \Omega$, i.e. f has a convergent power series at z_0 with radius of convergence $R_{z_0} > 0$.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad |z - z_0| < R_{z_0}, \quad a_n \in X$$

which converges in the norm of X .

3. $f : \Omega \rightarrow X$ is continuous (in the norm) and for every piecewise smooth closed contour Γ contained in a disk D ($\Gamma \subseteq D \subseteq \Omega$).

$$\int_{\Gamma} f(z) dz = 0$$



Definition: (Strongly) Holomorphic Function

If (1)-(3) hold, then f is (strongly)-holomorphic.

Remarks: Integration of Vector-Valued Functions

A piecewise smooth contour Γ can be parameterized by $\sigma : [0, 1] \rightarrow \Omega$.

$$\int_{\Gamma} f(z) dz = \int_0^1 \underbrace{f(\sigma(t))\sigma'(t)}_{h(t) \text{ continuous}} dt$$

This is independent of the choice of parameterization.

Now $I = \int_0^1 h(t) dt$ can be defined via Riemann sums. Given a partition P , $h : [0, 1] \rightarrow X$ continuous.

$$\lim_{\text{mesh}(P) \rightarrow 0} \|S(h, P, \xi) - I\|_X = 0$$

where $S(h, P, \xi) = \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1})$, $P = \{x_0, x_1, \dots, x_n\}$, $\xi_i \in [x_{i-1}, x_i]$.

Note that h is uniformly continuous and $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\text{mesh}(P_1) < \delta$, $\text{mesh}(P_2) < \delta$ implies

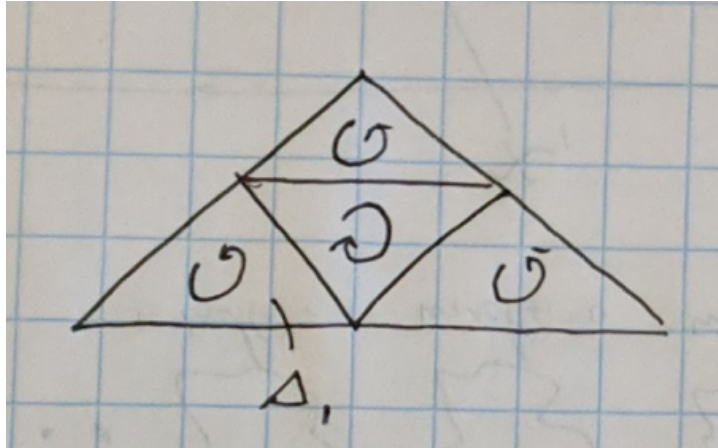
$$\|S(f, P_1, \xi^{(1)}) - S(f, P_2, \xi^{(2)})\| < \varepsilon$$

All usual properties of integrals hold.

- linear in integrand
- $\left\| \int_{\Gamma} f(z) dz \right\| \leq \int_{\Gamma} \|f(z)\| |dz| \leq (\text{length}(\Gamma)) \sup_{z \in \Gamma} \|f(z)\|.$

Sketch of Proof (1) to (3)

To show: $\int_{\Delta} f(z) dz = x_0 = 0$ by contradiction that $x_0 \neq 0$.

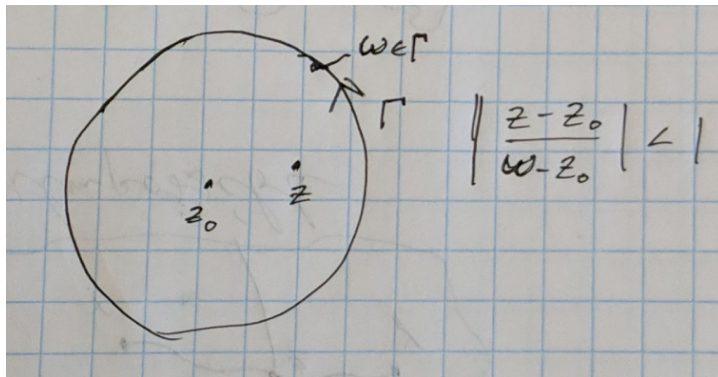


We have $\left| \int_{\Delta_1} f dz \right| \geq \frac{\|x_0\|}{4}, \left| \int_{\Delta_n} f dz \right| \geq \frac{\|x_0\|}{4^n}.$

Sketch of Proof (3) to (2)

$\int_{\Gamma} f dz = 0$ implies the Cauchy integral formula. Take

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\omega)}{\omega - z} d\omega$$



$$\frac{1}{\omega - z} = \frac{1}{(\omega - z_0) - (z - z_0)} = \frac{1}{\omega - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\omega - z_0} \right)^n$$

Therefore

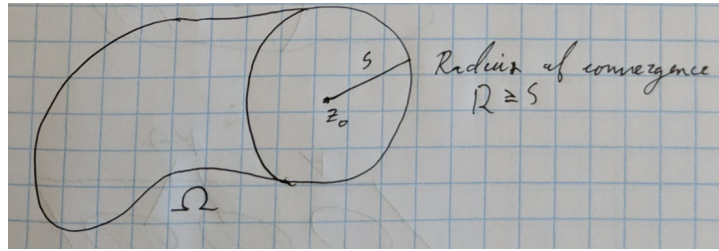
$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} f(\omega) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\omega - z)^{n+1}} d\omega = \sum_{n=0}^{\infty} (z - z_0)^n \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\omega)}{(\omega - z)^{n+1}} d\omega = \sum_{n=0}^{\infty} (z - z_0)^n a_n$$

with the sequence converging (in X) on $|z - z_0| < |\omega - z_0|$.

- Radius of Convergence

$$R^{-1} = \limsup_{n \rightarrow \infty} \|a_n\|^{1/n}$$

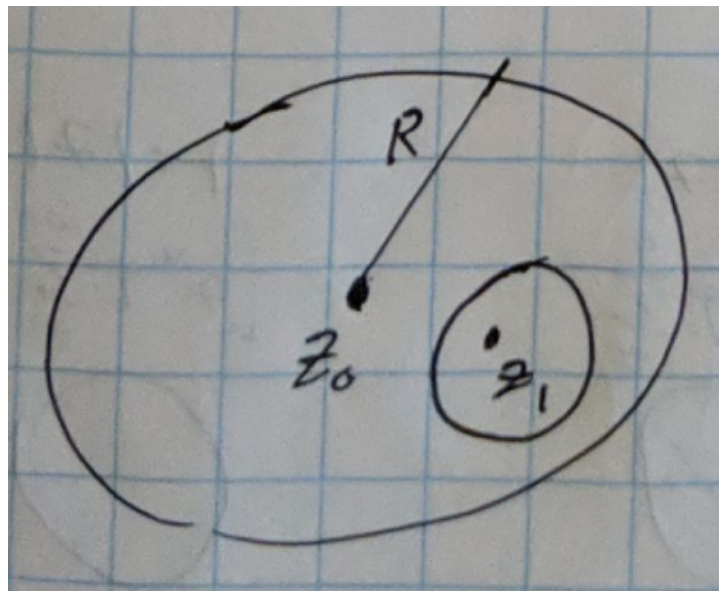
(Root Test: $|z - z_0| < R$ convergence; $|z - z_0| > R$ divergence)



Sketch of Proof (2) to (1)

One can show that a function defined by convergent power series is differentiable, $f(z) = \sum a_n(z - z_0)^n$, then $f'(z) = \sum a_n \cdot n(z - z_0)^{n-1}$.

The radius of convergence is the same. This also implies that f is infinitely differentiable.



Take $z - z_0 = (z - z_1) + (z_1 - z_0)$ and, by the binomial theorem,

$$f(z) = \sum_{k=0}^{\infty} (z - z_1)^k \left(\sum_{n=k}^{\infty} a_n \binom{n}{k} (z_1 - z_0)^{n-k} \right)$$

which converges for at least $|z - z_1| < R - |z_1 - z_0|$.

October 14, 2024

Theorem

Let $f : D_\varepsilon(z_0) \rightarrow X$ ($D_\varepsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$) be holomorphic.

Then $R = S$ where

1. R is the radius of convergence of $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ ($R^{-1} = \limsup_{n \rightarrow \infty} \|a_n\|^{\frac{1}{n}}$).
2. S is the radius of the largest open disk $D_S(z_0)$ such that there exists an analytic extension of f from $D_\varepsilon(z_0)$ to $D_S(z_0)$.

Proof

By definition, $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for $|z-z_0| < R$. Then $|z-z_0| < R$ if and only if $\limsup_{n \rightarrow \infty} ||a_n(z-z_0)^n||^{\frac{1}{n}} < 1$ if and only if $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges. Therefore, it converges to a holomorphic function on $R \leq S$.
If $f(z)$ has an analytic extension to $D_S(z_0)$, see step (3) \implies (2) of previous theorem.

IMAGE 2

Then $f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\omega)}{\omega-z} d\omega = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for $|z-z_0| < r < S$ with $a_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\omega)}{(z-\omega)^{n+1}} d\omega$.
From this, we conclude $R \geq S$.

Definition: (Weakly) Holomorphic Function

A function $f : \Omega \rightarrow X$ ($\Omega \subseteq \mathbb{C}$ open, X Banach) is called weakly holomorphic if $\phi \circ f : \Omega \rightarrow \mathbb{C}$ is holomorphic, $\forall \phi \in X^* = \mathcal{L}(X; \mathbb{C})$ bounded linear functionals.

A function $f : \Omega \rightarrow \mathcal{L}(X, Y)$ (X, Y Banach) is weakly-operator holomorphic if $h_{\phi, X} : \Omega \rightarrow \mathbb{C}$ is holomorphic for all $\phi \in Y^*$, $x \in X$ where $h_{\phi, X}(z) = \phi(f(z)x)$.

Remarks

Obviously: f strongly holomorphic $\implies f$ weakly holomorphic.

$$\left\| \frac{\phi(f(z+h)) - \phi(f(z))}{h} - \phi(f'(z)) \right\| \leq ||\phi|| \cdot \left\| \frac{f(z+h) - f(z)}{h} - f'(z) \right\|$$

For $f : \Omega \rightarrow \mathcal{L}(X, Y)$: f strongly holomorphic $\implies f$ weakly holomorphic $\implies f$ weakly operator holomorphic.

For $x \in X$, $\phi \in Y^*$, $\Lambda_{x, \phi} : \mathcal{L}(X, Y) \ni A \mapsto \phi(Ax) \in \mathbb{C}$ and $\Lambda_{x, \phi} \in (\mathcal{L}(X, Y))^*$.

All the converses are also true.

Theorem (Dunford)

Take X Banach, $\Omega \subseteq \mathbb{C}$ open.

If $f : \Omega \rightarrow X$ is weakly holomorphic, then it is strongly holomorphic.

Proof

We want to show that for any $z_0 \in \Omega$, $\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists in X .

Choose $\varepsilon > 0$ such that the disk $D_{\varepsilon}(z_0)$ and circle $C_{2\varepsilon}(z_0) = \Gamma$ are in Ω .

IMAGE 3

For $\phi \in X^*$, $\phi(f(z))$ is holomorphic in Ω .

$$\phi(f(z)) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\phi(f(\omega))}{z-\omega} d\omega, z \in D$$

Apply this to $z = z_0$, $z = z_0 + h_1$ and $z = z_0 + h_2$ with $0 < |h_1| < \varepsilon$, $0 < |h_2| < \varepsilon$, $h_1 \neq h_2$.

IMAGE 4

$$\begin{aligned}
A_{h_1, h_2} &= \frac{1}{h_1 - h_2} \left\{ \frac{f(z_0 + h_1) - f(z_0)}{h_1} - \frac{f(z_0 + h_2) - f(z_0)}{h_2} \right\} \\
\phi(A_{h_1, h_2}) &= \frac{1}{h_1 - h_2} \left\{ \frac{\phi(f(z_0 + h_1)) - \phi(f(z_0))}{h_1} - \frac{\phi(f(z_0 + h_2)) - \phi(f(z_0))}{h_2} \right\} \\
&= \frac{1}{2\pi i} \int_{\Gamma} \phi(f(\omega)) \frac{1}{h_1 - h_2} \left\{ \frac{1}{h_1} \left(\frac{1}{z_0 + h_1 - \omega} - \frac{1}{z_0 - \omega} \right) - \frac{1}{h_2} \left(\frac{1}{z_0 + h_2 - \omega} - \frac{1}{z_0 - \omega} \right) \right\} d\omega \\
&= \frac{1}{2\pi i} \int_{\Gamma} \phi(f(\omega)) \frac{1}{h_1 - h_2} \left\{ \frac{1}{(z_0 + h_1 - \omega)(z_0 - \omega)} - \frac{1}{(z_0 + h_2 - \omega)(z_0 - \omega)} \right\} d\omega \\
&= \frac{1}{2\pi i} \int_{\Gamma} \phi(f(\omega)) \frac{1}{(z_0 + h_1 - \omega)(z_0 + h_2 - \omega)(z_0 - \omega)} d\omega
\end{aligned}$$

Observe that the denominator is at least ε^3 , therefore $|\phi(A_{h_1, h_2})| \leq \frac{\varepsilon^3}{2\pi} \sup_{\omega \in \Gamma} \|f(\omega)\| \cdot \|\phi\|$ (so long as f continuous, which will be proven).

Therefore $\forall \phi \in X^*$,

$$\sup_{\substack{0 < |h_1| < \varepsilon \\ 0 < |h_2| < \varepsilon \\ h_1 \neq h_2}} |\phi(A_{h_1, h_2})| < +\infty.$$

By the uniform boundedness principle, identify $A_{h_1, h_2} \in X$ with $X^{**} = \mathcal{L}(X^*, \mathbb{C})$.

Then $\sup_{h_1, h_2} \|A_{h_1, h_2}\| < +\infty$ and

$$\left\| \frac{f(z_0 + h_1) - f(z_0)}{h_1} - \frac{f(z_0 + h_2) - f(z_0)}{h_2} \right\| \leq C \cdot |h_1 - h_2|.$$

Now, for any sequence $\{h_n\}_{n=3}^{\infty}$, $0 < |h_n| < \varepsilon$, $h_n \rightarrow 0$,

$$\frac{f(z_0 + h_n) - f(z_0)}{h_n}$$

is a Cauchy sequence. Therefore $\lim_{n \rightarrow \infty} \frac{f(z_0 + h_n) - f(z_0)}{h_n}$ exists in X independent of choice of $\{h_n\}$. That is

$$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists in X .

Chapter 4: Spectrum and Resolvent

Consider a unital Banach algebra B .

Definition: Spectrum

For $b \in B$, the spectrum of b in B $\sigma_B(b) = \{\lambda \in \mathbb{C} : \lambda e - b \text{ is not invertible in } B\}$.

Definition: Resolvent

The resolvent is a function $R(b; \lambda) = (\lambda e - b)^{-1}$. $R(b, \cdot) : \mathbb{C} \setminus \sigma_B(b) \rightarrow B$.

$\mathbb{C} \setminus \sigma_B(b)$ is the resolvent set.

Theorem

1. The spectrum $\sigma_B(b)$ is a non-empty, compact subset of \mathbb{C} .
2. The resolvent $R(b, \lambda)$ is an analytic, Banach valued function on $\mathbb{C} \setminus \sigma_B(b)$.

Proof of (a)

$\sigma_B(b)$ is bounded, because $\lambda e - b$ is invertible for $|\lambda| > \|b\|$.

$$\lambda e - b = \lambda \left(e - \frac{1}{\lambda} b \right)$$

has $\left\| \frac{1}{\lambda} b \right\| < 1$ for sufficiently large λ . Therefore, $\sigma_B(b) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|b\|\}$.

To show that $\sigma_B(b)$ is closed, if $\lambda \notin \sigma_B(b)$ then $\forall \mu$ such that $\|\lambda - \mu\| < \varepsilon$ we have that $\mu \notin \sigma_B(b)$.

$$\mu e - b = \lambda e - b + (\mu - \lambda)e = (\lambda e - b) \left\{ e + \underbrace{(\mu - \lambda)(\lambda e - b)^{-1}}_{\|\cdot\| < 1} \right\}$$

when $|\mu - \lambda| < \frac{1}{\|(\lambda e - b)^{-1}\|}$.

Therefore $\mathbb{C} \setminus \sigma_B(b)$ is open.

Proof of (b)

Take $\lambda \notin \sigma_B(b)$

IMAGE 5

$$\begin{aligned} \frac{R(b, \mu) - R(b, \lambda)}{\mu - \lambda} &= \frac{1}{\mu - \lambda} \left((\mu e - b)^{-1} - (\lambda e - b)^{-1} \right) \\ &= \frac{1}{-\mu - \lambda} (\mu e - b)^{-1} \{ (\lambda e - b) - (\mu e - b) \} (\lambda e - b)^{-1} \\ &= -(\mu e - b)^{-1} (\lambda e - b)^{-1} \end{aligned}$$

Using continuity with $GB \ni a \mapsto a^{-1} \in GB$ in the norm, $-(\mu e - b)^{-1} (\lambda e - b)^{-1} \rightarrow -((\lambda e - b)^{-1})^2$ as $\mu \rightarrow \lambda$.

Therefore $R^1(b, \lambda) = -(R(b, \lambda))^2$ and $R(b, \lambda)$ is analytic.

Proof of non-empty in (a)

Take $\sigma_B(b) \neq \emptyset$, otherwise $R(b, \lambda)$ is analytic on \mathbb{C} and bounded

$$(\lambda e - b)^{-1} = \frac{1}{\lambda} \left(e - \frac{1}{\lambda} b \right)^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\lambda^{n+1}} b^n$$

We can estimate

$$\|\cdot\| \leq \frac{1}{|\lambda| \left(1 - \frac{\|b\|}{|\lambda|} \right)} = \frac{1}{|\lambda| - \|b\|}$$

so $\lim_{\lambda \rightarrow \infty} \|(\lambda e - b)^{-1}\| = 0$.

By Liouville's theorem, bounded and entire functions are constant. But we may also proceed by weak analyticity.

If $\phi(R(b, \lambda))$ is analytic and bounded on \mathbb{C} , $\forall \phi \in B^*$, it follows that $\phi(R(b, \lambda)) \equiv 0$, $\forall \lambda$, $\forall \phi \in B^*$ and that $R(b, \lambda) \equiv 0$ for any λ a contradiction.

Definition: Spectral Radius

For $b \in B$, the spectral radius $r(b) = \max\{|\lambda| : \lambda \in \sigma_B(b)\}$.

IMAGE 6

Remark

Write $\frac{1}{r(b)} = \min\{|\lambda|^{-1} : \lambda e - b \text{ is not invertible}\} = \min\{|\mu| : e - \mu b \text{ is not invertible}\}$ with $\mu = \frac{1}{\lambda}$.

$$\underbrace{(e - \mu b)^{-1}}_{\text{analytic in } |\mu| < \frac{1}{\|b\|}} = \sum_{n=0}^{\infty} \mu^n b^n$$

converges for $|\mu| < \frac{1}{\|b\|}$.

Then the radius of convergence $R^{-1} = \limsup_{n \rightarrow \infty} \|b^n\|^{\frac{1}{n}}$ gives us that R is equal to the largest disk where $(e - \mu b)^{-1}$ has an analytic extension. Therefore $S = \frac{1}{r(b)}$.

IMAGE 7

Suppose we have an analytic extension $f(\mu)$ beyond S .

$$f(\mu)(e - \mu b) = (e - \mu b)f(\mu) = e$$

implies that and, if $(e - \mu_0 b)$ not invertible, $f(\mu_0)(e - \mu_0 b) = \dots = e$ a contradiction.

Theorem

$$r(b) = \lim_{n \rightarrow \infty} \|b^n\|^{\frac{1}{n}} = \inf_{n \in \mathbb{N}} \|b^n\|^{\frac{1}{n}}$$

Proof

To demonstrate existence, fix $n_0 \in \mathbb{N}$, $n = q \cdot n_0 + r$, $0 \leq r < n_0$.

$$\begin{aligned} \|b^n\| &\leq \|b^{n_0}\|^q \cdot \|b\|^r \\ \|b^n\|^{\frac{1}{n}} &\leq \|b^{n_0}\|^{\frac{q}{n}} \cdot \|b\|^{\frac{r}{n}} \\ \limsup_{n \rightarrow \infty} \|b^n\|^{\frac{1}{n}} &\leq \|b^{n_0}\|^{\frac{1}{n_0}} \cdot 1 \end{aligned}$$

Since $1 = \frac{q}{n} \cdot n_0 + \frac{r}{n}$. Take $n \rightarrow \infty$. Write

$$\limsup_{n \rightarrow \infty} \|b^n\|^{\frac{1}{n}} \leq \inf_{n_0 \in \mathbb{N}} \|b^{n_0}\|^{\frac{1}{n_0}} \leq \liminf_{n \rightarrow \infty} \|b^n\|^{\frac{1}{n}}$$