Manifolds II

January 6, 2025

Recall: Tangent Bundle

Given a chart (U,ϕ) about a point p, we have coordinates $(x^1,\dots,x^n)$ and a basis for T_qM of $\left(\frac{\partial}{\partial x^1}|_q,\dots,\frac{\partial}{\partial x^n}|_q\right)$ for $q\in U$.
Then given $TM\xrightarrow{\pi}M$, we may write $v_q = v^i\frac{\partial}{\partial x^i}|_q$.

Definition:

For M a topological manifold. A (real) vector bndle of rank k over M is a topological space E with a surjective continuous map $\pi: E\to M$ such that

1. $\forall\, p\in M$, the fiber $\pi^{-1}(p) =: E_p$ is endowed with the structure of a (real) vector space of dimension k.

2. $\forall\, p\in M$, there exists a neighborhood U of p in M and a homeomorphism $\Phi: \pi^{-1}(U)\to U\times\mathbb{R}^k$ called a local trivialization.

$$\begin{tikzcd} \Phi:\pi^{-1}(U) \arrow[rr] \arrow[dr, "\pi"] & & U\times\mathbb{R}^k \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd}$$

   and $\Phi|_{E_q}: E_q\to\{q\}\times\mathbb{R}^k$ is a linear isometry.

Examples

1. $TM\xrightarrow{\pi}M$

2. $E = M\times\mathbb{R}^k$ with a global trivialization.

3. The Mobius bundle over S^1. $\gamma:\mathbb{R}^2\to\mathbb{R}^2$ by $(x,y)\mapsto(x+1,(-1)\cdot y)$. Then $\langle\gamma\rangle\cong\mathbb{Z}$ a subgroup acting freely and isometrically on $\mathbb{R}^2$. Then $E=\mathbb{R}^2/\langle\gamma\rangle\xrightarrow{\pi}S^1=\mathbb{R}/\mathbb{Z}$ by $\overline{(x,y)}\mapsto\overline{x}$ is a vector bundle.

IMAGE 1

- We want to show that $\pi^{-1}(U)\cong U\times\mathbb{R}$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "q"] \arrow[d, "\pi"] & E \arrow[d, "\pi"] \\ \mathbb{R} \arrow[r, "\varepsilon"] & S^1 \end{tikzcd} \qquad \begin{tikzcd} (x,y) \arrow[r, mapsto] \arrow[d, mapsto] & \overline{(x,y)} \arrow[d, mapsto] \\ x \arrow[r, mapsto] & e^{(2\pi i)x} \end{tikzcd}$$

Then let $p\in S^1$. We choose U a neighborhood of p such that U is evenly covered by ε. This means $\varepsilon^{-1}(U)$ is a disjoint union of open sets difeomorphic to U.

IMAGE 2

Let $\tilde{U}$ be a component in $\pi^{-1}(U)$. Then $\pi_1^{-1}(\tilde{U})\cong\tilde{U}\times\mathbb{R}$ and $\pi^{-1}(U)$ is diffeomorphic to $U\times\mathbb{R}$.

Definition: Transition Function

Take $E \xrightarrow{\pi} M$ with $U, V \subseteq M$ admitting trivializations $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ and $\Psi : \pi^{-1}(V) \to V \times \mathbb{R}^k$. Let $w = U \cap V (\neq \varnothing)$.

$$\Phi \circ \Psi^{-1} : \quad \begin{tikzcd} W \times \mathbb{R}^k \arrow[r] \arrow[dr] & \pi^{-1}(W) \arrow[r] \arrow[d] & W \times \mathbb{R}^k \arrow[dl] \\ & W & \end{tikzcd}$$

Then $\Phi \circ \Psi^{-1}|_{\{p\} \times \mathbb{R}^k}$ by $\{p\} \times \mathbb{R}^k \to \{p\} \times \mathbb{R}^k$ is a linear isomorphism.
$\Phi \circ \Psi^{-1}(p, v) = (p, \tau(p)v)$ by $\tau : p \mapsto \tau(p)$ and $\tau(p) \in GL(k, \mathbb{R})$ gives a smooth map $W \to GL(k, \mathbb{R})$.

Definition:

Let $\{E_1, \ldots, E_k\}$ be a basis of $\mathbb{R}^k$. Then

$$\tau(p) \cdot E_i = \sum_j \tau(p)_i^j E_j$$

with $\tau(p) = (\tau(p)_i^j)$ and $\tau(p)_j^i \in \mathbb{R}$. It suffices to show each $\tau(*)_i^j$ mapping $W \to \mathbb{R}$ and $p \mapsto (\tau(p)_i^j)$ is smooth. Then if $\sigma(p, v) := \Phi \circ \Psi^{-1}(p, v)$, $\tau(p)_i^j = \pi_j(\sigma(p, E_i))$ and π_j is a projection to the j-th component in $\mathbb{R}^k$.

Lemma 10.6 (Vector Bundle Chart Lemma)

Given M a smooth manifold, suppose that $\forall p \in M$ we are given a vector space E_p of dimension k. Let $E = \coprod_{p \in M} E_p$ (as a set) and $\pi : E \to M$ a mapping E_p to p. Suppose also that we have

1. $\{U_\alpha\}_{\alpha \in A}$ an open cover of M with a countable subcover.
2. $\forall \alpha \in A$ we hav ea bijection $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ such that $\Phi_\alpha|_{E_p} : E_p \to \{p\} \times \mathbb{R}^k$ is a linear isomorphism.
3. $\forall \alpha, \beta \in A$ with $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \varnothing$ we have a smooth map $\tau_{\alpha\beta} : U_{\alpha\beta} \to GL(k, \mathbb{R})$ such that $\Phi_\alpha \circ \phi_\beta^{-1} : U_{\alpha\beta} \times \mathbb{R}^k \to U_{\alpha\beta} \times \mathbb{R}^k$ by $(p, v) \mapsto (p, \tau(p)v)$.

Then $E \xrightarrow{\pi} M$ is a vector bundle.

Example (Whitney Sum):

Suppose we have $E' \xrightarrow{\pi'} M$ and $E'' \xrightarrow{\pi''} M$ two vector bundles over M.
Define $E = E' \oplus E''$ a new vector bundle over M by $E_p = E'_p \oplus E''_p$. Let $\{U_\alpha\}_{\alpha \in A}$ be a countable open cover of M such that each U_α admits trivializations for E' and E''. Then for $\pi : E \to M$, define $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{k'} \times \mathbb{R}^{k''}$ by $(v', v'')_p \mapsto (p, \pi_2 \circ \Phi_\alpha^{-1}(v'), \pi_2 \circ \Phi''_\alpha(v''))$ where

$$\pi'(U_\alpha) \xrightarrow{\Phi'_\alpha} U_\alpha \times \mathbb{R}^{k'} \xrightarrow{\pi_2} \mathbb{R}^{k'}$$

Note that π_2 is the projection into the second component. Then $\tau : U_{\alpha\beta} \to G(k' + k'', \mathbb{R})$ by

$$p \mapsto \begin{pmatrix} \tau'(p) & 0 \\ 0 & \tau''(p) \end{pmatrix}$$

Example

For $\tau_{\alpha\beta} : U_{\alpha\beta} \to GL(k,\mathbb{R})$ by $p \mapsto \tau_{\alpha\beta}(p)$, we can write $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cup U_\gamma (\neq \varnothing)$ and get $\tau_{\alpha\beta} \cdot \tau_{\beta\gamma} = \tau_{\alpha\gamma}$.
Note that this is $\Phi_\alpha \circ (\phi_\beta^{-1} \circ \phi_\beta) \circ \Phi_\gamma^{-1}$.
Without loss of generality, we assume each U_α is a chart for M. Then we want to show that we satisfy Lemma 1.35 from Lee

$$\pi^{-1}(U_\alpha) \overset{\Phi_\alpha}{\to} U_{\alpha\times\mathbb{R}^k} \overset{\phi_\alpha\times\mathrm{id}}{\to} \phi_\alpha(U_\alpha)\times\mathbb{R}^k \subseteq \mathbb{R}^{n+k}$$

$(\pi^{-1}(U_\alpha)\cdot\tilde{\phi}_\alpha = (\phi_\alpha\times\mathrm{id})\circ\Phi_\alpha)_{\alpha\in A}$ which satisfies (1).
Since

$$\pi^{-1}(U_\alpha)\cap\pi^{-1}(U_\beta) = \pi^{-1}(U_\alpha\cap U_\beta) \to \phi_\alpha(U_{\alpha\beta})\times\mathbb{R}^K$$

we have that (2) is satisfied.
Finally, for (3),

$$\tilde{\phi_\beta}\circ\tilde{\phi_\alpha}^{-1} = (\Phi_\beta\circ(\phi_\beta\times\mathrm{id}))\circ((\phi_\alpha\times\mathrm{id})^{-1}\circ\Phi_\alpha^{-1}) = \Phi_\beta\circ((\phi_\beta\circ\phi_\alpha)\times\mathrm{id})\circ\Phi_\alpha^{-1}$$

gives $(x,c)\mapsto((\phi_\beta\circ\phi_\alpha^{-1})x,(\Phi_\beta\circ\Phi_\alpha^{-1})v)$ a diffeomorphism.
(4) and (5) are trivial, and this is indeed a smooth manifold. Now we wish to show that it is a vector bundle. To show that $\pi : E\to M$ is smooth,

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[r, "\pi"] & U_\alpha \arrow[d, "\phi_\alpha"] \\
\phi_\alpha(U_\alpha)\times\mathbb{R}^k \arrow[u, "\tilde{\phi}_\alpha^{-1}"] & \phi_\alpha(U_\alpha)
\end{tikzcd}$$

We have $\tilde{\phi}_\alpha^{-1} = (\phi_\alpha\times\mathrm{id})^{-1}\circ\Phi_\alpha^{-1}$.

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[r, "\Phi_\alpha"] & U_\alpha\times\mathbb{R}^k \arrow[d, "\phi_\alpha\times\mathrm{id}"] \\
\phi_\alpha(U_\alpha)\times\mathbb{R}^k \arrow[u, "\tilde{\phi}_\alpha^{-1}"] & \phi_\alpha(U_\alpha\times\mathbb{R}^k)
\end{tikzcd}$$

Definition: Section of a Bundle

A (smooth) section of $E\overset{\pi}{\to}M$ is a (smooth) map $\sigma : M\to E$ such that $\pi\circ\sigma = \mathrm{id}_M$.
$\Gamma(E) = \{$smooth sections of $E\overset{\pi}{\to}M\}$ and $\Gamma(E)$ is a $C^\infty(M)$-module.
The zero section $Z : M\to E$ is given by $p\mapsto 0_p\in E_p$.
If U has a local trivialization, $\Phi:\pi^{-1}(U)\to U\times\mathbb{R}^k$.

$$\begin{tikzcd}
\Phi: & \pi^{-1}(U) \arrow[rr] & & U\times\mathbb{R}^k & & (p,e_i) \arrow[ll, "\Phi^{-1}"'] \\
& & U \arrow[ul, dashed] \arrow[ur] & & p \arrow[ur, mapsto, "\tilde{e}_i"] &
\end{tikzcd}$$

Define $\sigma_i : U\to\pi^{-1}(U)$ by $\sigma_i = \Phi^{-1}\circ\tilde{e}_i$ gives a local section that is non-zero on U.
$\{\sigma_1,\dots,\sigma_n\}$ form a local frame on U (i.e. form a basis in E_p, $\forall\, p\in U$).

January 8, 2025

Recall

Last time we had a vector bundle $E \xrightarrow{\pi} M$ of rank k satisfying

1. $\pi^{-1}(p) = E_p$ has a (real) vector space structure of dimension k.

2. We have a local trivialization, $\forall\, p \in M$ there exists a neighborhood U and a diffeomorphism Φ

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^k \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd}$$

   and $\Phi|_{E_p} : E_p \to \{p\} \times \mathbb{R}^k$ is a linear isomorphism.

A section $\sigma : M \to E$ is a smooth map such that $\pi \circ \sigma = \mathrm{id}_M$.
We say that a collection of sections $\{\sigma_1, \dots, \sigma_k : U \to E\}$ is linearly independent if $\{\sigma_1(x), \dots, \sigma_k(x)\}$ is linearly independent for each $x \in U$. This is a (local) frame if it is a basis.
If $U \subseteq M$ admits a trivialization

$$\Phi : \quad \begin{tikzcd} \pi^{-1}(U) \arrow[rr] \arrow[dr] & & U \times \mathbb{R}^k \arrow[dl] \\ & U & \end{tikzcd}$$

then there is a local frame $\{\sigma_1, \dots, \sigma_k\}$ defined on U. Precisely, with $\tilde{e}_i(x) = (x, e_i)$, $\sigma_i = \Phi^{-1} \circ \tilde{e}_i$.

Proposition 10.19

If $U \subseteq M$ admits a local frame, then $\pi^{-1}(U)$ admits a local trivialization.

Remember

If $E \xrightarrow{\pi} M$ admits a global frame, then $E = \pi^{-1}(M)$ has a trivialization. In other words, E is diffeomorphic to a trivial vector bundle $M \times \mathbb{R}^k$.

Examples

Example 1

Mobius bundle over S^1.

IMAGE 1

To check whether it is a trivial bundle of S^1, it suffices to check whether there exists a nowhere zero (global) section. This cannot happen (by itermediate value theorem), hence it is not $S^1 \times \mathbb{R}$.

Example 2

TS^2 becasue there is no non-vanishing vector field over S^2, hence $TS^2 \neq S^2 \times \mathbb{R}^2$.

Example 3

Let G be a Lie group. Every $X \in T_eG(\cong \mathfrak{g})$ uniquely determines a (left-invariant) vector field $\tilde{X} \in \mathfrak{X}(G)$.
Starting with a basis $\{E_i\} \subseteq T_eG$ we get a global frame $\{\tilde{E}_i\}$ for TG. Hence TG is a trivial vector bundle $G \times \mathbb{R}^n$ ($n = \dim G$). In particular, $TS^1 = S^1 \times \mathbb{R}$, $TS^3 = S^3 \times \mathbb{R}^3$.

Proof of Proposition

Define $\Psi : (x, v^1, \ldots, v^k) \in U \times \mathbb{R}^k \to \pi^{-1}(U) \ni v_x$ where $v_x = v^i \sigma_i(x)$.
Ψ is a bijection. Note that $\Psi|_{E_x} : E_x \to \{x\} \times \mathbb{R}^k$is a linear isomorphism because $\{\sigma_i(x)\}$ is a basis.
Then to show that Ψ is a diffeomorphism, it suffices to show then that Ψ is a local diffeomorphism.
Let $x \in U$ and let V be a neighborhood of x such that $\pi^{-1}(V) \xrightarrow{\Phi} V \times \mathbb{R}^k$.

$$V \times \mathbb{R}^k \xrightarrow{\Psi|_{V\times\mathbb{R}^k}} \pi^{-1}(V) \xrightarrow{\Psi} V \times \mathbb{R}^k$$

We show that this composition is a diffeomorphism. Since $\Phi(\sigma_i(x)) = (x, \sigma_i^1(x), \ldots, \sigma_i^k(x))$

$$\begin{aligned}\Phi \circ \Psi|_{V\times\mathbb{R}^k}(x, v^1, \ldots, v^k) &= \Phi(v^i \sigma_i(x)) \\ &= (x, v^i \sigma_i^1(x), \ldots, v^i \sigma_i^k(x))\end{aligned}$$

Each $\sigma_i^j(x)$ is smooth. Hence $\Phi \circ \Psi|_{V\times\mathbb{R}^k}$ is smooth.
Let $\vec{v} = (v^1, \ldots, v^k)$ and $\sum(x) = (\sigma_i^j(x))$, then $\Phi \circ \Psi(x, \vec{v}) = (x, \vec{v} \cdot \sum(x))$. Its inverse

$$(\Phi \circ \Psi)^{-1}(x, \vec{w}) = \left(x, \vec{w} \cdot \sum(x)\right)$$

is also smooth. This shows that $\Phi \circ \Psi|_{V\times\mathbb{R}^k}$ is a diffeomorphism. Hence $\Psi|_{V\times\mathbb{R}^k}$ is a diffeomorphism ($V \subseteq U$) and $\Psi : U \times \mathbb{R}^k \to \pi^{-1}(U)$ is also a diffeomorphism.

Definition: Bundle Morphism

A bundle morphism between is a pair of smooth maps (f, F) such that this diagram commutes

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d, "\pi"] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "f"] & M' \end{tikzcd}$$

and $F|_{E_p} : E_p \to E'_{f(p)}$ is a linear map ($\forall p \in M$).
If it admits an inverse which is itself a bundle morphism, it is a unble isomorphism.
Remember that f is smooth because $f = \pi' \circ F \circ Z$

$$p \overset{Z}{\mapsto} 0_p \overset{F}{\mapsto} 0_{f(p)} \overset{\pi'}{\mapsto} f(p)$$

Remark

$$\begin{tikzcd} E \arrow[rr, "F"] \arrow[dr, "\pi"] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

commutes and $F|_{E_p} : E_p \to E'_p$ is linear ($\forall p$).

Remark

$\operatorname{rank}(F|_{E_p})$ may depend on $p \in M$.

$$\begin{tikzcd} TM \arrow[r, "Df"] \arrow[d, "\pi"] & TR \arrow[d, "\pi"] \\ M \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

e.g. $M = \mathbb{R}^2$, $E = E' = TR^2 (= \mathbb{R}^4)$, $F((u,v)_{(x,y)}) = (u, xv)$. For $x \neq 0$, $\operatorname{rank}(F|_{(x,y)}) = 2$ but for $x = 0$ $\operatorname{rank}(F|_{(0,y)}) = 1$.

Proposition 10.26

$$\begin{tikzcd} E \arrow[rr, "F"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

If F is a bijective, smooth bundle homomorphism, then it is a bundle isomorphism.
Proof left as an exercise. We need to show that F^{-1} is smooth.

Definition: Fiber Bundle

$F \to E \xrightarrow{\pi} M$ with fiber F such that $E_x = \pi^{-1}(x)$ is diffeomorphic to F. This diagram commutes.

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\pi_U"] \\ & U & \end{tikzcd}$$

Fact

If $N \xrightarrow{F} M$ is a submersion from compact manifolds, then F is a fiber bundle.

Chapter 11: Cotangent Bundles

Review: Linear Algebra

Suppose we have a real vector space V of dimension n. Then $V^* = \{f : V \to \mathbb{R} \text{ linear}\}$.

If V has a basis $\{E_1, \ldots, E_n\}$, then we may define the dual basis for V^* $\{\epsilon^1, \ldots, \epsilon^n\}$ by $\epsilon^j(E_i) = \delta_i^j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$.

Remember $V^{**} \cong V$ by $\xi : V \to V^{**}$ by $v \mapsto \xi(v) : V^* \to \mathbb{R}$ and $\omega \mapsto \omega(v)$.
Remember also that if A is a linear map $V \to W$ then we may define $A^* : \omega \in W^* \to V^* \ni A^*\omega$ by $v \in V \to \mathbb{R} \ni \omega(Av)$ (ie. $(A^*\omega)(v) = \omega(Av)$).

Definition: Cotangent Bundle

Let M^n be a smooth manifold, and let (U,ϕ) be a chart. Then T_pM has a basis

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \dots, \frac{\partial}{\partial x^n}\bigg|_p \right\}$$

for every $p \in U$. Take its dual basis

$$\left\{ \lambda^1|_p, \dots, \lambda^n|_p \right\}$$

for T_p^*M. The cotangent bundle $T^*M = \coprod_{p\in M} T_p^*M$.
Similar to the TM case, if $T^*M \xrightarrow{\pi} M$, then $\omega|_p \in \pi^{-1}(U) \xrightarrow{\Phi} U \times \mathbb{R}^n \ni (p, a_1, \dots, a_n)$ where a_i is given by $\omega|_p = a_i \lambda^i|_p$.
In other words, $a_i = \omega|_p \left(\frac{\partial}{\partial x^i}\Big|_p \right)$.

Computing Dual Transition

Suppose $(U, (x^1, \dots, x^n))$ and $(V, (y^1, \dots, y^n))$ are two charts $(W = U \cap V \neq \varnothing)$. Then $\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}$ gives a dual $\{\lambda^i|_p\}$ and $\left\{ \frac{\partial}{\partial y^i}\Big|_p \right\}$ gives $\{\mu^i|_p\}$.
Then, recall, $\frac{\partial}{\partial y^i}\Big|_p = \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j}\Big|_p$ and $x^j(y^1, \dots, y^n)$ is a j-component of $(y^1, \dots, y^n) \to M \to (x^1, \dots, x^n)$.
If $\omega \in T_p^*M$, $\omega = a_i \lambda^i|_p = b_j \mu^j|_p$

$$a_i = \omega|_p \left(\frac{\partial}{\partial x^i}\bigg|_p \right) = \omega_p \left(\frac{\partial y^j}{\partial x_i} \frac{\partial}{\partial y^j} \right) = \frac{\partial y^j}{\partial x^i} \omega \left(\frac{\partial}{\partial y^j} \right) = \frac{\partial y^j}{\partial x^i} b_j$$

In particular, $\mu^j = \omega$, then $a_i = \frac{\partial y^k}{\partial x^i} b_k = \frac{\partial y^j}{\partial x^i}$. Hence $\mu^j = \omega = a_i \lambda^i = \frac{\partial y^i}{\partial x^i} \lambda^i$.

Definition: Smooth Covector Field

A smooth covector field is a smooth section of T^*M, cal it $\Omega^1(M) = \Gamma(T^*M)$.
Given $f \in C^\infty(M)$, we can define a smooth covector field $df \in \Omega^1(M)$ by $df(v|_p) = (v_p)(f)$.
$df(X) = Xf$ is smooth if X and f are smooth.

Differential

Given a local chart $(U, (x^1, \dots, x^n))$ and a smooth function $f : U \to \mathbb{R}$, $df_p = a_i(p) \lambda^i|_p$.

$$\frac{\partial f}{\partial x^j} = df_p \left(\frac{\partial}{\partial x^j}\bigg|_p \right) = a_i(p) \lambda^i|_p \left(\frac{\partial}{\partial x^j}\bigg|_p \right) = a_i(p) \delta^i_j = a_j(p)$$

That is, $df_p = \frac{\partial f}{\partial x^j}(p) \lambda^j|_p$. In particular, if we consider the coordinate function $x^i : U \to \mathbb{R}$, then $dx^i|_p = \frac{\partial x^i}{\partial x^j}(p) \lambda^j|_p = \lambda^i|_p$ for each $p \in U$ (i.e. $dx^i = \lambda^i$ on U).
With this, we can write $df = \frac{\partial f}{\partial x^i} dx^i$ and $dy^j = \frac{\partial y^j}{\partial x^i} \partial x^i$.

Proposition 11.22

For $f \in C^{\infty}(M)$, then $df = 0$ if and only if f is constant on every compnent of M.

Proof

($\Longleftarrow$) is trivial.
($\Longrightarrow$) We assume M is connected. Fix $p \in M$, define $\mathcal{A} = \{q \in M \,:\, f(p) = f(q)\}$ is closed.
Now let $q \in A$ and U a local chart around q. Then $0 = df = \frac{\partial f}{\partial x^i} dx^i$ (i.e. $\frac{\partial f}{\partial x^i} \equiv 0$, $\forall\, i$).
Hence f is constant on U and $f(q) = f(p)$ for $U \in \mathcal{A}$.

Proposition 11.23

Take $\gamma : J \to M$ a smooth curve $f \in C^{\infty}(M)$. Then $(df|_{\gamma(t)})(\gamma'(t)) = (\gamma'(t))f = (f \circ \gamma)'(t)$.

IMAGE 2

Recall that if $v \in T_pM$ and $f \in C^{\infty}(M)$ then $vf = (f \circ \gamma)'(0)$ where $\gamma : (-\varepsilon, \varepsilon) \to M$, $\gamma(0) = p$ and $\gamma'(0) = v$ ($f \circ \gamma : \mathbb{R} \to \mathbb{R}$).

January 13, 2025

Recall

T^*M and $\Omega^{I}(M) = \Gamma(T^*M)$. Let $(U, (x^1, \ldots, x^n))$ be a chart. Then inside U, we may write $\omega = \omega_i dx^i$.
$\{dx^i|_p\}$ is a dual basis of $\left\{\frac{\partial}{\partial x^i} \subseteq T_pM\right\}$.
They are also $x^i : U \to \mathbb{R}$ coordinates functions where dx^i is the differential of x^i.
Given $f \in C^{\infty}(M)$ or $C^{\infty}(U)$, $df \in \Omega^{I}(M)$ or $\Omega^{I}(U)$ is defined by $df(X_p) = (Xf)(p)$.
Inside a chart, $df = \frac{\partial f}{\partial x^i} dx^i$.
We have a change of coordinates where $(U, (x^1, \ldots, x^n))$ and $(V, (y^1, \ldots, y^n))$ and $W = U \cap V \neq \varnothing$ gives $dy^j = \frac{\partial y^j}{\partial x^i} dx^i$.

Recall (Linear Algebra)

If $A : V \to W$ is a linear map with $w \in W^*$ and $v \in V$, then $A^* : W^* \to V^*$ is the dual map defined by $(A^* w)(v) := w(Av)$.

Dual of the Tangent Space

Let $F : M \to N$ be a smooth map between manifolds.

$$DF_p : T_pM \to T_{F(p)}N$$
$$(DF_p)^* : T^*_{F(p)}M \to T^*_p N$$

and $(DF_p^* \omega)(v) = \omega(DF_p(v))$ for $\omega \in T^*_{F(p)}N$ and $v \in T_pM$.

Definition: Pullback

Given $\omega \in \Omega^{I}(N)$, we can define $F^*\omega$, a section of T^*M, by $(F^*\omega)_p(v) = \omega(DF_p(v))$ or $(F^*\omega)_p = DF_p^*\omega$.
We call this the pullback of ω by F.

Recall that for $u \in C^\infty(N)$, $M \xrightarrow{F} N \xrightarrow{u} \mathbb{R}$. Then we can define $F^* u \in C^\infty(M)$ by $F^* u = u \circ F$.

Proposition

If $F : M \to N$ is smooth, $u \in C^\infty(N)$ and $\omega \in \Omega^I(N)$, then

1. $F^*(u\omega) = (F^* u)(F^* \omega)$.
2. $F^*(du) = d(F^* u)$.

Proof of 1

$\forall\, p \in M$, $\forall\, v \in T_p M$,

$$(F^*(u\omega))_p(v) = DF_p^*(u\omega)(v) = u_{F(p)}\omega_{F(p)}(DF_p(v)) = (u \circ F(p))\omega(DF_p(v)) = (F^* u)(F^* \omega)$$

Proof of 2

$$(F^*(du))(v) = du(DF_p(v)) = (DF_p(v))u = (du)_{F(p)} DF_p(v) = d(u \circ F)(v) = d(F^* u)(v)$$

Change of Coordinates

Locally, $F : M \to N$. Let $(U, (x^1, \ldots, x^n))$ be a chart around p and $(V, (y^1, \ldots, y^n))$ a chart around $F(p)$. For $\omega \in \Omega^I(N)$, in V $\omega = \omega_i dy^i$ and

$$F^* \omega = F^*(\omega_i dy^i) = (F^* \omega_i)(F^* dy^i) = (F^* \omega_i) d(F^* y^i) = (\omega_i \circ F)(dF^i)$$

where $F^i = y^i \circ F$ is the ith component of F.
When F is smooth and $\omega \in \Omega^I(N)$, then $F^* \omega \in \Omega^I(M)$. In fact, locally, $F^* \omega = (\omega_i \circ F) d(F^i)$. Hence $F^* \omega$ is smooth.

Example 1

Take $F : \mathbb{R}^3 \to \mathbb{R}^2$ by $(x, y, z) \mapsto (u(x, y, z), v(x, y, z)) = (x^2 y, y \sin(z))$.
Then $\omega = u\, dv + v\, du \in \Omega^I(\mathbb{R}^2)$. So

$$\begin{aligned}
F^* \omega &= F^*(u\, dv + v\, du) \\
&= (F^* u) d(F^* v) + (F^* v) d(F^* u) \\
&= x^2 y\, d(y \sin(z)) + (y \sin(z))\, d(x^2 y) \\
&= x^2 y(\sin(z)\, dy + y \cos(z)\, dz) + y \sin(z)(2xy\, dx + x^2\, dy)
\end{aligned}$$

Example 2

$M = \mathbb{R}^2 - \{0\}$ and $\gamma : [0, 2\pi] \to M$ by $t \mapsto (r \cos(t), r \sin(t))$ for $r > 0$. Take $\omega = \frac{x\, dy - y\, dx}{x^2 + y^2} \in \Omega^I(M)$

$$\begin{aligned}
\gamma^* \omega &= \frac{1}{r^2}(r \cos(t)\, d(r \sin(t)) - r \sin(t)\, d(r \cos(t)) \\
&= \cos(t)(\cos(t))\, dt - \sin(t)(\sin(t))\, dt \\
&= dt
\end{aligned}$$

Definition: Line Integral

If $\eta \in \Omega^{\prime}(\mathbb{R})$ or $\Omega^{\prime}(I)$ (where $I \subseteq \mathbb{R}$) is an interval), η can be written as $\eta(t) = f(t)\,dt$ and define

$$\int_I \eta = \int_a^b f(t)\,dt$$

Let $\gamma : [a,b] \to M$ be a smooth curve on M. Let $\omega \in \Omega^{\prime}(t)$. Define

$$\int_\gamma \omega = \int_a^b \gamma^* \omega$$

with $\gamma^*(\omega) \in \Omega^{\prime}([a,b])$.

Proposition 11.31

Take $\phi : I \to J$ a diffeomorphism between intervals with $\phi' > 0$. Then

$$\int_J \phi^* \omega = \int_{\phi(J)} \omega$$

Write s for coordinates on J and t for coordinates on I. Then $\omega = f(t)\,dt \in \Omega^1(I)$ and

$$\phi^*\omega = (\phi^* f)\,d(\phi^* t) = (f \circ \phi)\,d(t \circ \phi) = f(\phi(s))\,d(\phi(s)) = f(\phi(s))\phi'(s)\,ds$$

Then

$$\int_J \phi^*\omega = \int_J f(\phi(s))\phi'(s)\,ds \overset{t=\phi(s)}{=} \int_I f(t)\,dt = \int_I \omega$$

Proposition 11.37: Independence of Reparameterization

Suppose $\gamma : I \to M$ is a smooth curve and $\phi : J \to I$ is a diffeomorphism with $\phi' > 0$. Then $\tilde{\gamma} := \gamma \circ \phi : J \to M$ is a reparameterization of γ and

$$\int_\gamma \omega = \int_{\tilde{\gamma}} \omega$$

If $\phi' < 0$, then $\int_\gamma \omega = -\int_{\tilde{\gamma}} \omega$.

Proof

$$\int_\gamma \omega = \int_I \gamma^* \omega \int_J \phi^* \gamma^* \omega = \int_J (\gamma \circ \phi)^* \omega = \int_{\tilde{\gamma}} \omega$$

Example

Take $\gamma : [0, 2\pi] \to M = \mathbb{R}^2 - \{0\}$ by $t \mapsto (r\cos(t), r\sin(t))$ with $r > 0$. If $\omega = \frac{x\,dy - y\,dx}{x^2+y^2}$, then $\gamma^*\omega = dt$ and

$$\int_\gamma \omega = \int_0^{2\pi} \gamma^* \omega = \int_0^{2\pi} dt = 2\pi$$

Proposition 11.38

For $\gamma : I \to M$

$$\int_\gamma \omega = \int_I \omega_{\gamma(t)}(\gamma'(t))\, dt$$

Proof

In a local chart $(U, (x^1, \dots, x^n))$, we can write $\omega = \omega_i dx^i$. Then $\gamma(t) = (\gamma^1(t), \dots, \gamma^n(t))$ and

$$\begin{aligned}
\gamma^* \omega &= \gamma^*(\omega_i dx^i) \\
&= (\gamma^* \omega_i) d(\gamma^* x^i) \\
&= (\omega_i \circ \gamma) d\gamma^i \\
&= \omega_i(\gamma(t)) \frac{d\gamma^i}{dt} dt \\
&= \omega_i(\gamma(t)) \dot{\gamma}^i(t) dt
\end{aligned}$$

Since $\omega = \omega_i dx^i$ and $\dot{\gamma}(t) = (\dot{\gamma}^1(t), \dots, \dot{\gamma}^n(t)) = \dot{\gamma}^i(t) \frac{\partial}{\partial x^i}$, $\omega_{\gamma(t)}(\dot{\gamma}(t)) = \omega_i(\gamma(t))\dot{\gamma}^i(t)$ and

$$\omega_i(\gamma(t))\dot{\gamma}^i(t) dt = \omega_{\gamma(t)}(\dot{\gamma}(t)) dt$$

Hence $\int_\gamma \omega = \int_I \gamma^* \omega = \int_I \omega_{\gamma(t)}(\dot{\gamma}(t))\, dt$.

Corollary

Then, if $f : M \to \mathbb{R}$ is a smooth function,

$$\int_\gamma df = \int_I (df)_{\gamma(t)}(\dot{\gamma}(t))\, dt = \int_I (f \circ \gamma)'(t)\, dt = f(\gamma(b)) - f(\gamma(a))$$

Therefore $\int_\gamma df$ only depends on the value of f at the endpoints of γ.

Definition: Exact and Conservative Forms

Let $\omega \in \Omega^1(M)$. We say that ω is...

1. exact if there exists $f \in C^\infty(M)$ such that $\omega = df$.
2. conservative if $\int_C \omega = 0$ for any closed, piecewise-smooth curve in M

f is called the potential of ω.

Remark

If $\int_C \omega = 0$, we may write C as the concatenation of curves γ then $-\sigma$. Then

$$0 = \int_C \omega = \int_\gamma \omega + \int_{-\sigma} \omega = \int_\gamma \omega - \int_\sigma \omega$$

Remark

Exact implies conservative.

Theorem

If $\omega \in \Omega^1(M)$ is conservative, then it is exact.

Proof

Fix a bse point $p_0 \in M$.
We have that $\int_p^q \omega = \int_\gamma \omega$ is well-defined by the conservative assumption, and we define $f(p) = \int_{p_0}^p \omega$.
Let $q_0 \in M$ and let $(U, (x^1, \ldots, x^n))$ be a chart centered at q_0. Inside U, $\omega = \omega_i dx^i$ and $df = \frac{\partial f}{\partial x^i} dx^i$.
We need to show that $\frac{\partial f}{\partial x^i} = \omega_i$ for each i. Fix an index i and consider a curve $\sigma : (-\varepsilon, \varepsilon) \to U$ by $t \mapsto (0, \ldots, t, \ldots, 0)$.

IMAGE 1

Let $q_- = \sigma(-\varepsilon)$, then

$$f(q_0) = \int_{p_0}^{q} \omega = \int_{p_0}^{q_-} \omega + \int_{q_-}^{q} \omega =: \tilde{f}(q)$$

so $f(q_0) = \text{constant} + \tilde{f}(q)$. Hence $\frac{\partial f}{\partial x^j} = \frac{\partial \tilde{f}}{\partial x^j}$ in U. Therefore

$$\begin{aligned}
\tilde{f}(\sigma(s)) &= \int_{q_-}^{\sigma(s)} \omega \\
&= \int_{\sigma|_{[-\varepsilon, s]}} \omega \\
&= \int_{-\varepsilon}^{s} \omega_{\sigma(t)}(\dot{\sigma}(t))\, dt \\
&= \int_{-\varepsilon}^{s} \omega_{\sigma(t)}\left(\frac{\partial}{\partial x^i}\right) dt \\
&= \int_{-\varepsilon}^{s} \omega_i(\sigma(t))\, dt
\end{aligned}$$

and

$$\left.\frac{\partial f}{\partial x^i}\right|_{q_0} = \left.\frac{\partial \tilde{f}}{\partial x^i}\right|_{q_0} = (\tilde{f} \circ \sigma)'(0) = \left.\frac{d}{ds}\right|_{s=0} \left(\int_{-\varepsilon}^{s} \omega_i(\sigma(t))\, dt \right) = \omega_i(\sigma(0)) = \omega_i(q_0)$$

Remark

Take $\omega = df \in \Omega^1(M)$ which is $\omega_i dx^i$ locally or $\omega_i = \frac{\partial f}{\partial x^i}$ when exact.

$$\frac{\partial \omega_i}{\partial x^j} = \frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial \omega_j}{\partial x^i}$$

Note: $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$ does not, in general, imply $\omega = df$.

January 15, 2025

Recall

If $\omega \in \Omega^1(M)$ and $\gamma : \mathbb{R} \supseteq I \to M$ a (piecewise) smooth curve, then

$$\int_\gamma \omega = \int_I \gamma^* \omega$$

If df is the differential of a smooth function, then

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a))$$

only depends on endpoints. In particular, along a closed (piecewise) smooth curve

$$\int_C df = 0$$

We have that ω is exact if $\omega = df$ and conservative if $\int_C \omega = 0$ for every closed curve.
ω is exact if and only if it is also conservative.

Recall: Checking Exactness

Take $\omega \in \Omega^1(M)$,

$$\omega_i dx^i = \omega = df = \frac{\partial f}{\partial x^i} dx^i$$

Then

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2}{\partial x^j \partial x^i}$$

That is, $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$.

Definition: Closed 1-Form

We say $\omega \in \Omega^1(M)$ is closed if in every chart $(U, (x^i))$, $\omega = \omega_i dx^i$ satisfies $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$.
Exact implies closed, however the converse is not true in general.

Example

$\exists \omega \in \Omega^1(\mathbb{R}^2 - \{0\})$ such that ω is closed but $\int_C \omega = 2\pi$.

Corollary 11.50

If $\omega \in \Omega^1(M)$ is closed, then $\forall p \in M$ there exists a chart U at p such that $\omega_U = df$ for some $f \in C^\infty(U)$

Proposition 11.45

For $\omega \in \Omega^1(M)$, the following are equivalent

1. ω is closed.
2. ω satisfies $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$ in some chart at every point.
3. For every open $U \subseteq M$ and $X, Y \in \mathfrak{X}(U)$, it holds that

$$X(\omega(Y)) - Y(\omega(X)) = \omega([X, Y])$$

The proof that 1 implies 2 is trivial.

Proof 3 Implies 1

Pick U as a chart, $X = \frac{\partial}{\partial x^i}$, and $Y = \frac{\partial}{\partial x^j}$. Then, since $\omega = \omega_i dx^i$,

$$X(\omega(Y)) = X(\omega_j) = \frac{\partial w_j}{\partial x^i}$$

Similarly, $Y(\omega(X)) = \frac{\partial \omega_i}{\partial x^j}$. Then $[X, Y] = \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$ and

$$\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} = 0$$

Proof 2 Implies 3

Fix any $p \in U$. We have a chart $(V, (x^i))$ at p such that $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$. Then

$$X(\omega(y)) = X\left((\omega_i dx^i)\left(Y^j \frac{\partial}{\partial x^j}\right)\right) = X(\omega_i Y^i) = (X\omega_i)Y^i + \omega_i(XY^i) = x^j \frac{\partial w_i}{\partial x^j} Y^i + \omega_i(XY^i) = X^j Y^i \frac{\partial \omega_j}{\partial x^i}$$

Similarly,

$$Y(\omega(X)) = Y(\omega_i x^i) = Y^j \frac{\partial \omega_i}{\partial x^j} x^i - \omega_i(YX^i) = X^i Y^j \frac{\partial \omega_i}{\partial x^j}$$

which is equivalent under a change of indicies. Hence

$$X(\omega(Y)) - Y(\omega(X)) = \omega_i(XY^i) - \omega_i(YX^i) = \omega_i(XY^i - YX^i) = \omega([X, Y])$$

Lemma

Suppose $F : M \to N$ is a local diffeomorphism. Then $F^* : \Omega^1(N) \to \Omega^1(M)$ sends exact (or closed) 1-forms to exact (or closed) ones.

Proof of Exact

If $\omega = df \in \Omega^1(N)$, then $F^*\omega = F^*(df) = d(F^* f)$ is exact on M.

Proof of Closed

If $\omega \in \Omega^1(N)$ is closed, then $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$ in every chart of N.
For any $p \in M$, we consider a chart at p by $(V, \phi \circ F)$

IMAGE 1

Therefore $\phi \circ F \circ (\phi \circ F)^{-1} = \mathrm{id}$ and $F^* = \mathrm{id}$ so $F^* \omega$ is closed.

Poincaré Lemma

Let $\omega \in \Omega^1(M)$ be closed. Fix $p \in M$, and let (U, ϕ) be a chart at p such that $\phi(U) = B_1(0) \subseteq \mathbb{R}^n$.

IMAGE 2

Assuming the above, every closed 1-form on $B_1(0)$ is exact. $(\phi^{-1})^*(\omega|_U) = df$ for some $f \in C^\infty(B_1(0))$ where $\omega|_U = \phi^*(df) = d(\phi^* f) \in C^\infty(U)$

Definition: Star-Shaped Domain

We say that $U \subseteq \mathbb{R}^n$ open is star-shaped with a center $c \in U$ (wlog $c = 0$) if for any $x \in U$, the segment γ_x from c to x is contained in U.

IMAGE 3

If $x = (x^i)$, then $\gamma_x(t) = (tx^i)$.

Theorem 11.49 (Poincaré Lemma)

If $U \subseteq \mathbb{R}^n$ is star-shaped, then every closed 1-form is exact.

Recall

If ω is an exact 1-form, then $f(q) = \int_{p_0}^{p} \omega$ is a potential.
We also have that $\int_\gamma \omega = \int_I \omega_{\gamma(t)}(\dot{\gamma}(t))\, dt$.

Proof

Let $\omega \in \Omega^1(U)$ be a closed 1-form.
We need to construct $f \in C^\infty(U)$ such that $df = \omega$. That is, for all i, $\frac{\partial f}{\partial x^i} = \omega^i$. Define

$$f(x) = \int_{\gamma_x} \omega = \int_0^1 \omega_{\gamma_x(t)}(\dot{\gamma}_x(t))\, dt = \int_0^1 \omega_i|_{\gamma_x(t)} dx^i(x^1, \dots, x^n)\, dt = \int_0^1 \omega_i|_{tx} x^i\, dt$$

Since everything is smooth,

$$\begin{aligned}
\frac{\partial f}{\partial x^j}(x) &= \int_0^1 \frac{\partial}{\partial x^j}(\omega_i(tx)\cdot x^i)\,dt \\
&= \int_0^1 \frac{\partial \omega_i(tx)}{\partial x^j}\cdot x^i + \omega_i(tx)\frac{\partial x^i}{\partial x^j}\,dt \\
&= \int_0^1 \left(\frac{\partial w_i}{\partial x^j}\right)\Big|_{(tx)} tx^i + \omega_j(tx)\,dt \\
&= \int_0^1 \frac{\partial \omega_j}{\partial x^i}\Big|_{tx} tx^i + \omega_j(tx)\,dt \\
&= \int_0^1 \frac{d}{dt}\big(t\omega_j(tx)\big)\,dt \\
&= t\omega_j(tx)\big|_0^1 \\
&= \omega_j(x)
\end{aligned}$$

Tensors: Multilinear Maps

All vector spaces will be finite dimensional in our consideration.

$$F: V_1\times\cdots\times V_k \to W$$

linear in every component. Denote $L(V_1,\dots,V_k;W)$ to be the set of all such multilinear maps.
Given $\omega\in L(V_1;\mathbb{R}) = V_1^*$ and $\eta\in V_2^*$, we can define $\omega\otimes\eta\in L(V_1,V_2;\mathbb{R})$ by $\omega\otimes\eta(v_1,v_2)=\omega(v_1)\cdot\eta(v_2)$.

- Remark

  $(2\omega)\otimes\eta = \omega\otimes(2\eta)$. We assume $\otimes_{\mathbb{R}}$.

  Similarly, given $\omega_i\in V_i^*$, we can define $\omega_1\otimes\cdots\otimes\omega_k\in L(V_1,\dots,V_K;\mathbb{R})$.

Proposition

Let V_j with dimension n_j $(j=1,\dots,k)$. Each V_j has a basis $\{E_1^{(j)},\dots,E_{n_j}^{(j)}\}$.
Its dual basis $\{\varepsilon_{(j)}^1,\dots,\varepsilon_{(j)}^{n_j}\}\subseteq V_j^*$. Then $L(V_1,\dots,V_k;\mathbb{R})$ has a basis

$$\mathcal{B} = \left\{\varepsilon_{(1)}^{i_1}\otimes\cdots\otimes\varepsilon_{(k)}^{i_k} \,:\, 1\le i_j\le n_j\right\}$$

Proof

For a multi-index $I=(i_1,\dots,i_k)$ with $i\le i_j\le n_j$, we write $\varepsilon^I = \varepsilon_{(1)}^{i_1}\otimes\cdots\otimes\varepsilon_{(k)}^{i_k}$.
For any $F\in L(V_1,\dots,V_k;\mathbb{R})$, define $F_I = F(E_{i_1}^{(1)},\dots,E_{i_k}^{(k)})$. We claim that $F=F_I\varepsilon^I$.
In fact, for $(v_1,\dots,v_k)\in V_1\times\cdots\times V_k$, $v_j = v_j^iE_i^{(j)}$. We may check that $F(v_1,\dots,v_k)=F_I\varepsilon^I(v_1,\dots,v_k)$.
Therefore $\mathcal{B}$ spans $L(V_1,\dots,V_k;\mathbb{R})$.
Then, if $F_I\varepsilon^I=0$, then applying it to $(E_{i_1}^{(1)},\dots E_{i_k}^{(k)})$ gives $F_I=0$. Therefore $\mathcal{B}$ is linearly independent.
In particular, $\dim\, L(V_1,\dots,V_k;\mathbb{R}) = \prod_{j=1}^k n_j = \prod_{j=1}^k \dim\, V_j$.

Definition: Formal Linear Combination

Let S be a set. Define

$$\mathcal{F}(S) = \left\{ \sum_{i=1}^{m} a_i s_i \;:\; a_i \in \mathbb{R},\; s_i \in S \right\}$$

This is the free (real) vector space on S containing formal linear combinations of elements of S.
Define $V_1 \otimes \cdots \otimes V_k = \mathcal{F}(V_1 \times \cdots \times V_k)/R$ where R is generated by

$$\begin{aligned}
(v_1, \ldots, v_j + v_j', \ldots, v_k) &\sim (v_1, \ldots, v_j, \ldots, v_k) + (v_1, \ldots, v_j', \ldots, v_k) \\
(v_1, \ldots, cv_j, \ldots, v_k) &\sim c(v_1, \ldots, v_k)
\end{aligned}$$

In other words, in the quotient $v_1 \otimes \cdots \otimes v_k = \prod(V_1, \ldots, v_k)$.

Proposition

$V_1 \otimes \cdots \otimes V_k$ has a basis $\left\{ E_{i_1}^{(1)} \otimes \cdots \otimes E_{i_k}^{(k)} \;:\; 1 \leq i_j \leq n_j \right\}$.

Proposition

There exists a canonical isomorphism $(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3)$ by sending $(v_1 \otimes v_2) \otimes v_3 \mapsto v_1 \otimes (v_2 \otimes v_3)$.

Proposition

$L(V_1, \ldots, V_k; \mathbb{R}) \cong V_1^* \otimes \cdots \otimes V_k^*$.

Proof Sketch

Define $\Phi : V_1^* \times \cdots \times V_k^* \to L(V_1, \ldots, V_k; \mathbb{R})$ by $(\omega^1, \ldots, \omega^k) \mapsto \omega^1 \otimes \cdots \otimes \omega^k$. By multilinear, this induces an isomorphism

$$\Phi : V_1^* \otimes \cdots \otimes V_k^* \cong L(V_1, \ldots, V_k; \mathbb{R})$$

Recall

$V^{**} \cong V$ for finite dimensional vector spaces, so $V_1 \otimes \cdots \otimes V_k = L(V_1^*, \ldots V_k^*; \mathbb{R})$.

Definition: Tensor

A tensor of (k, l)-type is an element in $\underbrace{V \otimes \cdots \otimes V}_{k} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{l}$.

The collection of such elements in $T^{(k,l)}V$. Most of the time we consider $T^{(0,l)}V$.

Examples

A vector in V is a $(1, 0)$-tensor.
A covector in V^* is a $(0, 1)$-tensor.
A linear map $A \in L(V)$ is a $(1, 1)$-tensor.
An inner product is a $(0, 2)$-tensor.

Symmetric Tensor

We say that $\alpha \in T^{(0,l)}V$ is symmetric if $\alpha(\dots, v_i, \dots, v_j \dots) = \alpha(\dots, v_j, \dots, v_i, \dots)$.

Alternating Tensor

We say that $\alpha \in T^{(0,l)}V$ is alternating if $\alpha(\dots, v_i, \dots, v_j \dots) = -\alpha(\dots, v_j, \dots, v_i, \dots)$.

January 22, 2024

Alternating/Symmetric Tensors

Let $\sigma \in S_l$ and $\alpha \in T^{(0,l)}V$.
Define σ_α or $(\sigma \cdot \alpha)$ as a new $(0,l)$-tensor by $(\sigma \cdot \alpha)(v_1, \dots, v_l) := \alpha(v_{\sigma(1)}, \dots, v_{\sigma(l)})$.
Then α is symmetric if and only if $\sigma \cdot \alpha = \alpha$.
α is alternating if and only if $\sigma \cdot \alpha = (\operatorname{sign} \sigma) \cdot \alpha$.
Define $\operatorname{Sym}: T^{(0,l)}V \to S^l V$ by

$$\operatorname{Sym}(\alpha) = \frac{1}{l!} \sum_{\sigma \in S^l} (\sigma \cdot \alpha)$$

Then $\operatorname{Sym}(\alpha)$ is symmetric for all $\tau \in S^l$.
Define $\operatorname{Alt}: T^{(0,l)}V \to \Lambda^l V$, the set of alternating (anti)-tensors by

$$\operatorname{Alt}(\alpha) = \frac{1}{l!} \sum_{\sigma \in S^l} (\operatorname{sign} \sigma)(\sigma \cdot \alpha)$$

Definition: Tensor Bundles

Recall that $T_pM \rightsquigarrow TM = \coprod_{p \in M} T_pM$ and $T_p^*M \rightsquigarrow T^*M$.
Then $T^{(k,l)}T_pM \rightsquigarrow T^{(k,l)}TM = \coprod_{p \in M} T^{(k,l)}T_pM$ a tensor bundle.
Mostly, we will consider $T^{(0,l)}TM$.
Inside a chart $(U, (x^1, \dots, x^n))$, $T^{(k,l)}TM$ has a local frame

$$\left\{ \frac{\partial}{\partial x^{i1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{ik}} \otimes dx^{j1} \otimes \cdots \otimes dx^{jl} \right\}$$

Definition: Smooth Tensor Field

A smooth tensor field of type (k,l) is a smooth section of $T^{(k,l)}TM$.
To check that a (o,l)-tensor field A is smooth, we can do either of the following

1. Write A in a local chart, then $A = A_I dx^I$ where A_I are functions in U and $dx^I = dx^{i1} \otimes dx^{il}$ with $I = (i1, \dots, il)$. Then A is smooth if and only if A_I is smooth for all I.

2. Check A testing on any l many smooth vector fields results in a smooth function.

Remark

Every $(0,l)$-tensor field A defines a map

$$\mathcal{A} = \underbrace{\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)}_{l} \to C^\infty(M)$$

by $A(x_1,\dots,X_l)(p) = A_p(X_1(p),\dots,X_l(p))$. This map $\mathcal{A}$ is $C^\infty(M)$-multilinear.

Lemma 12.24

Every $C^\infty(M)$-multilinear map $\mathcal{A} : \mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M) \to \mathbb{C}^\infty(M)$ defines a smooth $(0,l)$-tensor field

$$A_p(v_1,\dots,v_l) = (\mathcal{A}(X_1,\dots,X_l))(p)$$

Example

Given $\omega \in \Omega^1(M)$, define $\mathcal{A} : \mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M) \to \mathbb{C}^\infty(M)$ by $(X,Y) \mapsto \omega(L_X Y)$.
If X, Y and X', Y' only agree at a point p, then in general $(L_X Y)(p) \neq (L_{X'} Y')(p)$.

Proof

$\mathcal{A}$ acts locally only depending on the value of $X_1,\dots,X_l$ in a neighborhood of p, call it U.
It suffices to show that if $X_i = 0$ for some i on U, then $\mathcal{A}(X_1,\dots,X_l)(p) = 0$.
Let ψ be a bump function with $\operatorname{supp}\psi \subseteq U$ and $\psi(p) = 1$. Let also $V \subseteq U$ such that $\overline{V} \subseteq U$.
Then $\psi X_i \equiv 0$ on M. Then

$$0 = \mathcal{A}(X_1,\dots,\psi X_i,\dots,X_l)(p) = \psi(p)A(X_1,\dots,X_l)(p) = \mathcal{A}(X_1,\dots,X_l)(p)$$

Now $\mathcal{A}$ acts pointwisely. Write $X_i = a_i^j \frac{\partial}{\partial x^j}$ in U.
Extend each $\left.\frac{\partial}{\partial x^j}\right|_V$ to $E_j \in \mathfrak{X}(M)$ and each $a_i^j|_V$ to $f_i^j \in C^\infty(M)$.
Then inside V,

$$\mathcal{A}(X_1,\dots,X_l)(p) = \mathcal{A}(X_1,\dots,f_i^j E_j,\dots,X_l)(p) = f_i^j(p)\mathcal{A}(X_1,\dots,X_l)(p)$$

Now let $v_1,\dots,v_l \in T_pM$. Define A a $(0,l)$-tensor field by $A_p(v_1,\dots,v_l) = \mathcal{A}(X_1,\dots,X_l)$ where $X_i \in \mathfrak{X}(M)$ extends v_i.
By assumption, $A(X_1,\dots,X_l)$ is a smooth function if $X_1,\dots,X_l \in \mathfrak{X}(M)$ hence A is a smooth $(0,l)$-tensor field.

Definition:

Write $\mathcal{T}^{(0,l)}M = \Gamma(T^{(0,l)}TM)$ where Γ is the section.
Then for $F : M \to N$ a smooth map and $A \in \mathcal{T}^{(0,l)}N$, for $v_i \in T_pM$ define $F^*A \in \mathcal{T}^{(0,l)}M$ by

$$(F^*A)_p(v_1,\dots,v_l) := A_{F(p)}(DF_p(v_1),\dots,DF_p(v_l))$$

Lie Derivatives

Recall that if $X, Y \in \mathfrak{X}(M)$, we define $(L_X Y)_p$ where X generates a flow $\phi_t : M \to N$

IMAGE 1

$(\phi_{-t})_* Y_{\phi_t(p)} = ((\phi_{-t})_* Y)_p \in T_pM$ for $Y_p \in T_pM$. Then $L_X Y = \frac{d}{dt}\Big|_{t=0}((\phi_{-t})_* Y)_p$.
If $A \in \mathcal{T}^{(0,l)}M$,

IMAGE 2

$$(\phi_t^* A)_p = (\phi_t)^* (A_{\phi_t(p)} \in T^{(0,l)} T_pM$$

So $L_V A = \frac{d}{dt}\Big|_{t=0}(\phi_t^* A)_p$.

Properties

1. $L_V f = Vf$ (where $f \in C^\infty(M)$ can be thought of as a smooth $(0,0)$-tensor field). Then

$$(L_v f)(p) = \frac{d}{dt}\Big|_{t=0}(\phi_t^* f)_p = \frac{d}{dt}\Big|_{t=0}(f \circ \phi_t(p)) = (Vf)_p$$

1. $L_V(fA) = (Vf)A + fL_V A$.
2. $L_V(A \otimes B) = (L_V A) \otimes B + A \otimes (L_V B)$.
3. $L_V(A(X_1, \dots, X_l)) = (L_V A)(X_1, \dots, X_l) + A(L_V X_1, \dots, X_l) + \dots + A(X_1, \dots, L_V X_l)$ for $A \in \mathcal{T}^{(o,l)}M$ and $X_i \in \mathfrak{X}(M)$.

Proof of 2

We have $O := \{p \in M : V_p \neq 0\}$ open in M and $\operatorname{supp} V = \overline{\{p \in M : V_p \neq 0\}}$.

1. (2) holds on O.

   Recall that if $V_p \neq 0$, then there exists a local chart $(U, (x^i))$ centered at p such that on U, $V = \frac{\partial}{\partial x^1}$. In particular, its flow ϕ_t is $(x^1, \dots, x^n) \mapsto (x^1 + t, x^2, \dots, x^n)$.
   Then take some chart $U \subseteq O$ centered at p such that $V = \frac{\partial}{\partial x^1}$ in U. Inside U, write $A = A_I dx^I$, and

$$\begin{aligned}
\phi_t^*(fA) &= (\phi_t^* f)(\phi_t^* f)(\phi_t^* A) \\
&= (f \circ \phi_t)\phi_t^*(A_I dx^I) \\
&= f(x^1 + t, x^2, \dots, x^n) A_I(x^1 + t, \dots, x^n)\phi_t^* dx^I \\
&= f(x^1 + t, x^2, \dots, x^n) A_I(x^1 + t, \dots, x^n) dx^I
\end{aligned}$$

2. (2) holds on $\operatorname{supp} V$ by taking limits.
3. (2) holds outside $\operatorname{supp} V$, since $V \equiv 0$ on open $M \setminus \operatorname{supp} V$ and hence $\phi_t \equiv \mathrm{id}$. So both sides are identically zero.

January 27, 2025

Recall: Prop 12.32(2)

$$L_V(fA) = (Vf)A + fL_V A$$

Proof Step 1:

Show that hte equality holds on $\{p \in M : V(p) \neq 0\}$.

Let $p \in M$ with $V(p) \neq 0$.
Take any chart (U, x^i) centered at p such that $V = \frac{\partial}{\partial x^i}$ on U. Then its flow is

$$\theta_t : (x^1, \ldots, x^n) \mapsto (x^1 + t, x^2, \ldots, x^n)$$

in U. In U, we write $A = A_I dx^I$ (where $dx^I = dx^{i1} \otimes \cdots \otimes dx^{il}$). Recall that

$$\theta_t^*(dx^i) = d(\theta_t^* x^i) = d(x^i \theta_t) = \begin{cases} d(x^1 + t) = dx^1 & i = 1 \\ d(x^i) & i \neq 1 \end{cases}$$

Write the pullback of θ_t

$$\begin{aligned} \theta_t^*(fA) &= (\theta_t^* f)(\theta_t^* A_I dx^I) \\ &= (f \circ \theta_t)(A_I \circ \theta_t)(dx^I) \\ &= f(x^1 + t, x^2, \ldots, x^n) A_I(x^1 + t, \ldots, x^n) dx^I \end{aligned}$$

So for $p = (x^i)$,

$$\begin{aligned} (L_V(fA))_p &= \frac{d}{dt}\Big|_{t=0} f(x^1 + t, x^2, \ldots, x^n) A_I(x^1 + t, \ldots, x^n) dx^I \\ &= \underbrace{\frac{\partial f}{\partial x^1}(x^1, \ldots, x^n}_{Vf} \underbrace{A_I(x^1, \ldots, x^n) dX^I}_{\theta_t^* A} + f(x^1, \ldots, x^n) \frac{\partial A_I}{\partial x^1(x^1, \ldots, x^n) dx^I} \end{aligned}$$

inside U. Hence $Vf = \frac{\partial f}{\partial x^1}$.

Corollary

$L_V(df) = d(L_v f)$ for $f \in C^\infty(M)$.

- Proof

For all $X \in \mathfrak{X}(M)$,

$$(L_V(df))(X) = V(df(X)) - df(L_V X) = VXf - [V, X]f = VXf - (VXf - XVf) = XVf$$

and

$$(d(L_V f))(X) = X(L_V f) = XVf.$$

Proof Step 2:

Show that the equality holds on $\overline{\{p \in M : V(p) \neq 0\}}$.

Proof Step 3:

Show that the equality holds elsewhere.

Recall: Invariance

For two vector fields, X and Y, Y is invariant under the flow of X if $L_X Y \equiv 0$.
We say a $(0,l)$-tensor field A is invariant under a map $F: M \to M$ if $F^* A = A$. Equivalently, if under a flow $\theta_t : M \to M$ if $\theta_t^* A = A$ for all t.

Theorem 12.37

A is invariant under θ_t, $\forall\, t$, if and only if $L_V A = 0$.

Note

$$\frac{d}{dt}\bigg|_{t=t_0} (\theta_t^* A)_p = (\theta_{t_0}^*(L_v A))_p = \theta_{t_0})^* (L_V A)_{\theta_{t_0}^*(p)}$$

So

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=t_0} (\theta_t^* A)_p &= \frac{d}{dt}\bigg|_{t=t_0} (\theta_t^*) A_{\theta_t(p)} \\
&\overset{t=s+t_0}{=} \frac{d}{ds}\bigg|_{s=0} \theta_{s+t}^* A_{\theta_{s+t_0}(p)} \\
&= \frac{d}{ds}\bigg|_{s=0} \theta_{t_0}^* \circ \theta_s^* A_{\theta_{t_0}(\theta_s(p))} \\
&= \theta_{t_0}^* (L_V A)_{\theta_{t_0}^*(p)}
\end{aligned}$$

Therefore, if A is invariant under θ_t, then $\theta_t^* = A$ and

$$L_V A = \frac{d}{dt}\bigg|_{t=0} (\theta_t^* A)_p = \frac{d}{dt}\bigg|_{t=0} A_p = 0.$$

In the other direction, if $L_V A \equiv 0$, we show that $(\theta_t^* A)_p = A_p$ for every p and each t. From above,

$$\frac{d}{dt}\bigg|_{t=t_0} (\theta_t^* A)_p = \theta_{t_0}^* \underbrace{(L_V A)_{\theta_{t_0}(p)}}_{=0} = 0$$

Hence $(\theta_t^* A)_p$ is a constant A_p.

Special Tensors (for this course)

Riemannian Metric

g a $(0,2)$-tensor, symmetric and positive definite. That is, at each point p

$$g_p : T_p M \times T_p M \to \mathbb{R}$$

which is bilinear, symmetric and positive definite. This is an inner product.

K (Differential) Form

ω a $(0, k)$-tensor, alternating.

Riemannian Metric

In a chart $(U, (x^i))$, $g = g_{ij} dx^i \otimes dx^j$.
Since it is symmetric, $g(\partial_i, \partial_j) = g(\partial_j, \partial_i)$ (i.e. $g_{ij} = g_{ji}$). We write $dx^i dx^j = \mathrm{Sym}(dx^i \otimes dx^j)$. In this case

$$\mathrm{Sym}(dx^i \otimes dx^j) = \frac{1}{2}\left(dx^i \otimes dx^j + dx^j \otimes dx^i\right)$$

So we may write $g = g_{ij} dx^i dx^j$ and, sometimes, $(dx^1)^2 = dx^1 dx^1$.
We have also that g_{ij} correspinds to a positive definite, symmetric $n \times n$ matrix.

Example

In $\mathbb{R}^n$, $g_E = \delta_{ij} dx^i dx^j$. For $v = v^k \partial_k$ and $w = w^l \partial_l$,

$$g_E(v, w) = \delta_{ij} dx^i dx^j (v^k \partial_k w^l \partial_l) = v^k w^l \delta_{ij} \underbrace{dx^i(\partial_k)}_{\delta^i_k} \underbrace{dx^j(\partial_l)}_{\delta^i_l} = v^1 w^1 + \cdots + v^n w^n$$

Example

Consider $S^2 \subseteq \mathbb{R}^3$ embedded such that $T_p S^2 \hookrightarrow T_p \mathbb{R}^3 \cong \mathbb{R}^3$.
Then $g_p(v, w) = v \cdot w$ defines a Riemannian metric on S^2.

Proposition

Any smooth manifold admits a Riemannian metric.

Proof 1

Embed M into $\mathbb{R}^N$ with N sufficiently large. Then M is an embedded submanifold in $\mathbb{R}^N$ which induces a Riemannian metric on M.

Proof 2

Let $\{U_i\}$ be a countable cover of M (with each U_i a chart) and $\{\psi_i\}$ be a partition of unity with respect to this cover.

IMAGE 1

So $\phi_i^* g_E$ defines a Riemannian metric on U_i and we construct $\sum_i \psi_i (\phi_i^* g_E)$.

Example: Metric Product

Take (M_1, g_1) and (M_2, g_2) and construct $g_1 \oplus g_2$ on $M_1 \times M_2$ by either

$$g_1 \oplus g_2 = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$$

or

$$(g_1 + g_2)((v_1, v_1), (w_1, w_2)) = g_1(v_1, w_1) + g_2(v_2, w_2)$$

e.g. $S^1 \subseteq \mathbb{R}^2$ gives (S^1, g_1), then on the n-torus we construct $(\mathbb{T}^n, g_1 \oplus \cdots \oplus g_1)$.

Example: Warped Product

IMAGE 2

Take $f : M \to \mathbb{R}^+$ smooth, (M, g) and (N, h).
Define a new metric $\tilde{g}$ on $M \times N$ by

$$\tilde{g}_{(x,y)} = g_x + f(x)h_y$$

An example in polar coordinates is

$$(dx)^2 + (dy)^2 = (d(r\cos\theta))^2 + (d(r\sin\theta))^2 = (\cos\theta\, dr - r\sin\theta\, d\theta)^2 + (\sin\theta\, dr + r\cos\theta\, d\theta)^2 = dr^2 + r^2\, d\theta^2$$

Imagine fixing a direction r and at each point attaching a circle of radius r.

IMAGE 3

Recall: Gradient

If $f \in C^\infty(\mathbb{R}^n)$, then

$$\nabla f = \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}$$

Note that this violates our Einstein summation.
If $f \in C^\infty(M)$, its differential df is a 1-form and not a vector field. Why? Because in $\mathbb{R}^n$ we are implicitly using the Euclidean metric.
If we have an inner product on a TVS, say $(V, (\cdot, \cdot))$, then we can construct an isomorphism $V \cong V^*$ by $v \mapsto (v, \cdot)$.
On (M, g) we use g to construct a bundle isomorphism between TM and T^*M by $(p, v) \mapsto g_p(v, \cdot)$.
With this, given $df \in \Omega^1(M)$, we can define a vector field $\nabla f \in \mathfrak{X}(M)$ by

$$g(\nabla f, X) = (df)(X) = Xf$$

In a chart $(U, (x^i))$, set $\nabla f = b^i \frac{\partial}{\partial x^i}$. Then

$$g\left(\nabla f, \frac{\partial}{\partial x^j}\right) = g\left(b^i \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = b^i g_{ij} = (df)\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^j}$$

Let g^{ij} be the inverse of g_{ij}, then

$$b^k = b^i \delta_i^k = b^i g_{ij} g^{jk} = \frac{\partial f}{\partial x^j} g$$

so

$$\nabla f = b^k \frac{\partial}{\partial x^k} = \frac{\partial f}{\partial x^j} g^{jk} \frac{\partial}{\partial^k}$$

Then from above, we actually have

$$\nabla f = \frac{\partial f}{\partial x^i} \delta_{ij} \frac{\partial}{\partial x^j}$$

which satisfies our summation convention.

Example

If $g_E = dr^2 + r^2\, d\theta^2$ in polar coordinates,

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \quad \text{and} \quad (g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

So

$$\nabla f = \frac{\partial f}{\partial x^j} g^{jk} \frac{\partial}{\partial x^k} = \frac{\partial f}{\partial r} \frac{\partial}{\partial r} + \frac{\partial f}{\partial \theta} \frac{1}{r^2} \frac{\partial}{\partial \theta}$$

Isometric Metrics

We say that (M, g) and (N, h) are isometric if there is a diffeomorphism $F : M \to N$ such that $F^* h = g$.
With g, we can define (for $v \in T_p M$), $||v||_g = (g_p(v, v))^{1/2}$ and (for $v, w \in T_p M$)

$$\cos(v, w) = \frac{g_p(v, w)}{||w||_g ||w||_g}$$

Definition: Length

Let $\gamma : I \to M$ be a (piecewise) smooth curve.
Define $\mathrm{length}_g(\gamma) = \int_I ||\gamma'(t)||_g \, dt$.
Remember that $\mathrm{length}_g(\gamma)$ is independent of reparameterization. That is

$J \xrightarrow{\phi} I \xrightarrow{\gamma} M$ with $\tilde{\gamma} = \gamma \circ \phi$ we have

$$\begin{aligned} \int_J ||\tilde{\gamma}'(t)||\, dt &= \int_J ||(\gamma \circ \phi)'(t)||\, dt \\ &= \int_J ||\gamma'(\phi(t)) \cdot \phi'(t)||\, dt \\ &\overset{\phi' > 0}{=} \int_J ||\gamma'(\phi(t))|| \phi'(t)\, dt \\ &\overset{s = \phi(t)}{=} \int_I ||\gamma'(S)||\, ds \end{aligned}$$

Definition: Distance

Given (M, g), define

$$d_g(p, q) = \inf\left\{\text{length}_g(\gamma) : \gamma \text{ is piecewise smooth from } p \text{ to } q\right\}$$

Theorem

(M, d_g) is a metric space.
Moreover, it induces a metric topology that coincides with the manifold topology.

Theorem: Hopf-Rinow

The following are equivalent.

1. (M, d_g) is a complete metric space.
2. $\forall\, p, q \in M$, there exists a length-minimizing curve (a geodesic) from p to q.

Definition: Geodesic

A curve such that the second derivative along $\gamma \equiv 0$.

February 3, 2025

Recall: Wedge Product

$$\bigwedge^k V^* \times \bigwedge^l V^* \to \bigwedge^{k+l} V^*$$
$$(\omega, \eta) \mapsto \omega \wedge \eta$$

By $\frac{(k+l)!}{k!l!}\text{Alt}(\omega \otimes \eta) = \frac{1}{k!l!}\sum_{\sigma \in S_{k+l}}(\sigma \cdot (\omega \otimes \eta))$.
$\epsilon^I \in \bigwedge^k V^*$, so

$$\epsilon^I(v_1, \dots, v_k) = \det\begin{pmatrix} \epsilon^{i_1}(v_1) & \cdots & \epsilon^{i_1}(v_k) \\ \vdots & & \vdots \\ \epsilon^{i_k}(v_1) & \cdots & \epsilon^{i_k}(v_k) \end{pmatrix}$$

We have a V basis $\{E_I\}$ and a V^* dual basis $\{\epsilon^I\}$ with $I = (i_1, \dots, i_k)$. We also have that $\epsilon^I(E_{j_1}, \dots, E_{j_k}) = \delta^I_J$.
Then $\mathcal{B} = \{E^I : I \text{ is strictly increasing}\}$ is a basis for $\bigwedge^k V^*$.

Lemma 14.10

$\epsilon^I \wedge \epsilon^J = \epsilon^{IJ}$.

Proof

We show that $\epsilon^I \wedge \epsilon^J(E_{p_k}, \dots, E_{p_{k+l}}) = \epsilon^{IJ}(E_{p_1}, \dots, E_{p_{k+l}})$, $P = (p_1, \dots, p_{k+l})$.
If $I \cup J \neq P$, then both sides are zero.
If IJ or P has repeated index, then both sides are zero.
Then the only nontrivial case is when $P = IJ$ without repeated indecies. Write $IJ = \{i_1, \dots, i_k, j_1, \dots, j_l\}$ such that we can apply a permutation $\gamma \in S_{k+l}$ to generate a strictly increasing $P = \{p_1, \dots, p_{k+l}\}$. Then write $P_1 = \{p_1, \dots, p_k\}$ and $P_2 = \{p_{k+1}, \dots, p_{k+l}\}$, and compute

$$\begin{aligned}
\epsilon^P &= \epsilon^{P_1} \wedge \epsilon^{P_2} \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sign} \sigma) \cdot (\sigma(\epsilon^{P_1} \otimes \epsilon^{P_2})) \\
&= \frac{1}{k!l!} \sum_{\sigma' \in S_{k+l}} (\operatorname{sign} \sigma')(\operatorname{sign} \gamma)((\gamma \cdot \sigma')(\epsilon^{P_1} \otimes \epsilon^{P_2})) \\
&= \operatorname{sign} \gamma(\epsilon^I \wedge \epsilon^J)
\end{aligned}$$

Proposition 14.11

1. If $\omega^i \in V^*$ and $v_j \in V$, then $\omega^1 \wedge \cdots \wedge \omega^k(v_1, \dots, v_k) = \det(w^i(v_j))$.

Proof

It suffices to check (assuming I, J strictly increasing)

$$(\epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_k})(E_{j_1}, \dots, E_{j_k}) = \epsilon^I(E_{j_1}, \dots, E_{j_k}) = \delta^I_J = \det(\epsilon^{i_p}(E_{j_q})).$$

Definition: Graded Algebra

Write $\bigwedge V^* = \bigoplus_{k=0}^n \bigwedge^k V^*$ with $\dim \bigwedge V^* = 2^n$.
Remember that $\dim \bigwedge^k V^* = \binom{n}{k}$.
It is graded if $(\bigwedge^k) \wedge (\bigwedge^l) \subseteq \bigwedge^{k+l}$.

Differential Forms on Manifolds

Given a manifold M, a k-form on M $\bigwedge^k(T^*M) = \coprod_{p \in M} \left(\bigwedge^k T_p^* M\right)$ is a section of the bundle $\bigwedge^k(T^*M) \to M$.
$\Omega^k(M)$ is the collection of k-forms on M.
Locally, $\omega \in \Omega^k(M)$ may be written $\omega = \sum \omega_I dx^I$ for a chart $(U, (x^i))$.
Summing over strictly increasing I, $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ and $\omega_I = \omega\left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}}\right)$.

Pullback

For $F : M \to N$ and $\omega \in \Omega^k(N)$, we define $(F^*\omega) \in \Omega^K(M)$ by

$$(F^*\omega)(v_1, \dots, v_k) = \omega(DF(v_1), \dots, DF(v_k)).$$

It follows that

$$F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$$

and

$$F^*\left(\sum_I' \omega_I dx^I\right) = \sum_I' (F^*\omega_I)F^*(dx^{i_1} \wedge \cdots \wedge dx^{ik})$$
$$= \sum_I' (\omega_I \circ F)(d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F))$$
$$= \sum_I' (\omega_I \circ F) dF^{i_1} \wedge \cdots \wedge dF^{i_k}$$

Example

For $F : \mathbb{R}^2 \to \mathbb{R}^3$ by $F(u, v) = (u, v, u^2 - v^2)$ and $\omega = y\, dx \wedge dz \in \Omega^2(\mathbb{R}^3)$.

$$F^*\omega = F^*(y\, dx \wedge dz) = v\, du \wedge d(u^2 - v^2) = v\, du \wedge (2u\, du - 2v\, dv = -2v^2\, du \wedge dv$$

Proposition 14.20

For $F : M^n \to N^n$ with local coordinates (x^i) and (y^i) respectively, if $u \in C^\infty(N)$ then

$$F^*(u\, dy^1 \wedge \cdots \wedge dy^n) = (u \circ F) \det DF$$

Proof

Write F in components $(F^1, \ldots, F^n)$ where $F^i = y^i \circ F$

$$F^*(u\, dy^1 \wedge \cdots \wedge dy^n) = (u \circ F) dF^1 \wedge \cdots \wedge dF^n \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right)$$
$$= (u \circ F) \det\left(dF^i\left(\frac{\partial}{\partial x^j}\right)\right)$$
$$= (u \circ F) \det(DF)$$

If $(U, (x^i))$ and $(\tilde{U}, (\tilde{x}^i))$ are local charts with $U \cap \tilde{U} \neq \varnothing$, then using $F = \mathrm{id}_{U \cap \tilde{U}}$ we have that $F^* = \mathrm{id}$

$$d\tilde{x}^i \wedge \cdots \wedge d\tilde{x}^n = \det\left(\frac{\partial \tilde{x}^j}{\partial x^i}\right) dx^1 \wedge \cdots \wedge dx^n$$

Definition: Exterior Derivative

For $\omega \in \Omega^k(U)$, $U \subseteq \mathbb{R}^n$ open, $\omega = \sum_I' \omega_I dx^I$ define $d : \omega^k(U) \to \omega^{k+1}(U)$ by $\omega \mapsto d\omega$. Then

$$d\omega = \sum_I' \underbrace{d\omega_I}_{\in \Omega^1(U)} \wedge \underbrace{dx^I}_{\in \Omega^k(U)}$$

Example

$\omega \in \Omega^1(U)$, $\omega = \sum_{i=1}^n \omega_i dx^i$.

$$d\omega = \sum_{i=1}^n d\omega_i \wedge dx^i = \sum_{i,j=1}^n \frac{\partial \omega_i}{\partial x^j} dx^j \wedge dx^i = \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} \right) dx^i \wedge dx^j$$

For $\omega = df \in \Omega^1(M)$, $d(df) = \sum_{i<j} \left(\frac{\partial^2 f}{\partial x^j \partial x^i} - \frac{\partial^2 f}{\partial x^i \partial x^j} \right) dx^i \wedge dx^j = 0$. That is, $(d \circ d)(f) = 0$ for any smooth function $f \in C^\infty(M)$.

Proposition

1. d is $\mathbb{R}$-linear.
2. $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge (d\eta)$ with $k = \deg \omega$.
3. $d \circ d = 0$.
4. $F^*(d\omega) = d(F^*\omega)$.

Proof of b

Write $\omega = u\, dx^I$ and$\eta = v\, dx^J$.
Claim: $d(u\, dx^I) = du \wedge dx^I$ for any index (perhaps not strictly increasing) I.
If I has a repeated index, both sides are zero.
If not, let $\sigma \in S_k$ such that $I_\sigma = J$ strictly increasing.

$$d(u\, dx^I) = d((\operatorname{sign} \sigma) u\, dx^J) = \operatorname{sign} \sigma \cdot du \wedge dx^J = du \wedge (\operatorname{sign} \sigma \cdot dx^J) = du \wedge dx^I$$

Then

$$d(\omega \wedge \eta) = d(u\, dx^I \wedge v\, dx^J) = d(uv\, dx^I \wedge dx^J) = d(uv\, dx^{IJ}) = d(uv) \wedge dx^{IJ} = (u\, dv + v\, du) \wedge (dx^I \wedge dx^J)$$

So

$$d\omega \wedge \eta + (-1)^k \omega \wedge d\eta = du \wedge dx^I \wedge (v\, dx^J) + (-1)^k u\, dx^I \wedge (dv \wedge dx^J)$$

and it suffices to show that $dv \wedge dx^I \wedge dx^J = (-1)^k dx^I \wedge dv \wedge dx^J$.

Proof b Implies c

Write

$$d \circ d(\omega_I dx^I) = d(d\omega_I \wedge dx^I) = d(d\omega_I) \wedge dx^I + (-1)^1 d\omega_I \wedge d(dx^I) = 0$$

Proof of d

Write $\omega = u\,dx^I$ such that $d\omega = du \wedge dx^I$.

$$F^*(d\omega) = F^*(du \wedge dx^I) = d(u \circ F) \wedge dF^{i_1} \wedge \cdots \wedge dF^{i_k}$$

and

$$d(F^*\omega) = d((u \circ F)dF^{i_1} \wedge \cdots \wedge dF^{i_k} = d(u \circ F) \wedge dF^{i_1} \wedge \cdots \wedge dF^{i_k}$$

February 5, 2025

Theorem 14.24

There is a unique map $d : \Omega^*(M) \to \Omega^*(M)$ with $d(\Omega^k(M)) \subseteq \Omega^{k+1}(M)$ such that

1. d is $\mathbb{R}$-linear
2. $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$
3. $d \circ d = 0$
4. $df(X) = Xf$ for all $f \in \Omega^0(M) = C^\infty(M)$ and $X \in \mathfrak{X}(M)$.

Proof: Existence

Let $\omega \in \Omega^k(M)$. Then $\omega|_U \in \Omega^k(U)$. We have that $\varphi^{-1*}\omega \in \Omega^k(\varphi(U))$, $d(\varphi^{-1*}\omega) \in \Omega^{k+1}(\varphi(U))$, and $d\omega := \varphi^* d(\varphi^{-1*}\omega) \in \Omega^{k+1}(U)$ on U.

IMAGE 1

Proof: Well-defined

If (V, ψ) is another chart with $U \cap V \neq \varnothing$, we need to show that $\psi^*(d(\psi^{-1*}\omega)) = \varphi^*(d\varphi^{-1*}\omega))$. This is equivalent to

$$\begin{aligned} &\iff \quad d(\psi^{-1*}\omega) = \psi^{-1*}\varphi^*(d(\varphi^{-1*}\omega)) \\ &\iff \quad d(\psi^{-1*}\omega = (\varphi \circ \psi^{-1})^* d(\varphi^{-1*}\omega) \end{aligned}$$

where

$$(\varphi \circ \psi^{-1})^* d(\varphi^{-1*}\omega) = d((\varphi \circ \psi^{-1})^* \varphi^{-1*}\omega) = d(\psi^{-1*} \circ \varphi^* \circ \varphi^{-1*}\omega) = d(\psi^{-1*}\omega)$$

Proof: Uni!

For any $d : \Omega^*(M) \to \Omega^*(M)$ with the property $(d\omega)_p$ only depends on $\omega|_U$ where $p \in U$.
Suppose $\omega_1 = \omega_2$ on U. We need to show that $(d\omega_1)_p = (d\omega_2)_p$.
So set $\eta = \omega_1 - \omega_2$. Then $\omega \equiv 0$ on U, and we need to show that $(d\eta)_p = 0$.
Let ψ be a bump function such that $\operatorname{supp}\psi \subseteq U$ and $\psi(p) = 1$.

Then $\psi\eta = 0 \in \Omega^k(M)$.

$$0 = d(\psi\eta) = d\psi \wedge \eta + (-1)^0 \psi \wedge d\eta$$

At point p, it reads

$$0 = 0 \wedge \eta_p + \overbrace{\psi(p)}^{=1} \wedge d\eta_p$$

That is, $0 = d\eta_p$. Let $p \in M$, U a chart around p, say $(U, (x^i))$, and $\omega \in \Omega^k(U)$. We know that $(d\omega)_p$ only depends on $\omega|_U = \sum_I' \omega_I dx^I$. Then for $p \in V \subseteq \overline{V} \subseteq U$, $\omega|_U$ extends functtions $\omega_I, x^I \in C^\infty(V)$ to globally defined functions $\tilde{\omega}_I, \tilde{x}^I \in C^\infty(M)$. Therefore

$$\begin{aligned} d(\omega|_U) &= \sum_I' d(\omega_I dx^I) \\ &= \sum_I' d(\tilde{\omega}_I \tilde{x}^I) \\ &= \sum_I' (d\tilde{\omega}_I \wedge d\tilde{x}^I + \omega_I \wedge \overbrace{d(d\tilde{x}^{i_1} \wedge \cdots \wedge d\tilde{x}^{i_k})}^{=0} \\ &= \sum_I' d\omega_I \wedge dx^I \end{aligned}$$

which is the same formula for $\mathbb{R}^n$.

Proposition: 14.26

$F^*(d\omega) = d(F^*\omega)$.

Proposition: 14.32

$\mathcal{L}_V(\omega \wedge \eta) = (\mathcal{L}_V\omega) \wedge \eta + \omega \wedge (\mathcal{L}_V\eta)$.

Corollary

$\mathcal{L}_V(d\omega) = d(\mathcal{L}_V\omega)$.

Definition: Interior Multiplication

Given $\omega \in \bigwedge^k V^*$ and $v \in V$, define $\iota_V\omega \in \bigwedge^{k-1} V^*$ (sometimes written $V \lrcorner \omega$).

$$(\iota_v\omega)(u_1, \dots, u_{k-1}) = \omega(v, u_1, \dots, u_{k-1})$$

This defines $\iota_V : \bigwedge^k V^* \to \bigwedge^{k-1} V^*$, and we have $\iota_V \circ \iota_V = 0$.

$$\iota_v(\omega \wedge \eta) = (\iota_V\omega) \wedge \eta + (-1)^k \omega \wedge (\iota_V\eta)$$

Proof

It suffices to show that for $\omega^1, \ldots, \omega^k \in V^*$

$$\iota_V(\omega^1 \wedge \cdots \wedge \omega^k) = \sum_{i=1}^{k} (-1)^{i-1} \omega^i(v) \omega^1 \wedge \cdots \wedge \hat{\omega}^i \wedge \cdots \wedge \omega^k$$

Where $\hat{\omega}^i$ is meant to denots "forgetting" a term in the wedge product. That is, the first term has no ω^1, the second no ω^2, etc.
Assuming this, it suffices to consider $\omega = \omega^1 \wedge \cdots \wedge \omega^k$ and $\eta = \eta^1 \wedge \cdots \wedge \eta^l$. Then

$$\begin{aligned}
\iota_V(\omega \wedge \eta) &= \iota_V(\omega^1 \wedge \cdots \omega^k \wedge \eta^1 \wedge \cdots \wedge \eta^l) \\
&= \sum_{i=1}^{k} (-1)^{i-1} \omega^i(v) \omega^1 \wedge \cdots \wedge \hat{\omega}^i \wedge \cdots \wedge \omega^k \wedge \eta^1 \wedge \cdots \wedge \eta^l + \sum_{i=1}^{l} (-1)^{k+i-1} \eta^i(v) \omega^1 \wedge \cdots \wedge \omega^k \wedge \eta^1 \wedge \cdots \wedge \hat{\eta}^i \wedge \cdots \wedge \eta^l \\
&= (\iota_V \omega) \wedge \eta + (-1)^k \omega \wedge (\iota_V \eta)
\end{aligned}$$

Write $v_1 = v$, and apply both sides to $(v_2, \ldots, v_k)$. The left hand side is

$$\omega^1 \wedge \cdots \omega^k(v_1, \ldots, v_k) = \det(\omega^i(v_j)) = \det \begin{pmatrix} \omega^1(v_1) & \cdots & \omega^i(v_1) & \cdots & \omega^k(v_1) \\ \vdots & & & & \vdots \\ \omega^1(v_k) & \cdots & \omega^i(v_1) & \cdots & \omega^k(v_k) \end{pmatrix}$$

The right hand side is given by

$$\sum_{i=1}^{k} (-1)^{i-1} \omega^i(v_1)(\omega^1 \wedge \cdots \wedge \hat{\omega}^i \wedge \cdots \wedge \omega^k)(v_1, \ldots, v_k)$$

which, when expanded, gives $\det(\omega^i(v_j))$ along the first row.

Proposition 14.35 (Cartan)

If $V \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$, then

$$\mathcal{L}_V \omega = V \lrcorner (d\omega) + d(V \lrcorner \omega)$$

Corollary

$$\mathcal{L}_V(d\omega) = d(\mathcal{L}_V \omega)$$

Proof

By assuming Cartan's formula, the left hand side is

$$V \lrcorner \overbrace{(d \circ d\omega)}^{=0} + d(V \lrcorner d\omega)$$

and the right hand side is

$$d(V \lrcorner d\omega + d(V \lrcorner \omega)) = d(V \lrcorner d\omega) + \overbrace{d \circ d(v \lrcorner \omega)}^{=0}$$

Proof (of Cartan's Formula)

We prove by induction on $\deg(\omega)$. When ω is a function $f \in C^\infty(M) = \Omega^0(M)$, the left hand side is

$$\mathcal{L}_V f = Vf$$

and the right hand side is

$$V \lrcorner (df) + d\overbrace{(V \lrcorner f)}^{=0} = df(V) = Vf$$

since ι_V maps Ω^k to Ω^{k-1}.
Assuming it holds for $k-1$ forms, we consider $\omega \in \Omega^k(M)$ and locally write $\omega = \sum' \omega_I dx^I$.
It suffices to show that the formula holds for $\omega = du \wedge \beta$, $u \in C^\infty(M)$, $\beta \in \Omega^{k-1}(M)$.

$$(\omega_I dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \underbrace{dx^{i_1}}_{du} \wedge \underbrace{(\omega_I dx^{i_1} \wedge \cdots \wedge dx^{i_k})}_{\beta}$$

The left hand side is

$$\begin{aligned}\mathcal{L}_V(du \wedge \beta) &= \mathcal{L}_V du) \wedge \beta + du \wedge \mathcal{L}_V \beta \\ &= d(\mathcal{L}_V u) \wedge \beta + du \wedge (V \lrcorner d\beta + d(V \lrcorner \beta)) \\ &= d(Vu) \wedge \beta + du \wedge (V \lrcorner d\beta) + du \wedge d(V \lrcorner \beta)\end{aligned}$$

and the right hand side is

$$\begin{aligned}V \lrcorner (d(du \wedge \beta)) + d(V \lrcorner (du \wedge \beta)) &= V \lrcorner (\overbrace{(d \circ du)}^{=0} \wedge \beta + (-1) du \wedge d\beta + d(\overbrace{(V \lrcorner du)}^{=Vu} \wedge \beta + du \wedge (V \lrcorner \beta)) \\ &= (-1)(Vu) d\beta + d(Vu) \wedge \beta + (Vu) d\beta\end{aligned}$$

Proposition 14.32

$$d\omega(X_1, \ldots, X_{k+1}) = \sum_{1 \le i \le k+1} (-1)^{i-1} X_i(\omega(X_1, \ldots, \hat{x}_i, \ldots, X_{k+1})) + \sum_{1 \le i \le j \le k+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1})$$

When $\omega \in \Omega^1$, it reads

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$$

In particular, for ω closed,

$$X(\omega(Y)) - Y(\omega(X)) = \omega([X, Y])$$

Proof

It suffices to prove that for $\omega = udv,\ u, v \in C^\infty(M)$ that

$$d(\omega) = d(udv) = du \wedge dv$$

The left hand side

$$(du \wedge dv)(X, Y) = \det\begin{pmatrix} du(X) & du(Y) \\ dv(X) & dv(Y) \end{pmatrix} = \det\begin{pmatrix} X_u & Y_u \\ X_v & Y_v \end{pmatrix}$$

and the right hand side

$$\begin{aligned} X(udv(Y)) - Y(udv(X)) - u(dv([X,Y]) &= X(u(Yv)) - Y(u(Xv)) - u([X,Y]v) \\ &= (Xu)(Yv) + u(XYv) - (Yu)(Xv) - u(YXv) - u([X,Y]v) \\ &= \det\begin{pmatrix} X_u & Y_u \\ X_v & Y_v \end{pmatrix} \end{aligned}$$

Example

For $f \in \Omega^*(\mathbb{R}^3)$ and $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \in \Omega^{*+1}(\mathbb{R}^3)$, write $Pdx + Qdy + Rdz$ and

$$\begin{aligned} d(Pdx + Qdy + Rdz) &= \left(\frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz\right) \wedge dx + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial z}dz\right) \wedge dy + \left(\frac{\partial R}{\partial x}dx + \frac{\partial R}{\partial y}dy\right) \wedge dz \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}dx \wedge dy\right) + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}dy \wedge dz\right) + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}dz \wedge dx\right) \end{aligned}$$

Recall that for $X = (P, Q, R) \in \mathfrak{X}(\mathbb{R}^3)$, this is the curl of X.
Let $\omega = udx \wedge dy + vdy \wedge dz + wdz \wedge dx$, then

$$\begin{aligned} d\omega &= \frac{\partial u}{\partial z}dz \wedge dx \wedge dy + \frac{\partial v}{\partial z}dx \wedge dy \wedge dz + \frac{\partial w}{\partial z}dy \wedge dz \wedge dx \\ &= \left(\frac{\partial V}{\partial x} + \frac{\partial W}{\partial y} + \frac{\partial U}{\partial z}\right)dx \wedge dy \wedge dz \end{aligned}$$

Recall that this is divergence. We can also look at the gradient

$$\operatorname{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

we have

$$\operatorname{grad} f \cdot X = Xf = df(X) = \sum \frac{\partial f}{\partial x^i} \cdot x^i$$

Putting this together,

$$\begin{tikzcd} C^\infty(\mathbb{R}^3) \arrow[r, "\mathrm{grad}"] \arrow[d] & \mathfrak{X}(M) \arrow[r, "\mathrm{curl}"] \arrow[d] & \mathfrak{X}(M) \arrow[r, "\mathrm{div}"] \arrow[d] & C^\infty(\mathbb{R}^3) \arrow[d] \\ \Omega^0(\mathbb{R}^3) \arrow[r, "d"] & \Omega^1(\mathbb{R}^3) \arrow[r, "d"] & \Omega^2(\mathbb{R}^3) \arrow[r, "d"] & \Omega^3(\mathbb{R}^3) \end{tikzcd}$$

February 10, 2025

Orientation. Lee pages 378 to 390.

February 12, 2025

Recall

$$[E_1, E_2, \dots, E_n]$$

and $\omega \in \Lambda^n V^* - \{0\}$
On a manifold, we say that $\omega \in \Omega^n(M)$ is nonvanishing if and only if

- the manifold has an orientation if and only if
- the manfiold admits an ordered atlas

For $S^{n-1} \hookrightarrow M^n$, if N is a vector field along S and nowhere tangent to S and M has an orientation given by $\omega \in \Omega^n(M)$, then S has an induced orientation $(N \lrcorner \omega) \in \Omega^{n-1}(S)$.
In particular, $\partial M \to M$ is oriented for N outwarding vector field aong ∂M, we have induced orientation given by $N \lrcorner \omega) \in \Omega^{n-1}(\partial M)$.

$$F : (M^n, O_m) \to (N^n, O_N)$$

is a local diffeormorphism and orientation preserving if $F^* O_N = O_M$. It is orientation reserving if $F^* O_N = -O_M$.
$F^* O_N$ is gvien the pullback $F^* \omega$, where $\omega \in \Omega^n(N)$ is non-vanishing and matching with O_N.

Example 1

For example, $A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by $(x^i) \mapsto (-x^i)$ has orientation $[E_1, \dots, E_{n+1}]$. Then

$$[AE_1, \dots, AE_{n+1}] = [E_1, \dots, E_{n+1}] = (-1)^{n+1}[E_1, \dots, E_n]$$

and A is orientation preserving if and only if n is odd. Instead, if we consider forms $\omega = \varepsilon^1 \wedge \cdots \wedge \varepsilon^{n+1}$ then we have

$$A^* \omega(X_1, \dots, X_{n+1}) = \omega(AX_1, \dots, AX_{n+1}) = (\det A)(\omega(X_1, \dots, X_{n+1}))$$

so $A^* \omega = (\det A)\omega = (-1)^{n+1}\omega$.

Example 2

Consider $S^N \hookrightarrow \mathbb{R}^{n+1}$ and $A : S^n \to S^n$ by $x \mapsto -x$.

IMAGE 1

$A_* N = N$.

Then S^n has an induced orientation $(N \lrcorner \omega) \in \Omega^{n-1}(S)$ where $\omega = \varepsilon^1 \wedge \cdots \wedge \varepsilon^{n+1} \in \Omega^{n+1}\mathbb{R}^{n+1}$. Compute

$$\begin{aligned}
A^*(N \lrcorner \omega)(X_1, \ldots, X_n) &= (N \lrcorner \omega)(A_* X_1, \ldots, A_* X_n) \\
&= \omega(N, A_* X_1, \ldots, A_* X_n) \\
&= \omega(A_* N, A_* X_1, \ldots, A_* X_n) \\
&= \det(DA)\omega(N, X_1, \ldots, X_n) \\
&= (-1)^{n+1}(N \lrcorner \omega)(X_1, \ldots, X_n)
\end{aligned}$$

Therefore $A^*(N \lrcorner \omega) = (-1)^{n+1}(N \lrcorner \omega)$ and $A : S^n \to S^n$ is orientation preserving when n is odd.

An aside about covering maps

Consider all φ such that this diagram

$$\begin{tikzcd}
\hat{M} \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & \hat{M} \arrow[dl, "\pi"] \\
& M &
\end{tikzcd}$$

commutes. Then take $\mathrm{Aut}(\pi) = \{\varphi : \hat{M} \to \hat{M} \text{ diffeomorphic } : \pi = \pi \circ \varphi\}$. Then $\varphi \in \mathrm{Aut}(\pi)$ preserves the preimage $\pi^{-1}(x)$.

IMAGE 2

IMAGE 3

So $\mathrm{Aut}(\pi) = \mathbb{Z}_2$. For example, $S^n \xrightarrow{\pi} \mathbb{R}P^n$, $\mathrm{Aut}(\pi) = \mathbb{Z}_2 = \{\mathrm{id}, A\}$. By theorem, $\mathbb{R}P^n$ is orientable if and only if

- $A : S^n \to S^n$ is orientation perserving if and only if
- n is odd.

In the case of the Mobius band,

IMAGE 4

$\mathrm{Aut}(\pi) = \langle \gamma \rangle$ where $\gamma : (x, y) \mapsto (x + 1, -y)$ is orientation reversing. This implies that M is not orientable.

Theorem 15.36

Let $\pi : \hat{M} \to M$ be a covering map.

1. If M is orientable, then $\hat{M}$ is orientable.
2. If $\hat{M}$ is orientable, what about M?

M is orientable if and only $\mathrm{Aut}(\pi)$ acts as an orientation preserving idffeomorphism on $\hat{M}$.

Proof

($\Longleftarrow$) On M, we start with an atlas $\{V_\beta\}$ such that each V_β is evenly covered by π with $\pi^{-1}(V) = \bigcup_i U_i$

IMAGE 5

Each U_i carries an orientation (coming from $O_{\hat{M}}$).
Define an orientation by V such that $\pi|_{U_i} : U_i \to V$ is orientation preserving (i.e. $\pi^* O_V = O_{U_i}$). For a different U_j,

$$\pi^* O_v = (\pi \circ \varphi)^* O_V = \varphi^* \pi^* O_V = \varphi^* O_{U_i} = O_{U_j}$$

($\Longrightarrow$) As M is orientable, it has two orientations. Fix $\hat{p} \in \hat{M}$, $p = \pi(\hat{p}) \in M$. Choose the orientation on M such that $d\pi_{\hat{p}} : T_{\hat{p}}\hat{M} \to T_p M$ is orientation perserving. With this orientation O_M, we have

$$O_{\hat{M}} = \pi^* O_M = (\pi \circ \varphi)^* O_M = \varphi^* \pi^* O_M \varphi^* O_{\hat{M}}$$

so any $\varphi \in \mathrm{Aut}(\pi)$ is orientation preserving.

Orientation Covering Space

If M is a connected un-orientable manifold, then there exists $\pi : \hat{M} \to M$ a 2-folded (2-sheet) covering map – in the sense that $\#\pi^{-1}(x) = 2$ (e.g. $S^2 \to \mathbb{R}P^2$) – such that $\hat{M}$ is orientable.

Example: Mobius Band

We have $\pi / \langle \gamma \rangle \to M$

IMAGE 6

and $\gamma^2 : (x, y) \mapsto (x+2, y)$ which gives a cylinder with $\overline{\gamma} : (\theta, y) \mapsto (-\theta, -y)$.

Construction

Let M be connected. We construct

$$\hat{M} = \{(p, O_p) \ : \ p \in M,\ O_p \text{ is an orientation on } T_p M\}$$

where $\pi : \hat{M} \to M$ is given by $(p, O_p) \mapsto p$ which is 2-folded.

1. $\hat{M}$ has a smooth structure.

2. with this smooth structure, π is a smooth covering map.

3. $U \subseteq M$ (not necessariy a chart) is evenly covered by π if and only if U is orientable.

Given (U, O) where U is a chart in M and O is an orientation on U, we define $\hat{U}_O \subseteq \hat{M}$ by

$$\hat{U}_O = \{(p, O_p) \in \hat{M} \ : \ p \in U \text{ and } O_p \text{ matches with } O\}$$

Consider a basis

$$\mathcal{B} = \{\hat{U}_O \ : \ U \subseteq M \text{ a chart, and } O \text{ an orienation on } U\}$$

1. $\mathcal{B}$ covers $\hat{M}$
2. For $\hat{U}_O \cap \hat{U}'_O \neq \varnothing$, we have (p, O_p) such that $p \in U \cap U'$ and O_p matches with both $O_{U'}$ and O_U. Choose $V \subseteq U \cap U'$ and an orienation O_V such that O_V matches with O_p. Then O_V matches with oth O_U and $O_{U'}$, $\hat{V}_0 \subseteq \hat{U}_O \cap \hat{U}'_{O'}$.

So $\pi : \hat{U}_O \to U$ by $(p, O_p) \mapsto p$ is a bijective homeomorphism, and it defines a smooth structure on $\hat{M}$ such that $\{\hat{U}_O\}$ is an atlas. Then π is a smooth covering map. In fact, every chart $U \subseteq M$ is evenly covered by $\hat{U}_O$ and $\hat{U}_{-O}$.
To show that $\hat{M}$ is orientable, at each point $\hat{p} = (p, O_p) \in \hat{M}$ we give an orientation at $T_{\hat{p}}\hat{M}$ such that $d\pi_{\hat{p}} : T_{\hat{p}}\hat{M} \to (T_pM, O_p)$ is orientation preserving. We need to show that this pointwise orientation is continuous.
We have that $\hat{p} = (p, O_p) \in \hat{U}_O$ for the orientation of O on U matching with O_p. Then $\pi : \hat{U}_O \to (U, O)$ is orientation perserving (i.e. the orientation on $\hat{U}_O$ is $\pi^* O$).
Finally, we need to show that if $U \subseteq M$ is open and evenly covered, then U is orientable. In fact, $\pi^{-1}(U) = V_1 \cup V_2 \subseteq \hat{M}$ where $\pi : V_i \to U$ is a diffeomorphism. Since $\hat{M}$ is orientable, it induces an orientation on V_1. Then we get an orientation on U through the diffeomorphism π.
Conversely, if U is orientable then it has two orientations – call them O and $-O$. So we can construct $\hat{U}_O$ and $\hat{U}_{-O}$ not necesasrily charts where $\pi^{-1}(U) = \hat{U}_O \cup \hat{U}_{-O}$.

Connectedness

So far, we have $\pi : \hat{M} \to M$ a 2-folded covering with M connected.

1. if M is orientable, then $\hat{M}$ is two copies of M (i.e. $\hat{M}$ is not connected).

   From above, we have that $\pi^{-1}(M)$ is the disjoint union of two copies of M.

2. if instead M is un-orientable, then $\hat{M}$ is connected.

   Fact: $\pi : \hat{M} \to M$ a covering map with M connected, then $\#\pi^{-1}(x)$ is constant on M.
   Suppose $\hat{M}$ is not connected, then let W be components with $\pi|_W : W \to M$ covering maps. $\#(\pi|_W)^{-1}(x)$ is either one or two. If it is one, then $\pi|_W : W \to M$ is a diffeomorphism. However W is orientable while M is not, a contradiction. If instead the cardinality is two, then $W = \hat{M}$ and hence $\hat{M}$ is connected.

Corollary

If M is simply connected (i.e. $\pi_1 = \{e\}$), then M is orientable. In fact, if M is orientable then $\pi : \hat{M} \to M$ is a 2-folded covering with $\hat{M}$ connected. If M is simply connected, then $\hat{M} = M$ a contradiction.

Remark

If $\pi_1(M)$ does not have a subgroup of index 2, then M is orientable.
For example, $\pi_1(\mathbb{R}P^2) = \mathrm{Aut}(\pi) = \mathbb{Z}^2$ with $\pi : S^2 \to \mathbb{R}P^2$ and, for the Mobius band M, $\pi_1(M) = \mathrm{Aut}(\pi) = \mathbb{Z} = \langle \gamma \rangle$ has a subgroup $\left\langle \gamma^2 \right\rangle$ and $2\mathbb{Z} \leq \mathbb{Z}$ is a subgroup with index 2.

February 19, 2025

Integration in Rn

In $\mathbb{R}^n$, let $\omega \in \Omega^n(\mathbb{R}^n)$ and suppose that a domain D is “good” and compact. Then $\omega = f\, dx^1 \wedge \cdots \wedge dx^n$ and

$$\int_D \omega := \int_D f\, dx^1 \wedge \cdots \wedge dx^n.$$

Proposition 16.1

Suppose we have domains $D, E \in \mathbb{R}^n$ and a diffeomorphism $G : \overline{D} \to \overline{E}$. If $\omega \in \Omega^n(\overline{E})$, then $G^* \omega \in \Omega^n(\overline{D})$ and

$$\int_D G^* \omega = \pm \int_E \omega$$

where $\pm$ depends on whether G preserves orientations (i.e. $\det(DG) > 0$ or $\det(DG) < 0$).

Proof

Write $G : \overline{D} \to \overline{E}$ as $(x^1, \dots, x^n) \mapsto (y^1, \dots, y^n)$ and $\omega = f(y^1, \dots, y^n)\, dy^1 \wedge \cdots \wedge dy^n$. Then since

$$y^i = G^i(x^i, \dots, x^n \quad \text{and} \quad dy^1 \wedge \cdots \wedge dy^n = dG^1 \wedge \cdots \wedge dG^n,$$

we have

$$\begin{aligned}
\int_E \omega &= \int_E f(y^1, \dots, y^n)\, dy^1 \wedge \cdots \wedge dy^n \\
&\overset{y^i = y^i(x^1, \dots, x^n)}{=} \int_D f \circ G(x^1, \dots, x^n) |\det(DG)|\, dx^1 \wedge \cdots \wedge dx^n \\
&= \pm \int_D (f \circ G) \cdot \det(DG)\, dx^1 \wedge \cdots \wedge dx^n \\
&= \pm \int_D G^* \omega \\
&= G^*(f\, dy^1 \wedge \cdots \wedge dy^n) \\
&= (f \circ G) G^*(dy^1 \wedge \cdots \wedge dy^n)
\end{aligned}$$

More Generally

If $\omega \in \Omega^n(\mathbb{R}^n)$ with compact suppoert, then we can pick a “good” domain D such that $\operatorname{supp} \omega \subseteq D$ and $\overline{D}$ is compact. Define

$$\int_{\mathbb{R}^n} \omega := \int_D \omega$$

This works similarly on any open set $U \supseteq \operatorname{supp} \omega$. Pick a good domain D such that $\operatorname{supp} \omega \subseteq D \subseteq U$ with $\overline{D}$ compact. Then

$$\int_U \omega := \int_D \omega$$

where U may be chosen to be an open ball $B_r^n(0)$.

Integration on Manifolds

On a manifold M^n with $\omega \in \Omega^n(M)$, we first consider the case where $\operatorname{supp} \omega \subseteq U$ for U a chart.

IMAGE 1

$$\int_M \omega := \pm_{\phi(U)} (\phi^{-1})^* \omega$$

where $\pm$ depends on whether $\phi : (U, O|_U) \to (\phi(U), O_E)$ is orientation preserving. This is well defined

IMAGE 2

Since $\psi(W) = \psi \circ \phi^{-1}(\phi(W))$,

$$\int_{\psi(W)} (\psi^{-1})^* \omega = \int_{\psi\circ\phi^{-1}(\phi(W))} (\psi^{-1})^* \omega = \int_{\phi(W)} (\psi \circ \phi^{-1})^* (\psi^{-1})^* \omega = \int_{\phi(W)} (\phi^{-1})^* \omega$$

General Case

Suppose M^n is oriented with $\omega \in \Omega^n(M)$ having comapct support.
Let $\{U_i\}$ be a finite open cover of $\operatorname{supp}\omega$ such that each U_i is a chart, and ψ_i a partition of unity subordinated to U_i (i.e. $\operatorname{supp}\psi_i \subseteq U_i$). Assume further that $\phi_i : (U_i, O|_{U_i}) \to (\phi_i(U_i), O_E)$ is orientation preserving (reversing introduces a sign). Define

$$\int_M \omega := \sum_{i=1}^{n} \int_M \psi_i \omega$$

This is well defined. Suppose $\{\tilde{U}_j\}$ is another open cover and $\tilde{\psi}_j$ another partition of unity with respect to $\{\tilde{U}_j\}$. Then

$$\int_M \psi_i \omega = \int_M \left(\sum_j \tilde{\psi}_j \right) \psi_i \omega = \sum_j \int_M \tilde{\psi}_j \psi_i \omega.$$

Summing over i,

$$\sum_i \int_M \psi_i \omega = \sum_{i,j} \int_M \tilde{\psi}_j \psi_i \omega = \sim_j \int_M \tilde{\psi}_j \left(\sum_i \psi_i \right) \omega = \sum_j \int_M \tilde{\psi}_j \omega.$$

Integration over Parameterizations

Take M^n oriented and $\omega \in \Omega^n(M^n)$ with comapct support. Suppose $D_1, \ldots, D_k$ are open domains in $\mathbb{R}^n$and $F_i : \overline{D}_i \to M$ such that

1. $F_i|_{D_i}$ is a diffeomorphism onto its image $W_i := F_i(D_i)$.
2. $W_i \cap W_j = \varnothing$, $\forall i, j$, and
3. $\bigcup_i \overline{W}_i = M$.

Then

$$\int_M \omega = \sum_{i=1}^{n} \int_{W_i} \omega = \sum_{i=1}^{n} \int_{D_i} F_i^* \omega.$$

Example

$$\omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$$

on $S^2 \subseteq \mathbb{R}^3$. Parameterize S^2 by $F : [0,\pi] \times [0,2\pi] \to S^2$ by $(\varphi,\theta) \mapsto (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$.

IMAGE 3

Orient S^2 by an outward normal vector field N (i.e. the induced orientation on S^2 is $N \lrcorner (e^1 \wedge e^2 \wedge e^3)$.
Then we need to show that $(N \lrcorner (e^1 \wedge e^2 \wedge e^3)\left(DF\left(\frac{\partial}{\partial\varphi}\right), DF\left(\frac{\partial}{\partial\theta}\right)\right) > 0$.

$$\begin{aligned}
DF\left(\frac{\partial}{\partial\varphi}\right) &= \frac{\partial F}{\partial\phi} = (\cos\phi\cos\theta, \cos\phi\sin\phi, -\sin\phi) \\
DF\left(\frac{\partial}{\partial\theta}\right) &= \frac{\partial F}{\partial\theta} = (-\sin\phi\sin\theta, \sin\phi\cos\phi, 0)
\end{aligned}$$

At $q = (0,1,0) \in S^2$, $q = F\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ so

$$\begin{aligned}
DF\left(\frac{\partial}{\partial\varphi}\right) &= (0,0,-1) \\
DF\left(\frac{\partial}{\partial\theta}\right) &= (-1,0,0)
\end{aligned}$$

while $N = (0,1,0)$. So we compute $(e^1 \wedge e^2 \wedge e^3)\left(N, DF\left(\frac{\partial}{\partial\varphi}\right), DF\left(\frac{\partial}{\partial\theta}\right)\right)$ is

$$\det\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix} = 1$$

and preserves orientation. So $\int_{S^2} \omega = \int_D F^*\omega$ and

$$F^*dx = d(F^*x) = d(x \circ F) = d(\sin\varphi\cos\theta) = \sin\varphi\; d\cos\theta + \cos\theta\; d\sin\varphi = -\sin\varphi\sin\theta\; d\theta + \cos\varphi\cos\theta\; d\varphi$$

Similarly,

$$F^*(dy) = d(\sin\varphi\sin\theta) = \sin\varphi\; d\sin\theta + \sin\theta\; d\sin\varphi = \sin\varphi\cos\theta\; d\theta + \cos\varphi\sin\theta\; d\varphi$$

Finally, $F^*dz = d\cos\varphi = -\sin\varphi\; d\phi$, so

$$\begin{aligned}
F^*\omega &= (\sin\varphi\cos\theta)\cdot(\sin^2\varphi\cos\theta\; d\varphi \wedge d\theta) + (\sin\varphi\sin\theta)\cdot(\sin^2\varphi\sin\theta\; d\varphi \wedge d\theta) \\
&\quad + \cos\varphi(\sin^2\theta\sin\varphi\cos\varphi\; d\varphi \wedge d\theta) + \cos^2\theta\sin\varphi\cos\varphi\; d\varphi \wedge d\theta \\
&= (\sin^3\varphi\cos^2\theta + \sin^3\varphi\sin^2\theta)\; d\varphi \wedge d\theta + (\cos^2\varphi\sin\phi)\; d\phi \wedge d\theta \\
&= \sin\varphi\; d\varphi \wedge d\theta
\end{aligned}$$

We conclude

$$\int_{S^2} \omega = \int_D F^*\omega = \int_D \sin\varphi\; d\phi d\theta = \int_0^\pi \sin\phi\; d\phi \int_0^{2\pi} d\theta = 2 \cdot 2\pi = 4\pi.$$

Stokes' Theorem

For M^n with boundary ∂M $(\dim \partial M = n-1)$,

$$\int_M d\omega = \int_{\partial M} \omega$$

for all $\omega \in \Omega^{n-1}(M)$ where ∂M has outward orientation.

Example

Take $\omega \in \Omega^2(B_1^3)$, then

$$d\omega = dx \wedge dy \wedge dz + dy \wedge dz \wedge dx + dx \wedge dx \wedge dy = 3 dx \wedge dy \wedge dz.$$

Since $S^2 = \partial B_1^3$,

$$\int_{S^2} \omega = \int_{\partial B_1^3} \omega = \int_{B_1^3} d\omega = \int_{B_1^3} 3 dx \wedge dy \wedge dz = 3 \cdot \mathrm{vol}(B_1^3) = 3 \cdot \frac{4}{3}\pi = 4\pi.$$

Example

Take $M = [a, b] \subseteq \mathbb{R}^1$ with orientation dt

IMAGE 4

We have that $\partial M = \{a\} \cup \{b\}$. So, at a $\left(-\frac{\partial}{\partial t}\right) \lrcorner (dt) = -1$ and at b $\left(\frac{\partial}{\partial t}\right) \lrcorner (dt) = 1$. So

$$\int_a^b f'(t)\, dt = \int_M d\omega = \int_{\partial M} \omega = -f(a) + f(b).$$

Example

Take a line integral along $\gamma : [0, 1] \to M$ with $\omega \in \Omega^1(M)$.
Suppose $\omega = df$. Then

$$\int_\gamma \omega = \int_\gamma df = \int_{\partial \gamma} f = f(\gamma(b)) - f(\gamma(a)).$$

Consequences

If M^n is compact, oriented and without boundary, then

$$\int_m d\omega = \int_{\partial M} \omega = 0$$

for $\omega \in \Omega^{n-1}(M)$. That is to say integrating an exact form over a closed manifold returns zero.
If M^n is compact and oriented with $\omega \in \Omega^{n-1}(M)$ satisfying $d\omega = 0$ (i.e. closed), then

$$\int_{\partial M} \omega = \int_M d\omega = 0.$$

Remark

If we write $(M,\omega) := \int_M \omega$, then Stokes' theorem says $(\partial M,\omega) = (M, d\omega)$.

Proof

In the special case that $M = \mathbb{R}^n$ with $\omega \in \Omega^{n-1}(\mathbb{R}^n)$ having compact support.
Cover the support of ω by a large cube $[-R,R]^n$. Then

$$\begin{aligned}
\omega &= \omega_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\
d\omega &= \sum_i \frac{\partial \omega_i}{\partial x^i} dx^i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\
&= \sum_{i=1}^n (-1)^{i-1} \frac{\partial \omega_i}{dx^i} dx^1 \wedge \cdots \wedge dx^n.
\end{aligned}$$

It follows that from Frobenius and the Fundamental Theorem of Calculus that

$$\begin{aligned}
\int_{\mathbb{R}^n} d\omega &= \sum_{i=1}^n (-1)^{i-1} \int_{[-R,R]^n} \frac{\partial \omega_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n \\
&= \sum_{i=1}^n (-1)^{i-1} \int_{-R}^R \cdots \left(\int_{-R}^R \frac{\partial \omega_i}{\partial x^i}\, dx^i \right) \cdots \\
&= \cdots (\omega_i(\cdots, R, \cdots) - \omega_i(\cdots, -R, \cdots)) \cdots \\
&= 0
\end{aligned}$$

In the special case that $M = \mathbb{H}^n$ with $\omega \in \Omega^{n-1}(\mathbb{H}^n)$ having compact support.
Covering the support of ω by $[-R,R]^{n-1} \times [0,R]$,

$$\begin{aligned}
\int_{\mathbb{H}^n} d\omega &= \sum_{i=1}^n (-1)^{i-1} \int_{[-R,R]^{n-1}\times[0,R]} \frac{\partial \omega_i}{\partial x^i} dx^1 \cdots dx^n \\
&= (-1)^{n-1} \int_{[-R,R]^{n-1}\times[0,R]} \frac{\partial \omega_n}{\partial x^n} dx^1 \cdots dx^n \\
&= (-1)^{n-1} \int_{-R}^R \cdots \int_{-R}^R \left(\int_0^R \frac{\partial \omega_n}{\partial x^n} dx^n) dx^1 \cdots dx^n \right) \\
&= (-1)^n \int_{-R}^R \cdots \int_{-R}^R \omega_n(x^1, \ldots, x^{n-1}, 0) dx^1 \cdots dx^{n-1} \\
&= (-1)^n \int_{\partial\mathbb{H}^n \cap \operatorname{supp}\omega} \omega_n dx^1 \wedge \cdots \wedge dx^{n-1}
\end{aligned}$$

Recall that the induced orientation on the boundary $\partial\mathbb{H}^n$ matches with the standard orientation on $\mathbb{R}^{n-1}$ if and only if n is even. So

$$\begin{aligned}
\int_{\partial\mathbb{H}^n} \omega &= \sum_i \int_{\partial\mathbb{H}^n} \omega_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\
&= \int_{\partial\mathbb{H}^n} \omega_n dx^1 \wedge \cdots \wedge dx^{n-1}
\end{aligned}$$

which matches our previous calculation since $(-1)^n = 1$ for n even.

Green's Theorem

If $D \subseteq \mathbb{R}^2$ is a domain with $\overline{D}$ compact, then

$$\int_{\partial D} P\,dx + q\,dy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

and $\omega = P\,dx + Q\,dy \in \Omega^1(\mathbb{R}^2)$ so

$$d\omega = \frac{\partial P}{\partial y}\,dy \wedge dx + \frac{\partial Q}{\partial x}\,dx \wedge dy = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$$

Therefore

$$\int_{\partial D} \omega \int_D d\omega = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right); dxdy$$

with ∂D outward oriented.

February 24, 2025

Recall: Stoke's Theorem

For M^n a smooth manifold and $\omega \in \Omega_C^{n-1}(M)$ with compact support,

$$\int_M d\omega = \int_{\partial M} \omega.$$

1. $\omega \in \omega_C^{n-1}(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} d\omega = 0$.
2. $\omega \in \omega_C^{n-1}(\mathbb{H}^n)$, $\int_{\mathbb{H}^n} d\omega = \int_{\partial\mathbb{H}^n} \omega$.

Special Case

If $\operatorname{supp}\omega \subseteq (U, \phi)$ a chart, then $\operatorname{supp}(d\omega) \subseteq U$.

IMAGE 1

$$\int_M d\omega = \int_U \varphi\omega = \int_{\varphi(U)} (\varphi^{-1})^* d\omega = \int_{\varphi(U)} d(\varphi^{-1*}\omega).$$

So

$$\int_{\mathbb{R}^n} d(\varphi^{-*}\omega) = 0$$

and

$$\int_{\mathbb{H}^n} d(\varphi^{-1*}\omega \int_{\partial\mathbb{H}^n} \varphi^{-1*}\omega = \int_{\partial\mathbb{H}^n \cap \varphi(U)} \varphi^{-1*}\omega = \int_{\partial M \cap U} \omega = \int_{\partial M} \omega$$

Stoke's Theorem: General CAse

In general, $\omega \in \Omega_C^{n-1}(M)$.
Let $\{\psi_i\}_i$ be a partition of unity with respect to a countable cover of M by charts. Then, recalling that $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge d\eta$ with $k = \deg \omega$,

$$\int_{\partial M} \omega = \sum_i \int_{\partial M} \psi_i \omega = \sum_i \int_M d(\psi_i \omega) = \sum_i \int_M d\psi_i \wedge \omega + \psi_i d\omega = \int_M d\left(\overbrace{\sum_i \psi_i}^{=1}\right) \wedge \omega + \int_M \left(\overbrace{\sum_i \psi_i}^{=1}\right) d\omega = \int_M d\omega$$

Integration on Riemannian Manifolds

Recall

For (M^n, g) oriented, the volume form ω_g is an n-form such that $\omega_g(E_1, \ldots, E_n) = 1$ for all positively oriented orthonormal frame $\{E_1, \ldots, E_n\}$.
Inside a chart $(U_i, (x^1, \ldots, x^n))$ it has the formula

$$\omega_g = \sqrt{\det g} \cdot dx^1 \wedge \cdots \wedge dx^n$$

where $\det g = \det(g_{ij})$.

Definition:

Let $f \in C_C^\infty(M)$. Define

$$\int_M f = \int_m f \omega_g$$

Remarks

1. $\text{vol}(M) = \int_M 1$

2. $\omega_g \in \Omega^n(M)$ is usually written as dV_g or $d\,\text{vol}_g$.

Proposition

For (M, g) oriented and $f \in C_C^\infty(M)$, if $f \geq 0$ then $\int_M f \geq 0$. Equality holds if and only if $f \equiv 0$ on M.

Proof

$$\int_M f = \int_m f\, \omega_g = \sum_i \int_{U_i} \psi_i f \omega_g = \sum_i \int_{U_i} \psi_i f \sqrt{\det g_{ij}} dx^1 \wedge \cdots \wedge dx^n$$

where each term is greater than or equal to zero (assuming positive orientation on each U_i).

On Manifolds with Boundary

Take $\partial M \subseteq M^n$ with outward orientation.
Recall that for N an outward pointing vector field along ∂M, if M has an orientation n-form ω, then ∂M has an induced orientation given by

$$(N \lrcorner \omega) \in \omega^{n-1}(\partial M).$$

If (M, g) is an oriented Riemannian manifold with boundary ∂M, ω_g a volume form and N a unit outward pointing vector field orthogonal to ∂M.
Let $\tilde{g}$ be the induced Riemannian metric on ∂M, we observe that

$$\omega_{\tilde{g}} = N \lrcorner \omega_g$$

Let $\{E_1, \ldots, E_{n-1}\}$ be a (locally defined) orthonormal frame on ∂M. $\{E_1, \ldots, E_n\}$ being positively oriented on ∂M means that

$$(N \lrcorner \omega_g)(E_1, \ldots, E_n) = 1$$

Lemma 16.30

For (M, g) oriented and $(\partial M, \tilde{g})$, if $X \in \mathfrak{X}(\partial M)$, then $(X \lrcorner \omega_g)|_{\partial M} = g(X, N)\omega_{\tilde{g}}$.

Proof

Decompose $X = X^T + X^\perp$ where $X^\perp = g(X, N)N$ and $X^T = X - X^\perp$. Write

$$(X^\perp \lrcorner \omega_g)|_{\partial M} = g(X, N)(N \lrcorner \omega_g)|_{\partial M} = g(X, N)\omega_{\tilde{g}}$$

and

$$(X^T \lrcorner \omega_g)|_{\partial M}(E_1, \ldots, E_{n-1}) = \omega_g(X^T, E_1, \ldots, E_{n-1}) = 0$$

Generalized Stokes on Manifold with Boundary

Take $X \in \mathfrak{X}(M)$, $(X \lrcorner \omega_g) \in \Omega^{n-1}(M)$ and $d(X \lrcorner \omega_g) \in \Omega^n(M)$. Write

$$\int_M d(X \lrcorner \omega_g) = \int_{\partial M} X \lrcorner \omega_g = \int_{\partial M} g(X, N)\omega_{\tilde{g}} = \int_{\partial M} g(X, N).$$

Definition: Divergence

Let $\operatorname{div} X \in C^\infty(M)$ defined by $d(X \lrcorner \omega_g) = \operatorname{div} X \cdot \omega_g$. Then

$$\int_M d(X \lrcorner \omega_g) = \int_M \operatorname{div} X \cdot \omega_g = \int_M \operatorname{div} X$$

Theorem: Divergence Theorem

$$\int_X \operatorname{div} X = \int_{\partial M} g(X,N)$$

Remark

Inside $\mathbb{R}^n$, $X = X^i \frac{\partial}{\partial X^i} \in \mathfrak{X}(\mathbb{R}^n)$, then $\operatorname{div} X = \frac{\partial}{\partial X^i}(X^i)$.

Problem 16-11

$$\operatorname{div}\left(X^i \frac{\partial}{\partial X^i}\right) = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial X^i}\left(X^i \sqrt{\det g}\right)$$

For $(\mathbb{R}^n, g_E)$, $g_{ij} = \delta_{ij}$ and $\sqrt{\det g} = 1$. Then $\operatorname{div}\left(X^i \frac{\partial}{\partial X^i}\right) = \frac{\partial}{\partial X^i}(X^i)$.

Problem 16-9

$$\omega = |x|^{-n} \sum_{i=1}^{n} (-1)^{i-1} x^i dx^1 \wedge \cdots \hat{dx^i} \wedge \cdots \wedge dx^n \in \Omega^{n-1}(\mathbb{R}^n - \{0\})$$

and

$$\omega|_{S^{n-1}} = \sum_{i=1}^{n} (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge \hat{dx^i} \wedge \cdots \wedge dx^n$$

For example

$$\begin{aligned} n=2 \quad & \omega|_{S^1} = x\,dy - y\,dx \\ n=3 \quad & \omega|_{S^2} = x\,dy \wedge dz \underbrace{- y\,dx \wedge dz}_{+y\,dz \wedge dx} + z\,dx \wedge dy \end{aligned}$$

Claim: $\omega|_{S^{n-1}}$ is the standard volume form on S^{n-1} ($S^{n-1} \hookrightarrow \mathbb{R}^n$ or $S^{n-1} = \partial B_1^n$). We need to check that $\omega_{S^{n-1}} = (N \lrcorner \omega_E)$ We have that N is $(x^1, \ldots, x^n)$ at the point $(x^1, \ldots, x^n)$ (i.e. $N = x^i \frac{\partial}{\partial x^i}$ on S^{n-1}). Write

$$(N \lrcorner \omega_E) = \left(x^i \frac{\partial}{\partial x^i}\right) \lrcorner (dx^1 \wedge \cdots \wedge dx^n) = x^i \left(\frac{\partial}{\partial x^i} \lrcorner (dx^1 \wedge \cdots \wedge dx^n\right)$$

Compute

$$\begin{aligned} \left(\frac{\partial}{\partial x^1} \lrcorner (dx^1 \wedge \cdots \wedge dx^n\right)(E_1, \ldots, E_{n-1}) &= dx^1 \wedge \cdots \wedge dx^n \left(\frac{\partial}{\partial x^1}, E_1, \ldots, E_{n-1}\right) \\ &= \det \begin{pmatrix} dx^1\left(\frac{\partial}{\partial x^1}\right) & \overbrace{dx^2\left(\frac{\partial}{\partial x^1}\right)}^{=0} & \cdots & \overbrace{dx^n\left(\frac{\partial}{\partial x^1}\right)}^{=0} \\ dx^1(E_1) & \cdots dx^2(E_1) & \cdots & dx^n(E_1) \\ \vdots & & & \vdots \\ dx^1(E_{n-1}) & \cdots dx^2(E_{n-1}) & \cdots & dx^n(E_{n-1}) \end{pmatrix} \\ &= dx^2 \wedge \cdots \wedge dx^n (E_1, \ldots, E_{n-1}) (-1)^{i-1} \end{aligned}$$

In general,

$$\frac{\partial}{\partial x^i} \lrcorner \left(dx^1 \wedge \cdots \wedge dx^n\right) = \frac{\partial}{\partial x^i} \lrcorner (dx^i \wedge dx^1 \wedge \cdots \wedge \hat{dx^i} \wedge \cdots \wedge dx^n) = (-1)^{i-1} dx^1 \wedge \cdots \wedge \hat{dx^i} \wedge \cdots \wedge dx^n$$

Conclusion

$\omega|_{S^{n-1}}$ is the volume form on S^{n-1}, $0 < \int_{S^{n-1}} \omega|_{S^{n-1}}$.

1. $\omega|_{S^{n-1}} \in \Omega^{n-1}(S^{n-1})$ is not exact (if it is, $\omega = d\eta$ and $\int_{S^{n-1}} \omega = \int_{S^{n-1}} d\eta = 0$)
2. $\omega|_{S^{n-1}}$ is closed (By direct calculation on $d\omega$ on $\mathbb{R}^n - \{0\}$).

Proposition 16.33

Let (M, g) be an oriented Riemannian manifold and $X \in \mathfrak{X}(M)$ a complete vector field.
Let θ be the flow of X. Then $\operatorname{div} X \equiv 0$ if and only if θ_t is volume preserving for all time.

Proof

Let $D \subseteq M$ be any compact domain.

$$\operatorname{vol}(\theta_t(D)) = \int_{\theta_t(D)} \omega_g = \int_D \theta_t^* \omega_g$$

Recall Cartan's Formula: $\mathcal{L}_X = i_X \circ d + d \circ i_X$. So

$$\mathcal{L}_X(\omega_g) = X \lrcorner (\overbrace{d\omega_g}^{=0}) + d(X \lrcorner \omega_g) = \operatorname{div} X \cdot \omega g$$

Therefore

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=t_0} \operatorname{vol}(\theta_t(D)) &= \int_D \frac{d}{dt}\Big|_{t=t_0} \theta_t^* \omega_g \\
&= \int_D \theta_t^* (\mathcal{L}_X \omega_g) \\
&= \int_D \theta_{t_0}^* (\operatorname{div} X \cdot \omega_g) \\
&= \int_{\theta_{t_0}(D)} \operatorname{div} X \cdot \omega_g
\end{aligned}$$

If $\operatorname{div} X \equiv 0$ on M, then the right hand side is zero. Hence $\operatorname{vol}(\theta_t(D))$ is a constant function (i.e. θ_t is volume preserving everywhere).
If instead θ_t is assumed to be volume preserving, then the left hand side is zero for all times t_0 and any domain D. Then, without loss of generality for $t_0 = 0$, $\int_D \operatorname{div} X = 0$ (i.e. $\operatorname{divX} \equiv 0$).

Remark

For $f \in C^\infty(M)$, $\operatorname{grad} f \in \mathfrak{X}(M)$, $\Delta f := \operatorname{div}(\operatorname{grad} f) \in C^\infty(M)$.
In $(\mathbb{R}^n, g_E)$, $\operatorname{grad} f = \frac{\partial f}{\partial x^i} \cdot \frac{\partial}{\partial x^i}$ and $\Delta f := \operatorname{div}(\operatorname{grad} f) = \sum_{i=1}^n \frac{\partial^2 f}{\partial (x^i)^2}$.

Recall: Poincaré Lemma

Recall that if $U \subseteq \mathbb{R}^n$ is star-shaped, then $\omega \in \Omega^1(U)$ is closed if and only if ω is exact.
For ($\Longleftarrow$), this is always true; for ($\Longrightarrow$) we need star-shaped.

Definition: Path-homotopic

$\gamma_0, \gamma_1 : I \to M$ continuous such that $\gamma_0(a) = \gamma_1(a) = p$ and $\gamma_0(b) = \gamma_1(b) = q$.

IMAGE 2

A path-homotopy between γ_0 and γ_1 is a continuous map $H : I \times [0,1] \to M$ such that

$$\begin{aligned} H(\cdot, 0) &= \gamma_0 \quad H(a, \cdot) = p \\ H(\cdot, 1) &= \gamma_1 \quad H(b, \cdot) = q \end{aligned}$$

Proposition

Let $\gamma_0, \gamma_1 : [a,b] \to M$ be smooth path-homotopic, and let $\omega \in \Omega^1(M)$ be closed. Then $\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$.

Proof

Assume $a = 0$ and $b = 1$, then noting that faces 2 and 4 (see above) collapse to points with integral zero,

$$\begin{aligned} 0 = \int_{H(I)} \overbrace{d\omega}^{=0} &= \int_{I^2} H^*(d\omega) \\ &= \int_{I^2} d(H^*\omega) \\ &= \int_{\partial I^2} H^*\omega \\ &= \int_{i=1}^{4} \int_{F^i} H^*\omega \\ &= \sum_{i=1}^{4} \int_{H(F_i)} \omega \\ &= \int_{H(F_1)} \omega + \int_{H(F_3)} \omega \\ &= \int_{\gamma_0} \omega - \int_{\gamma_1} \omega \end{aligned}$$

Corollary

For M with $\pi_1(M) = e$ (i.e. every closed curve is path-homotopic to a point), then every closed 1-form is exact.

February 26, 2025

Corollary

If $\omega \in \Omega^1(M)$ is closed with γ_0 and γ_1 path-homotopic to each other, then $\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$.\

Corollary

If $\pi_1(M) = e$ (i.e. every closed curve in M is path-homotopic to a point), then every closed 1-form on M is exact.

Definition: Manifold with Corners

Let $\mathbb{R}_+ = (0, +\infty)$, $\overline{\mathbb{R}}_+^n = ([0, +\infty))^n = \{(x^1, \dots, x^n) \,:\, x^1 \geq 0, \dots, x^n \geq 0\}$, and $\partial\overline{\mathbb{R}}_+^n = \bigcup_{i=1}^n H_i$ where $H_i = \{(x^1, \dots, x^n) \in \overline{\mathbb{R}}_+^n \,:\, x^i = 0\}$.
In $\overline{\mathbb{R}}_+^n$, a corner point is $(x^1, \dots, x^n) \in \mathbb{R}_n^+$ such that at least two components are zero.

IMAGE 1

Definition: Corner Chart

Let M be a Hausdorff, second countable topological space. A corner chart (U, φ) where $U \subseteq M$ open and $\varphi : U \to \mathbb{R}_+^n$ homeomorphic to $\varphi(U)$.
A point p on M is called a corner point if it has a chart (U, φ) centered at p such that $\varphi(p)$ is a corner point in $\overline{\mathbb{R}}_+^n$.

Proposition: Invariance of Corner Points

IMAGE 2

If the above happens, $\psi(p) \in \mathbb{H}^n$ with $\psi(W)$ an open set in $\mathbb{H}^n$, and $\varphi(p) \in \overline{\mathbb{R}}_+^n$ as a corner point.
Let S be an open subset of a $(n-1)$-dimensional plane through $\psi(p)$ such that $\psi(W) \supseteq S$. Then $F = \varphi \circ \psi^{-1}$ is a diffeomorphism and, at $\psi(p)$, $d(F|_S) : T_{\psi(p)}S \to T_{\phi(p)}(F(S)) \subseteq \mathbb{R}^n$ is injective. We have also that $\dim(\operatorname{im} dF|_S) = \dim T_{\psi(p)}S = n-1$. Therefore we may pick a vector $v \in \mathbb{R}^n$ such that $v = (v^1, \dots, v^n)$ with $v^{n-1} \cdot v^n \neq 0$ and $v \in \operatorname{im} dF|_S$. Without loss of genreality, we may assume $v^n < 0$. There is $w \in T_{\psi(p)}S$ such that $dF(w) = v$. Let $\gamma : (-\epsilon, \epsilon) \to S$ be a curve with $\gamma(0) = \psi(p)$ and $\gamma'(0) = w$. Then $\beta = F \circ \gamma$ is a smooth curve with $\beta(0) = \phi(p)$ ($\phi(p) = (x^1, \dots, x^{n-1}, 0, 0)$) and $\beta'(0) = v = (v^1, \dots, v^n)$ with $v^n < 0$. Then by calculus there exists $\delta \in (0, \epsilon)$ such that $\beta(\delta) \notin \overline{\mathbb{R}}_+^n$. This is a contradiction.

Integration on Manifolds with Corners

Observe that $\partial\overline{\mathbb{R}}_+^n = \bigcup_{i=1}^n H_i$ where $H_i = \{(x^1, \dots, x^n) \in \overline{\mathbb{R}}_+^n \,:\, x^i = 0\} \cong \overline{\mathbb{R}}_+^{n-1}$.
Suppose $\omega \in \Omega_C^{n-1}(M)$ for M a manifold with corners, and consider the special case where $\operatorname{supp} \omega \subseteq (U, \varphi)$ is a corner chart.

$$\int_{\partial M} \omega := \sum_{i=1}^{n} (\phi^{-1})^* \omega$$

The general case may be done by partitions of unity.
In the orientation case, H_i has induced outward orientation (i.e. $-\frac{\partial}{\partial x^i} = N$).

$$\left(-\frac{\partial}{\partial x^i}\right) \lrcorner (dx^1 \wedge \cdots \wedge dx^n)$$

Where $H_i = \{(x^1, \dots, x^n \in \overline{\mathbb{R}}_+^n \,:\, x^i = 0\} \cong \overline{\mathbb{R}}_+^{n-1} \subseteq \mathbb{R}^{n-1}$ carries the normal orientation by $dx^1 \wedge \cdots \widehat{dx^i} \wedge \cdots \wedge dx^n$.

$$(dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n)\left(\frac{\partial}{\partial x^1}, \cdots, \widehat{\frac{\partial}{\partial x^i}}, \cdots, \frac{\partial}{\partial x^n}\right) = 1$$

and

$$\left(\left(-\frac{\partial}{\partial x^i}\right) \lrcorner (dx^1 \wedge \cdots \wedge dx^n)\right)\left(\frac{\partial}{\partial x^1}, \ldots, \widehat{\frac{\partial}{\partial x^i}}, \ldots, \frac{\partial}{\partial x^n}\right) = (-1) dx^1 \wedge \cdots \wedge dx^n \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^1}, \ldots, \widehat{\frac{\partial}{\partial x^i}}, \ldots, \frac{\partial}{\partial x^n}\right)$$
$$= (-1) \cdot (-1)^{i-1} = (-1)^i$$

Standard orientation on H_i and induced boundary orientation on H_i agree if and only if i is even. Then for

$$\int_M d\omega = \int_{\partial M} \omega$$

with induced boundary orienation, it suffices to consider a corner chart. $\omega \in \Omega_C^{n-1}(M)$ with $\operatorname{supp} \omega \subseteq (U, \varphi)$ and $\varphi : U \to \overline{\mathbb{R}}^n_+$.
It suffices to consider $M = \overline{\mathbb{R}}^n_+$ and $\omega \in \Omega_C^{n-1}(\overline{\mathbb{R}}^n_+)$.

$$\omega = \omega_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$
$$d\omega = \frac{\partial \omega_i}{\partial x^i} dx^i \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n = \sum_i (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n$$

Pick $R > 0$ large such that $\operatorname{supp} \omega \subseteq [0, R]^n$, then

$$\begin{aligned}
\int_{\overline{\mathbb{R}}^{n}_{+}} d\omega &= \sum_{i=1}^n (-1)^{i-1} \int_{[0,R]^n} \frac{\partial \omega_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n \\
&= \sum_{i=1}^n (-1)^{i-1} \int_0^R \cdots \left(\int_0^R \frac{\partial \omega_i}{\partial x^i} dx^i\right) dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\
&= \sum_{i=1}^n (-1)^i \int_0^R \cdots \int_0^R \omega_i(x^1, \ldots, 0, \ldots, x^n) dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\
&= \sum_{i=1}^n (-1)^i \int_{H_i} \omega_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \quad \text{(with standard orientation)} \\
&= \sum_{i=1}^n \int_{H_i} \omega_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n = \int_{\partial \mathbb{R}^n_+} \omega
\end{aligned}$$

Example

Let $M = I^2$ and $\omega \in \Omega^1(M)$ closed (i.e. $\int_{\partial_M} \omega = \int_M d\omega = 0$)

IMAGE 3

$$\int_{\partial M} \omega = \sum_{i=1}^4 \int_{F_i} \omega$$

$$N \lrcorner (dx \wedge dy) = (dx \wedge dy)(N, \_)$$

Definition: Homotopy

We say that $F, G : M \to N$ are (smoothly) homotopic if there is a smooth homotopy $H : M \times I \to N$ such that

$$H(\cdot, 0) = F(\cdot) \quad \text{and} \quad H(\cdot, 1) = G(\cdot).$$

Write $F \simeq G$.

Example: Problem 16-5

Let M^n, N^n be oriented, compact, connected without boundary. Take $F, G : M \to N$ local diffeomorphisms and suppose $F \simeq G$. Then F is orientation preserving if and only if G is orientation preserving.

Proof

Let ω_N be the orientation form on N^n with $d\omega_N = 0$. The homotopy $H : M \times I \to N$

$$0 = \int_{M\times I} H^*(d\omega_N) = \int_{M\times I} d(H^*\omega) = \int_{\partial(M\times I)} H^*\omega = \int_{M\times\{0\}} F^*\omega + \int_{M\times\{1\}} G^*\omega$$

IMAGE 4

Let $\omega_{M\times I}$ be the orientation form on $M \times I$ ($\omega_{M\times I} = \omega_M \wedge dt$).
On $M \times \{0\}$ orientable, $-\frac{\partial}{\partial t} \lrcorner\, \omega_{M\times I}$ and on $M \times \{1\}$ $\frac{\partial}{\partial t} \lrcorner\, \omega_{M\times I}$. Therefore $\int_M F^*\omega = \int_M G^*\omega$.

Example: Problem 16-6

S^n admits a nonvanishing vector field if and only if n is odd.

Proof

($\Longleftarrow$) suppose n odd. In the $n = 1$ case

IMAGE 5

Write $V(x^1, x^2) = (-x^2, x^1)$. In general, when $S^n \subseteq \mathbb{R}^{n+1}$ for n odd

$$\vec{z} = (x^1, y^1, x^2, y^2, \dots, x^{2k}, y^{2k})$$

gives

$$V(\vec{z}) = -y^1, x^1, -y^2, x^2, \dots, -y^{2k}, x^{2k})$$

with $V \in \mathfrak{X}(S^n)$ nonvanishing.
($\Longrightarrow$) Suppose $V \in \mathfrak{X}(S^n)$ nonvanishing. Then for any v, rewrite as $\frac{v}{||v||}$ such that without loss of generality $||1|| = 1$.

IMAGE 6

Next, we use $V(x)$ to construct a homotopy between id_{S^n} and (the antipodal map) $-\mathrm{id}_{S^n}$.
Construct a homotopy $H : S^n \times I \to S^n$ by $H(x,t) = (\cos t)x + (\sin t)V(x)$ with $||H(x,t)|| = 1$, $H(x,0) = x$, $H(x,\pi) = -x$.
Hence H is a smooth homotpy between id_{S^n} and $-\mathrm{id}_{S^n}$. Hence the antipodal map on S^n is orientation preserving and n is odd.

March 3, 2025

Chapter 7: De Rahm Cohomology

Let M^n be smooth and write $Z^k(M) = \{\omega \in \Omega^k(M) \,:\, d\omega = 0\}$, the set of closed k-forms, with $B^k(M) = \{\omega \in \Omega^k(M) \,:\, \omega = d\eta,\ \eta \in \Omega^{k-1}(M)\}$, the set of exact k-forms. Note that $B^k(M) \subseteq Z^k(M)$, since $d(d\eta) = 0$. We may also write

$\Omega^*(M) = \bigoplus_{k=0}^n \Omega^k(M)$ and

$$0 \xrightarrow{d} \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0$$

with $d^2 = 0$. Finally, we have the k-th de Rahm cohomology group $H^k_{\text{dR}}(M) = Z^k(M)/B^k(M)$ as a $\mathbb{R}$-vector space.
Fact: If M^n is closed, then $H^k_{\text{dR}}(M)$ is finite dimensional for all k.

Example

If M^n is connected and has $\pi_1(M) = \{e\}$ (i.e. every smooth loop is contractible to a point), then $\omega \in \Omega^1(M)$ is closed if and only if ω is exact. That is to say that $Z^1(M) = B^1(M)$ and $H^1_{\text{dR}}(M) = 0$.

Example

If $M = S^1 \subseteq \mathbb{R}^2$ and $\omega = \frac{x\,dy - y\,dx}{x^2+y^2} \in \Omega^1(\mathbb{R}^2 - \{0\})$, then ω is closed but not exact ($\int_{S^1} \omega \neq 0$).
Hence, ω gives a non-trivial element in $H^1(S^1)$ (i.e. $H^1(S^1) \neq \{0\}$.
Similarly, on $S^{n-1} \subseteq \mathbb{R}^n$ with $\omega = |x|^{-n} \sum_{i=1}^n (-1)^{i-1} x^i dx^i \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \in \Omega^{n-1}(\mathbb{R}^n - \{0\})$, we have that $d\omega = 0$, ω is not exact ($\int_{S^n} \omega \neq 0$) and that $H^{n-1}(S^{n-1}) \neq \{0\}$.

Notation

Given $\omega \in Z^k(M)$, we write the de Rahm cohomology class $[\omega]$. The corresponding element in $H^k_{\text{dR}}(M)$, $[\omega_1] = [\omega_2]$ in $H^k_{\text{dR}}(M)$ means ω_1 and ω_2 differ by an exact form (i.e. $\omega_2 = \omega_1 + d\eta$ for some $\eta \in \Omega^{k-1}(M)$.

Proposition

If $F : M \to N$ is a diffeomorphism, it induces $F^* : \Omega^*(N) \to \Omega^*(M)$ which maps $Z^*(N) \to Z^*(M)$ and $B^*(N) \to B^*(M)$.

- Proof

  - For $\omega \in Z^*(N)$ with $d\omega = 0$, $d(F^*\omega) = F^*(d\omega) = 0$. So $F^*\omega$ is closed.

  - For $\omega \in B^*(N)$ with $\omega = d\eta$, $F^*\omega = F^*(d\eta) = d(F^*\eta)$. So $F^*\omega$ is exact.

  Therefore, $F^* : H^k_{\text{dR}}(N) \to H^k_{\text{dR}}(M)$.

For $F \circ G = \text{id}$ and $G \circ F = \text{id}$, the descend to $F^* \circ G^* = \text{id}$ and $G^* \circ F^* = \text{id}$ on H^k_{dR}. Hence $F^* : H^*_{\text{dR}}(N) \to H^*_{\text{dR}}(M)$ is an isomorphism.

Proposition 17.5

Let $M^n = \coprod_j M_j$ be a disjoint union of at most countably many connected manifolds, (the inclusion map) $\iota_j : M_j \to M$ induces $\iota_j^* : \Omega^k(M) \to \Omega^k(M_j)$ by $\omega \mapsto \omega|_{M_j}$. Define $\Phi : \Omega^k(M) \to \prod_j \Omega^k(M_j)$ by $\omega \mapsto (\iota_1^*\omega, \ldots, \iota_j^*\omega, \ldots) = (\omega|_{M_1}, \ldots, \omega|_{M_j}, \ldots)$. Φ induces an isomorphism $\Phi : H^k_{\text{dR}}(M) \to \prod_j H^k_{\text{dR}}(M_j)$.

- Proof

  - Φ is injective. If $\Phi[\omega] = 0$, then $\left[\omega|_{M_j}\right] = 0$. So ω is exact on M_j for each j, exact on M and $[\omega] = 0$.

– Φ is surjective. Given any $([\omega_1],\dots,[\omega_j],\dots)$, define $\omega \in \Omega^k(M)$ by $\omega|_{M_j} = \omega_j$. Then $\Phi[\omega] = ([\omega_1],\dots,[\omega_j],\dots)$.

Proposition 17.6

If M^n is connected, then $H^0_{\text{dR}}(M) \cong \mathbb{R}$.

Proof

$H^0_{\text{dR}}(M) = Z^0(M)/B^0(M)$ where $Z^0(M) = \{f \in C^\infty(M) : df = 0\} = \{f \in c^\infty(M) : f \equiv c\}$ and $B^0(M) = \{0\}$.

$0 \xrightarrow{d} \Omega^0(M) \xrightarrow{d} \Omega^1(M)$

Hence $H^0_{\text{dR}}(M) \cong \mathbb{R}$.

Homotopy Invariance

Given $F, G : M \to N$, we say F and G are smoothly homotopic to eachother if there exists a smooth map $H : M \times [0,1] \to N$ such that $H(\cdot, 0) = F(\cdot)$ and $H(\cdot, 1) = G(\cdot)$.
They induce $F^*, G^* : H^*_{\text{dR}}(N) \to H^*_{\text{dR}}(M)$.

Proposition 17.10

For $F, G : M \to N$, if $F \simeq G$, then $F^* = G^* : H^*_{\text{dR}}(N) \to H^*_{\text{dR}}(M)$.

Goal

$[F^*\omega] = F^*[\omega] = G^*[\omega] = [G^*\omega]$ with ω closed in N. That is, $F^*\omega$ and $G^*\omega$ differ by an exact form, $G^*\omega - F^*\omega = d\eta$ with $\eta \in \Omega^{k-1}(M)$.
This gives a map $h : Z^k(N) \to \Omega^{k-1}(M)$ by $\omega \mapsto \eta$.
In fact, we will construct a map $h : \Omega^k(N) \to \Omega^{k-1}(M)$ such that $G^*\omega - F^*\omega = d(h(\omega)) + h(d\omega)$. Then for any closed k-form ω, $G^*\omega - F^*\omega = d(h(\omega)) + 0$, $[G^*\omega] = [F^*\omega]$ in $H^k_{\text{dR}}(M)$ and $G^* = F^*$.

Lemma 17.9

Given $\iota_0, \iota_1 : M \hookrightarrow M \times [0,1]$ (where clearly $\iota_0 \simeq \iota_1$), then there exists $h : \Omega^k(M \times [0,1]) \to \Omega^{k-1}(M)$ such that $\iota_1^*\omega - \iota_0^*\omega = d(h(\omega)) + h(d\omega)$ for all $\omega \in \Omega^k(M \times [0,1])$.
Assuming that 17.9 holds, 17.10 follows.

IMAGE 1

$F = h \circ \iota_0$, $G = H \circ \iota_1$. At the H^*_{dR} level,

$$F^* = (h \circ \iota_0)^* = \iota_0^* \circ h^* = \iota_1^* \circ h^* = (h \circ \iota_1)^* = G^*.$$

Proof of 17.9

Consider $V = \frac{\partial}{\partial t} \in \mathfrak{X}(M \times [0,1]$ with flow $\theta_t(x,s) = (x, s+t)$, so $\theta_t \circ \iota_0 = \iota_t$ and $\iota_0^* \circ \theta_t^* = \iota_t^*$ at the Ω^*-level. Compute

$$
\begin{aligned}
\iota_1^* \omega - \iota_0^* \omega &= \int_0^1 \frac{d}{dt}(\iota_t^* \omega)\, dt \\
&= \int_0^1 \frac{d}{dt}\left(i_0^* \circ \theta_t^*(\omega)\right)\, dt \\
&= \int_0^1 \iota_0^* \left(\frac{d}{dt}\theta_t^*(\omega)\right)\, dt && \left.\frac{d}{dt}\right|_{t=t_0} \theta_t^* \omega = \theta_{t_0}^*(\mathcal{L}_V \omega) \\
&= \int_0^1 \iota_0^* \left(\theta_t^*(\mathcal{L}_V \omega)\right)\, dt \\
&= \int_0^1 \iota_t^* (\mathcal{L}_V \omega)\, dt && \mathcal{L}_V \omega = d \circ i_V(\omega) + i_V \circ d(\omega) \\
&= \int_0^1 \iota_t^* (d(V \lrcorner \omega) + V \lrcorner (d\omega))\, dt \\
&= \int_0^1 d(\iota_t^*(V \lrcorner \omega))\, dt + \int_0^1 \iota_t^* (V \lrcorner d\omega)\, dt \\
&= d\left(\int_0^1 \iota_t^*(V \lrcorner \omega)\, dt\right) + \int_0^1 \iota_t^*(V \lrcorner d\omega)\, dt
\end{aligned}
$$

Then we may define $h : \Omega^k(M \times [0,1]) \to \Omega^{k-1}(M)$ by $h(\omega) = \int_0^1 \iota_t^*(V \lrcorner \omega)\, dt$. Then

$$\iota_1^* \omega - \iota_0^* \omega = d(h(\omega)) + h(d\omega).$$

More precisely, for $q \in M$,

$$h(\omega)_q = \int_0^1 \underbrace{\iota_t^* \overbrace{(V \lrcorner \omega_{(q,t)})}^{\in \Lambda^{k-1} T_{(q,t)}(M \times [0,1])}}_{\in \Lambda^{k-1} T_q M}\, dt$$

Corollary

If M and N are homotopic to each other, then $H^k_{\mathrm{dR}}(M) \cong H^k_{\mathrm{dR}}(N)$. That is, there exist maps $F : M \to N$, $G : N \to M$ such that $G \circ F \simeq \mathrm{id}_M$ and $F \circ G \simeq \mathrm{id}_N$. Therefore,

$$
\begin{aligned}
F^* \circ G^* &= (G \circ F)^* = (\mathrm{id}_M)^* = \mathrm{id}_{H^*_{\mathrm{dR}}(M)} \\
G^* \circ F^* &= (F \circ G)^* = (\mathrm{id}_N)^* = \mathrm{id}_{H^*_{\mathrm{dR}}(N)}
\end{aligned}
$$

and both F^* and G^* are isomorphisms.

Example

$\mathbb{R}^n$ is homotopic to $\{0\}$

$$
\begin{aligned}
F : \mathbb{R}^n &\to 0 & \qquad G : 0 &\to \mathbb{R}^n \\
x &\mapsto 0 & 0 &\mapsto 0
\end{aligned}
$$

so $F\circ G: 0\to 0$ (id_0), $G\circ F:\mathbb{R}^n\to\mathbb{R}^n$ by $x\mapsto 0$ $(\simeq \mathrm{id}_{\mathbb{R}^n})$.
Consider $H:\mathbb{R}^n\times[0,1]\to\mathbb{R}^n$ by $(x,t)\mapsto tx$ with $H(\cdot,0)=0$ and $H(\cdot,1)=\mathrm{id}_{\mathbb{R}^n}$. More generally, if $U\subseteq\mathbb{R}^n$ is star shaped then U is homotopic to $\{p\}$.

Definition: Contractible

We say that M is contractible if M is homotopic to a point

$$H^k_{\text{dR}}(p)=\begin{cases}0 & k\neq 0\\ \mathbb{R} & k=0\end{cases}.$$

Corollary

M is contractible (e.g. $M=\mathbb{R}^n$ or $M=\mathbb{H}^n$), then

$$H^k_{\text{dR}}(M)=\begin{cases}0 & k\neq 0\\ \mathbb{R} & k=0\end{cases}.$$

In particular, on such an M, $\omega\in\Omega^k(M)$ $(k\geq 1)$ is closed if and only if ω is exact.
In fact, $H^k_{\text{dR}}(M)=0$ $(k\geq 1)$ means $B^k(M)=Z^k(M)$.

Mayer-Vietoris Sequence

Setup

Take M covered by two open sets U,V.

$$\begin{tikzcd} & U \arrow[dr, "k"] & \\ U\cap V \arrow[ur, "i"] \arrow[dr, "j"'] & & M \\ & V \arrow[ur, "l"'] & \end{tikzcd} \qquad \begin{tikzcd} & \Omega^k(U) \arrow[dr, "i^*"] & \\ \Omega^k(M) \arrow[ur, "k^*"] \arrow[dr, "l^*"'] & & \Omega^k(U\cap V) \\ & \Omega^k(V) \arrow[ur, "j^*"'] & \end{tikzcd}$$

Consider a short exact sequence

$$0\longrightarrow\Omega^k(M)\xrightarrow{k^*\oplus l^*}\Omega^k(U)\oplus\Omega^k(V)\xrightarrow{i^*-j^*}\Omega^k(U\cap V)\longrightarrow 0$$

$$\omega\longmapsto(\omega|_U,\omega|_V)\longrightarrow 0$$

$$(\omega,\eta)\longmapsto(\omega|_{U\cap V}-\eta|_{U\cap V})$$

To show $0\mapsto\Omega^k(M)\mapsto\Omega^k(U)\oplus\Omega^k(V)$ is exact, we need to show that $k^*\oplus l^*$ is injective.
Suppose $(\omega|_U,\omega|_V)=(0,0)$. Since $U\cap V=M$, $\omega\equiv 0$ on M. Therefore $k^*\oplus l^*$ is injective.
To show $\Omega^k(M)\mapsto\Omega^k(U)\oplus\Omega^k(V)\mapsto\Omega^K(U\cap V)$, $\ker(i^*-j^*)\supseteq\mathrm{im}(k^*\oplus l^*)$. In fact, if $(\omega|_U,\omega|_V)\in\mathrm{im}(k^*\oplus l^*)$, then $\omega|_{U\cap V}=\omega|_{U\cap V}$ and $(i^*-j^*)(\omega|_U,\omega|_V)=0$.
For $\mathrm{im}(k^*\oplus l^*)\supseteq\ker(i^*-j^*)$, let $(\omega,\eta)\in\ker(i^*-j^*)$. Then $\omega|_{U\cap V}-\eta|_{U\cap V}=0$. Define $\sigma\in\Omega^k(M)$ by

$$\sigma = \begin{cases} \omega & \text{on } U \\ \eta & \text{on } V \end{cases}$$

Then $(\omega, \eta) = (k^* \oplus l^*)(\sigma)$.
Finally, to show $\Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V) \to 0$, we need to show that $i^* - j^*$ is surjective.
Let $\omega \in \Omega^k(U \cap V)$, and let $\{\varphi_U, \varphi_V\}$ be a partiation of unity with respect to $\{U, V\}$.

IMAGE 2

Define $\eta_U = \varphi_U \omega \in \Omega^k(U)$ on U and $\eta_V = -\phi_V \omega \in \Omega^k(V)$ on V. Then on $U \cap V$,

$$\eta_U - \eta_V = (\varphi_U + \varphi_V)\omega = \omega$$

That is, $(i^* - j^*)(\eta_u, \eta_v) = \omega$.

March 5, 2025

Recall

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0$$

With $d \circ d = 0$, $Z^k(M)$ the set of closed k-forms, $B^k(M)$ the set of exact k-forms, and the de Rahm cohomology $H^k_{\mathrm{dR}}(M) = Z^k(M)/B^k(M)$.

1. M is connected, then $H^0_{\mathrm{dR}}(M) = \mathbb{R}$.
2. If M is contractible, then $H^k_{\mathrm{dR}}(M) = H^k_{\mathrm{dR}}(p)$ for p a point in M.

Recall also the Mayer-Vietoris setup (see above).

Mayer-Vietoris

The short exact sequence

$$0 \longrightarrow \Omega^k(M) \xrightarrow{k^* \oplus l^*} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{i^* - j^*} \Omega^k(U \cap V) \longrightarrow 0$$

induces a long exact sequence

$$\cdots \xrightarrow{\delta} H^k_{\mathrm{dR}}(M) \longrightarrow H^k_{\mathrm{dR}}(U) \oplus H^k_{\mathrm{dR}}(V) \longrightarrow H^k_{\mathrm{dR}}(U \cap V)$$

$$\xrightarrow{\delta} H^{k+1}_{\mathrm{dR}}(M) \longrightarrow H^{k+1}_{\mathrm{dR}}(U) \oplus H^{k+1}_{\mathrm{dR}}(V) \longrightarrow H^{k+1}_{\mathrm{dR}}(U \cap V)$$

$$\longrightarrow \cdots$$

Definition: Chain COmplex

A chain complex A^i is a $\mathbb{R}$-vector group

$$0 \longrightarrow A^n \xrightarrow{\partial} A^{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} A^1 \xrightarrow{\partial} A^0 \xrightarrow{\partial} 0$$

with $\partial \circ \partial = 0$.

A cochain complex is

$$0 \longrightarrow A^0 \xrightarrow{d} A^1 \xrightarrow{d} \cdots \xrightarrow{d} A^{n-1} \xrightarrow{d} A^n \xrightarrow{d} 0$$ with $d \circ d = 0$ and the k-th cohomology is $\ker / \operatorname{im}$ in A^i.

We write the cochain complex as A^*. A short exact sequence of cochain complexes

$$0 \longrightarrow \Omega^*(M) \longrightarrow \Omega^*(U) \oplus \Omega^*(V) \longrightarrow \Omega^*(U \cap V) \longrightarrow 0$$

$$0 \longrightarrow A^* \longrightarrow B^* \longrightarrow C^* \longrightarrow 0$$

Theorem

A short exact sequence of cochain complexes

$$0 \longrightarrow A^* \longrightarrow B^* \longrightarrow C^* \longrightarrow 0$$ induces a long exact sequence of cohomoology groups

$$\cdots \longrightarrow H^k(A) \longrightarrow H^k(B) \longrightarrow H^k(C)$$

$$\longrightarrow H^{k+1}(A) \longrightarrow H^{k+1}(B) \longrightarrow H^{k+1}(C)$$

$$\longrightarrow \cdots$$

Proof

We want $\delta : H^k(C) \to H^{k+1}(A)$
Given $a \in C^k$ with $dc = 0$, we need to come up with some $a \in A^{k+1}$ with $da = 0$.

$$\begin{tikzcd}
& b \arrow[r, mapsto] \arrow[d, mapsto] & c \arrow[r, mapsto] \arrow[d, mapsto, "d"] & 0 \\
a' \arrow[r, mapsto] & b' \arrow[r, mapsto] & 0
\end{tikzcd}$$

So define $\delta(c) = a'$.

Cochain Complexes

The full picture is given by

$$\begin{tikzcd}
& \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & \Omega^{k-1}(M) \arrow[r] \arrow[d, "d"] & \Omega^{k-1}(U) \oplus \Omega^k(V) \arrow[r] \arrow[d, "d"] & \Omega^{k-1}(U \cap V) \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & \Omega^k(M) \arrow[r, "k^* \oplus l^*"] \arrow[d, "d"] & \Omega^k(U) \oplus \Omega^k(V) \arrow[r, "i^* - j^*"] \arrow[d, "d"] & \Omega^k(U \cap V) \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & \Omega^{k+1}(M) \arrow[r] \arrow[d] & \Omega^{k+1}(U) \oplus \Omega^k(V) \arrow[r] \arrow[d] & \Omega^{k+1}(U \cap V) \arrow[r] \arrow[d] & 0 \\
& \vdots & \vdots & \vdots &
\end{tikzcd}$$

Then we have for $\omega = \eta_U - \eta_V$ on $U \cap V$.

$$\begin{tikzcd} & (\eta_u,\eta_v) \arrow[r, mapsto] \arrow[d, mapsto] & \omega \in Z^k(U\cap V) \\ \sigma \arrow[r, mapsto] & (d\eta_U, d\eta_V) & \end{tikzcd}$$

Since $\sigma|_U = d\eta_U$ and $\sigma|_V = d\eta_V$.

Example

Let $M = S^n$. Then $U = S^n - \{\text{north pole}\}$, $V = S^n - \{\text{south pole}\}$ and U, V are diffeomorphic to $\mathbb{R}^n$. It follows that $U \cap V = S^n - \{\text{two poles}\} \cong \mathbb{R}^n - \{0\} \simeq S^{n-1}$ and

$$H^k_{\mathsf{dR}}(U) = H^k_{\mathsf{dR}}(V) = \begin{cases} 0 & k \neq 0 \\ \mathbb{R} & k = 0 \end{cases}.$$

Then for $k \geq 1$,

$$\cdots \longrightarrow H^k_{\mathsf{dR}}(S^n) \longrightarrow \overbrace{H^k_{\mathsf{dR}}(U)}^{=0} \oplus \overbrace{H^k_{\mathsf{dR}}(V)}^{=0} \longrightarrow H^k_{\mathsf{dR}}(U\cap V)$$

$$\longrightarrow H^{k+1}_{\mathsf{dR}}(S^n) \longrightarrow \overbrace{H^{k+1}_{\mathsf{dR}} \oplus H^{k+1}_{\mathsf{dR}}(V)}^{=0} \longrightarrow \cdots$$

and we have a short exact sequence $0 \to A \to B \to 0$ such that $A \cong B$.
It follows that $H^{k+1}_{\mathsf{dR}}(S^n) \cong H^k_{\mathsf{dR}}(U\cap V) \cong H^k_{\mathsf{dR}}(S^{n-1})$.

IMAGE 1

Then

$$0 \longrightarrow H^0_{\mathsf{dR}}(S^1) \longrightarrow H^0_{\mathsf{dR}}(U) \oplus H^0_{\mathsf{dR}}(V) \longrightarrow H^0_{\mathsf{dR}}(U\cap V)$$

$$\longrightarrow H^1_{\mathsf{dR}}(S^1) \longrightarrow \overbrace{H^1_{\mathsf{dR}}(U) \oplus H^1_{\mathsf{dR}}(V)}^{=0}$$

Which gives

$$0 \longrightarrow \mathbb{R} \xrightarrow{\mathrm{im}\cong\mathbb{R}} \mathbb{R}^2 \xrightarrow[\mathrm{im}\cong\mathbb{R}]{\ker\cong\mathbb{R}} \mathbb{R}^2 \xrightarrow{\ker\simeq\mathbb{R}} \overbrace{H^1_{\mathsf{dR}}(S^1)}^{\cong\mathbb{R}} \longrightarrow 0$$

and therefore that

$$H^k_{\mathsf{dR}}(S^1) = \begin{cases} \mathbb{R} & k \in \{0,1\} \\ 0 & k \notin \{0,1\} \end{cases}.$$

For $n \geq Z$, $U \cap V$ continuous

$$0 \longrightarrow H^0_{\mathsf{dR}}(S^n) \longrightarrow H^0_{\mathsf{dR}}(U) \oplus H^0_{\mathsf{dR}}(V) \longrightarrow H^0_{\mathsf{dR}}(U\cap V)$$

$$\longrightarrow H^1_{\mathsf{dR}}(S^n) \longrightarrow \overbrace{H^1_{\mathsf{dR}}(U) \oplus H^1_{\mathsf{dR}}(V)}^{=0}$$

and

$$0 \longrightarrow \mathbb{R} \xrightarrow{\mathrm{im}\cong\mathbb{R}} \mathbb{R}^2 \xrightarrow[\mathrm{im}\cong\mathbb{R}]{\ker\cong\mathbb{R}} \mathbb{R} \xrightarrow{\ker\simeq\mathbb{R}} \overbrace{H^1_{\mathsf{dR}}(S^n)}^{=0} \longrightarrow 0$$

Therefore, $H^0_{\mathsf{dR}}(S^3) = \mathbb{R}$, $H^1_{\mathsf{dR}}(S^3) = 0$, $H^2_{\mathsf{dR}}(S^3) \cong H^1_{\mathsf{dR}}(S^2) =$

0 and $H^3_{\mathrm{dR}}(S^3) \cong H^2_{\mathrm{dR}}(S^2) \cong \mathbb{R}$. By induction, we conclude that

$$H^k_{\mathrm{dR}}(S^n) = \begin{cases} \mathbb{R} & k \in \{0, n\} \\ 0 & k \notin \{0, n\} \end{cases}.$$

Corollary

Take $\omega \in \Omega^n(S^n)$ closed where $\omega = |x|^{-n} \sum_i (-1) x^i \, dx^i \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$. Then $\omega|_{S^n}$ is closed but not exact.
Hence $[\omega] \in H^n_{\mathrm{dR}}(S^0) = \mathbb{R}$ is a non-trivial element.
Since $H^n_{\mathrm{dR}} S^n = \mathbb{R}$, and element in $H^n_{\mathrm{dR}}(S^n)$ is of the form $[c\omega]$ for $c \in \mathbb{R}$.

Corollary

$\omega \in \Omega^n(S^n)$ is exact if and only if $\int_{S^n} \omega = 0$.

Proof

$\Longrightarrow$ if $\omega = d\eta$, then $\int_{S^n} d\eta = \int_{\partial S^n} \eta = 0$ by Stokes' theorem.
$\Longleftarrow$ If $I : \Omega^n(S^n) \to \mathbb{R}$ by $\omega \mapsto \int_{S^n} \omega$ then, since $\Omega^n(S^n) = Z^n(S^n)$ and $I(B^n(S^n)) = 0$ by Stokes', it induces

$$\begin{aligned} I : \overbrace{H^n_{\mathrm{dR}}(S^n)}^{=\mathbb{R}} &\to \mathbb{R} \\ [\omega] &\to \int_{S^n} \omega \end{aligned}$$

I is surjective, hence I is an isomorphism. In particular $\ker I = \{0\}$. That is, $\int_{S^n \omega = 0}$ implies ω is exact.

Corollary

Let $U \subseteq \mathbb{R}^n$ be an open subset and $x \in U$. Then $H^{n-1}_{\mathrm{dR}}(U - \{x\} \neq 0$.

Proof

Let S^{n-1} be a sphere in $U - \{x\}$ which encloses x. Then we have inlcusion $\iota : S \to (U - \{x\})$ and radial projection $r : (U - \{x\}) \to S$.

IMAGE 2

So $r \circ \iota = \mathrm{id}_S$ and

$$\iota^* \circ r^* = (r \circ \iota)^* = \mathrm{id} : H^{n-1}_{\mathrm{dR}}(S) \to H^{n-1}_{\mathrm{dR}}(S^{n-1})$$

which implies that

$$r^* = \overbrace{H^{n-1}_{\mathrm{dR}}(S)}^{\cong\mathbb{R}} \to H^{n-1}_{\mathrm{dR}}(U - \{x\})$$

is injective.

Theorem 17.26: Topological Invariance of Dimension

Let $U \subseteq \mathbb{R}^n$ and $V \in \mathbb{R}^m$ be open $(n < m)$. Then U is nothomeomorphic to V.

Proof

Suppose U is homeomoprhic to V by φ. Then $U - \{x\}$ is homeomorphic to $V - \{\varphi(x)\}$.
We have that if $W = B_r^n(0) \subseteq U$, then $\varphi(W)$ is open in $\mathbb{R}^m$ and, therefore, $W = B_r^n(0)$ is homeomorphic to both $\mathbb{R}^n$ and $\varphi(W) \subseteq \mathbb{R}^m$.
Therefore $H_{\text{dR}}^{m-1}(\mathbb{R}^n - \{x\}) = H_{\text{dR}}^{m-1}(S^{n-1}) = 0$ but $H_{\text{dR}}^{m-1}(V - \{\varphi(x)\}) \neq 0$.

Compactly Supported de Rahm Cohomology

Let $\Omega_C^k(M) = \{\omega \in \Omega^k(M) : \omega \text{ is compactly supported}\}$.

$$0 \xrightarrow{d} \Omega_C^0(M) \longrightarrow \cdots \longrightarrow \Omega_C^n(M) \longrightarrow 0$$

If $\omega = d\eta$, can we choose $\eta \in \Omega_C^{k-1}(M)$?

Lemma 17.27: Poincaré Lemma

Let $\omega \in \Omega_C^k(\mathbb{R}^n)$ be a closed k-form and, for $k = n$, further assume that $\int_{\mathbb{R}^n} \omega = 0$.
Then there exists $\eta \in \Omega_C^{k-1}(M)$ such that $d\eta = \omega$.

Proof

If $n = k = 1$ and $\omega \in \Omega_C^1(\mathbb{R})$, $\omega = f(t)\,dt$ for $f \in C_C^\infty(M)$ and $\int_{\mathbb{R}} f = 0$.
We need to show $F \in C_C^\infty(M)$ such that $dF = \omega$ (i.e. $F'(t)\,dt = f(t)\,dt$ or $F'(t) = f(t)$). Set

$$F(t) = \int_{-\infty}^t f(t)\,dt \left(= \int_{-R}^t f(t)\,dt \right).$$

where $\operatorname{supp} f \subseteq (-R, R)$. $F'(t) = f(t) - f(-R) = f(t)$. So $\operatorname{supp} F \subseteq (-R, R)$.
For $n \geq 2$, $\omega \in \Omega_C^k(M)$ closed and $\operatorname{supp}\omega \subseteq B_R(0)$, by the usual Poncaré lemma, there is $\eta_0 \in \Omega^{k-1}(M)$ such that $d\eta_0 = \omega$.
Our goal is to find $\eta \in \Omega_C^{k-1}(M)$ such that $d\eta = d\eta_0 (= \omega)$.
If $k = 1$, $\omega \in \Omega_C^1(M)$, $\eta_0 \in C_C^\infty(M)$ such that $d\eta_0 = \omega$, and $\operatorname{supp}\omega \subseteq B_R(0)$. Hence outside $B_R(0)$, $d\eta_0 = \omega = 0$ and $\eta_0 = c$ on $\mathbb{R}^n - B_R(0)$.
Consider $\eta = \eta_0 - c \in C_C^\infty(\mathbb{R}^n)$. Then $d\eta = d\eta_0 = \omega$.
If $1 \leq k \leq n-1$, $\omega \in \Omega_C^k(\mathbb{R}^n)$ closed, and $\eta_0 \in \Omega^{k-1}(\mathbb{R}^n)$ such that $d\eta_0 = \omega$, on $\mathbb{R}^n - B_R(0)$ where $\operatorname{supp}\omega \subseteq B_R(0)$ we have that $d\eta_0 = \omega = 0$. That is, $\eta_0 \in Z^{k-1}(\mathbb{R}^n - B_R(0))$. We know that $\mathbb{R}^n - B_R(0) \simeq S^{n-1}$ and $H_{\text{dR}}^{k-1}(\mathbb{R}^n - B_r(0)) = H_{\text{dR}}^{k-1}(S^{n-1}) = 0$.
Therefore, every closed $(k-1)$-form on $\mathbb{R}^n - B_R(0)$ is exact. Then there exists $\sigma \in \Omega^{k-2}(\mathbb{R}^n - B_R(0))$ such that $d\sigma = \eta_0$.
PROOF TO BE CONTINUED

March 10, 2025

Recall

Poincaré lemma with compact support, $\omega \in \Omega_C^k(\mathbb{R}^n)$ closed.
If $k = n$, we also assume that $\int_{\mathbb{R}^n} \omega = 0$. Then $\eta \in \Omega_C^{k-1}(M)$ such that $d\eta = \omega$.
By Poincaré lemma, there is $\eta \in \Omega^{k-1}(M)$ such that $d\eta = \omega$. We need to modify this η.

Cases (1) $k = n = 1$; and (2) $n \geq 2$, $k = 1$ are above.
If $\omega = 0$ on $\mathbb{R}^n - B_R(0)$, then $dF = \omega\varnothing$ on $\mathbb{R}^n - B_R(0)$ with F constant on $\mathbb{R}^n - B_R(0)$. Then also $F - c \in \Omega^0_C(\mathbb{R}^n)$ such that $d(F - c) = dF = \omega$ on $\mathbb{R}^n$.

Poincaré Lemma (Continued)

Proof (Continued)

For $n \geq 2$ and $2 \leq k \leq n-1$, $\omega \in \Omega^k_C(\mathbb{R}^n)$ and $\operatorname{supp}\omega \subseteq B_r(0) \subseteq B_R(0)$.
By Poincaré lemma, there exists $\eta \in \Omega^{k-1}(\mathbb{R}^n)$ such that $d\eta = \omega$, $d\eta = \omega = 0$ on $\mathbb{R}^n - B_r(0)$ with $\eta \in \Omega^{k-1}(\mathbb{R}^n - B_r(0))$ closed.
We know that $(\mathbb{R}^n - B_r(0)) \simeq S^{n-1}$ and $H^{k-1}_{\mathrm{dR}}(S^{n-1}) = 0$. Hence, $\eta \in \Omega^{k-1}(\mathbb{R}^n - B_r(0))$ is exact (i.e. $\eta = d\sigma$ for $\sigma \in \Omega^{k-2}(\mathbb{R}^n - B_r(0))$.
Let ψ be a bump function where $\psi \equiv 1$ on $\mathbb{R}^n - B_R(0)$. Define $\eta_0 = \eta - d(\psi\sigma)$. Then $d\eta_0 = d\eta - d^2(\psi\sigma) = \omega$.
On $\mathbb{R}^n - B_R(0)$, $\eta_0 = \eta - d(\psi\sigma) = \eta - d\sigma = 0$. Hence $\eta_0 \in \Omega^{k-1}_C(\mathbb{R}^n)$.
In the final case, $n \geq 2$, $k = n$, $\omega \in \Omega^n_C(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \omega = 0$. Here the previous proof does not work because $H^{k-1}_{\mathrm{dR}}(S^{n-1}) = \mathbb{R} \neq 0$.
Let $R > r > 0$ such that $\operatorname{supp}\omega B_r(0) \subseteq B_R(0)$.

$$0 \int_{B_r(0)} \omega = \int_{B_r(0)} d\eta = \int_{\partial B_r(0)} \eta.$$

That is, we have $\eta \in \Omega^{n-1}(\mathbb{R}^n)$ such that $d\eta = \omega$ and $\int_{\partial B_r(0)} \eta = 0$. Recall that

$$\begin{aligned} H^{n-1}(S^{n-1}) &\to \mathbb{R} \\ [\eta] &\mapsto \int_{S^{n-1}} \eta \end{aligned}$$

Hence $[\eta] = 0 \in H^{n-1}_{\mathrm{dR}}(\mathbb{R}^n - B_r(0))$. Hence $\eta = d\sigma$ for some $\sigma \in \Omega^{n-2}(\mathbb{R}^n - B_r(0))$ and the proof proceeds as in the previous case.

Definition: Compactly Supported de Rahm Cohomology Group

For M^n,

$$0 \longrightarrow \Omega^0_C(M) \xrightarrow{d} \Omega^1_C(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_C(M) \xrightarrow{d} 0$$
where

$$H^k_C(M) = \frac{\text{closed } k\text{-forms with compact support}}{\text{exact } k\text{-forms with compact support}}.$$

Theorem 17.28

$$H^k_C(\mathbb{R}^n) = \begin{cases} 0 & 0 \leq k \leq n-1 \\ \mathbb{R} & k = n \end{cases}.$$

Remark

For $k = n$,

$$\begin{aligned} I : H^n_C &\to \mathbb{R} \\ [\omega] &\mapsto \int_{\mathbb{R}^n} \omega \end{aligned}$$

is an isomorphism.

Remark

H^k_{dR} is a homotopic invariance, but H^k_C is not.

Theorem 17.30

Let M^n be connected, oriented and without boundary. Then $H^n_C(M) = \mathbb{R}$. In particular, if M is closed (i.e. compact and without boundary), then $H^n_{\text{dR}}(M) = H^n_C(M) = \mathbb{R}$.

Proof

Write

$$I : \Omega^n_C(M) \to \mathbb{R}$$
$$\omega \mapsto \int_M \omega$$

If $\omega = d\eta$ is exact, then

$$\int_M \omega = \int_M d\eta = \int_{\partial M} \eta = 0.$$

I induces

$$I : H^n_C \to \mathbb{R}$$
$$[\omega] \mapsto \int_M \omega$$

We want to show that I is an isomorphism. In the trivial case, $n = 0$, $M = \{\text{point}\}$ so $I(f) = f(\text{point})$.

$$H^0_C(\text{point}) = \Omega^0_C(\text{point}) = \{f : \text{point} \to \mathbb{R}\} \cong \mathbb{R}.$$

If $n \geq 1$, let $(U, (x^i))$ be a chart in M, $\theta \in \Omega^n_C(U)$ by $\theta = f\, dx^1 \wedge \cdots \wedge dx^n$ and $f \geq 0$ but not constantly zero on U. So $\int_U \theta = c > 0$ and $\theta \in \Omega^n_C(M)$ by extending as 0 outside of U. So I is surjective.
For injectivity, we need to show that if $\int_M \omega = 0$ then $\omega = d\eta$ for some $\eta \in \Omega^{n-1}_C(M)$.
Cover M by open sets $\{U_i\}$ such that

1. each U_i is diffeomorphic to $\mathbb{R}^n$,
2. $\operatorname{supp} \omega \subseteq \bigcup_{i=1}^k U_i$, and
3. relable $\{U_i\}_{i=1}^k$ if necessary.

Then write $M_j = \bigcup_{i=1}^j U_i$ which satisfies $M_j \cap U_{j+1} \neq \varnothing$. We will prove by induction that for each $j = 1, \ldots, k$ such that if $\omega \in \Omega^n_C(M_j)$ and $\int_{M_j} \omega = 0$, there is $\eta \in \Omega^{n-1}_C(M_j)$ such that $d\eta = \omega$.
When $j = 1$, $M_1 \cong \mathbb{R}^n$ and this follows from the Poincaré lemma with compact support.
Consider the $j+1$ case with $\omega \in \Omega^n_C(M_{j+1})$ and $\int_{M_j} \omega = 0$. Let $\{\varphi, \psi\}$ be a partition of unity with respect to $\{M_j, U_{j+1}\}$

($\operatorname{supp}\varphi \subseteq M_j$ and $\operatorname{supp}\psi \subseteq U_{j+1}$). Then $\varphi\omega \in \Omega^n_C(M_j)$. If $\int_{M_j}\varphi\omega = 0$, then by induction there exists $\alpha \in \Omega^{n-1}_C(M_j)$ such that $d\alpha = \varphi\omega$. By assumption

$$\int_{U_{j+1}} \psi\omega = \int_{M_{j+1}} \psi\omega = \int_{M_{j+1}} (1-\varphi)\omega = \int_{M_{j+1}} \omega - \int_{M_j} \varphi\omega = 0.$$

Then there exists $\beta \in \Omega^n_C(U_{j+1})$ such that $d\beta = \psi\omega$, and $\alpha+\beta \in \Omega^n_C(M_{j+1})$ has $d(\alpha+\beta) = (\varphi+\psi)\omega = \omega$.
In general, $\int_{M_j}\varphi\omega = c$. Construct $\theta \in \Omega^n_C(M_j \cap U_{j+1})$ such that $\int_{M_j\cap U_{j+1}} \theta = 1$. Then $\int_{M_j}\varphi\omega - c\theta = 0$. By induction, there exists $\alpha \in \Omega^{n-1}_C(M_j)$ such that $d\alpha = \varphi\omega - c\theta$. Then for $\psi\omega + c\theta \in \Omega^n_C(U_{j+1})$,

$$\int_{U_{j+1}} \psi\omega + c\theta = \int_{M_{j+1}} \omega - \int_{M_j} \varphi\omega + \int_{U_{j+1}} c\theta = 0 - c + c = 0$$

Then there exists $\beta \in \Omega^n_C(U_{j+1})$ such that $d\beta = \psi\omega + c\theta$ and $\alpha+\beta \in \Omega^n_C(M_j+1)$ has $d(\alpha+\beta) = (\varphi+\psi)\omega = \omega$.

Remark

For M^n oriented, connected and without boundary,

1. $H^n_C(M) \cong \mathbb{R}$ (in particular, if M is closed then $H^n_{\mathrm{dR}}(M) \cong \mathbb{R}$).
2. If M is non-compact, then $H^n_{\mathrm{dR}}(M) = 0$.

Proof of 2

The proof requires an "exhaustion function". That is, a smooth function $f : M \to \mathbb{R}$ such that

1. $\inf f > -\infty$ and
2. $f^{-1}(-\infty, c]$ is compact for every c.

This means $M = \bigcup_{k=0}^{\infty} f^{-1}(-\infty, k]$. As an example, consider $M = \mathbb{R}^n$ and $f(x) = x_1^2 + \cdots + x_n^2$. Then $f^{-}(\infty, c] = \overline{B_C}(0)$ is compact.
Without loss of generality, let $\inf_M f = 0$. Then $M = f^{-1}([0,+\infty))$. Let $V_i = f^{-1}((i-2,i))$ for $i \in \mathbb{N}$. Then V_i only intersects V_{i-1} and V_{i+1}.
Let $\omega \in \Omega^n(M)$. Our goal is to find η such that $d\eta = \omega$. Let $\{\varphi_i\}$ be a partition of unity with respect to $\{V_i\}$. Then let $\omega_i = \varphi_i\omega \in \Omega^n_C(V_i)$. On V_1, if $\int_{V_1}\omega_1 = 0$, then since $H^n_C(V_1) \cong \mathbb{R}$ we have that $\omega_1 = d\eta_1$ for some $\eta_1 \in \Omega^{n-1}_C(V_1)$.
If $\int_{V_1}\omega_1 = c_1 \neq 0$, we construct $\theta_1 \in \Omega^n_C(V_1 \cap V_2)$ such that $\int_{V_1\cap V_2}\theta_1 = 1$. Then $\int_{V_1}\omega_1 - c_1\theta_1 = 0$. Hence there exists $\eta_1 \in \Omega^{n-1}_C(V_1)$ such that $d\eta_1 = \omega_1 - c\theta_1$.
In general, on each $V_i \cap V_{i+1}$, we may construct $\theta_i \in \Omega^n_C(V_i \cap V_{i+1})$ such that $\int_{V_i\cap V_{i+1}}\theta_i = 1$. For $i = 2$, we choose c_2 suc that $\int_{V_2}\omega_2 + c_1\theta_1 - c_2\theta_2 = 0$. Then there exists $\eta_2 \in \Omega^{n-1}_c(V_2)$ such that $d\eta_2 = \omega_2 + c_1\theta_1 - c_2\theta_2$.
Inductively, we have $\omega_i = \varphi_i\omega$ with $\theta_i \in \Omega^n_C(V_i \cap V_{i+1})$ and $\eta_i \in \Omega^{n-1}_C(V_i)$ such that $d\eta_i = \omega_i + c_i\theta_i - c_{i+1}\theta_{i+1}$.
Consider $\eta = \sum_{i=1}^{\infty}\eta_i$ which is a finite sum at any given point. This $\eta \in \Omega^{n-1}(M)$ satisfies $d\eta = d(\sum\eta_i) = d(\sum\varphi_i\omega) = \omega$.

Recall

If M is nonorientable, then there is a double cover $\pi : \hat{M} \to M$ such that $\hat{M}$ is connected and orientable.

Lemma: 17.33

$\pi^* : H^k_{\mathrm{dR}}(M) \to H^k_{\mathrm{dR}}(\hat{M})$ is injective. The same is true of $\pi^* : H^k_C(M) \to H^k_C(\hat{M})$.

Theorem: 17.34

If M^n is connected, non-oriented and without boundary, then $H^n_{\mathrm{dR}}(M) = 0 = H^n_C(M)$.

Proof of First Equality

From above, if $\hat{M}$ is non-compact, $H^n_{\mathrm{dR}}(\hat{M}) = 0$. Because $\pi^* : H^n_{\mathrm{dR}}(M) \to H^n_{\mathrm{dR}}(\hat{M})$ is injective and $H^n_{\mathrm{dR}}(M) = 0$.

March 12, 2025

Lemma

If $\pi : \hat{M} \to M$ is a double cover, then $\pi^* : H^k_{\mathrm{dR}}(M) \to H^k_{\mathrm{dR}}(\hat{M})$ is injective (holds equally for H^*_C).

Proof

Goal: If $[\omega] \in H^k_C(M)$ has $\pi^*[\omega] = 0 \in H^k_C(\hat{M})$, then $[\omega] = 0$. In other words, if $\omega \in \Omega^k_C(M)$ is closed and $\pi^*\omega = d\eta$ for some $\eta \in \Omega^{k-1}_C(\hat{M})$ then ω is exact.
Let $\Gamma = \{\mathrm{id}_{\hat{M}}, \alpha\}$ be the group of covering transformations where $\alpha^2 = \mathrm{id}_{\hat{M}}$ and

$$\begin{tikzcd} \hat{M} \arrow[rr, "\alpha"] \arrow[dr, "\pi"'] & & \hat{M} \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

commutes. Then $\pi \circ \alpha = \pi$. Consider $\tilde{\eta} = \frac{1}{2}(\eta + \alpha^*\eta)$ which satisfies $\alpha^*\tilde{\eta} = \frac{1}{2}(\alpha^*\eta + \alpha^2\eta) = \tilde{\eta}$.
Compute

$$d\tilde{\eta} = \frac{1}{2}(d\eta + \alpha^* d\eta) = \frac{1}{2}(\pi^*\omega + \alpha^*\pi^*\omega) = \frac{1}{2}(\pi^*\omega + (\pi \circ \alpha)^*)\omega = \pi^*\omega.$$

Excercise: since $\tilde{\eta}$ satisfies $\alpha^*\tilde{\eta} = \tilde{\eta}$, it descends to some well-defile $\sigma \in \Omega^{k-1}_C(M)$ such that $\pi^*\sigma = \tilde{\eta}$.
For $U \in M$ open and $V_1, V_2 \in \hat{M}$ in the covers over U, write diffeomorphisms $\tau : U \to V_1$ and $\alpha \circ \tau : U \to V_2$.
Then on V_1 we have $\tau^*\left(\tilde{\eta}|_{V_1}\right) = \sigma$ and

$$d\sigma = d(\tau^*\tilde{\eta}) = \tau^*(d\tilde{\eta}) = \tau^*\pi^*\omega = (\overbrace{\pi \circ \tau}^{\mathrm{id}})^*\omega = \omega.$$

Remark

This lemma also holds for $\pi : \hat{M} \to M$ with finite sheets.

Example

$S^n \xrightarrow{\pi} \mathbb{RP}^n$, $\pi^* : H^k_{\mathrm{dR}}(\mathbb{RP}^n) \to H^k_{\mathrm{dR}}(S^n)$ is injective when $k \neq 0$ or $k \neq n$.
Hence $H^k_{\mathrm{dR}}(\mathbb{RP}^n) = 0$. What about $H^n_{\mathrm{dR}}(\mathbb{RP}^n)$?
When n is odd, $\mathbb{RP}^n$ is orientable, then $H^n_{\mathrm{dR}}(\mathbb{RP}^n) \cong \mathbb{R}$.
When n is even, $\mathbb{RP}^n$ is non-orientable, then $H^n_{\mathrm{dR}}(\mathbb{RP}^n) = 0$.

Hint for Exam Problem 4

if $F : M \to N$ is a diffeomorphism, then

$$\int_M F^* \omega = \pm \int_{F(M)} \omega.$$

Theorem

Let M^n be a connected, non-orientable manifold without boundary, then $H^n_C(M^n) = 0 = H^n_{\mathrm{dR}}(M^n)$.

Proof

Let $\pi : \hat{M} \to M$ be the orientation double cover where $\hat{M}$ is connected and orientable.
$\pi^* : H^n_C(M) \to H^n_C(\hat{M}) \cong \mathbb{R}$ is injective, so we want to show that π^* is the zero map. Therefore, we need to show that $\pi^*[\omega] = 0$ or equivalently that $\int_{\hat{M}} \pi^* \omega = 0$.
Let $\omega \in \Omega^n_C(M)$ and write $\pi^* \omega = \hat{\omega}$. Write $\alpha : \hat{M} \to \hat{M}$ a non-trivial covering transformation which reverses orientation.
Then $\alpha^* \hat{\omega} = \alpha^* \pi^* \omega = \pi^* \omega = \hat{\omega}$ and we compute

$$\int_{\hat{M}} \pi^* \omega = \int_{\hat{M}} \hat{\omega} = \int_{\hat{M}} \alpha^* \hat{\omega} = -\int_{\hat{M}} \hat{\omega} = -\int_{\hat{M}} \pi^* \omega.$$

So it must be the case that $\int_{\hat{M}} \pi^* \omega = 0$.
If M is compact, then $H^n_{\mathrm{dR}}(M) = H^n_C(M) = 0$.
If M is non-compact, then $\pi^* : H^n_{\mathrm{dR}}(M) \to H^n_{\mathrm{dR}}(\hat{M}) = 0$ is injective.

Degree Theory

Suppose $F : M \to N$ for M^n, N^n closed and orientable.
Then F induces $F^* : \mathbb{R} \cong H^n_{\mathrm{dR}}(N) \to H^n_{\mathrm{dR}}(M) \cong \mathbb{R}$, so F^* is the multiplication by a number $k \in \mathbb{R}$.

$$\begin{tikzcd}
\omega \arrow[r, "F^*"] \arrow[d, mapsto, "I"] & F^* \omega \arrow[d, mapsto, "I"] \\
\int_N \omega \arrow[r, mapsto, "k \cdot"] & k \cdot \int_N \omega = \int_M F^* \omega
\end{tikzcd}$$

To show that $k \in \mathbb{Z}$, we prove that if $q \in N$ is a regular value (i.e. $\forall\, p \in F^{-1}(q)$, DF_p is surjective). Then $k = \sum_{p \in F^{-1}(q)} \operatorname{sgn}(p)$ where

$$\operatorname{sgn}(p) = \begin{cases} 1 & \text{if } DF_p \text{ preserves orientation,} \\ -1 & \text{if } DF_p \text{ reverses orientation} \end{cases}.$$

Remark: $F^{-1}(q)$ is an embedded 0 dimensional submanifold (i.e. $F^{-1}(q)$ is a disjoint union of points).

Proof

Let $q \in N$ be a regular value. For each $p \in F^{-1}(p)$, $DF_p : T_p M \to T_q N$ is a linear isomorphism.
Then there is $p \in U$ and $q \in W$ such that $F : U \to W$ is a diffeomorphism.
Without loss of generality, for each $p_i \in F^{-1}(q)$ there is U_i such that $F : U_i \to W$ is a diffeomorphism.
Let $\omega \in \Omega^n(N)$ such that $\operatorname{supp} \omega \subseteq W$ and $\int_N \omega = 1$. Then $\operatorname{supp} F^* \omega \subseteq \bigcup_i U_i$, and

$$\int_M F^* \omega = \sum_i \int_{U_i} F^* \omega = \sum_i \operatorname{sgn}(p_i) \int_{F(U_i)} \omega = \sum_i \operatorname{sgn}(p_i) \overbrace{\int_W \omega}^{=1}.$$

On the other hand, $\int_M F^* \omega = k \cdot \int_N \omega = k$, so $k = \sum_i \operatorname{sgn}(p_i)$.

Remarks

(1)

If $F : M \to N$ is not surjective (e.g. $F : S^n \to S^n$ by sending everything to the north pole), then let $W \subseteq N$ open such that $F(M) \cap W = \varnothing$.
Let $\omega \in \Omega^n(N)$ such that $\operatorname{supp} \omega \subseteq W$ and $\int_N \omega = 1$. Then

$$(F^* \omega)_p(X_1, \dots, X_n) = \omega_F(p)(F_* X_1, \dots, F_* X_n) = 0.$$

Hence $F^* \omega = 0$ and

$$0 = \int_M F^* \omega = \deg(F) \cdot \int_N \omega = \deg(F).$$

(2)

For $M \xrightarrow{F} N \xrightarrow{G} P$, $\deg(G \circ F) = (\deg G)(\deg F)$.

(3)

If F is a diffeomorphism, then $\deg F = \pm 1$.

(4)

If $F, G : M \to N$ and $F \simeq G$, then $\deg F = \deg G$ (because $F^* = G^*$ on H^n_{dR}).

Theorem (Hopf)

If $F, G : S^n \to S^n$ such that $\deg F = \deg G$, then $F \simeq G$.

Remarks: Whitney Approximation

If $F : M \to N$ is continuous, then by Whitney approximation there exists $F \simeq F' : M \to N$. Then define $\deg F = \deg F'$ such that $F'' \simeq F \simeq F'$ is well-defined.

Theorem

Let N^n be closed and orientable and X^{n+1} compact, oriented and with boundary ∂X.
If $f : \partial X \to N$ has an extension $\tilde{f} : X \to N$, then $\deg f = 0$.

Proof

Let $\omega \in \Omega^n(N)$, $d\omega = 0$ and $\int_N \omega = 1$. Then

$$\deg f = \deg f \cdot \int_N \omega = \int_{\partial X} f^* \omega = \int_X d(f^* \omega) = \int_X f^*(d\omega) = 0.$$

This can be first proved in the smooth case and then approximated to the continuous case by Whiteny approximation.

Theorem

Any continuous map $F : \overline{B^n} \to \overline{B^n}$ has a fixed point.

Proof

Suppose F has no fixed points (i.e. $F(x) \neq x,\ \forall x \in \overline{B^n}$).
Define $G : B^n \to S^{n-1}$ by $x \mapsto \frac{x-F(x)}{||x-F(X)||}$. Let $g = G|_{S^{n-1}} : S^{n-1} \to S^{n-1}$, and note that $g \simeq \mathrm{id}_{S^{n-1}}$. In fact, we have $H : S^{n-1} \times [0,1] \to S^{n-1}$ by $H(x,t) = \frac{x-tF(x)}{||x-tF(x)||}$ well defined since $|x| = 1$ and by assumption $||tF(x)|| \leq t < 1$. Then $H(\cdot, 0) = \mathrm{id}_{S^{n-1}}$ and $H(\cdot, 1) = g$.
Therefore $\deg g = \deg(\mathrm{id}_{S^{n-1}}) = 1$. On the other hand, $g : S^{n-1} \to S^{n-1}$ has an extension $G : \overline{B^n} \to S^{n-1}$. Hence $\deg g = 0$, a contradiction.

Laplacian on Forms

For $H^k_{\mathrm{dR}}(M)$ with M closed and orientable, if M has additional structure then we can pick a unique representative in each class.
For (M, g) with a Riemannian metric and $\Delta f = -\operatorname{div}(\operatorname{grad} f)$, we can take $\omega \in \Omega^k(M)$.
Locally for $\{E_1, \dots, E_n\}$ and $\{\epsilon^1, \dots, \epsilon^n\}$ orthonormal frames, we can declare $\{\epsilon^I \ : \ I = (i_1, \dots, i_k) \text{ strictly increasing}\}$ to be an orthonorma frame.
With this, we can calculate $g(\omega, \eta)$ for $\omega, \eta \in \Omega^k(M)$.
Define $\langle \omega, \eta \rangle = \int_M g_p(\omega, \eta)\, d\mathrm{Vol}_g$; define also $d^* : \Omega^{k+1} \to \Omega^k$ the adjoint to $d : \Omega^k \to \Omega^{k+1}$ (i.e. $\langle d\omega, \eta \rangle = \left\langle \omega, d^* \eta \right\rangle$.
Then define the Lapalcian on forms $\Delta : \Omega^k \to \Omega^k$ by $\Delta = dd^* + d^* d$.

1. Δ is self-adjoint (i.e. $\Delta^* = \delta$).

2. $\Omega^k = \ker \Delta \oplus \operatorname{im} \Delta$.

3. We say ω is harmonic if $\Delta\omega = 0$, if and only if $d\omega = 0$ or $d^*\omega = 0$.

So H^k_{dR} is the set of harmonic k-forms (Hodge).

Lie Group Invariance

If $M = G$ is a Lie group, $\omega \in \Omega^k(M)$ may be left-invariant, right-invariant or bi-invariant ($L_g^* \omega = R_h^* \omega = \omega$.
We have that $H^k_{\mathrm{dR}}(G)$ is the set of bi-invariant k-forms (Cartan).
If G is a compact Lie group with a bi-invariant Riemannian metric, $\omega \in \Omega^k(M)$ is harmonic if and only if ω is bi-invariant.