

# Partial Differential Equations I

January 8, 2024

Homework

Assigned exercises and concept maps. Graded by completion.

Presentations

Assigned topics; responsible for giving a class.

Definition: Partial Differential Equation(s) (PDE)

An identity relating an unknown function, its partial derivatives and its variables.

$$F(D^k u, \dots, D^2 u, Du, u, x) = 0, \quad x \in U \subseteq \mathbb{R}^n$$

where  $U$  is an open subset of  $\mathbb{R}^n$ ,  $u : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $Du = (\partial_{x_1} u, \dots, \partial_{x_n} u)$ .

Then  $F : \mathbb{R}^{n^k} \times \dots \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ , where  $F$  is given.

$x = (x_1, \dots, x_n)$  is (are) the independent variable(s).

$u$  is the unknown function or dependent variable.

$k$  is the order of the PDE.

Goal

Given a PDE, we consider

- Existence
- Uniqueness
- Stability

Recall: Multiindex Notation

$\alpha = (\alpha_1, \dots, \alpha_n)$  a vector such that  $\alpha_i \in \mathbb{Z}_{\geq 0}$ .

Then we say that  $\alpha$  is a multiindex with order  $|\alpha| = \alpha_1 + \dots + \alpha_n$ .

Notation

$u : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\alpha = (\alpha_1, \dots, \alpha_n)$ .

$u^\alpha := D^\alpha u = \partial_{x_n}^{\alpha_n} \dots \partial_{x_1}^{\alpha_1} u$ , where  $\partial^0 u = u$ .

Definition: Linear Partial Differential Equation

A linear PDE of order  $k$  is of the form

$$(*) \sum_{|\alpha|=k} a_\alpha(x) D^\alpha u = f(x)$$

Remark

This means that  $F$  is multilinear in the first  $n^k + n^{k-1} + \dots$  variables.

## Definition: Homogeneous Linear Partial Differential Equation

A linear given by  $(*)$  is homogeneous if  $f(x) \equiv 0$ .  
Otherwise, it is non-homogeneous.

### Example 1: Linear Transport Equation

$$\nabla u \cdot (1, b) = u_t + b \cdot Du = f(t, x)$$

This is a linear PDE of order 1 on  $\mathbb{R} \times \mathbb{R}^n \equiv \mathbb{R}^{n+1}$  where  $(t, x)$  are independent variables and  $u$  is dependent. Here,  $x$  is the spatial variable while  $t$  is time and  $Du$  represents the gradient.  
 $\nabla u = (\partial_t u, \nabla u)$ ,  $b \cdot Du = \sum_{i=1}^n b_i \partial_{x_i} u$ ,  $(b_1, \dots, b_n) \in \mathbb{R}^n$  is fixed.

### Example 2: Laplace Equation

$$\Delta u := \sum_{i=1}^n \partial_{x_i}^2 u = 0$$

This is a linear, homogeneous PDE of order 2.

### Example 3: Poisson Equation

$$-\Delta u := f(u)$$

This is a nonlinear PDE of order 2.

Consider  $f(u) = u^2$ .

### Example 4: Heat Equation (Diffusion Equation)

$$u_t - \Delta u = 0$$

This is a linear, homogeneous PDE of order 2.

### Example 5: Wave Equation

$$u_{tt} - \Delta u = 0$$

This is a linear, homogeneous PDE of order 2.

### Transport Equation

$u : \mathbb{R}^n(0, \infty) \rightarrow \mathbb{R}$  given by

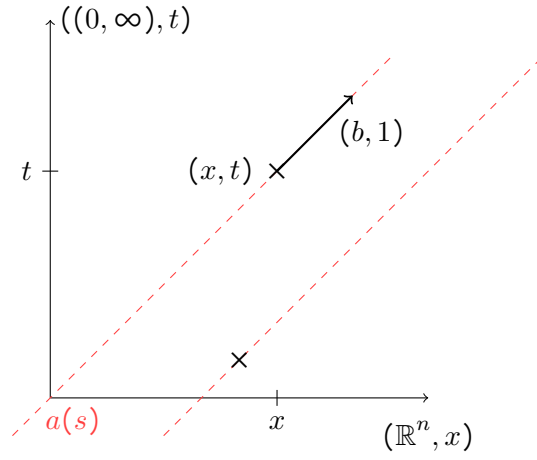
$$u_t + b \cdot Du = 0, \quad b \in \mathbb{R}^n$$

In order to get a solution, first assume that there exists a “nice” (e.g. smooth,  $C^1$ , differentiable, etc.) solution.

#### Step 1

Notice that the PDE is equivalent to

$$\nabla u \cdot (b, 1) = 0$$



Step 2

Consider a curve on  $\mathbb{R}^{n+1}$  with velocity  $(1, b)$  which passes through  $(x, t)$ . i.e.

$$\alpha(s) = (x + sb, t + s)$$

Notice  $\alpha'(s) = (b, 1)$ .

Then, let us study  $u$  along the curve  $\alpha(s)$ .

$$z(s) := u(\alpha(s))$$

Taking the derivative with respect to  $s$ ,

$$z'(s) := \frac{d}{ds}(u \circ \alpha(s)) = \nabla u|_{\alpha(s)} \cdot \alpha'(s) = \nabla u|_{\alpha(s)} \cdot (b, 1) = 0$$

That is  $z'(s) = 0$ ,  $z(s)$  is constant, and  $u$  along  $\alpha(s)$  is constant.

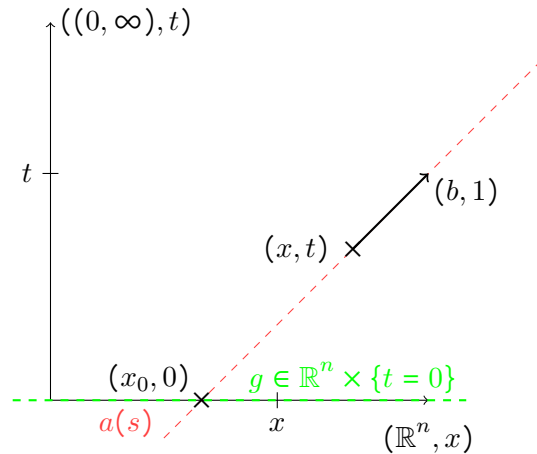
Conclusion

If we know some value of  $u$  along  $\alpha(s)$ , then we know all values along  $\alpha(s)$ .

If we have some value of  $u$  along every  $\alpha(s)$ , then we know  $u$  on  $\mathbb{R}^n \times (0, \infty)$ .

Transport Equation - Homogeneous Initial Value Problem

$$(*) \begin{cases} \nabla u \cdot (b, 1) = 0, & \mathbb{R}^n \times (0, \infty) \\ u = g, & \mathbb{R}^n \times \{t = 0\} \end{cases}$$



Here,  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  is given.

Consider  $(x, t)$ ; we want to find  $(x_0, 0)$ .

We know  $\alpha(s) = (x + sb, t + s) = (x_0, 0)$ , therefore

$$\begin{cases} x + sb = x_0 & (1) \\ t + s = 0 & \implies s = -t \end{cases} \quad (2)$$

Then, by replacing (2) in (1),

$$x_0 = x - tb$$

Then from the conclusion

$$u(x, t) = u(x_0, 0) = g(x_0) = g(x - tb)$$

Therefore,  $u(x, t) := g(x - tb)$  (♥).

Remark

1. If there exists a regular (differentiable or  $C^1$ ) solution  $u$  for  $*$ , then  $u$  should look like ♥.
2. If  $g$  is (differentiable or  $C^1$ ), then  $u$  defined by ♥ is a (differentiable or  $C^1$ ) solution for my problem.

Homework

Show that ♥ satisfies  $*$ .

Transport Equation - Non-homogeneous Initial Value Problem

$$(*) \begin{cases} \nabla u \cdot (b, 1) = f(x, t), & \mathbb{R}^n \times (0, \infty) \\ u = g, & \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Where  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  and  $f : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{R}$  are given.

Solution

Notice that the PDE is equivalent to

$$\nabla u \cdot (b, 1) = f(x, t)$$

Define the “characteristic curve”

$$\alpha(s) = (x + sb, t + s)$$

and

$$z(s) := u(\alpha(s))$$

Taking  $\frac{d}{ds}$ ,

$$z'(s) = \nabla u|_{\alpha(s)} \cdot (b, 1) = f(\alpha(s)) \implies z'(s) = f(x + sb, t + s) \quad (c)$$

Notice that  $c$  is an ordinary differential equation. Integrating from  $-t$  to  $0$ .

$$\begin{aligned} \int_{-t}^0 z'(s) \, ds &= \int_{-t}^0 f(x + sb, t + s) \, ds \\ z(0) - z(-t) &= \int_{-t}^0 f(x + sb, t + s) \, ds \end{aligned}$$

Notice that  $z(0) = u(x, t)$  and  $z(-t) = u(\alpha(-t)) = u(x - tb, 0)$ .

$$u(x, t) = u(x - tb, 0) + \int_{-t}^0 f(x + sb, t + s) \, ds$$

Then

$$\begin{aligned} u(x, t) &= g(x - tb) + \int_{-t}^0 f(x + sb, t + s) \, ds \\ &\stackrel{\bar{s}=s+t}{=} g(x - tb) + \int_0^t f(x + (\bar{s} - t)b, \bar{s}) \, d\bar{s} \\ &= g(x - tb) + \int_0^t f(x + (s - t)b, s) \, ds \end{aligned}$$

Remark: Method of Characteristics

Try to vert the PDE into an ODE and solve using characteristic curves.

January 10, 2024

Definition: Harmonic Function

If  $u \in C^2$  such that  $\Delta u = 0$ , then  $u$  is a harmonic function.

Laplace Equation

Consider  $u : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  with  $U$  open, then the homogeneous (Laplace) form is given by

$$\Delta u = 0$$

and the non-homogeneous (Poisson) form is

$$-\Delta u = f$$

where  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is given.

Remark / Exercise

The Laplace equation is invariant under translation and rotation.

That is, if  $\Delta u(x) = 0$  and  $v(x) = u(x - y)$ , then  $\Delta v = 0$ .

Similarly, if  $w(x) = u(O(x))$  then  $\Delta w = 0$  where  $O$  is an orthogonal matrix.

## Fundamental Solution of the Laplace Equation

Remark: since the Laplace equation is invariant under rotation, we can consider a function in terms of the radius  $v(x) = v(|x|)$ .

Recall  $r = |x| = (x_1^2 + \dots + x_n^2)^{1/2}$ .

Because of this remark, assume that  $u(x) = v(|x|) = v(r(x))$  (\*) where  $v : (0, \infty) \rightarrow \mathbb{R}$ .

Therefore, we need

$$\frac{\partial r}{\partial x_i} = \frac{1}{2} \cdot \frac{2x_i}{(x_1^2 + \dots + x_n^2)^{1/2}} = \frac{x_i}{r}$$

Replace (\*) in the PDE

$$\frac{\partial u}{\partial x_i} = \frac{\partial}{\partial x_i} v(r(x)) = v'(r(x)) \cdot \frac{\partial r}{\partial x_i} = v'(r(x)) \cdot \frac{x_i}{r}$$

and

$$\begin{aligned} \frac{\partial^2 u}{\partial x_i^2} &= \frac{\partial}{\partial x_i} \left( v'(r(x)) \cdot \frac{x_i}{r} \right) \\ &= \frac{\partial}{\partial x_i} (v'(r(x))) \cdot \frac{x_i}{r} + v'(r(x)) \cdot \frac{\partial}{\partial x_i} \left( \frac{x_i}{r} \right) \\ &= v''(r(x)) \cdot \frac{x_i^2}{r^2} + v'(r(x)) \left[ \frac{1}{r} + x_i \frac{\partial}{\partial x_i} \left( \frac{1}{r} \right) \right] \\ &= v'' \frac{x_i^2}{r^2} + v' \left[ \frac{1}{r} - \frac{x_i^2}{r^3} \right] \end{aligned}$$

Then, summing across  $i$ ,

$$\Delta u = v'' + v' \left[ \frac{n}{r} + \frac{1}{r} \right] = 0$$

Then the PDE is equivalent to

$$v''(r) + \frac{v'}{r}(n+1) = 0 \quad (\square)$$

We need to find a solution for  $\square$ .

$$v''(r) = -\frac{(n+1)v'}{r}$$

Assume, without loss of generality, that  $v' \neq 0$  such that

$$\frac{v''(r)}{v'(r)} = -\frac{n+1}{r} \implies (\log(|v'|))' = -\frac{n+1}{r}$$

Then, integrating,

$$\log(|v'|) = -(n+1) \log(r) + C = \log(r^{-(n+1)}) + C$$

such that

$$|v'| = Cr^{-(n+1)} \implies v' = Cr^{-(n+1)} \implies v(r) = Cr^{-n} + D = Cr^{2-n} + D$$

Definition: Fundamental Solution of the Laplace Equation

The function  $\Phi$  given by

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log |x|, & n = 2 \\ \frac{1}{n(n-2)\alpha(n)} \cdot \frac{1}{|x|^{n-2}}, & n \geq 3 \end{cases}$$

where  $\alpha(n)$  is the volume of the unit ball in  $\mathbb{R}^n$  is called the fundamental solution.

Remark

$\Phi$  solves the Laplace equation away from 0.

Lemma: Estimates for the Fundamental Solution

- First Estimate

$$|D\Phi(x)| \leq \frac{C}{|x|^{n-1}}, \text{ for } x \neq 0.$$

$$\frac{\partial \Phi}{\partial x_i} = C \frac{\partial}{\partial x_i} (|x|^{2-n}) = \frac{C(2-n)}{1-n} |x|^{2-n-1} \frac{\partial |x|}{\partial x_i} = |x|^{1-n} \cdot \frac{x_i}{|x|} = C x_i |x|^{-n}$$

Therefore

$$|D\Phi(x)| \leq C|x||x|^{-n} \implies |D\Phi(x)| \leq C|X|^{1-n}$$

– Exercise

Compute for  $n = 2$ .

- Second Estimate

$$|D^2\Phi(x)| \leq \frac{C}{|x|^n}, \text{ for } x \neq 0.$$

$$\begin{aligned} \frac{\partial^2}{\partial x_J \partial x_i} \Phi &= C \frac{\partial}{\partial x_J} (x_i |x|^{-n}) \\ &= C \left[ \delta_{iJ} |x|^{-n} + x_i \frac{\partial}{\partial x_J} |x|^{-n} \right] \\ &= C \left[ \delta_{iJ} |x|^{-n} + (-n) \cdot \frac{x_i |x|^{-n-1} x_J}{|x|} \right] \\ &= C \left[ \frac{\delta_{iJ} |x|}{|x|^n} + \frac{C x_i x_J}{|x|^{n+1}} \right] \end{aligned}$$

Then

$$\left| \frac{\partial \Phi}{\partial x_i \partial x_J} \right| \leq \frac{C}{|x|^n} + \frac{C|x_i||x_J|}{|x|^{n+2}} \leq \frac{2C}{|x|^n} = \frac{C}{|x|^n}$$

Then, we are done since

$$|D^2\Phi(x)| = \sqrt{\sum_J \left( \frac{\partial \Phi}{\partial x_i \partial x_J} \right)^2}$$

## Poisson Equation

### Motivation

Suppose we have  $\Phi(x)$ , the fundamental solution.

Then for an arbitrary, fixed element  $y \in \mathbb{R}^n$ , then we have  $x \rightarrow \Phi(x - y)$  harmonic for  $x \neq y$ .

Consider  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $y \rightarrow f(y)$  then  $x \rightarrow f(y)\Phi(x - y)$  is similarly harmonic for  $x \neq y$ .

Now, if given  $\{y_1, \dots, y_m\}$  where  $y_i \in \mathbb{R}^n$ , then  $x \rightarrow \sum_{i=1}^m f(y_i)\Phi(x - y_i)$  is harmonic  $\forall x \neq \{y_1, \dots, y_m\}$ .

Then, what happens if we consider

$$u(x) := \int_{\mathbb{R}^n} f(y)\Phi(x - y) dy \quad (\square_3)$$

Is  $u$  harmonic? No, since  $\Delta\Phi(x - y)$  is not summable in  $\mathbb{R}^n$  we may not pass the limit into the integral.

(to be covered later) However, since  $\Delta\Phi(x - y)$  acts as  $\delta_{xy}$  in distribution, this may solve the Poisson equation.

### Remark / Exercise

Assume that  $f \in C_c^2(\mathbb{R}^n)$  (i.e  $f$  is twice continuously differentiable with compact support on  $\mathbb{R}^n$ ).

The function  $\Phi$  is integrable near the singularity on compact sets.

Prove using spherical coordinates.

Therefore,  $u$  defined by  $\square_3$  is well defined

$$|u| = \left| \int_{\mathbb{R}^n} f(y)\Phi(x - y) dy \right| = \left| \int_K \Phi(x - y) dy \right| < \infty$$

### Theorem: Solving the Poisson Equation

If  $f \in C_c^2(\mathbb{R}^n)$  and  $u$  is defined by  $\square_3$ , then

1.  $u \in C^2(\mathbb{R}^n)$
2.  $-\Delta u = f$ , in  $\mathbb{R}^n$

#### • Proof of 1

Since  $\Phi$  presents a problem at  $x = y$  but  $f$  is well behaved, we will change variables such that  $\bar{y} = x - y$ ,  $y = x - \bar{y}$ , and  $\frac{dy}{d\bar{y}}(-1)I_{m \times m}$  and then redefine  $\bar{y} = y$ .

$$u(x) = \int_{\mathbb{R}^n} f(y)\Phi(x - y) dy = \int_{\mathbb{R}^n} f(x - \bar{y})\Phi(\bar{y}) d\bar{y} = \int_{\mathbb{R}^n} f(x - y)\Phi(y) dy$$

In short, we have sent the problem from  $\Phi$  to  $f$ .

Now, let us consider  $e_i = (0, \dots, 1, \dots, 0)$ .

Then for  $h > 0$ ,

$$\frac{u(x + he_i) - u(x)}{h} = \frac{1}{h} \int_{\mathbb{R}^n} \Phi(y) [f(x + he_i - y) - f(x - y)] dy$$

Now, the limit as  $h \rightarrow 0$

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{u(x + he_i) - u(x)}{h} &= \lim_{h \rightarrow 0} \int_{\mathbb{R}^n} \overbrace{\Phi(y) \left[ \frac{f(x + he_i - y) - f(x - y)}{h} \right]}^{H(h,y)} dy \\ &= \int_{\mathbb{R}^n} \Phi(y) \cdot \frac{\partial f(x - y)}{\partial x_i} dy \end{aligned}$$



To justify passing the limit into the integral, take an arbitrary sequence  $h_m \xrightarrow{0} 0$  and consider

$$f_m(y) := H(h_m, y)$$

We want to consider

$$\begin{aligned} |H(h_m, y)| &\leq \Phi(y) \left[ \frac{f(x + h_m e_i - y) - f(x - y)}{h} \right] \\ &\leq \Phi(y) f'(c) \end{aligned}$$

Where  $c$  is along the curve between  $f(x + h_m e_i - y)$  and  $f(x - y)$  and chosen by mean value theorem.

– Exercise

$$|H(h_m, y)| \leq \Phi(y) \|f'\|_{L^\infty \chi_{B(x, R)}(y)}$$

Note that

$$C \int_{\mathbb{R}^n} |\Phi(y)| \chi_{B(x, R)}(y) dy = \int_{B(x, R)} |\Phi(y)| dy < \infty$$

– Exercise

Using the fact that a continuous function is uniformly continuous on a compact set, show that  $u \in C^2(\mathbb{R}^n)$ .

## Dominated Convergence Theorem

If  $f_m(x)$  such that  $f_m(x) \xrightarrow[\text{pointwise}]{m \rightarrow \infty} f(x)$ , and  $|f_m(x)| \leq g(x)$  for  $g \in L^1$ , then  $f$  is integrable and

$$\lim_{m \rightarrow \infty} \int f_m(x) dx = \int f(x) dx$$

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Recall: Averages

$$\begin{aligned} f &: \{1, \dots, n\} \rightarrow \mathbb{R} \\ i &\rightarrow a(i) \end{aligned}$$

The average is given as  $\frac{a(1) + \dots + a(n)}{n}$ .

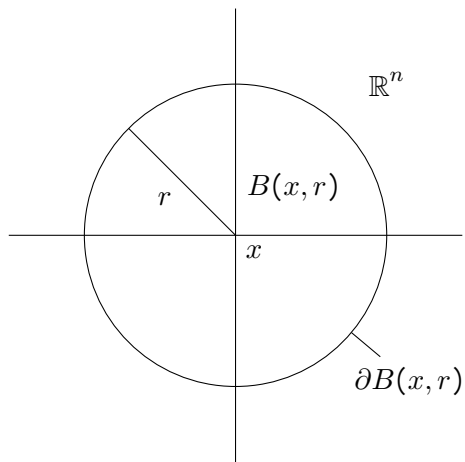
Then for  $f : \Omega \rightarrow \mathbb{R}$ , the average is given as

$$\frac{1}{|\Omega|} \int f(y) dy := \oint_{\Omega} f d\mu$$

In our case,  $f : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$

$$\oint_{B(x, n)} f d\mu \equiv \frac{1}{|B(x, n)|} \oint_{B(x, n)} f d\mu$$

$$\oint_{\partial B(x, n)} f d\mu = \frac{1}{|\partial B(x, n)|} \oint_{\partial B(x, n)} f d\mu$$

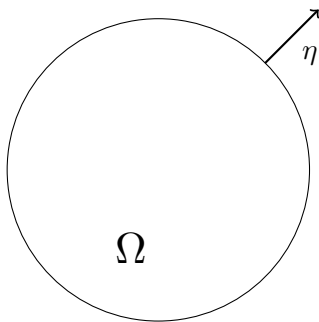


Theorem: Lebesgue Differentiation

$$u|_x = \lim_{r \rightarrow 0} \int_{B(x, r)} u \, d\mu = \lim_{r \rightarrow 0} \int_{\partial B(x, r)} u \, d\mu$$

Integration by Parts

$$\int_{\Omega} u \Delta v = - \int_{\Omega} \nabla u \cdot \nabla v + \int_{\partial \Omega} u \cdot \frac{\partial v}{\partial \eta}$$



Recall: Poisson's PDE

$$f \in C_c^2(\mathbb{R}^n), \quad u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) \, dy.$$

$$\Phi(x) = \left\{ \frac{1}{n(n-2)|\alpha(n)|} \frac{1}{|X|^{(n-2)}} \right.$$

$$u(x) = \int_{\mathbb{R}^n} f(x-y) \Phi(y) \, dy$$

Part A

$$u \in C^2(\mathbb{R}^n)$$

Then

$$\frac{\partial u}{\partial x_1} = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_1}(x-y) \Phi(y) \, dy$$

$$\frac{\partial^2 u}{\partial x_1 \partial x_T} = \int_{\mathbb{R}^n} \frac{\partial^2 f}{\partial x_1 \partial x_T}(x-y) \Phi(y) \, dy$$

Notice that if we prove that

$$g(x) = \int_{\mathbb{R}^n} h(x-y)\Phi(y) dy$$

– where  $h$  is continuous with compact support – is continuous then we are done.

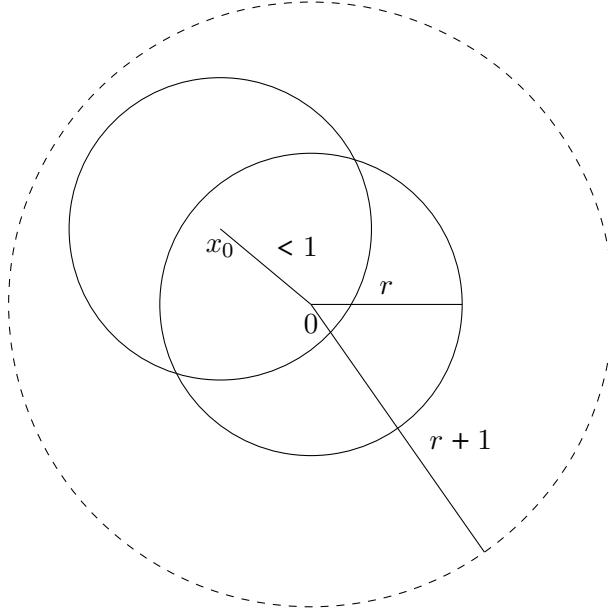
Let us prove that  $g$  is continuous.

Let  $\varepsilon > 0$ ,

$$|g(x) - g(x_0)| \leq \int_{\mathbb{R}^n} \Phi(y) |h(x-y) - h(x_0-y)| dy$$

Without loss of generality,  $h$  has compact support on  $B(0, r)$  for some radius  $r$ .

Therefore  $h(x, y)$  has compact support on  $B(x, r)$  and  $h(x_0, y)$  has compact support on  $B(x_0, r)$ .



Consider  $|x - x_0| < 1$ , then  $|h(x-y) - h(x_0-y)|$  has compact support on  $B(x_0, r+1)$ . Then

$$|g(x) - g(x_0)| \leq \int_{B(x_0, r+1)} \Phi(y) |h(x-y) - h(x_0-y)| dy$$

Since  $h$  is continuous on a compact domain, it is uniformly continuous.

Therefore  $\exists \delta > 0$  such that  $|w - z| < \delta \implies |h(w) - h(z)| < \epsilon$ .

Set  $w = x - y$  and  $z = x_0 - y$  such that  $|w - z| = |x - x_0| < \delta$ , then  $|h(x-y) - h(x_0-y)| < \epsilon$ . Thus,

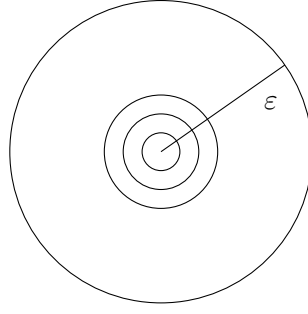
$$|g(x) - g(x_0)| \leq \varepsilon \int_{B(x_0, r+1)} \Phi(y) dy$$

Part B

$$-\Delta u = f$$

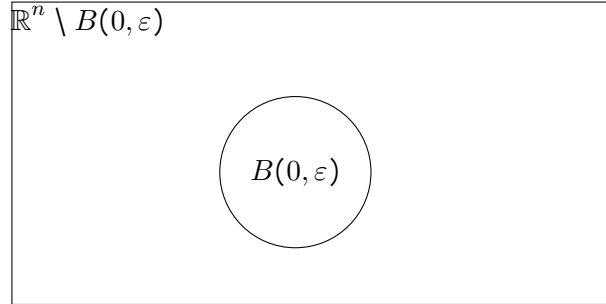
Letting  $\varepsilon > 0$  and taking the Laplacian of both sides,

$$\begin{aligned} \Delta_x u(x) &= \int_{\mathbb{R}^n} \Delta_x f(x-y)\Phi(y) dy \\ &= \overbrace{\int_{B(0, \varepsilon)} \Delta_x f(x-y)\Phi(y) dy}^{I_\varepsilon} + \overbrace{\int_{\mathbb{R}^n \setminus B(0, \varepsilon)} \Delta_x f(x-y)\Phi(y) dy}^{J_\varepsilon} \end{aligned}$$



Then

$$\begin{aligned}
|I_\varepsilon| &\leq \int_{B(0,\varepsilon)} |\Delta_x f(x-y)| \Phi(y) \, dy \\
&\leq \| |\nabla^2 f| \|_{L^\infty} \int_{B(0,\varepsilon)} \Phi(y) \, dy \\
&\leq c \int_0^\varepsilon \int_{\partial B(0,r)} \Phi(y) \, dS(y) \, dr \\
&\leq c \int_0^\varepsilon \int_{\partial B(0,r)} \frac{1}{|y|^{n-2}} \, dS(y) \, dr \\
&= c \int_0^\varepsilon \int_{\partial B(0,r)} \frac{1}{r^{n-2}} \, dS(y) \, dr \\
&= c \int_0^\varepsilon \frac{1}{r^{n-2}} \int_{\partial B(0,r)} dS(y) \, dr \\
&\leq c \int_0^\varepsilon \frac{r^{n-1}}{r^{n-2}} \, dr \\
&= c \int_0^\varepsilon r \, dr = c\varepsilon^2
\end{aligned}$$



As an exercise, attempt the same argument with  $n = 2$ .

Therefore,

$$\Delta_x u = I_\varepsilon + J_\varepsilon$$

and  $\lim_{\varepsilon \rightarrow 0} I_\varepsilon = 0$ .

Now, we need to control  $J_\varepsilon$ .

$$J_\varepsilon = \int_{\mathbb{R}^n} \Delta_x f(x-y) \Phi(y) \, dy$$

$$\Delta_x f(x-y) = \sum \frac{\partial^2 f}{\partial x^2} f(x-y)$$

$$\begin{aligned}\frac{\partial f}{\partial x}(x-y) &= \nabla f|_{z=(x-y)} \cdot e_i = \frac{\partial f}{\partial z_i}|_{z=(x-y)} \\ \frac{\partial^2 f}{\partial x_i^2} &= \frac{\partial^2 f}{\partial z_i^2}|_{z=(x-y)}\end{aligned}$$

$$\begin{aligned}\Delta_y f(x-y) &= \sum \frac{\partial^2 f}{\partial y_i^2}(x-y) \\ \frac{\partial f}{\partial y_i}(x-y) &= \nabla f|_{z=(x-y)} \cdot -e_i = -\frac{\partial f}{\partial z_i}|_{z=(x-y)} \\ \frac{\partial^2 f}{\partial y_i^2} &= \frac{\partial^2 f}{\partial y_i^2}|_{z=x-y}\end{aligned}$$

So

$$\begin{aligned}J_\varepsilon &= \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Delta_y f(x-y) \Phi(y) dy \\ &= \overbrace{- \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \nabla_y f(x-y) \nabla \Phi(y) dy}^{K_\varepsilon} + \overbrace{\int_{\partial(\mathbb{R}^n \setminus B(0,\varepsilon))} \frac{\partial f}{\partial \eta} \Phi(y) dS(y)}^{L_\varepsilon}\end{aligned}$$

and

$$\Delta_x u = I_\varepsilon + K_\varepsilon + L_\varepsilon$$

To control  $L_\varepsilon$ , since

$$\begin{aligned}|L_\varepsilon| &\leq \int_{\partial B(0,\varepsilon)} \frac{|\partial f|}{\partial \eta} \Phi(y) dy \\ &\leq \int_{\partial B(0,\varepsilon)} |\nabla f| \Phi(y) dy \\ &\leq \|\nabla f\|_{L^\infty} \int_{\partial B(0,\varepsilon)} \Phi(y) dy \\ &\leq c \int_{\partial B(0,\varepsilon)} \frac{1}{|y|^{n-2}} dy \\ &= \frac{c}{\varepsilon^{n-2}} \int_{\partial B(0,\varepsilon)} dy \\ &\leq \frac{c\varepsilon^{n-1}}{\varepsilon^{n-2}} \\ &= c\varepsilon\end{aligned}$$

and  $K_\varepsilon$ , since  $\frac{\partial \Phi}{\partial \eta} = \nabla \phi \cdot \eta = \frac{-y}{n\alpha(n)|y|^n} \cdot \eta = \frac{|y|^2}{n\alpha(n)|y|^n|y|} = \frac{1}{n\alpha(n)|y|^{n-1}}$

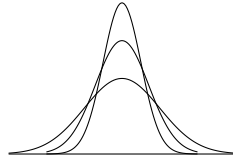
$$\begin{aligned}
|K_\varepsilon| &= - \int_{\mathbb{R}^n \setminus B(0, \varepsilon)} \nabla_y f(x-y) \nabla_y \Phi(y) dy \\
&= \overbrace{\int_{\mathbb{R}^n \setminus B(0, \varepsilon)} f(x-y) \Delta_y \Phi(y) dy}^0 - \int_{\partial(\mathbb{R}^n \setminus B(0, \varepsilon))} f(x-y) \frac{\partial \Phi}{\partial \eta} \\
&= - \int_{\partial B(0, \varepsilon)} f(x-y) \frac{\partial \Phi}{\partial \eta} \\
&= - \int_{\partial B(0, \varepsilon)} f(x-y) \frac{1}{n\alpha(n)|y|^{n-1}} dS(y) \\
&= - \frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B(0, \varepsilon)} f(x-y) dS(y) \\
&= \underbrace{- \frac{1}{n\alpha(n)\varepsilon^{n-1}}}_{\text{volume}} \int_{\partial B(0, \varepsilon)} f(z) dS(z) \\
&= \frac{1}{|\partial B(0, \varepsilon)|} \int_{\partial B(0, \varepsilon)} f(z) dz \\
&= - \oint_{\partial B(x, \varepsilon)} f(z) dz
\end{aligned}$$

Laplacian as a Distribution

$$-\Delta \Phi(y) = \delta(y)$$

Define the Dirac delta “function” as

$$\delta(y) = \begin{cases} \infty & y = 0 \\ 0 & y \neq 0 \end{cases}$$



such that  $\int_{\mathbb{R}^n} \delta = 1$ .

Translate the Dirac delta as

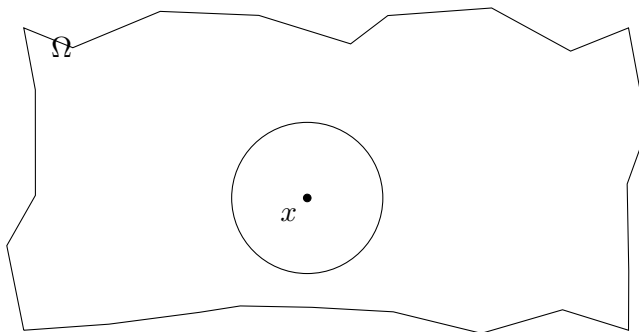
$$\delta_x = \begin{cases} \infty & y = x \\ 0 & y \neq x \end{cases}$$

and recall that

$$\begin{aligned}
\Delta u(x) &= \Delta \left( \int_{\mathbb{R}^n} \Phi(x-y) f(y) dy \right) \\
&= \int_{\mathbb{R}^n} \overbrace{\Delta \Phi(x-y)}^{-\delta_x(y)} f(y) dy \\
&= - \int_{\mathbb{R}^n} \delta_x(y) f(y) dy \\
&= - \int_{\mathbb{R}^n} \delta_x(y) f(x) dy \\
&= -f(x) \overbrace{\int_{\mathbb{R}^n} \delta_x(y) dy}^1 \\
&= -f(x)
\end{aligned}$$

## Harmonic Functions

Suppose  $u$  is harmonic



$u : \Omega \rightarrow \mathbb{R}^n$  harmonic.

## Mean-value Formulas

Let  $U$  be an open set in  $\mathbb{R}^n$ ,  $u : U \rightarrow \mathbb{R}$  such that  $\Delta u = 0$  in  $U$ . Then

$$\begin{aligned}
u(x) &= \oint_{\partial B(x,r)} -u(y) dS(y) \\
&= \oint_{B(x,r)} u(y) dy
\end{aligned}$$

where  $B(x, r) \subseteq U$ .

IMAGE HERE

Proof

Consider  $\phi(r) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y) dS(y)$ .

If  $\phi'(r) = 0$ , when we are done since that would make  $\phi$  constant and  $\phi(r) = \lim_{s \rightarrow 0} \phi(s) = u(x)$ . Then

$$\begin{aligned}
\phi(r) &= \frac{1}{|\partial B(x, r)|} \int_{\partial B(x, r)} u(y) dS(y) \\
&\stackrel{y=x+rz}{=} \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(0,1)} u(x+rz)r^{n-1} dS(z) \\
&= \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} u(x+rz) dS(z)
\end{aligned}$$

Therefore

$$\begin{aligned}
\phi'(r) &= \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} \nabla u(x+rz) \cdot z dS(z) \\
&\stackrel{y=x+rz}{=} \frac{1}{|\partial B(0, r)|} \int_{\partial B(x, r)} \nabla u(y) \cdot \frac{y-x}{r} dS(y) \\
&= \frac{1}{|\partial B(0, r)|} \int_{\partial B(x, r)} \nabla u(y) \cdot \eta dS(y) \\
&= \frac{1}{|\partial B(0, r)|} \int_{\partial B(x, r)} \frac{\partial y}{\partial \eta} dS(y) \\
&= \frac{1}{|\partial B(0, r)|} \int_{B(x, r)} \Delta u \\
&= 0
\end{aligned}$$

January 22, 2024

Mean Value Formula

For  $U \subseteq \mathbb{R}^n$ ,  $U$  open with  $u : U \rightarrow \mathbb{R}$  such that  $u \in C^2(U)$ ,  $\Delta u = 0$ , we have

$$u(x) \underset{(a)}{=} \oint_{\partial B(x, r)} u \underset{(b)}{=} \oint_{B(x, r)} u$$

for all  $B(x, r) \subseteq U$ .

Recall that (a) was proven above by setting  $\phi(r) = \oint_{\partial B(x, r)} u(y) dS(y)$  and showing  $\phi'(r) = 0$ .

For (b), we again apply spherical coordinates such that

$$\begin{aligned}
\int_{B(x, r)} u(y) dy &= \int_0^r \int_{\partial B(x, s)} u(y) dS(y) ds \\
&= \int_0^r |\partial B(x, s)| \overbrace{\oint_{\partial B(x, s)} u(y) dS(y)}^{u(x)} ds \\
&= u(x) \int_0^r |\partial B(x, s)| ds \\
&= u(x) \int_0^r n\alpha(n)S^{n-1} ds \\
&= \frac{u(x)n\alpha(n)S^n}{n} \Big|_0^r \\
&= u(x) \overbrace{\alpha(n)r^n}^{|B(x, r)|}
\end{aligned}$$



Converse

Recall that

$$\phi'(r) = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u(y) dy$$

Suppose then that we do not know that  $\Delta u = 0$  but we have

$$u(x) = \oint_{\partial B(x,r)} u, \quad \forall B(x,r) \subseteq U$$

Then, necessarily,  $\Delta u = 0$  in  $U$ .

- Proof

Suppose, for sake of contradiction, that  $\Delta u \neq 0$ . Then, without loss of generality, there exists  $y \in U$  such that  $\Delta u(x) > 0$  for  $x \in B(y,n) \subseteq U$ .

IMAGE HERE

$$\phi(r) = \oint_{\partial B(x,r)} u(x) dS(x)$$

and

$$\phi'(r) = \frac{1}{n\alpha(n)r^{n-1}} \int_{B(y,r)} \Delta u(x) dS(x) > 0$$

which contradicts  $\phi'(x) = 0$ .

## Strong Maximum Principle

Let  $U \subseteq \mathbb{R}^n$  be a bounded open set,  $u \in C^2(U) \cap C(\overline{U})$ ,  $\Delta u = 0$  on  $U$ . Then

1.  $\max_{\overline{U}}(u) = \max_{\partial U}(u)$ .
2. If  $U$  is connected and  $u$  has its maximum in an interior point, then  $u$  is constant on  $\overline{U}$ .

IMAGE HERE - 2

## Proof of A

Since  $\partial U \subseteq \overline{U}$ ,  $\max_{\partial U}(u) \leq \max_{\overline{U}}(u)$ .

Let  $x_0 \in \overline{U}$  such that  $u(x_0) = \max_{\overline{U}}(u)$ .

IMAGE HERE - 4

So either  $x_0 \in \partial U$  or  $x_0 \in U$ .

Let  $U^I$  be the connected component which contains  $x_0$ . Then  $x_0 \in U^I$ , so by part (b)  $u$  is constant on  $\overline{U^I}$ . So

$$\max_{\overline{U}}(u) = u(x_0) = \max_{\partial U^I}(u) \leq \max_{\partial U}(u)$$

## Proof of B

Then there exists  $x_0 \in U$  such that  $\max_{\overline{U}}(u) = u(x_0) = M$ .

Let us define  $\Omega = \{y \in U \mid u(y) = M\}$ . Then

1.  $\Omega \neq \emptyset$ ,  $B \setminus x_0 \in \Omega$ .
2.  $\Omega$  open set.

IMAGE HERE - 3

1.  $\Omega$  is closed, since  $\Omega = u^{-1}(\{M\})$ .

It follows that  $\Omega = U$  and, therefore,  $u(y) = M$ ,  $\forall y \in U$ .

- Proof of 2

Let  $y \in \Omega$ ,  $y \in U$ ,  $u(y) = M$ . Then there exists  $B(y, r) \subseteq U$ , and

$$M = u(y) = \oint_{B(y,r)} u(x) dS(x) \leq M$$

Then

$$\oint_{B(y,r)} u(x) dx = M$$

so  $u(x) = M$ ,  $\forall x \in B(y, r)$  and, therefore  $B(y, r) \subseteq \Omega$  and  $\Omega$  is open.

Remark: Boundedness Is Important

1. Consider  $f(x) = x$  on  $\mathbb{R}_{\geq 0}$ .
2. IMAGE HERE - 5

Remark: Strong Minimum Principle Is Equivalent

Consequences

1. Positivity of harmonic functions.
2. Uniqueness of the Poisson problem.

Corollary: Positivity of Harmonic Functions

Suppose that  $U$  is connected and  $u : U \rightarrow \mathbb{R}$ ,  $u \in C^2(U) \cap C(\overline{U})$  solves the following problem

$$\begin{cases} \Delta u = 0 & \text{on } U \\ u = g & \text{on } \partial U \end{cases}$$

If  $g \geq 0$  on  $\partial U$ , then  $u$  is positive on  $U$  as long as  $g$  is positive in some point.

Proof

Assume  $\exists x_0 \in \partial U$  where  $x_0$  is the minimum. Then  $u(x_0) = \min_{\overline{U}}(u)$  and,  $\forall x \in U$ ,

$$0 \leq u(x_0) = \min_{\overline{U}}(u) \leq u(x)$$

so  $u$  is non-negative. If  $u(x) = 0$ , then  $u(x_0) = 0$  and the minimum is achieved in the interior. That would mean  $u(x) = 0$ ,  $\forall x \in \overline{U} \supseteq \partial U$  and  $g(x) = 0$ ,  $\forall x \in \partial U$  which would be a contradiction.

Theorem: Uniqueness of the Poisson Problem

Suppose  $U \subseteq \mathbb{R}^n$  is open, connected and bounded. Then, there exists at most one solution to

$$(*) \begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

where  $u \in C^2(U) \cap C(\overline{U})$ .

Proof

Let  $u_1$  and  $u_2$  be two solutions of  $*$ .

Consider  $w = u_1 - u_2$  and observe that

$$\Delta w = \Delta u_1 - \Delta u_2 = -f + f = 0, \quad \text{in } U$$

and  $w|_{\partial U} = g - g = 0$  on  $\partial U$ . Therefore

$$\begin{cases} \Delta w = 0 & \text{in } U \\ w = 0 & \text{on } \partial U \end{cases}$$

By applying the minimum and maximum principles,

$$w(x_0) = \min_{\overline{U}}(w) \leq w(x) \leq \max_{\overline{U}}(w) = w(x)$$

so  $w(x) = 0$ ,  $\forall x \in \overline{U}$  and therefore  $u_1 = u_2$ .

Example

Let's consider  $f : \mathbb{C} \rightarrow \mathbb{C}$  analytic (i.e.  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  for  $a_n, z \in \mathbb{C}$ ). Then

$$f(z) = u(z) + v(z)$$

If  $\mathbb{C} \cong \mathbb{R}^2$ ,

$$f(x + y) = u(x, y) + v(x, y)$$

for  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $v : \mathbb{R}^2 \rightarrow \mathbb{R}$ .

Claim:  $u$  and  $v$  are Harmonic.

$$u(x, y) + v(x, y) = \sum_{n=0}^{\infty} a_n (x + iy)^n$$

and

$$\begin{aligned} \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} &= \sum_{n=1}^{\infty} a_n n |x + iy|^{n-1} \\ \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} &= \sum_{n=1}^{\infty} a_n n |x + iy|^{n-1} i \end{aligned}$$

So

$$i \left( \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) = \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

Recall: Convolution and smoothing

Let  $U \subseteq \mathbb{R}^n$  be an open set.

For  $\varepsilon > 0$ , define  $U_\varepsilon = \{x \in U \mid d(x, \partial U) > \varepsilon\}$ .

IMAGE HERE - 6

Define

$$\eta(x) = \begin{cases} ce^{\left(\frac{1}{|x|^2-1}\right)} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

with  $c$  such that  $\int_{\mathbb{R}^n} \eta(x) \, dx = 1$ ,  $\eta \in C^\infty(\mathbb{R}^n)$

IMAGE HERE - 7

Note that  $\text{supp}(\eta) = B(0, 1)$  and take

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right), \quad \eta_\varepsilon \in C^\infty(\mathbb{R}^n)$$

Then

$$\int_{\mathbb{R}^n} \eta_\varepsilon(x) \, dx = 1$$

and  $\text{supp}(\eta_\varepsilon) \subseteq B(0, \varepsilon)$ .

If  $f$  is locally integrable on  $U$ , define its mollification

$$f^\varepsilon(x) = \int_U \eta_\varepsilon(x - y) f(y) \, dy \quad \forall x \in U_\varepsilon$$