

Manifolds I

September 26, 2024

Class Organization

1 Takehome Midterm

1 Takehome Final

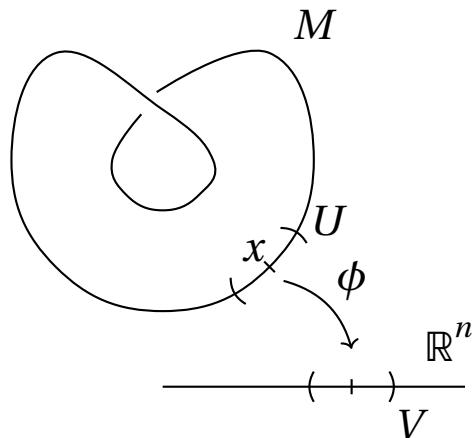
Homeworks assigned, but not graded.

<https://ginzburg.math.ucsc.edu/teaching/208manifolds1-2024/syl.html>

Definition: Topological Manifolds

For M a topological space, M is a topological manifold if $\forall x \in M, \exists M \ni U \ni x$ and homeomorphism $\phi: U \rightarrow V \subset \mathbb{R}^n$ for V open.

To avoid problems (see below), further assume that M is Hausdorff and second countable.

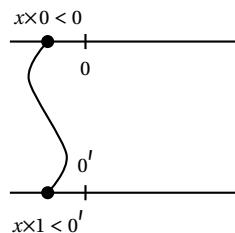


Exercise

We can require V to be an open ball.

Problems

- M need not be Hausdorff.



With $(\mathbb{R} \times 0 \coprod \mathbb{R} \times 1) / \sim$.

- M need not be second countable.

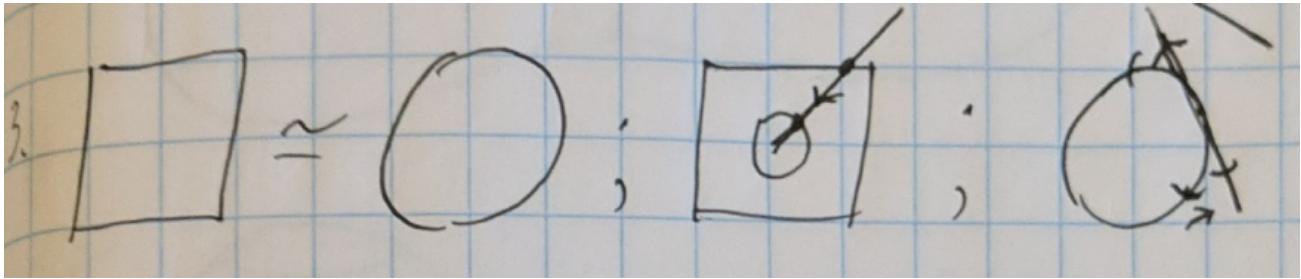
Take $\coprod_S \mathbb{R}_S$ where S is an uncountable index.

Examples

Example 1

If $N \underset{\text{homeo}}{\simeq} M$, this implies N is a manifold.

Example 2



Example 3

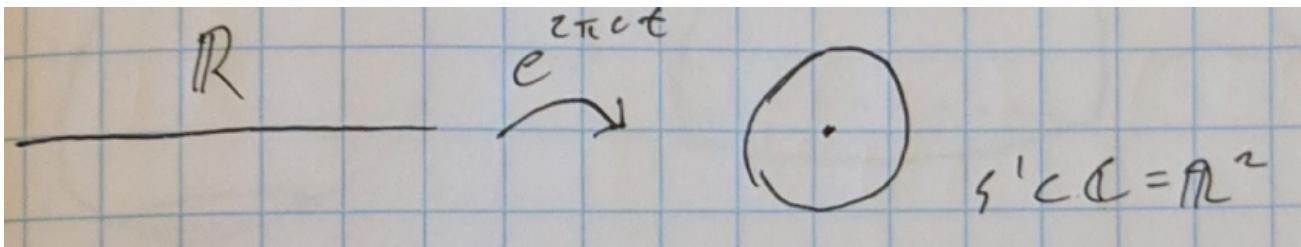
An open subset of a manifold is a manifold.

Example 4

M, N manifolds implies $M \times N$ is a manifold.

Example 5

Take \mathbb{R}/\mathbb{Z} by the equivalence relation $t \sim t'$ iff $t' - t \in \mathbb{Z}$.



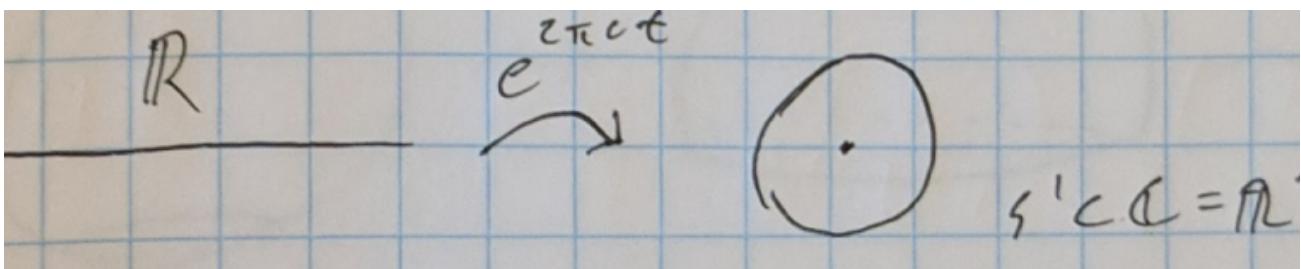
Then $C^0(S^1)$ relates to periodic functions with period 1.

Example 6

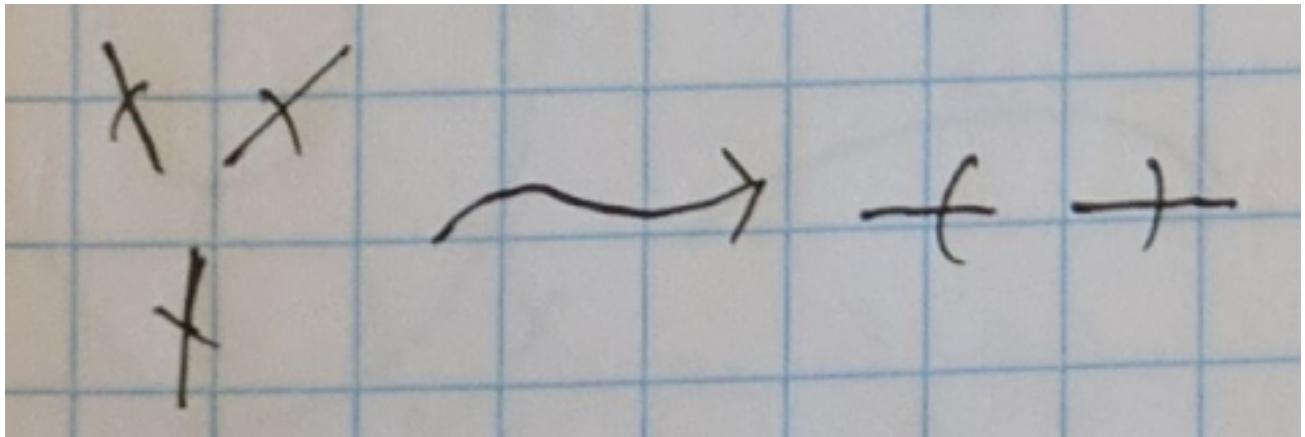
$$\mathbb{T}^n = S^1 \times \cdots \times S^1.$$

Counterexample 1

$[0, 1]$ is not a manifold.

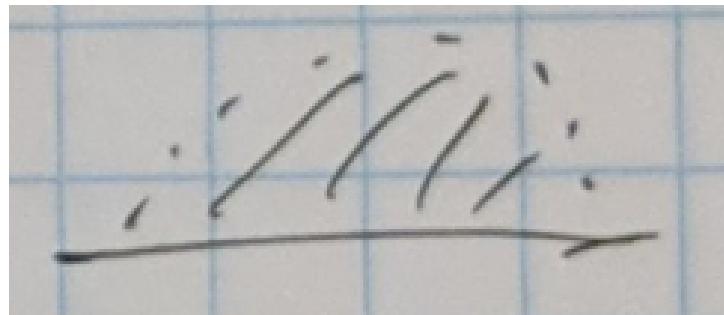


Since 0 must map somewhere in the open interval, its deletion results in a connected space in the former case but a disconnected one in the latter. Similarly, the following breaks into three and two connected components respectively.



Definition: Manifold with Boundary

There exists a neighborhood $\forall x \in M$ homeomorphic to either the open ball or the half-closed half-ball.



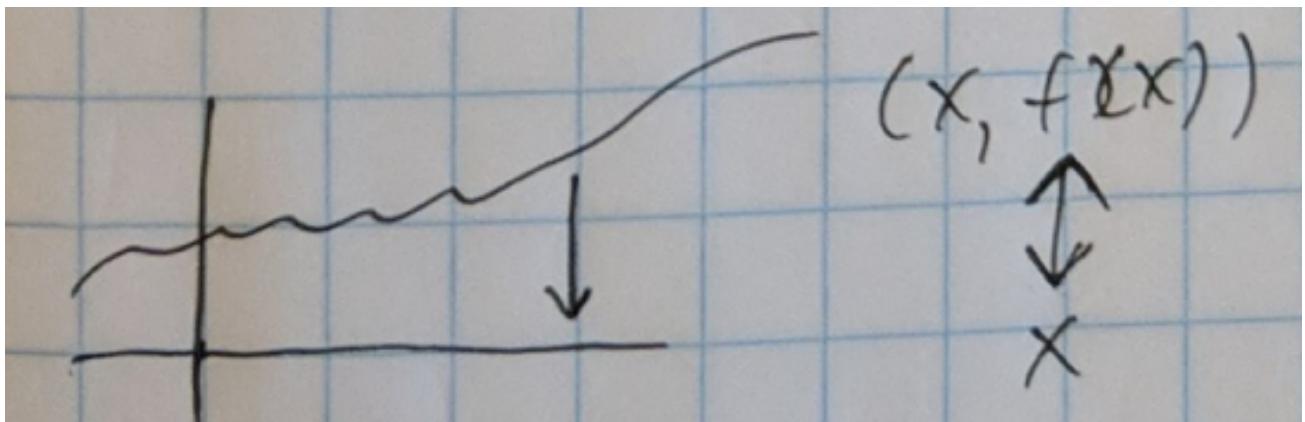
Exercise

A connected manifold is path-connected.

Examples

Example 7

Take $f : \mathbb{R}^n \xrightarrow{C^0} \mathbb{R}$ with graph $\Gamma_f = \{(x, f(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^n \times \mathbb{R}$.

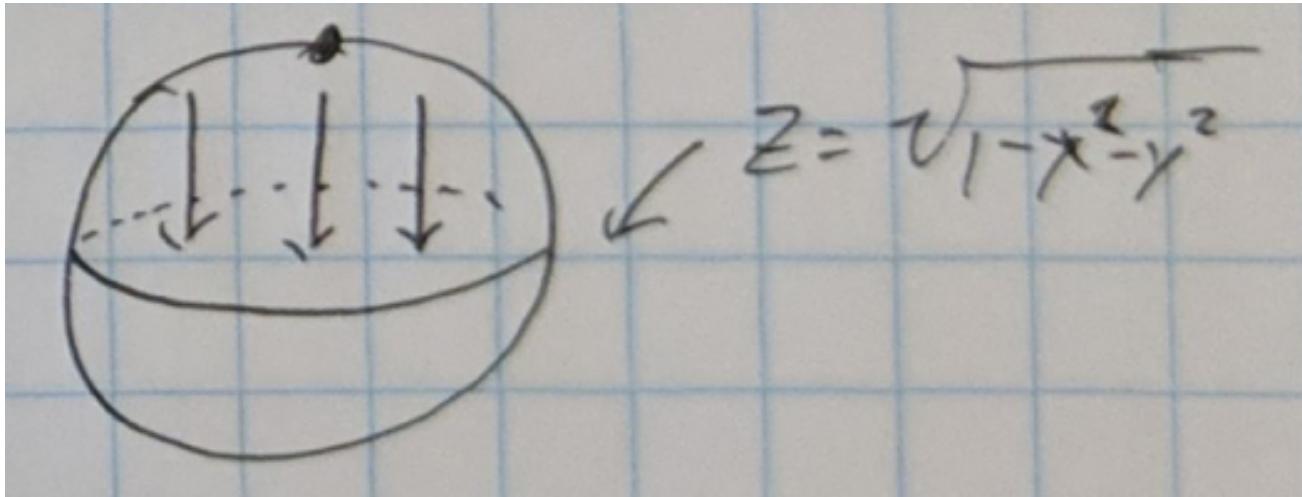


Example 8

Take $f : M \rightarrow N$ between manifolds, then $M \simeq \Gamma_f \subseteq M \times N$.

Example 9

$$S^n \subset \mathbb{R}^{n+1}.$$



Definition: Real Projective Spaces

Take $\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim$ where $x \sim y \iff x = \lambda y$ for $\lambda \neq 0$.

Informally, the collection of lines through the origin.

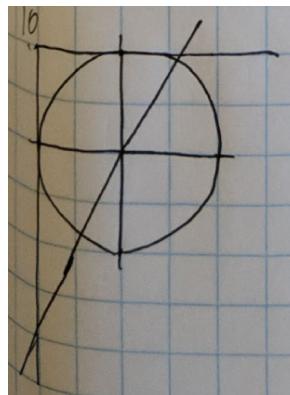
Alternatively, $\mathbb{RP}^n = S^n / \sim$ where $x \sim -x$.

That is, identifying the antipodal points of the unit sphere.

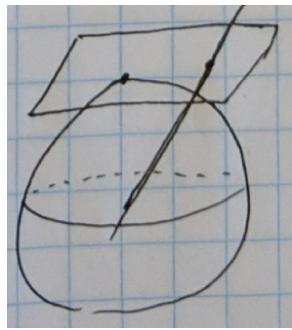
We may also consider $\mathbb{RP}^n = SO(n+1)/SO(n)$.

Claim

\mathbb{RP}^n is a manifold.



$$\mathbb{RP}^1 \setminus \{x\text{-axis}\} \xrightarrow{\text{homeo}} \mathbb{R}.$$



$$\mathbb{RP}^2 \setminus \mathbb{RP}^1 \xrightarrow{\text{homeo}} \mathbb{R}^2$$

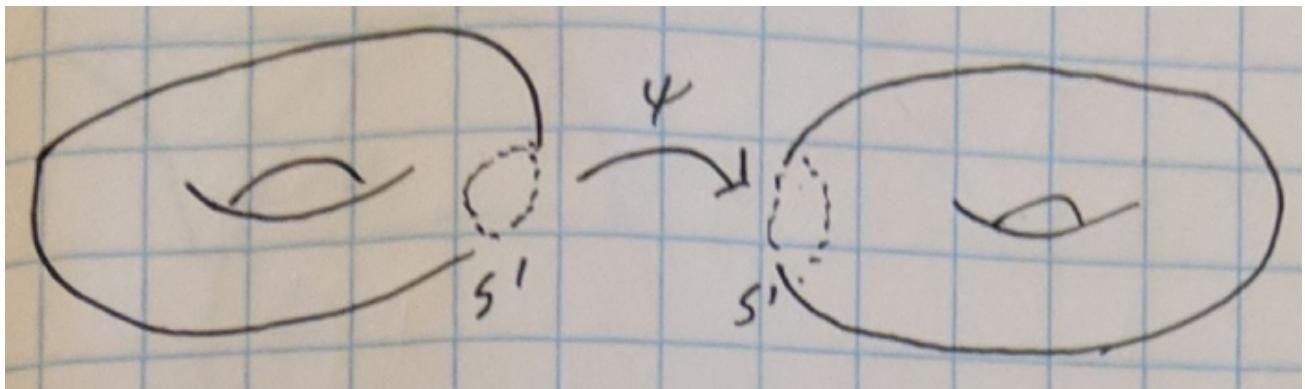
We have that \mathbb{RP}^1 is homeomorphic to the circle, and $\mathbb{RP}^n = \mathbb{RP}^{n-1} \cup B^n$.

Take $x = (x_0, \dots, x_n)$, $y = (y_0, \dots, y_n) = (\lambda x_0, \dots, \lambda x_n)$ and $[x] = [x_0 : x_1 : \dots : x_n]$.

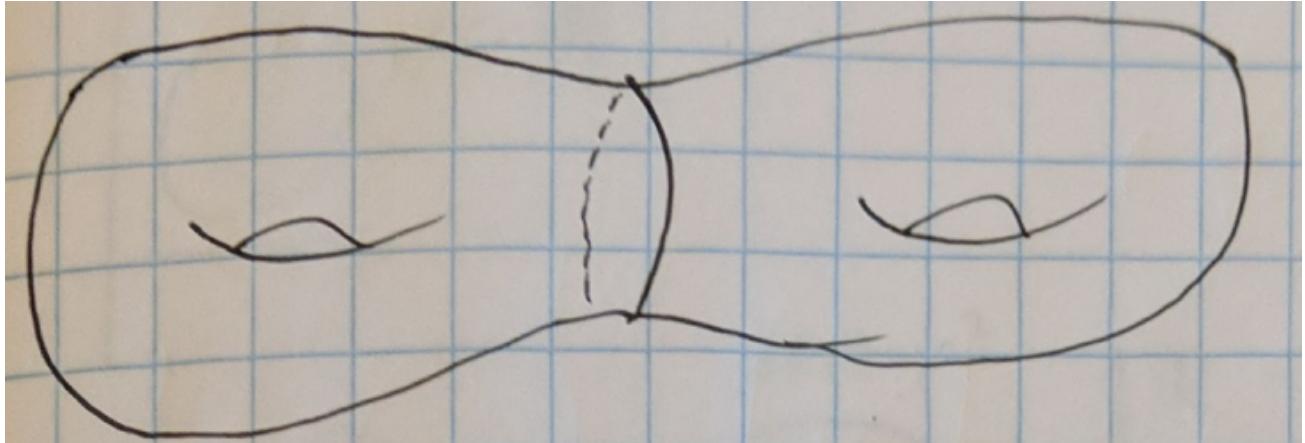
Then for $U_k \subset \mathbb{RP}^n$ with $U_k = \{[x] : x_k \neq 0\}$, we have that U_0, \dots, U_n covers \mathbb{RP}^n .

Then define $U_k \rightarrow \mathbb{R}^n$ by $[x_0 : \dots : x_n] \mapsto \left(\frac{x_0}{x_k}, \dots, \frac{x_{k-1}}{x_k}, \frac{x_{k+1}}{x_k}, \dots, \frac{x_n}{x_k}\right)$.

Connected Sum of Manifolds

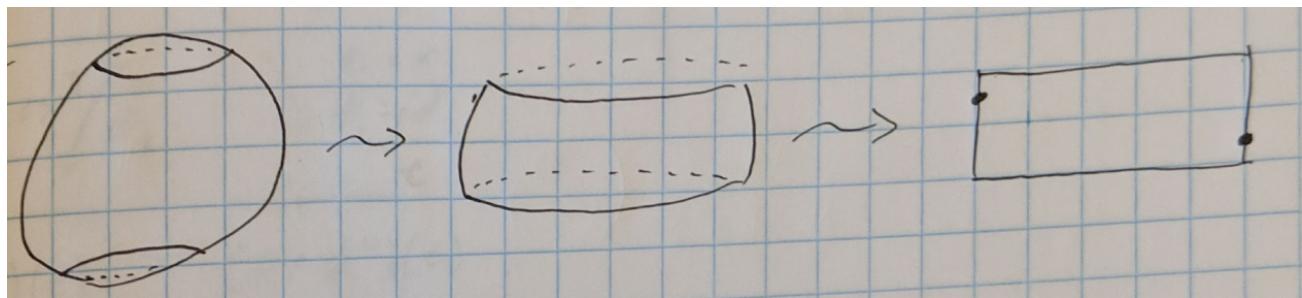


$$M \setminus B^n \coprod N \setminus B^n$$



$$M \# N.$$

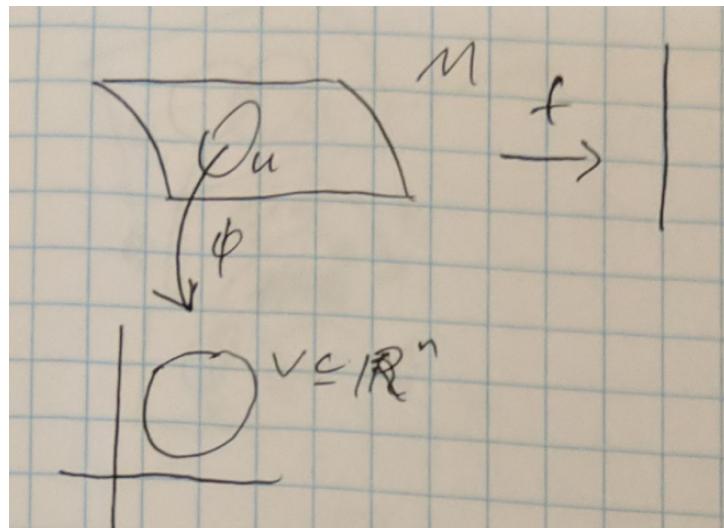
Möbius Band



October 1, 2024

A Failed Definition

$$f \in C^{r \geq 1}; f \circ \phi^{-1} : V \xrightarrow{C^r} \mathbb{R}.$$



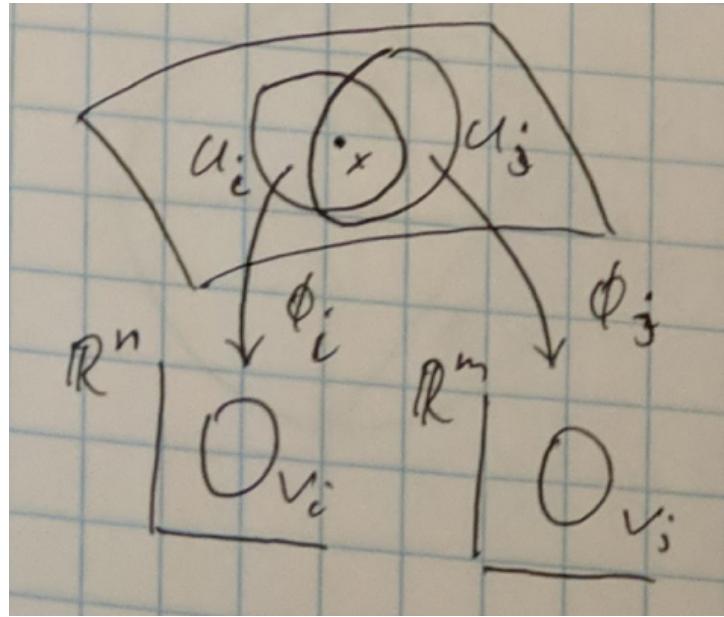
Example

$$\begin{array}{c}
 2. \quad M = \mathbb{R} \xrightarrow{x} \mathbb{R} \\
 t = x^3 \xrightarrow{\phi_2} t = x \xrightarrow{f(x) = x^2} \mathbb{R} \\
 \underline{t} \quad \underline{t} \\
 (f \circ \phi_2^{-1})(t) = t^{2/3} \\
 \text{Not } C^1 \\
 \underline{(f \circ \phi_1^{-1})(t)} = t^2 \in C^\infty
 \end{array}$$

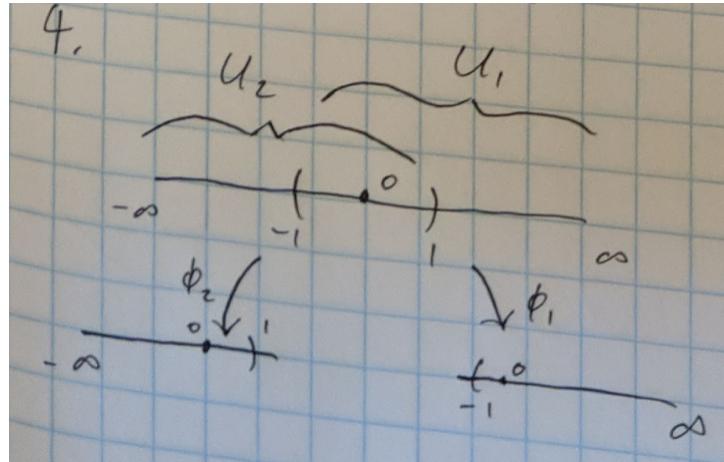
Definition: Charts

Say there exists a cover U_i by open sets and $U_i \xrightarrow{\phi_i} V_i \subseteq \mathbb{R}^n$ fixed.
Then the pair (U_i, ϕ_i) is a chart.

What if a point belongs to two charts?



With f smooth at x , $f \circ \phi_i^{-1}$ smooth at $\phi_i(x)$ and $f \circ \phi_j^{-1}$ smooth at $\phi_j(x)$.

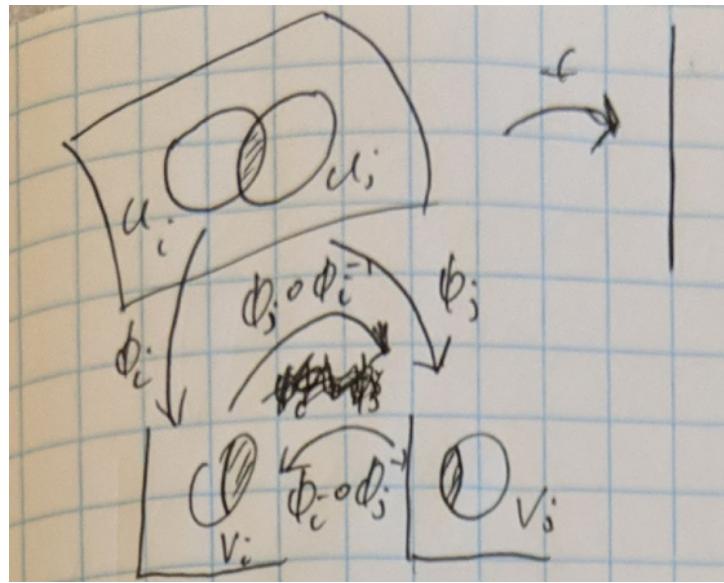


Notation

The notation C^r will be used interchangably with the term smooth.

Definition: Smooth Atlas

Let M be a topological manifold. A smooth atlas on M is a cover $(U_i, \phi_i : U_i \xrightarrow{\sim} V_i \subset \mathbb{R}^n)$ where $\phi_j \circ \phi_i^{-1}$ and $\phi_i \circ \phi_j^{-1}$ are smooth for every i and j .



Say that the charts are (smooth) compatible.

Definition: Smooth Function

Say that f is smooth at $x \in M$ if there exists a chart $U_i \ni x$ such that $f \circ \phi_i$ is smooth at $\phi_i(x)$. Equivalently, if for every chart $U_i \ni x$ we have that $f \circ \phi_i$ is smooth at $\phi_i(x)$.

- Proof

$$f \circ \phi_j^{-1} = (f \circ \phi_i^{-1}) \circ \underbrace{(\phi_i \circ \phi_j^{-1})}_{C^r}$$

Definition: Compatibility (Equivalence) of Atlases

Atlases A_1 and A_2 are compatible or equivalent if every chart in A_1 is compatible with every chart in A_2 . Equivalently, $A_1 \cup A_2$ is also an atlas.

- Claim: This is an equivalence relation.

Example

Consider \mathbb{R} .

Atlas 1: $U = \mathbb{R}$ and $\phi = \text{id}$.

Atlas 2: $U_1 = (1, \infty)$, $\phi_1(x) = x^2$, $U_2 = (-\infty, 2)$ and $\phi_2(x) = x$.

Definition: Diffeomorphism

$\mathbb{R}^n \supset V \xrightarrow{F} W \subset \mathbb{R}^n$ is a diffeomorphism if

- F is C^r ,
- F is invertible, and
- F^{-1} is C^r

Counterexample

$y = x^3$ is a smooth homeomorphism but not a diffeomorphism.

Definition: Smooth Structure / Maximal Atlas

Given an atlas, we may take all compatible atlases and define a smooth structure by the union of all such objects (i.e. the maximal atlas).

Lemma:

Every smooth manifold has a countable, locally finite atlas of precompact charts.

Examples

- Zero dimensional manifolds (i.e. a point).
- \mathbb{R}^n and open subsets of \mathbb{R}^n .
- If M, N are smooth manifolds, then $M \times N$ is a smooth manifold.

That is, if we have atlases (U_i, ϕ_i) and (W_j, ψ_j) , we may generate $(U_i \times W_j, \phi_i \times \psi_j)$.

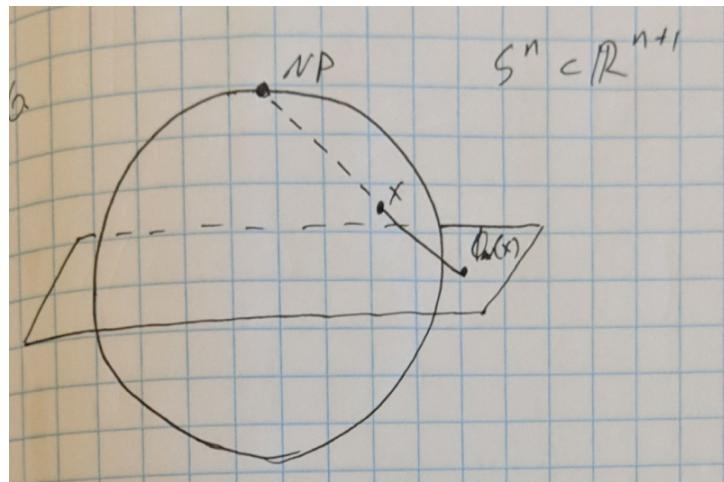
- Take $F: M \xrightarrow{\text{homeo}} N$ with N a smooth manifold. Then M is smooth.

Take an atlas A on N and the pullback $F^{-1}A = \{(F^{-1}(U_i), \phi_i \circ F)\}$.

- An open subset of a smooth M is a smooth manifold.
- $\text{GL}(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$.

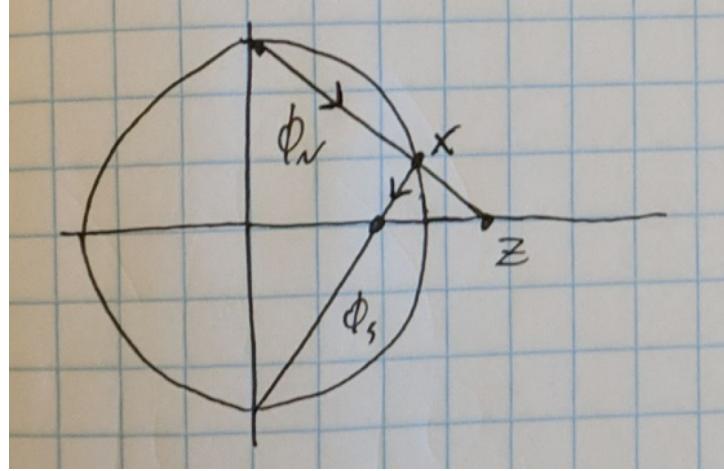
The n-Sphere

- S^n is a manifold



$$U_N = S^n \setminus NP \xrightarrow{\phi_N} \mathbb{R}^n$$

$$U_S = S^n \setminus SP \xrightarrow{\phi_S} \mathbb{R}^n$$



$$\phi_S \phi_N^{-1}(z) = \frac{z}{|z|^2}.$$

– A different construction for S^n .

Take hemispheres $U \xrightarrow{\text{orthogonal projection}} B^n$.

Projective Space

$$\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus 0 / \sim \text{ where } x \sim \lambda x \text{ for } \lambda \neq 0.$$

$$[x] = [x_0 : x_1 : \dots : x_n] = [\lambda x_0 : \lambda x_1 : \dots : \lambda x_n].$$

Take $U_i = \{x_i \neq 0\}$ and open cover, and maps $U_i \rightarrow \mathbb{R}^n$ given by $[x_0 : \dots : x_n] \mapsto \left(\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, 1, \frac{x_i}{x_i}, \dots, \frac{x_n}{x_i} \right)$. Then for $j < i$ take

$$\phi_j \phi_i^{-1}(y_1, \dots, y_n) = \left(\frac{y_0}{y_j}, \dots, \frac{y_{j-1}}{y_j}, \frac{y_{j+1}}{y_j}, \dots, \frac{y_{i-1}}{y_1}, 1, \frac{y_i}{y_i}, \dots, \frac{y_n}{y_1} \right)$$

Definition: Diffeomorphism

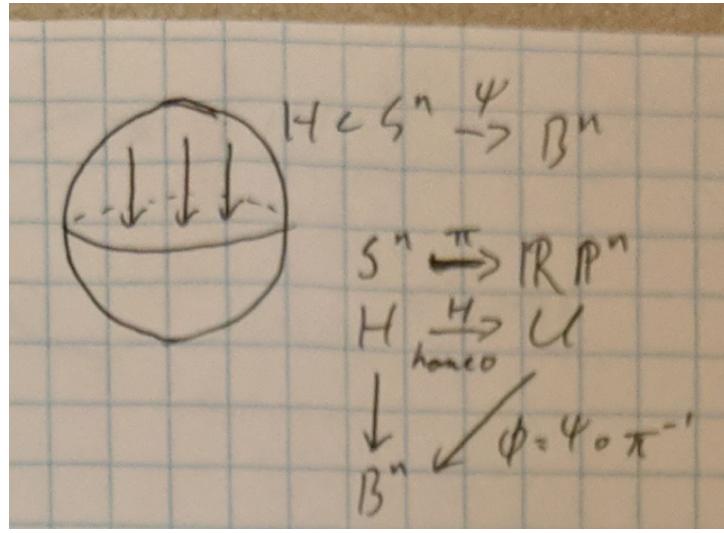
$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ B \subset B_{\max} & & A \supset A_{\max} \end{array}$$

F is a diffeomorphism if F is a homoeomorphism and $F^{-1} A_{\max} = B_{\max}$ ($F^{-1} A \sim B$).

October 3, 2024

Recall

$$\mathbb{RP}^n = \begin{cases} \mathbb{R}^{n+1} \setminus 0 / \sim & x \mapsto \lambda x \\ S^n / x \sim -x \end{cases}$$



Note

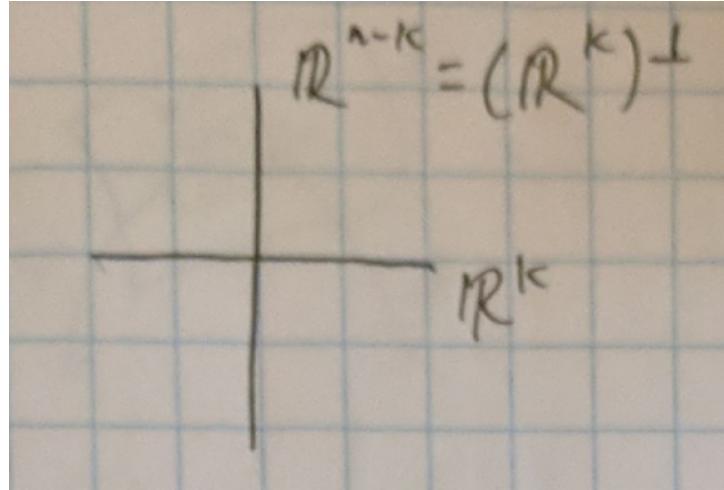
Given a manifold M and A a smooth atlas, we generate a continuum of smooth atlases not equivalent to each other. That is, given $M \xrightarrow[\text{homeo}]{} M$, $F^{-1}A \neq A$.

Confer With Groups

$$G \xrightarrow{F} G, a * b = F^{-1}(F(a)F(b)).$$

Definition: Grassmannians

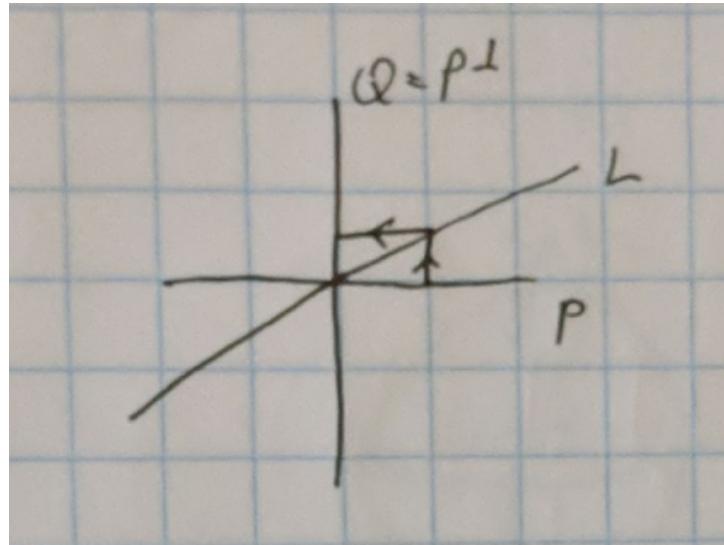
Write $G_k(n)$, the collection of all k -dimensional subspace L in \mathbb{R}^n .



Observe that if $O(i)$ is the collection of orthogonal transformations in dimension i ,

$$G_k(n) = \frac{O(n)}{O(k) \times O(n-k)}$$

with $X \sim Y$ when $Y = XA = X(O(K) \times O(n-k))$.

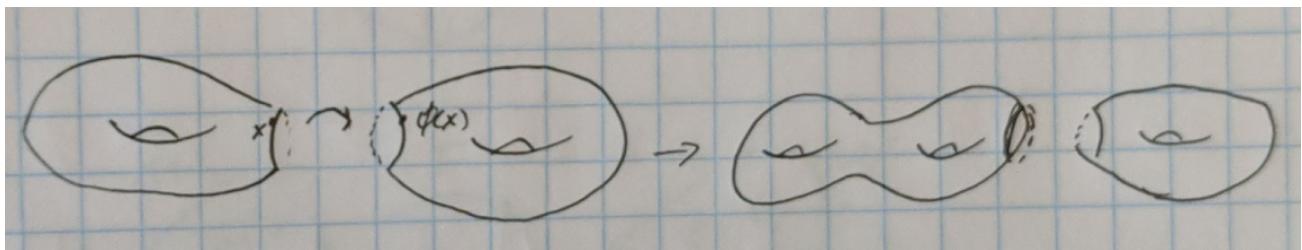


Where $\dim(L) = k$, $U_p = \{L : L \cap Q = \{0\}\}$, $L = \text{graph}(A : P \rightarrow Q)$, and we have a homeomorphism

$$U_p \xrightarrow{\phi} \underbrace{\{\text{linear maps } P \rightarrow Q\}}_{\mathbb{R}^{k \times (n-k)}}.$$

Surfaces

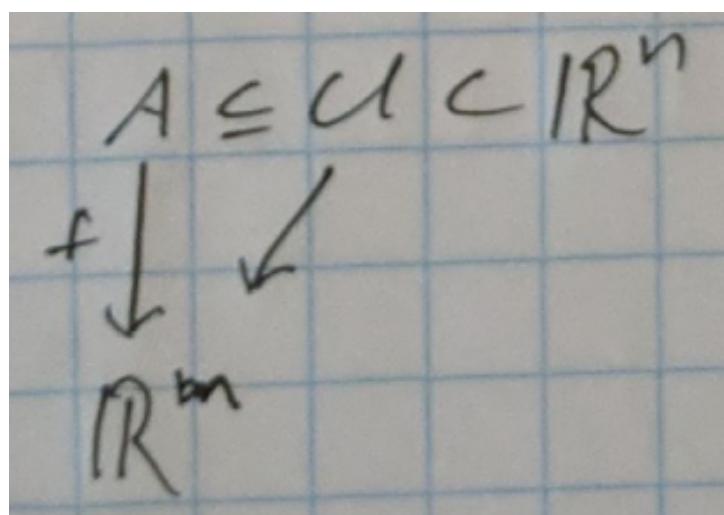
We have explored S^2 , \mathbb{RP}^2 , $\mathbb{T}^2 = S^1 \times S^1$. We have also connected sums.



Terminological Remark

Let $\mathbb{R}^N \ni A \xrightarrow{f} \mathbb{R}^m$.

Then f is smooth if it extends to a smooth map



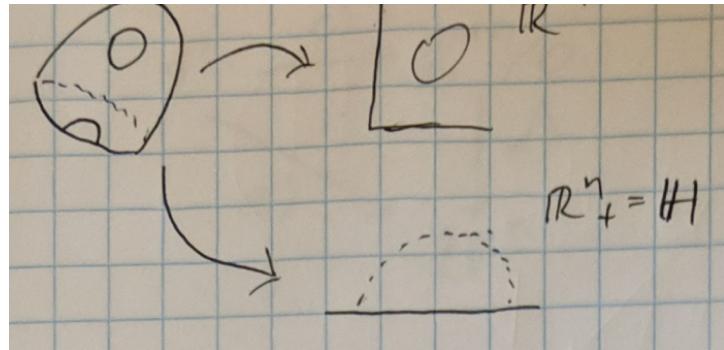
Exercise

Let $A = [0, \infty) \subset \mathbb{R}$. $f : A \rightarrow \mathbb{R}$ is smooth if and only if it is infinitely differentiable.
Construct $(-\varepsilon, \infty)$.

Definition: Smooth Manifold with Boundary

A smooth manifold with boundary is a topological space along with an atlas A with charts of two types

$$\begin{aligned}\phi : U \rightarrow B^n &\quad (\text{open ball}) \\ \phi : U \rightarrow B^n \cap H\end{aligned}$$



As before, $\phi_i \circ \phi_j^{-1}$ must be smooth.

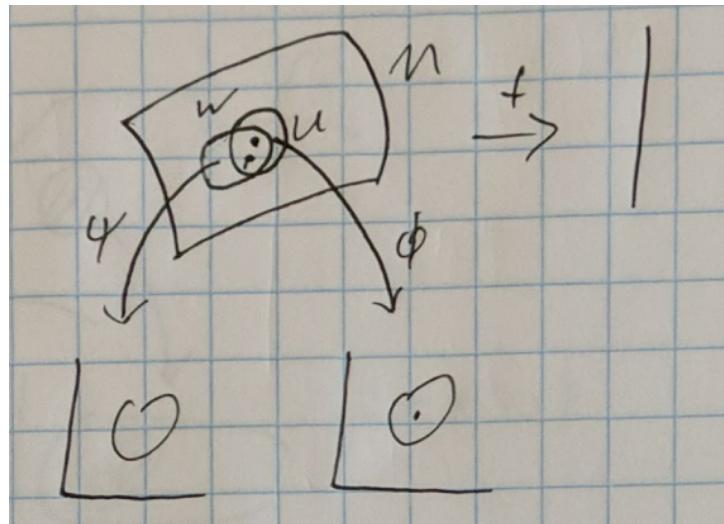
Examples

- $M \setminus \text{open ball}.$
- The upper half space.

Definition-Lemma: Smooth Function

A function $f : M \rightarrow \mathbb{R}$ is smooth at $p \in M$ if either of the following equivalent conditions is satisfied

1. \exists a chart $(U, \phi) \ni p$ such that $f \circ \phi^{-1}$ is smooth at $\phi(p)$.
2. \forall a chart $(U, \phi) \ni p$ such that $f \circ \phi^{-1}$ is smooth at $\phi(p)$.



Where $f \circ \phi^{-1} = f \circ \psi^{-1}(\psi \circ \phi^{-1})$.

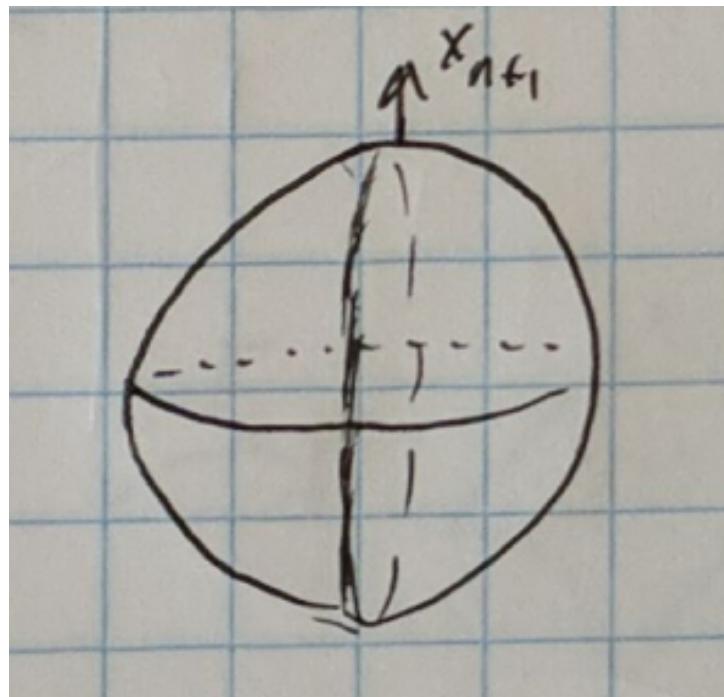
If the above hold for each $p \in M$, then f is smooth.

Remark

f smooth implies f is C^0

Exercise / Sketch

The height function on S^n is smooth.

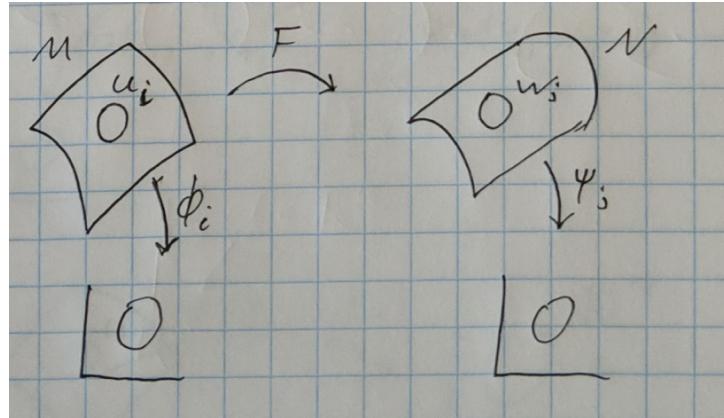


$$\phi : (x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_n).$$

$$f \circ \phi^{-1} = \pm \sqrt{1 - x_1^2 - \dots - x_n^2}.$$

Note that handling the equator requires examining the Eastern and Western hemisphere.
The stereographic projection leads to a simpler proof.

Definition: Smooth Function Between Manifolds



$F : M \rightarrow N$ is smooth if F is C^0 and one of the following equivalent conditions is satisfied

1. \exists an atlas $A \subset A_{\max}$ on M and an atlas $B \subset B_{\max}$ on N such that $\psi_j \circ F \circ \phi_i^{-1}$ is smooth on $F^{-1}(W_j) \cap U_i$.
2. The same as a., but for A_{\max} and B_{\max} .

Consider as an example $S^n \rightarrow \mathbb{RP}^n$.

Properties of Smooth Maps

$$C^\omega \implies C^\infty \implies C^r \implies C^{r-1} \implies C^1 \implies C^0.$$

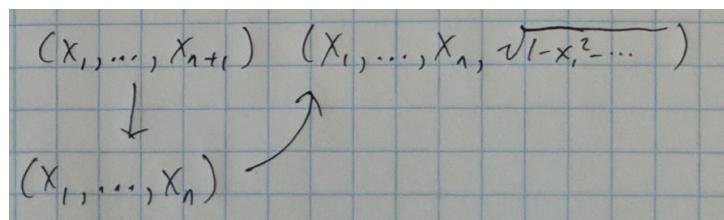
The sum and product of smooth functions is smooth.

Exercise

The composition of smooth maps is smooth.

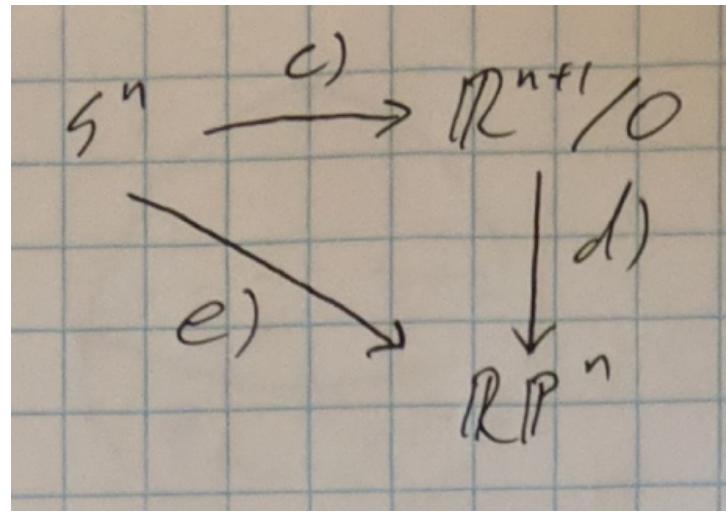
Examples

1. $M \times N \xrightarrow{pr} M$ is smooth.
2. $\underbrace{M \xrightarrow{(F_1, F_2)} N_1 \times N_2}_{\text{smooth}}$ if and only if F_1 and F_2 are smooth.
3. $S^n \hookrightarrow \mathbb{R}^{n+1}$ is smooth.



$$1. \mathbb{R}^{n+1} \setminus 0 \rightarrow \mathbb{RP}^n \text{ is smooth with } [x_0 : \dots : x_n] \mapsto \left(\frac{x_0}{x_i}, \dots, \frac{\hat{x}_i}{x_i}, \dots, \frac{x_n}{x_i} \right).$$

$$2. S^n \rightarrow \mathbb{RP}^n.$$



Definition: Diffeomorphism

$$F: M \xrightarrow[A_{\max}]^{\text{diffeo}} N \text{ if } B_{\max}$$

- F is smooth.
- F is invertible.
- F^{-1} is smooth.

Previous Definition

$$F^{-1}(B_{\max}) = A_{\max}.$$

Exercise

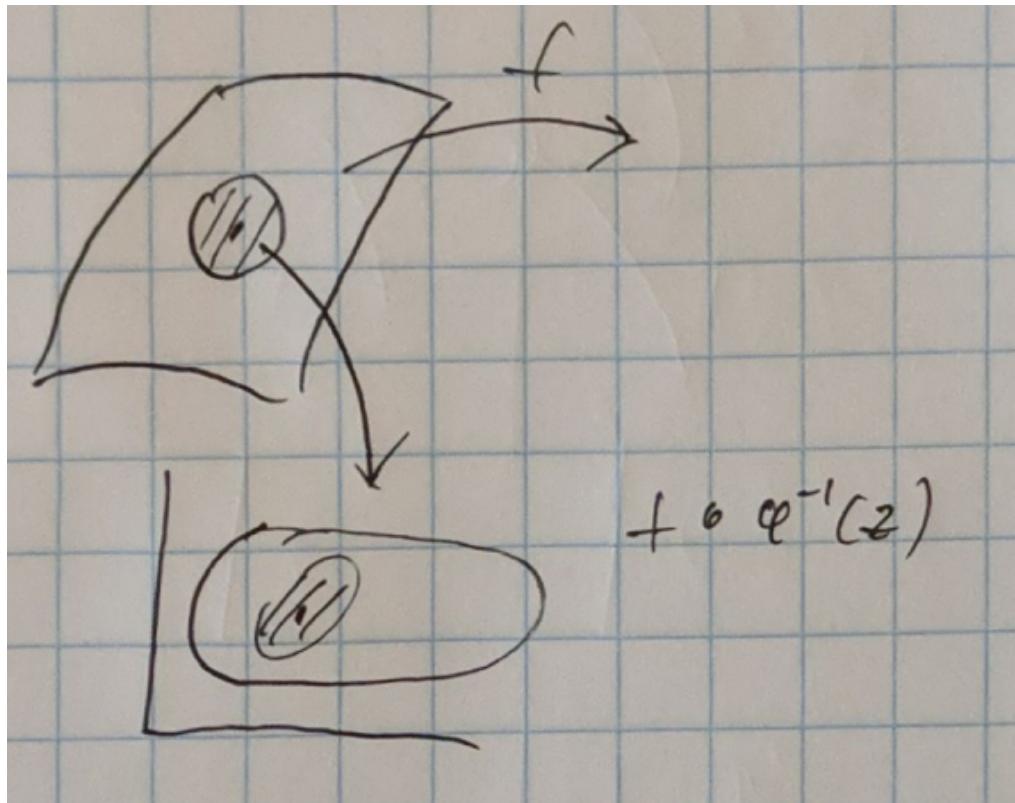
Prove that the definitions are equivalent.

Examples

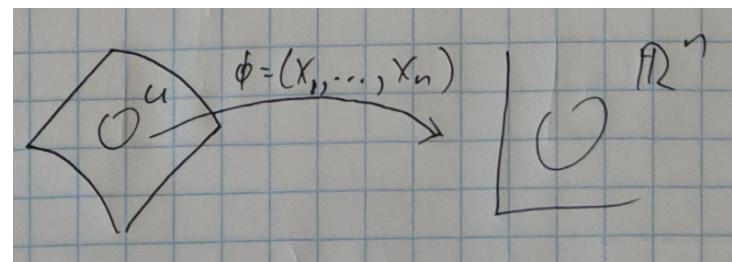
1. $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \xrightarrow{\text{diffeo}} \mathbb{R}$.
2. $x \mapsto x^3$, $\mathbb{R} \rightarrow \mathbb{R}$ is not a diffeomorphism.
3. $G_k(n) \leftrightarrow G_{n-k}(n)$ with $P \leftrightarrow P^\perp$.

Example 4

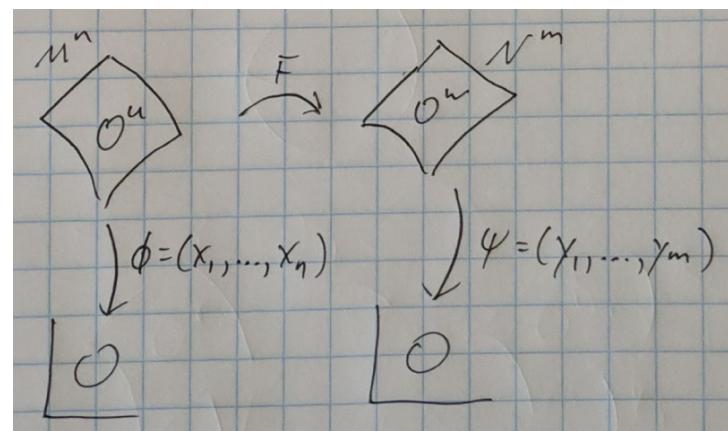
A compact, analytic manifold admits only constant smooth functions by the maximum modulus principle.



Example 5

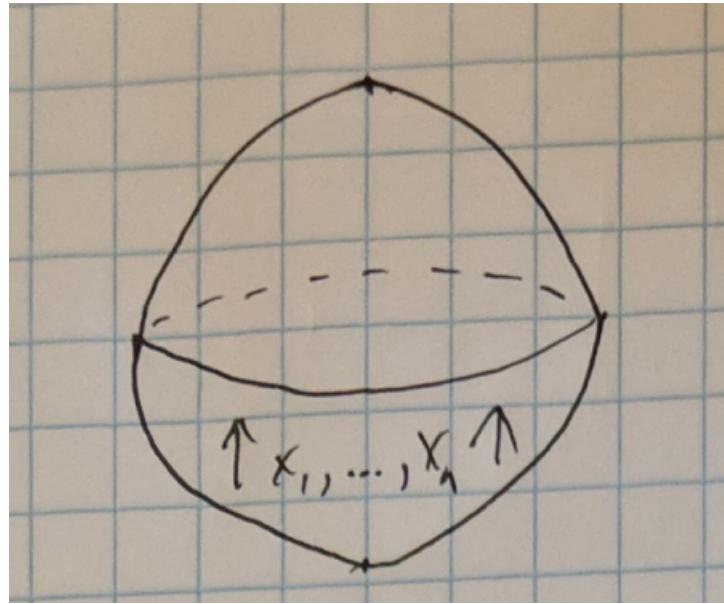


Where $\phi = (x_1, \dots, x_n)$ and each x_i is a real-valued function.



$$\psi \circ F \circ \phi^{-1} = (y_1(x_1, \dots, x_n), \dots, y_m(x_1, \dots, x_n)).$$

Then $S^n \hookrightarrow \mathbb{R}^{n+1}$



where $y_1 = x_1, \dots, y_n = x_n$, and $y_{n+1} = -\sqrt{1 - x_1^2 - \dots - x_n^2}$.

Example 6

$$\mathbb{R}^{n+1} \setminus 0 \xrightarrow{F} \mathbb{RP}^n.$$

Need to check that $\psi_j \circ F$ is smooth.

$$[t_0 : \dots : t_n] \xrightarrow{\psi_0} \left(\frac{t_1}{t_0}, \dots, \frac{t_n}{t_0} \right) \text{ with } U_0 : t_0 \neq 0$$

October 8, 2024

Questions

Question 1

Given M smooth and $x \neq y \in M$, does there exist $f \in C^r(M)$ such that $f(x) = 0$ and $f(y) = 1$.

Question 2

Given $K \subset U \subset M$ with K compact and U open and $g : K \xrightarrow{C^r} \mathbb{R}$, does there exist a C^r extension f of g on M such that $\text{supp } f \subset U$.

Definition: Partitions of Unity

Let W_i be a locally finite open cover.

A partition of unity subordinated to W_i , is a collection of functions $f_i : M \xrightarrow{C^r} \mathbb{R}$ satisfying

- $0 \leq f_i \leq 1$
- $\text{supp } f_i \subset W_i$
- $\sum f_i \equiv 1$

Definition: Refinement

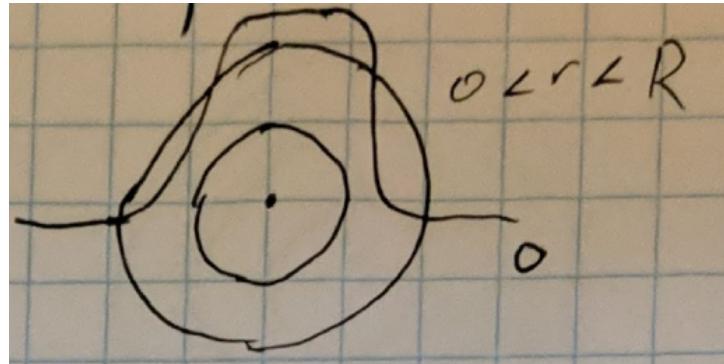
Given covers U_j and W_i , U_j is a refinement if for each j we may find i such that $U_j \subset W_i$.

Theorem

There exists a partition of unity subordinated to W_i .

Lemma 1

Take $B(r) \subset B(R) \subset \mathbb{R}^n$.



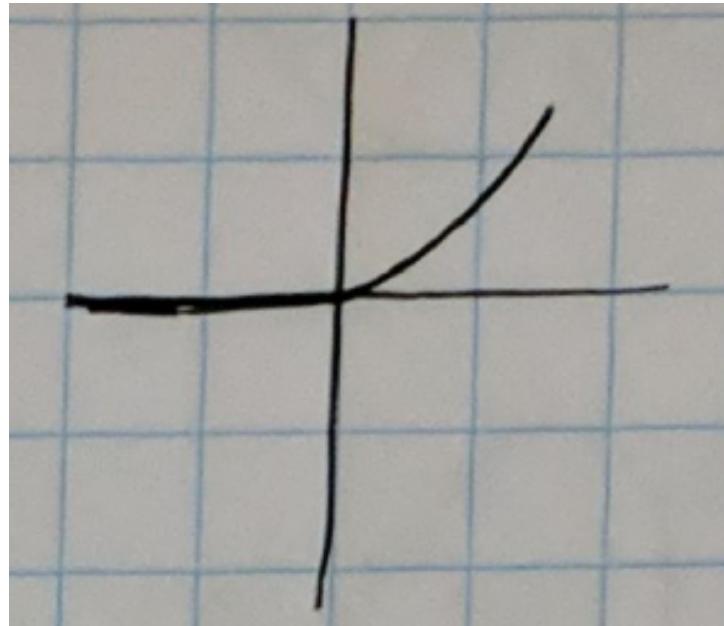
Then there exists $g \in C^\infty$ such that

- $0 \leq g \leq 1$
- $g|_{\overline{B(r)}} \equiv 1$
- $\text{supp } g \subset B(R)$

Proof

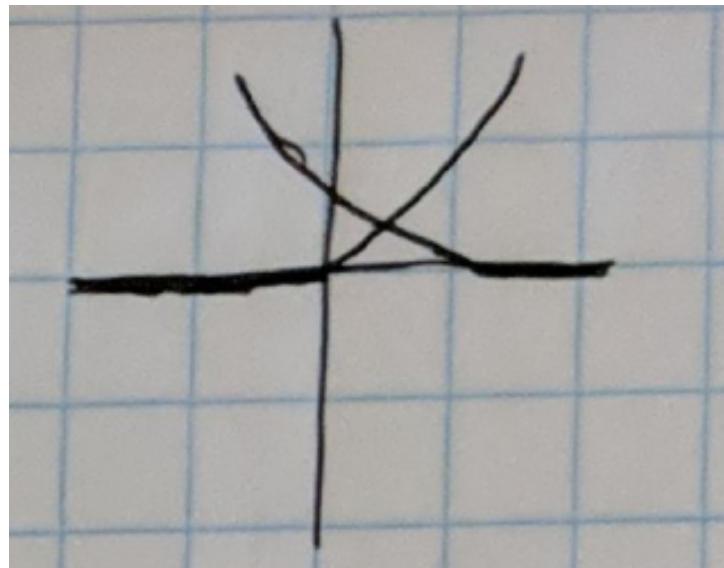
Take

$$h_0(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$



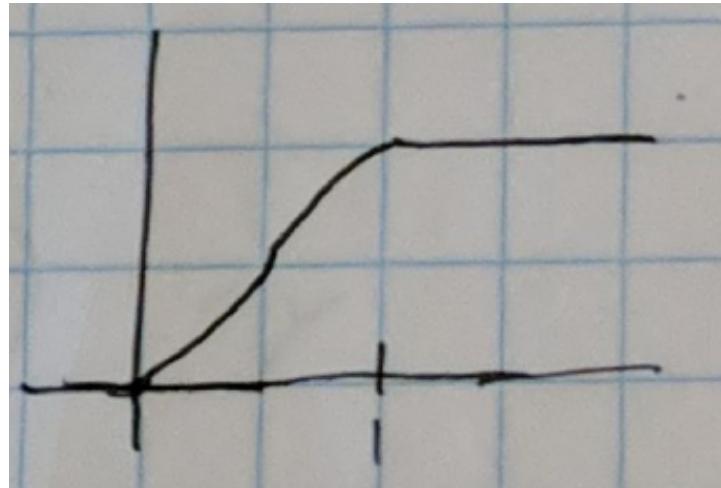
- Exercise: show that $h_0 \in C^\infty(\mathbb{R})$ (Hint: show that derivatives agree at zero from the right.)

Then take $h_1(x) = h_0(x) \cdot h_0(1-x)$.

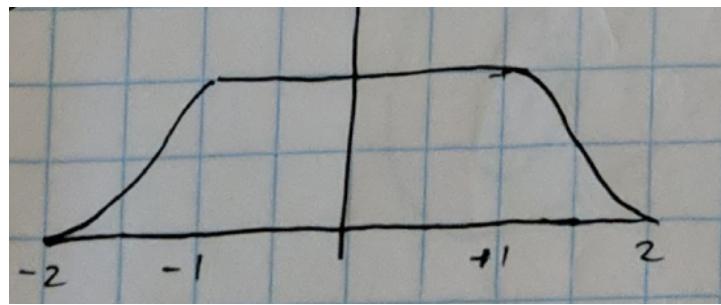


Then define

$$h_2(x) = \frac{1}{\int_{-\infty}^{\infty} h_1(t) dt} \int_{-\infty}^x h_1(t) dt$$



Finally, we define $h(x) = h_2(x+2) \cdot h_2(2-x)$.



Then $g(x) = h(||x||)$ satisfies our requirements.

Lemma 2

There exists a refinement (U_j, ϕ_j) by coordinate charts such that

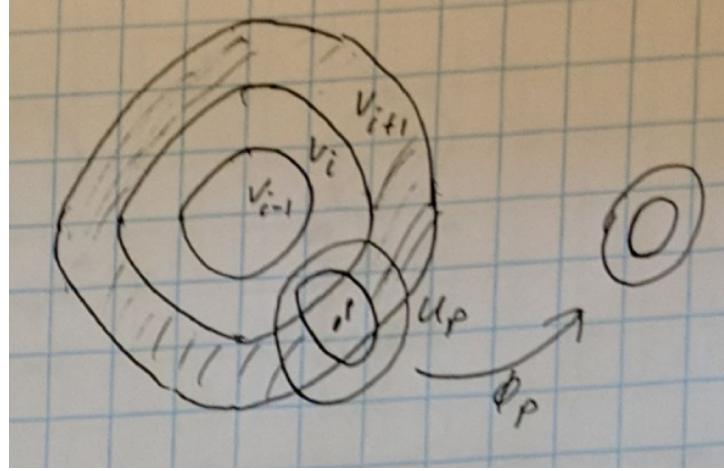
1. U_j is locally finite.
2. $\phi_i : U_j \xrightarrow{\text{diffeo}} B^n(2)$.
3. $\phi_j^{-1}(B(1))$ is also a cover.

Proof

There exists a compact exhaustion of M , $C_1 \subset C_2 \subset C_3 \subset \dots$ where $\bigcup C_i = M$.

There exists also an open exhaustion by precompact open sets $\emptyset = V_0 \subset V_1 \subset V_2 \subset \dots$ where $\overline{V_i} \subset V_{i+1}$.

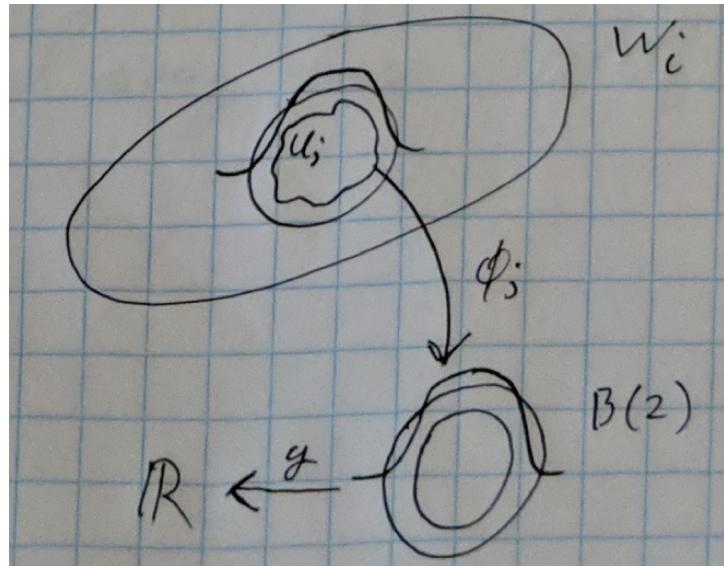
Then define $V_i := \text{nbhd}(C_i \cup \overline{V}_{i-1})$.



Take $A_i = \overline{V}_{i+1} \setminus V_i$ compact, $p \in A_i$. Then we have a map $U_p \xrightarrow{\phi_p} B(2)$.

- $U_p \subset W_i$ for some i . (Refinement)
- $U_p \cap V_{i-1} = \emptyset$. (Locally Finite)
- There exists a finite subcover such that $\phi_p^{-1}(B(1))$ is also an open cover.

Proof of Theorem



Set $g_j = g \circ \phi_j \in C^r$, extended to 0 outside of U_j .

- $\text{supp } g_j \subset U_j$
- $\forall p \in M, \exists g_j(p) = 1 \neq 0 \implies \sum g_j > 0$
- $\text{supp } g_j$ is locally finite.

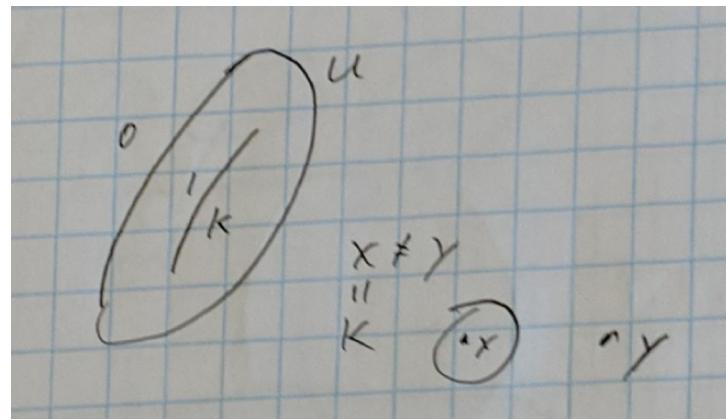
Then we have

$$f_j = \frac{g_j}{\sum g_j}$$

Corollary 1

For $K \subset U$, K compact and U open, there exists $h \in C^r(M)$ such that

- $h|_K \equiv 1$
- $\text{supp } h \subset U$



Proof

Take $U_0 = U$ and $U_1 = M \setminus K$.

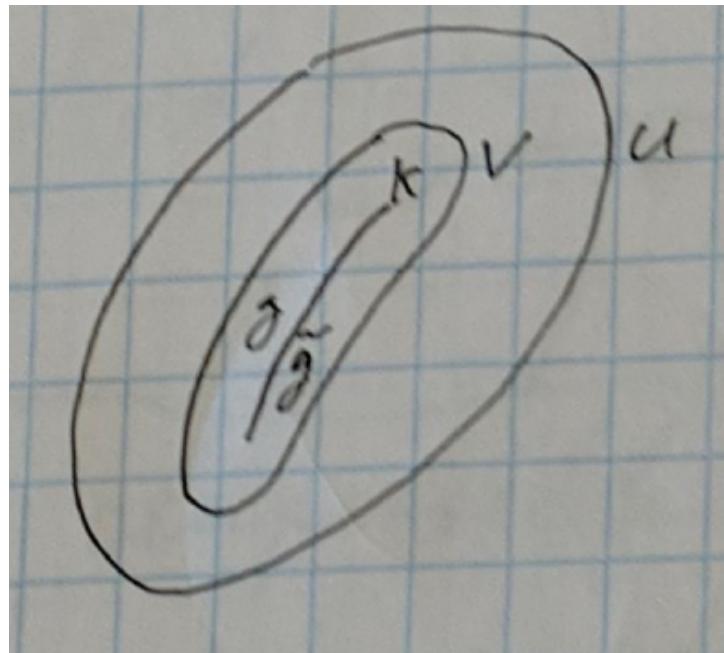
Then there exists a partition of unity f_0 and f_1 where $f_0 + f_1 = 1$. Therefore f_0 has support in U and f_1 has support in $M \setminus K$ this occurs if and only if $f_1|_K = 0$.

Corollary 2

For $K \subset U$, K compact and U open, and $g : K \xrightarrow{C^r} \mathbb{R}$, there exists an extension $f : M \rightarrow \mathbb{R}$ of g : $f|_K = g$ such that $\text{supp } f \subset U$.

Proof

For $g \in C^r(K)$, there exists a neighborhood V of K and a function $\tilde{g} : V \rightarrow \mathbb{R}$ such that $\tilde{g}|_K = g$.



By corollary 1, there exists $h \in C^r(M)$ such that

- $\text{supp } h \subset V$
- $h|_K = 1$

Therefore $f = h \cdot \tilde{g}$.

Corollary 3

There exists $f : M \xrightarrow{C^r} \mathbb{R}$ bounded from below and proper.

Definition: Proper Function

f is proper if and only if $f^{-1}(K)$ is compact for K compact.

Proof

See Textbook.

Consequence

Take $\{x : f(x) \leq C_i\} =: E_i$ as $i \rightarrow \infty$. Then we get a compact exhaustion $E_1 \subset E_2 \subset E_3 \subset \dots$
e.g. $f(x) = \|x\|^2$.

October 10, 2024

Algebra = Analysis (Geometry)

Take M to be either a compact metrizable space ($A = C^0(M)$) or a copact manifold ($A = C^\infty(M)$).

$$M \leftrightarrow A$$

Let $I \subset A$ be an ideal and take $V(I) = \{x : f(x) = 0, \forall f^n \in I\}$.

Take also $Y \subset M$ closed and consider $I_Y = \{f : f|_Y = 0\}$ which is also an ideal.

$$\begin{array}{c} Y \rightarrow I_y \\ V(I) \leftarrow I \end{array}$$

Denote $I_x = \{f : f(x) = 0\}$.

Theorem

- I_x is a maximal ideal.
- Every maximal ideal is of this form, and x is unique.

$$M \leftrightarrow \text{Maximal Ideals of } A$$

Proof

Maximal Ideal

Take I_x and $g \notin I_x$.

We want to show that g with I_x generates A .

Then take $f \in A$ defined as $f = h + ag$ for some $h \in I_x$.

Since $g \notin I_x$, $g(x) \neq 0$. Take $a = \frac{f(x)}{g(x)}$ and define $h := f - ag$.

Then $f(x) - \frac{f(x)}{g(x)}g(x) = 0 \in I_x$.

Every Maximal Ideal

Let I be a proper ideal such that $V(I) \neq 0$, $\exists x$ such that $f(x) = 0, \forall f \in I$.

Maximal $\implies V(I) = \{x\}$.

By contradiction, assume not: $\forall x \in M, \exists f_x \in I, f_x(x) \neq 0$.

It follows that $f_x|_{U_x} \neq 0$ where $U_x \ni x$ is a neighborhood from an open cover.

Then, by compactness, we have a finite subcover

$I \ni f_i = f_{x_i} \neq 0$ on U_i .

$I \ni \underbrace{\sum f_i^2}_{g>0} > 0$ on M . But then $1 = g^{-1}g \in I$.

Uniqueness

$$I_{x_1} = I_{x_2} \iff x_1 = x_2$$

(\iff) is obvious.

(\implies) $x_1 \neq x_2$ implies that there exists $f \in A$ such that $f(x_1) = 1$ and $f(x_2) = 0$.

Then $f \in I_{x_2}$ while $f \notin I_{x_1}$ implies that $I_{x_1} \neq I_{x_2}$.

Reading the Topology / Smooth Structure

We have a correspondence

$$\text{closed sets} \leftrightarrow \text{ideals}$$

Algebra Homomorphisms

Take $\phi : A \rightarrow \mathbb{R}$. Then $\ker \phi$ is a maximal ideal, and

$$\begin{aligned} A/I_x &\xrightarrow{\delta_x} \mathbb{R} \\ x &\mapsto \pm f(x) \end{aligned}$$

Then

$$\begin{aligned} M &\leftrightarrow \text{Alg. Hom. } A \rightarrow \mathbb{R} \\ 1 &\mapsto 1 \\ A > 0 &\mapsto \text{pos. \#} \end{aligned}$$

Counterexample

Take instead $M = \mathbb{R}$. Claim: there exist maximal ideals other than I_x .

Proof

Take J to be the ideal of all compactly supported functions such that $J \subset M$ for some maximal ideal M . However, $\forall x \in \mathbb{R}$ there exists a compactly supported function $f(x) \neq 0$. So $M \neq I_x$.

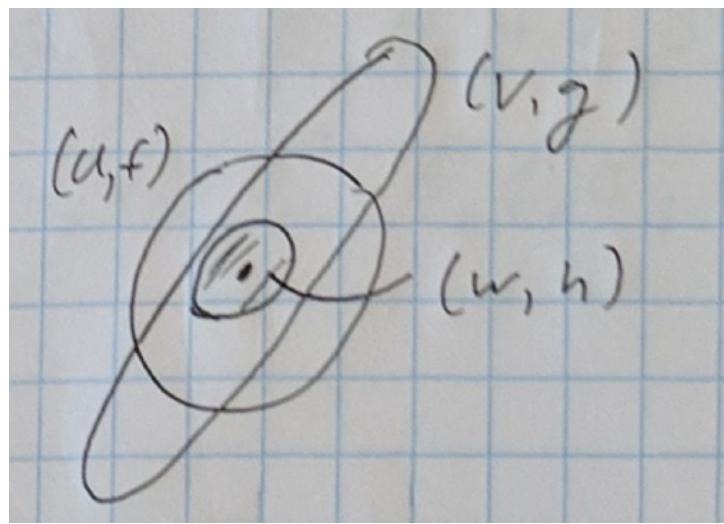
Functorial Considerations

Take $F : M \rightarrow N$, M and N compact. Then

$$\begin{aligned} F : M &\rightarrow N \\ C^{0/\infty}(M) &\xleftarrow{F^A} C^0(N) \\ f \circ F &\leftrightarrow f \end{aligned}$$

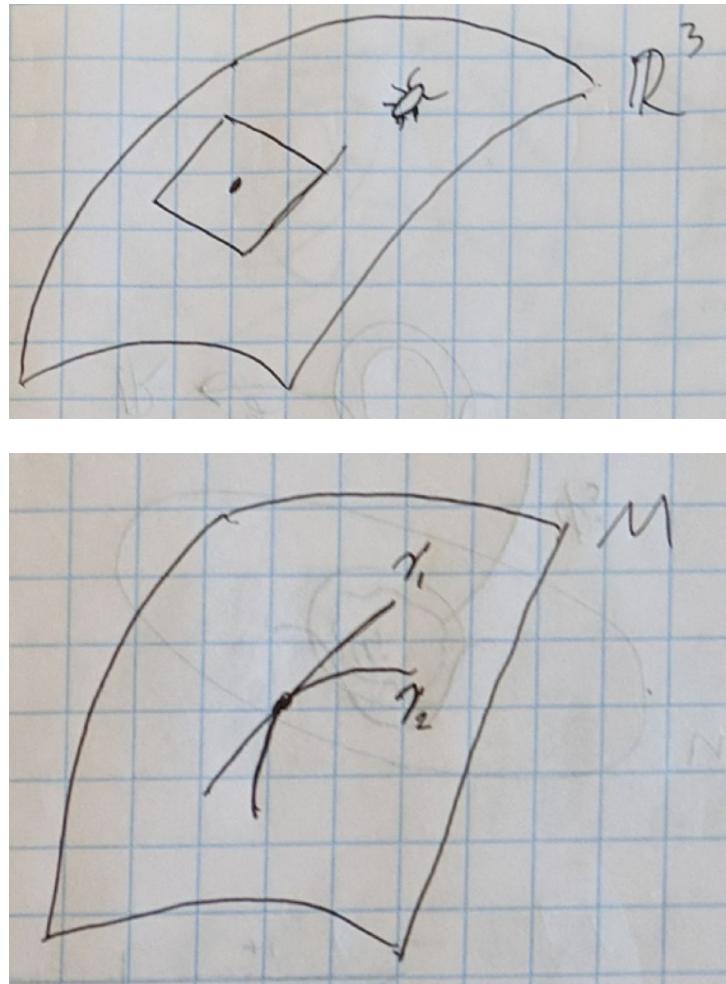
Definition: Germ

Take $M \ni P$.



Where $(U, f) \sim (V, g)$ if $f \equiv g$ on some neighborhood of p .

Definition: Tangent Spaces



With $\gamma_1 : (-\varepsilon, \varepsilon) \rightarrow M$, $\gamma_2 : (-\varepsilon, \varepsilon) \rightarrow M$, and $\gamma_1 \sim \gamma_2 \iff \gamma'_1(0) = \gamma'_2(0)$.

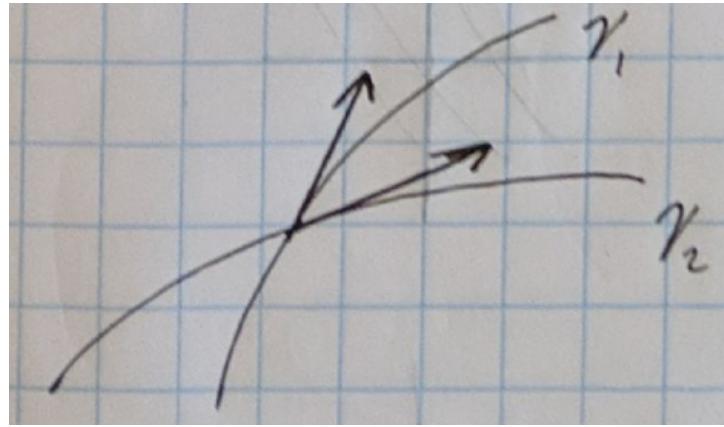
$\gamma(t) = ((x(t), y(t), z(t))$ and $\gamma'(t) = (x'(t), y'(t), z'(t))$.

The tangent space to M at p , written $T_p M$, is the set of equivalence classes.

Remarks: Tangent Vectors

Take $V \ni p$ a finite-dimensional vector space ($= \mathbb{R}^n$).

$$\begin{aligned} 0 &\mapsto p \\ \gamma_1, \gamma_2 &: (-\varepsilon, \varepsilon) \rightarrow V \quad (\text{germs}) \\ \gamma'_1 \sim \gamma'_2 &: \gamma'_1(0) = \gamma'_2(0) \end{aligned}$$



Write

$$\gamma'(0) = \lim_{t \rightarrow 0} \frac{\gamma(t) - \gamma(0)}{t}$$

$\gamma = (x_1, \dots, x_n)$ and $\gamma'(0) = (x'_1(0), \dots, x'_n(0))$.
 A tangent vector to V at p is an equivalence class $T_p V$.
 Claim: $T_p V$ is a vector space.

Operations

Take $p = 0$. Write

$$[\gamma_1] + [\gamma_2] = [\gamma_1 + \gamma_2]$$

$$\lambda[\gamma] = [\lambda\gamma]$$

When $p \neq 0$, instead

$$[\gamma_1] + [\gamma_2] = [\gamma_1 + \gamma_2 - p]$$

$$\lambda[\gamma] = [\lambda\gamma + (1 - \lambda)p]$$

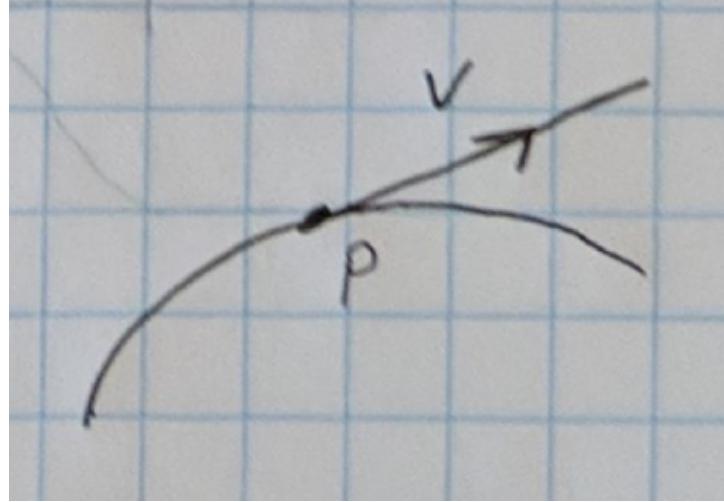
Claim: $T_p V$ is canonically isomorphic to V

$$[\gamma] \mapsto \gamma'(0)$$

$$[\gamma = (x_1, \dots, x_n)] \mapsto (x'_1(0), \dots, x'_n(0))$$

$$T_p V \rightarrow V$$

$$p + tv \leftrightarrow v$$



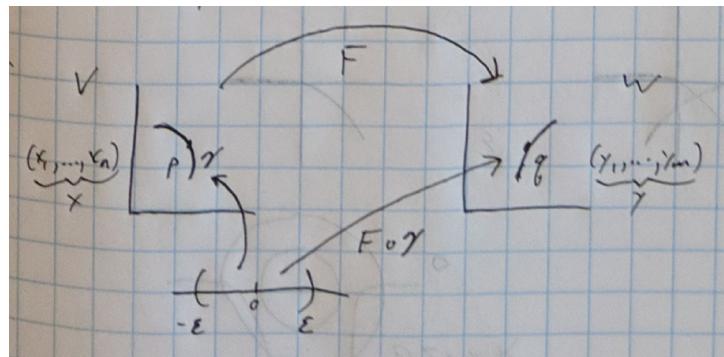
Proposition

Take

$$\begin{array}{ccc} V & \xrightarrow{C^r} & W \\ \mathbb{R}^n & & \mathbb{R}^m \\ p & \mapsto & q \end{array}$$

Then

$$\begin{aligned} V &\xrightarrow{\sim} W \\ DF_p = F_* : T_p V &\rightarrow T_q W \\ [\gamma] &\mapsto [F \circ \gamma] \end{aligned}$$



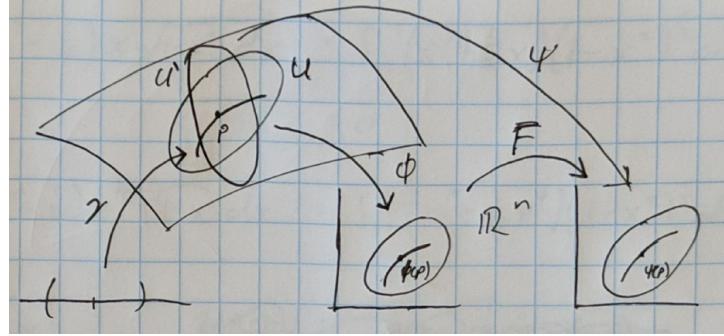
Then F is well-defined and linear.

$F = (F_1, \dots, F_m)$ and $F \circ \gamma(F_1(\gamma_1, \dots, \gamma_n), \dots, F_m(\gamma_1, \dots, \gamma_n))$.

We have that $[\gamma] = \gamma'(0)$ and $[F \circ \gamma] = \frac{d}{dt} F \cdot \gamma(t)|_{t=0}$. By chain rule,

$$\frac{d}{dt}(F \circ \gamma)|_{t=0} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{pmatrix} \begin{pmatrix} \gamma'_1(0) \\ \vdots \\ \gamma'_n(0) \end{pmatrix}$$

Tangent Space



$\gamma_1 \sim \gamma_2 \iff \phi \circ \gamma_1 \sim \phi \circ \gamma_2$ and $\gamma_1 \sim \gamma_2 \iff \psi \circ \gamma_1 \sim \psi \circ \gamma_2$, so

$$(\psi \circ \phi^{-1})(\phi \circ \gamma_1) \sim (\psi \circ \phi^{-1})(\phi \circ \gamma_2)$$

Now, take $\{[\gamma]\} = T_p M$. Claim: this is a vector space.

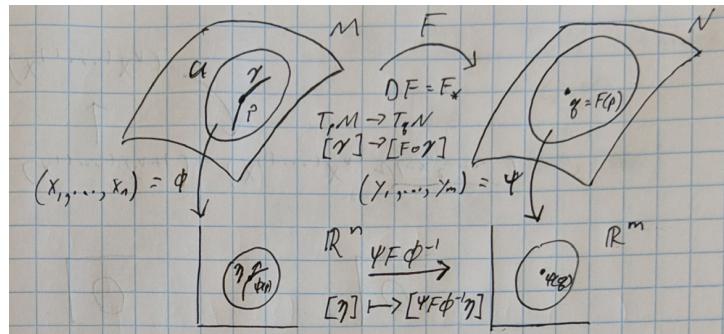
$$\begin{array}{ccc} T_p M & \xrightarrow{D\phi} & T_{\phi(p)} \mathbb{R}^n \\ & \searrow D\psi & \downarrow D(\psi \circ \phi^{-1}) \\ & & T_{\psi(p)} \mathbb{R}^n \end{array}$$

$$[\gamma_1] + [\gamma_2] = [\phi^{-1}(\phi \circ \gamma_1 + \phi \circ \gamma_2)].$$

October 15, 2024

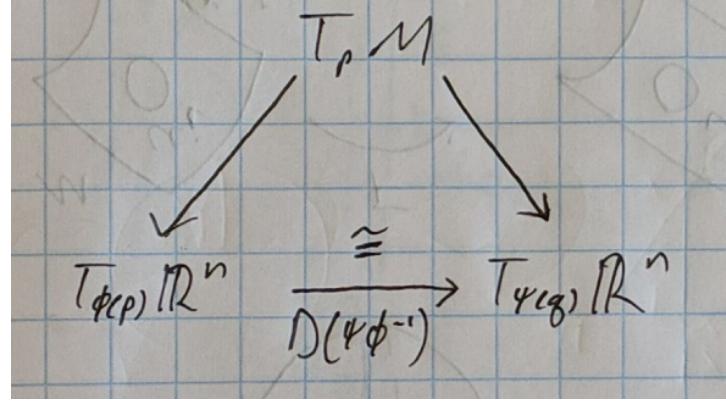
Recall: Tangent Space by Equivalence Classes

$\gamma : (-\varepsilon, \varepsilon) \rightarrow M, \gamma(0) = p, \gamma_1 \sim \gamma_2 \iff \phi \circ \gamma'_1 \sim \phi \circ \gamma'_2 \iff (\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0).$
 $T_p M = \{[\gamma]\} \xrightarrow{D\phi=\phi_*} T_{\phi(p)} \mathbb{R}^n = \mathbb{R}^n.$



Then for (F_1, \dots, F_m) , we have $D(\psi F \phi^{-1}) : \underbrace{T_{\phi(p)} \mathbb{R}^n}_{\mathbb{R}^n} \rightarrow \underbrace{T_{\psi(q)} \mathbb{R}^m}_{\mathbb{R}^m}$ where

$$D(\psi F \phi^{-1}) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$



Chain Rule

We have that $D(FG) = DF \circ DG$ since $(FG) \circ \gamma = F(G \circ \gamma)$.

Example

Take

$$f: \underbrace{M}_{(x_1, \dots, x_n)} \rightarrow \mathbb{R}$$

and $Df = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$.

Definition: Directional Derivatives

Take $v = [\gamma] \in T_p M$ and $f \in C^r(M)$.

The directional derivative is given by

$$L_v f = \frac{d}{dt} f(\gamma(t))|_{t=0} = \frac{d}{dt} \underbrace{(f \circ \phi^{-1}) \phi \circ \gamma}_{g \circ \eta}(t)|_{t=0}$$

Then

$$\frac{d}{dt} g(\eta(t))|_{t=0} = \sum \frac{\partial g}{\partial x_i}(\phi) \eta'_i(0)$$

which is determined by $(\eta'_1(0), \dots, \eta'_n(0)) = \eta'(0)$.

Properties

$$L_v : C^\infty(M) \rightarrow \mathbb{R}$$

1. Linear over \mathbb{R}
2. $L_v(fg) = (L_v f)g(\phi) + f(\phi)(L_v g)$ (product rule)
3. Linear in v .

Derivations

The collection $\mathcal{D}_p = \{C^r(M) \rightarrow \mathbb{R} : (1) \text{ and } (2) \text{ hold}\}$ is called the derivations at p .

Algebraic Aside

$\delta_p : A \rightarrow \mathbb{R}$ given by $f \mapsto f(p)$ yields

$$D(fg) = Df\delta_p(g) + \delta_p(f)Dg$$

Theorem

$T_p M \rightarrow \mathcal{D}_p$ given by $v = [\gamma] \mapsto L_v$ is a linear isomorphism.

Recall: Germ

Take $(f, U) \ni p$ and $(g, V) \ni p$. Then $(f, U) \sim (g, V)$ if and only iff $f \equiv g$ on $W \subset U \cap V$. The equivalence classes of this relation are germs.

Hadamard's Lemma

Take $f \in C^r(\mathbb{R}^n, 0)$ on \mathbb{R}^n with $f(0) = 0$.

There exists $g_1, \dots, g_n \in C^{r-1}(\mathbb{R}^n, 0)$ such that

$$f(x) = \sum x_i g_i(x)$$

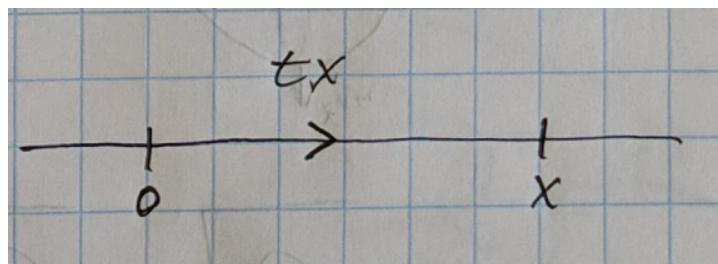
and $g_i(0) = \frac{\partial f}{\partial x_i}(0)$.

Example

For $n = 1$ and $f : \mathbb{R} \xrightarrow{C^r} \mathbb{R}$ with $f(0) = 0$. Then we have $f(x) = x \overbrace{g(x)}^{c^{r-1}}$ given by

$$g(x) = \begin{cases} \frac{f(x)}{x} & x \neq 0 \\ f'(0) & x = 0 \end{cases}.$$

Proof of Example



Take

$$\begin{aligned} \int_0^1 \underbrace{\frac{d}{dt} f(tx) dt}_{f'(tx) \cdot x} &= x \underbrace{\int_0^1 f'(tx) dt}_{g(x)} \\ &= f(1 \cdot x) - f(0 \cdot x) \end{aligned}$$

Proof of Lemma

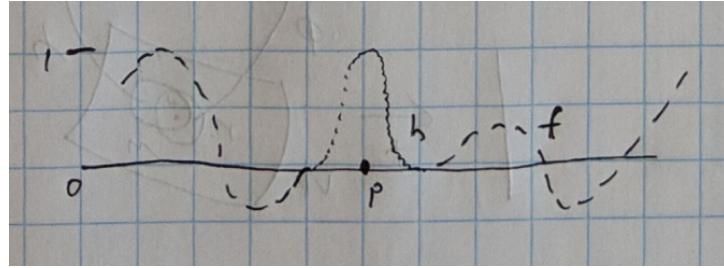
$$\int_0^1 \underbrace{\frac{d}{dt} f(tx) dt}_{\sum \frac{\partial f}{\partial x_i}(tx) \cdot x_i} = \sum x_i \underbrace{\int_0^1 \frac{\partial f}{\partial x_i}(tx) dt}_{g_i}$$

Lemma

For $D \in \mathcal{D}_p$, Df depends only on the germ of f .

Proof

Need to show that if $f \equiv 0$ near p , $Df = 0$.



Where $\text{supp } h \subset \{x : f(x) = 0\}$, $h(p) = 1$ and, consequently, $f \circ h = 0$.

So $D(0 = f \circ h)$, $D0 = Df \cdot h(p) + f(p)Dh$ and $0 = Df$.

Lemma

$D(k) = 0$ for k constant.

Proof

$1 \cdot 1 = 1 \implies D(1) = D(1 \cdot 1) = D(1) \cdot 1 + 1 \cdot D(1) = 2 \cdot D(1)$, so $D(1) = 0$.

Arbitrary constants follow from the fact that D is linear.

Proof: One-to-One

Let $L_v f = L_w f$ for all $f = x_i$, then

$$\sum v_i \frac{\partial f}{\partial x_i} = \sum w_i \frac{\partial f}{\partial x_i}$$

for each f and $v_i = w_i$.

Lemma

$$\text{Der}(C^r(m)@p) = \text{Der}(C^r(M, p))$$

Proof

Take $M = \mathbb{R}^n$ and $p = 0$.

Given D , we need $v = \sum v_i \frac{\partial}{\partial x_i}$ with

$$L_v f = \sum v_i \frac{\partial f}{\partial x_i} = Df$$

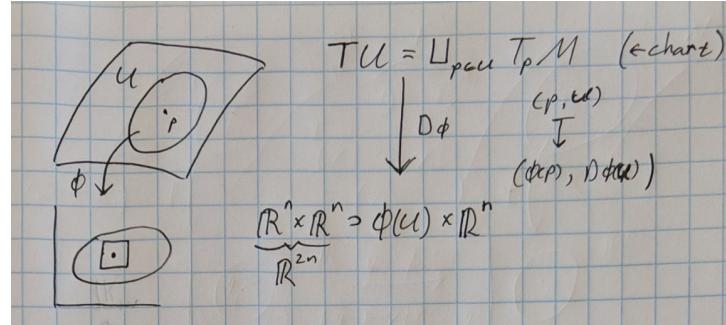
for every f .

By Hadamard's lemma, write $f = \sum x_i g_i(x)$. Then

$$\begin{aligned} Df &= \sum (Dx_i)g_i(0) + \overbrace{x_i(0)}^{=0} Dg_i \\ &= \sum \underbrace{(Dx_i)}_{v_i} \frac{\partial f}{\partial x_i}(0) \end{aligned}$$

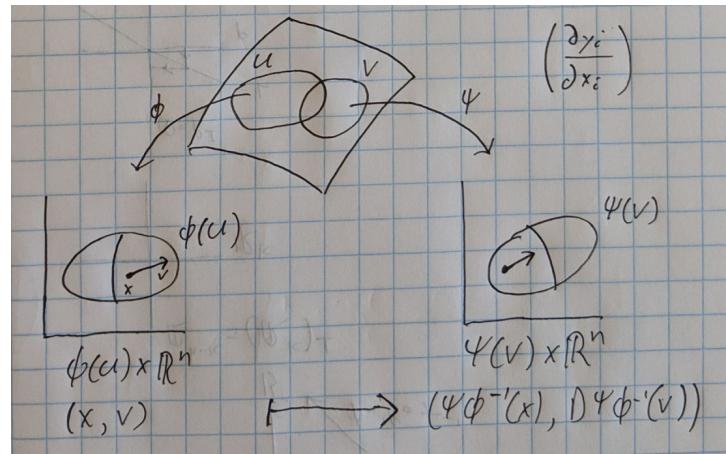
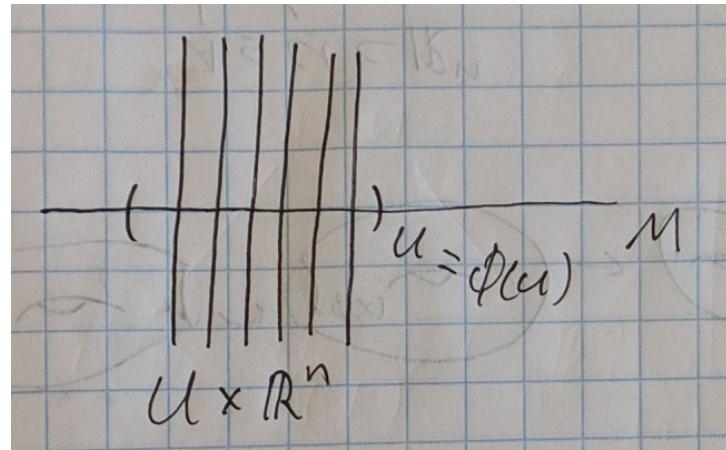
That is, since $v_i = L_v x_i$, we have $v_i = Dx_i$.

Definition: Tangent Bundle



For every $p \in M$, we have $T_p M$.

The tangent bundle $TM = \coprod_{p \in M} T_p M$.



When M is C^r , TM is C^{r-1} .