

# Analysis II

January 9, 2024

(Real) Analysis

- Calculus
  - Differential
  - Integral (Riemann)
- Functions and Maps
  - Measure Theory
  - (Lebesgue) Integration
- Topology
  - Completeness (as a metric space)
  - Compactness (Bolzano-Weierstrass theorem [real]) (Arzela-Ascoli)
  - Paracompactness / Metrizable / Baire Category Theorem
  - Algebraic / Combinatoric (continuous maps or functions)

Definition: Cardinality

For sets  $A, B$ ,  $\text{Card}(A) = \text{Card}(B)$  if there exists a one-to-one correspondence  $q : A \leftrightarrow B$ .

Counting, labelling, indexing, etc.

$\text{Card}(A) \leq \text{Card}(B)$  if  $A \subset B$  or there exists a one-to-one mapping  $A \rightarrow B$ .

Definition: Countable

If  $A \hookrightarrow \mathbb{N}$ , then  $A$  is countable.

Theorem

The countable union of countable sets is countable.

Proof

Let  $A_i = \{a_j\}_{j=1}^{\infty}$ ,  $i = 1, 2, \dots$

$$\begin{array}{cccc} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ \vdots & & & \\ a_{k1} & a_{k2} & a_{k3} & \cdots \end{array}$$

Index by diagonalization.

## Theorem

The cartesian product of countable sets is countable.

## Proof

$$X \times Y = \{(x_i, y_j) \mid x_i \in X, y_j \in Y\}$$

$$\begin{array}{cccc}
(x_1, y_1) & (x_1, y_2) & (x_1, y_3) & \cdots \\
(x_2, y_1) & (x_2, y_2) & (x_2, y_3) & \cdots \\
\vdots & & & \\
(x_k, y_1) & (x_k, y_2) & (x_k, y_3) & \cdots
\end{array}$$

## Theorem

$\text{Card}(2^X) > \text{Card}(X)$ , where  $2^X = \{A \subset X\}$  is the power set of  $X$ .

## Proof

For all  $x \in X$ ,  $\{x\} \subset 2^X$ , so  $\text{Card}(X) \leq \text{Card}(2^X)$ .

Assume, for sake of contradiction, that  $\text{Card}(X) = \text{Card}(2^X)$ .

Then, by definition, there exists a one-to-one correspondence  $\phi : X \leftrightarrow 2^X$ .

Set  $A = \{x \in X \mid x \notin \phi(x)\}$ , and let  $a = \phi^{-1}(A)$  (i.e.  $A = \phi(a)$ ).

If  $a \in A$ , then  $a \notin A \subset \phi(a)$ ; but if  $a \notin A$ , then  $a \in A$ , a contradiction.

## Theorem

$$\text{Card}(\mathbb{R}) = \text{Card}(2^{\mathbb{N}}).$$

## Topology of the Real Line

Completeness (as a metric space)

$$d(a, b) = |a - b|, \quad \forall a, b \in \mathbb{R}.$$

1.  $x_i \rightarrow x$  if  $\forall \varepsilon > 0, \exists n \in \mathbb{N}$  such that  $|x_i - x| < \varepsilon, \forall i \geq n$ .
2.  $\{x_i\}$  is Cauchy if  $\forall \varepsilon > 0, \exists n \in \mathbb{N}$  such that  $|x_i - x_j| < \varepsilon, \forall i, j \geq n$ .

## Definition: Open Interval

$(a, b)$  is an open set on the real line.

There exist interior points for any subset  $A$  of real numbers.

$\forall x \in A$ ,  $x$  is interior if  $\exists (a, b)$  such that (1)  $x \in (a, b)$  and (2)  $(a, b) \subset A$ .

- Theorem

The union of open sets is open.

The intersection of finitely many open sets is open.

$\emptyset$  and  $\mathbb{R}$  are open.

Definition: Limit Point

A limit point  $x \in \mathbb{R}$  of a subset  $A$  is a limit point in  $A$  if for every open neighborhood  $U$  of  $x$ ,  $(U \setminus \{x\}) \cap A \neq \emptyset$ .

Definition: Closed

$A$  is closed if  $A$  contains all of its limit points.

- Theorem

$A$  is closed if and only if  $A^c = \mathbb{R} \setminus A$  is open.

- Proof

$A$  closed  $\implies A^c$  open.

Otherwise,  $\exists x \in A^c$  such that for every neighborhood  $U$  of  $x$ ,  $(U \setminus \{x\}) \cap A \neq \emptyset$  which would make it a limit point of  $A$  not in  $A$ . By assumption,  $A$  contains all its limit points so this is a contradiction.

$A^c$  open  $\implies A$  closed.

For any  $x$  a limit point of  $A$ , assume otherwise that  $x \in A^c$ .

Then there exists some neighborhood  $U$  of  $x$  such that  $U \subset A^c$  (since  $A^c$  is open).

It follows that  $(U \setminus \{x\}) \cap A = \emptyset$  and  $x$  is not a limit point of  $A$ , which is a contradiction.

Definition: Sequential Compactness

$A$  is compact if  $\forall \{x_i\}$ ,  $x_i \in A$  there exists a convergent subsequence  $\{x_{i_k}\}$  and  $x_{i_k} \rightarrow x \in A$ .

- Theorem: Bolzano-Weierstrass

For  $A \subseteq \mathbb{R}$ ,  $A$  is compact if and only if  $A$  is closed and bounded.

- Proof

$A$  compact  $\implies A$  closed and bounded.

Assume that  $A$  is not bounded from above.

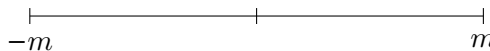
Then there exists a sequence  $\{x_i\}$ ,  $x_i \in A$  where  $x_{i+1} > x_i + 1$  and  $\{x_i\}$  has no convergent subsequences.

Then compactness implies closedness.

$A$  closed and bounded  $\implies A$  (sequentially) compact.

Let any  $\{x_i\}$ ,  $x_i \in A$ .

Claim:  $\forall \{x_i\}$  of reals, if there exists  $m \in \mathbb{R}$  such that  $|x_i| \leq m$ ,  $\forall m$  then there is some convergent subsequence.



Divide and conquer: dividing the interval in half necessitates that at least one half contains infinitely many points. Repeat indefinitely.

- Theorem: Heine-Borel

$A \subseteq \mathbb{R}$  is (sequentially) compact if and only if any open cover has a finite subcover.

- Proof

Heine-Borel Property  $\implies$  closed and bounded.

Assume that  $A$  is unbounded,  $U_n = (-n, n)$  and  $\{U_n\}_{n=1}^{\infty}$  an open cover for  $A \subseteq \mathbb{R}$  has no finite subcover.

Assume  $A$  is not closed, then  $x \in \dot{A}$  (where  $\dot{A}$  is the limit set of  $A$ ) and  $x \notin A$ ,  $U_n \left\{ \left( -\infty, x - \frac{1}{n} \right) \cup \left( x + \frac{1}{n}, +\infty \right) \right\}$ .

Then  $\{U_n\}$  covers  $\mathbb{R} \setminus \{x\} \supset A$  has no finite subcover of  $A$ .

$A$  is bounded and closed  $\implies A$  is Heine-Borel

Divide and conquer: using open sets with respect to open covers.

Definition: Cantor Set

$C = \{x \in [0, 1] \mid \text{the ternary expansion of } x \text{ has only the digits } \{0, 2\}\}.$

Equivalently, let  $C_0 = [0, 1]$ ,  $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$ ,  $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$ .

Then  $C_n = \bigcup_{k=1}^{2^n} C_n^k$  and  $C = \bigcap_{n=1}^{\infty} C_n$ .

$|C_n| = 2^n \left(\frac{1}{3}\right)^n \rightarrow 0.$

Definition: Perfectly Symmetric Sets

Let  $\{\xi_n\}$  where  $\xi_n \in \left(0, \frac{1}{2}\right).$

$E_0 = [0, 1]$ ,  $E_1 = [0, \xi_1] \cup [1 - \xi_1, 1]$ ,  $E_2 = [0, \xi_1 \xi_2] \cup [\xi_1 - \xi_1 \xi_2, \xi_1] \cup [1 - \xi_1, 1 - \xi_1 + \xi_1 \xi_2] \cup [1 - \xi_1 \xi_2, 1]$ .

Then the cantor set is given by  $\xi_n = \frac{1}{3}$ .

$E_n = \bigcup_{k=1}^{2^n} E_n^k$ ,  $|E_n^k| = \xi_1 \xi_2 \cdots \xi_n$ , and  $|E_n| = \sum |E_n^k| = 2^n \xi_1 \xi_2 \cdots \xi_n$ .

Therefore,  $E = \bigcap_{n=1}^{\infty} E_n$  and we define  $|E| = \lim_{n \rightarrow \infty} |E_n| = \lim_{n \rightarrow \infty} (2^n \xi_1 \xi_2 \cdots \xi_n) = \lambda$  where  $\lambda \in [0, 1)$ .

Let

$$2\xi_n = \frac{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n-1}\right)^{n-1}}{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n}\right)^n} < 1$$

, then

$$2^n \xi_1 \cdots \xi_n = \frac{1}{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n}\right)^n} \rightarrow \lambda.$$

Proof

$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^{n/x} = e^x$ , then  $\lim_{y \rightarrow 0} (1 + y)^{1/y} = e$ ,  $\log(1 + y)^{1/y} = \frac{\log(1+y)}{y} \xrightarrow{y \rightarrow 0} 1.$

Observe that

$$\left(\frac{\log(1 + y)}{y}\right)' = \frac{\frac{y}{1+y} - \log(1 + y)}{y^2} = \left(1 + \frac{1}{1 + y} - \log(1 + y)\right)' = \frac{1}{(1 + y)^2} - \frac{1}{1 + y} = -\frac{y}{(1 + y)^2} < 0$$

Theorem

Cantor sets and perfect symmetric sets are closed, perfect, uncountable, and nowhere dense.

January 11, 2024

Last Week

Cardinality.

Topology of the reals.

- Cantor (perfect symmetric sets)

$$\begin{aligned}
C_0 &= [0, 1] \\
C_1 &= [0, 1/3] \cup [2/3, 1] \\
C_2 &= [0, 1/9] \cup [2/9, 3/9] \cup [6/9, 7/9] \cup [8/9, 1] \\
C_n &= \bigcup_{k=1}^{2^n} C_n^k \\
|C_n^k| &= \left(\frac{1}{3}\right)^n \\
C &= \bigcap_{n=1}^{\infty} C_n \\
|C_n| &= 2^n \frac{1}{3^n} = \left(\frac{2}{3}\right)^n \implies |C| = \lim_{n \rightarrow \infty} |C_n| = 0 \\
&\text{Closed, no interior points and uncountable.}
\end{aligned}$$

- Perfect Symmetric Sets

$$\begin{aligned}
\{\xi_k\} &\in \left(0, \frac{1}{2}\right) \\
E_0 &= [0, 1] \\
E_1 &= [0, \xi_1] \cup [1 - \xi_1, 1] \\
E_2 &= [0, \xi_1 \xi_2] \cup [\xi_1 - \xi_1 \xi_2, \xi_1] \cup [1 - \xi_1, 1 - \xi_1 + \xi_1 \xi_2] \cup [1 - \xi_1 \xi_2, 1] \\
E_n &= \bigcup_{k=1}^{2^n} E_n^k \\
|E_n^k| &= \xi_1 \xi_2 \cdots \xi_n \\
|E_n| &= 2^n \xi_1 \xi_2 \cdots \xi_n \\
&= \left(1 + \frac{\log\left(\frac{1}{n}\right)}{n-1}\right)^{n-1} \\
2\xi_n &= \frac{1}{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n}\right)^n} < 1 \\
|E_n| &= \frac{1}{\left(1 + \frac{\log\left(\frac{1}{n}\right)}{n}\right)^n} \\
|E| &= \lim_{n \rightarrow \infty} |E_n| = \frac{1}{e^{\log\left(\frac{1}{n}\right)}} = \lambda, \quad \lambda \in (0, 1)
\end{aligned}$$

Volterra's Function

$$\phi(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

IMAGE HERE - graph of phi(x)

$$\phi'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$f(x) = \begin{cases} 0 & x \in E \\ \phi(x-a) & x \in (a, a+y) \\ -\phi(b-x) & x \in (b-y, b) \\ \phi(y) & x \in (a+y, b-y) \end{cases}, \quad (a, b) \in E^c$$

IMAGE HERE - f interval (a,b)

Propositions

1.  $f'(x) = 0$  for  $x \in E$ .

2.  $f'(x)$  discontinuous on  $E$ .
3.  $f'$  exists on  $[0, 1]$  and is bounded.

Since  $|E| > 0$ ,  $f'(x)$  is not Riemann integrable and, therefore, the fundamental theorem of calculus does not apply.

## Lebesgue Outer Measure

$$|(a, b)| = b - a.$$

Let  $A \subseteq \mathbb{R}$ , then  $m^*(A) = \inf \left\{ \sum_{n=1}^{\infty} |I_n| \mid A \subseteq \bigcup_{n=1}^{\infty} I_n \right\}$

Question:  $m^*(A \cup B) \stackrel{?}{=} m^*(A) + m^*(B)$  for  $A \cap B \neq \emptyset$ ?

## Properties

1.  $A \subseteq B \implies m^*(A) \leq m^*(B)$ .
2.  $m^*(\emptyset) = 0$ .
3. If  $I$  is an interval, then  $m^*(I) = |I|$ .
4. If  $\{A_i\}$  is countable,  $m^*\left(\bigcup A_i\right) \leq \sum m^*(A_i)$ .

### • Proof of 4

$\forall A_i, \exists \{I_n\}$  open intervals such that  $\sum_n |I_n| < m^*(A_i) + \frac{\varepsilon}{2^i}$ .

Then  $\bigcup_i \bigcup_n I_n^i \supset \bigcup_i A_i$ , and  $\sum_{n,i} |I_n^i| = \sum_i \left( \sum_n |I_n^i| \right) \leq \sum_i \left( m^*(A_i) + \frac{\varepsilon}{2^i} \right)$ .

– Corollary

If  $A$  is countable, then  $m^*(A) = 0$ .

Thus, by contraposition, every interval is uncountable.

## Proposition

For  $A \subseteq \mathbb{R}$ ,  $\forall \varepsilon > 0$ ,  $\exists U$  open such that  $A \subseteq U$  and  $m^*(U) \leq m^*(A) + \varepsilon$ .

## Corollary

There exists  $G$  in the intersection of countable open sets such that  $m^*(G) = m^*(A)$  and  $G \supseteq A$ .

## Caratheodory Criteria

If  $\forall E, m^*(E \cap A) + m^*(E \cap A^c) = m^*(E)$ , then  $A$  is Lebesgue measurable.

- Remark:  $m^*(E \cap A) + m^*(E \cap A^c) \leq m^*(E) \leq +\infty$

## Propositions

1. If  $A$  is measurable, then  $A^c$  is measurable.

2.  $m^*(A) = 0$ , then  $A$  is measurable.
3. If  $A, B$  are measurable, then  $A \cup B, A \cap B, A \setminus B$  are measurable.
4. If  $\{A_i\}_{i=1}^k$  are disjoint and measurable, then  $m^*\left(\bigcup_{i=1}^k A_i\right) = \sum_{i=1}^k m^*(A_i)$ .

• Proof of 3

$$\begin{aligned}
m^*(E \cap (A \cup B)) + m^*(E \cap (A \cup B)^c) &= m^*((E \cap A) \cup (E \cap B)) + m^*(E \cap A^c \cap B^c) \\
&= m^*(E \cap A) + m^*((E \cap A^c) \cap B) + m^*((E \cap A^c) \cap B^c) \\
&\leq m^*(E)
\end{aligned}$$

Since  $(A \cap B)^C = A^c \cup B^c$ , this holds from before; similarly,  $A \setminus B = A \cap B^c = A^c \cup B$ .  
If  $A, B$  disjoint, then

$$\begin{aligned}
m^*(A \cup B) &= m^*(E \cap A) + m^*(E \cap A^c) \\
&= m^*(A) + m^*(B)
\end{aligned}$$

Theorem

If  $\{A_i\}$  is a countable collection of disjoint and measurable sets, then

1.  $\bigcup_i A_i$  is measurable.
2.  $m^*\left(\bigcup_i A_i\right) = \sum_i m^*(A_i)$ .

Proof of 1

Want to show:

$$m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) + m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right) \leq m^*(E)$$

By assumption, since the measure of  $E$  is finite,  $m^*(E \cap \bigcup_{i=1}^{\infty} A_i) < +\infty$ .

Claim:  $\forall \varepsilon > 0, \exists k$  such that

Therefore  $m^*\left(E \cap \bigcup_{i=1}^k A_i\right) \geq m^*(E \cap \bigcup_{i=1}^{\infty} A_i) - \varepsilon$ .

$$m^*(E) \leq m^*\left(E \cap \bigcup_{i=1}^k A_i\right) + \varepsilon + m^*\left(E \cap \left(\bigcup_{i=1}^k A_i\right)^c\right) \leq m^*(E) + \varepsilon$$

Proof of 2

We have shown  $m^*\left(\bigcup_i A_i\right) \leq \sum_{i=1}^{\infty} m^*(A_i)$ .

Assume  $m^*\left(\bigcup_i A_i\right) < +\infty$ , then

$$\sum_{i=1}^k m^*(A_i) = m^*\left(\bigcup_{i=1}^k A_i\right) \leq m^*\left(\bigcup_i A_i\right) \implies \sum_{i=1}^{\infty} m^*(A_i) \leq m^*\left(\bigcup_i A_i\right)$$

January 16, 2024

Office Hours Tuesday / Thursday 10 AM - 11:30 AM

A note on notation: Latin characters are to be understood as countable indices; greek as possible uncountable.

## Lebesgue Outer Measure

$A \subset \mathbb{R}$

$$m^*(A) = \inf \left\{ \sum_{i=1}^{\infty} |I_i| \mid \bigcup_{i=1}^{\infty} I_i \supset A, I_i \text{ open intervals} \right\}$$

### Properties

1.  $A \subset B \implies m^*(A) \leq m^*(B)$ .
2.  $m^*(\emptyset) = 0$ .
3.  $m^*(I) = |I|$  for  $I$  an interval.
4. Countable Subadditivity:  $\{A_i\}_{i=1}^{\infty} \implies m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} m^*(A_i)$ .
5.  $\forall A \subset \mathbb{R}, \forall \varepsilon > 0, \exists$  open neighborhood  $U \supseteq A$  such that  $m^*(U) \leq m^*(A) + \varepsilon$ .
6.  $\exists G \in \bigcap_{n=1}^{\infty} U_n, U_n \text{ open}, U_n \supseteq A \implies G \supseteq A$ , such that  $m^*(G) = m^*(A)$ .

### Measurable (Caratheodory Criterion)

$\forall A \subseteq \mathbb{R}$  is Lebesgue measurable if

$$m^*(A) = m^*(E \cap A) + m^*(E \cap A^c)$$

Essentially,  $m^*(E \cap A) + m^*(E \cap A^c) \leq m^*(E) \leq +\infty$ .

#### • Propositions

1.  $A$  measurable  $\implies A^c$  measurable.
2.  $m^*(A) = 0 \implies A$  measurable.
3.  $\{A_i\}_{i=1}^{\infty}$  countable with  $A_i$  measurable, then
  - (a)  $\bigcap_{i=1}^{\infty} A_i$  are measurable.
  - (b) Moreover,  $A_i \cap A_j = \emptyset \implies m^*\left(\bigcup_{i=1}^{\infty} (A_i)\right) = \sum_{i=1}^{\infty} m^*(A_i)$ .
  - (c)  $A, B$  measurable  $\implies A \cup B, A \cap B, A \setminus B$  measurable.
  - (d)  $A \cap B = \emptyset \implies m^*(A \cup B) = m^*(A) + m^*(B)$ .
  - (e)  $\{A_i\}_{i=1}^{\infty}$  with  $A_i$  measurable, then  $\bigcup_{i=1}^{\infty} A_i$  is measurable and  $A_i \cap A_j = \emptyset \implies m^*\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m^*(A_i)$ .
- Proof of e  $\forall E \subset \mathbb{R}, m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) + m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right)$ .



Claim:  $m^*(E \cap (\bigcup_{i=1}^{\infty} A_i)) = \sum_{i=1}^{\infty} m^*(E \cap A_i)$  for  $A_i \cap A_j = \emptyset$ .  
Then,  $\forall \varepsilon > 0, \exists n \in \mathbb{N}$ ,

$$\begin{aligned} m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) &= \sum_{i=1}^{\infty} m^*(E \cap A_i) \leq \sum_{i=1}^n m^*(E \cap A_i) + \varepsilon \\ \implies m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) + m^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)^c\right) &\leq m^*\left(E \cap \left(\bigcup_{i=1}^n A_i\right)\right) + m^*\left(E \cap \left(\bigcup_{i=1}^n A_i\right)^c\right) + \varepsilon \leq m^*(E) + \varepsilon \\ &\implies \bigcup_{i=1}^{\infty} A_i \text{ measurable} \end{aligned}$$

Proof of Claim:

Step 1:  $A, B$  measurable and  $A \cap B = \emptyset$ . Since  $A$  is measurable,

$$\begin{aligned} m^*(E \cap (A \cup B)) &= m^*((E \cap (A \cup B)) \cap A) + m^*((E \cap (A \cup B)) \cap A^c) \\ &= m^*(E \cap A) + m^*(E \cap A^c) \end{aligned}$$

For  $\{A_i\}_{i=1}^{\infty}, \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} A'_i$  with  $A_1 = A'_1$  and  $A'_i = A_i \setminus \bigcup_{k=1}^{i-1} A_k, \forall i \geq 2$ .  
Therefore  $A'_i \cap A'_j = \emptyset$  and  $A'_i$  is measurable.

$$\begin{aligned} m^*\left(\bigcup_{i=1}^n A_i\right) &\leq m^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} m^*(A_i) \\ m^*\left(\bigcup_{i=1}^n A_i\right) &= \sum_{i=1}^n m^*(A_i) \leq m^*\left(\bigcup_{k=1}^{\infty} A_k\right) < +\infty \implies \sum_{i=1}^{\infty} m^*(A_i) \leq m^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{i=1}^{\infty} m^*(A_i) \end{aligned}$$

## Sigma Algebra and Borel Sets

Definition: Sigma Algebra

Let  $S \subset 2^X$  for some set  $X$ . Then  $S$  is said to be a  $\sigma$ -algebra if

1.  $\emptyset \in S$ .
2.  $A^c \in S$  if  $A \in S$ .
3.  $\bigcup_{i=1}^{\infty} A_i \in S$  if  $A_i \in S$ .

- Equivalently,  $\bigcap_{i=1}^{\infty} A_i \in S$  if  $A_i \in S$ .

Theorem:

The collection  $\mathcal{L}$  of all Lebesgue measurable sets is a  $\sigma$ -algebra.

Definition: Borel Set

Let  $B$  be the  $\sigma$ -algebra generated by open sets of reals (i.e. the smallest  $\sigma$ -algebra containing all open sets of reals).  
Then  $b \in B$  is called a Borel set.

Remark

$B$  is generated by  $\{(a, +\infty) \mid a \in \mathbb{R}\}$ .

1.  $(a, +\infty)^c = (-\infty, a]$ .
2.  $\bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, +\infty\right) = [a, +\infty)$ .
3.  $[a, +\infty)^c = (-\infty, a)$ .
4.  $(-\infty, b) \cap (a, +\infty) = (a, b)$ .
5.  $(-\infty, b] \cap [a, +\infty) = [a, b]$ .

Theorem:

Any Borel set is Lebesgue measurable.

Proof

It suffices to demonstrate that  $(a, +\infty)$  is measurable  $\forall a \in \mathbb{R}$ .

$\forall E \subset \mathbb{R}$ , we want to show that  $m^*(E \cap (a, +\infty)) + m^*(-\infty, a] \leq m^*(E)$ .

Then,  $\forall \varepsilon > 0$ ,  $\exists \mathcal{C} = \{I_i\}$  with  $I_i$  open intervals such that  $\sum_{I_i \in \mathcal{C}} |I_i| \leq m^*(E) + \varepsilon/2$ . Set

$$\begin{aligned}\mathcal{C}^\ell &= \{I \in \mathcal{C} \mid x < a, \forall x \in I\} \\ \mathcal{C}^r &= \{I \in \mathcal{C} \mid x > a, \forall x \in I\} \\ \mathcal{C}^m &= \{I \in \mathcal{C} \mid a \in I\} = \{I_k\}\end{aligned}$$

Then  $\mathcal{AC} = \mathcal{C}^\ell \cup \mathcal{C}^r \cup \mathcal{C}^m$ .

$\forall I_k \in \mathcal{C}^m = \{I_k\}$ ,  $I_k = (c_k, d_k)$  for some  $c_k, d_k \in \mathbb{R}$ , define

$$\begin{aligned}I_k^\ell &= \left(c_k, a + \frac{\varepsilon}{2^{k+1}}\right) \\ I_k^r &= (a, d_k)\end{aligned}$$

Let  $\mathcal{C}^m = \{I_k^\ell\} \cup \{I_k^r\} = \overline{\mathcal{C}}^{m\ell} \cup \overline{\mathcal{C}}^{mr}$ . Then

$$\begin{aligned}\mathcal{C}^\ell \cup \overline{\mathcal{C}}^{m\ell} &\text{ covers } E \cap (-\infty, k] \\ \mathcal{C}^r \cup \overline{\mathcal{C}}^{mr} &\text{ covers } E \cap (k, +\infty) \\ \mathcal{C}^\ell \cup \mathcal{C}^r \cup \mathcal{C}^m &\text{ covers } E\end{aligned}$$

Observe that

$$|I_k^\ell| + |I_k^r| \leq |I_k| + \frac{\varepsilon}{2^{k+1}}$$

Therefore

$$m^*(E \cap (a, +\infty)) \leq \sum_{I \in \mathcal{C}^R + \bar{\mathcal{C}}^{mr}} |I|$$

$$m^*(E \cap [-\infty, a]) \leq \sum_{I \in \mathcal{C}^\ell + \bar{\mathcal{C}}^{m\ell}} |I|$$

Therefore

$$\begin{aligned}
m^*(E \cap (a, +\infty)) + m^*(E \cap (-\infty, a]) &\leq \sum_{I \in \mathcal{C}^r \cup \bar{\mathcal{C}}^{mr}} |I| + \sum_{I \in \mathcal{C}^\ell \cup \bar{\mathcal{C}}^{m\ell}} |I| \\
&= \sum_{I \in \mathcal{C}^r} |I| + \sum_{I \in \mathcal{C}^\ell} |I| + \sum_k (|I_k^\ell| + |I_k^r|) \\
&\leq \sum_{I \in \mathcal{C}} |I| + \sum_{k=1}^{+\infty} \frac{\varepsilon}{2^{k+1}} \\
&\leq m^*(E) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&\leq m^*(E) + \varepsilon
\end{aligned}$$

Lebesgue Measurable vs Borel

Theorem

The following statements are equivalent

1.  $A$  is measurable.
2.  $\forall \varepsilon > 0, \exists U$  open,  $U \supset A$  such that  $m(U \setminus A) < \varepsilon$ .
3.  $\forall \varepsilon > 0, \exists C$  closed,  $C \subset A$  such that  $m(A \setminus C) < \varepsilon$ .
4.  $\forall A \in \mathbb{R}, \exists \bigcap_{n=1}^{\infty} U_n = F \in B, U_n$  open,  $U_n \supset A$  such that  $F \supset A$  and  $m(F \setminus A) = 0$ .
5.  $\exists \{C_n\}, C_n$  closed and  $C_n \subset A$  such that  $G = \bigcup_{n=1}^{\infty} C_n \subset A$  and  $m(A \setminus G) = 0$ .

Corollary

Every measurable set is the union of a Borel set and a measure zero set.

Proof 1 Implies 2

Step 1: if  $m(A) < \infty$ , then for  $\varepsilon > 0, \exists U$  open and  $U \supset A$ , then

$$m(U) \leq m(A) + \varepsilon \iff m(U \setminus A) = m(U) - m(A) \leq \varepsilon$$

Step 2: let  $A_n = A \cap (-n, n), n \in \mathbb{N}$ .

Then  $m(A_n) \leq 2n < +\infty$ .

For each  $A_n, \exists U_n$  open with  $U_n \supset A_n$  and  $m(U_n \setminus A_n) < \frac{\varepsilon}{2^n}$

Let  $U = \bigcup_{n=1}^{\infty} U_n$  and  $A = \bigcup_{n=1}^{\infty} A_n$ .

Now verify that

$$m(U \setminus A) = m\left(\bigcup_{n=1}^{\infty} U_n \setminus \bigcup_{n=1}^{\infty} A_n\right) = m\left(\bigcup_{n=1}^{\infty} U_n \setminus A_n\right) \leq \sum_{n=1}^{\infty} m(U_n \setminus A_n) \leq \varepsilon$$

Proof 2 Implies 3

Write

$$A \setminus C = A \cap C^c = C^c \cap A = C^c \setminus A^c$$

Apply (2).

Proof 3 Implies 4

$U_n$  comes from 2.

Proof 4 Implies 5

Follows from 4.

Proof 5 Implies 1

$A = G \cup (A \setminus G) \implies A$  is measurable.

Example: Non-measurable Set

Define  $x \sim y$  if  $x - y \in \mathbb{Q}$ ,  $\forall x, y \in \mathbb{R}$ .

Let  $A = \{x \in (0, 1) \mid x \text{ is a representative of each class } \mathbb{R}/\sim\} \subset (0, 1) \subset \mathbb{R}$ .

Claim:  $A$  is not Lebesgue measurable.

Let  $(-1, 1) \supset S = \bigcup_{r \in \mathbb{Q} \cap (0, 1)} (A + r) \supset (0, 1)$ , and observe that  $\mathbb{Q} \cap (0, 1)$  is countable.

So  $(A + r) \cap (A + s) = \emptyset$  for  $s \neq r$ .

Then  $1 < m(S) < 2$ , so  $m(A) = 0$  and  $m(A) > 0$  are both contradictions.

January 18, 2024

Abstract measure theory.

Definition: Topological Space

A set  $X$  equipped with a collection of subsets  $\tau \subset 2^X$  where  $\tau$  is a topology if

1.  $\emptyset, X \in \tau$
2. Union of subsets in  $\tau$  remains in  $\tau$ .
3. Intersection of finitely many subsets in  $\tau$  remains in  $\tau$ .

Any subset of  $\tau$  is called an open set of  $X$ .

## Definition: Measure Space

For a set  $X$  with  $\Lambda \subset 2^X$  a  $\sigma$ -algebra such that

1.  $\emptyset \in \Lambda$
2.  $A^c \in \Lambda$  if  $A \in \Lambda$ .
3.  $\bigcup_{i=1}^{\infty} A_i \in \Lambda$  if  $A_i \in \Lambda$ .
4. Remark: Borel Sigma Algebra

The  $\sigma$ -algebra generated by  $\tau$  for a topological space  $(X, \tau)$ .

The measure space  $(X, \Lambda, \mu)$ ,  $\Lambda \subset 2^X$  a  $\sigma$ -algebra equipped with set function  $\mu : \Lambda \rightarrow [0, +\infty]$  such that

1.  $\mu(\emptyset) = 0$
2.  $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$  for  $A_i \in \Lambda$  and  $A_i \cap A_j = \emptyset$  for all  $i \neq j$  (countable additivity).

Proposition: Monotonicity

$$A, B \in \Lambda, A \subseteq B \implies \mu(A) \leq \mu(B).$$

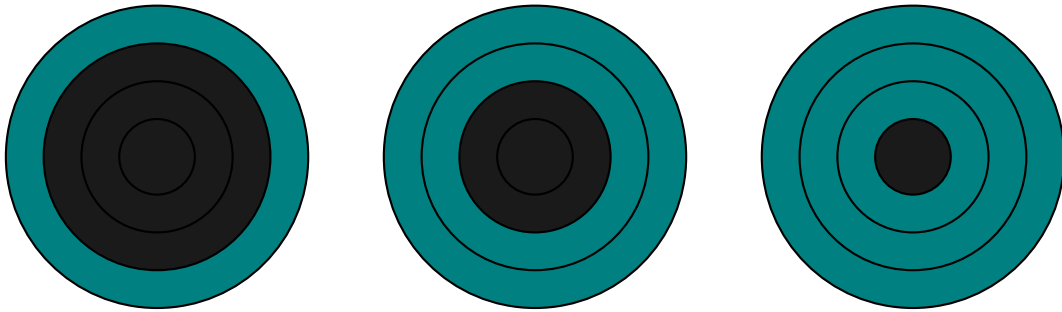
Proposition: Countable Subadditivity

$$\mu\left(\bigcup A_i\right) \leq \sum \mu(A_i) \text{ if } A_i \in \Lambda$$

Proposition: Monotone Convergence

Given  $A_i \in \Lambda$  such that  $A_i \subset A_{i+1}$  where  $A = \bigcup_{i=1}^{\infty} A_i$ , then  $\mu(A_i) \rightarrow \mu(A)$ .

Similarly, if  $A_i \supset A_{i+1}$  such that  $A = \bigcap_{i=1}^{\infty} A_i$ , then  $\mu(A_i) \rightarrow \mu(A)$  if  $\mu(A_k) < +\infty$  for some  $k = 1, 2, 3, \dots$



$$\text{Given } A'_i = \begin{cases} A_1 & i = 1 \\ A_i \setminus \bigcup_{j=1}^{i-1} A_j & i > 1 \end{cases}, \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} A'_i \text{ and}$$

$$\mu(A) \sum_{i=1}^{\infty} A'_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(A'_i)$$

and

$$\sum_{i=1}^n \mu(A'_i) = \mu(A_1) + (\mu(A_2) - \mu(A_1)) + (\mu(A_3) - \mu(A_2)) + \dots + (\mu(A_n) - \mu(A_{n-1})) = \mu(A_n)$$

Similarly,  $A_1 \setminus A = \bigcup_{i=2}^{\infty} (A_i \setminus A_{i-1})$  where  $\mu(A_1) < +\infty$  gives

$$\mu(A_1) - \mu(A) = \mu(A_1) + \sum_{i=2}^{\infty} (\mu(A_i) - \mu(A_{i-1})) = \lim_{n \rightarrow \infty} \mu(A_n)$$

Definition: Complete Measure Space

A measure space  $(X, \Lambda, \mu)$  is complete if  $\forall A \in \Lambda$  with  $\mu(A) = 0$ , then  $\forall B \subset A$  and  $B \in \Lambda$ .

Example

The Lebesgue measure space on the reals  $(\mathbb{R}, \mathcal{L}, m)$  is complete.

Theorem: Completion of a Measure Space

Given a measure space  $(X, \Lambda, \mu)$ , then there exists  $(X, \overline{\Lambda}, \overline{\mu})$  such that

1.  $\Lambda \subset \overline{\Lambda}$ .
2. If  $A \in \Lambda$ , then  $\overline{\mu}(A) = \mu(A)$ .
3.  $(X, \overline{\Lambda}, \overline{\mu})$  is complete.

Proof (Construction)

Let  $\overline{\Lambda} = \{A \cup Z \mid A \in \Lambda, \exists D \in \Lambda, \mu(D) = 0, Z \subset D\}$  and  $\overline{\mu}(A \cup Z) := \mu(A)$ .

Verify:

1.  $\overline{\Lambda}$  is a  $\sigma$ -Algebra.
  - (a) If  $A \cup Z \in \overline{\Lambda}$ , then  $(A \cup Z)^c \in \overline{\Lambda}$ .
  - (b) If  $A_i \cup Z_i \in \overline{\Lambda}$ , then  $\bigcup (A_i \cup Z_i) \in \overline{\Lambda}$ .

2.  $\overline{\mu}$  is a well-defined measure on  $\overline{\Lambda}$ .

3.  $(X, \overline{\Lambda}, \overline{\mu})$  is complete.

• Proof of 1

Given  $A \in \Lambda$  and  $Z \subset D$  where  $\mu(D) = 0$  and  $D \in \Lambda$ , we know  $D^c \subset Z^c$  and  $Z^c = D^c \cup (Z^c \cap D)$ . Therefore

$$(A \cup Z)^c = A^c \cap Z^c = A^c \cap (D^c \cup (Z^c \cap D)) = (A^c \cap D^c) \cup (A^c \cap Z^c \cap D) \in \overline{\Lambda}$$

Since  $A^c \cap D^c \in \Lambda$  and  $A^c \cap Z^c \cap D \in D$

Since  $\bigcup A_i \in \Lambda$  and  $\bigcup Z_i \subset \bigcup D_i$ ,

$$\bigcup_{i=1}^{\infty} (A_i \cup Z_i) = \left( \bigcup_{i=1}^{\infty} A_i \right) \cup \left( \bigcup_{i=1}^{\infty} Z_i \right) \in \overline{\Lambda}$$

- Proof of 2

Given  $A_1 \cup Z_1 = A_2 \cup Z_2$ ,  $A_1 \subset A_2 \cup Z_2 \subset A_2 \cup D_2$  implies  $\mu(A_1) \leq \mu(A_2)$ .

Then,  $\mu(A_2) \leq \mu(A_1) \implies \mu(A_1) = \mu(A_2)$ . So  $\bar{\mu}$  is well defined.

Given  $\{A_i \cup Z_i\}$  with  $(A_i \cup Z_i) \cap (A_j \cup Z_j) = \emptyset$  for all  $i \neq j$ ,

$$\bar{\mu}\left(\bigcup_{i=1}^{\infty} (A_i \cup Z_i)\right) = \bar{\mu}\left(\left(\bigcup_{i=1}^{\infty} A_i\right) \cup \bigcup_{i=1}^{\infty} Z_i\right) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty} \bar{\mu}(A_i \cup Z_i)$$

So  $\bar{\mu}$  is countably additive and therefore a measure.

## Borel Measure and Radon Measure

Given a measure space  $(X, \Lambda, \mu)$  and an underlying topology  $(X, \tau)$ ,

Definition: Borel Measure

$\mu$  is a Borel measure if all borel sets  $\tau \subset \Lambda$ .

Definition: Locally Finite Measure

$\mu$  is locally finite if  $\forall x \in X, \exists U \subset X$  a neighborhood such that  $\mu(U) < +\infty$ .

Definition: Borel Regularity

$\mu$  is Borel regular if  $\forall A \in \Lambda, \exists B$  a Borel set such that  $B \supseteq A$  and  $\mu(B) = \mu(A)$ .

Definition: Radon Measure

$\mu$  is a Radon measure if

1. it is a Borel measure.
2.  $\mu(K) \leq +\infty$  for  $K$  compact.
3.  $\mu(V) = \sup\{\mu(K) \mid K \subset V, K \text{ compact}\}, V$  open.
4.  $\mu(A) = \inf\{\mu(V) \mid A \subset V, V \text{ open}\}, \forall A \in \Lambda$ .

- Example 1

Lebesgue measure.

- Example 2

Point charge:  $\mu(\{x\}) = 1$  and  $\mu(A) = 0$  if  $x \notin A$ .

Theorem:

Let  $(X, \Lambda, \mu)$  be a Borel regular measure space where the underlying topology  $(X, \tau)$  is a metric space. Then

1. For  $A \in \Lambda$  with  $\mu(A) < +\infty, \forall \varepsilon > 0, \exists C \subseteq A$  closed such that  $\mu(A \setminus C) < \varepsilon$ .
2. For  $A \in \Lambda, \exists \{V_i\}$  open sets such that  $A \subset \bigcup_{i=1}^{\infty} V_i$  and  $\mu(V_i) < +\infty$ . Then  $\forall \varepsilon > 0, \exists U$  open with  $A \subset U$  and  $\mu(U \setminus A) < \varepsilon$ .

Proof

Given  $\mu(A) < +\infty$ ,  $\nu(B) = \mu(B \cap A) < +\infty$ ,  $\forall B \in \Lambda$  and  $(X, \Lambda, \nu)$ .

Let  $F = \{B \in \Lambda \mid \forall \varepsilon > 0, \exists C \subset B \text{ closed, with } \nu(B \setminus C) < \varepsilon\}$ .

Note that closed sets are in  $F$ .

Claim 1: the Borel  $\sigma$ -algebra is in  $F$ .

Claim 2: if  $A_i \in F$ ,  $\bigcup A_i, \bigcap A_i \in F$ .

Given claim 2,  $\forall U$  open,  $U^c$  is closed. Then  $U_\varepsilon = \{x \in U \mid \text{dist}(x, U^c) \leq \varepsilon\}$  is closed and, therefore,  $U = \bigcup_{i=1}^{\infty} U_{1/i}$ .

So, given  $A_i \in F$ ,  $\exists C_i \subset A_i$  closed where  $\nu(A_i \setminus C_i) < \varepsilon/2^{i+1}$ . We want to show that  $\nu(\bigcap A_i \setminus \bigcap C_i) < \varepsilon$ .

Then, for  $x \in \bigcap A_i \setminus \bigcap C_i$ ,  $x \in A_i$  for all  $i$  and  $x \notin C_{i_0}$  for some  $i_0$ .

Therefore  $x \in A_{i_0}$ ,  $x \notin C_{i_0}$ , and  $x \in A_{i_0} \setminus C_{i_0}$ . It follows that

$$\begin{aligned} \bigcap_{i=1}^{\infty} A_i \setminus \bigcap_{i=1}^{\infty} C_i &\subset \bigcup_{i=1}^{\infty} (A_i \setminus C_i) \\ \nu\left(\bigcap_{i=1}^{\infty} A_i \setminus \bigcap_{i=1}^{\infty} C_i\right) &\leq \sum_{i=1}^{\infty} \nu(A_i \setminus C_i) < \varepsilon \end{aligned}$$

Therefore

$$\nu\left(\bigcup_{i=1}^{\infty} A_i \setminus \bigcup_{i=1}^n C_i\right) \rightarrow \nu\left(\bigcup_{i=1}^{\infty} A_i \setminus \bigcup_{i=1}^{\infty} C_i\right) \leq \nu\left(\bigcup_{i=1}^{\infty} (A_i \setminus C_i)\right) < \frac{\varepsilon}{2}$$

so  $\exists N \gg 1$  such that  $\nu\left(\bigcup_{i=1}^{\infty} \setminus \bigcup_{i=1}^N C_i\right) < \varepsilon$  with  $\bigcup_{i=1}^N C_i$  closed.

Restatement

For  $A$  Borel,

$$\varepsilon > \nu(A \setminus C) = \mu((A \setminus C) \cap A) = \mu(A \setminus C)$$

January 23, 2024

Review - Abstract Measure

Given  $(X, \Lambda, \mu)$  where  $\Lambda \subseteq 2^X$  is a  $\sigma$ -algebra,  $\mu : \Lambda \rightarrow [0, +\infty]$

1.  $\mu(\emptyset) = 0$ .
2.  $m(\bigcup A_i) = \sum \mu(A_i)$ ,  $A_i \cap A_j = \emptyset$ .

Properties of a Measure

Monotonicity

$$\mu(A) \leq \mu(B), A, B \in \Lambda, A \subseteq B$$

Countable Subadditivity

$$\mu\left(\bigcup A_i\right) \leq \sum \mu(A_i)$$



## Monotone Convergence

$$A_i \subset A_{i+1}, A_i \rightarrow \bigcup A_i \implies \mu(A) = \mu(\bigcup A_i).$$

$$A_i \supset A_{i+1}, A_i \rightarrow \bigcap A_i \implies \mu(A_i) \rightarrow \mu(\bigcap A_i) \text{ if } \mu(A_1) < \infty$$

- Example

$$A_n = (n, +\infty) \text{ gives } \bigcap A_n = \emptyset$$

## Completeness of a Measure

$(X, \Lambda, \mu)$  is complete if  $\forall A \in \Lambda$  with  $\mu(A) = 0$ , then  $\forall B \in \Lambda$  if  $B \subseteq A$ .

Theorem:

Given  $(X, \Lambda, \mu)$ , there exists  $(X, \bar{\Lambda}, \bar{\mu})$  such that  $\Lambda \subset \bar{\Lambda}$  and  $\bar{\mu}(A) = \mu(A)$  if  $A \in \Lambda$ .

$$\bar{\Lambda} = \{A \cup Z \mid A \in \Lambda, Z \subset D, D \in \Lambda \text{ with } \mu(D) = 0\}$$

$$\bar{\mu}(A \cup Z) = \mu(A)$$

$(X, \bar{\Lambda}, \bar{\mu})$  is complete.

## Measure Space with Topology

Given a topological space  $(X, \tau)$ , a measure space  $(X, \Lambda, \mu)$

Definition: Locally Finite

The measure  $\mu$  is locally finite if  $\forall x \in X$ , there exists an open neighborhood  $U$  of  $x$  such that  $U \in \Lambda$  and  $\mu(U) < +\infty$ .

Definition: Borel Measure

$\mu$  is a Borel measure if the Borel  $\sigma$ -algebra generated by  $\tau$ ,  $\mathcal{B}$ , is a subset of  $\Lambda$ .

Definition: Borel Regular

$$\forall A \in \Lambda, \exists B \in \mathcal{B} \text{ such that } B \supset A \text{ and } \mu(B) = \mu(A).$$

Definition: Radon Measure

1. Borel.
2.  $\mu(K) < +\infty$  for  $K$  compact.
3.  $\mu(V) = \sup\{\mu(K) \mid K \text{ compact}, K \subset V\}, \forall V \text{ open}.$
4.  $\mu(A) = \inf\{\mu(V) \mid V \text{ open}, A \subset V\}, \forall A \in \Lambda.$

Theorem:

If  $X$  is a metric space equipped with a Borel regular  $(X, \Lambda, \mu)$ , then

1.  $\forall A \in \Lambda, \mu(A) < +\infty, \forall \varepsilon > 0, \exists C$  closed where  $C \subset A$  and  $\mu(C \setminus A) < \varepsilon$ .
2. If  $\exists \{V_i\}$ ,  $V_i$  open and  $\mu(V_i) < +\infty$ , and  $A \in \Lambda$  with  $A \subset \bigcup V_i$ , then  $\exists U$  open such that  $A \subset U$  and  $\mu(U \setminus A) < \varepsilon$ .

Proof of 1

Define  $\nu(B) = \mu(B \cap A)$  such that  $(X, \Lambda, \nu)$  is a new measure space.

Define  $F = \{B \in \Lambda \mid \forall \varepsilon > 0, \exists C \text{ closed}, C \subset B, \nu(B \setminus C) < \varepsilon\}$ , all closed sets in  $F$ .

Claim 1:  $\bigcap A_i, \bigcup A_i \in F$  if  $A_i \in F$ .

Claim 2:  $U$  is open.

$U = \bigcup U_i, U_i = \{x \in U \mid \text{dist}(x, U^c) \leq \frac{1}{i}\}$ , therefore  $B \subset F$ .

IMAGE HERE - 1

If  $A$  is Borel, then  $\forall \varepsilon > 0, \exists C$  closed with  $C \subset A$  and  $\mu(A \setminus C) < \varepsilon$ .

To finish,  $\forall A \in \Lambda$  by Borel Regularity of  $\mu$ ,  $\exists B \in \mathcal{B}$  such that  $B \supset A$  and  $\mu(B) = \mu(A)$ .

Note also that this requires  $\mu(B \setminus A) = 0$  since  $\mu(A) < +\infty$ .

IMAGE HERE - 2

Then  $B \setminus A \in \Lambda$ ,  $\exists D \in \mathcal{B}$  such that  $DB \setminus A$  and  $\mu(D) = \mu(B \setminus A) = 0$ . Then

$$\begin{aligned} B \cap A^c &= B \setminus A \subset D \\ (B \cap A^c)^c &\supset D^c \\ B \cap (B^c \cup A) &\supset D^c \cap B \\ A &\supset B \setminus D \end{aligned}$$

$$A \setminus (B \setminus D) = A \cap (B \cap D^c)^c = A \cap (B^c \cup D) = \overbrace{(A \cap B^c)}^{\emptyset} \cup A \cap D = A \cap D \subset D$$

Therefore  $B \setminus D \subset A$ , and  $\mu(A \setminus (B \setminus D)) = 0$ .

$B \setminus D \in \mathcal{B}$ ,  $\forall \varepsilon > 0, \exists C$  closed such that  $C \subset B \setminus D \subset A$ ,  $\mu((B \setminus D) \setminus C) < \varepsilon$ .

This implies that  $\mu(A \setminus C) = \mu(A \setminus (B \setminus D)) + \mu((B \setminus D) \setminus C) < \varepsilon$ .

Proof of 2

Consider  $V_i \setminus A$  where  $\mu(V_i \setminus A) \leq \mu(V_i) < +\infty$ .

By (1),  $\exists C_i$  closed with  $C_i \subset V_i \setminus A$  and  $\mu((V_i \setminus A) \setminus C_i) < \varepsilon/2^{i+1}$ . Write

$$(V_i \setminus A) \setminus C_i = (V_i \setminus A) \cap C_i^c = V_i \cap A^c \cap C_i^c = (V_i \cap C_i^c) \cap A^c = (V_i \setminus C_i) \setminus A$$

Note that  $V_i \setminus C_i$  is open, since  $C_i$  is closed.

Define  $U = \bigcup (V_i \setminus C_i) \supset A$ . Then,

$$U \setminus A = \left( \bigcup (V_i \setminus C_i) \right) \setminus A = \bigcup ((V_i \setminus C_i) \setminus A)$$

Therefore  $\mu(U \setminus A) \leq \varepsilon_{\sum_{i=1}^{\infty} 2^{i+1}} = \varepsilon$ .

Remark

$X = \bigcup V_i$ ,  $V_i$  open and  $\mu(V_i) < +\infty$ .

Then  $\forall A \in \Lambda$ ,  $\forall \varepsilon > 0$ ,  $\exists U$  open such that  $U \supset A$  and  $\mu(U \setminus A) < \varepsilon$ .

For  $A^c$ ,  $\exists U \supset A^c$  ( $\implies U^c \subset A$ ),  $\mu(U \setminus A^c) < \varepsilon$ . So

$$U \cap A = U \setminus A^c = A \setminus U^c = A \cap U$$

and  $\mu(A \setminus U^c) < \varepsilon$ ,  $U^c \subset A$  with  $U^c$  closed.

## Corollary

For  $\mathbb{R}^n$ , a measure is Radon if and only if it is locally finite and Borel regular.

- Proof

( $\implies$ )

Let  $B(r, x_0) = \{x \in \mathbb{R}^n \mid |x - x_0| < r\}$  and  $\overline{B(r, x_0)} = \{x \in \mathbb{R}^n \mid |x - x_0| \leq r, \text{ compact}\}$ .

Then  $\mu(B(r, x_0)) \leq \mu(\overline{B(r, x_0)}) < +\infty$ . So  $\mu$  is locally finite.

For  $A \in \Lambda$ , we may assume without loss of generality that  $\mu(A) < +\infty$ .

Then  $\forall i, \exists U_i$  open where  $U_i \supset A$  and  $\mu(A) \leq \mu(U_i) \leq \mu(A) + \frac{1}{i} < +\infty$ .

Set  $G = \bigcap U_i \in \mathcal{B}$ , then  $\mu(G) = \mu(A)$ .

( $\impliedby$ )

1. Borel regular implies Borel.

2. For  $K$  compact,  $\forall x \in K \ni U_x$  open where  $\mu(U_x) < +\infty$ .

$\{U_\lambda\}_{\lambda \in k}$  is an open cover. Therefore there is a finite subcover  $\{U_{\lambda_i}\}_{i=1}^\lambda$  where

$$\mu(K) \leq \mu\left(\bigcup_{i=1}^k U_{\lambda_i}\right) \leq \sum_{i=1}^k \mu(U_{\lambda_i}) < +\infty$$

3.  $\forall V$  open,  $B(i) = B(i, 0)$ ,  $V \cap B(i)$ ,  $\mu(V \cap B(i)) < +\infty$ ,  $\exists C_i$  closed where  $C_i \subset V \cap B(i)$  so  $C_i$  is bounded and therefore compact.

So  $\mu(C_i) \leq \mu((V \cap B(i)) \setminus C_i) < \frac{1}{i}$  and  $\mu(V \cap B(i)) \leq \mu(C_i) + \frac{1}{i}$ .

Then  $\mu(V) = \lim_{i \rightarrow \infty} \mu(V \cap B(i)) = \lim_{i \rightarrow \infty} \mu(C_i)$ , and  $C_i \subset V \cap B(i) \subset V$  compact.

Therefore  $\mu(V) = \sup\{\mu(K) \mid K \text{ compact}, K \subset V\}$ .

4.  $\forall A \in \Lambda$ ,  $\forall i$ ,  $\exists U_i$  open where  $U_i \supset A$  and  $\mu(U_i \setminus A) < \frac{1}{i}$

This implies that  $\mu(A) \leq \mu(U_i) \leq \mu(A) + \frac{1}{i}$  and therefore  $\mu(A) = \inf\{\mu(U) \mid U \supset A, U \text{ open}\}$ .

## Caratheodory Construction

Definition: Outer Measure

$\mu^*(A), \forall A \in 2^X$

1.  $\mu^*(\emptyset) = 0$ .

2.  $\mu^*(A) \leq \mu^*(B)$  if  $A \subseteq B$ .

3.  $\mu^*\left(\bigcup A_i\right) \leq \sum \mu^*(A_i), \forall A_i \in 2^X$  (countable subadditivity)

Define  $\Lambda = \{A \in 2^X \mid \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c), \forall E \in 2^X\}$ .

Then  $\mu(A) = \mu^*(A)$  if  $A \in \Lambda$ .

$(X, \Lambda, \mu)$  is complete.