Manifolds I

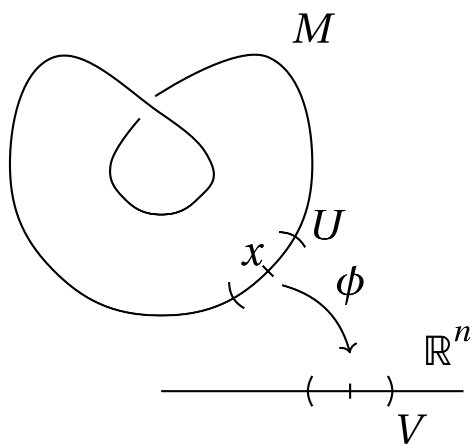
September 26, 2024

Class Organization

1 Takehome Midterm
1 Takehome Final
Homeworks assigned, but not graded.
`https://ginzburg.math.ucsc.edu/teaching/208manifolds1-2024/syl.html`

Definition: Topological Manifolds

For M a topological space, M is a topological manifold if $\forall x \in M$, $\exists M \supset U \ni x$ and homeomorphism $\phi : U \to V \subset \mathbb{R}^n$ for V open.
To avoid problems (see below), further assume that M is Hausdorff and second countable.

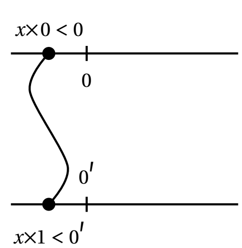


Exercise

We can require V to be an open ball.

Problems

- M need not be Hausdorff.



With $(\mathbb{R} \times 0 \coprod \mathbb{R} \times 1)/\sim$.

- M need not be second countable.

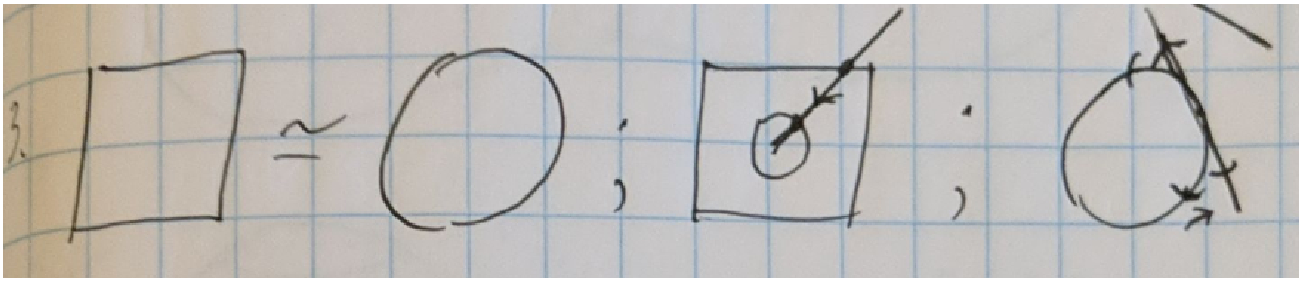
  Take $\coprod_S \mathbb{R}_S$ where S is an uncountable index.

Examples

Example 1

If $N \underset{\text{homeo}}{\simeq} M$, this implies N is a manifold.

Example 2



Example 3

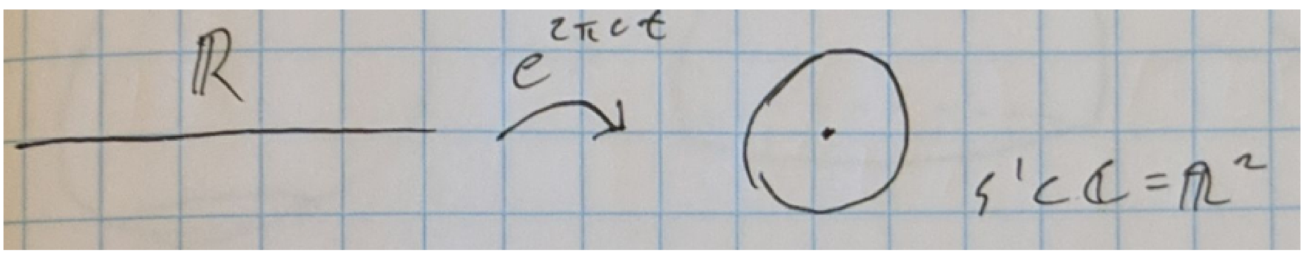
An open subset of a manifold is a manifold.

Example 4

$M,\ N$ manfiolds implies $M \times N$ is a manifold.

Example 5

Take $\mathbb{R}/\mathbb{Z}$ by the equivalence relation $t \sim t'$ iff $t' - t \in \mathbb{Z}$.



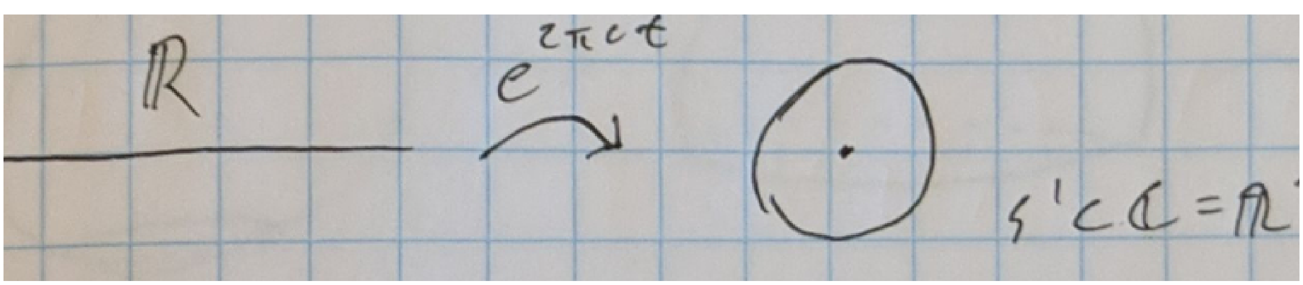
Then $C^0(S^1)$ relates to periodic functions with period 1.

Example 6

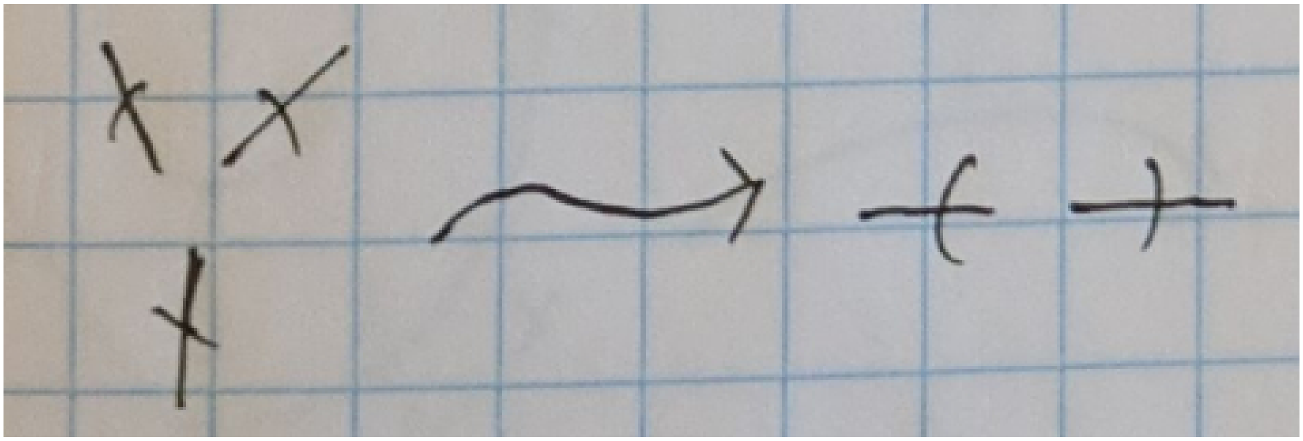
$\mathbb{T}^n = S^1 \times \cdots \times S^1$.

Counterexample 1

$[0,1]$ is not a manifold.

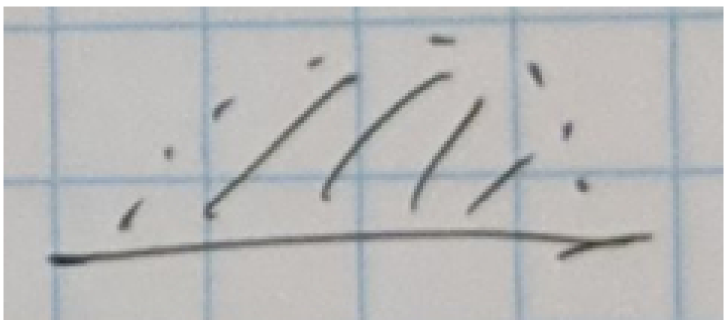


Since 0 must map somewhere in the open interval, its deletion results in a connected space in the former case but a disconnected one in the latter. Similarly, the following breaks into three and two connected components respectively.



Definition: Manifold with Boundary

There exists a neighborhood $\forall x \in M$ homeomorphic to either the open ball or the half-closed half-ball.



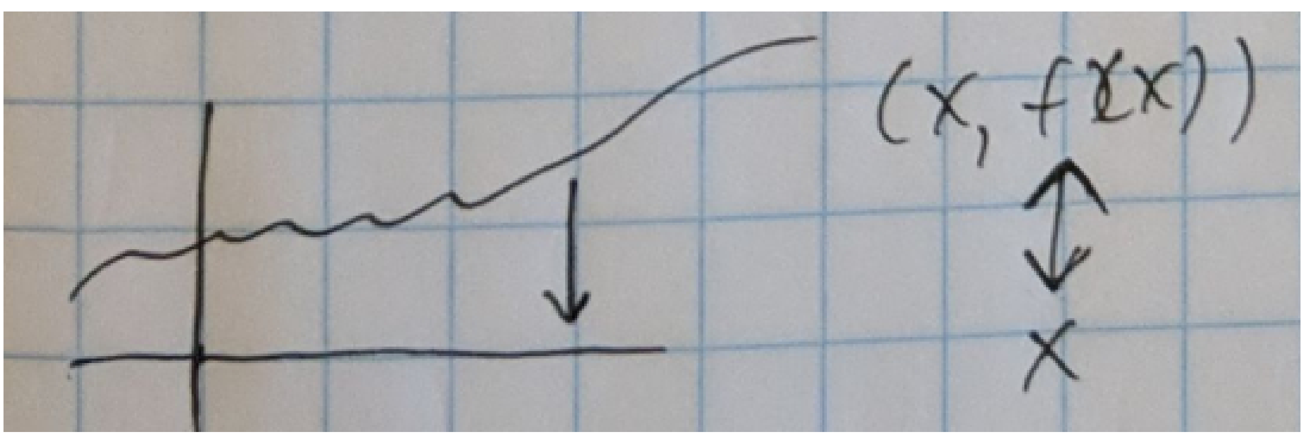
Exercise

A connected manifold is path-connected.

Examples

Example 7

Take $f : \mathbb{R}^n \overset{C^0}{\to} \mathbb{R}$ with graph $\Gamma_f = \{(x, f(x)) \,:\, x \in \mathbb{R}^n\} \subset \mathbb{R}^n \times \mathbb{R}$.

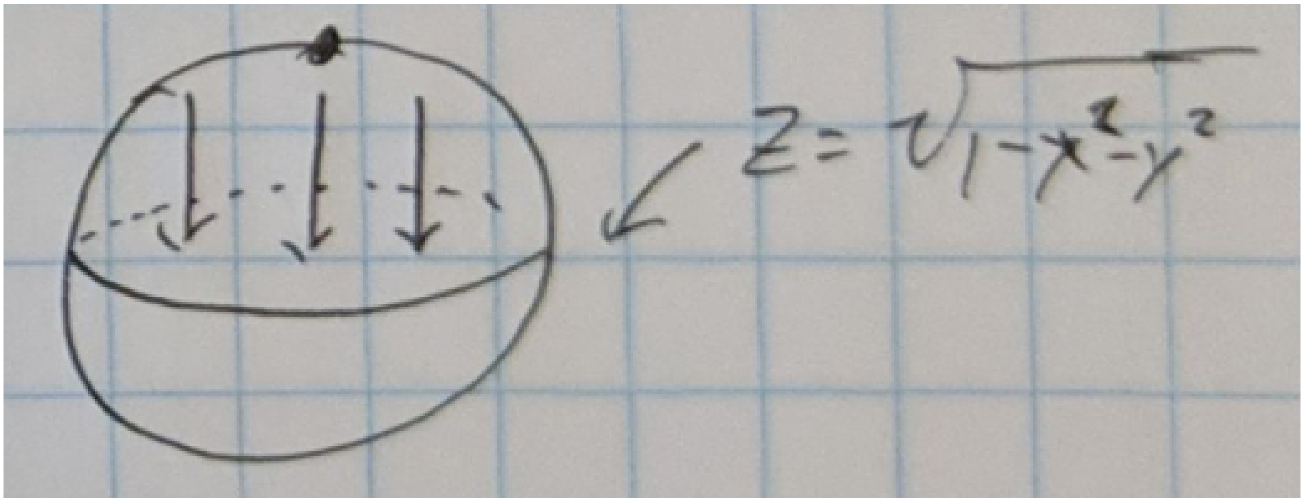


Example 8

Take $f : M \to N$ between manfiolds, then $M \simeq \Gamma_f \subseteq M \times N$.

Example 9

$S^n \subset \mathbb{R}^{n+1}$.

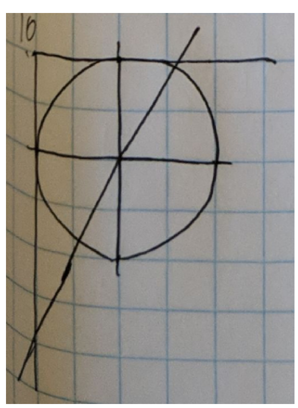


Definition: Real Projective Spaces

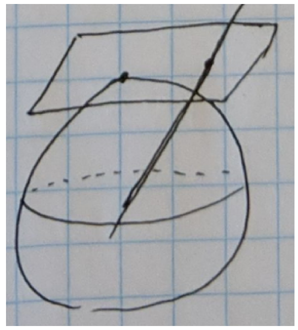
Take $\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus \{0\} / \sim$ where $x \sim y \iff x = \lambda y$ for $\lambda \neq 0$.
Informally, the collection of lines through the origin.
Alternatively, $\mathbb{RP}^n = S^n / \sim$ where $x \sim -x$.
That is, identifiying the antipodal points of the unit sphere.
We may also consider $\mathbb{RP}^n = SO(n+1)/SO(n)$.

Claim

$\mathbb{RP}^n$ is a manifold.

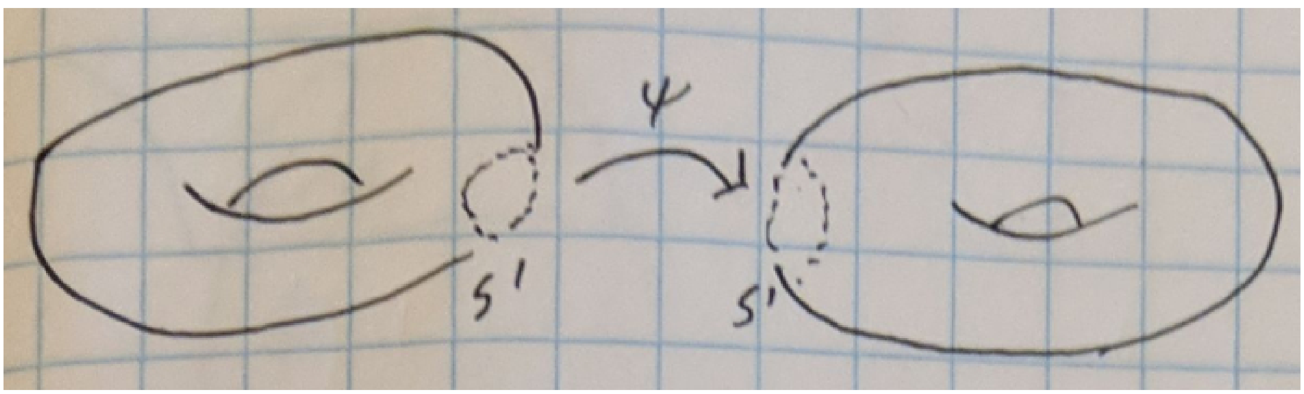


$\mathbb{RP}^1 \setminus \{x\text{-axis}\} \overset{\text{homeo}}{\to} \mathbb{R}$.

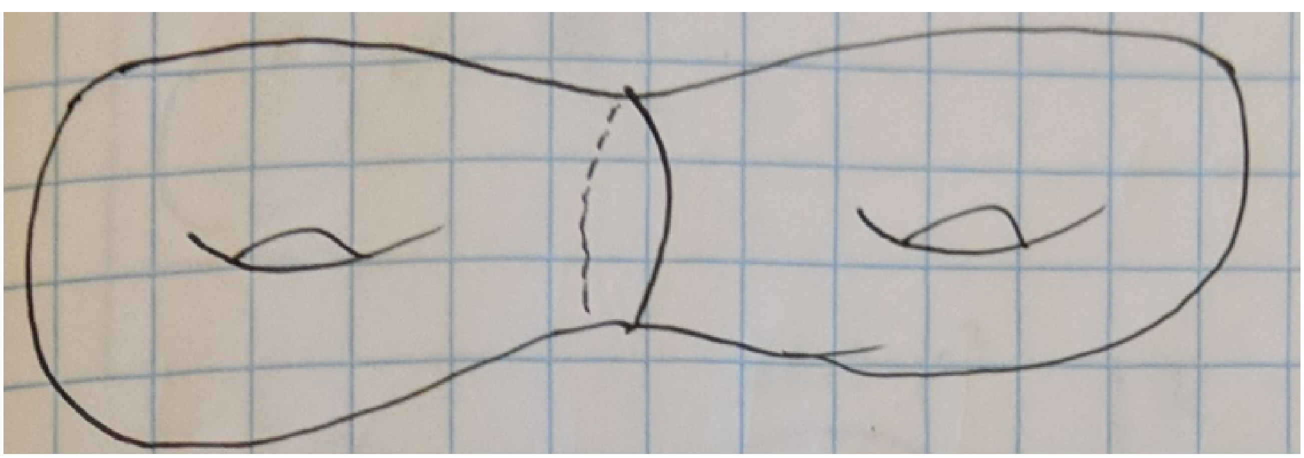


$\mathbb{RP}^2 \setminus \mathbb{RP}^1 \overset{\text{homeo}}{\to} \mathbb{R}^2$
We have that $\mathbb{RP}^1$ is homeomorphic to the circle, and $\mathbb{RP}^n = \mathbb{RP}^{n-1} \cup B^n$.
Take $x = (x_0, \dots, x_n)$, $y = (y_0, \dots, y_n) = (\lambda x_0, \dots, \lambda x_n)$ and $[x] = [x_0 : x_1 : \cdots : x_n]$.
Then for $U_k \subset \mathbb{RP}^n$ with $U_k = \{[x] \,:\, x_k \neq 0\}$, we have that $U_0, \dots, U_n$ covers $\mathbb{RP}^n$.
Then define $U_k \to \mathbb{R}^n$ by $[x_0 : \cdots : x_n] \to \left(\frac{x_0}{x_k}, \dots, \frac{x_k}{x_k}, \dots, \frac{x_n}{x_k}\right)$.

Connected Sum of Manfiolds

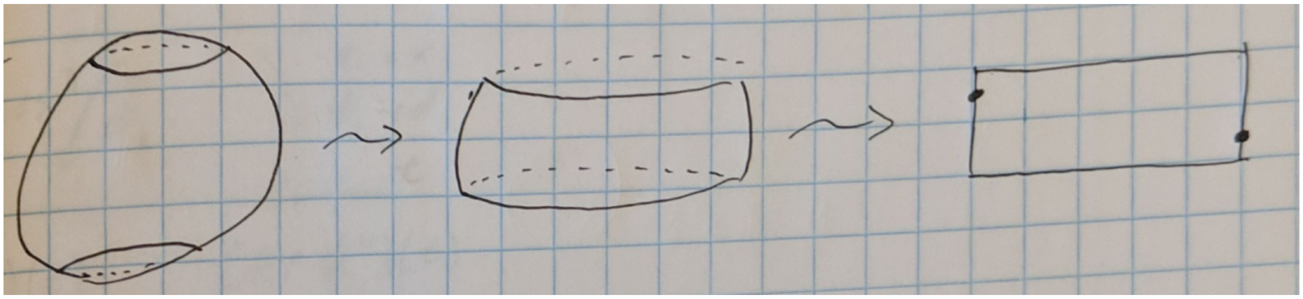


$M \setminus B^n \coprod N \setminus B^n$



$M \# N$.

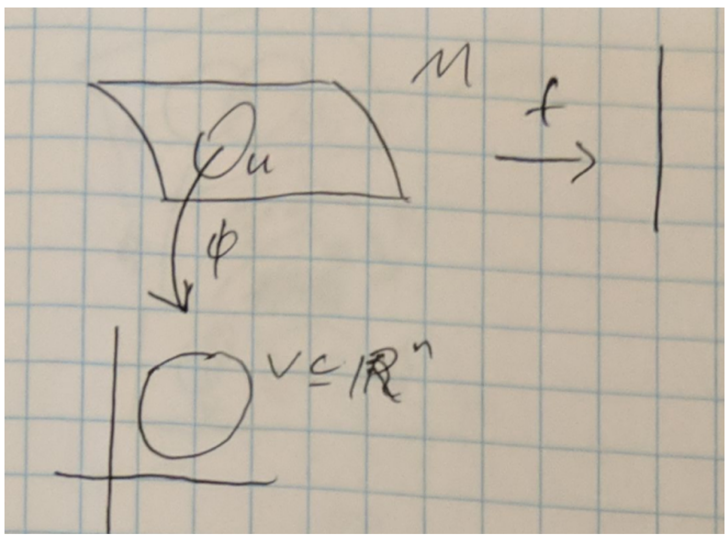
Mobius Band



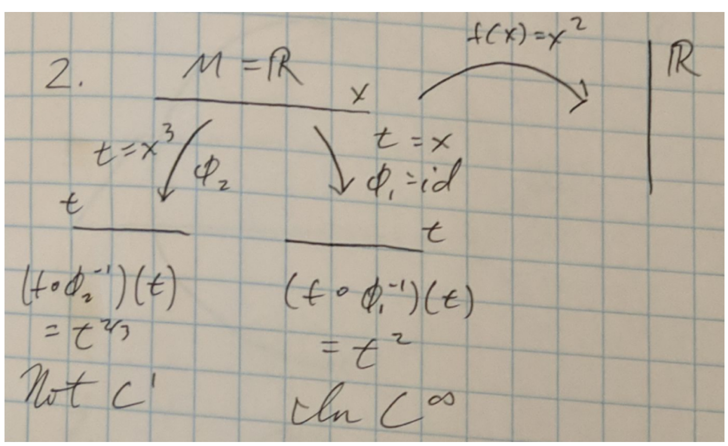
October 1, 2024

A Failed Definition

$f \in C^{r \geq 1}$; $f \circ \phi^{-1} : V \xrightarrow{C^r} \mathbb{R}$.



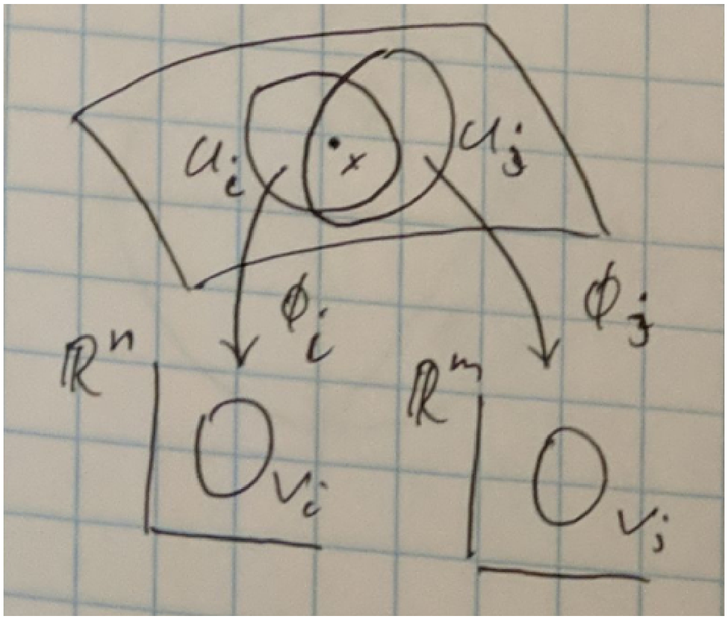
Example



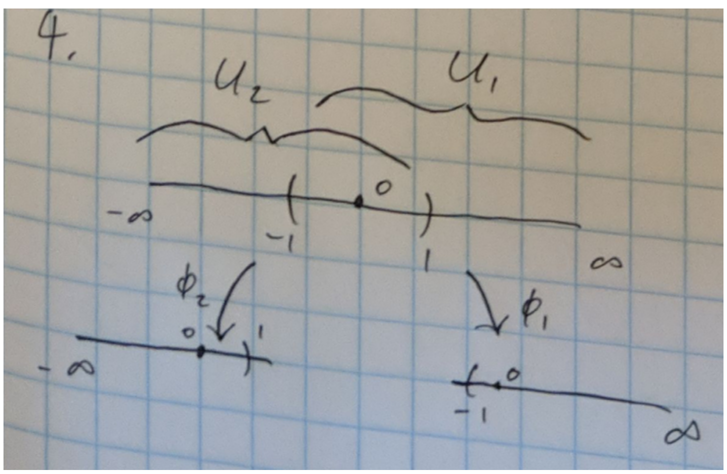
Definition: Charts

Say there exists a cover U_i by open sets and $U_i \xrightarrow{\phi_i} V_i \subseteq \mathbb{R}^n$ fixed.
Then the pair (U_i, ϕ_i) is a chart.

What if a point belongs to two charts?



With f smooth at x, $f \circ \phi_i^{-1}$ smooth at $\phi_i(x)$ and $f \circ \phi_j^{-1}$ smooth at $\phi_j(x)$.

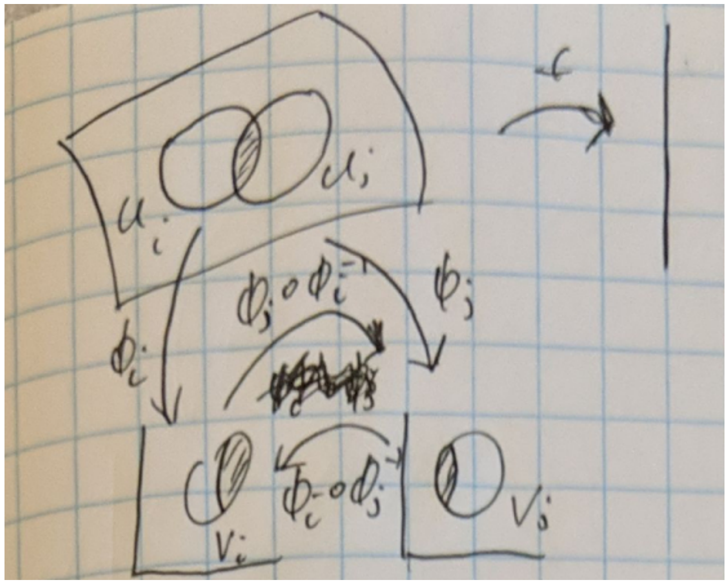


Notation

The notation C^r will be used interchangably with the term smooth.

Definition: Smooth Atlas

Let M be a topological manifold. A smooth atlas on M is a cover $(U_i, \phi_i : U_i \xrightarrow{\sim} V_i \subset \mathbb{R}^n)$ where $\phi_j \circ \phi_i^{-1}$ and $\phi_i \circ \phi_j^{-1}$ are smooth for every i and j.



Say that the charts are (smooth) compatible.

Definition: Smooth Function

Say that f is smooth at $x \in M$ if there exists a chart $U_i \ni x$ such that $f \circ \phi_i$ is smooth at $\phi_i(x)$.
Equivalently, if for every chart $U_i \ni x$ we have that $f \circ \phi_i$ is smooth at $\phi_i(x)$.

- Proof

$$f \circ \phi_j^{-1} = (f \circ \phi_i^{-1}) \circ \underbrace{(\phi_i \circ \phi_j^{-1})}_{C^r}$$

Definition: Compatibility (Equivalence) of Atlases

Atlases A_1 and A_2 are compatible or equivalent if every chart in A_1 is compatible with every chart in A_2.
Equivalently, $A_1 \cup A_2$ is also an atlas.

- Claim: This is an equivalence relation.

Example

Consider $\mathbb{R}$.
Atlas 1: $U = \mathbb{R}$ and $\phi = \mathrm{id}$.
Atlas 2: $U_1 = (1, \infty)$, $\phi_1 = (x) = x^2$, $U_2 = (-\infty, 2)$ and $\phi_2(x) = x$.

Definition: Diffeomorphism

$\mathbb{R}^n \supset V \xrightarrow{F} W \subset \mathbb{R}^n$ is a diffeomorphism if

- F is C^r,
- F is invertible, and
- F^{-1} is C^r

Counterexample

$y = x^3$ is a smooth homemorphism but not a diffeomorphism.

Definition: Smooth Structure / Maximal Atlas

Given an atlas, we may take all compatible atlases and define a smooth structure by the union of all such objects (i.e. the maximal atlas).

Lemma:

Every smooth manifold has a countable, locally finite atlas of precompact charts.

Examples

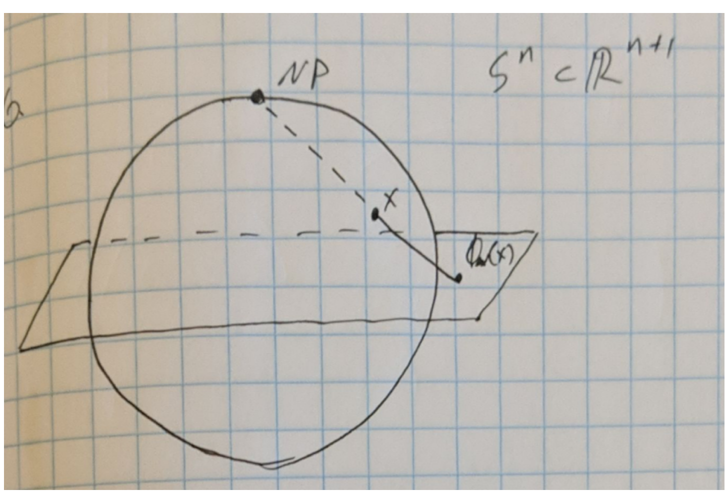
- Zero dimensional manfiolds (i.e. a point).
- $\mathbb{R}^n$ and open subsets of $\mathbb{R}^n$.
- If M, N are smooth manfiolds, then $M \times N$ is a a smooth manfiold.

  That is, if we have atlases (U_i, ϕ_i) and (W_j, ψ_j), we may generate $(U_i \times W_j, \phi_i \times \psi_j)$.
- Take $F : M \overset{\text{homeo}}{\to} N$ with N a smooth manfiold. Then M is smooth.

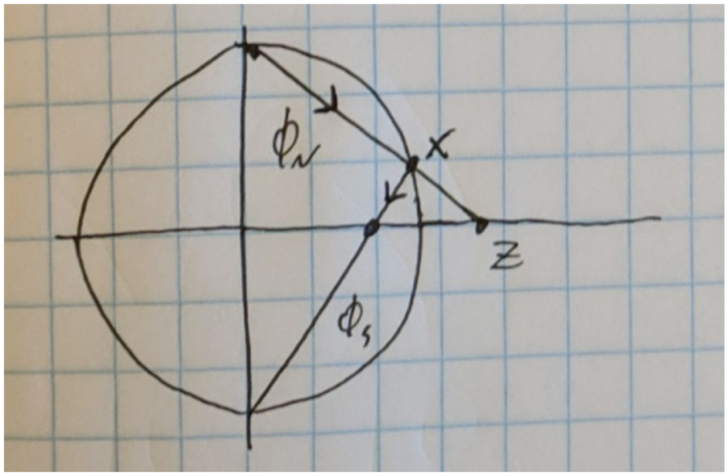
  Take an atlas A on N and the pullback $F^{-1}A = \{(F^{-1}(U_i), \phi_i \circ F)\}$.
- An open subset of a smooth M is a smooth manifold.
- $\mathrm{GL}(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$.

The n-Sphere

- S^n is a manifold



$$U_N = S^n \setminus NP \overset{\phi_N}{\to} \mathbb{R}^n$$
$$U_S = S^n \setminus SP \underset{\phi_S}{\to} \mathbb{R}^n$$



$\phi_S\phi_N^{-1}(z) = \frac{z}{|z|^2}$.

- A different construction for S^n.

  Take hemispheres $U \xrightarrow{\text{orthogonal projection}} B^n$.

Projective Space

$\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus 0/\sim$ where $x \sim \lambda x$ for $\lambda \neq 0$.
$[x] = [x_0 : x_2 : \cdots : x_n] = [\lambda x_0 : \lambda x_2 : \cdots : \lambda x_n]$.
Take $U_i = \{x_i \neq 0\}$ and open cover, and maps $U_i \to \mathbb{R}^n$ given by $[x_0 : \cdots : x_n] \mapsto \left(\frac{x_0}{x_i}, \ldots, \frac{x_i}{x_i}, \ldots, \frac{x_n}{x_i}\right)$. Then for $j < i$ take

$$\phi_j\phi_i^{-1}(y_1, \ldots, y_n) = \left(\frac{y_0}{y_j}, \ldots, \frac{y_{j-1}}{y_j}, \frac{y_{j+1}}{y_j}, \ldots, \frac{y_{i-1}}{y_1}, \frac{1}{y_i}, \frac{y_i}{y_i}, \ldots, \frac{y_n}{y_1}\right)$$

Definition: Diffeomorphism

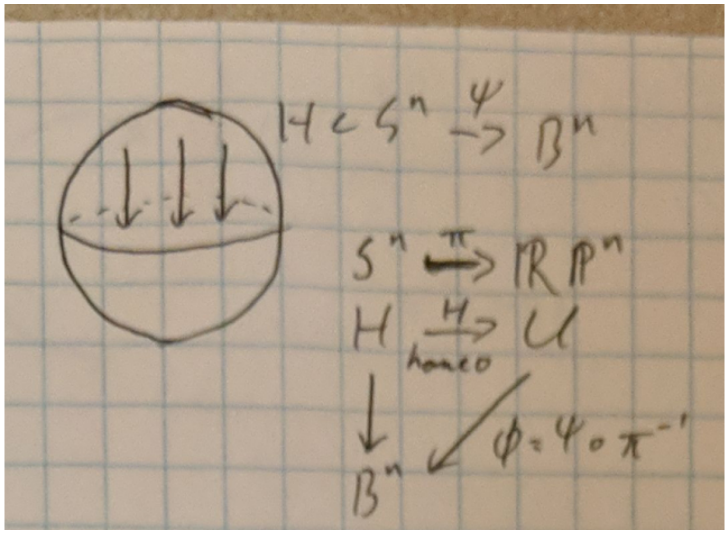
$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ B \subset B_{\text{max}} & & A \supset A_{\text{max}} \end{array}$$

F is a diffeomorphism if F is a homoeomorphism and $F^{-1}A_{\text{max}} = B_{\text{max}}$ ($F^{-1}A \sim B$).

October 3, 2024

Recall

$$\mathbb{RP}^n = \begin{cases} \mathbb{R}^{n+1} \setminus 0/\sim & x \mapsto \lambda x \\ S^n/x \sim -x \end{cases}$$



Note

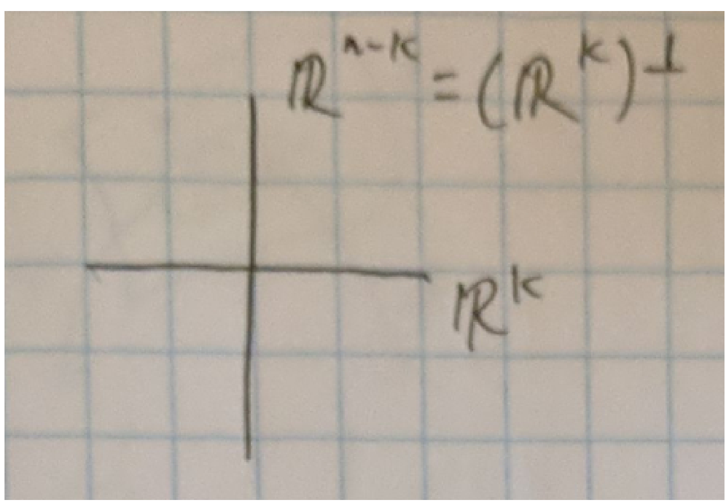
Given a manifold M and A a smooth atlas, we generate a continuum of smooth atlases not equivalent to each other. That is, given $M \underset{\text{homeo}}{\overset{F}{\to}} M$, $F^{-1}A \nsim A$.

Confer With Groups

$G \overset{F}{\to} G$, $a * b = F^{-1}(F(a)F(b))$.

Definition: Grassmannians

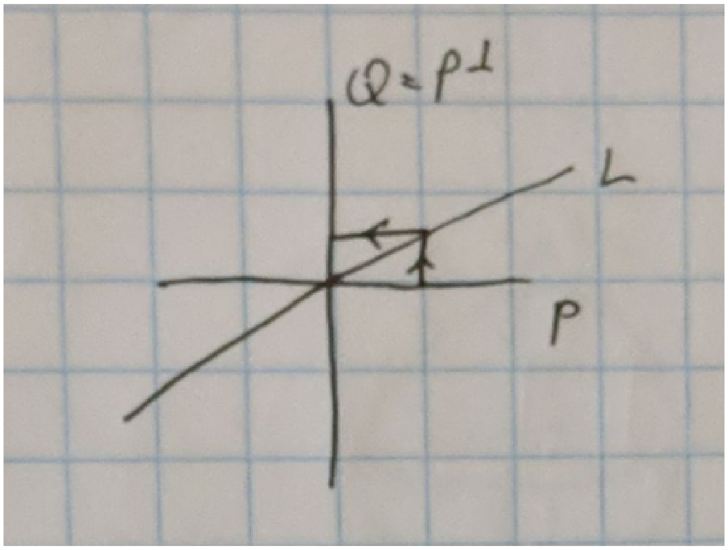
Write $G_k(n)$, the collection of all k-dimensional subspace L in $\mathbb{R}^n$.



Observe that if $O(i)$ is the collection of orthogonal transformations in dimension i,

$$G_k(n) = \frac{O(n)}{O(k) \times O(n-k)}$$

with $X \sim Y$ when $Y = XA = X(O(K) \times O(n-k))$.

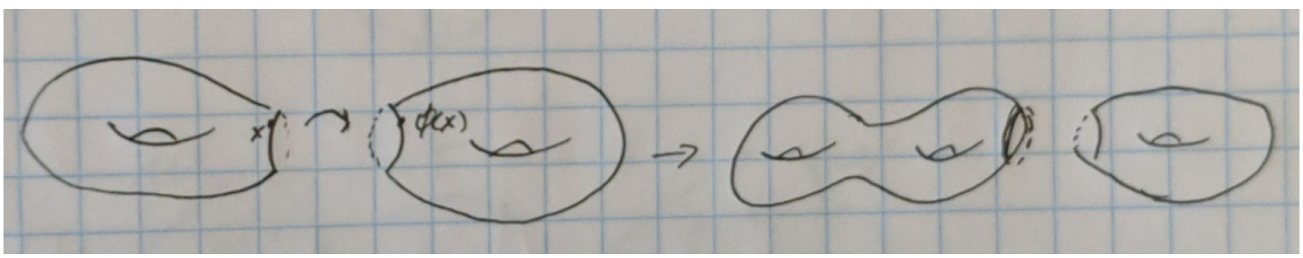


Where $\dim(L) = k$, $U_p = \{L : L \cap Q = \{0\}\}$, $L = \mathrm{graph}(A : P \to Q)$, and we have a homemorphism

$$U_p \xrightarrow{\phi} \underbrace{\{\text{linear maps } P \to Q\}}_{\mathbb{R}^{k\times(n-k)}}.$$

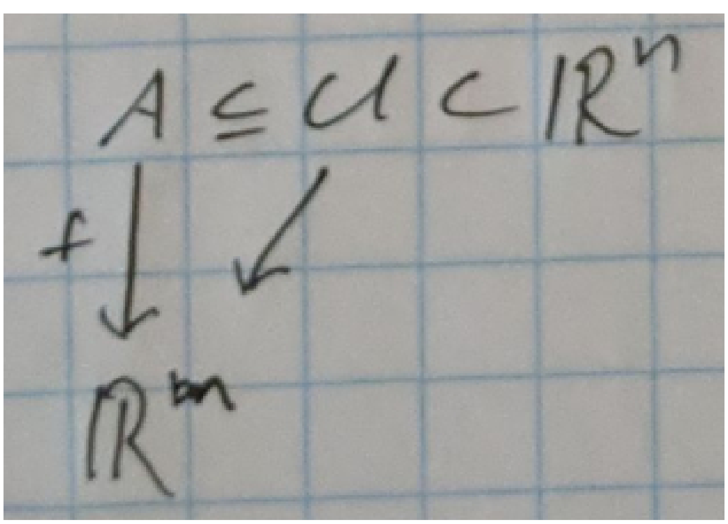
Surfaces

We have explored S^2, $\mathbb{RP}^2$, $\mathbb{T}^2 = S^1 \times S^1$. We have also connected sums.



Terminological Remark

Let $\mathbb{R}^N \supset A \xrightarrow{f} \mathbb{R}^m$.
Then f is smooth if it extends to a smooth map



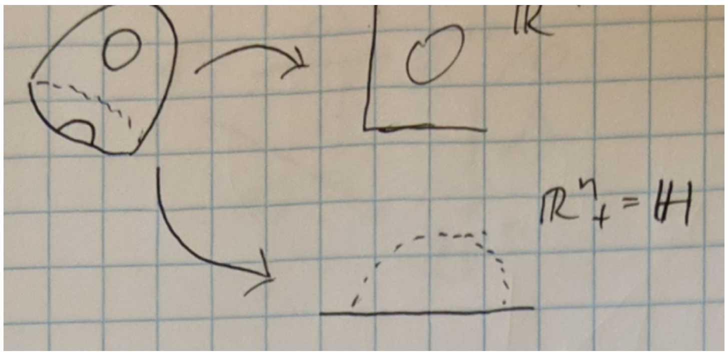
Exercise

Let $A = [0, \infty) \subset \mathbb{R}$. $f : A \to \mathbb{R}$ is smooth if and only if it is infinitely differentiable.
Construct $(-\varepsilon, \infty)$.

Definition: Smooth Manifold with Boundary

A smooth manifold with boundary is a topological space along with an atlas A with charts of two types

$$\phi : U \to B^n \quad \text{(open ball)}$$
$$\phi : U \to B^n \cap H$$



As before, $\phi_i \circ \phi_j^{-1}$ must be smooth.

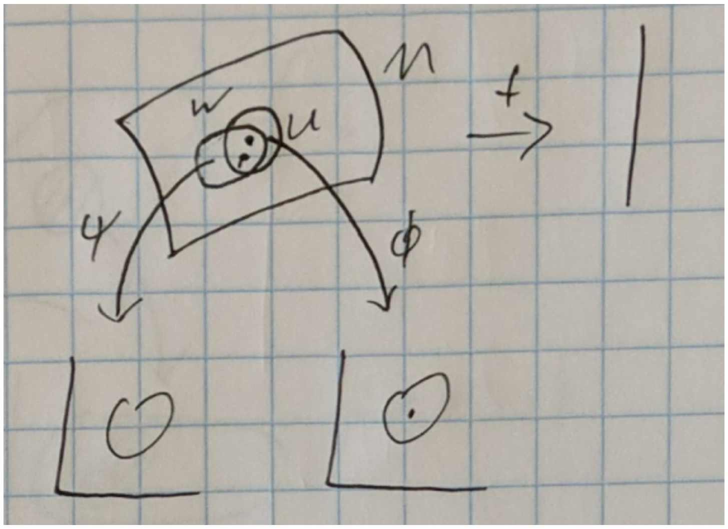
Examples

- $M \backslash$open ball.
- The upper half space.

Definition-Lemma: Smooth Function

A function $f : M \to \mathbb{R}$ is smooth at $p \in M$ if either of the following equivalent conditions is satisfied

1. $\exists$ a chart $(U, \phi) \ni p$ such that $f \circ \phi^{-1}$ is smooth at $\phi(p)$.
2. $\forall$ a chart $(U, \phi) \ni p$ such that $f \circ \phi^{-1}$ is smooth at $\phi(p)$.



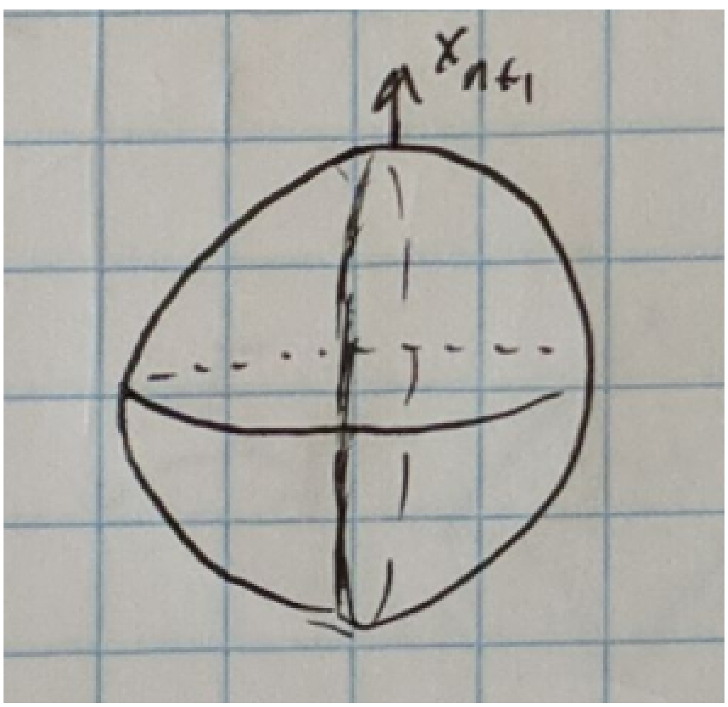
Where $f \circ \phi^{-1} = f \circ \psi^{-1}(\psi \circ \phi^{-1})$.
If the above hold for each $p \in M$, then f is smooth.

Remark

f smooth implies f is C^0

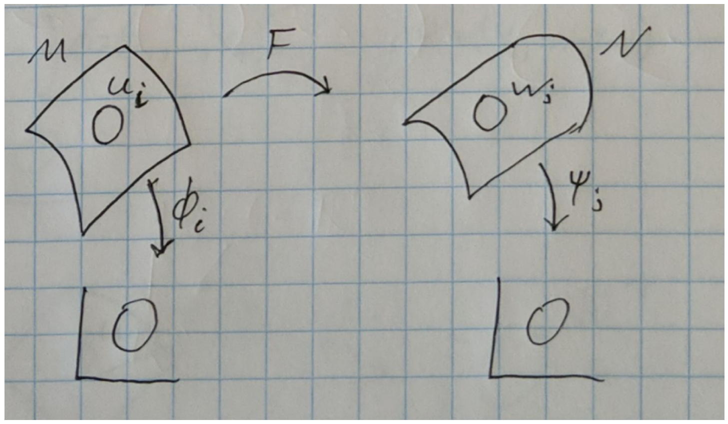
Exercise / Sketch

The height function on S^n is smooth.



$\phi : (x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_n)$.
$f \circ \phi^{-1} = \pm\sqrt{1 - x_1^2 - \cdots - x_n^2}$.
Note that handling the equator requires examining the Eastern and Western hemisphere.
The stereographic projection leads to a simpler proof.

Definition: Smooth Function Between Manifolds



$F: M \to N$ is smooth if F is C^0 and one of the following equivalent conditions is satisfied

1. $\exists$ an atlas $A \subset A_{\max}$ on M and an atlas $B \subset B_{\max}$ on N such that $\psi_j \circ F \circ \phi_i^{-1}$ is smooth on $F^{-1}(W_j) \cap U_i$.
2. The same as a., but for $A_{\max}$ and $B_{\max}$.

Consider as an example $S^n \to \mathbb{RP}^n$.

Properties of Smooth Maps

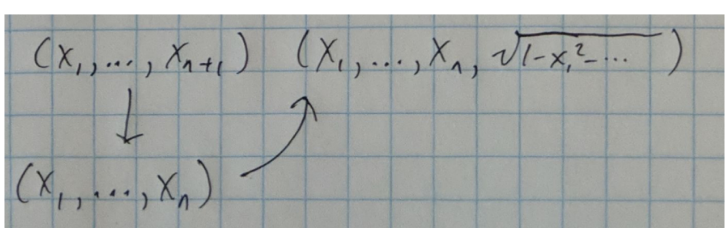
$C^\omega \implies C^\infty \implies C^r \implies C^{r-1} \implies C^1 \implies C^0$.
The sum and product of smooth functions is smooth.

Exercise

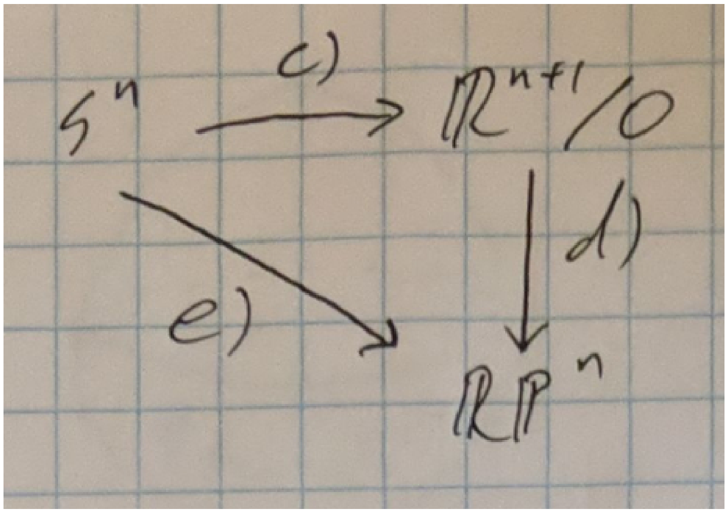
The composition of smooth maps is smooth.

Examples

1. $M \times N \overset{pr}{\to} M$ is smooth.
2. $\underbrace{M \overset{(F_1,F_2)}{\to} N_1 \times N_2}_{\text{smooth}}$ if and only if F_1 and F_2 are smooth.
3. $S^n \hookrightarrow \mathbb{R}^{n+1}$ is smooth.



1. $\mathbb{R}^{n+1} \setminus 0 \to \mathbb{RP}^n$ is smooth with $[x_0 : \cdots : x_n] \mapsto \left(\frac{x_0}{x_i}, \dots, \frac{\hat{x}_i}{x_i}, \dots, \frac{x_n}{x_i}\right)$.
2. $S^n \to \mathbb{RP}^n$.



Definition: Diffeomorphism

$F : \underset{A_{\max}}{M} \xrightarrow{\text{diffeo}} \underset{B_{\max}}{N}$ if

- F is smooth.
- F is invertible.
- F^{-1} is smooth.

Previous Definition

$F^{-1}(B_{\max}) = A_{\max}$.

Exercise

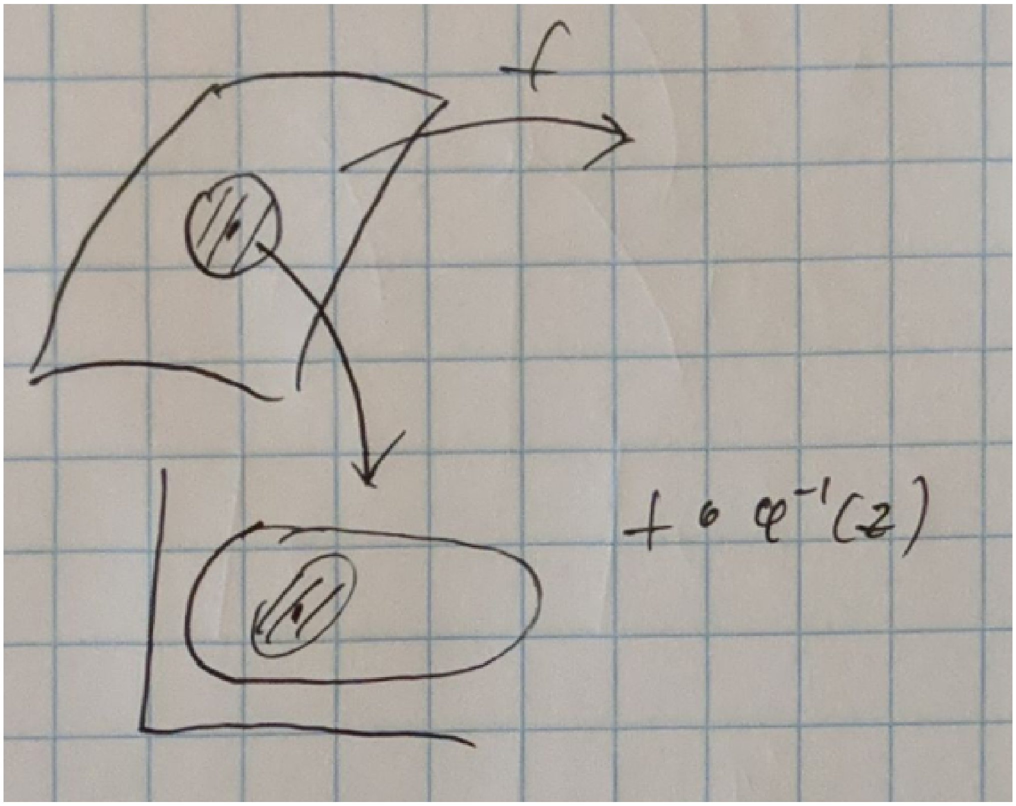
Prove that the definitions are equivalent.

Examples

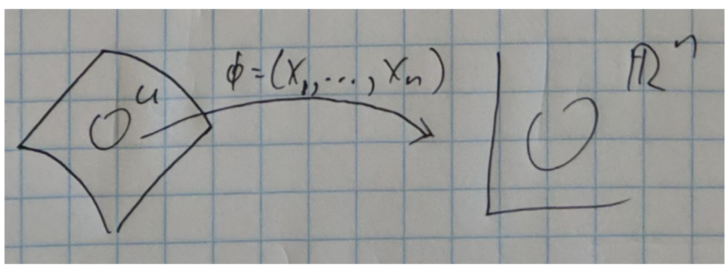
1. $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \xrightarrow{\text{diffeo}} \mathbb{R}$.
2. $x \mapsto x^3$, $\mathbb{R} \to \mathbb{R}$ is not a diffeomorphism.
3. $G_k(n) \leftrightarrow G_{n-k}(n)$ with $P \leftrightarrow P^{\perp}$.

Example 4

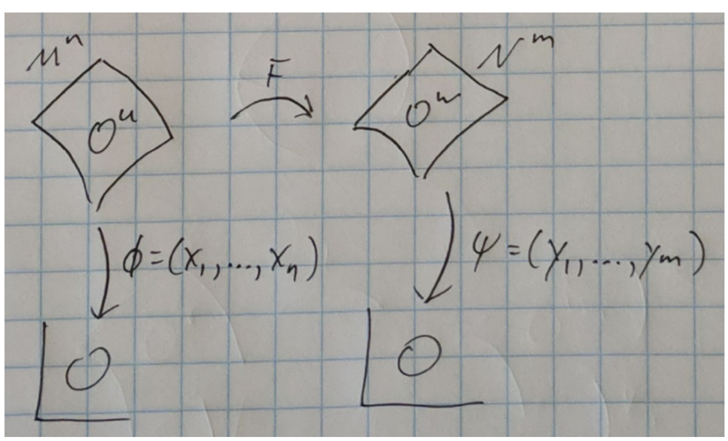
A compact, analytic manifold admits only constant smooth functions by the maximum modulus principle.



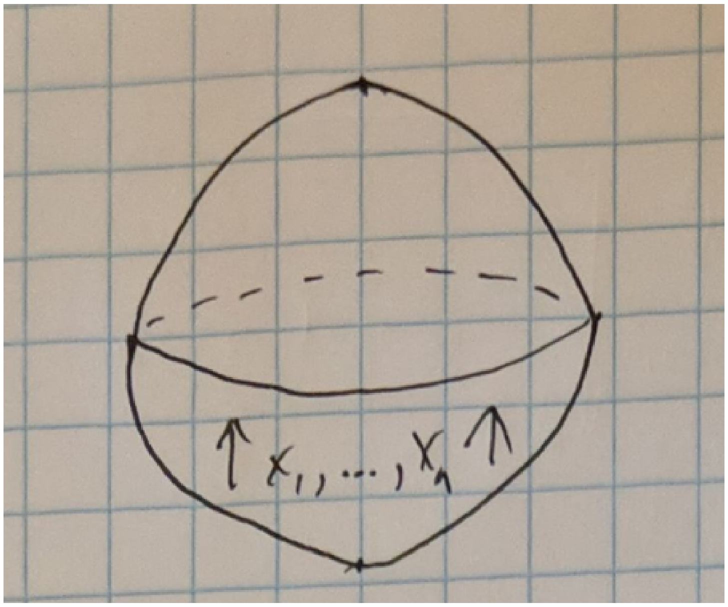
Example 5



Where $\phi = (x_1, \ldots, x_n)$ and each x_i is a real-valued function.



$\psi \circ F \circ \phi^{-1} = (y_1(x_1, \ldots, x_n), \ldots, y_m(x_1, \ldots, x_n))$.
Then $S^n \hookrightarrow \mathbb{R}^{n+1}$



where $y_1 = x_1, \ldots, y_n = x_n$, and $y_{n+1} = -\sqrt{1 - x_1^2 - \cdots - x_n^2}$.

Example 6

$\mathbb{R}^{n+1} \setminus 0 \xrightarrow{F} \mathbb{RP}^n$.
Need to check that $\psi_j \circ F$ is smooth.
$[t_0 : \cdots : t_n] \xrightarrow{\psi_0} \left(\frac{t_1}{t_0}, \ldots, \frac{t_n}{t_0}\right)$ with $U_0 : t_0 \neq 0$