Manifolds III

March 31, 2025

Review

If X, Y are topological spaces and $f, g : X \to Y$ continuous maps, we say f and g are homotopic (written $f \simeq g$) if there is a homotopy $H : X \times I \to Y$ (where $I = [0,1]$) such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$.
We say that f is null-homotopic if it is homotopic to a constant map.

Proposition

Homotopy is an equivalence relation on the collection of continuous maps.

1. $f \simeq f$ by $H(x,t) := f(x)$.
2. $f \overset{\tilde{H}}{\simeq} g \implies g \simeq f$ by defining $\tilde{H}(x,t) := H(x, 1-t)$.
3. $(f \overset{F}{\simeq} g \wedge g \overset{G}{\simeq} h) \implies f \simeq h$ by

$$H(x,t) := \begin{cases} F(x,2t) & 0 \le t \le 1/2 \\ G(x,2t-1) & 1/2 \le t \le 1 \end{cases}.$$

Proposition

For $f_0, f_1 : X \to Y$ and $g_0, g_1 : Y \to Z$, if $f_0 \overset{F}{\simeq} f_1$ and $g_0 \overset{G}{\simeq} g_1$, then $g_0 \circ f_0 \simeq g_1 \circ f_1$.

Proof

Define $H(x,t) := G(F(x,t),t)$ such that $H(x,0) = G(F(x,0),0) = G(f_0(x),0) = g_0 \circ f_0(x)$.
Similarly, $H(x,1) = g_1 \circ f_1(x)$.

Definition: Homotopic Spaces

We say that two spaces X and Y are homotopic to each other ($X \simeq Y$) if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.

Example

$\mathbb{R}^n$ is homotopic to $\{0\}$ (or any single point) by $\iota : 0 \to \mathbb{R}^n$ and $r : \mathbb{R}^n \to 0$. Then $r \circ \iota : 0 \to 0$ is id_0 and $\iota \circ r : \mathbb{R}^n \ni x \mapsto 0 \in \mathbb{R}^n$ is homotopic to $\mathrm{id}_{\mathbb{R}^n}$. In fact, consider $H : \mathbb{R}^n \times I \to \mathbb{R}^n$ where $H(x,t) = tx$, $H(x,1) = x = \mathrm{id}_{\mathbb{R}^n}(x)$ and $H(x,0) = 0$.

Definition: Path

A path in X from p to q is a continuous map $f : I \to X$ such that $f(0) = p$ and $f(1) = q$.

Definition: Path Homotopic

Let $f, g : I \to X$ be two paths in X from p to q.
We say that f and g are path homotopic (write $f \sim g$) if there is a homotopy $H : I \times I \to X$ such that $H(s,0) = f(s)$, $G(s,1) = g(s)$, $H(0,t) = p$ and $H(1,t) = q$.

Proposition

Path homotopy is an equivalence relation on the collection of paths from p to q.
Write $[f]$, the equivalence class of f in the quotient.

Definition: Loop

In the special case that $p = q$, we say that $f : I \to X$ is a loop

Definition: Fundamental Group

Given (X, p), $\pi_1(X, p)$ (the fundamental group of X at the point p) is the set of all loops at p modulo the path homotopy.

$$\{\text{loops at } p\}/\sim$$

Equivalently, $(S^1, 1)$, $\{\text{loops at } p\} = \{\text{continuous maps } f : (S^1, 1) \to (X, p)\}$ with $f(1) = p$. We say this is the homotopy "relative to $1 \in S^1$". We have $H : S^1 \times I \to X$ such that $H(s, 0) = f(s)$, $H(s, 1) = g(s)$ and $H(1, t) = p$.

Definition: Free Homotopy

For two loops $f, g : S^1 \to X$, we say that f and g are free homotopic if $f \simeq g$.

Lemma

When $f : I \to X$ is a path from p to q, if $f \circ \varphi$ is a reparameterization of f then $(f \circ \varphi) \sim f$ where $\varphi : I \to I$ satisfies $\varphi(0) = 0$ and $\varphi(1) = 1$.

Proof

Note that φ is homotopic to the identity map id_I through $H(s, t) = ts + (1 - t)\varphi(s)$ since $H(s, 0) = \varphi(s)$ and $H(s, 1) = s = \mathrm{id}_I(s)$.
Then consider $f \circ H : I \times I \to X$ which is a path homotopy between f and $f \circ \varphi$.

Fundamental Group

Let $f, g : I \to X$ be two paths with $f(1) = g(0)$.
Then we can "compose" (concatenate) f and g together $(f \cdot g) : I \to X$ by

$$(f \cdot g)(s) := \begin{cases} f(2s) & 0 \le s \le 1/2 \\ g(2s - 1 & 1/2 \le s \le 1 \end{cases}.$$

Lemma

If $f_0 \overset{F}{\sim} f_1$, $g_0 \overset{G}{\sim} g_1$ and $f_0(1) = f_1(1) = g_0(0) = g_1(0)$, then $f_0 \cdot g_0 \sim f_1 \cdot g_1$.

Proof

Define

$$H(s, t) := \begin{cases} F(2s, t) & 0 \le s \le 1/2 \\ G(2s - 1, t) & 1/2 \le s \le 1 \end{cases}.$$

Then

$$H(s,0) = \begin{cases} F(2s,0) = f_0(2s) & 0 \le s \le 1/2 \\ G(2s-1, 0 = g_0(2s-1) & 1/2 \le s \le 1 \end{cases}.$$

Similarly $H(s,1) = (f_1 \cdot g_1)(s)$, hence $f_0 \cdot g_0 \sim f_1 \cdot g_1$.
With this, we have a well-defined $[f] \cdot [g] := [f \cdot g]$.

Simple Properties

For f from p to q where c_p is the constant map at p,

1. $[c_p] \cdot [f] = [f] = [f] \cdot [c_q]$ since $c_p \cdot f$ is a reparameterization of f.

2. Let $\overline{f}$ be the inverse path of f (i.e. $\overline{f}(s) = f(1-s)$). Then $[f] \cdot [\overline{f}] = [c_p]$ and $[\overline{f}] \cdot [f] = [c_q]$.

$$H(s,t) := \begin{cases} f(2s) & 0 \le s \le t/2 \\ f(t) & t/2 \le s \le 1 - t/2 \\ f(2-2s) & 1 - t/2 \le s \le 2 \end{cases}.$$

1. $([f] \cdot [g]) \cdot [h] = [f] \cdot ([g] \cdot [h])$, since these are reparameterizations of the same path.

Group Structure

$\pi_1(X,p) = \{\text{loops at } p\} / \sim$.
Define $[f] \cdot [g] := [f \cdot g]$.
It has an identity element $[c_p] = e$.
For any $f \in \pi_1(X,p)$, it has an inverse $[\overline{f}]$ such that $[f] \cdot [\overline{f}] = [\overline{f}] \cdot [f] = [c_p]$.
Finally, it is associative by (3) above.

Proposition

Suppose $p, q \in X$ with X path-connected.
Then $\pi_1(X,p)$ is isomorphic to $\pi_1(X,q)$.
Remark: this isomorphism is not canonical.

Proof

We define a path γ from q to p and $\Phi_\gamma : \pi_1(X,p) \to \pi_1(X,q)$ by $[f] \mapsto [\gamma \cdot f \cdot \overline{\gamma}]$.
Φ_γ is a group homomorphism.

$$\begin{aligned} \Phi_\gamma[f] \cdot \Phi_\gamma[g] &= [\gamma \cdot f \cdot \overline{\gamma}] \cdot [\gamma \cdot g \cdot \overline{\gamma}] \\ &= [\gamma \cdot f \cdot \overline{\gamma} \cdot \gamma \cdot g \cdot \overline{\gamma}] \\ &= [\gamma \cdot f] \cdot \overbrace{[\overline{\gamma} \cdot \gamma]}^{=e} \cdot [g \cdot \overline{\gamma}] \\ &= [\gamma \cdot (f \cdot g) \cdot \overline{\gamma}] \\ &= \Phi_\gamma[f \cdot g]. \end{aligned}$$

Φ_γ has an inverse, $\Phi_{\overline{\gamma}} : \pi_1(X,q) \to \pi_1(X,p)$.

$$\Phi_{\overline{\gamma}} \circ \Phi_\gamma[f] = \Phi_{\overline{\gamma}}[\gamma \cdot f \cdot \overline{\gamma}] = [\overline{\gamma} \cdot \gamma \cdot f \cdot \overline{\gamma} \cdot \gamma] = [f].$$

Induced Homomorphism

$\varphi : (X, p) \to (Y, q)$ induces

$$\begin{aligned} \varphi_* : \pi_1(X, p) &\to \pi_1(Y, q) \\ [f] &\mapsto [\varphi \circ f]. \end{aligned}$$

φ_* is a homomorphism.

$$(\varphi_*[f]) \cdot (\varphi_*[g]) = [\varphi \circ f] \cdot [\varphi \circ g] = [(\varphi \circ f) \cdot (\varphi \circ g)] = [\varphi \circ (f \cdot g)] = \varphi_*[f \cdot g].$$

Proposition

If $\varphi, \psi : (X, p) \to (Y, q)$ are homotopic, then $\varphi_* = \psi_* : \pi_1(X, p) \to \pi_1(Y, q)$.

Proof

Let $[f] \in \pi_1(X, p)$, $\varphi_*[f] = [\varphi \circ f]$ and $\psi_*[f] = [\psi \circ f]$ and $H : X \times I \to Y$ a homotopy between φ and ψ.
Then define $\tilde{H} := I \times I \to Y$ by $\tilde{H}(s, t) = H(f(s), t)$ such that

$$\begin{aligned} \tilde{H}(s, 0) &= H(f(s), 0) = \varphi \circ f(s) \\ \tilde{H}(s, 1) &= H(f(s), 1) = \psi \circ f(s). \end{aligned}$$

Corollary

If $X \simeq Y$, then $\pi_1(X) \simeq \pi_1(Y)$.

Examples

$\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(S^n) = 0$ for $n \geq 2$.
For $n \geq 2$, write $S^n = A_+ \cup A_-$ where A_+ and A_- are large balls centered at the north and south pole respectively.
Then A_+ and A_- are both homeomorphic to $\mathbb{R}^n$ and $A_+ \cap A_-$ (their intersection about the equator) is homemorphic to $S^{n-1} \times \mathbb{R}$.
We fix a base point $p \in A_+ \cap A_-$ and let $f : I \to S^n$ be a loop based at p.
There exists a partition of I, $0 = s_0 < s_1 < \cdots < s_k = 1$, such that $f|_{[s_i, s_{i+1}]}$ is contained in A_- or A_+.
Draw a path γ_i from p to $f(s_i)$ such that $\gamma_i \subseteq A_+ \cap A_-$. Let $f_i = f|_{[s_i, s_{i+1}]}$ such that $f = f_0 \cdot f_1 \cdots f_k$. Then this is path homotopic to

$$(f_0 \cdot \overline{\gamma}_1) \cdot (\gamma_1 \cdot f \cdot \overline{\gamma}_2) \cdots (\gamma_{k-1} \cdot f_{k-1} \cdot \overline{\gamma}_k) \cdot (\gamma_k \cdot f_k).$$

Each $\gamma_i \cdot f_i \cdot \overline{\gamma}_i$ is contained in A_- or A_+, hence $\gamma_i \cdot f_i \overline{\gamma}_{i+1} \sim c_p$, $f \simeq c_p$ and $[f] = e$.

April 2, 2025

Correction

For $\varphi, \psi : (X, x_0) \to (Y, y_0)$ where $\varphi \simeq \psi$, we say a homotopy H between φ and ψ is base point preserving if $H(x_0, t) = y_0$ for all $t \in [0, 1]$.

Proposition

If $\varphi \simeq \psi$ through a base point preserving homotopy, then $\varphi_* = \psi_*$, $\pi_1(X, x_0) \to \pi_1(Y, y_0)$.
For $X \simeq Y$, $\varphi : X \to Y$ and $\psi : Y \to X$ where $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$, in general $\psi \circ \varphi(x_0) \neq x_0$ and $\varphi \circ \psi(y_0) \neq y_0$.
Set up: $\varphi_0, \varphi_1 : X \to Y$ with $\varphi_0 \simeq \varphi_1$ through a homotopy H.
Write $\varphi_t = H(\cdot, t) : X \to Y$ and fix a base point $x_0 \in X$ and set $\gamma(t) = \varphi_t(x_0)$ for $t \in [0,1]$.

Proposition 1

$(\varphi_0)_* = \Phi_\gamma \circ (\varphi_1)_* : \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$.

Proof

Let f be a loop at x_0.

IMAGE 1

Let γ_t be $\gamma|_{[0,t]}$ and then, by rescaling the domain $[0, t]$ to $[0, 1]$ i.e.

$$\begin{aligned} \gamma_t : [0,1] &\to Y \\ s &\mapsto \gamma(ts). \end{aligned}$$

from $\varphi_0(x_0)$ to $\gamma(t) = \varphi_t(x_0)$. Then $\gamma_t \cdot (\phi_t \circ f) \cdot \overline{\gamma}_t$ is a homotopy between $(\varphi_0 \circ f)$ and $\gamma \cdot (\varphi_1 \circ f) \cdot \overline{\gamma}$. Hence

$$(\varphi_0)_*[f] = [\varphi_0 \circ f] = [\gamma] \cdot [\varphi_1 \circ f] \cdot [\overline{\gamma}] = \Phi_\gamma \circ (\varphi_1)_*[f].$$

Proposition 2

If $X \simeq Y$, then $\pi_1(X) \cong \pi_1(Y)$.

Proof

Since $(\psi \circ \varphi) \simeq \mathrm{id}_X$, by Proposition 1

$$\psi_* \circ \varphi_* = (\psi \circ \varphi)_* = \Phi_\gamma \circ (\mathrm{id}_x)_* = \Phi_\gamma.$$

Hence $\psi_* \circ \varphi_*$ is an isomorphism (as is $\varphi_* \circ \psi_*$). Therefre φ_* and ψ_* are isomorphisms.

Recall: Covering Map

For $X, \tilde{X}$ connected, $\pi : \tilde{X} \to X$ is a covering map if for each $p \in X$ there exists a neighborhood $U \subset X$ such that $\pi^{-1}(U)$ is a disjoint union

$$\pi^{-1}(U) = \dot{\bigcup}_{\alpha \in A} U_\alpha$$

such that $\pi|_{U_\alpha} : U_\alpha \to U$ Is a homeomorphism.

Lifting Properties

A lift is a map $\tilde{f}$ such that $f = \pi \circ \tilde{f}$.

1. Path Lifting: Let $f : I \to X$ be a path from x_0. Then, for any $\tilde{x}_0 \in \pi^{-1}(x_0)$, there is a unique lift $\tilde{f}$ of f with $\tilde{f}(0) = \tilde{x}_0$.
2. Homotopy Lifting: Let $f_0, f_1 : I \to X$ be paths in X with $f_0(0) = f_1(0) = x_0$ and $f_0(1) = f_1(1)$. Suppose H is a path homotopy between f_0 and f_1. Then for any $\tilde{x}_0 \in \pi^{-1}(x_0)$, there is a unique lift $\tilde{H} : I \times I \to \tilde{X}$ of H. In particular, $\tilde{H}$ is a path homotopy between $\tilde{f}_0$ and $\tilde{f}_1$. That is if $H(0,t) = x_0$ then $\tilde{H}(0,t) \in \pi^{-1}(x_0)$ for all t. Hence $\tilde{H}(0,t) = \tilde{x}_0$, $\forall\, t \in [0,1]$. Similarly, $\tilde{H}(1,t)$ is identically constant. In particular, $\tilde{f}_0(1) = \tilde{H}(1,0) = \tilde{H}(1,1) = \tilde{f}_1(1)$.

Fundamental Group of the Circle

$\pi_1(S^1) = \mathbb{Z}$.

Example

$\pi : \mathbb{R} \to S^1$ by $s \mapsto e^{2\pi i \cdot s}$.

Proof

Take as a base point $1 = x_0 \in S^1 \subseteq \mathbb{C}$. For each $n \in \mathbb{Z}$, we define a loop $\omega_n : [0,1] \to S^1$ by $s \mapsto e^{2\pi i \cdot ns}$. Let f be a loop at $x_0 \in S^1$. We can lift f to $\tilde{f} : I \to \mathbb{R}$ at $0 \in \mathbb{R}$. Then $\tilde{f}(1) \in \pi^{-1}(x_0) = \mathbb{Z} \subseteq \mathbb{R}$. This defines a map φ that sends a loop f to $\tilde{f}(1) \in \mathbb{Z}$. This φ induces $\varphi : \pi_1(S^1, x_0) \to \mathbb{Z}$ well-defined. If $f_0, f_1 : I \to S^1$ at x_0 are path homotopic via H, then we may lift H to $\tilde{H} : I \times I \to \mathbb{R}$ which implies $\tilde{f}_0(1) = \tilde{f}_1(1)$.
φ is surjective, since for any $n \in Z$ we may consider the loop ω_n where $\tilde{\omega}_n(1) = n$.
φ is a group homorphism since $\varphi[f \cdot g] = \widetilde{f \cdot g}(1) = \tilde{g} + \tilde{f}(1) = \varphi[f] + \varphi[g]$.
φ is injective, since if $\varphi[f] = 0$ (i.e. $\tilde{f}(0) = 0$) then $\tilde{f}$ is a loop in $\mathbb{R}$ and $\tilde{f}$ is null-homotopic to c_0 by H. Therefore $\pi \circ \tilde{H}$ is a path-homotopy between f and c_{x_0} (i.e. $[f] = e$).

Path-Lifting

For $f : I \to X$, we have a special case where $\operatorname{im} f \subseteq U$ evenly covered. Write $\pi^{-1}(U) = \dot{\bigcup} \tilde{U}_\alpha$ and pick the $\tilde{U}_\alpha$ which contains $\tilde{x}_0$. Since $\pi|_{\tilde{U}_\alpha} : \tilde{U}_\alpha \to U$ is a homemorphism, $\tilde{f} := (\pi|_{\tilde{U}_\alpha})^{-1} \circ f$ is the unique lift of f at $\tilde{x}_0$.
In general, pick a partititon of $I = [0,1]$, $0 = t_0 < t_1 < \cdots < t_m = 1$, such that $\operatorname{im} f|_{[t_i, t_{i+1}]} \subseteq U_i$ evenly covered. We can lift $f|_{[0,t_1]}$ at $\tilde{x}_0$, giving $\tilde{f} : [0,t] \to \tilde{X}$. Next, we lift $f|_{t_1, t_2]}$ at $\tilde{f}(t_1) \in \tilde{X}$. Since the partition is finite, we may repeat the process until f is entirely lifted. This lift is unique.

Homotopy Lifting

For each fixed $(y_0, t_0) \in I \times I$, by continuity, there is a neighborhood $N(y_0) \times (t_0 - \varepsilon, t_0 + \varepsilon)$ such that H sends $N(y_0) \times (t_0 - \varepsilon, t_0 + \varepsilon)$ inside an evenly covered neighborhood. By compactness of $\{y_0\} \times [0,1]$, there is a finite collection of $N_{t_i}(y_0) \times (t_i - \varepsilon_i, t_i + \varepsilon_i)$ such that they cover $\{y_0\} \times I$ and the image of each under H is contained in an evenly covered neighborhood. Set $N = \bigcap_i N_{t_i}(y_0)$, a neighborhood of y_0, and construct a partition $0 = t_0 < t_1 < \cdots < t_m = 1$ such that $H(N \times [t_i, t_{i+1}] \subseteq U_i$ evenly covered. Then we can start with $H|_{N \times [0,t_1]}$ and lift it at $\tilde{x}_0$ by some $(\pi|_{\tilde{U}_\alpha})^{-1}$. Then lift each $H|_{N \times [t_i, t_{i+1}]}$ one by one. Eventually, we have $\tilde{H} : N \times [0,1] \to \tilde{X}$ that lifts $H : N \times [0,1] \to \tilde{X}$ at $\tilde{x}_0$. This lift holds for any $y_0 \in I$ and, if two strips overlap, then the lift must agree there by the uniqueness of path lifting. This assures that $\tilde{H} : I^2 \to \tilde{X}$ is continuous.

Remark

Given a continuous map $F : Y \times I \to X$ and a covering $\pi : \tilde{X} \to X$, suppose that we have a map $\tilde{F} : Y \times \{0\} \to \tilde{X}$ that lifts $F|_{Y\times\{0\}} : Y \times \{0\} \to X$. Then there is a unique lift $\tilde{F} : Y \times I \to \tilde{X}$ of F which extends $\tilde{F} : Y \times \{0\} \to \tilde{X}$.

Theorem: Fundamental Theorem of Algebra

A polynomial $p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ (with $a_i \in \mathbb{C}$) has a root in $\mathbb{C}$.

Proof

Suppose otherwise. Then $p(z) \neq 0$, $\forall z \in \mathbb{C}$. Consider $f_r : [0,1] \to S^1$ ($r \geq 0$) by

$$f_r(s) = \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}.$$

Then $f_0(s) \equiv 1$ is a constant loop at $1 \in \mathbb{C}$, and $f_r \simeq f_0$ for each $r \geq 0$. Consider $R \geq 1$ large such that $R \gg \sum_{i=1}^n |a_i|$. On $\{z : |z| = R\}$, we have

$$|z^n| > \left(\sum_{i=1}^n |a_i|\right) \cdot |z^{n-1}| \geq \sum_{i=1}^n |a_i| \cdot |z^{n-i}| = \left|\sum_{i=1}^n |a_i z^{n-i}\right|.$$

This implies that p does not have any roots on $\{|z| = R\}$. Moreover, for $p_t(z) = z^n + t(a_1 z^{n-1} + \cdots a_{n-1} z + a_n)$ with $0 \leq t \leq 1$, p_t does not have any roots on $\{|z| = R\}$. Consider

$$f_{R,t}(s) = \frac{p_t(Re^{2\pi i s})/p_t(R)}{|p_t(Re^{2\pi i s})/p_t(R)|}.$$

Then

$$f_{R,0}(s) = \frac{(Re^{2\pi i s})^n / R^n}{|(Re^{2\pi i s})^n / R^n|} = (e^{2\pi i s})^n = \omega_n(s).$$

Therefore $f_{R,1}(s) \simeq f_R(s)$ and $f_R \simeq \omega_n$. But since $\omega_n \not\simeq$ constant so this is a contradiction.