

# Analysis I

**October 2, 2023**

## Lecture Notes

Class will not have dedicated lecture notes. Many are available already.

Undergraduate notes are available on Canvas.

Lecture 1 overview available on Canvas (lecture1.pdf).

## Tentative Office Hours

Mondays 2-3pm and Tuesday 1-2pm.

## Homework

Nominally due at beginning of class; ask for leeway if needed.

First week homework will be review of undergraduate proofs.

First homework due Wednesday, October 11.

## Notation

Natural Numbers:  $\mathbb{N} = \{1, 2, 3, \dots\}$

Non Negative Integers:  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$

Integers:  $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$

Rationals:  $\mathbb{Q} = \left\{ \frac{p}{q}, p \in \mathbb{Z}, q \in \mathbb{Z} \right\} = \mathbb{Z} \times \mathbb{N} / \sim$

- Equivalent representation of rationals:  $(p_1, q_1) \sim (p_2, q_2)$  iff  $p_1 q_2 = p_2 q_1$

Sequence of Rationals:  $\{u_n\}_{n \in \mathbb{N}}, u_n \in \mathbb{Q}, \forall n.$

## Properties of the Rationals

$(\mathbb{Q}, +, \cdot)$  is a (ii) totally ordered (i) field satisfying the (iii) Archimedean property.

### (i) Field

1.  $+$  is associative:  $(a + b) + c = a + (b + c)$

2.  $+$  is commutative:  $a + b = b + a$

3.  $\cdot$  is associative and commutative.
4.  $\exists 0 \in \mathbb{Q}$  such that  $\forall a \in \mathbb{Q}, 0 + a = a + 0$
5.  $\exists 1 \in \mathbb{Q} \setminus \{0\}$  such that  $\forall a \in \mathbb{Q}, 1 \cdot a = a \cdot 1 = a$
6.  $\forall a \in \mathbb{Q} \setminus \{0\} \exists b \in \mathbb{Q}, a \cdot b = b \cdot a = 1$

- $b = a^{-1} = \frac{1}{a}$

## (ii) Totally Ordered

$\exists$  a set  $\mathbb{Q}_+ \subseteq \mathbb{Q}$  of “Positive Numbers” stable under  $+$  and  $\cdot$  such that  $\forall A \in \mathbb{Q}$  either  $a > 0$  ( $a \in \mathbb{Q}_+$ ),  $-a > 0$  (also  $a < 0$ ) or  $a = 0$ .

- Ordering:  $\forall a, b \in \mathbb{Q}, a < b$  if and only if  $b - a > -0$ .
- Trichotomy:  $\forall a, b \in \mathbb{Q}$  either  $a < b$ ,  $a > b$ , or  $a = b$ .
- $\max(a, b) = \begin{cases} a & \text{if } a > b \\ b & \text{otherwise} \end{cases}$ .
- $|a| = \max(a, -a)$  (helps measure distance in  $\mathbb{Q}$ ).
- $\text{dist}(a, b) := |b - a|$
- Triangle Inequality:  $|u \pm v| \leq |u| + |v|$
- Observe also:  $||u| - |v|| \leq |u \pm v|$ . The triangle inequality may be used to prove this.
- Proof of Triangle Inequality  $-|u| \leq u \leq |u|$  and  $-|v| \leq v \leq |v|$ , therefore  $-|u| - |v| \leq u + v \leq |u| + |v|$ .  
Therefore  $u + v \leq |u| + |v|$  and  $-(u + v) \leq |u| + |v|$  implies  $|u + v| \leq |u| + |v|$ .

## (iii) Archimedian Property:

$$\forall \epsilon > 0, \exists N, \forall n \geq N, \frac{1}{n} < \epsilon.$$

## Bounded Sequence of Rationals

$\{u_n\}_{n \in \mathbb{N}}$  is bounded if  $\exists m \in \mathbb{Q}_+$  such that  $|u_n| \leq m, \forall n$ .

$\{u_n\}_{n \in \mathbb{N}}$  converges to  $a \in \mathbb{Q}$  ( $\lim_{n \rightarrow \infty} u_n = a$ ) if  $\forall \epsilon > 0, \exists N, \forall n \geq N, |u_n - a| < \epsilon$ .

## Famous Limits

### Decaying Rational

1.  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$

- $\forall \epsilon \in \mathbb{Q}_+, \exists n \in \mathbb{N}, 0 < \frac{1}{n} < \epsilon$

- $\forall n \in \mathbb{N}, \exists n \in \mathbb{N}, n \geq N$

– b. and c. are equivalent.

### Decaying Exponential Rational

$r \in \mathbb{Q}, 0 < r < 1, \lim_{n \rightarrow \infty} r^n = 0.$

- Proof: Write  $r = \frac{1}{1+k}$  for some  $k > 0$ . Then  $r^n = \frac{1}{(1+k)^n} \stackrel{\text{Bernoulli}}{\leq} \frac{1}{1+nk}.$

### Geometric

1.  $r \in \mathbb{Q}, 0 < r < 1, u_n = 1 + r + \dots r^n = \frac{1-r^{n+1}}{1-r} \rightarrow \frac{1}{1-r}$

## Features of Limits

### Limits are Unique

If the limit of a sequence exists, it is unique.

### Squeezing Lemma

If  $\{a_n\}, \{b_n\}$  are such that  $0 \leq a_n \leq b_n$ , and  $b_n \rightarrow 0$  as  $n \rightarrow \infty$ , then  $a_n \rightarrow 0$ .

### Limits Preserve Order

If  $a_n \leq b_n \forall n$  and  $a_n$  and  $b_n$  converge, then  $\lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n$ .

### Limit Algebraic Rules

$\lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} (a_n + b_n)$  when  $a_n$  and  $b_n$  converge.

If  $\lim_{n \rightarrow \infty} b_n \neq 0$ , then  $\frac{a_n}{b_n} \rightarrow \frac{\lim a_n}{\lim b_n}.$

## Peculiarity of the Rationals

$\mathbb{Q}$  lacks completeness.

## Examples

Consider  $u_1 = 1$  and  $u_{n+1} = \frac{1}{2}(u_n + \frac{2}{u_n})$ .

Then  $u_n \in \mathbb{Q}$ ,  $\forall n \in \mathbb{N}$ .

It can further be proven, by induction, that  $u_n \geq 1$ ,  $\forall n$ .  $\left(u_{n+1} - 1 = \frac{1}{2}(u_n + \frac{1}{u_n}) - 1 = \frac{1}{2u_n}((u_n - 1)^2 + 1)\right)$ .  
 $\lim_{n \rightarrow \infty} u_n^2 = 2$ .

$$\begin{aligned} u_{n+1}^2 - 2 &= \left(\frac{1}{2}\left(u_n + \frac{2}{u_n}\right)\right)^2 - 2 \\ &= \left(1 \frac{1}{2u_n}(u_n^2 + 2)^2 - 4u_n\right) \\ &= 1 \frac{4}{u_n^2}(u_n^2 - 2)^2 \\ &\leq \frac{1}{4}(u_n^2 - 2)^2 \end{aligned}$$

If  $u_n$  converged in  $\mathbb{Q}$  to  $L$ , by algebraic limit rules,  $2 = \lim u_n^2 = (\lim u_n)^2 = L^2$ , yet  $\sqrt{2} \notin \mathbb{Q}$ .

## Cauchy Criterion

A sequence  $\{u_n\}_{n \in \mathbb{N}}$  of rationals is Cauchy if  $\forall \epsilon > 0$ ,  $\exists n \in \mathbb{N}$ ,  $\forall p, q \geq n$ ,  $|u_p - u_q| < \epsilon$ .

## Visual Justification



## Example 1

The sequence from before is Cauchy.

$$|u_p - u_q| = \frac{|u_p^2 - u_q^2|}{|u_p + u_q|} \leq \frac{1}{2}|u_p^2 - u_q^2|$$

## Example 2

$$u_n = \sum_{k=0}^n \frac{1}{k!} \in \mathbb{Q}.$$

- This is increasing.
- It is bounded above by 3:

$$\begin{aligned} 1 + 1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{2 \cdots n} &\leq 1 + 1 + \cdots \frac{1}{2^{n-1}} \\ &\leq 1 + \frac{1 - 2^{-n}}{1 - \frac{1}{2}} \\ &\leq 3 \end{aligned}$$

## Convergence, Cauchy and Boundedness.

Given a sequence  $\{u_n\}_{n \in \mathbb{N}}$ ,

$\{u_n\}$  converges  $\implies \{u_n\}$  is Cauchy  $\implies \{u_n\}$  is bounded.

Note that in  $\mathbb{Q}$  none of these implications may be reversed.

## Construction of the Real Numbers

Short version: If the limit of a sequence isn't in the set, then define the limit as the sequence itself.

Let  $C_{\mathbb{Q}} = \{\text{Cauchy sequences of rationals.}\}$ .

### Two Operations

- Termwise Addition  $\{u_n\}_n + \{v_n\}_n := \{u_n + v_n\}_{n \in \mathbb{N}}$
- Termwise Multiplication  $\{u_n\}_n \cdot \{v_n\}_n := \{u_n \cdot v_n\}_{n \in \mathbb{N}}$

### Closure of Cauchy Sequence

If  $\{u_n\}_n, \{v_n\}_n \in C_{\mathbb{Q}}$ , then  $\{u_n\}_n + \{v_n\}_n \in C_{\mathbb{Q}}$  and  $\{u_n\}_n \cdot \{v_n\}_n \in C_{\mathbb{Q}}$ .

### Example

Infinite decimal expansion.

Fix  $N \in \mathbb{Z}$ ,  $a_1 \cdots a_n \in \{0, \dots, 9\}$ .

Then let  $u_n = N + \sum_{k=1}^n a_k (10)^{-k}$  (that is the number  $N.a_1 a_2 \dots a_n$ ).

This is always increasing and bounded above by  $N + \sum_{k=1}^n 9 \cdot (10)^{-k} = N + \frac{9}{10} \cdot \sum_{k=1}^n (10)^{-(k+1)} \leq N + 1$ .

Hence, it is Cauchy.

### Increasing and Bounded Above Implies Cauchy

By contrapositive, increasing and not Cauchy implies not bounded.

By the negation of Cauchy and letting  $p \geq q$  without loss of generality, we can force  $u_p > u_q + \epsilon$ .

### Negation of Cauchy

$\exists \epsilon > 0, \forall N, \exists p, q \geq N, |u_p - u_q| > \epsilon$ .

## Real Numbers as Equivalence Classes of Cauchy Sequences

On  $C_{\mathbb{Q}}$  define the relation  $\{x_n\}_n \sim \{y_n\}_n$  if and only if  $\lim_{n \rightarrow \infty} |x_n - y_n| = 0$ .

### Equivalence Relation

Reflexive:  $x_n - x_n = 0$

Transitive: Uses algebraic limit rules.  $x_n - z_n = x_n - y_n + y_n - z_n$ .

Symmetric.

## Definition of the Reals

$\mathbb{R} := C_{\mathbb{Q}} / \sim$

Then  $x \in \mathbb{R}$ ,  $x = [\{x_n\}_n]$ .

## Addition and Multiplication of Reals

- Addition  $x + y := [\{x_n + y_n\}_n]$ .
- Multiplication  $x \cdot y := [\{x_n \cdot y_n\}_n]$ .

## Operations Do Not Depend on Choice of Representative

If  $\{x_n\}_n \sim \{x'_n\}_n$  and  $\{y_n\}_n \sim \{y'_n\}_n$ , then  $\{x_n\}_n + \{y_n\}_n \sim \{x'_n\}_n + \{y'_n\}_n$ .

If  $\{x_n\}_n \sim \{x'_n\}_n$  and  $\{y_n\}_n \sim \{y'_n\}_n$ , then  $\{x_n\}_n \cdot \{y_n\}_n \sim \{x'_n\}_n \cdot \{y'_n\}_n$ .

## The Reals are a Field

There are nine properties to check, eight of which are “obvious”:

### Commutativity of Addition (and Other “Obvious” Features)

$$[\{x_n\}_n] + [\{y_n\}_n] = [\{x_n + y_n\}_n] = [\{y_n + x_n\}_n] = [\{y_n\}_n] + [\{x_n\}_n]$$

That is, the Reals inherit most field features from the Rationals.

- Zero Element  $0_{\mathbb{R}} = [\{0_{\mathbb{Q}}\}_n]$
- One Element  $1_{\mathbb{R}} = [\{1_{\mathbb{Q}}\}_n]$

## Multiplicative Inverses

How to define  $x^{-1}$  for  $x \in \mathbb{R}$  where  $x \neq 0$ ?

- Idea If  $x = [\{x_n\}_n]$  choose  $x^{-1} = [\{\frac{1}{x_n}\}_n]$ .  
If  $x \in \mathbb{R}$ ,  $x \neq 0$  then

1.  $\exists \{x_n\}_n \in C_{\mathbb{Q}}$  representing  $x$  with non zero entries.
  2.  $\{\frac{1}{x_n}\}_n$  is Cauchy.
- Proof of 1 Pick any  $\{x_n\}_n$  representing  $x$ .

\*  $x \neq 0$ , so NOT  $(\lim_{n \rightarrow \infty} x_n = 0: \exists \epsilon_0 > 0, \forall N, \exists n \geq N, |x_n| > \epsilon_0)$ .

\*  $\{x_n\}$  is Cauchy:  $\forall \epsilon > 0, \exists N, \forall p, q \geq N, |x_p - x_q| < \epsilon$ .

Therefore,  $\exists N$  such that  $\forall p, q \geq N_1, |x_p - x_q| < \frac{\epsilon_0}{2}$

And  $\exists N_2 \geq N, |x_{N_2}| > \epsilon_0$ .

For  $q \geq N_2$ , the Cauchy Criterion states that  $|x_q| = |x_q - x_{N_2} + x_{N_2}| \geq |x_{N_2}| - |x_{N_2} - x_q| \geq \epsilon_0 - \frac{\epsilon_0}{2} \geq \frac{\epsilon_0}{2}$ .

Therefore, the sought sequence is  $\{x_{N_2} + k\}_{k \in \mathbb{N}}$ .

– Proof of  $2 \left| \frac{1}{x_p} - \frac{1}{x_q} \right| = \frac{|x_p - x_q|}{|x_p||x_q|} \leq \frac{4}{\epsilon_0^2} |x_p - x_q|$ .

## Order on the Reals

Let  $x \neq 0$ ,  $\exists \{x_n\}_{n \in \mathbb{N}}$  be a representation of  $x$  and  $\epsilon_0 > 0$ .

Then for  $|x_n| > \epsilon_0$ ,  $\forall n \in \mathbb{N}$ , there is a dichotomy:

- Either  $\exists N \in \mathbb{N}$ ,  $x_n > \epsilon_0$ ,  $\forall n \geq N$  (in which case we write  $x > 0$ )
- Or  $\exists N \in \mathbb{N}$ ,  $x_n < -\epsilon_0$ ,  $\forall n \geq N$  (in which case we write  $x < 0$ )

Thus the Reals are totally ordered.

## October 4, 2023

### Overview

Completeness of  $\mathbb{R}$ .

Topology of the Real Line.

### Non-zero Reals Are Either Positive or Negative

Given  $x \in \mathbb{R} \setminus \{0\}$ ,  $\exists \delta \in \mathbb{Q}_+$  such that  $\forall \{x_n\}_n$  representing  $x$ ,  $\exists N \in \mathbb{N}$  such that  $|x_n| > \delta$ ,  $\forall n \geq N$ .

Moreover, one of the following (but not both) holds:

1.  $\forall \{x_n\}_n \in x$ ,  $\exists, x_n > \delta$ ,  $\forall n \geq N$  (i.e.  $x > 0$ )
2.  $\forall \{x_n\}_n \in x$ ,  $\exists, x_n < -\delta$ ,  $\forall n \geq N$  (i.e.  $x < 0$ )

Recall that  $x \in \mathbb{R} \setminus \{0\}$  is an equivalence class of Cauchy sequences.

### Total Ordering of the Reals

$x > 0$  produces a total ordering of  $\mathbb{R}$  where  $x < y$  if and only if  $y - x > 0$ .

$$\leadsto \max(x, y) = \begin{cases} x & \text{if } x > y \\ y & \text{otherwise} \end{cases}$$

$|x| = \max(x, -x)$  (which satisfies the triangle inequality)

### Lemma A

Let  $x, y \in \mathbb{R}$ . If  $\{x_n\}_n, \{y_n\}_n$  represent  $x, y$  and satisfy  $x_n < y_n$ ,  $\exists N \in \mathbb{N}$ ,  $\forall n \geq N$ , then  $x \leq y$ .

- Proof By contradiction, suppose  $x > y$  and  $\exists \{x_n\}_n, \{y_n\}_n$  representing  $x, y$  such that  $x_n \leq y_n$ ,  $\forall n \geq N_1$ . Then, by definition,  $x - y > 0 \implies \exists \delta > 0$ ,  $\exists N_2$ ,  $x_n - y_n > \delta$  for  $n \geq N_2$ . But  $x_n \leq y_n$  contradicts  $x_n - y_n > \delta$ .

### Sequences of Reals

$\{x_n\}_n$ ,  $x_n \in \mathbb{R}$

The definition of bounded, convergent and Cauchy sequences are the same as in  $\mathbb{Q}$ .

### Injection of Rationals

$\iota : \mathbb{Q} \rightarrow \mathbb{R}$  such that  $r \mapsto [\{u_n = r\}_n]$

This is isometric in the sense that  $|\iota(r) - \iota(s)|_{\mathbb{R}} = |r - s|_{\mathbb{Q}}$

### Theorem (Completeness 1)

Let  $\{x_n\}_n \in C_{\mathbb{Q}}$  and  $x = [\{x_n\}_n]$ , then  $\{\iota(x_n)\}_n$  converges to  $x$ .

### Proof

What to show:  $\forall \epsilon > 0$ ,  $\exists N$ ,  $\forall n \geq N$ ,  $|\iota(x_n) - x| < \epsilon$ .

Let  $\epsilon \in \mathbb{Q}_+$ . By the Cauchy criterion,  $\exists N$ ,  $\forall q, p \geq N$ ,  $|x_p - x_q| < \epsilon$ .

This is equivalent to  $x_q - \epsilon \leq x_p \leq x_q + \epsilon$  where  $p$  is frozen.

Then by Lemma A,  $x - \epsilon \leq \iota(x_p) \leq x + \epsilon$ .

It follows that  $\forall p \geq N$ ,  $|\iota(x_p) - x| \leq \epsilon$ .

### Corollary

$\mathbb{Q} \cong \iota(\mathbb{Q})$  is dense in  $\mathbb{R}$ . That is,  $\forall \epsilon > 0$ ,  $\forall x \in \mathbb{R}$ ,  $\exists r \in \mathbb{Q}$ ,  $|\iota(r) - x| < \epsilon$ .

### The Isometric Copy of Rationals

For brevity, the  $\iota$  notation will be dropped and the  $\mathbb{Q}$  will be understood as  $\iota(\mathbb{Q})$ .

### Completeness of the Real Numbers

A sequence of real numbers converges in  $\mathbb{R}$  if and only if it is Cauchy.

### Proof

( $\implies$ ) This is clear.

( $\impliedby$ ) Take a Cauchy sequence of reals  $\{x_n\}_n$ . Then  $\forall \epsilon > 0$ ,  $\exists N$ ,  $\forall p, q \geq N$ ,  $|x_p - x_q| < \epsilon$ .

Using the density of  $\mathbb{Q}$ ,  $\forall n \in \mathbb{N}$ ,  $\exists r_n \in \mathbb{Q}$  such that  $|x_n - r_n| < \frac{1}{n}$ .



Claim:  $\{r_n\}_n$  is Cauchy. Indeed,

$$\begin{aligned} |r_p - r_q| &= |r_p - x_p + x_p - x_q + x_q - r_q| \\ &\leq |r_p - x_p| + |x_p - x_q| + |x_q - r_q| \\ &\leq \frac{1}{p} + |x_p - x_q| + \frac{1}{q} \end{aligned}$$

Take  $\epsilon > 0$ .  $\{x_n\}$  cauchy implies  $\exists N_1, \forall p, q \geq N, |x_p - x_q| \leq \frac{\epsilon}{3}$  and  $\exists N_2, \forall p, q \geq N_2, \frac{1}{p} \leq \frac{\epsilon}{3}, \frac{1}{q} \leq \frac{\epsilon}{3}$  for  $p, q \geq \max(N_1, N_2)$   $|r_p - r_q| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$ .

Then, for Cauchy  $\{r_n\}_n$ , call  $r = [\{r_n\}_n]$ , then  $\lim_{n \rightarrow \infty} r_n = r$  by the above theorem.

Then my algebraic limit rules,  $x_n(x_n - r_n) + r_n$  where  $(x_n - r_n) \rightarrow 0$  and  $r_n \rightarrow r$  as  $n \rightarrow \infty$ . So  $\{x_n\}$  converges.

### Example

Let  $x_1 = 1, x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n})$ .

Then  $\{x_n\}_n \in C_{\mathbb{Q}}$ , and it converges to  $L \in \mathbb{R}$ .

By algebraic limit rules,  $L^2(\lim x_n)^2 = \lim x_n^2 = 2$ .

## Subsets of the Reals, Infimum and Supremum

### Notation

Subset:  $S \subseteq \mathbb{R}$

Inclusion:  $x \in S$

Open Interval:  $(a, b) = \{x \in \mathbb{R} | a < x < b\}$

Semiclosed Interval:  $(a, b] = \{x \in \mathbb{R} | a < x \leq b\}$

Closed Interval:  $[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$

Unbounded Semiclosed Interval:  $(-\infty, a] = \{x \in \mathbb{R} | x \leq a\}$

Unbounded Open:  $(-\infty, a) = \{x \in \mathbb{R} | x < a\}$

### Supremum

$S \subseteq \mathbb{R}$  is bounded above (respectively below) if  $\exists M \in \mathbb{R}, \forall x \in S, x \leq M$  (respectively  $\exists L \in \mathbb{R}, \forall x \in S, L \leq x$ )

$S$  admits a least upper bound, LUB, supremum or  $\sup M$  if

1.  $\forall x \in S, x \leq M$

2.  $\forall M' \in \mathbb{R}, \text{upper bound of } S, M \leq M'$

If  $\sup S$  exists, it is unique.

If  $x \in S$  and  $x$  is an upper bound for  $S$ , then  $x = \sup S$ .

### Example 1

$$\sup(0, 1) = \sup[0, 1] = 1$$

### Example 2

$S = \{x \in \mathbb{Q}, x^2 < 2\}$  does not have a greatest element in  $\mathbb{Q}$ , nor a least upper bound in  $\mathbb{Q}$ .

### Theorem (Completeness 2)

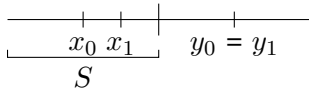
Every subset  $S \subseteq \mathbb{R}$ , nonempty and bounded above, has a supremum in  $\mathbb{R}$ .

#### Proof

By dichotomy.

$S \neq \emptyset \implies \exists x_0 \in S$  and  $S$  bounded above implies  $\exists y_0 \in \mathbb{R}, \forall x \in S, x \leq y_0$  (in particular  $x_0 \leq y_0$ ).

If  $x_0 = y_0$ , done. Otherwise, consider  $m_0 = \frac{x_0 + y_0}{2}$ .



Two options exist: if  $m_0$  is an upper bound for  $S$ , set  $y_1 = m_0$  and  $x_1 = x_0$ .

Otherwise,  $\exists x_1 \in S$ , such that  $m_0 < x_1$  so set  $y_1 = y_0$ .

Repeat this process forever to construct two sequences  $x_n, y_n$ .

$\forall n, x_n \in S, y_n$  is an upper bound for  $S$ .

- $x_n \leq y_n$
- $x_n$  is increasing and bounded above by  $y_0$ , so it must be Cauchy and converging to  $x$ .
- $y_n$  is decreasing and bounded below by  $x_0$ , so it must be Cauchy and converging to  $y$ .
- $|x_{n+1} - y_{n+1}| \leq \frac{|x_n - y_n|}{2}$  which implies  $|x_n - y_n| \leq \frac{1}{2^n} |x_0 - y_0|$  and  $x = y = z$ .

Therefore, the process may be understood as  $x_0 \leq \dots \leq x_n \leq x_{n+1} \leq y_{n+1} \leq y_n \leq \dots \leq y_0$ .

There remain two things to check: (1)  $z$  is an upper bound for  $S$  and (2)  $z$  is no larger than any other upper bound for  $S$ .

1. Take  $x \in S, \forall n, x \leq y_n \xrightarrow{n \rightarrow \infty} x \leq z$ .
2. Take upper bound for  $S, z', x_n \leq z', \forall n \xrightarrow{n \rightarrow \infty} z \leq z'$ .

So  $z = \sup S$ .

### Monotone Convergence Theorem (Completeness 3)

An increasing sequence of reals,  $\{x_n\}_n$ , that is bounded above, converges to  $\sup X = \sup\{x_n | n \in \mathbb{N}\}$ .

To prove that this converges, since it is monotone and bounded above it is Cauchy and therefore must be convergent.

### Proof

Call  $x$  the limit, then  $\forall n, x_n \leq x$ . To see this, suppose  $\exists n_0, x < x_{n_0}$  then  $\forall m \geq m_0, x < x_{m_0} \leq x_m \implies |x_m - x| \geq |x_{n_0} - x| > 0, \forall m \geq n_0$  is a contradiction.

Let  $M$  be an upper bound of  $X$ . Then  $x_n \leq M, \forall n \xrightarrow{n \rightarrow \infty} x \leq M \implies x = \sup X$ .

### Theorem (Existence of Roots)

$\forall x \in \mathbb{R}$  where  $x > 0, p \in \{2, 3, \dots\}, \exists! y > 0$  such that  $y^p = x$ .

### Proof

Left as an exercise.

Either by dichotomy or consider  $S = \{y \in \mathbb{R} | y^p < x\}$ , show:  $S \neq \emptyset$ , bounded above and  $(\sup S)^p = x$ .

For uniqueness, show  $y_1^p = y_2^p = x \iff 0 = y_1^p - y_2^p = (y_1 - y_2)(\dots \neq 0) \implies y_1 = y_2$ .

### Topological Properties

$S \subseteq \mathbb{R}$  is open if  $\forall x \in S, \exists a, b \in \mathbb{R}, x \in (a, b) \subset S$ .

$x$  is an accumulation or limit point of  $S$  if  $\forall \epsilon > 0, \exists y \in S, 0 < |x - y| < \epsilon$ .

$S \subseteq \mathbb{R}$  is closed if it contains all its limit points.

A set may be both open closed, just open, just closed or neither.

Given  $S \subseteq \mathbb{R}$ , the interior of  $S$  is  $\bigcup_{S' \text{ open} \subset S} S' = S^{\text{int}} = S^0$ .

The closure is  $\bigcap_{F \text{ closed} \supseteq S} F = \overline{S} \stackrel{\text{wts}}{=} S \cup \{\text{limit points of } S\}$ .

### Example

$\{x\}$  is not open, but, since the limit points of  $x$  are  $\emptyset$ , it is closed.

### Propositions

1. Arbitrary unions and finite intersections of open sets are open.
2.  $S$  is open if and only the complement  $S^c = \mathbb{R} \setminus S$  is closed.
3. Arbitrary intersections and finite unions of closed sets are closed.

### Bolzano-Weierstrass Theorem

A bounded sequence in  $\mathbb{R}$  admits a convergent (Cauchy) subsequence.  $\exists M, |x_n| \leq M, \forall n$

### Proof by Dichotomy

Suppose  $I_0 = [a, b]$  contains the sequence.

Construct a sequence of intervals by indicators: if  $\left[a, \frac{a+b}{2}\right]$  contains infinitely terms of  $\{x_n\}_n$ , choose  $n$  such that  $x_{n_1} \in \left[a, \frac{a+b}{2}\right]$  and call  $I_1 = \left[a, \frac{a+b}{2}\right]$ .

Otherwise,  $\left[\frac{a+b}{2}, b\right]$  must contain infinitely many terms. Choose  $n$  in a similar fashion as above such that  $I_1 = \left[\frac{a+b}{2}, b\right]$ .

This process may be repeated to create a sequence of intervals such that  $I_k \supseteq I_{k+1} \supseteq I_{k+2}$  and  $l(I_k) = \frac{b-a}{2^k}$ . A subsequence  $\{u_{n_k}\}_k$  such that  $u_{n_k} \in I_l$  for  $k \geq l$ .

## Exercise

Extract a Cauchy criterion out of the above.

## October 9, 2023

### Overview

- Topology of  $\mathbb{R}$  continued.
- Numerical series.

Next Wednesday

- Absolute and Conditional Convergence.
- Rearrangement theorem.

## Last Time

Finished with Bolzano-Weierstrass.

## Limits

### Limit Point

We say  $x \in \mathbb{R}$  is a limit point of  $\{x_n\}_n$  if a subsequence of  $\{x_n\}_n$  converges to  $x$ .

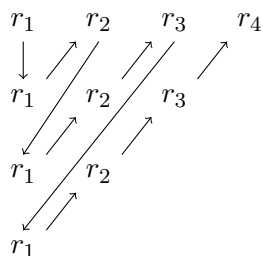
Equivalently,  $\forall \epsilon > 0, \forall n_0 \in \mathbb{N}, \exists n \geq n_0, |x_n - x| < \epsilon$ .

That is, the sequence revisits an epsilon neighborhood of  $x$  infinitely many times.

### Limit Set

The limit set of  $\{x_n\}_n$  :  $LS(\{x_n\}_n)$  = the set of limit points of  $\{x_n\}_n$ .

- Comments
  - if  $\lim_{n \rightarrow \infty} \{x_n\} = x$ , then  $LS(\{x_n\}_n) = \{x\}$ .
  - The limit set can be as big as  $\mathbb{R}$ !



– What Bolzano-Weierstrass says is that if  $\{x_n\}$  is bounded, then  $\text{LS}(\{x_n\}) \neq \emptyset$ .

- Examples  $\text{LS}(\{x_n\}) = \emptyset$ .  
 $\text{LS}(\{x_n\})$  is closed (good exercise).

## Limit Superior

If  $\{x_n\}_n \in [a, b]$  is bounded,  $\forall k \in \mathbb{N}$ ,  $\sup\{x_j | j \geq k\}$  exists in  $\mathbb{R}$ .

Because

$$a \leq \sup\{x_j | j \geq k+1\} = y_{k+1} \leq \sup\{x_j | j \geq k\} = y_k$$

by the Monotone Convergence Theorem,  $\{y_k\}_k$  converges. Call its limit  $\limsup_n x_n = \inf_n \sup\{x_j | j \geq n\}$ .

## Limit Inferior

Similarly, define  $\liminf_n x_n = \sup_n \inf\{x_j | j \geq n\}$ .

## Limit Superior and Limit Inferior Always Exist

What to show:  $\limsup x_n, \liminf x_n \in \text{LS}(\{x_n\})$ .

Left as an exercise.

## Convergence at the Limit

A bounded sequence  $\{x_n\}_n$  converges if and only if  $\liminf_n x_n = \limsup_n x_n$ .

- Proof Technique Often it is useful to structure a proof such that

$$L < \liminf_n x_n \leq \limsup_n x_n < L$$

## Topology of the Reals Continued

### Compactness

Let  $A \subseteq \mathbb{R}$ .

$A$  is (sequentially) compact if every sequence in  $A$  has a limit point in  $A$ .

$A$  is (Heine-Borel) compact if every open cover of  $A$  has a finite subcover.

- Open Cover  $\{O_\alpha\}_{\alpha \in I}$ , with  $O_\alpha$  open, is an open cover of  $A$  if  $A \subseteq \bigcup_{\alpha \in I} O_\alpha$ .
- Finite Subcover  $O_1, \dots, O_n, n \in \mathbb{N}$ .

## Heine-Borel Theorem

Let  $A \subseteq \mathbb{R}$ .

The following are equivalent

1.  $A$  is Heine-Borel compact.
2.  $A$  is closed and bounded.
3.  $A$  is sequentially compact.

## Proof

(1)  $\implies$  (2)  $\implies$  (3)  $\implies$  (1)

- Heine-Borel Compact Implies Closed and Bounded Suppose  $A$  satisfies the Heine-Borel property.  
Consider  $\{(-n, n)\}_{n \in \mathbb{N}}$ . Clearly  $\bigcup_n (-n, n) = \mathbb{R} \supseteq A$ .  
By Heine-Borel,  $\exists n_0, \dots, n_p$  such that  $A \subseteq \bigcup_{j=0}^p (-n_j, n_j) = (-N, N)$ ,  $N = \max(n_0, \dots, n_p)$ . So  $A$  is bounded.  
 $A$  is closed if  $y \notin A \implies y$  is not a limit point of  $A$ .  
Take  $y \in A^c$ , then  $A \subseteq \mathbb{R} \setminus \{y\} = \bigcup_{n \in \mathbb{N}} (-\infty, y - \frac{1}{n}) \cup (y + \frac{1}{n}, \infty)$ .  
By the Heine-Borel property,

$$\begin{aligned} A &\subseteq \bigcup_{n_0, \dots, n_p} (-\infty, y - \frac{1}{n}) \cup (y + \frac{1}{n}, \infty) \\ &= (-\infty, y - \frac{1}{N}) \cup (y + \frac{1}{N}, \infty) \end{aligned}$$

Which implies  $A \cap [y - \frac{1}{N}, y + \frac{1}{N}] = \emptyset$  and  $y$  is not a limit point of  $A$ .  
That is,  $A$  contains its limit points.

- Closed and Bounded Implies Sequential Compactness Suppose  $A$  is both closed and bounded.  
Let  $\{x_n\}_n \in A$ . Then  $\{x_n\}_n$  is bounded. By Bolzano-Weierstrass, it has a limit point  $x$  and a subsequence  $\{x_{n_k}\}_k$  converging to  $x$ .  
Since  $A$  is closed,  $\lim_{k \rightarrow \infty} x_{n_k} = x \in A$ . ■
- Sequential Compactness Implies Heine-Borel Suppose  $A \subseteq \mathbb{R}$  is sequentially compact.  
Consider an open cover of  $A$ ,  $\{O_\alpha | \alpha \in I\}$ .  
First, turn it into a countable cover:

$$- \forall \alpha \in I, O_\alpha \subseteq (r_\alpha^1, r_\alpha^2), r_\alpha^1, r_\alpha^2 \in \mathbb{Q}$$

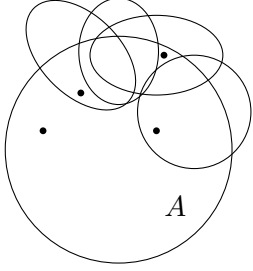
Assume that  $\{O_\alpha\}_\alpha$  can be made countable  $(O_1, \dots, O_n)$

By contradiction, suppose  $\forall n \in \mathbb{N}, A \setminus \left(\bigcup_{j=1}^n O_j\right) \neq \emptyset$ .

Take  $x_n \in A \setminus \left(\bigcup_{j=1}^n O_j\right)$ . Since  $A$  is sequentially compact,  $\exists \{x_{n_k}\}_k$  subsequence of  $\{x_n\}_n$  converging to  $x \in A$ .

Since  $A \subset \bigcup_{j \in \mathbb{N}} O_j$ ,  $\exists j_0, x \in O_{j_0}$ ,  $O_{j_0}$  is open:  $\exists \delta > 0, (x - \delta, x + \delta) \subseteq O_{j_0}$ .

Then  $\exists N, k \geq N \implies x_{n_k} \in (x - \delta, x + \delta) \subseteq O_{j_0}$ . But if  $k$  is such that  $n_k > j_0$ , we also have  $x_{n_k} \notin O_{j_0}$  which is a contradiction!



## Structure of Open and Closed Sets

$A$  is open in  $\mathbb{R}$  if and only if it can be written as an at most countable, disjoint union of open intervals.

### TODO Proof

For  $x \in A$ ,  $\exists (a, b)$ , such that  $x \in (a, b) \subseteq A$ .

Let  $I_x = \bigcup_{\text{open interval containing } x, I \subseteq A} I$ . This is the maximal interval containing  $x$  in  $A$ .

Then,  $A \subseteq \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} I_x \subseteq A$ .

That is,  $A = \bigcup_{x \in A} I_x$  (\*).

Next, if  $x, y \in A$ , then  $\begin{cases} \text{either } I_x = I_y \\ \text{or } I_x \cap I_y = \emptyset \end{cases}$

IMAGE HERE

The union (\*), as a disjoint union, is at most countable because each distinct one must contain a distinct rational and  $\mathbb{Q}$  is countable.

### Long Story Short

The topology of the reals avoids exotic sets. Closed sets, on the other hand, can be quite complicated.

### TODO Cantor Set

$C := \bigcap_{k \in \mathbb{N}_0} I_k$ .  $I_{k+1}$  is obtained by removing the middle open third of each interval making  $I_k$ .

IMAGE HERE - CANTOR

$I_0 = [0, 1]$ . One interval of length 1.

$I_1 = [0, 1/3] \cup [2/3, 1]$ . Two intervals of length  $2/3$ .

$I_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$ . Four intervals of  $(2/3)^2$

$I_k$  is  $2^k$  intervals of length  $(2/3)^k$ .

$I_{k+1} \subseteq I_k \implies C \subseteq I_k, \forall k \implies l(C) \leq l(I_k) = (2/3)^k \implies l(C) = 0$ .

### TODO Triadic Expansions

Goal:

1.  $C$  is perfect (i.e. every point in  $C$  is a limit point of  $C$ ).
2.  $C$  contains no open intervals.

Property 2 is easy because  $C \subseteq I_k$ , which does contain interval of length greater than  $(1/3)^k$ .

1.  $C$  is uncountable.

Every  $x \in [0, 1]$  can be written in the form  $x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$ ,  $a_k \in \{0, 1, 2\}$ .

That is,  $x = 0.a_1a_2\dots$  in base 3. This is not always unique (e.g.  $1/3 = 0.100\dots = 0.022\dots$ ).

IMAGE HERE - THIRDS OF INTERVAL

Note that the Cantor set is removing all decimal expansions with leading 1s. That is,  $x \in C$  if and only if it has a triadic expansion only made of 0s and 2s.

- Proof of 1 If  $x \in C$ ,  $x = \sum_{k \geq 1} \frac{a_k}{3^k} = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{a_k}{3^k}$ , then  $x_n \in C$ ,  $\forall n$  and  $x_n = 0.a_1\dots a_n0000\dots$  where  $a_1, a_n \in \{0, 2\}$ .

Unique representation can be maintained by forcing the behavior of the  $n + 1$ th digit.

- Proof of 3 Every point in  $[0, 1]$  can also be written as  $x = \sum_{n=1}^{\infty} \frac{b_n}{2^n}$ ,  $b_n \in \{0, 1\}$  (i.e. a binary expansion). Then  $C \mapsto [0, 1]$  gives  $x = \sum \frac{a_k}{3^k} \mapsto \sum \frac{b_k}{2^k}$ ,  $b_k = \frac{a_k}{2}$  for  $a_k \in \{0, 2\}$  is a bijection!

## October 11, 2023

### Overview: Numeric Series

- Series with non-negative terms.
- Series with general terms.
- Convergence criteria.
- Algebraic rules.
- Rearrangements.

### General Notation

Sequence  $\{x_n\}_{n \geq n_0}$  (often  $n_0 \in \{0, 1\}$ )

### Definition: Partial Sum

$$S_n = \sum_{k=n_0}^n x_k \quad (x_n = S_n - S_{n-1})$$

We say  $\sum_n x_n$  converges if  $\lim_{n \rightarrow \infty} S_n$  exists.

We denote  $\sum_{k=n_0}^{\infty} x_k = \lim_{n \rightarrow \infty} S_n$



- Example: Geometric Series  $\sum_{k=0}^n r^k = S_n$ ,  $r \in (0, 1)$   
 $\frac{1-r^{n+1}}{1-r} \rightarrow \frac{1}{1-r}$
- Example: P Series  $\sum_{k=1}^n \frac{1}{k^p}$ ,  $p > 0$
- Example: Exponential  $\sum_{k=0}^n \frac{1}{k!}$

### Series without Non-negative Terms

The series has non-negative terms if  $x_n \geq 0$ ,  $\forall n$ .

### Obvious Algebraic Limit Rules

If  $\sum_{n \geq n_0} a_n$  and  $\sum_{n \geq n_0} b_n$  converge and  $\alpha \in \mathbb{R}$ , then  $\sum_{n \geq n_0} (a_n + \alpha b_n)$  converges to

$$\sum_{n=n_0}^{\infty} a_n + \alpha \sum_{n=n_0}^{\infty} b_n = \sum_{n=n_0}^{\infty} (a_n + \alpha b_n)$$

- Proof (Sketch) Reason on the partial sums; use algebraic limit rules on sequences.

### Proposition

If  $\sum_n x_n$  converges in  $\mathbb{R}$ , then  $\lim_{n \rightarrow \infty} x_n = 0$ .

- Proof  $x_n = S_n - S_{n-1} \xrightarrow{n \rightarrow \infty} S - S = 0$   
 Since  $S_n \xrightarrow{n \rightarrow \infty} S$  and  $S_{n-1} \xrightarrow{n \rightarrow \infty} S = \sum_{n=n_0}^{\infty} x_n$ .

### Series with Non-negative Terms

If  $x_n \geq 0$ ,  $\forall n$ ,  $S_n = \sum_{k=n_0}^n x_k$  is non-decreasing.

By monotone convergence theorem,  $S_n$  is either bounded, and therefore converges, or unbounded from above where

$$\forall m > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, S_n \geq m$$

This is “diverging to  $+\infty$ .”

### Theorem: Convergence Criteria

- Term Test If  $0 \leq a_n \leq b_n$ ,  $\forall n \geq n_0$  and  $\sum_n b_n$  converges, then  $\sum_n a_n$  converges.
  - Proof Suppose  $0 \leq a_n \leq b_n$ , and  $t_n = \sum_{k=n_0}^n b_k$  converges and, therefore, is bounded above by  $B = \sum_{k=n_0}^{\infty} b_k$ .  
 Then  $\forall n$ ,  $\sum_{k=n_0}^n a_k \leq \sum_{k=n_0}^n b_k \leq B$ .  
 Thus, by monotone convergence theorem,  $\sum_{k=n_0}^{\infty} a_k$  converges.

- Ratio Test If  $a_n > 0$ ,  $\forall n$  and  $\exists n_0 \in \mathbb{R}$  such that  $\frac{a_{n+1}}{a_n} \leq r < 1$ ,  $\forall n \geq n_0$ , then  $\sum_n a_n$  converges.
  - Clarification The harmonic series has ratio  $\frac{k}{k+1} < 1$  but since  $\frac{k}{k+1} \xrightarrow{k \rightarrow \infty} 1$ , there is no  $r$  which satisfies the ratio test.
  - Proof Suppose  $a_{n+1} \leq r a_n$  for  $n \geq n_0$ .  
Then  $a_{m_0+p} \leq a_{m_0+(p-1)} r \leq a_{m_0+(p-2)} r^2 \leq \dots \leq a_{m_0} r^p$ .  
Then for  $n \geq n_0$ ,
 
$$\sum_{k=n_0}^n a_k = \sum_{k=n_0}^{m_0} a_k + \sum_{k=m_0+1}^n a_k \leq \sum_{k=m_0}^{m_0+(n-m_0)} a_{m_0} r^{n-m_0} \leq a_{m_0} \sum_{k=m_0}^{n-m_0} r^{n-m_0} \leq \frac{1}{1-r}$$
  - Rate of Convergnce The above proof shows that the ratio test implies a geometric rate of convergence.
- Root Test If  $\exists n_0 \in \mathbb{N}$  and  $r \in (0, 1)$  such that  $a_n^{1/n} \leq r$ , then  $\sum_n a_n$  converges.
  - Proof (Sketch) Same story as the ratio test:  $a_n^{1/n} \leq r \implies a_n \leq r^n$ .
- Rejection of Ratio/Root If  $\exists n_0 \in \mathbb{N}$  such that either  $\frac{a_{n+1}}{a_n} \geq 1$  for  $n \geq n_0$  or  $a_n^{1/n} \geq 1$  for  $n \geq n_0$ , then  $\sum_n a_n$  diverges to  $+\infty$ .
  - Proof (Sketch) In either case,  $a_n$  cannot converge to zero. Therefore the series cannot converge.

## Prototype Scales

### Geometric Rates

$\sum_{n \geq 1} \frac{1}{n^\alpha}$  converges if and only if  $\alpha > 1$  (to  $\zeta(\alpha)$ )

$$a_k = \frac{1}{k^\alpha} \rightsquigarrow 2^k a_{2^k} = \frac{2^k}{2^{k\alpha}} = \left(\frac{1}{2^{\alpha-1}}\right)^k \implies t_n = \sum_{k=1}^n 2^k a_{2^k} \text{ converges if and only if } \frac{1}{2^{\alpha-1}} < 1 \text{ if and only if } \alpha > 1.$$

### Log Geometric Case

$\sum_{n \geq 1} \frac{1}{n(\log(n))^\beta}$  converges if and only if  $\beta > 1$ .

$$a_k = \frac{1}{k(\log(k))^\beta} \rightsquigarrow 2^k a_{2^k} = \frac{2^k}{2^k (\log(2^k))^\beta} = \frac{1}{(\log(2))^\beta k^\beta} \text{ converges if and only if } \beta > 1.$$

### Lemma:

Suppose  $a_n$  decreases to 0.

Then the sequence  $S_n = \sum_{k=1}^n a_k$  converges if and only if  $t_n = \sum_{k=1}^n 2^k a_{2^k}$  converges.

$$S_{2^n} = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \dots$$

• Proof

$$a_3 + a_3 \leq \underbrace{\quad} \leq a_2 + a_3$$

$$S_n = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \dots$$

$$= a_1 + \sum_{k=1}^n \sum_{p=1}^{2^k-1} a_{2^k+p}$$

$$\leq a_1 + 2^k a_{2^{k+1}} +$$

This gives

$$\frac{1}{2}(t_n - a_1) \leq S_{2^n} - a_1 \leq t_{n-1}$$

Therefore  $S_{2^n}$  converges, which implies that  $t_n$  converges, and, since  $S_n$  is monotone,  $S_n$  itself converges.

## Series with General Terms

General term is signed.

### Trick

Write  $a_n = a_n^+ - a_n^-$  and  $a_n^\pm = \max(0, \pm a)$ . Then

$$S_n = \sum_{k=n_0}^n a_k = \left( \sum_{k=n_0}^n a_k^+ \right) - \left( \sum_{k=n_0}^n a_k^- \right)$$

### Convergence Outcomes

	$\sum_{k=n_0}^\infty a_k^+ < \infty$	$\sum_{k=n_0}^\infty a_k^+ = \infty$	If
$\sum_{k=n_0}^\infty a_k^- < \infty$	absolute convergence	$\lim S_n = +\infty$	
$\sum_{k=n_0}^\infty a_k^+ = \infty$	$\lim S_n = -\infty$	lots of things can happen; divergence, convergence, any limit sequence	

$S_n^+$  and  $S_n^-$  converge, we can return to algebraic limit rules.

$S_n$  converges to  $\lim_{n \rightarrow \infty} S_n^+ - \lim_{n \rightarrow \infty} S_n^-$

### Definition: Absolute Convergence

We say  $\sum_n a_n$  converges absolutely if and only if  $\sum_n |a_n|$  converges.

### Note

$$|a_n| = a_n^+ + a_n^-$$

### Proposition: Absolute Convergence Implies Convergence

#### Proof

Absolute convergence  $\implies \sum |a_n|$  converges  $\implies \sum a_n^+$  and  $\sum a_n^-$  converges  $\implies \sum (a_n^+ - a_n^-)$  converges.

### Definition: Conditional Convergence

$\sum_n a_n$  converges conditionally if and only if  $\sum_n a_n$  converges while  $\sum_n |a_n|$  diverges.

### Criteria for Convergence

For absolute convergence, run root/ratio/term test on  $\sum_n |a_n|$ .  
Other criteria which might indicate conditional convergence.

### Alternating Series Test

If  $a_n(-1)^n b_n$ ,  $b_n \geq 0$  decreases to zero, the series is conditionally convergent.

—++•++—

### Dirichlet Test

If  $a_n = b_n c_n$ , where  $b_n$  decreases to zero and  $c_n$  satisfies  $|c_0 + c_1 + \dots + c_n| \leq C$ ,  $\exists C \in \mathbb{R}, \forall n \in \mathbb{N}$ , then  $\sum_{n \geq 0} a_n$  converges conditionally.

- Applications  $\sum_{n \geq 1} \frac{(-1)^n}{n}$   
 $\sum_{n \geq 1} \frac{\cos(n)}{n}$
- Proof Write  $C_n = c_0 + c_1 + \dots + c_n$ , such that  $|C_n| \leq C, \forall n$ .  
Then  $c_n = C_n - C_{n-1}$ , and

$$\sum_{k=0}^n b_k c_k = \sum_{k=0}^n b_k (C_k - C_{k-1}) = \sum_{k=0}^n b_k C_k - \sum_{k=0}^n b_k C_{k-1} \stackrel{l=k-1}{=} \sum_{k=0}^n b_k C_k - \sum_{l=0}^{n-1} b_{l+1} C_l = b_n C_n + \sum_{k=0}^{n-1} (b_k - b_{k+1}) C_k$$

Then, since  $b_n C_n \xrightarrow{n \rightarrow \infty} 0$ , we only need to show that the final term converges absolutely. Consider

$$\sum_{k=0}^{n-1} |b_k - b_{k+1}| |C_k| \leq C \sum_{k=0}^{n-1} (b_k - b_{k+1}) = C(b_0 - b_n) \leq C(b_0)$$

independent of  $n$ . Hence,  $\sum_{k=0}^n b_k c_k$  converges.

### Definition: Rearrangement

Take  $\sigma : \mathbb{N} \rightarrow \mathbb{N}$  a bijection and  $\sum_{n \geq 1} a_n$  a series such that  $S_n = \sum_{k=1}^n a_k$ .

Then define a rearranged sum  $S_n^{(\sigma)} = \sum_{k=1}^n a_{\sigma(k)}$ .

### Q: When does the rearranged sum converge; to where?

- Theorem: Rearrangement of Absolute Convergence If  $\sum a_n$  converges absolutely, then  $\forall \sigma, \lim_{n \rightarrow \infty} S_n^{(\sigma)} = \lim_{n \rightarrow \infty} S_n$ .
- Theorem: Rearrangement of Conditional Convergence If  $\sum a_n$  converges conditionally, then  $\forall x \in \mathbb{R}, \exists \sigma$  such that  $\lim_{n \rightarrow \infty} S_n^{(\sigma)} = x$ .

October 16, 2023

## Overview

Sequences and Series of Functions

Things that will be glossed over for time

- Limits
- Continuity
- Differentiability
- Integrability

## Why care about sequences and series?

Extending features of functions.

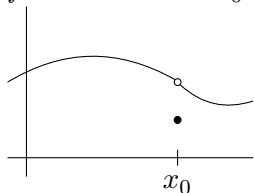
Approximations.

## Limits and Continuity

Let  $I \subseteq \mathbb{R}$  be an interval and  $f : I \rightarrow \mathbb{R}$ ,  $x_0 \in I$ .

### Definition: Limit

$f$  has a limit at  $x_0$  if  $\exists \ell \in \mathbb{R}, \forall \epsilon > 0, \exists \delta > 0, 0 < |x - x_0| < \delta \implies |f(x) - \ell| < \epsilon$

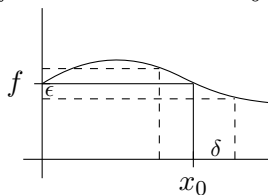


- Equivalently

For every sequence  $\{x_n\}_n$  in  $I$  converging to  $x$  (but distinct to  $x$ ),  $\lim_{n \rightarrow \infty} f(x_n) = \ell$ .

### Definition: Continuous

$f$  is continuous at  $x_0$  if  $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ .



- Modulus of Continuity  $\forall \epsilon > 0, \exists \delta > 0, \forall x \in I, |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$   
Then  $\delta(x_0, \epsilon)$  is the modulus of continuity.

### Definition: Uniform Continuity on I

$f$  is uniformly continuous on  $I$  if  $\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in I, |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$ .  
Where  $\delta$  is  $\delta(\epsilon)$ . That is, the modulus of continuity does not depend on the points.

### Special Types of Uniform Continuity

#### Hölder Continuous

$f$  is  $\alpha$ -Hölder continuous on  $I$  for  $\alpha \in (0, 1]$ , if  $\exists c > 0$  such that  $\forall x, y \in I, |f(x) - f(y)| \leq c|x - y|^\alpha$   
 $\alpha = 1$  implies that  $f$  is “Lipschitz-continuous”

- Example

If  $f'$  exists and is bounded on  $[a, b]$  by  $M$ , then by the Mean Value Theorem:  
 $|f(x) - f(y)| = |f'(\xi)||x - y| \leq M|x - y|$ , where  $x \leq \xi \leq y$ .

### Continuity on Compact Sets

Let  $K \subseteq \mathbb{R}$  be a compact set and  $f : K \rightarrow \mathbb{R}$  be continuous.  
Then

1.  $f(K)$  is compact. In particular,  $f$  is bounded on  $K$ .
2.  $f$  achieves its extrema on  $K$ . (e.g.  $\exists M \in K$  such that  $f(M) = \sup\{f(x) \mid x \in K\}$ ).
3.  $f$  is uniformly continuous on  $K$ .

Note: the proofs of these features are good practice. In particular, proofs that exploit the Heine-Borel property.

#### Proof 1: Compact

Let  $y_n$  be a sequence in  $f(K)$ .

Then,  $\forall n, y_n = f(x_n)$  for  $x_n \in K$ .

It follows that there exists a subsequence  $\{x_{n_k}\}_k$  converging to  $x$  in  $K$ .

By continuity,  $y_{n_k} = f(x_{n_k}) \xrightarrow{k \rightarrow \infty} f(x) \in f(K)$ .

#### Proof 2: Achieves Its Extrema

Construct  $M$ .

By the supremum property,  $S = \sup\{f(x) \mid x \in \mathbb{R}\}$ ,  $\forall n, \exists x_n \in K$  such that  $S - \frac{1}{n} \leq f(x_n) < S$ .

Since  $K$  is compact, there exists a subsequence  $\{x_{n_k}\}_k$  converging to  $x \in K$ .

Since  $f$  is continuous at  $x$ ,  $f(x_{n_k}) \xrightarrow{k \rightarrow \infty} f(x)$ , and also  $S - \frac{1}{n_k} \leq f(x_{n_k}) \leq S \xrightarrow{k \rightarrow \infty} S = f(x)$ .

### Proof 3: Uniformly Continuous

Suppose, for sake of contradiction, that  $\exists \epsilon > 0, \forall \delta > 0, \exists x_\delta, y_\delta \in K, |x_\delta - y_\delta| < \delta$  and  $|f(x_\delta) - f(y_\delta)| \geq \epsilon$ .

Letting  $\delta = \frac{1}{n}$ , we may write  $x_n, y_n \in K, |x_n - y_n| < \frac{1}{n}$  and  $|f(x_n) - f(y_n)| \geq \epsilon$ .

Since  $K$  is compact, there exists a subsequence  $\{x_{n_k}\}_k$  which converges to  $x \in K$ .

Since  $|x_{n_k} - y_{n_k}| < \frac{1}{n_k}$ , then  $\{y_{n_k}\}_k$  also converges to  $x$ .

By continuity of  $f$  at  $x$ ,  $\lim_{k \rightarrow \infty} f(x_{n_k}) - f(y_{n_k}) = 0$ . However, this contradicts the established fact that  $|f(x_n) - f(y_n)| \geq \epsilon$  for  $\epsilon > 0$ .

### Notation

Let  $I \subseteq \mathbb{R}$  be an interval.

### Sequence of Functions

$$f_n(x), x \in I, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$$

### Series of Functions

$$S_n(x) = \sum_{k=0}^n f_k(x)$$

### Definition: Pointwise Convergence

A sequence or series of functions converges pointwise on  $I$  if and only if  $\forall x \in I, \{f_n(x)\}_n$  is convergent.

Call  $f(x)$  the limit.

**Q: Under what conditions do properties of a sequence (e.g. continuity, differentiability, integrability) propagate to the limit?**

### Power Series

$$S_n(x) = \sum_{k=0}^n a_k (x - x_0)^k$$

$\underbrace{\hspace{10em}}_{x_0}$

### Fourier Series

$$S_n = a_0 + \sum_{n=1}^N a_n \cos(nx) + b_n \sin(nx) \text{ is } 2\pi\text{-periodic.}$$

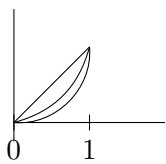
### Approximation

For purposes of approximation, it is useful to know if, for example, the integral may be approximated by taking integrals of the partial sums.

## Deficiencies of Pointwise Convergence

### Example 1

On  $[0, 1]$ ,  $f_n(x) = x^n \xrightarrow{n \rightarrow \infty} \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases}$ ,



$f_n$  is continuous on  $[0, 1]$ ,  $\forall n$ , but  $f$  is not.

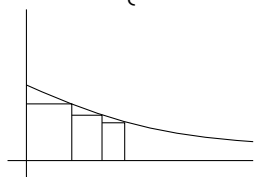
- Exercise

Show that there is no uniform convergence here.

Hint: negate uniform convergence and prove the negation.

### Example 2

$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$  is not Riemann-integrable on  $[0, 1]$ .



If  $r_n$  denotes a denumeration of rationals in  $[0, 1]$ , define  $f_n(x) = \begin{cases} 1 & \text{if } x \in \{r_0, \dots, r_n\} \\ 0 & \text{otherwise} \end{cases}$ .

So  $f_n$  converges pointwise on  $\chi_{\mathbb{Q}}$ .

Yet,  $\forall n$ ,  $f_n$  is Riemann-integrable and  $\int_0^1 f_n(x) dx = 0$ .

## Definition: Uniform Convergence

We say  $f_n : D \rightarrow \mathbb{R}$  (e.g.  $D$  an interval) converges uniformly to  $f$  on  $D$  (notation  $f_n \rightrightarrows f$  on  $D$ ) if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, n \geq N \implies \begin{cases} |f_n(x) - f(x)| < \epsilon, \forall x \in D \\ \sup_D |f_n - f| < \epsilon \end{cases}$$

## Compare with Pointwise Convergence

Compare to  $f_n \rightarrow f$  pointwise on  $D$ .

$$\forall x \in D, \forall \epsilon > 0, \exists N \in \mathbb{N}, n \geq N \implies |f_n(x) - f(x)| < \epsilon.$$

In this case, the behavior is primarily contingent upon the choice of  $x$ . That is  $N(x, \epsilon)$  is dependent on  $x$ .

## Theorem: Weierstrass M-Test

Let  $f_n : D \rightarrow \mathbb{R}$  be bounded by  $M_n$  on  $D$ .

If  $\sum_{n=1}^{\infty} M_n < \infty$ , then the series  $S_n(x) = \sum_{k=1}^n f_k(x)$  converges uniformly to  $S(x)$



**Proof**

$\forall x \in D, |S_n(x) - S(x)| = |\sum_{k=n+1}^{\infty} f_k(x)| \stackrel{\text{triangle inequality}}{\leq} \sum_{k=n+1}^{\infty} |f_k(x)| \leq \sum_{k=n+1}^{\infty} M_k$ , where  $\sum_{k=n+1}^{\infty} M_k$  is a uniform bound in  $x$ .

Let  $\epsilon > 0, \exists n, n \geq N \implies \sum_{k=n+1}^{\infty} M_k < \epsilon$ .

Then  $\forall x \in D, n \geq N, |S_n(x) - S(x)| \leq \sum_{k=n+1}^{\infty} M_k < \epsilon$ . ■

**Theorem: Continuity and Uniform Limits**

Let  $f_n : D \rightarrow \mathbb{R}$  be continuous on  $D$  for all  $n$  and  $f_n \rightarrow f$  on  $D$  ( $\lim_{n \rightarrow \infty} \sup_D |f_n - f| = 0$ ).

Then  $f$  is continuous on  $D$ .

**Proof**

Fix  $x \in D$ , with  $x_n$  converging to  $x$  in  $D$ .

What To Show:  $f(x_n) \xrightarrow{n \rightarrow \infty} f(x)$ .

Scratch:  $f(x_n) - f(x) = (f(x_n) - f_p(x_n)) + (f_p(x_n) - f_p(x)) + (f_p(x) - f(x))$ .

Let  $\epsilon > 0$  be given.

$f_n \rightarrow f : \exists N, n \geq N \implies |f_n(y) - f(y)| < \frac{\epsilon}{3}, \forall y \in D$ .

For  $p \geq N, |f_p(y) - f(y)| < \frac{\epsilon}{3}, \forall y \in D \implies \forall n \in \mathbb{N}, |f(x_n) - f(x)| \stackrel{\text{triangle inequality}}{\leq} \frac{2\epsilon}{3} + |f_p(x_n) - f_p(x)|$ .

With  $p = N$ , since  $f_p$  is continuous at  $x$ ,  $\exists N_1, n \geq N_1 \implies |f_p(x_n) - f_p(x)| < \frac{\epsilon}{3}$ .

Hence, for  $n \geq N_1, |f(x_n) - f(x)| \leq \epsilon$ . ■

**Riemann-Integrability**

Fix  $D = [a, b]$  and  $g : [a, b] \rightarrow \mathbb{R}$  bounded by  $|g(x)| \leq M, \forall x$ .

**Definition: Subdivision**

$$\sigma = \{a = x_0 < x_1 < x_2 < \dots < x_n = b\}$$

**Definition: Upper and Lower Riemann Sums**

$S^+(g, \sigma) = \sum_{k=1}^n (x_k - x_{k-1}) M_k$  is the upper sum.

$S^-(g, \sigma) = \sum_{k=1}^n (x_k - x_{k-1}) m_k$  is the lower sum.

Where  $M_k = \sup_{[x_{k-1}, x_k]} g$  and  $m_k = \inf_{[x_{k-1}, x_k]} g$ .

This gives  $-M(b-a) \leq S^-(g, \sigma) \leq S^+(g, \sigma) \leq (b-a)M$ .

If  $\mathfrak{S}[a, b] = \{\text{subdivisions of } [a, b]\}$ , then

$I^-(g) = \sup_{\sigma \in \mathfrak{S}[a, b]} S^-(g, \sigma)$  and  $I^+(g) = \inf_{\sigma \in \mathfrak{S}[a, b]} S^+(g, \sigma)$ .

**Definition: Riemann Integrable**

$g$  is Riemann integrable if  $I^+(g) = I^-(g)$  and we denote  $\int_a^b g(t) dt = I^+(g)$ .

**Lemma**

$g$  is Riemann integrable if and only if  $\forall \epsilon > 0, \exists \sigma \in \mathfrak{S}[a, b]$  such that  $S^+(g, \sigma) - S^-(g, \sigma) < \epsilon$ .

## Properties

1. Continuous functions and monotone functions are Riemann Integrable.
2.  $f \mapsto \int_a^b f(t) dt$  is linear.
3. If  $f, g$  are Riemann Integrable and  $f(x) \leq g(x), \forall x \in [a, b]$ , then  $\int_a^b f(t) dt \leq \int_a^b g(t) dt$ .

## Theorem:

If  $f_n \Rightarrow f$  on  $[a, b]$  and  $f_n$  is Riemann Integrable for all  $n$ , then  $f$  is Riemann Integrable on  $[a, b]$  and  $\lim_{n \rightarrow \infty} \int_a^b f_n(t) dt = \int_a^b \lim_{n \rightarrow \infty} f_n(t) dt = \int_a^b f(t) dt$ .

## Proof

$\forall n, \forall x \in [a, b], f_n(x) - \epsilon \leq f(x) \leq f_n(x) + \epsilon$  where  $\epsilon_n = \sup_{x \in [a, b]} |f_n(x) - f(x)|$  (by hypothesis  $\epsilon_n \xrightarrow{n \rightarrow \infty} 0$ )

Then, for any  $\sigma \in \mathfrak{S}[a, b]$ ,  $S^-(f_n, \sigma) - \epsilon_n(b-a) \leq S^-(f, \sigma) \leq S^+(f, \sigma) \leq S^+(f_n, \sigma) + \epsilon_n(b-a)$ .

It follows that  $S^+(f, \sigma) - S^-(f, \sigma) \leq S^+(f_n, \sigma) - S^-(f_n, \sigma) + 2\epsilon_n(b-a)$ .

Finishing the proof is left as an exercise.

## October 18, 2023

### Overview

- Sequences/Series
- Power Series
- Exponential and Logarithms

## Fundamental Theorems of Calculus

Full proofs in 105A lecture notes.

### Differentiation of the Integral

$f : [a, b] \rightarrow \mathbb{R}$  continuous.

$\forall x \in [a, b]$ , can define  $F(x) = \int_a^x f(t) dt$ .

Then  $F$  is continuously differentiable on  $[a, b]$

$F'(x) = f(x)$  for  $x \in [a, b]$ .

### Integration of the Derivative

$f \in C^1[a, b]$  with one-sided derivatives at  $a$  and  $b$  well defined. (e.g.  $\frac{f(a+h)-f(a)}{h} \xrightarrow{h>0; h \rightarrow 0} f'(a)$ ).

Then  $\forall x, y, a \leq x \leq y \leq b$ ,  $f(y) - f(x) = \int_x^y f'(t) dt$ .

## Theorem: Differentiability of Uniform Limits

Let  $f_n : (a, b) \rightarrow \mathbb{R}$  be a sequence in  $C^1[a, b]$ , and assume  $f_n(x) \rightarrow f(x)$  pointwise while  $f'_n(x) \rightrightarrows g(x)$  uniformly. Then  $f \in C^1(a, b)$  and  $f' = g$ .

### Proof

Fix  $a_0 \in (a, b)$ .

Then  $\forall x \in (a, b)$ , by the Fundamental Theorem of Calculus,

$$f_n(x) - f_n(a_0) = \int_{a_0}^x f'_n(t) dt$$

Observe that  $f_n(x) \xrightarrow{n \rightarrow \infty} f(x)$  and  $f_n(a_0) \xrightarrow{n \rightarrow \infty} f(a_0)$  pointwise, and  $\int_{a_0}^x f'_n(t) dt \rightarrow \int_{a_0}^x g(t) dt$  by the integrability of uniform limits. Then

$$f(x) - f(a_0) = \int_{a_0}^x g(t) dt, \quad \forall x \in (a, b)$$

which implies  $f \in C^1$  and  $f' = g$ . ■

## Interesting Applications

$$S_n(x) = \sum_{k=0}^n f_k(x).$$

Suppose pointwise convergence, that  $S'_n(x) = \sum_{k=0}^n f'_k(x)$  is continuous,  $|f'_k(x)| \leq M_k$  and  $\sum_{k=0}^{\infty} M_k < \infty$ . Long story short, this implies

$$\left( \sum_{k=0}^{\infty} f_k(x) \right)' = \sum_{k=0}^{\infty} f'_k(x)$$

### Example

$$f(x) = \sum_{n=0}^{\infty} \frac{\cos(nx)}{n^3}$$

Call  $u_n(x) = \frac{\cos(nx)}{n^3}$ , then  $|u_n(x)| \leq \frac{1}{n^3}$  summable and  $|u'_n(x)| = \left| \frac{-\sin(nx)}{n^2} \right| \leq \frac{1}{n^2}$  summable.

This implies  $f'(x) = -\sum_{n=0}^{\infty} \frac{\sin(nx)}{n^2}$ .

Repetition of this process informs us that  $f \in C^2$ .

## Power Series

$S_n(x) = \sum_{k=1}^n a_k(x - x_0)^k$  for,  $x_0 \in \mathbb{R}$  fixed, is 'centered at  $x_0$ .' Note that each term is  $C^\infty(\mathbb{R})$ .

### Example 1

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \text{ for } |x| < 1.$$

**Example 2**

$\exp(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!}$  converges  $\forall x \in \mathbb{R}$ .

- Why?

Ratio Test.

$$\frac{a_{k+1}}{a_k} = \frac{x^{k+1}}{(k+1)!} \cdot \frac{k!}{x^k} = \frac{x}{k+1}$$

So  $\left| \frac{a_{k+1}}{a_k} \right| \xrightarrow[k \rightarrow \infty]{} 0$

**Lemma: Radius of Convergence**

Suppose a power series  $\sum_{n \geq 0} a_n x^n$  converges at  $b \in \mathbb{R}$ .

1. Converges absolutely  $\forall x, |x| < |b|$ .
2.  $\forall a \in (0, b)$  converges uniformly on  $[-a, a]$ .

- Proof of 1

Suppose  $\sum_{n \geq 0} a_n b^n$  converges.

Then  $a_n b^n \rightarrow 0$ .

Let  $x$  such that  $|x| < b$ , then

$$|a_n x^n| = \left| a_n b^n \left( \frac{x}{b} \right)^n \right| \leq M \left( \frac{|x|}{b} \right)^n$$

By term test,  $\sum_{n=0}^{\infty} |a_n x^n| < \infty \implies \sum a_n x^n$  converges absolutely.

- Proof of 2

If  $|x| \leq a < b$ ,

$$|a_n x^n| \leq M \left( \frac{|x|}{b} \right)^n \leq M \left( \frac{a}{b} \right)^n$$

Thus, by  $M$ -test for  $x \in [-a, a]$ , the series converges uniformly on  $[-a, a]$ .

- Upshot

The set where a power series converges is an interval centered at  $x_0$ .

**Theorem: Radius of Convergence**

Given a power series, define  $R$  to be such that  $\frac{1}{R} = \limsup_n |a_n|^{1/n}$ . Then

1.  $\forall a \in (0, R)$ , the series converges uniformly on  $[-a, a]$ .
2. If  $|x| > R$ , the series diverges.

## Proof

IMAGE HERE - RADIUS OF CONVERGENCE

Fix  $x$ . As an exercise,  $\limsup_n |a_n x^n|^{1/n} = |x| \cdot \limsup_n |a_n|^{1/n} = \frac{|x|}{R}$ .

Recall that  $\limsup_n |a_n x^n|^{1/n} = \lim_{n \rightarrow \infty} y_n$  where  $y_n = \sup_{k > n} \{|a_k x^k|^{1/k}\}$ .

If  $\frac{|x|}{R} < 1$ , then  $\exists N_0, n \geq N_0 \implies y_n < \frac{1 + \frac{|x|}{R}}{2} < 1$ .

This implies  $\forall k \geq N_0, |a_k x^k|^{1/k} \leq \frac{1 + \frac{|x|}{R}}{2} < 1$  and, by the root test, the series converges.

If  $\frac{|x|}{R} > 1$ ,  $\forall n, \sup_{k > n} \{|a_k x^k|^{1/k}\} \geq \frac{|x|}{R}$ .

By the properties of the supremum with  $\epsilon = \left(\frac{|x|}{R} - 1\right)/2 > 0$ ,

$$\forall n, \exists k, 1 \leq \frac{\frac{|x|}{R} + 1}{2} \leq y_n - \epsilon \leq |a_k x^k|^{1/k} \leq y_n$$

Therefore  $\forall n, \exists k > n, |a_k x^k|^{1/k} \geq 1$ . ■

## Observation: Behavior at Endpoints

At the endpoints of  $(-R, R)$ , a series might

### Converge Absolutely

e.g.  $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$ ,  $R = 1$ ,  $\frac{1}{R} = \limsup_n \left(\frac{1}{n^2}\right)^{1/n} \xrightarrow{n \rightarrow \infty} 1$

### Converge Conditionally

e.g.  $\sum_{k=1}^{\infty} \frac{x^k}{k}$ ,  $R = 1 \implies \frac{1}{R} = \limsup_n \left(\frac{1}{n}\right)^{1/n} = 1$   
Converges conditionally at  $x = -1$ .

### Diverge

e.g.  $\sum_{k=0}^{\infty} x^k$ ,  $R = 1$

## Theorem: Power Series Differentiation

Let  $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$  converge on  $(x_0 - R, x_0 + R)$ .

Then  $\forall k > 0, f \in C^k(x_0 - R, x_0 + R)$  and  $f^{(k)}(x) = \sum_{n=k}^{\infty} a_n n(n-1) \cdots (n-k+1) (x - x_0)^{n-k}$ ,  $\forall x \in (x_0 - R, x_0 + R)$

## Exercise

Show that if  $a_n \rightarrow a > 0$ , then  $\limsup a_n b_n = a \limsup b_n$ .

### Proof (by Induction)

Consider the series  $S_n(x) = \sum_{n=1}^{\infty} a_n n(x - x_0)^{n-1} = \sum_{n=0}^{\infty} a_{n+1}(n+1)(x - x_0)^n$ .  
Then

$$(x - x_0) \frac{1}{R \text{ of series of derivatives}} = \limsup_{n \rightarrow \infty} (a_n n)^{1/n} \limsup_{n \rightarrow \infty} a_n^{1/n} n^{1/n} = \limsup_{n \rightarrow \infty} a_n^{1/n} = \frac{1}{R}$$

This implies  $\sum_{k=0}^{\infty} \frac{d}{dx} (a_k(x - x_0)^k)$  converges uniformly on  $[x_0 - a, x_0 + a]$ ,  $\forall a \in (0, R)$ .

By the Theorem on Differentiability of Uniform Limits,  $f'(x)$  exists and  $\forall x \in (x_0 - R, x_0 + R)$

$$f'(x) = \sum_{n=1}^{\infty} a_n n(x - x_0)^{n-1}$$

Repeat to get higher derivatives.

### Integration

It is similarly possible to integrate term by term.

### Famous Power Series

- $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ ,  $|x| < 1$
- PSE of  $\frac{1}{x}$  centered at  $x_0 > 0$

IMAGE HERE - GRAPH

$$\frac{1}{x} = \frac{1}{x - x_0 + x_0} = \frac{1}{x_0} \cdot \frac{1}{1 + \frac{x-x_0}{x_0}} = \frac{1}{x_0} \sum_{k=0}^{\infty} \left( -\frac{x-x_0}{x_0} \right)^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{x_0^{k+1}} (x - x_0)^k \text{ if } |x - x_0| < |x_0|, x \in (0, 2x_0)$$

- $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$
- $\exp(0) = 1$
- $\exp'(x) = \sum_{k=1}^{\infty} \frac{kx^{k-1}}{k!} = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} = \exp(x)$

### Law of Exponents

$\exp(a) \exp(b) = \exp(a + b)$ ,  $\forall a, b \in \mathbb{R}$

## Proof

Special case of the “Cauchy product of convergent series.”

If  $\sum_{n \geq 0} a_n$  converges absolutely to  $A$  and  $\sum_{n \geq 0} b_n$  converges to  $B$ , then  $\sum_{n \geq 0} c_n$  converges to  $AB$ , where

$$c_n = \sum_{k=0}^n a_k b_{n-k} = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$$

- Heuristics

$$\left( \sum_{p=0}^{\infty} a_p x^p \right) \left( \sum_{l=0}^{\infty} b_l x^l \right) = \sum_{p,l \in \mathbb{N}_0^2} a_p b_l x^{p+l}$$

IMAGE HERE - CIRCLES FROM L TO P

$$\{(p, l) : p + l = n, p, l \in \mathbb{N}_0\} = \{(0, n), (1, n-1), \dots, (n, 0)\}$$

## Proof Continued

$\text{Aexp}(a) = \sum_{k=0}^{\infty} \frac{a^k}{k!}$  and  $\exp(b) = \sum_{l=0}^{\infty} \frac{b^l}{l!}$ , thus  $\exp(a) \exp(b) = \sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} \frac{(a+b)^n}{n!} = \exp(a+b)$  since

$$c_n = \frac{1}{n!} \sum_{k=0}^n \frac{a^k}{k!} \cdot \frac{b^{n-k}}{(n-k)!} \text{ and } n! = \frac{1}{n!} (a+b)^n$$

## Power Series Expansion of Exponential

Centered at  $x_0$ , we have

$$\exp(x) = \exp(x - x_0) \exp(x_0) = \exp(x_0) \sum_{k=0}^{\infty} \frac{(x - x_0)^k}{k!}$$

## Observation:

$\exp$  is the only  $C^1(\mathbb{R})$  solution of  $\begin{cases} \exp'(x) = \exp(x) \\ \exp(0) = 1 \end{cases}$

- Proof If  $f$  solves the above, then for some constant  $c$

$$\frac{d}{dx} (f(x) \exp(-x)) = f'(x) \exp(-x) - f(x) \exp(-x) = 0 \xRightarrow{\text{MVT}} f(x) \exp(-x) = c = f(0) \exp(-0) = 1$$

this implies

$$f(x) = \exp(x) f(x) \exp(-x) = \exp(x)$$

## Exponential Features

$$\exp(x) > 0, \forall x \in \mathbb{R} \implies \begin{cases} \text{if } x \geq 0, \exp(x) \geq 1 > 0 \\ \text{if } x < 0, \exp(x) = \frac{1}{\exp(-x)} > 0 \end{cases}$$

## Theorem: Exponential and $e$

$\exp(x) = (\exp(1))^x \forall x \in \mathbb{R}$  and  $e = \exp(1)$

### Proof

Using law of exponents for

$$x \in \mathbb{N} : \quad \exp(n) = \exp(1 + (n-1)) = e \cdot \exp(n-1) = \cdots = e^n \exp(0)$$

$$x = \frac{1}{q}, q \in \mathbb{N} : \quad \left( \exp\left(\frac{1}{q}\right) \right)^q = \exp\left(\frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}\right) = \exp(1) = e$$
$$\therefore \exp\left(\frac{1}{q}\right) = e^{1/q}$$

$$x = \frac{p}{q}, p, q \in \mathbb{N} : \quad \exp\left(\frac{p}{q}\right) = \exp\left(\overbrace{\frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}}^p\right) = \left(e^{1/q}\right)^p = e^{p/q}$$

$$x \in -\mathbb{N}, \mathbb{Q} < 0 : \quad \text{left as an exercise}$$

Therefore, the functions  $x \mapsto \begin{cases} \exp(x) \\ e^x \end{cases}$  are continuous on  $\mathbb{R}$  and agree on  $\mathbb{Q}$ . This implies that they must be equal everywhere.

**October 23, 2023**

### Today

Exp and log.

Real-analytic functions. (Newest bit of information.)

Trig functions.

**Wednesday, October 25, 2023**

Analytic vs  $C^\infty$

Approximation by polynomials.

### Next Week

Fourier series.

## Exponential and Log

### Covered Last Lecture

Law of Exponents

$\exp(x) = e^x$  and  $e = \exp(1) = \sum_{k=0}^{\infty} \frac{1}{k!}$



## Error Estimate

$e = \lim_{n \rightarrow \infty} S_n$  where  $S_n = \sum_{k=0}^{\infty} \frac{1}{k!}$  (increases).

$$e - S_n = \sum_{k=n+1}^{\infty} \frac{1}{k!}$$

For  $k = n + 1 + p$ ,  $p \geq 0$ ,  $e - S_n = \sum_{p=0}^{\infty} \frac{1}{(n+1+p)!}$ .

Then,

$$\begin{aligned} \frac{1}{(n+1+p)!} &= \frac{1}{(n+1)!} \cdot \frac{1}{\underbrace{(n+2)(n+3)\cdots(n+p+1)}_{p \text{ factors}}} \\ &\leq \frac{1}{(n+1)!} \cdot \frac{1}{(n+1)^p} \end{aligned}$$

and

$$\begin{aligned} e - S_n &= \sum_{k=n+1}^{\infty} \frac{1}{k!} \\ &= \sum_{p=0}^{\infty} \frac{1}{(n+1+p)!} \\ &\leq \frac{1}{(n+1)!} \cdot \sum_{p=0}^{\infty} \left( \frac{1}{n+1} \right)^p \\ &= \frac{1}{(n+1)!} \cdot \frac{1}{1 - \frac{1}{n+1}} \\ &= \frac{1}{(n+1)!} \cdot \frac{n+1}{n} \end{aligned}$$

Therefore,

$$0 \leq e - S_n \leq \frac{1}{n!} \cdot \frac{1}{n}$$

## Theorem: e is Irrational

### Proof

Suppose  $e = \frac{p}{q}$ ,  $q > 2$ , and  $p$  and  $q$  coprime. Consider

$$\begin{aligned} 0 &< e - S_q \leq \frac{1}{q!} \cdot \frac{1}{q} \\ 0 &< q!(e - S_q) \leq \frac{1}{q} \\ 0 &< q! \left( \frac{p}{q} - \sum_{k=0}^q \frac{1}{k!} \right) \leq \frac{1}{q} < \frac{1}{2} \end{aligned}$$

where  $q! \left( \frac{p}{q} - \sum_{k=0}^q \frac{1}{k!} \right) \in \mathbb{N}$ .

This is a contradiction. Thus,  $e$  must be irrational. ■

## Exponential Decay

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\lim_{x \rightarrow +\infty} x^k e^{-x} = 0, \forall k \in \mathbb{N}$$

$$\text{For } x > 0, \exp(x) \geq \frac{x^{k+1}}{(k+1)!} \text{ if and only if } x^k \exp(-x) \leq \frac{(k+1)!}{x} \xrightarrow{x \rightarrow +\infty} 0.$$

## Exponential Strictly Positive Over Reals

$$\exp(x) > 0, \forall x \in \mathbb{R}$$

$x > 0$  is obvious.

$$x \leq 0, \exp(x) = \frac{1}{\exp(-x)} > 0$$

$$\lim_{x \rightarrow -\infty} \exp(x) = \lim_{x \rightarrow -\infty} \frac{1}{\exp(-x)} = 0.$$

## Proposition: Exponential is a Bijection

$\exp : \mathbb{R} \rightarrow (0, \infty)$  is a  $C^\infty$  ( $\exp' = \exp$ ) bijection (diffeomorphism in the sense that  $\exp'(x) > 0, \forall x \in \mathbb{R}$ ).

By Inverse Function Theorem then, define  $\log : (0, \infty) \rightarrow \mathbb{R}$  such that  $\exp(\log(x)) = x$ .

$$\text{By MATH 105A, } \frac{d}{dx}(\log(x)) = \frac{d}{dx}(\exp^{-1}(x)) = \frac{1}{\exp'(\exp^{-1}(x))} = \frac{1}{\exp(\log(x))} = \frac{1}{x}.$$

$\log(1) = 0$  (since  $\exp(0) = 1$ ) which implies, by the Fundamental Theorem of Calculus, that  $\log(x) - \log(1) = \int_1^x \frac{dt}{t}$ .

## Properties (left as an exercise)

- $\log(xy) = \log(x) + \log(y), x, y > 0$
- Power Series Expansion:  $\log(1-x) = -\sum_{k=0}^{\infty} \frac{x^k}{k}, x \text{ near } 0, \text{ radius of convergence: } 1.$
- $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = \exp(x)$

## Definition: Real-Analytic Functions

A function  $f : (a, b) \rightarrow \mathbb{R}$  is real-analytic on  $(a, b)$  if  $\forall x_0 \in (a, b), \exists r > 0$  and a power series  $\sum_{n \geq 0} (x - x_0)^n$  converging to  $f$  on  $(x_0 - r, x_0 + r)$ .

When such a power series exists,  $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ , then

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

The radius of convergence is related by  $\frac{1}{R} = \limsup_n |a_n|^{1/n}$  which provides a constraint on rate of divergence.

## Example 1: Polynomial

For every polynomial,  $p : \mathbb{R} \rightarrow \mathbb{R}$ , and  $\forall x_0 \in \mathbb{R}$ ,

$$p(x) = \sum_{k=0}^{\infty} \frac{p^{(k)}(x_0)}{k!} (x - x_0)^k, \forall x \in \mathbb{R}$$

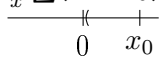
## Example 2: Exponential

$$\exp(x) = \exp(x - x_0 + x_0) = \sum_{k=0}^{\infty} \frac{e^{x_0}}{k!} (x - x_0)^k$$

and the radius of convergence,  $R = \infty$ .

## Example 3: $1/x$

$$\frac{1}{x} \text{ analytic on } (0, \infty)$$

$$\frac{1}{x} \sum_{k=0}^{\infty} (x - x_0)^k \text{ and } R = |x_0|.$$


## Remark: Analyticity Implies Smoothness

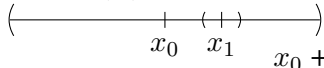
$f$  analytic on  $(a, b) \implies f$  smooth ( $C^\infty$ ) on  $(a, b)$

The converse is not true. (Example Wednesday)

## Proposition:

Suppose  $\sum_{n \geq 0} a_n (x - x_0)^n$  converges to  $f(x)$  on  $(x_0 - R, x_0 + R)$ .

Then  $f(x)$  is analytic on  $(x_0 - R, x_0 + R)$ .



That is to say,  $\forall x_1 \in (x_0 - R, x_0 + R)$ , there exists some power series expansion for  $f$ , centered at  $x_1$ , with positive radius of convergence.

## Proof

Let  $x_0 = 0$  for simplicity and  $x_1 \in (-R, R)$ .

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n (x - x_1 + x_1)^n = \sum_{n=0}^{\infty} a_n \sum_{k=0}^n \binom{n}{k} (x - x_1)^k x_1^{n-k}$$

Assuming that rearrangement is possible, this is

$$\sum_{n, k, n \geq 0} a_n \binom{n}{k} (x - x_1)^k x_1^{n-k} = \sum_{k=0}^{\infty} \underbrace{\left( \sum_{n=k}^{\infty} a_n \binom{n}{k} x_1^{n-k} \right)}_{b_k} (x - x_1)^k$$

Need to prove two things:

1.  $b_k$  is well-defined
2. Interchange of sums valid.

- Proof of 1

For  $k$  fixed,  $\binom{n}{k}$  is a degree  $k$  polynomial in  $n$ .  
Letting

$$b_k = \sum_{p=0}^{\infty} a_{p+k} \binom{p+k}{k} x_1^p$$

where  $p = n - k$ , we have

$$\limsup_{p \rightarrow \infty} \left( |a_{p+k}| \binom{p+k}{k} \right)^{1/p} = \limsup_{p \rightarrow \infty} |a_p|^{1/p}$$

since  $x_1 \in (-R, R)$ ,  $b_k < \infty$ ,  $\forall k$ .

- Proof of 2

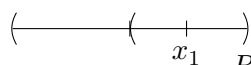
The proof requires invoking Fubini's Theorem to allow rearrangement.  
Need to check that

$$\sum_{n,k, n \geq k} |a_n| \binom{n}{k} |(x - x_1)^k x_1^{n-k}|$$

converges.

Consider

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} |a_n| \binom{n}{k} |x - x_1|^k |x_1|^{n-k} = \sum_{n=0}^{\infty} |a_n| (|x - x_1| + |x_1|)^n$$



Make sure that  $|x - x_1| = r < R - |x_1|$ , then

$$\sum_{n=0}^{\infty} |a_n| r^n$$

where  $r < R$  which, by absolute convergence of the original power series, is finite. ■

## Remark: Analytic Continuation

The process of recentering a power series is also called “analytic continuation.”

The radius of convergence of the new series might actually be larger and allow the original function.

## Example

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

IMAGE HERE - Decaying curve.

## Facts: Analytic Functions

- If  $f, g$  are analytic on  $(a, b)$ , then so is  $f \cdot g$ .
- If  $f, g$  are analytic and  $g$  does not vanish on  $(a, b)$ , then  $\frac{f}{g}$  is analytic.
- If  $f$  is analytic on  $(x_0 - R, x_0 + R)$  and  $g$  is analytic on  $(f(x_0) - \delta, f(x_0) + \delta)$ , then  $g \circ f$  is analytic on a neighborhood of  $x_0$ . (Proof in ; page number in lecture notes).

## Remark: No Analytic Bump Functions

IMAGE HERE - BUMP FUNCTION  $-|x|$

## Trig Functions

IMAGE HERE – UNIT CIRCLE

We want  $(\cos(\theta), \sin(\theta))$  to be the point on the unit circle making an arclength  $\theta$  from  $(1, 0)$ .

For  $x$  in the right-half plane,  $\cos(\theta) \geq 0$ .

For  $x$  in top right quadrant,

$$\theta = \int_0^{\sin(\theta)} \sqrt{1 + (f'(y))^2} dy$$

Then,  $y \mapsto (\underbrace{\sqrt{1 - y^2}}_{f(y)}, y)$ ,  $y \in (-1, 1)$ . It follows that

$$\theta = \lim_{y \rightarrow 0}^{\sin(\theta)} \frac{dy}{\sqrt{1 - y^2}} \xrightarrow{\text{FTC}} \arcsin'(x) = \frac{1}{\sqrt{1 - x^2}} \in C^\infty((-1, 1))$$

and

$$\arcsin(x) = \lim_{y \rightarrow 0}^x \frac{dy}{\sqrt{1 - y^2}}$$

Therefore,  $\arcsin$  is a diffeomorphism from  $(-1, 1) \rightarrow (\lim_{x \rightarrow -1} \arcsin(x), \lim_{x \rightarrow 1} \arcsin(x))$ .

Since  $\frac{1}{\sqrt{1 - x^2}}$  is integrable near  $\pm 1$ , these limits are finite.

## Definition: Pi

$$\pi = 2 \lim_{x \rightarrow 1} \arcsin(x)$$

## Inverse Function Theorem

$\sin : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow (-1, 1)$  exists as a  $C^1$  inverse of  $\arcsin$ .

On  $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ , define  $\cos(\theta) = +\sqrt{1 - \sin^2(\theta)}$ . Then

$$\sin'(\theta) = \frac{1}{\arcsin'(\sin(\theta))} = \sqrt{1 - \sin^2(\theta)} = \cos(\theta).$$

Similarly,  $\cos'(\theta) = -\sin(\theta) \rightsquigarrow \sin, \cos$  are  $C^\infty$  on  $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ .

## Extension to the Reals

By graphical or geometric arguments, for  $\theta \in (0, \frac{\pi}{2})$ ,

$$\begin{aligned}\cos(\theta) &= -\sin\left(\theta - \frac{\pi}{2}\right) \\ \sin(\theta) &= -\cos\left(\theta - \frac{\pi}{2}\right)\end{aligned}$$

This helps extend to  $\mathbb{R}$ , with  $2\pi$ -periodicity such that

$$\begin{cases} \cos' &= -\sin \\ \sin' &= \cos \\ \cos(0) &= 1 \\ \sin(0) &= 0 \end{cases}$$

Therefore, you get all derivatives at  $x = 0$  and the corresponding Taylor expansion looks like

$$C(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \qquad S(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

We find that  $R = \infty$  for both, and

$$C(0) = 1, \qquad S(0) = 0, \qquad C'(x) = -S(x), \qquad S'(x) = C(x).$$

Take

$$\epsilon(x) = (C(x) - \cos(x))^2 + (S(x) - \sin(x))^2$$

with  $\epsilon(0) = 0$ . Then, finally,

$$\epsilon'(x) = 0 \implies \epsilon = \text{some constant} = 0.$$

## October 25, 2023

### Today

Analytic vs  $C^\infty$

Approximation by Polynomials

### Definition: Real Analytic

$f$  is real analytic on  $(a, b)$  if  $\forall x_0 \in (a, b), \exists \delta > 0, \{a_n\}_n$  such that  $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \forall x \in (x_0 - \delta, x_0 + \delta)$ .

### Proposition: Analyticity Implies Smoothness

Analytic on  $(a, b) \implies C^\infty$  smooth on  $(a, b)$ .

$$\sum_{n=0}^{\infty} (x - x_0)^n \rightsquigarrow a_n - \frac{f^n(x_0)}{n!}$$

Note:  $C^w(a, b) \not\subseteq C^\infty(a, b)$

The converse is not true.

### Example

Let  $x \in \mathbb{R}$  and  $f(x) = \begin{cases} 0 & x < 0 \\ \exp\left(\frac{-1}{x^2}\right) & x > 0 \end{cases}$

IMAGE HERE - FUNCTION

$x \neq 0, f \in C^\infty(\mathbb{R} \setminus \{0\})$ .

- What about at  $x = 0$ ?

$$\lim_{x \rightarrow 0; x < 0} f(x) = 0 = \lim_{x \rightarrow 0; x > 0} e^{-\frac{1}{x^2}}$$

So we can define  $f(0) = 0$ , the resulting function is continuous on  $\mathbb{R}$ .

- What about higher derivatives?

Claim:  $\forall k > 0, \lim_{x \rightarrow 0; x > 0} \frac{d^k}{dx^k} \left( e^{-\frac{1}{x^2}} \right) = 0$

- Proof (Sketch)

$$\frac{d}{dx} \left( e^{-x^2} \right) = 2x^{-3} e^{-x^{-2}}$$

$$\lim_{x \rightarrow 0} \frac{e^{-\frac{1}{x^2}}}{x^3} \stackrel{y=\frac{1}{x}}{=} \lim_{y \rightarrow \infty} y^3 e^{-y^2} = 0$$

Claim by induction:

$$\frac{d^k}{dx^k} \left( e^{-\frac{1}{x^2}} \right) = p_k(1/x) e^{-\frac{1}{x^2}}$$

for some polynomial  $p_k$ .

If the claim is true, then

$$\lim_{x \rightarrow 0} p_k \left( \frac{1}{x} \right) e^{-\frac{1}{x^2}} = \lim_{y \rightarrow +\infty} p_k(y) e^{-y^2} = 0 \quad \blacksquare$$

Then we can extend  $f^{(k)}$  as a continuous function on  $\mathbb{R}$  such that  $f^{(k)}(0) = 0$ .

- Claim

$f(x)$  is not analytic on any neighborhood of  $x_0 = 0$ .

If it were, it would equal  $\sum_{n=0}^{\infty} a_n x^n$  on  $(-\delta, \delta)$  for some  $a_k$ s. But,

$$a_k = \frac{f^{(k)}(0)}{k!} = 0 \quad \text{then} \quad \sum_{n=0}^{\infty} a_n x^n = 0, \forall x \in (-\delta, \delta)$$

which is impossible, since  $f(x) \neq 0$  whenever  $x > 0$ .  $\blacksquare$

### Remark: Contraposition Can Disprove Analyticity

The existence of a non-zero radius of convergence for  $\sum a_k(x - x_0)^k$  means

$$\frac{1}{R} = \limsup_n |a_n|^{1/n} = \left( \frac{f^{(n)}(x_0)}{n!} \right)^{1/n} < \infty$$

and

$$\left( \frac{f^{(n)}(x_0)}{n!} \right)^{1/n} \rightsquigarrow f^{(n)}(x_0) \leq n! \left( \frac{c}{R} \right)^n$$

for some constant  $c$ .

### Remark: Analyticity is Not Guaranteed

The conditions

$$\begin{cases} h \in C^\infty(R) \\ \limsup_n \left( \frac{h^{(n)}(0)}{n!} \right)^{1/n} < \infty \end{cases}$$

are not sufficient to claim  $h$  is analytic on any neighborhood of 0.

Indeed, if  $h$  is analytic then  $h(x) + f(x)$  will not be for otherwise

$$f(x) = -(h(x) + f(x)) - h(x)$$

would also be analytic, which it isn't.

### Definition: Exponential Blip Function

Let  $g(x) = \frac{f(x+1)f(1-x)}{f(1)^2}$ , where  $f$  is the “exponential glue” function.

IMAGE HERE - FUNCTION

Smooth on  $\mathbb{R}$ ;  $g(x) \geq 0$ .

### TODO Theorem: Borel

TODO - Name for theorem?

Given any sequence  $\{a_n\}_n$  of reals and any  $\begin{cases} x_0 \in \mathbb{R} \\ \lambda > 0 \end{cases}$ ,  $\exists f \in C^\infty(\mathbb{R})$  such that

$$\begin{cases} f^{(k)}(x_0) = a_k & \forall k \\ f(x) = 0 & \text{if } |x - x_0| > \lambda \end{cases}$$

IMAGE HERE - BUMPY MOUNTAIN CLOSE TO X0

### Proof

Reductions:  $x_0 = 0$  and  $\lambda = 1$ .

Ansatz:  $f(x) = \sum_{k=0}^{\infty} b_k x^k g\left(\frac{x}{\lambda_k}\right)$  where  $b_k$ s and  $\lambda_k$ s need to be tuned.

IMAGE HERE - G(X) and G(X/LAMBDA K)



$$g(x) = 0 \iff |x| \geq 1 \text{ and } g\left(\frac{x}{\lambda_k}\right) = 0 \iff \left|\frac{x}{\lambda_k}\right| \geq 1 \iff |x| \geq \lambda_k$$

Observations: if  $\lambda_k \xrightarrow[k \rightarrow \infty]{} 0$ , then  $\forall x \neq 0$  the series is actually finite!

Since  $g\left(\frac{x}{\lambda_k}\right) = 0$  once  $\lambda_k < |x|$ .

Therefore, convergent and  $C^\infty$  on  $\mathbb{R} \setminus \{0\}$ .

Constraints:

$$\begin{aligned} a_0 &= f(0) = b_0 \\ a_1 &= f'(0) = \frac{d}{dx} \left( b_0 g\left(\frac{x}{\lambda_0}\right) \right) \Big|_{x=0} + b_1 \end{aligned}$$

Generally,

$$a_n = \sum_{k=0}^{n-1} \frac{d^n}{dx^n} \left( b_k x^k g\left(\frac{x}{\lambda_k}\right) \right) \Big|_{x=0} + n! b_n$$

If  $\lambda_n$  are chosen, these constraints uniquely determine the  $b_n$ s.

## How to Choose Lambdas?

Want to enforce

$$\max_{0 \leq k \leq n-1} \sup_{x \in \mathbb{R}} \left| \frac{d^k}{dx^k} \left( b_n x^n g\left(\frac{x}{\lambda_n}\right) \right) \right| \leq 2^{-n}$$

- Example

Determine  $\lambda_2$ :

$$\begin{aligned} k=0 : \left| b_n x^n g\left(\frac{x}{\lambda_n}\right) \right| &\leq |b_n| \lambda_n^n 2^{-n} \\ k=1 : \left| b_n \left( n x^{n-1} g\left(\frac{x}{\lambda_n}\right) \right) + b_n x^n \frac{1}{\lambda_n} g'\left(\frac{x}{\lambda_n}\right) \right| &\leq |b_n| \lambda_n^{n-1} (n + \|g'\|_\infty) \leq 2^{-n} \end{aligned}$$

In general,

$$a \lambda_n^p < 2^{-n}$$

for  $p > 0$ .

So we construct  $b_0$ , then  $\lambda_0$ , then  $b_1$ , then  $\lambda_1, \dots$

## Claim: Produces Uniform Convergence

When

$$\max_{0 \leq k \leq n-1} \sup_{x \in \mathbb{R}} \left| \frac{d^k}{dx^k} \left( b_n x^n g\left(\frac{x}{\lambda_n}\right) \right) \right| \leq 2^{-n}$$

is satisfied,  $\forall k \in \mathbb{N}$

$$\sum_{n=0}^{\infty} \frac{d^k}{dx^k} \left( b_n x^n g\left(\frac{x}{\lambda_n}\right) \right)$$

satisfies the Weierstrass M-Test. Therefore it is uniformly convergent. Because

$$\sum_{n=0}^{\infty} \left| \frac{d^k}{dx^k} \left( b_n x^n g\left(\frac{x}{\lambda_n}\right) \right) \right| \leq \underbrace{\sum_{n=0}^k \left| \frac{d^k}{dx^k} \left( b_n x^n g\left(\frac{x}{\lambda_n}\right) \right) \right|}_{\text{finite sum, uniformly bounded}} + \sum_{n=k+1}^{\infty} 2^{-n}$$

## Approximation by Polynomials

### Goal (Weierstrass Approximation Theorem):

If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous on the compact set  $[a, b]$ , then there exists a sequence of polynomials  $p_n$  such that  $\lim_{n \rightarrow \infty} \sup_{x \in [a, b]} |f(x) - p_n(x)| = 0$ .

That is, polynomials are dense in  $(C([a, b]), \|\cdot\|_{\infty})$ , where  $\|f\|_{\infty} := \sup_{x \in [a, b]} |f(x)|$ .

How to do this?

### Lagrange Interpolation

Give  $f \in C([a, b])$ .

Idea: subdivide  $[a, b]$  with  $a = x_0 < x_1 < \dots < x_n < b$  where  $x_k = x_0 + k \left( \frac{b-a}{n} \right)$ .

IMAGE HERE - UNIFORM SUBDIVISION

Let  $p_n(x) = \sum_{k=0}^n f(x_k) \prod_{j \neq k} \frac{x - x_j}{x_k - x_j}$ .

Problem: the Runge phenomenon.

IMAGE HERE - SMOOTHEST FUNCTION I CAN THINK OF (use the bump again)  $1/(1+25x^2)$

### Definition: Convolution

Take  $f, g : \mathbb{R} \rightarrow \mathbb{R}$ , define

$$h(x) = f * g(x) = \int_{\mathbb{R}} f(t)g(x-t) dt \underset{y=x-t}{=} \int_{\mathbb{R}} f(x-y)g(y) dy = g * f(x)$$

Take  $f, g \in C(\mathbb{R})$  with compact support ( $C_C(\mathbb{R})$ ). That is, they vanish outside a compact set.

IMAGE HERE - F AND G CONVOLVED

### Definition: Approximation of Identity

An approximation of the identity is a sequence  $\{g_n\}_n$ , all piecewise continuous, defined on  $\mathbb{R}$  such that

$$\begin{cases} g_n(x) \geq 0 & \forall x \\ \int_{\mathbb{R}} g_n(x) dx = 1 \\ \forall \delta > 0, & \lim_{n \rightarrow \infty} \int_{|x| > \delta} g_n(x) dx = 0 \end{cases}$$

IMAGE HERE - CONVOLUTION ACCUMULATING BETWEEN -DELTA AND DELTA

### Example

Let  $g_n(x) = \frac{n \cdot g(nx)}{\int_{\mathbb{R}} g(x) dx}$ .

### Lemma:

If  $\{g_n\}_n$  is an approximation of identity, then  $\forall f \in C_C(\mathbb{R})$

$$g_n * f \rightrightarrows f$$

on  $\mathbb{R}$ .

**October 30, 2023**

### Today

Approximation by polynomials.  
Fourier Series.

### Recall: Convolution

$f, g \in C_C(\mathbb{R})$ ,  $f * g(x) = \int_{\mathbb{R}} f(x-y)g(y) dy$ .

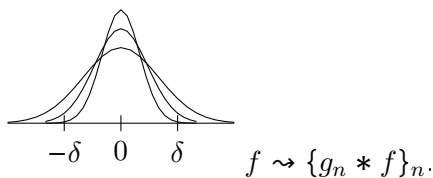
### Recall: Approximation of Identity

$\{g_n\}_n$  where  $g_n : \mathbb{R} \rightarrow \mathbb{R}$  is piecewise continuous (this is overkill but sufficient).

1.  $\int g_n dx = 1$ .

2.  $g_n(x) \geq 0$ .

3.  $\forall \delta > 0, \lim_{n \rightarrow \infty} \int_{|x| > \delta} g_n(x) dx = 0$ .



### Example

Take any  $g(x) \geq 0$  (piecewise continuous) with  $\int_{\mathbb{R}} g(x) dx = 1$ .

Define  $g_n(x) = n \cdot g(nx)$ .

Claim: this defined an approximation of identity.

## Lemma: Convolution of Approximation of Identity Converges Uniformly

Suppose  $\{g_n\}_n$  is an approximation of identity.

Then, for any  $f \in C_C(\mathbb{R})$ ,

$$g_n * f \quad \text{converges uniformly to } f \text{ on } \mathbb{R}$$

That is to say,  $\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}} |g_n * f(x) - f(x)| = 0$ .

### Proof

Since  $\int_{\mathbb{R}} g_n(y) dy = 1$ ,

$$\begin{aligned} g_n * f(x) - f(x) &= \int_{\mathbb{R}} g_n(y) f(x-y) dy - f(x) \cdot \int_{\mathbb{R}} g_n(y) dy \\ &= \int_{\mathbb{R}} g_n(y) (f(x-y) - f(x)) dy \\ &= \int_{|y| \geq \delta} g_n(y) \overbrace{(f(x-y) - f(x))}^{\leq 2M} dy + \int_{|y| < \delta} g_n(y) \overbrace{(f(x-y) - f(x))}^{\leq \epsilon} dy \end{aligned}$$

By assumption,  $f \in C_C(\mathbb{R})$  so  $f$  is bounded by  $M$  on  $\mathbb{R}$ .

$f$  is continuous on  $\text{supp}(f)$ , which is compact, so  $f$  is uniformly continuous on  $\mathbb{R}$ .

Let  $\epsilon > 0$  be given.

By uniform continuity,  $\exists \delta > 0, \forall x, y \in \mathbb{R}, |x-y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{2}$ .

By the Approximation of Identity property,  $\exists N, \forall n \geq N, \int_{|y| \geq \delta} g_n(y) dy < \frac{\epsilon}{4M}$ .

For  $n \geq N$ ,

$$\begin{aligned} |g_n * f(x) - f(x)| &= \left| \int_{\mathbb{R}} g_n(y) (f(x-y) - f(x)) dy \right| \\ &\leq \int_{|y| \geq \delta} g_n(y) \overbrace{|f(x-y) - f(x)|}^{\leq 2M} dy + \int_{|y| < \delta} g_n(y) \overbrace{|f(x-y) - f(x)|}^{\leq \frac{\epsilon}{2} \text{ since } |x-y-x|=|y| < \delta} dy \\ &\leq 2M \frac{\epsilon}{4M} + \frac{\epsilon}{2} \overbrace{\int_{|y| < \delta} g_n(y) dy}^{\leq 1} \\ &\leq \epsilon, \quad \forall x \in \mathbb{R} \quad \blacksquare \end{aligned}$$

## Recall: Riemann Integral Properties

If  $f$  is Riemann integrable, then

$$\left| \int f dx \right| \leq \int |f| dx$$

$$\left| \sum_{n=1}^{\infty} S_n \right| \leq \sum_{n=1}^{\infty} |S_n|$$

$$\left| \int f^+ dx - \int f^- dx \right| \leq \int f^+ dx + \int f^- dx = \int (f^+ + f^-) dx$$

## Theorem: Weierstrass Approximation Theorem

If  $[a, b]$  is compact, then  $\forall f \in C([a, b])$ , there exists a sequence of polynomials  $p_n(x)$  converging uniformly to  $f$ .

### Step 1

Extend  $f$  into  $F \in C_c(\mathbb{R})$ .

IMAGE HERE - EXTEND FUNCTION

$$F(x) = \begin{cases} 0 & \text{on } (-\infty, a-1] \cup [b+1, \infty) \\ f(x) & \text{on } [a, b] \\ f(a)(x - (a-1)) & \text{on } [a-1, a] \\ f(b)(b+1-x) & \text{on } [b, b+1] \end{cases}$$

### Step 2

Note:  $\forall \{g_n\}_n$  Approximation of Identity,  $g_n * f \rightrightarrows F(x)$  on  $\mathbb{R}$  (by previous lemma),

and  $\sup_{x \in [a, b]} |g_n * F(x) - f(x)| \leq \sup_{x \in \mathbb{R}} |g_n * F(x) - F(x)|$ .

Trick: Construct  $g_n$  such that  $g_n * F$  is a polynomial on  $[a, b]$ .

Answer:

$$g_n(x) = \begin{cases} a_n \left(1 - \frac{x^2}{(b-a+1)^2}\right)^n & \text{if } x \in [-(b-a+1), b-a+1] \\ 0 & \text{otherwise} \end{cases}$$

where  $a_n$  is chosen such that  $\int_{\mathbb{R}} g_n(x) dx = 1$ .

IMAGE HERE - NARROWING GAUSSIAN WITH PEAK AT (0,1) BETWEEN -(b-a+1) and b-a+1

If  $x \in [a, b]$  and  $y \in [a-1, b+1]$ , then

$$-b-1 \leq -y \leq -a+1 \implies -(b-a+1) \leq x-y \leq b-a+1$$

Then

$$\begin{aligned} g_n * F(x) &= \int_{a-1}^{b+1} F(y) \underbrace{g_n(x-y)}_{a_n \left(1 - \frac{(x-y)^2}{(b-a+1)^2}\right)^n = \sum_{p=0}^{2n} x^p a_{p,n(y)}} dy \\ &= \sum_{p=0}^{2n} x^p \int_{a-1}^{b+1} F(y) a_{p,n(y)} dy \quad \blacksquare \end{aligned}$$

## Background: Fourier Series

### Historical Perspective

In Strichartz.

Associated with solving the wave equation on  $[0, L]_x \times [0, T]_t$  (Bernoulli) and the heat equation (Fourier).

## Wave Equation

On  $[0, L]_x \times [0, T]_t$ ,  $u(x, t)$  displacement field.

IMAGE HERE - WAVE FROM 0 to L PEAK OF FIRST OSCILLATION AT U(X,T)

---

$$\frac{\partial^2 u}{\partial t^2}(x, t) = c^2 \frac{\partial^2 u}{\partial x^2}(x, t)$$

where  $c$  is a fixed coefficient.

Plus Initial Conditions and Boundary Conditions

$$\text{Initial Condition : } u|_{t=0}(x) = f(x)$$

$$\frac{\partial u}{\partial t}|_{t=0}(x) = 0$$

$$\text{Boundary Conditions : } u(0, t) = u(L, t) = 0$$

Observation: if  $f(x) = \sin\left(\frac{k\pi x}{L}\right)$ ,

IMAGE HERE - THREE SINUSOIDAL WAVES OVERLAPPING

Ansatz:  $u(x, t) = \sin\left(\frac{k\pi x}{L}\right)g(t)$ .

Plug into the PDE:

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= \sin\left(\frac{k\pi x}{L}\right)g''(t) \\ c^2 \frac{\partial^2 u}{\partial x^2} &= -\frac{k^2 \pi^2}{L^2} c^2 \sin\left(\frac{k\pi x}{L}\right)g(t)\end{aligned}$$

Setting

$$\sin\left(\frac{k\pi x}{L}\right)g''(t) = -\frac{k^2 \pi^2}{L^2} c^2 \sin\left(\frac{k\pi x}{L}\right)g(t) \xrightarrow[g]{\text{ode for}} g'' = -\frac{k^2 \pi^2}{L^2} c^2 g$$

Which gives a general solution

$$g(t) = A \cos\left(\frac{k\pi ct}{L}\right) + B \sin\left(\frac{k\pi ct}{L}\right).$$

Initial conditions imply that  $g(0) = 1$  and  $g'(0) = 0$  which gives

$$g(t) = \cos\left(\frac{k\pi ct}{L}\right).$$

Thus

$$u(x, t) = \sin\left(\frac{k\pi x}{L}\right) \cos\left(\frac{k\pi ct}{L}\right)$$

Solves the PDE!

## Wave Equation Superposition

Consider instead

$$f(x) = \sum_{k=0}^n \sin\left(\frac{k\pi x}{L}\right) a_k$$

Then

$$u(x, t) = \sum_{k=0}^n a_k \sin\left(\frac{k\pi x}{L}\right) \cos\left(c \frac{k\pi x}{L}\right)$$

### Next Question:

What if  $f$  is more general?

$\implies$  existence of Fourier series?

In what sense do they converge?

### Definition: Fourier Series

Context:  $f : [-\pi, \pi) \rightarrow \mathbb{R}$  Riemann-Integrable or

$f : \mathbb{R} \rightarrow \mathbb{R}$   $2\pi$ -periodic. ( $f(x + 2\pi) = f(x)$ ,  $\forall x$ ).

The Fourier series of  $f$ :

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^n a_k[f] \cos(kx) + b_k[f] \sin(kx)$$

where  $a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx$  and  $b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx$ .

Alternatively,

$$S_n(x) = \sum_{k=-n}^n c_k e^{ikx}$$

where  $c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$ .

As an exercise: relate  $c_k$ s to  $a_k$ s and  $b_k$ s and prove that these are equivalent.

### Question:

In what sense does  $S_n(x)$  converge to  $f(x)$ ? That is

- For what topology?

– Uniform Convergence:  $\sup_{x \in [-\pi, \pi)} |S_n(x) - f(x)| \xrightarrow{n \rightarrow \infty} 0$

–  $L^2$  Convergence:  $\int_{-\pi}^{\pi} |S_n(x) - f(x)|^2 dx \xrightarrow{n \rightarrow \infty} 0$

- What are the (smoothness) requirements on  $f$ ?

– Observation: if  $f(x) = \sum_{k=-N}^N f_k e^{ikx}$  is a trigonometric polynomial, then, for  $n \geq N$ ,  $S_n(x) = f(x)$ .

## Lemma: The Kronecker Delta

Fix  $N \in \mathbb{N}$

If  $\sum_{k=-N}^N f_k e^{ikx} = \sum_{k=-N}^N c_k e^{ikx}$ , then  $f_k = c_k, \forall k$ .

Note

$$\int_{-\pi}^{\pi} e^{ikx} e^{-imx} dx = \int_{-\pi}^{\pi} e^{i(k-m)x} dx = \begin{cases} 2\pi & \text{if } k = m \\ \left[ \frac{1}{i(k-m)} e^{i(k-m)x} \right]_{-\pi}^{\pi} = 0 & \text{otherwise} \end{cases}$$

Why  $-imx$ ?

$$\begin{aligned} \langle if, g \rangle &= i \langle f, g \rangle \\ \langle f, ig \rangle &= -i \langle f, g \rangle \end{aligned}$$

and

$$\int_{-\pi}^{\pi} f(x) \overline{g(x)} dx$$

**November 1, 2023**

## Fourier Series

For  $f$  Riemann-integrable on  $(-\pi, \pi)$ , define

$$S_n(x) = \sum_{k=-n}^n c_k e^{ikx}$$

with

$$c_k := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$$

Then  $f : [-\pi, \pi) \rightarrow \mathbb{R}$ .

## Recall

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikx} e^{-ilx} dx = \begin{cases} 0 & \text{if } k \neq l \\ 1 & \text{if } k = l \end{cases} = \delta_{kl} \text{ (the Kronecker delta)}$$

## Definition: Norm

$|| \cdot || : E \rightarrow \mathbb{R}_{\geq 0}$  is a “norm” on  $E$  if

1.  $||f|| = 0 \iff f \equiv 0$
2.  $||\lambda f|| = |\lambda| \cdot ||f||, \forall \lambda \in \mathbb{R}, f \in E$
3.  $||f + g|| \leq ||f|| + ||g||$



**Definition: Normed Space**

$(E, || \cdot ||)$  is a normed space.  
 e.g.  $(\mathbb{R}, | \cdot |)$  or  $(\mathbb{Q}, | \cdot |)$

**Definition: Complete Space**

$(E, || \cdot ||)$  is complete if every cauchy sequence in  $E$  converges in  $E$ .

**In what sense does a Fourier series converge?**

Depends on regularity of  $f$  and the topology used.

**Note**

On  $C([-\pi, \pi])$ , can put 2 norms.

- $||f||_{\infty} = \sup_{x \in [-\pi, \pi]} |f(x)|$

$d(f, g) = ||f - g||_{\infty}$ : “ $f_n$  converges uniformly to  $f$ ”  $\leftrightarrow \lim_{n \rightarrow \infty} ||f_n - f||_{\infty} = 0$ .  
 $(C([-\pi, \pi]), || \cdot ||_{\infty})$  is complete.

- $||f||_2 := \left( \int_{-\pi}^{\pi} f^2(x) dx \right)^{1/2}$

“ $f_n$  converges to  $f$  in  $L^2$ ”  $\leftrightarrow \lim_{n \rightarrow \infty} ||f_n - f||_2 = 0$ .  
 $(C([-\pi, \pi]), || \cdot ||_2)$  is not complete.

**Example**

Take  $f(x) = \begin{cases} 1 & \text{if } |x| \leq \pi/2 \\ 0 & \text{if } |x| > \pi/2 \end{cases}$

IMAGE HERE - BOX FUNCTION FROM  $-\pi/2$  to  $\pi/2$



$$c_k = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{-ikx} dx = \frac{1}{2\pi} \left[ \frac{1}{-ik} e^{-ikx} \right]_{-\pi/2}^{\pi/2} = \frac{1}{2\pi} \frac{1}{-ik} \left[ e^{-ik(\pi/2)} - e^{ik(\pi/2)} \right] = \frac{1}{k\pi} \sin(k(\pi/2))$$

So  $c_k = 0$  and for  $k = 2p + 1$ :  $c_{2p+1} = \frac{(-1)^p}{\pi(2p+1)}$ .

IMAGE HERE - BOX FUNCTION WITH SINUSOIDALS APPROXIMATING

However, the approximation will over and undershoot at the boundaries. This is the “Gibbs Phenomenon”, and the discrepancy is roughly 12%.

For  $k < 0$ :

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{ikx} dx = \overline{c_{-k}} = c_k$$

In the end,

$$S_{2p+1}(x) = \sum_{l=1}^p \frac{(-1)^p}{\pi(2p+1)} \underbrace{\left( e^{i(2p+1)x} + e^{-i(2p+1)x} \right)}_{2 \cos((2p+1)x)}$$

### Theorem: Uniform Convergence of Continuously Differentiable Continuous Functions

1. If  $f$  is  $C^2$ ,  $2\pi$ -periodic, then  $S_n \Rightarrow f$  on  $[-\pi, \pi)$ .

Moreover,  $\|S_n - f\|_\infty \leq \frac{c}{n}$  for some  $c > 0$ .

1. If  $f \in C^1$ ,  $2\pi$ -periodic, same conclusion with  $\|S_n - f\|_\infty \leq \frac{c}{\sqrt{n}}$  for some  $c > 0$ .

### Proof of Part 1

Write

$$S_n(x) = \sum_{k=-n}^n c_k e^{ikx} = \sum_{k=-n}^n \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iky} dy e^{ikx} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \underbrace{\left[ \sum_{k=-n}^n e^{ik(x-y)} \right]}_{D_n(x-y)} dy$$

Where  $D_n(t) = \sum_{k=-n}^n e^{ikt}$  is the “Dirichlet kernel.”

That is  $S_n$  is a convolution of  $f(y)$  with some kernel.

$$e^{it} \cdot D_n(t) = \sum_{k=-n}^n e^{i(k+1)t} = \sum_{l=k+1}^{n+1} e^{ilt} = D_n(t) + e^{i(n+1)t} - e^{-int}$$

Therefore

$$D_n(t) = \frac{e^{i(n+1)t} - e^{-int}}{e^{it} - 1} = \frac{e^{(it)/2} \left( e^{i(n+(1/2))t} - e^{-i(n+(1/2))t} \right)}{e^{(it)/2} \left( e^{(it)/2} - e^{-(it)/2} \right)} = \frac{\sin((n+1/2)t)}{\sin(t/2)}$$

IMAGE HERE - DN(T) OSCILLATING WITH MANY ZEROS THEN PEAKING TO  $2N+1$  at  $X=0$   
 \_\_\_\_\_ So

$$\int_{-\pi}^{\pi} D_n(t) dt = 2\pi$$

Then

$$\begin{aligned} S_n(x) - f(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) D_n(x-y) dy - f(x) \\ &\stackrel{z=x-y}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-z) D_n(z) dz - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) D_n(z) dz \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x-z) - f(x)) D_n(z) dz \end{aligned}$$

and

$$S_n(x) \cdot f(x) = \frac{1}{2\pi} \underbrace{\frac{(f(x-y) - f(x))}{\sin(y/2)}}_{\text{call } g_x(y) = \frac{f(x-y) - f(x)}{\sin(y/2)}} \sin((n + (1/2)y) dy$$

If  $g_x(y)$  was differentiable (in fact  $C^1$ ), then integrating by parts

$$S_n(x) - f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_x(y) \sin((n + (1/2)y) dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} g'_x(y) \frac{\cos((n + (1/2)y)}{n + (1/2)} dy$$

Then

$$|S_n(x) - f(x)| \leq \sup_{y \in [-\pi, \pi]} |g'_x(y)| \frac{1}{n + (1/2)}$$

- Claim

If  $f \in C^2$ ,  $2\pi$ -periodic, then  $\sup_{x \in [-\pi, \pi]} |g'_x(y)| < \infty$ . Then the first part of the theorem is proved.

– Proof of Claim

$f \in C^2 \implies g_x \in C^2$  away from  $y = 0$ . ( $g''_x(y)$  = differentiation rules).

At  $y = 0$ , write

$$f(x-y) - f(x) = \int_x^{x-y} f'(t) dt$$

Changing variables such that  $t = x + u(x - y - x) = x - uy$  for  $u \in [0, 1]$  gives  $dt = -y du$

$$= -y \int_0^1 f'(x - uy) du$$

Therefore

$$g_x(y) = \underbrace{\left( \frac{-y}{\sin(y/2)} \right)}_{\text{smooth near } y=0} \int_0^1 f'(x - uy) du$$

Calling the smooth piece  $h(y)$ ,

$$g_x(y) = h(y) \int_0^1 f'(x - yu) du$$

is differentiable at 0 if and only if  $\frac{d}{dy} \left( \int_0^1 f'(x - yu) du \right) = \int_0^1 f''(x - yu)(-u) du$  exists. ■

## Proof of Part 2 (Sketch)

$$S_n(x) - f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_x(y) \sin(n + (1/2)y) dy$$

If  $f$  is only  $C^1$ , then  $g$  is  $C^1$  away from 0, so it is unclear near  $y = 0$ . So, for some  $\delta$  to be chosen later

$$S_n(x) - f(x) = \underbrace{\frac{1}{2\pi} \int_{[-\delta, \delta]} g_x(y) \sin((n + (1/2))y) dy}_{\leq \frac{2\delta}{2\pi} (\|f'\|_\infty + \|f\|_\infty)} + \underbrace{\frac{1}{2\pi} \int_{[-\pi, \pi] \setminus [-\delta, \delta]} g_x(y) \sin((n + (1/2))y) dy}_{\text{integrate by parts}}$$

Study  $\int_\delta^\pi$  (study of  $\int_{-\pi}^{-\delta}$  is similar)

$$\begin{aligned} \int_\delta^\pi g_x(y) \sin((n + (1/2))y) dy &= \int_\delta^\pi g_x(y) \frac{d}{dy} \left( \frac{-\cos((n + (1/2))y)}{n + (1/2)} \right) dy \\ &= \int_\delta^\pi \frac{d}{dy} \left( g_x(y) \frac{-\cos((n + (1/2))y)}{n + (1/2)} \right) - \int_\delta^\pi g'_x(y) \frac{\cos((n + (1/2))y)}{n + (1/2)} dy \\ &= -g_x(\pi) \frac{\cos((n + (1/2))\pi)}{n + (1/2)} + g_x(\delta) \frac{\cos((n + (1/2))\delta)}{n + (1/2)} - \int_\delta^\pi g'_x(y) \frac{\cos((n + (1/2))y)}{n + (1/2)} dy \end{aligned}$$

Problem:

$$g'_x(y) = \frac{-f'(x - y) \sin(y/2) - (1/2) \cos(y/2)(f(x - y) - f(x))}{(\sin(y/2))^2} \approx \frac{c}{y} \text{ near } y = 0$$

So

$$\left| \int_\delta^\pi g_x(y) \sin((n + (1/2))y) dy \right| \leq \frac{1}{n + (1/2)} \cdot \frac{1}{\delta}$$

Combining all estimates, for  $\delta > 0$

$$|S_n(x) - f(x)| \leq C_1 \delta + C_2 \frac{1}{n\delta}$$

Since we are free to choose  $\delta$ , we may optimize over  $\delta$ .

Balancing out the terms is done by choosing  $\delta = \delta(n)$  such that

$$\delta \stackrel{n \rightarrow \infty}{\sim} \frac{1}{n\delta} \iff n\delta^2 \sim 1 \iff \delta \sim \frac{1}{\sqrt{n}}$$

which gives

$$|S_n(x) - f(x)| \leq C_1 \delta + C_2 \frac{1}{n\delta} = \frac{C_1}{\sqrt{n}} + C_2 \frac{1}{n \frac{1}{\sqrt{n}}} \leq \frac{C_1 + c_2}{\sqrt{n}}$$

- Comment on the Sketch

Morally, we want  $|g'_x(y)| \leq \frac{c}{y}$  for some constant  $c$ .

Numerator:

$$\left| -f'(x - y) \sin(y/2) - (1/2) \cos(y/2)(f(x - y) - f(x)) \right| \leq \|f'\|_\infty (y/2) + (\dots)y \leq Cy$$

Since  $|\sin(y/2)| \leq (y/2)$ ,

$$\begin{aligned} |\sin(x) - \sin(0)| &= |\cos(\xi)| |x - 0| \\ &= 1|x| \end{aligned}$$

Denominator

$$(\sin(y/2))^2 \geq \left(\frac{2y}{2\pi}\right)^2 = \frac{y^2}{\pi}$$

So,

$$\left|g'_x(y)\right| \leq \frac{Cy}{\left(\frac{y}{\pi}\right)^2} \leq \frac{C^1}{y}$$

### Theorem: Continuous, Periodic Functions Converge in L2

If  $f$  is continuous,  $2\pi$ -periodic, then  $\lim_{n \rightarrow \infty} \|S_n - f\|_2 = 0$ .

That is,  $\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} |S_n - f(x)|^2 dx = 0$ .

IMAGE HERE - PERIODIZE  $f(x)=x$  THEN APPROXIMATE WITH FOURIER

**November 6, 2023**

**Recall: Fourier Series**

$$f : [-\pi, \pi] \rightarrow \mathbb{R} \text{ or } \mathbb{C}$$

Fourier Coefficient:

$$c_k(f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{ikx} dx, \quad k \in \mathbb{Z}$$

$$s_n(x) = \sum_{k=-n}^n c_k e^{ikx} = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(x-y) f(y) dy$$

Dirichlet Kernel:

$$D_n(y) := \frac{\sin((n+1/2)y)}{\sin((1/2)y)}$$

### Theorem: L2 Convergence of $S_n$ to $N$

If  $f$  is  $C^0$ ,  $2\pi$ -periodic, then

$$\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} |s_n(x) - f(x)|^2 dx = 0$$

## Recall: Kronecker Delta

For  $m, n \in \mathbb{Z}$ ,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{imx} e^{-inx} dx = \delta_{m,n} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

That is  $\{e^{inx}\}_{n \in \mathbb{Z}}$  is an orthonormal system for the inner product

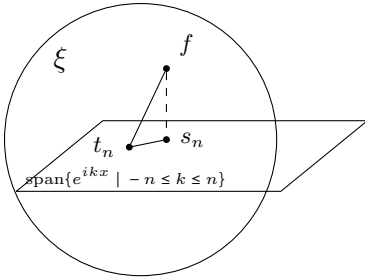
$$\begin{aligned} \xi \times \xi &\rightarrow \mathbb{C} \\ (f, g) &\mapsto \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx \end{aligned}$$

where  $\xi = \{f : \mathbb{R} \rightarrow \mathbb{C}, 2\pi\text{-periodic, continuous}\}$ .

## Example

For  $f \in \xi$ , fixing  $n \in \mathbb{N}_0$ , consider the map

$$\begin{aligned} \mathbb{C}^{2n+1} &\rightarrow \mathbb{R} \\ (d_{-n}, \dots, d_n) &\mapsto \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - \sum_{k=-n}^n d_k e^{ikx}|^2 dx \end{aligned}$$



- Claim:

$F_n$  is minimal if and only if  $\lambda_k = c_k(f)$ ,  $\forall -n \leq k \leq n$ .

- Proof:

Take any  $\lambda_n, \lambda_{n+1}, \dots, \lambda_n$  and set  $t_n(x) = \sum_{k=-n}^n \lambda_k e^{ikx}$ . Then

$$\int_{-\pi}^{\pi} |f(x) - t_n(x)|^2 dx = \int_{-\pi}^{\pi} |f(x) - s_n(x) + s_n(x) - t_n(x)|^2 dx$$

Then, since

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) = |z_1|^2 + |z_2|^2 + 2 \cdot \Re(z_1 \overline{z_2})$$

$$\begin{aligned} &\int_{-\pi}^{\pi} |f(x) - t_n(x)|^2 dx \\ &= \int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 dx + \int_{-\pi}^{\pi} |s_n(x) - t_n(x)|^2 dx + 2 \cdot \Re \int_{-\pi}^{\pi} (f(x) - s_n(x)) \overline{(s_n(x) - t_n(x))} dx \end{aligned}$$

What to Show: Integral on real part is zero.

$$\begin{aligned}
A &= \int_{-\pi}^{\pi} (f(x) - s_n(x)) \overline{\sum_{k=-n}^n (c_k - \lambda_k) e^{ikx}} dx \\
&= \sum_{k=-n}^n \overline{(c_k - \lambda_k)} \underbrace{\int_{-\pi}^{\pi} (f(x) - s_n(x)) e^{-ikx} dx}_{2\pi(c_k - c_k)=0}
\end{aligned}$$

Since

$$\int_{-\pi}^{\pi} s_n(x) e^{-ikx} dx = \int_{-\pi}^{\pi} \sum_{p=-n}^n c_p e^{ipx} e^{-ikx} dx = 2\pi c_k$$

It follows that

$$\begin{aligned}
\underbrace{\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - t_n(x)|^2 dx}_{F_n(\lambda_{-n}, \dots, \lambda_n)} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 dx + \frac{1}{2\pi} \int_{-\pi}^{\pi} |t_n(x) - s_n(x)|^2 dx \\
&\geq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_n(x)| dx \\
&\geq F_n(c_{-n}, \dots, c_n)
\end{aligned}$$

Moreover:

$$\begin{aligned}
\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \underbrace{t_n(x) - s_n(x)}_{\underbrace{(t_n - s_n)}_{\sum_{p=-n}^n (\lambda_p - c_p) e^{ipx}}} \right|^2 dx &= \frac{1}{2\pi} \sum_{p,l=-n}^n (\lambda_p - c_p) \overline{(\lambda_l - c_l)} \underbrace{\int_{-\pi}^{\pi} e^{ipx} e^{-ilx} dx}_{\delta_{p,l}} \\
&= \frac{1}{2\pi} \sum_{p=-n}^n |\lambda_p - c_p|^2
\end{aligned}$$

Conclusion:

- \*  $\forall (\lambda_{-n}, \dots, \lambda_n \neq (c_{-n}, \dots, c_n), F_n(\lambda_{-n}, \dots, \lambda_n) > F_n(c_{-n}, \dots, c_n)$
- \*  $F_n(c_{-n}, \dots, c_n) = F_n(c_{-n}, \dots, c_n)$
- \* Lemma

For all trigonometric polynomials of degree at most  $n$ , of the form  $\sum_{k=-n}^n \lambda_k e^{ikx} = t_n(x)$ ,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 dx \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - t_n(x)|^2 dx$$

and

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |t_n(x)|^2 dx = \sum_{k=-n}^n |\lambda_k|^2$$

Apply this to  $(\lambda_{-n}, \dots, \lambda_n) = (0, \dots, 0)$ :

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 dx + \sum_{k=-n}^n |c_k|^2$$

As a consequence, for all  $n$ ,

$$\sum_{k=-n}^n |c_k|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$$

which implies that  $\sum_{k=-n}^n |c_k|^2$  converges absolutely and, in particular,  $c_k \rightarrow 0$  as  $k \rightarrow \infty$ .

### Riemann-Lebesgue Lemma

The above proves that if  $f \in \xi$  (more generally, if  $f$  is Riemann-integrable), then

$$\lim_{k \rightarrow \infty} \int_{-\pi}^{\pi} f(x) e^{\pm i k x} dx = 0$$

Moreover, sending  $n \rightarrow \infty$ , we get

$$\lim_{n \rightarrow \infty} \sum_{k=-n}^n |c_k|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$$

Importantly, there is equality whenever  $\lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 dx = 0$ .  
When does that happen?

### Theorem:

If  $f \in \xi$ , then

$$\lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 dx = 0$$

### Proof

For  $n \geq 0$ , define  $\sigma_n = \frac{1}{n+1} \sum_{k=0}^n s_k$  (the “Cesano sum”).  
Then

$$\sigma_n \in \text{span}\langle e^{-inx}, \dots, e^{inx} \rangle.$$

In particular,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 dx \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - \sigma_n(x)|^2 dx \leq \left( \sup_{[-\pi, \pi]} |f - \sigma_n| \right)^2$$

What to show:  $\sigma_n \rightrightarrows f$  on  $[-\pi, \pi]$ .

Recall that



$$s_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(x-y) f(y) dy$$

$$\begin{aligned} \sigma_n(x) &= \frac{1}{n+1} \sum_{k=0}^n s_k(x) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(x-y) f(y) dy \end{aligned}$$

Where

$$\begin{aligned} K_n(y) &= \frac{1}{n+1} \sum_{k=0}^n D_k(y) \\ &= \frac{1}{n+1} \frac{1}{\sin(y/2)} \sum_{k=0}^n \sin((k+1/2)y) \end{aligned}$$

Using  $2 \sin((k+1/2)y) \sin(y/2) = \cos(ky) - \cos((k+1)y)$ .

$$\begin{aligned} &= \frac{1}{n+1} \frac{1}{(\sin(y/2))^2} \frac{1}{2} \sum_{k=0}^n \underbrace{\cos(ky) - \cos((k+1)y)}_{\substack{\frac{1-\cos((n+1)y)}{2} \\ \underbrace{\qquad\qquad\qquad}_{\sin^2((\frac{n+1}{2})y)}}} \\ &= \frac{1}{n+1} \left( \frac{\sin\left(\left(\frac{n+1}{2}\right)y\right)}{\sin(y/2)} \right)^2 \end{aligned}$$

This is the Féjer kernel.

IMAGE HERE - FÉJER KERNEL

Claims:

1.  $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(y) dy = 1$
2.  $K_n(y) \geq 0$  on  $[-\pi, \pi]$  (obvious)
3.  $\forall \delta > 0, K_n \rightrightarrows 0$

- Proof of 1

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(y) dy = \frac{1}{2\pi} \frac{1}{n+1} \sum_{k=0}^n \underbrace{\int_{-\pi}^{\pi} D_k(y) dy}_{2\pi} = 1$$

- Proof of 3 If  $|y| \geq \delta$ ,

$$|K_n(y)| = \frac{1}{n+1} \frac{\overbrace{|\sin((n+1)y/2)|^2}^{\leq 1}}{|\sin(y/2)|^2}$$

Recall  $|\sin(x)| \geq \frac{2|x|}{\pi}$

$$\begin{aligned} &\leq \frac{1}{n+1} \frac{1}{(|y|/\pi)^2} \\ &\leq \frac{1}{n+1} \frac{1}{(\delta/\pi)^2} \end{aligned}$$

Which goes uniformly to 0 on  $[-\pi, \pi] \setminus [-\delta, \delta]$  as  $n \rightarrow \infty$ .

What to show:  $K_n * f \rightrightarrows f$  on  $[-\pi, \pi]$ .

The proof scheme is identical to: if  $f \in C_c(\mathbb{R})$  and  $K_n$  is an approximation of identity, then  $K_n * f \rightrightarrows f$  on  $\mathbb{R}$ .

Left as an exercise.

## Corollary: Parseval's Equality

$\forall \delta \in \xi$ ,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \lim_{n \rightarrow \infty} \sum_{k=-n}^n |c_k|^2$$

## Remark:

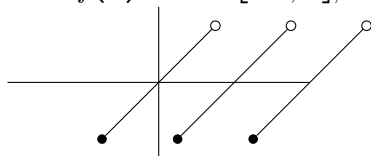
This should hold for a larger class of function.

- Piecewise Continuous

- $L^2$  functions

## Example

Take  $f(x) = x$  on  $[-\pi, \pi]$ ,  $2\pi$ -periodized



Then  $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ .

## Application to Solving the Heat Equation

On  $[0, L]_x \times \mathbb{R}_+$ ,  $u(x, t)$  is the “heat distribution”

IMAGE HERE - ONE DIMENSIONAL ROD HEAT EQUATION YADA YADA

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right)$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( -\frac{\partial u}{\partial x} \right) = 0$$

## Problem

$$\begin{array}{lll}
 \text{PDE} & \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} & \text{on } [0, L] \times [0, T] \\
 \text{Boundary Conditions} & u(0, t) = u(L, t) = 0 & \\
 \text{Initial Conditions} & u(x, 0) = f(x) & f \text{ continuous, } f(0) = f(L) = 0
 \end{array}$$

IMAGE HERE - POSITION TIME PLANE

- Step 1: Separation of Variables

Seek an ansatz of the form

$$u(x, t) = g(x)h(t)$$

Where

$$\begin{aligned}
 \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} &\iff g(x)h'(t) = g''(x)h(t) \\
 &\iff \frac{h'(t)}{h(t)} = \frac{g''(x)}{g(x)} = c
 \end{aligned}$$

Left Solving:

$$g''(x) = cg(x) \quad g(0) = 0 = g(L)$$

$$h'(t) = ch(t) \rightsquigarrow h(t) = h(0)e^{ct}$$

Then

$$\begin{aligned}
 g''(x) - cg(x) = 0 &\rightsquigarrow c = 0. \quad g(x) = a + bx \\
 c > 0. &\quad g(x) = ae^{\sqrt{c}x} + be^{-\sqrt{c}x} \\
 c < 0. &\quad g(x) = a \cos(\sqrt{-c}x) + b \sin(\sqrt{-c}x)
 \end{aligned}$$

and

$$g(0) = 0 = g(L) \implies \begin{cases} c = 0 : & g \equiv 0 & \text{(no solution)} \\ c > 0 : & g \equiv 0 & \text{(no solution)} \\ c < 0 : & a = 0. \quad g(x) = b \sin(\sqrt{-c}x) \end{cases}$$

$$\begin{aligned}
 g(L) = 0 &\implies \sin(\sqrt{-c}L) = 0 \\
 &\implies L\sqrt{-c} = k\pi \\
 &\implies c = -\left(\frac{k\pi}{L}\right)^2, k \in \mathbb{N}_0
 \end{aligned}$$

$$\text{For } c = -\left(\frac{k\pi}{L}\right)^2 = \lambda_k,$$

$$g_k(x) = \sin\left(\frac{k\pi x}{L}\right)$$

$$h_k(x) = h_k(0) \exp\left(-\left(\frac{k\pi}{L}\right)^2 t\right)$$

For all  $k \in \mathbb{N}_0$ ,

$$u_k(x, t) = g_k(x)h_k(t)$$

solves the heat equation with boundary conditions.

Initial conditions  $g_k(x)$ , fix  $h_k(0) = 1$ .

Ansatz for a solution:

$$u(x, t) = \sum_{k=0}^{\infty} a_k g_k(x) h_k(t) \implies u(x, 0) = \sum_{k=0}^{\infty} a_k \sin\left(\frac{k\pi x}{L}\right) = f(x)$$

Thus, the left hand side is the solution.

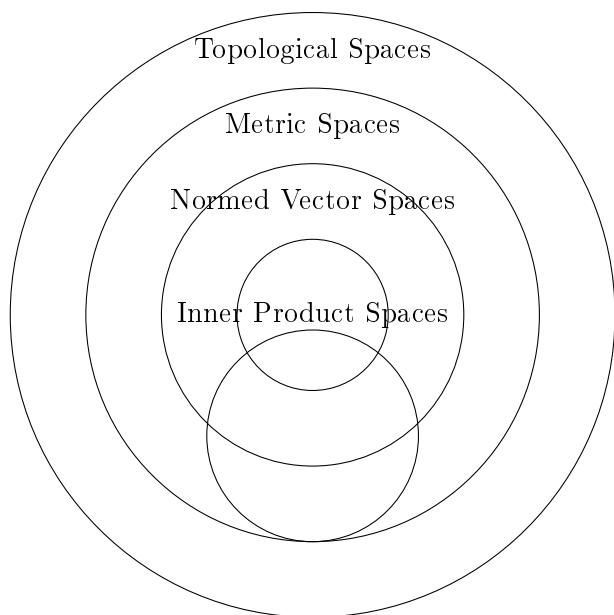
**November 8, 2023**

## Topology of Metric Spaces

### Definition: Topological Space

A pair  $(X, \tau)$  is a topological space if

- $X$  is a set.
  - $\tau \subseteq \mathcal{P}(X)$  and satisfies
    1.  $\emptyset, X \in \tau$
    2.  $\tau$  is stable under arbitrary unions and finite intersections.
  - Elements of  $\tau$  are called “open sets”.
- IMAGE HERE - ADD COMPLETE BANACH HILBERT



### Definition: Vector Space

$(E, +, \cdot)$  is a vector space (over  $\mathbb{R}$ ) if

There are two operations  $+: E \times E \rightarrow E$  and  $\cdot: \mathbb{R} \times E \rightarrow E$  ( $(\lambda, x) \mapsto \lambda x$ ) such that

$$(E, +) \text{ is a commutative group } \begin{cases} x + y = y + x & \forall x, y \in E \\ (x + y) + z = x + (y + z) & \forall x, y, z \in E \\ \exists 0 \in E \text{ such that } x + 0 = 0 + x = x & \forall x \in E \\ \forall x \in E, \exists -x \in E, \text{ such that } x + (-x) = (-x) + x = 0 \end{cases}$$

$$\text{and } \begin{cases} \lambda(x + y) = \lambda \cdot x + \lambda \cdot y & \forall \lambda \in \mathbb{R}, x, y \in E \\ (\lambda \cdot \mu) \cdot x = \lambda \cdot (\mu \cdot x) & \forall \lambda, \mu \in \mathbb{R}, x \in E \\ (\lambda + \mu) \cdot x = \lambda \cdot x + \mu \cdot x & \forall \lambda, \mu \in \mathbb{R}, x \in E \end{cases}$$

### Example 1

$$(\mathbb{R}, +, \cdot)$$

### Example 2

$$\begin{aligned} \mathbb{R}^n : x &= (x_1, \dots, x_n) \\ x + y &= (x_1 + y_1, \dots, x_n + y_n) \\ \lambda \cdot x &= (\lambda x_1, \dots, \lambda x_n) \end{aligned}$$

### Example 3

Functions from  $\mathbb{R} \rightarrow \mathbb{R}$

$$\begin{aligned} (f + g)(x) &= f(x) + g(x) \\ \lambda \cdot f(x) &= \lambda f(x) \end{aligned}$$

Sequences  $\mathbb{N}_0 \rightarrow \mathbb{R}$   
 $C(\mathbb{R}); C^k(\mathbb{R}), \forall k; C^\infty(\mathbb{R}); C^w(\mathbb{R})$  (real-analytic functions).

### Definition: Normed Vector Space

A norm on a vector space  $E$  is a map  $||\cdot|| : E \rightarrow \mathbb{R}$  such that

1.  $||x|| \geq 0, \forall x \in E$ , with equality if and only if  $x = 0$ .
2.  $||\lambda x|| = |\lambda| ||x||, \forall \lambda \in \mathbb{R}, \forall x \in E$ .
3.  $||x + y|| \leq ||x|| + ||y||, \forall x, y \in E$  (triangle inequality).

$(E, ||\cdot||)$  is a normed vector space.

### Example 1

$$\text{On } \mathbb{R}^n : ||x||_\infty = \max_{1 \leq i \leq n} |x_i|$$

$$1 \leq p \leq \infty ||x||_p = \left( \sum_{i=1}^n |x_i|^p \right)^{1/p}$$

### Example 2

$$\text{On } \mathbb{C}([a, b]) : ||f||_\infty = \sup_{x \in [a, b]} |f(x)|$$

$$[a, b] \text{ compact } ||f||_p = \left( \int_a^b |f(x)|^p dx \right)^{1/p}$$

### Definition: Inner Product Space

An inner product on  $\mathbb{R}$ -vector space  $E$  is a map  $\langle \cdot, \cdot \rangle : E \times E \rightarrow \mathbb{R}$  such that

1. It is bilinear:  $\langle \lambda f + \mu g, h \rangle = \lambda \langle f, h \rangle + \mu \langle g, h \rangle$ , for all  $\lambda, \mu \in \mathbb{R}, f, g, h \in E$
2. It is symmetric:  $\langle f, g \rangle = \langle g, f \rangle, \forall f, g \in E$
3. It is positive definite:  $\langle f, f \rangle \geq 0$ , with equality if and only if  $f = 0$ .

The pair  $(E, \langle \cdot, \cdot \rangle)$  is called an inner product space (or a pre-hilber space).

### Example 1

$$\text{On } \mathbb{R}^n : \langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n \\ x = (x_1, \dots, x_n)$$

- Proof
  1. Satisfied by properties of  $\mathbb{R}$ .
  2. Satisfied by multiplicative commutativity.
  3.  $\langle x, x \rangle = \sum_{i=1}^n x_i^2 = 0$  if and only if  $x_i = 0, \forall i$ .

### Example 2

On  $\mathbb{R}^n : A(a_{ij})_{i,j=1}^n$  symmetric positive definite matrix.  
Then  $\langle x, y \rangle_A := \langle x, Ay \rangle$  is an inner product.  
Notice  $\langle x, x \rangle = ||x||_2^2$ .

### Example 3

On  $C([a, b])$ ,  $\langle f, g \rangle = \int_a^b f(t)g(t) dt$ .

### Fact: Every Inner Product Gives Rise to a Norm

If the inner product is  $\langle \cdot, \cdot \rangle$ , then the norm is  $||x|| = \langle x, x \rangle^{1/2}$ .  
But not every norm comes from an inner product.

### Proposition:

Let  $(E, \langle \cdot, \cdot \rangle)$  an inner product space.  
Denote  $||x|| := \langle x, x \rangle^{1/2}$  for  $x \in E$ .  
Then

1.  $\forall x, y \in E, |\langle x, y \rangle| \leq ||x|| ||y||$  (Cauchy-Schwarz)
2.  $\forall x, y \in E, ||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2)$  (Parallelogram Identity)
3.  $\forall x, y \in E, ||x + y|| \leq ||x|| + ||y|| \implies ||\cdot||$  is a norm.

- Proof of 1
$$\forall t \in \mathbb{R}, \langle x + ty, x + ty \rangle \geq 0$$

$$0 \leq \langle x + ty, x + ty \rangle = \langle x, x \rangle + 2t\langle x, y \rangle + t^2\langle y, y \rangle$$

Therefore the discriminant is less than 0 and

$$(2\langle x, y \rangle)^2 - 4\langle x, x \rangle \langle y, y \rangle \leq 0$$

implies that  $\langle x, y \rangle^2 \leq ||x||^2 ||y||^2$ .

- Proof of 2

$$\begin{aligned} ||x+y||^2 + ||x-y||^2 &= \langle x+y, x+y \rangle + \langle x-y, x-y \rangle \\ &= ||x||^2 + 2\langle x, y \rangle + ||y||^2 + ||x||^2 - 2\langle x, y \rangle + ||y||^2 \\ &= 2(||x||^2 + ||y||^2) \end{aligned}$$

- Proof of 3

$$\begin{aligned} || &= ||x||^2 + 2\langle x, y \rangle + ||y||^2 \\ &\stackrel{\text{CS}}{\leq} ||x||^2 + 2||x|| ||y|| + ||y||^2 \\ &\leq (||x|| + ||y||)^2 \\ &\xrightarrow{\sqrt{\phantom{x}}} ||x+y|| \leq ||x|| + ||y|| \end{aligned}$$

### Proposition:

Let  $(E, || \cdot ||)$  be a normed space such that  $|| \cdot ||$  satisfies the parallelogram law, then

$$\langle x, y \rangle := \underbrace{\frac{1}{4} (||x+y||^2 - ||x-y||^2)}_{\text{"polarization identity"}}$$

is an inner product on  $E$ .

### Definition: Metric Space

A pair  $(X, d)$  is a metric space if

- $X$  is a set.
- $d : X \times X \rightarrow \mathbb{R}$  such that

1.  $d(x, y) \geq 0$ ,  $\forall x, y \in X$  with equality if  $x = y$ .
2.  $d(x, y) = d(y, x)$ ,  $\forall x, y \in X$ .
3.  $d(x, y) \leq d(x, z) + d(z, y)$ ,  $\forall x, y, z \in X$ .

$d$  is a "distance function."



**Example 1**

On  $\mathbb{R}$ ,  $d(x, y) = |x - y|$ .

**Example 2**

On  $(E, \|\cdot\|)$ ,  $d(x, y) = \|x - y\|$  is a distance function.

Note that  $d(x + z, y + z) = \|x + z - y - z\| = \|x - y\|$  (translation-invariance).

$\|x - y\| = \|x - z + z - y\| \leq \|x - z\| + \|z - y\|$ .

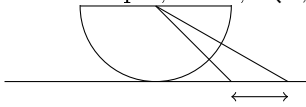
Therefore, every normed space gives rise to a metric space.

**Proposition:**

Not every metric space is a normed space.

If  $E$  is a vector space, might be interested in non-translation-invariant distances.

For example, on  $\mathbb{R}$ ,  $d(x, y) = |\tan^{-1}(x) - \tan^{-1}(y)|$ .

**Proposition:**

Also,  $E$  might not be a vector space.

For example  $S^1$ , manifolds, graphs, etc.

**Definition: Open Ball**

Let  $(X, d)$  be a metric space.

$x \in X, \delta > 0$  define  $B_\delta(x) := \{y \in X, d(x, y) \leq \delta\}$  (and “open ball”).

We say  $A \subseteq X$  is open if and only if  $\forall x \in A, \exists \delta > 0, B_\delta(x) \subseteq A$ .

**Definition: Open Neighborhood**

An open neighborhood of  $x \in X$  is any open set  $A$  containing  $x$ .

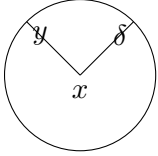
**Proposition:**

Let  $(X, d)$  be a metric space.

1. Open balls in  $X$  are open sets.
2. Arbitrary unions and finite intersections of open sets are open.
3.  $\emptyset, X$  are open.

### Proof of 1

Take  $B_\delta(x)$ ,  $x \in X$ ,  $\delta > 0$ .



Take  $y \in B_\delta(x)$ , then  $d(x, y) < \delta$ . Set  $\epsilon = \frac{\delta - d(x, y)}{2}$

Consider  $B_\epsilon(y)$ : if  $z \in B_\epsilon(y)$ ,  $d(z, y) < \frac{\delta - d(x, y)}{2}$ , then

$$d(x, z) \leq d(x, y) + \underbrace{d(y, z)}_{\frac{\delta - d(x, y)}{2}} < \frac{d(x, y) + \delta}{2} < \delta$$

Hence  $B_\epsilon(y) \subseteq B_\delta(x)$ .

### Proof of 2

Arbitrary Union:

Suppose  $A_\alpha$ ,  $\alpha \in I$  are all open.

$x \in \bigcup_\alpha A_\alpha$ ,  $\exists \alpha_0$ ,  $x \in A_{\alpha_0}$ .

$\exists \delta > 0$ ,  $B_\delta(x) \subseteq A_0 \subseteq \bigcup_\alpha A_\alpha$

Finite Intersection:

Suppose  $A_1 \cdot A_n$  are open.

$$x \in \bigcap_{j=1}^n A_j, \quad \forall j, \quad \exists \delta_j > 0$$

$B_{\delta_j}(x) \subseteq A_j$ .

Take  $\delta = \min(\delta_1, \dots, \delta_n)$ , then

$$B_\delta \subseteq \bigcap_{j=1}^n A_j$$

### Definitions that Generalize on a Metric Space

Limits of sequences:  $x_n$  converges to  $x$  if  $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ .

Cauchy sequences.

Boundedness

### Definition: Limit Points of Sets

$a \in X$  is a limit point of  $A \subseteq X$  if  $\forall \epsilon > 0$ ,  $\exists b \in A \setminus \{a\}$ ,  $d(a, b) < \epsilon$ .

Equivalently, every open neighborhood of  $a$  has a point in  $A \setminus \{a\}$ .

### Definition: Closed Set in Metric Space

$A$  is closed if and only if it contains all its limit points.

**Proposition:**

On a metric space  $(X, d)$ ,

1.  $A \subseteq X$  is closed if and only if  $A^C$  is open ( $A^C = X \setminus A$ ).
2. Finite unions and countable intersections of closed sets are closed.

**Proof of 1**

Suppose  $A$  is open. Want to show that  $A^C$  is closed.

If  $x \notin A^C$ , then  $x \in A$ , then  $\exists \delta > 0$  such that  $B_\delta(x) \subseteq A$  (i.e.  $B_\delta(x) \cap A^C = \emptyset$ ).

Therefore  $x$  is not a limit point of  $A^C$ .

Suppose  $A$  not open.

Then  $\exists x \in A$  such that  $\forall \delta > 0, B_\delta(x) \cap A^C \neq \emptyset$ .

Therefore  $x \in A^C$  and  $x$  is a limit point of  $A^C$ , so  $A^C$  is not closed.

**Proof of 2 (Finite Union)**

Take  $F_1, \dots, F_n$  closed sets and consider

$$\bigcup_{j=1}^n F_j = \left( \bigcup_{j=1}^n F_j \right)^{CC} = \left( \bigcap_{j=1}^n \underbrace{F_j^C}_{\text{open by 1}} \right)^C$$

is closed.

**Definition: Completeness**

A metric space  $(X, d)$  is complete if and only if every Cauchy sequence converges to a point in  $X$ .

If  $(X, d)$  comes from a normed vector space  $(X, \|\cdot\|)$ , it is called Banach.

If  $(X, d)$  comes from an inner product space, it is called Hilbert.

**Examples**

These are complete.

$(\mathbb{R}, |\cdot|)$

$(\mathbb{R}^n, \|\cdot\|_2)$

$(C([a, b]), \|f - g\|_\infty)$

**Counter Examples**

This is not complete.

$(C([a, b]), \|f - g\|_2)$  where  $\|f - g\|_2 = \left( \int_a^b (f(t) - g(t))^2 dt \right)^{1/2}$ .

Consider  $x^n$  on  $[0, 1]$ .

$$\left( \int_0^1 (x^n - 0)^2 dx \right)^{1/2} = \frac{1}{\sqrt{2n+1}}.$$

**Theorem:**

$\forall n \in \mathbb{N}$ ,  $(\mathbb{R}^n, || \cdot ||_2)$  is complete.

Let  $x_p$  be a cauchy sequence in  $\mathbb{R}^n$ ,  $x_p = (x_{p,1}, \dots, x_{p,n})$ .

Note for  $1 \leq j \leq n$ ,  $|x_{p,j} - x_{q,j}| \leq ||x_p - x_q||_2$ .

Therefore  $\forall 1 \leq j \leq n$ ,  $\{x_{p,j}\}_p$  is Cauchy in  $\mathbb{R}$ .

**November 13, 2023**

Topology of metric spaces.

- Induced Topology
- Continuous Maps
- Connectedness
- Compact Sets / Separability

**Induced Topology**

Let  $(X, d)$  be a metric space, and  $M \subseteq X$ .

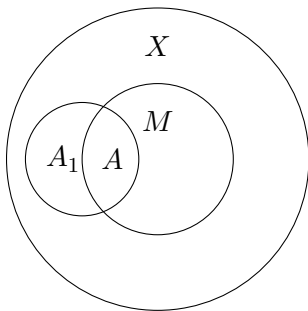
Can restrict the distance function to  $M \times M$ , then  $(M, d|_{M \times M})$  is a metric space.

$U \subseteq M$  is open if  $\forall a \in U$ ,  $\exists \delta > 0$ ,  $\{x \in M \mid d(a, x) < \delta\} = B_\delta(a) \subset U$ .

Note: if  $a \in M \subseteq X$ ,

$$B_\delta^M(a) = \{x \in M \mid d(a, x) < \delta\} = B_\delta^X(a) \cap M$$

$$B_\delta^X(a) = \{x \in X \mid d(a, x) < \delta\}$$



Open sets for both topologies may differ!

Note:  $M$  is always open for  $(M, d|_{M \times M})$ .

e.g.  $X = \mathbb{R}$  and  $M = [0, 1]$ .

**Proposition:**

In the setting above,  $A \subset M$  is open in  $(M, d)$  if and only if  $\exists A_1 \subseteq X$  open in  $(X, d)$  such that  $A = A_1 \cap M$ .

- Proof

( $\implies$ ) Suppose  $A$  open in  $(M, d)$ ,  $\forall a \in A$ ,  $\exists \delta_a > 0$ ,  $B_{\delta_a}^M(A) \subset A$ .

$$A = \bigcup_{a \in A} B_{\delta_a}^M(a) = \bigcup_{a \in A} (B_{\delta_a}^X(a) \cap M) = \underbrace{\left( \bigcup_{a \in A} B_{\delta_a}^X(a) \right)}_{A_1} \cap M$$

( $\impliedby$ ) Suppose  $A = A_1 \cap M$ ,  $A_1$  open in  $(X, d)$ . Let  $a \in A_1$ .

$a \in A_1 : \exists \delta > 0$ ,  $B_\delta^X(a) \subseteq A_1$ .

Then

$$B_\delta^M(a) = \underbrace{B_\delta^X(a)}_{\subseteq A_1} \cap M \subseteq A_1 \cap M = A \quad \blacksquare$$

### Proposition:

A closed subspace  $M$  of a complete metric space  $(X, d)$  is also complete.

- Proof

Take a Cauchy sequence in  $M$ ,  $\{x_k\}_k$ , then it is also Cauchy in  $X$ .

Therefore it converges to  $x \in X$ .

Since  $M$  contains its limit points,  $x \in M$ .

### Theorem:

Let  $f : (M, d_M) \rightarrow (N, d_N)$ , where  $(M, d_m)$  and  $(N, d_n)$  are metric spaces. The following are equivalent.

1.  $\forall x \in M$ ,  $\forall \epsilon > 0$ ,  $\exists \delta > 0$

$$d_m(x, y) < \delta \implies d_N(f(x), f(y)) < \epsilon$$

(in short:  $\forall \epsilon > 0$ ,  $\exists \delta > 0$ ,  $f(B_\delta^M(x)) \subset B_\epsilon^N(f(x))$ )

2.  $\forall x \in M$  and  $\{x_n\}_n$  convergin to  $x \in M$ ,  $f(x_n)$  converges to  $f(x)$  in  $N$ .

3.  $\forall O$  open in  $N$ ,  $f^{-1}(O)$  is open in  $M$ .

### Definition: Continuity

When (1), (2) or (3) is satisfied, we say “ $f$  is continuous on  $M$ ”.

### Proof that 1 Implies 2

Let  $x \in M$ ,  $\{x_n\}_n$  converges to  $x$ .

What to show:  $\lim_{n \rightarrow \infty} d_N(f(x_n), f(x)) = 0$ .

Let  $\epsilon > 0$ , by (1),  $\exists \delta > 0$ ,  $f(B_\delta^M(x)) \subseteq B_\epsilon^N(f(x))$  (\*).

Since  $x_n \rightarrow x$ ,  $\exists n_0, n \geq n_0 \implies d_M(x_n, x) < \delta$  (i.e.  $x_n \in B_\delta^M(x)$ ).

Then, by (\*),  $f(x_n) \in B_\epsilon^N(f(x))$  (i.e.  $d_N(f(x_n), f(x)) < \epsilon$ ,  $\forall n \geq n_0$ )

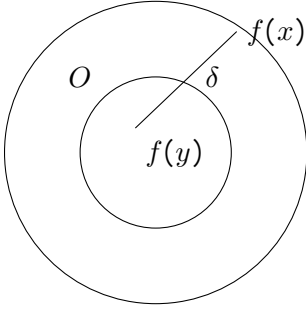
### Proof that 2 Implies 3

Assume (2) and let  $\exists O$  open in  $N$  such that  $f^{-1}(O)$  is not open:

$$\exists y \in f^{-1}(O), \forall \delta = \frac{1}{n} > 0, \exists x_n, d_M(x_n, y) < \frac{1}{n} \text{ and } f(x_n) \notin O$$

Then  $x_n$  converges to  $y$ , but  $f(x_n)$  cannot converge to  $f(y)$ .

Completion of proof left as an exercise.



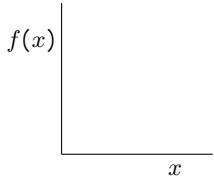
### Proof that 3 Implies 1

Let  $x \in M$  and  $\epsilon > 0$ .  $B_\epsilon^N(f(x))$  is open.

By (3),  $f^{-1}(B_\epsilon^N(f(x)))$  is open and contains  $x$ .

Then  $\exists \delta > 0$ ,  $B_\delta^M(x) \subseteq f^{-1}(B_\epsilon^N(f(x)))$ , which implies

$$f(B_\delta^M(x)) \subseteq B_\epsilon^N(f(x))$$



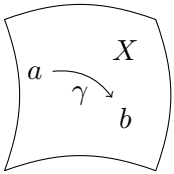
### Definition: Path Connected

Let  $(X, d)$  be a metric space.

We say  $X$  is path-connected if  $\forall a, b \in X$ ,

$$\exists \gamma : [0, 1] \rightarrow X$$

continuous with  $\gamma(0) = a$  and  $\gamma(1) = b$ .



### Definition: Connected

We say  $X$  is connected if the only subsets of  $X$  that are both open and closed are  $\emptyset$  and  $X$ .

Alternatively,  $X = A_1 \cup A_2$ ,  $A_1 \neq \emptyset$ ,  $A_2 \neq \emptyset$ ,  $A_1 \cap A_2 \neq \emptyset$ ,  $A_1, A_2$  open.

### Proposition: Path-connectedness Implies Connectedness

Let  $(X, d)$  be a metric space. If  $(X, d)$  is path-connected, then it is connected.

#### Remarks

If  $(X, d) = (\mathbb{R}, |\cdot|)$ ,  $A \subseteq \mathbb{R}$  is connected if and only if it is path-connected if and only if it is an interval. In general, connectedness does not imply path connectedness.

- Counterexample

In  $\mathbb{R}^2$ , the topologist's sine wave

$$A = \left\{ (0, 0) \right\} \cup \left\{ \left( x, \sin \frac{1}{x} \right) \mid x \in \mathbb{R} \setminus \{0\} \right\}$$

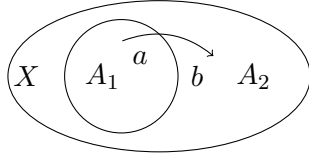
IMAGE HERE - TOPOLOGISTS SINE WAVE

#### Proof

Suppose  $X$  is path-connected but not connected. That is

$X = A_1 \cup A_2$ ,  $A_1, A_2$  open and nonempty,  $A_1 \cap A_2 = \emptyset$ .

Pick  $a \in A_1$ ,  $b \in A_2$  and consider  $\gamma : [0, 1] \rightarrow X$  continuous with  $\gamma(0) = a$  and  $\gamma(1) = b$ .



$a \in A_1$ , which is open, so  $\exists \delta > 0$ ,  $B_\delta(a) \subseteq A_1$ .

Then  $\gamma^{-1}(B_\delta(a))$  is open and contains 0, therefore it contains  $[0, \epsilon)$  for some  $\epsilon > 0$ .

Let  $l = \sup\{\epsilon > 0 \mid \gamma([0, \epsilon)) \subset A_1\}$ .

Then  $\gamma([0, l]) \subset A_1$ .

If  $l < 1$ ,  $\gamma(l) \in A_1$  open. Then  $\exists \delta' > 0$ ,  $B_{\delta'}(\gamma(l)) \subseteq A_1$ , and  $\gamma^{-1}(B_{\delta'}(\gamma(l)))$  is open in  $[0, 1]$ , of the form  $(l - \epsilon', l + \epsilon')$ .

This contradicts the supremum property of  $l$ , so  $l = 1$ .

But then  $\gamma(l) \in A_1$  and  $\gamma(l) = \gamma(1) = b \in A_2$ , which is a contradiction.

#### Definition: Compact

Let  $(X, d)$  be a metric space.

1. We say that  $A \subseteq X$  is compact if every sequence in  $A$  has a limit point in  $A$ .
2. We say  $A \subseteq X$  satisfies Heine-Borel property if every open cover of  $A$  has a finite subcover, still covering  $A$ .

#### Definition: Dense

We say  $A$  is dense in  $X$  if  $\forall x \in X, \forall \epsilon > 0, \exists a \in A, d(x, a) < \epsilon$ .

## Definition: Separable

We say  $B$  is separable if  $B$  has a countable dense subset.

i.e.  $\exists \{x_n\}_n \in B$  such that every point in  $B$  is a limit point of  $\{x_n\}_n$ .

### Example 1

$(\mathbb{R}, |\cdot|)$ , with dense subset  $\mathbb{Q}$ .

### Example 2

$(C([a, b]), \|\cdot\|_\infty)$ , with dense subset polynomials with rational coefficients.

Proof left as an exercise.

## Theorem:

Suppose  $(X, d)$  is a compact metric space. Then it is separable.

### Distance Between Sets.

Given a finite collection  $\{x_1, \dots, x_n\}$ , write  $d(x, \{x_1, \dots, x_n\}) = \min_{1 \leq j \leq n} d(x, x_j)$ .

### Proof

Pick  $x_1 \in X$ .

Look at  $R_1 := \sup\{d(x, x_1) \mid x \in X\}$ .

Claim:  $R_1 < \infty$ . Otherwise, construct a sequence with no convergent subsequence.

Then, pick  $x_2$  such that  $d(x_1, x_2) > \frac{1}{2}R_1$ .

Look at  $R_2 := \sup\{d(x, \{x_1, x_2\}) \mid x \in X\} < \infty$ .

Pick  $x_3$  such that  $d(x_3, \{x_1, x_2\}) > \frac{1}{2}R_2$ .

Repeat: if  $x_1, \dots, x_k$  are constructed, look at  $R_k := \sup\{d(x, \{x_1, \dots, x_k\}) \mid x \in X\} < \infty$  and set  $x_{k+1}$  such that  $d(x_{k+1}, \{x_1, \dots, x_k\}) > \frac{1}{2}R_k$ .

Claim:  $R_k \xrightarrow[k \rightarrow \infty]{} 0$ , otherwise  $\{x_n\}_n$  has no convergent subsequences.

But then,  $\forall x \in X, d(x, \{x_1, \dots, x_k\}) \leq R_k$ . Hence  $\lim_{k \rightarrow \infty} d(x, \{x_1, \dots, x_k\}) = 0$ .

Then for  $\epsilon > 0, \exists k$  such that  $d(x, \{x_1, \dots, x_k\}) < \epsilon$ .

i.e.  $\exists k_0 \in \{1, \dots, k\}, d(x, x_{k_0}) < \epsilon$ . ■

## Theorem:

A subsets  $A$  of a metric space  $(X, d)$  is compact if and only if it satisfies the Heine-Borel property.

### Proof

( $\Leftarrow$ ) (Note: true even if  $A$  is not separable)

Take a sequence  $\{x_n\}_n$  in  $A$  and argue by contradiction.

Case 1:  $\{x_n\}_n$  has a limit point  $b \notin A$ .

Then  $U_k = \{x \in X \mid d(x, b) > \frac{1}{k}\}$ .

IMAGE HERE - COVERS AROUND B WITH X AND A

$\bigcup_k U_k = X \setminus \{b\}$  covers  $A$ , but no finite subcover covers  $A$  since  $b$  is a limit point of  $A$ .



Case 2:  $\{x_n\}_n$  has no limit points at all.

$V_k = X \setminus \{x_k, x_{k+1}, \dots\}$  is open and  $\bigcup_k V_k$  covers  $A$ , but no finite subcover covers  $A$ . ■

( $\implies$ ) Sketch

To do: assume compactness show that it leads to Heine-Borel.

Take an arbitrary cover.

Separability implies one can extract a countable subcover of  $\mathcal{O}$  covering  $A$ .

## November 15, 2023

Contraction Mapping Theorem

Application 1: Local Existence Theorem for ODEs

Application 2: Implicit Function Theorem

### Definition: Contraction

$f : M \rightarrow N$ ,  $(M, d_M)$ ,  $(N, d_N)$  two metric spaces.

$f$  is a contraction if  $\exists C \in [0, 1)$  such that

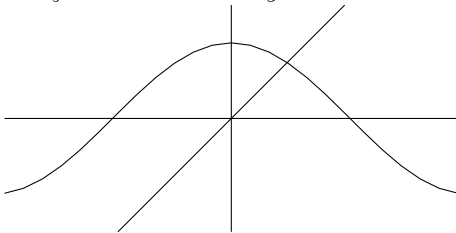
$$\forall x, y \in M \quad d_N(f(x), f(y)) \leq C d_M(x, y)$$

**Example:**

$$f(x) = \frac{1}{2} \cos(x), \quad f : \mathbb{R} \rightarrow \mathbb{R}.$$

$$|f(x) - f(y)| \stackrel{\substack{\text{Mean Value Theorem} \\ \underbrace{\quad}}}{=} \underbrace{|f'(\xi)|}_{|\frac{1}{2} \sin(\xi)|} |x - y| \leq \frac{1}{2} |x - y|$$

for  $\xi$  between  $x$  and  $y$ .



### Theorem: Contraction Mapping Theorem

Suppose  $(M, d)$  is a complete metric space and  $f : M \rightarrow M$  a contraction. Then

$$\exists! x \in M, f(x) = x$$

(i.e.  $f$  has a unique fixed point).

### Proof

- Existence.

Pick any  $x_0 \in M$ . Consider the sequence  $x_{k+1} = f(x_k)$ .

Claim 1:  $\{x_n\}_n$  is Cauchy (then, by completeness, it converges to some  $x \in M$ )

Claim 2: If Claim 1 is true, by continuity of  $f$  at  $x$ ,

$$f(x) = f\left(\lim_{k \rightarrow \infty} x_k\right) \stackrel{\text{continuity @ } x}{=} \lim_{k \rightarrow \infty} \underbrace{f(x_k)}_{x_{k+1}} = \lim_{k \rightarrow \infty} x_{k+1} = x$$

What to show:  $\{x_n\}_n$  is Cauchy.

$\forall \epsilon > 0, \exists N, \forall p \geq N, k \geq 0, d(x_{p+k}, x_p) < \epsilon$

– Scratch Work

$$\begin{aligned} d(x_{p+k}, x_p) &\leq d(x_{p+k}, x_{p+k-1}) + d(x_{p+k-1}, x_{p+k-2}) + \cdots + d(x_{p+1}, x_p) \\ &\leq \sum_{q=0}^{k-1} \underbrace{d(x_{p+q+1}, x_{p+q})}_{C^q d(x_{p+1}, x_p)} \\ &\leq \underbrace{d(x_{p+1}, x_p)}_{C^p d(x_1, x_0)} \cdot \underbrace{\sum_{q=0}^{k-1} C^q}_{\leq \frac{1}{1-C}} \end{aligned}$$

$$d(x_{p+2}, x_{p+1}) = d(f(x_{p+1}), f(x_p)) \leq C d(x_{p+1}, x_p)$$

$$d(x_2, x_1) = d(f(x_1), f(x_0)) \leq C d(x_1, x_0)$$

$$d(x_3, x_2) \leq C d(x_2, x_1) \leq C C d(x_1, x_0)$$

Ultimately,

$$(*) \quad d(x_{p+k}, x_p) \leq \frac{d(x_1, x_0)}{1-C} C^p$$

- Proof of Cauchy

Let  $\epsilon > 0$ , since  $\lim_{p \rightarrow \infty} \frac{d(x_1, x_0)}{1-C} C^p = 0$ ,

$$\exists N, \forall p \geq N, \frac{d(x_1, x_0)}{1-C} C^p < \epsilon$$

Then, for  $p \geq N$  and  $k \geq 0$ ,

$$d(x_{p+k}, x_p) \stackrel{(*)}{\leq} \frac{d(x_1, x_0)}{1-C} C^p < \epsilon \quad \blacksquare$$

- Uniqueness

If  $x, y$  satisfy  $f(x) = x$  and  $f(y) = y$

$$d(x, y) = d(f(x), f(y)) \leq C d(x, y)$$

If  $d(x, y) \neq 0$ , then  $1 \leq C$  is a contradiction.

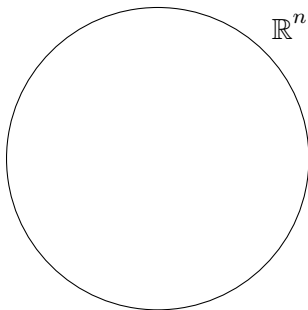
## Application to ODEs

A system of 1st order Ordinary Differential Equations takes the form

$$\begin{cases} \frac{dx}{dt} = f(t, x(t)) & (*) \\ x(0) = x_0 \end{cases}$$

$$x : [0, b]_t \rightarrow \mathbb{R}^n$$

$$f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$$



Under what conditions on  $f$  does  $(*)$  have unique solution  $x(t)$  on some interval  $[0, \epsilon]$ ?

### Example

$$\frac{d^2 x}{dt^2} = \sin(x(t))$$

IMAGE HERE - PENDULUM WITH GRAVITY VECTOR, LENGTH AND SIN(T) PERIOD

### Meta-Principle:

An ODE of order  $k$  in  $\mathbb{R}^n$ :  $\frac{d^k x}{dt^k} x(t) = f(t, x(t), x^1(t), \dots, x^{k-1}(t))$  can be rephrased as a 1st-order system in  $\mathbb{R}^{nk}$  upon introducing variables

$$\begin{aligned} x_1(t) &= \frac{dx}{dt}(t) \\ x_2(t) &= \frac{dx_1}{dt}(t) \\ &\vdots \\ x_{k-1}(t) &= \frac{dx_{k-2}}{dt}(t) \end{aligned}$$

The ODE becomes

$$\frac{d}{dt} \underbrace{\begin{bmatrix} x(t) \\ x_1(t) \\ \vdots \\ x_{k-1}(t) \end{bmatrix}}_{U(t)} = \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{k-1}(t) \\ f(t, x(t), \dots, x_{k-1}(t)) \end{bmatrix}}_{F(U(t))}$$

### Theorem: Local Existence Theorem of ODEs

Take  $x_0 \in \mathbb{R}^n$ . Suppose  $\exists \delta > 0$  such that  $f$  is continuous on  $\mathbb{R} \times \overline{B_{\delta}(x_0)}$ .

Where  $\overline{B_{\delta}(x_0)} = \{x \in \mathbb{R}^n \mid ||x - x_0||_2 < \delta\}$

And  $f$  is Lipschitz with respect to  $x : |f(t, x) - f(t, y)| \leq C|x - y|, \forall t \in \mathbb{R}, \forall x, y \in \overline{B_\delta(x_0)}$ .

Notation:  $|x| = ||x||_2$ .

Then  $\exists > 0$  and  $\exists! x \in C^1([0, \epsilon]; \mathbb{R}^n)$  solution to  $(*)$ .

### Proof

If  $x$  solves  $(*)$ ,

$$\int_0^t \rightsquigarrow x(t) - x_0 = \int_0^t f(u, x(u)) \, du \iff x(t) = x_0 + \underbrace{\int_0^t f(u, x(u)) \, du}_{Tx(t)}$$

This is a fixed point problem.

Goal:

1. find  $(E, d)$  a complete metric space
2. such that  $T(E) \subseteq E$  and  $T$  is a contraction.
3. Then, by the Contraction Mapping Theorem,  $\exists! x \in E$ .
4. Finally, show that  $x$  is actually  $C^1$ .

#### • Part A

$\overline{B_\delta(x_0)}$  is closed in  $(\mathbb{R}^n, ||\cdot||_2)$  which is complete  $\implies (\overline{B_\delta(x_0)}, ||\cdot||_2)$  is complete.

Then  $\forall \epsilon > 0, C([0, \epsilon], \overline{B_\delta(x_0)})$ , with norm  $||x(t)||_\infty = \sup_{t \in [0, \epsilon]} ||x(t)||_2$ , is complete.

Set

$$E = C([0, \epsilon], \overline{B_\delta(x_0)})$$

#### • Part B

When is  $T(E) \subseteq E$ ?

Suppose  $x \in E$ , i.e.  $||x(t) - x_0||_2 \leq \delta, \forall t \in [0, \epsilon]$ .

$$||x(t) - x_0||_2 = \left\| \int_0^t f(u, x(u)) \, du \right\|_2 \leq \int_0^t \underbrace{||f(u, x(u))||_2}_{\text{Note}} \, du \leq tM \leq \epsilon M \quad \forall t \in [0, \epsilon]$$

Note: since  $f$  is continuous on  $\mathbb{R} \times \overline{B_\delta(x_0)} \implies f$  bounded by  $M$ .

Upon making  $\epsilon M \leq \delta$ , then  $\sup_{t \in [0, \epsilon]} ||Tx(t) - x_0||_2 \leq \delta$

$\implies T(E) \subseteq E$  if  $\epsilon \leq \frac{\delta}{M}$ .

Is  $T$  continuous on  $E$ ?

$$\begin{aligned}
\|Tx(t) - Ty(t)\|_2 &= \left\| \int_0^t (f(u, x(u)) - f(u, y(u))) du \right\|_2 \\
&\leq \underbrace{\int_0^t \|f(u, x(u)) - f(u, y(u))\|_2 du}_{\substack{\leq L \|x(u) - y(u)\|_2 \\ L \sup_{u \in [0, \epsilon]} \|x(u) - y(u)\|_2 \\ \|x - y\|_E}} \\
&\leq tL \|x - y\|_E \leq \epsilon L \|x - y\|_E
\end{aligned}$$

Therefore  $\|Tx - Ty\|_E \leq \epsilon L \|x - y\|_E$  is a contraction if  $\epsilon L < 1$ , i.e.  $\epsilon < \frac{1}{L}$ .

- Part C

By *CMT*,  $\exists! x \in E(\epsilon)$  as long as  $\epsilon \leq \frac{\delta}{M}$  and  $\epsilon < \frac{1}{L}$ .

- Part D

$x(t) \in C([0, t], \overline{B_\delta(x_0)})$ , but also notice

$$\begin{aligned}
x(t) &= \underbrace{x_0 + \int_0^t f(u, x(u)) du}_{\text{differentiable with derivative } f(t, x(t))}
\end{aligned}$$

which is continuous.

Therefore  $x$  is  $C^1$ ,  $x'(t) = f(t, x(t))$ , and  $x(0) = x_0 + \int_0^0 = x_0$ .

### Remark: Lipschitz Requirement for Uniqueness

$f$  being locally Lipschitz is necessary for uniqueness purposes.

### Example

$$\begin{cases} \frac{dx}{dt} = 3x^{2/3} \\ x(0) = 0 \end{cases}$$

This ODE has two distinct solutions:

$$\begin{cases} x(t) = 0 \\ x(t) = t^3 \rightsquigarrow \frac{dx}{dt} = 3t^2 = 3x^{2/3} \end{cases}$$

### Remark:

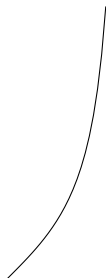
The time of existence is not sharply controlled (could be infinite, finite, very small)

## Example

$$\begin{cases} \frac{dx}{dt} = \overbrace{x^2}^{\text{Lipschitz}} \leadsto \frac{1}{x^2} \frac{dx}{dt} = 1 \implies -\frac{d}{dt} \left( \frac{1}{x} \right) \implies \frac{-1}{x(t)} + \frac{1}{x_0} = t \implies x(t) = \frac{x_0}{1-x_0 t} \\ x(0) = x_0 \end{cases}$$

Lifetime:  $t^\star = \frac{1}{x_0}$ .

IMAGE HERE - FROM X0 EXPLODING AT 1/X0; FASTER FOR 2X0



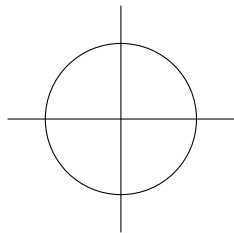
## Theorem: Implicit Function Theorem / Inverse Function Theorem

### Example

$$F : \mathbb{R}^2 \rightarrow \mathbb{R}, F(x, y) = x^2 + y^2.$$

Question: Can the set  $F(x, y) = 1$  be described by equations “ $x(y)$ ” or “ $y(x)$ ” ?

IMAGE HERE - CIRCLE AS TWO EQUATIONS WITH VERTICAL LINE TEST



Implicit Function Theorem gives situations where it is possible.

### Setting

$F : \mathbb{R}_x^p \times \mathbb{R}_y^q \rightarrow \mathbb{R}^q$  continuously differentiable.

Fix  $(a, b) \in \mathbb{R}^p \times \mathbb{R}^q$  and set  $F(a, b) = c$ .

If the  $q \times q$  matrix

$$\frac{\partial F}{\partial y}(a, b) = \begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} & \dots & \frac{\partial F_1}{\partial y_q} \\ \vdots & \ddots & & \vdots \\ \frac{\partial F_q}{\partial y_1} & \dots & \dots & \frac{\partial F_q}{\partial y_q} \end{bmatrix}$$

is invertible, then  $\exists \Omega$  neighborhood of  $a$  and a function  $y : \Omega \rightarrow \mathbb{R}^q$  such that  $\forall x \in \Omega$ ,

$$F(x, y(x)) = c$$

Moreover:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial x} = 0 \implies \frac{\partial y}{\partial x} = \left[ \frac{\partial F}{\partial y}(x, y(x)) \right]^{-1} \frac{\partial F}{\partial x}(x, y(x))$$

Parallel: solve  $f(x) = 0$  by Newton's iteration.

$x_{n+1}$  = the zero of the line  $(f(x_n) + (x - x_n)f'(x_n) = 0) = x_n - (f'(x_n))^{-1}f(x_n)$ .

Left iterating

$$\phi(x) = x - (f'(x))^{-1}f(x)$$

Claim:  $\phi$  is a contraction near fixed points  $x^*$  whenever  $f'(x^*) \neq 0$ .

- Proof (Sketch)

Uses CMT.

Freeze  $x$ ; approximately solve for  $y$  the equation  $F(x, y) - c = 0$ .

Iterate  $\phi_x(y) = y - \frac{\partial F}{\partial Y}^{-1}(a, b)(F(x, y) - c)$ .

## November 20, 2023

Continuity and compactness.

Uniform equicontinuity and Arzela-Ascoli.

Bolzano's example of a continuous, nowhere differentiable function.

## Continuity on Compact Domains

Context:

$(M, d_m)$ ,  $(N, d_n)$  two metric spaces.

$f : M \rightarrow N$  continuous.

$M$  compact.

### Proposition:

In the context above, we can deduce:

1.  $f$  is uniformly continuous in the sense that

$$\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in M, d_M(x, y) < \delta \implies d_n(f(x), f(y)) < \epsilon$$

2.  $f(M)$  is compact.

- Corollary:

If  $(M, d_M)$  is a compact metric space and  $f(M) \rightarrow (\mathbb{R}, |\cdot|)$  continuous, then  $f$  is bounded and achieves its extrema.

$$\exists x_{\pm} \in M \text{ such that } f(x_+) = \sup\{|f(x)| \mid x \in M\} \text{ and } f(x_-) = \inf\{|f(x)| \mid x \in M\}$$

- Proof of 1

Alternative proof in lecture notes.

By contradiction, suppose

$$\exists \epsilon > 0, \forall \underbrace{\delta}_{1/n} > 0, \exists \underbrace{x, y}_{x_n, y_n} \in \underbrace{M, d_M(x, y) < \delta}_{d_M(x_n, y_n) < 1/n} \text{ and } d_N(f(x), f(y)) \geq \epsilon$$

$x_n$  lives in  $M$  compact, therefore a subsequence  $x_{n_k}$  converges to  $x$  in  $M$ .

$$\begin{aligned} d_M(x_{n_k}, y_{n_k}) < \frac{1}{n_k} &\implies d_M(y_{n_k}, x) \leq d_M(y_{n_k}, x_{n_k}) + d_M(x_{n_k}, x) \\ &\leq d_M(x_{n_k}, x) + \frac{1}{n_k} \xrightarrow{k \rightarrow \infty} 0 \end{aligned}$$

So  $y_{n_k}$  converges to  $x$ .

Since  $f$  is continuous at  $x$ ,  $f(x_{n_k}) \rightarrow f(x)$  and  $f(y_{n_k}) \rightarrow f(x)$  as  $k \rightarrow \infty$ .

Hence  $d_N(f(x_{n_k}), f(y_{n_k})) \xrightarrow{k \rightarrow \infty} d_N(f(x), f(y)) = 0$  (cf. homework problem).

This contradicts the assumption.

- Proof of 2

What to show:  $\forall \mathcal{O}$  open covers of  $f(M)$ , there exists a finite subcover  $\mathcal{O}' \subset \mathcal{O}$  still covering  $f(M)$ .

Take  $\mathcal{O} = \{O_\alpha \mid \alpha \in I\}$  covering  $f(M)$ .

Set  $V_\alpha = f^{-1}(O_\alpha)$  open by the continuity assumption.

Claim:  $\mathcal{V} = \{V_\alpha \mid \alpha \in I\}$  covers  $M$ : let  $x \in M$ ,  $f(x) \in f(M)$ , then  $f(x) \in O_\alpha$  for some  $\alpha \in I$ .

By Heine-Borel Compactness,

$$\exists \{ \underbrace{V_1}_{f^{-1}(O_1)}, \dots, \underbrace{V_n}_{f^{-1}(O_n)} \}$$

subcollection of  $\mathcal{V}$  covering  $M$ .

Claim:  $(O_1, \dots, O_n)$  covers  $f(M)$ : Pick  $y \in f(M)$ ,  $y = f(x)$  for  $x \in M$

$M$  covered by

$$V_1, \dots, V_n \implies \exists j, x \in V_j \implies \underbrace{f(x)}_y \in O_j$$

## Arzela-Ascoli

Q: What are the compact sets of  $(C([a, b]), \|\cdot\|_\infty)$ ?

Context:

$[a, b] \subset \mathbb{R}$  a compact interval.

$C([a, b])$  continuous functions on  $[a, b]$ .

$\|f\|_\infty := \sup_{x \in [a, b]} |f(x)|$  is a norm.

$(C([a, b]), \|\cdot\|_\infty)$  is complete.

### Definition: Uniform Boundedness / Uniform Equicontinuity

For  $\mathcal{F} \subset C([a, b])$ , we say

1.  $\mathcal{F}$  is uniformly bounded if  $\exists M \geq 0, \forall x \in [a, b], \forall f \in \mathcal{F}, |f(x)| \leq M$

2.  $\mathcal{F}$  is uniformly equicontinuous (UEC) if  $\forall \epsilon > 0, \exists \delta > 0,$

$$\forall x, y \in [a, b], |x - y| < \delta \implies |f(x) - f(y)| < \epsilon, \quad \forall f \in \mathcal{F}$$



- Example 1

Any finite  $\mathcal{F} = \{f_1, \dots, f_n\}$  is uniformly bounded and uniformly equicontinuous.

$$M = \max\{M_1, \dots, M_n\}$$

$$\delta(\epsilon) = \min\{\delta_1(\epsilon), \dots, \delta_n(\epsilon)\}$$

- Example 2

Sets satisfying a uniform Lipschitz/Hölder criterion, i.e.  $\exists L > 0, \alpha \in (0, 1]$ ,

$$|f(x) - f(y)| \leq L|x - y|^\alpha, x, y \in [a, b], f \in \mathcal{F}, \left(\delta(\epsilon) = \left(\frac{\epsilon}{L}\right)^{1/\alpha}\right)$$

- Example 3

Sets satisfying a uniform bound on first derivatives (assuming they exist).

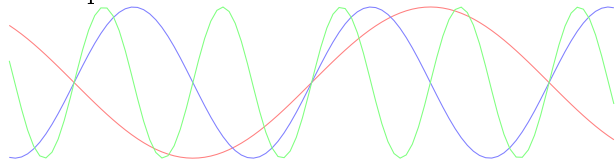
$$\mathcal{F} = \{f \in C^1([a, b]) \mid |f'(x)| \leq M, x \in [a, b]\}$$

- Non-example

$$\mathcal{F} = \{\sin(nx) \mid x \in [0, \pi], n \in \mathbb{N}\}$$

Uniformly bounded by 1.

Not equicontinuous.



### Proposition:

If  $f_n \in C([a, b])$  converges uniformly to  $f \in C([a, b])$ , then  $\{f_n \mid n \in \mathbb{N}\}$  is uniformly bounded and UEC.

### Proof

- Uniform Boundedness

Since  $\|f_n - f\|_\infty \xrightarrow{n \rightarrow \infty} 0$  and  $f$  is continuous on  $[a, b]$  compact, then  $f$  bounded implies  $\|f\|_\infty < \infty$ .

For

$$\begin{aligned} &\implies \|f_n - f\|_\infty < 1 \\ \implies \|f_n\|_\infty &= \|f_n - f + f\|_\infty \end{aligned}$$

Then  $M = \max\{\|f_1\|_\infty, \dots, \|f_n\|_\infty, \|f\|_\infty + 1\}$  is a uniform bound.

- Uniform Equicontinuity

$$\forall \epsilon > 0, \exists \delta > 0, \forall x, y \in [a, b], |x - y| < \delta \implies |f_n(x) - f_n(y)| < \epsilon \quad \forall n \in \mathbb{N}$$

$f$  is continuous, hence uniform continuous on  $[a, b]$  with modulus of continuity  $d_f(\epsilon)$ .

Let  $\epsilon > 0$ , by uniform convergence, (a)  $\exists N, n \geq N \implies |f_n(x) - f(x)| \leq \epsilon/3 \quad \forall x \in [a, b]$ .

For  $x, y, |x - y| < \delta_f$ , and  $n \geq N$ ,

$$|f_n(x) - f_n(y)| = \underbrace{|f_n(x) - f(x)|}_{< \frac{\epsilon}{3} \text{ by (a)}} + \underbrace{|f(x) - f(y)|}_{< \frac{\epsilon}{3} \text{ by uniform convergence of } f} + \underbrace{|f(y) - f_n(y)|}_{< \frac{\epsilon}{3} \text{ by (a)}}$$

Therefore  $\mathcal{F} = \{f_n\}_{n \geq N}$  is UEC.

$$\delta(\epsilon) = \min\{d_f(\epsilon/3), \delta_1(\epsilon), \dots, \delta_N(\epsilon)\}.$$

### Theorem: Arzela-Ascoli

Let  $[a, b] \subset \mathbb{R}$  be compact and  $f_k \in C([a, b])$  be uniformly bounded and uniformly equicontinuous. Then a subsequence of  $f_k$  converges uniformly.

### Proof

Step A: construct a subsequence converging at all rationals in  $[a, b]$ .

(uniform boundedness is enough)

Step B: show that the subsequence is uniformly Cauchy.

$$\forall \epsilon > 0, \exists N, \forall p, q \geq N, \|f_p - f_q\|_\infty < \epsilon$$

(uses UEC)

- Step A

Let  $r_k$  be a denumeration of all rationals in  $[a, b]$ .

$\{f_n(r_1)\}_n$  has a convergent subsequence (Bolzano-Weierstrass),  $\{f_{1,n}(r_1)\}_n$ .

$\{f_{1,n}(x)\}_n$  converges at  $r_1$ ; a subsequence  $f_{2,n}$  converges at  $r_2 \implies f_{2,n}$  converges at  $r_1, r_2$ .

Repeat  $\{f_{2,n}(r_3)\}_n$  is bounded  $\implies$  a subsequence  $\{f_{3,n}(r_3)\}_n$  converges  $\implies f_{3,n}$  converges at  $r_1, r_2, r_3$ .

$\forall p, \{f_{p+1,n}\}_n$  subsequence of  $\{f_{p,n}\}_n$  converges at  $r_1, \dots, r_p, r_{p+1}$ .

Consider  $\{f_{k,k}\}_{k \in \mathbb{N}}$  subsequence of  $\{f_n\}_n$ .

$\forall p, f_{k,k}(r_p)$  converges because  $\{f_{k,k}(r_p)\}_{k \geq p}$  is a subsequence of  $\{f_{p,n}\}_n$ .

- Step B

Simplified statement: if  $f_n$  is uniformly bounded, uniformly equicontinuous and converges at all rationals in  $[a, b]$ , then  $f_n$  is uniformly Cauchy.

Let  $\epsilon > 0$ . By UEC,  $\exists \delta_0, \forall x, y \in [a, b], |x - y| < \delta \implies |f_n(x) - f_n(y)| < \delta \quad \forall n \in \mathbb{N}$ .

$\exists r_{i_1}, \dots, r_{i_l}$  rationals such that  $\forall x \in [a, b], \min_{1 \leq j \leq l} |x - r_{i_j}| < \delta$ .

Since  $f_n(r_{i_j})$  converges  $\forall j$ ,

$$\exists N, p, q \geq N, |f_p(r_{i_j}) - f_q(r_{i_j})| < \frac{\epsilon}{3}$$

Now for any  $x \in [a, b]$  and  $p, q \geq N$ ,

$$|f_p(x) - f_q(x)| \leq \underbrace{|f_p(x) - f_p(r_{i_j})|}_{< \frac{\epsilon}{3} \text{ by UEC}} + \underbrace{|f_p(r_{i_j}) - f_q(r_{i_j})|}_{< \frac{\epsilon}{3}} + \underbrace{|f_q(r_{i_j}) - f_q(x)|}_{< \frac{\epsilon}{3} \text{ by UEC}}$$

We showed  $\forall \epsilon > 0, \exists N, p, q \geq N \implies \|f_p - f_q\|_\infty < \epsilon$ . ■

### Consequence

$\mathcal{F} \subseteq C([a, b])$  is compact if and only if it is closed, uniformly bounded and uniformly equicontinuous.

### Interesting Functional Analytic Consequence

Consider  $(C^1([a, b]), f \mapsto \|f\|_\infty + \|f'\|_\infty)$

One can show that it is complete.

Arzela-Ascoli  $\implies$  bounded sets in  $C^1([a, b])$  are precompact (i.e. have compact closures) in  $(C([a, b]), \|\cdot\|_\infty)$ .

We say the injection

$$C^1([a, b]) \hookrightarrow C([a, b])$$

is compact.

**November 22, 2023**

Topological spaces.

- Definitions
- Hausdorff
- Pathological Cases
- Base for a Topology

### Definition:

Given  $X$  a set and  $\tau \subset \mathcal{P}(X)$ , we say that  $\tau$  is a topology if

1.  $\emptyset, X \in \tau$
2.  $\tau$  is stable under finite intersection and arbitrary union.

- $O_1, \dots, O_n \in \tau \implies \bigcap_{j=1}^n O_j \in \tau$
- $\{O_\alpha\}_\alpha \in \tau \implies \bigcup_\alpha O_\alpha \in \tau$

We say  $(X, \tau)$  is a topological space and elements of  $\tau$  are called open sets.

### Examples

1.  $(X, d)$  a metric space, then  $\tau = \{\text{open sets defined by } \forall x \in O, \exists \delta > 0, B_{\delta(x)} \subset O\}$
2.  $(X, \{\emptyset, X\})$
3.  $(X, \mathcal{P}(X))$

### Topological Definitions

Fix  $(X, \tau)$  a topological space.

#### Definition: Open Neighborhood

An open neighborhood of  $x \in X$  is an open set  $U \ni x$ .

#### Definition: Interior Point

$A \subseteq X$ , then  $a \in A$  is an interior point of  $A$  if  $\exists U$  an open neighborhood of  $x$  such that  $x \in U \subseteq A$ .

#### Definition: Interior

The interior of  $A \subseteq X$ , denoted  $A^\circ$ , is the set of all interior points.

#### Definition: Convergence

$x_n$  converges to  $x$  if and only if for every neighborhood  $U$  of  $x$ ,  $\exists N, n \geq N \implies x_n \in U$ .

#### Definition: Limit Point

$x$  is a limit point of  $A$  if for every neighborhood  $U$  of  $x$ ,  $A \cap (U \setminus \{x\}) \neq \emptyset$ .

#### Definition: Closed Set

$A$  is closed if it contains all its limit points.

#### Definition: Closure

The closure of  $A$ , called  $\overline{A}$ , is  $A \cup \{\text{limit points of } A\}$ .

### Proposition: Induced Topology

Given  $(X, \tau)$  a topological space and  $A \subseteq X$ , then  $\tau_A = \{U \cap A \mid U \in \tau\}$  is a topology on  $A$  called the induced topology.

### Topological Propositions

Take  $(X, \tau)$  a topological space (fixed) and  $A \subseteq X$ .

1.  $A \in \tau$  if and only if every point in  $A$  is an interior point of  $A$ . (Then  $A$  deserves to be called open)
2.  $A$  is closed if and only if  $A^c$  is open
3. Arbitrary intersection and finite union of closed sets are closed.
4.  $A^\circ$  is open and  $A^\circ = \bigcup_{\substack{O \subseteq A \\ O \in \tau}} O$  (which implies  $A$  open if and only if  $A = A^\circ$ ).
5.  $\overline{A}$  is closed and  $\overline{A} = \bigcap_{\substack{F \supseteq A \\ F^c \in \tau}} F$  (which implies  $A$  is closed if and only if  $\overline{A} = A$ )

#### Proof of 1

( $\Leftarrow$ ) Take  $A$ .

$$\begin{aligned} \forall x \in A, \exists U_x \in \tau, x \in U_x \subseteq A &\implies A = \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} U_x \subseteq A \\ &\implies A = \bigcup_{x \in A} U_x \in \tau \end{aligned}$$

( $\Rightarrow$ )  $A \in \tau$ ,  $x \in A$ , take  $U \in \tau$ , containing  $x$ , then  $A \cap U \in \tau$ ,  $x \in A \cap U \subseteq A$ .

#### Proof of 2

$A$  open  $\implies A^c$  closed.

If  $x \notin A^c$ , then  $x \in A$ .

Therefore  $\exists U \in \tau, x \in U \subseteq A \implies U \cap A^c = \emptyset \implies x$  not a limit point of  $A^c$ .

So  $A^c$  contains its limit points.

Converse left as an exercise.

#### Proof of 3 (Technique)

De Morgan's Laws.

#### Proof of 4

Take  $x \in A^\circ : \exists U \in \tau, x \in U \subseteq A$ .

Claim  $U \subseteq A^\circ$ . Why?

If  $y \in U$ ,  $U \in \tau$  so  $\exists V \in \tau, y \in V \subseteq U \subseteq A \implies y \in A^\circ$ .

Therefore  $A^\circ$  is open.

$$A^\circ = \bigcup_{\substack{O \subseteq A \\ O \in \tau}} O:$$

$$(\subset) A^\circ \subseteq A, \text{ so } A^\circ \subseteq \bigcup_{\substack{O \subseteq A \\ O \in \tau}} O.$$

(<sup>\supset</sup>) Claim: if  $O \in \tau, O \subseteq A$ , then  $O \subseteq A^\circ$ .

Pick  $x \in O, \exists U \in \tau, x \in U \subseteq A \implies x \in A^\circ$ .

Therefore  $\bigcup_{\substack{O \subseteq A \\ O \in \tau}} O \subseteq A^\circ$ .

### Proof of 5 (Exercise)

$(\bar{A})^c = (A^c)^\circ$ ? Prove or disprove.

### Definition: Continuous Maps

Fix  $(X, \tau), (Y, \sigma)$  topological spaces.

$f : X \rightarrow Y$  is continuous if  $\forall O \in \sigma, f^{-1}(O) \in \tau$ .

Can also define continuity at  $x \in X$ :

$\forall V$  open neighborhood of  $f(x), \exists U$  open neighborhood of  $x$  such that  $f(U) \subseteq V$ .

IMAGE HERE - EPSILON DELTA NEIGHBORHOODS ON  $\mathbb{R}^2$

### Proposition:

$f$  is continuous on  $X$  if and only if it is continuous at every  $x \in X$ .

### Proposition:

$f$  continuous at  $x$  if and only if  $\forall x_n$  such that  $\lim_{n \rightarrow \infty} x_n = x$ , then  $\lim_{n \rightarrow \infty} f(x_n) = f(x)$ .

### Definition: Homeomorphism

$f : (X, \tau) \rightarrow (Y, \sigma)$  is a homeomorphism if and only if  $f$  is continuous, bijective and  $f^{-1} : (Y, \sigma) \rightarrow (X, \tau)$  is continuous.

A homeomorphism induces a one to one correspondence between  $\tau$  and  $\sigma$ .

$$O \longrightarrow f(O) = (f^{-1})^{-1}(O)$$

i.e.  $(X, \tau) \sim (Y, \sigma)$  if and only if  $\exists f : X \rightarrow Y$  homomorphic is an equivalence relation.

### Examples

1.  $\tan : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), |\cdot| \rightarrow (\mathbb{R}, |\cdot|)$  is a homeomorphism.

2.  $X, d_1, d_2$  two metrics such that  $\exists C_1, C_2 > 0, \forall x, y \in X, C_1 d_2(x, y) \leq d_1(x, y) \leq C_2 d_2(x, y)$ , then

$$\text{id} : (X, \tau(d_1)) \rightarrow (X, \tau(d_2))$$

is a homeomorphism.

## Topological Connectedness

### Definition: Connected

$(X, \tau)$  is connected if  $X = A \cup B$ ,  $A \cap B = \emptyset$ , and  $A, B \in \tau$  implies  $A = \emptyset$  or  $B = \emptyset$ .  
Equivalently, the only two subsets of  $X$  that are open and closed are  $\emptyset$  and  $X$ .

### Definition: Path Connected

$(X, \tau)$  is path-connected under the same definition as before.

### Definition: Connected Subspace

$A$  is connected if and only if  $(A, \tau_A)$  is connected where  $\tau_A$  is the induced topology).

### Proposition:

Let  $f : (X, \tau) \rightarrow (Y, \sigma)$  be continuous.

1. If  $X$  is connected, so is  $f(X)$ .
2. If  $X$  is path-connected, so is  $f(X)$ .

## Topological Compactness

Keep the Heine-Borel definition only.

### Definition: Compact

A set  $A \subseteq X$  is (HB)-compact if  $\forall$  open covers  $\mathcal{O}$  of  $A$ , a finite subcollection of  $\mathcal{O}$  still covers  $A$ .

### Proposition:

If  $K \subset X$  is compact and  $F \subseteq K$  is closed, then  $F$  is compact.

- Proof

Take  $\{O_\alpha\}_\alpha$  open cover of  $F$ . Then  $\{O_\alpha\}_\alpha, F^c$  covers  $K$ .

By HBP, a finite subcover (e.g.  $\{O_\alpha, \dots, O_{\alpha_n}, F^c\}$ ) covers  $K$ , in particular  $F$ .

And since  $F \cap F^c = \emptyset$ ,  $O_\alpha, \dots, O_{\alpha_n}$  covers  $F$ . ■

### Proposition:

If  $f : X \rightarrow Y$  continuous and  $K$  compact in  $X$ , then  $f(K)$  compact.

### Definition: Hausdorff Property

We say that a topological space  $(X, \tau)$  is Hausdorff if and only if “it separates points”:

$$\forall x, y \in X, x \neq y, \exists U_x \ni x, \exists U_y \ni y, U_x, U_y \in \tau, U_x \cap U_y = \emptyset$$

IMAGE HERE - HAUSDORFF SEPARABILITY DRAWING

### Example 1

Any  $(X, \tau(d))$  induced by a metric  $d$  is Hausdorff.

- Proof

IF  $x \neq y$ , let  $\delta = d(x, y) > 0$ , take  $U_x = B_{\delta/3}(x)$  and  $U_y = B_{\delta/3}(y)$ .

IMAGE HERE – POINTS X Y WITH DELTA/3 BALLS

### Non-example 2

$(X = \{0, 1\}, \tau = \{\emptyset, X\})$  not Hausdorff.

### Example 3

$(X, \mathcal{P}(X))$  is Hausdorff.

### Example 4

If  $X$  is not Hausdorff, the singleton  $\{x\}$  need not be closed.

In Non-example 2,  $\{0\}$  is neither open nor closed.

### Theorem:

If  $(X, \tau)$  is Hausdorff, then for  $K \subseteq X$  compact and  $x \notin K$ ,  $\exists U, V$  open sets such that  $U \cap V = \emptyset$ ,  $x \in U$ ,  $K \subset V$ .

### Proof

$\forall y \in K, \exists U_y, V_y \in \tau, x \in U_y, y \in V_y, U_y \cap V_y = \emptyset$ .

Then  $\{V_y \mid y \in K\}$  is an open cover of  $K$ , so  $\exists y_1, \dots, y_n$  such that  $K$  is covered by  $V_{y_1}, \dots, V_{y_n}$ .

$\tau \ni \bigcup_{j=1}^n V_{y_j} \supseteq K$  and  $U = \bigcap_{j=1}^n U_{y_j} \in \tau$ , contains  $x$ .

$$\forall j, U \cap V_{y_j} = \bigcap_{p=1}^n U_{y_p} \cap V_{y_j} \subseteq U_{y_j} \cap V_{y_j} = \emptyset$$

Therefore  $U \cap \bigcup_{j=1}^n V_{y_j} = \emptyset$ ,



### Corollary

Let  $(X, \tau)$  be Hausdorff.

1.  $K \subseteq X$  compact implies  $K$  closed.
2.  $K$  compact and  $F$  closed implies  $F \cap K$  compact (i.e.  $K^c$  contains no limit points of  $K$ ).

### Theorem:

Let  $(X, \tau)$  be Hausdorff.

Suppose that  $\{K_\alpha\}_\alpha$  is a collection of compact sets in  $X$  such that  $\bigcap_\alpha K_\alpha = \emptyset$ .

Then  $\exists \alpha_1, \dots, \alpha_n, \bigcap_{j=1}^n K_{\alpha_j} = \emptyset$ .

### Proof

Single out  $K_{\alpha_1}$ .

Then  $K_{\alpha_1} \cap \bigcap_{\alpha \neq \alpha_1} K_\alpha = \emptyset$  i.e.  $K_{\alpha_1} \subset \left(\bigcap_{\alpha \neq \alpha_1} K_\alpha\right)^c = \bigcup_{\alpha \neq \alpha_1} K_\alpha^c$ .

Therefore  $\{K_\alpha^c\}_{\alpha \neq \alpha_1}$  is an open cover of  $K_{\alpha_1}$ .

Take a finite subcover  $K_{\alpha_2}, \dots, K_{\alpha_n}$ , then  $K_{\alpha_1} \subseteq \bigcup_{j=2}^n K_{\alpha_j}^c = \left(\bigcap_{j=2}^n K_{\alpha_j}\right)^c$ .

Therefore  $\bigcap_{j=1}^n K_{\alpha_j} = \emptyset$ . ■

### Some Heuristic Statements

One usually seeks a topology that is not too small (coarse) and not too large (fine).

On  $X$ , given  $\tau_1$  and  $\tau_2$ , we say  $\tau_1$  is finer (contains more open sets) than  $\tau_2$  if  $\tau_2 \subseteq \tau_1$ .

The finest of all is  $\mathcal{P}(X)$ .

- Hausdorff.
- Completely disconnected (every set is open and closed).
- Very few compact sets (only the finite sets).

The coarser sets of all is  $\{\emptyset, X\}$ .

- Very few open sets.
- Not Hausdorff (as soon as  $X$  has more than one element)
- Compact
- Connected

In general, due to Heine-Borel, more open sets means fewer compact sets and vice versa.

IMAGE HERE - PRODUCT AND QUOTIENT TOPOLOGIES

**November 27, 2023**

Quotient Topology

Base for a topology.

Topology on finite products.

### Definition: Equivalence Relation on X

Let  $X$  be a set and  $\sim$  an equivalence relation on  $X$ :

- $\sim \in \mathcal{P}(X \times X)$ .
- Reflexive:  $\forall x \in X, x \sim x$ .
- Symmetric:  $\forall x \in X, y \in X, x \sim y \implies y \sim x$
- Transitive  $\forall x, y, z \in X, x \sim y \wedge y \sim z \implies x \sim z$

### Definition: Equivalence Class on X

$[x] := \{y \in X \mid y \sim x\}$

### Lemma

Two equivalence classes are either disjoint or equal.

### Definition: Quotient Space

The equivalence relation induces a partitioning of  $X$  into equivalence classes.

Define  $X/\sim = \{[x] \mid x \in X\}$  (the quotient space), then there exists a natural projection map  $\pi : X \rightarrow X/\sim$ ,  $x \mapsto [x]$ .

### Question:

If  $X$  carries a topology,  $\tau$ , can we induce one on  $X/\sim$ ?

Answer: Yes. We say  $U$  is open (i.e.  $U \in \tau_\sim$ ) in  $X/\sim$  if and only if  $\pi^{-1}(U)$  is open in  $X$ .

- Claim:  
 $\tau_\sim$  is a topology.

– Proof

$$\pi^{-1}(\emptyset) = \emptyset \in \tau \text{ so } \emptyset \in \tau_\sim.$$

$$\pi^{-1}(X/\sim) = X \in \tau \text{ so } X/\sim \in \tau_\sim.$$

Stability under finite intersection and arbitrary union:

$$\pi^{-1}\left(\bigcup_\alpha U_\alpha\right) = \bigcup_\alpha \pi^{-1}(U_\alpha).$$

$$\pi^{-1}\left(\bigcap_{n=1}^k V_n\right) = \bigcap_{n=1}^k \pi^{-1}(V_n).$$

- Claims:  
 $\tau_{\sim}$  makes  $\pi$  continuous.  
 $\tau_{\sim}$  is the finest topology making  $\pi$  continuous.

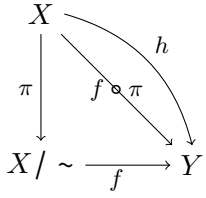
## Obvious Corollaries

If  $X$  is compact/connected/path-connected, so is  $X/\sim$ .

- Proof:

$\backslash (X/\sim = \pi(x) \backslash), \pi$  continuous.

IMAGE HERE - COMMUTATIVE DIAGRAM



## Proposition

Let  $X, \tau, \sim$  be as above.

Let  $f : X/\sim \rightarrow (Y, \tau_Y)$ .

Then  $f$  is continuous if and only if  $f \circ \pi$  is continuous.

## Proof

( $\implies$ ) Obvious (composition of continuous maps).

( $\impliedby$ ) Suppose  $f \circ \pi$  is continuous.

Take  $U \in \tau_Y$ , then  $(f \circ \pi)^{-1}(U) = \pi^{-1}(f^{-1}(U))$  is open.

Then  $f^{-1}(U)$  is open. Therefore  $f$  is continuous.

## Proposition:

Let  $X, \sim, \tau, (Y, \tau_Y)$  as above and  $h : X \rightarrow Y$ .

Then  $\exists f : X/\sim \rightarrow Y$  such that  $h = f \circ \pi$  if and only if  $\forall [a] \in X/\sim, \forall x \in [a], h(x) = h(a)$ .

Moreover (by previous propositions),  $f$  continuous if and only if  $h$  is continuous.

## Proof:

( $\implies$ ) If  $h = f \circ \pi, [a] \in X/\sim, x \in [a]$

$$h(x) = f(\pi(x)) = f([x]) = f([a]) = f(\pi(a)) = h(a)$$

( $\impliedby$ ) If  $\forall [a] \in X/\sim, \forall x \in [a], h(x) = h(a)(*)$ , define  $f([a]) := h(a)$  ro  $h$ (any representative of  $[a]$ ).  
 $f$  is well defined thanks to  $(*)$ .

## Note:

Hausdorff property can be lost in a quotient construction.

IMAGE HERE - ON TWO UNIT INTERVALS SEND ALL POSITIVE VALUES BUT NOT ZERO

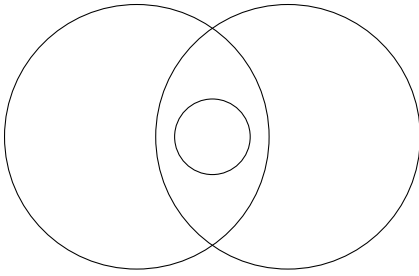
## Definition: Base for a Topology

Take  $X$  a set.

Suppose  $\sigma \subset \mathcal{P}(X)$  satisfies

1.  $\sigma$  covers  $X$  (i.e.  $X \subseteq \bigcup_{A \in \sigma} A$ )
2.  $\forall A, B \in \sigma, x \in A \cap B, \exists C \in \sigma$  such that  $x \in C \subset A \cap B$ .

IMAGE HERE - VENN DIAGRAM WITH A, B, INTERSECT SIGMA AND C IN INTERSECT



Then  $\tau := \{\text{textarbitrary unionsofelementin}\sigma\} \cup \{\emptyset\} = \mathcal{T}(\sigma)$  is a topology.

$\mathcal{T} : \mathcal{P}(\mathcal{P}(X)) \rightarrow \mathcal{P}(\mathcal{P}(X))$  if  $\tau$  is a topology,  $\mathcal{T}(\tau) = \tau$ .

$\sigma$  is called a base for  $\tau$ .

## Proof

$\tau$  is stable under arbitrary unions.

$\tau$  is stable under intersection: let  $A, B \in \sigma$ .

Then  $A \cap B = \bigcup_{x \in A \cap B} C_x$  where  $C_x$  is given by (2), so  $A \cap B \in \tau$ .

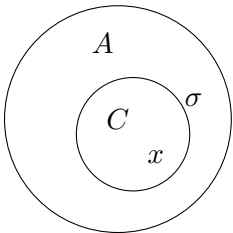
If  $A, B \in \tau$ ,  $A = \bigcup_{\alpha} A_{\alpha} \in \sigma$  and  $B = \bigcup_{\beta} B_{\beta} \in \sigma$ , so

$$A \cap B = \left( \bigcup_{\alpha} A_{\alpha} \right) \cap \left( \bigcup_{\beta} B_{\beta} \right) = \bigcup_{\alpha, \beta} \underbrace{(A_{\alpha} \cap B_{\beta})}_{\in \tau} \underbrace{\hspace{1cm}}_{\in \tau}$$

## Proposition: Criterion for a Basis

$(X, \tau)$  topological space and  $\sigma \subset \tau$ .

Then  $\sigma$  is a base for  $\tau$  if and only if  $\forall A \in \tau, \forall x \in A, \exists C \in \sigma, x \in C \subset A$ .



**Proof**

( $\implies$ ) Let  $A \in \tau$ , then  $A = \bigcup_{\alpha} C_{\alpha}$ ,  $C_{\alpha} \in \sigma$ . If  $x \in A$ , then  $x \in C_{\alpha_0}$  for some  $\alpha_0$ .  
 ( $\impliedby$ ) Let  $A \in \tau$ ,

$$A = \bigcup_{x \in A} C_x$$

Where  $C_x \in \sigma$  comes from the hypothesis.

**Proposition:**

Let  $\sigma_1, \sigma_2 \in \mathcal{P}(\mathcal{P}(X))$ , then  $\mathcal{T}(\sigma_1) \subset \mathcal{T}(\sigma_2)$  if and only if  $\forall A \in \sigma_1, \forall x \in A, \exists B \in \sigma_2, x \in B \subset A$ .  
 This helps give a criterion for when  $\mathcal{T}(\sigma_1) = \mathcal{T}(\sigma_2)$ .

**Example**

A base for the standard topology on  $\mathbb{R}$  is

$$\sigma = \left\{ \left( r - \frac{1}{n}, r + \frac{1}{n} \right) \mid r \in \mathbb{Q}, n \in \mathbb{N} \right\}$$

Note: it's countable.

**Definition: Dense Space**

Take  $(X, \tau)$  a topological space. We say  $A \subset X$  is dense in  $X$  if  $\forall x \in X, \forall U$  open neighborhood of  $x$ ,  $U \cap A \neq \emptyset$ .

**Definition: Separable Space**

Take  $(X, \tau)$  a topological space. We say  $X$  is separable if it has a countable dense subset.

**Proposition:**

Given  $(X, \tau)$  a topological space, if  $\tau$  has a countable base then

1.  $X$  is separable.
2. Any cover of  $X$  has a countable subcover.

**Proof of 1**

Call  $\sigma = \{C_k \mid k \in \mathbb{N}\}$  a countable base.

$\forall k$ , let  $x_k \in C_k$ .

Claim:  $A := \{x_k\}_k$  is dense in  $X$ .

Indeed, if  $x \in X$ ,  $U$  open neighborhood of  $x$ ,  $U \in \tau$ ,  $U = \bigcup_{j \in \mathbb{N}} C_{k_j}$  then  $x_{k_j} \in U$  for all  $j$ .

### Proof of 2

Write  $\mathcal{O} = \{O_\alpha\}_\alpha$  a cover of  $X$ .

Construct  $\mathcal{O}'$  as follows:

$\forall k \in \mathbb{N}$ , if  $C_k \subset O_\alpha$  for some  $\alpha_k$ , adjoin  $O_{\alpha_k}$  to  $\mathcal{O}'$ .

That is at most countably many.

Let  $x \in X$ .

Since  $\mathcal{O}$  covers  $X$ ,  $\exists O_\alpha$  such that  $x \in O_\alpha$ .

$$O_\alpha \in \tau \implies O_\alpha = \bigcup_{j \in \mathbb{N}} C_{k_j}^\alpha \implies \exists j, x \in C_{k_j}^\alpha$$

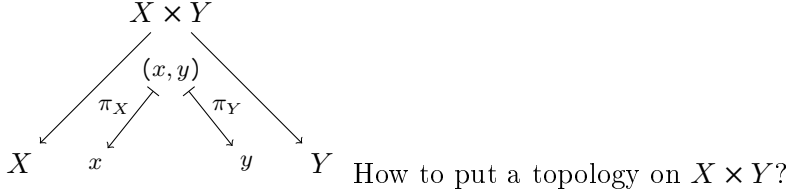
Then either  $O_\alpha \in \mathcal{O}'$  or another  $x \in O_{\alpha_k} \in \mathcal{O}'$ .

Either way,  $\exists V \in \mathcal{O}', x \in V$ .

### Topology of Finite Products

Setup:  $(X, \tau_X)$  and  $(Y, \tau_Y)$ ,  $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$ .

Two natural projections  $\pi_X$  and  $\pi_Y$ .



It should make  $\pi_X, \pi_Y$  continuous.

That is,  $\forall U \in \tau_X, \pi_X^{-1}(U) = U \times Y \in \tau$  and  $\forall V \in \tau_Y, \pi_Y^{-1}(V) = X \times V \in \tau$ .

By stability under intersection,  $\tau$  should contain

$$(U \times Y) \cap (X \times V) = (U \cap X) \times (Y \cap V) = U \times V$$

Now define  $\sigma := \{U \times V \mid U \in \tau_X, V \in \tau_Y\}$ .  $\sigma$  satisfies

1. It covers  $X \times Y$ , since  $X \times Y \in \sigma$ .

2.  $A, B \in \sigma \implies A \cap B \in \sigma$ . Indeed,

$$U_1 \times V_1 \cap U_2 \times V_2 = \underbrace{(U_1 \cap U_2)}_{\in \tau_X} \times \underbrace{(V_1 \cap V_2)}_{\in \tau_Y}$$

### Definition: Product Topology

The product topology on  $X \times Y$  is  $\mathcal{T}_{X \times Y}$ .

It is the coarsest topology making  $\pi_X, \pi_Y$  continuous (i.e. if  $\tau$  is a topology on  $X \times Y$  such that  $\pi_X, \pi_Y$  continuous, then  $\tau \supset \mathcal{T}(\sigma)$ ).

### Proposition:

Given  $(Z, \tau_Z)$  and  $f : Z \rightarrow (X \times Y, \mathcal{T}_{X \times Y})$ ,  $f$  is continuous if and only if  $\pi_X \circ f$  and  $\pi_Y \circ f$  are continuous.

### Proposition

( $\implies$ ) Clear: composition of continuous functions.

( $\impliedby$ ) Take  $U \in \mathcal{T}_{X \times Y}$ , write  $U = \bigcup_{\alpha} U_{\alpha} \times V_{\alpha}$ , then

$$\begin{aligned}
f^{-1}(U) &= f^{-1}\left(\bigcup_{\alpha} U_{\alpha} \times V_{\alpha}\right) \\
&= \bigcup_{\alpha} f^{-1}(U_{\alpha} \times V_{\alpha}) \\
&= \bigcup_{\alpha} \underbrace{(\pi_X \circ f)(U_{\alpha})}_{\in \tau_Z} \cap \underbrace{(\pi_Y \circ f)(V_{\alpha})}_{\in \tau_Z}
\end{aligned}$$

where

$$(U_{\alpha} \times V_{\alpha}) = \{z \in Z \mid f(z) \in U_{\alpha} \times V_{\alpha}\} = \{z \in Z \mid \pi_X \circ f(z) \in U_{\alpha}, \pi_Y \circ f(z) \in V_{\alpha}\}$$

### Proposition:

Let  $X, Y$  be two topological spaces and  $X \times Y$  with the product topology.

1. If  $X, Y$  are Hausdorff, so is  $X \times Y$ .
2. If  $X, Y$  are connected, so is  $X \times Y$ .
3. If  $X, Y$  are compact, so is  $X \times Y$ .

### Proof of 1

Pick  $(x_1, y_1), (x_2, y_2) \in X \times Y$ .

If  $x_1 \neq x_2$ , then  $\exists U_1, U_2 \in \tau_X, U_1 \cap U_2 = \emptyset, x_1 \in U_1, x_2 \in U_2$ .

Then  $U_1 \times Y$  and  $U_2 \times Y$  separate  $(x_1, y_1)$  and  $(x_2, y_2)$ .

IMAGE HERE - X Y PLANE WITH VERTICAL LINES DEMONSTRATING

If  $x_1 = x_2$ , then  $y_1 \neq y_2$  by a similar construction.

### Proof of 2

Suppose  $A \neq \emptyset$  both open and closed in  $X \times Y$ .

What to show:  $A = X \times Y$ .

Will follow from:  $(x, y) \in A \implies \{x\} \times Y \subset A$  and  $X \times \{y\} \in A$ .

Claim:  $\{x\} \times Y \cap A$  is both open and closed in  $\{x\} \times Y$ .

Since  $\{x\} \times Y$  is connected, then  $\{x\} \times Y \cap A = \{x\} \times Y$ .

Therefore  $\{x\} \times Y \subset A, \forall x \in A$ .

Similarly,  $X \times \{y\} \subset A, \forall y \in A$ .

**November 29, 2023**

Compcatness of products.

- Finite products
- Zorn's lemma
- Arbitrary products (Tychonoff's Theorem)
- Mapping spaces

### Theorem: Compactness of Product

Let  $(X, \tau_X)$ ,  $(Y, \tau_Y)$  be topological spaces,  $\sigma = \{U \times V \mid U \in \tau_X, V \in \tau_Y\}$  and on  $X \times Y$  equip the topology  $\mathcal{T}(\sigma) = \tau_{X \times Y}$ .

If  $X, Y$  are compact, then so is  $X \times Y$ .

### Proof

Take  $(C_\alpha)_{\alpha \in A} = \mathcal{C}$  a cover of  $X \times Y$ .

Since  $\tau_{X \times Y}$  is generated by  $\sigma$ , each  $C_\alpha = \bigcup_\beta U_{\alpha, \beta} \times V_{\alpha, \beta}$  where  $U_{\alpha, \beta} \in \tau_X$  and  $V_{\alpha, \beta} \in \tau_Y$ .

Then  $X \times Y$  is covered by  $\mathcal{C}' = \{U_{\alpha, \beta} \times V_{\alpha, \beta} \mid \alpha, \beta\}$ .

Extract a finite subset of  $\mathcal{C}' \stackrel{\text{rewrite}}{=} \{U_\alpha \times V_\alpha \mid \alpha \in A\}$ .

Then,  $\forall y \in Y$ ,  $X \times \{y\}$  is compact. So  $X \times \{y\}$  is covered by  $\{U_{\alpha_j}^y \times V_{\alpha_j}^y \mid i \leq j \leq k_y\}$ .

Then  $V_y = \bigcap_{j=1}^{k_y} V_{\alpha_j}^y$  is an open neighborhood of  $y$ . Then  $\{V_y\}_{y \in Y}$  is an open cover of  $Y$  which is compact.

Therefore,  $Y$  is covered by  $V_{y_1}, \dots, V_{y_p}$  for some  $p \in \mathbb{N}$ ,  $y_1, \dots, y_p \in Y$ .

Check that:  $X \times Y$  is covered by  $\{U_{\alpha_i}^{y_j} \times V_{\alpha_i}^{y_j} \mid 1 \leq j \leq p, 1 \leq i \leq k_{y_j}\}$ .

Take  $(x, y) \in X \times Y$ ,  $y \in V_{\alpha_{j_0}}$  for some  $j_0$  (i.e.  $y \in V_{\alpha_i}, \forall i \in 1, 2, \dots, k_{y_{j_0}}$ ).

Since  $X \subseteq \bigcup_{i=1}^{k_{y_{j_0}}} U_{\alpha_i}^{y_{j_0}}$ ,  $\exists i_0$  such that  $(x, y) \in U_{\alpha_{i_0}}^{y_{j_0}} \times V_{\alpha_{i_0}}^{y_{j_0}}$ .

### Recall: Axiom of Choice

Given an arbitrary collection of nonempty sets  $\{S_\alpha\}_{\alpha \in A}$ , there exists a function defined on  $A$ ,  $f$ , such that  $\forall \alpha \in A, f(\alpha) \in S_\alpha$ .

### Definition: Partially Ordered Set

Take  $X \neq \emptyset$  a set. A relation " $\leq$ " on  $X$  is a "partial order" if

1.  $x \leq x, \forall x \in X$ .
2.  $x \leq y$  and  $y \leq z$  implies  $x \leq z, \forall x, y, z \in X$ .
3.  $x \leq y$  and  $y \leq x$  implies  $x = y, \forall x, y \in X$ .

It is a "total order" if additionally

1.  $\forall x, y \in X$  either  $x \leq y$  or  $y \leq x$ .



$(X, \leq)$  is a partially ordered set or “poset”.

### Example

$(\mathbb{R}, \leq)$  is totally ordered.

$X \neq \emptyset$ , consider  $\mathcal{P}(X), \subseteq$ .

e.g.  $X = \{0, 1\}$  and  $\mathcal{P}(X) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$ .

### Definition: Totally Ordered Subset

If  $(X, \leq)$  is a poset, we say  $A \subset X$  is a totally ordered subset if  $\leq|_{A \times A}$  is a total order.

IMAGE HERE - ILLUSTRATION OF INCLUSION AND  $A = \{0, 1, 0, 1\}$  A TOTALLY ORDERED SUBSET

Given  $(X, \leq)$  a poset,  $A \subset X$ , we say

- $A$  has an upper bound if  $\exists m \in X, a \leq m, \forall a \in A$ .
- $X$  has a maximum element if  $\exists m \in X, \forall x \in X, m \leq x \implies m = x$ .

a maximum element need not be unique.

### Lemma: Zorn's Lemma

Let  $(X, \leq)$  be a poset.

If every totally ordered subset of  $X$  has an upperbound, then  $X$  has a maximum element.

### Example

Zorn's lemma true for  $(X, \mathcal{P}(X))$ : a totally ordered subset of  $X$  might look like  $A_1 \subseteq A_2 \subseteq \dots \subseteq X$ .

Write  $\mathcal{A} = \{A_\alpha \mid \alpha \in J\}$ .

Guess for upperbound:  $m_{\mathcal{A}} = \bigcup_{\alpha \in J} A_\alpha$ .

### Infinite Product

Take  $A$  an index set. For  $\alpha \in A$ ,  $(X_\alpha, \tau_\alpha)$  is a topological space.

Define  $X = \prod_{\alpha \in A} X_\alpha$  the set of maps  $x : A \rightarrow \bigcup_{\alpha} X_\alpha$  such that  $\forall \alpha \in A, x(\alpha) \in X_\alpha$ .

Topology on  $\prod_{\alpha} X_\alpha$ ?

$\forall \alpha \in A$ , there is a natural projection map  $\pi_\alpha : X \rightarrow X_\alpha$  ( $\pi_\alpha : x \mapsto x(\alpha)$ ).

On  $X$ , we want  $\tau$  making  $\pi_\alpha$  continuous for every  $\alpha$ .

Therefore  $\tau$  should contain  $\pi_\alpha^{-1}(U_\alpha)$  (“slabs”),  $\forall U_\alpha \in \tau_\alpha$ .

By stability under finite intersection, it should contain  $\bigcap_{i=1}^n \pi_{\alpha_i}^{-1}(U_{\alpha_i}), \forall n \in \mathbb{N}, \alpha_1, \dots, \alpha_n \in A, U_{\alpha_j} \in \tau_{\alpha_j}, \forall j = 1, \dots, n$ .

Then  $\sigma = \{\bigcap_{i=1}^n \pi_{\alpha_i}^{-1}(U_{\alpha_i}) \mid n \in \mathbb{N}, \alpha_1, \dots, \alpha_n \in A, U_{\alpha_j} \in \tau_{\alpha_j}, j = 1, \dots, n\}$  satisfies the axioms of a base.

Define  $\tau_X : \mathcal{T}(\sigma)$ , the coarsest topology on  $X$  making  $\pi_\alpha$  continuous for every  $\alpha$ .

### Notation:

$C$  is a subset of  $X$  (i.e.  $\in \mathcal{P}(X)$ )

$\mathcal{C}$  is a cover of  $X$  (i.e.  $\in \mathcal{P}(\mathcal{P}(X))$ )

$\mathbf{C}$  is a set of covers (i.e.  $\in \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$ )

## Theorem: Tychonoff's Theorem

Following from above: if  $X_\alpha$  is compact  $\forall \alpha \in A$ , then  $X$  is compact.

An arbitrary product of compact spaces (equipped with the product topology) is compact.

### Proof

- Step 1

If  $\mathcal{C}$  is an open cover of  $X$  made of slabs  $(\pi_\alpha^{-1}(U_\alpha), \alpha \in A, U_\alpha \in \tau_\alpha)$ , then it admits a finite subcover.

Suppose not and write  $\mathcal{C} = \coprod_{\alpha \in A} \mathcal{C}_\alpha$  where  $\mathcal{C}_\alpha = \{O \in \mathcal{C} \mid O = \pi_\alpha^{-1}(U_\alpha) \text{ for some } U_\alpha \in \tau_\alpha\}$  and  $\mathcal{U}_\alpha = \{U_\alpha \mid \pi_\alpha^{-1}(U_\alpha) \in \mathcal{C}_\alpha\}$ .

Note that  $\mathcal{U}_\alpha$  covers  $X_\alpha$  if and only if  $\mathcal{C}_\alpha$  covers  $X$ .

If  $\mathcal{U}_\alpha$  covers  $X_\alpha$ , since  $X_\alpha$  is compact, then  $\exists U_1^\alpha, \dots, U_n^\alpha$  covering  $X_\alpha$ .

Then  $\{\pi_\alpha^{-1}(U_j^\alpha) \mid j = 1, \dots, n\}$  covers  $X$ , a contradiction.

Then  $\mathcal{U}_\alpha$  does not cover  $X_\alpha, \forall \alpha \implies \exists x_\alpha$  not covered by  $\mathcal{U}_\alpha$ .

Define  $x \in X$  by  $x(\alpha) = x_\alpha, \forall \alpha \in A$ . Then  $X$  is not covered by  $\mathcal{C}_\alpha, \forall \alpha \implies$  not covered by  $\mathcal{C}$ .

- Step 2

Take  $\mathcal{B}$  an arbitrary open cover of  $X$ . Suppose  $\mathcal{B}$  has no finite subcover. We will construct a subcover made of slabs and appeal to step 1, making a contradiction.

Let  $\mathbf{P} = \{\text{open covers } \mathcal{A} \supset \mathcal{B} \text{ with no finite subcover}\}$ , poset for set-inclusion ( $\mathcal{B} \in \mathbf{P} \neq \emptyset$ ).

Claim: by Zorn,  $\mathbf{P}$  has a maximum element  $\mathcal{O}$ .

To show this, prove: if  $\mathbf{W}$  is totally ordered subset of  $\mathbf{P}$ , it has an upper bound.

- Proof

Let  $\mathbf{W}$  as above, let  $\mathcal{A}_\mathbf{W} := \bigcup_{\mathcal{A} \in \mathbf{W}} \mathcal{A}$ .

Crux to prove:  $\mathcal{A}_\mathbf{W} \in \mathbf{P}$ .

By contradiction: if  $\mathcal{A}_\mathbf{W}$  has a finite subcover, call it  $\{A_1, \dots, A_n\}, \forall j \in \{1, \dots, n\}, \exists \mathcal{A}_j \in \mathbf{W}, A_j \in \mathcal{A}_j$ .

Since  $\mathbf{W}$  is totally ordered,  $\exists j_0, \forall j, A_j \in \mathcal{A}_j \subset \mathcal{A}_{j_0}$ .

But then  $\{A_1, \dots, A_n\}$  is a finite subcover of  $\mathcal{A}_{j_0}$ , a contradiction.

Note: " $\mathcal{O}$  max element in  $\mathbf{P}$ " means  $\forall U \notin \mathcal{O}, \mathcal{O} \cup \{u\}$  has a finite subcover.

- Final Step

Let  $\mathcal{O}' = \{\text{slabs in } \mathcal{O}\}$ .

Claim:  $\mathcal{O}'$  covers  $X$  ( $\implies$  Step 1  $\implies$  Contradiction)

Let  $x \in X$ , since  $\mathcal{O}$  covers  $X$ ,  $\exists O \in \mathcal{O}$  containing  $x$ .

Since the product topology is generated by  $\sigma$ ,  $\exists n, \alpha_1, \dots, \alpha_n, U_{\alpha_j} \in \tau_{\alpha_j}$  such that  $x \in \bigcap_{j=1}^n \pi_{\alpha_j}^{-1}(U_{\alpha_j})$ .

Suppose by contradiction that  $\forall j = 1, \dots, n, \pi_{\alpha_j}^{-1}(U_{\alpha_j}) \notin \mathcal{O}$ .

Then, by maximality of  $\mathcal{O}$ ,  $\exists O_{j,1}, \dots, O_{j,k} \in \mathcal{O}$  such that  $X$  is covered by  $O_{j,1}, \dots, O_{j,k} \cup \pi_{\alpha_j}^{-1}(U_{\alpha_j})$ .

$\forall j, X \setminus (O_{j,1}, \dots, O_{j,k}) \subset \pi_{\alpha_j}^{-1}(U_{\alpha_j})$ , so  $X \setminus (\bigcup_{j=1}^n \bigcup_{i=1}^{k_j} O_{j,i}) \subset \bigcap_{j=1}^n \pi_{\alpha_j}^{-1}(U_{\alpha_j})$ .

Therefore  $X$  is covered by  $U = \bigcap_{j=1}^n \pi_{\alpha_j}^{-1}(U_{\alpha_j})$  and  $\{O_{i,j} \mid j = 1, \dots, n, i = 1, \dots, k_j\}$ , hence  $O \supseteq U$ .

December 4, 2023

Bookkeeping

Final on Friday, December 15 at a 12:00.

Last homework will not be due; neither will its content be on the final.

## Today

Locally Compact Hausdorff

## Wednesday

Baire Category Theorem

## Partition of Unity

IMAGE HERE - MANIFOLD WITH COORDINATE MAPS CONNECTED TO  $\mathbb{R}^n$

$0 \leq h_i(x) \leq 1$ ,  $\text{supp } h_i \subseteq O_i \rightsquigarrow g_i$ ,  $\sum_{i=1}^n h_i(x) = 1$ ,  $\sum_{i=1}^n h_i(x)g_i(x)$ .  $h_i$  continuous,  $C^\infty$ , ?

## Definition: Locally Compact Hausdorff Space

A topological space  $X$  is Hausdorff if  $\forall x, y \in X, x \neq y, \exists U, V$  open such that  $U \cap V = \emptyset, x \in U, y \in V$ .

A Hausdorff space  $X$  is locally compact if every  $x \in X$  has a neighborhood  $U$  with compact closure ( $\overline{U}$  compact).

Notation:  $X$  is LCH Space.

## Examples

$\mathbb{R}^n$ : if  $x \in \mathbb{R}^n, B_1(x), \overline{B_1(x)}$  compact (since closed and bounded).

Any compact, Hausdorff space ( $[0, 1] \subseteq \mathbb{R}$ , Cantor  $\subseteq \mathbb{R}$ )

## Counterexamples (Hausdorff but not Locally Compact)

Example to be digested later: infinite-dimensional Hilbert space.

$\mathbb{Q}$ .

## Theorem: (Target)

Let  $V_1, \dots, V_n$  be open sets in an LCH space  $X$ .

Suppose that  $K \subset X$  is compact and covered by  $V_1, \dots, V_n$ .

Then there exist functions  $h_i \in C(X; \mathbb{R})$  with  $\text{support}(h_i) \subset V_i$  such that  $\sum_{i=1}^n h_i(x) = 1, \forall x \in K$ .

We call  $(h_1, \dots, h_n)$  a partition of unity on  $K$  subordinate to the cover  $V_1, \dots, V_n$ .

Proof to Follow

## Recall

Let  $X$  be a Hausdorff topological space.

R1. If  $x \in X, K \subseteq X$  compact,  $\exists U, V$  open such that  $U \cap V = \emptyset, x \in U, K \subseteq V$ .

R2. If  $\{K_\alpha\}_{\alpha \in A}$  is a collection of compact sets such that  $\bigcap_\alpha K_\alpha = \emptyset$ , then  $\exists \alpha_1, \dots, \alpha_n$  such that  $\bigcap_{i=1}^n K_{\alpha_i} = \emptyset$ .

**Lemma: (K-U-V)**

Let  $X$  be a LCH space,  $K \subset X$  compact,  $U$  open and  $K \subset U$ .

Then there exists  $V$  open with compact closure, such that  $K \subset V \subset \overline{V} \subset U$ .

**Proof**

Step 1:  $K$  has an open neighborhood with compact closure.

$\forall x \in K$ ,  $x$  has an open neighborhood  $W_x$  with compact closure.

Then  $K \subset \bigcup_{x \in K} W_x \implies$ , by compactness,  $\exists x_1, \dots, x_n \in K$  such that

$$K \subset W_{x_1} \cup \dots \cup W_{x_n} := W$$

with  $\overline{W_{x_1}} \cup \dots \cup \overline{W_{x_n}}$  compact.

Step 2: Build  $V$ .

$\forall q \in U^c$ ,  $q \notin K$ . By (R1) above,  $\exists V_q$  open such that  $K \subseteq V_q$  and  $q \notin V_q$ .

Now consider the collection  $\{U^c \cap \overline{W} \cap \overline{V_q}\}_{q \in U^c}$ .

Observe that this collection is compact since  $\overline{W}$  is compact, and  $(\bigcap_{q \in U^c} \overline{V_q}) \cap U^c = \emptyset$ .

Then

$$\bigcap_{q \in U^c} U^c \cap \overline{W} \cap \overline{V_q} = \emptyset \xrightarrow{(R2)} \exists q_1, \dots, q_p \text{ such that } U^c \cap \overline{W} \cap \overline{V_{q_1}} \cap \dots \cap \overline{V_{q_p}} = \emptyset$$

Recall that  $A \subset B$  if and only if  $A \cap B^c = \emptyset$ .

Hence  $\overline{W} \cap \overline{V_{q_1}} \cap \dots \cap \overline{V_{q_p}} \subset U$ .

Set  $V = W \cap V_{q_1} \cap \dots \cap V_{q_p}$  open, containing  $K$ , and  $\overline{V} = \overline{W} \cap \overline{V_{q_1}} \cap \dots \cap \overline{V_{q_p}}$ . ■

**Definition / Notation:**

Given a topological space  $X$ ,

1. if  $f : X \rightarrow \mathbb{R}$ , define the support of  $f$ :  $\text{supp}(f) = \overline{\{x \in X \mid f(x) \neq 0\}}$ .  
(e.g.  $\text{supp}\left(e^{-\frac{1}{x^2}}\right) = [0, \infty)$ )
2. if  $K$  compact,  $k < f$  means  $f$  has compact support and  $f = 1$  on  $K$ .
3. if  $V$  open,  $f < V$  means  $f : X \rightarrow [0, 1]$  such that  $\text{supp}(f) \subseteq V$ .

**Definition: Semi-continuity**

Let  $f : X \rightarrow \mathbb{R}$  with  $X$  a topological space.

We say  $f$  is lower semi-continuous if  $f^{-1}((a, \infty)) = \{x \in X \mid f(x) > a\}$  is open  $\forall a$ .

We say  $f$  is upper semi-continuous if  $f^{-1}((-\infty, b))$  is open  $\forall b$ .

**Examples**

If  $U$  is open in  $X$ ,  $\chi_U = \begin{cases} 1 & \text{on } U \\ 0 & \text{outside} \end{cases}$  is lower semi-continuous.

$\chi_{(0,\infty)}$  is lower semi-continuous.

If  $C$  closed in  $X$ ,  $\chi_C = 1 - \chi_{C^c}$  is lower semi-continuous.

### Lemma

1. If  $f_\alpha$  is lower semi-continuous  $\forall \alpha$ , then  $f = \sup_\alpha f_\alpha$  is lower semi-continuous.
2. If  $g_\alpha$  is upper semi-continuous  $\forall \alpha$ , then  $g = \inf_\alpha g_\alpha$  is upper semi-continuous.

#### • Proof of 1

$\forall a \in \mathbb{R}, f^{-1}((a, \infty)) = \bigcup_\alpha f_\alpha^{-1}((a, \infty))$ , open since  $f_\alpha$  is lower semi-continuous.  
 $f(x) > a$  if and only if  $\exists \alpha, f_\alpha(x) > a$ .

### Continuity

A function which is both upper and lower semi-continuous is continuous. Look at

$$f^{-1}((a, b)) = f^{-1}((a, \infty) \cap (-\infty, b)) = f^{-1}((a, \infty)) \cap f^{-1}((-\infty, b))$$

which is open.

### Urysohn's Lemma

Let  $X$  be an LCH space,  $K$  compact,  $U$  open,  $K \subset U$ .

Then  $\exists f \in C(X; [0, 1])$  with  $K < f < U$ .

### Notation

For  $S \subseteq X$ , let  $\chi_S(X) = \begin{cases} 0 & x \notin S \\ 1 & x \in S \end{cases}$ , the characteristic function of  $S$ . Then

$$K < f < U \iff f \in C(X; [0, 1]), X_U(x) \geq f(x) \geq X_K(x)$$

### Metric Space Version

$A$  closed,  $d_A(x) = \inf\{d(x, y) \mid y \in A\}$ .

$$f(x) = \frac{d_{U^c}(x)}{d_{U^c}(x) + d_K(x)}$$

### Proof

Fix  $K, U$  as in the statement.

Apply the (K-U-V) Lemma to  $K, U$  to get  $V_0$  such that  $K \subset V_0 \subset \overline{V_0} \subset U$ .

Apply the lemma to  $K, V_0$  to get  $V_1$  such that  $K \subset V_1 \subset \overline{V_1} \subset V_0 \subset \overline{V_0} \subset U$ .

Next Goal: for every  $r \in \mathbb{Q} \cap [0, 1]$ , construct  $V_r$  open with compact closure such that

$$s > r \implies \overline{V_s} \subset V_r, \forall s, r \in \mathbb{Q} \cap [0, 1]$$

To Do So: set  $r_0 = 0, r_1 = 1, \{r_n\}_{n \geq 2}$  a denumeration of  $\mathbb{Q} \cap (0, 1)$ .

Construct, by induction,  $\{V_n\}_{n \geq 2}$ : for fixed  $n$ , suppose  $V_{r_0}, \dots, V_{r_{n-1}}$  have been constructed such that

$$r_p > r_q \implies \overline{V_{r_p}} \subset \overline{V_{r_q}}, \forall p, q \in \{0, \dots, n-1\}$$

Let  $p_0 \in \{0, \dots, n-1\}$  such that  $r_{p_0}$  is the largest rational less than  $r_n$ .

Let  $q_0 \in \{0, \dots, n-1\}$  such that  $r_{q_0}$  is the smallest rational greater than  $r_n$ .

$$r_{p_0} < r_n < r_{q_0}$$

Apply (K-U-V) Lemma to  $K' = \overline{V_{q_0}}$  and  $U' = V_{r_{p_0}}$  to produce  $V_{r_n}$  such that

$$\overline{V_{r_{q_0}}} \subset V_{r_n} \subset \overline{V_{r_n}} \subset V_{r_{p_0}}$$

thereby propagating the induction.

Next: for any  $r \in \mathbb{Q} \cap [0, 1]$ , let  $f_r := r \cdot \chi_{V_r} = \begin{cases} r & \text{on } V_r \\ 0 & \text{outside} \end{cases}$ .

$$g_r := r + (1-r)\chi_{\overline{V_r}} = \begin{cases} 1 & \text{on } \overline{V_r} \\ 0 & \text{outside} \end{cases}.$$

$$\text{Let } \begin{cases} f(x) := \sup_r f_r(x) \\ g(x) = \inf_r g_r(x) \end{cases}$$

IMAGE HERE - PROCESS ON REAL LINE

Claim:  $f(x) = g(x)$ :  $\forall r \in \mathbb{Q} \cap [0, 1], f_r(x) \leq g_r(x) \implies f(x) \leq g(x)$ .

By contradiction, if  $x$  is such that  $f(x) < g(x)$ , then  $\exists r < s$  two rationals such that  $f(x) < r < s < g(x)$ .

Then  $f(x) < r \implies x \notin V_r$  and  $g(x) > s \implies x \in \overline{V_s}$ .

Since  $\overline{V_s} \subset V_r$ , this is a contradiction.

Bottom Line:  $f = g$ . Urysohn's Lemma will be proved if  $f$  is continuous.

For all  $r$ ,  $f_r = r\chi_{V_r}$  is lower semi-continuous and  $g_r = r + (1-r)\chi_{\overline{V_r}}$  is upper semi-continuous.

Then  $f = g$  is both upper and lower semi-continuous and, subsequently, continuous.

## Proof of (Target) Theorem

First, find compact sets in each  $V_i$  (and apply Urysohn's Lemma).

$\forall x \in X, \exists i$  such that  $x \in V_i$ .

By (K-U-V) Lemma, with  $K = \{x\}$  and  $U = V_i$ ,  $\exists W_x$  with compact closure such that  $x \in W_x \subset \overline{W_x} \subset V_i$ .

By compactness of  $K$ ,  $K \subset W_{x_1} \cup \dots \cup W_{x_n}$ .

For each  $i$ , set  $K_i := \bigcup \{\overline{W_{x_j}} \mid W_{x_j} \subseteq V_i\}$  where  $K_i$  compact,  $K_i \subseteq V_i$  and  $K \subset \bigcup_{i=1}^n K_i$ .

By Urysohn's Lemma, let  $g_i$  such that  $K_i < g_i < V_i$ .

Now let  $h_1(x) = g_1(x)$ ,  $h_2(x) = g_2(x)(1 - g_1(x))$ ,  $\dots$ ,  $h_n(x) = g_n(x)(1 - g_1(x)) \cdots (1 - g_{n-1}(x))$ .

Then  $\text{supp}(h_i) \subseteq \text{supp}(g_i) \subseteq V_i$  and  $h_i$ , as the product of continuous functions, is continuous.

Finally

$$\begin{aligned} 1 - \sum_{i=1}^n h_i(x) &= 1 - g_1(x) + g_2(x)(1 - g_1(x)) + \dots + [g_n(x)(1 - g_1(x)) \cdots (1 - g_{n-1}(x))] \\ &= 1 - g_1(x) + (1 - g_1(x))(g_2(x) + (1 - g_2(x))(g_3(x) + (1 - g_3(x))) \cdots \\ &= \prod_{i=1}^n (1 - g_i(x)) \end{aligned}$$