Problem 1

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a) $dS = rS\,dt + \sigma S\,dW$

Using Itos lemma to $\ln S_t$,

$$d\ln S_t = \left(\frac{dS_t}{S_t} - \frac{1}{2}\left(\frac{\sigma S_t\,dw}{S_t}\right)^2\right) = \left(r - \frac{\sigma}{2}\right)dt + \sigma dw_t$$

than integrating both sides,

$$\ln S_t = \ln S_t + \left(r + \frac{\sigma^2}{2}\right)(t - t) + \sigma(W_t - W_t$$

Since $W_t - W_t$ is normally distributed with $\mu = 0$ and $\sigma^2 = T - t$. the expected value is,

$$E[\ln S_t] = \ln S_t + \left(r + \frac{\sigma^2}{2}\right)(T - t)$$

Than by using the risk nuetral evaluation, the price $f(S_t, t)$ is,

$$f(S_t, t) = e^{-r(t-t)} E(\ln S] = e^{-r(T-t)}[\ln S_t + \left(r - \frac{\sigma^2}{2}\right)(T - t)$$

b) From a we know $f(S_t, t)$. Than by black sholes,

$$\frac{\partial f}{\partial t} = r e^{-r(T-t)}[\ln S_t + \left(r - \frac{\sigma^2}{2}\right)(T - t) - e^{-r(T-t)}\left(r - \frac{\sigma^2}{2}\right)$$

$$\frac{\partial f}{\partial S} = \frac{e^{-r(T-t)}}{S}$$

$$\frac{\partial^2 f}{\partial S^2} = \frac{-e^{-r(t-t)}}{S^2}$$

Than substituting in,

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - rf = e^{-r(T-t)}[r\ln S + r\left(r - \frac{\sigma^2}{2}\right)(t - t) - \left(r - \frac{\sigma^2}{2}\right) + r - \frac{\sigma^2}{2}$$

$$= e^{-r(t-t)}[r\ln s + r\left(r - \frac{\sigma^2}{2}\right](t - t)$$

$$= rf$$

Thus the equation is satisfied