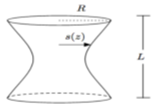
Consider a surface connecting two rings of equal radii R, separated by a distance L, as shown in the figure to the right. Your task in this assignment is to find the shape of the surface that has the *minimal* area that connects both rings. Soap films find these minimal areas automatically due to surface tension, which lets them minimize their total surface area in order to reach the lowest possible energy configuration.



Theory

If the function $s(z)$ describes the radius of the surface as a function of height z (with $0 < z < L$) then the area A of the surface is given by:

$$A = 2\pi \int_0^L s(z)\sqrt{1+\left(\frac{ds}{dz}\right)^2}\,dz \qquad (1)$$

Taking the derivative of the above expression and setting it to zero, we obtain the following differential equation for the function s:

$$1+\left(\frac{ds}{dz}\right)^2 - s\frac{d^2s}{dz^2} = 0 \qquad (2)$$

with boundary conditions $s(0) = R$ and $s(L) = R$. It can be shown that the general solution of the above differential equation is given by:

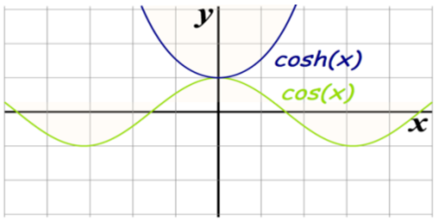
$$s(z) = \alpha\cosh\left(\frac{z-\beta}{\alpha}\right) \qquad (3)$$

where α and β are independent real constants (think about which units they have) and **cosh** is the hyperbolic cosine function.

Starting with the general solution to the referenced differential equation:

$$s(z) = \alpha\cosh\left(\frac{z-\beta}{\alpha}\right) \qquad (1)$$

Applying the boundary conditions, s(0) = R = s(L), along with recognising the fact that cosh(x) is an even function:



cosh vs cos

It is found that (1) becomes:

$$s(0) = \alpha\cosh\left(\frac{-\beta}{\alpha}\right) = \alpha\cosh\left(\frac{\beta}{\alpha}\right) = R \qquad (2)$$

$$s(L) = \alpha\cosh\left(\frac{L-\beta}{\alpha}\right) = \alpha\cosh\left(\frac{-L+\beta}{\alpha}\right) = R \qquad (3)$$

From (2), we simply take the inverse of the sinusoidal term to find the constant β:

$$\beta = \alpha cosh^{-1}\left(\frac{R}{\alpha}\right) \qquad (4)$$

Likewise from (3), we can isolate the α constant:

$$\alpha = \frac{L-\beta}{cosh^{-1}\left(\frac{R}{\alpha}\right)} \qquad (5)$$

Which, using the result from (4) for the constant, may be simplified to:

$$\alpha = \frac{L-\beta}{\frac{\beta}{\alpha}} = \frac{\alpha(L-\beta)}{\beta} = \frac{\alpha L}{\beta} - \alpha \qquad (6)$$

Choosing (6) now, pursuant to trying to solve a homogeneous(RHS = 0) polynomial, of the form $f(\alpha, R, L)$ it then makes sense to try and use the explicit form of β in (6):

$$2\alpha = \frac{\alpha L}{\beta} = \frac{\alpha L}{\alpha cosh^{-1}\left(\frac{R}{\alpha}\right)} \qquad (7)$$

I think it is just noteworthy to point out that at this point, we do actually glean from (7), that the two constants can be written:

$$\beta = \frac{L}{2}, \alpha = \frac{L}{2cosh^{-1}\left(\frac{R}{\alpha}\right)} \qquad (8)$$

But otherwise, continuing from (7), we arrive at the homogeneous polynomial of the form:

$$2\alpha cosh^{-1}\left(\frac{R}{\alpha}\right) - \alpha L = 0 \qquad (9)$$