Applied Optimization for Wireless, Machine Learning, Big Data.
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Lecture - 09
Eigenvalue Decomposition of Hermitian Matrices and Properties

Hello, welcome to another module in this massive open online course. Let us discuss the Eigenvalue decomposition of Hermitian Symmetric Matrix.

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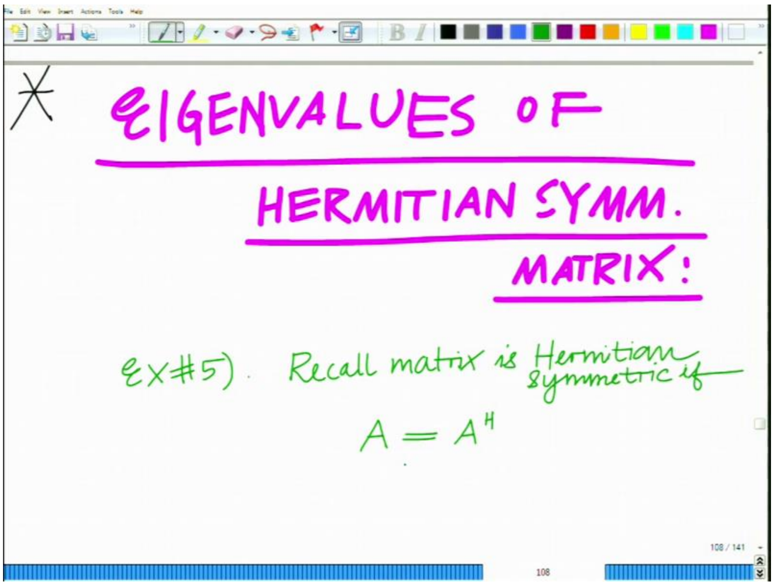

So, recall that the Hermitian Symmetric matrix is the matrix which is equal to the Hermitian of itself and that is;

$$A = A^H$$

Here A is a complex matrix that is $A \in \mathbb{C}^{m\times n}$.

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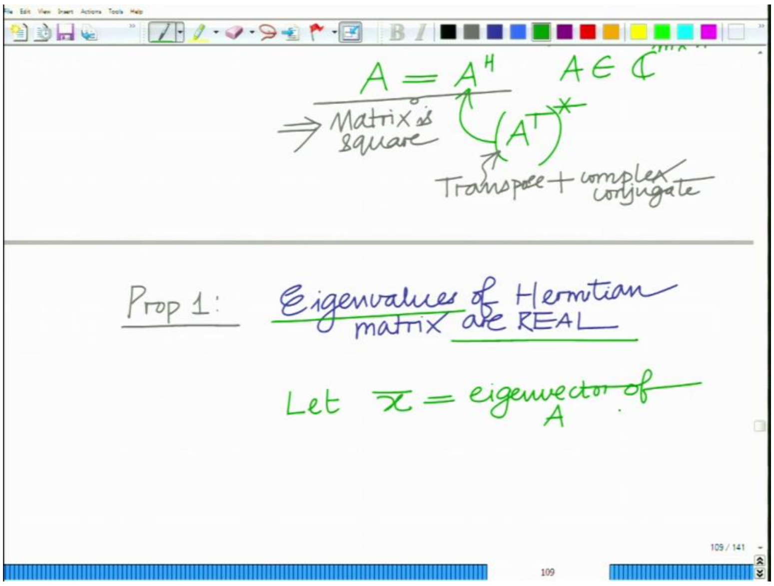

Also to make a Hermitian matrix of a complex matrix, first take the transpose of the matrix and then take the complex conjugate of its elements. And for being a matrix equal to its Hermitian matrix, it will be possible only if the matrix is a square matrix. Therefore the above illustration of Hermitian symmetric matrix is only possible if $m = n$.

Let us discuss some properties of Hermitian symmetric matrix. The first property is that the eigenvalues of a Hermitian matrix are all real. To prove this, let us consider a vector $\bar{x}$ be the Eigenvector of matrix A and λ is the Eigenvalue of matrix A corresponding to the Eigenvector $\bar{x}$.

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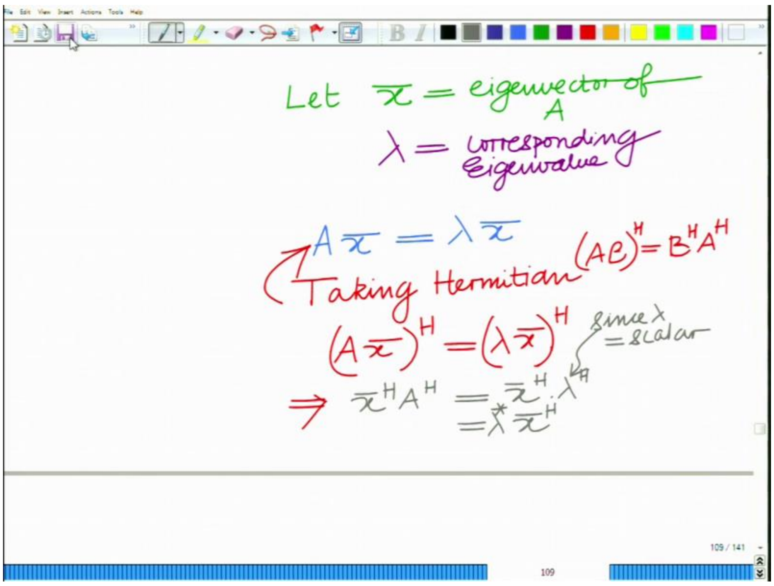

Therefore,

$$A\bar{x} = \lambda\bar{x}$$

Take Hermitian of the above equation.

$$\left(A\bar{x}\right)^H = \left(\lambda\bar{x}\right)^H$$

And recall the following property.

$$\left(AB\right)^H = B^H A^H$$

Therefore;

$$\begin{aligned}\bar{x}^H A^H &= \bar{x}^H \lambda^H \\ &= \lambda^* \bar{x}^H\end{aligned}$$

Since λ is a scalar quantity and for a scalar quantity, the Hermitian of the quantity is simply equal to its complex conjugate.

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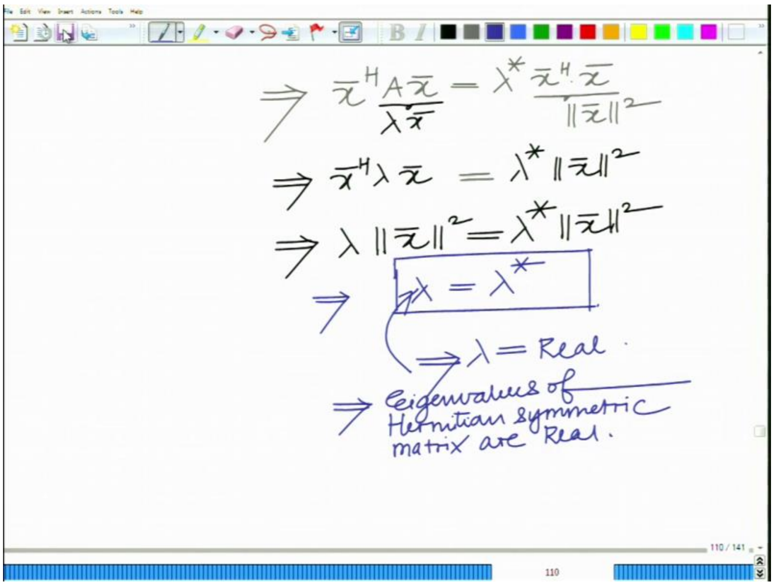

On multiplying $\bar{x}$ on right of above equation in both the sides,

$$\overline{x}^H A^H \overline{x} = \lambda^* \overline{x}^H \overline{x}$$

$$\overline{x}^H \lambda \overline{x} = \lambda^* \|\overline{x}\|^2$$

$$\lambda \|\overline{x}\|^2 = \lambda^* \|\overline{x}\|^2$$

And hence it implies that;

$$\lambda = \lambda^*$$

And above equation is only possible if λ is a real quantity. Therefore it leads to the conclusion that the Eigenvalue of a Hermitian symmetric matrix always a real quantity.

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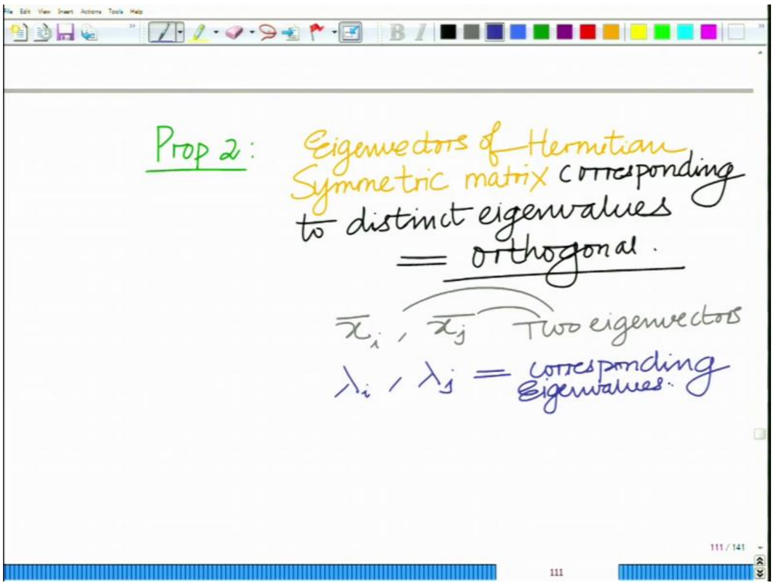

The second property of a Hermitian symmetric matrix is that the Eigenvectors of a Hermitian symmetric matrix corresponding to distinct Eigenvalues are orthogonal. Orthogonality implies zero inner product. To demonstrate this property, consider two Eigenvectors $\overline{x}_i$ and $\overline{x}_j$ of matrix A and its corresponding Eigenvalues λ_i and λ_j which are distinct in nature. This implies that $\lambda_i \neq \lambda_j$.

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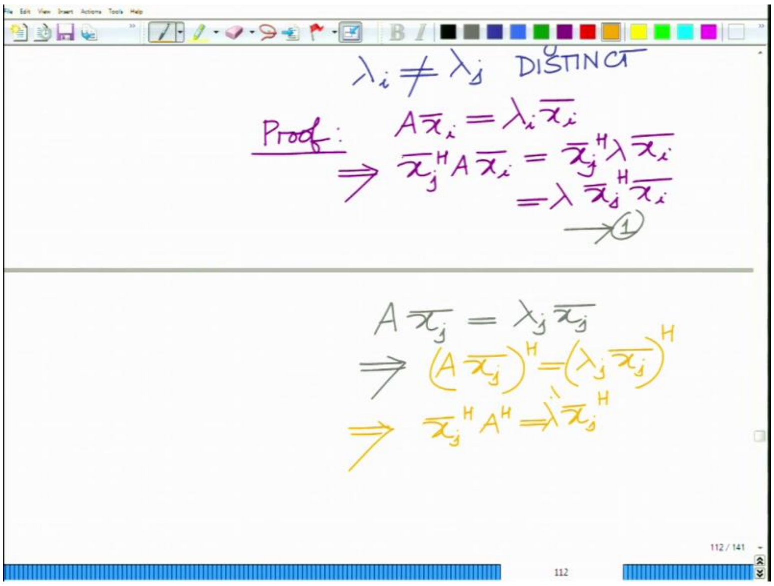

So, to prove the second property, the inner product of both the Eigenvectors must be zero. Thus, the proof can proceed as follows. First write the characteristic equation of A with one Eigenvector $\bar{x}_i$ and its corresponding Eigenvalue λ_i and multiply other Eigenvector $\bar{x}_j$ on the left of equation in both the sides.

$$A\bar{x}_i = \lambda_i \bar{x}_i$$
$$\bar{x}_j^H A\bar{x}_i = \bar{x}_j^H \lambda_i \bar{x}_i$$

And hence

$$\bar{x}_j^H \lambda \bar{x}_i = \lambda_i \bar{x}_j^H \bar{x}_i \qquad \text{Eq. 1}$$

Now write the characteristic equation of A with the Eigenvector $\bar{x}_j$ and its corresponding Eigenvalue λ_j and take Hermition of both the sides.

$$A\bar{x}_j = \lambda_j \bar{x}_j$$
$$\left(A\bar{x}_j\right)^H = \left(\lambda_j \bar{x}_j\right)^H$$

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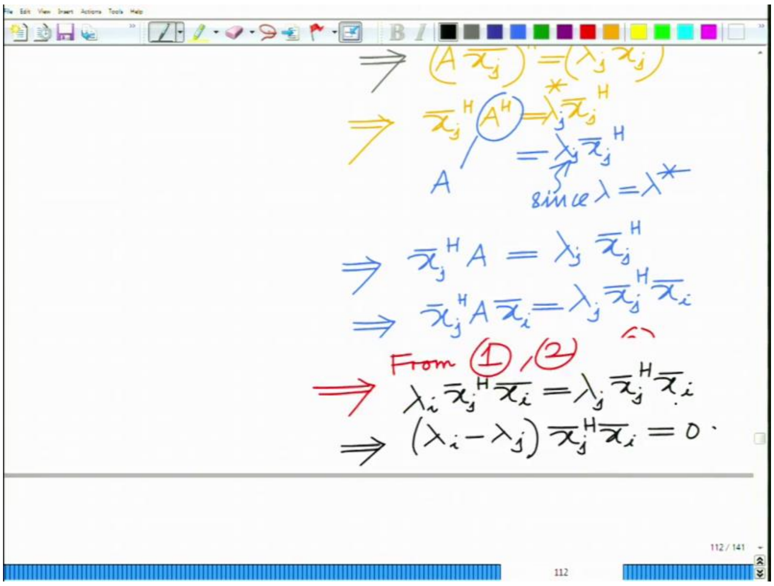

As λ is a scalar quantity and also it is a real number, therefore;

$$\bar{x}_j^H A^H = \lambda_j \bar{x}_j^H$$

Also it is a Hermitian symmetric matrix and $A = A^H$ therefore

$$\bar{x}_j^H A = \lambda_j \bar{x}_j^H$$

Now multiply the above equation by other eigenvalue $\bar{x}_i$ on right.

$$\bar{x}_j^H A \bar{x}_i = \lambda_j \bar{x}_j^H \bar{x}_i \qquad \text{Eq. 2}$$

On solving equation 1 and 2;

$$\lambda_i \bar{x}_j^H \bar{x}_i = \lambda_j \bar{x}_j^H \bar{x}_i$$

$$(\lambda_i - \lambda_j) \bar{x}_j^H \bar{x}_i = 0$$

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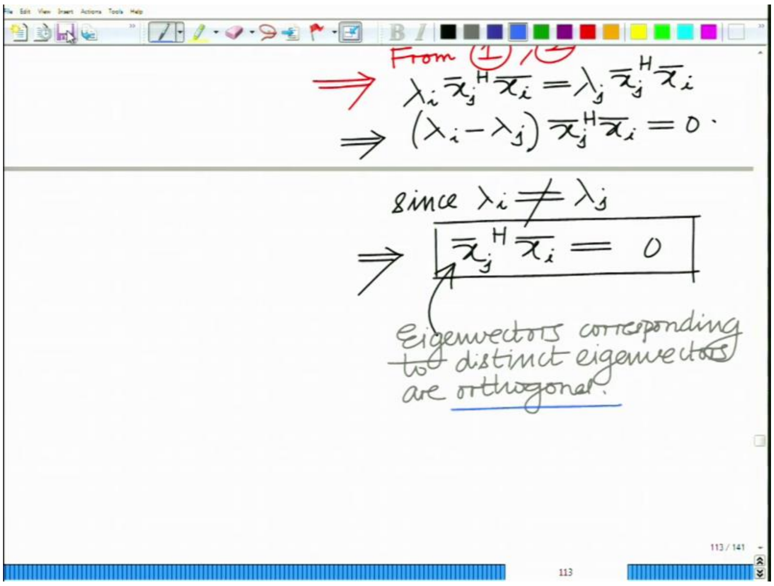

Since eigenvalues λ_i and λ_j are distinct that is $\lambda_i \neq \lambda_j$; therefore it implies that

$$\bar{x}_j^H \bar{x}_i = 0$$

Hence it is verified that the eigenvectors of a Hermitian symmetric matrix corresponding to distinct Eigenvalues are orthogonal. So, these are two important properties of Hermitian symmetric matrices that are frequently going to be used during the development of various techniques for optimization.

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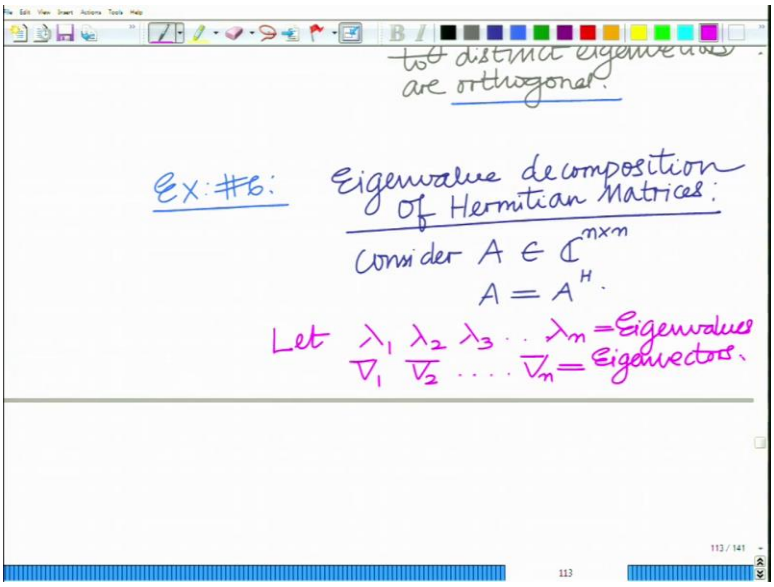

Let us discuss another example showing Eigenvalue decomposition of Hermitian matrices. Again consider a Hermitian matrix $A \in \mathbb{C}^{n \times n}$, having Eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ and corresponding Eigenvectors $\bar{V}_1, \bar{V}_2, \cdots, \bar{V}_n$.

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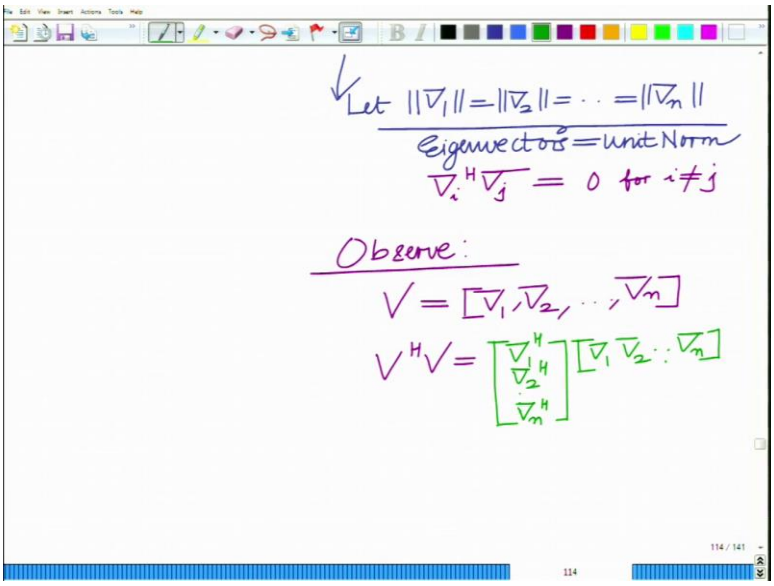

So observe that these Eigenvectors are unit norm by simply normalize them. It is because normalization is nothing but scaling of a vector in terms of its norm which is simply a constant and hence after normalization, these Eigenvectors will still remain Eigenvectors. Therefore

$$\|\bar{V}_1\| = \|\bar{V}_2\| = \cdots = \|\bar{V}_n\|$$

Further, from the previous property, assume that the Eigenvalues are distinct which implies

$$\bar{V}_i^H \cdot \bar{V}_j = 0 \qquad \text{for} \quad i \neq j$$

Notice that if consider that matrix V as

$$V = \left[\bar{V}_1, \bar{V}_2, \cdots, \bar{V}_n\right]$$

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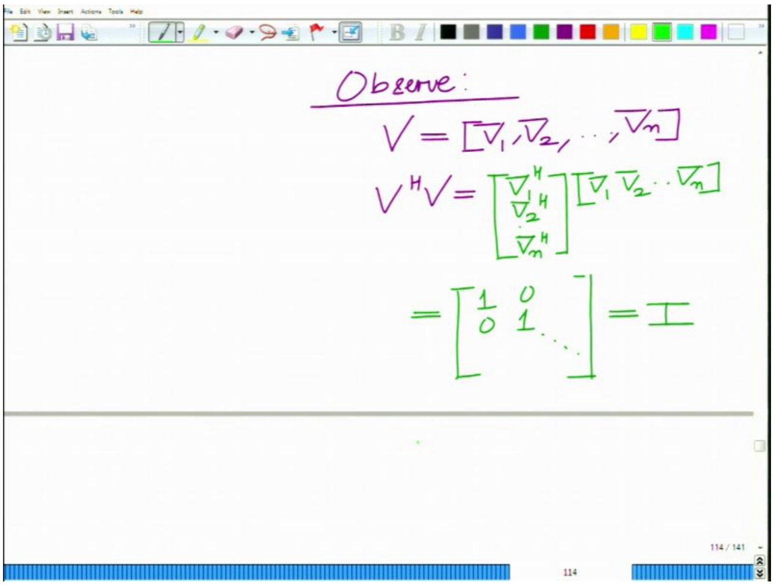

And then

$$
\begin{aligned}
V^H V &= \begin{bmatrix} \bar{V}_1^{\,H} \\ \bar{V}_2^{\,H} \\ \vdots \\ \bar{V}_n^{\,H} \end{bmatrix} \begin{bmatrix} \bar{V}_1 & \bar{V}_2 & \cdots & \bar{V}_n \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \\
&= I
\end{aligned}
$$

So, this is simply an identity matrix which implies V is the inverse of V^H.

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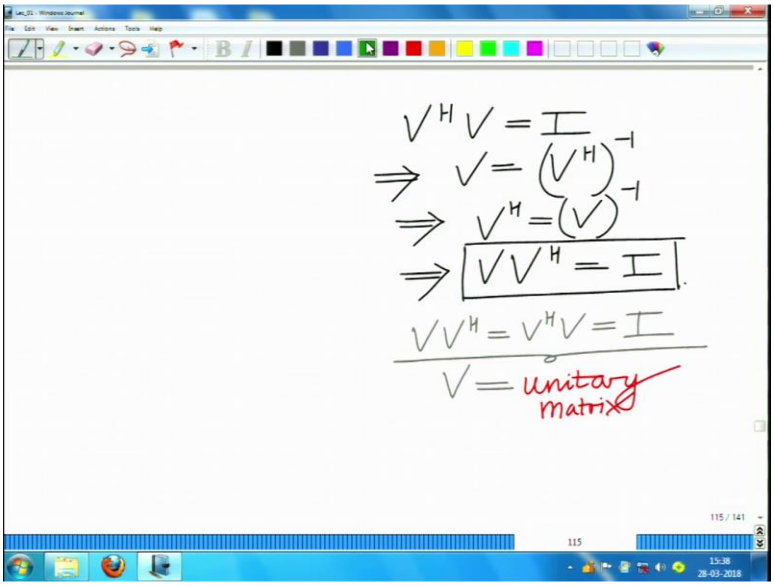

And since the inverse of a square matrix is unique and this also implies V^H equals to V^{-1}. So this also implies that if matrix A is B^{-1}, then if matrix AB is an identity matrix then matrix BA is also an identity matrix.

$$V^H V = I$$

$$V = \left(V^H\right)^{-1}$$

$$VV^H = I$$

Such matrix is termed as a unitary matrix.

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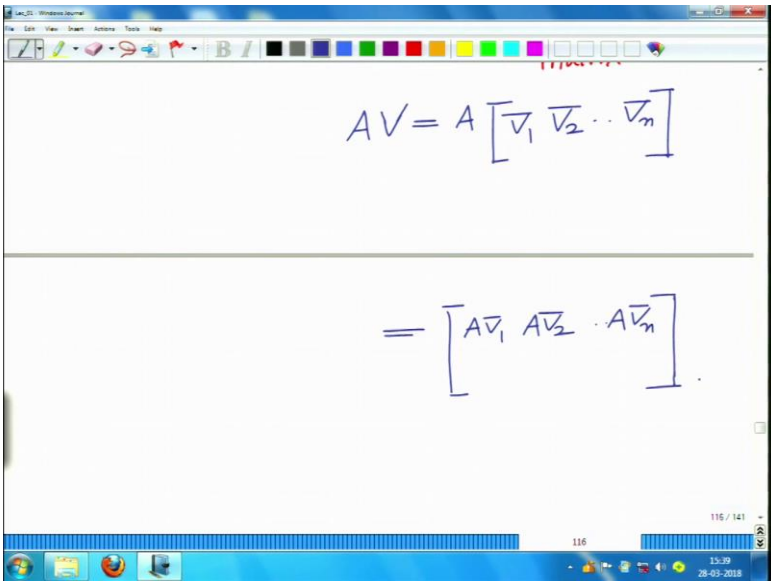

Now let us look at the product of matrix A and its Eigenvector space V that is

$$AV = A\begin{bmatrix} \bar{V}_1 & \bar{V}_2 & \cdots & \bar{V}_n \end{bmatrix}$$
$$= \begin{bmatrix} A\bar{V}_1 & A\bar{V}_2 & \cdots & A\bar{V}_n \end{bmatrix}$$

As $\bar{V}_1, \bar{V}_2, \cdots, \bar{V}_n$ are Eigenvectors so $AV_i = \lambda V_i$.

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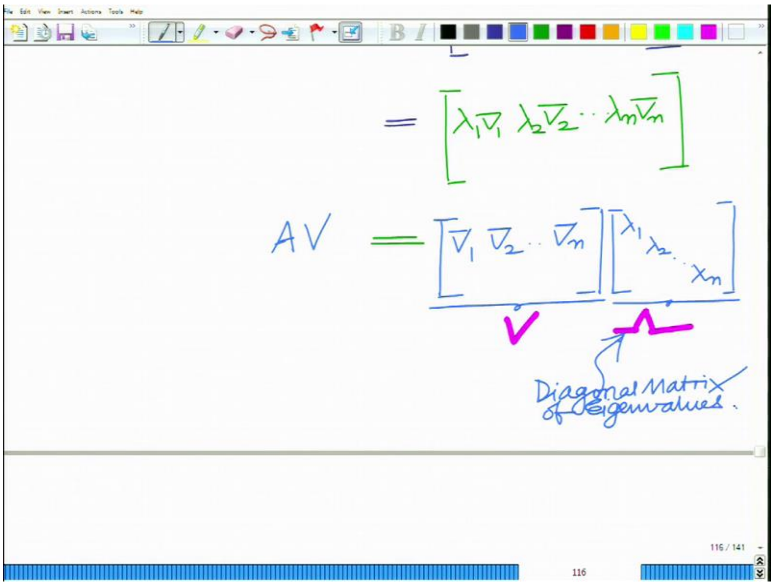

Therefore,

$$AV = \begin{bmatrix} \lambda_1 \bar{V}_1 & \lambda_2 \bar{V}_2 & \cdots & \lambda_n \bar{V}_n \end{bmatrix}$$

This can also be written as

$$AV = \begin{bmatrix} \bar{V}_1 & \bar{V}_2 & \cdots & \bar{V}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & \cdots & \cdots & \cdots \\ \vdots & \lambda_2 & \cdots & \cdots \\ \vdots & \vdots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \lambda_n \end{bmatrix}$$
$$= V\Lambda$$

Here Λ is the diagonal matrix of Eigenvalues and simply known as Lambda.

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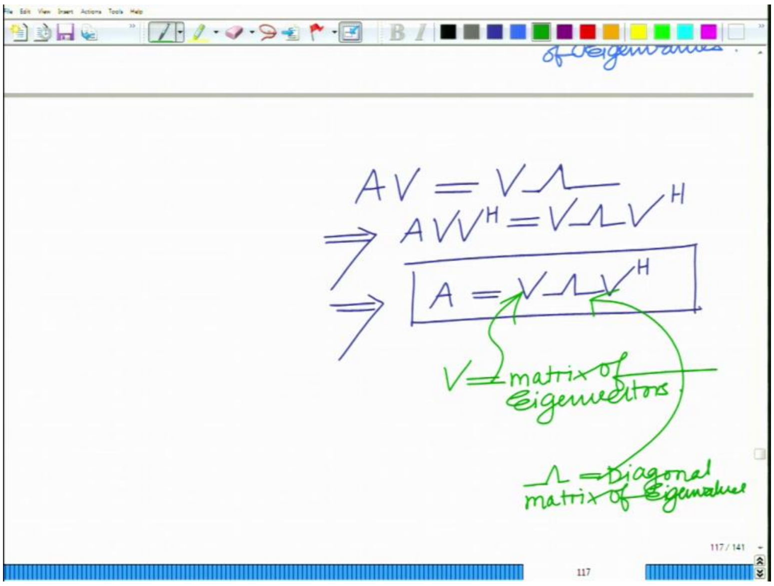

So it can be written that

$$\begin{aligned} AV &= V\Lambda \\ AVV^H &= V\Lambda V^H \\ A &= V\Lambda V^H \end{aligned}$$

This is termed as the Eigenvalue Decomposition of a matrix A.

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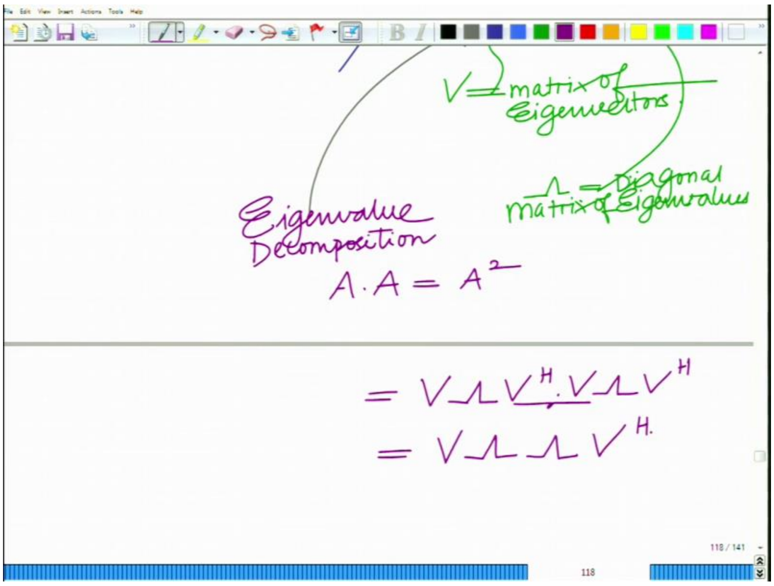

It has several properties. First is that to compute the square of a matrix A, one can also write it as

$$
\begin{aligned}
A^2 &= A \cdot A \\
&= V\Lambda V^H \cdot V\Lambda V^H \\
&= V\Lambda\Lambda V^H \\
&= V\Lambda^2 V^H
\end{aligned}
$$

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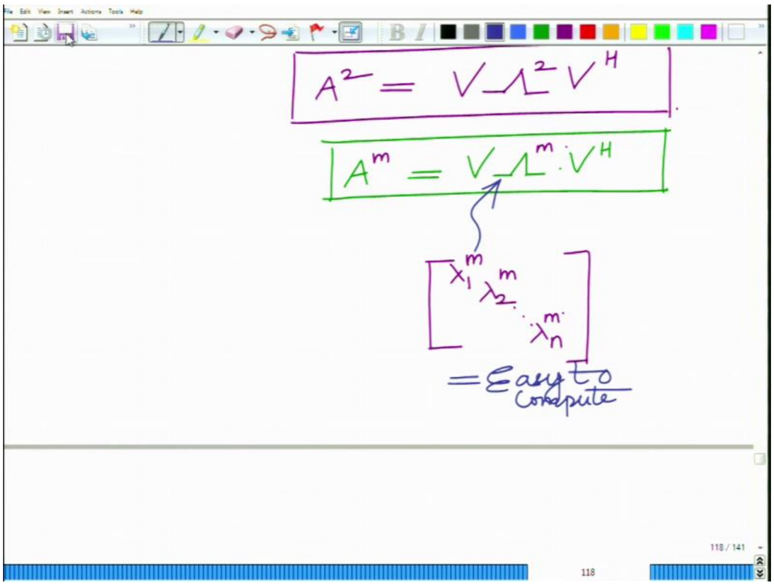

Similarly on generalizing the above equation;

$$A^m = V\Lambda^m V^H$$

Where Λ^m is the diagonal matrix containing m[th] power of Eigenvalues λ_i and is easy to compute.

$$\Lambda^m = \begin{bmatrix} \lambda_1^{\ m} & \cdots & \cdots & \cdots \\ \vdots & \lambda_2^{\ m} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \lambda_n^{\ m} \end{bmatrix}$$

So Eigenvalue Decomposition of a matrix is one of the fundamental decomposition or property of a matrix. Let us continue with other aspects in the subsequent modules.

Thank you very much.