**Applied Optimization for Wireless, Machine Learning, Big Data**
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**Lecture – 06**
**Properties of Norm, Gaussian Elimination, Echelon form of matrix**

Hello. Welcome to another module in this massive open online course. So, we are looking at the concept of inner product and its various properties. In particular, we have also looked at the Cauchy-Schwarz inequality.

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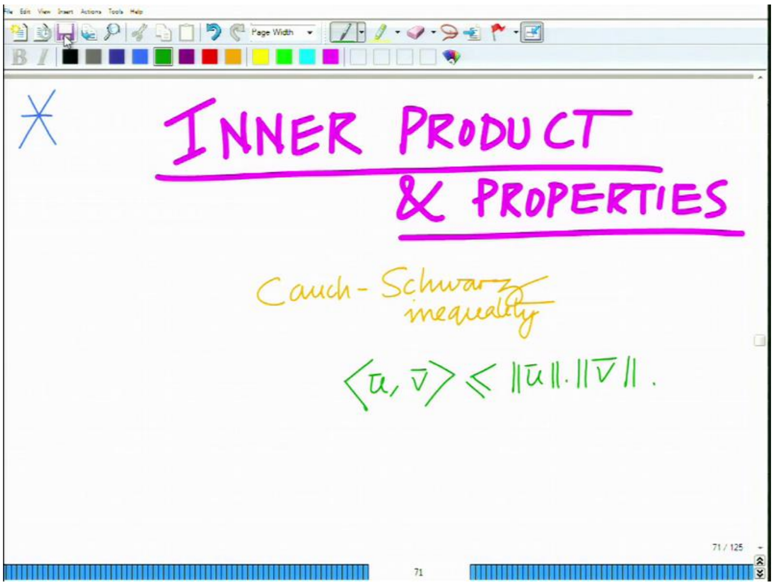


The Cauchy-Schwarz inequality states that

$$\langle \bar{u}, \bar{v} \rangle \leq \|\bar{u}\| \cdot \|\bar{v}\|$$

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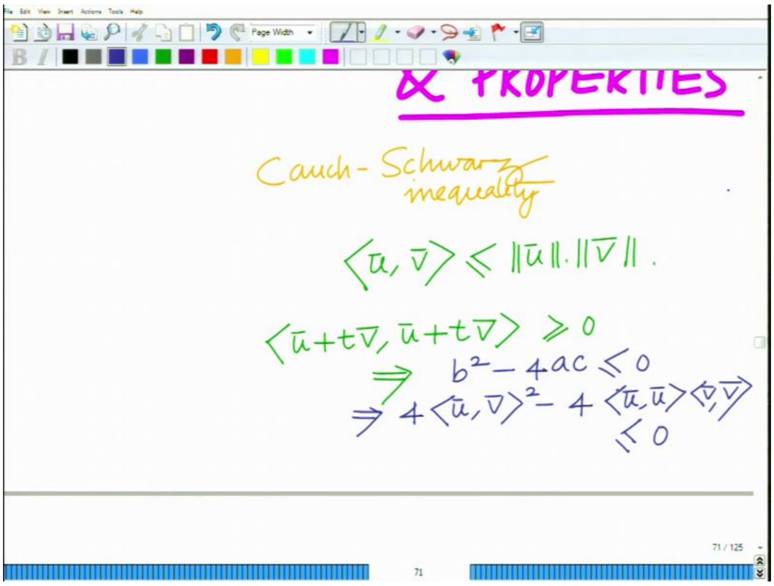


And in fact, in the derivation of the Cauchy-Schwarz inequality, we have seen that

$\langle \bar{u}+t\bar{v}, \bar{u}+t\bar{v} \rangle \geq 0$

And this implies that quadratic in

$$b^2 - 4ac \leq 0$$

$$4\langle \bar{u},\bar{v} \rangle^2 - 4\langle \bar{v},\bar{v} \rangle \langle \bar{u},\bar{u} \rangle \leq 0$$

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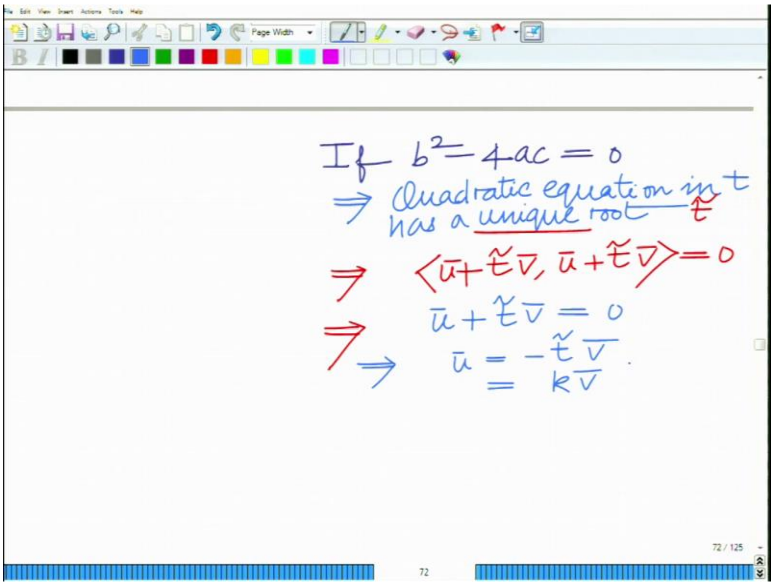

In fact, if quadratic equation has a unique root then it means

$$b^2 - 4ac = 0$$

And now,

$$\langle \bar{u} + \tilde{t}\bar{v}, \bar{u} + \tilde{t}\bar{v} \rangle = 0$$
$$\bar{u} + \tilde{t}\bar{v} = 0$$
$$\bar{u} = -\tilde{t}\bar{v} = k\bar{v}$$

And we can denote $\tilde{t}$ as some constant $k$.

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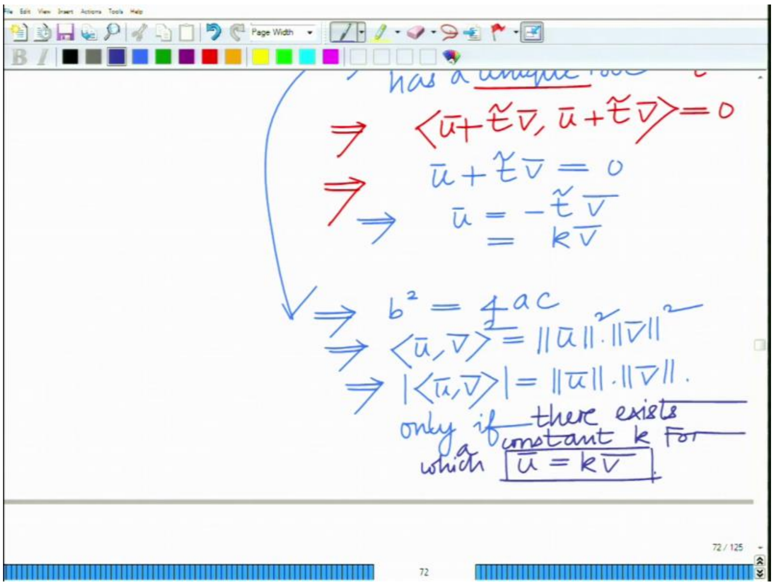


So, this implies that,

$$\langle \bar{u}, \bar{v} \rangle^2 = \|\bar{u}\|^2 \cdot \|\bar{v}\|^2$$

$|\langle \bar{u}, \bar{v} \rangle| = \|\bar{u}\| \cdot \|\bar{v}\|$ is true only if there exists a constant k for which $\bar{u} = k\bar{v}$.

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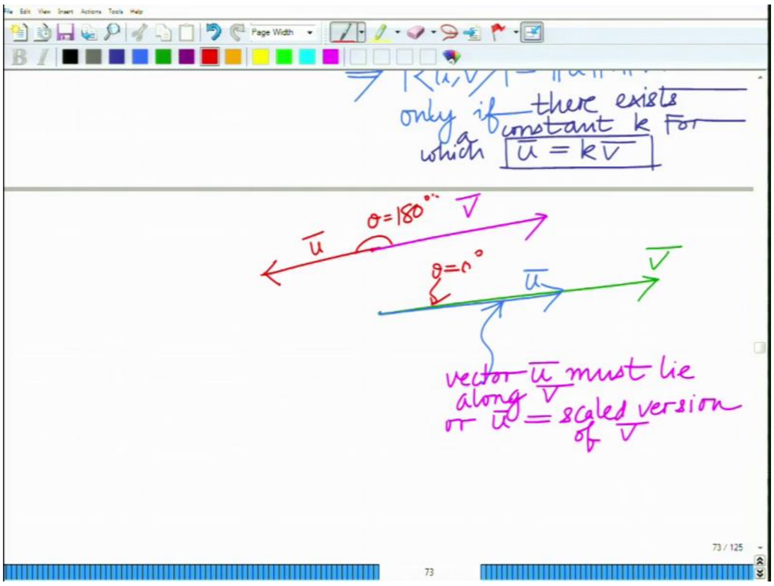


This means, that the vector $\bar{u}$ must also lie along $\bar{v}$. Which also implies that $\bar{u}$ is simply a scaled version of $\bar{v}$. Also, both the vectors are in the same direction or in eactly opposite direction (for the negative values of *k*).

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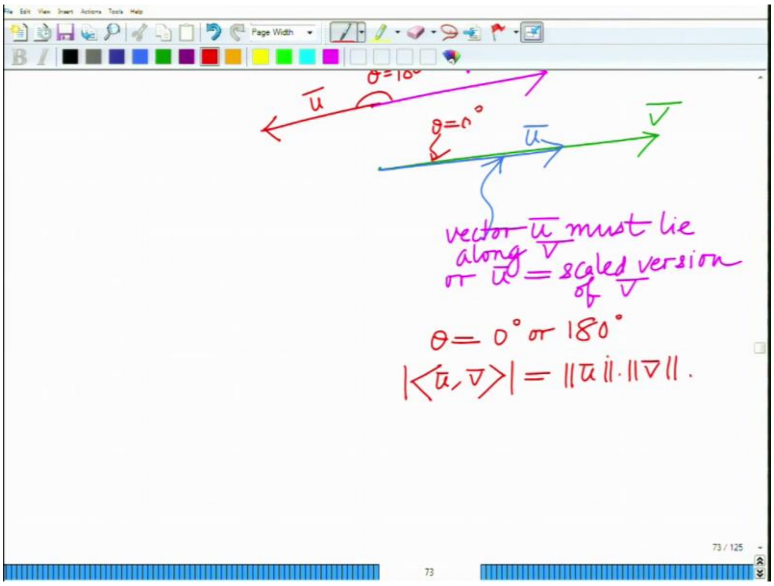


Also, in both the cases $\theta = 0^0$ or $180^0$ the magnitude of inner product of two vectors is same. So conclusively we can say that magnitude of the inner product equal to the product of the norms but these vectors have to be either aligned or exactly anti aligned.

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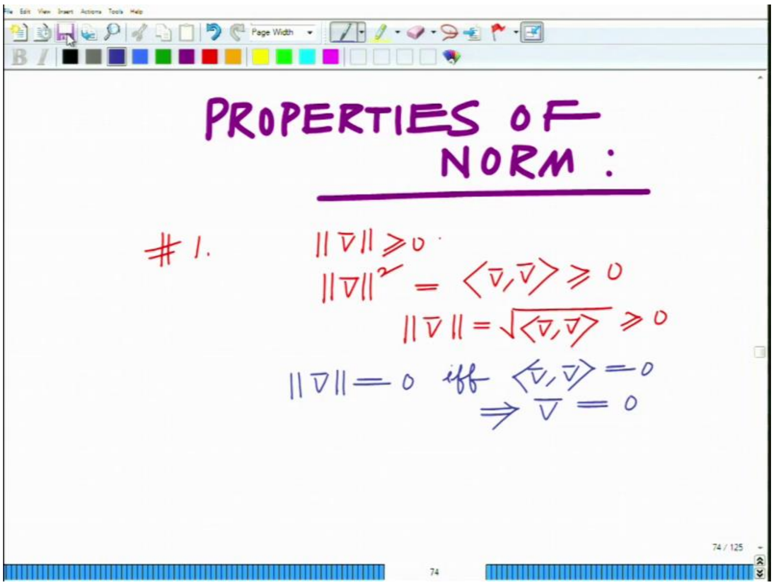


This is an interesting property that is related to the inner product which we will frequently invoke at instances during our discussion on optimization.

Let us now look at other various properties of the norm and these are as follows. The first property is that

$$\|\bar{v}\| \geq 0$$

So further we can write it as

$$\|\bar{v}\|^2 = \langle \bar{v}, \bar{v} \rangle \geq 0$$
$$\|\bar{v}\| = \sqrt{\langle \bar{v}, \bar{v} \rangle} \geq 0$$

Therefore, we can infer that

$\|\bar{v}\| = 0$ if and only if $\langle \bar{v}, \bar{v} \rangle = 0$.

This implies that the inner product of a vector with itself is zero only when the vector is a zero vector.

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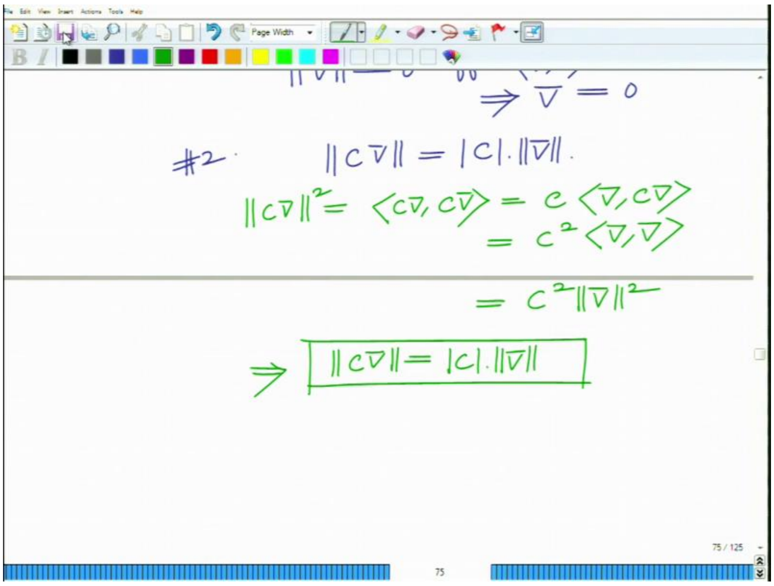


Now, the second aspect is that

$$\|c\overline{v}\| = |c| \cdot \|\overline{v}\|$$

In fact, we can also see that

$$\begin{aligned}
\|c\overline{v}\|^2 &= \langle c\overline{v}, c\overline{v}\rangle \\
&= c\langle \overline{v}, c\overline{v}\rangle \\
&= c^2\langle \overline{v}, \overline{v}\rangle \\
&= c^2\|\overline{v}\|^2
\end{aligned}$$

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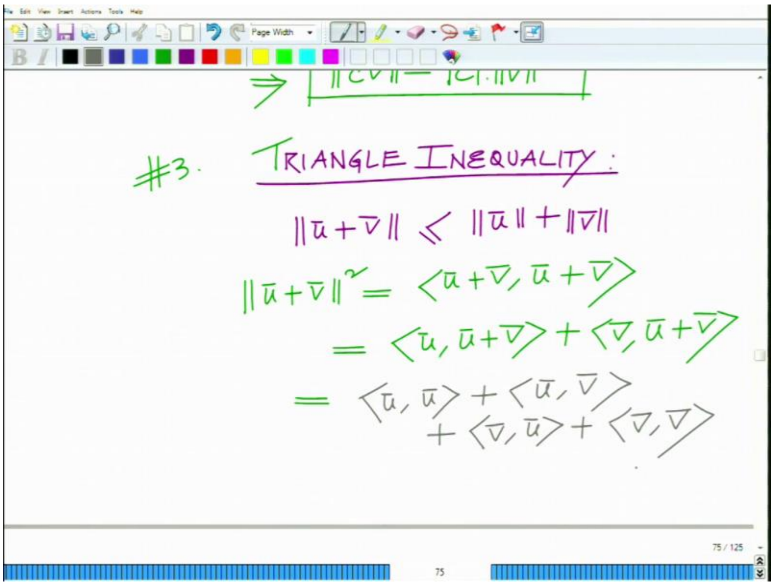


Now the third property is an important property. This is known as the triangle inequality. This states that

$$\|\bar{u}+\bar{v}\| \leq \|\bar{u}\| + \|\bar{v}\|$$

This follows as

$$\begin{aligned}
\|\bar{u}+\bar{v}\|^2 &= \langle \bar{u}+\bar{v}, \bar{u}+\bar{v} \rangle \\
&= \langle \bar{u}, \bar{u}+\bar{v} \rangle + \langle \bar{v}, \bar{u}+\bar{v} \rangle \\
&= \langle \bar{u}, \bar{u} \rangle + \langle \bar{u}, \bar{v} \rangle + \langle \bar{v}, \bar{u} \rangle + \langle \bar{v}, \bar{v} \rangle \\
&= \|\bar{u}\|^2 + 2\langle \bar{u}, \bar{v} \rangle + \|\bar{v}\|^2
\end{aligned}$$

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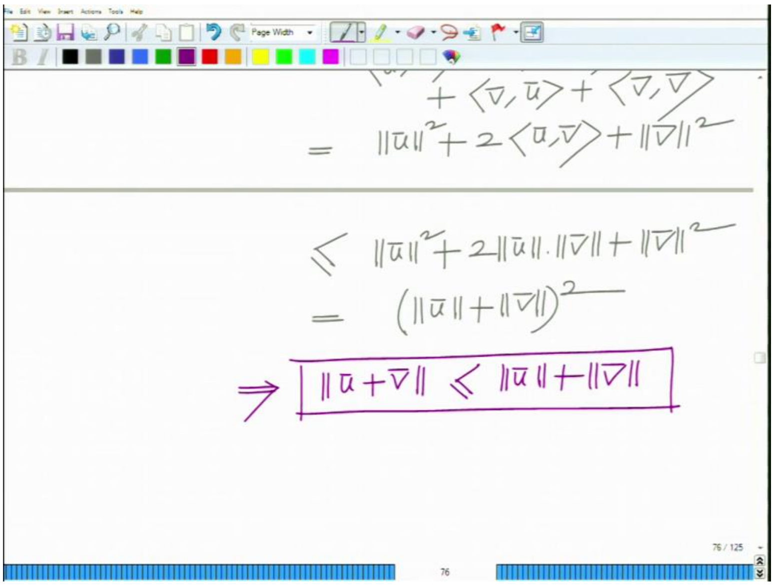


Now, applying the Caushy-Schwardz inequality, we know that

$$\begin{aligned}\|\bar{u}+\bar{v}\|^2 &\le \|\bar{u}\|^2 + 2\|\bar{u}\|\cdot\|\bar{v}\| + \|\bar{v}\|^2 \\ &\le \left(\|\bar{u}\| + \|\bar{v}\|\right)^2\end{aligned}$$

And therefore

$$\|\bar{u}+\bar{v}\| \le \|\bar{u}\| + \|\bar{v}\|$$

This is basically the triangle inequality of the norm. This is valid for any norm that is induced by the inner product.

So, that basically concludes our discussion on the various aspects of the norm and the properties of the norm that is induced by the inner product. So, now, let us start doing some examples on these mathematical preliminaries.

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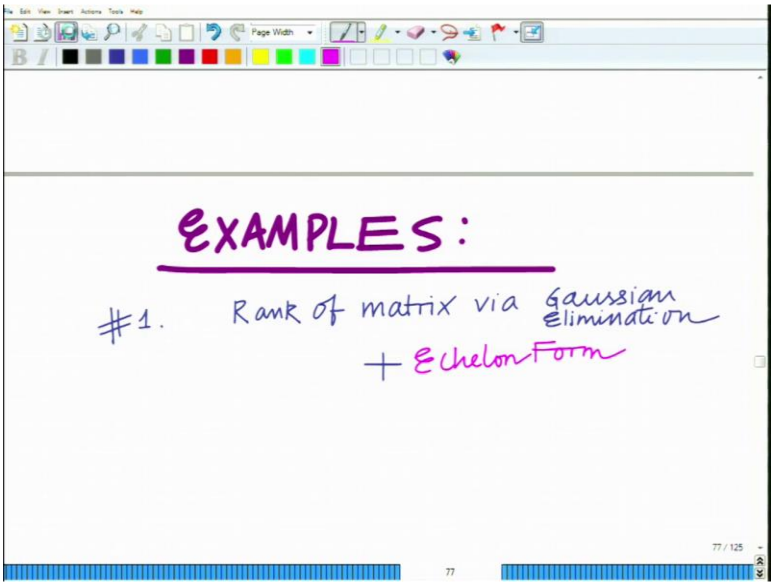


Consider a simple example to discuss the row elimination via Gaussian Elimination to obtain the row echelon form.

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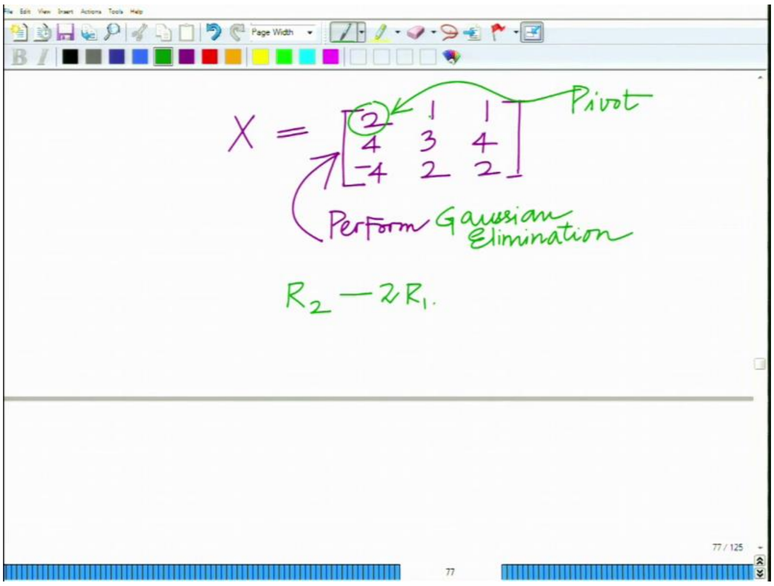


Take a matrix X as

$$X = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 4 \\ -4 & 2 & 2 \end{pmatrix}$$

In Gaussian elimination, we use a pivot which is going to be subtracted in an operation to reduce an element to zero and hence this pivot helps in converting a matrix into upper triangular matrix. This pivot is chosen for an operation according to element that needs to be zero in this operation. So, first perform $R_2 - 2R_1 \to R_2$, here $R_1$ and $R_2$ are denoting row 1 and row 2 respectively. Also $2R_1$ is used as the pivot to reduce $x_{21}$ to zero. Therefore,

$$X \equiv \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ -4 & 2 & 2 \end{pmatrix}$$

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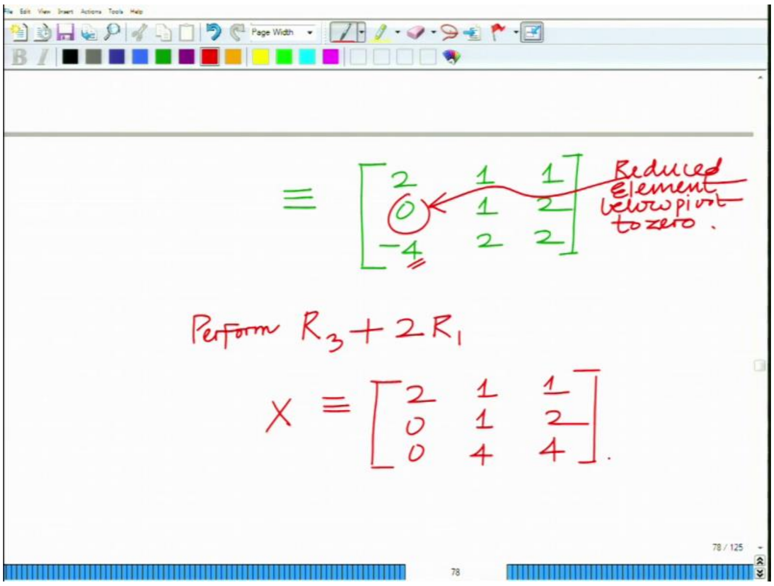


Now perform $R_3 + 2R_1 \to R_3$.

$$X \equiv \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 4 & 4 \end{pmatrix}$$

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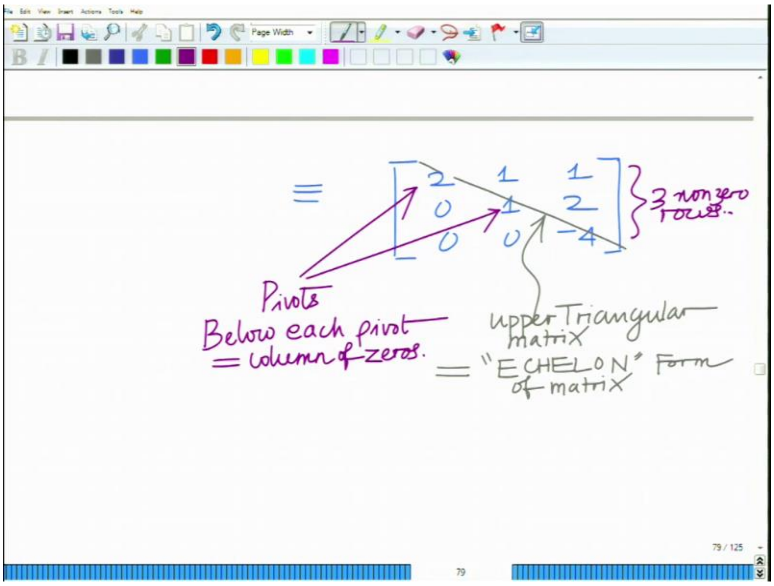


Now perform $R_3 - 4R_1 \to R_3$.

$$X \equiv \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -4 \end{pmatrix}$$

*X* is now an upper triangular matrix and this is termed as the Echelon form of the matrix. We can also observe that below each pivot, we have a column of zeros. And here the number of nonzero rows is equal to the rank of the matrix.

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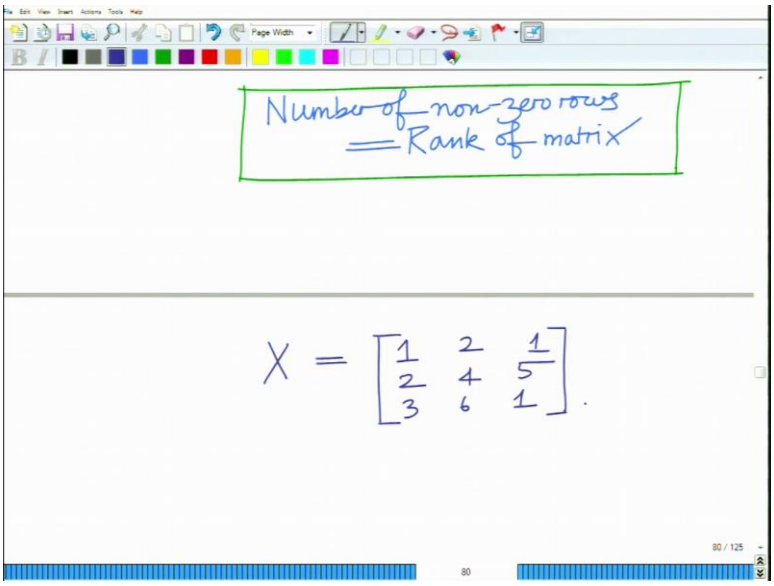


Let us now consider another matrix.

$$X \equiv \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 3 & 6 & 1 \end{pmatrix}$$

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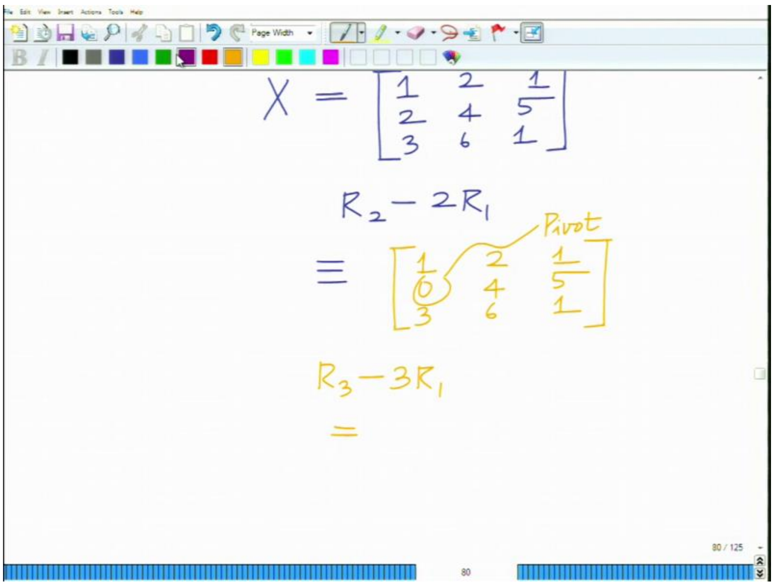


So now perform $R_2 - 2R_1 \to R_2$,

$$X \equiv \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 3 & 6 & 1 \end{pmatrix}$$

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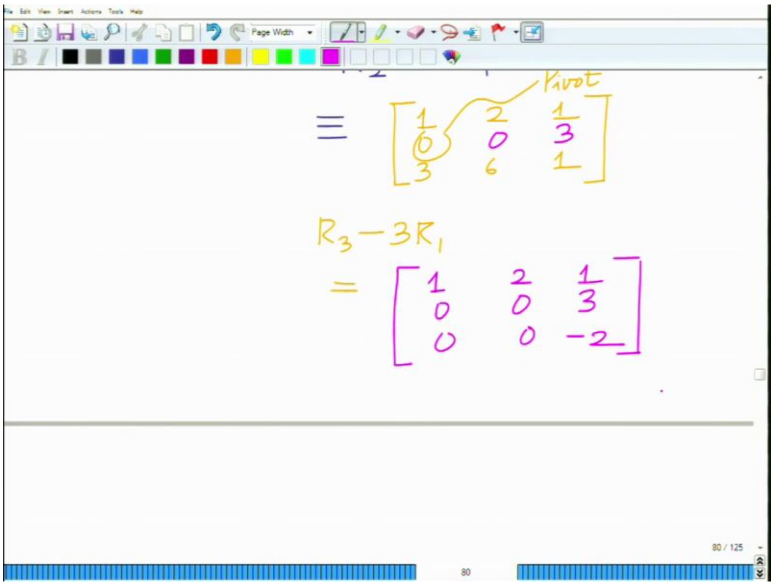


So now perform $R_3 - 3R_1 \rightarrow R_3$,

$$X \equiv \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & -2 \end{pmatrix}$$

And now, you can interestingly see that all the possible pivots in the second column are 0.

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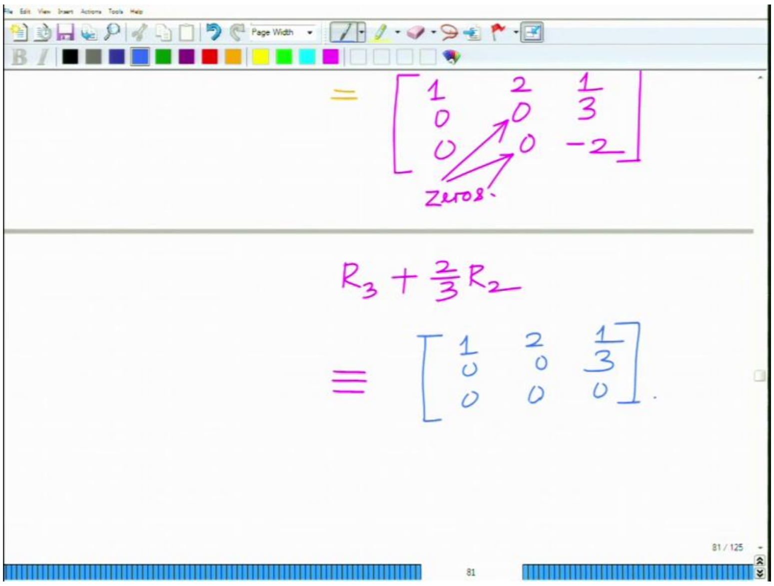


So, these entries which can be used as possible pivots, these are zeros. Therefore use $R_2$ as pivot which gives us something interesting. So now perform $R_3 + \frac{2}{3}R_1 \to R_3$,

$$X \equiv \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

Now we finally have a row of all 0s.

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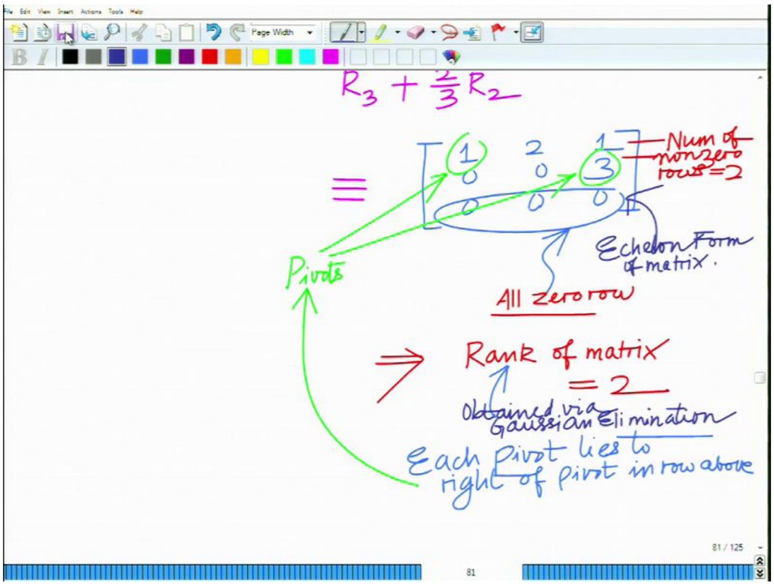

So in this Echelon form there is an all zero row. Therefore, number of nonzero rows is 2 and this implies that rank of this matrix is 2.

Also note that each chosen pivot for the current operation must has to lie to the right of the pivot in the above row.

This is the simple procedure of Gaussian elimination using pivoting and it can be used to reduce a matrix to the Echelon form and also to determine the rank of the matrix and this makes it much easier to solve a system of linear equations.

Thank you very much.