Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture – 15
Norm Cone, Polyhedron and its Applications: Base Station Cooperation

Hello, welcome to another module in this massive open online course. Let us continue the discussion by looking another class of convex sets that is the convex cone or norm cone.

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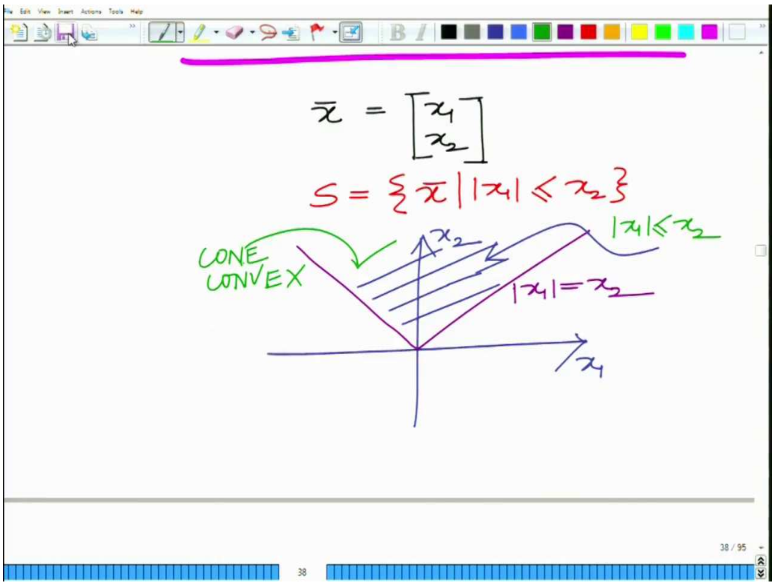

Consider a 2-dimensional vector $\bar{x}$ as

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Such that

$$S = \{\bar{x} \mid |x_1| \leq x_2\}$$

Such a set represents the area above the line segments $|x_1| = x_2$. This region is a cone in 2- dimensions and this is basically also convex. Thus this region is also termed as convex cone or norm cone.

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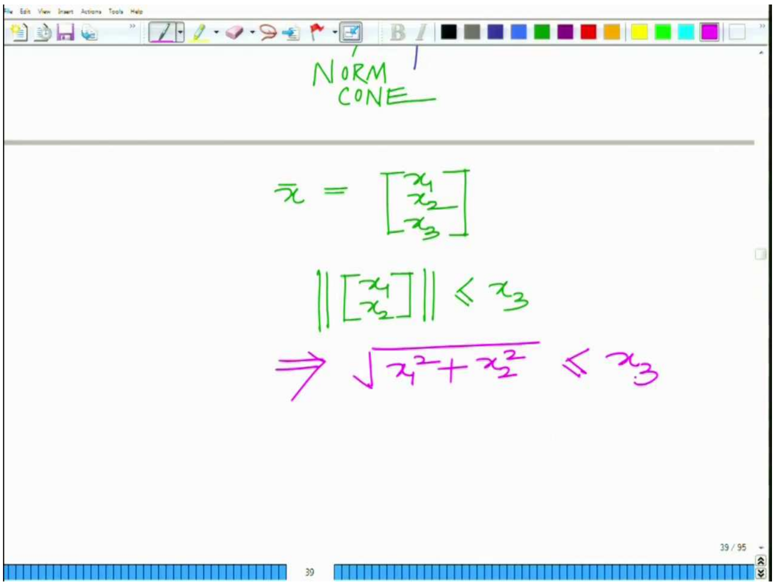

In a 3-dimensional scenario this vector $\bar{x}$ is

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Such that

$$\left\| \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\| \leq x_3$$

$$\sqrt{{x_1}^2 + {x_2}^2} \leq x_3$$

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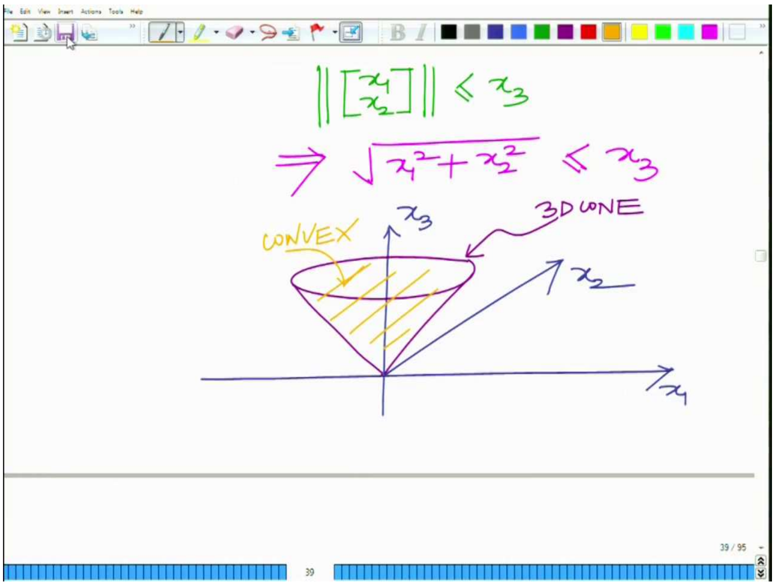

This region has the shape of a classical cone and this is the 3D cone. The interior of this cone is convex which is also reasonably easy to show.

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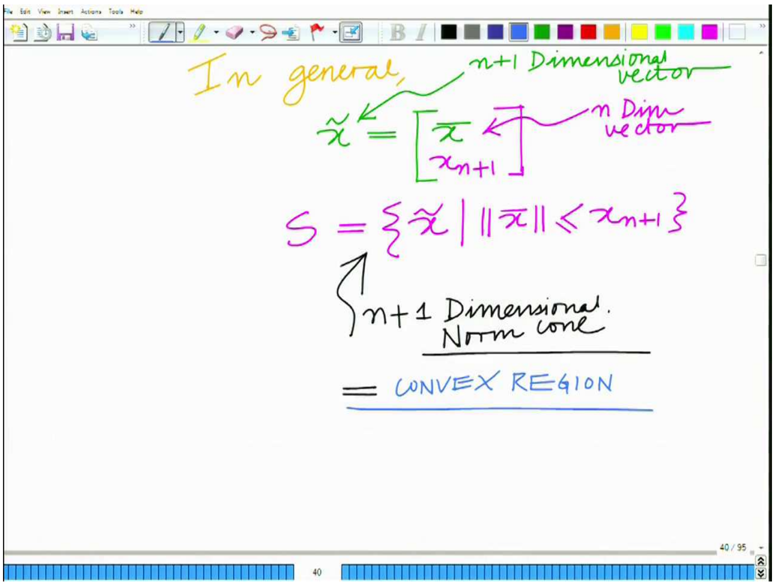

Hence, In general, let us have $(n+1)$ -dimensional vector $\tilde{x}$ as

$$\tilde{x} = \begin{bmatrix} \bar{x} \\ x_{n+1} \end{bmatrix}$$

Such that

$$S = \{ \tilde{x} \mid \|\bar{x}\| \le x_{n+1} \}$$

Where $\bar{x}$ is a n-dimensional vector.

Such a set represents an $(n+1)$-dimensional norm cone which is a convex region. It is a fairly important class of convex regions, and in fact it is slightly difficult to describe a practical application of the convex cone in the context of wireless communications or signal processing at this point, although it has very prominent applications which will be explored in the subsequent modules of this course.

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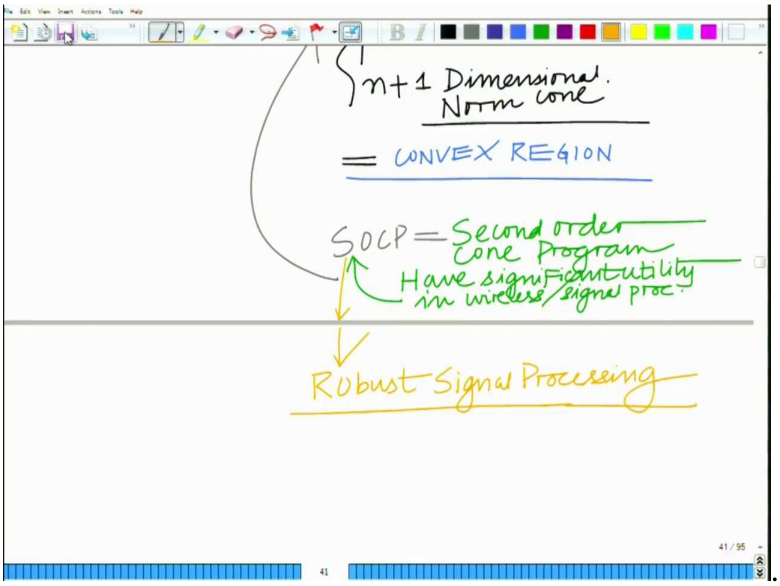

Some problems which have conic constraints are commonly known as Second Order Cone Programs (SOCP) which has significant application and utility in the context of wireless communication and robust signal processing. One of the most prominent applications of this SOCP paradigm in the context of robust signal processing is robust beam forming in a multiple antenna wireless communication system.

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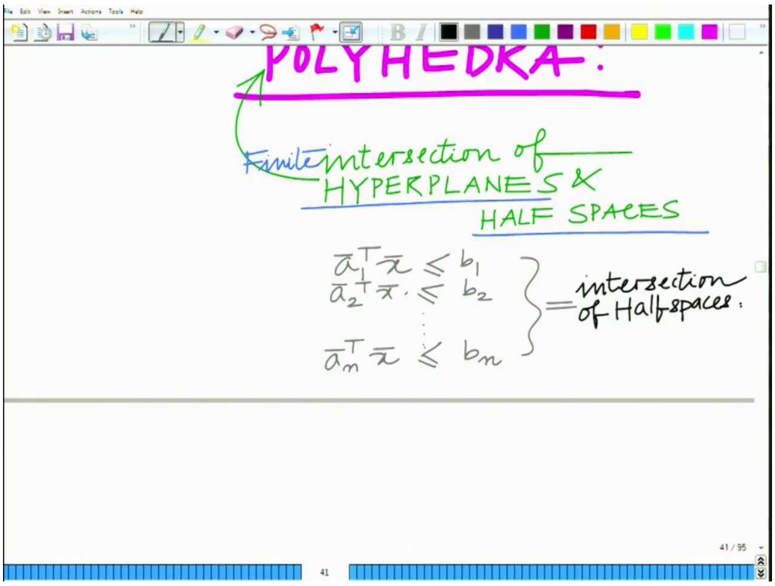

Now let us discuss another interesting convex set normally termed as Polyhedra. Polyhedra are basically formed from the finite intersection of hyperplanes and half spaces.

Let us say there are n half spaces represented as follows.

$$\begin{aligned}
\bar{a}_1^T\bar{x} &\leq b_1 \\
\bar{a}_2^T\bar{x} &\leq b_2 \\
&\vdots \\
\bar{a}_n^T\bar{x} &\leq b_n
\end{aligned}$$

So the vector $\bar{x}$ is an intersection of half spaces.

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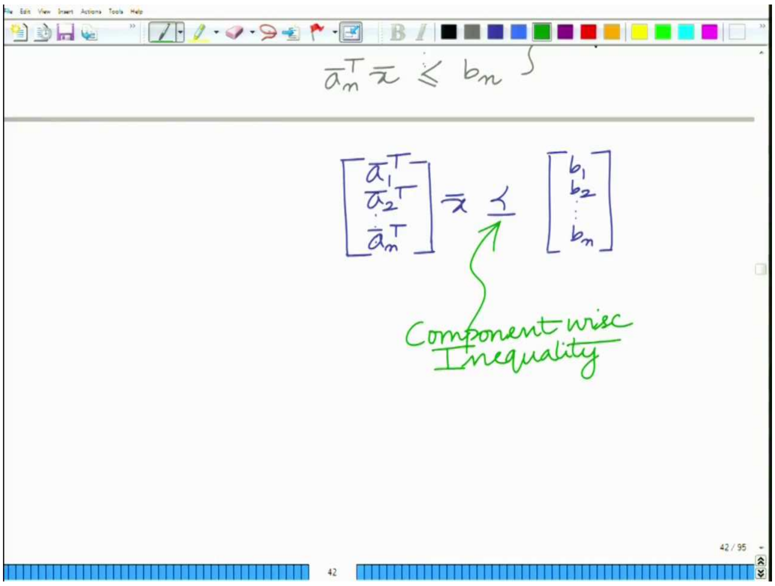

On concatenating all the above half spaces in a matrix,

$$\begin{bmatrix} \bar{a}_1^T \\ \bar{a}_2^T \\ \vdots \\ \bar{a}_n^T \end{bmatrix} \bar{x} \leq \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Here the inequality is a component wise inequality, which means that each component on the left is less than or equal to each component on the right.

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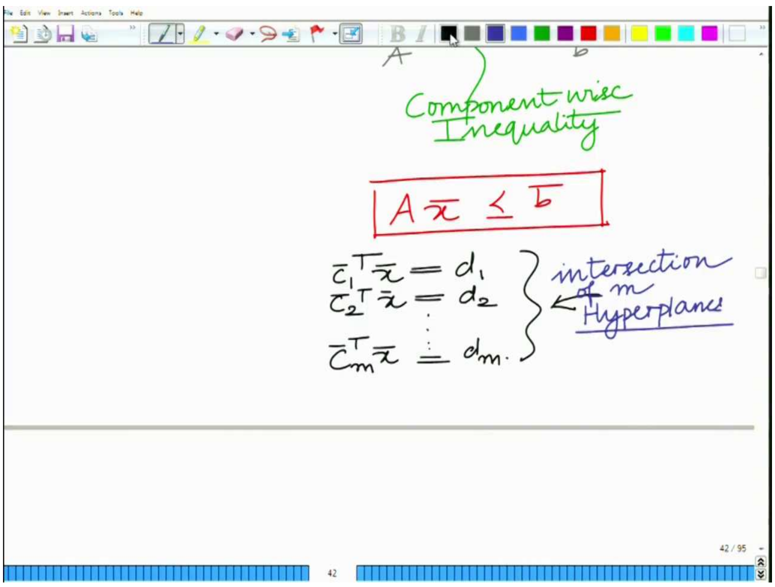

So, this can be written as

$$A\bar{x} \le \bar{b}$$

So, this basically represents an intersection of half spaces.

Similarly formulate the m hyperplanes as follows.

$$\begin{aligned}
\bar{c}_1^T \bar{x} &= d_1 \\
\bar{c}_2^T \bar{x} &= d_2 \\
&\vdots \\
\bar{c}_m^T \bar{x} &= d_m
\end{aligned}$$

Again here the vector $\bar{x}$ is an intersection of m hyperplanes.

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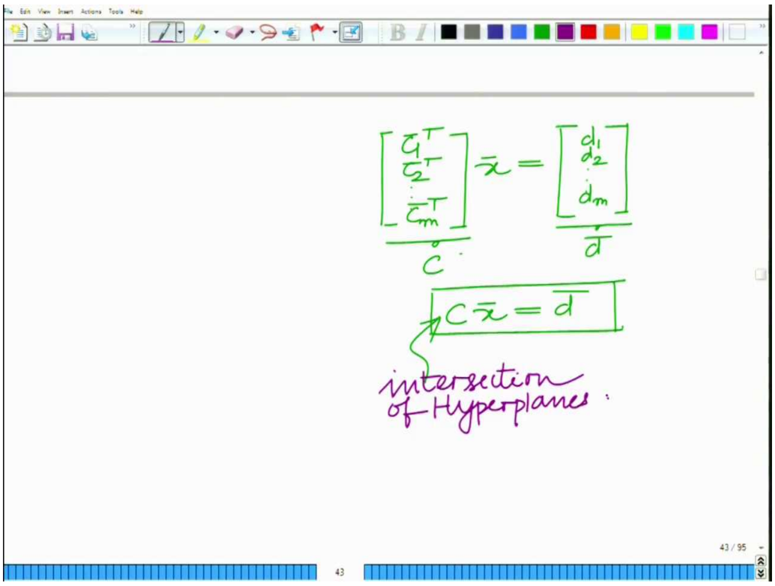

On concatenating all the above hyperplanes in a matrix,

$$\begin{bmatrix} \bar{c}_1^T \\ \bar{c}_2^T \\ \vdots \\ \bar{c}_m^T \end{bmatrix} \bar{x} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix}$$

Similarly, this can be written as

$C\overline{x} = \overline{d}$

And this basically represents an intersection of hyperplanes.

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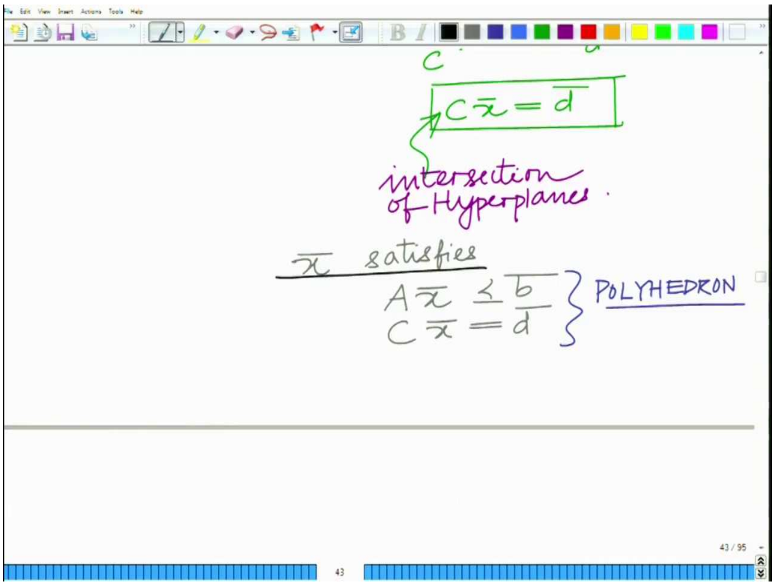

So on putting intersection of half spaces and intersection of hyperplanes together, one will have an intersection of hyperplanes and half spaces which can be written as follows.

$$A\overline{x} \leq \overline{b}$$
$$C\overline{x} = \overline{d}$$

That is vector $\overline{x}$ satisfies both the above expressions and such a region represents a polyhedral or polyhedron.

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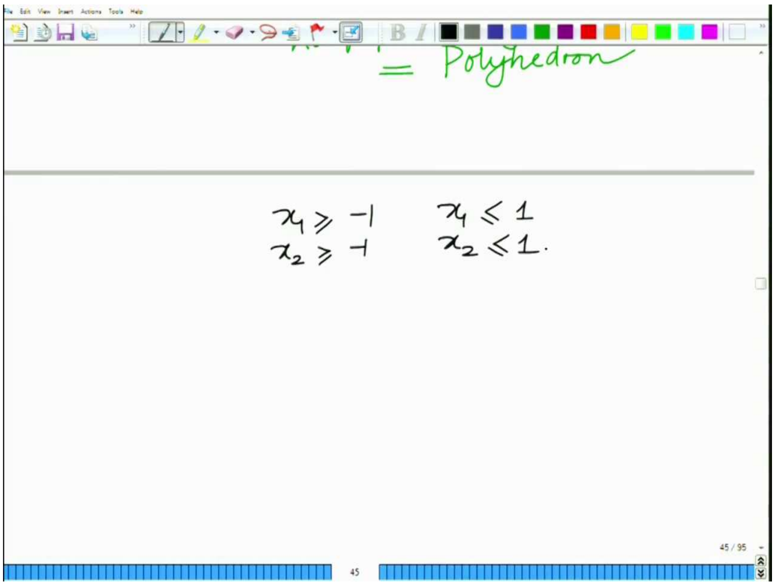

For instance let us take a simple example. Consider a polyhedron form by four halfspaces which are

$x_1 \geq -1,$
$x_2 \geq -1,$
$x_1 \leq 1,$
$x_2 \leq 1$

Where x_1 is conventional x-coordinate and x_2 is conventional y-coordinate.

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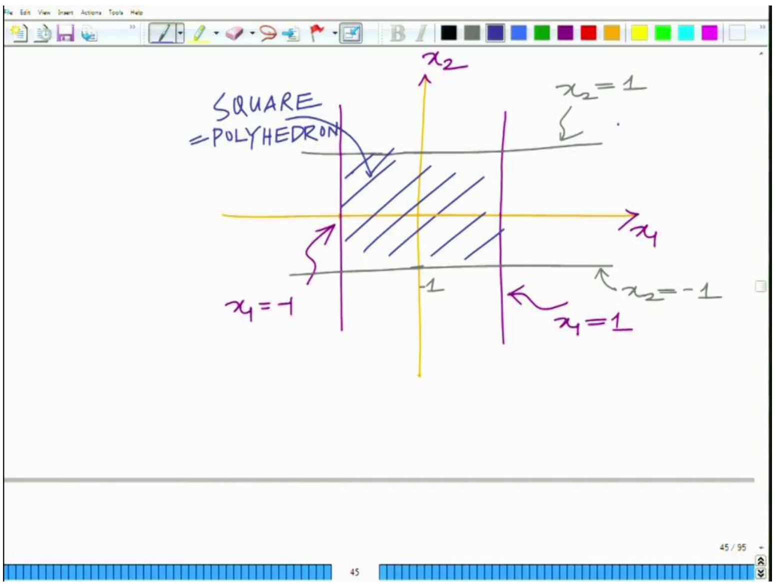

The intersection region of above four half spaces is square in shape. This square is a polyhedron which is convex because intersection of convex sets is convex and each half space is convex.

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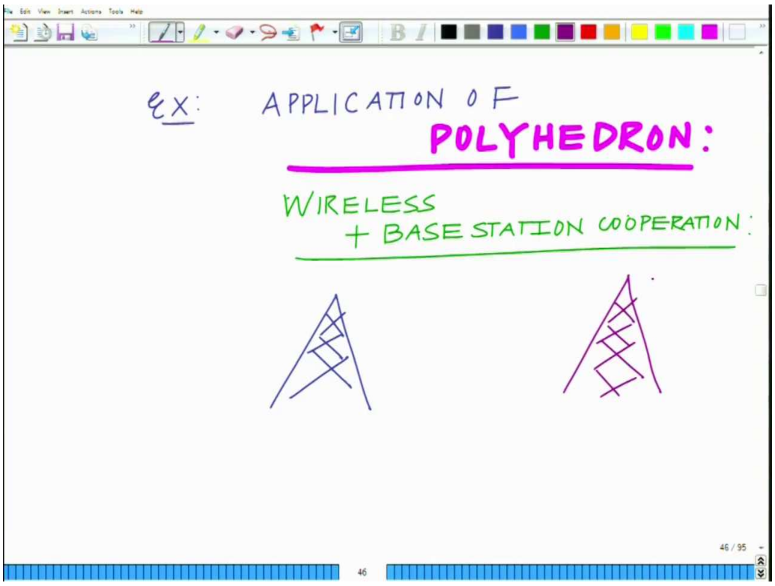

Let us look at an application of this concept of polyhedron in the context of cooperative wireless communication. Conventionally a single base station transmits to a single mobile, but in cooperative wireless communication or cooperative base station transmission, several base stations cooperating to transmit to a single user or a group of users. This is especially possible if the users are at the edge of the cell or in the region between multiple cells, where they can be simultaneously served by several base stations.

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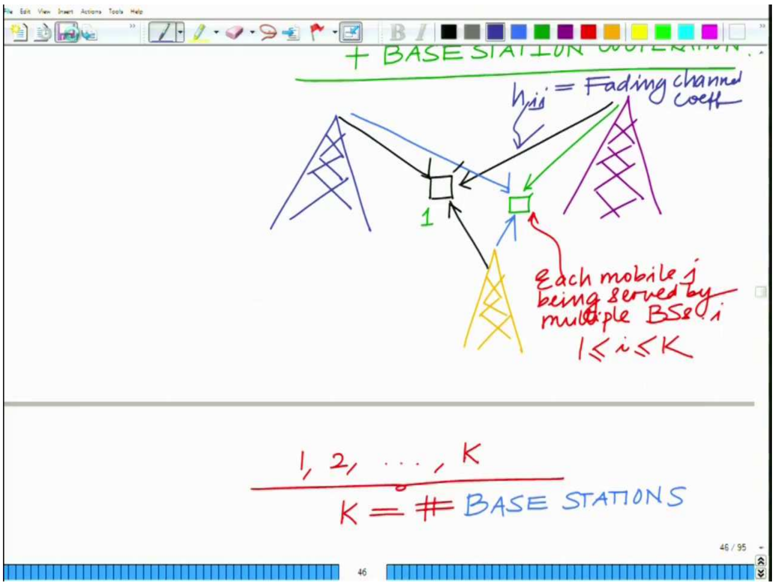

So, let say there are K base stations where each mobile is cooperatively served by i number of base stations.

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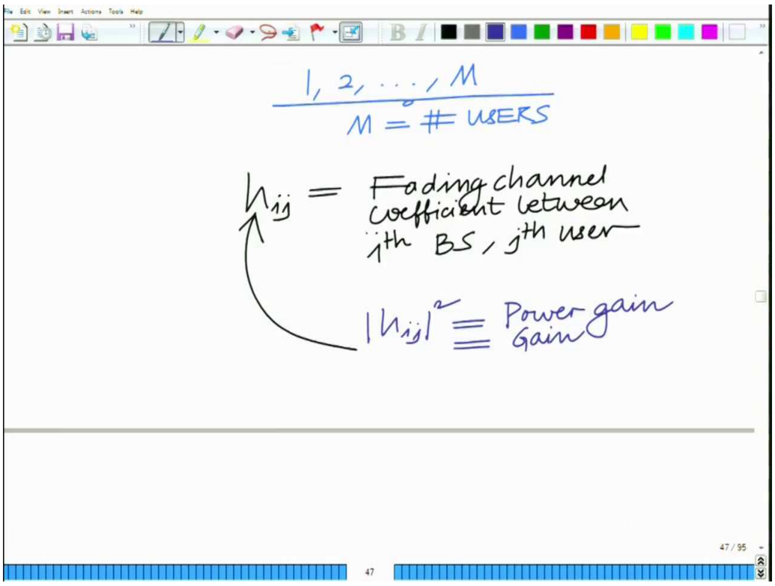

Also let say there are total M users. This means that each jth user is served by i number of BSs cooperatively such that $1 \leq i \leq K$

As this is a wireless transmission scenario, this means there is a fading channel between the each transmitter and receiver. These channels are characterized by fading channel coefficient h_{ij} between i^{th} base station and j^{th} user.

So, this is the fading channel coefficient therefore the magnitude of square of h_{ij} that is $\left|h_{ij}\right|^2$ represents the power gain or sometimes referred as the gain. This simply means that if $\left|h_{ij}\right|^2$ is strong then this received signal at the j^{th} user corresponding to the signal transmitted by i^{th} base station will to be strong.

Again, in case of deep fade, there is a lot of interference in the channel and hence $\left|h_{ij}\right|^2$ is very low and hence the power received by j^{th} user corresponding to the signal transmitted by i^{th} base station is going to be very low.

So, naturally the power that has to be transmitted by the various base stations such that each user receives the desired amount of power must be optimized. Such optimization problem is related to the polyhedron that is a convex set.

So let us continue this practical application in the context of a cooperative base station transmission in the next module.