

**Applied Optimization for Wireless, Machine Learning, Big Data**  
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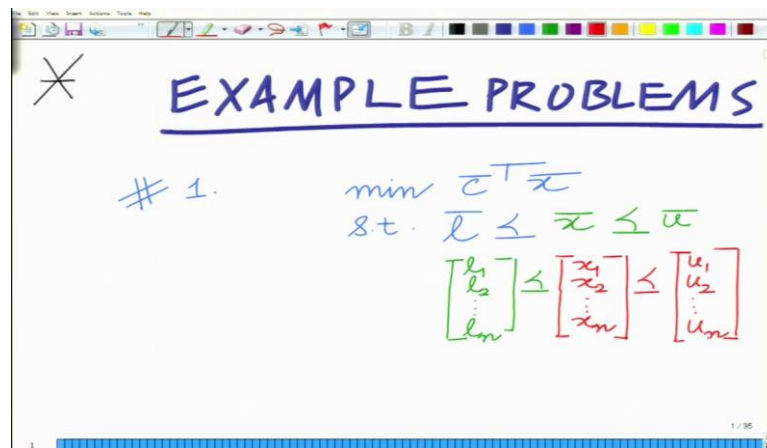
**Lecture – 70**

**Examples: Linear objective with box constraints, Linear Programming**

**Keywords:** *Linear objective, Box constraints, Linear Programming*

Hello, welcome to another module in this massive open online course. So we are looking at example problems in convex optimization.

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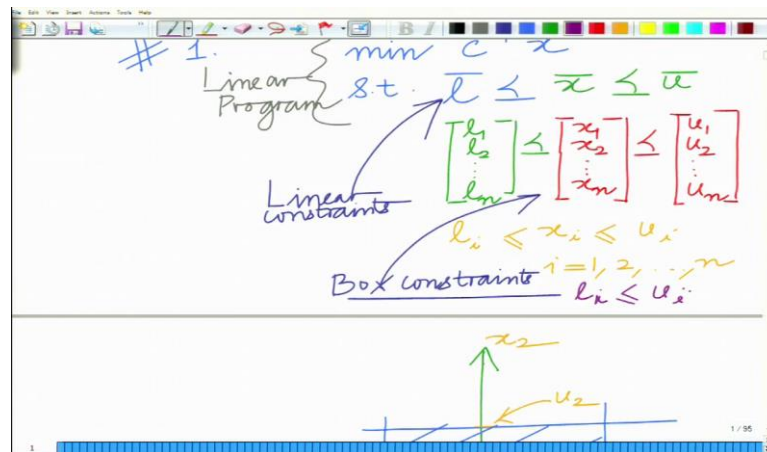


So we are looking at example problems and let us look at problem number 1. So we have

$$\min c^T x$$
$$s.t. \quad l \leq x \leq u$$
 which means that if you have the elements of  $x$  are component wise less than

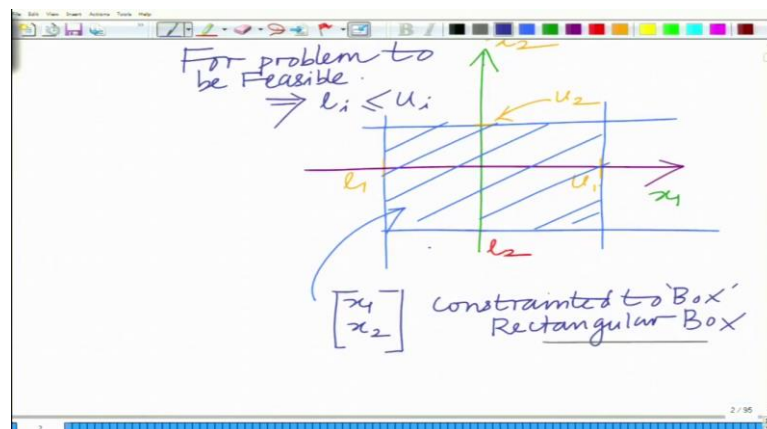
or equal to the other elements.

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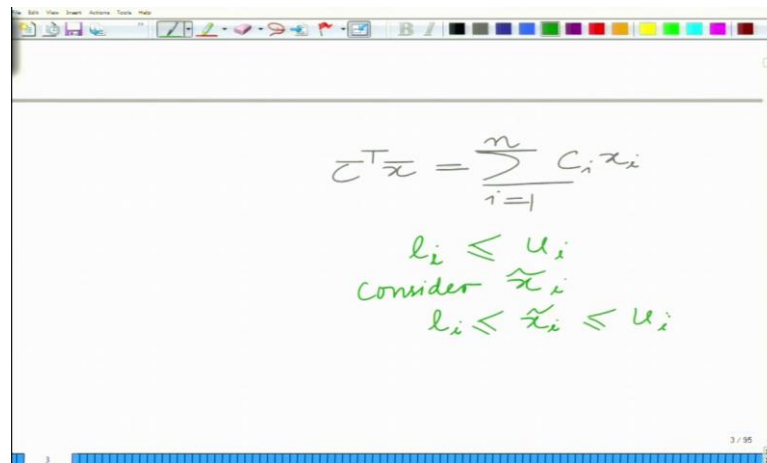
Each  $l_i \leq x_i \leq u_i$ , so this is also known as box constraints and this is as shown in slide.

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So therefore,  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  are confined to this box as shown in slide and hence this is also termed as box type constraint. In fact, it is a simple linear program. So the solution for this is fairly straight forward.

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$$c^T x = \sum_{i=1}^n c_i x_i$$

Consider  $\tilde{x}_i$

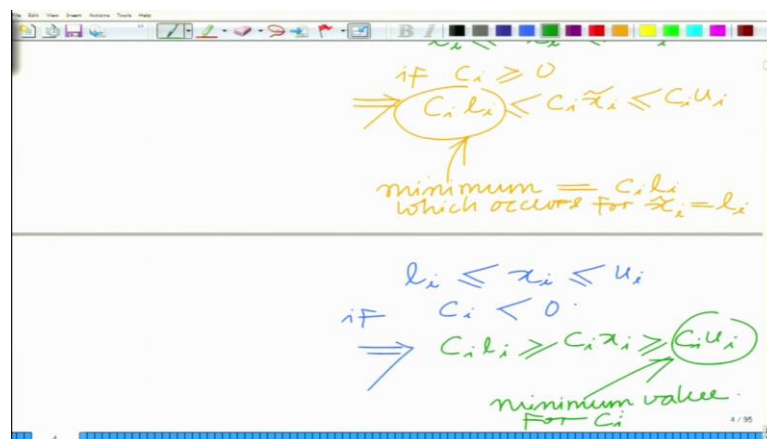
$$l_i \leq \tilde{x}_i \leq u_i$$

So you have  $c^T x = \sum_{i=1}^n c_i x_i$ . Now, these box type constraints make sense only if  $l_i \leq u_i$ .

So we assume here that  $l_i \leq u_i$ . If this is not so, then the problem becomes infeasible.

Now, consider any  $x_i$  such that  $l_i \leq x_i \leq u_i$ .

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if  $c_i \geq 0$

$$c_i l_i \leq c_i x_i \leq c_i u_i$$

minimum =  $c_i l_i$   
which occurs for  $x_i = l_i$

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$l_i \leq x_i \leq u_i$

if  $c_i < 0$

$$c_i l_i \geq c_i x_i \geq c_i u_i$$

minimum value for  $c_i$

Now, if  $c_i \geq 0$ , this implies  $c_i l_i \leq c_i x_i \leq c_i u_i$ . So minimum value for  $x_i$  lying in this box is

$c_i l_i$  which occurs when  $x_i = l_i$ . On the other hand, if  $c_i < 0$  this implies  $c_i l_i \geq c_i x_i \geq c_i u_i$ .

Now the minimum value is  $c_i u_i$ .

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minimum value if  $c_i < 0$

Optimal value of each  $x_i$

$$= \begin{cases} l_i & \text{if } c_i \geq 0 \\ u_i & \text{if } c_i < 0 \end{cases}$$

$$\min \bar{c}^T \bar{x} = \min \sum_{i=1}^n c_i x_i$$

And therefore, the optimal value of each  $x_i$  is  $\begin{cases} l_i & \text{if } c_i \geq 0 \\ u_i & \text{if } c_i < 0 \end{cases}$ .

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$$\min \bar{c}^T \bar{x} = \min \sum_{i=1}^n c_i x_i$$

$$= \sum_{i=1}^n c_i^+ l_i + c_i^- u_i$$

where  $c_i^+ = \max\{c_i, 0\}$

$$= \begin{cases} c_i & \text{if } c_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

So  $\min \bar{c}^T \bar{x} = \min \sum_{i=1}^n c_i x_i = \sum_{i=1}^n c_i^+ l_i + c_i^- u_i$  where  $c_i^+ = \max\{c_i, 0\} = \begin{cases} c_i & \text{if } c_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$  and

$$c_i^- = \begin{cases} c_i & \text{if } c_i < 0 \\ 0 & \text{otherwise} \end{cases}.$$

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$$c_i^- = \begin{cases} c_i & \text{if } c_i < 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\min_{\substack{\bar{c}^T \bar{x} \\ \text{s.t. } \bar{l} \leq \bar{x} \leq \bar{u}}} \bar{c}^T \bar{x} = \sum_{i=1}^n c_i^+ l_i + c_i^- u_i$$

You can also write this as  $\min_{\substack{\bar{c}^T \bar{x} \\ \text{s.t. } \bar{l} \leq \bar{x} \leq \bar{u}}} \bar{c}^T \bar{x} = \sum_{i=1}^n c_i^+ l_i + c_i^- u_i$  where  $c_i^+$  contains all positive elements of  $\bar{c}$  and  $c_i^-$  contains only negative elements of  $\bar{c}$  and the rest are 0. So that is the optimal value of this problem.

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$$\min_{\substack{\bar{c}^T \bar{x} \\ \text{s.t. } \bar{l} \leq \bar{x} \leq \bar{u}}} \bar{c}^T \bar{x} = \sum_{i=1}^n c_i^+ l_i + c_i^- u_i$$

$$= \bar{l}^T \bar{c}^+ + \bar{u}^T \bar{c}^-$$

Contains only non-negative elements of  $\bar{c}$   
Rest = 0.

Contains only -ve elements of  $\bar{c}$   
Rest = 0.

Let us proceed to a slightly more sophisticated example for which the solution might not be very obvious and that is problem number 2 where we have  $\min_{\substack{\bar{c}^T \bar{x} \\ \text{s.t. } A \bar{x} \leq b}} \bar{c}^T \bar{x}$ . This is a linear program, but slightly more sophisticated and the solution depends on the nature of  $A$  which is a square full rank matrix. This implies that  $A$  is invertible.

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# 2.

$$\min. \bar{c}^T \bar{x}$$

$$\text{s.t. } A \bar{x} \leq \bar{b}$$

$A$  is square Full rank matrix  
 $\Rightarrow A$  is invertible.

Substitute  $A \bar{x} = \bar{y}$   
 $\Rightarrow \boxed{\bar{x} = A^{-1} \bar{y}}$

Now we substitute,  $A \bar{x} = \bar{y} \Rightarrow \bar{x} = A^{-1} \bar{y}$ .

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Substitute  $\Rightarrow \boxed{\bar{x} = A^{-1} \bar{y}}$

Therefore, optimization problem can be formulated in terms of  $\bar{y}$

$$\bar{c}^T \bar{x} = \bar{c}^T A^{-1} \bar{y}$$

$$= \tilde{c}^T \bar{y}$$

$$\bar{c}^T A^{-1} = \tilde{c}^T$$

$$\Rightarrow (A^{-1})^T \bar{c} = \tilde{c}$$

$$\Rightarrow \boxed{A^{-T} \bar{c} = \tilde{c}}$$

Now, we will write the equivalent optimization problem in terms of  $\bar{y}$ . So we have the

objective  $\bar{c}^T \bar{x} = \bar{c}^T A^{-1} \bar{y} = \tilde{c}^T \bar{y} \Rightarrow A^{-T} \bar{c} = \tilde{c}$ .

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$$\begin{aligned} \min \quad & \tilde{c}^T \bar{y} \\ \text{s.t.} \quad & A \bar{y} \leq \bar{b} \\ \Rightarrow & \bar{y} \leq \bar{b} \end{aligned}$$

$$\begin{aligned} \min \quad & \tilde{c}^T \bar{y} = \sum_{i=1}^n \tilde{c}_i y_i \\ \text{s.t.} \quad & \bar{y} \leq \bar{b} \end{aligned}$$

So the objective becomes  $\min_{\bar{y} \leq \bar{b}} \tilde{c}^T \bar{y}$ . So now, we have  $\tilde{c}^T \bar{y} = \sum_{i=1}^n \tilde{c}_i y_i$ .

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$$\begin{aligned} \text{s.t.} \quad & \bar{y} \leq \bar{b} \\ \Rightarrow & y_i \leq b_i \\ \text{if} \quad & \tilde{c}_i > 0 \\ & y_i \leq b_i \Rightarrow \tilde{c}_i y_i \leq \tilde{c}_i b_i \\ \Rightarrow & \tilde{c}_i y_i \rightarrow -\infty \text{ as } y_i \rightarrow -\infty \\ \text{if any } \tilde{c}_i > 0 & \Rightarrow \tilde{c}_i y_i \rightarrow -\infty \text{ as } y_i \rightarrow -\infty \\ \Rightarrow & \text{min. value} = -\infty \end{aligned}$$

And the constraint will be component wise constraint, this implies that each component of this vector  $\bar{y}$  is less than or equal to each component of vector  $\bar{b}$ . Now we consider if any  $\tilde{c}_i > 0 \Rightarrow y_i \leq b_i \Rightarrow \tilde{c}_i y_i \leq \tilde{c}_i b_i$ . So this implies that  $\tilde{c}_i y_i \rightarrow -\infty$  as  $y_i \rightarrow -\infty$ . So objective becomes  $-\infty$ . So it is unbounded below.

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if any  $\tilde{c}_i > 0 \Rightarrow C_i y_i \rightarrow -\infty$   
 if  $y_i \rightarrow -\infty$   
 $\Rightarrow$  min. value  $= -\infty$

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if all  $\tilde{c}_i \geq 0$   
 $\Rightarrow y_i \leq b_i$   
 $\Rightarrow \tilde{c}_i y_i \geq \tilde{c}_i b_i$   
 minimum occurs for  $y_i = b_i$

Now, if all  $\tilde{c}_i \leq 0 \Rightarrow y_i \leq b_i \Rightarrow \tilde{c}_i y_i \geq \tilde{c}_i b_i$ . So the minimum occurs for  $y_i = b_i$ .

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minimum occurs for  $y_i = b_i$

$\Rightarrow \min C^T y$   
 $= \sum_{i=1}^n \tilde{c}_i b_i$   
 $= \tilde{C}^T \bar{b}$

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$= \tilde{C}^T A^{-1} \bar{b}$   
 if  $\tilde{C} \leq 0$   
 $\Rightarrow A^{-T} \tilde{C} \leq 0$

And therefore, the net minimum implies

$$\min_{\bar{c}} \bar{c}^T \bar{y} = \sum_{i=1}^n \tilde{c}_i b_i = \bar{c}^T \bar{b} = \bar{c}^T A^{-1} \bar{b} \text{ if } \bar{c} \leq 0 \Rightarrow A^{-T} \bar{c} \leq 0.$$



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$$= c^T b$$

$$= c^T A^{-1} b$$

if  $c \leq 0$   
 $\Rightarrow A^{-T} c \leq 0$   
 otherwise  $\min c^T x = -\infty$

$$\min_{s.t. Ax \leq b} c^T x = \begin{cases} c^T A^{-1} b & \text{if } A^{-T} c \leq 0 \\ -\infty & \text{otherwise} \end{cases}$$

Therefore, the minimum value is  $\min_{s.t. Ax \leq b} c^T x = \begin{cases} c^T A^{-1} b & \text{if } A^{-T} c \leq 0 \\ -\infty & \text{otherwise} \end{cases}$ .

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$$\min_{s.t. Ax \leq b} c^T x = \begin{cases} c^T A^{-1} b & \text{if } A^{-T} c \leq 0 \\ -\infty & \text{otherwise} \end{cases}$$

if any element of  $A^{-T} c$  is  $> 0$ .

So that is basically the solution to this optimization problem. So let us stop here. Thank you very much.