Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture - 67
Application of KKT conditions : Optimal MIMO Power allocation (Waterfilling)

***Keywords**: Karush-Kuhn-Tucker (KKT) conditions, Optimal MIMO Power allocation Waterfilling Algorithm*

Hello, welcome to another module in this massive open online course. So we have looked at the KKT conditions to solve an optimization problem. Let us look at an application to better understand how one can use the KKT conditions to solve an optimization problem.

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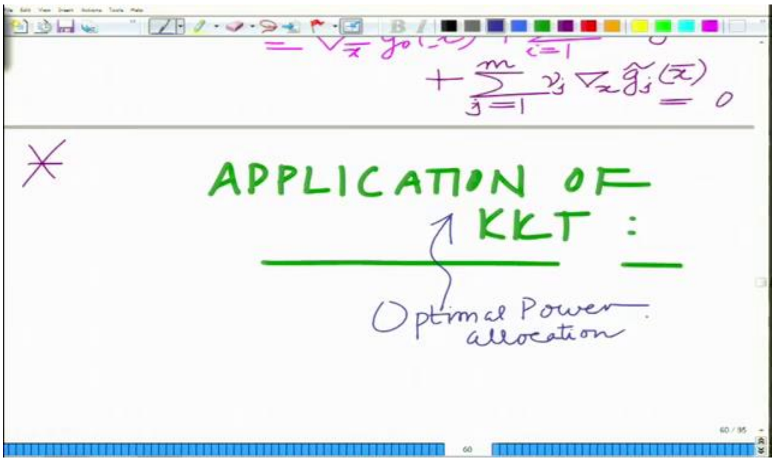

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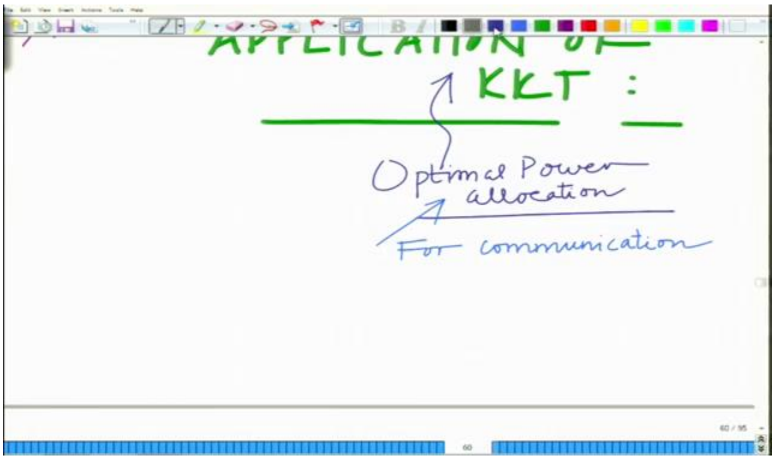

So let us say you have a set of parallel channels.

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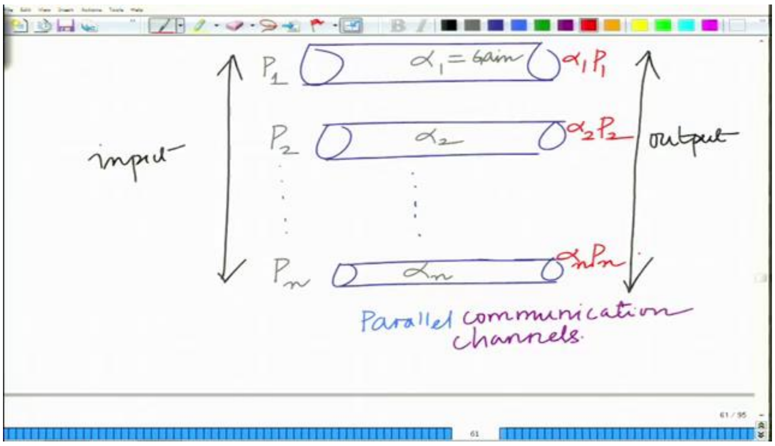

So let assume that these are arranged in decreasing order of gains. So you can transmit at a certain bit rate over each of these communication channels and bit rate depends on the power that is allocated to that particular channel. So let us say the power allocated for first channel is P_1, second channel is P_2 and so on and for n^{th} channel is P_n. Let us say the gain of first channel is α_1, gain of channel 2 is α_2 and so on gain of channel n is α_n. So as shown in the slide, this is the input and this is the output or you can think of it as a transmitter and the receiver. The received power across channel 1 will be $\alpha_1 P_1$, similarly across channel 2 will be $\alpha_2 P_2$ and so on across channel n will be $\alpha_n P_n$. Now in addition for every communication at the receiver we will have thermal noise or Gaussian noise which is typically modelled as additive Gaussian noise.

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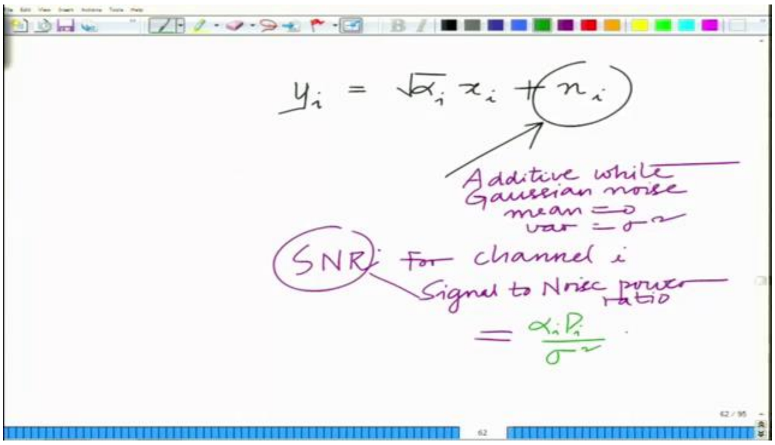

So we have $y_i \sqrt{\alpha_i} x_i + n_i$ where this quantity n_i is the additive white Gaussian noise, with mean 0 and variance σ^2. So noise power is σ^2 for each channel. The SNR for channel i is $\frac{\alpha_i P_i}{\sigma^2}$.

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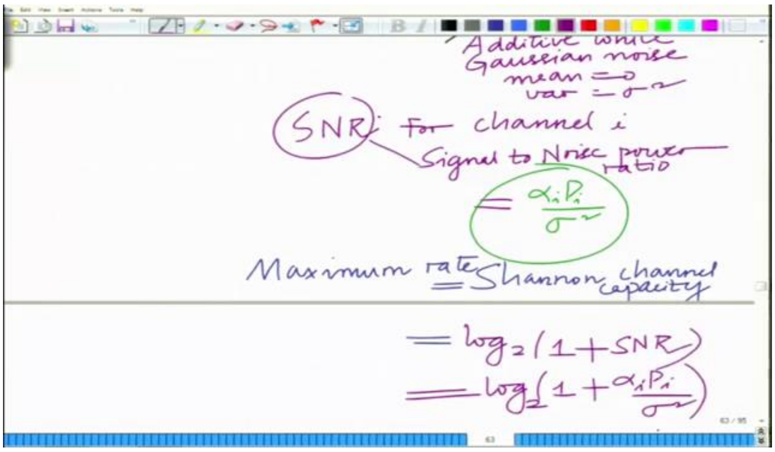

And now the maximum information rate is given by the Shannon's formula for the capacity of the channel. So this is given as $\log_2\left(1+\frac{\alpha_i P_i}{\sigma^2}\right)$.

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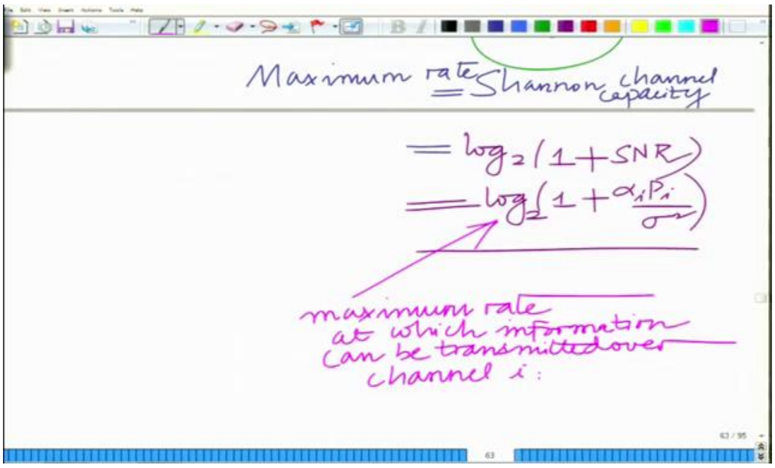

So this is the maximum rate at which information can be transmitted over the channel i. And therefore the maximum sum rate of information transmitted across all these n parallel channels will be given by the sum of the individual rates across each of these n parallel channels.

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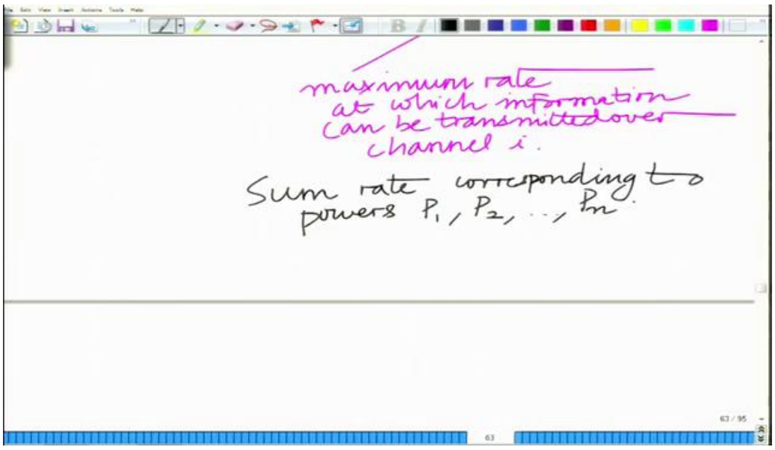

So the maximum sum rate corresponding to powers P_1, P_2,..,P_n is to be calculated.

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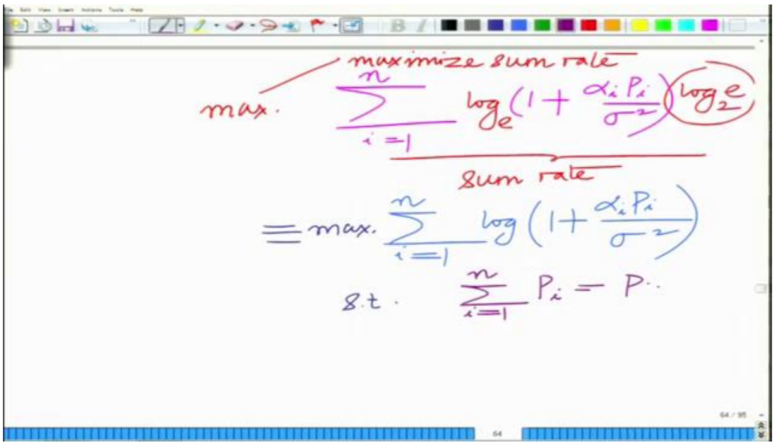

We want to maximize this sum rate, so we have $\max\sum_{i=1}^{n}\log_2\left(1+\frac{\alpha_i P_i}{\sigma^2}\right)$. Now we are making a minor modification here $\max\sum_{i=1}^{n}\log_2\left(1+\frac{\alpha_i P_i}{\sigma^2}\right)\log_2 e$. So this then becomes simply the natural logarithm, so we have $\max\sum_{i=1}^{n}\log\left(1+\frac{\alpha_i P_i}{\sigma^2}\right)$, so instead of maximizing the objective function times a constant, we can simply ignore the constant factor. Now the constraint is that the total transmit power is a fixed quantity.

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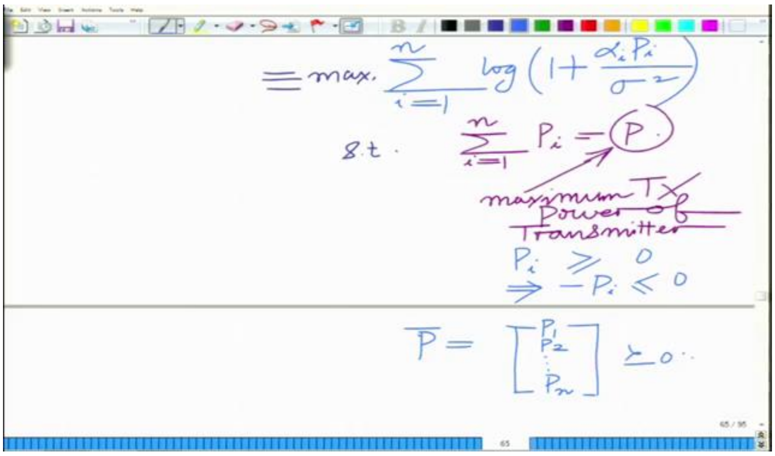

So the optimization problem is

$$\max \sum_{i=1}^{n} \log\left(1+\frac{\alpha_i P_i}{\sigma^2}\right)$$
$$s.t \sum_{i=1}^{n} P_i = P$$
$$P_i \geq 0$$

. This log is a concave function and the sum of log is also a concave function. So this is the maximization of concave objective function.

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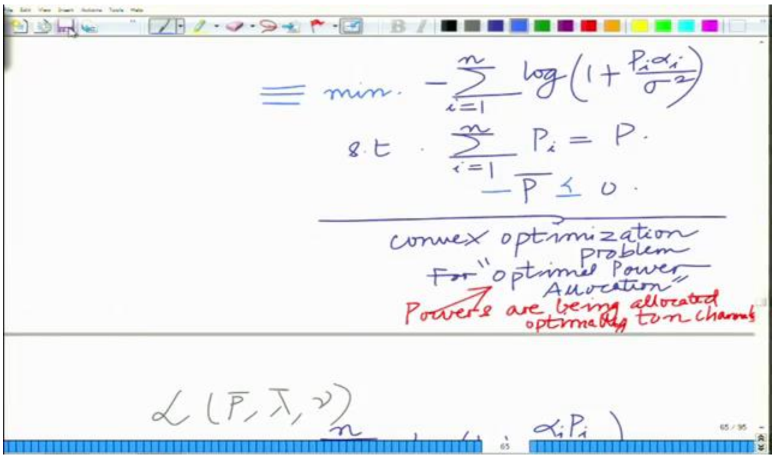

So this can equivalently be written as

$$\max -\sum_{i=1}^{n} \log\left(1+\frac{\alpha_i P_i}{\sigma^2}\right)$$
$$s.t \sum_{i=1}^{n} P_i = P$$
$$-P_i \le 0$$

. So this is the convex optimization problem for optimal power allocation. You are allocating the powers optimally and hence it is termed as optimal power allocation.

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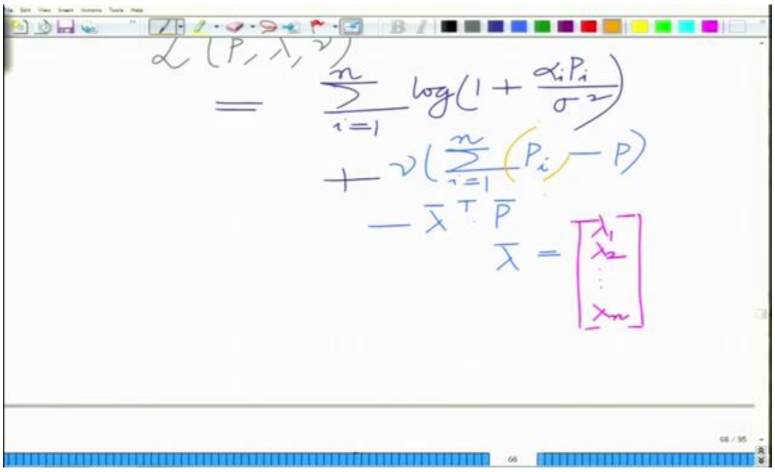

Now we will use the KKT conditions to solve this, let us start with the Lagrangian. So we have $L\left(\bar{P},\bar{\lambda},\nu\right) = \log\left(1+\frac{\alpha_i P_i}{\sigma^2}\right) + \nu\left(\sum_{i=1}^{n} P_i - P\right) - \bar{\lambda}^T \bar{P}$. So you have one Lagrange multiplier for each inequality constraint.

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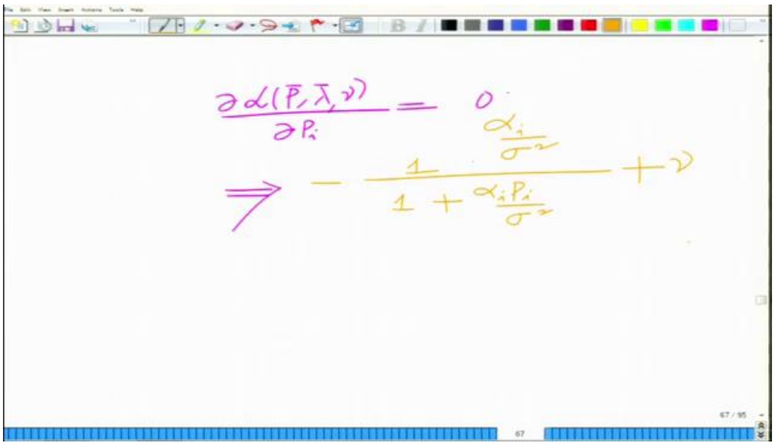

Now $\nabla_{\bar{P}} L\left(\bar{P}, \bar{\lambda}, \nu\right) = 0$ is one of the KKT conditions. So on solving this we get

$$\nu = \frac{\frac{\alpha_i}{\sigma^2}}{1 + \frac{\alpha_i P_i}{\sigma^2}} + \lambda_i .$$

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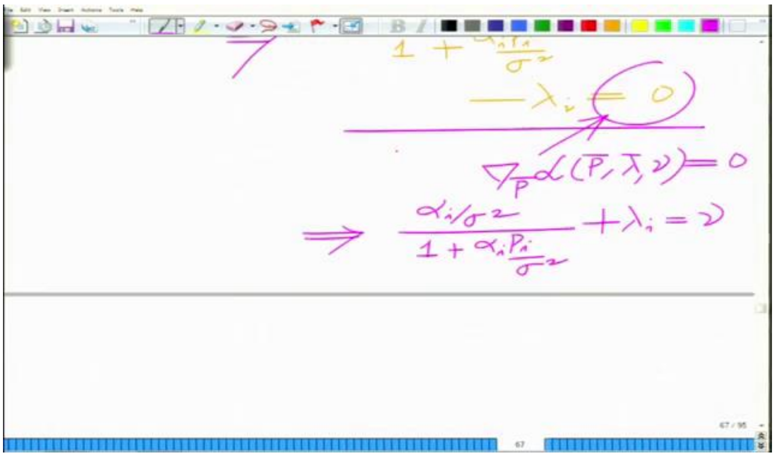

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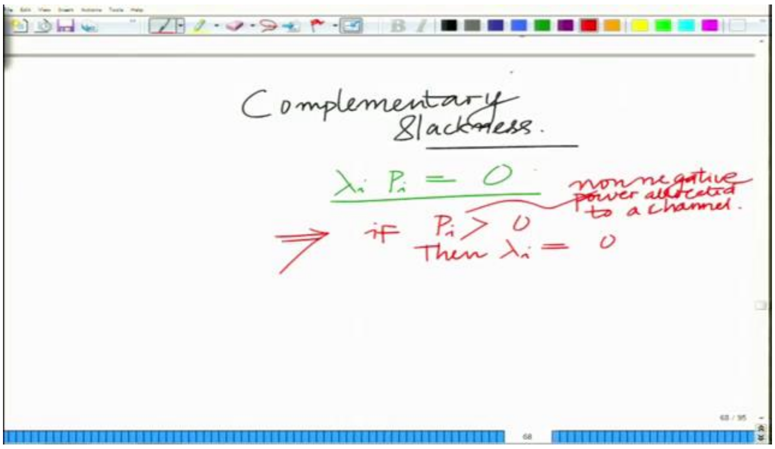

And now from the complementary slackness we have $\lambda_i P_i = 0$ that is either the constraint is slack or the Lagrange multiplier is slack, but not both. So let us consider these two conditions. If $P_i > 0$, that is power allocated to a channel is non-negative, then $\lambda_i = 0$.

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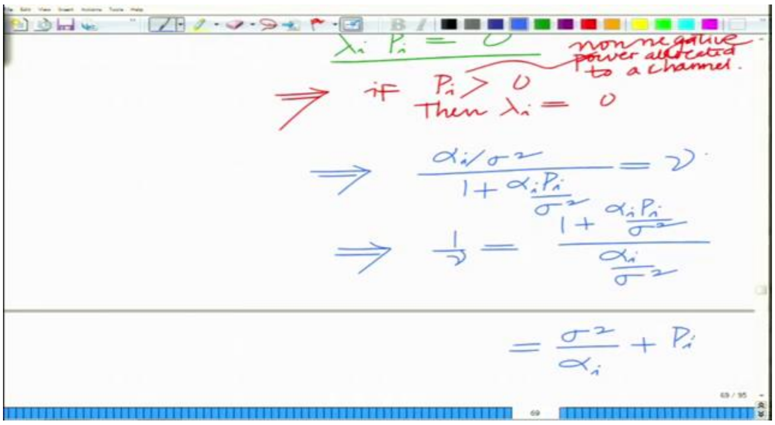

So this implies that $\frac{1}{\nu} = \frac{\sigma^2}{\alpha_i} + P_i$.

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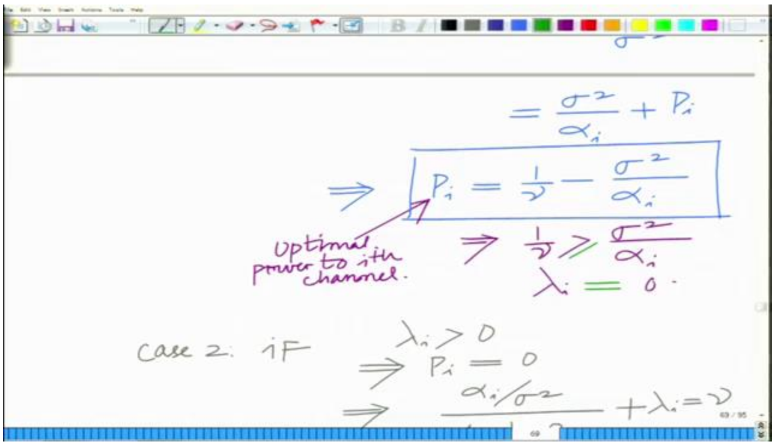

This implies that $P_i = \frac{1}{\nu} - \frac{\sigma^2}{\alpha_i} \Rightarrow \frac{1}{\nu} \geq \frac{\sigma^2}{\alpha_i}$ and the corresponding eigen value is 0. So this is the optimal power allocated to the i[th] channel.

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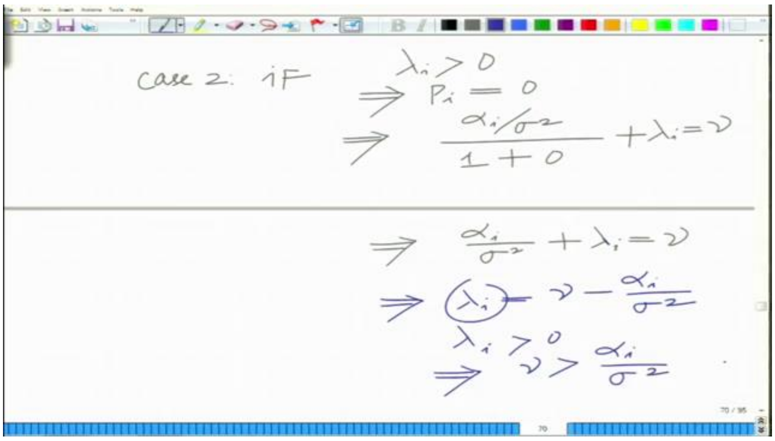

On the other hand, if you consider the case 2, if $\lambda_i > 0$, that is Lagrange multiplier is slack which implies that $P_i = 0$. Now this implies $\frac{1}{\nu} < \frac{\sigma^2}{\alpha_i}$ as shown in the slide.

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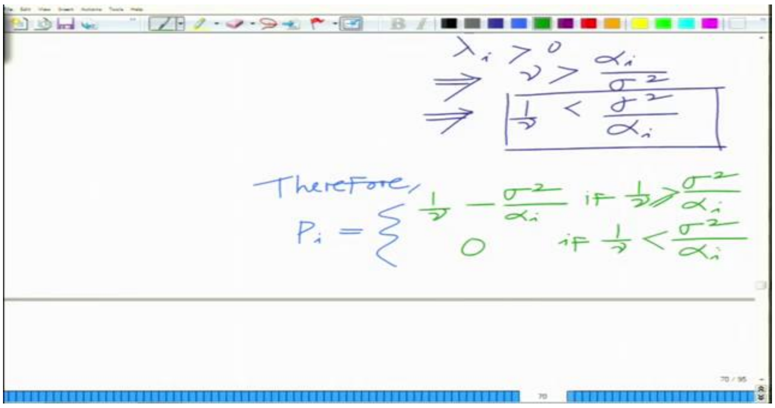

So there are two cases and therefore if you summarize it we have

$$P_i = \begin{cases} \frac{1}{\nu} - \frac{\sigma^2}{\alpha_i} & if \ \frac{1}{\nu} \geq \frac{\sigma^2}{\alpha_i} \\ 0 & if \ \frac{1}{\nu} < \frac{\sigma^2}{\alpha_i} \end{cases}.$$

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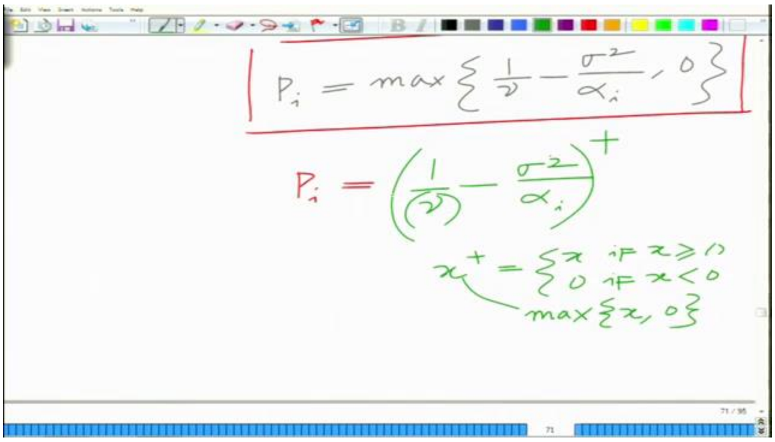

And therefore, you can write this P_i as $P_i = \left(\frac{1}{\nu} - \frac{\sigma^2}{\alpha_i}\right)^+$. So $P_i = \max\left\{\frac{1}{\nu} - \frac{\sigma^2}{\alpha_i}, 0\right\}$.

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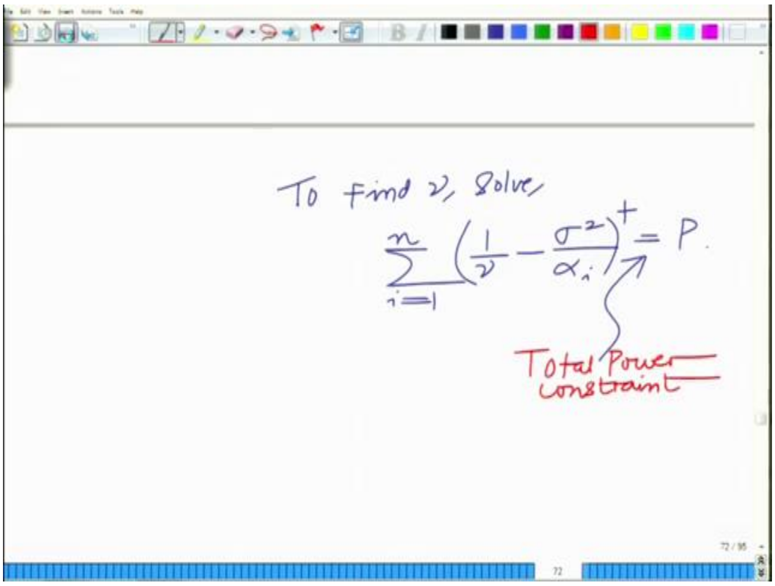

So to find nu solve $\sum_{i=1}^{n} \left(\frac{1}{\nu} - \frac{\sigma^2}{\alpha_i}\right)^+ = P$ that is the total power constraint.

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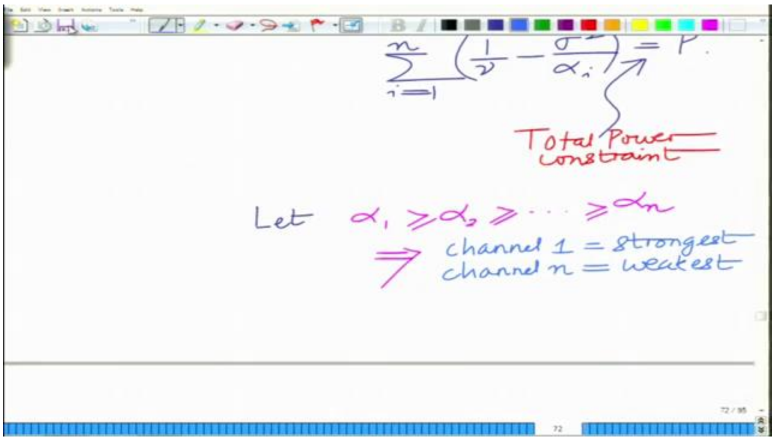

Let us let us assume that these are ordered as the first channel is the strongest and the last one is the weakest.

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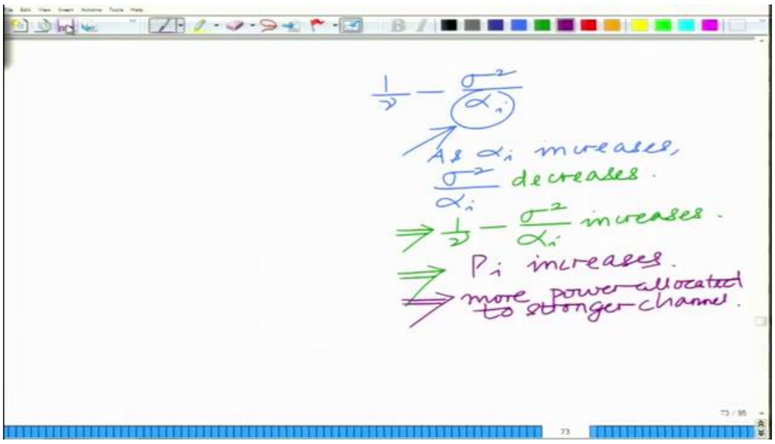

Now what this implies is that more power is allocated to the stronger channel as shown in slides.

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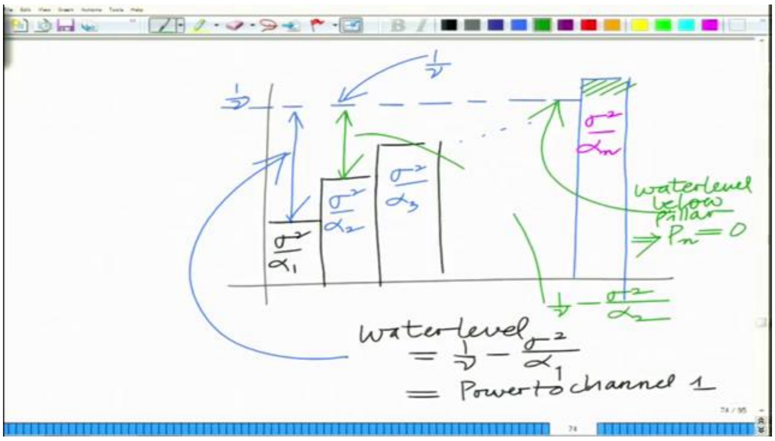

Now, let us look at this representation, for instance a sort of bowl or you can call it an area with this kind of pillars. So the first pillar is corresponding to first channel and then it decreases. Now if you draw here the level $\frac{1}{\nu}$, you can think of this as a water level, now the power allocated to the first channel is basically the amount of water.

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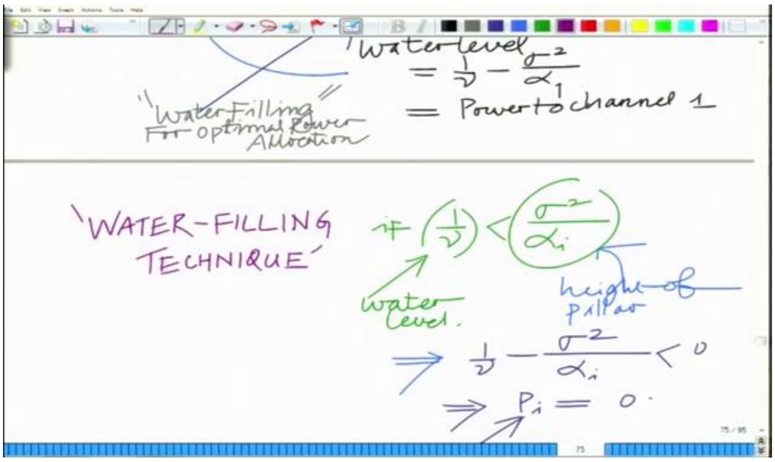

This is as shown in slide.

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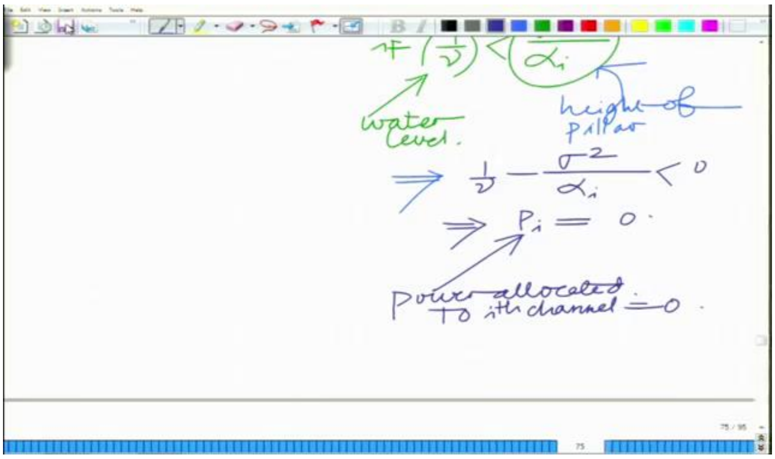

Therefore, this scheme is known as the optimal water filling algorithm. So you can think of this as a water level. So it is a solution of a convex optimization problem derived or obtained using the KKT conditions and the complementary slackness plays a very key role. So this is a nice scheme or this is the optimal scheme to allocate power across the parallel channels that maximizes the sum rate of communication between the transmitter and the receiver. So we will stop here and continue in the subsequent module. Thank you very much.