Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture -12
Affine Set Examples - Line, Halfspace, Hyperplane and Application - Power Allocation for Users in Wireless Communication

Hello, welcome to another module in this massive open online course. So, let us continue this discussion by looking at the affine set.

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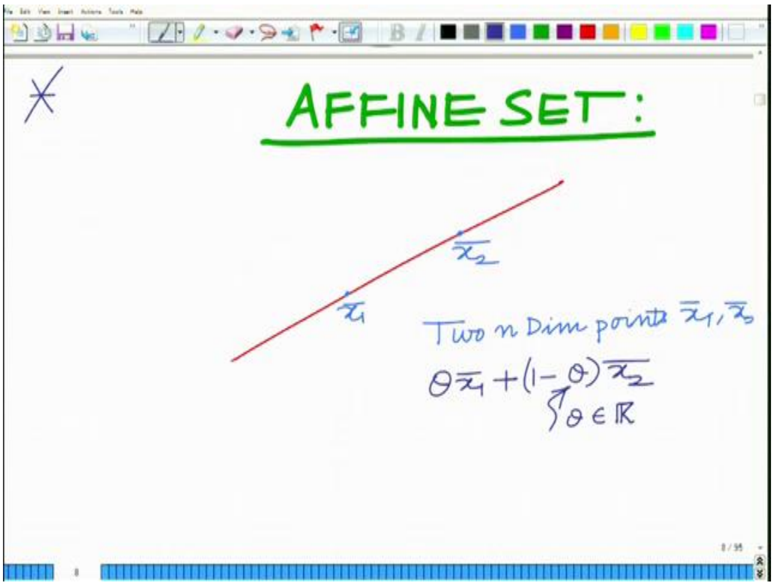

In the previous module, convex set has been discussed. So, to define a affine set, consider any two n dimensional points $\bar{x}_1$ and $\bar{x}_2$ such that the linear combination of $\bar{x}_1$ and $\bar{x}_2$ is

$$\theta\bar{x}_1 + (1-\theta)\bar{x}_2 = \bar{x}$$

In case of a convex set, θ is restricted to have any value in $[0,1]$, but in affine set θ can have any real value that is $\theta \in \mathbb{R}$.

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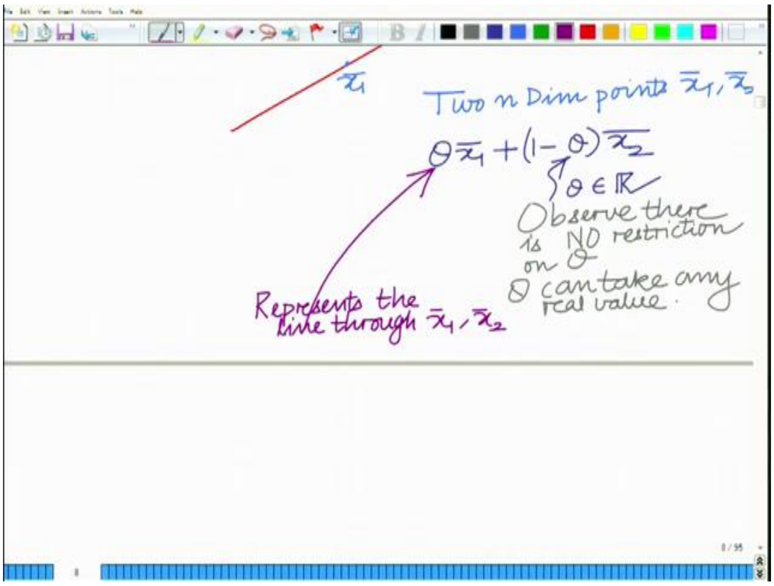

Therefore, an affine set is a convex set in which θ can have any real value. This means that for a set being an affine set, instead of considering the line segment that joins any two n-dimensional points $\overline{x}_1$ and $\overline{x}_2$, the whole line passing through both the points $\overline{x}_1$ and $\overline{x}_2$ that belongs to the set must lie inside the set.

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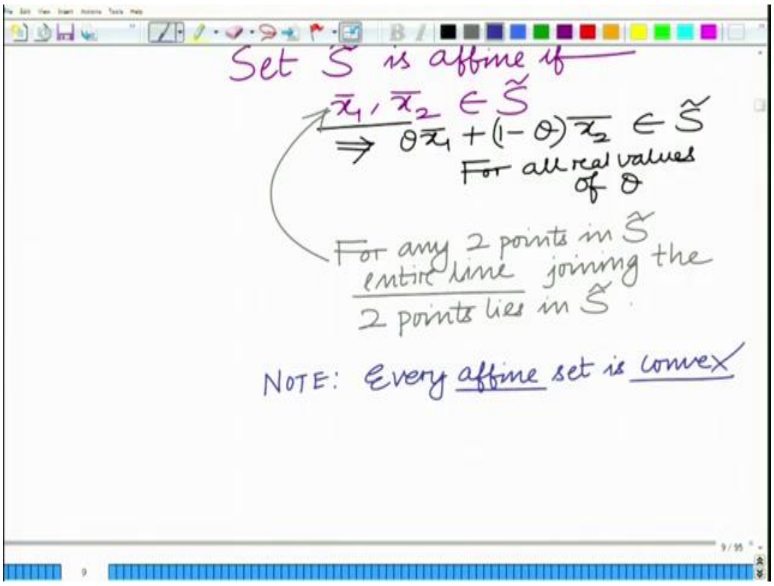

That is if set $\tilde{S}$ is an affine set, then for any $\overline{x}_1, \overline{x}_2 \in \tilde{S}$

$\theta \bar{x}_1 + (1-\theta)\bar{x}_2 \in \tilde{S}$ for all $\theta \in \mathbb{R}$

And note that every affine set is a convex set. Its reason is that if it contains the entire line joining the two points, it naturally contains the line segment also. But vice versa of this statement is not true. So, every affine set is a convex set, but all the convex sets are not affine. So, the affine set is a special case of a convex set.

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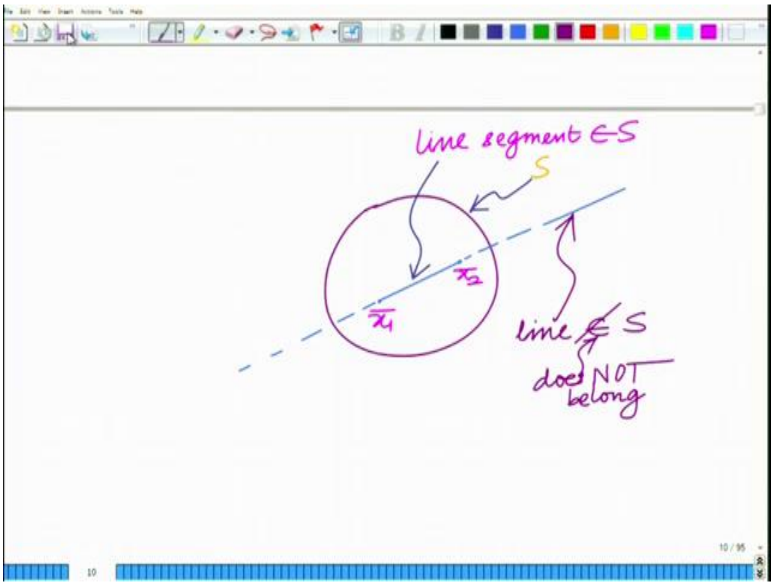

To verify the above statement let us take a simple example. Consider a circle which is a convex set because the whole line segment joining the any two points contained in the set lies within the set. But as shown in the above figure, if this line segment is extended in both direction then it forms the line and after a certain limit, the portion of this line does not belong inside the circle.

This is an interesting relation between affine sets and convex sets. let us look at some examples to understand this better.

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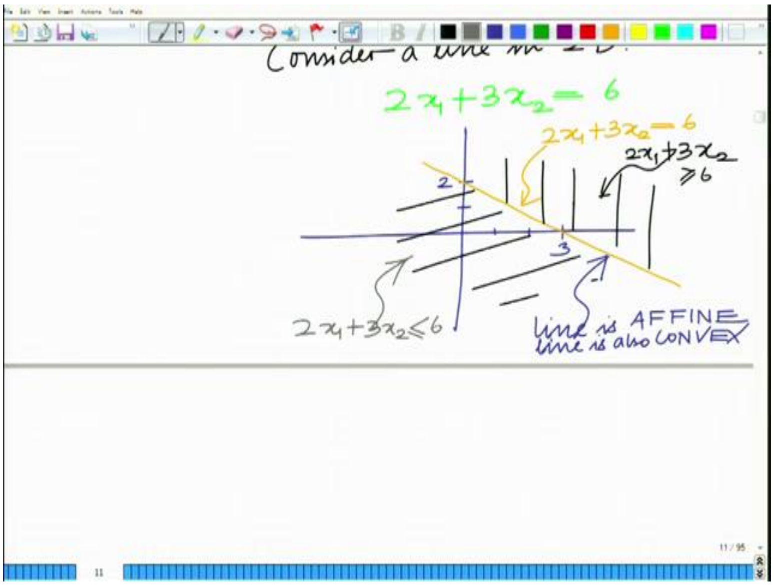

A line is a trivial example of an affine set because if we take any two points on the line and join them to make a line then this entire line overlaps the original line. So consider a line in 2-Dimentinal space and it is given by the following equation.

$$2\overline{x}_1 + 3\overline{x}_2 = 6$$

So the area above this line can be denoted by

$$2\overline{x}_1 + 3\overline{x}_2 \geq 6$$

And the area below this line can be denoted by

$$2\overline{x}_1 + 3\overline{x}_2 \leq 6$$

These regions above and below the lines are known as half spaces.

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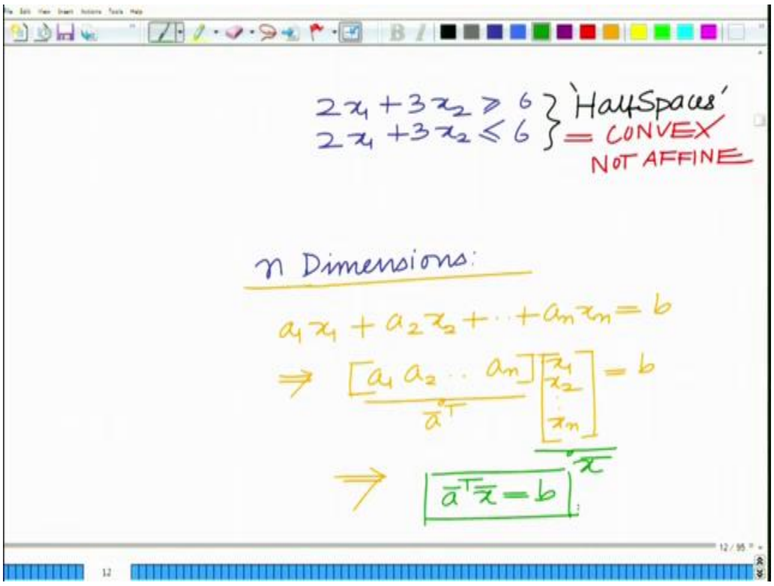

So, the line divides the plane into two regions known as half spaces. Also the line is convex and also affine. But these half spaces are only convex but not affine.

Consider an n-dimensional equation which is of the following form.

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

In the terms of row vector and column vector, the above equation will be written as

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = b$$

$$\bar{a}^T \bar{x} = b$$

This equation represents a hyper plane in n-dimensions and it is an affine set. This also implies that this is a convex set S. So it can be written as follows. For a convex set S such that $\bar{x}_1, \bar{x}_2 \in S$,

$$\bar{a}^T \bar{x}_1 = b$$
$$\bar{a}^T \bar{x}_2 = b$$

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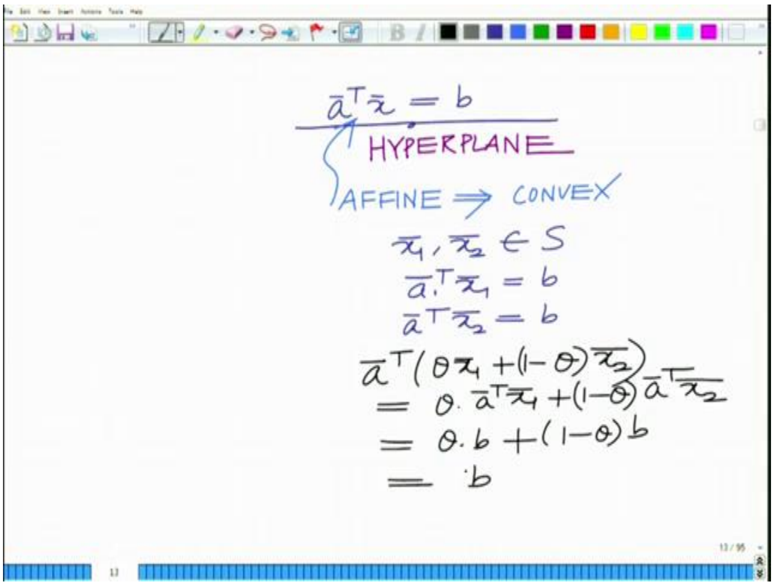

Therefore,

$$\bar{a}^T\left(\theta\bar{x}_1 + (1-\theta)\bar{x}_2\right)$$
$$= \theta\bar{a}^T\bar{x}_1 + (1-\theta)\bar{a}^T\bar{x}_2$$
$$= \theta b + (1-\theta)b$$
$$= b$$

Also note that there is no restriction on θ. Thus,

$$\theta \in R$$

And this further implies that this an affine set.

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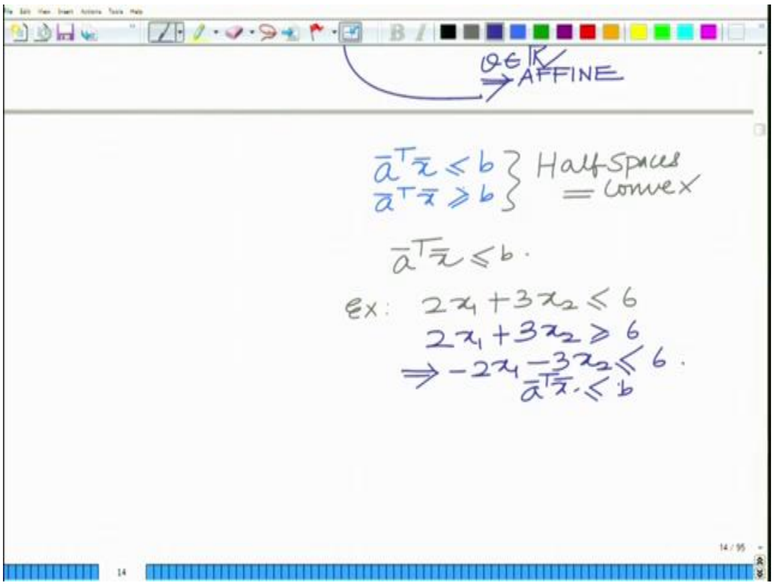

So, this hyper plane divides the space into two n-dimensional spaces known as half spaces. These two regions are defined as follows.

$$\left.\begin{array}{l} \bar{a}^T\bar{x} \leq b \\ \bar{a}^T\bar{x} \geq b \end{array}\right\} \text{Half Spaces}$$

This is already shown in the previous example of line $2\bar{x}_1 + 3\bar{x}_2 = 6$. The first half space equation is

$$2\bar{x}_1 + 3\bar{x}_2 \leq 6$$

Similarly the second half space equation is

$$2\bar{x}_1 + 3\bar{x}_2 \geq 6$$

This equation can also be written as

$$\begin{aligned} -2\bar{x}_1 - 3\bar{x}_2 &\leq 6 \\ -\left(2\bar{x}_1 + 3\bar{x}_2\right) &\leq 6 \end{aligned}$$

Therefore, the general equation of a half space is of the form

$$\bar{a}^T\bar{x} \leq b$$

Thus the important thing to realize here is that hyper planes are affine and hence convex also but half spaces are only convex.

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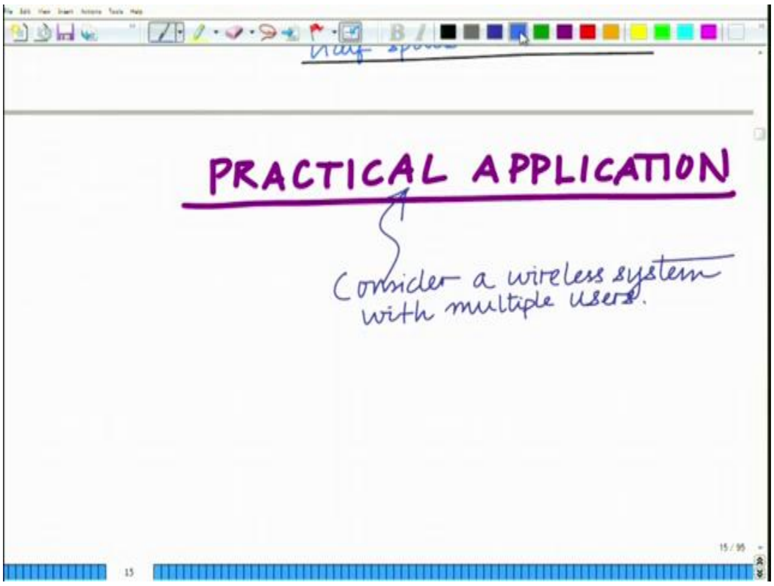

Let us explore a practical application regarding convexity. So, consider a wireless system with k multiple users.

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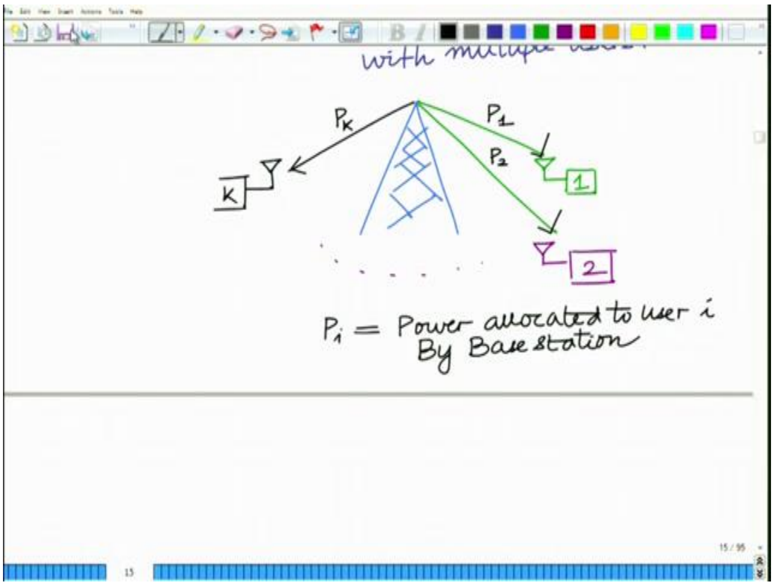

Consider a downlink scenario with a base station is transmitting signals to k users. Let us have the set of user as $U_1, U_2, \ldots, U_k$.

The signal power transmitted to different base stations is different and is represented as

$P_1, P_2, \ldots, P_k$

So, P_i is the power of signal allocated to user U_i by the base station. Thus the total power allocated to different users has to be less than or equal to the total power transmitted by the base station to these users. So, this is a constraint in a practical wireless scenario.

Let us say that the total power transmitted by the base station to k users is P. Thus, the above constraint will be shown as

$P_1 + P_2 + \ldots + P_k \le P$

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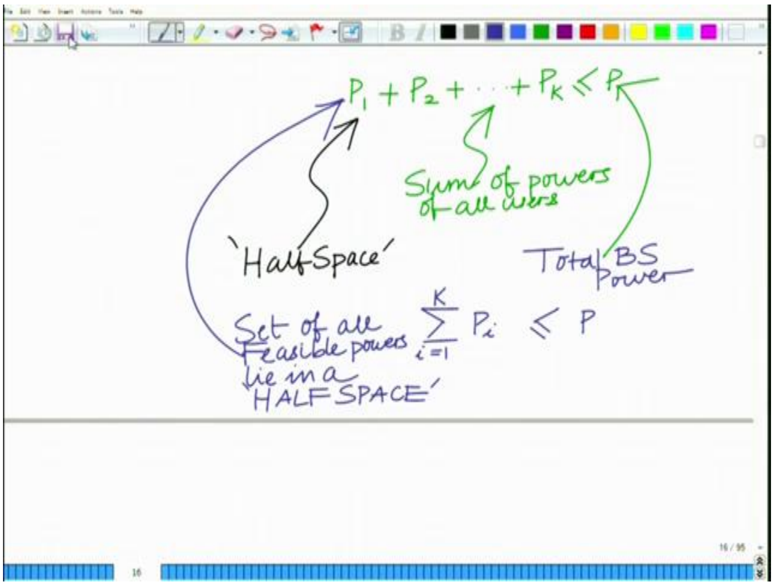

And in general this can be written as

$$\sum_{i=1}^{k} P_i \le P$$

Observe that this constraint is basically a half space constraint because it is a linear combination of $P_1 + P_2 + \ldots + P_k$. For such a linear combination this can be considered that weighting coefficients $a_1, a_2, \ldots, a_k$ are unity.

So, basically the set of all feasible powers in the wireless scenario that satisfy this constraint, lie in a half space.

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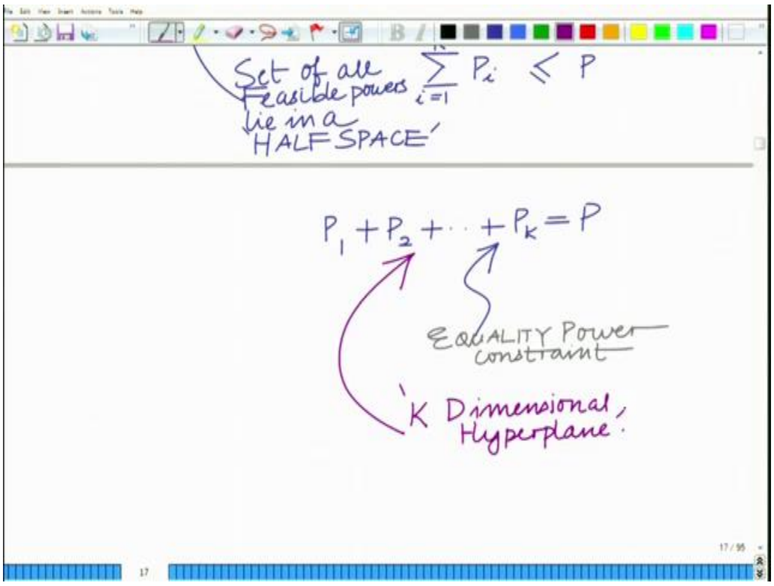

On the other hand, another constraint is also defined in wireless scenario which is known as equality power constraint. This means to nullify the wastage of signal power, the power of all the users is set in such a way that

$$P_1 + P_2 + \ldots + P_k = P$$

And this simply represents a k-dimensional hyper plane. That means all the feasible power in wireless scenario lie on a hyper plane.

So, this is an interesting practical perspective to the theoretical concepts of convex sets and affine sets. Several more links between the various theoretical concepts or the theoretical building blocks of optimization and its relation to practical applications in several fields as wireless communications will be explored in the subsequent modules.