Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture - 49
Stochastic Linear Program, Gaussian Uncertainty

Keywords: *Stochastic Linear Program, Gaussian Uncertainty*

Hello, welcome to another module in this massive open online course. So we have looked at linear programs and also demonstrated the practical application of linear program. And in this module let us start looking at an extension of it known as the robust linear program.

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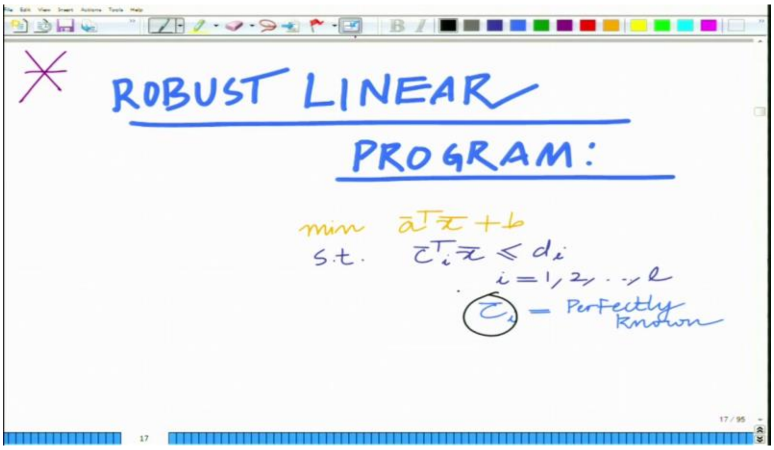

So the linear program can be formulated as $\begin{matrix} \min \bar{a}^T\bar{x}+b \\ s.t\ \bar{c}_i^T\bar{x} \leq d_i \\ i=1,2,\ldots,l \end{matrix}$. While formulating this linear program, we have assumed these constraints to be perfectly known. So these have to be estimated in practice for any particular problem which means there can be a certain level of uncertainty in this.

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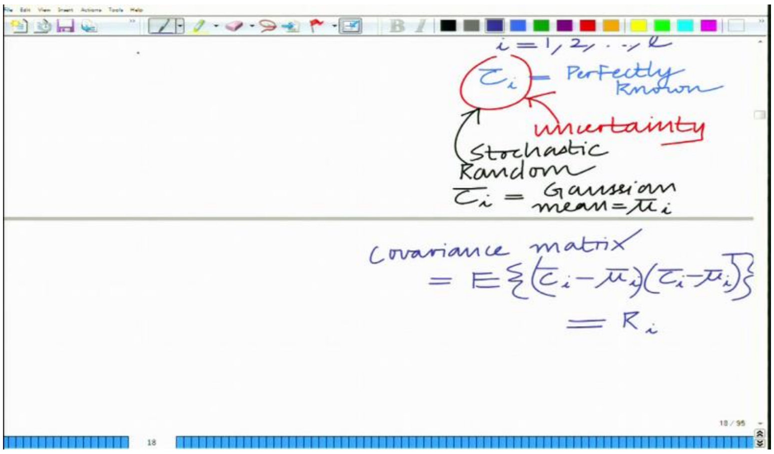

So one practically useful model for such scenarios where there is uncertainty in $\bar{c}_i$ is to assume that these are random in nature. In particular you can assume that these $\bar{c}_i$'s are Gaussian random vectors with their mean $\bar{\mu}_i$ and the covariance matrix $E\left\{\left(\bar{c}_i - \bar{\mu}_i\right)\left(\bar{c}_i - \bar{\mu}_i\right)^T\right\} = R_i$.

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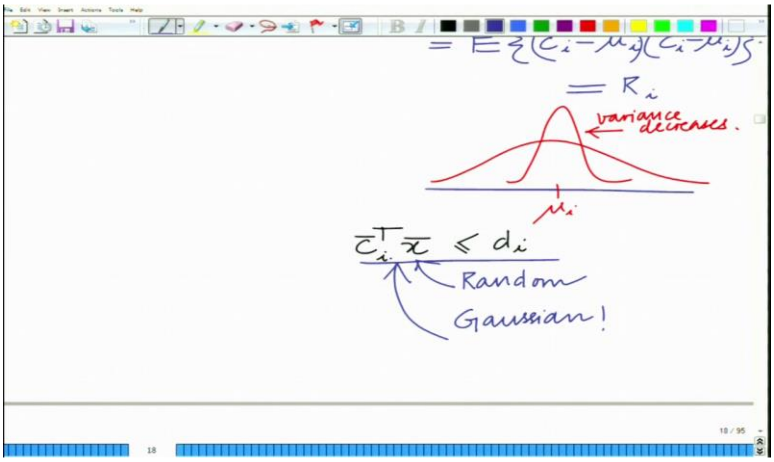

The covariance characterizes the spread around the mean. Let us simplify this in case of a scalar, for instance we have a Gaussian case with mean μ_i and as the variance decreases, it becomes more and more concentrated on the mean. When the covariance matrix tends to 0 or becomes very close to 0, it reduces to the deterministic linear program that we have seen before. So for this stochastic linear program in which the $\bar{c}_i$'s are random, if you look at the constraint $\bar{c}_i^T \bar{x} \le d_i$, this is also random. There is a linear transformation of Gaussian random variables yields another Gaussian random variable.

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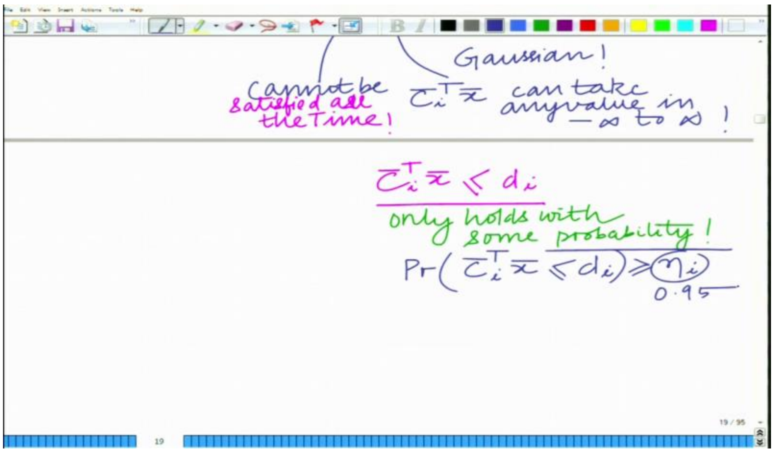

And since this is a Gaussian random variable this can take any value in $-\infty$ *to* ∞ . So this means that one cannot hope that this constraint is always satisfied in the optimization problem. So this means that it only holds with some probability let us say η_i . So let us say $Pr\left(\bar{c}_i^T \bar{x} \le d_i\right) \ge \eta_i$ which means this constraint need not hold all the time, but it can hold with a very high probability.

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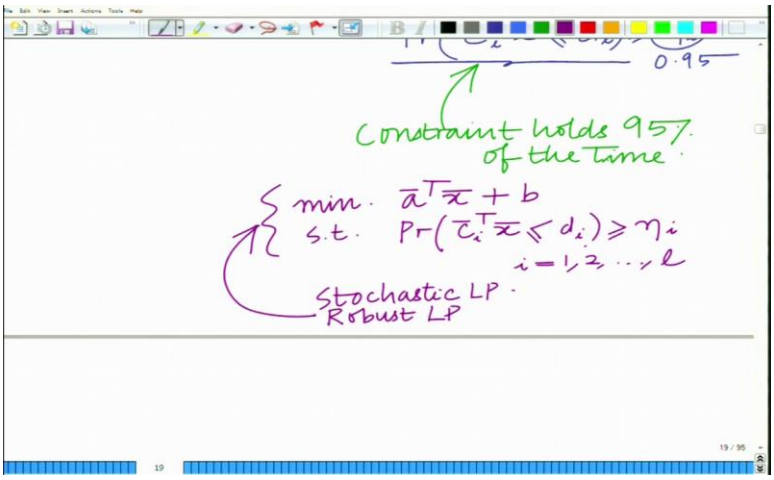

And therefore I can modify this linear program as $\begin{array}{l} \min \bar{a}^T \bar{x} + b \\ s.t \ \Pr\left(\bar{c}_i^T \bar{x} \le d_i\right) \ge \eta_i \\ \quad i = 1, 2, \ldots, l \end{array}$. So this is the robust LP or you call it as a stochastic one because the constraint is random in nature or it holds with a certain probability. It is robust since you are ensuring that you are taking the uncertainty in $\bar{c}_i$ into account.

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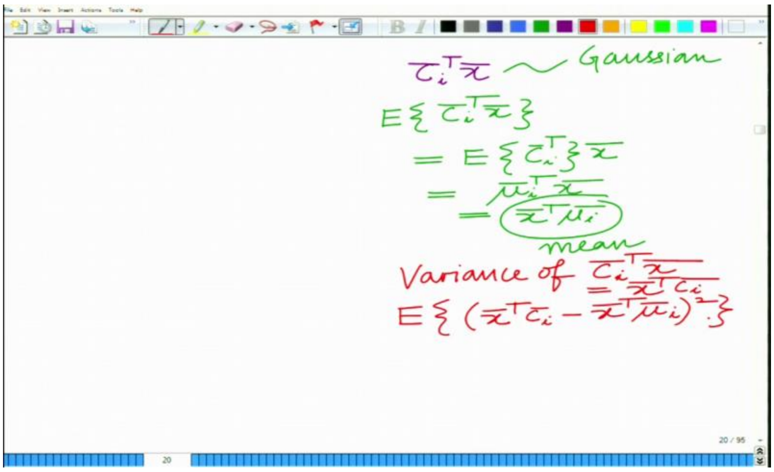

Now, let us modify this problem further, let us look at this quantity $\bar{c}_i^T \bar{x}$, we have already said that this quantity is a Gaussian random variable and if we find the mean and variance of this Gaussian random variable, we get the mean as $\bar{x}^T \bar{\mu}_i$ and the variance as $\bar{x}^T E\left\{\left(\bar{c}_i - \bar{\mu}_i\right)\left(\bar{c}_i - \bar{\mu}_i\right)^T\right\}\bar{x} = \bar{x}^T R_i \bar{x}$.

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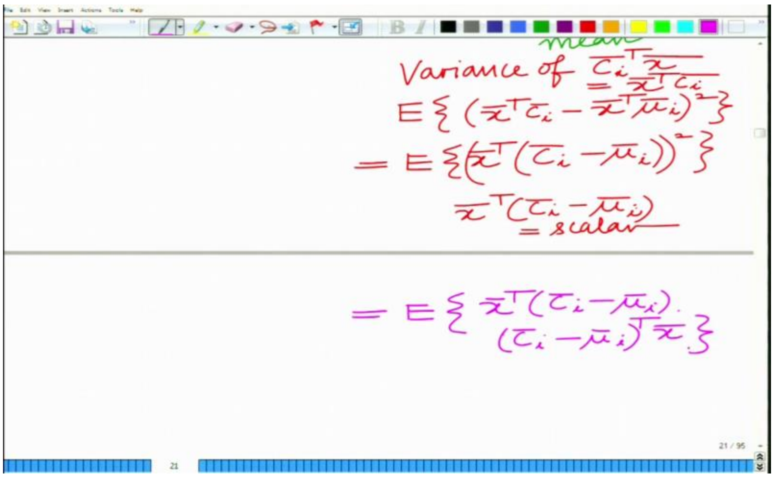

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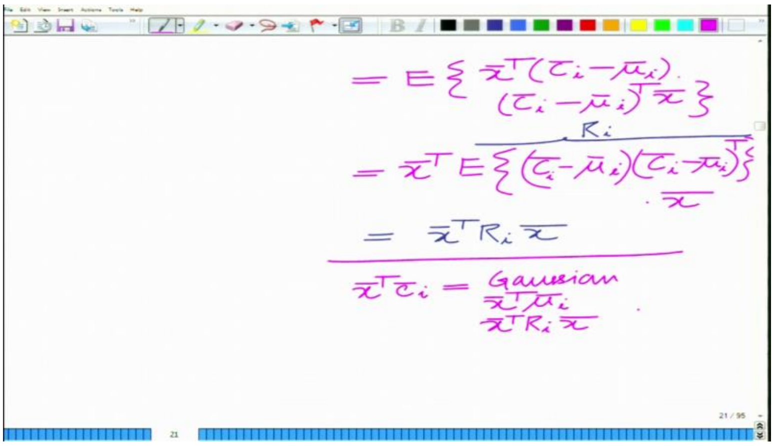

And now let us find the $\Pr\left(\bar{c}_i^T \bar{x} \le d_i\right)$.

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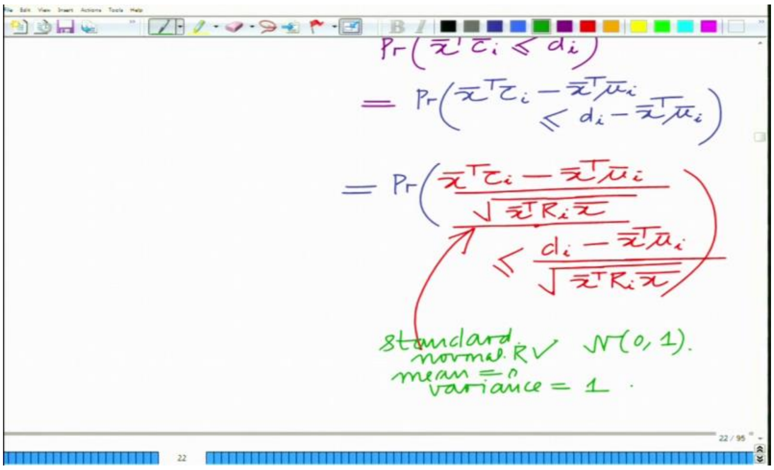

Now let us simplify this as shown in slides and we will divide it by the standard deviation σ which is $\sqrt{\bar{x}^T R_i \bar{x}}$.

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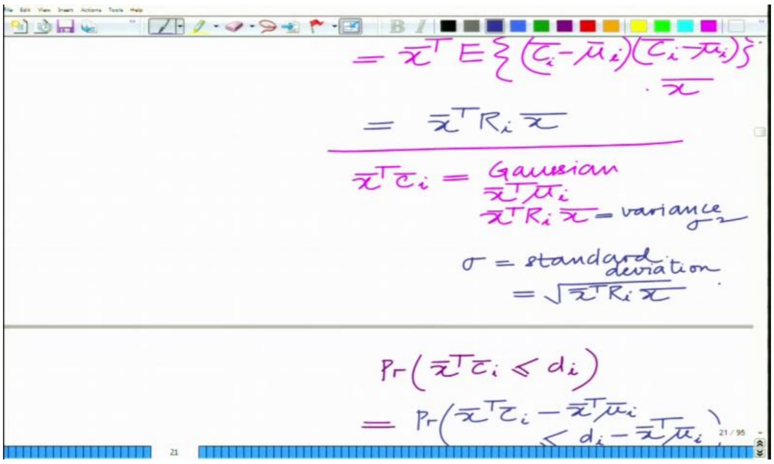

So from a Gaussian random variable we have basically subtracted the mean and divided by the standard deviation that gives us a zero mean unit variance Gaussian random variable which is nothing but the standard normal random variable. So we have

$$\Pr\left(\bar{c}_i^T \bar{x} \le d_i\right) = \Pr\left(\frac{\bar{x}^T \bar{c}_i - \bar{x}^T \bar{\mu}_i}{\sqrt{\bar{x}^T R_i \bar{x}}} \le \frac{d_i - \bar{x}^T \bar{\mu}_i}{\sqrt{\bar{x}^T R_i \bar{x}}}\right) = 1 - \Pr\left(\frac{\bar{x}^T \bar{c}_i - \bar{x}^T \bar{\mu}_i}{\sqrt{\bar{x}^T R_i \bar{x}}} \ge \frac{d_i - \bar{x}^T \bar{\mu}_i}{\sqrt{\bar{x}^T R_i \bar{x}}}\right) = 1 - Q\left(\frac{d_i - \bar{x}^T \bar{\mu}_i}{\sqrt{\bar{x}^T R_i \bar{x}}}\right)$$

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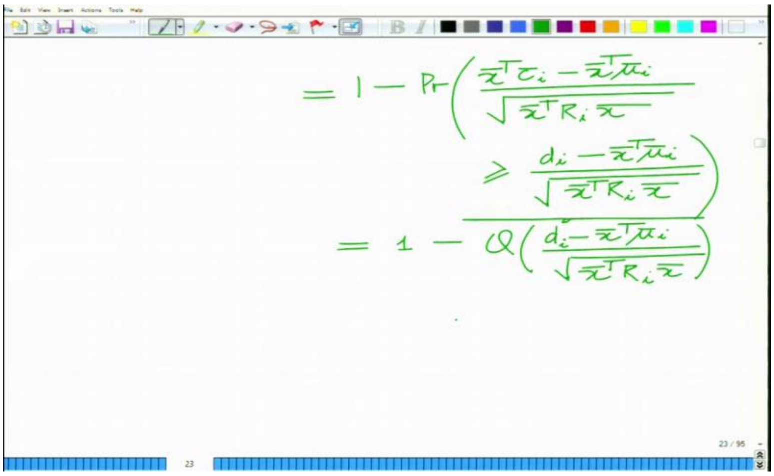

So this quantity is nothing but the tail probability of the standard normal random variable.

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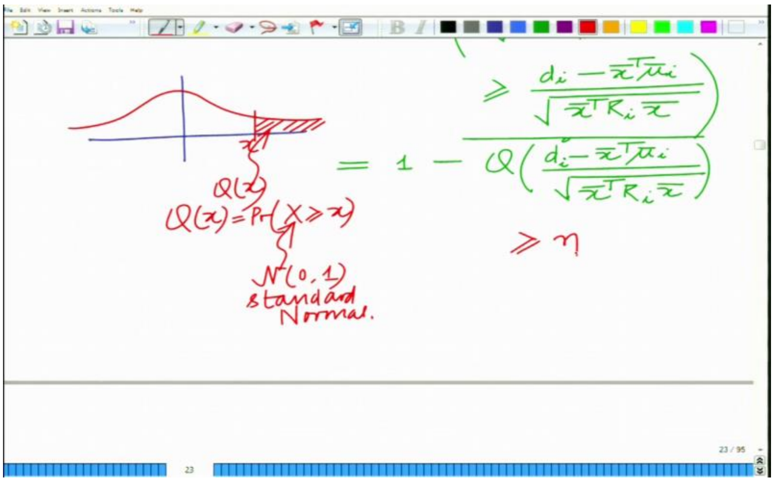

And we have $Q(x) = \Pr(X \geq x)$ where X is the Gaussian random variable with mean 0 and variance 1. Now we need $1 - Q\left(\frac{d_i - \bar{x}^T \bar{\mu}_i}{\sqrt{\bar{x}^T R_i \bar{x}}}\right) \geq \eta_i$.

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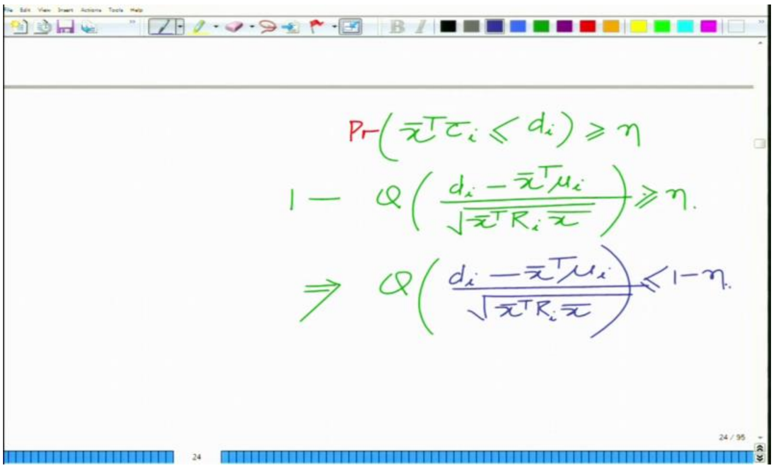

Now on solving this we have $\frac{d_i - \bar{x}^T \bar{\mu}_i}{\sqrt{\bar{x}^T R_i \bar{x}}} \geq Q^{-1}(1 - \eta_i)$.

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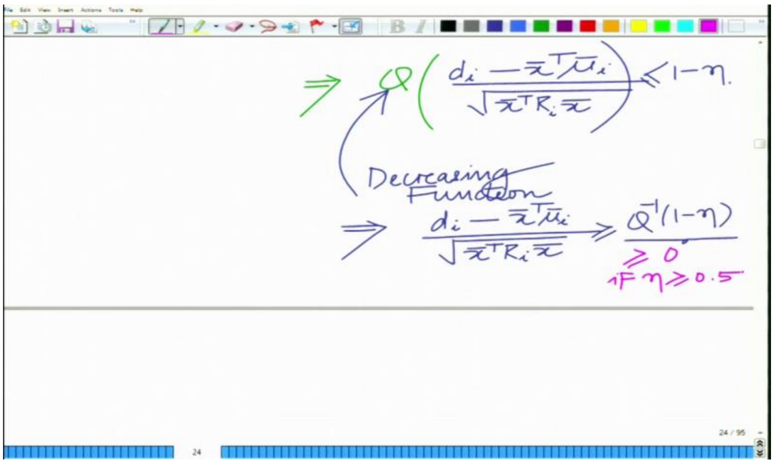

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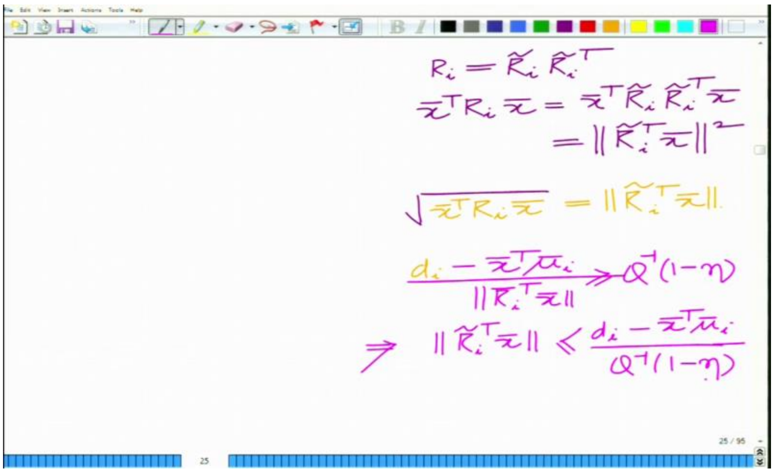

Let me just simplify this now further, remember R_i is the covariance matrix and so it is a positive semi definite matrix. So this can be factored as $R_i = R_i R_i^T$ so this implies $\bar{x}^T R_i \bar{x} = \bar{x}^T R_i R_i^T \bar{x} = \left\| R_i^T \bar{x} \right\|^2$. And therefore, the condition above this can be simplified as

$$\left\| R_i^T \bar{x} \right\| \le \frac{d_i - \bar{x}^T \bar{\mu}_i}{Q^{-1}(1-\eta_i)} \quad .$$

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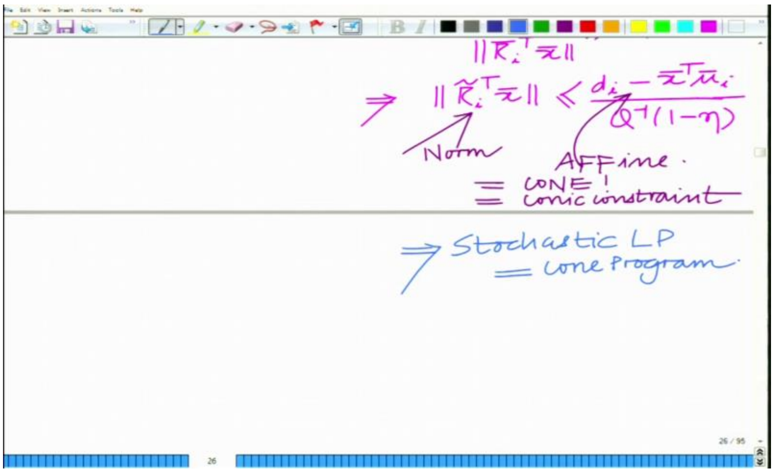

And now you have a norm of a vector less than or equal to an affine function of the same vector which represents the cone. So this is the conic constraint and hence this will become a cone program. So this implies that the stochastic LP becomes a cone program.

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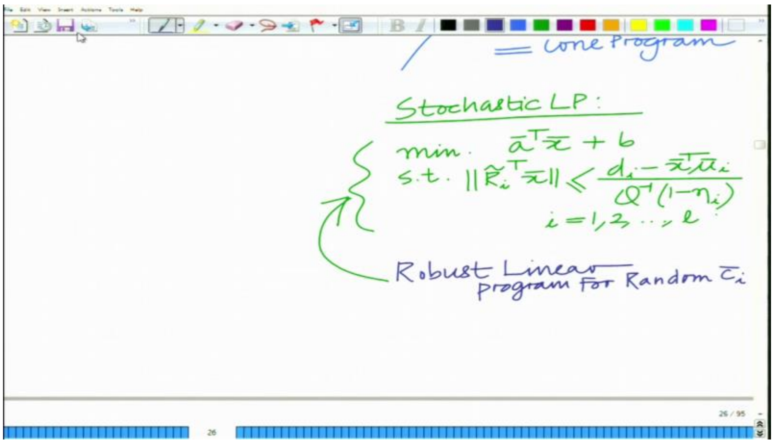

Now you can formulate this stochastic LP as

$$\begin{aligned} &\min \bar{a}^T \bar{x} + b \\ &s.t \left\| R_i^T \bar{x} \right\| \le \frac{d_i - \bar{x}^T \bar{\mu}_i}{Q^{-1}(1-\eta_i)} \\ &\qquad i = 1, 2, \ldots, l \end{aligned}$$

. So you can think of this as a robust version for the scenario when these vectors are random in nature and this is a practical flavour of the traditional linear program. In particular this is the robust stochastic LP or robust LP. We will stop here. Thank you very much.