Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture – 02
Eigenvectors and Eigenvalues of Matrices and their Properties

Hello, welcome to another module in this massive open online course.

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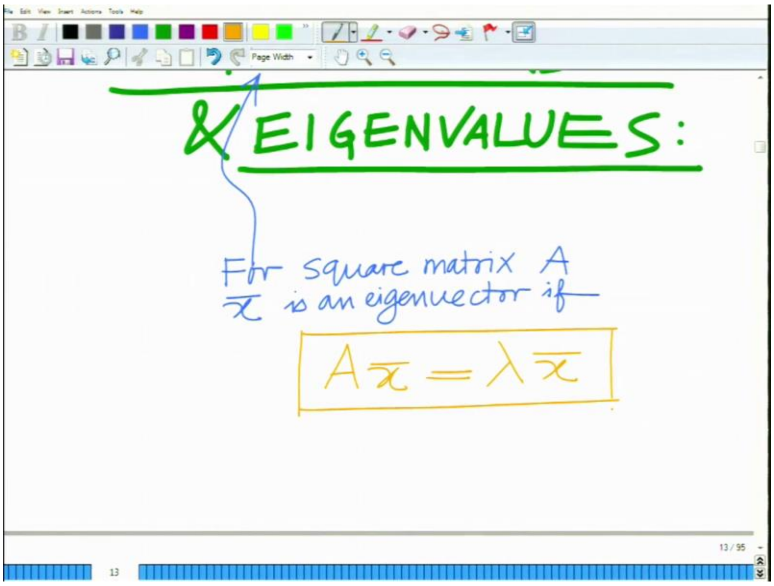

Let us continue our discussion regarding the mathematical preliminaries, for the framework of convex optimization by looking at another very important concept that is of the Eigenvalues, the eigenvectors and eigenvalues of square matrices.

So, let's talk about the concepts of eigenvectors and eigenvalues. Eigenvalue is defined only for a square matrix. So, for a square matrix A, $\bar{x}$ is an eigenvector, if

$$A\bar{x} = \lambda\bar{x}$$

Where λ is known as the eigenvalue of matrix A. This is the fundamental equation of a matrix for the eigenvector.

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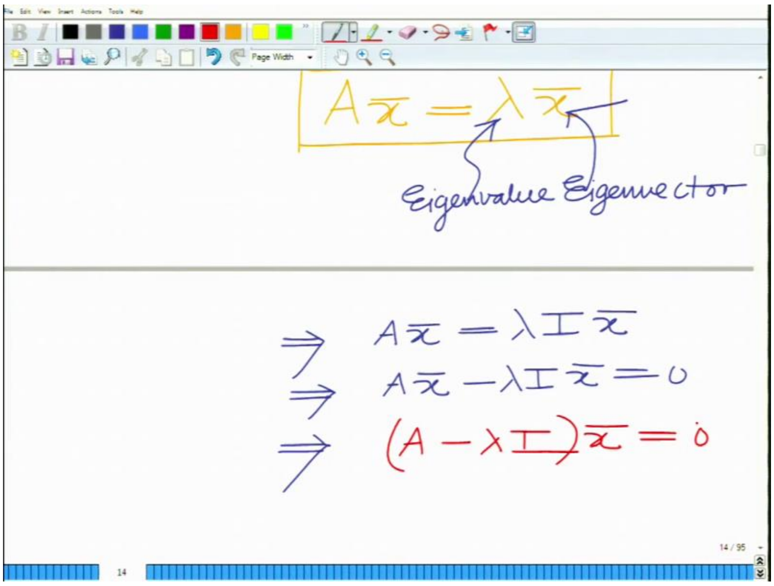

And, now we can also write

$$A\bar{x} = \lambda I\ \bar{x}$$
$$A\bar{x} - \lambda I\ \bar{x} = 0$$
$$(A - \lambda I)\ \bar{x} = 0$$

So, here, the matrix $(A - \lambda I)$ is a singular matrix.

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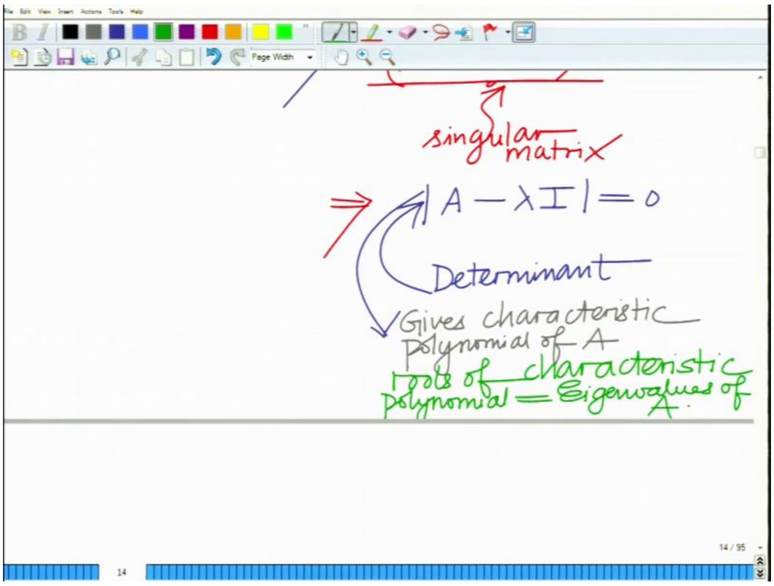

This implies that if λ is an eigenvalue of A then

$$|A-\lambda I|=0$$

Above equation is known as the characteristic equation corresponding to the matrix A and it gives the characteristic polynomial. So, the roots of this characteristic polynomial are eigenvalues of A.

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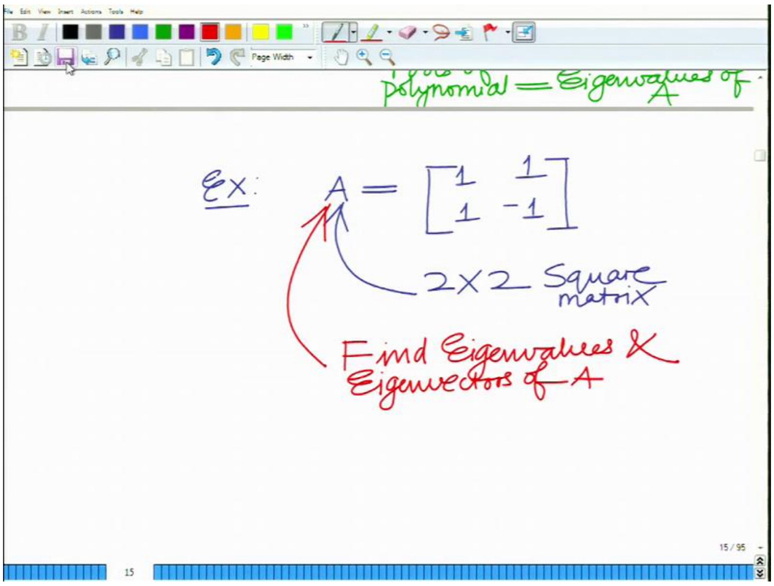

For example, let us say A is a 2×2 matrix.

$$A=\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

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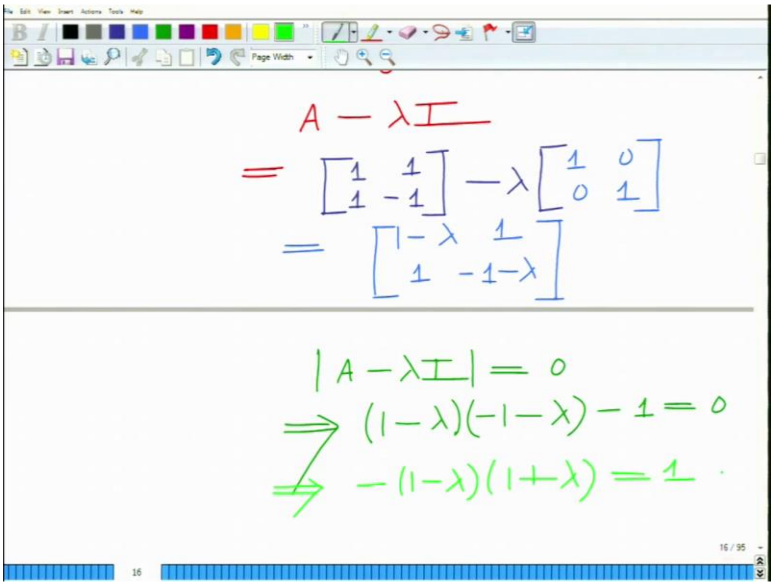

Now to find the eigenvalues λ and the corresponding eigenvectors $\bar{x}$ of this square matrix A, write down its characteristic equation as follows.

$$
\begin{aligned}
A - \lambda I &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix}
\end{aligned}
$$

Now consider the determinant of this is

$$
\begin{aligned}
|A - \lambda I| &= 0 \\
(1-\lambda)(-1-\lambda) - 1 &= 0 \\
-(1-\lambda)(1+\lambda) &= 1 \\
\lambda^2 - 1 &= 1 \\
\lambda &= \pm\sqrt{2}
\end{aligned}
$$

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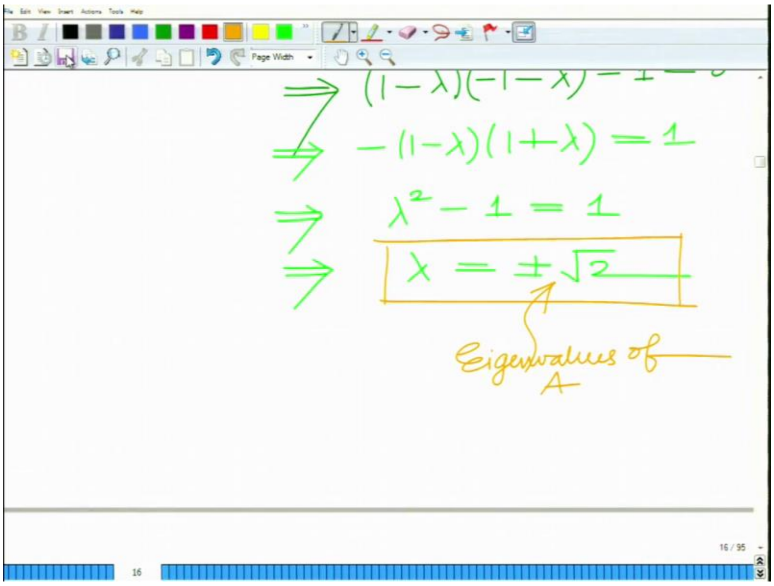

So, we have got two eigenvalues; $\sqrt{2}$ and $-\sqrt{2}$. Now let us find the eigenvectors of the matrix A corresponding to both of these eigenvalues.

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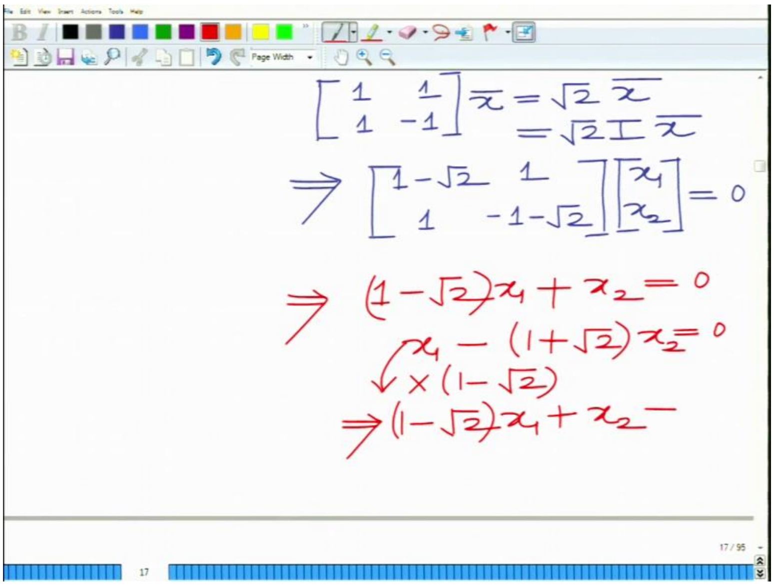

So, for $\lambda = \sqrt{2}$;

$$A\bar{x} = \lambda I\ \bar{x}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\bar{x} = \sqrt{2}I\ \bar{x}$$

$$\begin{pmatrix} 1-\sqrt{2} & 1 \\ 1 & -1-\sqrt{2} \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\left(1-\sqrt{2}\right)x_1 + x_2 = 0$$

Now if we multiply this equation with constant $\left(1-\sqrt{2}\right)$ then, So it gives another equation identical to the last equation. So, basically we have just one equation and therefore, this is an infinite number of solutions and that is kind of obvious, because if the eigenvector corresponding to eigenvalue is not unique. So, if $\bar{x}$ is an eigenvector, then $\bar{x}$ scaled by any constant k is also an Eigen vector corresponding to the same eigenvalue. And therefore, there are infinite number of eigenvectors.

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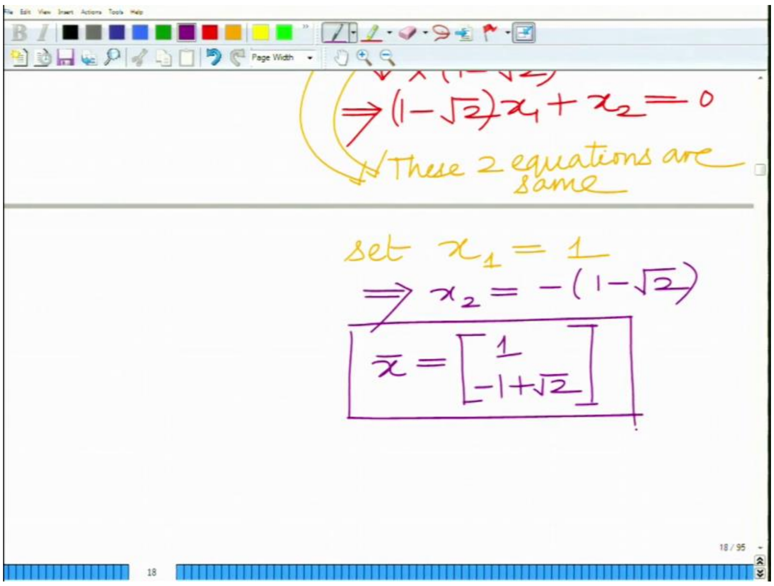

Now, to derive a solution, set $x_1 = 1$. Therefore,

$$x_2 = -\left(1-\sqrt{2}\right)$$

And hence we get,

$$\bar{x} = \begin{bmatrix} 1 \\ -1+\sqrt{2} \end{bmatrix}$$

This is one of the eigenvectors of A corresponding to eigenvalue $\sqrt{2}$.

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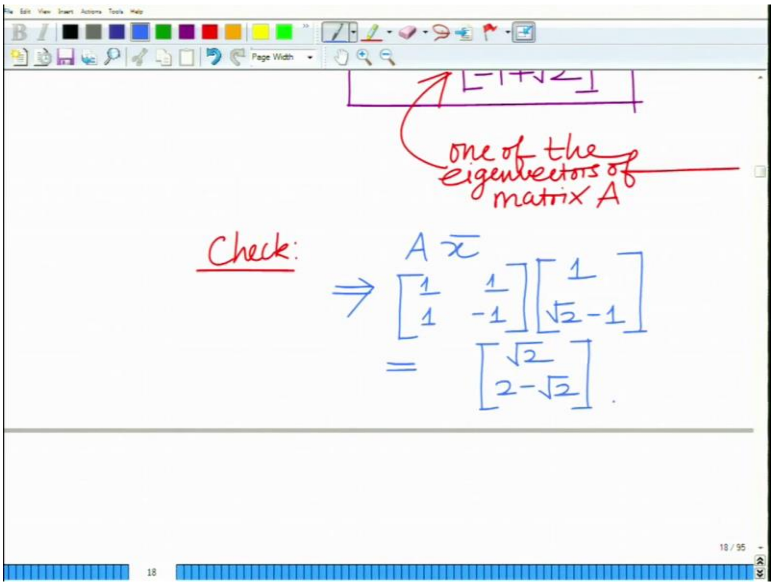

To check this

$$\begin{aligned} A\bar{x} &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{bmatrix} 1 \\ -1+\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{2} \\ 2-\sqrt{2} \end{bmatrix} \\ &= \sqrt{2}\begin{bmatrix} 1 \\ -1+\sqrt{2} \end{bmatrix} \\ &= \lambda\bar{x} \end{aligned}$$

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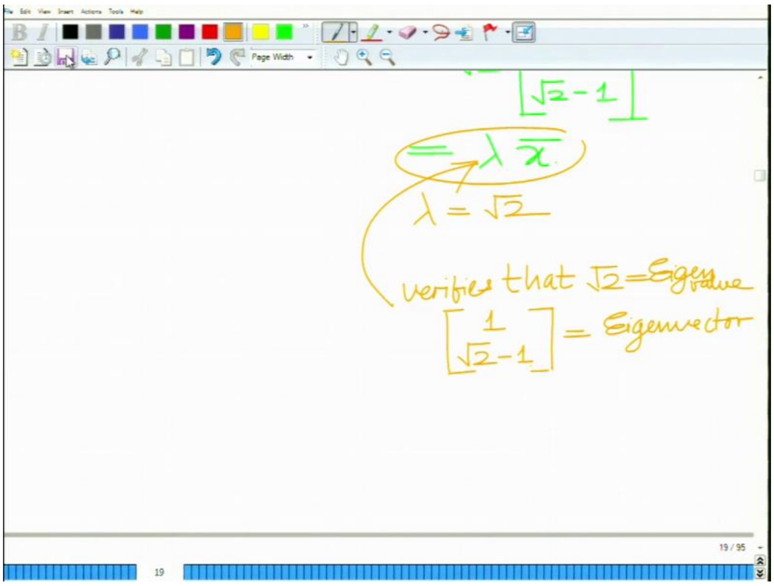

So for $\lambda = \sqrt{2},\ \bar{x} = \begin{bmatrix} 1 \\ -1+\sqrt{2} \end{bmatrix}$

So, this verifies basically both; the eigenvalue and the eigenvector of this matrix A.

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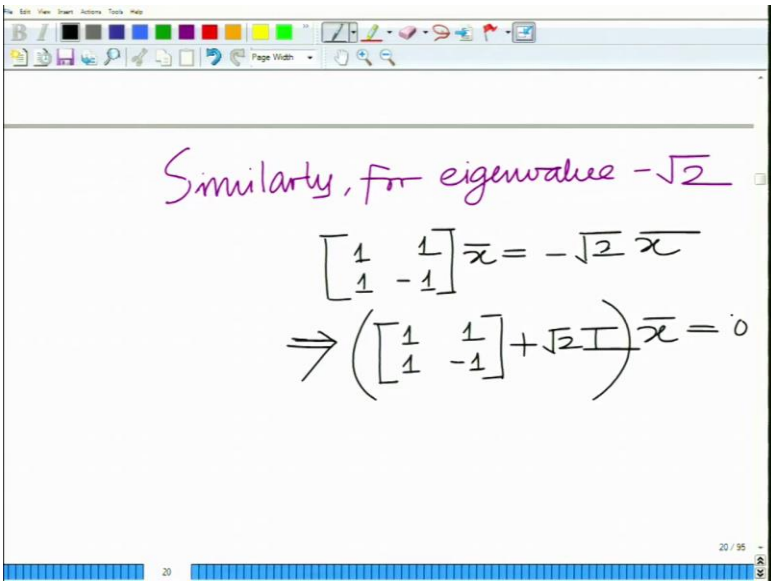

Now, similarly for $\lambda = -\sqrt{2}$;

$$A\overline{x} = \lambda I\ \overline{x}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\overline{x} = -\sqrt{2}I\ \overline{x}$$

$$\begin{pmatrix} 1+\sqrt{2} & 1 \\ 1 & -1+\sqrt{2} \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\left(1+\sqrt{2}\right)x_1 + x_2 = 0$$

And basically you can see both the equations will reduce to the same thing.

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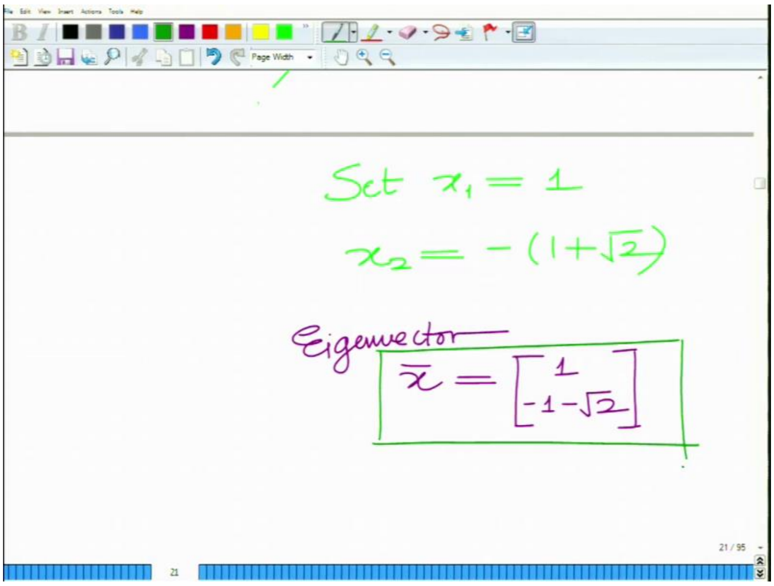

And now once again; set $x_1 = 1$

This implies $x_2 = -\left(1+\sqrt{2}\right)$

And therefore the eigenvector

$$\overline{x} = \begin{bmatrix} 1 \\ -\left(1+\sqrt{2}\right) \end{bmatrix}$$

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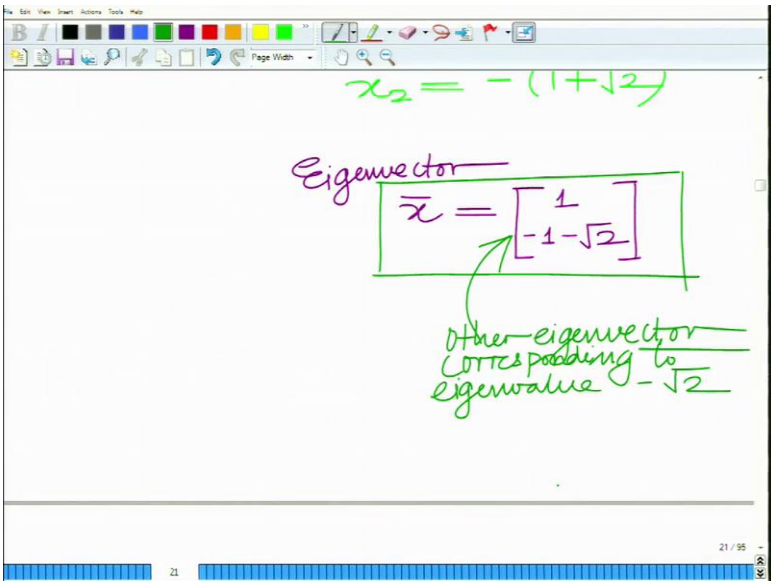

This is the other eigenvector, corresponding to the other eigenvalue $-\sqrt{2}$. This is the brief introduction to the concept of eigenvectors and eigenvalues of the matrix.

Let us look at another important concept which is symmetric and Hermitian matrices.

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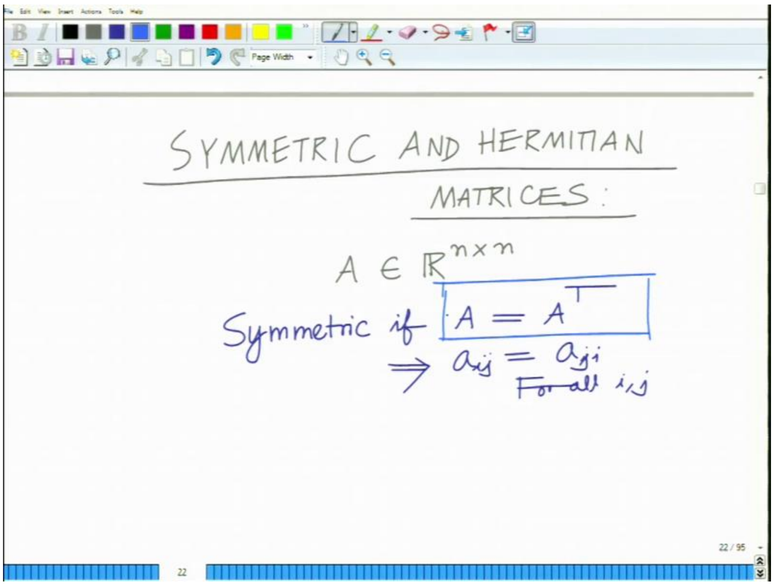

So, what we want to look at now is basically the notion of symmetric and Hermitian. Symmetric and Hermitian matrices. So, let us say A is a real $n \times n$ matrix that is

$A \in \Re^{n \times n}$

So, matrix A is symmetric, only if $A = A^T$

which implies that

$a_{ij} = a_{ji}$ for all i,j

And naturally it implies that this must be a square matrix because the symmetry is only preserved if matrix is a square matrix.

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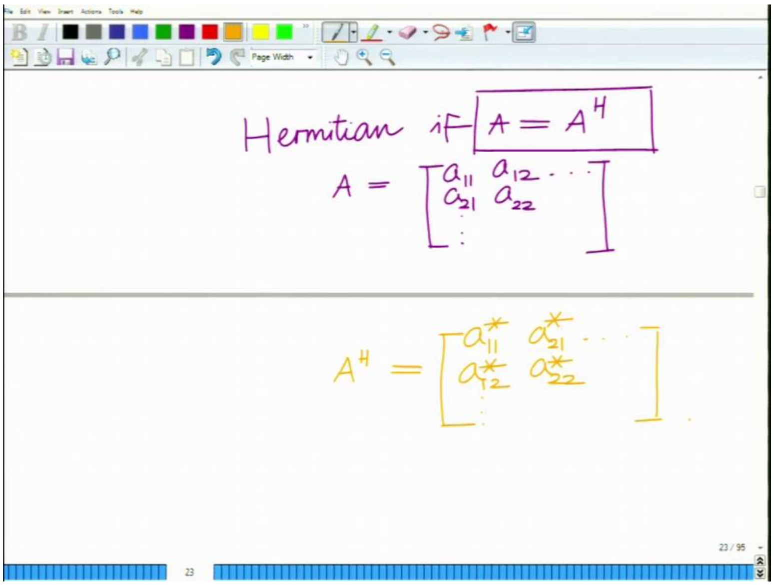

Also the matrix is a Hermitian symmetric matrix if $A = A^H$

Now, let us say A is matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

And the Hermitian matrix is

$$A^H = \begin{pmatrix} a_{11}^* & \cdots & a_{1n}^* \\ \vdots & \ddots & \vdots \\ a_{m1}^* & \cdots & a_{mn}^* \end{pmatrix}$$

Here a_{mn}^* is the transpose conjugate of a_{mn}.

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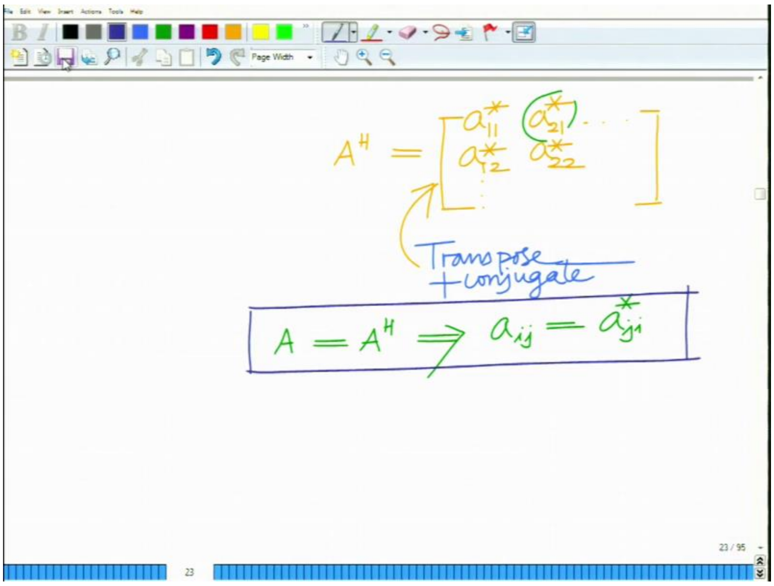

This implies that

$A = A^H \Rightarrow a_{ij} = a_{ji}^*$ for all i,j

Now, there are several interesting properties of Hermitian and symmetric matrices.

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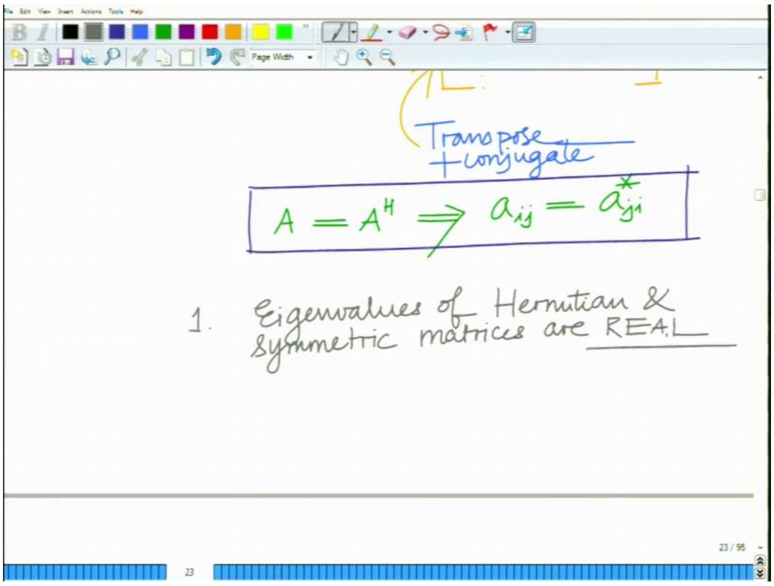

So, the first property is that the eigenvalues of Hermitian and symmetric matrices are real.

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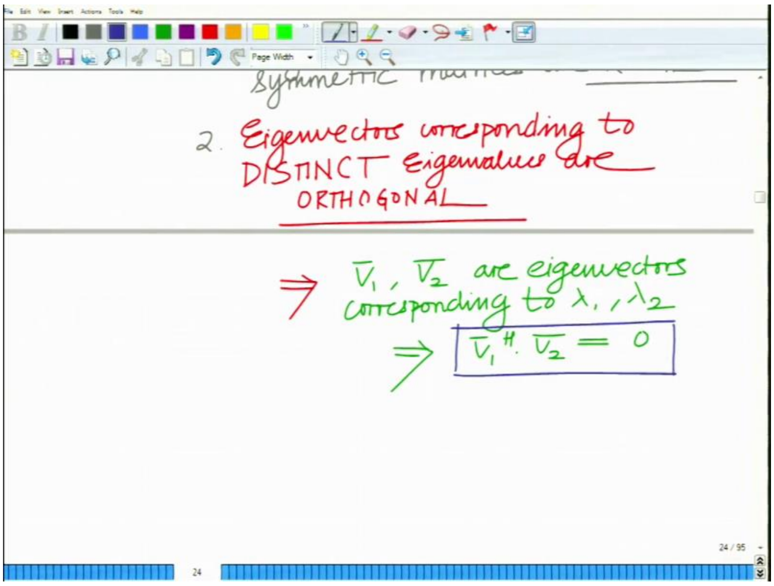

Second property is another interesting property. Eigen vectors corresponding to distinct eigenvalues are orthogonal. This implies that if $\bar{V}_1$ and $\bar{V}_2$ are the eigenvectors corresponding to distinct eigenvalues λ_1 and λ_2 then it implies that, for a symmetric matrix,

$$\bar{V}_1^H \cdot \bar{V}_2 = 0$$

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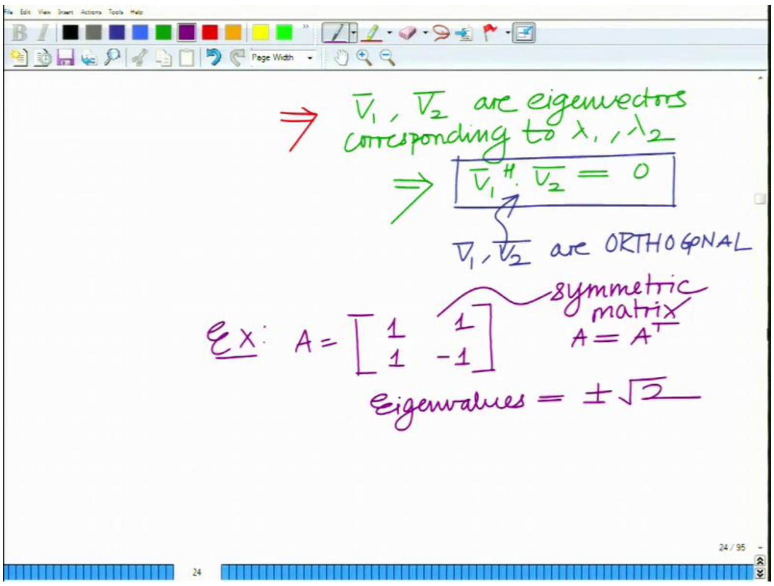

So, this is about the orthogonality of vectors. Now, let us go back to our earlier example to illustrate it. Our matrix A is

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Which is a symmetric matrix and its eigenvalues are real quantities.

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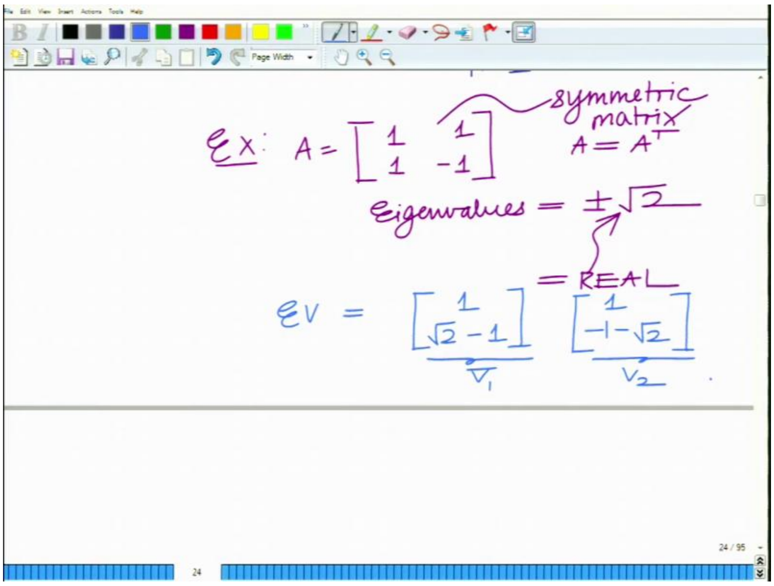

So, now let us look at the eigenvectors and we will show that the eigenvectors are orthogonal. The eigenvectors are

$$V_1 = \begin{bmatrix} 1 \\ \sqrt{2}-1 \end{bmatrix} \text{ and } V_2 = \begin{bmatrix} 1 \\ -\sqrt{2}-1 \end{bmatrix}$$

Now, since these vectors are real we can simply take its transpose.

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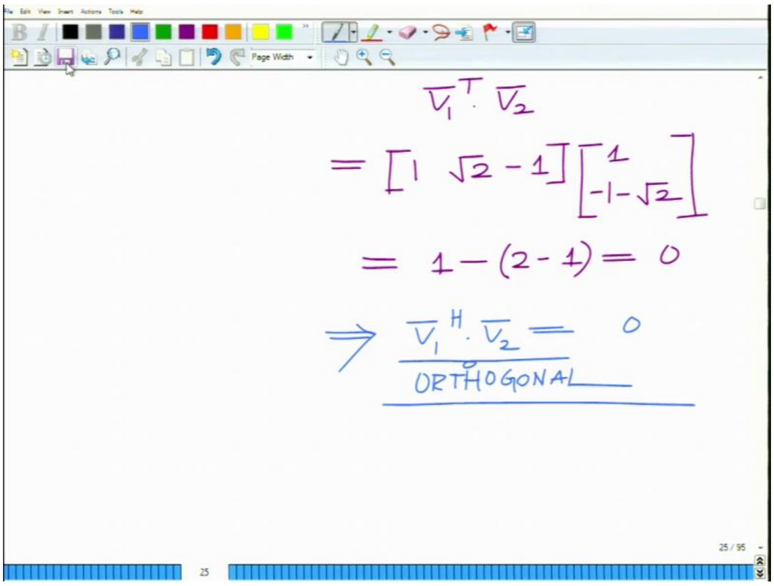

Therefore

$$\bar{V}_1^T \cdot \bar{V}_2 = \begin{bmatrix} 1 & \sqrt{2}-1 \end{bmatrix} \begin{bmatrix} 1 \\ -\sqrt{2}-1 \end{bmatrix}$$
$$= 0$$

Now, as transpose or Hermitian will give the same thing for real vectors, therefore

$$\bar{V}_1^H \cdot \bar{V}_2 = 0$$

Which means that these vectors are orthogonal.

That is a very interesting property, because the matrix is symmetric. So, in this module we have looked at various interesting and also very important concepts of eigenvalues, eigenvectors and symmetric matrices. And these are going to be used frequently in our discussion and the development of the framework of optimization for various applications.

Thank you.