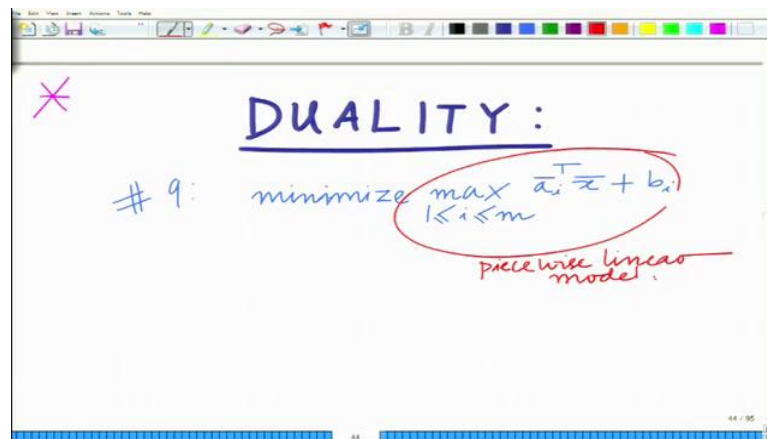


Applied Optimization for Wireless, Machine Learning, Big Data  
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Lecture - 74  
Examples on Duality: Min-Max problem, Analytic Centering

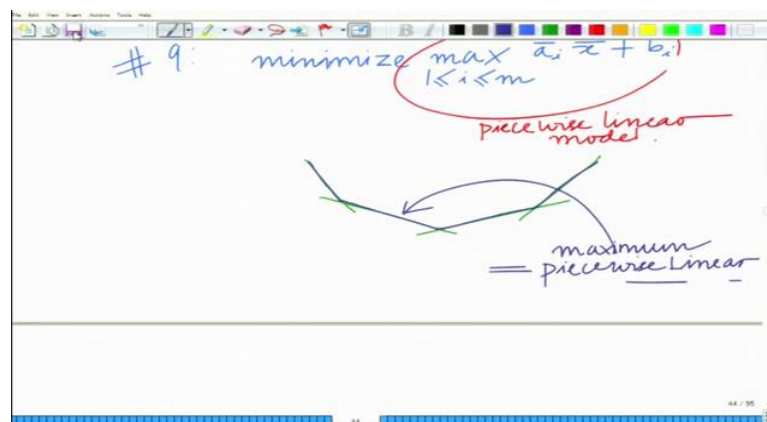
**Keywords:** Min-Max problem, Analytic Centering

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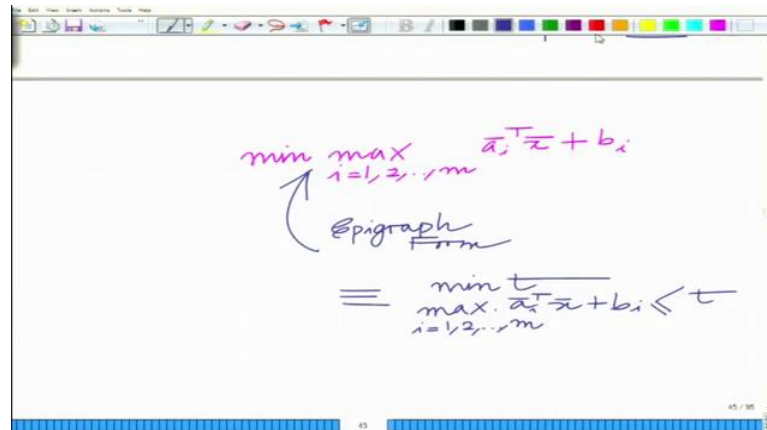
Hello, welcome to another module in this massive open online course. So we are looking at example problems in duality. Let us look at example problem number 9 where we want to find the dual of the problem, minimize  $\max_{1 \leq i \leq m} \bar{a}_i^T \bar{x} + b_i$  and this is known as a piecewise linear model.

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For instance, each of these represents a line, therefore if you look at these  $m$  different lines and you take the maximum, it will look something like as shown in slide. So this is piecewise linear and now we want to find the dual problem.

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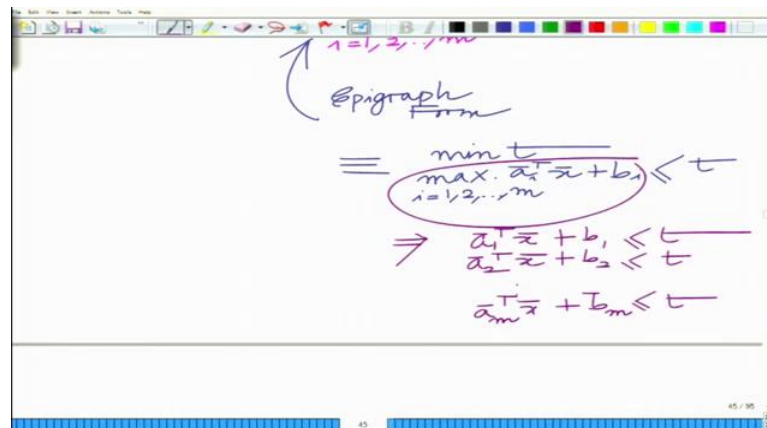
$$\min \max_{i=1,2,\dots,m} \bar{a}_i^T \bar{x} + b_i$$

Epigraph Form

$$\equiv \min_t \max_{i=1,2,\dots,m} \bar{a}_i^T \bar{x} + b_i \leq t$$

So using the epigraph form, this can be equivalently written as  $\min_t \max_{1 \leq i \leq m} \bar{a}_i^T \bar{x} + b_i \leq t$ . So this implies that each of this is less than or equal to  $t$  which is as shown in slide.

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$$\equiv \min_t \max_{i=1,2,\dots,m} \bar{a}_i^T \bar{x} + b_i \leq t$$

$$\Rightarrow \begin{aligned} \bar{a}_1^T \bar{x} + b_1 &\leq t \\ \bar{a}_2^T \bar{x} + b_2 &\leq t \\ \bar{a}_m^T \bar{x} + b_m &\leq t \end{aligned}$$

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Handwritten notes on a whiteboard showing the dual problem formulation. The text reads:

$$\equiv \begin{array}{ll} \min & t \\ \text{s.t.} & \bar{a}_i^T \bar{x} + b_i \leq t \\ & i=1, 2, \dots, m \end{array}$$

An arrow points from the text "DUAL of problem" to the above formulation. Below this, the Lagrangian function is written as:

$$L(\bar{x}, t, \bar{\lambda})$$

So I can write this basically as an equivalent optimization problem  $\min t$   
 $\text{s.t. } \bar{a}_i^T \bar{x} + b_i \leq t$ . The  
 $i=1, 2, \dots, m$

dual of this problem is obtained as follows, first you form the Lagrangian that is we have

$$L(\bar{x}, t, \bar{\lambda}) = t + \sum_{i=1}^m \lambda_i (\bar{a}_i^T \bar{x} + b_i - t), \text{ one Lagrange multiplier for each constraint.}$$

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Handwritten notes on a whiteboard showing the expansion of the Lagrangian function. The text reads:

$$\begin{aligned} L(\bar{x}, t, \bar{\lambda}) &= t + \sum_{i=1}^m \lambda_i (\bar{a}_i^T \bar{x} + b_i - t) \\ &= t + \left( \sum_{i=1}^m \lambda_i \bar{a}_i^T \right) \bar{x} \\ &\quad + \sum_{i=1}^m \lambda_i b_i - \left( \sum_{i=1}^m \lambda_i \right) t \end{aligned}$$

Now, we want to group all the terms corresponding to each and this is as shown in slide.

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$$\begin{aligned}
 & + \frac{\sum_{i=1}^m \lambda_i b_i}{\lambda^T \bar{b}} - \frac{\left(\sum_{i=1}^m \lambda_i\right)t}{I^T \bar{x}} \\
 & \bar{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{bmatrix} \quad \bar{\lambda} \geq 0 \\
 & = (1 - I^T \bar{\lambda})t + \bar{x}^T \sum_{i=1}^m \bar{a}_i \lambda_i + \bar{\lambda}^T \bar{b}
 \end{aligned}$$

Affine in  $t, \bar{x}$

And therefore, if you simplify this as shown we have

$$L(\bar{x}, t, \bar{\lambda}) = (1 - \bar{I}^T \bar{\lambda})t + \bar{x}^T \sum_{i=1}^m \bar{a}_i \lambda_i + \bar{\lambda}^T \bar{b}.$$

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$$\begin{aligned}
 & = (1 - I^T \bar{\lambda})t + \bar{x}^T \sum_{i=1}^m \bar{a}_i \lambda_i + \bar{\lambda}^T \bar{b} \\
 & \text{Affine in } t, \bar{x} \\
 & g(\bar{\lambda}) = \min L(\bar{x}, t, \bar{\lambda})
 \end{aligned}$$

Now, this is affine in  $t, \bar{x}$  which means it is a hyperplane. So now the dual is getting the minimum of with respect to  $t, \bar{x}, \bar{\lambda}$  and this is as shown.

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$$g(\lambda) = \min_{\bar{x}, t} L(\bar{x}, t, \lambda)$$

$$= \min_{\bar{x}, t} \left( (1 - \mathbf{1}^T \lambda) t + \bar{x}^T \sum_{i=1}^m \lambda_i \bar{a}_i + \bar{\lambda}^T \bar{b} \right)$$

Affine in  $t, \bar{x}$  NOT interesting

if  $1 - \mathbf{1}^T \lambda \neq 0$   
or  $\sum_{i=1}^m \lambda_i \bar{a}_i \neq 0$

$$\Rightarrow g(\lambda) = (-\infty)$$

Now we proceed as shown in slides.

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when  $1 - \mathbf{1}^T \lambda = 0$   
and  $\sum_{i=1}^m \lambda_i \bar{a}_i = 0$

$$\Rightarrow \begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \dots & \bar{a}_m \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{bmatrix} = 0$$

$$\Rightarrow A\lambda = 0$$

$$g(\lambda) = \min_{\bar{x}, t} L(\bar{x}, \lambda, \bar{t})$$

$$=$$

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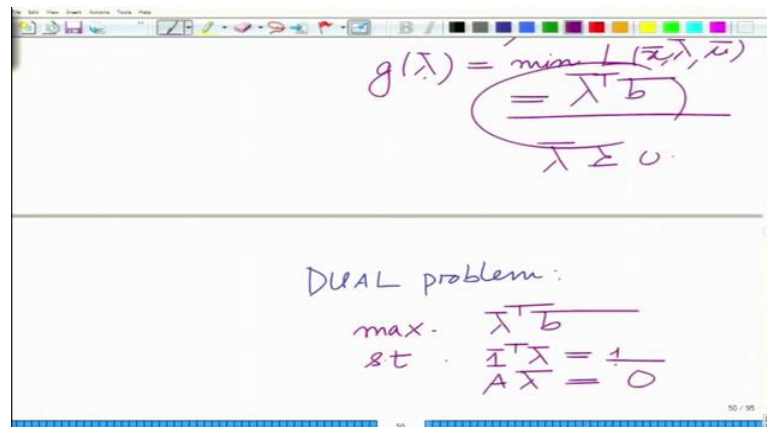
$$\Rightarrow \begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \dots & \bar{a}_m \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{bmatrix} = 0$$

$$\Rightarrow A\lambda = 0$$

$$g(\lambda) = \min_{\bar{x}, t} L(\bar{x}, \lambda, \bar{t})$$

$$= \lambda^T \bar{b}$$

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Handwritten notes on a whiteboard:

$$g(\lambda) = \min_{(x, u)} L(x, \lambda, u)$$

$$= \lambda^T b$$

$$\lambda \geq 0$$

DUAL problem:

$$\max. \lambda^T b$$

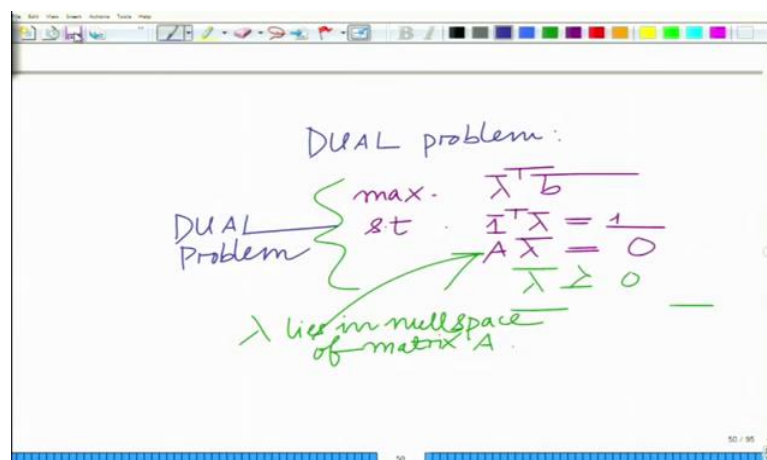
$$s.t. \quad \begin{aligned} 1^T \lambda &= 1 \\ A \lambda &= 0 \end{aligned}$$

$$\max \lambda^T b$$

And therefore the dual problem can be formulated as

$$s.t. \quad \begin{aligned} 1^T \lambda &= 1 \\ A \lambda &= 0 \end{aligned}$$

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Handwritten notes on a whiteboard:

DUAL problem:

$$\max. \lambda^T b$$

$$s.t. \quad \begin{aligned} 1^T \lambda &= 1 \\ A \lambda &= 0 \\ \lambda &\geq 0 \end{aligned}$$

DUAL Problem {

$\lambda$  lies in nullspace of matrix A.

This is the dual problem for the given original min max problem.

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# 11: Derive dual of

$$\min - \sum_{i=1}^m \log(b_i - \bar{a}_i^T \bar{x})$$

Analytic centering

Domain is  $b_i \geq \bar{a}_i^T \bar{x}$   
 $\Rightarrow \bar{a}_i^T \bar{x} \leq b_i$   
 $i=1, 2, \dots, m$

Let us look at another interesting application, we want to derive the dual of  $\min - \sum_{i=1}^m \log(b_i - \bar{a}_i^T \bar{x})$ . This problem is termed as analytic centering problem. So the domain of this is  $b_i \geq \bar{a}_i^T \bar{x}$ . So this is an intersection of half spaces and this is a polyhedron.

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$y_i = b_i - \bar{a}_i^T \bar{x}$

Polyhedron

$$\equiv \min - \sum_{i=1}^m \log y_i$$

So to develop the dual we will use a simple substitution. We will substitute  $y_i = b_i - \bar{a}_i^T \bar{x}$ .

So the optimization problem can be equivalently written as  $\min - \sum_{i=1}^m \log y_i$ .

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$$\equiv \min - \sum_{i=1}^m \log y_i$$

$$\text{s.t. } \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} - \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix} x$$

$$\bar{y} = \bar{b} - A^T \bar{x}$$

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix}$$

And then we proceed as shown in slides below.

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$$\equiv \min - \sum_{i=1}^m \log y_i$$

$$\bar{y} = \bar{b} - A^T \bar{x}$$

$$L(\bar{x}, \bar{y}, \bar{v})$$

$$= - \sum_{i=1}^m \log y_i + \sum_{i=1}^m v_i (y_i - b_i - a_i^T \bar{x})$$

So we have  $L(\bar{x}, \bar{y}, \bar{v}) = - \sum_{i=1}^m \log y_i + \sum_{i=1}^m v_i (y_i - b_i - a_i^T \bar{x})$ .



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$$= \sum_{i=1}^m (-\log y_i + \lambda_i y_i) + \left( \sum_{i=1}^m \lambda_i a_i^T \right) \bar{x} - \sum_{i=1}^m \lambda_i b_i$$

Now, once again collecting all the terms we proceed as shown in slides. And now we have to take the infimum with respect to the primal variables  $\bar{x}, \bar{y}$ .

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$$g(\bar{\lambda}) = \min_{\bar{x}, \bar{y}} L(\bar{x}, \bar{y}, \bar{\lambda})$$

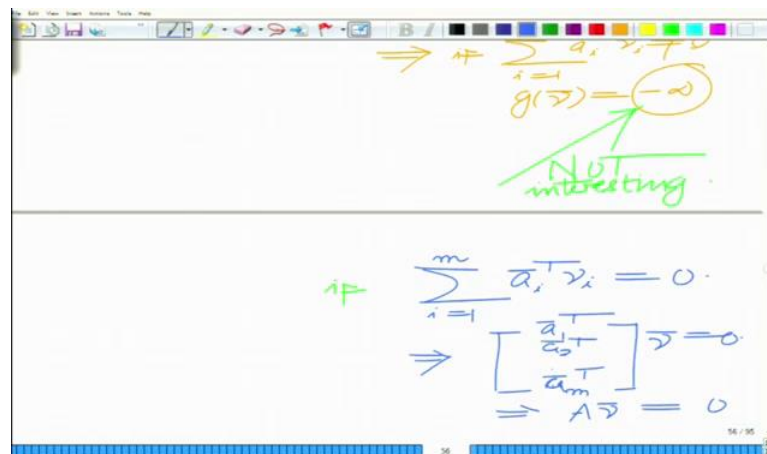
$$= \min \sum_{i=1}^m (-\log y_i + \lambda_i y_i)$$

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$$g(\bar{\lambda}) = \min_{\bar{x}, \bar{y}} \sum_{i=1}^m (-\log y_i + \lambda_i y_i) + \left( \sum_{i=1}^m a_i^T \lambda_i \right) \bar{x} - \bar{\lambda}^T \bar{b}$$

Affine in  $\bar{x}$   
 $\Rightarrow$  if  $\sum_{i=1}^m a_i^T \lambda_i \neq 0$   
 $g(\bar{\lambda}) = (-\infty)$

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Handwritten notes on a whiteboard:

$$\Rightarrow \sum_{i=1}^m a_i^T x_i = 0$$

$$g(x) = -\infty$$

NOT interesting.

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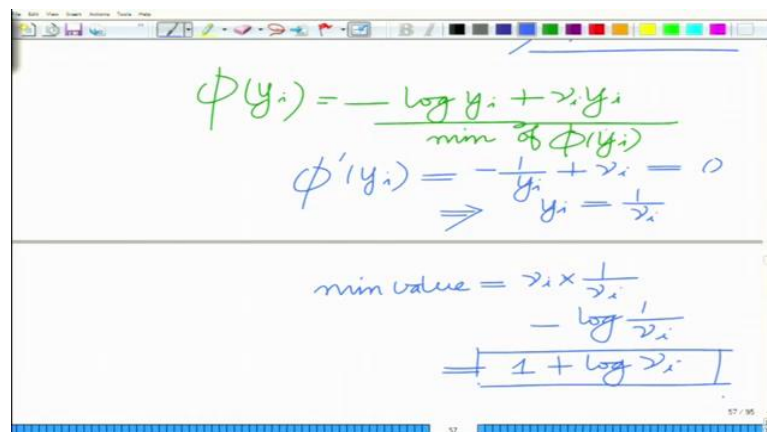
Handwritten notes on a whiteboard:

$$\sum_{i=1}^m a_i^T x_i = 0$$

$$\Rightarrow \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix} x = 0$$

$$\Rightarrow Ax = 0$$

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Handwritten notes on a whiteboard:

$$\phi(y_i) = -\log y_i + x_i y_i$$

min of  $\phi(y_i)$

$$\phi'(y_i) = -\frac{1}{y_i} + x_i = 0$$

$$\Rightarrow y_i = \frac{1}{x_i}$$


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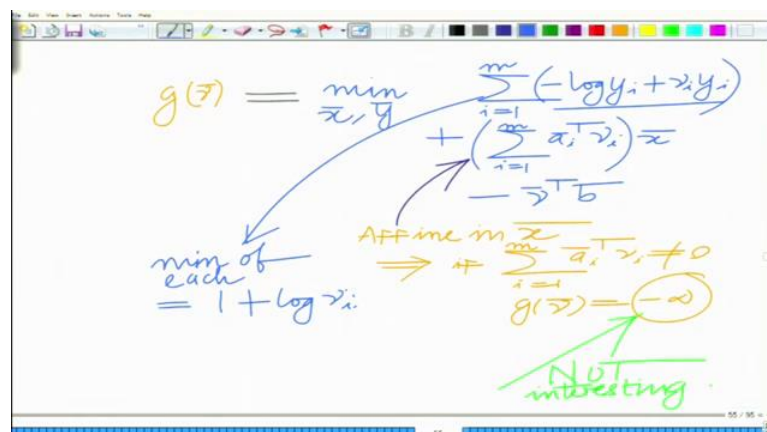
Handwritten notes on a whiteboard:

$$\text{min value} = x_i \times \frac{1}{x_i} - \log \frac{1}{x_i}$$

$$= 1 + \log x_i$$

Now, we proceed as shown in the slides.

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Handwritten notes on a whiteboard:

$$g(x) = \min_{x, y} \left( \sum_{i=1}^m (-\log y_i + x_i y_i) + \left( \sum_{i=1}^m a_i^T x_i \right) x - x^T b \right)$$

min of each =  $1 + \log x_i$

Affine in  $x$

$$\Rightarrow \sum_{i=1}^m a_i^T x_i \neq 0$$

$$g(x) = -\infty$$

NOT interesting.

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$$\begin{aligned}
 g(\vec{v}) &= \min L(\vec{x}, \vec{y}, \vec{v}) \\
 &= \sum_{i=1}^m \frac{\log v_i + 1}{-b^T \vec{v}} \\
 &= m + \sum_{i=1}^m \log v_i - b^T \vec{v} \\
 &\quad \text{s.t. } A\vec{v} = 0
 \end{aligned}$$

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DUAL Problem

$$\begin{aligned}
 \max. \quad & m + \sum_{i=1}^m \log v_i - b^T \vec{v} \\
 \text{s.t.} \quad & A\vec{v} = 0
 \end{aligned}$$

$\vec{v}$  lies in NULL space of matrix A.

And therefore the dual problem of this analytical centering is

$$\begin{aligned}
 \max \quad & m + \sum_{i=1}^m \log v_i - b^T \vec{v} \\
 \text{s.t.} \quad & A\vec{v} = 0
 \end{aligned}$$

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$\text{s.t. } A\vec{v} = 0$

$\vec{v}$  lies in NULL space of matrix A.

$$\begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix}$$

$$\begin{aligned}
 A\vec{v} &= 0 \\
 \Rightarrow a_i^T \vec{v} &= 0 \\
 \Rightarrow \vec{v} &\text{ is orthogonal to each } a_i
 \end{aligned}$$

So these are some examples of various convex optimization problems and how to formulate their dual problem which often yields very useful insights and these are often very useful in practice. So let us stop here and continue in the subsequent modules. Thank you very much.