Applied Optimization for Wireless, Machine Learning Big Data

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Lecture - 56
Practical Application

***Keywords**: Compressive Sensing, Sparsity*

Hello welcome to another module in this Massive Open Online Course. So we are looking at compressive sensing where we try to compress during the sensing process itself by making much fewer number of measurements in comparison to the dimension of the signal and then later try to reconstruct the signal from the very few measurements made.

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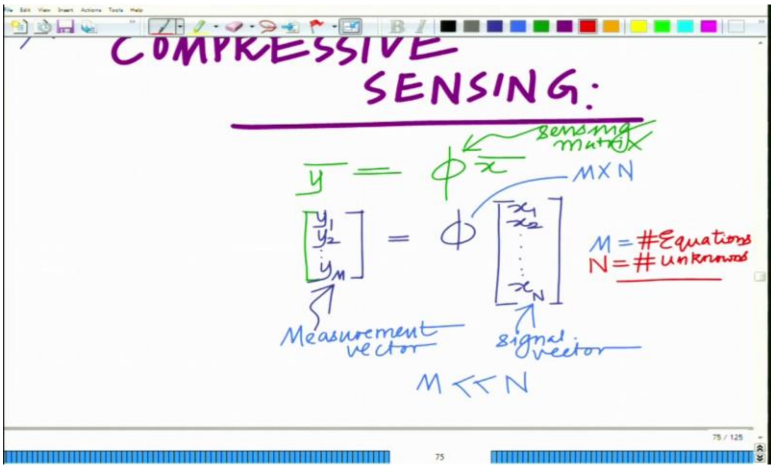

So we have a measurement vector $\bar{y} = \phi \bar{x}$ and there are M measurements as shown in slide. The sensing matrix is $M \times N$ and we make significantly fewer measurements that is $M \ll N$. Now if you view this as a system of equation then we have M number of equations and N number of unknowns. So simple linear algebra tells us that one cannot reconstruct the vector $\bar{x}$ of length N from M equations, since the number of equations is much lower than the number of unknowns. So this is an underdetermined system. Therefore, this sensing system has to satisfy certain special properties in order to recover $\bar{x}$.

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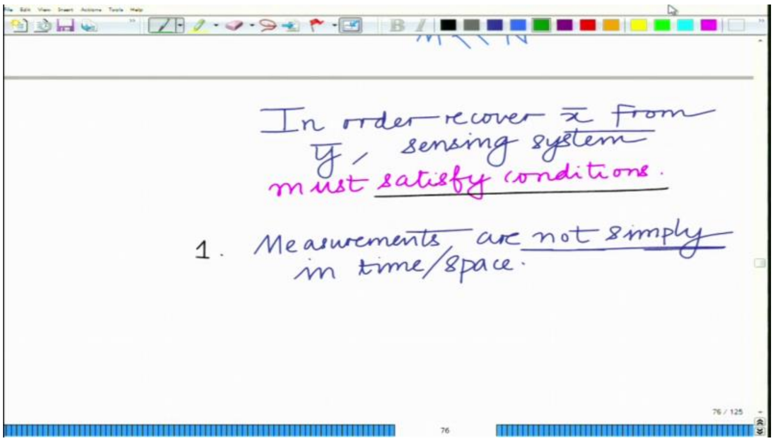

Now the first condition states that the measurements are not simply in the time or space. Rather, they have to employ noise like or one can say pseudo noise like waveform.

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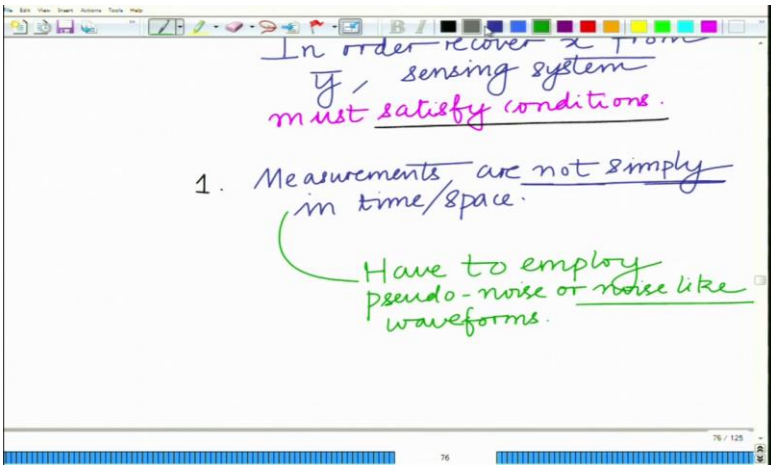

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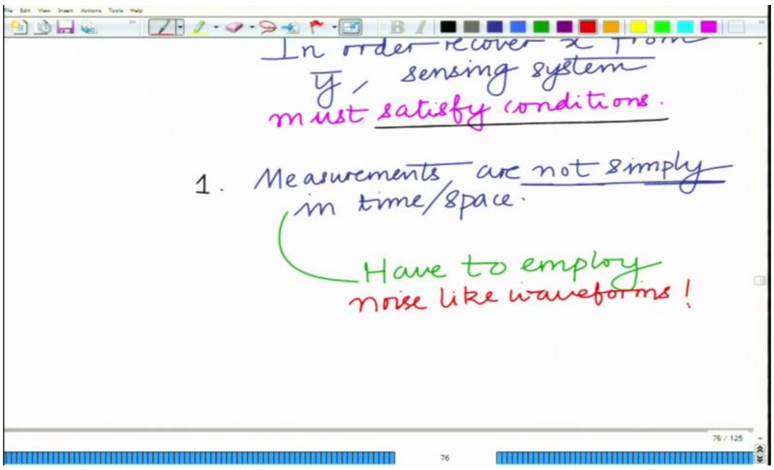

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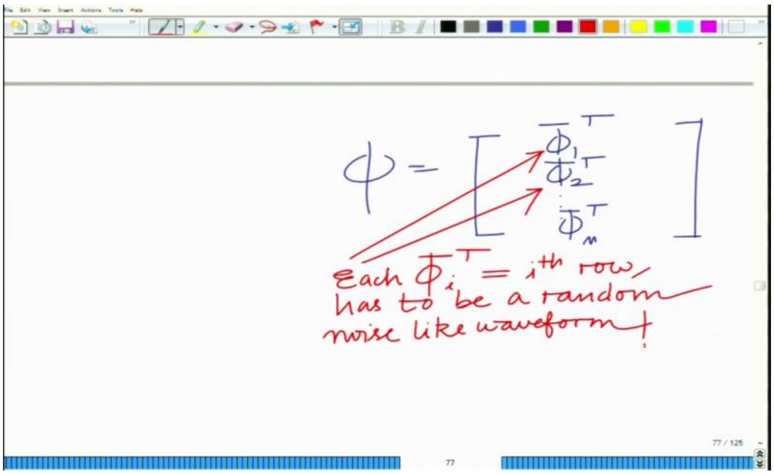

If you look at each row of the sensing matrix which we are denoting by $\overline{\phi_1}^T, \overline{\phi_2}^T, \overline{\phi_M}^T$ this has to be a noise like waveform which means that it has to be something very random, either can be a random sequence of - 1, 1 or it has to be some random noise like waveform such as Gaussian noise.

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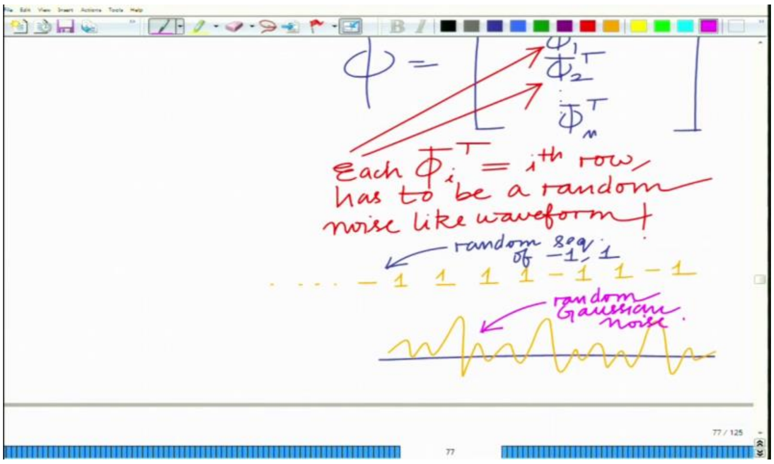

So these rows have to look like independent realization of the noise waveforms. And when you are making the measurement you are taking the projections of the signal on this noise like waveform.

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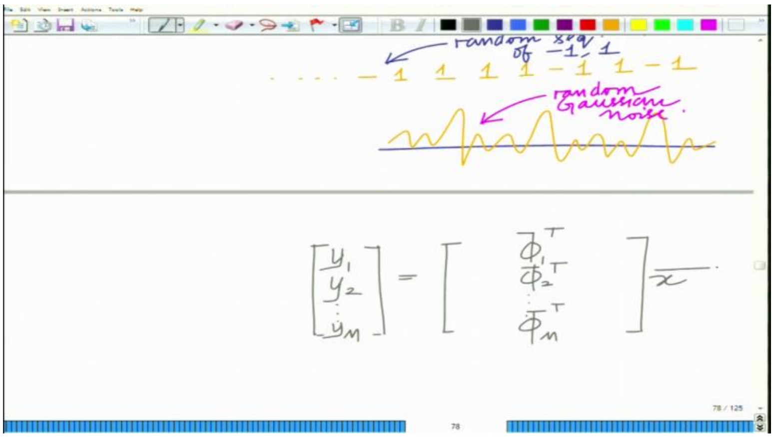

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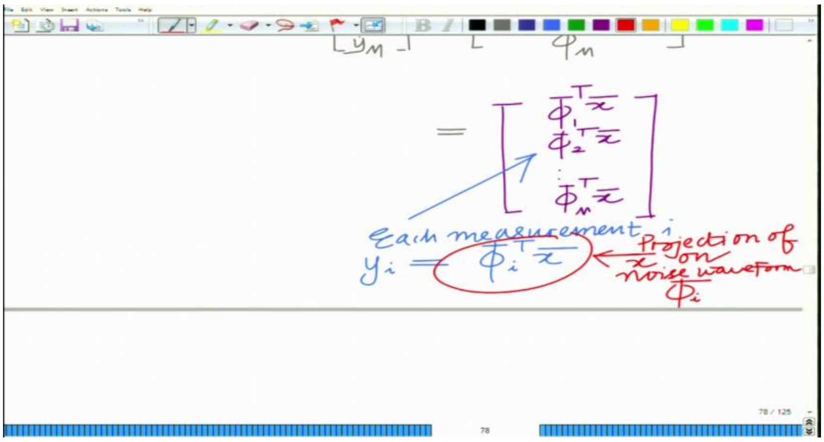

So each $\bar{\phi}_i^T \bar{x}$ is a projection of $\bar{x}$ and we are taking the linear combination of $\bar{x}$ using this noise like waveform.

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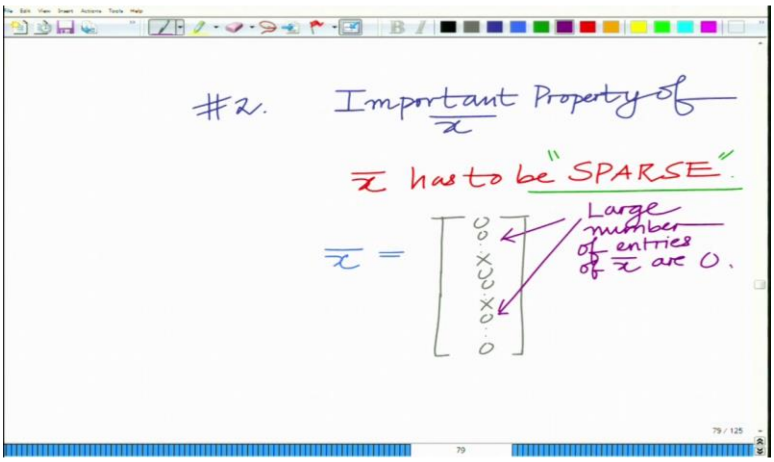

Now the vector $\bar{x}$ itself has to satisfy an important property that is $\bar{x}$ has to be sparse and this is a very important property. Now $\bar{x}$ is sparse implies that a large number of entries of $\bar{x}$ are 0 and only very few entries are non-zero.

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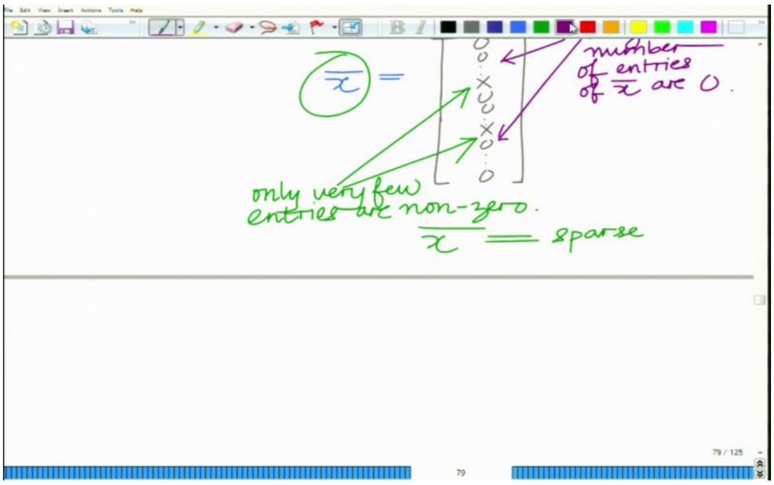

So $\bar{x}$ is sparse and typically $\bar{x} = \varphi \bar{\alpha}$.

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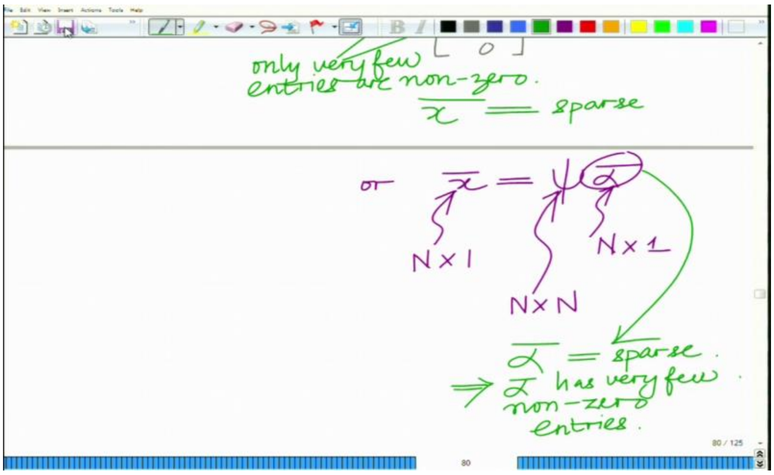

So $\bar{x}$ is N × 1, this $\bar{\alpha}$ is N × 1 and φ is an N × N basis such that $\bar{\alpha}$ is a sparse vector. For instance, we take an image and if you look at the wavelet coefficients of an image then they are sparse.

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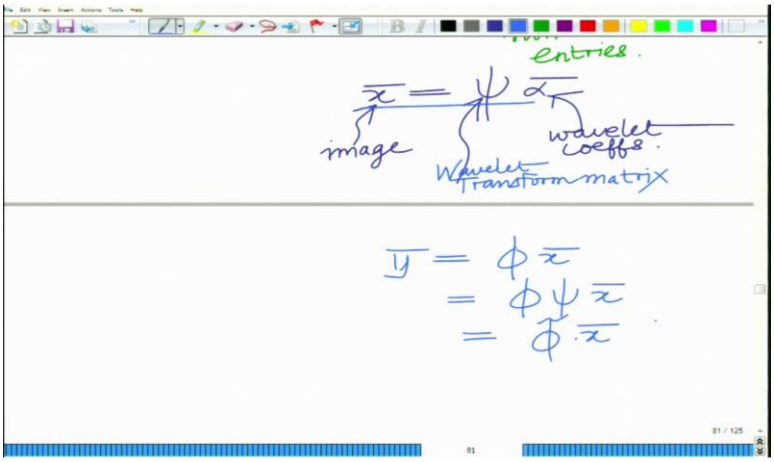

So $\bar{x}$ is sparse can be expressed in terms of $\bar{\alpha}$ which is sparse and therefore, now if you substitute this, the sensing model becomes $\bar{y} = \phi \bar{x} = \phi \varphi \bar{\alpha} = \phi \bar{\alpha}$.

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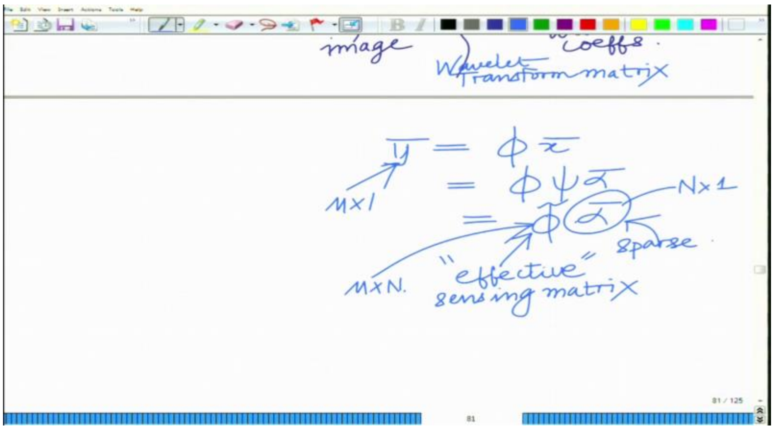

Now ϕ becomes your effective sensing matrix. Now, it is as if you are trying to sense the vector $\bar{\alpha}$ which is in the wavelet domain. Now once you get the wavelet coefficients you can reconstruct the image because image and wavelet have a 1 to 1 correspondence. But the wavelet coefficient is sparse and that is very amenable to compressive sensing.

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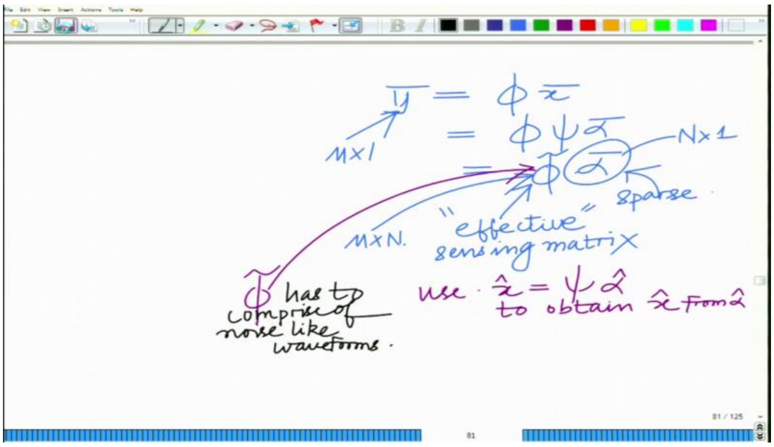

So once you get $\bar{\alpha}$ use $x = \varphi \alpha$ to obtain the estimate x from α and this ϕ matrix has to contain noise like waveforms.

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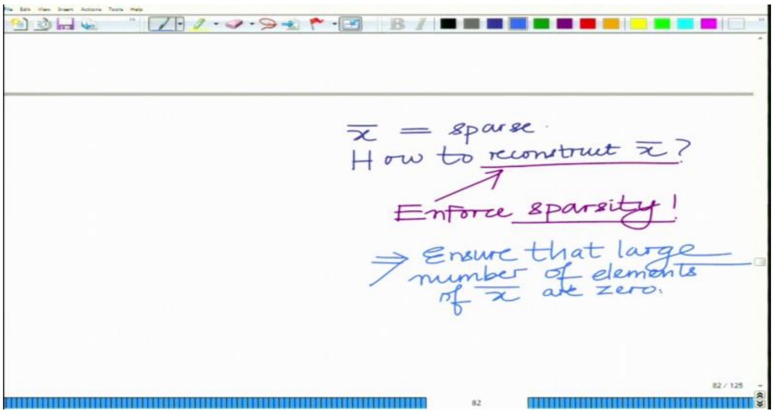

Now to reconstruct $\bar{x}$ enforce sparsity which implies that we have to ensure that the reconstructed vector $\bar{x}$ is such that a large number of elements are 0's and only some elements are non-zero. This is precisely what we call the l_0 norm that is if you denote the l_0 norm of a vector this equals the number of non-zero elements of $\bar{x}$.

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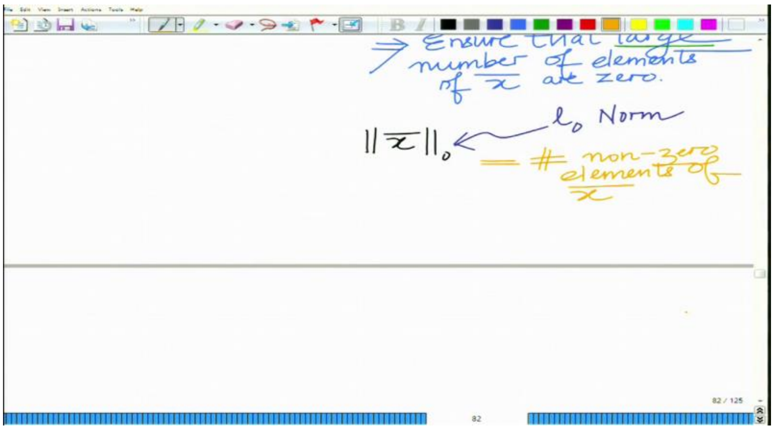

So we want to minimize the number of non-zero elements of $\bar{x}$ which is l_0 norm.

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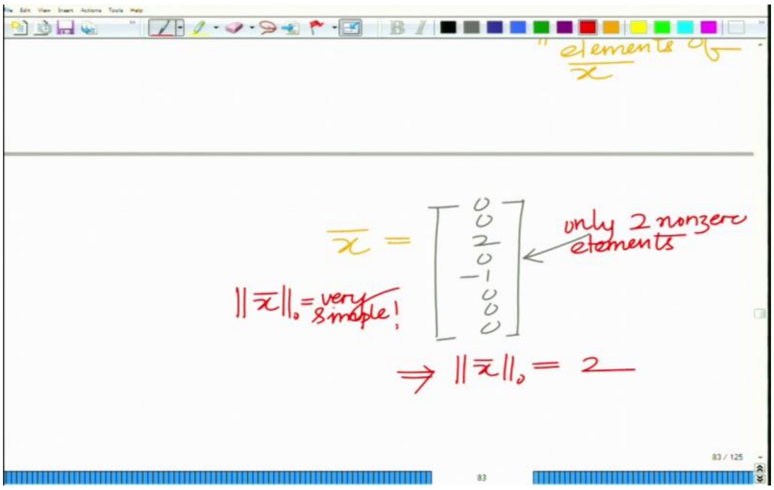

For instance, let us say you have the vector $\bar{x}$ as shown in slide and there are 8 elements, but only 2 non-zero elements, which implies the l_0 norm of $\bar{x}$ is 2.

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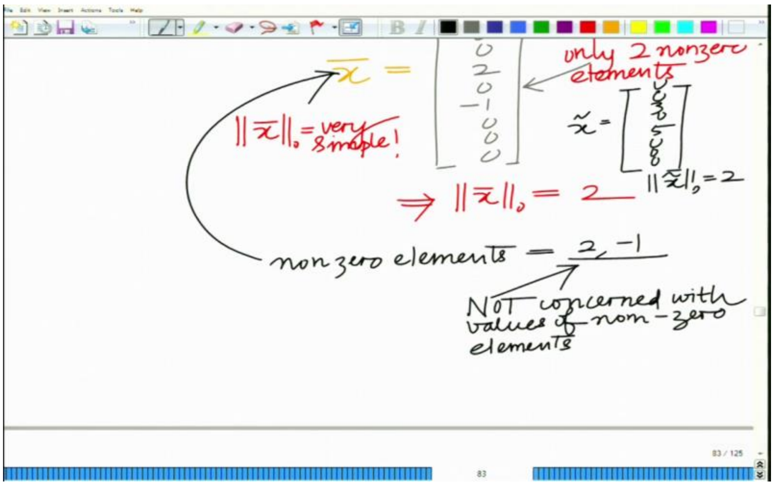

We are not concerned with the values of the non-zero elements we just have to consider the number of non-zero elements.

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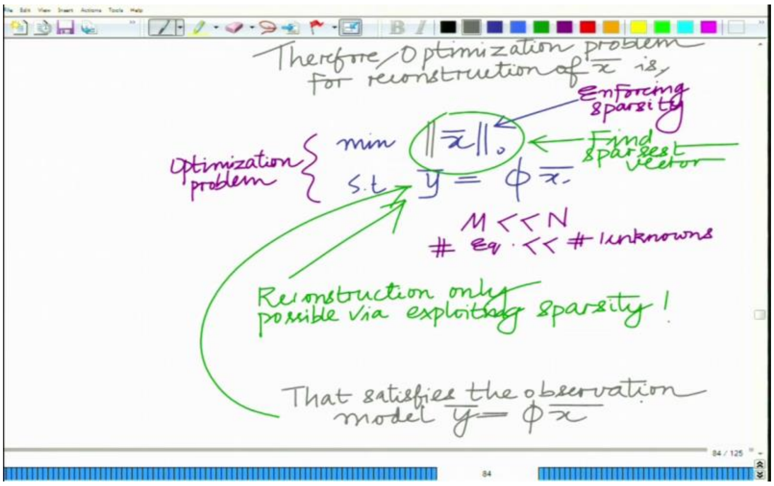

Therefore, the optimization problem for reconstruction of $\bar{x}$ can be given as $\begin{matrix} \min \|\bar{x}\|_0 \\ s.t\ \bar{y} = \phi \bar{x} \end{matrix}$.

And we need this because $M \ll N$ and therefore, one has to exploit sparsity. So you are trying to find the sparsest vector which satisfies this observation model. Now the problem with this optimization problem is that not only the objective is non-differentiable, this optimization problem is highly non convex.

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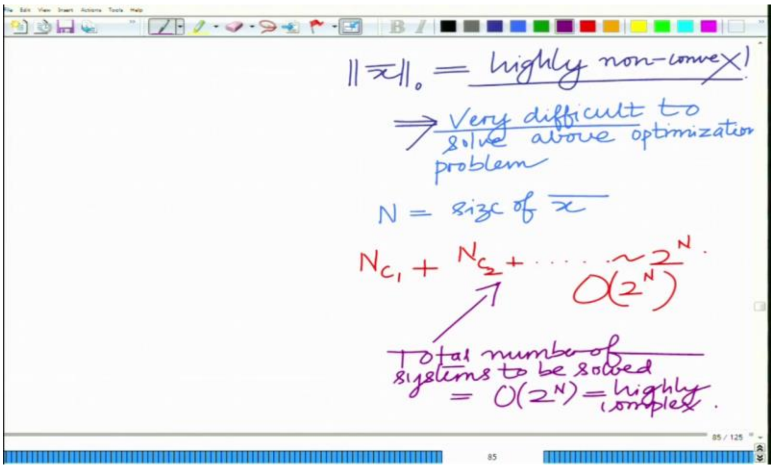

So this implies it is very difficult to solve the optimization problem. So the point is that although it is very simple to state the optimization problem it is an extremely complicated one.

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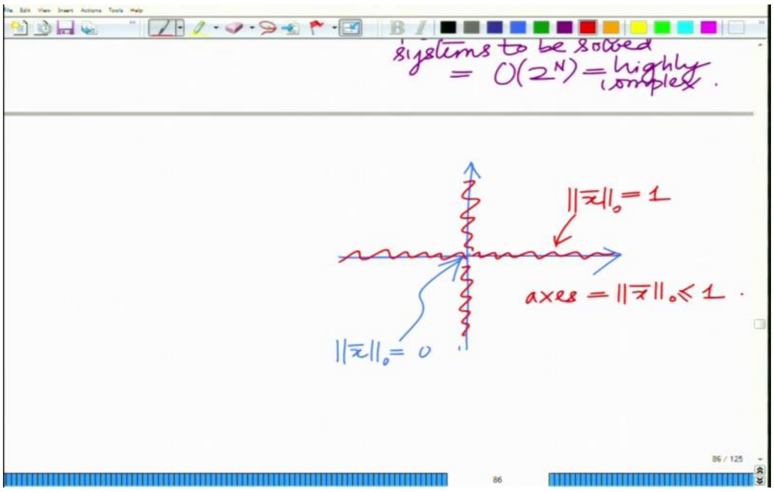

Now, if you look at l_0 norm $\left\|\bar{x}\right\|_0 \leq 1$ that comprises only of the axis, so it is highly non convex.

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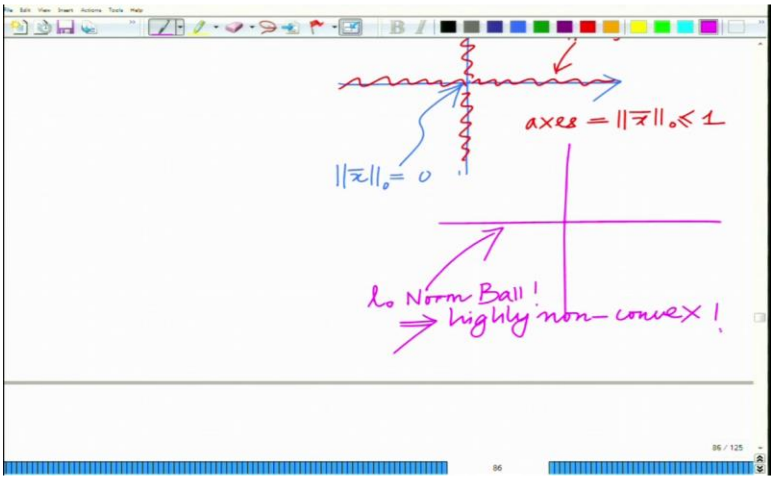

Therefore, one has to come up with other intelligent techniques to solve this optimization problem and that forms the basis for compressive sensing. So let us stop here and continue in the subsequent modules. Thank you very much.