Applied Optimization for Wireless, Machine Learning, Big Data

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Lecture - 74
Examples on Duality: Min-Max problem, Analytic Centering

***Keywords**: Min-Max problem, Analytic Centering*

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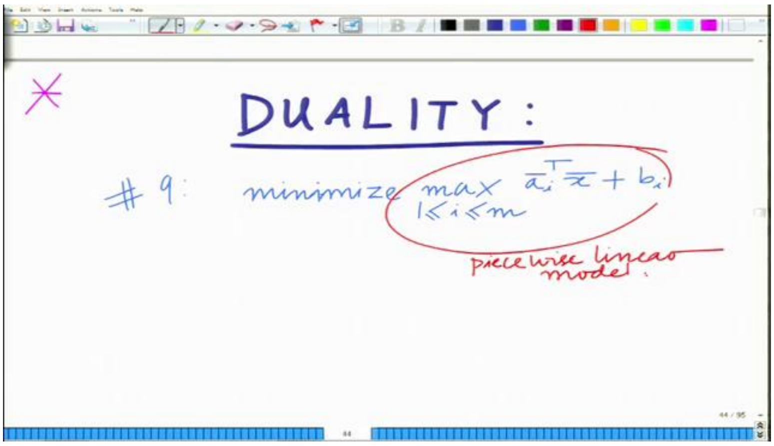

Hello, welcome to another module in this massive open online course. So we are looking at example problems in duality. Let us look at example problem number 9 where we want to find the dual of the problem, minimize $\max_{1 \le i \le m} \bar{a}_i^T \bar{x} + b_i$ and this is known as a piecewise linear model.

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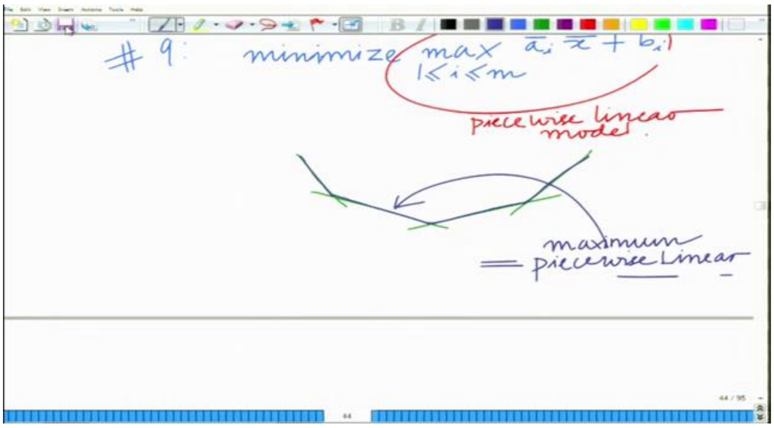

For instance, each of these represents a line, therefore if you look at these m different lines and you take the maximum, it will look something like as shown in slide. So this is piecewise linear and now we want to find the dual problem.

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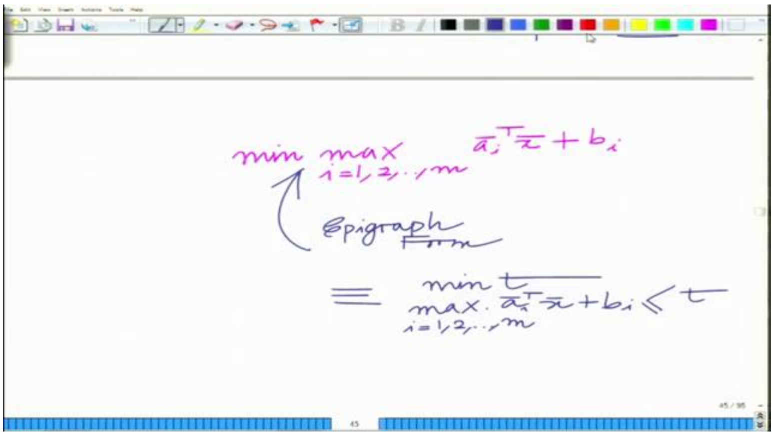

So using the epigraph form, this can be equivalently written as $\begin{array}{l} \min t \\ \max\limits_{1 \le i \le m} \bar{a}_i^T \bar{x} + b_i \le t \end{array}$. So this implies that each of this is less than or equal to t which is as shown in slide.

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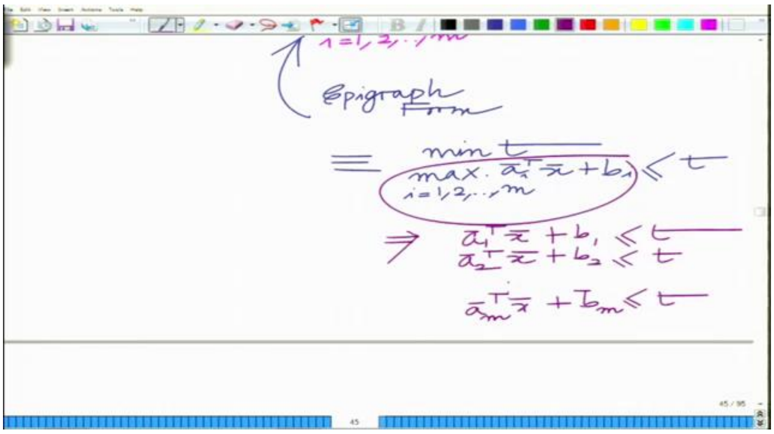

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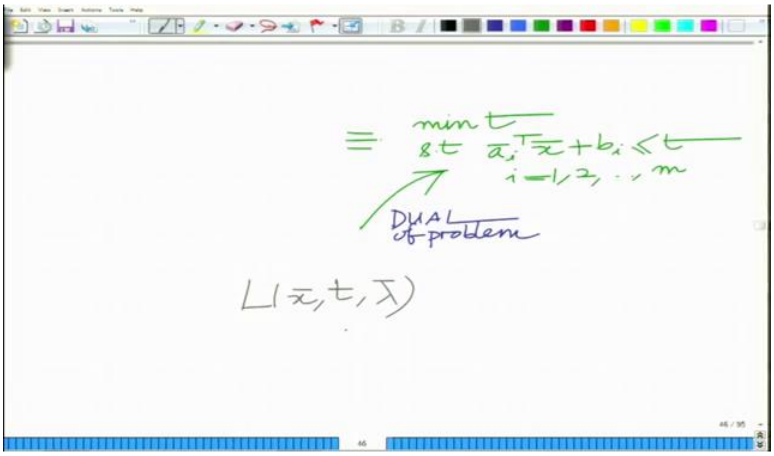

So I can write this basically as an equivalent optimization problem $\begin{array}{l} \min t \\ s.t\ \bar{a}_i^T \bar{x} + b_i \leq t \\ \quad i = 1, 2, \ldots, m \end{array}$. The dual of this problem is obtained as follows, first you form the Lagrangian that is we have $L\left(\bar{x}, t, \bar{\lambda}\right) = t + \sum_{i=1}^{m} \lambda_i \left(\bar{a}_i^T \bar{x} + b_i - t\right)$, one Lagrange multiplier for each constraint.

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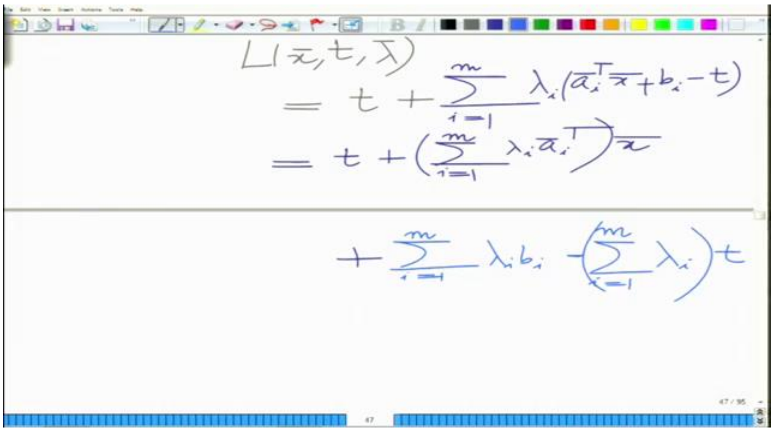

Now, we want to group all the terms corresponding to each and this is as shown in slide.

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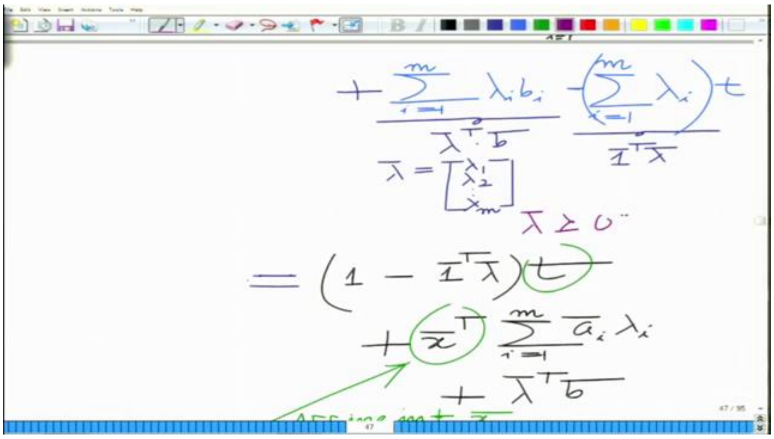

And therefore, if you simplify this as shown we have $L\left(\bar{x}, t, \bar{\lambda}\right) = \left(1 - \bar{1}^T \bar{\lambda}\right) t + \bar{x}^T \sum_{i=1}^{m} \bar{a}_i \lambda_i + \bar{\lambda}^T \bar{b}$.

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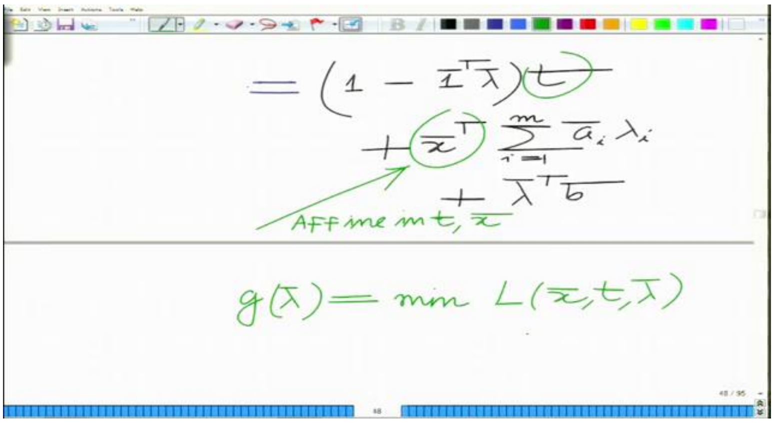

Now, this is affine in t, $\bar{x}$ which means it is a hyperplane. So now the dual is getting the minimum of with respect to t, $\bar{x}$, $\bar{\lambda}$ and this is as shown.

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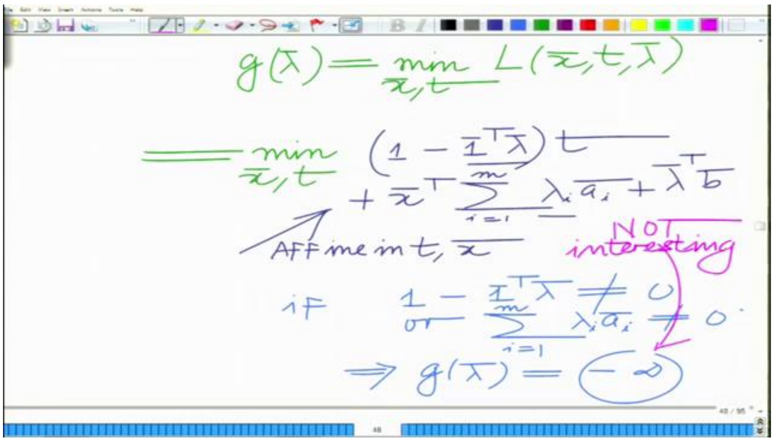

Now we proceed as shown in slides.

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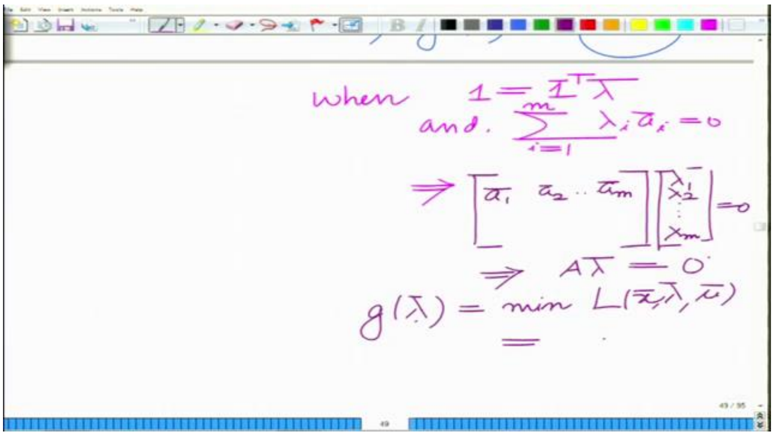

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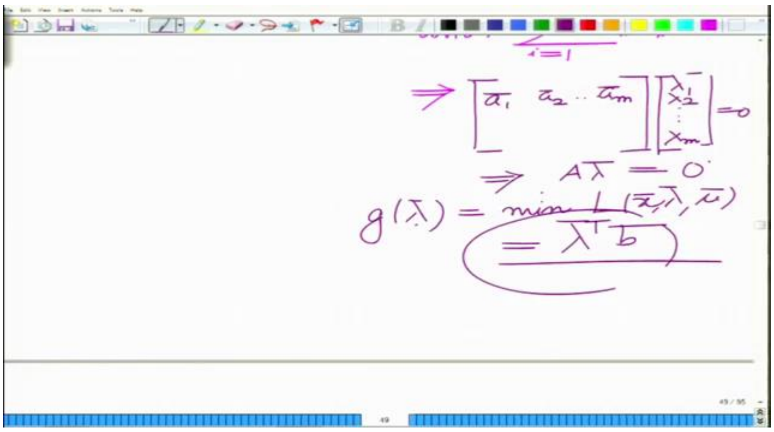

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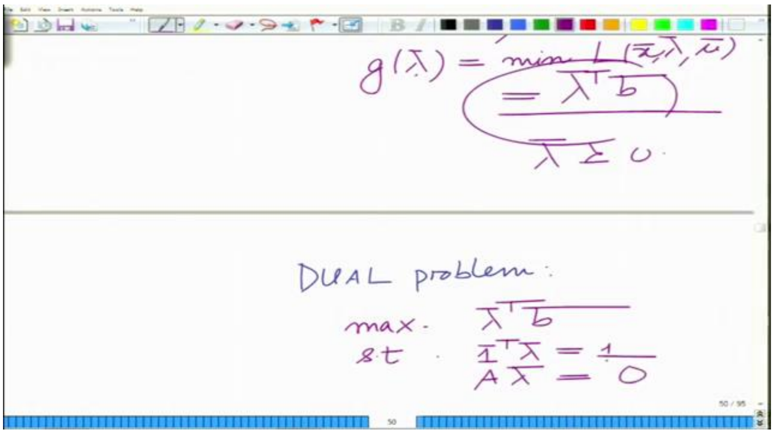

And therefore the dual problem can be formulated as $\begin{array}{l} \max \bar{\lambda}^T \bar{b} \\ s.t\, \bar{1}^T \bar{\lambda} = 1 \\ A\bar{\lambda} = 0 \end{array}$.

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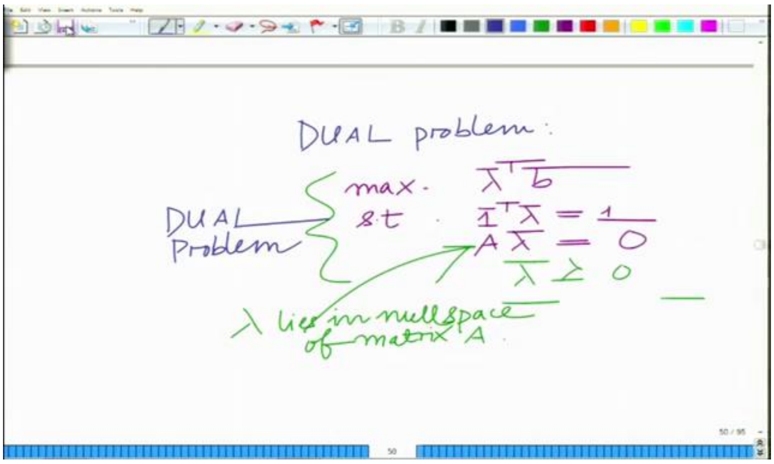

This is the dual problem for the given original min max problem.

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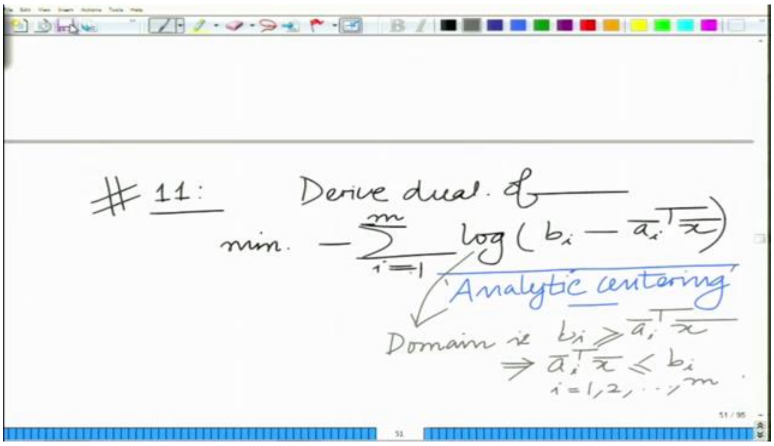

Let us look at another interesting application, we want to derive the dual of $\min - \sum_{i=1}^{m} \log\left(b_i - \bar{a}_i^T \bar{x}\right)$. This problem is termed as analytic centering problem. So the domain of this is $b_i \geq \bar{a}_i^T \bar{x}$. So this is an intersection of half spaces and this is a polyhedron.

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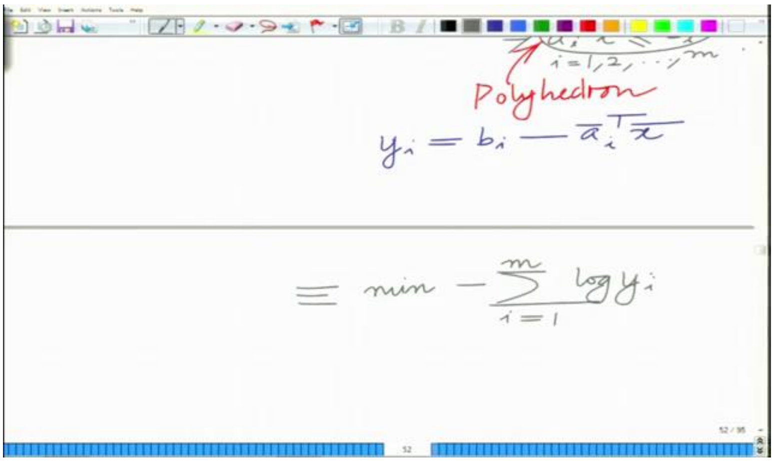

So to develop the dual we will use a simple substitution. We will substitute $y_i = b_i - \bar{a}_i^T \bar{x}$.

So the optimization problem can be equivalently written as $\min - \sum_{i=1}^{m} \log y_i$.

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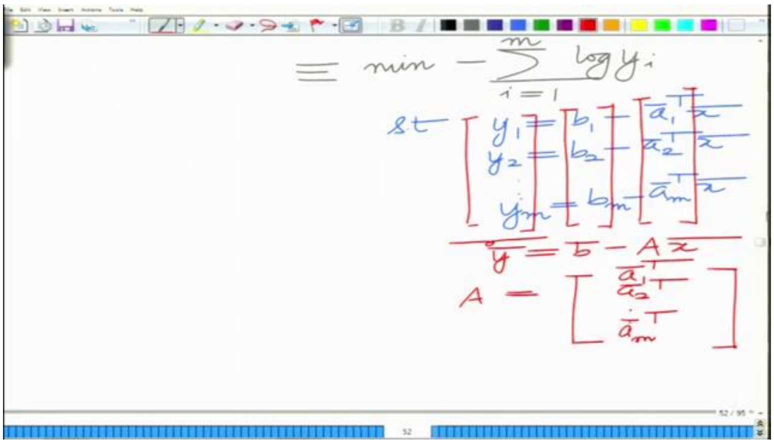

And then we proceed as shown in slides below.

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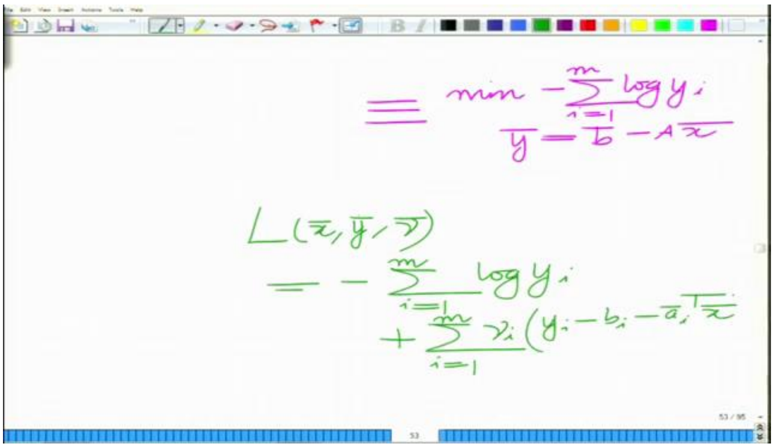

So we have $L\left(\bar{x}, \bar{y}, \bar{\nu}\right) = -\sum_{i=1}^{m} \log y_i + \sum_{i=1}^{m} \nu_i \left(y_i - b_i - \bar{a}_i^T \bar{x}\right)$.

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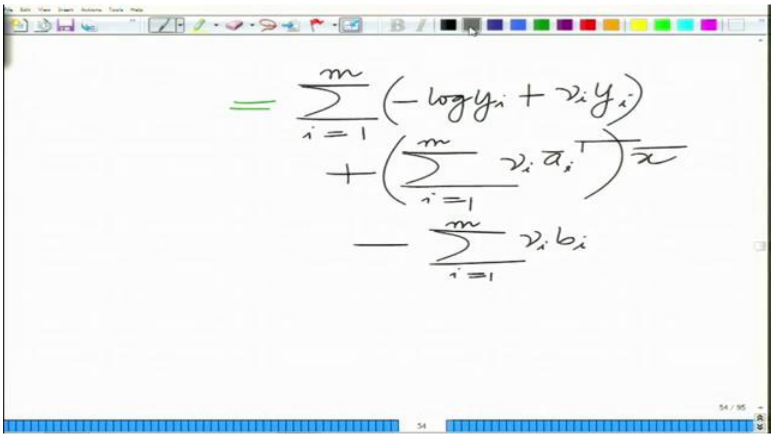

Now, once again collecting all the terms we proceed as shown in slides. And now we have to take the infimum with respect to the primal variables $\bar{x}, \bar{y}$.

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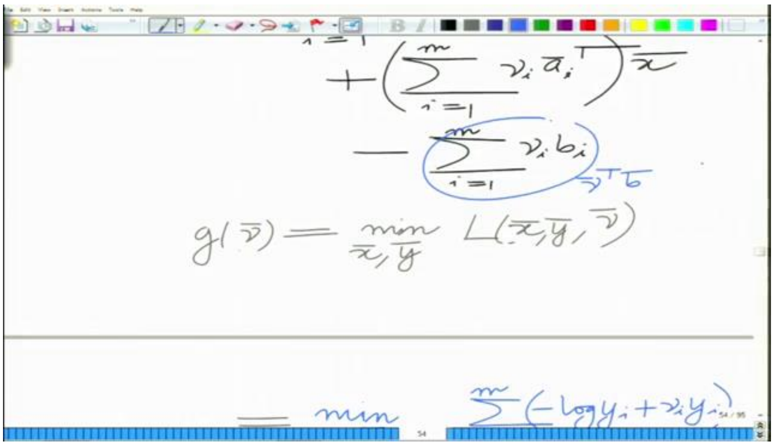

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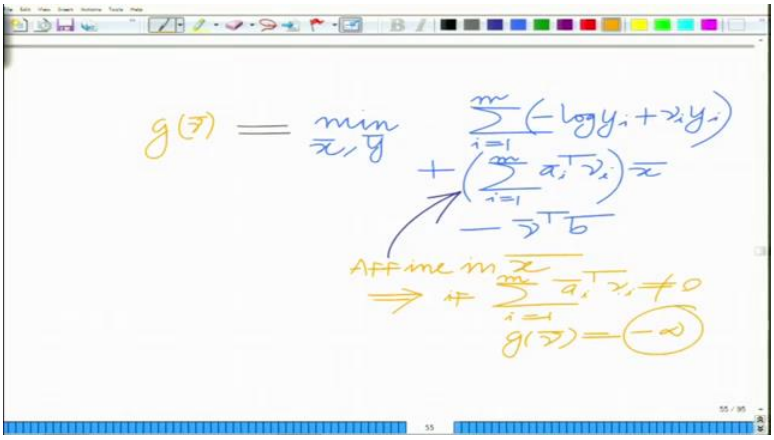

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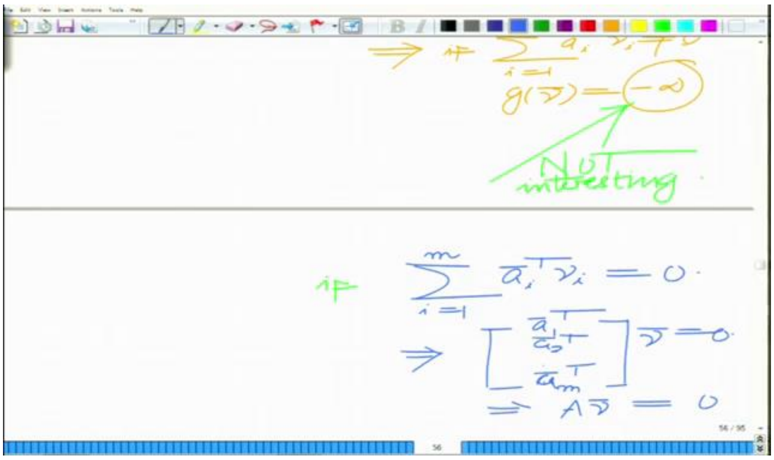

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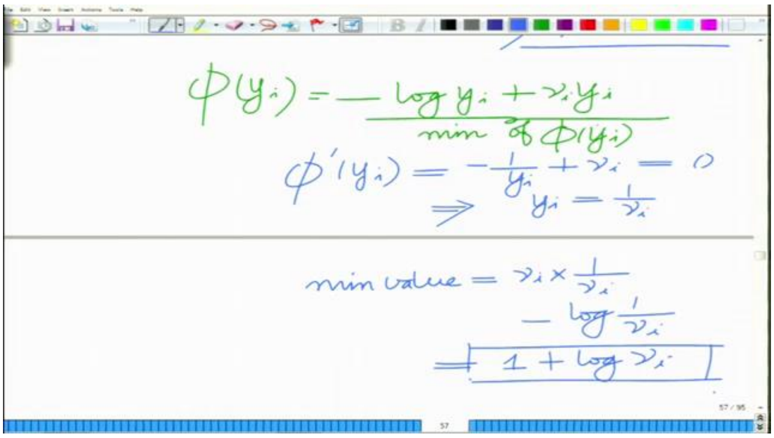

Now, we proceed as shown in the slides.

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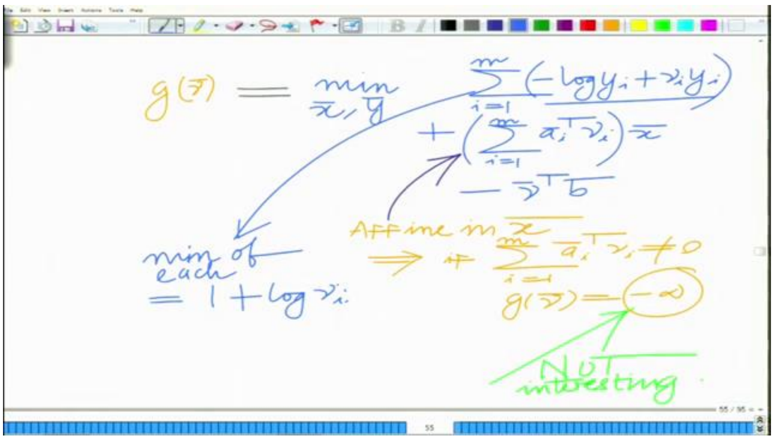

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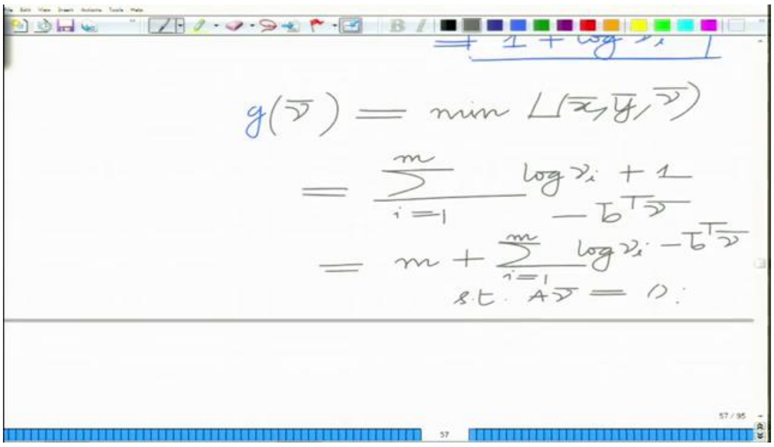

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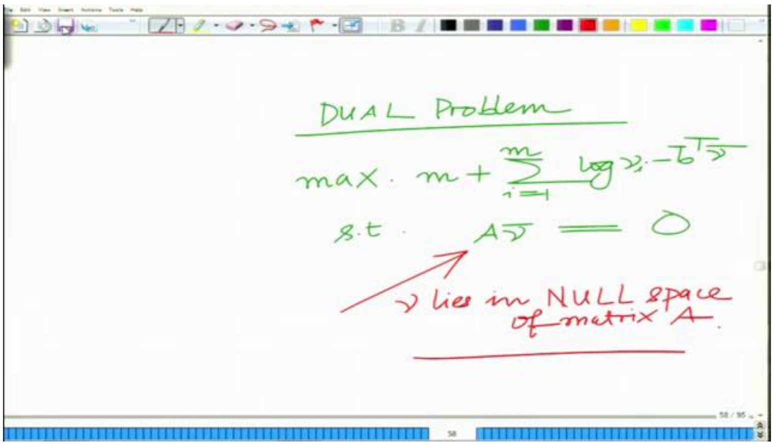

And therefore the dual problem of this analytical centering is

$$\begin{aligned} &\max\ m + \sum_{i=1}^{m} \log v_i - \bar{b}^T \bar{v} \\ &s.t \quad A\bar{v} = 0 \end{aligned}.$$

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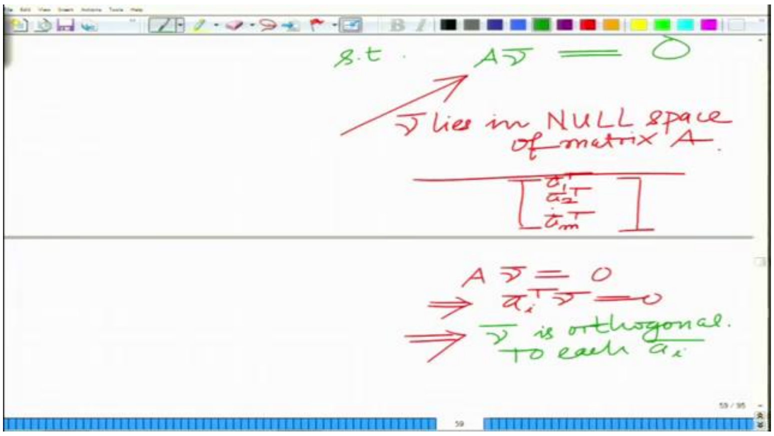

So these are some examples of various convex optimization problems and how to formulate their dual problem which often yields very useful insights and these are often very useful in practice. So let us stop here and continue in the subsequent modules. Thank you very much.