Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture – 25
Test for Convexity: Positive Semidefinite Hessian Matrix, example problems

Hello, welcome to another module in this massive open online course. Let us extend the test for convex functions of a vector or a multidimensional variable.

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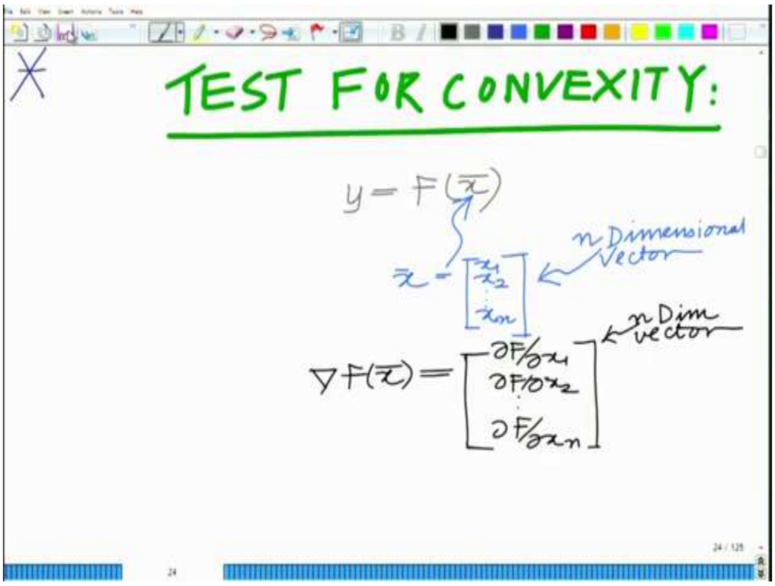

For this, consider a function of an *n*-dimensional vector $\bar{x}$.

$$y = F\left(\bar{x}\right)$$

Such that

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

The gradient of this function is also an *n*-dimensional vector and it is defined as the vector which contains the partial derivative of function with respect to each component of the vector $\bar{x}$.

$$\nabla F(\bar{x}) = \begin{bmatrix} \dfrac{\partial F}{\partial x_1} \\ \dfrac{\partial F}{\partial x_2} \\ \vdots \\ \dfrac{\partial F}{\partial x_n} \end{bmatrix}$$

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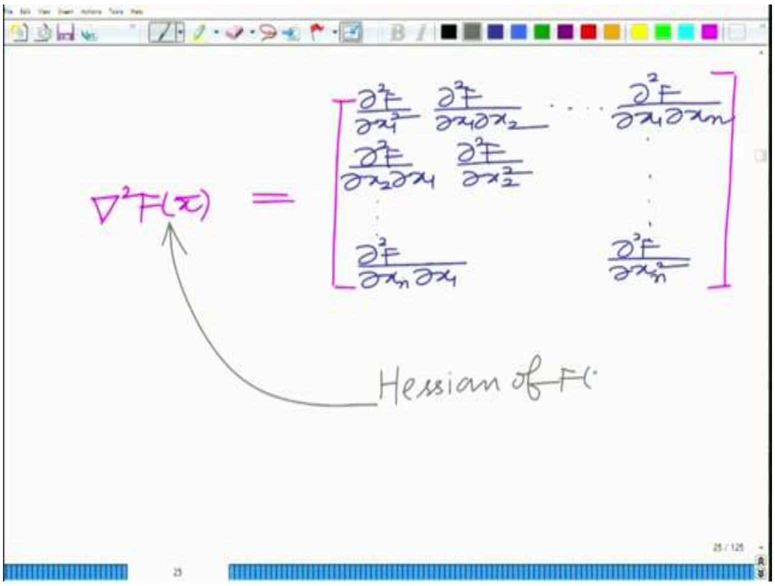

Consider the second order derivative of this function which is known as the Hessian of this function $\nabla^2 F(\bar{x})$ and it is a $n \times n$ matrix. So, this is defined as follows.

$$\nabla^2 F(\bar{x}) = \begin{bmatrix} \dfrac{\partial^2 F}{\partial x_1^2} & \dfrac{\partial^2 F}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 F}{\partial x_1 \partial x_n} \\ \dfrac{\partial^2 F}{\partial x_2 \partial x_1} & \dfrac{\partial^2 F}{\partial x_2^2} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 F}{\partial x_n \partial x_1} & \cdots & \cdots & \dfrac{\partial^2 F}{\partial x_n^2} \end{bmatrix}$$

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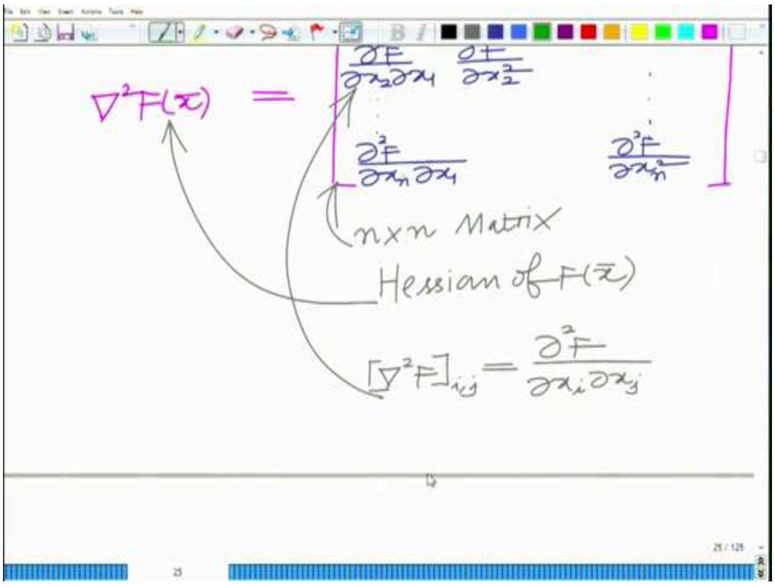

So, the $(i,j)^{th}$ component of this hessian of the function $\left[\nabla^2 F\right]_{i,j}$ can be defined as

$$\left[\nabla^2 F\right]_{i,j} = \frac{\partial^2 F}{\partial x_i \partial x_j}$$

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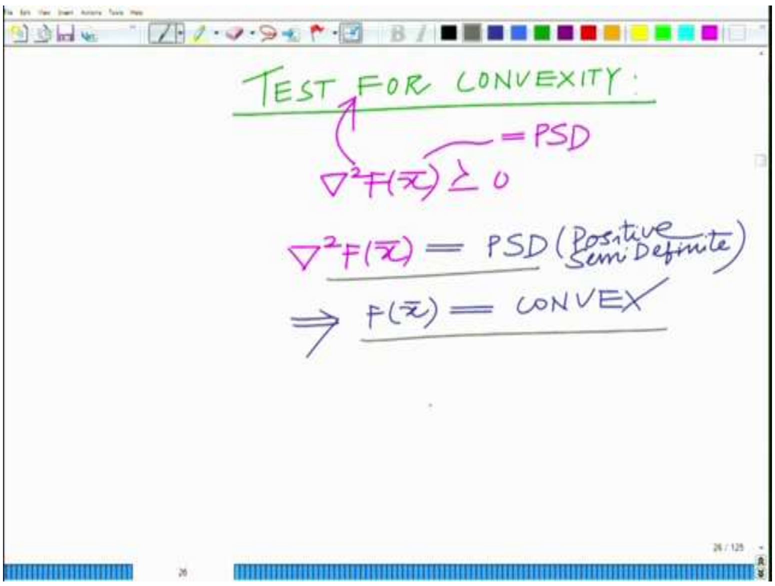

Now, the condition the test for convexity of a function of a multidimensional vector is

$$\nabla^2 F(\bar{x}) \geq 0$$

This means that $\nabla^2 F(\bar{x})$ is a positive semi definite matrix. So it concludes that if the Hessian of a function of a multidimensional vector is a positive semi definite matrix then this function is a convex function.

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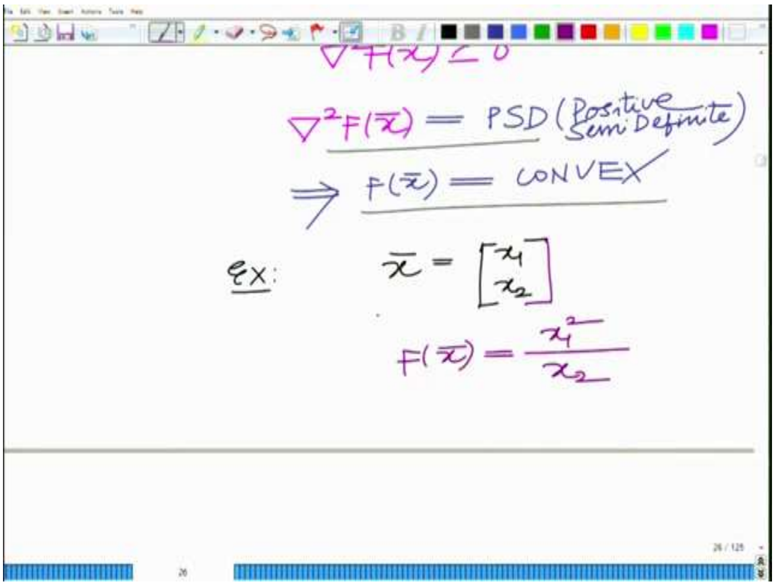

Let us look at a simple example. Consider a 2-D vector $\bar{x}$.

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{where } x_1, x_2 \geq 0$$

The function of this vector is given as

$$F(\bar{x}) = \frac{x_1^2}{x_2}$$

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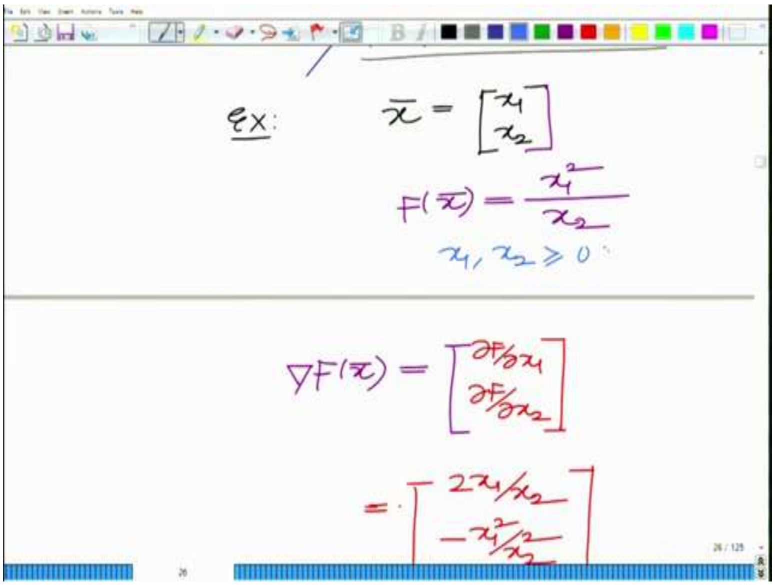

So the first gradient of this function is

$$\nabla F(\bar{x}) = \begin{bmatrix} \dfrac{\partial F}{\partial x_1} \\ \dfrac{\partial F}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \dfrac{2x_1}{x_2} \\ \dfrac{-x_1}{x_2^2} \end{bmatrix}$$

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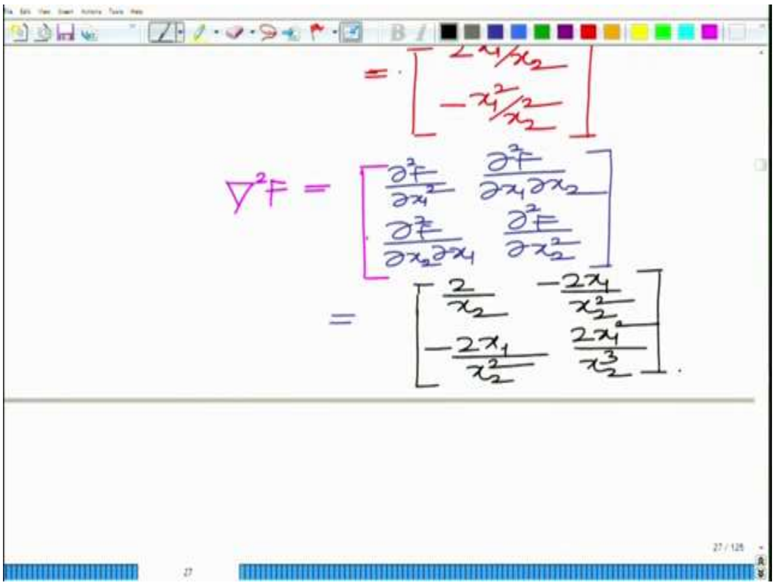

The Hessian of this function is

$$\nabla^2 F = \begin{bmatrix} \dfrac{\partial^2 F}{\partial x_1^2} & \dfrac{\partial^2 F}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 F}{\partial x_2 \partial x_1} & \dfrac{\partial^2 F}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} \dfrac{2}{x_2} & \dfrac{-2x_1}{x_2^2} \\ \dfrac{-2x_1}{x_2^2} & \dfrac{2x_1^2}{x_2^3} \end{bmatrix}$$

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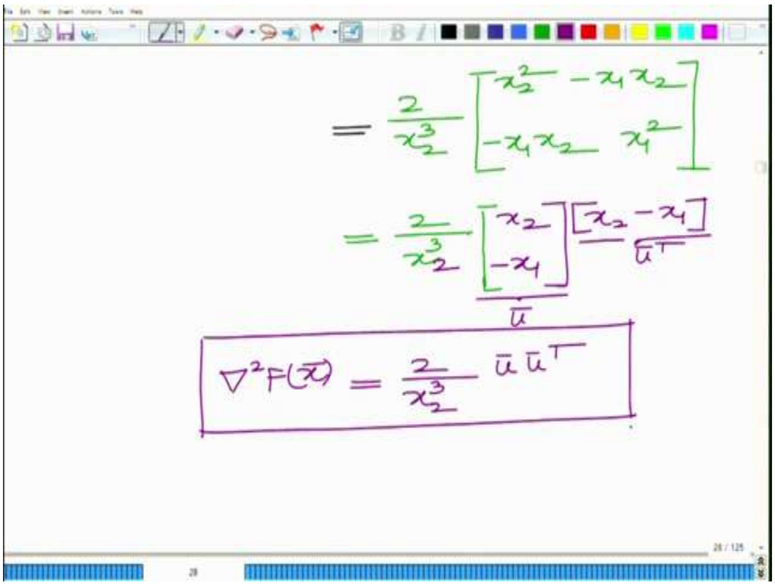

And this matrix can be decomposed as follows.

$$\begin{aligned} \nabla^2 F &= \frac{2}{x_2^3}\begin{bmatrix} x_2^2 & -x_1x_2 \\ -x_1x_2 & x_1^2 \end{bmatrix} \\ &= \frac{2}{x_2^3}\begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}\begin{bmatrix} x_2 & -x_1 \end{bmatrix} \\ &= \frac{2}{x_2^3}\bar{u}\cdot\bar{u}^T \end{aligned}$$

Where $\bar{u} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$.

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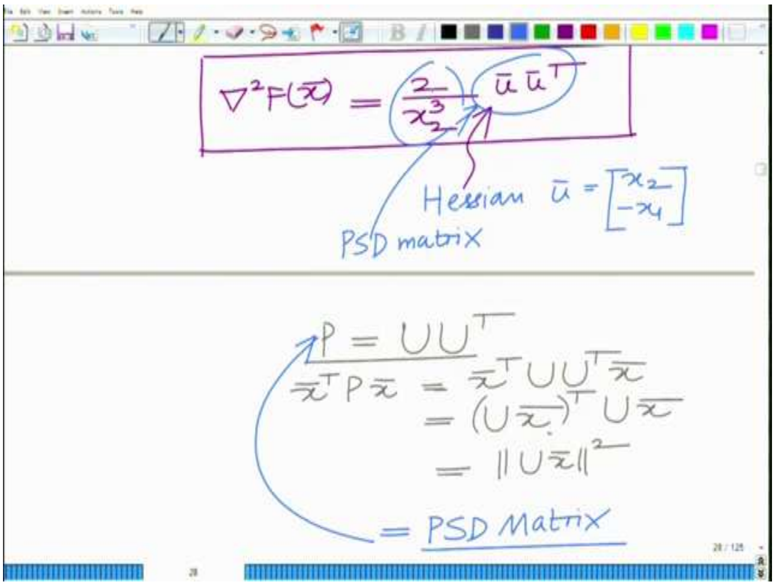

Let us look at matrix $\bar{u} \cdot \bar{u}^T$.

$P = UU^T$

Let us now check whether this matrix P is a positive semi definite matrix.

$$\begin{aligned}
\bar{x}^T P \bar{x} &= \bar{x}^T U U^T \bar{x} \\
&= (U\bar{x})^T U\bar{x} \\
&= \|U\bar{x}\|^2
\end{aligned}$$

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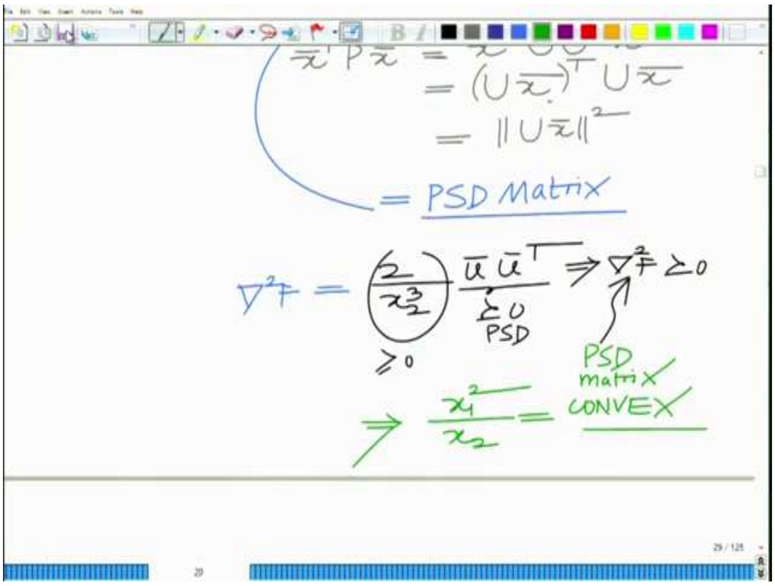

This shows that matrix P is a positive semi definite matrix. And this implies that

$$\nabla^2 F \geq 0$$

The function $F\left(\bar{x}\right) = \frac{x_1^2}{x_2}$ is a convex function with the restricted domain $x_1, x_2 \geq 0$.

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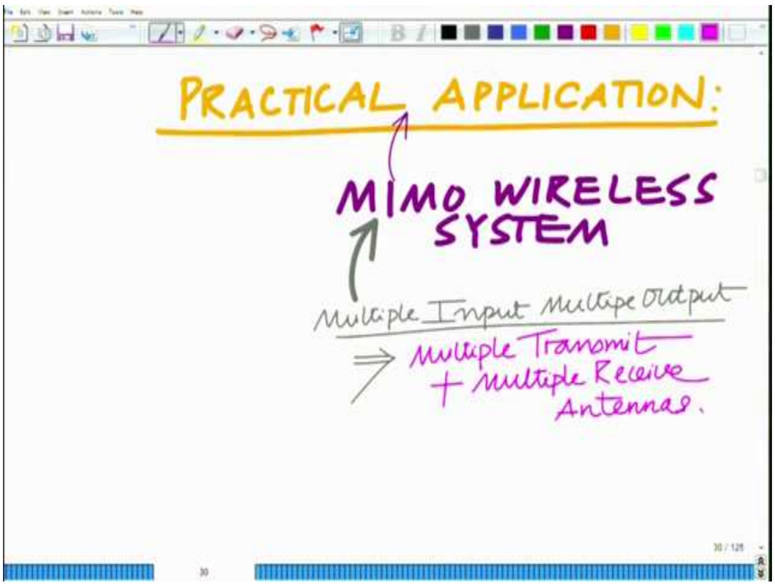

Let us look at an interesting practical application of the convexity of a function of a vector. Consider a Multiple Input Multiple Output (MIMO) wireless communication system. This implies that there are multiple transmit and multiple receive antennas. This significantly increases the data rate of a wireless communication system.

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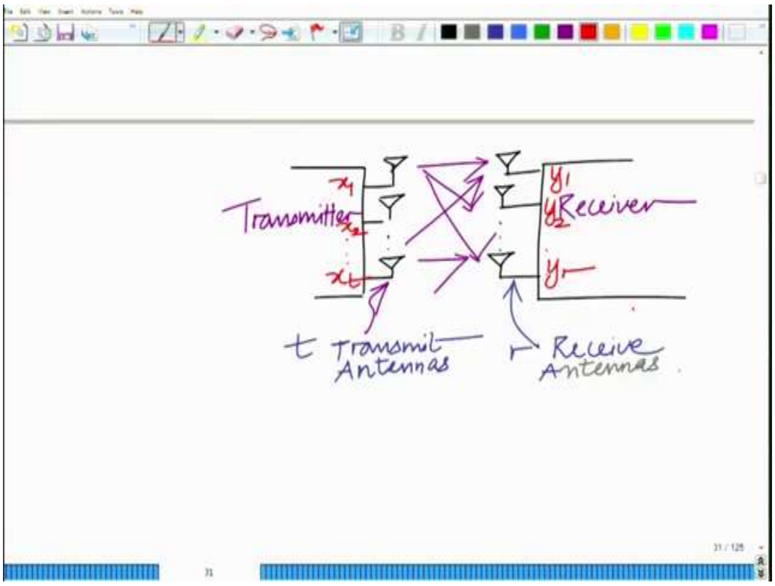

Let there are t transmit antennas and r receive antennas. So the transmit symbols from t transmitters are $x_1, x_2, \cdots, x_t$ and receive symbols at r receivers are $y_1, y_2, \cdots, y_r$ and there are so many possible channels between each transmitter and receiver.

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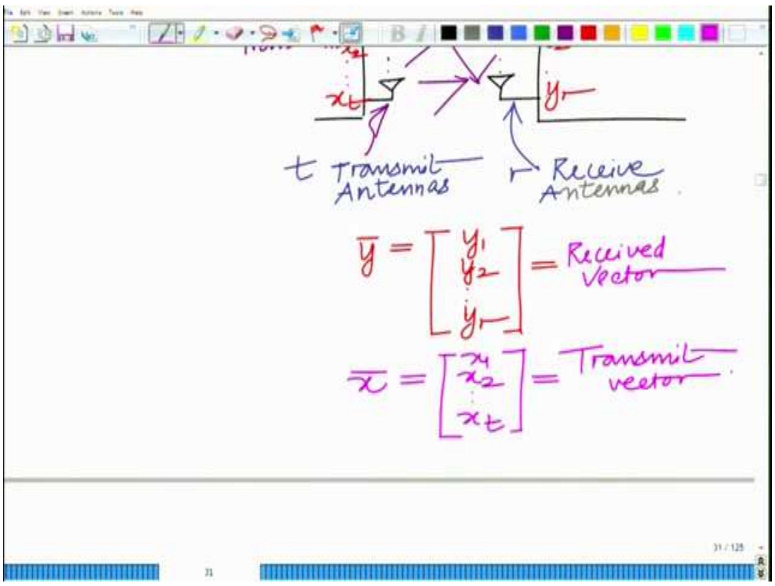

So, the $t \times 1$ transmit vector is

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{bmatrix}$$

And the $r \times 1$ received vector is

$$\bar{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \end{bmatrix}$$

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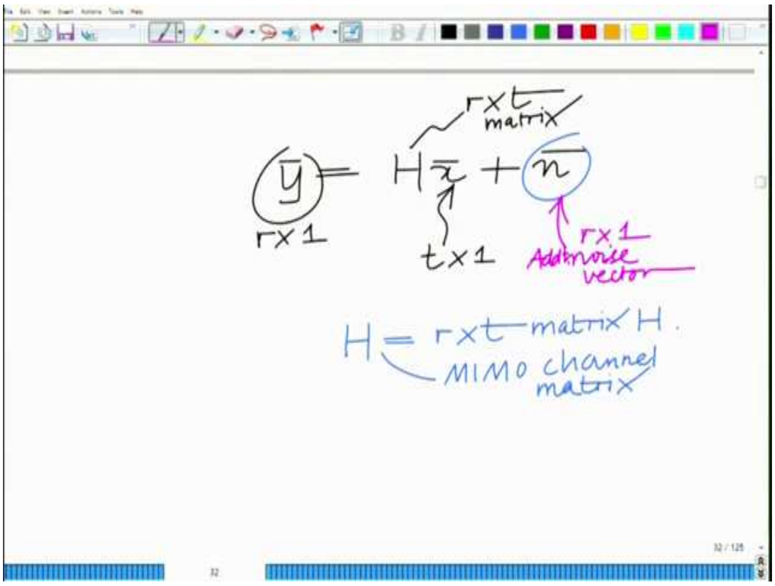

Therefore the model for this MIMO system is given as

$$\bar{y} = H\bar{x} + \bar{n}$$

Where H is the $r \times t$ MIMO channel matrix and $\bar{n}$ is the $r \times 1$ additive noise vector.

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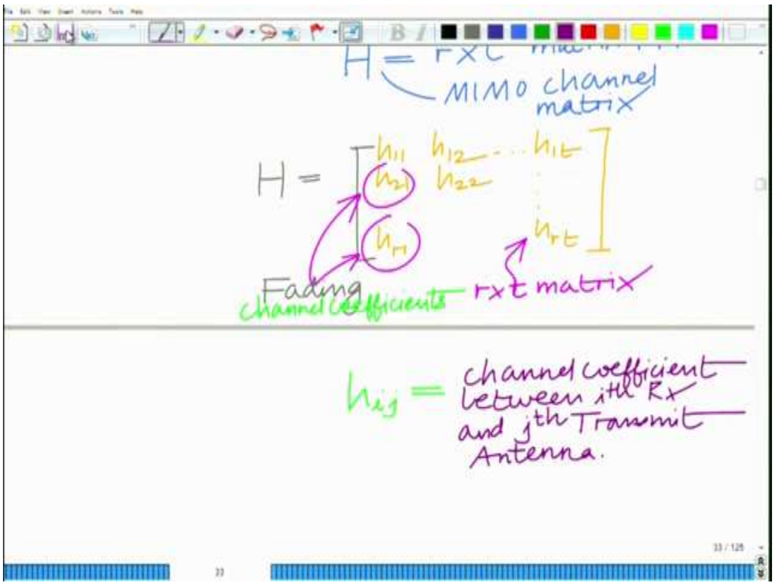

This channel matrix H has the following structure.

$$H = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1t} \\ h_{21} & h_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ h_{r1} & \cdots & \cdots & h_{rt} \end{bmatrix}$$

And each of quantity h_{ij} is the fading channel coefficient of channel between i^{th} transmitter and j^{th} reciever.

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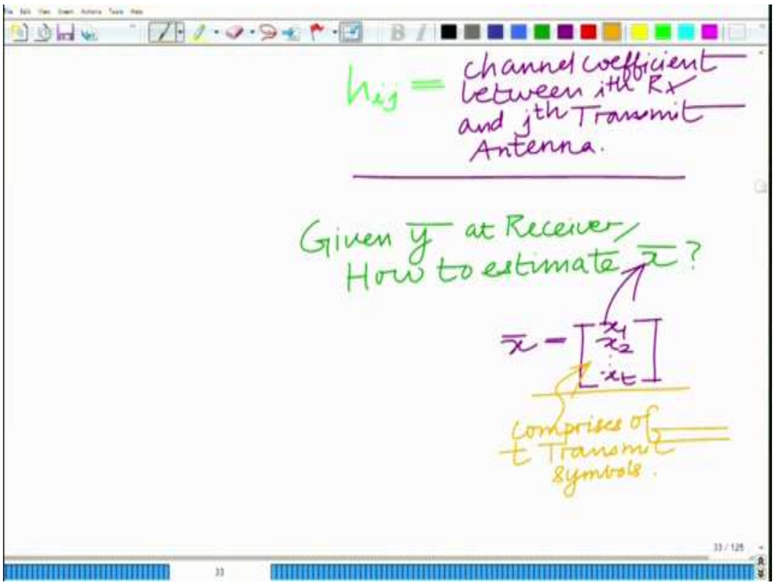

So the main challenge of a wireless communication system is the estimation of $\bar{x}$ from the received vector $\bar{y}$ at the receiver and this forms the problem of MIMO receiver design. So, one has to design a suitable algorithm or a technique for the MIMO receiver which recovers the transmit vector from received vector.

Let us explore the problem of MIMO receiver design and its relation to convex optimization in the subsequent module.