Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture – 21
Example Problems: Property of Norms, Problems on Convex Sets

Hello, welcome to another module in this massive open online course. Let us continue our discussion.

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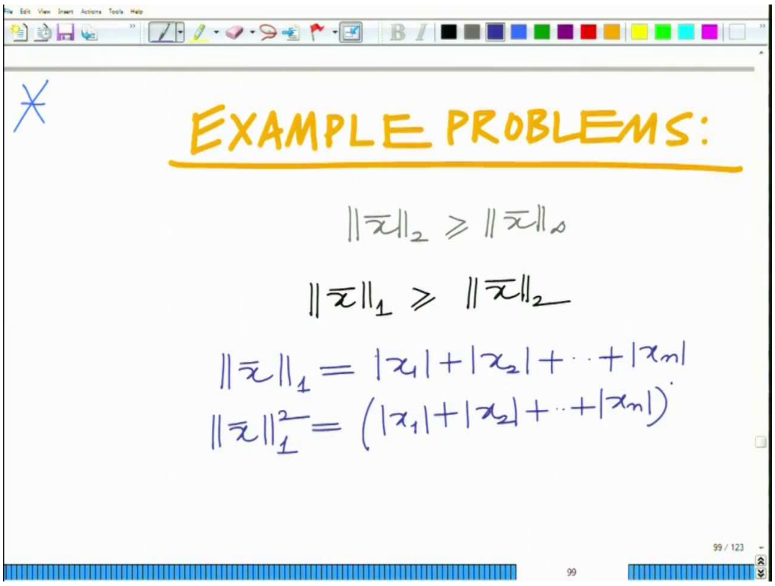

So, in the previous module, it was seen that the l_∞ norm of vector $\bar{x}$ is less than l_2 norm of vector $\bar{x}$

$$\|\bar{x}\|_2 \geq \|\bar{x}\|_\infty$$

Now, in the same way and one can also show that the l_1 norm of a vector $\bar{x}$ is greater than or equal to the l_2 norm of vector $\bar{x}$. This can be shown simply as follows. Let us take an n-dimensional vector $\bar{x}$ and its l_1 norm is as follows.

$$\|\bar{x}\|_1 = |x_1| + \ldots + |x_n|$$

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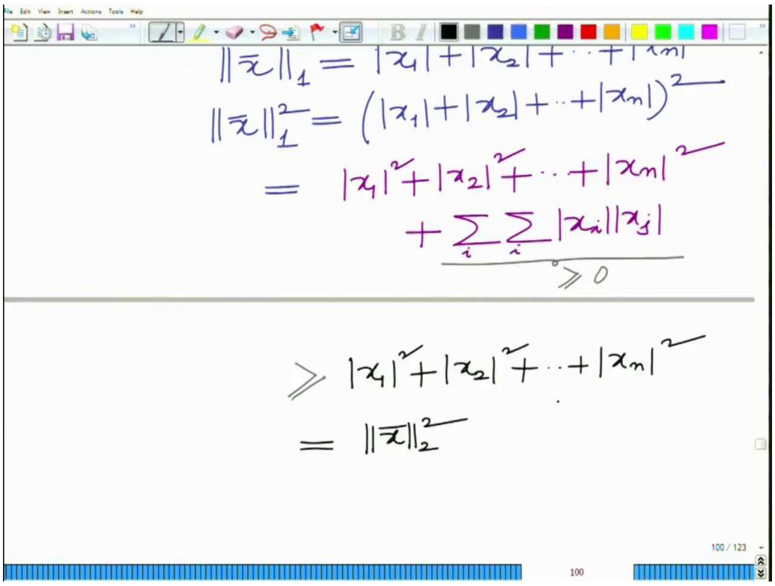

So the square of this l_1 norm is

$$\|\bar{x}\|_1^2 = \left(|x_1| + |x_2| + \ldots + |x_n|\right)^2$$
$$= |x_1|^2 + |x_2|^2 + \ldots + |x_n|^2 + \sum_i \sum_j |x_j||x_i|$$

The magnitudes of all the elements of the vector are positive, therefore the sum of all of their cross products is also positive and therefore

$$\sum_i \sum_j |x_j||x_i| \geq 0$$

And therefore,

$$\|\bar{x}\|_1^2 \geq |x_1|^2 + |x_2|^2 + \ldots + |x_n|^2$$
$$\|\bar{x}\|_1^2 \geq \|\bar{x}\|_2^2$$
$$\|\bar{x}\|_1 \geq \|\bar{x}\|_2$$

So it is shown that the l_1 norm of a vector $\bar{x}$ is greater than or equal to the l_2 norm of vector $\bar{x}$.

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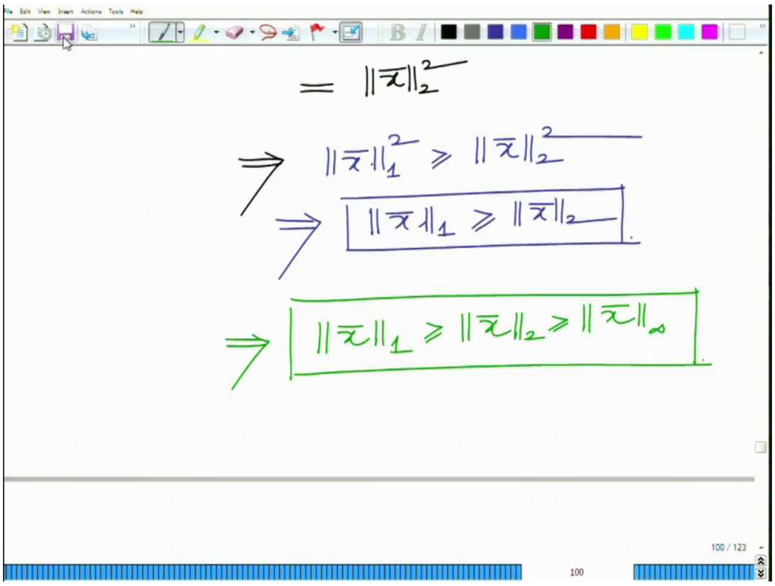

On putting both of these observations together, this can be concluded that the l_1 norm of a vector is greater than or equal to its l_2 norm and in same way it is greater than or equal to its l_∞ norm.

$$\|\bar{x}\|_1 \geq \|\bar{x}\|_2 \geq \|\bar{x}\|_\infty$$

So, the l_1 norm of a vector is the sum of the magnitude values of its element. The l_2 norm of a vector is the length of the vector in Euclidian space and the l_∞ norm of a vector is the maximum value of the magnitudes of the different elements of the vector.

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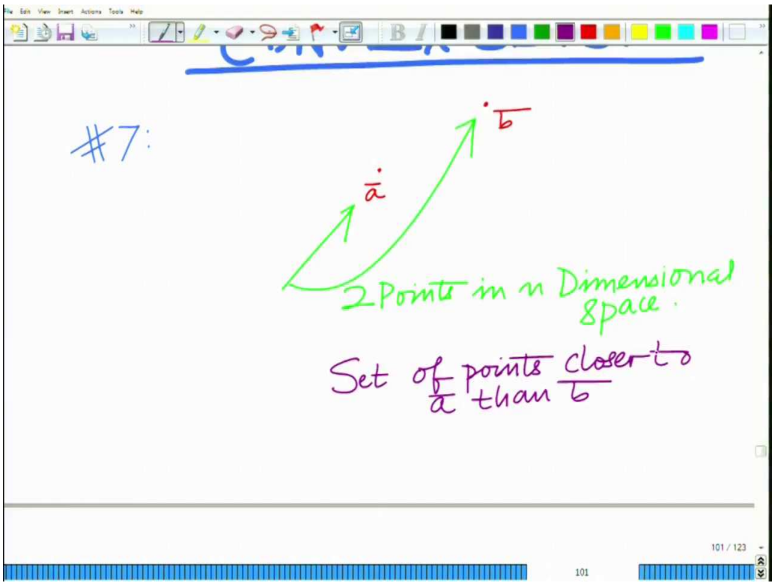

Let us move on to the example problems related to convex sets and their applications. Consider two points $\bar{a}$ and $\bar{b}$ in an n-dimensional space. Take a set S as a set of points in this n-dimensional space that are closer to the point $\bar{a}$ than point $\bar{b}$.

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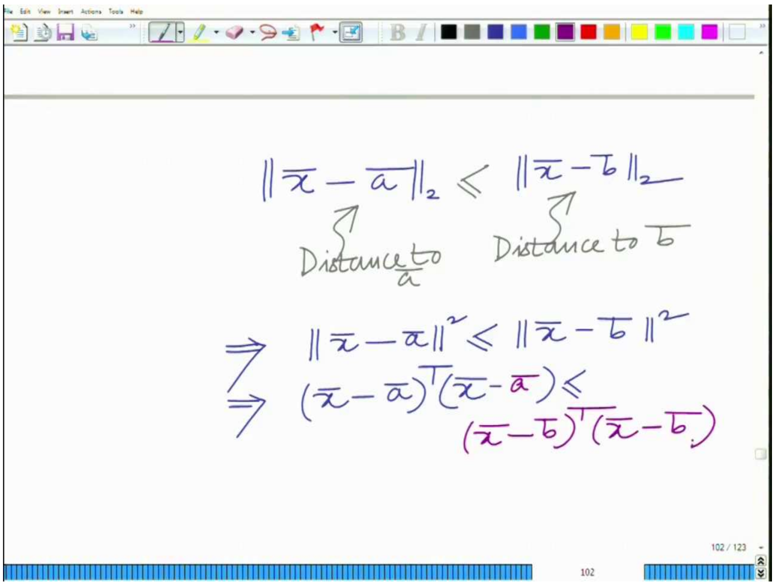

So, for a point $\bar{x} \in S$, the distance of $\bar{x}$ to point $\bar{a}$ is $\|\bar{x}-\bar{a}\|_2$ and the distance of $\bar{x}$ to point $\bar{b}$ is $\|\bar{x}-\bar{b}\|_2$. Therefore point $\bar{x} \in S$ satisfies

$$\left\| \overline{x} - \overline{a} \right\|_2 \le \left\| \overline{x} - \overline{b} \right\|_2$$

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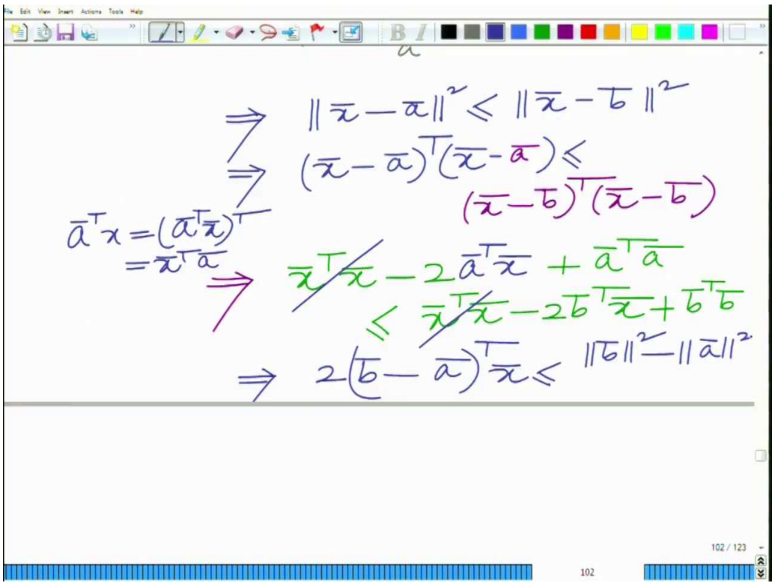

This set S is a convex set and to show this, take a square of above inequality on both the sides.

$$\left\| \overline{x} - \overline{a} \right\|^2 \le \left\| \overline{x} - \overline{b} \right\|^2$$

$$\left(\overline{x} - \overline{a} \right)^T \left(\overline{x} - \overline{a} \right) \le \left(\overline{x} - \overline{b} \right)^T \left(\overline{x} - \overline{b} \right)$$

$$\overline{x}^T \overline{x} - 2\overline{x}^T \overline{a} + \overline{a}^T \overline{a} \le \overline{x}^T \overline{x} - 2\overline{x}^T \overline{b} + \overline{b}^T \overline{b}$$

$$-2\overline{x}^T \overline{a} + \overline{a}^T \overline{a} \le -2\overline{x}^T \overline{b} + \overline{b}^T \overline{b}$$

From the basic property of a matrix

$$\overline{x}^T \overline{a} = \left(\overline{a}^T \overline{x} \right)^T$$
$$= \overline{a}^T \overline{x}$$

Also that both points and are scalar therefore

$$\left(\overline{a}^T \overline{x} \right)^T = \overline{a}^T \overline{x}$$

Therefore on substituting $\overline{x}^T \overline{a} = \overline{a}^T \overline{x}$ in the above inequality, it becomes

$$\|\bar{x}-\bar{a}\|^2 \le \|\bar{x}-\bar{b}\|^2$$
$$-2\bar{a}^T\bar{x}+\bar{a}^T\bar{a} \le -2\bar{x}^T\bar{b}+\bar{b}^T\bar{b}$$
$$2\left(\bar{b}-\bar{a}\right)^T\bar{x} \le \|\bar{b}\|^2-\|\bar{a}\|^2$$
$$\left(\bar{b}-\bar{a}\right)^T\bar{x} \le \frac{1}{2}\left(\|\bar{b}\|^2-\|\bar{a}\|^2\right)$$

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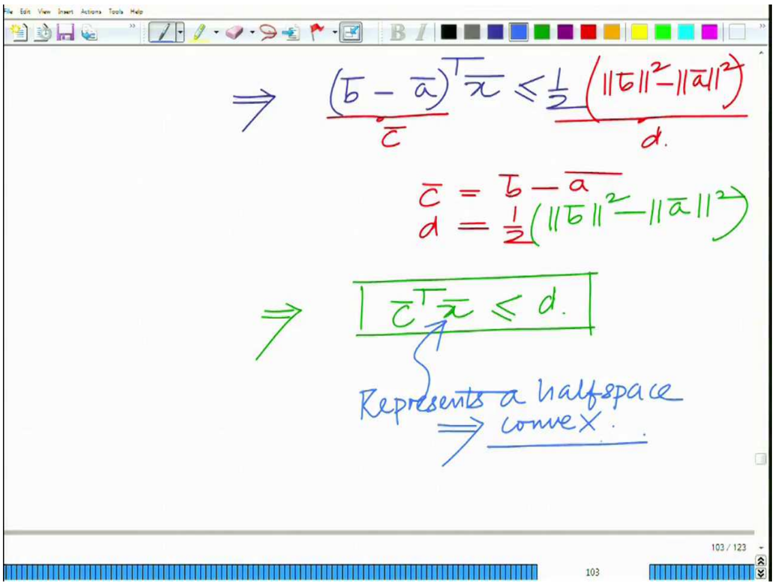

Now if consider two scalar points $\bar{c}$ and d such as

$$\bar{c}=\bar{b}-\bar{a}$$
$$d=\frac{1}{2}\left(\|\bar{b}\|^2-\|\bar{a}\|^2\right)$$

Then above inequality becomes

$$\bar{c}^T\bar{x} \le d$$

This represents a halfspace which is convex. This shows that the set S is a convex set.

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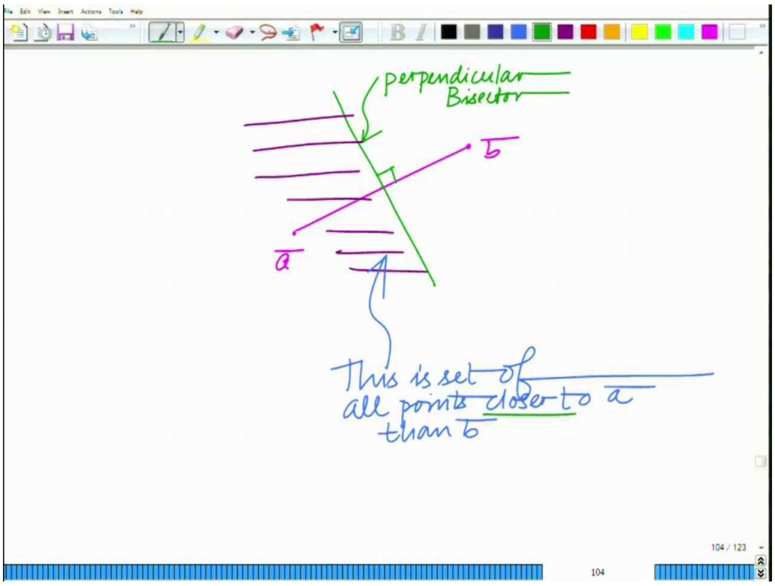

Interestingly, if a line segment is drawn joining the points $\bar{a}$ and $\bar{b}$; then this half space is the region of one side of the perpendicular bisector of this line segment and this side of perpendicular bisector is the same side in which point $\bar{a}$ lies.

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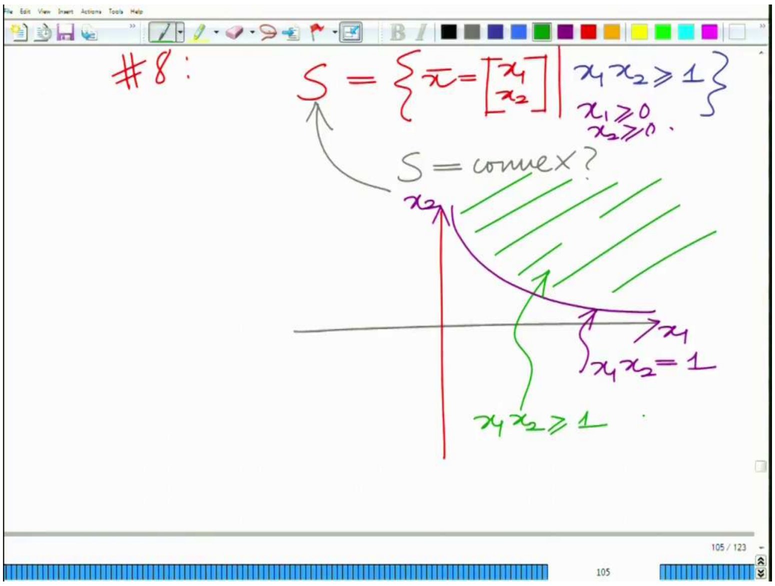

Let us look at another interesting set. Let set S is the set of 2-dimensional vectors $\bar{x}$ such that

$$S = \left\{ \overline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_1 x_2 \geq 1, x_1 \geq 0, x_2 \geq 0 \right\}$$

This set S is a convex set. This set S is region lies right to the hyperbola $x_1 x_2 = 1$ in the plot of axes x_1 and x_2. So, visually one can see that this set is convex.

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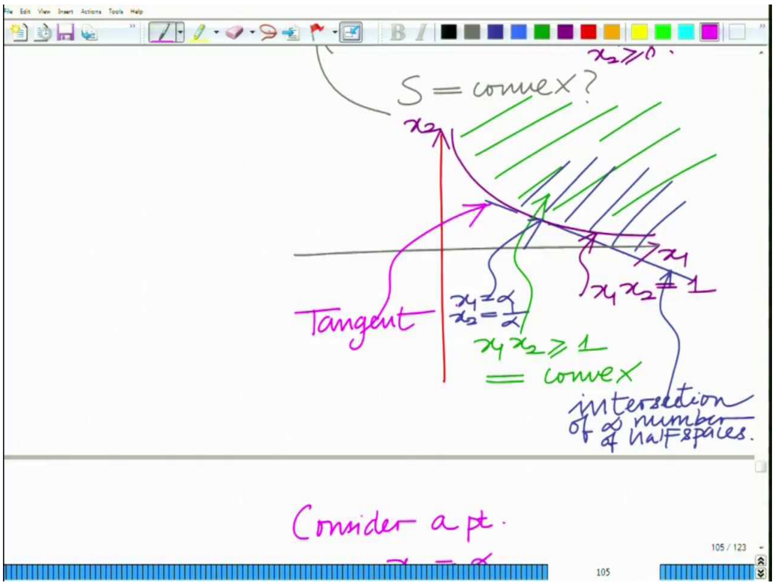

Let us take an alternative approach that this half space can be represented as the intersection of infinite number of half spaces. So, in this way, as the intersection of half spaces is essentially a polyhedral, therefore this will show that this set is a convex set.

Let us draw a tangent on the parabola. The region on the right hand side of this tangent defines a half space. So the infinite tangents drawn on the infinite points of parabola makes infinite number of half spaces and the intersection of these half spaces represents set S.

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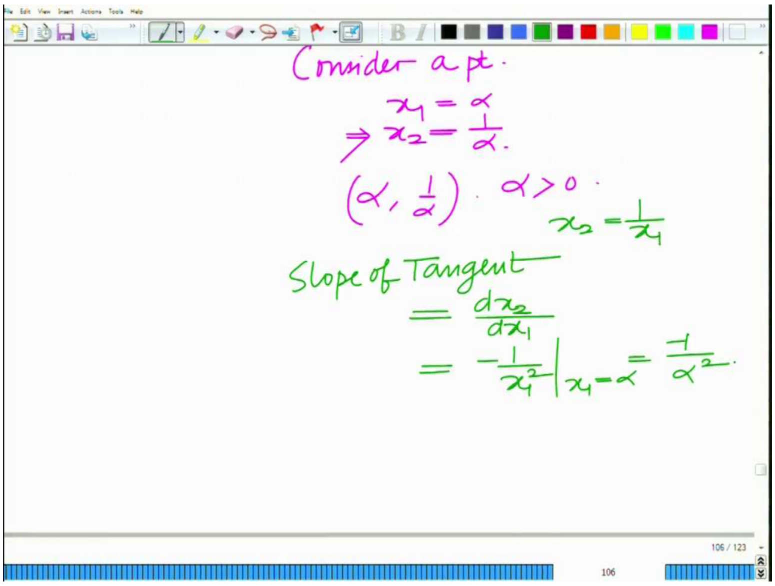

Let us look at any point where $x_1 = \alpha$ and $x_2 = \frac{1}{\alpha}$ where α is strictly greater than zero.

So the slope of tangent at $\left(\alpha, \frac{1}{\alpha}\right)$ is

$$\frac{dx_2}{dx_1} = -\frac{1}{x_1^2}\Big|_{x_1=\alpha}$$

$$= -\frac{1}{\alpha^2}$$

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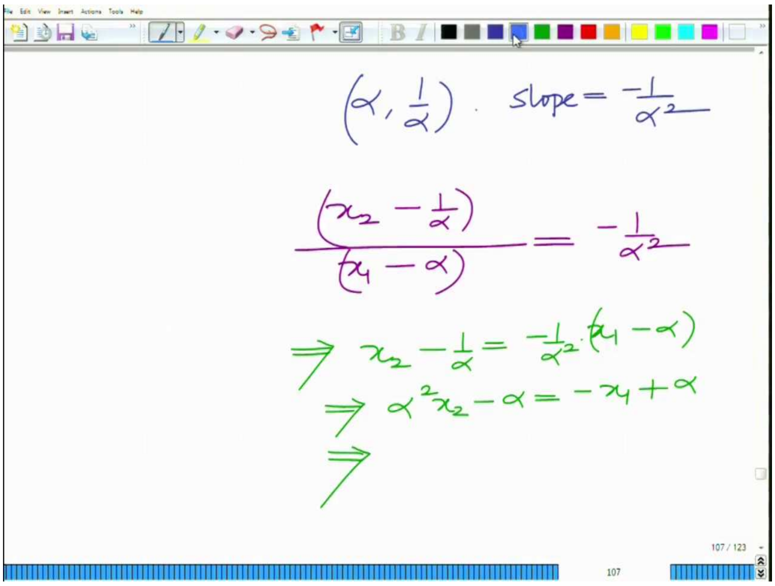

So for the point $\left(\alpha, \frac{1}{\alpha}\right)$, the slope of the tangent is $-\frac{1}{\alpha^2}$. From basic geometry, the slope of the tangent at this same point is

$$\frac{\left(x_2 - \frac{1}{\alpha}\right)}{(x_1 - \alpha)} = -\frac{1}{\alpha^2}$$

$$\left(x_2 - \frac{1}{\alpha}\right)\alpha^2 = -(x_1 - \alpha)$$

$$\alpha^2 x_2 - \alpha = -x_1 + \alpha$$

$$\alpha^2 x_2 + x_1 - 2\alpha = 0$$

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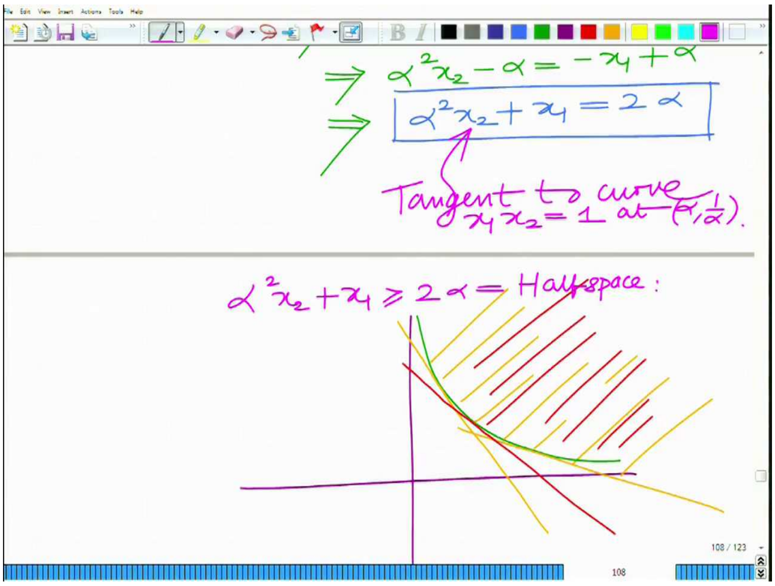

Thus the equation of these tangents is

$$\alpha^2 x_2 + x_1 - 2\alpha = 0$$

And hence the equation of these half spaces is

$$\alpha^2 x_2 + x_1 \geq 2\alpha$$

This means that there exists such half spaces whose intersection represents the set S and hence it shows that the set S is convex.

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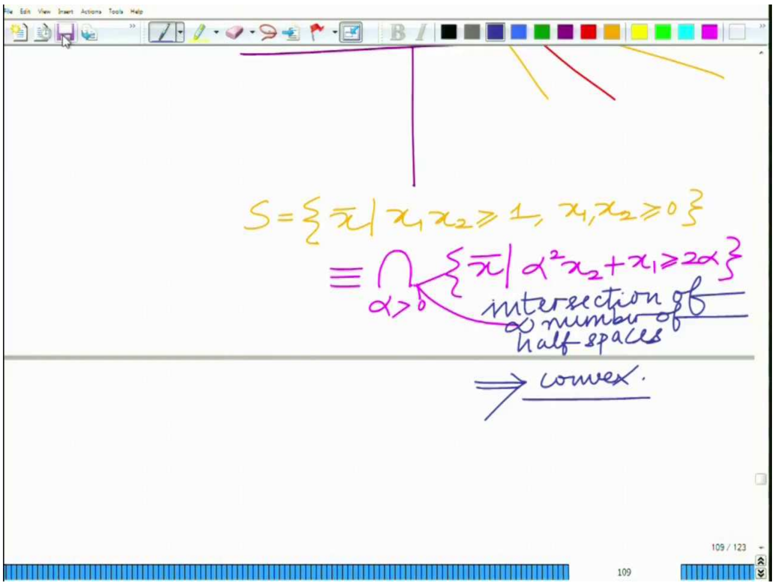

Hence the set S can also be defined as follows.

$$S = \left\{ \bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \;\middle|\; \alpha^2 x_2 + x_1 \geq 2\alpha, \alpha \geq 0 \right\}$$

These are some interesting application problems which demonstrate the convexity of a set. Let us continue this discussion in the subsequent modules.