Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture – 11
Introduction to Convex Sets and Properties

Hello. Welcome to another module in this massive open online course. So, let us discuss the next topic which is the convex sets.

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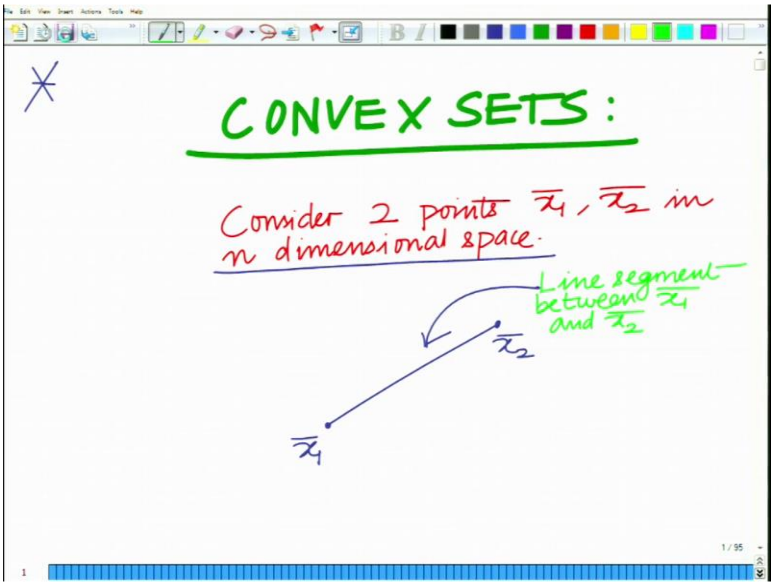

One of the important concepts to understand in convex optimization is a convex set. So, to define a convex set, let us start with the following setup. Consider two n-dimensional points $\bar{x}_1$ and $\bar{x}_2$. This means that these $\bar{x}_1$ and $\bar{x}_2$ are vectors in n-dimensional space. So there will be a line segment that is joining $\bar{x}_1$ and $\bar{x}_2$.

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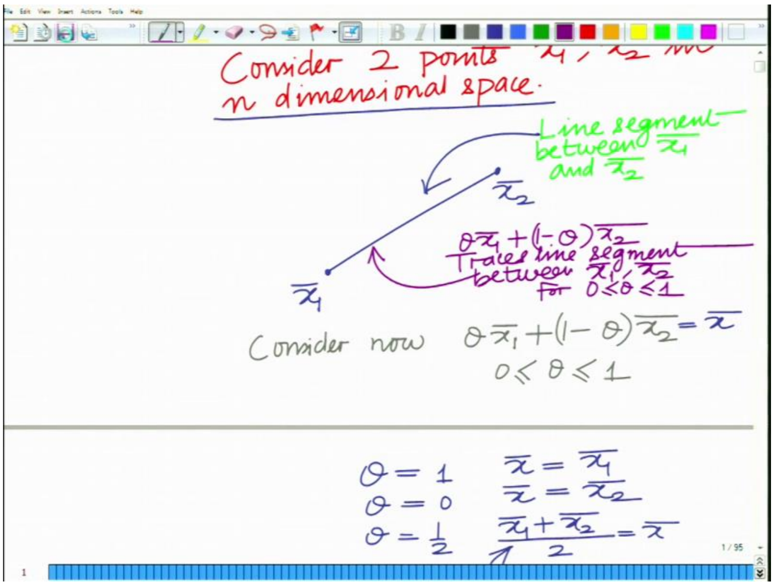

Let us say that $\bar{x}$ is the expression of this line segment which is nothing but a linear combination.

$$\theta\bar{x}_1+(1-\theta)\bar{x}_2=\bar{x}$$

Where θ is the weight and $0\le\theta\le1$. The important aspect is the values outside $[0,1]$ are not allowed to be equal to the θ.

So for instance let us take a look at the following points generated by this combination.

$$If\ \ \theta=1,\ \text{then}\ \bar{x}=\bar{x}_1.$$
$$If\ \ \theta=0,\ \text{then}\ \bar{x}=\bar{x}_2.$$
$$If\ \ \theta=\frac{1}{2},\ \text{then}\ \bar{x}=\frac{\bar{x}_1+\bar{x}_2}{2}.$$

It is clear from the above observations that for the midpoint of the line segment between $\bar{x}_1$ and $\bar{x}_2$, the value of θ must be equal to $\frac{1}{2}$.

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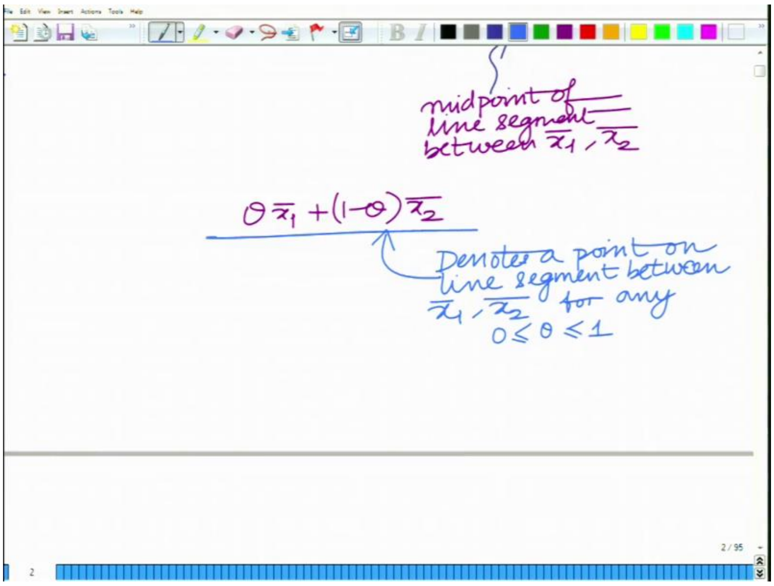

So, $\theta \bar{x}_1 + (1-\theta)\bar{x}_2$ traces line segment between $\bar{x}_1$ and $\bar{x}_2$ and such a linear combination is termed as a convex combination. This denotes a point on the line segment between $\bar{x}_1$ and $\bar{x}_2$ for any $0 \leq \theta \leq 1$.

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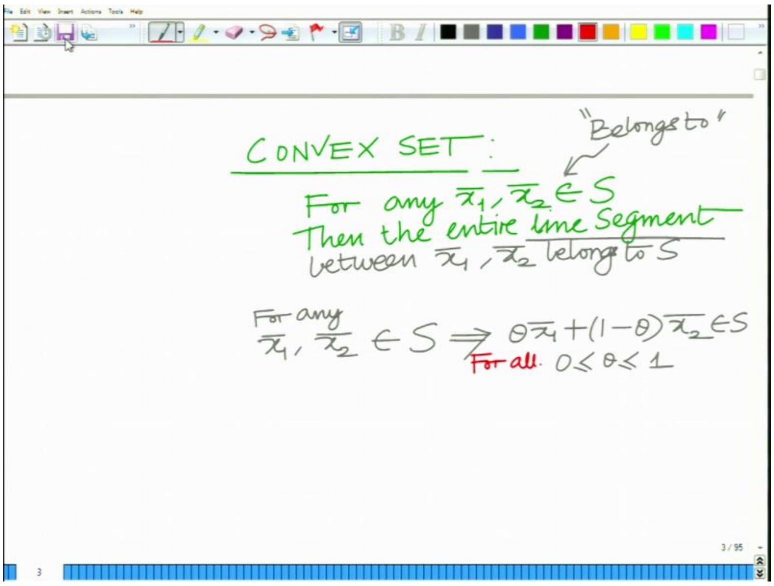

Therefore, if any two points $\bar{x}_1$ and $\bar{x}_2$ belong to a set S and also the entire line segment drawn between $\bar{x}_1$ and $\bar{x}_2$ belongs to S then such set of points $\bar{x}_1$ and $\bar{x}_2$ is known as a convex set. So the mathematical way of stating a convex set is as follows.

For any points $\bar{x}_1, \bar{x}_2 \in S$; for all $0 \le \theta \le 1$

$$\theta \bar{x}_1 + (1-\theta)\bar{x}_2 \in S$$

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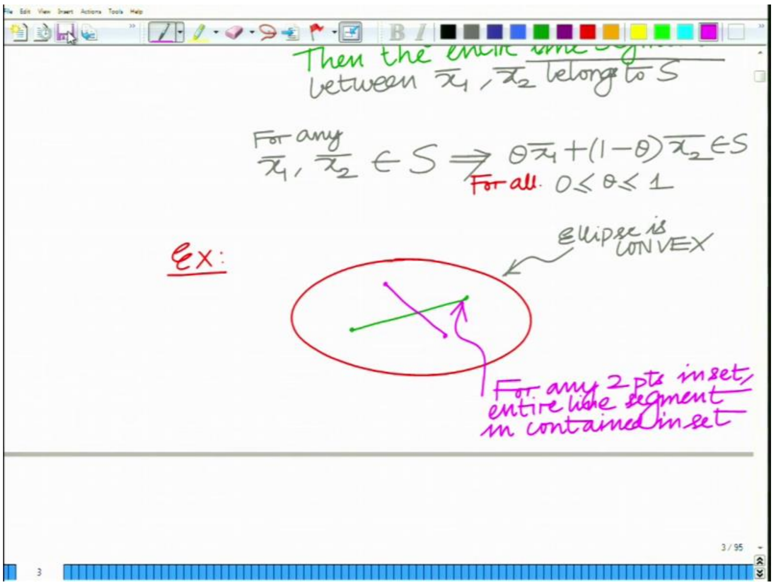

Let us take a simple example to illustrate this. For instance, it can be seen that an ellipse is a convex set. So choose any two points. So the entire line segment joining these two points lies in the set. So, for any two points, the entire line segment is contained in set and hence the ellipse is a convex set.

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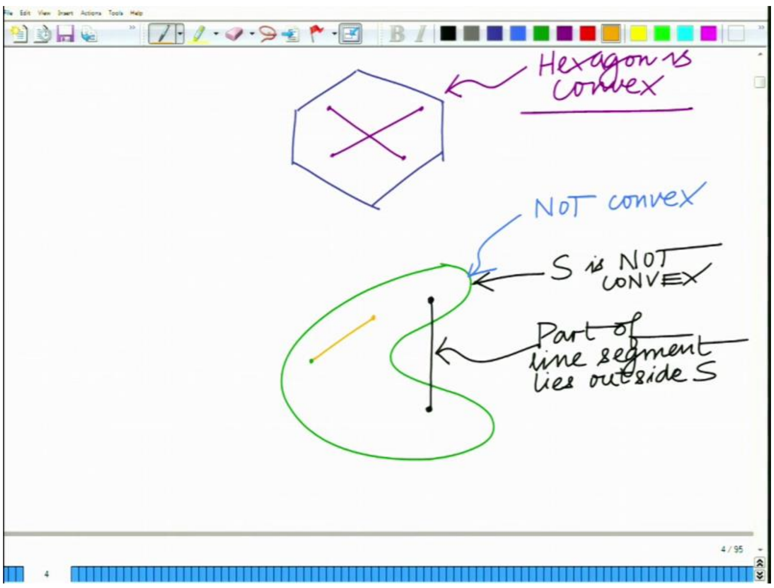

Similarly, in a hexagon, all the points on the line segment drawn between the pairs of points which are contained in the hexagon, lies in the hexagon and hence Hexagon is also a convex set.

On the other hand, consider a region S shown in the above image. It is clear that there are few pair of points contained in S, which does not satisfy the criteria of a convex set. That is, the line segment drawn between few pairs of point contained in the set S is not entirely inside the region. Therefore it implies that the above shown set S is not a convex set.

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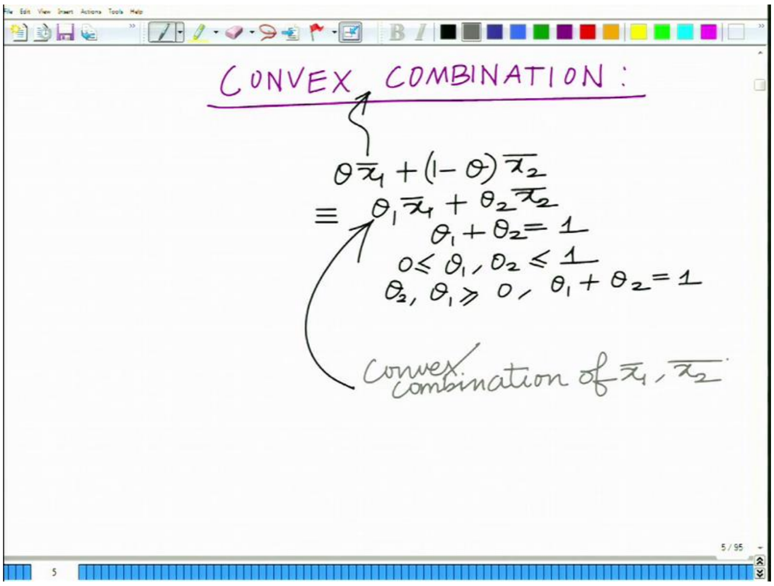

Another important concept associated to the convex set is the convex combination. To define a convex combination, let us recall the linear combination of $\bar{x}_1$ and $\bar{x}_2$ which was used to define the convex set. Therefore the linear combination $\theta\bar{x}_1 + (1-\theta)\bar{x}_2 \in S$ can be equivalently represented as follows.

$$\theta\bar{x}_1 + (1-\theta)\bar{x}_2 \equiv \theta_1\bar{x}_1 + \theta_2\bar{x}_2$$

Such that

$$\theta_1 + \theta_2 = 1$$

Also that θ_1 and θ_2 satisfies the property of a convex set, therefore

$0 \le \theta_1, \theta_2 \le 1$

Thus, such combinations of θ_1, θ_2 are known as the convex combination of $\bar{x}_1$ and $\bar{x}_2$ and the notion of this convex combination is $\theta_1\bar{x}_1 + \theta_2\bar{x}_2$.

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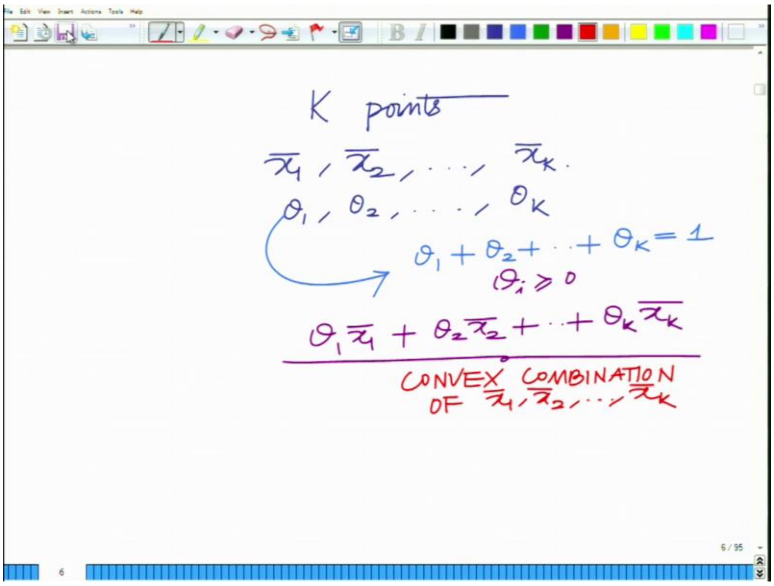

Therefore, the generalized notion of convex combination of that includes k points $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k$ in n dimensional space, is

$$\theta_1\bar{x}_1 + \theta_2\bar{x}_2 + \ldots + \theta_k\bar{x}_k$$

where $\theta_1, \theta_2, \ldots, \theta_k$ are defined as $\theta_1 + \theta_2 + \ldots + \theta_k = 1$ for each $i \in R,\ \theta_i \ge 0$.

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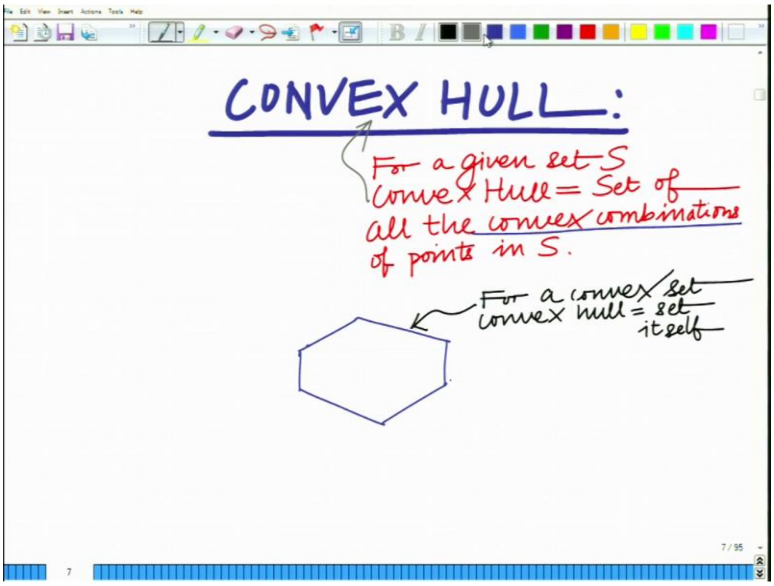

Another concept associated to the convex set is the convex hull of a set. The convex hull is the set of all of the convex combination of points in S.

Now naturally observe that for any convex set S, the convex hull is the set itself; because if S is a convex set; then it already contains all the convex combinations of the points in S. So, for a convex set, for instance we saw yesterday that the hexagon is a convex set, correct.

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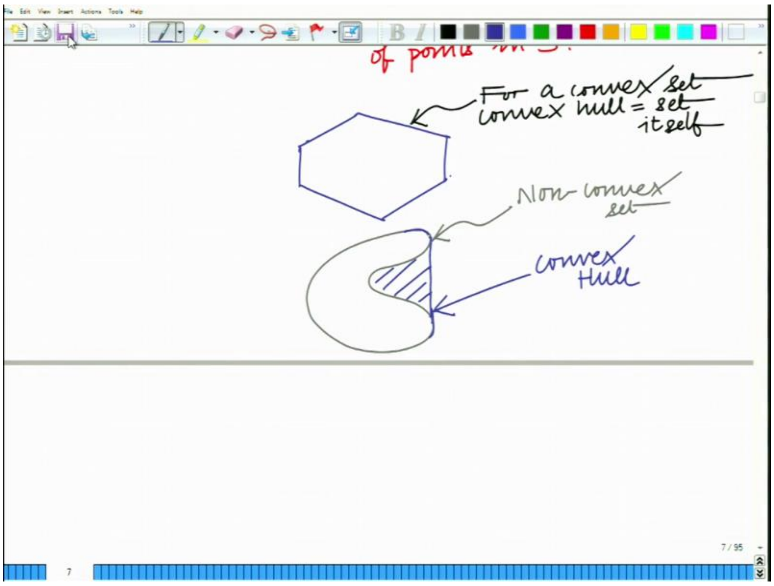

For a non-convex set.

In case of non-convex set, consider the non-convex set shown in the above image. In such type of sets, convex hull is the set which contains the whole non-convex set as well as all the possible points which lies on the different line segments drawn between any two points of the non-convex set. Therefore it can be said that the convex hull simply fills this region around the non-convex set to make it a convex set.

So, let us look at other aspects in the subsequent modules.