Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture – 66
Karush-Kuhn-Tucker (KKT) conditions

Keywords: *Karush-Kuhn-Tucker (KKT) conditions, Complimentary Slackness*

Hello, welcome to another module in this massive open online course. In this module you want to start looking at KKT conditions, the Karush-Kuhn-Tucker conditions, which are convenient for solving any optimization problem.

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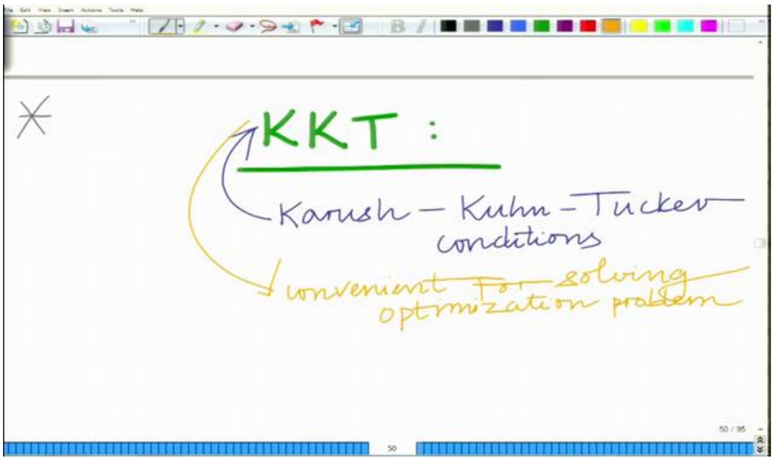

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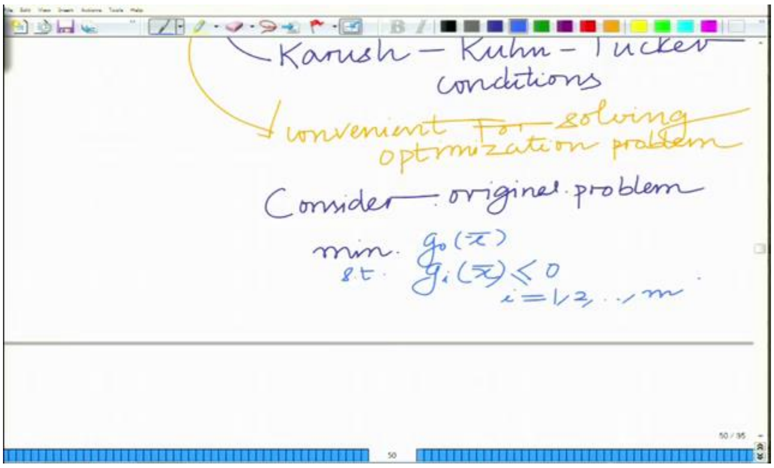

So now consider again the original or the primal optimization problem that is we have

$$\min \quad g_0(\bar{x})$$
$$\text{s.t} \quad \begin{array}{l} g_i(\bar{x}) \le 0 \\ i = 1, 2, .., l \\ g_j(\bar{x}) = 0 \\ j = 1, 2, .., m \end{array}$$

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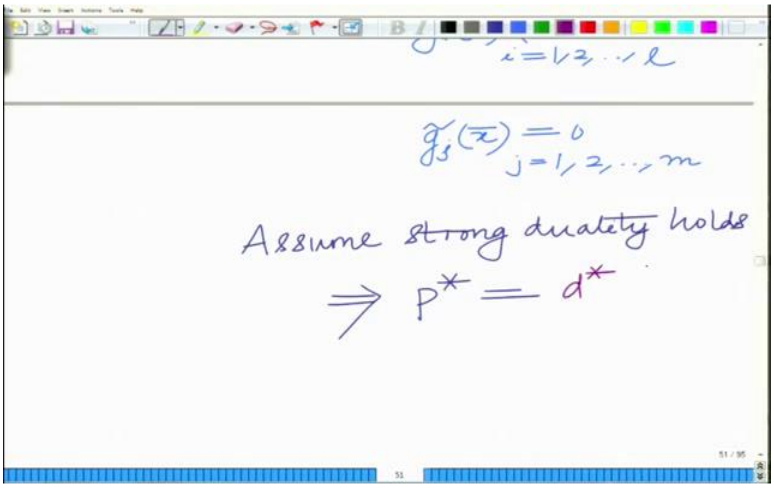

And in addition assume that strong duality holds which implies $P^* = d^*$.

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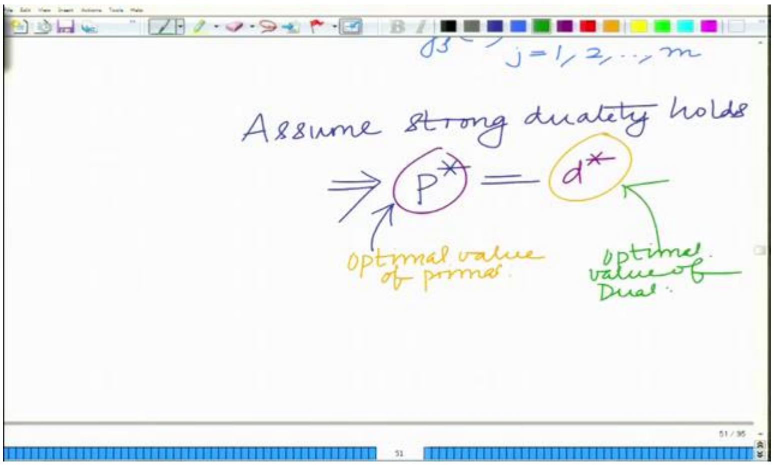

So P^* is the optimal value of primal problem and d^* is the optimal value of the dual problem.

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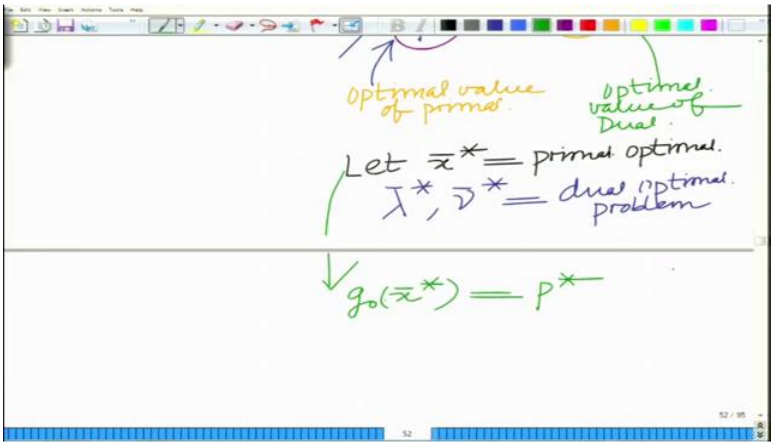

Now, let $\bar{x}^*$ is the primal optimal solution and let $\bar{\lambda}^*, \bar{\nu}^*$ be the solution of the dual problem. Then by strong duality, $P^* = d^*$, $g_0(\bar{x}^*) = P^*$ and $g_d(\bar{\lambda}^*, \bar{\nu}^*) = d^*$.

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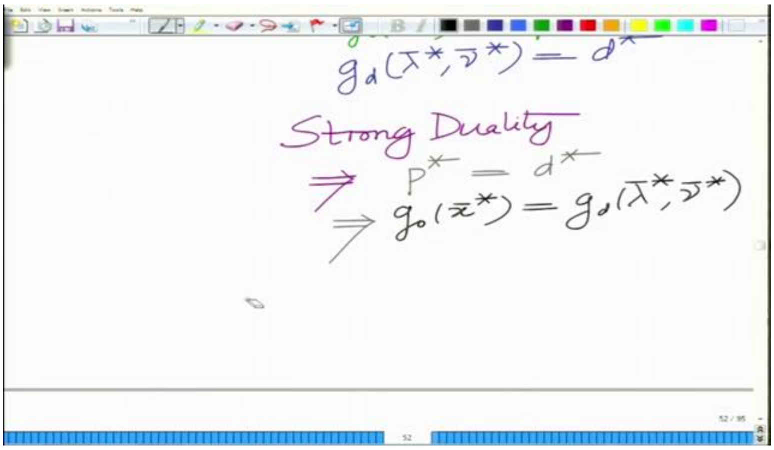

So this implies $g_0(\bar{x}^*) = g_d(\bar{\lambda}^*, \bar{\nu}^*)$.

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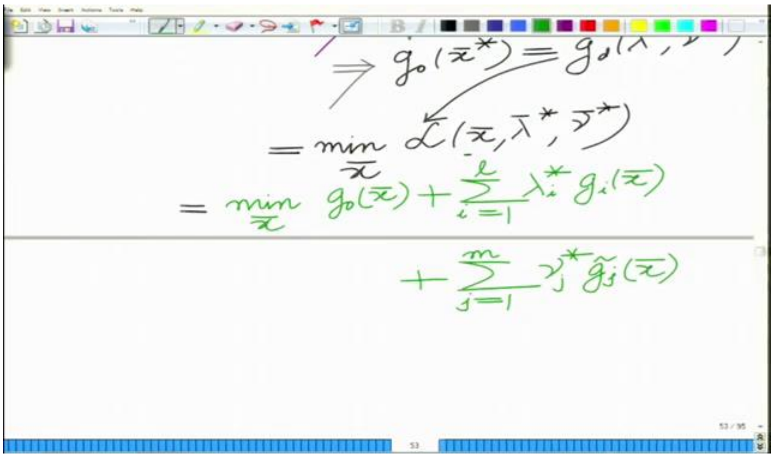

Now pn solving this as shown in the slides we have $g_0(\bar{x}^*) \le g_0(\bar{x}^*)$.

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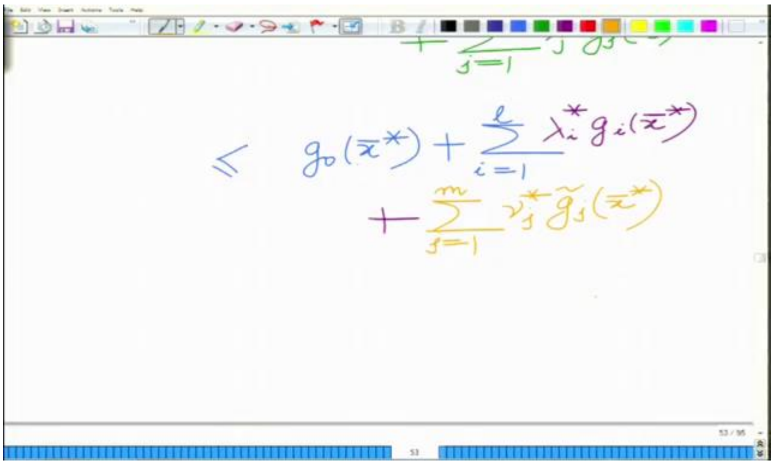

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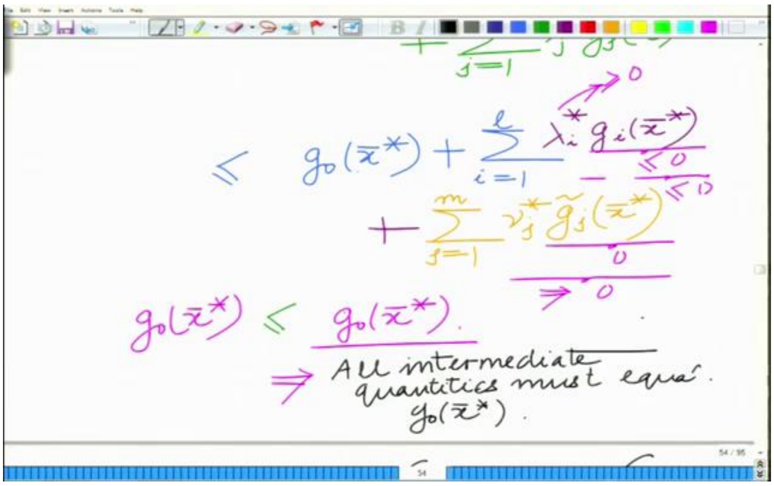

This implies all the intermediate quantities which are sandwiched in between must be also equal to $g_0(\bar{x}^*)$.

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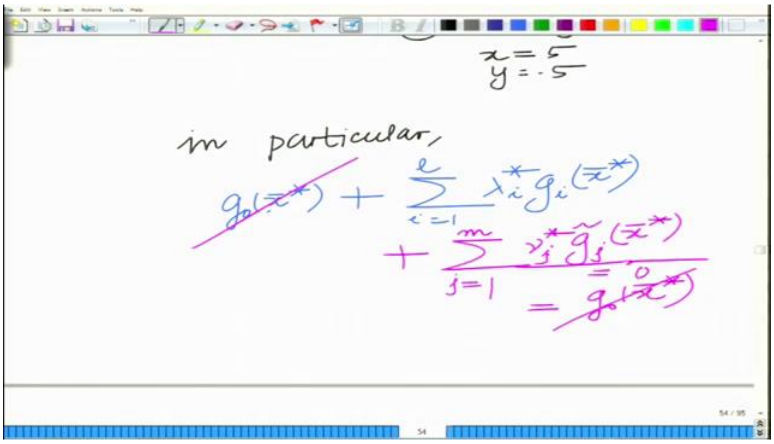

And therefore, this implies proceeding further as shown in slides, we have $\lambda_i g_i(\bar{x}) = 0$.

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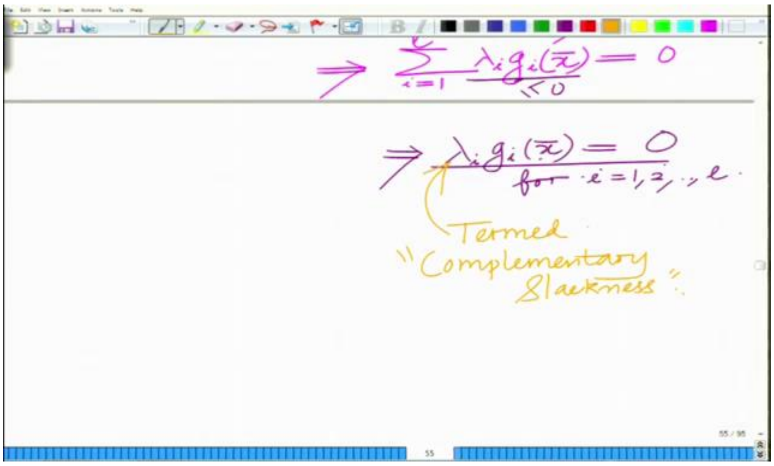

So this is the very interesting property, this is termed as complimentary slackness.

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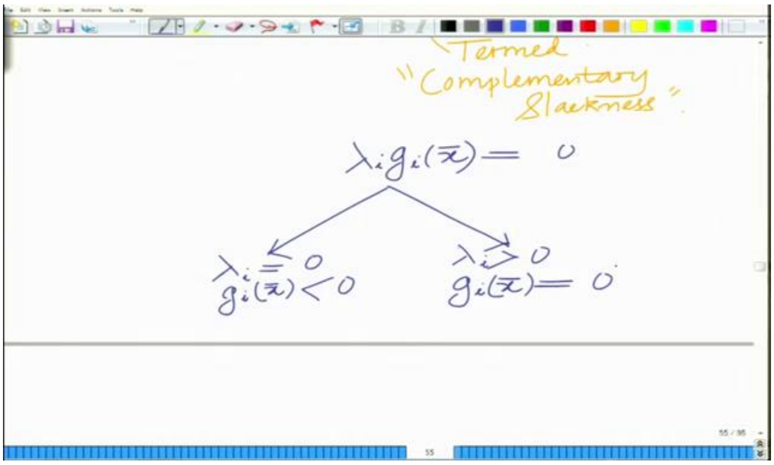

So this implies two things either $\lambda_i = 0$, which means $g_i(\bar{x}) < 0$ or $\lambda_i > 0$ and $g_i(\bar{x}) = 0$.

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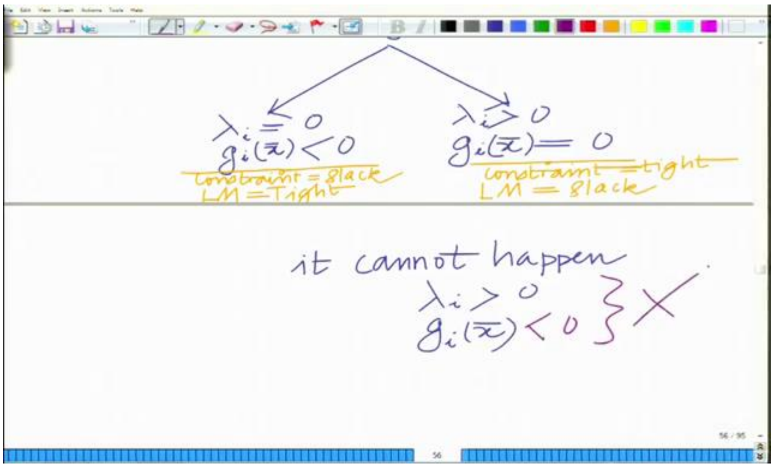

So in the first case the constraint is slack and Lagrange multiplier is tight. In the second case, the constraint is tight and Lagrange multiplier is slack. So this is the meaning of the complimentary slack that is either the Lagrange multiplier is slack or the constraint is slack. It cannot happen that both the Lagrange multiplier and the constraint are slack.

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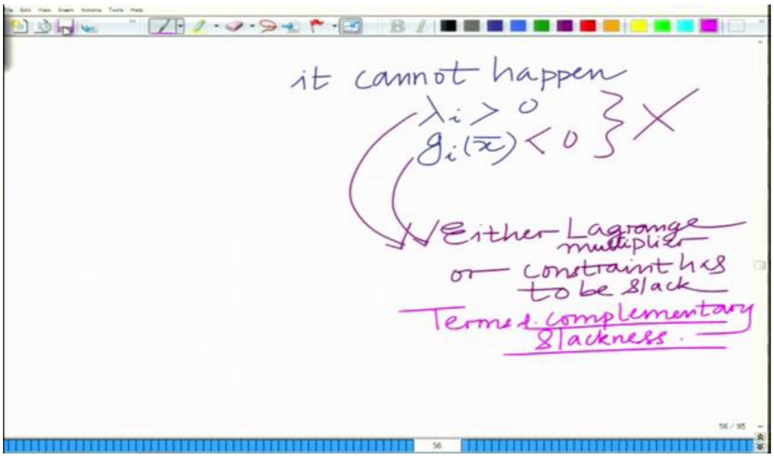

So these complement each other and this is termed as complimentary. This property is termed as complimentary slackness. And this is a unique aspect of the KKT conditions.

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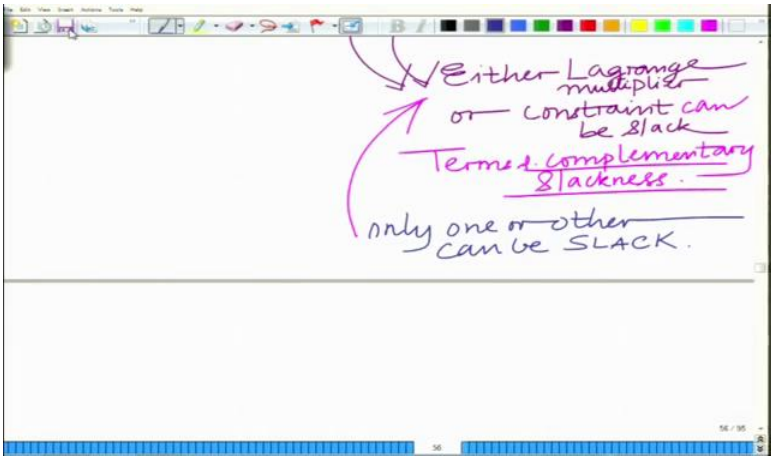

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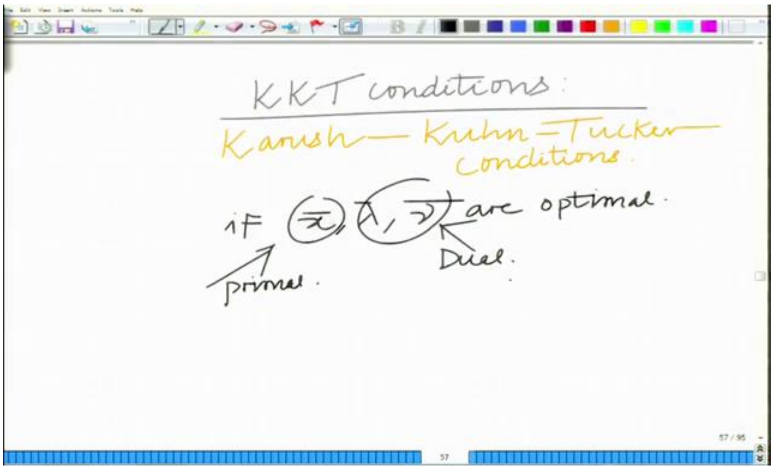

And the KKT conditions can be finally stated as follows, if $\bar{x}, \bar{\lambda}, \bar{\nu}$ are optimal and strong duality holds, this implies that $g_0(\bar{x}) = g_d(\bar{\lambda}, \bar{\nu})$.

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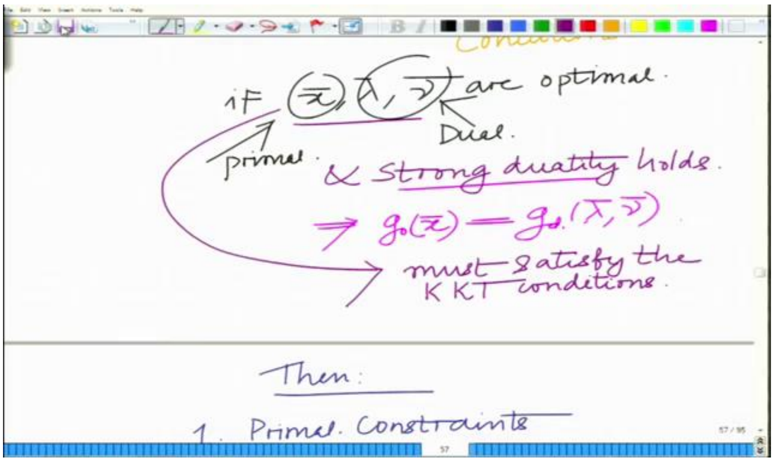

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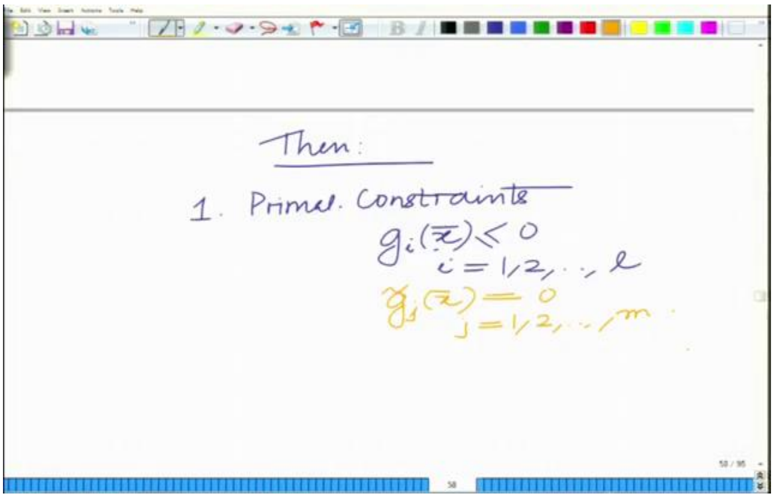

Then it must be that, first, the primal constraints are basically $\begin{matrix} g_i(\bar{x}) \leq 0 \\ i = 1, 2, \ldots, l \\ g_j(\bar{x}) = 0 \\ j = 1, 2, \ldots, m \end{matrix}$.

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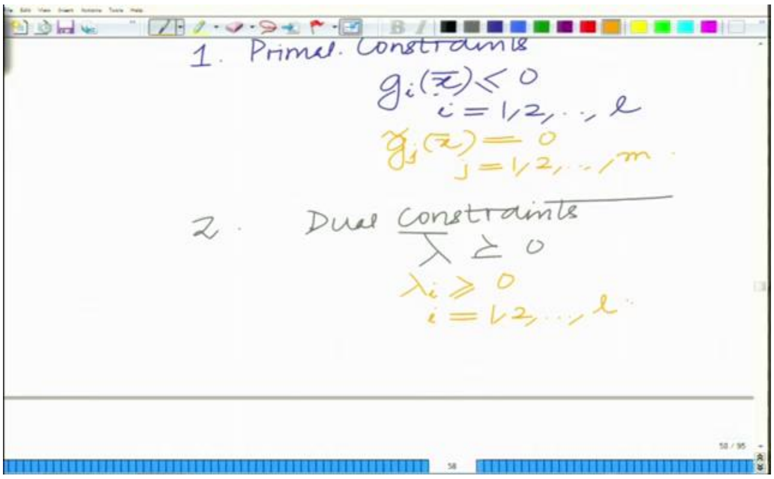

Then the dual constraints $\begin{matrix} \bar{\lambda} \geq 0 \\ \lambda_i \geq 0 \\ i = 1, 2, \ldots, l \end{matrix}$ must hold.

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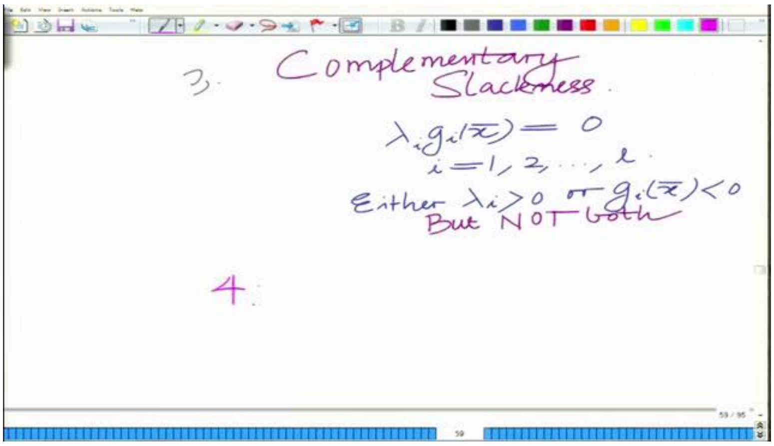

The third condition is complimentary slackness that is $\lambda_i g_i(\bar{x}) = 0$, $i = 1, 2, .., l$ that is either $\lambda_i > 0$ or $g_i(\bar{x}) < 0$ but not both. And finally, since at $\bar{x}$ you have the minimum of the Lagrangian function, the gradient with respect to $\bar{x}$ of the Lagrangian must vanish at this point $\bar{x}$ and this is as shown in the slide and we have $\nabla_{\bar{x}} L(\bar{x}, \bar{\lambda}, \bar{\nu}) = 0$.

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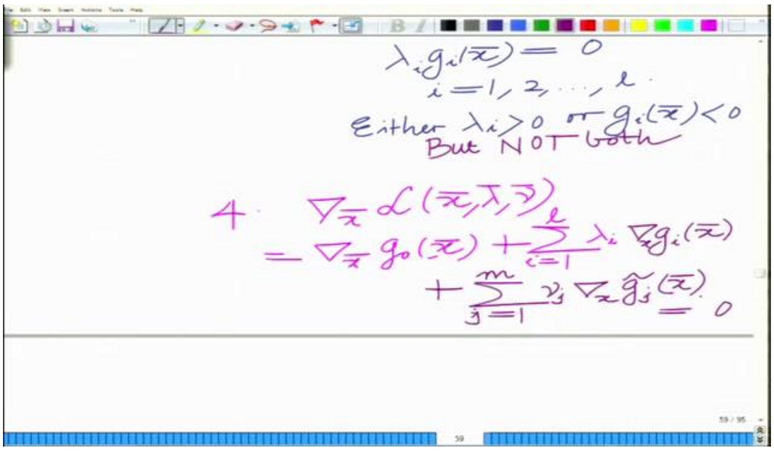

So these are the four KKT conditions that must be satisfied by the solution of the primal optimization problem and the dual optimization. So let us stop here and continue in the subsequent modules. Thank you very much.