Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture – 04
Inner Product Space and its Properties: Linearity, Symmetry and Positive Semi-definite

Hello, welcome to another module in this massive open online course. We are looking at the mathematical preliminaries for optimization; let us continue our discussion with another concept namely an inner product space.

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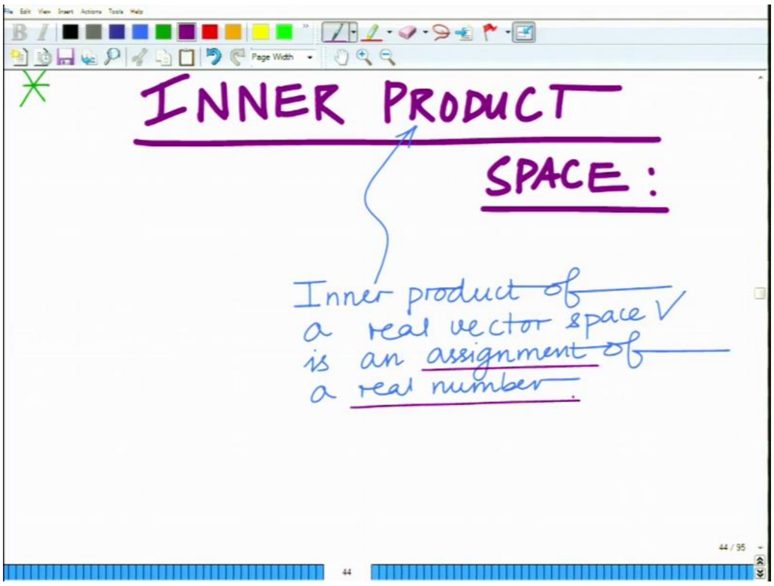

An inner product space of a real vector space is an assignment of a real number.

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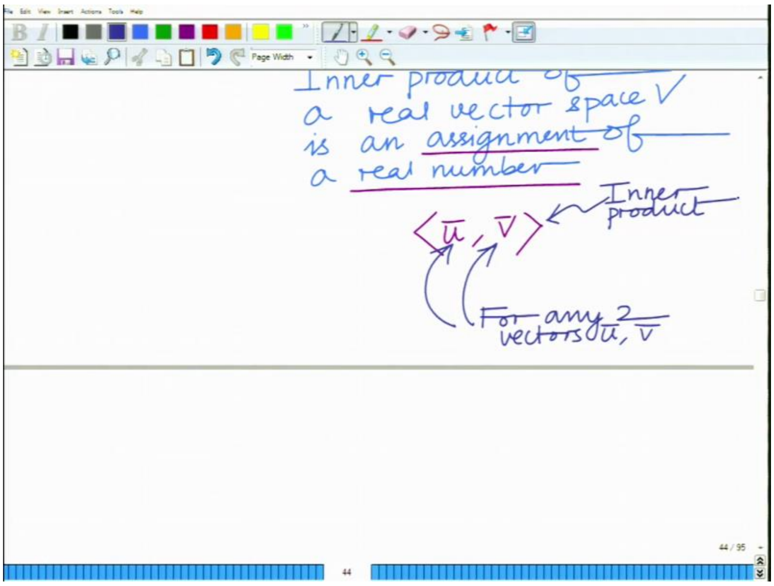

And for any two real vectors $\bar{u}$ and $\bar{v}$, the inner product space is denoted as $\langle \bar{u}, \bar{v} \rangle$ and it is a real number. Therefore the inner product satisfies the following properties.

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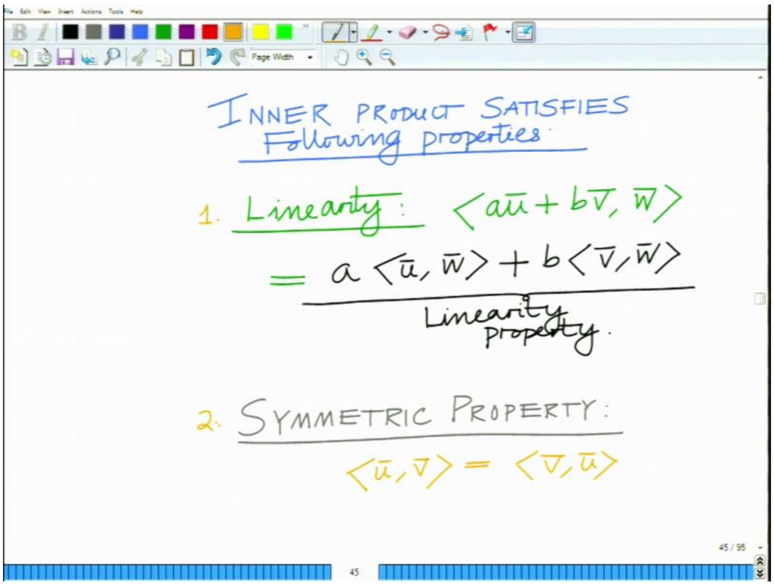

First property is Linearity property. That is the inner product of a linear combination of two real vectors is the linear combination of the inner products. In other words, $\bar{u}$, $\bar{v}$ and $\bar{w}$ are linear if it satisfies the following for some real constants a and b.

$$\langle a\bar{u} + b\bar{v}, \bar{w} \rangle = a\langle \bar{u}, \bar{w} \rangle + b\langle \bar{v}, \bar{w} \rangle$$

The second property is symmetric property. Any two real vectors $\bar{u}$ and, $\bar{v}$ are symmetric if their inner product space satisfies the following.

$$\langle \bar{u}, \bar{v} \rangle = \langle \bar{v}, \bar{u} \rangle$$

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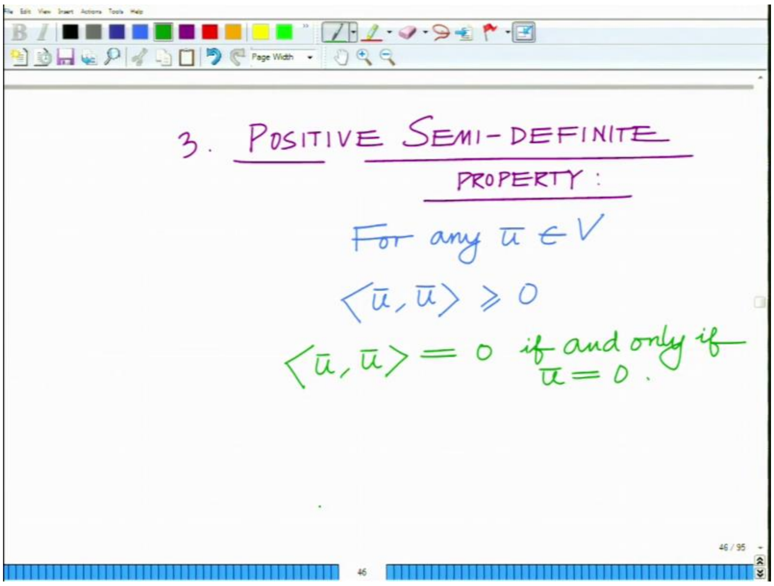

Now the third property is the positive semi definite property. It is that for any vector $\bar{u}$ of the vector space V, the inner product of $\bar{u}$ with itself must be greater than or equal to 0 and also, more importantly, this inner product is zero if and only if the vector $\bar{u}$ is a zero vector. Therefore we can say that for any $\bar{u} \in V$

$\langle \bar{u}, \bar{u} \rangle \geq 0$ and only if $\langle \bar{u}, \bar{u} \rangle = 0$.

So, it is an assignment for a real vector space as it satisfies the linearity, symmetric, symmetry and the positive semi definite properties. Let us look at a simple example to understand this. For instance, let us consider the standard dot product of two vectors.

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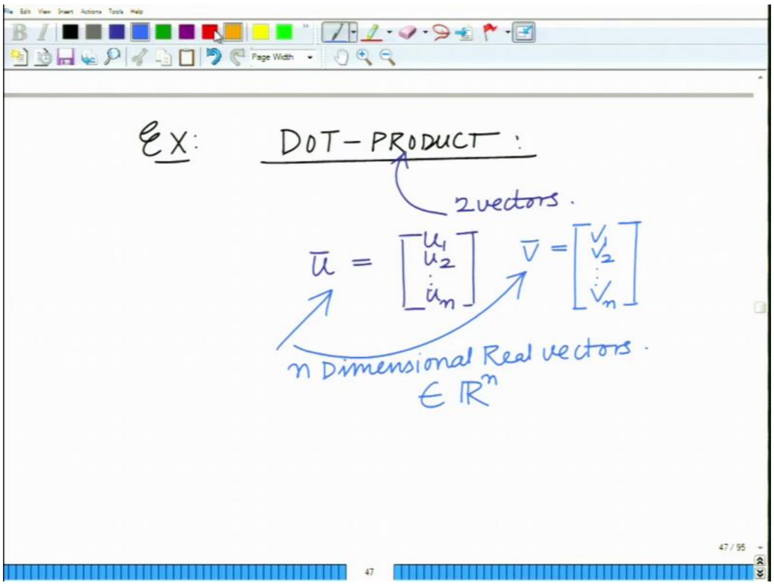

Consider two vectors $\bar{u} \in \mathbb{R}^n$ and $\bar{v} \in \mathbb{R}^n$.

$$\bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

These are real n-dimensional vectors and this is also termed as the Euclidean n-space.

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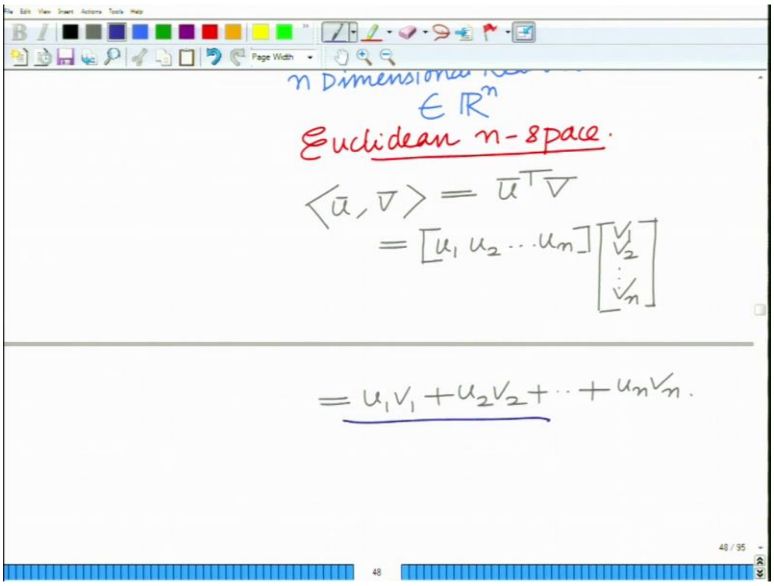

So the inner product in this Euclidean n-space between two vectors is defined as

$$
\begin{aligned}
\langle \bar{u}, \bar{v} \rangle &= \bar{u}^T \bar{v} \\
&= \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \\
&= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n
\end{aligned}
$$

So, this is the dot product between two n-dimensional real vectors.

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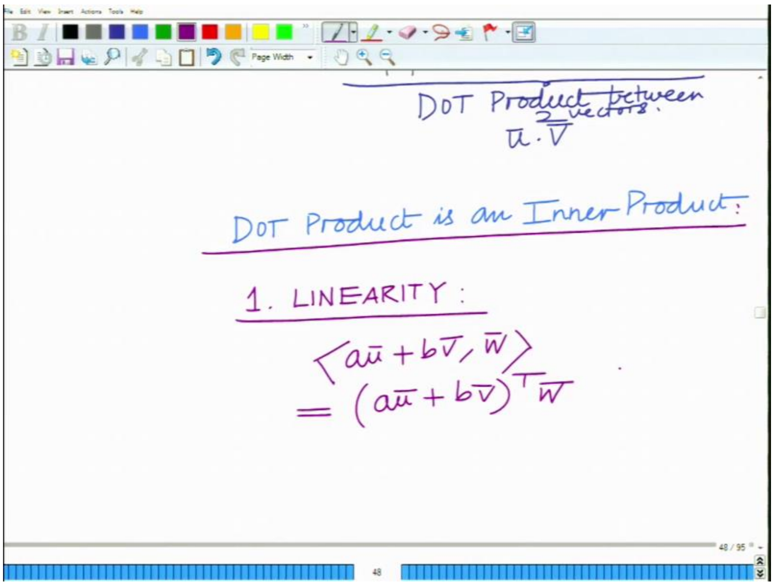

Now we will show that the dot product is an inner product. So let us look at the linearity property. As the definition of the dot product;

$$\langle a\bar{u} + b\bar{v}, \bar{w} \rangle = (a\bar{u} + b\bar{v})^T \bar{w}$$

On simplifying this

$$
\begin{aligned}
\langle a\bar{u} + b\bar{v}, \bar{w} \rangle &= (a\bar{u} + b\bar{v})^T \bar{w} \\
&= a \cdot \bar{u}^T \bar{w} + b \cdot \bar{v}^T \bar{w} \\
&= a \langle \bar{u}, \bar{w} \rangle + b \langle \bar{v}, \bar{w} \rangle
\end{aligned}
$$

Therefore, it satisfies the linearity property.

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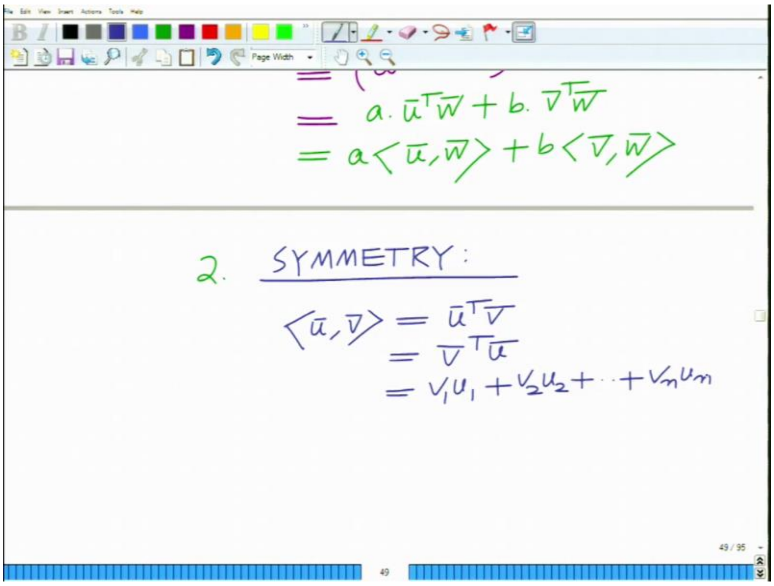

Now coming to the symmetry property, we have

$$
\begin{aligned}
\langle \bar{u}, \bar{v} \rangle &= \bar{u}^T \bar{v} \\
&= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \\
&= \bar{v}^T \bar{u} \\
&= \langle \bar{v}, \bar{u} \rangle
\end{aligned}
$$

Therefore it satisfies the symmetric property also.

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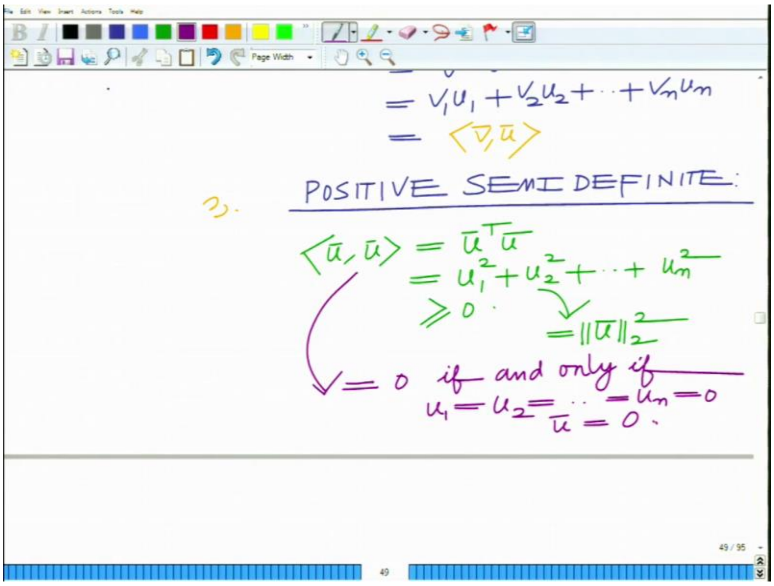

Now for the positive semi definite property,

$$\begin{aligned}\langle \bar{u}, \bar{u} \rangle &= \bar{u}^T \bar{u} \\ &= u_1^{\ 2} + u_2^{\ 2} + \cdots + u_n^{\ 2} \\ &= \|\bar{u}\|_2^2 \qquad \geq 0\end{aligned}$$

And also $\langle \bar{u}, \bar{u} \rangle = 0$ if and only if

$$u_1 = u_2 = \cdots = u_n = 0$$

Or we can say

$$\bar{u} = 0$$

Therefore the dot product of two vectors is a positive semi definite. And hence, it is verified that the dot product is an inner product.

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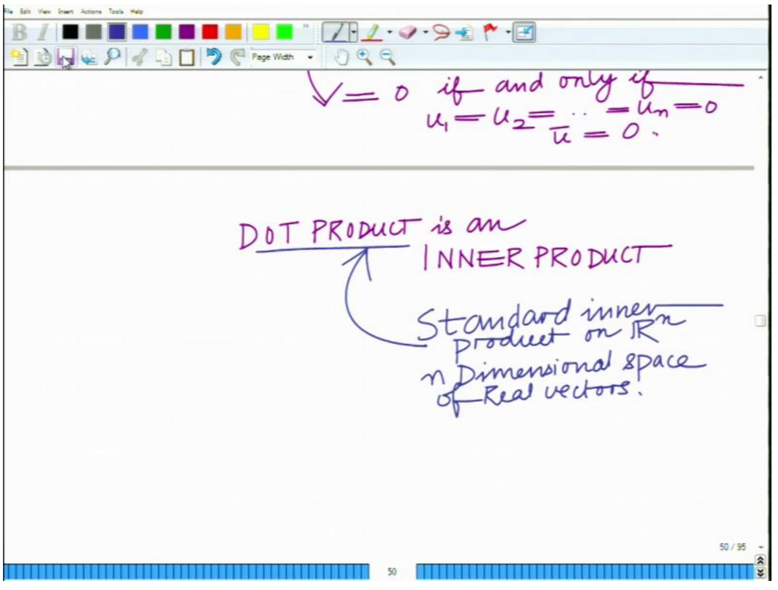

And in fact, this is also termed as the standard inner product on $\mathbb{R}^n$ that is the Euclidean n-space or the n-dimensional set of n-dimensional space of real vectors.

Let us now consider another example for 2 dimensional vectors.

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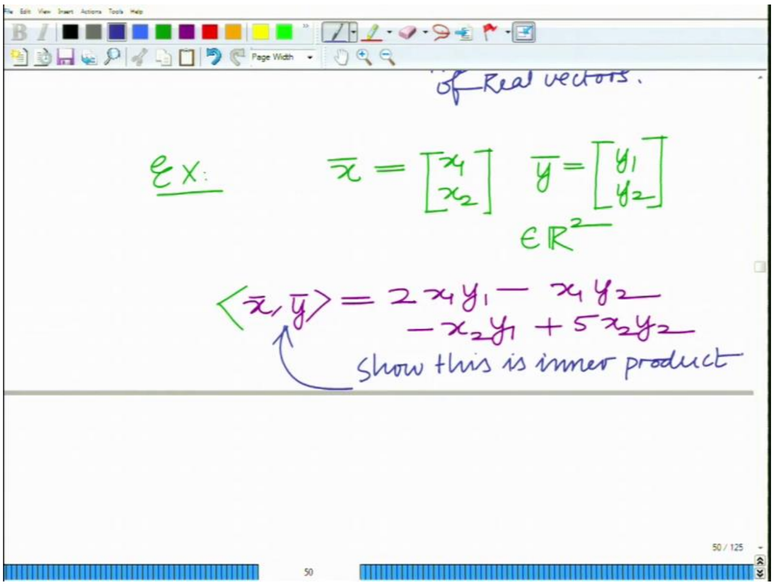

Consider two 2D vectors such as

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \quad , \quad \bar{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$$

So, both of these are basically 2D vectors that is there belong to the 2 dimensional Euclidean space. And let us define this assignment as

$$\langle \bar{x}, \bar{y} \rangle = 2x_1y_1 - x_1y_2 - x_2y_1 + 5x_2y_2$$

This assignment is a valid inner product and this can be shown as follows. Let us start with the linearity property.

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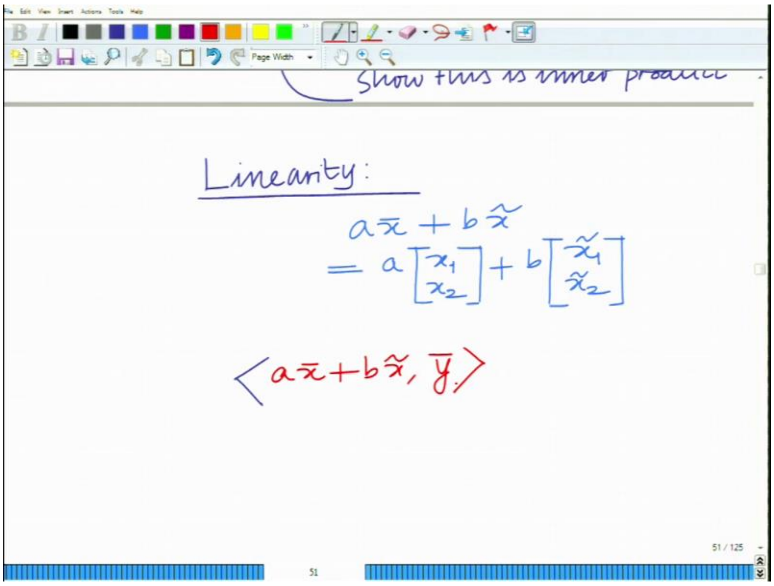

Now let us consider a vector that is

$$a\bar{x} + b\tilde{x} = a\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

This is a linear combination of the 2 vectors. Now for this to be an inner product let us consider

$$\begin{aligned}\langle a\bar{x} + b\tilde{x}, \bar{y}\rangle &= 2(ax_1 + b\tilde{x}_1)y_1 - (ax_1 + b\tilde{x}_1)y_2 - (ax_2 + b\tilde{x}_2)y_1 + 5(ax_2 + b\tilde{x}_2)y_2 \\ &= a\langle \bar{x}, \bar{y}\rangle + b\langle \tilde{x}, \bar{y}\rangle\end{aligned}$$

Which means it is linear. Therefore, it implies that this assignment is linear.

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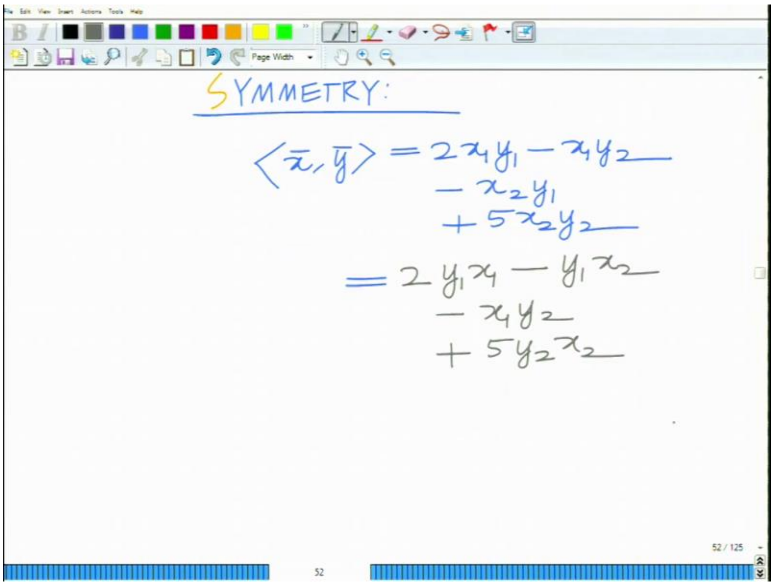

Now let us come to the symmetric property and this can be shown as follows.

$$
\begin{aligned}
\langle \bar{x}, \bar{y} \rangle &= 2x_1y_1 - x_1y_2 - x_2y_1 + 5x_2y_2 \\
&= 2y_1x_1 - y_1x_2 - y_2x_1 + 5y_2x_2 \\
&= \langle \bar{y}, \bar{x} \rangle
\end{aligned}
$$

And hence, it implies that it satisfies the symmetry property.

Now let us come to the positive semi definite property.

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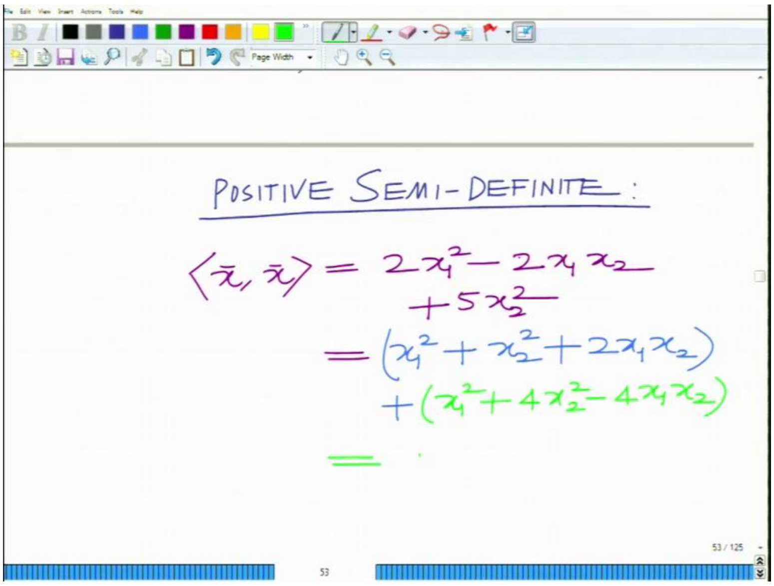

$$
\begin{aligned}
\langle \bar{x}, \bar{x} \rangle &= 2x_1x_1 - x_1x_2 - x_2x_1 + 5x_2x_2 \\
&= 2x_1^{\ 2} - 2x_1x_2 + 5x_2^{\ 2} \\
&= \left(x_1^{\ 2} + x_2^{\ 2} + 2x_1x_2\right) + \left(x_1^{\ 2} + 4x_2^{\ 2} - 4x_1x_2\right) \\
&= \left(x_1 + x_2\right)^2 + \left(x_1 - 2x_2\right)^2
\end{aligned}
$$

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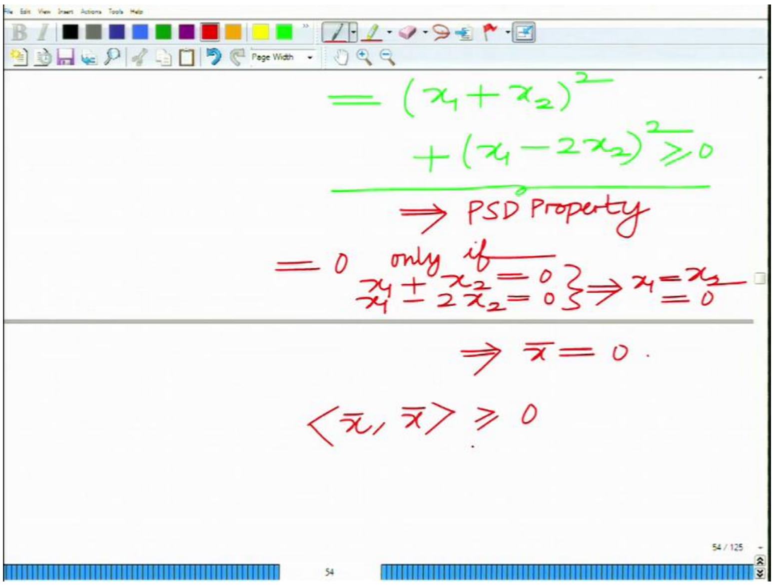

Here $\langle \bar{x}, \bar{x} \rangle \geq 0$ only if

$$
\left.\begin{aligned} x_1 + x_2 &= 0 \\ x_1 - 2x_2 &= 0 \end{aligned}\right\} \Rightarrow x_1 = x_2 = 0 \text{ that is } \bar{x} = 0.
$$

So it is shown that it also satisfies the PSD property. And because it satisfies the linearity, symmetry and PSD properties, hence this is an valid inner product space.

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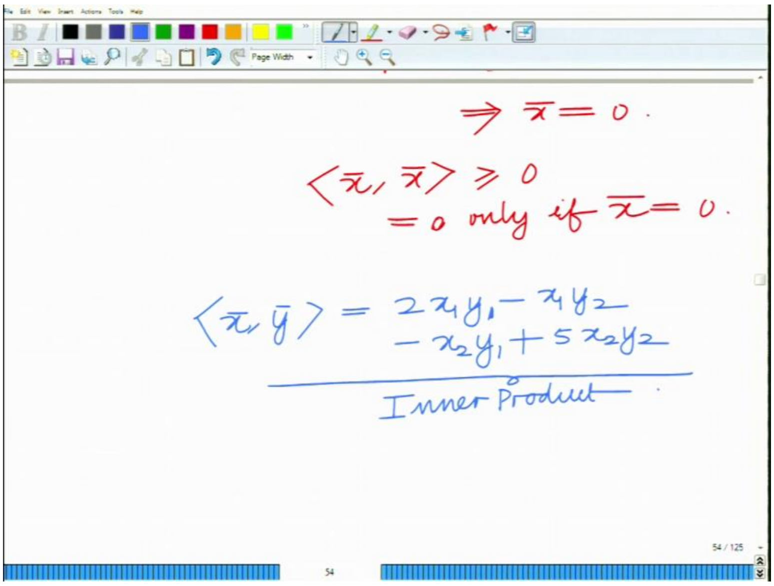

So, we can write it as

$$
\begin{aligned}
\langle \bar{x}, \bar{y} \rangle &= 2x_1y_1 - x_1y_2 - x_2y_1 + 5x_2y_2 \\
&= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\
&= \bar{x}^T A \bar{y}
\end{aligned}
$$

And now we claim that

$$
A = \begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix}
$$

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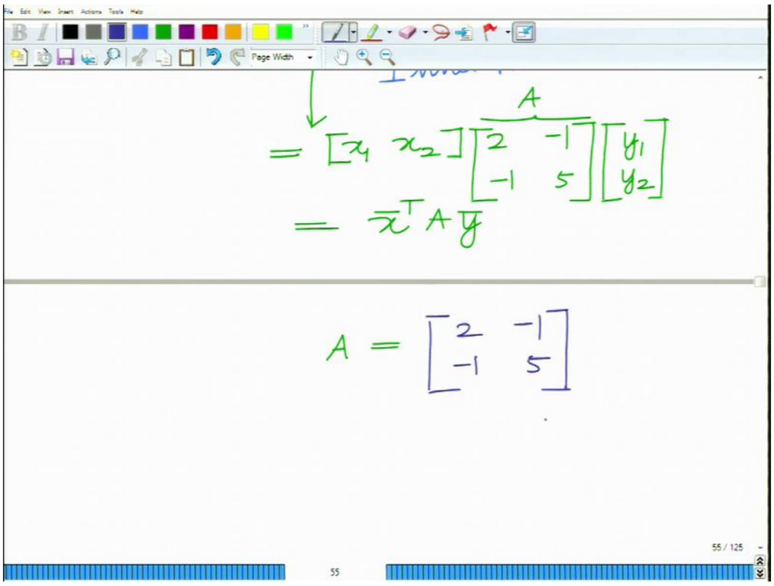

And now a very interesting property of this matrix is that matrix A is a positive definite matrix.

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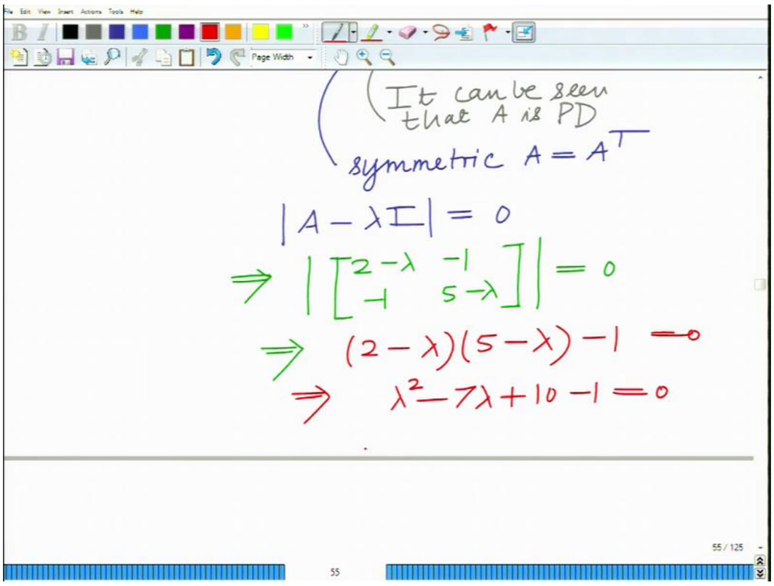

To show this we write the characteristic polynomial of A equals to 0. And hence we get

$$|A-\lambda I|=0$$

$$\left|\begin{pmatrix} 2-\lambda & -1 \\ -1 & 5-\lambda \end{pmatrix}\right|=0$$

$$(2-\lambda)(5-\lambda)-1=0$$

$$\lambda^2-7\lambda+9=0$$

$$\lambda=\frac{7\pm\sqrt{13}}{2}$$

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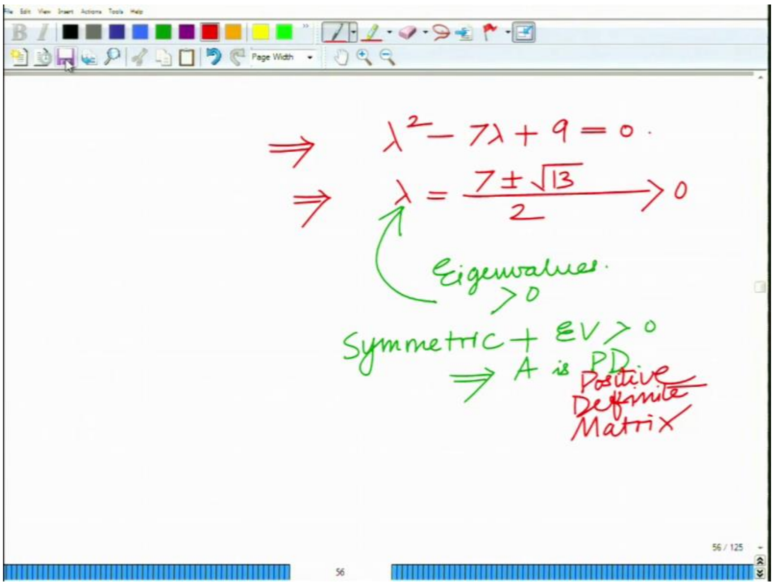

And we can see that both the Eigen values are greater than 0 which means that Eigen values are strictly greater than 0. Also A is a symmetric matrix. Therefore, A is a positive definite matrix.

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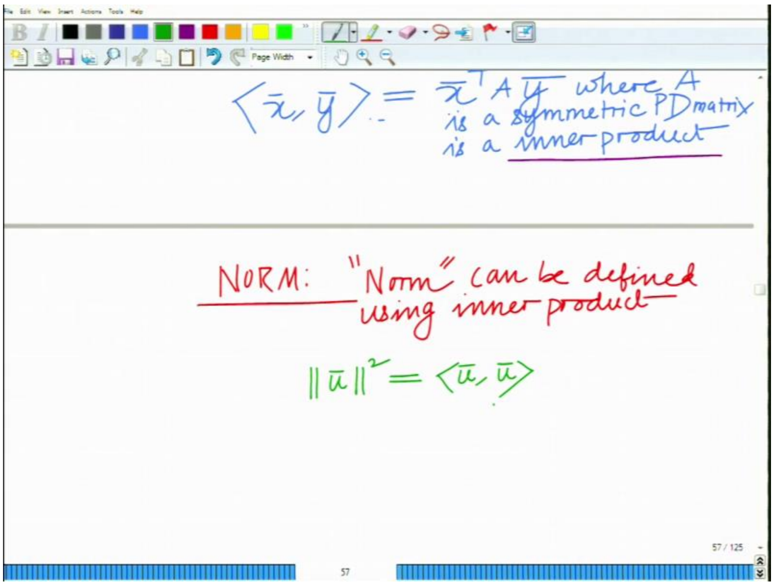

Therefore if we define that $\langle \bar{x}, \bar{y} \rangle = \bar{x}^T A \bar{y}$ where A is a symmetric positive matrix, then A is an inner product.

One of the other interesting aspects of the inner product is that it can also be used to define a norm. It induces a norm and given as follows.

$$\|\bar{u}\|^2 = \langle \bar{u}, \bar{u} \rangle$$

So, the norm is an inner product.

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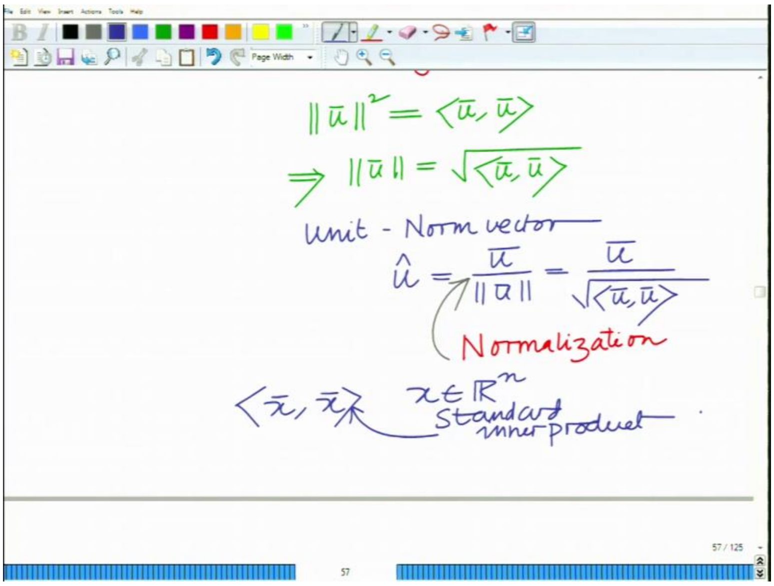

This means that the norm of vector $\bar{u}$ is equal to

$$\|\bar{u}\|^2 = \langle \bar{u}, \bar{u} \rangle$$
$$\|\bar{u}\| = \sqrt{\langle \bar{u}, \bar{u} \rangle}$$

And the unit norm vector can now be defined as

$$\hat{u} = \frac{\bar{u}}{\|\bar{u}\|}$$
$$= \frac{\bar{u}}{\sqrt{\langle \bar{u}, \bar{u} \rangle}}$$

This process is termed as the normalization that is, when a vector is divided by its norm then it means that the vector is normalized. And this is true for the standard inner product on $\mathbb{R}^n$ that is if we look at $\langle \bar{x}, \bar{x} \rangle$ where $\bar{x} \in \mathbb{R}^n$; this is the standard inner product.

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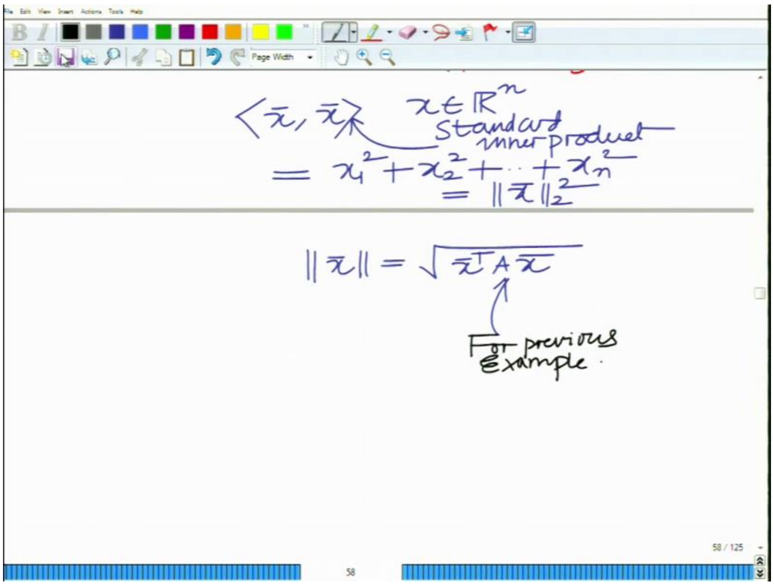

We already see in that inner product of xBar with itself is basically this is

$$
\begin{aligned}
\langle \bar{x}, \bar{x} \rangle &= x_1^{\ 2} + x_2^{\ 2} + \cdots + x_n^{\ 2} \\
&= \|\bar{x}\|_2^2
\end{aligned}
$$

And therefore, this is the square of the l_2 norm and we have already seen that the norm is given as the square root of the inner product of vector with itself; which means the norm of a vector under that inner product is given as

$$
\|\bar{x}\| = \sqrt{\bar{x}^T A \bar{x}}
$$

Let us look at other examples of inner products.

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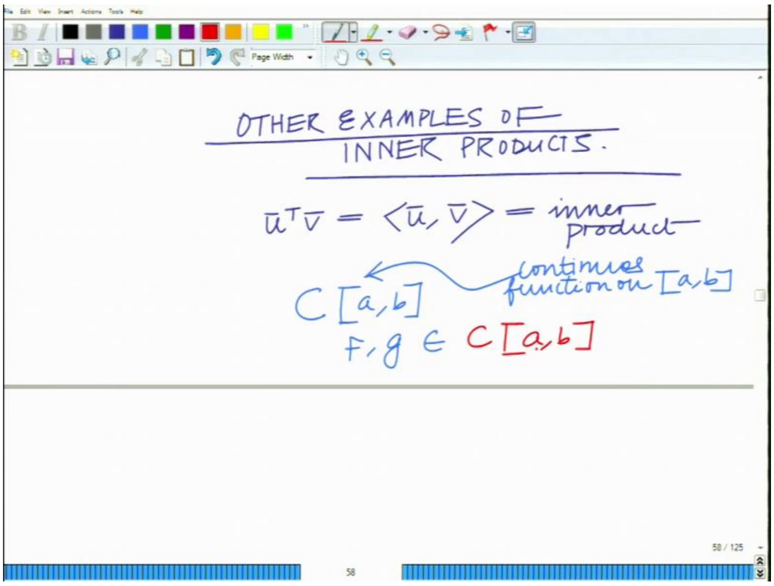

Let us consider the space of continuous functions on an interval [a,b] denoted by $C[a,b]$. Let us say we have two functions f and g such that

$$f \in C[a,b], \quad g \in C[a,b]$$

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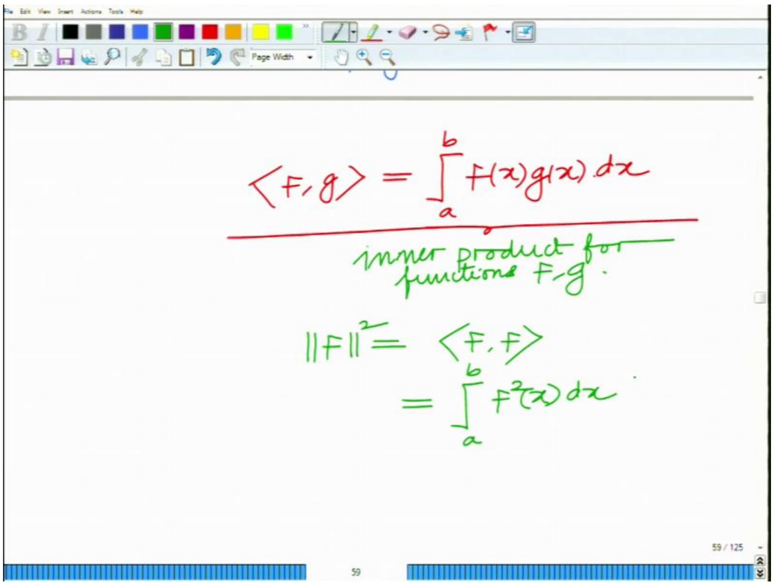

Then the assignment defined as

$$\langle f,g\rangle=\int_a^b f(x)g(x)dx$$

This is an inner product for functions *f* and *g*, and in fact, the norm that arises is basically nothing but

$$\|f\|^2=\langle f,f\rangle$$
$$=\int_a^b f^2(x)dx$$

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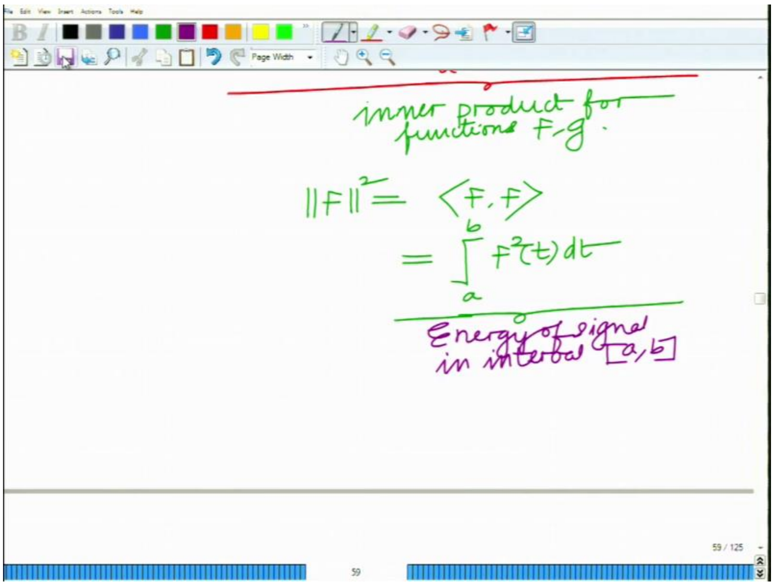

Now if we look at *f(x)* as signal in time, then $f^2(x)$ is nothing but the energy of the signal in interval [a,b] This is an important application of inner product space on the space of continuous functions.

Lets discuss an another interesting example of this inner product space. Consider the space of $m\times n$ matrices.

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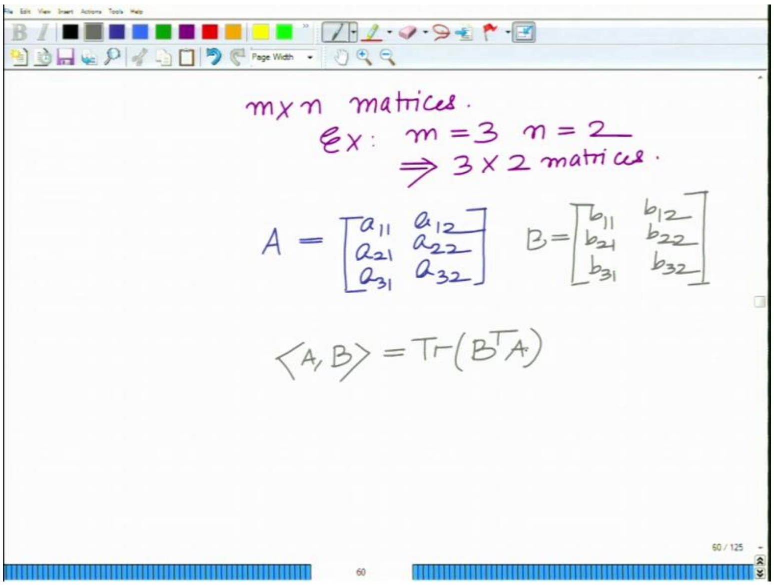

Such as

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \quad , \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

And the inner product of is defined as trace of $\left(B^T A\right)$ that is

$$\langle A, B \rangle = Tr\left(B^T A\right)$$

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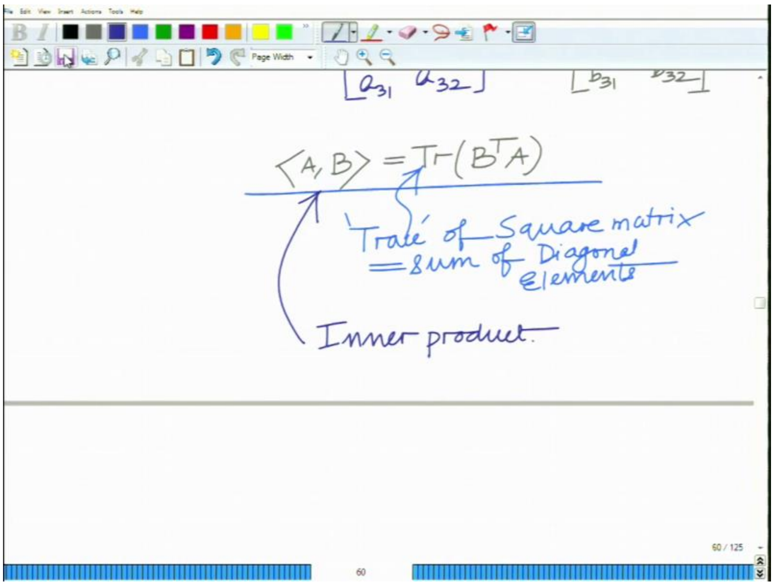

The trace of a square matrix is defined as the sum of the diagonal elements of a square matrix and this can be shown to be an inner product.

So, in this module is we have looked at the inner product, its definition, the various properties and several examples. We will continue this discussion in the next module.

Thank you very much.