**Applied Optimization for Wireless, Machine Learning, Big Data**
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**Lecture – 05**
**Inner Product Space and its Properties: Cauchy Schwarz Inequality**

Welcome to another module in this massive open online course. So, we are looking at the inner product of the matrices and we already have defined the inner product of two matrices in the previous module.

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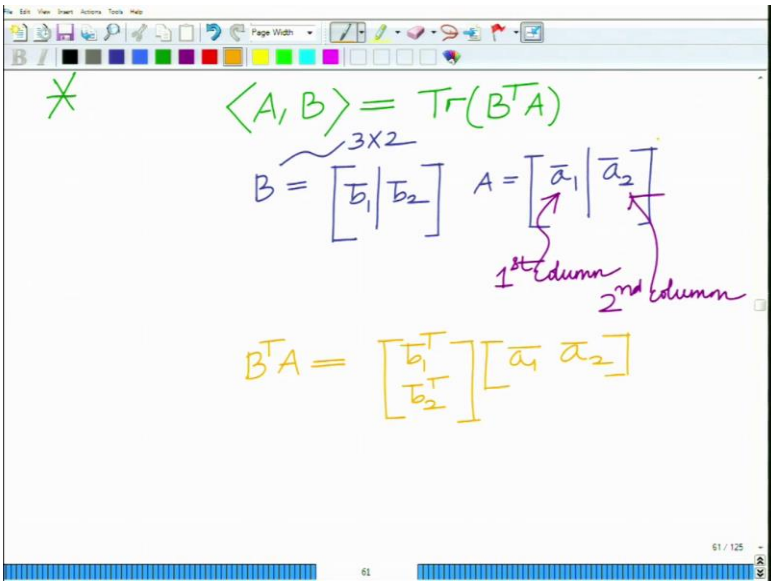


In fact, it is shown that the inner product of two matrices A and B of the same size is equal to the trace of $\left(B^T A\right)$. That is

$$\langle A, B\rangle = Tr\left(B^T A\right)$$

For instance, for

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \quad , \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

Let us say that first and second columns of A as $a_1$ and $a_2$; and similarly first and second columns of B as $b_1$ and $b_2$. Then;

$$A = [\bar{a}_1 \mid \bar{a}_2]$$
$$B = [\bar{b}_1 \mid \bar{b}_2]$$

And hence;

$$B^T A = \begin{bmatrix} \bar{b}_1^T \\ \bar{b}_2^T \end{bmatrix} [\bar{a}_1 \mid \bar{a}_2]$$

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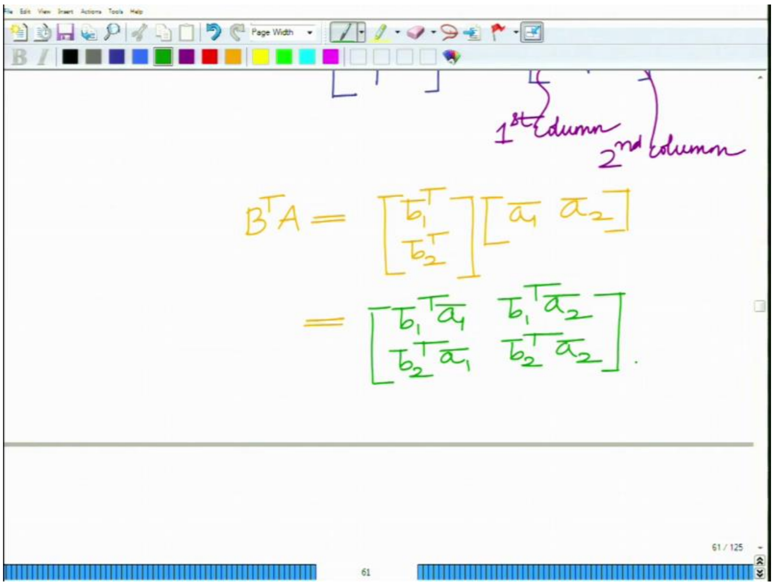


And it will be a $2 \times 2$ matrix and is as follows.

$$B^T A = \begin{bmatrix} \bar{b}_1^T \\ \bar{b}_2^T \end{bmatrix} [\bar{a}_1 \mid \bar{a}_2]$$
$$= \begin{bmatrix} \bar{b}_1^T \bar{a}_1 & \bar{b}_1^T \bar{a}_2 \\ \bar{b}_2^T \bar{a}_1 & \bar{b}_2^T \bar{a}_2 \end{bmatrix}$$

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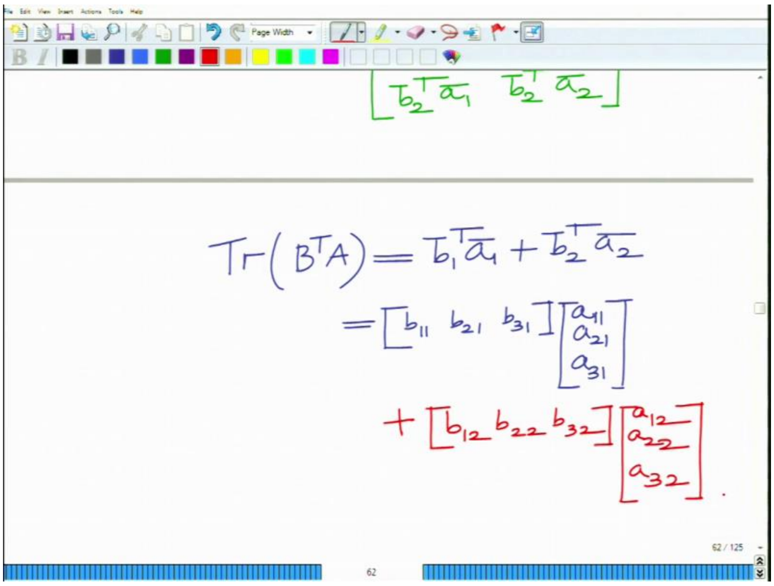


And therefore, the trace of $\left(B^T A\right)$ is

$$
\begin{aligned}
Tr\left(B^T A\right) &= \bar{b}_1^T \bar{a}_1 + \bar{b}_2^T \bar{a}_2 \\
&= \begin{bmatrix} b_{11} & b_{21} & b_{31} \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + \begin{bmatrix} b_{12} & b_{22} & b_{32} \end{bmatrix} \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \\
&= b_{11}a_{11} + b_{21}a_{21} + b_{31}a_{31} + b_{12}a_{12} + b_{22}a_{22} + b_{32}a_{32} \\
&= \langle A, B \rangle
\end{aligned}
$$

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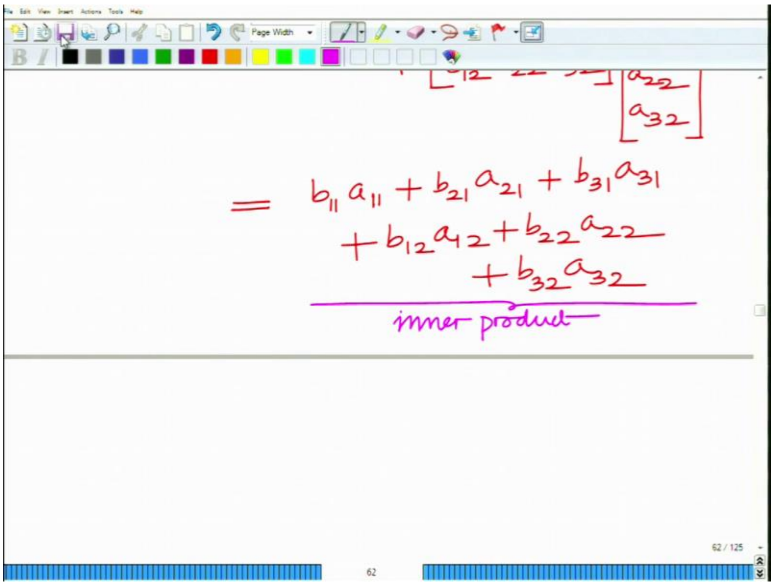

So this is the inner product of A and B. And in general this can be generalized for $m \times n$ matrices.

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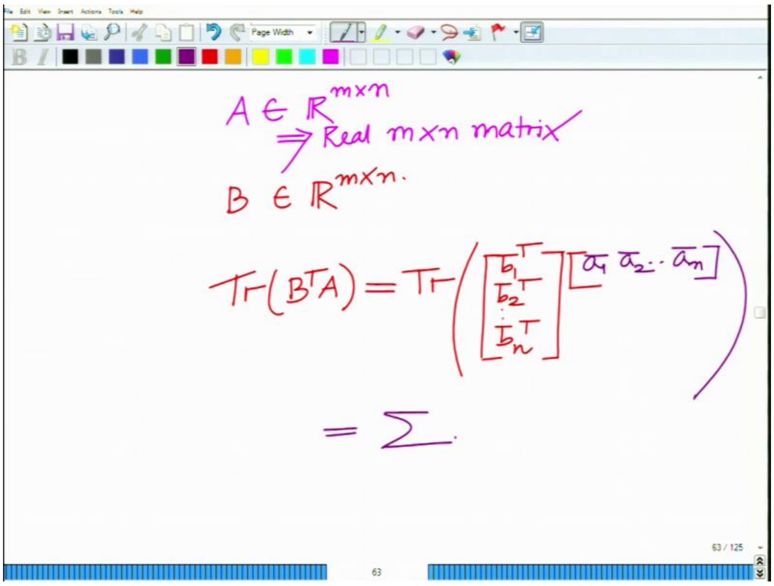


So, In general for $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{m \times n}$; we can say that

$$Tr\left(B^T A\right) = Tr\left(\begin{bmatrix} \bar{b}_1^T \\ \bar{b}_2^T \\ \vdots \\ \bar{b}_n^T \end{bmatrix} \begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_n \end{bmatrix}\right)$$

$$= \sum_{i=1}^{n} \bar{b}_i^T \bar{a}_i$$

Here, n equals the number of columns.

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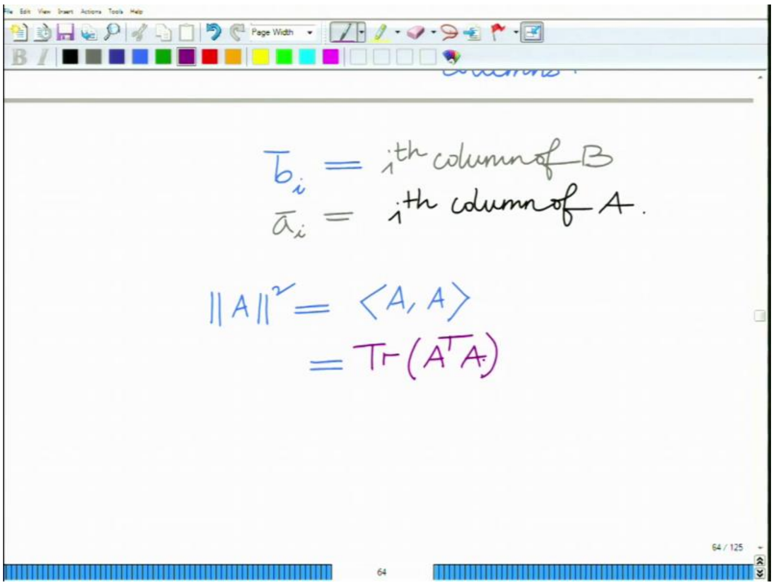


The various quantities are

$\bar{b}_i = i^{th}$ column of $B$

$\bar{a}_i = i^{th}$ column of $A$

And as we know that every inner product induces a norm. Therefore,

$$
\begin{aligned}
\|A\|^2 &= \langle A, A \rangle \\
&= Tr\left(A^T A\right)
\end{aligned}
$$

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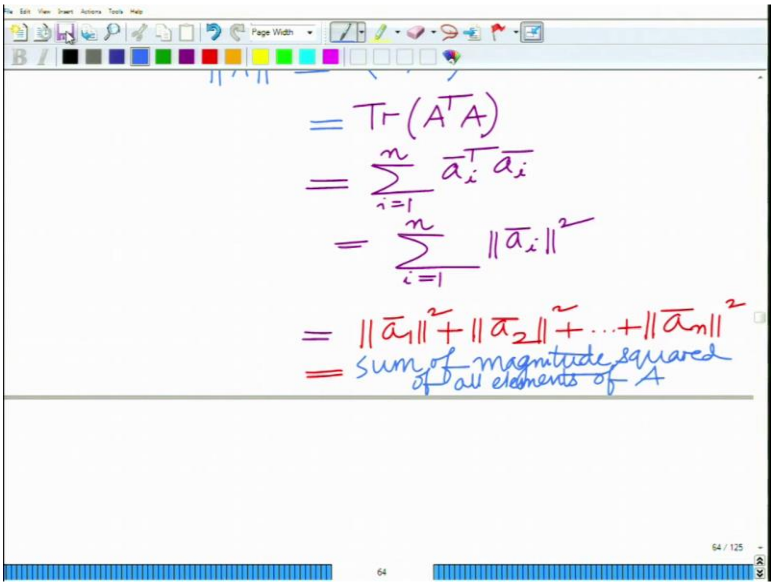


This further can be written as

$$\begin{aligned}
\|A\|^2 &= Tr\left(A^T A\right) \\
&= \sum_{i=1}^{n} \bar{a}_i^T \bar{a}_i \\
&= \sum_{i=1}^{n} \|\bar{a}_i\|^2 \\
&= \|\bar{a}_1\|^2 + \|\bar{a}_2\|^2 + \dots + \|\bar{a}_n\|^2
\end{aligned}$$

which is sum of the squares of the norms of all the columns of matrix A. This is also equal to the sum of magnitude square of all elements of A. This is termed as the Frobenius norm and is denoted as $\|A\|_F$ . So we can it as;

$$\|A\|_F^{\ 2} = \|\bar{a}_1\|^2 + \|\bar{a}_2\|^2 + \dots + \|\bar{a}_n\|^2$$

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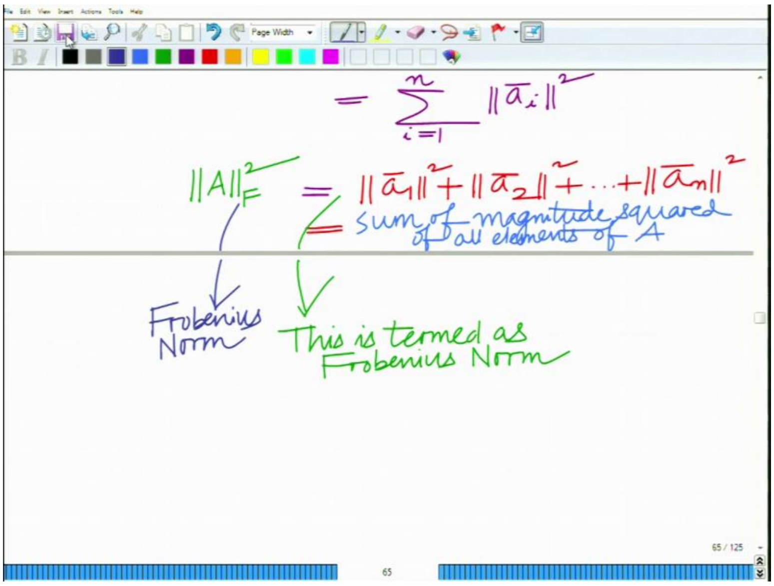


This is the matrix Frobenius norm which is the sum of the norm square of all the columns of the matrix or sum of the magnitude square of all the elements of the matrix. So, this is norm that is also induced by the definition of the inner product corresponding to matrices. The next important aspect is the Cauchy Schwarz inequality.

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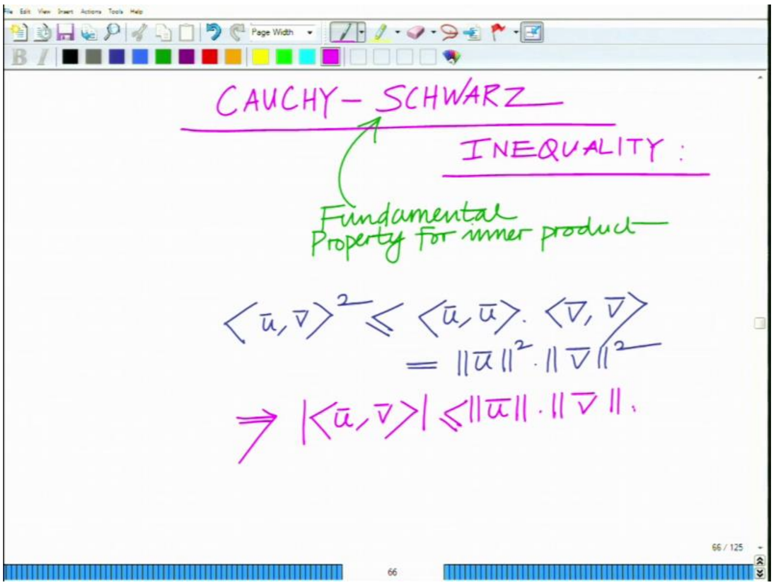


Cauchy Schwarz inequality is very fundamental property of a inner product. It states that

$$\langle \bar{u},\bar{v}\rangle^2 \le \langle \bar{u},\bar{u}\rangle \cdot \langle \bar{v},\bar{v}\rangle$$
$$\le \|\bar{u}\|^2 \cdot \|\bar{v}\|^2$$

This basically implies that the magnitude of the inner product of two matrices is less than or equal to the product of square of individual norms of two matrices.

$$|\langle \bar{u},\bar{v}\rangle| \le \|\bar{u}\| \cdot \|\bar{v}\|$$

This is the Cauchy Schwarz inequality. This is valid for either the inner product for the vectors or inner product of the functions. So, this is valid for any general definition of the inner product.

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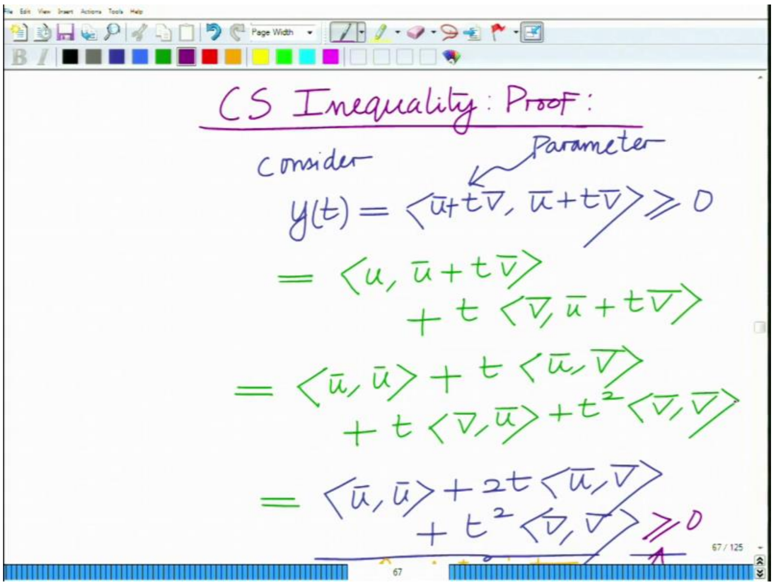


The Cauchy Schwarz inequality can be proved as follows. Consider a function $y(t)$ such as;

$$y(t) = \langle \bar{u}+t\bar{v}, \bar{u}+t\bar{v}\rangle \qquad \ge 0$$

This t is a parameter. And now on expanding the inner product, we get

$$y(t) = \langle \bar{u}, \bar{u}+t\bar{v}\rangle + t\langle \bar{v}, \bar{u}+t\bar{v}\rangle$$
$$= \langle \bar{u},\bar{u}\rangle + t\langle \bar{u},\bar{v}\rangle + t\langle \bar{v},\bar{u}\rangle + t^2\langle \bar{v},\bar{v}\rangle$$
$$= \langle \bar{u},\bar{u}\rangle + 2t\langle \bar{u},\bar{v}\rangle + t^2\langle \bar{v},\bar{v}\rangle$$

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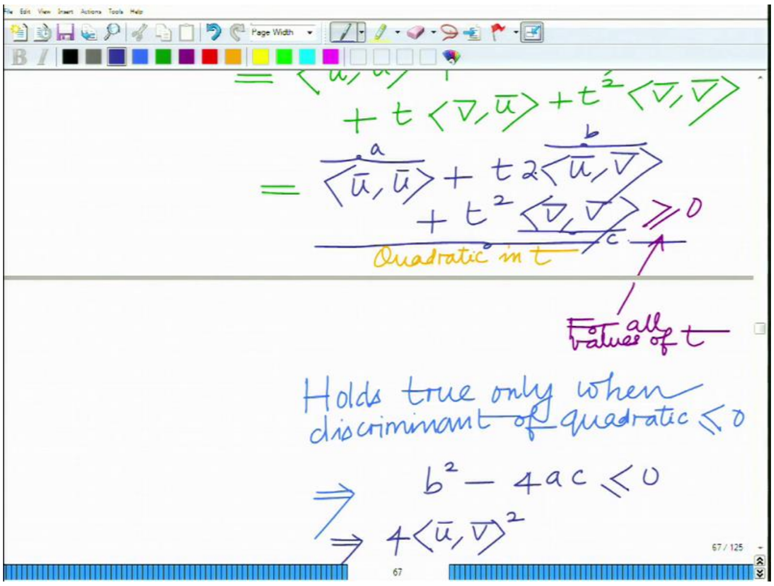


Now observe that this is a quadratic equation in t and according to the property of inner product space, this is always greater than or equal to 0 for all values of t. So, this holds true only when discriminant of the quadratic is less than or equal to 0.

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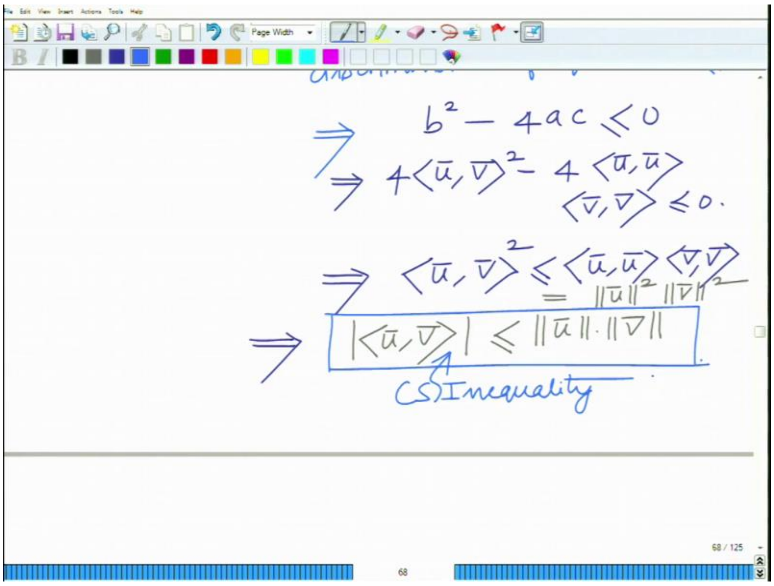


This implies that

$$b^2 - 4ac \le 0$$
$$4\langle \bar{u}, \bar{v} \rangle^2 - 4\langle \bar{v}, \bar{v} \rangle \langle \bar{u}, \bar{u} \rangle \le 0$$
$$\langle \bar{u}, \bar{v} \rangle^2 \le \langle \bar{v}, \bar{v} \rangle \langle \bar{u}, \bar{u} \rangle$$
$$|\langle \bar{u}, \bar{v} \rangle| \le \|\bar{u}\| \cdot \|\bar{v}\|$$

This is basically Cauchy Schwarz inequality. So, it verifies the Cauchy Schwarz inequality that the magnitude of the inner product between two vectors is less than or equal to the product of the their individual norms.

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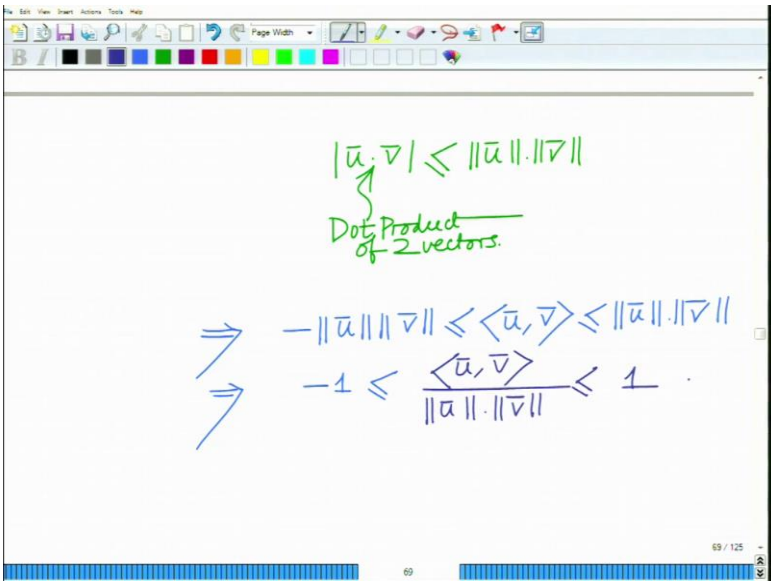


Also we can write it as the magnitude of the dot product. Therefore;

$$|\langle \bar{u}, \bar{v} \rangle| \le \|\bar{u}\| \cdot \|\bar{v}\|$$
$$|\bar{u} \cdot \bar{v}| \le \|\bar{u}\| \cdot \|\bar{v}\|$$

And hence we can also conclude that

$$-\|\bar{u}\| \cdot \|\bar{v}\| \le \langle \bar{u}, \bar{v} \rangle \le \|\bar{u}\| \cdot \|\bar{v}\|$$
$$-1 \le \frac{\langle \bar{u}, \bar{v} \rangle}{\|\bar{u}\| \cdot \|\bar{v}\|} \le 1$$

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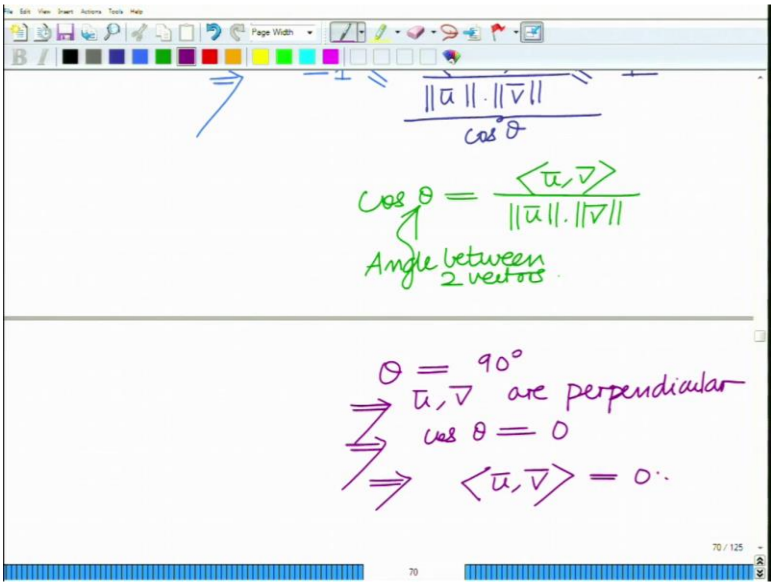


So, this quantity can be defined as the cosine of an angle $\theta$ because cosine $\theta$ lies between minus 1 and 1.

$$\cos\theta = \frac{\langle \bar{u}, \bar{v} \rangle}{\|\bar{u}\| \cdot \|\bar{v}\|}$$

And here $\theta$ is the angle between the two vectors $\bar{u}$ and $\bar{v}$ .

So, if $\theta = 90^0$ that implies the vectors are perpendicular. This implies $\bar{u}$ and $\bar{v}$ are perpendicular to each other. This implies that

$$\langle \bar{u}, \bar{v} \rangle = 0$$

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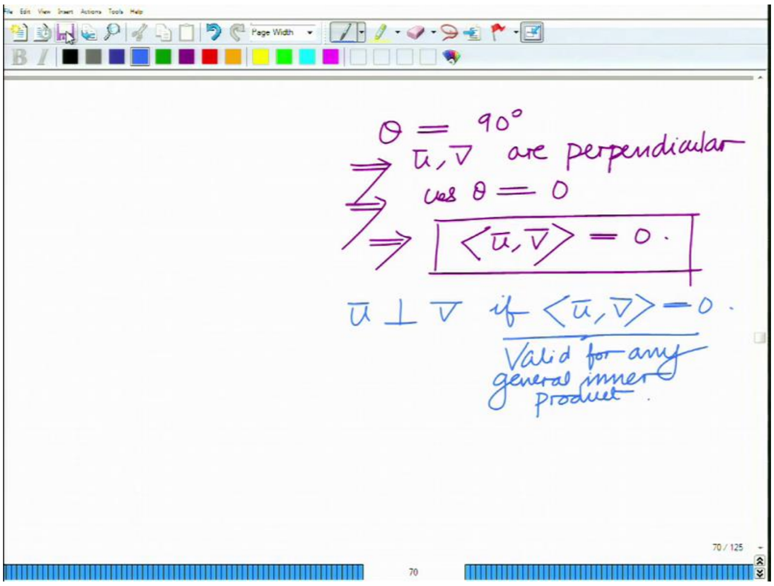


This shows that if the inner product of two vectors is equal to 0 then this means both the vectors are perpendicular to each other. This is valid for any general inner product. This concept of inner product is a very interesting and powerful concept which has a large number of applications and yields several interesting insights. So, we will stop here and continue with other aspects in the subsequent modules.

Thank you very much.