

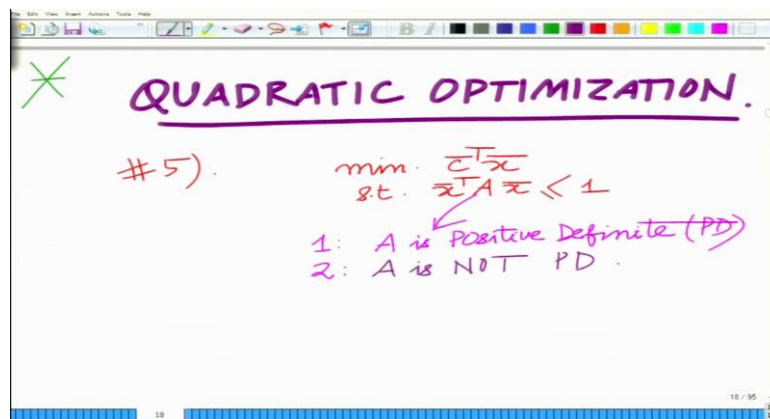
Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture - 72
Examples on Quadratic Optimization

Keywords: Quadratic Optimization

Hello, welcome to another module in this massive open online course. So we are looking at example problems for Convex Optimization. Let us look at another problem that is Quadratic Optimization.

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So the quadratic optimization objective is as follows $\min_{x} c^T x$ and we will consider $s.t. x^T A x \leq 1$

two cases for this, that is when A is positive definite and when A is not positive definite. Let us start with case 1, A is positive definite.

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Case 1: A is PD For P.D matrix L^{-1} exist

$$A = LL^T$$

$$L = A^{1/2}$$

$$y = L^T x$$

$$\Rightarrow x^T A x = x^T L L^T x = y^T y = \|y\|^2$$

When A is positive definite, you can write $A = LL^T$ where $L = A^{\frac{1}{2}}$, this is obtained by the Cholesky decomposition. So for a positive definite matrix in addition, this L is invertible. So we have $\bar{y} = L^T \bar{x}$ and $\bar{x}^T A \bar{x} = \bar{x}^T L L^T \bar{x} = \bar{y}^T \bar{y} = \|\bar{y}\|^2$.

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$$x^T A x = x^T L L^T x = y^T y = \|y\|^2$$

$$\Rightarrow x = (L^T)^{-1} y = L^{-T} y$$

$$c^T x = c^T L^{-T} y = (L^{-1} c)^T y = \tilde{c}^T y$$

Now as shown in slide we have $\bar{x} = (L^T)^{-1} \bar{y} = L^{-T} \bar{y}$. Therefore,

$$\bar{c}^T \bar{x} = \bar{c}^T L^{-T} \bar{y} = (L^{-1} \bar{c})^T \bar{y} = \tilde{c}^T \bar{y}.$$

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Handwritten slide content:

$$= \tilde{c}^T \bar{y}$$

$$\boxed{\tilde{c} = L^T c}$$

min $\tilde{c}^T \bar{y}$
s.t. $\|\bar{y}\| \leq 1$
 $\Rightarrow \|\bar{y}\| \leq 1$

maximum occurs for
 $\bar{y} = \frac{\tilde{c}}{\|\tilde{c}\|}$

So we can write the optimization problem as $\min_{\bar{y}} \tilde{c}^T \bar{y}$ s.t. $\|\bar{y}\| \leq 1$. Now the maximum occurs for

$\bar{y} = \frac{\tilde{c}}{\|\tilde{c}\|}$ which is given as shown in the following slides.

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Handwritten slide content:

$$= \frac{\tilde{c}}{\|\tilde{c}\|}$$

$$= \frac{L^T c}{\sqrt{(L^T c)^T L^T c}}$$

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Handwritten derivation of the optimal \bar{y} expression:

$$\begin{aligned} &= \frac{L^{-1} \bar{c}}{\sqrt{(L^{-1} \bar{c})^T L^{-1} \bar{c}}} \\ &= \frac{L^{-1} \bar{c}}{\sqrt{\bar{c}^T L^{-T} L^{-1} \bar{c}}} \\ &= \frac{L^{-1} \bar{c}}{\sqrt{\bar{c}^T \underbrace{(L L^T)^{-1}}_A \bar{c}}} \end{aligned}$$

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Handwritten derivation of the optimal \bar{x} expression:

$$\begin{aligned} \bar{y} &= \frac{L^{-1} \bar{c}}{\sqrt{\bar{c}^T A^{-1} \bar{c}}} \\ \bar{x} &= \frac{L^{-T} \bar{y}}{\sqrt{\bar{c}^T A^{-1} \bar{c}}} \\ &= \frac{L^{-T} L^{-1} \bar{c}}{\sqrt{\bar{c}^T A^{-1} \bar{c}}} \end{aligned}$$

So you have the optimal $\bar{y} = \frac{L^{-1} \bar{c}}{\sqrt{\bar{c}^T A^{-1} \bar{c}}}$ and the optimal $\bar{x} = \frac{A^{-1} \bar{c}}{\sqrt{\bar{c}^T A^{-1} \bar{c}}}$ as shown in slides.

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Handwritten derivation of the optimal \bar{x} expression, highlighting the final result:

$$\begin{aligned} \bar{x} &= \frac{L^{-T} \bar{y}}{\sqrt{\bar{c}^T A^{-1} \bar{c}}} \\ &= \frac{L^{-T} L^{-1} \bar{c}}{\sqrt{\bar{c}^T A^{-1} \bar{c}}} \\ \boxed{\bar{x} = \frac{A^{-1} \bar{c}}{\sqrt{\bar{c}^T A^{-1} \bar{c}}}} & \text{optimal } \bar{x} \end{aligned}$$

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Handwritten derivation of the optimal value of the objective function:

$$\begin{aligned} \text{Optimal value of objective} &= \bar{c}^T \bar{x} \\ &= \bar{c}^T A^{-1} \bar{c} \\ &= \sqrt{\bar{c}^T A^{-1} \bar{c}} \\ &= \sqrt{\bar{c}^T A^{-1} \bar{c}} \end{aligned}$$

The expression $\sqrt{\bar{c}^T A^{-1} \bar{c}}$ is circled in purple.

So the optimal value of objective equals $\sqrt{\bar{c}^T A^{-1} \bar{c}}$ as shown in slide.

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Handwritten note about the positive definiteness of matrix A:

$$A \text{ is PD} = \text{Positive Definite}$$

The expression $\sqrt{\bar{c}^T A^{-1} \bar{c}}$ is circled in purple.

But remember that this entire case is when A is PD. Now when A is not positive definite, then it cannot be decomposed as LL^T .

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Handwritten derivation of the spectral decomposition of matrix A:

$$\begin{aligned} A \text{ is NOT PD:} \\ A &= Q \Lambda Q^T \\ &= \begin{bmatrix} q_1 & q_2 & \dots & q_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \dots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix} \end{aligned}$$

The matrix Q is labeled below the first bracket, and the matrix Q^T is labeled below the last bracket.

So in that scenario let us say you have an eigenvalue decomposition of A which is $A = Q \Lambda Q^T$. So this can be written as a matrix of eigenvectors as shown in slide.

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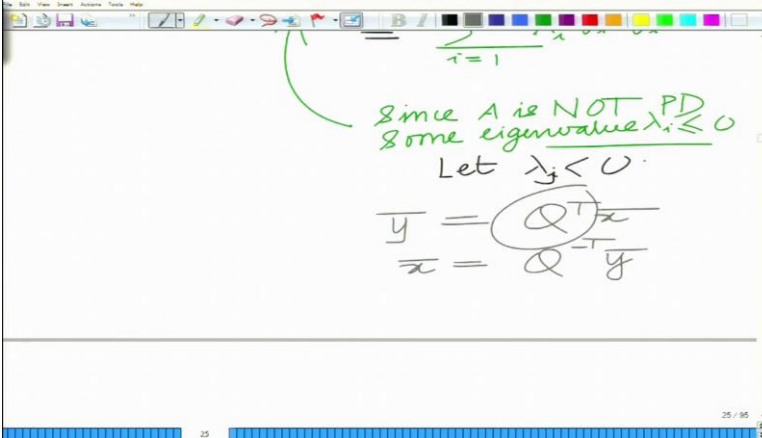
Slide 24 shows the eigenvalue decomposition $A = Q \Lambda Q^T$. The matrix Q is defined as the matrix of eigenvectors, with columns q_1, q_2, \dots, q_n . The matrix Λ is a diagonal matrix containing the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. The slide also shows the transpose of Q , Q^T , which contains the transposed eigenvectors $q_1^T, q_2^T, \dots, q_n^T$.

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Slide 25 shows the eigenvalue decomposition $A = Q \Lambda Q^T$. It states that since A is NOT PD, some eigenvalue $\lambda_i \leq 0$. Let $\lambda_j < 0$.

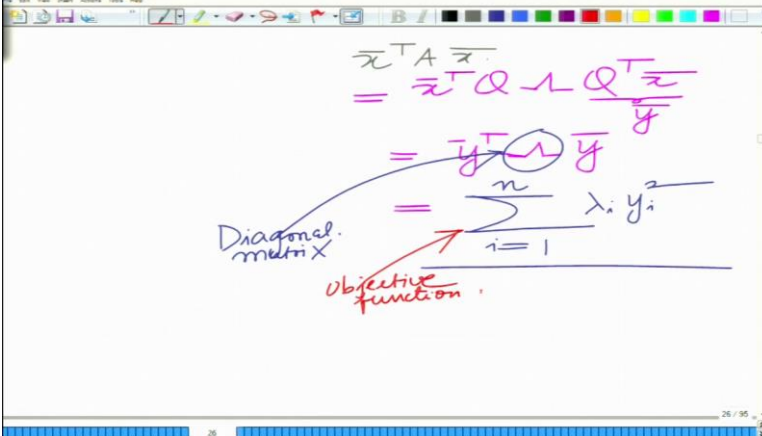
Now you can multiply it out and you can write this as $A = \sum_{i=1}^n \lambda_i \bar{q}_i \bar{q}_i^T$. Now, since A is not PD, you have some eigenvalue $\lambda_j < 0$. In this case let us set $\bar{y} = Q^T x \Rightarrow \bar{x} = Q^{-T} \bar{y}$.

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$i=1$
 since A is NOT PD
 Some eigenvalue $\lambda_i \leq 0$
 Let $\lambda_j < 0$.
 $\bar{y} = Q^T \bar{x}$
 $\bar{x} = Q^{-T} \bar{y}$

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$\bar{x}^T A \bar{x}$
 $= \bar{x}^T Q \Lambda Q^T \bar{x}$
 $= \bar{y}^T \Lambda \bar{y}$
 $= \sum_{i=1}^n \lambda_i y_i^2$
 Diagonal matrix
 objective function

And therefore, now if we look at $\bar{x}^T A \bar{x} = \bar{x}^T Q \Lambda Q^T \bar{x} = \bar{y}^T \Lambda \bar{y} = \sum_{i=1}^n \lambda_i y_i^2$. So this is the objective function.

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Handwritten derivation on a whiteboard:

constraint.

$$\text{Objective} = \bar{c}^T \bar{x}$$

$$= \bar{c}^T Q^{-T} \bar{y}$$

$$= \bar{b}^T \bar{y}$$

$\bar{b} = Q^T \bar{c}$

$$= \sum_{i=1}^n b_i y_i$$

Now we have $\bar{c}^T \bar{x} = \bar{c}^T Q^{-T} \bar{y} = \bar{b}^T \bar{y} = \sum_{i=1}^n b_i y_i$. So I can recast this optimization problem.

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Handwritten reformulation on a whiteboard:

$$= \sum_{i=1}^n b_i y_i$$

$$\equiv \min \sum_{i=1}^n b_i y_i$$

$$\text{s.t. } \sum_{i=1}^n \lambda_i y_i^2 \leq 1$$

$\lambda_j < 0$ since A is NOT PD.
if $b_j > 0$, then set $y_j \rightarrow -\infty$

So we have $\min \sum_{i=1}^n b_i y_i$. Now we are assuming that one particular $\lambda_j < 0$, since A is not positive definite. Now if $b_j > 0$, then set y_i to be a very large negative value.

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$\lambda_j < 0$ since A is NOT PD.
 if $b_j > 0$, then set $y_j \rightarrow -\infty$
 $\lambda_j = -ve$
 $\Rightarrow \lambda_j y_j^2 \rightarrow -\infty \leq 1$
 \Rightarrow constraint satisfied.
 Objective $b_j y_j$
 $b_j y_j \rightarrow -\infty$
 $\Rightarrow b_j y_j \rightarrow -\infty$

Now, since λ_j is negative implies $\lambda_j y_j^2$ equals negative implies the constraint is satisfied.

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Objective $b_j y_j$
 $b_j y_j \rightarrow -\infty$
 $\Rightarrow (b_j) y_j \rightarrow -\infty$
 since $b_j < 0$
 \Rightarrow problem is unbounded below.
 $\lambda_j < 0$ if $b_j < 0$
 then $y_j \rightarrow \infty$
 $\Rightarrow b_j y_j \rightarrow -\infty$
 \Rightarrow unbounded Below!

So basically by setting $y_j = -\infty$ we can make the optimization objective as small as possible. Now, consider another scenario, when $\lambda_j < 0$ and if $b_j < 0$, then set y_j to a large positive value. This again implies that the constraint is always satisfied as shown in slide. Let us look at another example, example number 6.

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#6. $\bar{x}^T \bar{x} \leq yz, y, z \geq 0$
 $\Rightarrow \|\bar{x}\|^2 \leq yz$
 $\Rightarrow 4\|\bar{x}\|^2 \leq 4yz$
 $\Rightarrow 4\|\bar{x}\|^2 \leq (y+z)^2 - (y-z)^2$
 $\Rightarrow 4\|\bar{x}\|^2 + (y-z)^2 \leq (y+z)^2$
 $\Rightarrow \left\| \begin{bmatrix} 2\bar{x} \\ y-z \end{bmatrix} \right\| \leq y+z, y, z \geq 0$

So show that $\bar{x}^T \bar{x} \leq yz, y, z \geq 0$. This implies that

$\|\bar{x}\|^2 \leq yz \Rightarrow 4\|\bar{x}\|^2 \leq 4yz \Rightarrow 4\|\bar{x}\|^2 \leq (y+z)^2 - (y-z)^2$. So this implies

$4\|\bar{x}\|^2 + (y-z)^2 \leq (y+z)^2 \Rightarrow \left\| \begin{bmatrix} 2\bar{x} \\ y-z \end{bmatrix} \right\| \leq y+z, y, z \geq 0$.

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$\bar{x}^T \bar{x} \leq yz$
 $\Rightarrow \left\| \begin{bmatrix} 2\bar{x} \\ y-z \end{bmatrix} \right\| \leq y+z, y, z \geq 0$
 Equivalence condition
 $\max \left(\sum_{i=1}^m \frac{1}{a_i^T \bar{x} - b_i} \right)^{-1}$

The condition that $\bar{x}^T \bar{x} \leq yz$, can be equivalently written as shown in the above slide.

Now, let us say we want to maximize the harmonic mean that is

$$\max \left(\sum_{i=1}^m \frac{1}{a_i^T \bar{x} - b_i} \right)^{-1}$$

s.t $a_i^T \bar{x} - b_i > 0$

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$$\max \left(\sum_{i=1}^m \frac{1}{\frac{a_i^T \bar{x} - b_i}{a_i^T \bar{x} - b_i}} \right)^{-1}$$

Harmonic mean of $\frac{1}{a_i^T \bar{x} - b_i}$

$$\begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix} \bar{x} - \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} > 0$$

$$\Rightarrow A \bar{x} - b > 0$$

This can be stacked in the form of a matrix which implies that $A \bar{x} - b > 0$.

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$$\begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix} \bar{x} - \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} > 0$$

$$\Rightarrow A \bar{x} - b > 0$$

$$\max \left(\sum_{i=1}^m \frac{1}{\frac{a_i^T \bar{x} - b_i}{a_i^T \bar{x} - b_i}} \right)^{-1}$$

$$\equiv \min \sum_{i=1}^m \frac{1}{a_i^T \bar{x} - b_i}$$

Now this is equivalent to $\min \left(\sum_{i=1}^m \frac{1}{a_i^T \bar{x} - b_i} \right)$ because everything is non-negative. Now,

let us write this in an epigraph form.

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$$\frac{1}{a_i^T \bar{x} - b_i} \leq t_i$$

$$\Rightarrow \frac{1}{\bar{x}} \leq \frac{(a_i^T \bar{x} - b_i) t_i}{\bar{z}}$$

$$\Rightarrow \left\| \begin{bmatrix} 2 \\ a_i^T \bar{x} - b_i - t_i \end{bmatrix} \right\| \leq a_i^T \bar{x} - b_i + t_i$$

So we have $\frac{1}{a_i^T x - b_i} \leq t_i \Rightarrow 1 \leq (a_i^T x - b_i) t_i \Rightarrow \left\| \begin{bmatrix} 2 \\ a_i^T x - b_i - t_i \end{bmatrix} \right\| \leq a_i^T x - b_i + t_i$.

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$$\left\| \begin{bmatrix} 2 \\ a_i^T \bar{x} - b_i - t_i \end{bmatrix} \right\| \leq a_i^T \bar{x} - b_i + t_i$$

Conic constraints m constraints one for each i.
 \Rightarrow SOCP.

So we will have m constraints, one for each i and this is a second order conic constraint.

So the resulting optimization problem will be a second order cone program.

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Handwritten notes on a whiteboard showing the reformulation of a minimization problem into an equivalent second-order cone programming (SOCP) problem.

The original problem is:

$$\min \sum_{i=1}^m t_i$$

subject to:

$$\left\| \begin{bmatrix} 2 \\ \overline{a_i^T \bar{x}} - b_i - t_i \end{bmatrix} \right\| \leq \overline{a_i^T \bar{x}} - b_i + t_i$$

and

$$t_i \geq 0, \quad \overline{a_i^T \bar{x}} - b_i \geq 0, \quad i = 1, 2, \dots, m$$

The notes state that this is equivalent to a problem for maximizing the H.M. (Harmonic Mean) and is an SOCP (Second-Order Cone Programming) problem.

So the equivalent optimization problem can be written as

$$\begin{aligned} & \min \sum_{i=1}^m t_i \\ & s.t. \left\| \begin{bmatrix} 2 \\ \overline{a_i^T \bar{x}} - b_i - t_i \end{bmatrix} \right\| \leq \overline{a_i^T \bar{x}} - b_i + t_i. \text{ So once you write this as a second order cone} \\ & t_i \geq 0 \\ & \overline{a_i^T \bar{x}} - b_i \geq 0 \\ & i = 1, 2, \dots, m \end{aligned}$$

program, you can use the convex solvers readily available to solve this. So let us stop here and continue in the subsequent module. Thank you very much.