

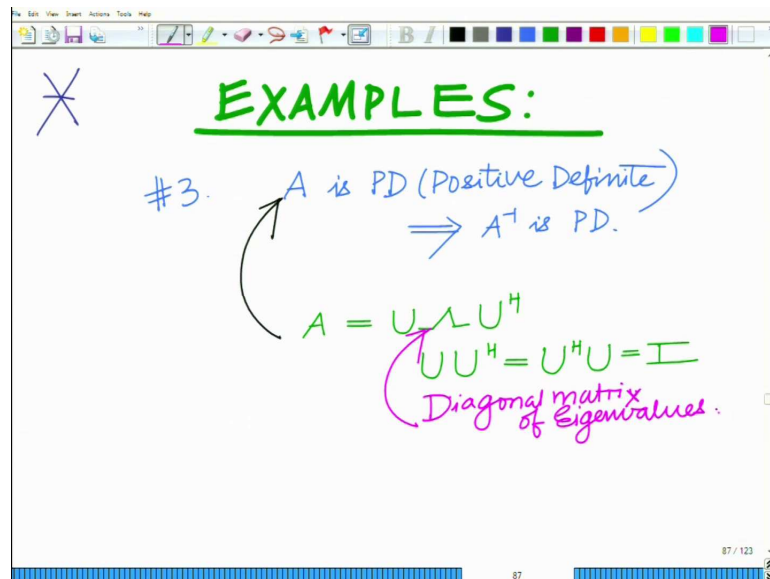
Applied Optimization for Wireless, Machine Learning, Big data
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Lecture-20

Inverse of a Positive Define Matrix, Eigenvalue Properties and Relation between different norms

Hello, welcome to another module in this massive open online course. Let us look at another example related to Positive Definite Matrices.

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So, another property of a positive definite matrix is that if a matrix A is a positive definite matrix then inverse matrix of matrix A is also a positive definite matrix. To verify this property, Let us write matrix A as follows.

$$A = U\Lambda U^H$$

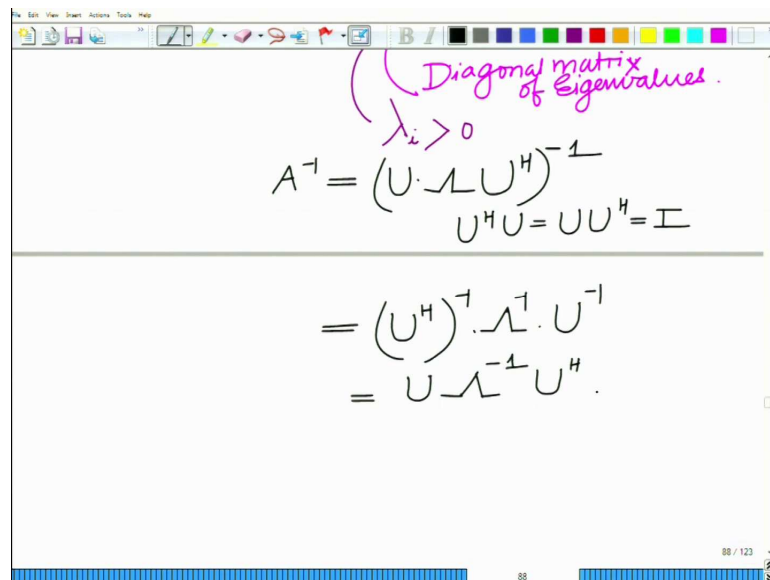
Where Λ is the diagonal matrix of eigenvalues of matrix A and U is a unitary matrix which satisfies that

$$UU^H = U^H U = I$$

I is the identity matrix.

Also this has been seen previously that the eigenvalues of any positive definite matrix are strictly greater than zero i.e. $\lambda_i > 0$. Therefore matrix A has all positive eigenvalues.

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Handwritten derivation on a whiteboard showing the inverse of matrix A using its eigenvalue decomposition. The text is written in black ink, with a note in pink ink stating "Diagonal matrix of Eigenvalues." and " $\lambda_i > 0$ ".

$$A^{-1} = (U \Lambda U^H)^{-1}$$

$$U^H U = U U^H = I$$

$$= (U^H)^{-1} \Lambda^{-1} U^{-1}$$

$$= U \Lambda^{-1} U^H$$

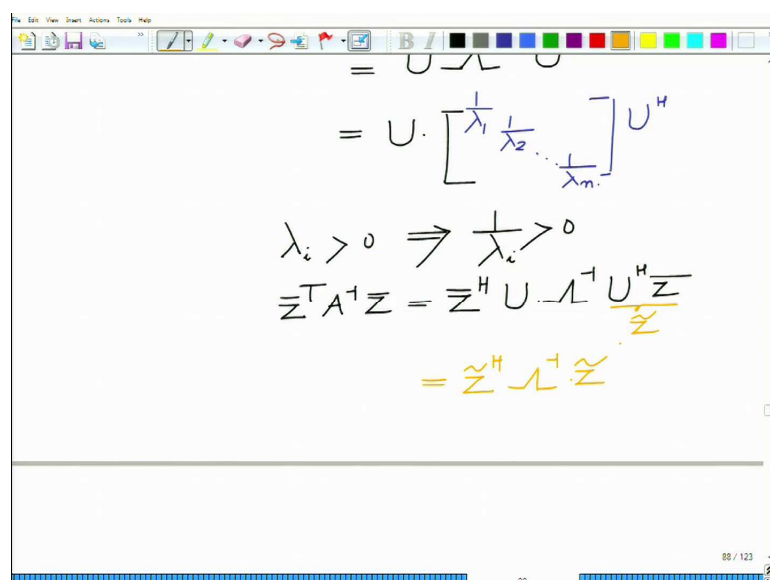
So, inverse of matrix A is

$$A^{-1} = (U \Lambda U^H)^{-1}$$

$$= (U^H)^{-1} \Lambda^{-1} U^{-1}$$

$$= U \Lambda^{-1} U^H$$

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Handwritten derivation on a whiteboard showing the quadratic form of the inverse of matrix A . The text is written in black ink, with a note in pink ink stating "Diagonal matrix of Eigenvalues." and " $\lambda_i > 0$ ".

$$= U \Lambda^{-1} U^H$$

$$= U \cdot \begin{bmatrix} \frac{1}{\lambda_1} & & \\ & \frac{1}{\lambda_2} & \\ & & \ddots \\ & & & \frac{1}{\lambda_n} \end{bmatrix} U^H$$

$$\lambda_i > 0 \Rightarrow \frac{1}{\lambda_i} > 0$$

$$\bar{z}^T A^{-1} \bar{z} = \bar{z}^H U \Lambda^{-1} U^H \bar{z}$$

$$= \bar{z}^H \Lambda^{-1} \bar{z}$$

This can also be written as

$$A^{-1} = U \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\lambda_n} \end{bmatrix} U^H$$

As all the eigenvalues λ_i are greater than zero therefore $\frac{1}{\lambda_i}$ are also greater than zero.

So, eigenvalues of A^{-1} are also greater than zero. And therefore, matrix A^{-1} is also a positive definite matrix.

This can also be checked as follows. Consider for any real vector \bar{Z} ;

$$\bar{Z}^T A^{-1} \bar{Z} = \bar{Z}^T U \Lambda^{-1} U^H \bar{Z}$$

Let us say

$$U^H \bar{Z} = \tilde{Z}$$

Therefore,

$$\bar{Z}^T A^{-1} \bar{Z} = \tilde{Z}^H \Lambda^{-1} \tilde{Z}$$

$$= \begin{bmatrix} \tilde{Z}_1^* & \tilde{Z}_2^* & \cdots & \tilde{Z}_n^* \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\lambda_n} \end{bmatrix} \begin{bmatrix} \tilde{Z}_1 \\ \tilde{Z}_2 \\ \vdots \\ \tilde{Z}_n \end{bmatrix}$$

$$= \sum_{i=1}^n \frac{1}{\lambda_i} |\tilde{Z}_i|^2$$

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$$\begin{aligned}
 &= [\tilde{z}_1^* \tilde{z}_2^* \dots \tilde{z}_n^*] \\
 &\quad \times \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \vdots \\ \tilde{z}_n \end{bmatrix} \\
 &= \sum_{i=1}^n \frac{1}{\lambda_i} |\tilde{z}_i|^2 \\
 &\quad \uparrow \quad \quad \uparrow \\
 &\quad >0 \quad >0
 \end{aligned}$$

$\Rightarrow \bar{Z}^T A^{-1} \bar{Z} > 0$
 For all vectors \bar{Z}
 $\Rightarrow A^{-1}$ is PD matrix

Now, as $\frac{1}{\lambda_i} > 0$ and $|\tilde{Z}_i| > 0$; therefore this implies that for all vectors \bar{Z} ,

$$\bar{Z}^T A^{-1} \bar{Z} > 0$$

This implies that A^{-1} is a positive definite matrix and this verifies the above property.

(Refer Slide Time: 07:55)

#4) A, B — invertible $n \times n$ matrices

$\Rightarrow AB, BA$ have the same eigenvalues.

Characteristic Polynomial of AB .

$$\begin{aligned}
 &|AB - \lambda I| \\
 &= |ABAA^{-1} - \lambda AA^{-1}|
 \end{aligned}$$

Next property of a matrix is that for two invertible $n \times n$ matrices A and B ; matrix AB and BA have same eigenvalues. To verify this, let us start with the characteristic

polynomial of matrix AB . Using property $AA^{-1} = I$; the characteristic polynomial of matrix AB can be written as

$$\begin{aligned}
 |AB - \lambda I| &= |ABAA^{-1} - \lambda AA^{-1}| \\
 &= |A(BA - \lambda I)A^{-1}| \\
 &= |A||BA - \lambda I||A^{-1}| \\
 &= |A||BA - \lambda I|\frac{1}{|A|} \\
 &= |BA - \lambda I|
 \end{aligned}$$

So, this implies that the characteristic polynomial of matrix AB is equal the characteristic polynomial of matrix BA . This further implies that the roots of matrix AB and BA are identical and therefore the eigenvalues of AB are equal as of matrix BA .

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Handwritten derivation on a whiteboard:

$$\begin{aligned}
 AB &= U\Lambda U^{-1} \\
 BA &= V\Lambda V^{-1} \\
 \Rightarrow V^T BA \cdot V &= \Lambda \\
 \Rightarrow AB &= U V^T BA \cdot U^{-1} V \\
 &= \tilde{U} \cdot BA \cdot \tilde{U}^{-1}
 \end{aligned}$$

A green arrow points from the Λ in the second equation to the first, labeled "Same eigenvalue".

Therefore, matrix AB and BA can be expressed as

$$\begin{aligned}
 AB &= U\Lambda^{-1}U^{-1} \\
 BA &= V\Lambda^{-1}V^{-1}
 \end{aligned}$$

Where Λ is the same diagonal matrix of eigenvalues for both the matrices AB and BA .

So,

$$V^{-1}BAV = \Lambda$$

Therefore

$$\begin{aligned} AB &= U(V^{-1}BAV)U^{-1} \\ &= UV^{-1}(BA)VU^{-1} \\ &= \tilde{U}.BA.\tilde{U}^{-1} \end{aligned}$$

Where matrix $\tilde{U} = UV^{-1}$.

In general it is known that matrix C is similar to matrix D only if there exists a matrix M such that

$$C = M^{-1}DM$$

Therefore, it verifies that matrix AB is similar to the matrix BA .

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Handwritten derivation on a whiteboard:

#2) Eigen matrix U

$U = \text{unitary}$

$\Rightarrow U^H U = U U^H = I$

Let $\bar{x} = \text{eigenvector}$
 $\lambda = \text{eigenvalue}$

$\Rightarrow U\bar{x} = \lambda\bar{x}$

$\Rightarrow (U\bar{x})^H \cdot U\bar{x} = (\lambda\bar{x})^H \cdot \lambda\bar{x}$

$\Rightarrow \bar{x}^H \cdot U^H U \bar{x} = \lambda^* \cdot \bar{x}^H \lambda \bar{x}$

The next interesting property of matrices is that the eigenvalues of unitary matrix have unit magnitude. For the verification of this property, consider matrix U be a unitary matrix.

So as per the definition of a unitary matrix,

$$U^H U = U U^H = I$$

If \bar{x} and λ are the eigenvector and eigenvalue of matrix U , then

$$U\bar{x} = \lambda\bar{x}$$

Therefore as the definition of unitary matrix;

$$(U\bar{x})^H U\bar{x} = (\lambda\bar{x})^H \lambda\bar{x}$$

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The image shows a handwritten derivation on a whiteboard. The steps are as follows:

$$\begin{aligned} & \Rightarrow \bar{x}^H \cdot U^H U \bar{x} = \lambda^* \bar{x}^H \lambda \bar{x} \\ & \Rightarrow \bar{x}^H \cdot U^H U \bar{x} = |\lambda|^2 \bar{x}^H \bar{x} \\ & \Rightarrow \|\bar{x}\|^2 = |\lambda|^2 \|\bar{x}\|^2 \\ & \Rightarrow |\lambda|^2 = 1 \\ & \Rightarrow |\lambda| = 1 \end{aligned}$$

Below the equations, it is written: "Eigenvalues of unitary matrix have unit magnitude."

As λ is a number so $\lambda^H = \lambda^*$ and therefore, above expression is now

$$\bar{x}^H U^H U \bar{x} = \lambda^* \bar{x}^H \lambda \bar{x}$$

$$\|\bar{x}\|^2 = |\lambda|^2 \|\bar{x}\|^2$$

This means that

$$|\lambda|^2 = 1$$

$$|\lambda| = 1$$

Hence this verifies that the eigenvalues of unitary matrix have unit magnitude.

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The image shows a digital whiteboard with a toolbar at the top. The handwritten content is as follows:

- An arrow points to the equation $|\lambda| = 1$.
- Below it, green text says: "Eigenvalues of unitary matrix have unit magnitude."
- The determinant is expressed as a product: $|\det(U)| = \left| \prod_{i=1}^n \lambda_i \right|$
- This is simplified to: $= \prod_{i=1}^n |\lambda_i|$
- The final result is boxed: $|\det(U)| = 1$

At the bottom right of the whiteboard, the text "94 / 123" is visible.

And similarly, the magnitude of the determinant of unitary matrix U is

$$\begin{aligned} |\det(U)| &= \left| \prod_{i=1}^n \lambda_i \right| \\ &= \prod_{i=1}^n |\lambda_i| \\ &= \prod_{i=1}^n \underbrace{(1, 1, \dots, 1)}_{n \text{ times}} \\ &= 1 \end{aligned}$$

Therefore, the magnitude of the determinant of unitary matrix is equal to 1.

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The image shows a whiteboard with handwritten mathematical derivations. At the top left, it is labeled "#6)". The first equation is $\|\bar{x}\|_1 \leq \sqrt{n} \cdot \|\bar{x}\|_2$. Below this, the vector \bar{x} is defined as $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. The next line shows the definition of the l_1 norm: $\|\bar{x}\|_1 = |x_1| + \dots + |x_n|$. The final line shows the definition of the l_2 norm: $\|\bar{x}\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \dots + |x_n|^2}$. The whiteboard has a toolbar at the top and a status bar at the bottom indicating "95 / 123".

Another property is that the relation between the l_1 norm and l_2 norm of a $n \times 1$ vector \bar{x} is as follows.

$$\|\bar{x}\|_1 \leq \sqrt{n} \|\bar{x}\|_2$$

So l_1 norm of vector \bar{x} is

$$\|\bar{x}\|_1 = |x_1| + \dots + |x_n|$$

And l_2 norm of vector \bar{x} is

$$\|\bar{x}\|_2 = \sqrt{|x_1|^2 + \dots + |x_n|^2}$$

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$$\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \dots + |x_n|^2}$$

$$\bar{u} = \begin{bmatrix} |x_1| \\ |x_2| \\ \vdots \\ |x_n| \end{bmatrix} \quad \bar{v} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\langle \bar{u}, \bar{v} \rangle^2 \leq \langle \bar{u}, \bar{u} \rangle \cdot \langle \bar{v}, \bar{v} \rangle$$

Consider two different vectors \bar{u} and \bar{v} as follows.

$$\bar{u} = \begin{bmatrix} |x_1| \\ |x_2| \\ \vdots \\ |x_n| \end{bmatrix}, \quad \bar{v} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

The Cauchy–Schwarz inequality states that,

$$\langle \bar{u}, \bar{v} \rangle^2 \leq \langle \bar{u}, \bar{u} \rangle \cdot \langle \bar{v}, \bar{v} \rangle$$

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Handwritten derivation on a digital whiteboard:

$$\langle \bar{u}, \bar{v} \rangle^2 \leq \langle \bar{u}, \bar{u} \rangle \cdot \langle \bar{v}, \bar{v} \rangle$$

$$\Rightarrow (\bar{u}^T \bar{v})^2 \leq \|\bar{u}\|^2 \|\bar{v}\|^2$$

$$\Rightarrow \boxed{\bar{u}^T \bar{v} \leq \|\bar{u}\|_2 \|\bar{v}\|_2}$$

$$\bar{u}^T \bar{v} = |x_1| + |x_2| + \dots + |x_n|$$

$$= \|\bar{x}\|_1$$

$$\|\bar{u}\|_2 = \sqrt{|x_1|^2 + \dots + |x_n|^2}$$

$$= \|\bar{x}\|_2$$

Therefore,

$$(\bar{u}^T \bar{v})^2 \leq \|\bar{u}\|^2 \cdot \|\bar{v}\|^2$$

$$\bar{u}^T \bar{v} \leq \|\bar{u}\|_2 \cdot \|\bar{v}\|_2 \quad \text{Eq.1}$$

On solving $\bar{u}^T \bar{v}$;

$$\bar{u}^T \bar{v} = |x_1| + \dots + |x_n|$$

$$= \|\bar{x}\|_1$$

And

$$\|\bar{u}\|_2 = \sqrt{|x_1|^2 + \dots + |x_n|^2}$$

$$= \|\bar{x}\|_2$$

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Handwritten derivation on a whiteboard:

$$= \|\bar{x}\|_2$$

$$\|\bar{v}\|_2 = \sqrt{1 + 1 + \dots + 1}$$

$$= \sqrt{n}$$

using (1)

$$\|\bar{x}\|_1 \leq \sqrt{n} \cdot \|\bar{x}\|_2$$

property

Also as \bar{v} is an all one vector of length n , therefore

$$\|\bar{v}\|_2 = \sqrt{n}$$

Therefore on putting these values in equation 1; it verifies that

$$\|\bar{x}\|_1 \leq \sqrt{n} \|\bar{x}\|_2$$

This is an interesting property.

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Handwritten derivation on a whiteboard:

$$\|\bar{x}\|_2$$

$$= \sqrt{|x_1|^2 + |x_2|^2 + \dots + |x_n|^2}$$

$$\geq \sqrt{(\max\{|x_i|\})^2}$$

$$= \max\{|x_i|\} = \|\bar{x}\|_\infty$$

$$\|\bar{x}\|_2 \geq \|\bar{x}\|_\infty$$

Similarly; on comparing the l_∞ norm and l_2 norm of \bar{x} ; l_2 norm is a sum of the squares of the magnitude of all the elements of the vector while l_∞ norm is

$$\|\bar{x}\|_\infty = \sqrt{\left(\max\{|x_i|\}\right)^2}$$

Therefore it is clear that the l_∞ norm of vector \bar{x} is less than l_2 norm of vector \bar{x} i.e.

$$\|\bar{x}\|_2 \geq \|\bar{x}\|_\infty$$

Let us stop here and we will continue with other aspects in the subsequent modules.