Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture - 07
Gram Schmidt Orthogonalization Procedure

Hello, welcome to another module in this massive open online course. So, let us look at another example to understand the Gram Schmidt Orthonormalization Process.

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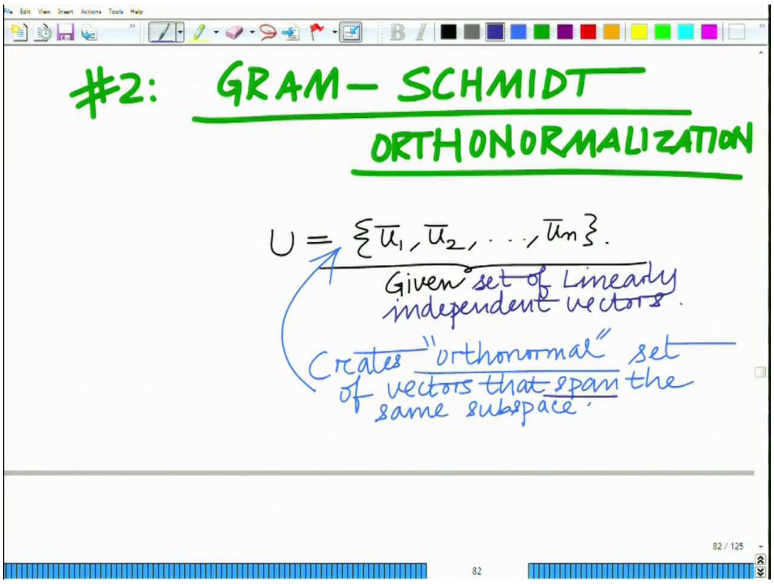

Let us take a set U of linearly independent vectors $\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n$; that is

$$U = \{\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n\}$$

The Gram Schmidt Orthonormalization procedure creates an orthonormal set of vectors that span the same subspace. An orthonormal set of vectors is a set in which each vectors has unit norm. The process of making the norm of a vector equals to unity is known as normalization. Also the term "orthogonal" represents that all the vectors in this set are pair-wise orthogonal to each other. So, Normalization and orthogonalization collectively converts a set of vectors into an orthonormal set of vectors.

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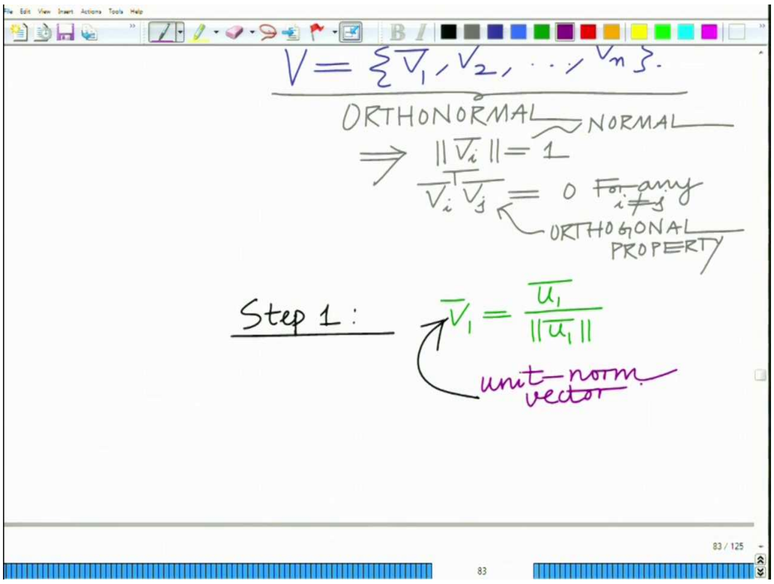

So, we have an orthonormal basis V as the basis of orthonormal vectors space $\overline{V}_1, \overline{V}_2, \cdots, \overline{V}_n$.

$$V = \left\{\overline{V}_1, \overline{V}_2, \cdots, \overline{V}_n\right\}$$

This implies that

$$\left\|\overline{V}_i\right\| = 1$$

This is basically the normal property. So, at any pair of real vectors $\overline{V}_i, \overline{V}_j$

$$\overline{V}_i^T \overline{V}_j = 0 \text{ if } i \neq j.$$

And this basically represents the orthogonal property of vector space *V*.

So, these are orthonormal if all the vectors in the set are orthogonal to each other and each vector has unit norm and also that they span the same subspace. So, to covert a vector space into orthonormal vector space, Gram Schmidt Orthonormalization procedure is used. This works in various steps and can be described as the follows.

The first step is to create a unit norm vector in the same subspace. So let us assume that vector space $\overline{V}$ is the orthonormal vector space of vector space $\overline{u}_1$. This means define $\overline{V}$

in terms of $\bar{u}_1$ such that $\bar{V}$ fulfills the criteria of orthonormal vectors. Therefore first define the first element of $\bar{V}$ that is $\bar{V}_1$ as

$$\bar{V}_1 = \frac{\bar{u}_1}{\|\bar{u}_1\|}$$

So, $\bar{V}_1$ is the unit norm of $\bar{u}_1$ and hence lies in the same subspace.

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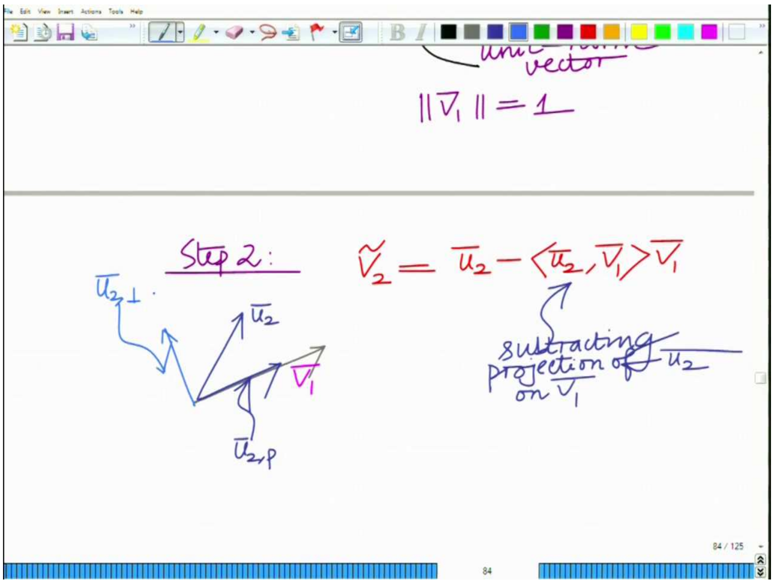

This means that

$$\|\bar{V}_1\| = 1$$

And so far it satisfies the criterion for Gram Schmidt Orthonormalization.

In step 2; create the second vector of $\bar{V}$ that is $\tilde{V}_2$ which is orthogonal to $\bar{V}_1$. Also $\tilde{V}_2$ must be derived from the vector $\bar{u}_2$, as $\bar{u}_2$ is the second vector in the vector space $\bar{u}$. Every vector can be represented as the sum of 2 components of another vector, one is the parallel component, and another is the perpendicular component. So here, find the projection of $\bar{u}_2$ on $\bar{V}_1$ that is the parallel component of $\bar{u}_2$ and denote it as $\bar{u}_{2,P}$. Here $\tilde{V}_2$ is the perpendicular component of $\bar{u}_2$ which is nothing but the $\bar{u}_2 - \bar{u}_{2,P}$.Therefore,

$$\tilde{V}_2 = \bar{u}_2 - \langle \bar{u}_2, \bar{V}_1 \rangle \bar{V}_1$$

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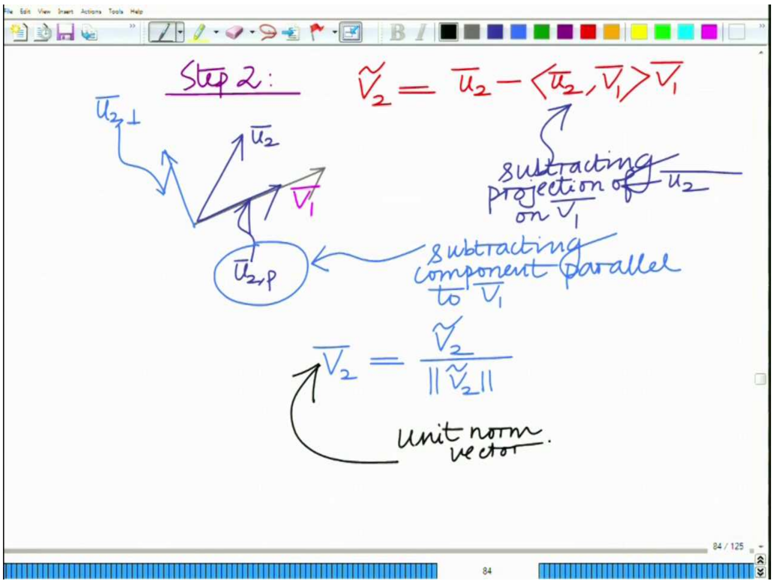

So, it fulfills the orthogonality property of orthonormal vector. Again to ensure the normality of $\tilde{V}_2$, Normalize it and generates $\bar{V}_2$ as the second vector of orthonormal vector space $\bar{V}$. And therefore

$$\bar{V}_2 = \frac{\tilde{V}_2}{\|\tilde{V}_2\|}$$

Hence the two vectors of the orthonormal vector spaces are generated using the Gram Schmidt Orthonormalization procedure. In the similar way, other vectors of this set are also generated.

To check the orthogonality of this vector space; consider the following demonstration.

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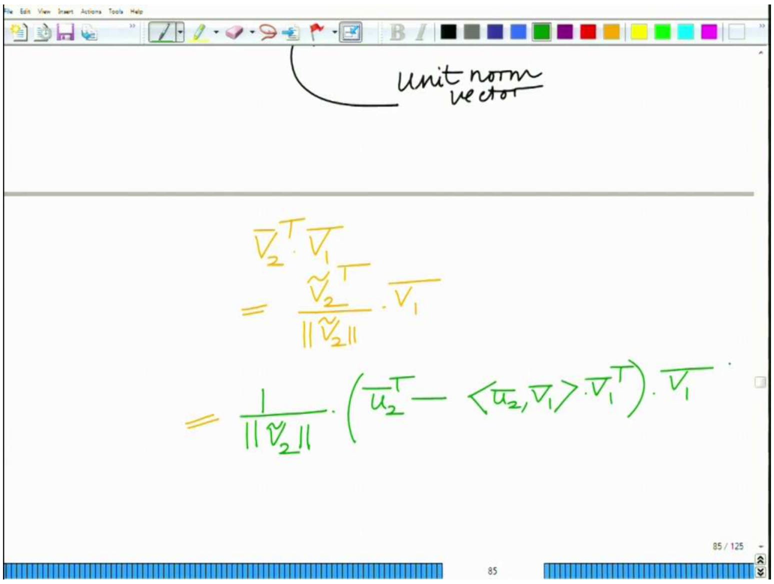

So,

$$\bar{V}_2^T \cdot \bar{V}_1 = \frac{\tilde{V}_2^T}{\|\tilde{V}_2\|} \cdot \bar{V}_1$$

$$= \frac{1}{\|\tilde{V}_2\|} \cdot \left(\bar{u}_2^T - \langle \bar{u}_2, \bar{V}_1 \rangle \bar{V}_1^T \right) \cdot \bar{V}_1$$

$$= \frac{1}{\|\tilde{V}_2\|} \cdot \left(\langle \bar{u}_2, \bar{V}_1 \rangle - \langle \bar{u}_2, \bar{V}_1 \rangle \|\bar{V}_1\|^2 \right)$$

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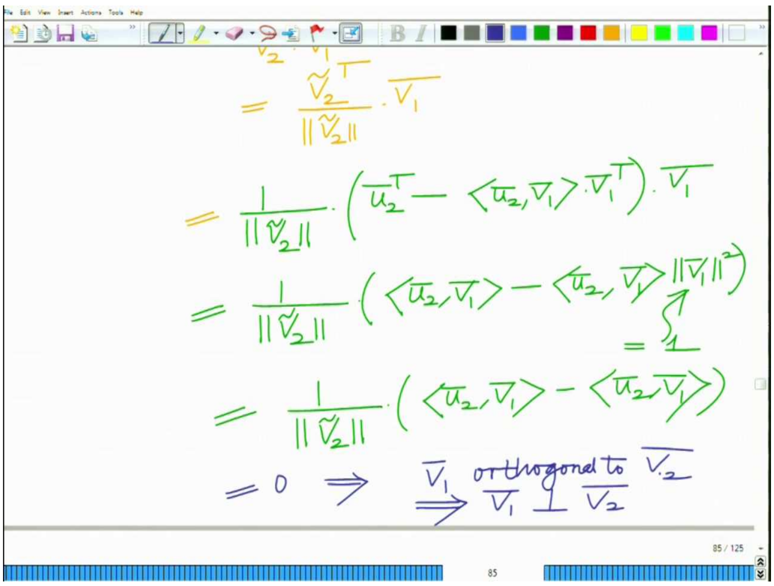

As $\bar{V}_1$ is a unit norm vector therefore $\|\bar{V}_1\| = 1$ and hence

$$\bar{V}_2^{\,T} \cdot \bar{V}_1 = \frac{1}{\|\tilde{V}_2\|} \cdot \left(\langle \bar{u}_2, \bar{V}_1 \rangle - \langle \bar{u}_2, \bar{V}_1 \rangle \right)$$
$$= 0$$

This implies that $\bar{V}_1 \perp \bar{V}_2$ because it is already been discussed that the cosine of the angle between two vectors is related to the inner product and if the inner product is 0, this means the cosine of the angle is 0,that is the angle between two vectors is 90 degrees and hence the vectors are perpendicular to each other.

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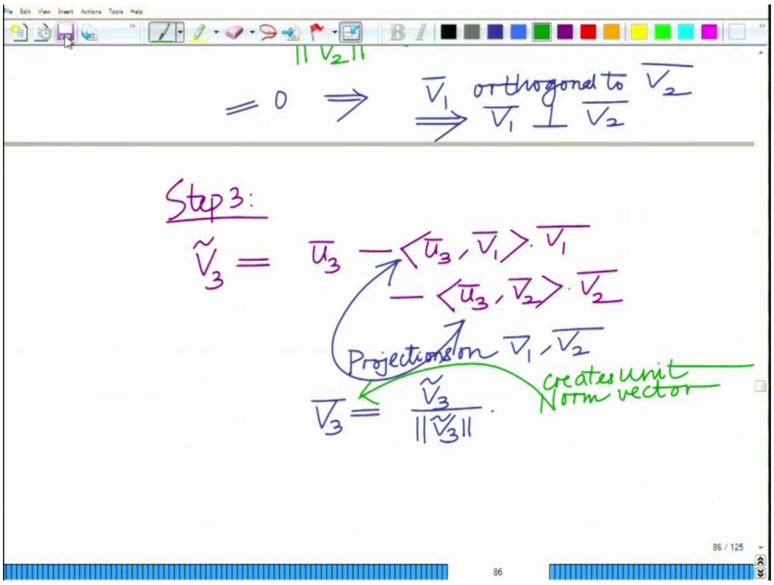

The further steps are same as the previous steps 1 and 2. Therefore step 3 is

$$\bar{V}_3 = \frac{\tilde{V}_3}{\|\tilde{V}_3\|}$$

And on further solving for $\tilde{V}_3$;

$$\tilde{V}_3 = \bar{u}_3 - \langle \bar{u}_3, \bar{V}_1 \rangle \bar{V}_1 - \langle \bar{u}_3, \bar{V}_2 \rangle \bar{V}_2$$

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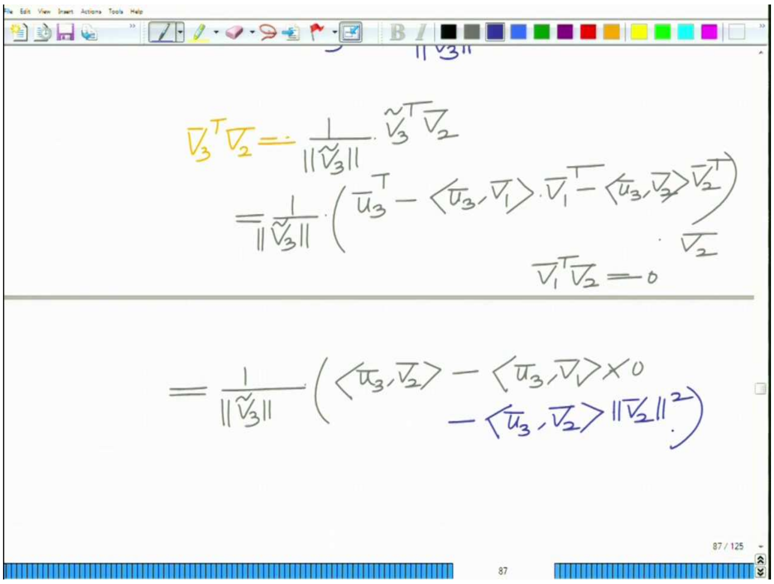

Lets check the orthogonality property.

$$\bar{V}_3^T \cdot \bar{V}_2 = \frac{\tilde{V}_3^{\,T}}{\|\tilde{V}_3\|} \cdot \bar{V}_2$$

$$= \frac{1}{\|\tilde{V}_3\|} \cdot \left(\bar{u}_3^{\,T} - \langle \bar{u}_3, \bar{V}_1 \rangle \bar{V}_1^T - \langle \bar{u}_3, \bar{V}_2 \rangle \bar{V}_2^T \right) \cdot \bar{V}_2$$

$$= \frac{1}{\|\tilde{V}_3\|} \cdot \left(\langle \bar{u}_3, \bar{V}_2 \rangle - \langle \bar{u}_3, \bar{V}_1 \rangle \left(\bar{V}_1^T \cdot \bar{V} \right)_2 - \langle \bar{u}_3, \bar{V}_2 \rangle \|\bar{V}_2\|^2 \right)$$

Also, $\bar{V}_1$ and $\bar{V}_2$ are orthogonal to each other.

$$\bar{V}_1^T \cdot \bar{V}_2 = 0$$

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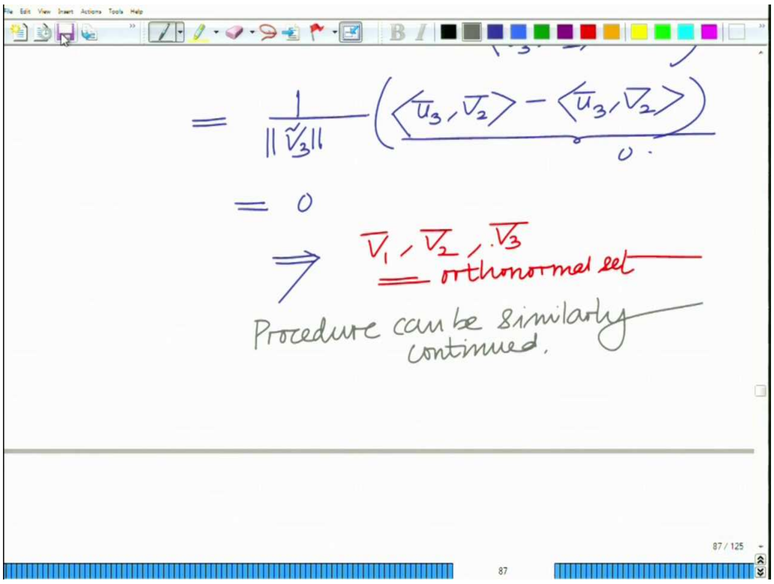

And $\bar{V}_2$ is a unit norm vector therefore $\|\bar{V}_2\| = 1$ and hence

$$\bar{V}_3^T \cdot \bar{V}_2 = \frac{1}{\|\tilde{V}_3\|} \cdot \left(\langle \bar{u}_3, \bar{V}_2 \rangle - \langle \bar{u}_3, \bar{V}_2 \rangle\right)$$
$$= 0$$

This implies that $\bar{V}_1 \perp \bar{V}_2 \perp \bar{V}_3$ and hence $\left(\bar{V}_1, \bar{V}_2, \bar{V}_3\right)$ is an orthonormal vector space and this procedure can similarly be continued.

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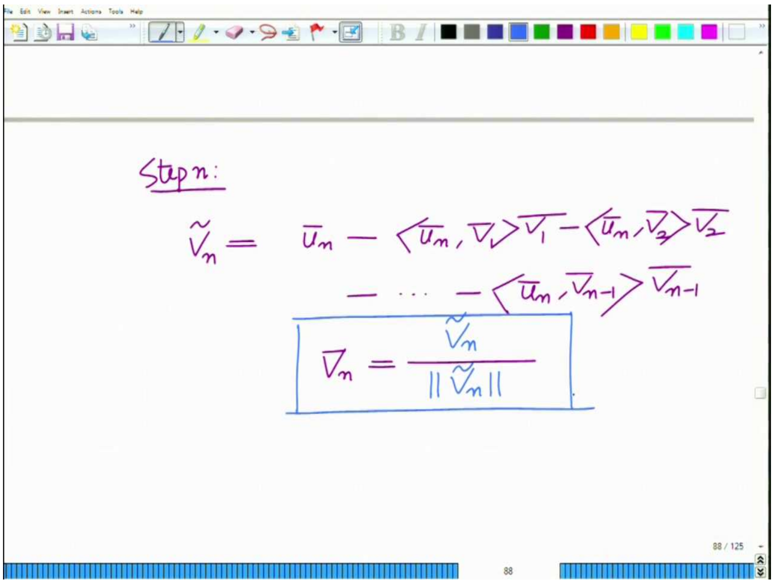

Therefore in nth step,

$$\bar{V}_n = \frac{\tilde{V}_n}{\|\tilde{V}_n\|}$$

and

$$\tilde{V}_n = \bar{u}_n - \langle \bar{u}_n, \bar{V}_1 \rangle \bar{V}_1 - \langle \bar{u}_n, \bar{V}_2 \rangle \bar{V}_2 - \ldots - \langle \bar{u}_n, \bar{V}_{n-2} \rangle \bar{V}_{n-1}$$

This basically summarizes the Gram Schmidt Orthonormalization procedure.

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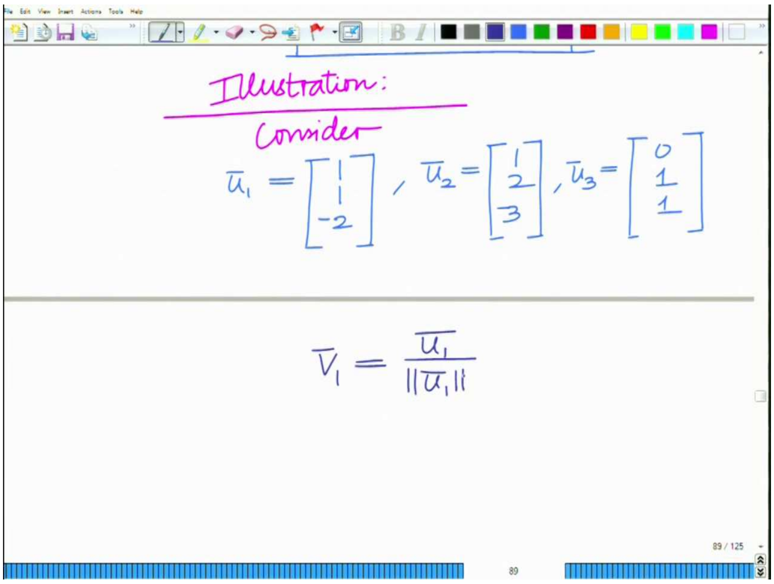

Let us take an example for a practical illustration of the Gram Schmidt Orthonormalization Procedure. Considering a set of vectors;

$$\bar{u}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \bar{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \text{ and } \bar{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Therefore, the unit norm vector of $\bar{u}_1$ is

$$\bar{V}_1 = \frac{\bar{u}_1}{\|\bar{u}_1\|}$$

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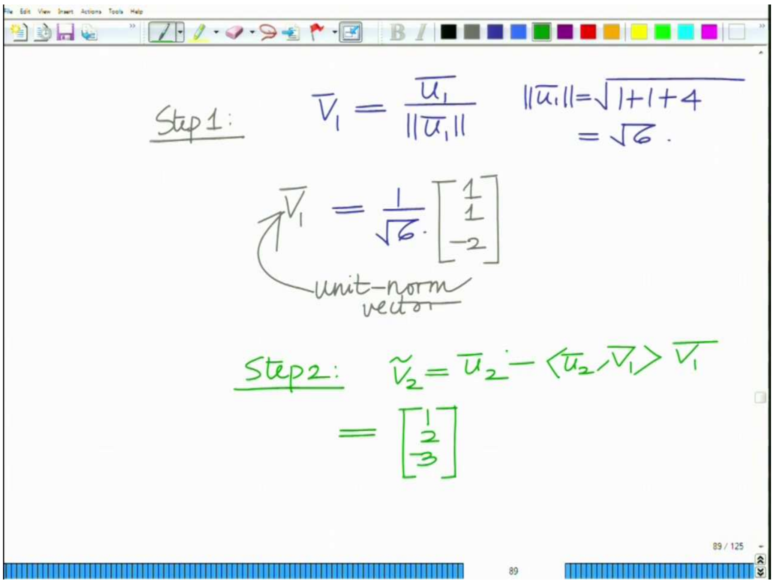

Therefore,

$$\bar{V}_1 = \frac{1}{\sqrt{1+1+4}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$
$$= \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

As there is no vector in the set $\bar{V}$, therefore no projection has been removed in this step.

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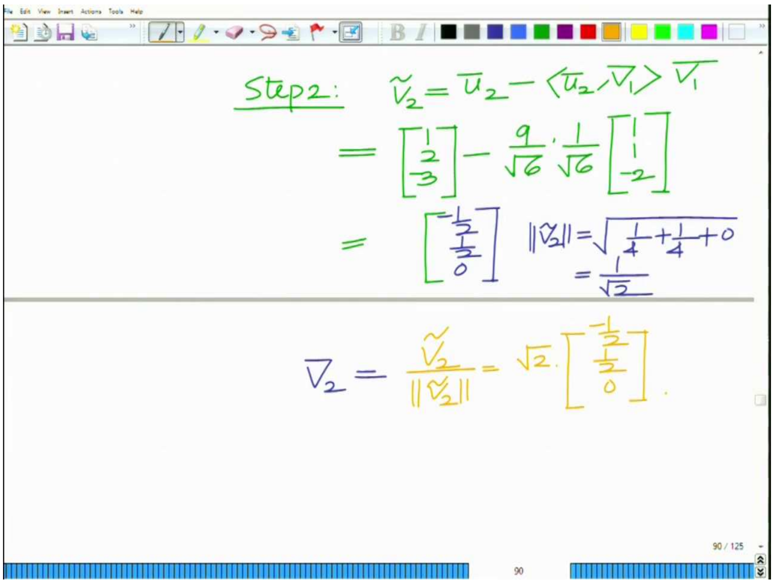

So moving forward to step 2;

$$\tilde{V}_2 = \bar{u}_2 - \langle \bar{u}_2, \bar{V}_1 \rangle \bar{V}_1$$
$$= \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} - \frac{9}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

So, on calculating $\|\tilde{V}_2\|$;

$$\|\tilde{V}_2\| = \sqrt{\frac{1}{4} + \frac{1}{4} + 0}$$
$$= \frac{1}{\sqrt{2}}$$

Therefore;

$$\bar{V}_2 = \frac{\tilde{V}_2}{\left\|\tilde{V}_2\right\|} = \sqrt{2}\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

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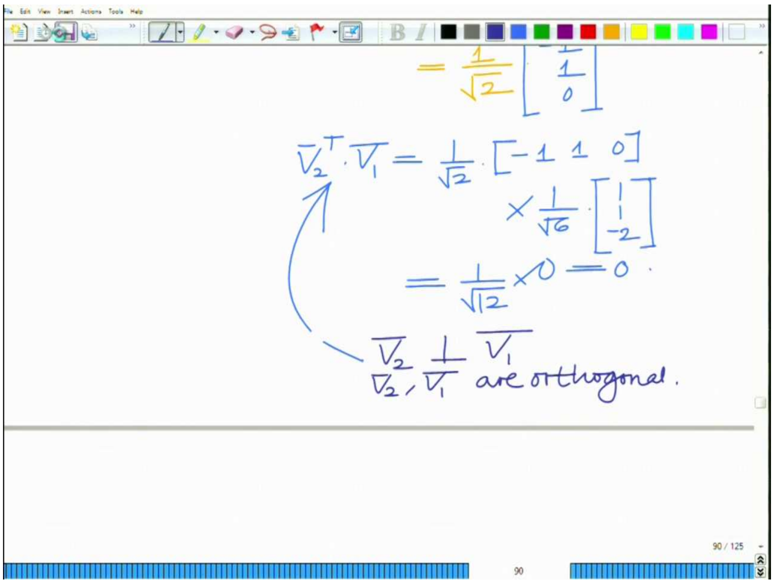

To check the orthogonality;

$$\bar{V}_2^T \cdot \bar{V}_1 = \frac{1}{\sqrt{2}}[-1 \quad 1 \quad 0] \times \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$
$$= \frac{1}{\sqrt{12}} \times 0$$
$$= 0$$

So, this implies that these are orthogonal. This means $\bar{V}_1$ and $\bar{V}_2$ are the vectors of an orthonormal vector space.

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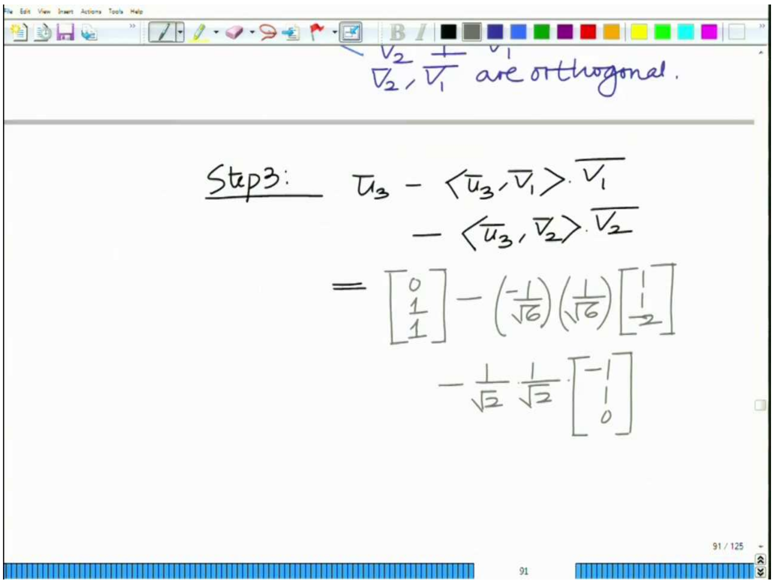

Now, in step 3;

$$\tilde{V}_3 = \bar{u}_3 - \langle \bar{u}_3, \bar{V}_1 \rangle \bar{V}_1 - \langle \bar{u}_3, \bar{V}_2 \rangle \bar{V}_2$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{-1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{6}}\right)\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \left(\frac{1}{6}\right)\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} - \left(\frac{1}{2}\right)\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

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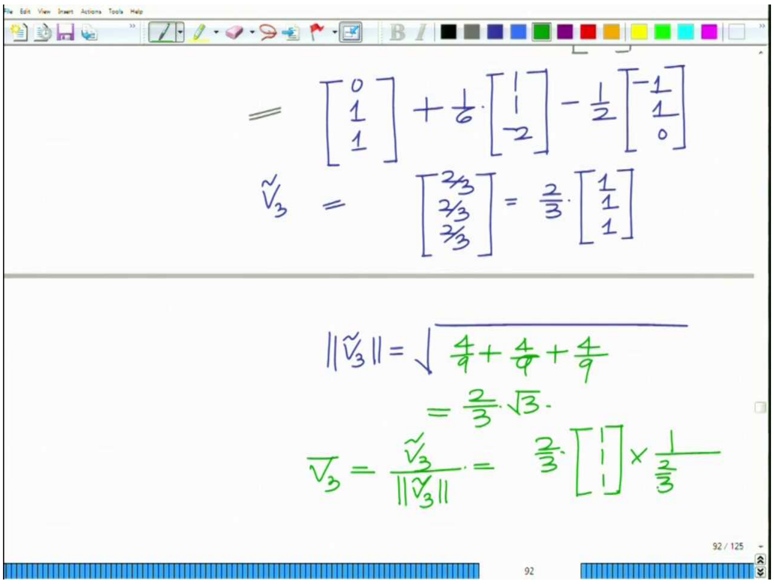

And then

$$\tilde{V}_3 = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} = \frac{2}{3} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

And its norm is

$$\left\| \tilde{V}_3 \right\| = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{4}{9}} = \frac{2}{3}\sqrt{3}$$

And hence the third orthonormal vector is

$$\begin{aligned} \bar{V}_3 &= \frac{\tilde{V}_3}{\left\| \tilde{V}_3 \right\|} \\ &= \frac{2}{3} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{\sqrt{3}}{2} \\ &= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

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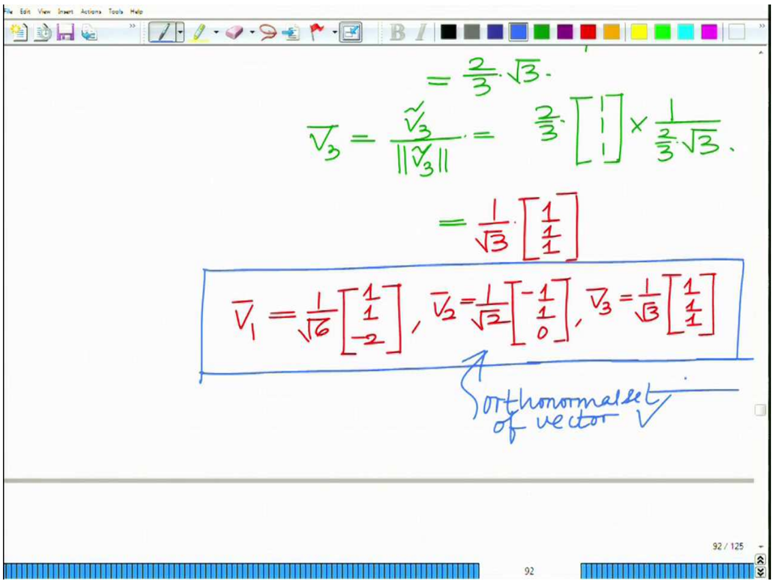

So, finally, the orthonormal set of vectors V, which has the same span space as given vectors $\bar{u}_1$, $\bar{u}_2$ and $\bar{u}_3$.

$$\bar{V}_1 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix},\quad \bar{V}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix},\quad \bar{V}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

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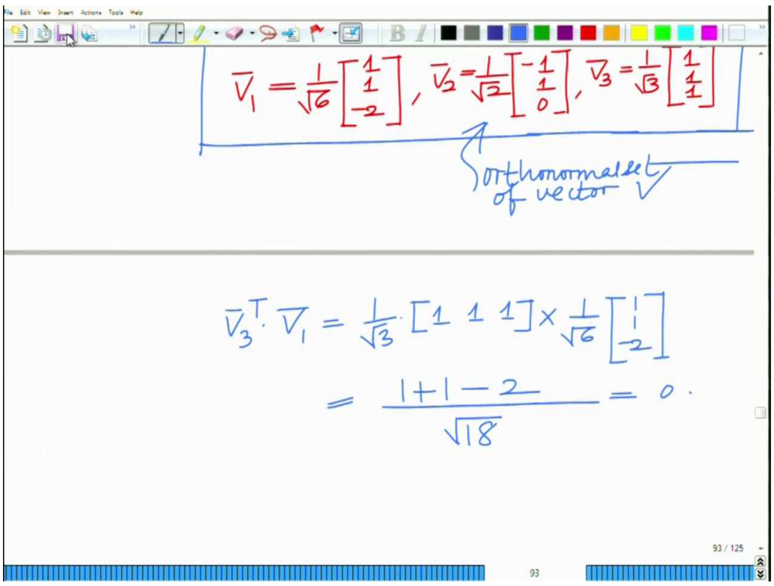

And to check the orthogonality of third vector;

$$\bar{V}_3^T \cdot \bar{V}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\end{bmatrix} \times \frac{1}{\sqrt{6}}\begin{bmatrix}1 \\ 1 \\ -2\end{bmatrix}$$

$$= \frac{1+1-2}{\sqrt{18}}$$

$$= 0$$

And therefore, $\bar{V}_3$ is also orthogonal to other two vectors. Therefore, $(\bar{V}_1, \bar{V}_2, \bar{V}_3)$ is an orthonormal set in the same subspace.

So, although both the given sets have the same subspace, it is very convenient to deal with $\bar{V}$ rather than $\bar{u}$ because $\bar{V}$ is an orthonormal set of vectors that spans the same subspace. And in fact, this can be used for any inner product space; as the set of continuous functions on the interval $[a,b]$ forms an inner product space.

So, for a given set of linearly dependent functions which spans subspace; one can similarly determine an orthonormal set of functions that span the same subspace of continuous functions on the interval $[a,b]$.

So, the Gram Schmidt Orthonormalization procedure is very convenient, handy and highly applicable because it is a low complexity procedure and also, it has immense utility in terms of simplifying; either deriving the span of a subspace or the representation of a new vector in the given subspace. So, we will stop here and we will continue with other aspects in the subsequent modules.

Thank you very much.