Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture - 08
Null Space and Trace of Matrices

Hello, welcome to another module in this massive open online course. In this module, let us discuss another important concept known as the Null Space of a Matrix.

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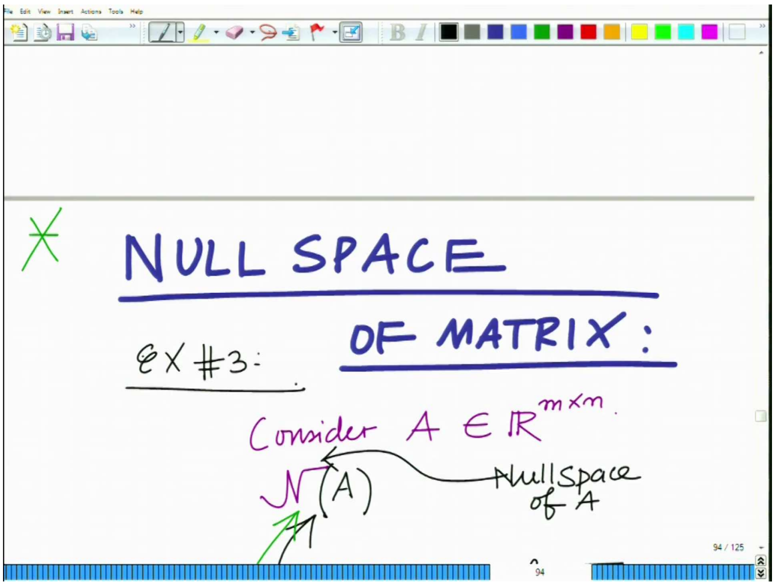

Consider a $m \times n$ matrix A such that $A \in \mathbb{R}^{m \times n}$.

Then the null space of A is denoted as $N(A)$.

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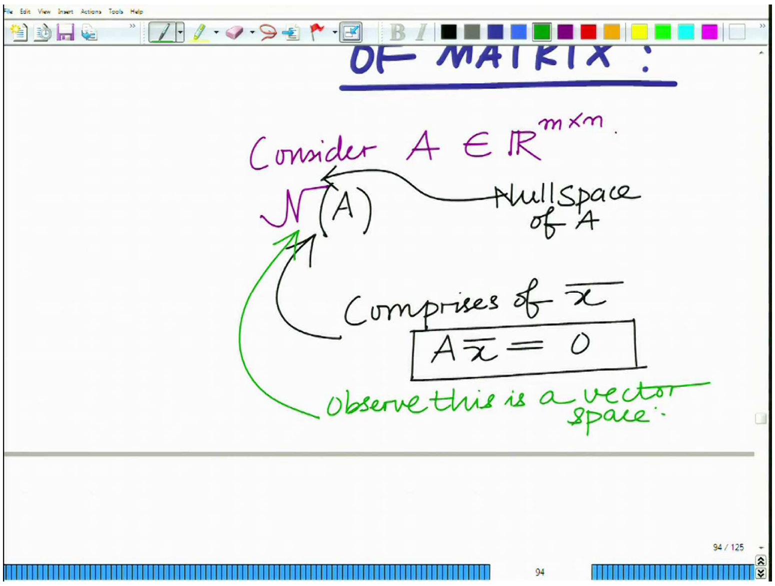

The null space of A; $N(A)$ comprises of all vectors $\bar{x}$ such that

$$A\bar{x} = 0$$

So the vector space of all vectors $\bar{x}$ is called the null space of the matrix A.

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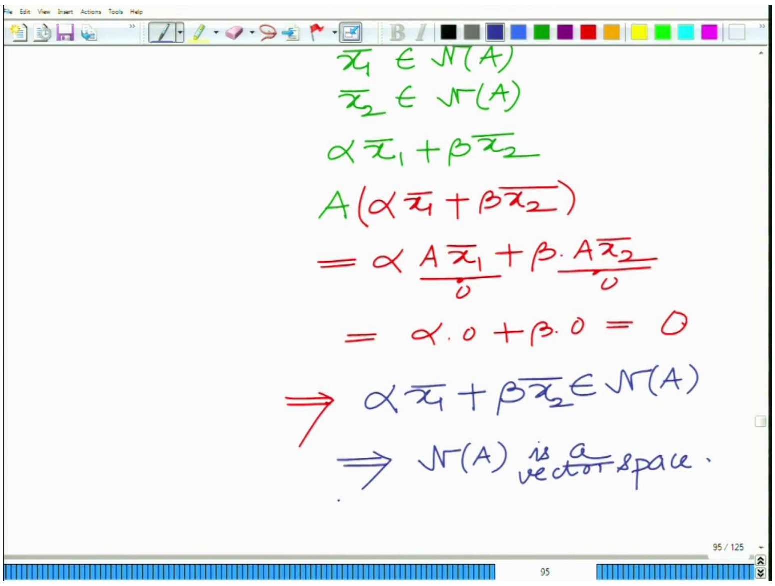

So, observe that the null space as a vector space can be seen as follows. If there are two vectors $\bar{x}_1$ and $\bar{x}_2$ have same null space $N(A)$.

$$\bar{x}_1 \in N(A) \text{ and } \bar{x}_2 \in N(A)$$

This also means that $A\overline{x}_1 = 0$ and $A\overline{x}_2 = 0$. Consider a linear combination of these two vectors is $\alpha\overline{x}_1 + \beta\overline{x}_2$ such that $\overline{x}_1$ and $\overline{x}_2$ both are the elements of the null space of matrix A.

Therefore

$$\begin{aligned} A(\alpha\overline{x}_1 + \beta\overline{x}_2) &= \alpha A\overline{x}_1 + \beta A\overline{x}_2 \\ &= \alpha \cdot 0 + \beta \cdot 0 \\ &= 0 \end{aligned}$$

This means that

$$\alpha\overline{x}_1 + \beta\overline{x}_2 \in N(A)$$

If $\overline{x}_1$ and $\overline{x}_2$ both belongs to the vector space then any linear combination of these vectors also belongs to the set known as a space or a subspace a vector space or a vector subspace. Therefore, the null space of A is a vector space.

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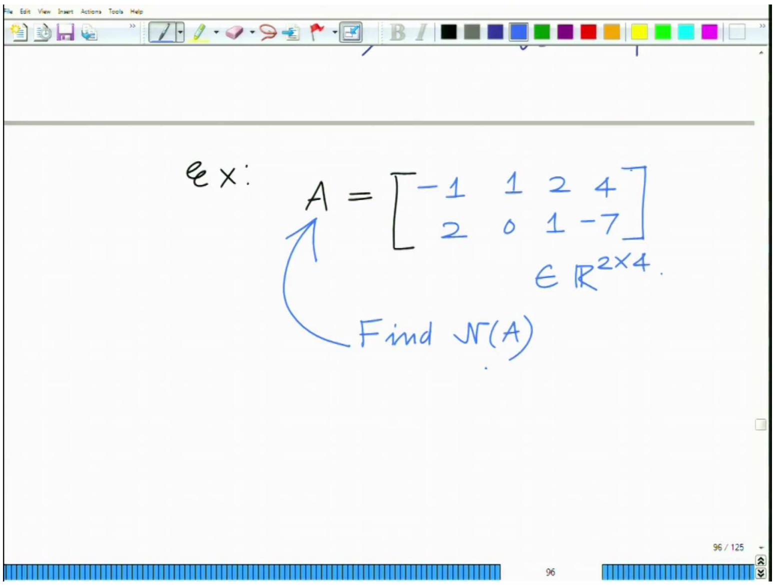

Take an example. Consider a matrix A as

$$A = \begin{bmatrix} -1 & 1 & 2 & 4 \\ 2 & 0 & 1 & -7 \end{bmatrix} \in \mathbb{R}^{2\times 4}$$

And for the given matrix A, find the null space of matrix A.

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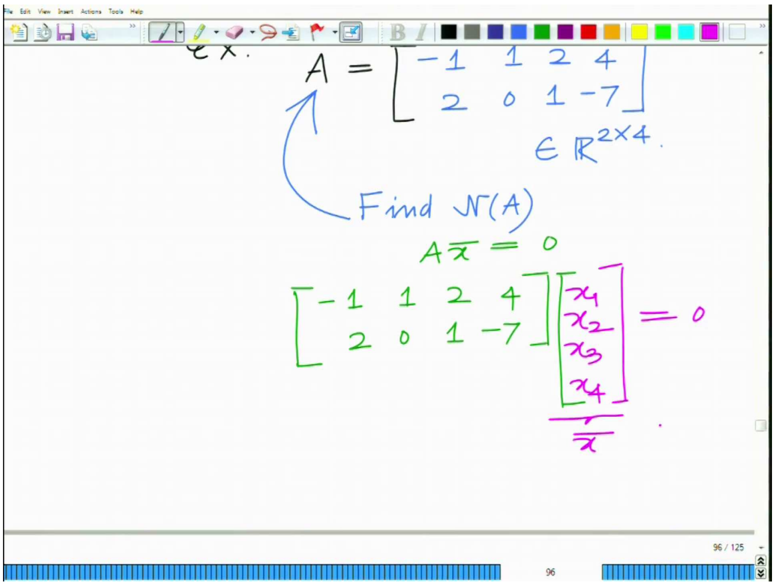

$$A = \begin{bmatrix} -1 & 1 & 2 & 4 \\ 2 & 0 & 1 & -7 \end{bmatrix} \in \mathbb{R}^{2\times 4}$$

So multiply matrix A by a 4 dimensional vector and put it equal to 0. Therefore consider a matrix $\overline{x}$ as

$$\overline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

And thus

$$A\overline{x} = 0$$

$$\begin{bmatrix} -1 & 1 & 2 & 4 \\ 2 & 0 & 1 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

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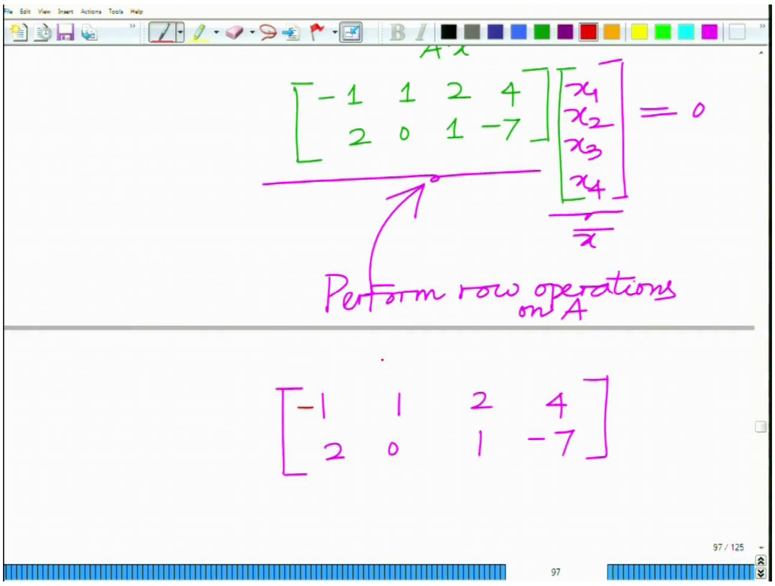

To solve the above equation, first perform row operations on the matrix A.

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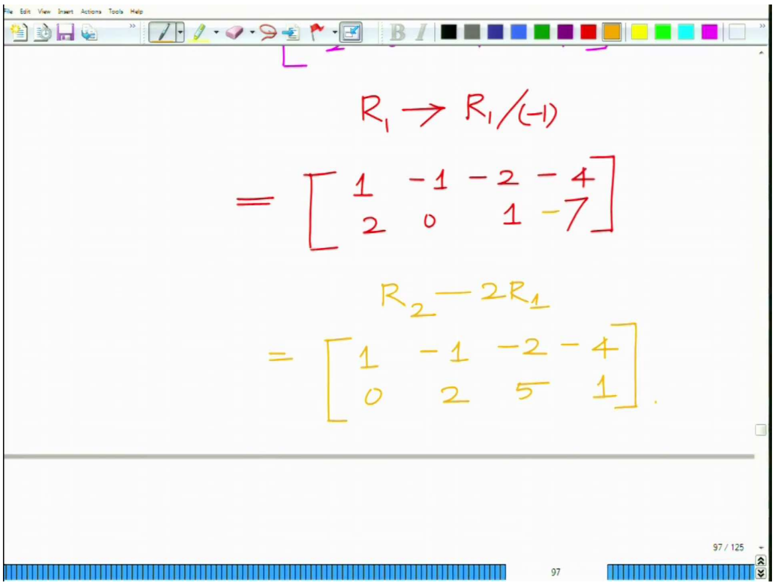

So, first perform we will divide $R_1 \to \frac{R_1}{-1}$.

$$A \equiv \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 0 & 1 & -7 \end{bmatrix}$$

Perform $R_2 \to R_2 - 2R_1$ and the matrix becomes

$$A \equiv \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 2 & 5 & 1 \end{bmatrix}$$

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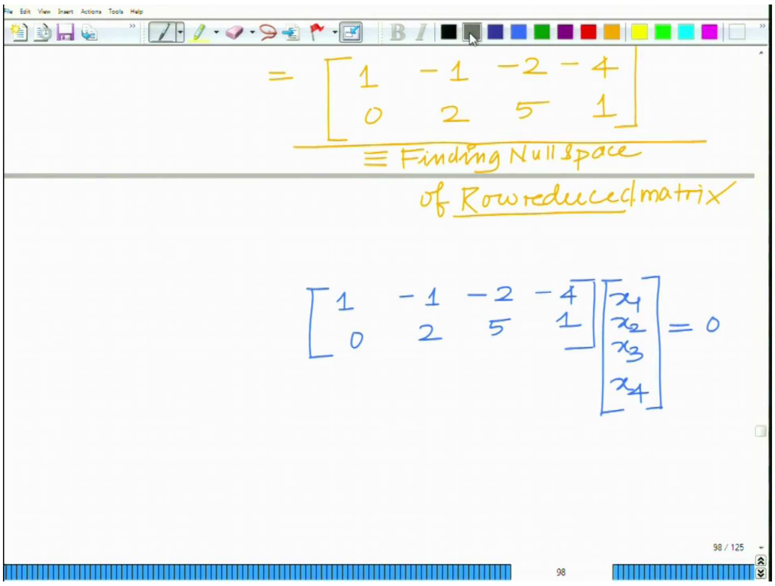

So now let’s find the null space of this row reduced matrix of A, because it is equivalent to finding null space of matrix A.

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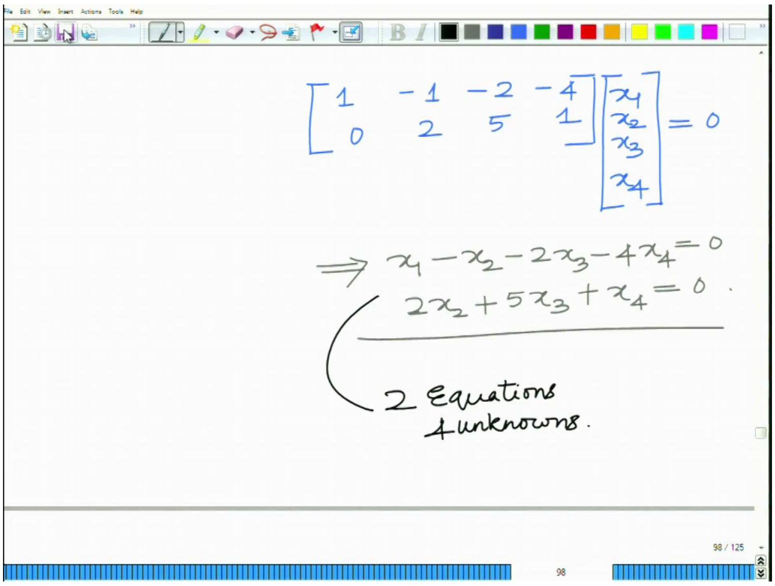

Therefore,

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 2 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

So on solving this, the two equations that came out are

$$\begin{aligned} x_1 - x_2 - 2x_3 - 4x_4 &= 0, \\ 2x_2 + 5x_3 + x_4 &= 0 \end{aligned}$$

This implies that there are two equations and four unknowns. So, set two unknown parameters as free variables. Thus set x_3 and x_4 as free variables and express x_1 and x_2 in terms of x_3 and x_4.

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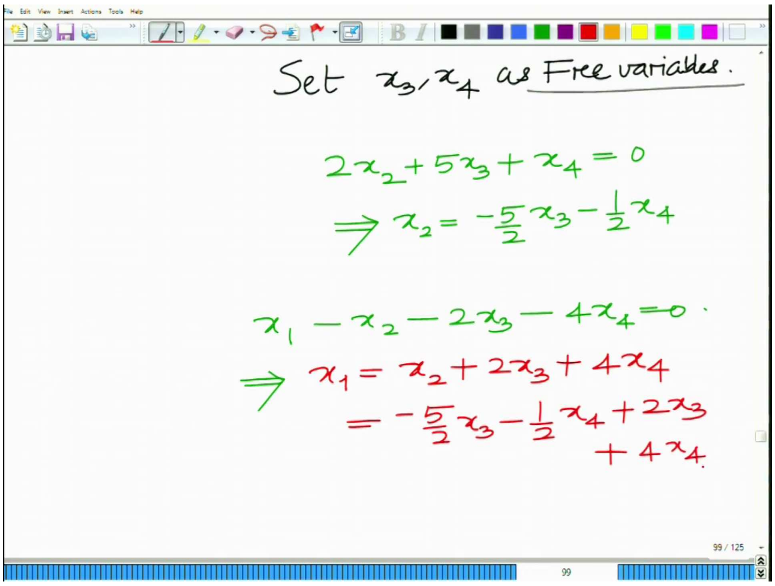

Therefore,

$$2x_2 + 5x_3 + x_4 = 0$$
$$x_2 = -\frac{5}{2}x_3 - \frac{1}{2}x_4$$

And also,

$$x_1 - x_2 - 2x_3 - 4x_4 = 0$$

$$x_1 = x_2 + 2x_3 + 4x_4$$

$$x_1 = -\frac{5}{2}x_3 - \frac{1}{2}x_4 + 2x_3 + 4x_4$$

$$x_1 = -\frac{1}{2}x_3 + \frac{7}{2}x_4$$

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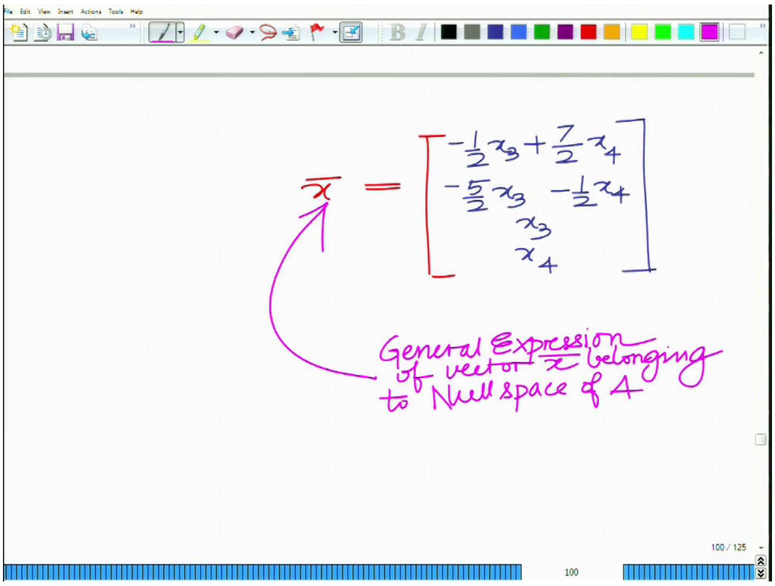

And therefore, the general structure of a null vector that belongs to null space of A will be as

$$\bar{x} = \begin{bmatrix} -\frac{1}{2}x_3 + \frac{7}{2}x_4 \\ -\frac{5}{2}x_3 - \frac{1}{2}x_4 \\ x_3 \\ x_4 \end{bmatrix}$$

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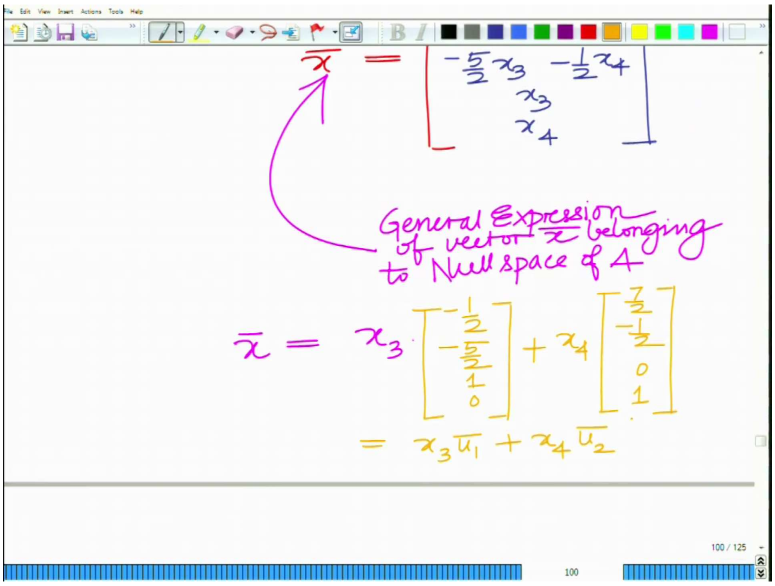

On simplifying this further;

$$\bar{x} = x_3 \begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

$$\bar{x} = x_3 \bar{u}_1 + x_4 \bar{u}_2$$

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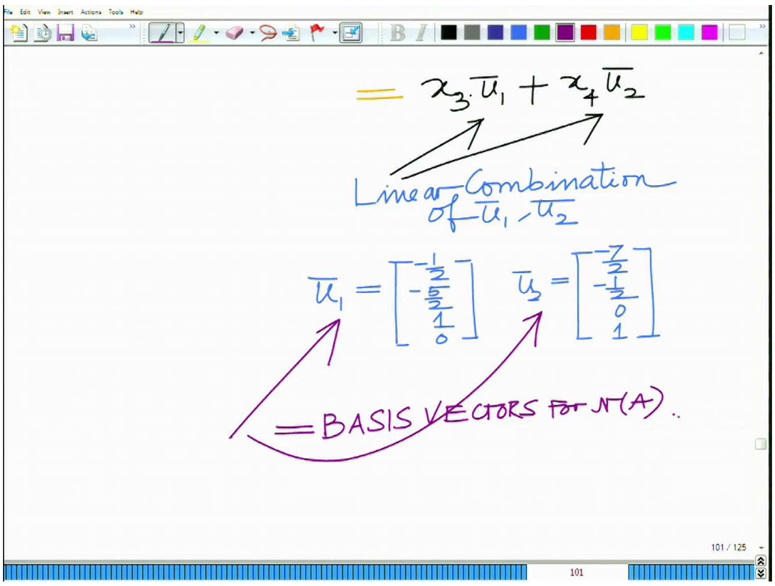

Observe this is a linear combination of two vectors $\bar{u}_1$ and $\bar{u}_2$. Therefore, this null space of A is formed by all linear combinations of these vectors $\bar{u}_1$ and $\bar{u}_2$ and therefore, $\bar{u}_1$ and $\bar{u}_2$ are the basis vectors for the null space of the matrix A. Therefore the basis vectors the null space of the matrix A are and

$$\bar{u}_1 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} \in N(A), \quad \bar{u}_2 = \begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \in N(A)$$

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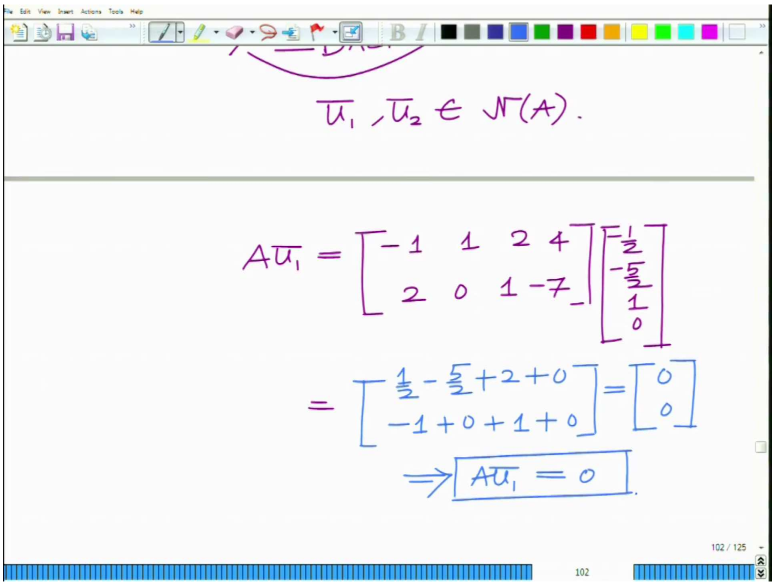

For instance consider $A\bar{u}_1$,

$$A\bar{u}_1 = \begin{bmatrix} -1 & 1 & 2 & 4 \\ 2 & 0 & 1 & -7 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} - \frac{5}{2} + 2 + 0 \\ -1 + 0 + 1 + 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So, this implies

$$A\bar{u}_1 = 0$$

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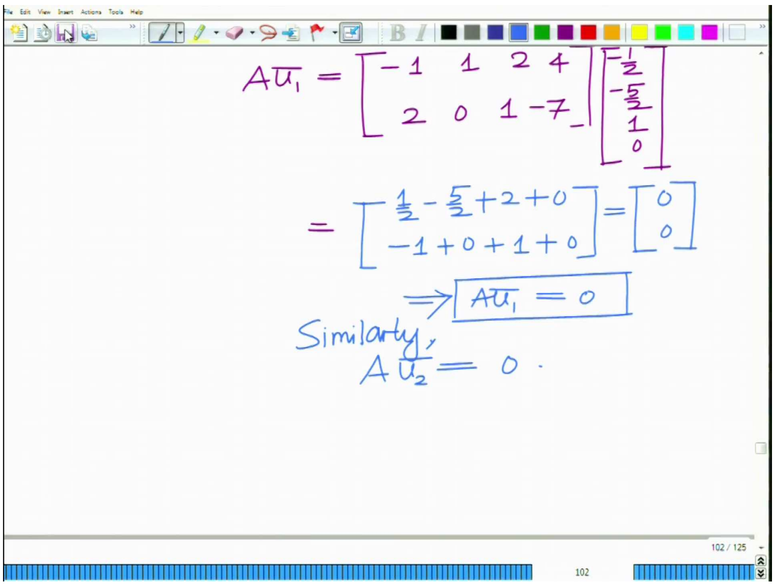

Similarly, consider $A\bar{u}_2$,

$$A\bar{u}_2 = \begin{bmatrix} -1 & 1 & 2 & 4 \\ 2 & 0 & 1 & -7 \end{bmatrix} \begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{7}{2} - \frac{1}{2} + 0 + 4 \\ 7 + 0 + 0 - 7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So, this implies

$$A\bar{u}_2 = 0$$

So this justifies that $\bar{u}_1$ and $\bar{u}_2$ are the null space of matrix A and together it is a vector space because the linear combination of these vectors also lies in this space.

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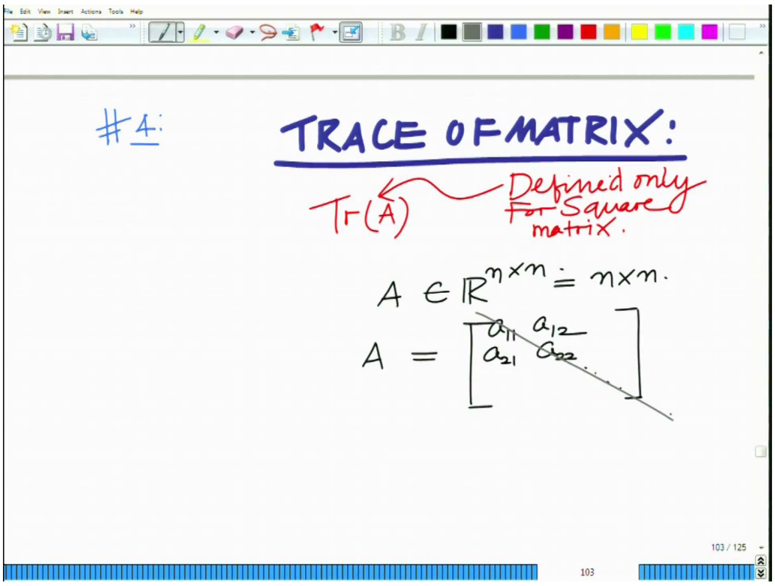

Let us look at another example regarding the trace of a matrix. The trace of a matrix is defined only for a square matrix and is the sum of its diagonal elements. So, take a $n \times n$ square matrix $A \in \mathbb{R}^{n\times n}$ as

$$a = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

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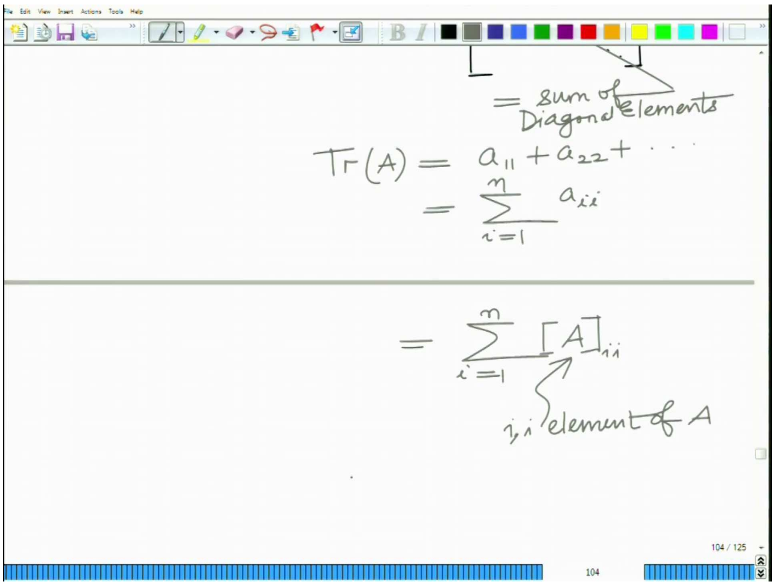

So, trace of matrix A is

$$\begin{aligned} Tr(A) &= a_{11} + a_{22} + \ldots + a_{nn} \\ &= \sum_{i=1}^{n} a_{ii} \\ &= \sum_{i=1}^{n} [A]_{ii} \end{aligned}$$

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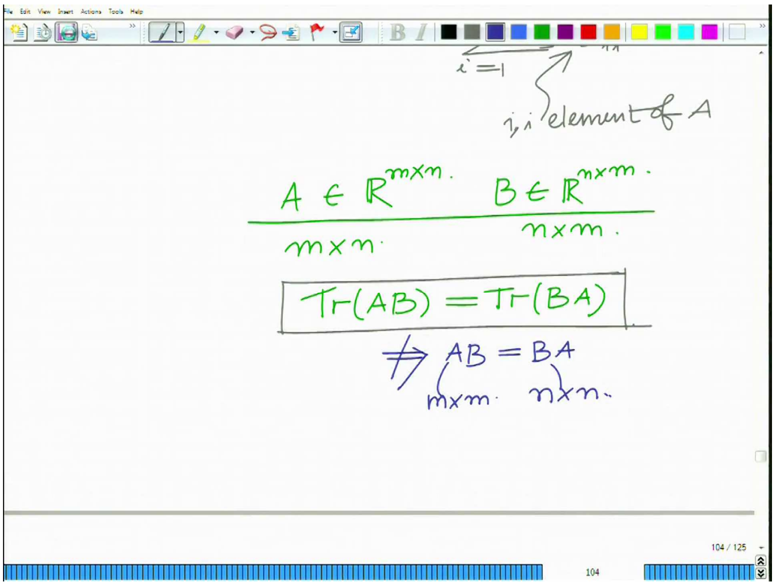

Now, one important property of trace of a matrix is that for two matrix A and B such that $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times m}$; it is true that

$$Tr(AB) = Tr(BA)$$

Also it does not imply that $AB = BA$. In fact, the sizes of AB and BA are not typically equal until $m = n$.

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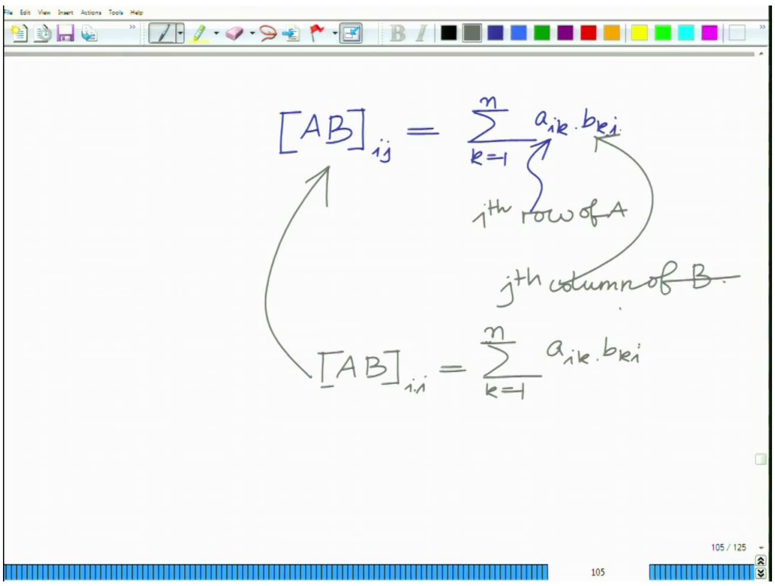

To prove the above property, write down matrix AB in terms of its elements and this is as

$$[AB]_{ij} = \sum_{k=1}^{n} a_{ik}.b_{kj}$$

Here $[AB]_{ij}$ is the ijth element of the matrix product *AB*, $\sum_{k=1}^{n} a_{ik}.b_{kj}$ is the summation of product of kth row elements of matrix A and jth column elements of matrix B.

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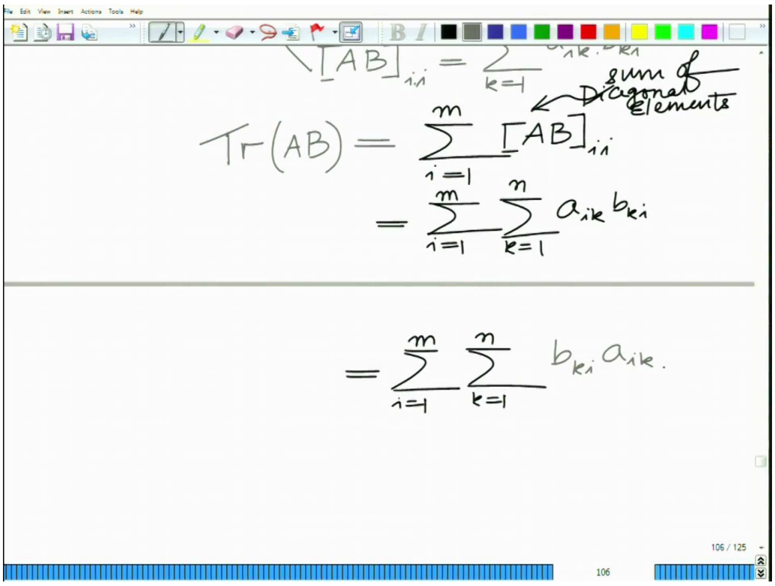

So now, the trace of matrix *AB* is the summation of all the diagonal elements of matrix AB and hence

$$Tr(AB) = \sum_{i=1}^{m} [AB]_{ii}$$
$$= \sum_{i=1}^{m}\sum_{k=1}^{n} a_{ik}.b_{ki}$$
$$= \sum_{i=1}^{m}\sum_{k=1}^{n} b_{ki}.a_{ik}$$
$$= \sum_{k=1}^{n}\sum_{i=1}^{m} b_{ki}.a_{ik}$$

Here b_{ki} is the kth row of B and a_{ik} is the kth column of A.

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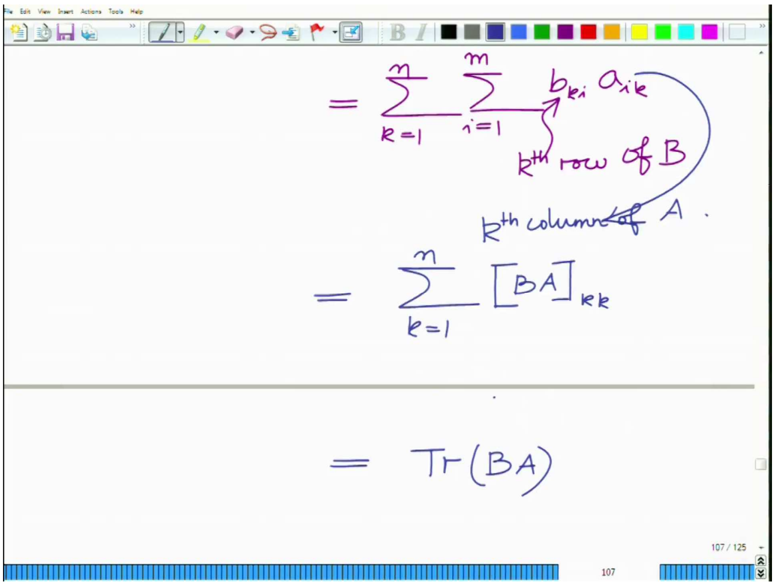

And therefore it can also be written as

$$\begin{aligned} Tr(AB) &= \sum_{k=1}^{n}\sum_{i=1}^{m} b_{ki}.a_{ik} \\ &= \sum_{k=1}^{n}[BA]_{kk} \\ &= Tr(BA) \end{aligned}$$

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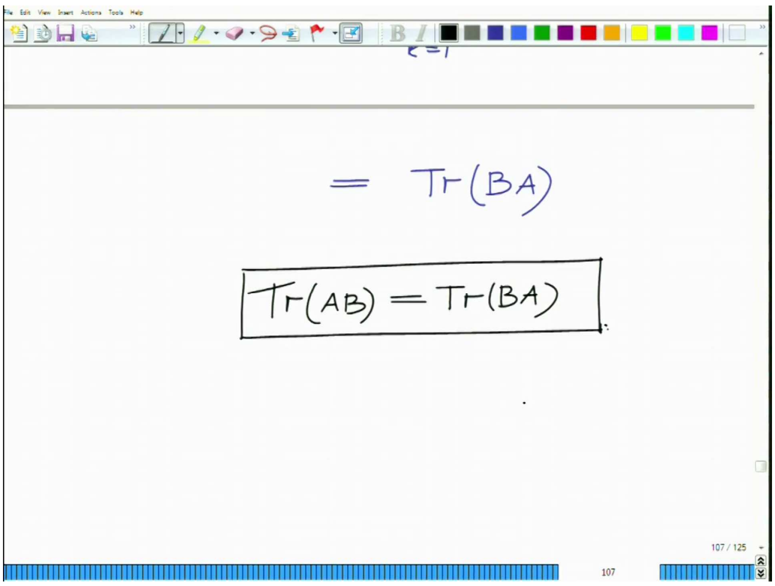

And therefore, the above property is verified. This means that in general the matrices do not have a commutative property, but trace of matrix AB equals the trace of the matrix

BA. This is an interesting property of matrices, which will come handy in several problems or several optimization problems, where matrices or the product of matrices will be needed to manipulate. So, let us stop here and we will continue with some other problems in the subsequent modules.

Thank you very much.