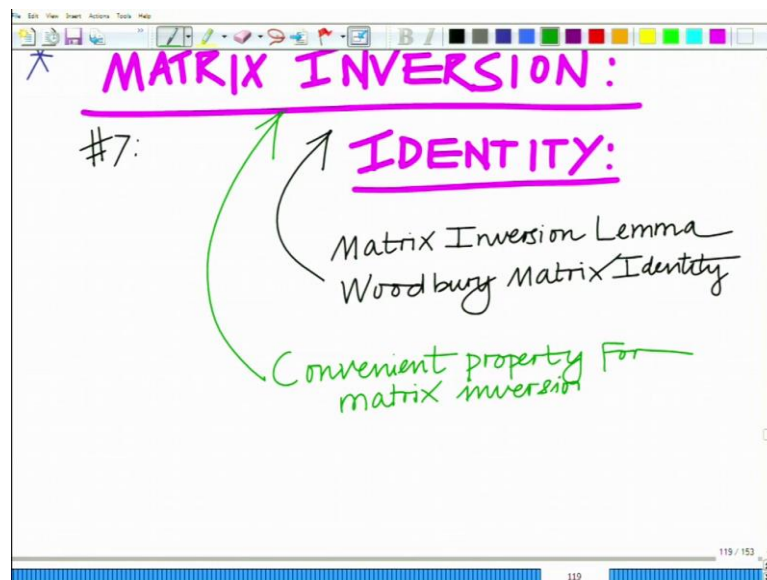


**Applied Optimization for Wireless, Machine Learning, Big Data**  
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**Lecture - 10**  
**Matrix Inversion Lemma (Woodbury identity)**

Hello. welcome to another module in this massive open online course. Let us continue this discussion with another important principle known as the Matrix Inversion Lemma or the Matrix Inversion Identity.

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The matrix inversion identity is also termed as the Woodbury matrix identity. It is a very convenient principle for the matrix inversions.

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Handwritten derivation of the Sherman-Morrison formula on a whiteboard. The formula is written as:

$$(A + UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}$$

Annotations include:

- An arrow from  $(A + UCV)^{-1}$  to the text "To compute inverse of matrix".
- An arrow from  $A^{-1}$  to the text " $A^{-1}$  is known".
- An arrow from  $UCV$  to the text " $UCV = \text{low rank Matrix}$ ".

Considering that the inverse of matrix  $A$  is known and also that matrix  $UCV$  is a low rank matrix.  $C$  is a constant value. Therefore according to this identity;

$$(A + UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}$$

So let us discuss it with an example.

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Handwritten example of the Sherman-Morrison formula on a whiteboard. The example is written as:

ex: Consider  $A = I$ ,  $U = \bar{x}$ ,  $C = 1$ ,  $V = \bar{x}^T$ . Then  $A + UCV = I + \bar{x}\bar{x}^T$ .

Annotations include:

- An arrow from  $A = I$  to the text "Rank = 1".
- An arrow from  $\bar{x}\bar{x}^T$  to the text "Rank = 1".
- An arrow from  $I + \bar{x}\bar{x}^T$  to the text "Easily compute inverse".
- The vector  $\bar{x}$  is defined as  $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ .
- The matrix  $A$  is defined as  $A = I$ .
- The matrix  $U$  is defined as  $U = \bar{x}$ .
- The scalar  $C$  is defined as  $C = 1$ .
- The matrix  $V$  is defined as  $V = \bar{x}^T$ .

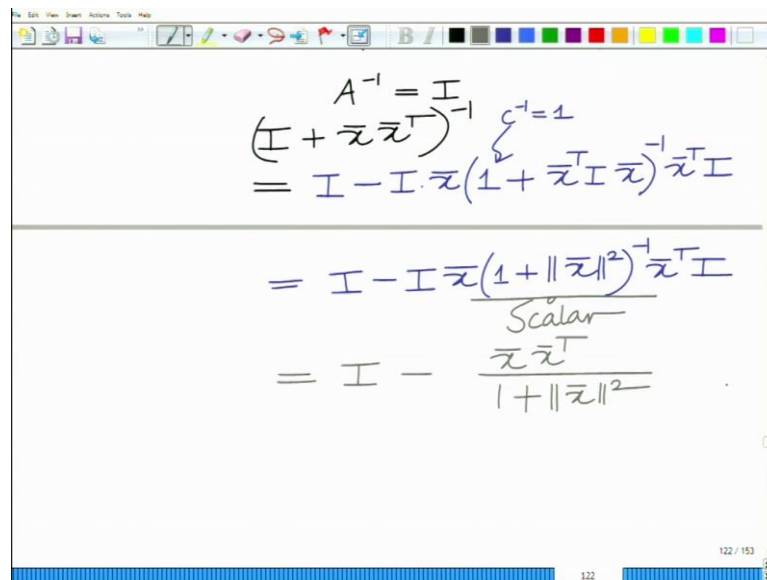
For example, compute the inverse of a  $n \times 1$  matrix  $\bar{x}$  defined as

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Let us use matrix  $I$  as the matrix  $A$  of the above equation because the inverse of  $I$  is known and it is  $I$  itself. Take  $C=1$ . Matrix  $U$  is  $\bar{x}$  and  $V$  is  $\bar{x}^T$ .

$$(A + UCV)^{-1} = (I + \bar{x}\bar{x}^T)^{-1}$$

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The image shows a handwritten derivation of the matrix inversion lemma. The steps are as follows:

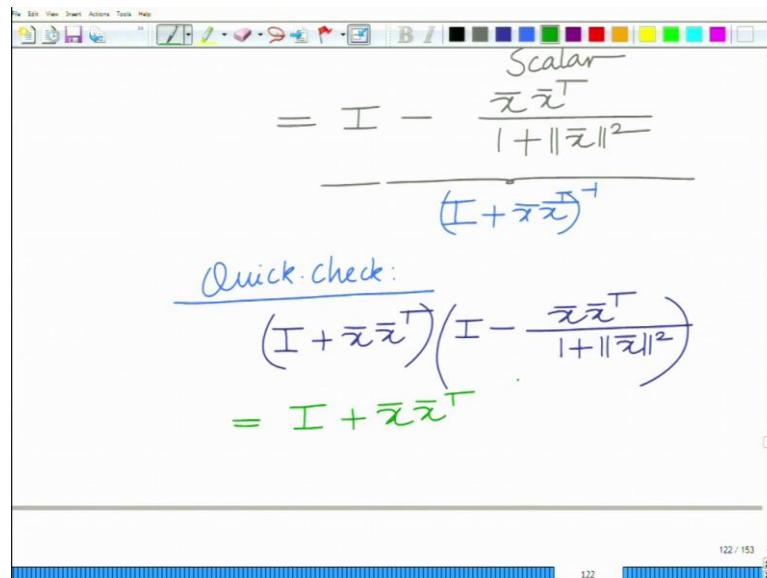
$$\begin{aligned} A^{-1} &= I \\ (I + \bar{x}\bar{x}^T)^{-1} &\stackrel{C^{-1}=1}{=} I - I \cdot \bar{x} (1 + \bar{x}^T I \bar{x})^{-1} \bar{x}^T I \\ &= I - I \bar{x} \frac{(1 + \|\bar{x}\|^2)^{-1}}{\text{Scalar}} \bar{x}^T I \\ &= I - \frac{\bar{x} \bar{x}^T}{1 + \|\bar{x}\|^2} \end{aligned}$$

So use the matrix inversion lemma property and this can be done as follows.

$$\begin{aligned} &(I + \bar{x}\bar{x}^T)^{-1} \\ &= I - I \cdot \bar{x} (1 + \bar{x}^T \cdot I \cdot \bar{x})^{-1} \bar{x}^T \cdot I \\ &= I - I \cdot \bar{x} (1 + \|\bar{x}\|^2)^{-1} \bar{x}^T \cdot I \\ &= I - \frac{\bar{x} \cdot \bar{x}^T}{(1 + \|\bar{x}\|^2)} \end{aligned}$$

Note that  $(1 + \|\bar{x}\|^2)$  is a scalar quantity.

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Handwritten derivation on a digital whiteboard. The first equation shows the inverse of  $I + \bar{x}\bar{x}^T$  as  $I - \frac{\bar{x}\bar{x}^T}{1 + \|\bar{x}\|^2}$ , with the term  $\frac{\bar{x}\bar{x}^T}{1 + \|\bar{x}\|^2}$  labeled as a "Scalar". Below this, a "Quick check" is performed by multiplying  $(I + \bar{x}\bar{x}^T)$  by the proposed inverse, resulting in  $I + \bar{x}\bar{x}^T$ .

$$= I - \frac{\bar{x}\bar{x}^T}{1 + \|\bar{x}\|^2}$$

$$(I + \bar{x}\bar{x}^T)^{-1}$$

Quick check:

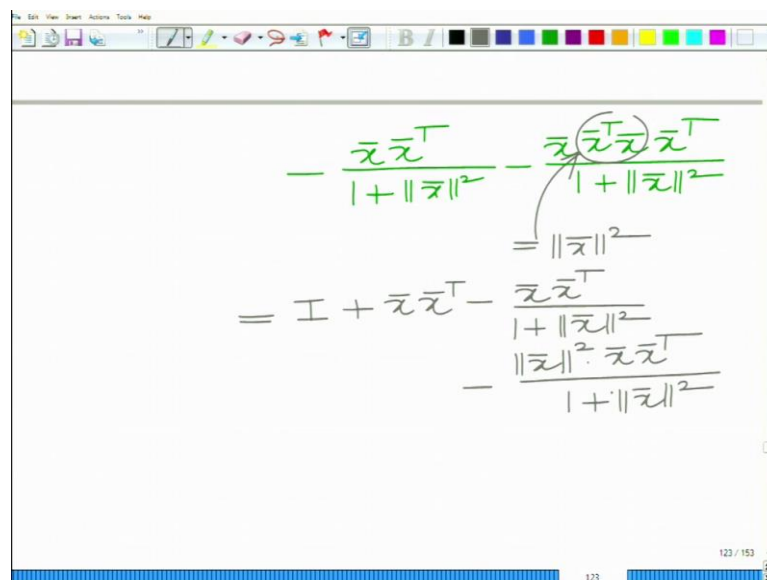
$$(I + \bar{x}\bar{x}^T) \left( I - \frac{\bar{x}\bar{x}^T}{1 + \|\bar{x}\|^2} \right)$$

$$= I + \bar{x}\bar{x}^T$$

So the inverse of  $I + \bar{x}\bar{x}^T$  is  $I - \frac{\bar{x} \cdot \bar{x}^T}{(1 + \|\bar{x}\|^2)}$ .

So, this is the simple trick that can be readily used to compute the inverse of such matrices.

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Handwritten derivation on a digital whiteboard showing the application of the Sherman-Morrison formula. The formula is written as  $I + \bar{x}\bar{x}^T - \frac{\bar{x}(\bar{x}^T\bar{x})\bar{x}^T}{1 + \|\bar{x}\|^2}$ . The term  $\bar{x}(\bar{x}^T\bar{x})\bar{x}^T$  is simplified to  $\|\bar{x}\|^2 \bar{x}\bar{x}^T$ , leading to the final expression  $I + \bar{x}\bar{x}^T - \frac{\|\bar{x}\|^2 \bar{x}\bar{x}^T}{1 + \|\bar{x}\|^2}$ .

$$= I + \bar{x}\bar{x}^T - \frac{\bar{x}(\bar{x}^T\bar{x})\bar{x}^T}{1 + \|\bar{x}\|^2}$$

$$= I + \bar{x}\bar{x}^T - \frac{\|\bar{x}\|^2 \bar{x}\bar{x}^T}{1 + \|\bar{x}\|^2}$$

Let's do a quick check of this.

$$\begin{aligned}
& (I + \bar{x}\bar{x}^T) \left( I - \frac{\bar{x}\bar{x}^T}{(1 + \|\bar{x}\|^2)} \right) \\
&= I + \bar{x}\bar{x}^T - \frac{\bar{x}\bar{x}^T}{(1 + \|\bar{x}\|^2)} - \frac{\bar{x}\bar{x}^T \bar{x}\bar{x}^T}{(1 + \|\bar{x}\|^2)} \\
&= I + \bar{x}\bar{x}^T - \frac{\bar{x}\bar{x}^T}{(1 + \|\bar{x}\|^2)} - \frac{\bar{x} \|\bar{x}\|^2 \bar{x}^T}{(1 + \|\bar{x}\|^2)} \\
&= I + \bar{x}\bar{x}^T - \frac{\bar{x}\bar{x}^T}{(1 + \|\bar{x}\|^2)} - \frac{\|\bar{x}\|^2 \bar{x}\bar{x}^T}{(1 + \|\bar{x}\|^2)}
\end{aligned}$$

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The image shows a handwritten derivation on a digital whiteboard. The steps are as follows:

$$\begin{aligned}
&= I + \bar{x}\bar{x}^T - \frac{\bar{x}\bar{x}^T}{1 + \|\bar{x}\|^2} - \frac{\|\bar{x}\|^2 \bar{x}\bar{x}^T}{1 + \|\bar{x}\|^2} \\
&= I + \bar{x}\bar{x}^T - \bar{x}\bar{x}^T \cdot \frac{1 + \|\bar{x}\|^2}{1 + \|\bar{x}\|^2} \\
&= I + \bar{x}\bar{x}^T - \bar{x}\bar{x}^T \\
&= I
\end{aligned}$$

And then it can further be simplified as

$$\begin{aligned}
& (I + \bar{x}\bar{x}^T) \left( I - \frac{\bar{x}\bar{x}^T}{(1 + \|\bar{x}\|^2)} \right) \\
&= I + \bar{x}\bar{x}^T - \frac{(1 + \|\bar{x}\|^2) \cdot \bar{x}\bar{x}^T}{(1 + \|\bar{x}\|^2)} \\
&= I + \bar{x}\bar{x}^T - \bar{x}\bar{x}^T \\
&= I
\end{aligned}$$

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The image shows a digital whiteboard interface with a toolbar at the top. The main content area contains a handwritten equation enclosed in a blue rectangular box. The equation is:

$$(I + \bar{x} \cdot \bar{x}^T)^{-1} = I - \frac{\bar{x} \cdot \bar{x}^T}{1 + \|\bar{x}\|^2}$$

Below the box, the text "Matrix Inversion Identity." is written in blue ink, with a blue arrow pointing from the text to the boxed equation. The whiteboard interface includes a status bar at the bottom with the page number "124 / 153" and a small "124" indicator.

Hence, the matrix inversion identity is verified.

$$(I + \bar{x} \bar{x}^T)^{-1} = \left( I - \frac{\bar{x} \bar{x}^T}{(1 + \|\bar{x}\|^2)} \right)$$

So, this, the Woodberry matrix inversion identity, is a very handy property. So the matrix inversion identity or the Woodberry matrix inversion or the Woodberry matrix inversion lemma is demonstrated.