Applied Optimization for Wireless, Machine Learning, Big data
Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture-20
Inverse of a Positive Define Matrix, Eigenvalue Properties and Relation between different norms

Hello, welcome to another module in this massive open online course. Let us look at another example related to Positive Definite Matrices.

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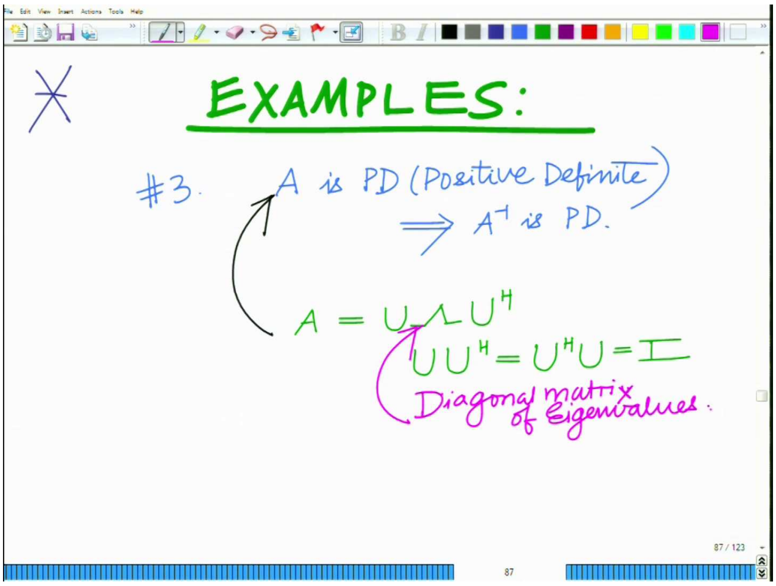

So, another property of a positive definite matrix is that if a matrix A is a positive definite matrix then inverse matrix of matrix A is also a positive definite matrix. To verify this property, Let us write matrix A as follows.

$$A = U \Lambda U^H$$

Where Λ is the diagonal matrix of eigenvalues of matrix A and U is a unitary matrix which satisfies that

$$UU^H = U^H U = I$$

I is the identity matrix.

Also this has been seen previously that the eigenvalues of any positive definite matrix are strictly greater than zero i.e. $\lambda_i > 0$. Therefore matrix A has all positive eigenvalues.

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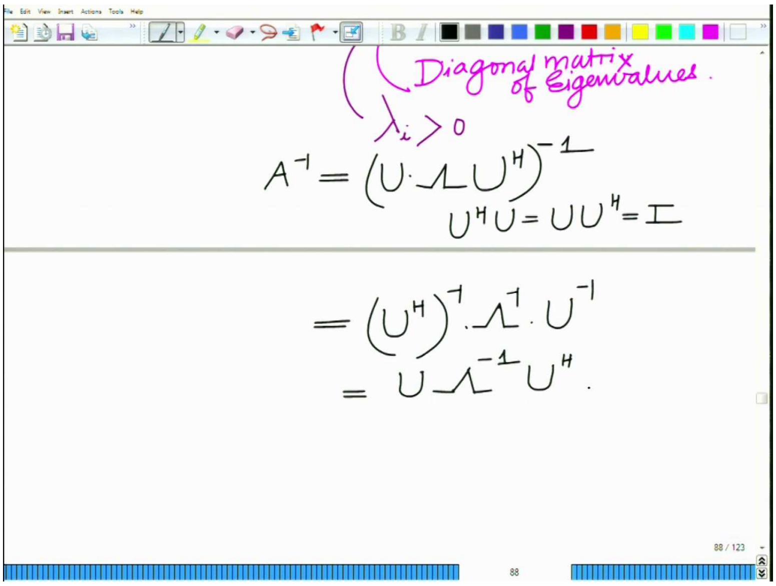

So, inverse of matrix A is

$$
\begin{aligned}
A^{-1} &= \left(U \Lambda U^H\right)^{-1} \\
&= \left(U^H\right)^{-1} \Lambda^{-1} U^{-1} \\
&= U \Lambda^{-1} U^H
\end{aligned}
$$

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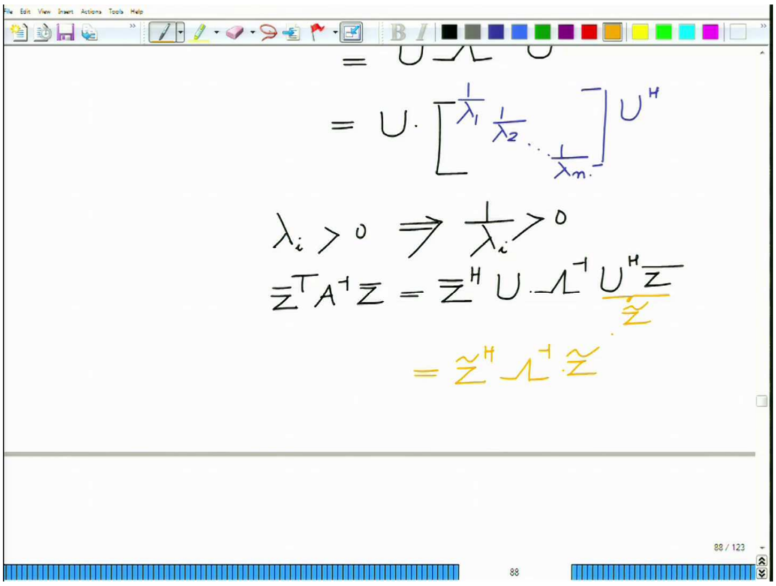

This can also be written as

$$A^{-1} = U \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\lambda_n} \end{bmatrix} U^H$$

As all the eigenvalues λ_i are greater than zero therefore $\frac{1}{\lambda_i}$ are also greater than zero. So, eigenvalues of A^{-1} are also greater than zero. And therefore, matrix A^{-1} is also a positive definite matrix.

This can also be checked as follows. Consider for any real vector $\overline{Z}$;

$$\overline{Z}^T A^{-1} \overline{Z} = \overline{Z}^T U \Lambda^{-1} U^H \overline{Z}$$

Let us say

$$U^H \overline{Z} = \tilde{Z}$$

Therefore,

$$\overline{Z}^T A^{-1} \overline{Z} = \tilde{Z}^H \Lambda^{-1} \tilde{Z}$$

$$= \begin{bmatrix} \tilde{Z}_1^* & \tilde{Z}_2^* & \cdots & \tilde{Z}_n^* \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\lambda_n} \end{bmatrix} \begin{bmatrix} \tilde{Z}_1 \\ \tilde{Z}_2 \\ \vdots \\ \tilde{Z}_n \end{bmatrix}$$

$$= \sum_{i=1}^{n} \frac{1}{\lambda_i} \left| \tilde{Z}_i \right|^2$$

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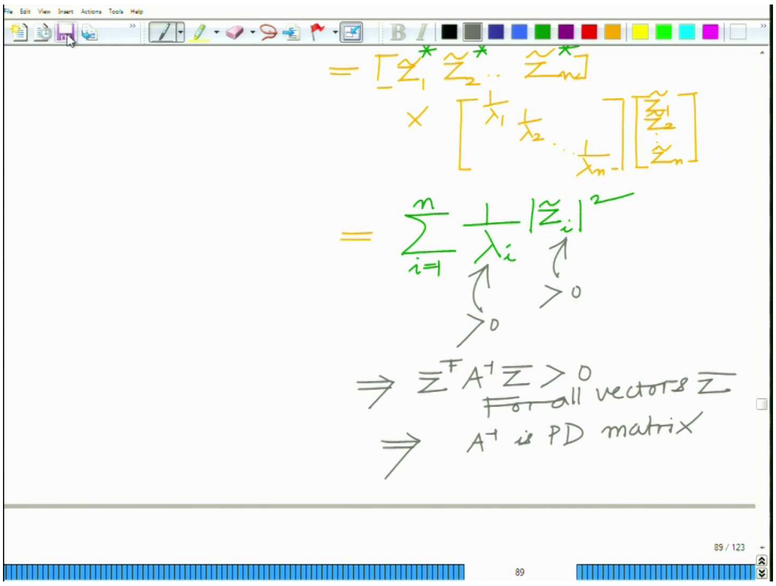

Now, as $\frac{1}{\lambda_i} > 0$ and $|\tilde{Z}_i| > 0$; therefore this implies that for all vectors $\bar{Z}$,

$$\bar{Z}^T A^{-1} \bar{Z} > 0$$

This implies that A^{-1} is a positive definite matrix and this verifies the above property.

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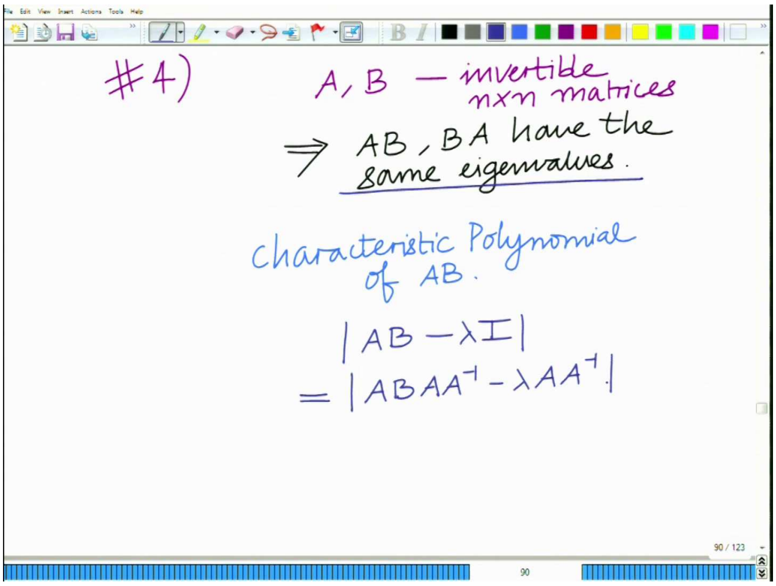

Next property of a matrix is that for two invertible $n \times n$ matrices A and B; matrix AB and BA have same eigenvalues. To verify this, let us start with the characteristic polynomial of matrix *AB*. Using property $AA^{-1} = I$; the characteristic polynomial of matrix *AB* can be written as

$$
\begin{aligned}
|AB - \lambda I| &= |ABAA^{-1} - \lambda AA^{-1}| \\
&= |A(BA - \lambda I)A^{-1}| \\
&= |A||BA - \lambda I||A^{-1}| \\
&= |A||BA - \lambda I|\frac{1}{|A|} \\
&= |BA - \lambda I|
\end{aligned}
$$

So, this implies that the characteristic polynomial of matrix *AB* is equal the characteristic polynomial of matrix *BA*. This further implies that the roots of matrix *AB* and *BA* are identical and therefore the eigenvalues of *AB* are equal as of matrix *BA*.

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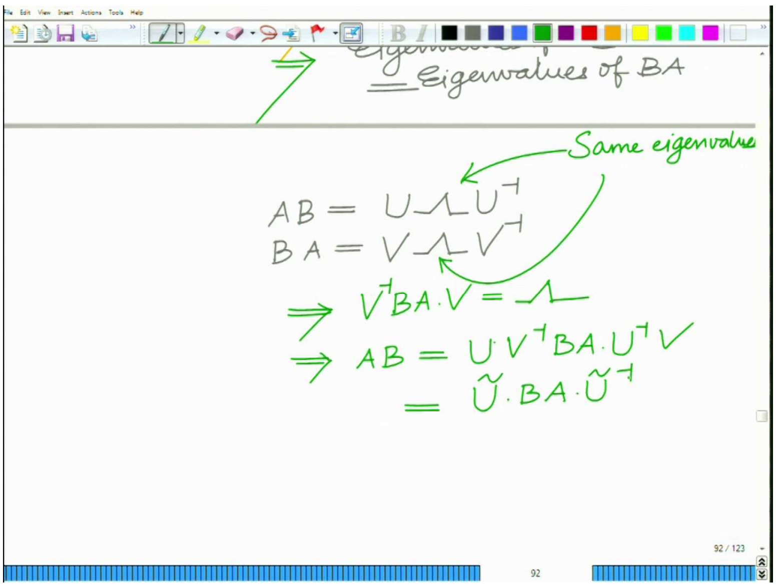

Therefore, matrix *AB* and *BA* can be expressed as

$$
\begin{aligned}
AB &= U\Lambda^{-1}U^{-1} \\
BA &= V\Lambda^{-1}V^{-1}
\end{aligned}
$$

Where Λ is the same diagonal matrix of eigenvalues for both the matrices *AB* and *BA*.

So,

$$V^{-1}BAV = \Lambda$$

Therefore

$$\begin{aligned} AB &= U\left(V^{-1}BAV\right)U^{-1} \\ &= UV^{-1}\left(BA\right)VU^{-1} \\ &= \tilde{U}.BA\tilde{U}^{-1} \end{aligned}$$

Where matrix $\tilde{U} = UV^{-1}$.

In general it is known that matrix C is similar to matrix D only if there exists a matrix M such that

$$C = M^{-1}DM$$

Therefore, it verifies that matrix AB is similar to the matrix BA.

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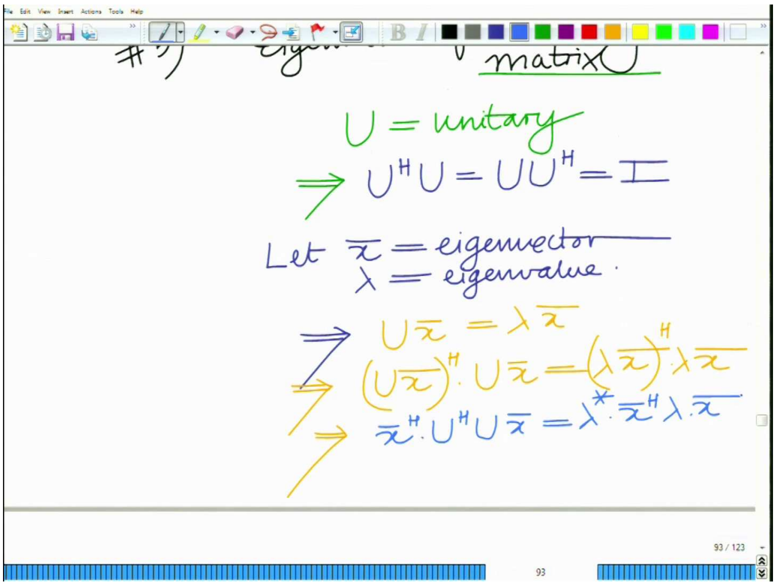

The next interesting property of matrices is that the eigenvalues of unitary matrix have unit magnitude. For the verification of this property, consider matrix U be a unitary matrix.

So as per the definition of a unitary matrix,

$$U^H U = UU^H = I$$

If $\overline{x}$ and λ are the eigenvector and eigenvalue of matrix U, then

$$U\overline{x} = \lambda\overline{x}$$

Therefore as the definition of unitary matrix;

$$(U\overline{x})^H U\overline{x} = (\lambda\overline{x})^H \lambda\overline{x}$$

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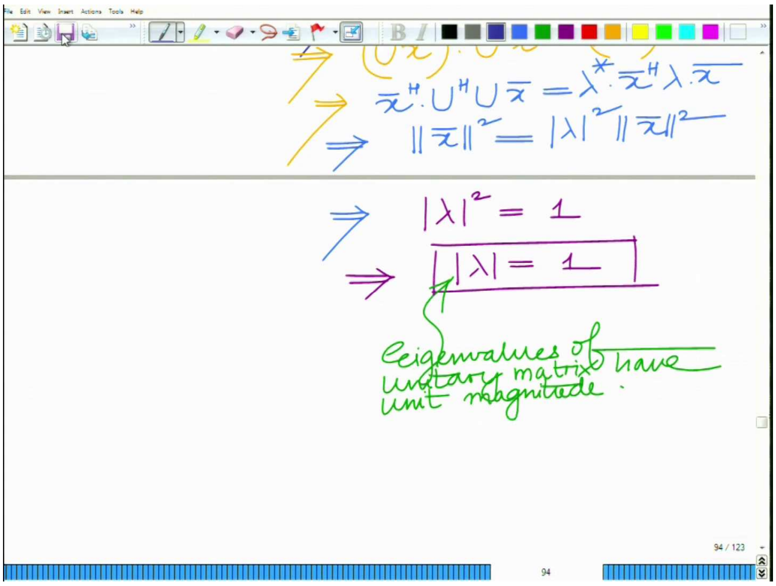

As λ is a number so $\lambda^H = \lambda^*$ and therefore, above expression is now

$$\overline{x}^H U^H U\overline{x} = \lambda^* \overline{x}^H \lambda\overline{x}$$
$$\|\overline{x}\|^2 = |\lambda|^2 \|\overline{x}\|^2$$

This means that

$$|\lambda|^2 = 1$$
$$|\lambda| = 1$$

Hence this verifies that the eigenvalues of unitary matrix have unit magnitude.

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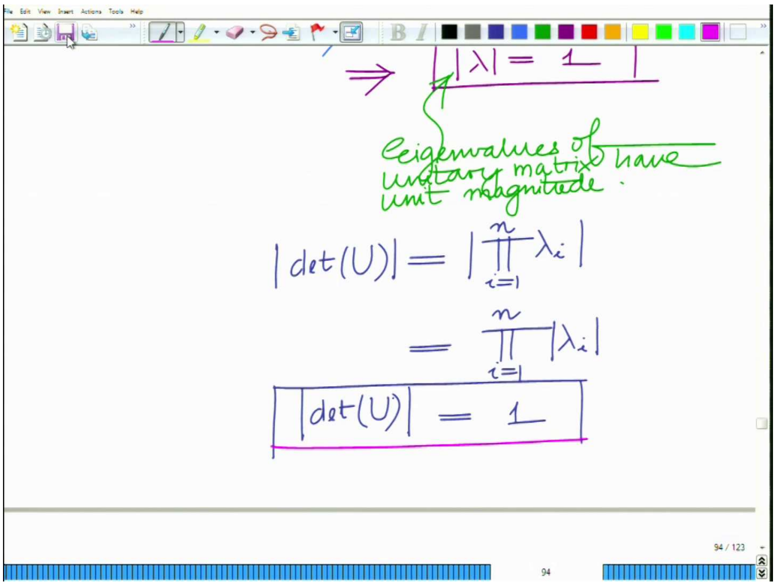

And similarly, the magnitude of the determinant of unitary matrix U is

$$\begin{aligned}
|\det(U)| &= \left|\prod_{i=1}^{n} \lambda_i\right| \\
&= \prod_{i=1}^{n} |\lambda_i| \\
&= \prod_{i=1}^{n} \left(\underbrace{1, 1, \cdots, 1}_{n \text{ times}}\right) \\
&= 1
\end{aligned}$$

Therefore, the magnitude of the determinant of unitary matrix is equal to 1.

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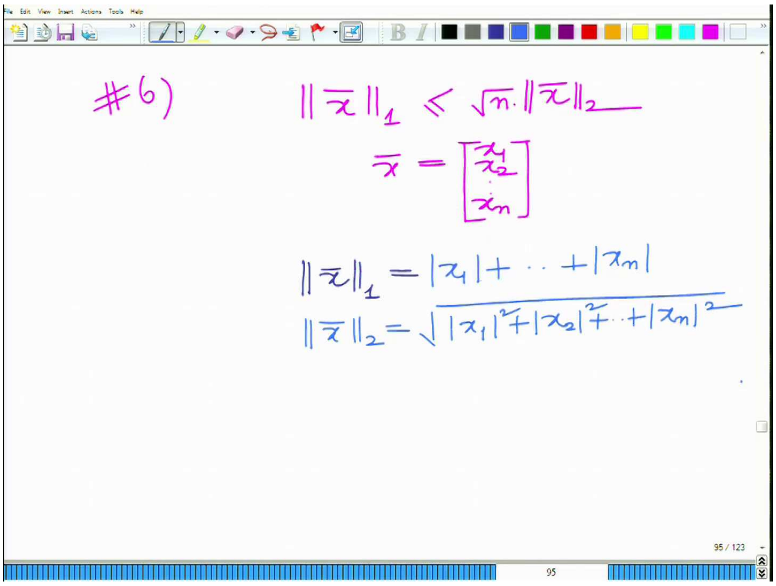

Another property is that the relation between the l_1 norm and l_2 norm of a $n \times 1$ vector $\bar{x}$ is as follows.

$$\|\bar{x}\|_1 \leq \sqrt{n}\|\bar{x}\|_2$$

So l_1 norm of vector $\bar{x}$ is

$$\|\bar{x}\|_1 = |x_1| + \ldots + |x_n|$$

And l_2 norm of vector $\bar{x}$ is

$$\|\bar{x}\|_2 = \sqrt{|x_1|^2 + \ldots + |x_n|^2}$$

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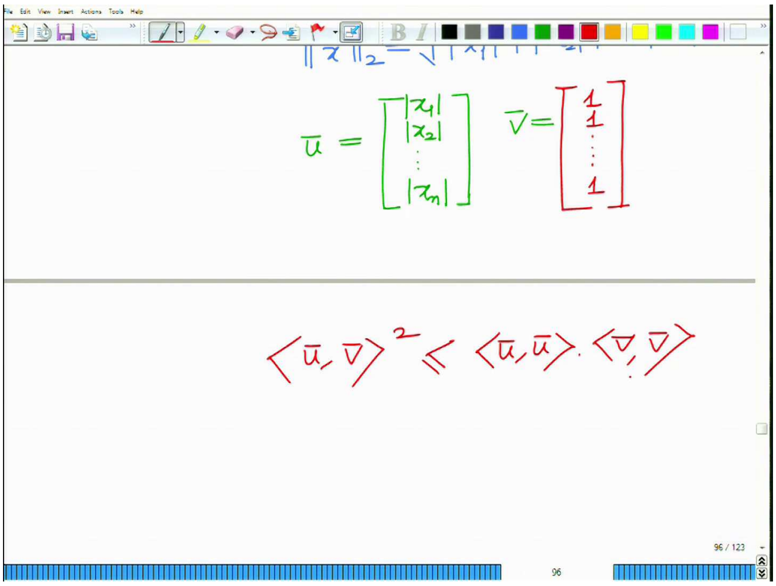

Consider two different vectors $\bar{u}$ and $\bar{v}$ as follows.

$$\bar{u} = \begin{bmatrix} |x_1| \\ |x_2| \\ \vdots \\ |x_n| \end{bmatrix}, \qquad \bar{v} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

The Cauchy–Schwarz inequality states that,

$$\langle \bar{u}, \bar{v} \rangle^2 \le \langle \bar{u}, \bar{u} \rangle \cdot \langle \bar{v}, \bar{v} \rangle$$

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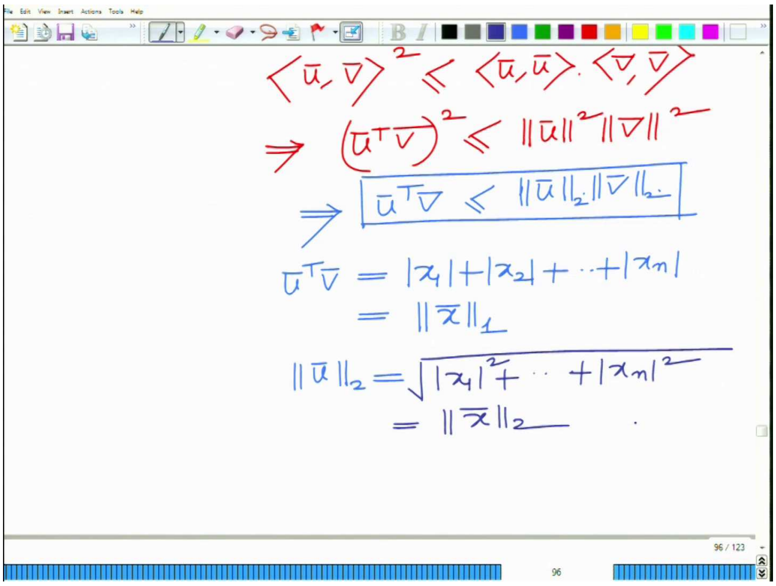

Therefore,

$$\left(\bar{u}^T\bar{v}\right)^2 \le \|\bar{u}\|^2 \cdot \|\bar{v}\|^2$$

$$\bar{u}^T\bar{v} \le \|\bar{u}\|_2 \cdot \|\bar{v}\|_2 \qquad \text{Eq.1}$$

On solving $\bar{u}^T\bar{v}$;

$$\bar{u}^T\bar{v} = |x_1| + \ldots + |x_n|$$
$$= \|\bar{x}\|_1$$

And

$$\|\bar{u}\|_2 = \sqrt{|x_1|^2 + \ldots + |x_n|^2}$$
$$= \|\bar{x}\|_2$$

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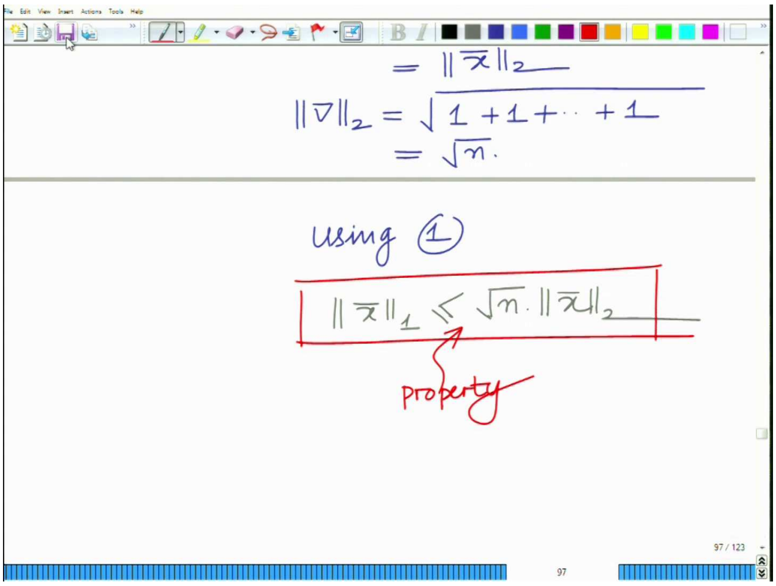

Also as $\bar{v}$ is an all one vector of length n, therefore

$$\|\bar{v}\|_2 = \sqrt{n}$$

Therefore on putting these values in equation 1; it verifies that

$$\|\bar{x}\|_1 \leq \sqrt{n}\,\|\bar{x}\|_2$$

This is an interesting property.

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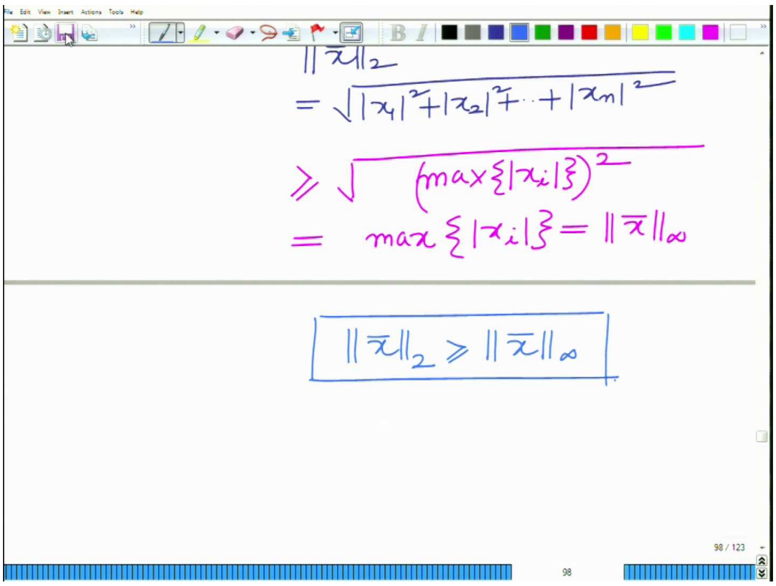

Similarly; on compairing the l_∞ norm and l_2 norm of $\overline{x}$; l_2 norm is a sum of the squares of the magnitude of all the elements of the vector while l_∞ norm is

$$\|\overline{x}\|_\infty = \sqrt{\left(\max\{|x_i|\}\right)^2}$$

Therefore it is clear that the l_∞ norm of vector $\overline{x}$ is less than l_2 norm of vector $\overline{x}$ i.e.

$$\|\overline{x}\|_2 \geq \|\overline{x}\|_\infty$$

Let us stop here and we will continue with other aspects in the subsequent modules.