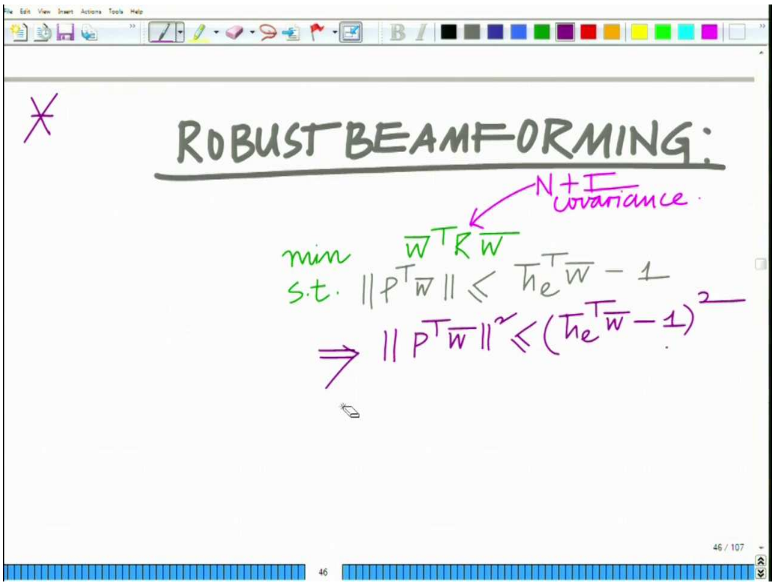
Applied Optimization for Wireless, Machine Learning, Big Data
Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture - 40
Practical Application: Detailed Solution for Robust Beamformer Computation in Wireless Systems

Hello, welcome to another module in this Massive Open Online Course. Robust Beamforming is discussed in the previous modules. Now let us derive the solution of optimal robust beam former.

(Refer Slide Time: 01:14)



So, the robust beam forming optimization problem is

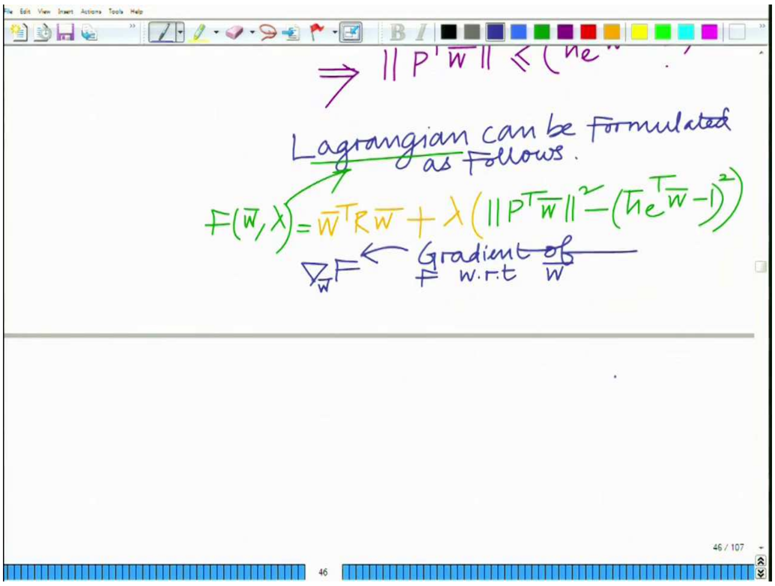
$$\min \ \bar{w}^T R \bar{w}$$
$$\text{such that } \ \left\|\bar{w}^T P\right\| \leq \bar{w}^T \bar{h}_e - 1$$

Where $\bar{h}_e$ is the nominal estimate of the channel, R is the noise plus interference covariance matrix and $\bar{w}$ is the beamforming vector.

This is the quadratic constraint. Therefore

$$\left\|P^T \bar{w}\right\|^2 \leq \left(\bar{h}_e^T \bar{w} - 1\right)^2$$

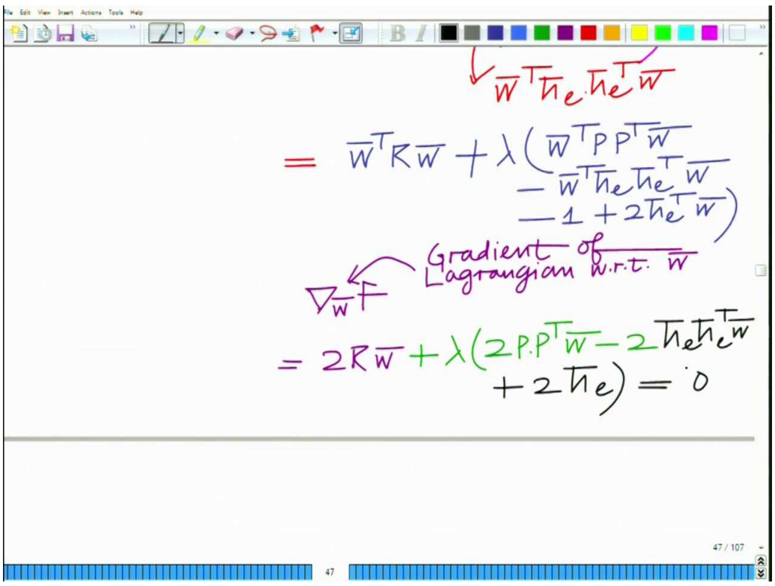
(Refer Slide Time: 03:06)



So, the Lagrangian can be formulated as follows.

$$
\begin{aligned}
F\left(\bar{w},\lambda\right) &= \bar{w}^T R\bar{w} + \lambda\left(\left\|P^T\bar{w}\right\|^2 - \left(\bar{h}_e^{\,T}\bar{w}-1\right)^2\right) \\
&= \bar{w}^T R\bar{w} + \lambda\left(\bar{w}^T PP^T\bar{w} - \left(\bar{h}_e^{\,T}\bar{w}\right)^2 - 1 + 2\bar{h}_e^{\,T}\bar{w}\right) \\
&= \bar{w}^T R\bar{w} + \lambda\left(\bar{w}^T PP^T\bar{w} - \bar{w}^T\bar{h}_e\bar{h}_e^{\,T}\bar{w} - 1 + 2\bar{h}_e^{\,T}\bar{w}\right)
\end{aligned}
$$

(Refer Slide Time: 06:11)

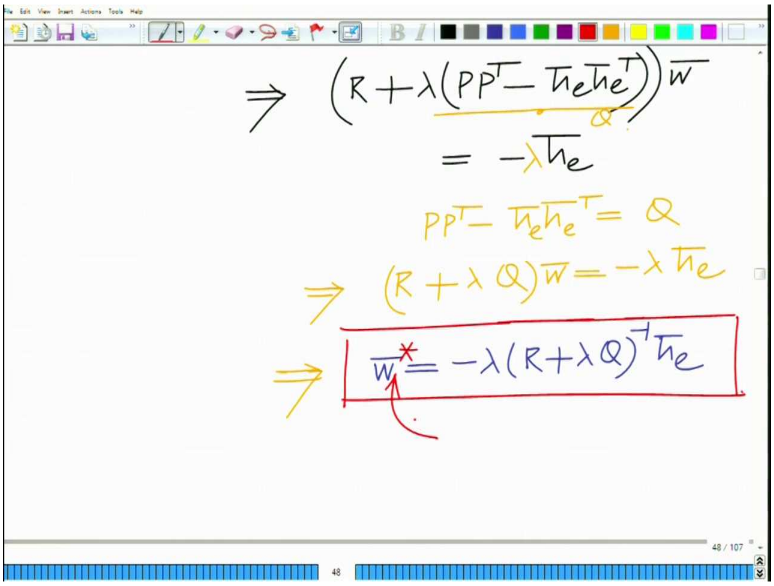


Let us take the gradient of this Lagrangian with respect to $\bar{w}$ and put it equal to zero.

$$\nabla_{\bar{w}} F = 0$$

$$R\bar{w} + \lambda\left(2PP^T\bar{w} - 2\bar{h}_e\bar{h}_e^{\,T}\bar{w} + 2\bar{h}_e\right) = 0$$

(Refer Slide Time: 08:19)



If matrix Q is defined as

$$PP^T - \bar{h}_e\bar{h}_e^{\,T} = Q$$

Then

$$\bar{w}(R + \lambda Q) = -\lambda\bar{h}_e$$

$$\bar{w}^* = -\lambda(R + \lambda Q)^{-1}\bar{h}_e$$

This is the optimal robust performer which depends on the Lagrange multiplier. So, this Lagrange multiplier has to be determined using the constraint. Therefore

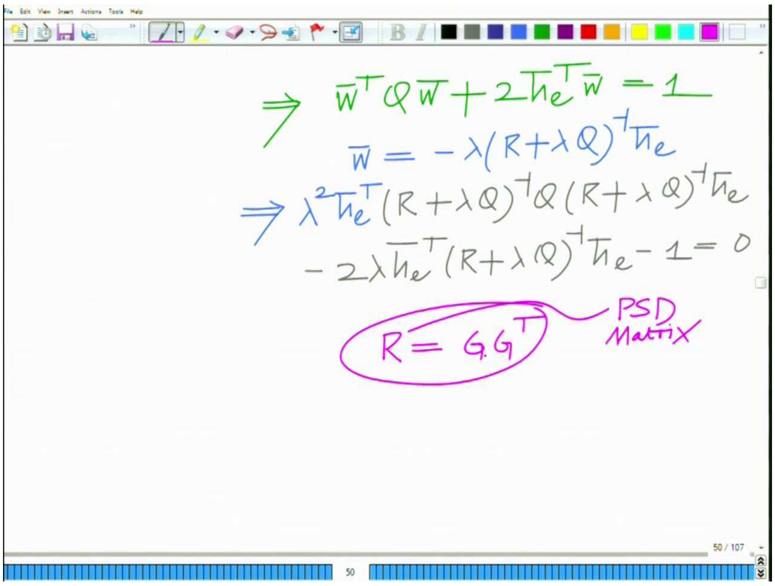
$$\left\|P^T\bar{w}\right\|^2 - \left(\bar{h}_e^{\,T}\bar{w} - 1\right)^2 = 0$$

$$\bar{w}^T PP^T\bar{w} - \bar{w}^T\bar{h}_e\bar{h}_e^{\,T}\bar{w} - 1 + 2\bar{h}_e^{\,T}\bar{w} = 0$$

$$\bar{w}^T\left(PP^T - \bar{h}_e\bar{h}_e^{\,T}\right)\bar{w} + 2\bar{h}_e^{\,T}\bar{w} = 1$$

$$\bar{w}^T Q\bar{w} + 2\bar{h}_e^{\,T}\bar{w} = 1$$

(Refer Slide Time: 13:18)



Put the value of $\bar{w}$ in the above expression.

$$\bar{w}^T Q \bar{w} + 2\bar{h}_e^T \bar{w} = 1$$

$$\lambda^2 \bar{h}_e^T \left(R+\lambda Q\right)^{-1} Q\left(R+\lambda Q\right)^{-1} \bar{h}_e - 2\lambda \bar{h}_e^T \left(R+\lambda Q\right)^{-1} \bar{h}_e - 1 = 0$$

Take the PSD matrix R as follows.

$$R = GG^T$$

Employ this in the above expression.

$$\lambda^2 \bar{h}_e^T \left(R+\lambda Q\right)^{-1} Q\left(R+\lambda Q\right)^{-1} \bar{h}_e - 2\lambda \bar{h}_e^T \left(R+\lambda Q\right)^{-1} \bar{h}_e - 1 = 0$$

$$\lambda^2 \bar{h}_e^T G^{-T} \left(I+\lambda G^{-1} Q G^{-T}\right)^{-1} G^{-1} Q G^{-T} \left(I+\lambda G^{-1} Q G^{-T}\right)^{-1} G^{-1} \bar{h}_e - 2\lambda \bar{h}_e^T G^{-T} \left(I+\lambda G^{-1} Q G^{-T}\right)^{-1} G^{-1} \bar{h}_e - 1 = 0$$

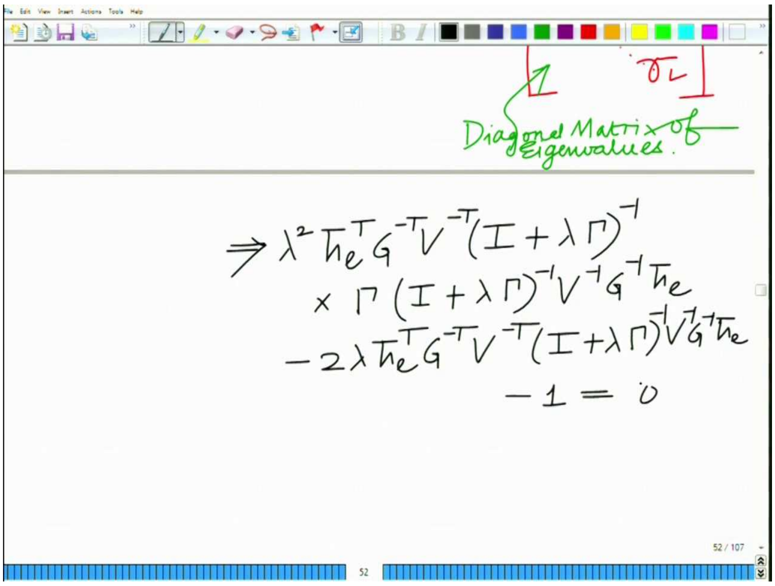
Let

$$G^{-1} Q G^{-T} = V \Gamma V^T$$

Where Γ is the diagonal matrix of eigenvalues.

$$\Gamma = \begin{bmatrix} \gamma_1 & & & \\ & \gamma_2 & & \\ & & \ddots & \\ & & & \gamma_L \end{bmatrix}$$

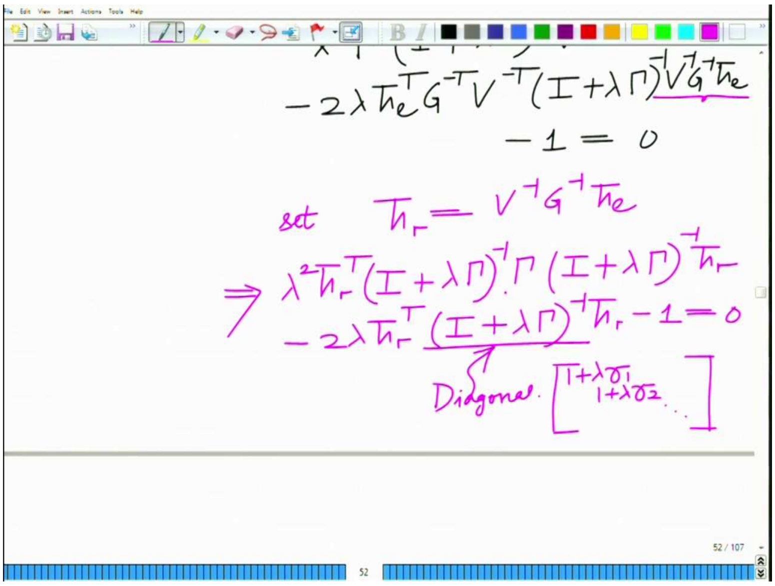
(Refer Slide Time: 19:04)



Therefore

$$\lambda^2 \overline{h}_e^T G^{-T} V^{-T} \left(I+\lambda\Gamma\right)^{-1} \Gamma \left(I+\lambda\Gamma\right)^{-1} V^{-1} G^{-1} \overline{h}_e - 2\lambda \overline{h}_e^T G^{-T} V^{-T} \left(I+\lambda\Gamma\right)^{-1} V^{-1} G^{-1} \overline{h}_e - 1 = 0$$

(Refer Slide Time: 21:02)



Now, set

$$\overline{h}_\Gamma = V^{-1} G^{-1} \overline{h}_e$$

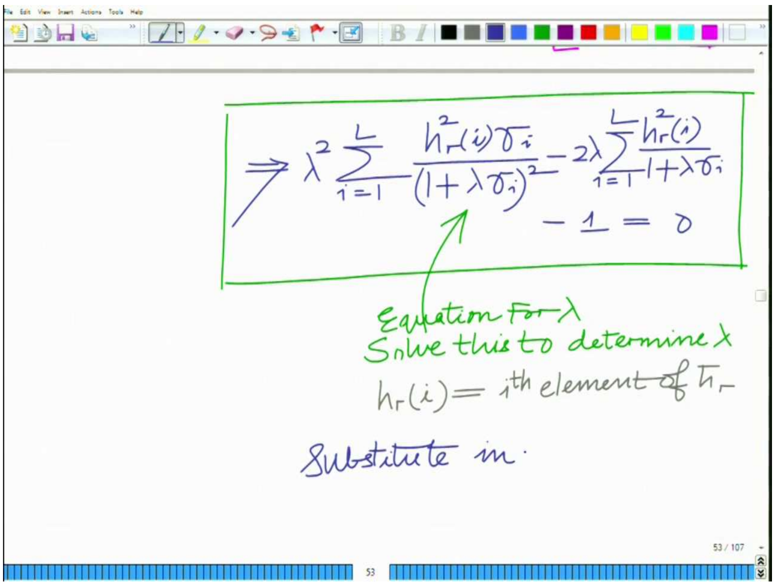
Therefore this can be simplified as follows.

$$\lambda^2 \bar{h}_\Gamma^T (I+\lambda\Gamma)^{-1} \Gamma (I+\lambda\Gamma)^{-1} \bar{h}_\Gamma - 2\lambda \bar{h}_\Gamma^T (I+\lambda\Gamma)^{-1} \bar{h}_\Gamma - 1 = 0$$

It is clear here that $(I+\lambda\Gamma)$ is the diagonal matrix.

$$(I+\lambda\Gamma) = \begin{bmatrix} I+\lambda\gamma_1 & & & \\ & I+\lambda\gamma_2 & & \\ & & \ddots & \\ & & & I+\lambda\gamma_L \end{bmatrix}$$

(Refer Slide Time: 22:44)



So the general equation for the Lagrange multiplier is given as follows.

$$\lambda^2 \sum_{i=1}^{L} \frac{\bar{h}_\Gamma^2(i)\gamma_i}{(1+\lambda\gamma_i)^2} - 2\lambda \sum_{i=1}^{L} \frac{\bar{h}_\Gamma^2(i)}{1+\lambda\gamma_i} - 1 = 0$$

Here $\bar{h}_\Gamma(i)$ is the ith diagonal element of matrix $\bar{h}_\Gamma$.

Hence to determine the value of λ, the above expression is used. And this value of λ is substituted in the expression of beamformer $\bar{w}^*$ which is

$$\bar{w}^* = -\lambda (R+\lambda Q)^{-1} \bar{h}_e$$

So this is the equation for the optimal robust beamformer which is robust to the uncertainty in the channel state information.