Applied Optimisation for Wireless, Machine Learning, Big Data

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Lecture – 73
Examples on Duality: Dual Norm, Dual of Linear Program (LP)

Keywords: *Dual Norm, Linear Program*

Hello, welcome to another module, in this massive open online course. Let us continue looking at examples and in this module let us start looking at examples pertaining to duality.

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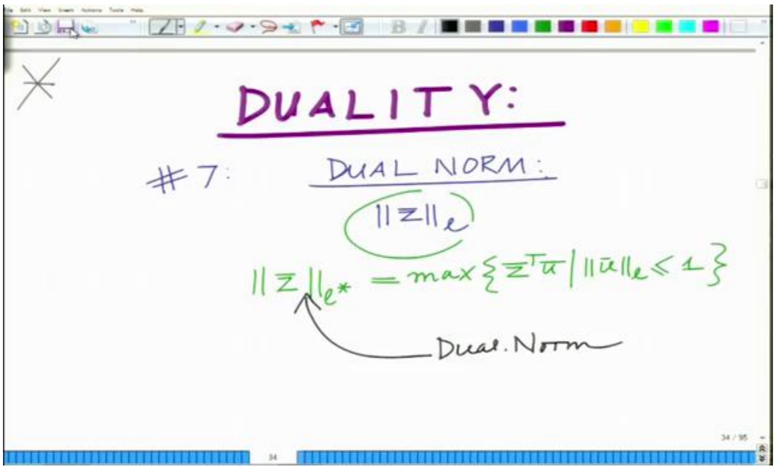

Let us start with the first example that is for instance if you have a vector $\bar{x}$ this $\|\bar{x}\|_l$ is the l norm, for instance, l can be 1, 2 and so on. Now, the dual norm of this is denoted by $\|\bar{z}\|_{l^*} = \max\left\{\bar{z}^T\bar{u} \mid \|\bar{u}\|_l \leq 1\right\}$. So this is basically the dual norm.

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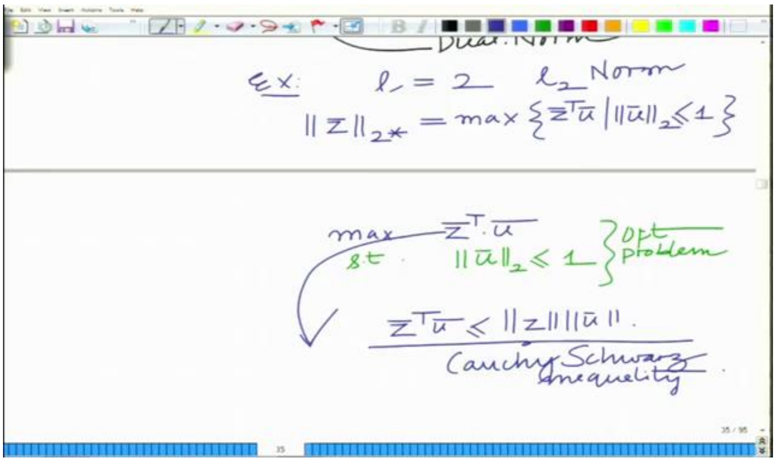

So let us look at some examples to understand this. Let us consider the dual norm of the l_2 norm that is $\left\|\bar{z}\right\|_{2^*} = \max\left\{\bar{z}^T\bar{u} \mid \left\|\bar{u}\right\|_2 \le 1\right\}$. Now $\begin{array}{l}\max\ \bar{z}^T\bar{u} \\ s.t \left\|\bar{u}\right\|_2 \le 1\end{array}$ is the pertinent optimization problem and this is convex in nature because, this is a linear objective, this is a convex constraint and now this is easy to solve. In fact, we know that, $\left|\bar{z}^T\bar{u}\right| \le \left\|\bar{z}\right\|\left\|\bar{u}\right\|$. So this follows from the Cauchy Schwarz Inequality.

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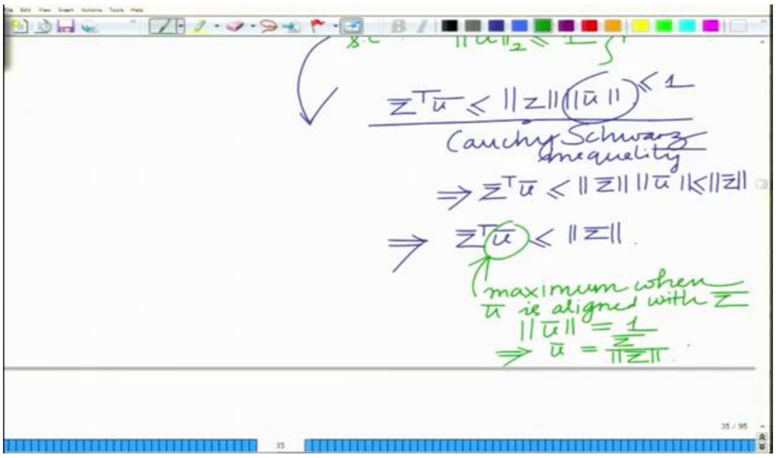

Now, we know that this $\left\|\bar{u}\right\| \leq 1$, which basically implies $\left|\bar{z}^T \bar{u}\right| \leq \left\|\bar{z}\right\|\left\|\bar{u}\right\| \leq \left\|\bar{z}\right\|$, which implies that $\bar{z}^T \bar{u} \leq \left\|\bar{z}\right\|$ and the maximum occurs when $\bar{u}$ is aligned with $\bar{z}$ which implies $\bar{u} = \frac{\bar{z}}{\left\|\bar{z}\right\|}$.

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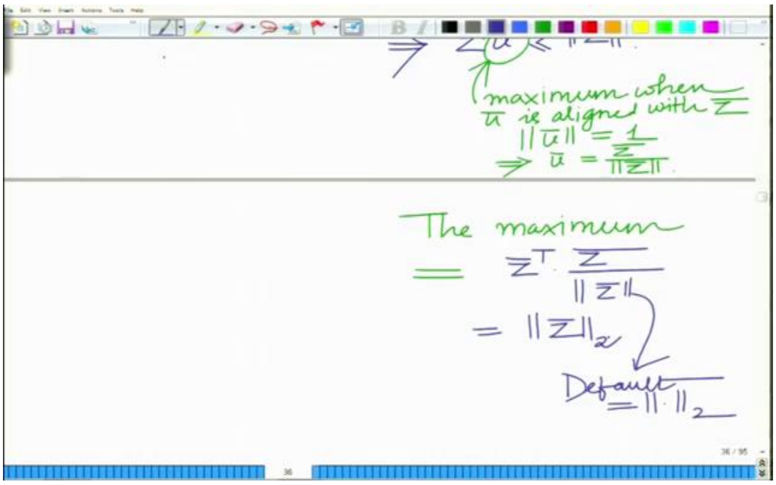

So the maximum is $\bar{z}^T \frac{\bar{z}}{\left\|\bar{z}\right\|} = \left\|\bar{z}\right\|_2$.

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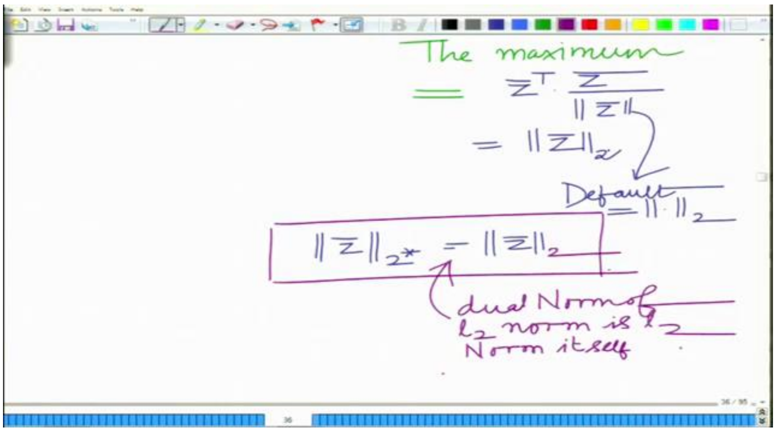

Therefore the dual norm of the l_2 norm is l_2 norm itself.

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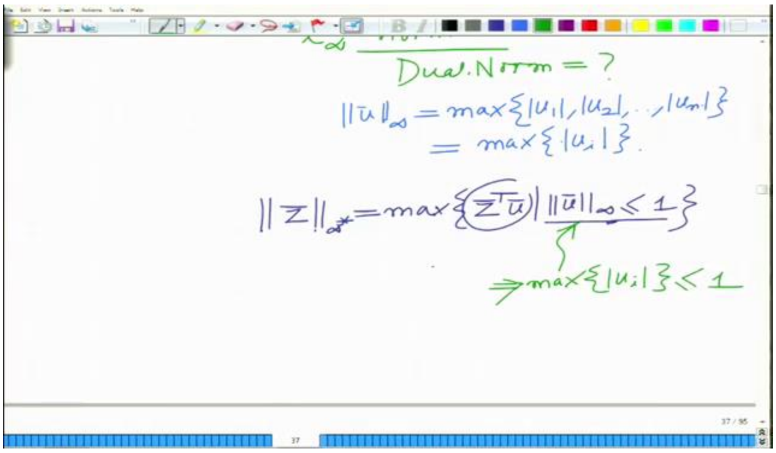

Now, we want to find the dual norm of the l_∞ norm. This is simply $\|\bar{z}\|_{\infty^*} = \max\{\bar{z}^T\bar{u} \mid \|\bar{u}\|_\infty \le 1\}$. Now, $\|\bar{u}\|_\infty \le 1 \Rightarrow \max\{|u_i|\} \le 1$.

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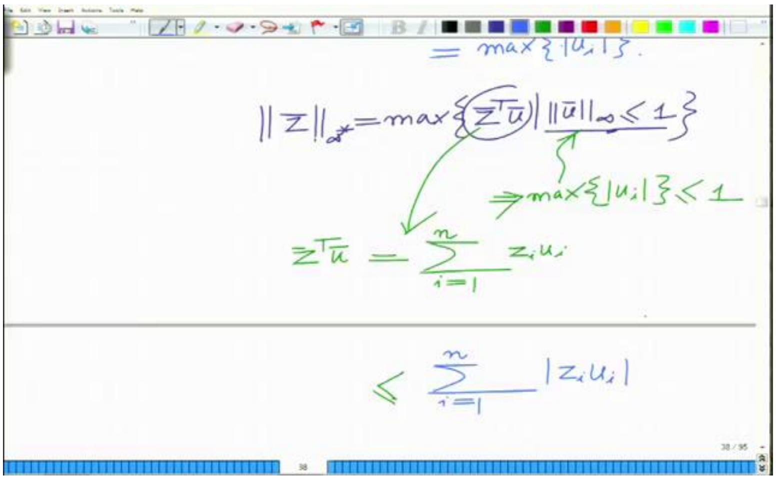

Now, assume $\bar{z}$ and $\bar{u}$ to be n dimensional vectors. Now $\bar{z}^T\bar{u} = \sum_{i=1}^{n} |z_i u_i|$ is simply the dot product between these two. This is as shown in slide.

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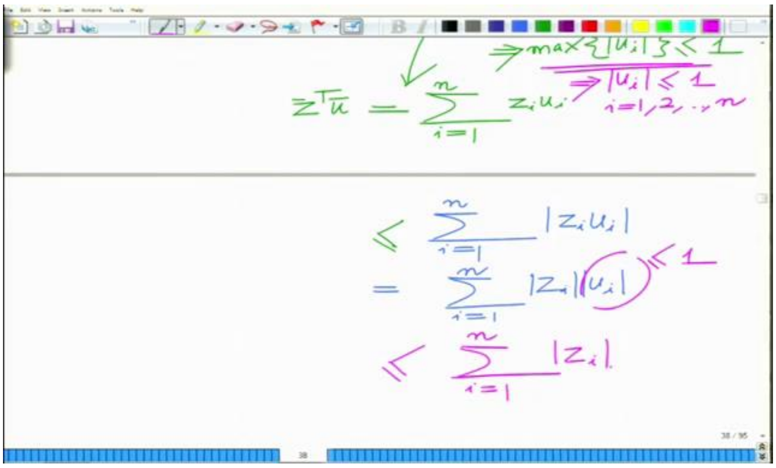

Now we have $\bar{z}^T \bar{u} \leq \sum_{i=1}^{n} |z_i|$.

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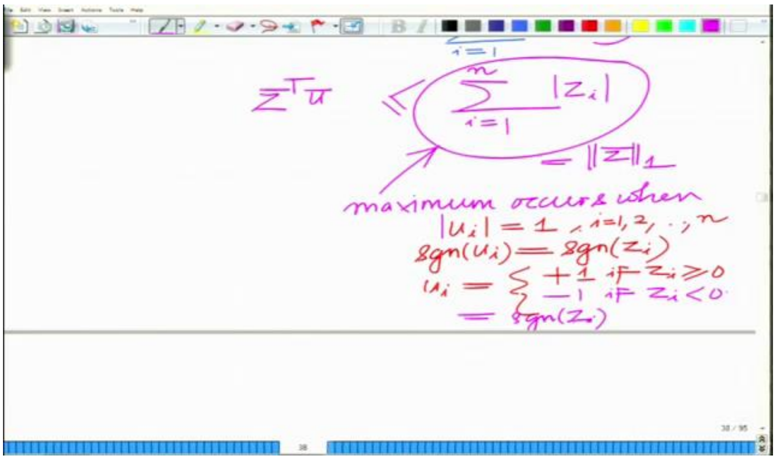

So the maximum occurs when $|u_i| = 1$ for each i and $sgn(u_i) = sgn(z_i)$ as shown in slide. The maximum value is nothing but the l_1 norm and therefore the dual norm of the infinity norm is the l_1 norm.

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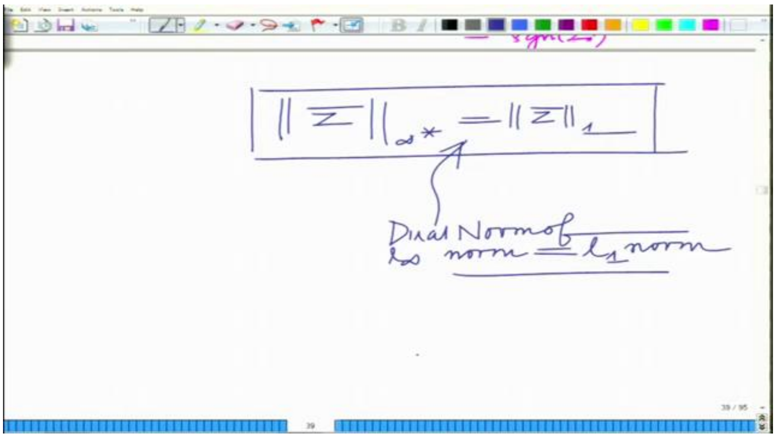

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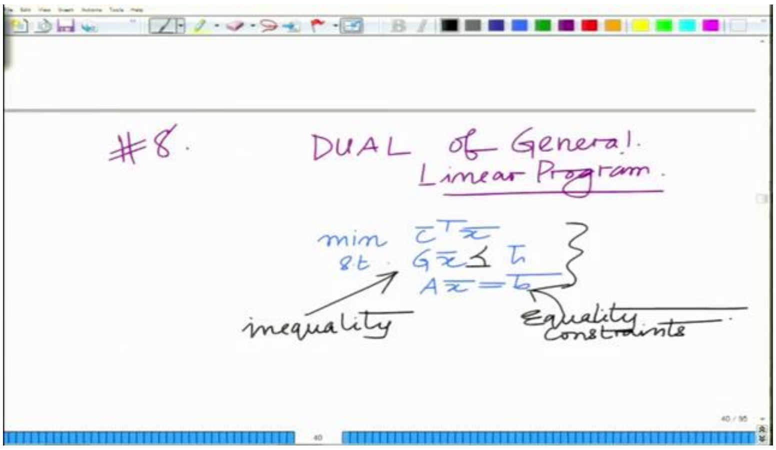

Let us look at another problem to derive the dual optimal problem corresponding to general LP. So consider the general linear program, that is $\begin{array}{l} \min \ \bar{c}^T \bar{x} \\ s.t \ G\bar{x} \leq \bar{h} \\ \quad A\bar{x} = \bar{b} \end{array}$. Now this is a general LP implies it has inequality constraints and equality constraints.

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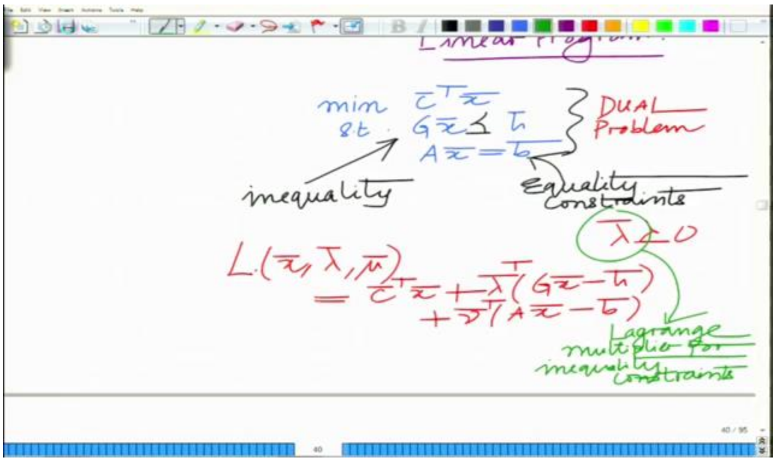

So we want to find the dual problem for this, so we have $L\left(\bar{x},\bar{\lambda},\bar{\mu}\right)=\bar{c}^{T}\bar{x}+\bar{\lambda}^{T}\left(G\bar{x}-\bar{h}\right)+\bar{\nu}\left(A\bar{x}-\bar{b}\right)$. These are the Lagrange multipliers for the both the inequality constraints and the equality constraints.

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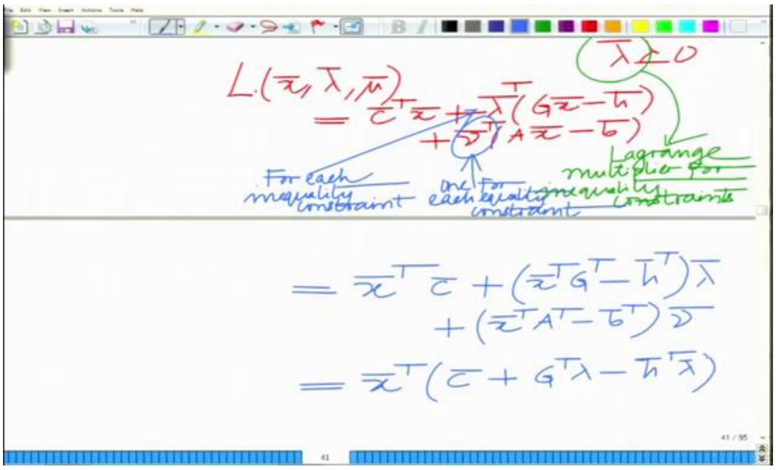

Now this is simplified as shown in the slide. So this can be simply rewritten as $L\left(\bar{x},\bar{\lambda},\bar{\mu}\right)=\bar{x}^{T}\left(\bar{c}+G^{T}\bar{\lambda}+A^{T}\bar{\nu}\right)-\left(\bar{h}^{T}\bar{\lambda}+\bar{b}^{T}\bar{\nu}\right)$.

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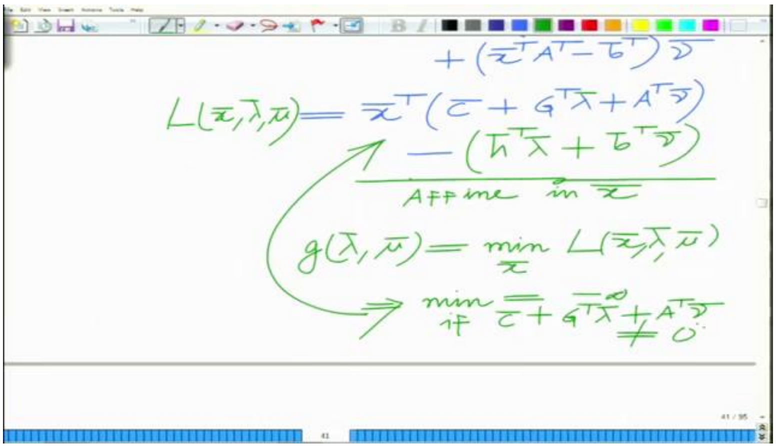

And now we have to find $g\left(\bar{\lambda}, \bar{\mu}\right) = \min_{x} L\left(\bar{x}, \bar{\lambda}, \bar{\mu}\right)$. This implies that the minimum is $-\infty$ if $\bar{c} + G^T \bar{\lambda} + A^T \bar{\nu} \neq 0$.

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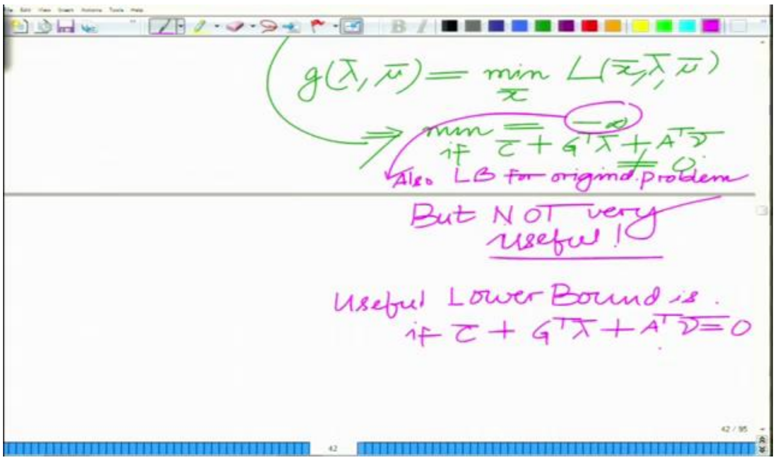

Now this is also a lower bound for the original problem, but it is not very useful. So instead we want a certain lower bound, which is more useful and that will be obtained if $\bar{c} + G^T \bar{\lambda} + A^T \bar{\nu} = 0$.

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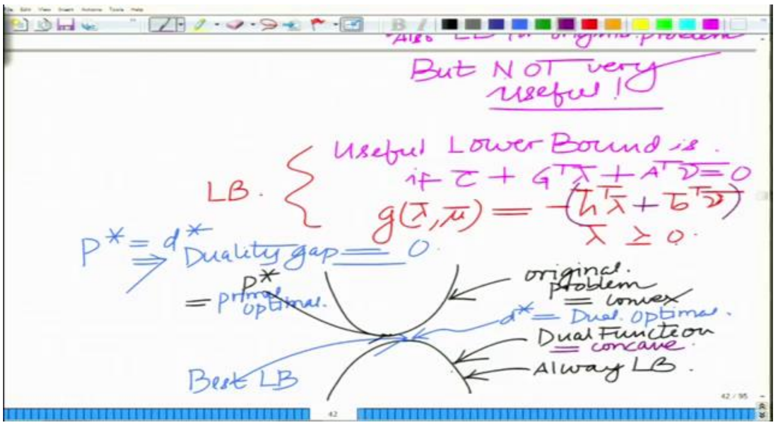

Therefore, in this case $g(\bar{\lambda}, \bar{\mu})$ reduces to the constant, which is $-\left(\bar{h}^T \bar{\lambda} + \bar{b}^T \bar{\nu}\right)$ and therefore this is a lower bound. This means that all the Lagrange multipliers associated with the inequality constraint have to be greater than or equal to 0 and this is a lower bound for the original optimization. So the best lower bound is the maximum value. So this is the primal optimal and this d^*, which is the dual optimal and this is what we call as the best lower bound because, it is the one that is closest to the optimum value p^* of the primal optimization problem and if $d^* = p^*$ implies that the duality gap is 0.

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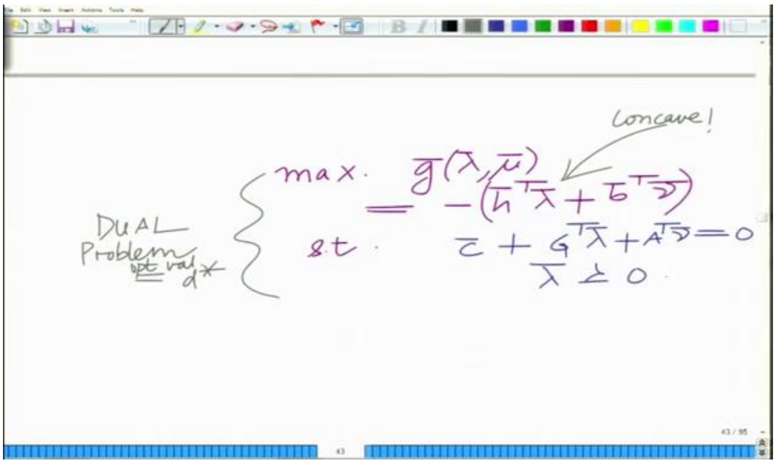

Therefore, the dual problem is basically the best lower bound, which is

$$\begin{aligned} &\max\ g(\bar{\lambda}, \bar{\mu}) = -\left(\bar{h}^T \bar{\lambda} + \bar{b}^T \bar{\nu}\right) \\ &s.t\ \ \bar{c} + G^T \bar{\lambda} + A^T \bar{\nu} = 0 \\ &\quad\ \ \bar{\lambda} \geq 0 \end{aligned}$$

. So this is concave and therefore, you can find the solution which is the optimum value d^*, where $d^* \leq p^*$. But in this case d^* will be exactly equal to p^* because this is a linear program. So in general for a convex optimization problem strong duality holds implies that $d^* = p^*$. So we will stop here and continue with other examples in the subsequent modules. Thank you very much.