Applied Optimization for Wireless, Machine Learning, Big Data

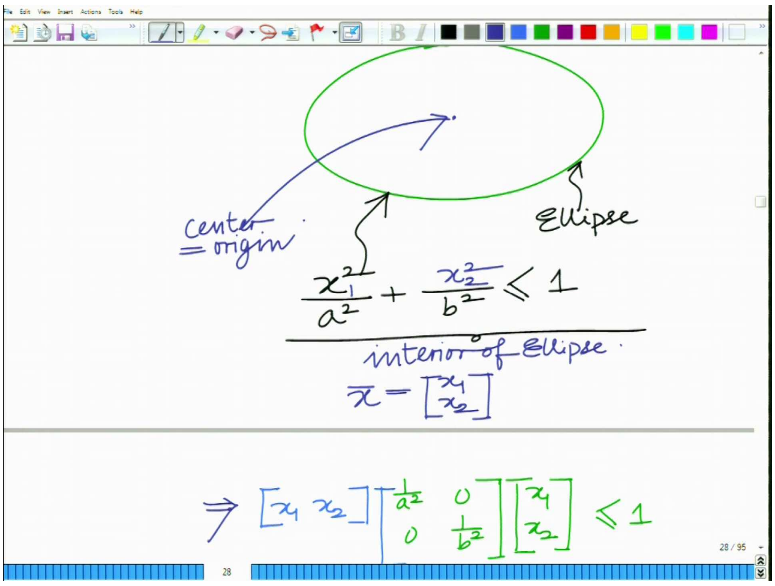
Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture – 14

Ellipsoid and its Practical Applications: Uncertainty Modeling for Channel State Information

Hello, welcome to another module in this massive open online course. Now let us continue this discussion by looking at another very important convex set that is the ellipse or the ellipsoid.

(Refer Slide Time: 00:29)



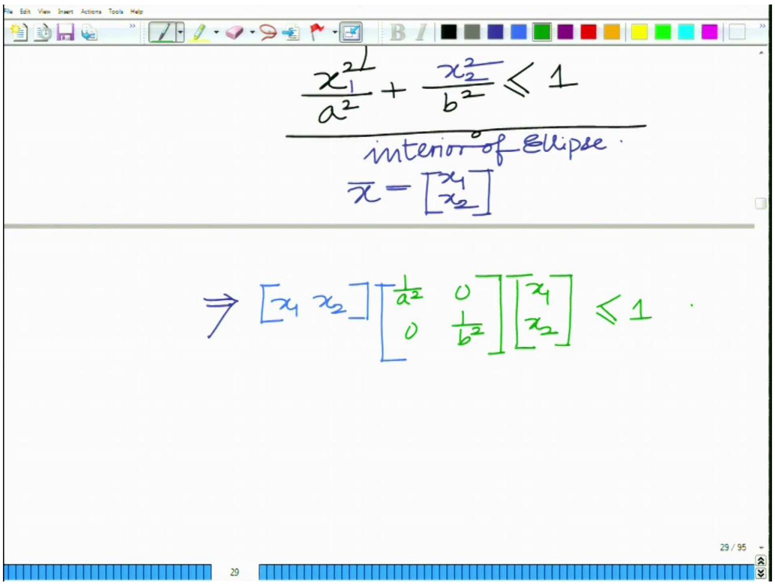
An ellipse is typically described by the following equation.

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$$

Thus, the equation of the interior of the ellipse including the boundary would be described by the following inequality.

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \leq 1$$

(Refer Slide Time: 01:33)

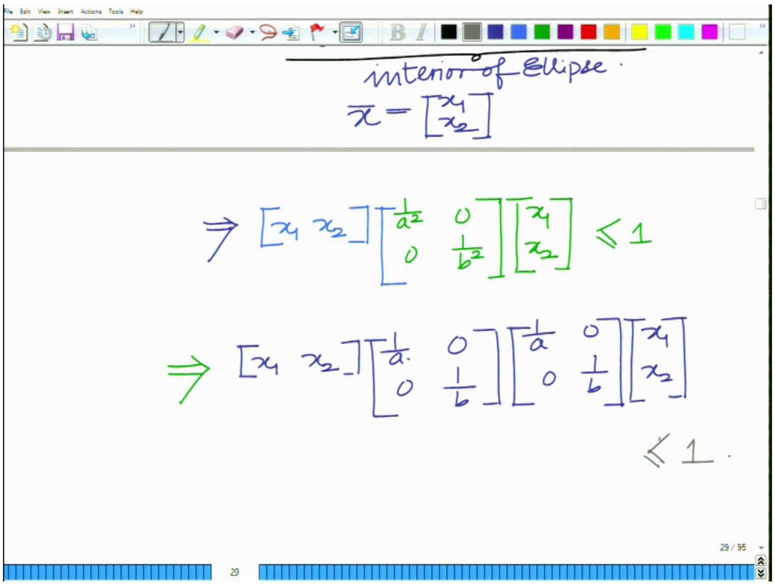


On simplifing the above inequality vector form, the general expression for an ellipse or an ellipsoidal region is as follows.

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq 1$$

Here x_1 is denoting conventional x-coordinate and x_2 is denoting conventional y-coordinate. Hence the above equation is in 2-dimentions and this can further be generalized in n-dimensions.

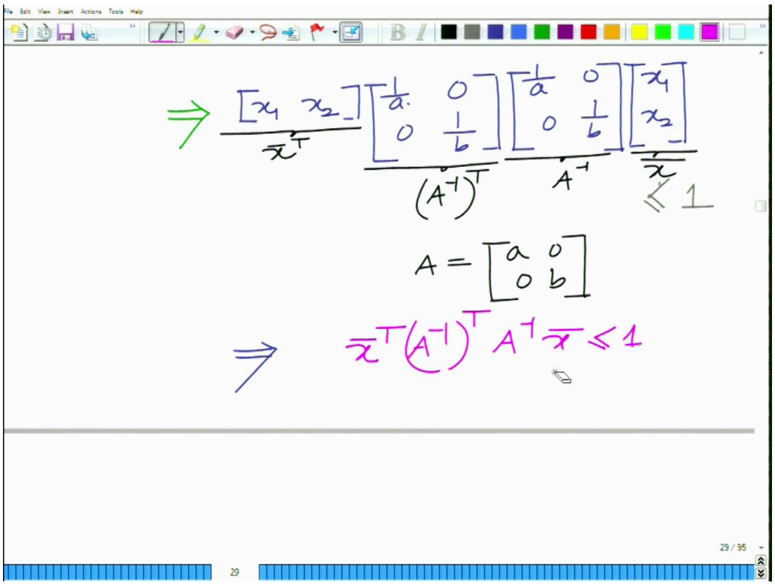
(Refer Slide Time: 03:07)



Further on simplifying this inequality,

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix} \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq 1$$

(Refer Slide Time: 04:06)



Also the matrix containing $\frac{1}{a}$ and $\frac{1}{b}$ as the diagonal elements is a diagonal matrix. Therefore above inequality can also be written as

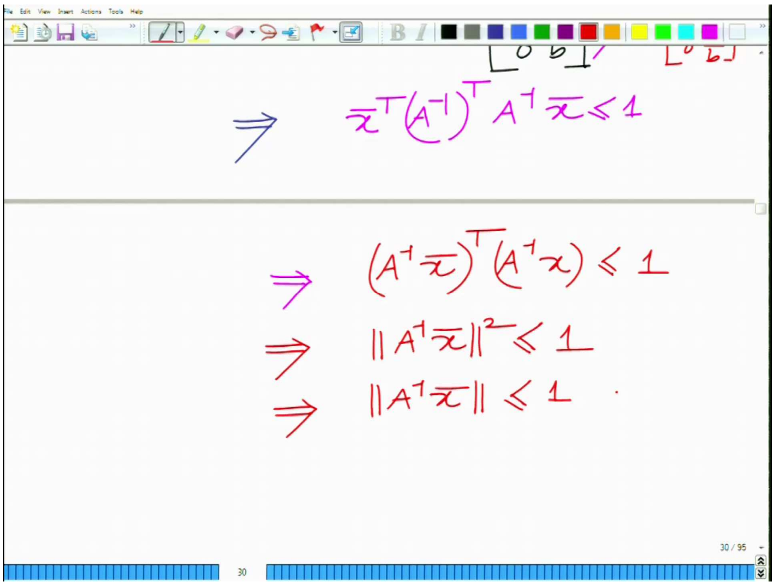
$$\bar{x}^T \left(A^{-1}\right)^T A^{-1}\bar{x} \leq 1$$

Here the diagonal matrix A is defined as

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

And $\bar{x}$ is the matrix denoting the coordinates.

(Refer Slide Time: 05:15)

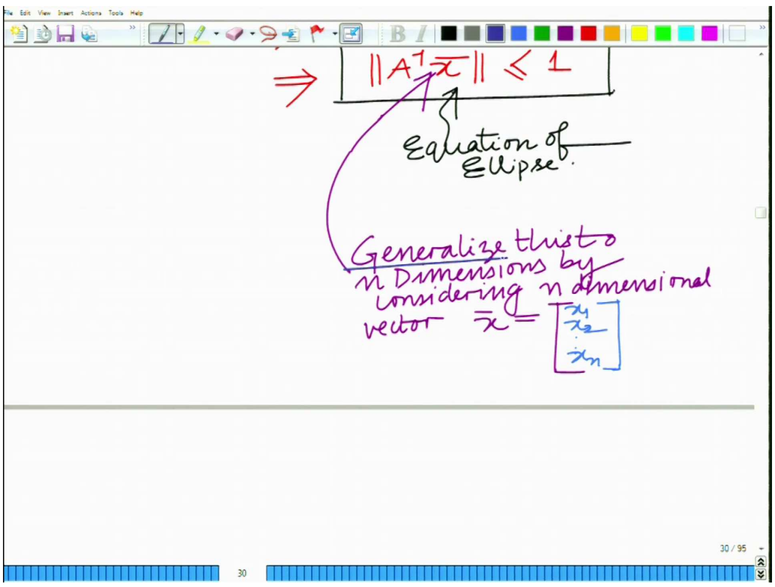


And, this further implies that

$$
\begin{aligned}
\left(A^{-1}\overline{x}\right)^T \left(A^{-1}\overline{x}\right) &\leq 1 \\
\left\|A^{-1}\overline{x}\right\|^2 &\leq 1 \\
\left\|A^{-1}\overline{x}\right\| &\leq 1
\end{aligned}
$$

This is the equation of ellipse.

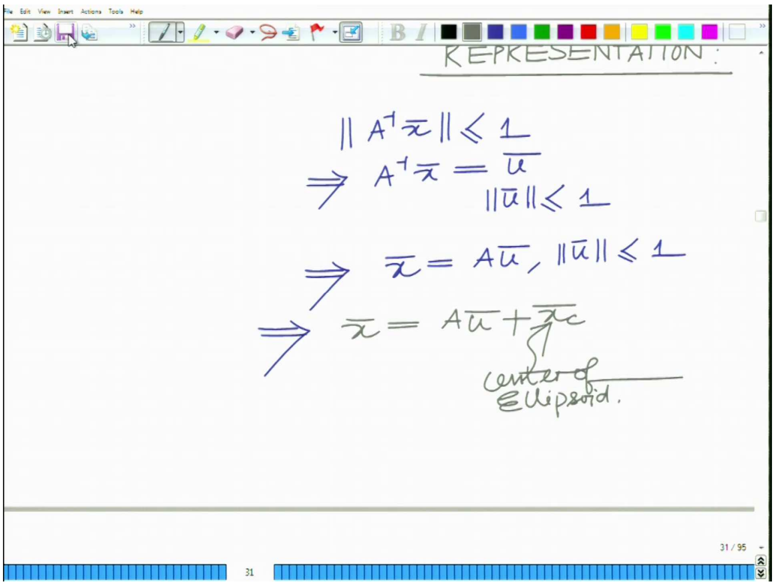
(Refer Slide Time: 06:22)



Now consider this region in n-dimensional vector. Now in n-dimensions, ellipse is an ellipsoid. Therefore consider coordinate matrix $\bar{x}$ in n-dimensions as

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

(Refer Slide Time: 07:53)



And the alternative representation of an ellipsoid similar to that of a norm ball can be derived as follows.

$$A^{-1}\bar{x} = \bar{u}$$
$$\bar{x} = A\bar{u}$$

Such that $\|\bar{u}\| \leq 1$.

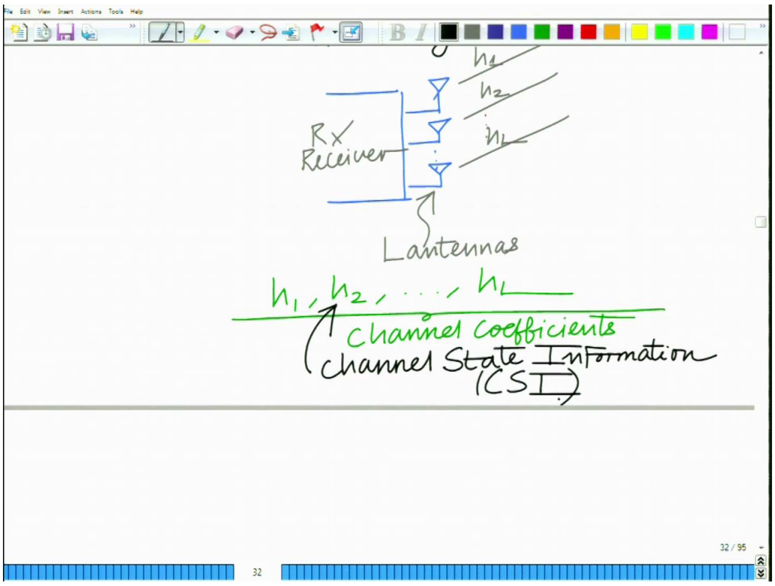
The above representation is for ellipsoid with centre at origin. So if the centre is at $\bar{x}_C$ which is not at the origin then the above equation of ellipsoid region is simply modified as

$$\bar{x} = A\bar{u} + \bar{x}_C$$

Therefore, the ellipsoid can be represented as

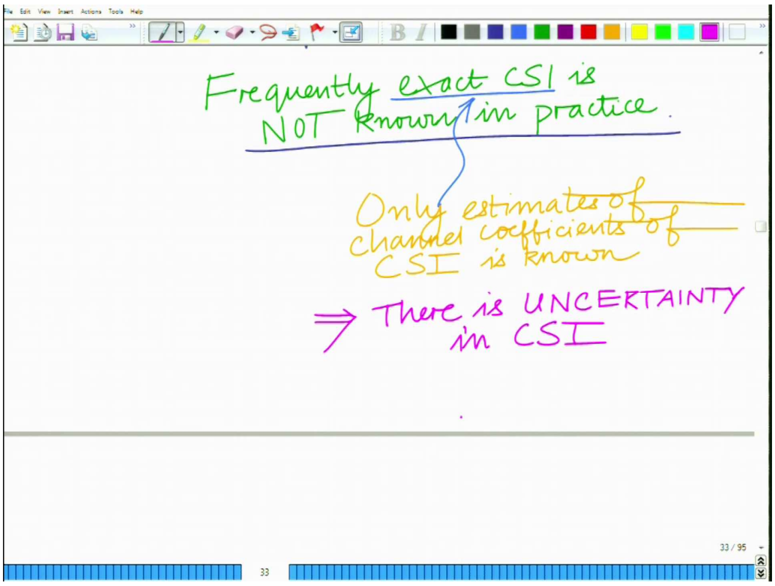
$$E(A, \bar{x}_C) = \{\bar{x}_C + A\bar{u} \ | \ \|\bar{u}\| \leq 1\}$$

(Refer Slide Time: 11:23)



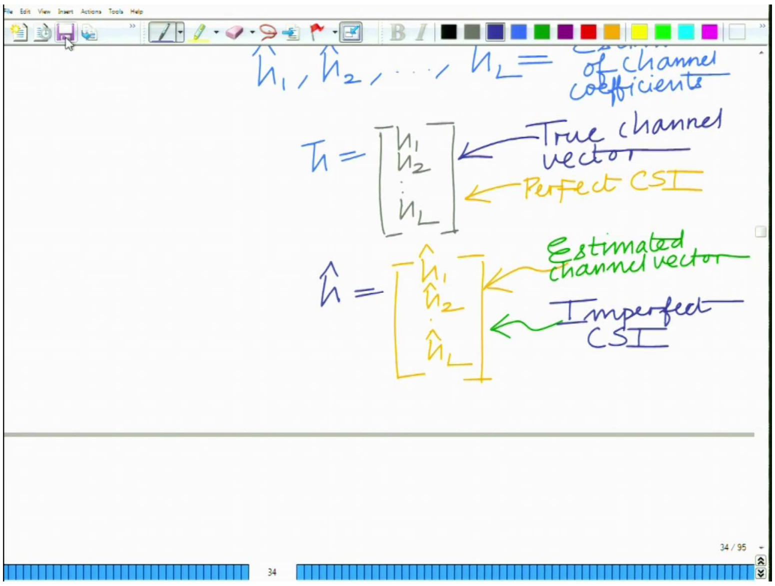
Now let us look at the practical application of ellipsoid region. Let us consider a multi antenna wireless system same as before. It basically has L antennas for improved performance of the system. The multiple channel coefficients are $h_1, h_2, \ldots, h_L$. These channel coefficients are also termed as Channel State Information (CSI) in wireless communication systems, because the channel coefficients characterizes the channel state and knowledge of the channel which is essential to develop enhanced signal processing schemes.

(Refer Slide Time: 15:39)



In practice, the exact channel state information is not known. Thus these channel state coefficients have to be estimated and this generates estimation error. So, only approximate or estimated CSI values are known. This simply implies that there is uncertainty in the CSI, which is commonly termed as CSI uncertainty, and this arises from the estimation errors.

(Refer Slide Time: 17:52)



Therefore the true channel coefficients are $h_1, h_2, \ldots, h_L$. Lets say the estimated channel coefficients are denoted by $\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_L$. Therefore the true channel vector is also known as perfect CSI and is defined as follows.

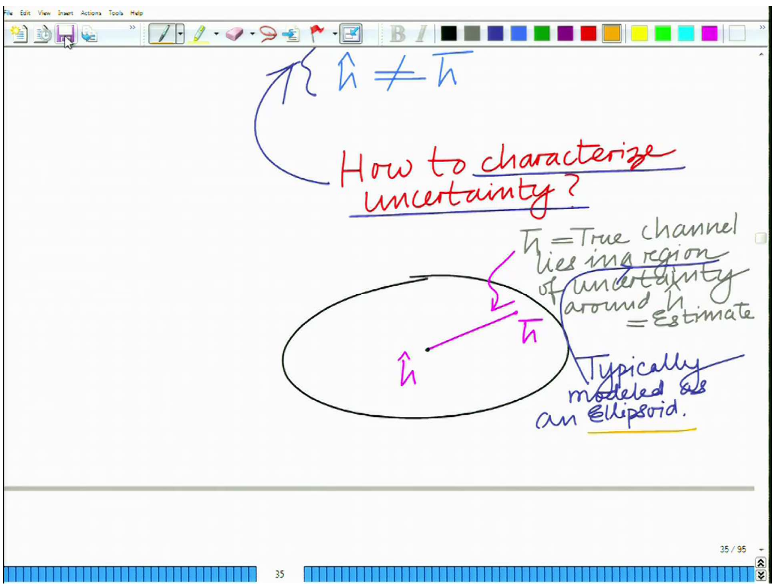
$$\bar{h} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_L \end{bmatrix}$$

Similarly the estimated channel vector is also known as imperfect CSI and is defined as follows.

$$\hat{h} = \begin{bmatrix} \hat{h}_1 \\ \hat{h}_2 \\ \vdots \\ \hat{h}_L \end{bmatrix}$$

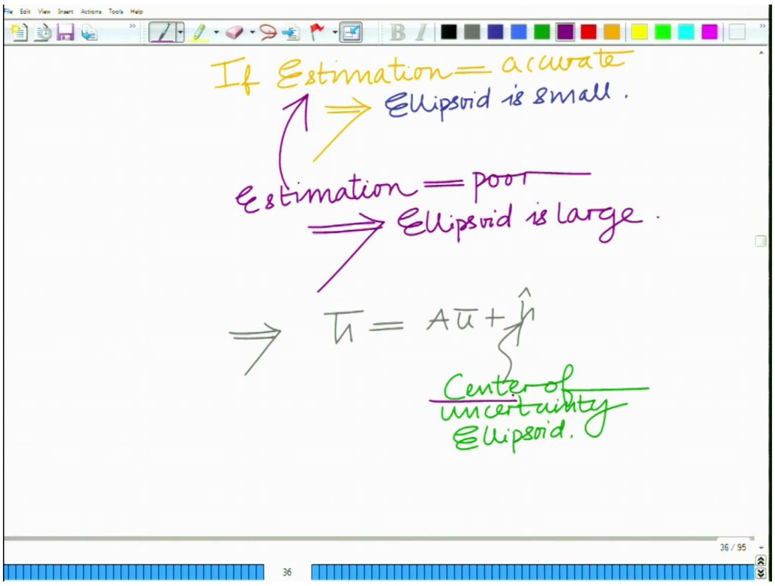
This imperfect CSI is close to the actual CSI but $\hat{h} \neq \bar{h}$.

(Refer Slide Time: 18:59)



So, this is an important consideration that $\hat{h}$ is approximately equal to $\bar{h}$ because in practice estimation of the underlying channel state coefficients with 100 percent accuracy is impossible. Therefore, one has to suitably characterize the phenomenon of uncertainty in CSI estimation to design signal processing schemes and yield improved performance. And, thus one can say that estimated CSI lies close to actual CSI or in other words, $\bar{h}$ lies in a region of uncertainty around $\hat{h}$ and this region is frequently modeled as an ellipsoid. Hence, if one considers an n-dimensional ellipsoidal region with the known estimate as its centre then the true channel lies within ellipsoid and this is a very important application of ellipsoid.

(Refer Slide Time: 22:23)

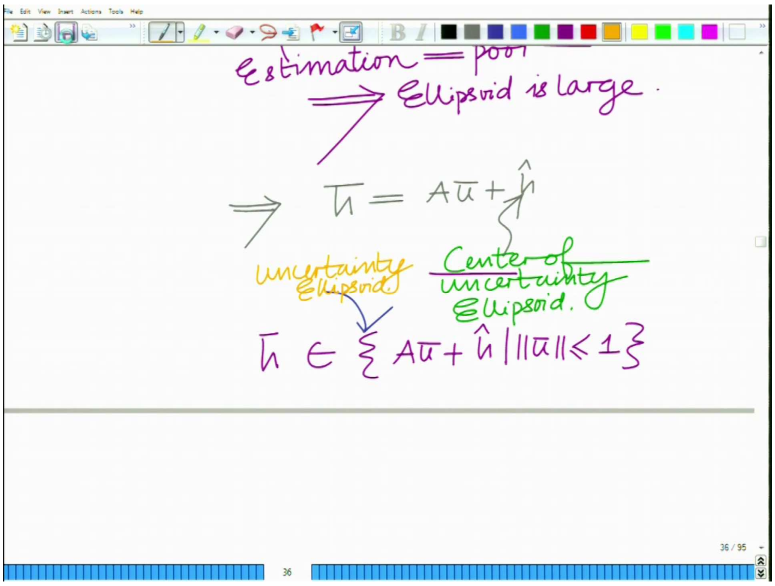


Now, if the ellipsoid is large; that means the uncertainty region is large, which further means that the estimation error is high. Similarly, if the ellipsoid is suitably small, this means the estimation error is low that is the estimation process is very good. In the later case, one can localize $\bar{h}$ to a much smaller region around $\hat{h}$.

Asymptotically when the estimation error becomes zero, the true channel $\bar{h}$ coincides with $\hat{h}$ and hence the SNR of estimation tends towards infinity. And therefore, the model to characterize the true channel vector can be represented as

$$\bar{h} = A\bar{u} + \hat{h}$$

(Refer Slide Time: 24:55)

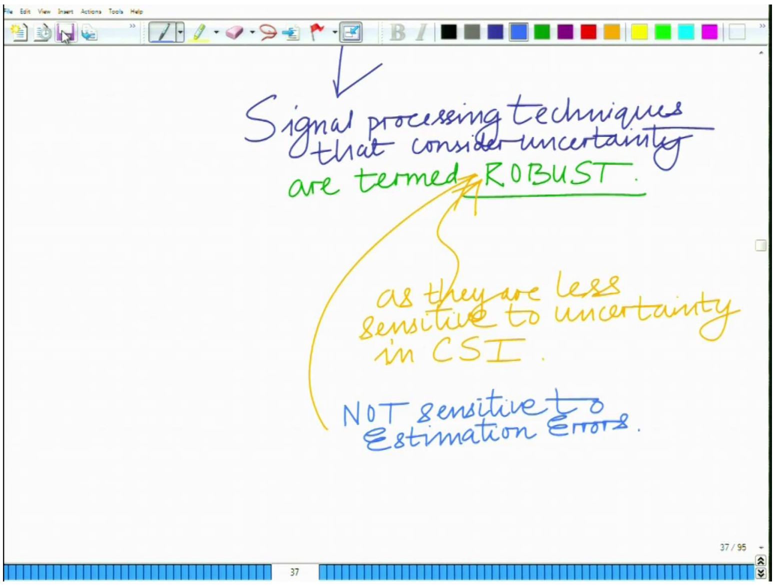


And, therefore, $\bar{h}$ belongs to the uncertainty ellipsoid which is given as

$$\bar{h} \in \left\{ A\bar{u} + \hat{h} \ \mid \ \|\bar{u}\| \leq 1 \right\}$$

And this is termed as the uncertainty ellipsoid.

(Refer Slide Time: 25:42)



Signal processing ellipsoid that consider uncertainty are termed as Robust, since they are less sensitive to the estimation errors in the channel state information, as they are not

So, this is an interesting application of the ellipsoidal region in wireless communication. Another important application of ellipsoidal region in signal processing is filter estimation. The true filter coefficients lie close to the estimated values within ellipsoidal region.