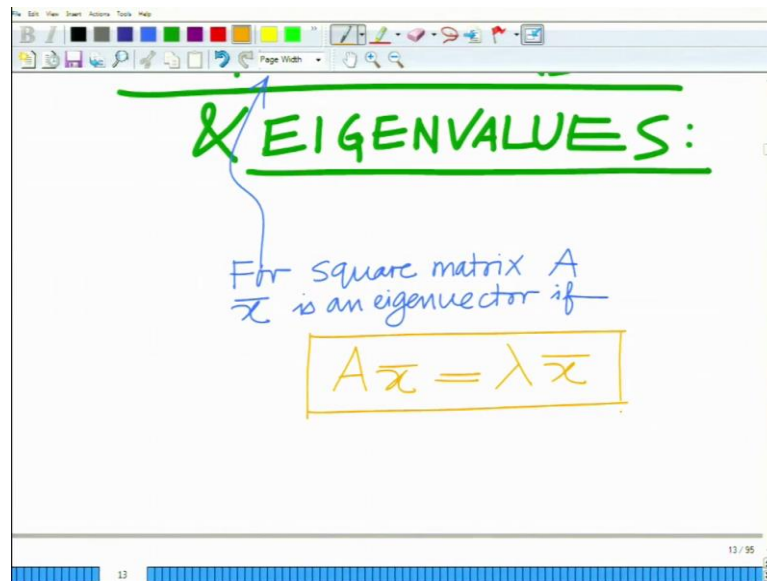


Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture – 02
Eigenvectors and Eigenvalues of Matrices and their Properties

Hello, welcome to another module in this massive open online course.

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Let us continue our discussion regarding the mathematical preliminaries, for the framework of convex optimization by looking at another very important concept that is of the Eigenvalues, the eigenvectors and eigenvalues of square matrices.

So, let's talk about the concepts of eigenvectors and eigenvalues. Eigenvalue is defined only for a square matrix. So, for a square matrix A , \bar{x} is an eigenvector, if

$$A\bar{x} = \lambda\bar{x}$$

Where λ is known as the eigenvalue of matrix A . This is the fundamental equation of a matrix for the eigenvector.

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The image shows a handwritten derivation of the eigenvalue equation. At the top, the equation $A\bar{x} = \lambda \bar{x}$ is boxed in yellow. Below it, two arrows point from the words "Eigenvalue" and "Eigenvector" to λ and \bar{x} respectively. Below this, three equations are listed, each preceded by a blue arrow pointing to the right:

$$\begin{aligned}\Rightarrow A\bar{x} &= \lambda I \bar{x} \\ \Rightarrow A\bar{x} - \lambda I \bar{x} &= 0 \\ \Rightarrow (A - \lambda I)\bar{x} &= 0\end{aligned}$$

The bottom equation is written in red ink. The slide number "14 / 95" is visible in the bottom right corner.

And, now we can also write

$$\begin{aligned}A\bar{x} &= \lambda I \bar{x} \\ A\bar{x} - \lambda I \bar{x} &= 0 \\ (A - \lambda I)\bar{x} &= 0\end{aligned}$$

So, here, the matrix $(A - \lambda I)$ is a singular matrix.

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The image shows a handwritten derivation of the characteristic equation. At the top, the word "singular" is written in red, with a red arrow pointing to the word "singular" in the phrase "singular matrix" below it. Below this, a blue arrow points to the equation $|A - \lambda I| = 0$. Below the equation, the word "Determinant" is written in blue. Below "Determinant", the text "Gives characteristic polynomial of A" is written in blue. Below this, the text "roots of characteristic polynomial = Eigenvalues of A" is written in green. The slide number "14 / 95" is visible in the bottom right corner.

This implies that if λ is an eigenvalue of A then

$$|A - \lambda I| = 0$$

Above equation is known as the characteristic equation corresponding to the matrix A and it gives the characteristic polynomial. So, the roots of this characteristic polynomial are eigenvalues of A.

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The screenshot shows a presentation slide with a white background and a blue border. At the top, there is a green header that reads "Polynomial = Eigenvalues of A". Below this, the word "Ex:" is written in blue. To the right of "Ex:", the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is written in blue. Below the matrix, the text "2x2 Square matrix" is written in blue. A red arrow points from the matrix to the text "Find Eigenvalues & Eigenvectors of A", which is written in red. The slide also features a toolbar at the top with various icons and a status bar at the bottom showing "15 / 95".

For example, let us say A is a 2×2 matrix.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

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The image shows a presentation slide with handwritten mathematical work. The top part shows the calculation of the matrix $A - \lambda I$ in blue ink:

$$A - \lambda I = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{bmatrix}$$

The bottom part shows the characteristic equation in green ink:

$$\begin{aligned} |A - \lambda I| &= 0 \\ \Rightarrow (1-\lambda)(-1-\lambda) - 1 &= 0 \\ \Rightarrow -(1-\lambda)(1+\lambda) &= 1 \end{aligned}$$

The slide has a toolbar at the top and a status bar at the bottom indicating slide 16 of 35.

Now to find the eigenvalues λ and the corresponding eigenvectors \bar{x} of this square matrix A, write down its characteristic equation as follows.

$$\begin{aligned} A - \lambda I &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix} \end{aligned}$$

Now consider the determinant of this is

$$\begin{aligned} |A - \lambda I| &= 0 \\ (1-\lambda)(-1-\lambda) - 1 &= 0 \\ -(1-\lambda)(1+\lambda) &= 1 \\ \lambda^2 - 1 &= 1 \\ \lambda &= \pm\sqrt{2} \end{aligned}$$

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A screenshot of a presentation slide showing the derivation of eigenvalues for a matrix A. The equations are written in green ink on a white background. The steps are as follows:

$$\begin{aligned} &\Rightarrow (1-\lambda)(-1-\lambda) - 1 = 0 \\ &\Rightarrow -(1-\lambda)(1+\lambda) = 1 \\ &\Rightarrow \lambda^2 - 1 = 1 \\ &\Rightarrow \boxed{\lambda = \pm\sqrt{2}} \end{aligned}$$

Below the boxed result, the text "Eigenvalues of A" is written in orange ink, with an arrow pointing from the box to the text.

16

So, we have got two eigenvalues; $\sqrt{2}$ and $-\sqrt{2}$. Now let us find the eigenvectors of the matrix A corresponding to both of these eigenvalues.

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A screenshot of a presentation slide showing the derivation of eigenvectors for the eigenvalue $\lambda = \sqrt{2}$. The equations are written in blue and red ink on a white background. The steps are as follows:

$$\begin{aligned} &\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \vec{x} = \sqrt{2} \vec{x} \\ &\quad = \sqrt{2} I \vec{x} \\ &\Rightarrow \begin{bmatrix} 1-\sqrt{2} & 1 \\ 1 & -1-\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \\ &\Rightarrow (1-\sqrt{2})x_1 + x_2 = 0 \\ &\quad x_1 - (1+\sqrt{2})x_2 = 0 \\ &\quad \downarrow \times (1-\sqrt{2}) \\ &\Rightarrow (1-\sqrt{2})x_1 + x_2 = 0 \end{aligned}$$

17

So, for $\lambda = \sqrt{2}$;

$$A\bar{x} = \lambda I \bar{x}$$

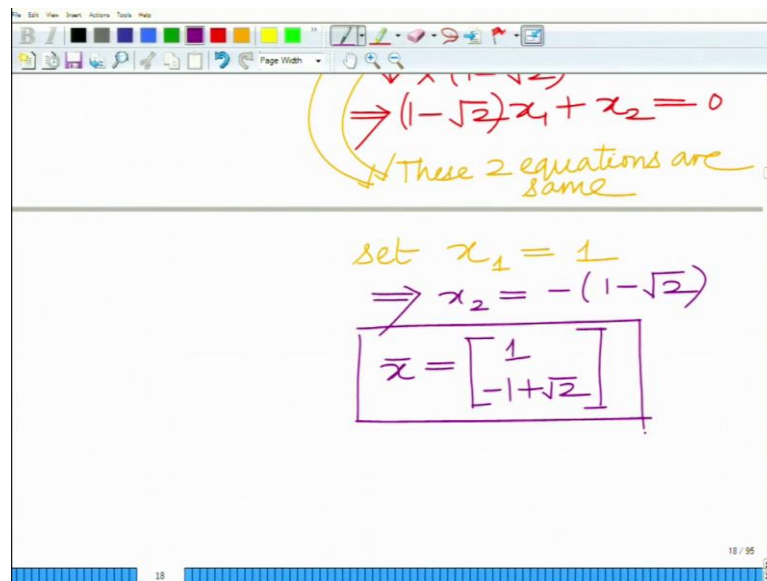
$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \bar{x} = \sqrt{2} I \bar{x}$$

$$\begin{pmatrix} 1-\sqrt{2} & 1 \\ 1 & -1-\sqrt{2} \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$(1-\sqrt{2})x_1 + x_2 = 0$$

Now if we multiply this equation with constant $(1-\sqrt{2})$ then, So it gives another equation identical to the last equation. So, basically we have just one equation and therefore, this is an infinite number of solutions and that is kind of obvious, because if the eigenvector corresponding to eigenvalue is not unique. So, if \bar{x} is an eigenvector, then \bar{x} scaled by any constant k is also an Eigen vector corresponding to the same eigenvalue. And therefore, there are infinite number of eigenvectors.

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Handwritten notes on a digital whiteboard:

$\Rightarrow (1-\sqrt{2})x_1 + x_2 = 0$

These 2 equations are same

set $x_1 = 1$

$\Rightarrow x_2 = -(1-\sqrt{2})$

$\bar{x} = \begin{bmatrix} 1 \\ -1+\sqrt{2} \end{bmatrix}$

Now, to derive a solution, set $x_1 = 1$. Therefore,

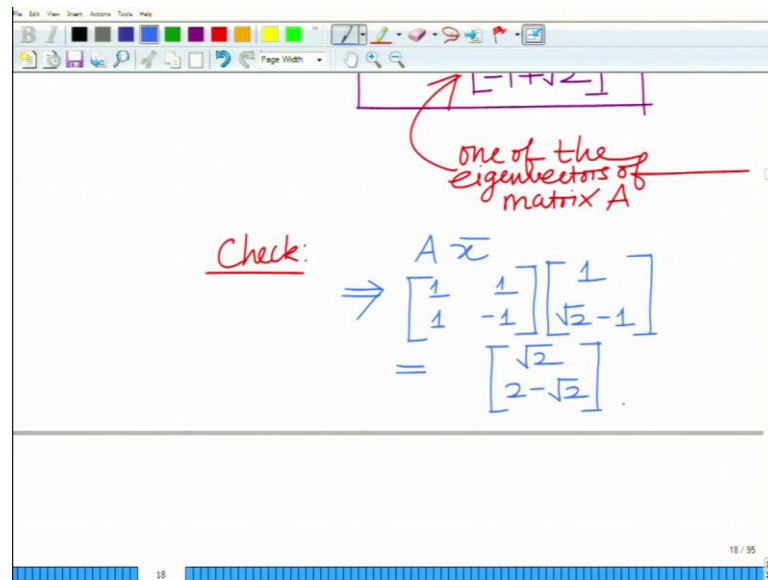
$$x_2 = -(1-\sqrt{2})$$

And hence we get,

$$\bar{x} = \begin{bmatrix} 1 \\ -1 + \sqrt{2} \end{bmatrix}$$

This is one of the eigenvectors of A corresponding to eigenvalue $\sqrt{2}$.

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Handwritten slide content showing a check of the eigenvector. It includes a purple box with the vector $\begin{bmatrix} 1 \\ -1 + \sqrt{2} \end{bmatrix}$ and a red arrow pointing to it with the text "one of the eigenvectors of matrix A". Below this, the calculation "Check: $A \bar{x}$ " is shown, resulting in a vector $\begin{bmatrix} \sqrt{2} \\ 2 - \sqrt{2} \end{bmatrix}$.

To check this

$$\begin{aligned} A\bar{x} &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{bmatrix} 1 \\ -1 + \sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{2} \\ 2 - \sqrt{2} \end{bmatrix} \\ &= \sqrt{2} \begin{bmatrix} 1 \\ -1 + \sqrt{2} \end{bmatrix} \\ &= \lambda \bar{x} \end{aligned}$$

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$$\begin{bmatrix} \sqrt{2} & -1 \\ 1 & \sqrt{2}-1 \end{bmatrix}$$
$$= \lambda \bar{x}$$
$$\lambda = \sqrt{2}$$

verifies that $\sqrt{2}$ = Eigenvalue
 $\begin{bmatrix} 1 \\ \sqrt{2}-1 \end{bmatrix}$ = Eigenvector

So for $\lambda = \sqrt{2}$, $\bar{x} = \begin{bmatrix} 1 \\ -1 + \sqrt{2} \end{bmatrix}$

So, this verifies basically both; the eigenvalue and the eigenvector of this matrix A.

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Similarly, for eigenvalue $-\sqrt{2}$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \bar{x} = -\sqrt{2} \bar{x}$$
$$\Rightarrow \left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} + \sqrt{2} I \right) \bar{x} = 0$$

Now, similarly for $\lambda = -\sqrt{2}$;

$$A\bar{x} = \lambda I \bar{x}$$

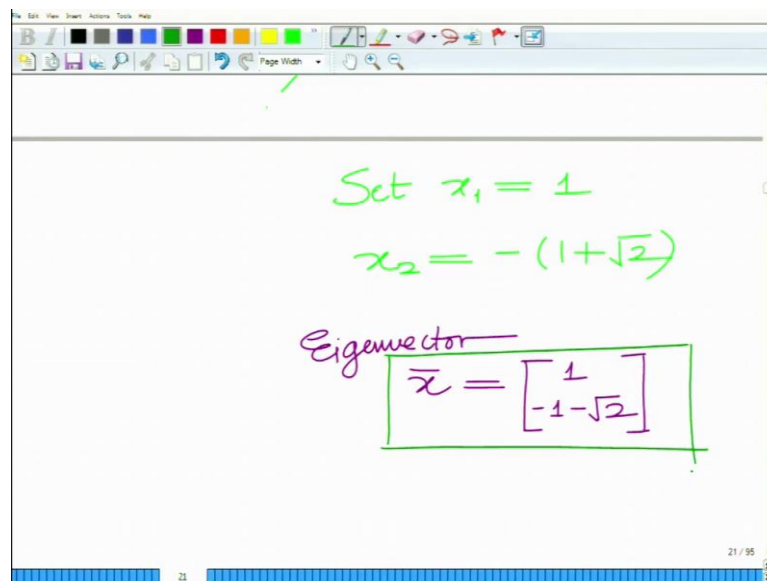
$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \bar{x} = -\sqrt{2} I \bar{x}$$

$$\begin{pmatrix} 1+\sqrt{2} & 1 \\ 1 & -1+\sqrt{2} \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$(1+\sqrt{2})x_1 + x_2 = 0$$

And basically you can see both the equations will reduce to the same thing.

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And now once again; set $x_1 = 1$

This implies $x_2 = -(1+\sqrt{2})$

And therefore the eigenvector

$$\bar{x} = \begin{bmatrix} 1 \\ -(1+\sqrt{2}) \end{bmatrix}$$

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The slide shows a handwritten note in a presentation software window. At the top, the equation $x_2 = -(1 + \sqrt{2})$ is written in green. Below it, the word "Eigenvector" is written in purple. Next to it, a purple vector is enclosed in a green box: $\vec{x} = \begin{bmatrix} 1 \\ -1 - \sqrt{2} \end{bmatrix}$. A green arrow points from the text "Other eigenvector corresponding to eigenvalue $-\sqrt{2}$ " to the vector box. The slide number "21" is visible in the bottom right corner.

$$x_2 = -(1 + \sqrt{2})$$

Eigenvector

$$\vec{x} = \begin{bmatrix} 1 \\ -1 - \sqrt{2} \end{bmatrix}$$

Other eigenvector corresponding to eigenvalue $-\sqrt{2}$

This is the other eigenvector, corresponding to the other eigenvalue $-\sqrt{2}$. This is the brief introduction to the concept of eigenvectors and eigenvalues of the matrix.

Let us look at another important concept which is symmetric and Hermitian matrices.

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The slide is titled "SYMMETRIC AND HERMITIAN MATRICES:" in blue. Below the title, it states $A \in \mathbb{R}^{n \times n}$. Then, it says "Symmetric if" followed by a blue box containing the equation $A = A^T$. Below the box, it says $\Rightarrow a_{ij} = a_{ji}$ and "For all i, j ". The slide number "22" is visible in the bottom right corner.

SYMMETRIC AND HERMITIAN
MATRICES:

$$A \in \mathbb{R}^{n \times n}$$

Symmetric if $A = A^T$

$$\Rightarrow a_{ij} = a_{ji}$$

For all i, j

So, what we want to look at now is basically the notion of symmetric and Hermitian. Symmetric and Hermitian matrices. So, let us say A is a real $n \times n$ matrix that is

$$A \in \mathbb{R}^{n \times n}$$

So, matrix A is symmetric, only if $A = A^T$

which implies that

$$a_{ij} = a_{ji} \quad \text{for all } i, j$$

And naturally it implies that this must be a square matrix because the symmetry is only preserved if matrix is a square matrix.

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The image shows a digital whiteboard with handwritten notes. At the top, it says 'Hermitian if $A = A^H$ '. Below this, matrix A is written as $A = \begin{bmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \\ \vdots & & \end{bmatrix}$. Below a horizontal line, the Hermitian conjugate matrix A^H is written as $A^H = \begin{bmatrix} a_{11}^* & a_{21}^* & \dots \\ a_{12}^* & a_{22}^* & \\ \vdots & & \end{bmatrix}$. The whiteboard interface includes a toolbar at the top and a status bar at the bottom showing '23 / 95'.

Also the matrix is a Hermitian symmetric matrix if $A = A^H$

Now, let us say A is matrix

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

And the Hermitian matrix is

$$A^H = \begin{pmatrix} a_{11}^* & \dots & a_{1n}^* \\ \vdots & \ddots & \vdots \\ a_{m1}^* & \dots & a_{mn}^* \end{pmatrix}$$

Here a_{mn}^* is the transpose conjugate of a_{mn} .

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A handwritten derivation on a whiteboard. At the top, a matrix A^H is shown with elements a_{11}^* , a_{21}^* , a_{12}^* , and a_{22}^* . A yellow arrow points from the matrix to the text "Transpose + conjugate". Below this, a boxed equation states $A = A^H \Rightarrow a_{ij} = a_{ji}^*$. The whiteboard interface includes a toolbar at the top and a status bar at the bottom showing "23 / 95".

This implies that

$$A = A^H \Rightarrow a_{ij} = a_{ji}^* \text{ for all } i, j$$

Now, there are several interesting properties of Hermitian and symmetric matrices.

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Handwritten notes on a whiteboard. At the top, the text "Transpose + conjugate" is written. Below it, a boxed equation states $A = A^H \Rightarrow a_{ij} = a_{ji}^*$. Below the box, the first property is listed: "1. Eigenvalues of Hermitian & Symmetric matrices are REAL". The whiteboard interface includes a toolbar at the top and a status bar at the bottom showing "23 / 95".

So, the first property is that the eigenvalues of Hermitian and symmetric matrices are real.

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2. Eigenvectors corresponding to
DISTINCT Eigenvalues are
ORTHOGONAL

⇒ \bar{v}_1, \bar{v}_2 are eigenvectors
corresponding to λ_1, λ_2
⇒ $\bar{v}_1^H \cdot \bar{v}_2 = 0$

Second property is another interesting property. Eigen vectors corresponding to distinct eigenvalues are orthogonal. This implies that if \bar{V}_1 and \bar{V}_2 are the eigenvectors corresponding to distinct eigenvalues λ_1 and λ_2 then it implies that, for a symmetric matrix,

$$\bar{V}_1^H \cdot \bar{V}_2 = 0$$

(Refer Slide Time: 23:42)

⇒ \bar{v}_1, \bar{v}_2 are eigenvectors
corresponding to λ_1, λ_2
⇒ $\bar{v}_1^H \cdot \bar{v}_2 = 0$
 \bar{v}_1, \bar{v}_2 are ORTHOGONAL

Ex: $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ symmetric matrix
 $A = A^T$
Eigenvalues = $\pm\sqrt{2}$

So, this is about the orthogonality of vectors. Now, let us go back to our earlier example to illustrate it. Our matrix A is

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Which is a symmetric matrix and its eigenvalues are real quantities.

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Handwritten notes on a digital whiteboard:

Ex: $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ symmetric matrix $A = A^T$

Eigenvalues = $\pm\sqrt{2}$

EV = $\begin{bmatrix} 1 \\ \sqrt{2}-1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ -1-\sqrt{2} \end{bmatrix}$ = REAL

V_1 V_2

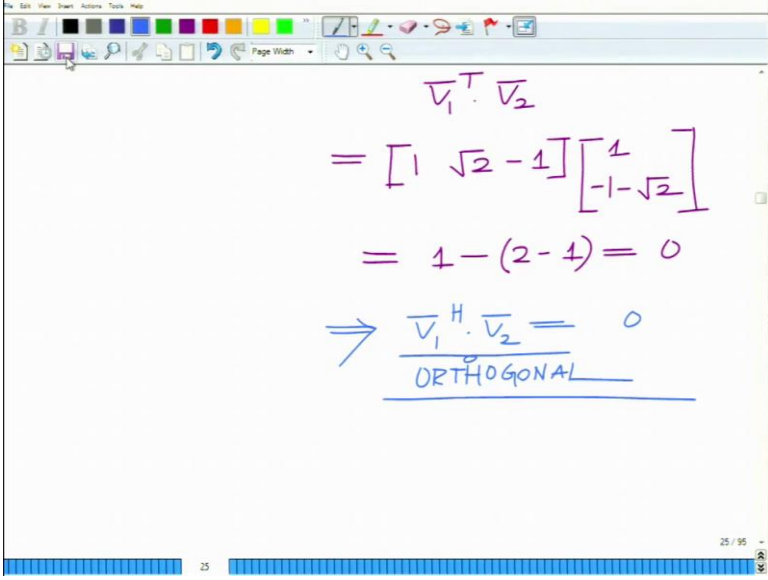
24 / 95

So, now let us look at the eigenvectors and we will show that the eigenvectors are orthogonal. The eigenvectors are

$$V_1 = \begin{bmatrix} 1 \\ \sqrt{2}-1 \end{bmatrix} \text{ and } V_2 = \begin{bmatrix} 1 \\ -\sqrt{2}-1 \end{bmatrix}$$

Now, since these vectors are real we can simply take its transpose.

(Refer Slide Time: 25:23)


$$\begin{aligned} & \bar{V}_1^T \cdot \bar{V}_2 \\ &= [1 \quad \sqrt{2}-1] \begin{bmatrix} 1 \\ -1-\sqrt{2} \end{bmatrix} \\ &= 1 - (2-1) = 0 \\ &\Rightarrow \bar{V}_1^H \cdot \bar{V}_2 = 0 \\ &\quad \underline{\text{ORTHOGONAL}} \end{aligned}$$

Therefore

$$\begin{aligned} \bar{V}_1^T \cdot \bar{V}_2 &= [1 \quad \sqrt{2}-1] \begin{bmatrix} 1 \\ -\sqrt{2}-1 \end{bmatrix} \\ &= 0 \end{aligned}$$

Now, as transpose or Hermitian will give the same thing for real vectors, therefore

$$\bar{V}_1^H \cdot \bar{V}_2 = 0$$

Which means that these vectors are orthogonal.

That is a very interesting property, because the matrix is symmetric. So, in this module we have looked at various interesting and also very important concepts of eigenvalues, eigenvectors and symmetric matrices. And these are going to be used frequently in our discussion and the development of the framework of optimization for various applications.

Thank you.