Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture - 10
Matrix Inversion Lemma (Woodbury identity)

Hello. welcome to another module in this massive open online course. Let us continue this discussion with another important principle known as the Matrix Inversion Lemma or the Matrix Inversion Identity.

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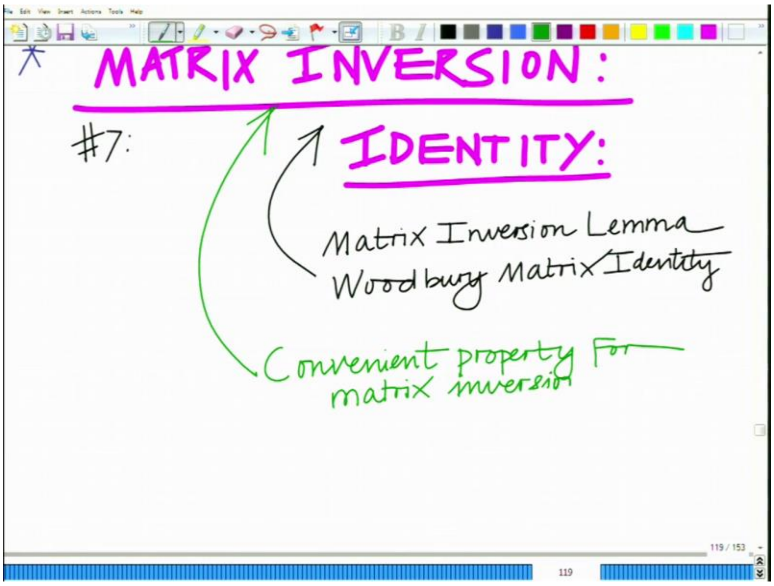

The matrix inversion identity is also termed as the Woodbury matrix identity. It is a very convenient principle for the matrix inversions.

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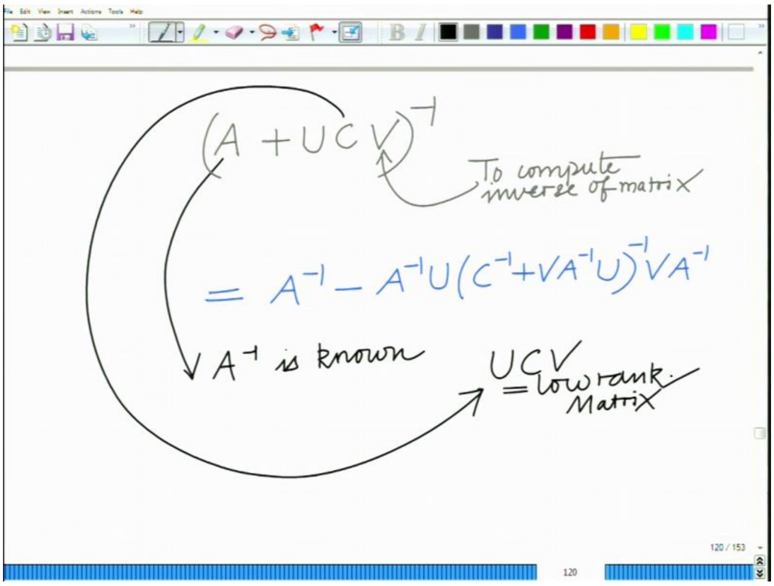

Considering that the inverse of matrix A is known and also that matrix UCV is a low rank matrix. C is a constant value. Therefore according to this identity;

$$\left(A+UCV\right)^{-1}$$
$$= A^{-1} - A^{-1}U\left(C^{-1} + VA^{-1}U\right)^{-1} VA^{-1}$$

So let us discuss it with an example.

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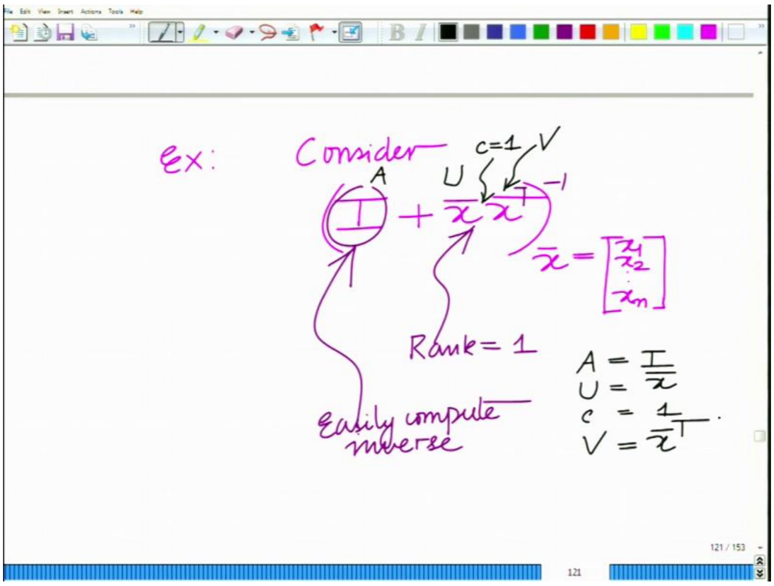

For example, compute the inverse of a $n \times 1$ matrix $\bar{x}$ defined as

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Let us use matrix I as the matrix A of the above equation because the inverse of I is known and it is I itself. Take C=1. Matrix U is $\bar{x}$ and V is $\bar{x}^T$.

$$(A+UCV)^{-1} = \left(I + \bar{x}\bar{x}^T\right)^{-1}$$

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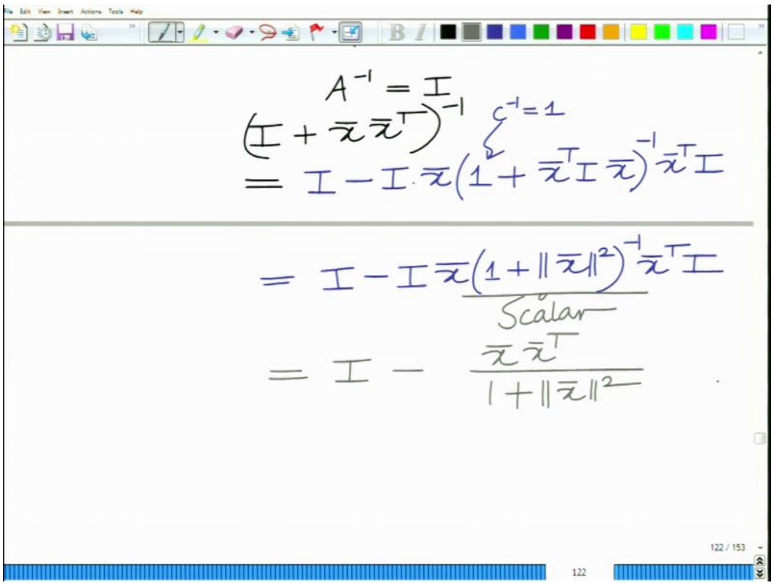

So use the matrix inversion lemma property and this can be done as follows.

$$\left(I + \bar{x}\bar{x}^T\right)^{-1}$$
$$= I - I\cdot\bar{x}\left(1 + \bar{x}^T \cdot I \cdot \bar{x}\right)^{-1}\bar{x}^T \cdot I$$
$$= I - I\cdot\bar{x}\left(1 + \|\bar{x}\|^2\right)^{-1}\bar{x}^T \cdot I$$
$$= I - \frac{\bar{x}\cdot\bar{x}^T}{\left(1 + \|\bar{x}\|^2\right)}$$

Note that $\left(1 + \|\bar{x}\|^2\right)$ is a scalar quantity.

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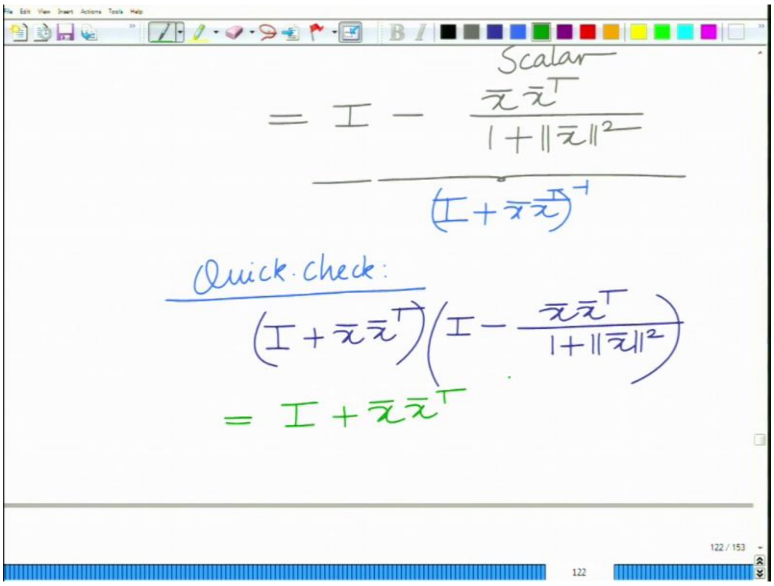

So the inverse of $I + \overline{x}\overline{x}^T$ is $I - \frac{\overline{x} \cdot \overline{x}^T}{\left(1 + \|\overline{x}\|^2\right)}$.

So, this is the simple trick that can be readily used to compute the inverse of such matrices.

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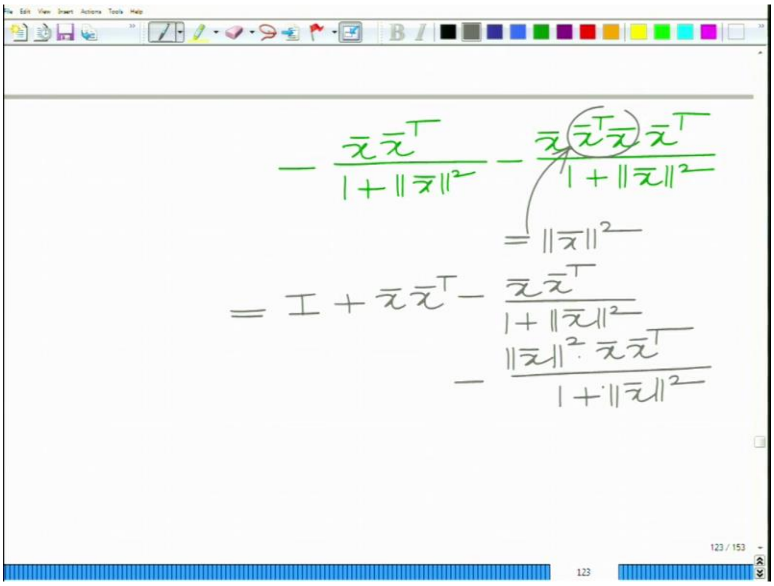

Let’s do a quick check of this.

$$\left(I+\overline{x}\overline{x}^T\right)\left(I-\frac{\overline{x}\overline{x}^T}{\left(1+\|\overline{x}\|^2\right)}\right)$$

$$=I+\overline{x}\overline{x}^T-\frac{\overline{x}\overline{x}^T}{\left(1+\|\overline{x}\|^2\right)}-\frac{\overline{x}\overline{x}^T\overline{x}\overline{x}^T}{\left(1+\|\overline{x}\|^2\right)}$$

$$=I+\overline{x}\overline{x}^T-\frac{\overline{x}\overline{x}^T}{\left(1+\|\overline{x}\|^2\right)}-\frac{\overline{x}\|\overline{x}\|^2\,\overline{x}^T}{\left(1+\|\overline{x}\|^2\right)}$$

$$=I+\overline{x}\overline{x}^T-\frac{\overline{x}\overline{x}^T}{\left(1+\|\overline{x}\|^2\right)}-\frac{\|\overline{x}\|^2\,\overline{x}\overline{x}^T}{\left(1+\|\overline{x}\|^2\right)}$$

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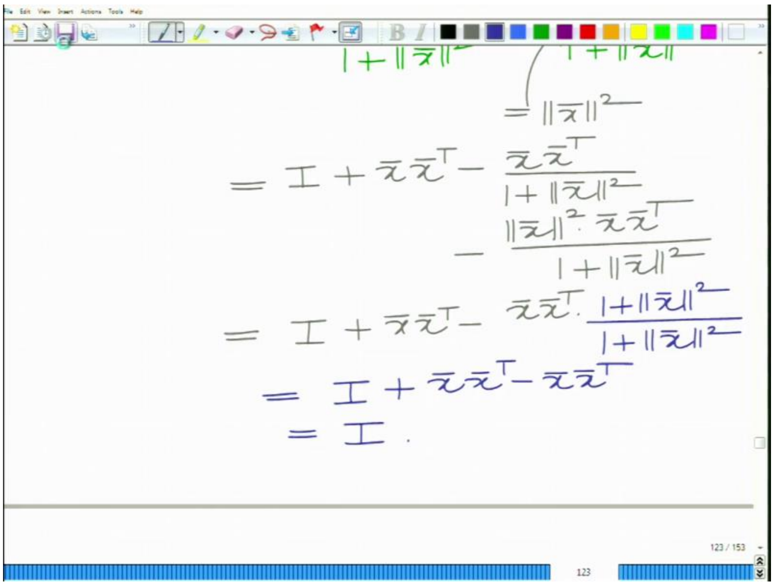

And then it can further be simplified as

$$\left(I+\overline{x}\overline{x}^T\right)\left(I-\frac{\overline{x}\overline{x}^T}{\left(1+\|\overline{x}\|^2\right)}\right)$$

$$=I+\overline{x}\overline{x}^T-\frac{\left(1+\|\overline{x}\|^2\right)\cdot\overline{x}\overline{x}^T}{\left(1+\|\overline{x}\|^2\right)}$$

$$=I+\overline{x}\overline{x}^T-\overline{x}\overline{x}^T$$

$$=I$$

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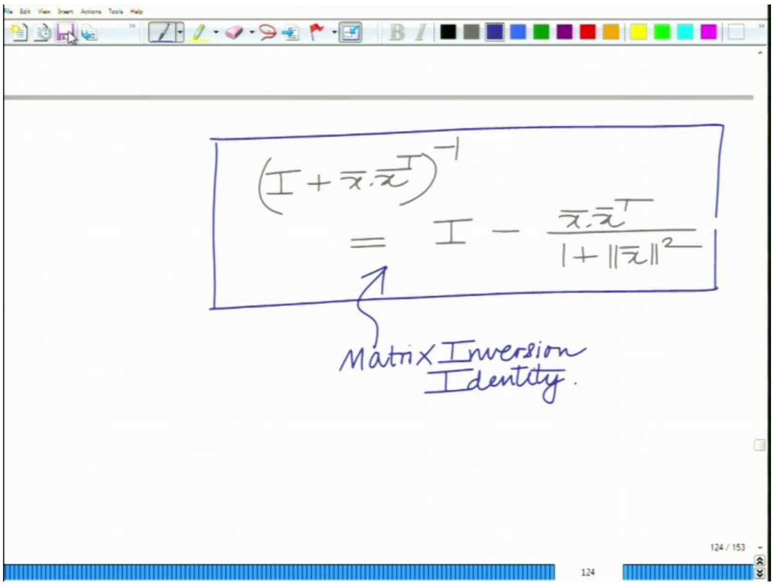

Hence, the matrix inversion identity is verified.

$$\left(I + \overline{x}\overline{x}^T\right)^{-1} = \left(I - \frac{\overline{x}\overline{x}^T}{\left(1 + \left\|\overline{x}\right\|^2\right)}\right)$$

So, this, the Woodberry matrix inversion identity, is a very handy property. So the matrix inversion identity or the Woodberry matrix inversion or the Woodberry matrix inversion lemma is demonstrated.