Applied Optimization for Wireless, Machine Learning, Big Data
Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture – 70
Examples: Linear objective with box constraints, Linear Programming

***Keywords**: Linear objective, Box constraints, Linear Programming*

Hello, welcome to another module in this massive open online course. So we are looking at example problems in convex optimization.

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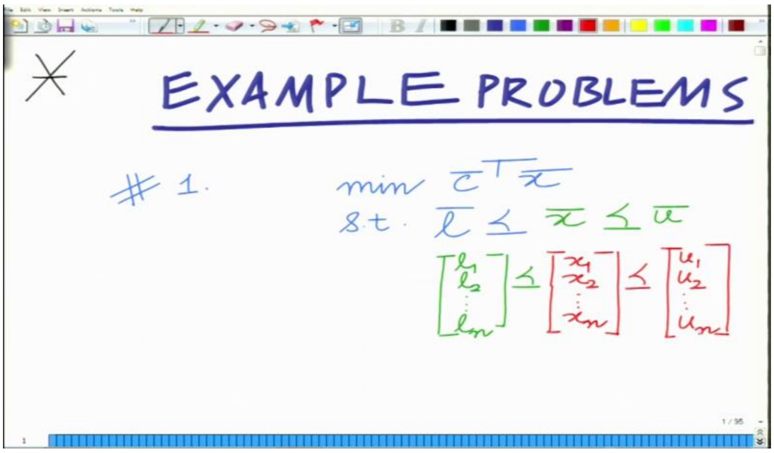

So we are looking at example problems and let us look at problem number 1. So we have $\begin{aligned} &\min \bar{c}^T \bar{x} \\ &s.t\ \bar{l} \le \bar{x} \le \bar{u} \end{aligned}$ which means that if you have the elements of are component wise less than or equal to the other elements.

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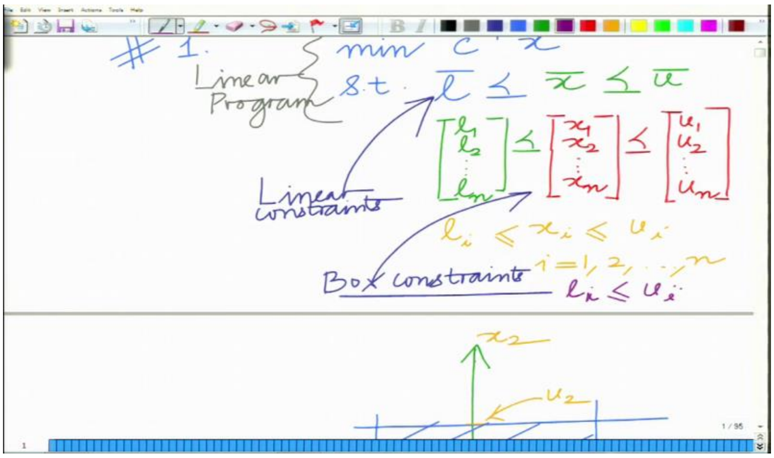

Each $l_i \le x_i \le u_i$, so this is also known as box constraints and this is as shown in slide.

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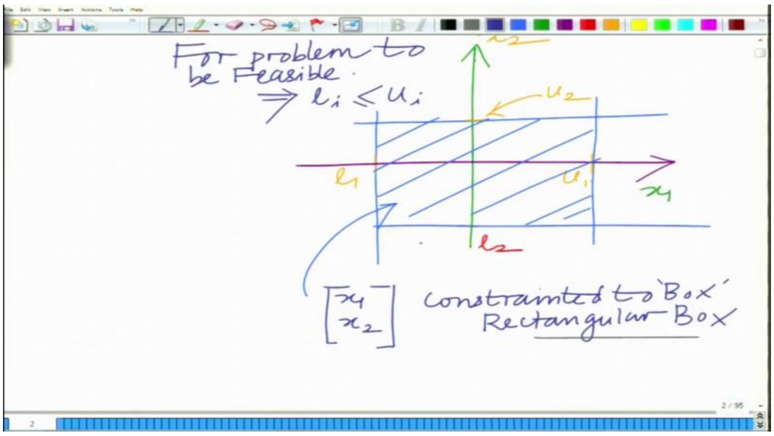

So therefore, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ are confined to this box as shown in slide and hence this is also termed as box type constraint. In fact, it is a simple linear program. So the solution for this is fairly straight forward.

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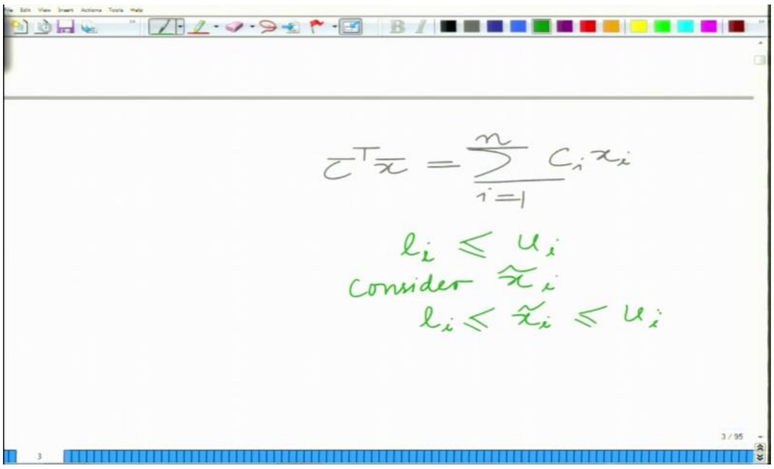

So you have $\bar{c}^T\bar{x} = \sum_{i=1}^{n} c_i x_i$. Now, these box type constraints make sense only if $l_i \le u_i$. So we assume here that $l_i \le u_i$. If this is not so, then the problem becomes infeasible. Now, consider any x_i such that $l_i \le x_i \le u_i$.

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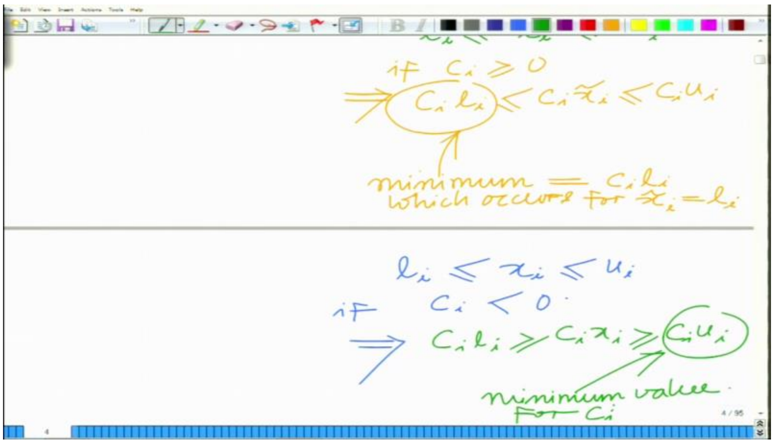

Now, if $c_i \ge 0$, this implies $c_i l_i \le c_i x_i \le c_i u_i$. So minimum value for x_i lying in this box is $c_i l_i$ which occurs when $x_i = l_i$. On the other hand, if $c_i < 0$ this implies $c_i l_i \ge c_i x_i \ge c_i u_i$. Now the minimum value is $c_i u_i$.

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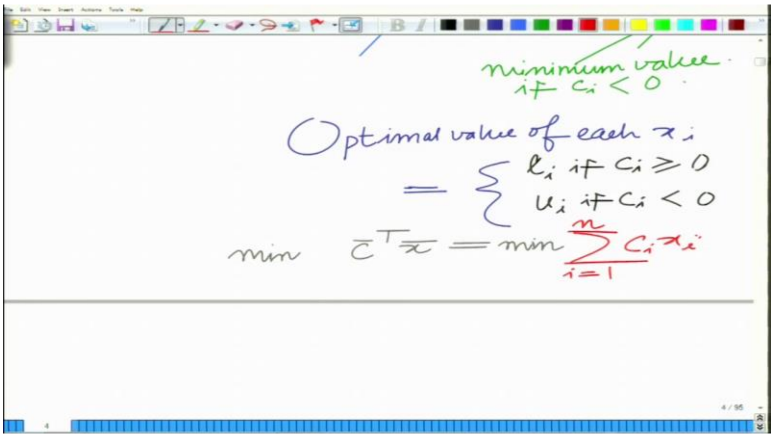

And therefore, the optimal value of each x_i is $\begin{cases} l_i \ \text{if}\ c_i \geq 0 \\ u_i \ \text{if}\ c_i < 0 \end{cases}$.

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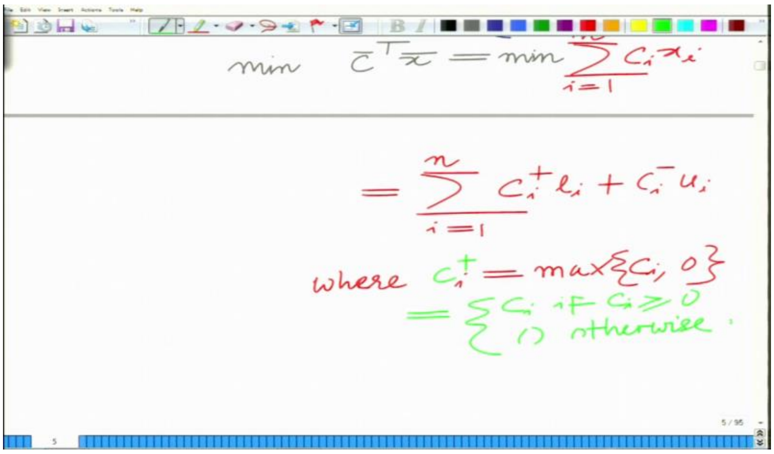

So $\min \bar{c}^T \bar{x} = \min \sum_{i=1}^{n} c_i x_i = \sum_{i=1}^{n} c_i^+ l_i + c_i^- u_i$ where $c_i^+ = \max\{c_i, 0\} = \begin{cases} c_i & \text{if}\ c_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$ and

$c_i^- = \begin{cases} c_i & \text{if}\ c_i < 0 \\ 0 & \text{otherwise} \end{cases}$.

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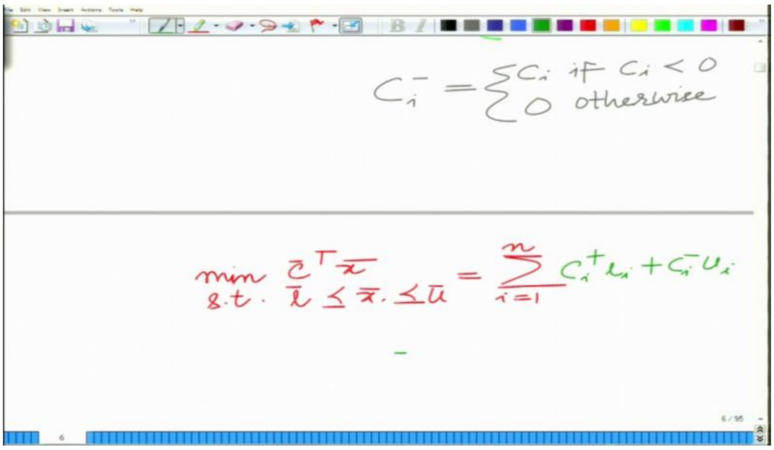

You can also write this as $\min_{s.t\ \underline{l} \le \underline{x} \le \underline{u}} \bar{c}^T \bar{x} = \sum_{i=1}^{n} c_i^+ l_i + c_i^- u_i$ where c_i^+ contains all positive elements of $\bar{c}$ and c_i^- contains only negative elements of $\bar{c}$ and the rest are 0. So that is the optimal value of this problem.

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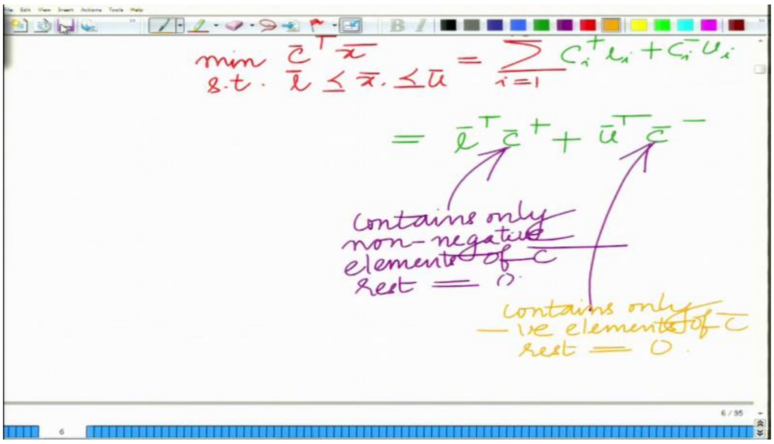

Let us proceed to a slightly more sophisticated example for which the solution might not be very obvious and that is problem number 2 where we have $\min_{s.t\ A\underline{x} \le \underline{b}} \bar{c}^T \bar{x}$. This is a linear program, but slightly more sophisticated and the solution depends on the nature of A which is a square full rank matrix. This implies that A is invertible.

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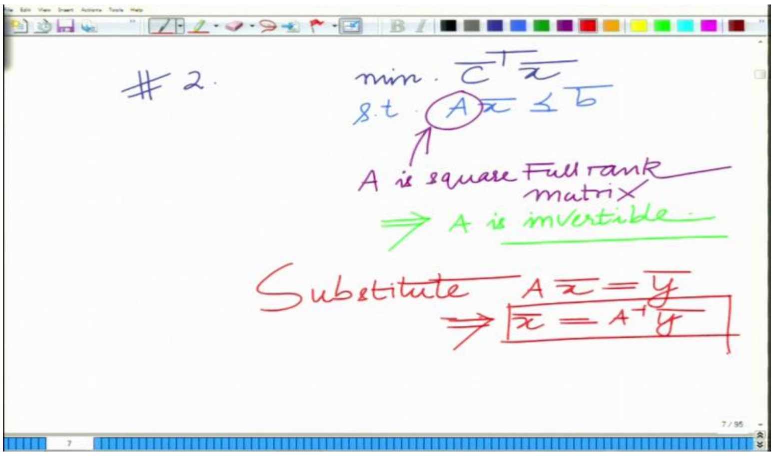

Now we substitute, $A\bar{x} = \bar{y} \Rightarrow \bar{x} = A^{-1}\bar{y}$.

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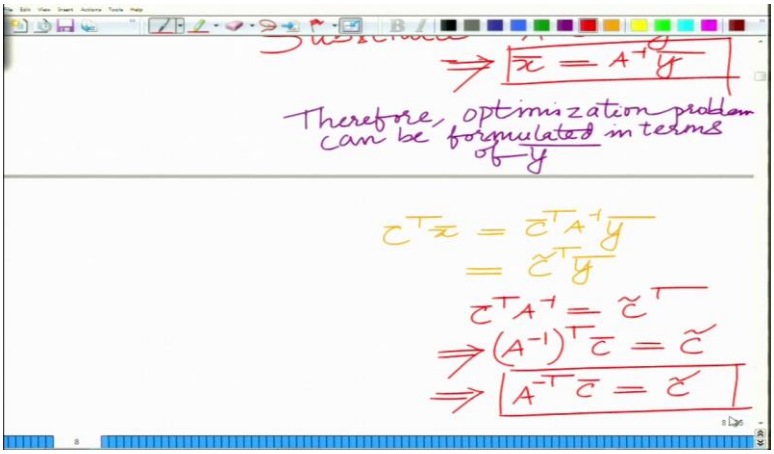

Now, we will write the equivalent optimization problem in terms of $\bar{y}$. So we have the objective $\bar{c}^T \bar{x} = \bar{c}^T A^{-1}\bar{y} = \tilde{c}^T \bar{y} \Rightarrow A^{-1}\bar{c} = \tilde{c}$.

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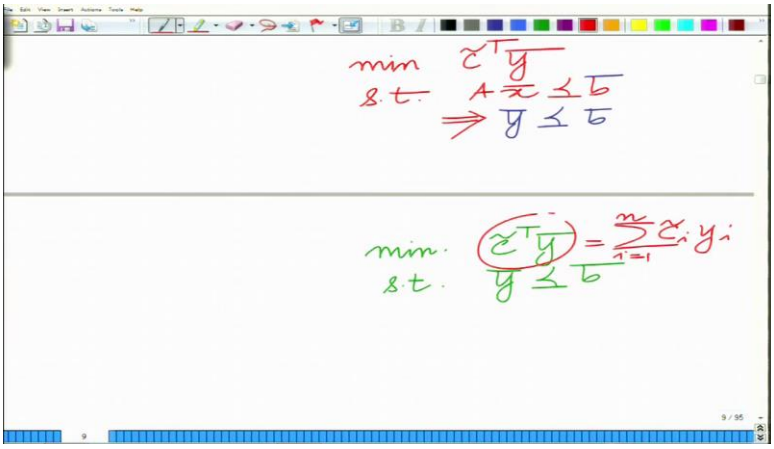

So the objective becomes $\begin{array}{l} \min \tilde{c}^T \bar{y} \\ s.t\ \bar{y} \le \bar{b} \end{array}$. So now, we have $\tilde{c}^T \bar{y} = \sum_{i=1}^{n} \tilde{c}_i y_i$.

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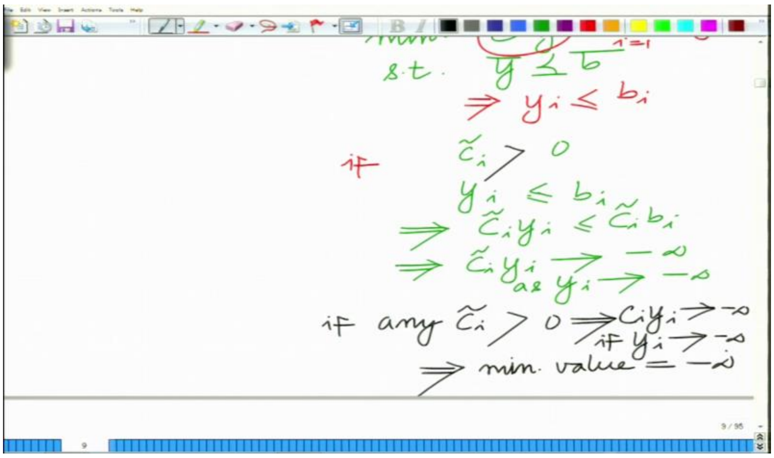

And the constraint will be component wise constraint, this implies that each component of this vector $\bar{y}$ is less than or equal to each component of vector $\bar{b}$. Now we consider if any $\tilde{c}_i > 0 \Rightarrow y_i \le b_i \Rightarrow \tilde{c}_i y_i \le \tilde{c}_i b_i$. So this implies that $\tilde{c}_i y_i \to -\infty \ as\ y_i \to -\infty$. So objective becomes $-\infty$. So it is unbounded below.

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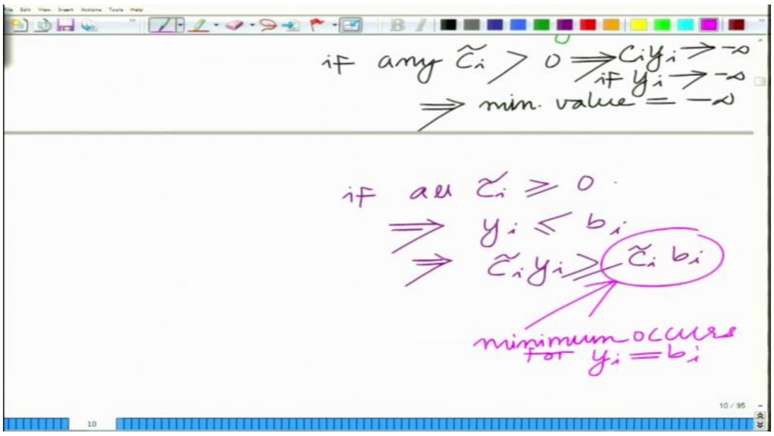

Now, if all $\tilde{c}_i \leq 0 \Rightarrow y_i \leq b_i \Rightarrow \tilde{c}_i y_i \geq \tilde{c}_i b_i$. So the minimum occurs for $y_i = b_i$.

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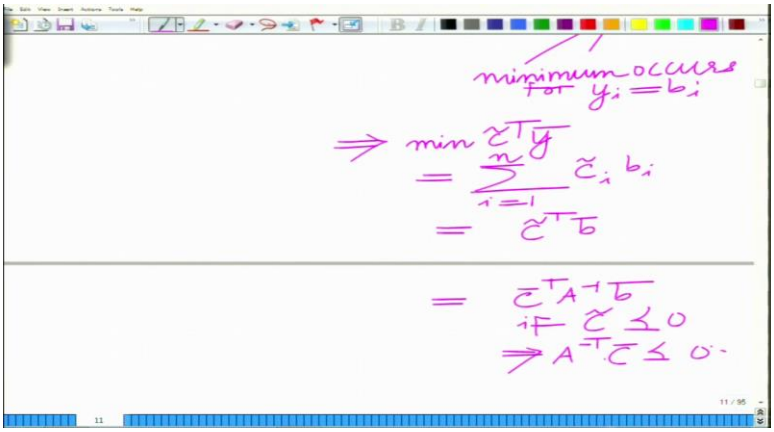

And therefore, the net minimum implies $\min \tilde{c}^T \bar{y} = \sum_{i=1}^{n} \tilde{c}_i b_i = \tilde{c}^T \bar{b} = \bar{c}^T A^{-1} \bar{b}$ if $\tilde{c} \leq 0 \Rightarrow A^{-T} \bar{c} \leq 0$.

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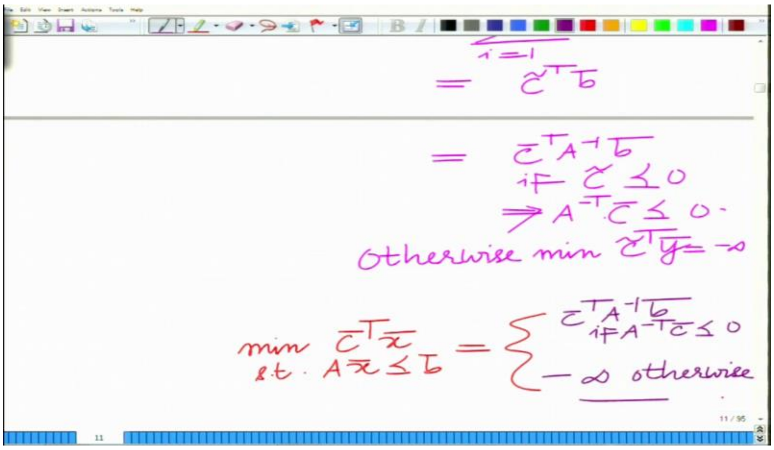

Therefore, the minimum value is $\min_{s.t.\; A\bar{x} \le \bar{b}} \bar{c}^T \bar{x} = \begin{cases} \bar{c}^T A^{-1} \bar{b} & if\ A^{-T}\bar{c} \le 0 \\ -\infty & otherwise \end{cases}$.

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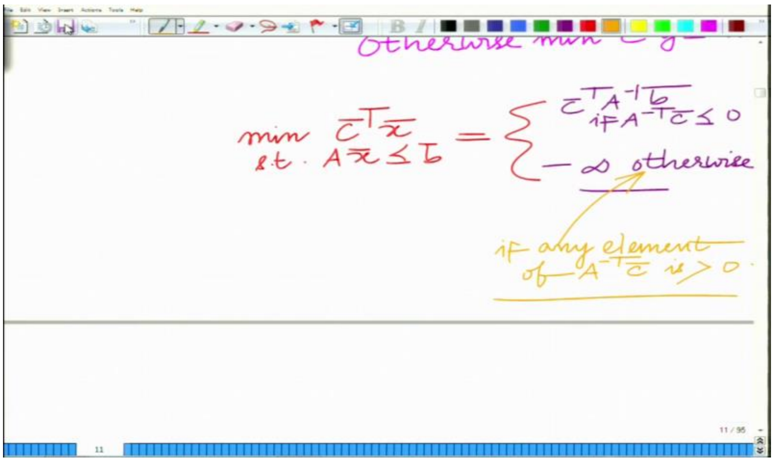

So that is basically the solution to this optimization problem. So let us stop here. Thank you very much.