Applied Optimization for Wireless, Machine Learning, Big data
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Lecture – 13
Norm Ball and its Practical Applications: Multiple Antenna Beamforming

Hello, welcome to another module in this massive open online course. Now let us look at the applications of convex sets and hyper spaces in wireless communication.

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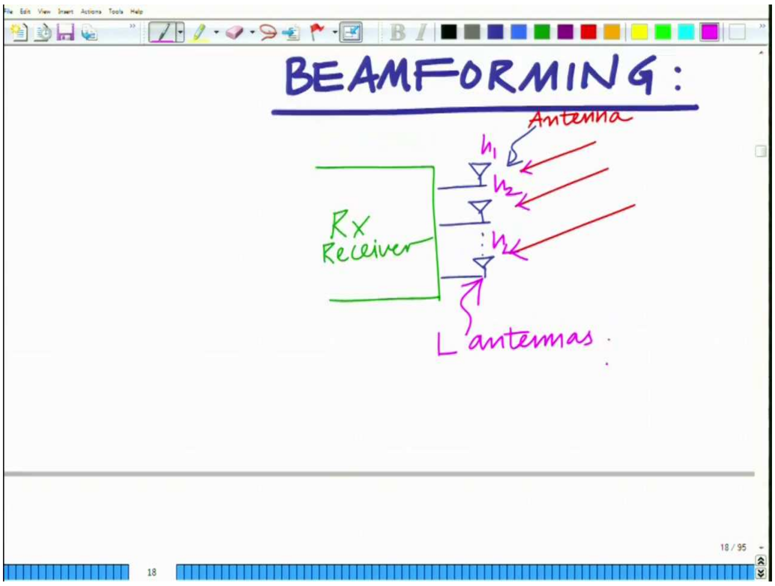

So, another application of the concept of hyper planes and half spaces is in the context of multi antenna beam forming. In multi antenna beam forming, there is a wireless receiver with multiple antennas. The signals with various channel coefficients are coming into these antennas. So, let us say there are *L* antennas and hence there are *L* different channel coefficients each corresponding to different antenna.

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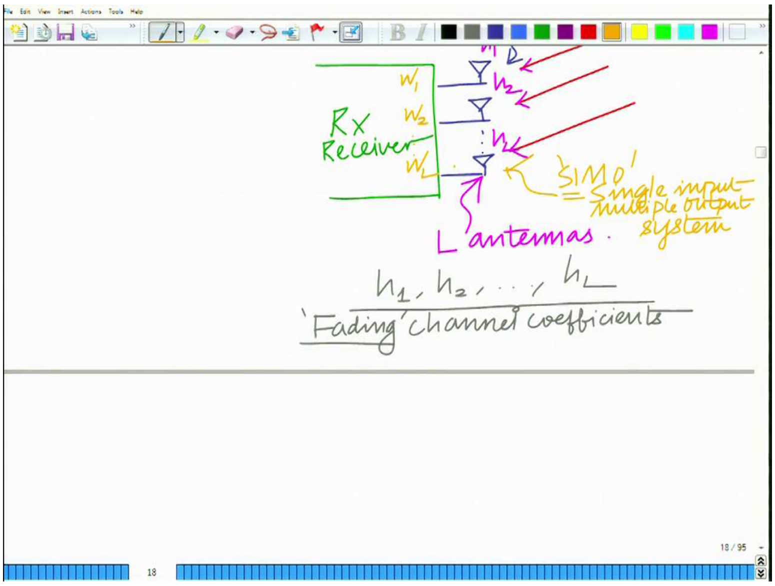

So let us have a set of L channel coefficients as $h_1, h_2, \ldots, h_L$. These are also known as the fading channel coefficients because the wireless channel is typically a fading channel which means that as a channel is varying with time, received power is increasing and decreasing. This type of system is also known as single input multiple output or a SIMO.

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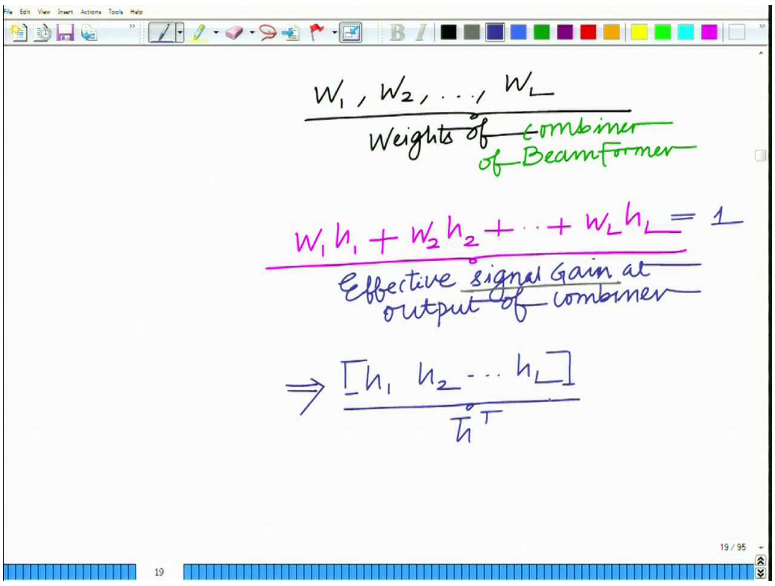

Now, consider linear combination of these signals at receiver with weights the combiner as $W_1, W_2, \ldots, W_L$. So in a sense, this combination is a beam former.

Thus, the effective signal gain at the output of the combiner will be $W_1h_1 + W_2h_2 + \ldots + W_Lh_L$. To normalize this effective signal gain, set this equal to 1. So,

$$W_1h_1 + W_2h_2 + \ldots + W_Lh_L = 1$$

So design a system such that its effective signal gain at the output of the combiner is unity and this is one of the types of constraint in multiple antenna processing.

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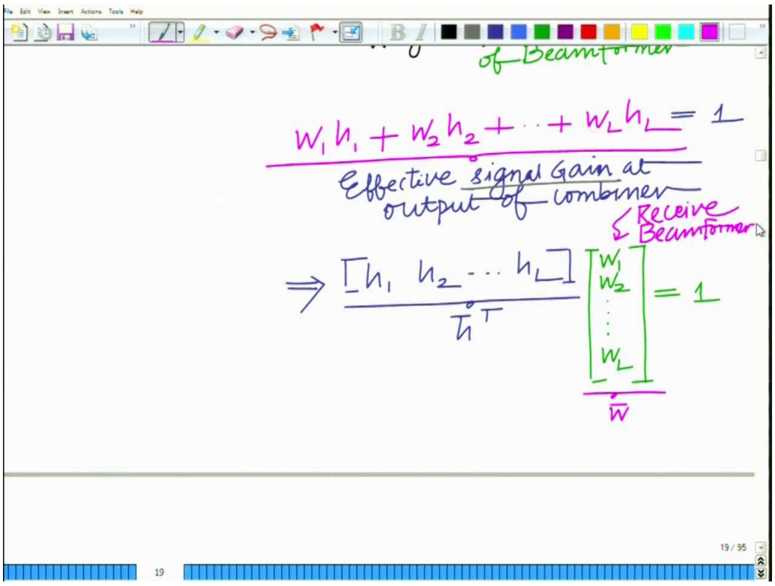

So, the above equation can be written as follows in vector form.

$$\begin{bmatrix} h_1 & h_2 & \cdots & h_L \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_L \end{bmatrix} = 1$$

$$\bar{h}^T \bar{W} = 1$$

And $\bar{W}$ is known as the beam forming vector or receive beam former.

The above equation is a hyper plane constraint. Hence, this is a practical application of the concept of hyper plane in a wireless communications which says that all beam forming vectors $\bar{W}$ lie on a hyper plane described by

$$\bar{h}^T \bar{W} = 1$$

This ensures unity gain for the desired user or desired signal at the output of the combiner.

So, typically, one can either suppress the noise or suppress the interfering signals of the interfering users.

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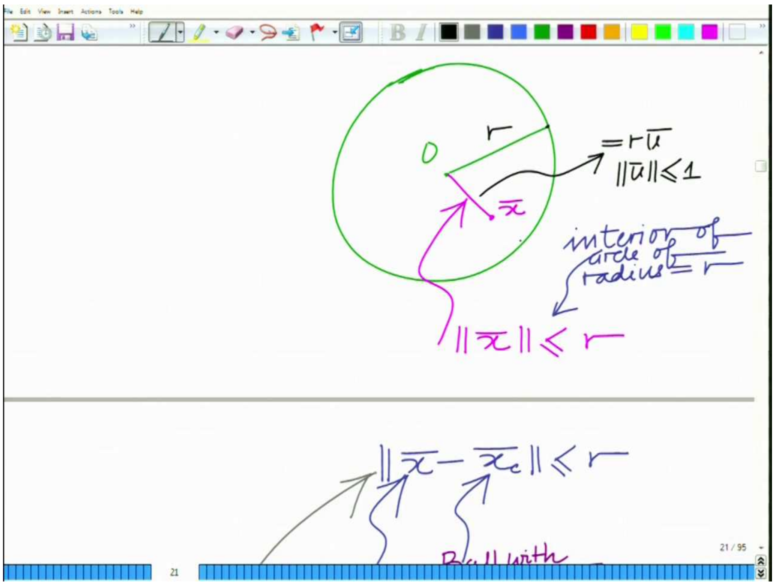

Let us look at spherical balls or the norm balls. It is also known as Euclidean ball. In 2-dimensions, it is basically a circle $\bar{x}$ with center at the origin $(0,0)$ and radius *r*. So, norm of any point within this circle $\bar{x}$ is the length of this point vector which naturally less than or equal to *r*.

$$\|\bar{x}\| \leq r$$

So, this describes the interior of a 2 dimensional ball or of a sphere in case of 3-dimensions.

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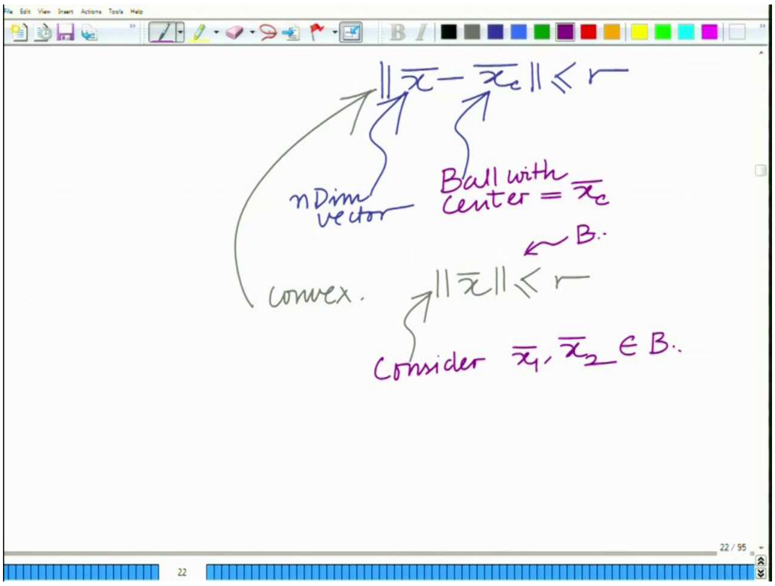

Also if the center is not the origin and at $\bar{x}_C$, then the general equation for the interior of this n-dimensional ball is

$$\|\bar{x} - \bar{x}_C\| \leq r$$

This is convex. To justify this, consider a ball with center at origin. Also consider two points to verify this such that these belong to a set *B*.

$$\bar{x}_1, \bar{x}_2 \in B$$

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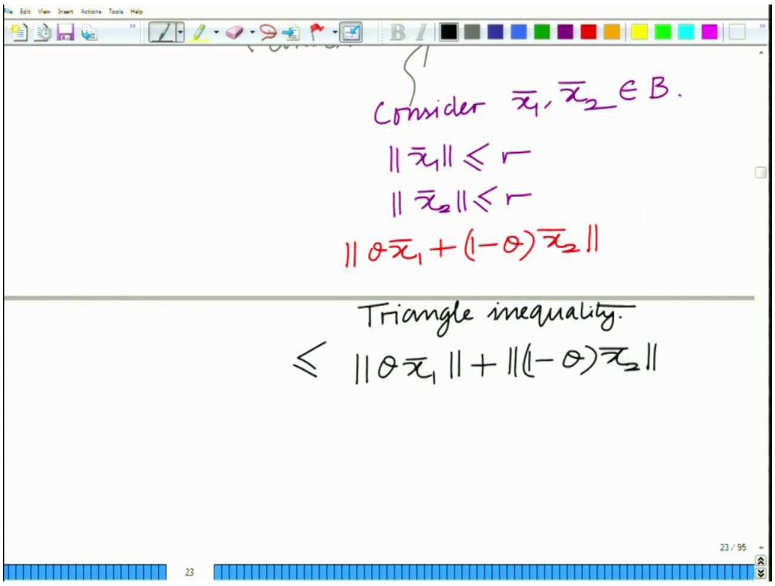

Then, by the definition of norm, if these points belong to the interior of the ball then

$\|\bar{x}_1\| \leq r,$
$\|\bar{x}_2\| \leq r$

Now, let us consider the convex combination of points $\bar{x}_1, \bar{x}_2 \in B$ as $\theta\bar{x}_1 + (1-\theta)\bar{x}_2$.

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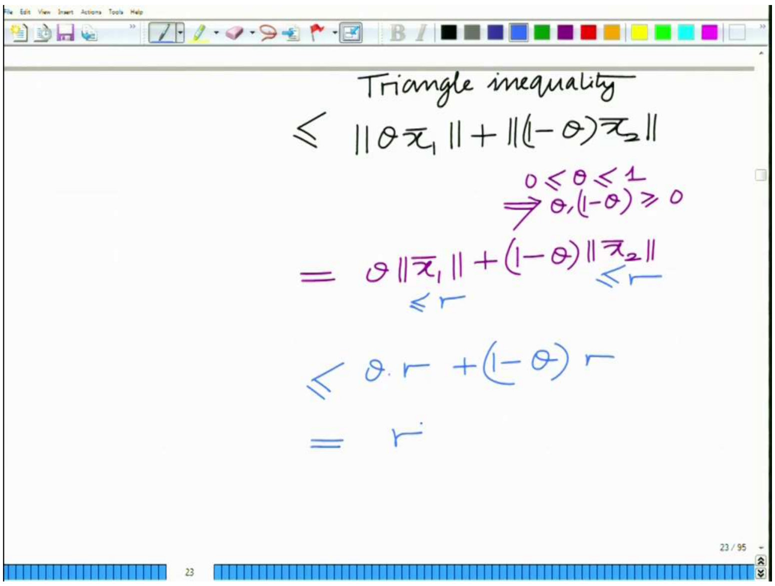

The above combination can also be written as,

$$\left\|\theta\overline{x}_1+(1-\theta)\,\overline{x}_2\right\|$$
$$=\left\|\theta\overline{x}_1\right\|+\left\|(1-\theta)\,\overline{x}_2\right\|$$

According to the triangle inequality,

$$0\leq\left\|\theta\overline{x}_1\right\|+\left\|(1-\theta)\,\overline{x}_2\right\|\leq 1$$

Also as this is the convex combination, then

$$0\leq\theta\leq 1$$
$$\Rightarrow\theta,(1-\theta)\geq 0$$

So,

$$\left\|\theta\overline{x}_1+(1-\theta)\,\overline{x}_2\right\|$$
$$=\left\|\theta\overline{x}_1\right\|+\left\|(1-\theta)\,\overline{x}_2\right\|$$
$$=\theta\left\|\overline{x}_1\right\|+(1-\theta)\left\|\overline{x}_2\right\|$$

And as $\left\|\overline{x}_1\right\|,\left\|\overline{x}_2\right\|\leq r$, this implies that

$$\left\|\theta\overline{x}_1+(1-\theta)\,\overline{x}_2\right\|$$
$$\leq\theta.r+(1-\theta).r$$
$$\leq r$$

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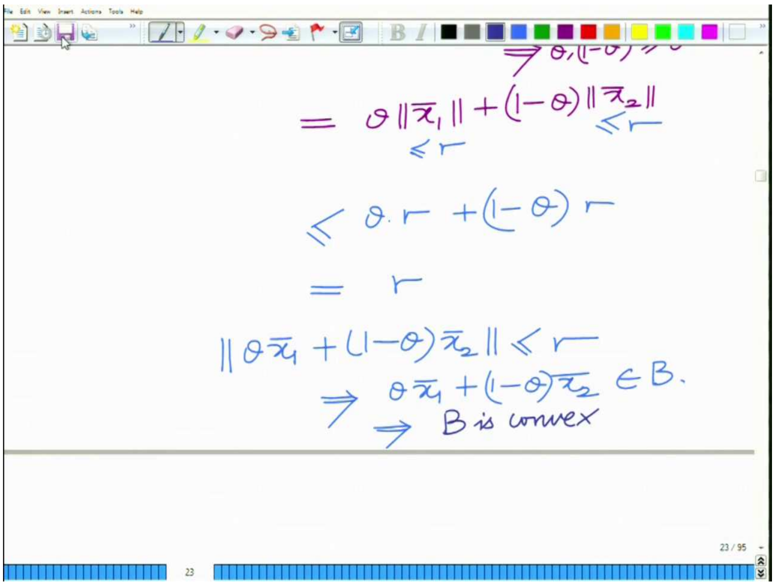

This further implies that convex combination of $\bar{x}_1, \bar{x}_2 \in B$ also lie in the interior of the ball and hence,

$$\left\| \theta \bar{x}_1 + (1-\theta)\bar{x}_2 \right\| \in B$$

And set B is a convex set.

This shows that if any two points belong to the ball, then all their convex combinations also belong to this ball. Therefore, the norm ball is convex.

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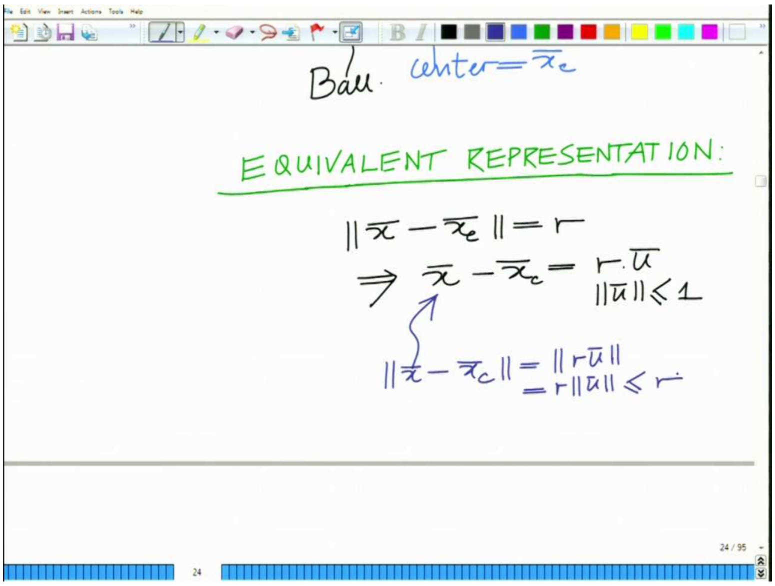

An equivalent representation of the norm ball with center at $\bar{x}_C$ and radius r is as follows.

$$\bar{x} - \bar{x}_C = r.\bar{u}$$

Where $\|\bar{u}\| \leq 1$.

Another equivalent way to represent the norm ball is

$$\begin{aligned}\|\bar{x} - \bar{x}_C\| &= \|r.\bar{u}\| \\ &= r.\|\bar{u}\| \\ &\leq r\end{aligned}$$

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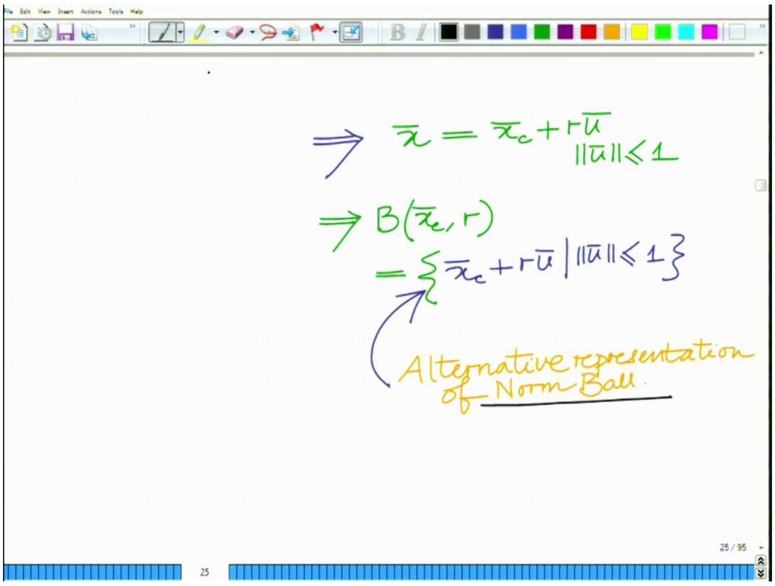

Also this can be written as

$\bar{x} = \bar{x}_C + r.\bar{u}$ with $\|\bar{u}\| \leq 1$

Again an alternative representation to this norm ball could be

$B(\bar{x}_C, r) = \{\bar{x}_C + r.\bar{u} \mid \|\bar{u}\| \leq 1\}$

where $\bar{u}$ is any vector such that $\|\bar{u}\| \leq 1$.

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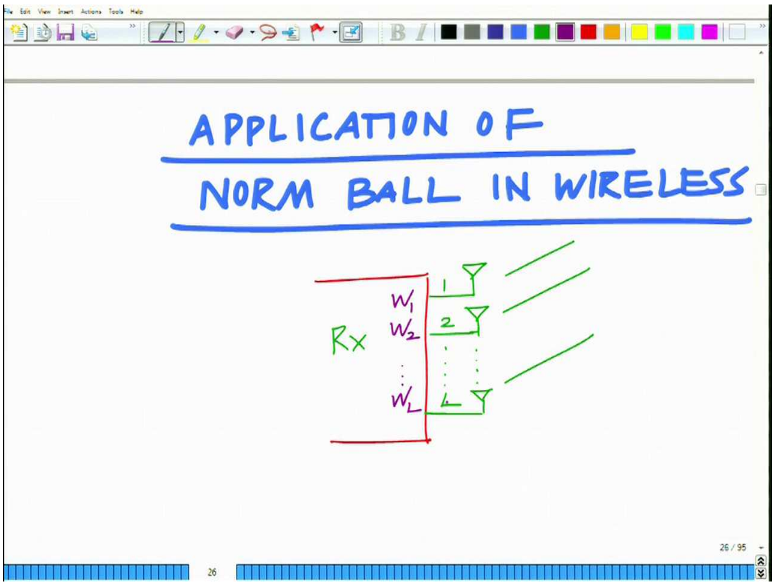

So, let us look at the application of norm ball in a wireless system. For instance, again go back to the multi antenna beamforming problem with L antenna reciever and beamforming weights as $W_1, W_2, \ldots, W_L$.

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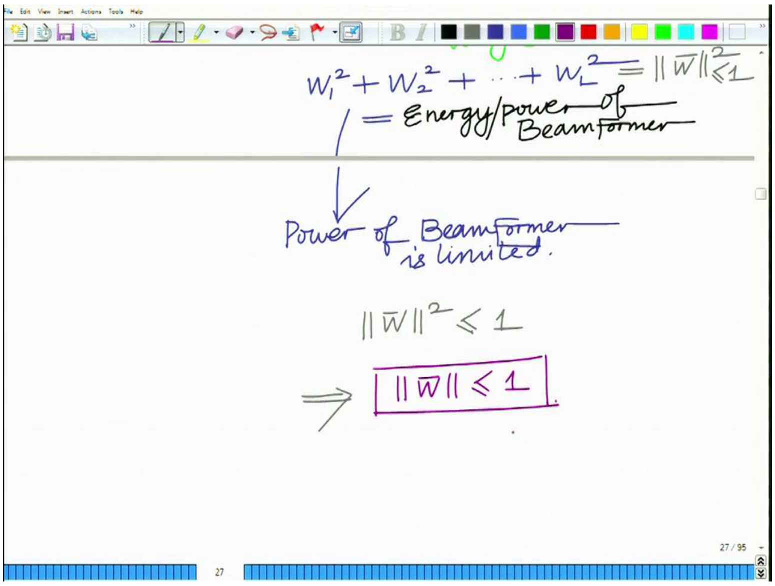

So lets look at the power of the beam former. As beamforming weights influence the output power of the combiner, therefore the power of the beam former is

$$W_1^{\,2} + W_2^{\,2} + \ldots + W_L^{\,2} = \|\bar{W}\|^2$$

In any wireless communication system the power of the beam former should be restricted because it influences the power of the signal at the output of the beam former. Therefore, to ensure stable beam former, this energy of the beam former is typically restricted to unity.

$$\left\|\bar{W}\right\|^2 \leq 1$$

And hence,

$$\left\|\bar{W}\right\| \leq 1$$

This is the beamformer power constraint which is a norm ball constraint.

So, that is an interesting application of the concept of a norm ball to a wireless communication scenario to design the constraint for a receive beam former. Lets continue in the subsequent modules.

Thank you very much.