

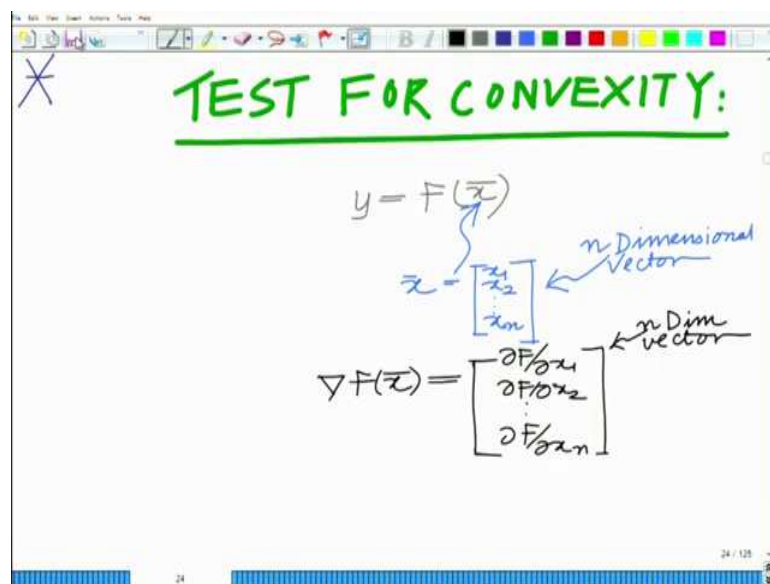
**Applied Optimization for Wireless, Machine Learning, Big Data**  
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**Lecture – 25**

**Test for Convexity: Positive Semidefinite Hessian Matrix, example problems**

Hello, welcome to another module in this massive open online course. Let us extend the test for convex functions of a vector or a multidimensional variable.

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For this, consider a function of an  $n$ -dimensional vector  $\bar{x}$ .

$$y = F(\bar{x})$$

Such that

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

The gradient of this function is also an  $n$ -dimensional vector and it is defined as the vector which contains the partial derivative of function with respect to each component of the vector  $\bar{x}$ .

$$\nabla F(\bar{x}) = \begin{bmatrix} \frac{\partial F}{\partial x_1} \\ \frac{\partial F}{\partial x_2} \\ \vdots \\ \frac{\partial F}{\partial x_n} \end{bmatrix}$$

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A digital whiteboard interface showing the definition of the Hessian matrix. The text  $\nabla^2 F(x) =$  is written in pink, followed by a large square bracket containing second-order partial derivatives:  $\frac{\partial^2 F}{\partial x_1^2}$ ,  $\frac{\partial^2 F}{\partial x_1 \partial x_2}$ , ...,  $\frac{\partial^2 F}{\partial x_1 \partial x_n}$  in the first row;  $\frac{\partial^2 F}{\partial x_2 \partial x_1}$ ,  $\frac{\partial^2 F}{\partial x_2^2}$ , ...,  $\frac{\partial^2 F}{\partial x_2 \partial x_n}$  in the second row; and  $\frac{\partial^2 F}{\partial x_n \partial x_1}$ , ...,  $\frac{\partial^2 F}{\partial x_n^2}$  in the last row. A curved arrow points from the text 'Hessian of F' below to the expression  $\nabla^2 F(x)$ . The whiteboard has a toolbar at the top and a status bar at the bottom showing '25 / 125'.

$$\nabla^2 F(x) = \begin{bmatrix} \frac{\partial^2 F}{\partial x_1^2} & \frac{\partial^2 F}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 F}{\partial x_1 \partial x_n} \\ \frac{\partial^2 F}{\partial x_2 \partial x_1} & \frac{\partial^2 F}{\partial x_2^2} & \cdots & \frac{\partial^2 F}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 F}{\partial x_n \partial x_1} & \cdots & \cdots & \frac{\partial^2 F}{\partial x_n^2} \end{bmatrix}$$

Hessian of F

Consider the second order derivative of this function which is known as the Hessian of this function  $\nabla^2 F(\bar{x})$  and it is a  $n \times n$  matrix. So, this is defined as follows.

$$\nabla^2 F(\bar{x}) = \begin{bmatrix} \frac{\partial^2 F}{\partial x_1^2} & \frac{\partial^2 F}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 F}{\partial x_1 \partial x_n} \\ \frac{\partial^2 F}{\partial x_2 \partial x_1} & \frac{\partial^2 F}{\partial x_2^2} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 F}{\partial x_n \partial x_1} & \cdots & \cdots & \frac{\partial^2 F}{\partial x_n^2} \end{bmatrix}$$

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$$\nabla^2 F(\bar{x}) = \begin{bmatrix} \frac{\partial^2 F}{\partial x_1 \partial x_1} & \frac{\partial^2 F}{\partial x_1 \partial x_2} & \dots & \frac{\partial^2 F}{\partial x_1 \partial x_n} \\ \frac{\partial^2 F}{\partial x_2 \partial x_1} & \frac{\partial^2 F}{\partial x_2 \partial x_2} & \dots & \frac{\partial^2 F}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 F}{\partial x_n \partial x_1} & \frac{\partial^2 F}{\partial x_n \partial x_2} & \dots & \frac{\partial^2 F}{\partial x_n \partial x_n} \end{bmatrix}$$

$n \times n$  Matrix

Hessian of  $F(\bar{x})$

$$[\nabla^2 F]_{i,j} = \frac{\partial^2 F}{\partial x_i \partial x_j}$$

So, the  $(i,j)^{\text{th}}$  component of this hessian of the function  $[\nabla^2 F]_{i,j}$  can be defined as

$$[\nabla^2 F]_{i,j} = \frac{\partial^2 F}{\partial x_i \partial x_j}$$

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TEST FOR CONVEXITY:

$\nabla^2 F(\bar{x}) \geq 0$  = PSD

$\nabla^2 F(\bar{x}) = \text{PSD (Positive Semi-Definite)}$

$\Rightarrow F(\bar{x}) = \text{CONVEX}$

Now, the condition the test for convexity of a function of a multidimensional vector is

$$\nabla^2 F(\bar{x}) \geq 0$$

This means that  $\nabla^2 F(\bar{x})$  is a positive semi definite matrix. So it concludes that if the Hessian of a function of a multidimensional vector is a positive semi definite matrix then this function is a convex function.

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Handwritten notes on a digital whiteboard:

- $\nabla^2 F(\bar{x}) \succeq 0$
- $\nabla^2 F(\bar{x}) = \text{PSD (Positive Semi Definite)}$
- $\Rightarrow f(\bar{x}) = \text{CONVEX}$
- EX:  $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
- $f(\bar{x}) = \frac{x_1^2}{x_2}$

Let us look at a simple example. Consider a 2-D vector  $\bar{x}$ .

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{where } x_1, x_2 \geq 0$$

The function of this vector is given as

$$F(\bar{x}) = \frac{x_1^2}{x_2}$$

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Ex:  $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$   
 $f(\bar{x}) = \frac{x_1^2}{x_2}$   
 $x_1, x_2 \geq 0$

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$$\nabla F(\bar{x}) = \begin{bmatrix} \frac{\partial F}{\partial x_1} \\ \frac{\partial F}{\partial x_2} \end{bmatrix}$$
$$= \begin{bmatrix} 2x_1/x_2 \\ -x_1^2/x_2^2 \end{bmatrix}$$

So the first gradient of this function is

$$\nabla F(\bar{x}) = \begin{bmatrix} \frac{\partial F}{\partial x_1} \\ \frac{\partial F}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{2x_1}{x_2} \\ -\frac{x_1^2}{x_2^2} \end{bmatrix}$$

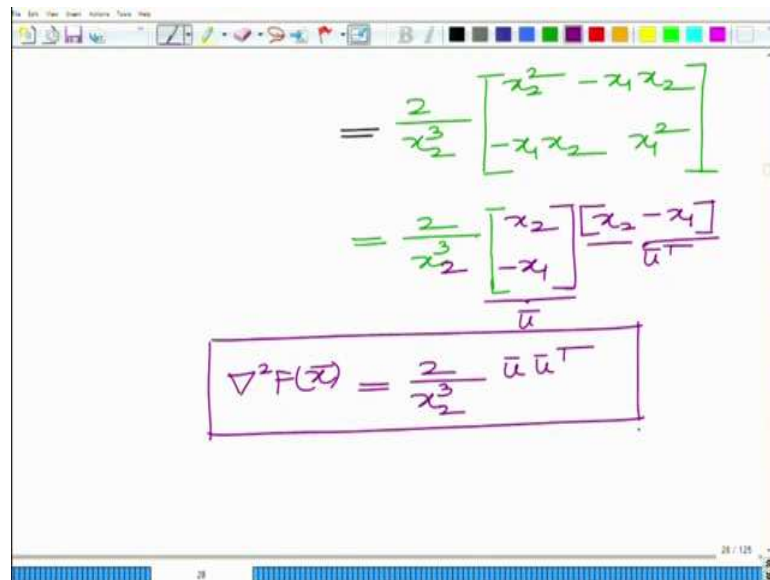
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$$\nabla^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{2}{x_2} & -\frac{2x_1}{x_2^2} \\ -\frac{2x_1}{x_2^2} & \frac{2x_1}{x_2^3} \end{bmatrix}$$

The Hessian of this function is

$$\nabla^2 F = \begin{bmatrix} \frac{\partial^2 F}{\partial x_1^2} & \frac{\partial^2 F}{\partial x_1 \partial x_2} \\ \frac{\partial^2 F}{\partial x_2 \partial x_1} & \frac{\partial^2 F}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} \frac{2}{x_2} & \frac{-2x_1}{x_2^2} \\ \frac{-2x_1}{x_2^2} & \frac{2x_1^2}{x_2^3} \end{bmatrix}$$

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The image shows a whiteboard with handwritten mathematical steps. The first step is a 2x2 matrix:  $\frac{2}{x_2^3} \begin{bmatrix} x_2^2 - x_1 x_2 & -x_1 x_2 \\ -x_1 x_2 & x_1^2 \end{bmatrix}$ . The second step shows this as a product:  $\frac{2}{x_2^3} \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} \begin{bmatrix} x_2 - x_1 \end{bmatrix} \frac{1}{x_2}$ . The third step defines  $\bar{u} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$  and shows the final boxed result:  $\nabla^2 F(\vec{x}) = \frac{2}{x_2^3} \bar{u} \bar{u}^T$ .

And this matrix can be decomposed as follows.

$$\begin{aligned} \nabla^2 F &= \frac{2}{x_2^3} \begin{bmatrix} x_2^2 & -x_1 x_2 \\ -x_1 x_2 & x_1^2 \end{bmatrix} \\ &= \frac{2}{x_2^3} \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} \begin{bmatrix} x_2 & -x_1 \end{bmatrix} \\ &= \frac{2}{x_2^3} \bar{u} \cdot \bar{u}^T \end{aligned}$$

Where  $\bar{u} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$ .

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$$\nabla^2 F(\bar{x}) = \left( \frac{2}{x_2^3} \right) \bar{u} \bar{u}^T$$

Hessian  $\bar{u} = \begin{bmatrix} x_2 \\ -x_4 \end{bmatrix}$   
PSD matrix

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$$P = U U^T$$

$$\begin{aligned} \bar{x}^T P \bar{x} &= \bar{x}^T U U^T \bar{x} \\ &= (U \bar{x})^T U \bar{x} \\ &= \|U \bar{x}\|^2 \\ &= \text{PSD Matrix} \end{aligned}$$

Let us look at matrix  $\bar{u} \cdot \bar{u}^T$ .

$$P = U U^T$$

Let us now check whether this matrix  $P$  is a positive semi definite matrix.

$$\begin{aligned} \bar{x}^T P \bar{x} &= \bar{x}^T U U^T \bar{x} \\ &= (U \bar{x})^T U \bar{x} \\ &= \|U \bar{x}\|^2 \end{aligned}$$

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The whiteboard shows the following derivations:

$$\begin{aligned}\bar{x}^T P \bar{x} &= \bar{x}^T U U^T \bar{x} \\ &= (U \bar{x})^T U \bar{x} \\ &= \|U \bar{x}\|^2 \\ &= \text{PSD Matrix}\end{aligned}$$
$$\nabla^2 F = \frac{2}{x_2^3} \bar{u} \bar{u}^T \Rightarrow \nabla^2 F \geq 0$$

Below this, it is noted that  $\bar{u} \bar{u}^T \geq 0$  is a PSD matrix, and the ratio  $\frac{x_1^2}{x_2^3}$  is a convex function.

This shows that matrix  $P$  is a positive semi definite matrix. And this implies that

$$\nabla^2 F \geq 0$$

The function  $F(\bar{x}) = \frac{x_1^2}{x_2}$  is a convex function with the restricted domain  $x_1, x_2 \geq 0$ .

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**PRACTICAL APPLICATION:**

**MIMO WIRELESS SYSTEM**

Multiple Input Multiple Output

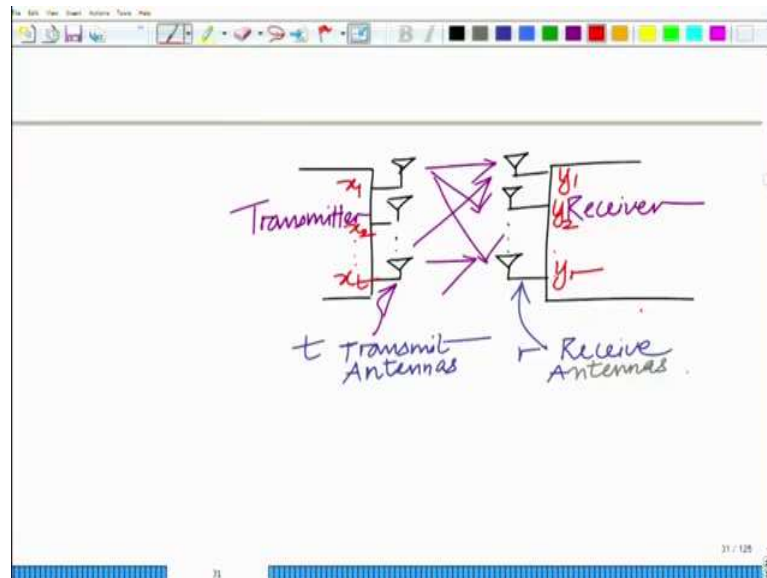
⇒ Multiple Transmit + Multiple Receive Antennas.

Let us look at an interesting practical application of the convexity of a function of a vector. Consider a Multiple Input Multiple Output (MIMO) wireless communication



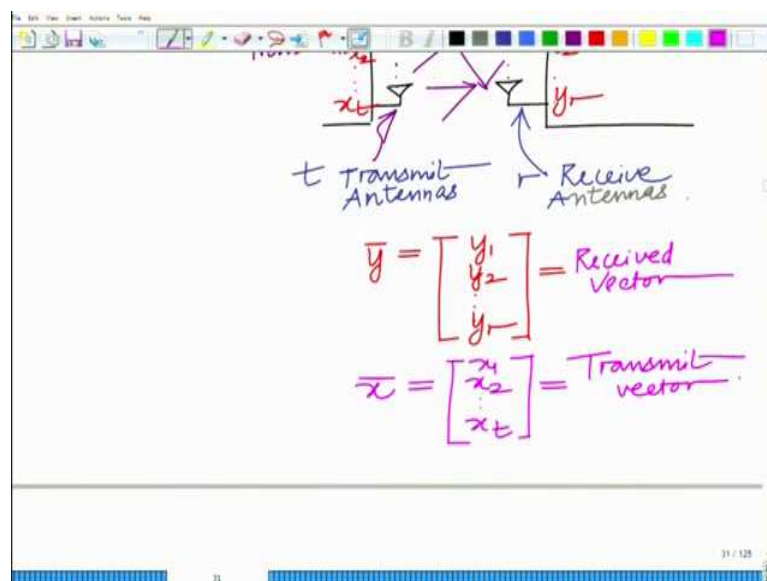
system. This implies that there are multiple transmit and multiple receive antennas. This significantly increases the data rate of a wireless communication system.

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Let there are  $t$  transmit antennas and  $r$  receive antennas. So the transmit symbols from  $t$  transmitters are  $x_1, x_2, \dots, x_t$  and receive symbols at  $r$  receivers are  $y_1, y_2, \dots, y_r$  and there are so many possible channels between each transmitter and receiver.

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So, the  $t \times 1$  transmit vector is

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{bmatrix}$$

And the  $r \times 1$  received vector is

$$\bar{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_r \end{bmatrix}$$

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The diagram shows the equation  $\bar{y} = H\bar{x} + \bar{n}$  with handwritten annotations. Above the  $H$  is a wavy line labeled  $r \times t$  matrix. Below the  $\bar{x}$  is a wavy line labeled  $t \times 1$ . To the right of the  $+$  is a circle around  $\bar{n}$  with a wavy line labeled  $r \times 1$  Additive vector. Below the equation,  $H$  is defined as  $H = r \times t$  matrix  $H$ . MIMO channel matrix.

Therefore the model for this MIMO system is given as

$$\bar{y} = H\bar{x} + \bar{n}$$

Where  $H$  is the  $r \times t$  MIMO channel matrix and  $\bar{n}$  is the  $r \times 1$  additive noise vector.

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Handwritten diagram illustrating the structure of the MIMO channel matrix  $H$ . The matrix is defined as  $H = F \times C$ , where  $F$  is the fading channel coefficients and  $C$  is the transmit matrix. The matrix  $H$  is shown as a  $r \times t$  matrix, where  $r$  is the number of receive antennas and  $t$  is the number of transmit antennas. The elements of the matrix are  $h_{ij}$ , representing the channel coefficient between the  $i$ th receive antenna and the  $j$ th transmit antenna.

$$H = \begin{bmatrix} h_{11} & h_{12} & \dots & h_{1t} \\ h_{21} & h_{22} & \dots & h_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ h_{r1} & \dots & \dots & h_{rt} \end{bmatrix}$$

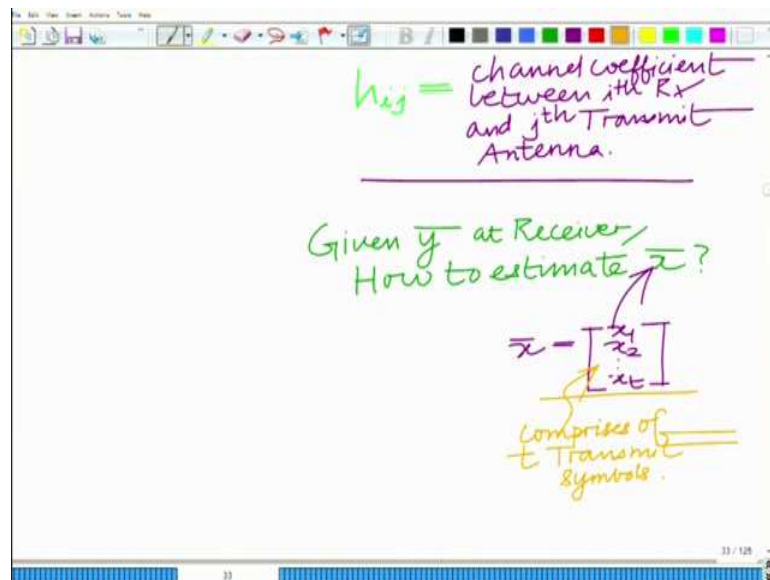
$h_{ij}$  = channel coefficient between  $i$ th Rx and  $j$ th Transmit Antenna.

This channel matrix  $H$  has the following structure.

$$H = \begin{bmatrix} h_{11} & h_{12} & \dots & h_{1t} \\ h_{21} & h_{22} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ h_{r1} & \dots & \dots & h_{rt} \end{bmatrix}$$

And each of quantity  $h_{ij}$  is the fading channel coefficient of channel between  $i^{\text{th}}$  transmitter and  $j^{\text{th}}$  receiver.

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So the main challenge of a wireless communication system is the estimation of  $\bar{x}$  from the received vector  $\bar{y}$  at the receiver and this forms the problem of MIMO receiver design. So, one has to design a suitable algorithm or a technique for the MIMO receiver which recovers the transmit vector from received vector.

Let us explore the problem of MIMO receiver design and its relation to convex optimization in the subsequent module.