

**Applied Optimisation for Wireless, Machine Learning, Big Data**  
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**Lecture – 73**

**Examples on Duality: Dual Norm, Dual of Linear Program (LP)**

**Keywords:** Dual Norm, Linear Program

Hello, welcome to another module, in this massive open online course. Let us continue looking at examples and in this module let us start looking at examples pertaining to duality.

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**DUALITY:**

# 7: DUAL NORM:

$\|z\|_e$

$\|z\|_{e^*} = \max \{ z^T u \mid \|u\|_e \leq 1 \}$

Dual Norm

Let us start with the first example that is for instance if you have a vector  $\bar{x}$  this  $\|\bar{x}\|_l$  is the  $l$  norm, for instance,  $l$  can be 1, 2 and so on. Now, the dual norm of this is denoted by  $\|\bar{z}\|_{l^*} = \max \{ \bar{z}^T \bar{u} \mid \|\bar{u}\|_l \leq 1 \}$ . So this is basically the dual norm.

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$$\text{ex: } l_1 = 2 \quad l_2 \text{ Norm}$$

$$\|z\|_{2*} = \max \{ z^T u \mid \|u\|_2 \leq 1 \}$$

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$$\begin{aligned} &\max_{s.t.} \quad z^T u \\ &\quad \|u\|_2 \leq 1 \end{aligned} \quad \left. \vphantom{\begin{aligned} &\max_{s.t.} \quad z^T u \\ &\quad \|u\|_2 \leq 1 \end{aligned}} \right\} \text{Opt problem}$$

$$\underline{z^T u \leq \|z\| \|u\|} \quad \text{Cauchy-Schwarz inequality}$$

So let us look at some examples to understand this. Let us consider the dual norm of the

$l_2$  norm that is  $\|z\|_{2*} = \max \{ z^T u \mid \|u\|_2 \leq 1 \}$ . Now  $\max_{s.t.} \{ z^T u \mid \|u\|_2 \leq 1 \}$  is the pertinent optimization

problem and this is convex in nature because, this is a linear objective, this is a convex constraint and now this is easy to solve. In fact, we know that,  $\left| z^T u \right| \leq \|z\| \|u\|$ . So this follows from the Cauchy Schwarz Inequality.

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$$\underline{z^T u \leq \|z\| (\|u\|)} \leq 1$$

$$\text{Cauchy-Schwarz inequality}$$

$$\Rightarrow z^T u \leq \|z\| \|u\| \leq \|z\|$$

$$\Rightarrow z^T u \leq \|z\|$$

maximum when  $u$  is aligned with  $z$   
 $\|u\| = 1$   
 $\Rightarrow u = \frac{z}{\|z\|}$

Now, we know that this  $\|\bar{u}\| \leq 1$ , which basically implies  $|\bar{z}^T \bar{u}| \leq \|\bar{z}\| \|\bar{u}\| \leq \|\bar{z}\|$ , which implies that  $\bar{z}^T \bar{u} \leq \|\bar{z}\|$  and the maximum occurs when  $\bar{u}$  is aligned with  $\bar{z}$  which implies

$$\bar{u} = \frac{\bar{z}}{\|\bar{z}\|}.$$

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$\Rightarrow$  maximum when  $\bar{u}$  is aligned with  $\bar{z}$   
 $\|\bar{u}\| = 1$   
 $\Rightarrow \bar{u} = \frac{\bar{z}}{\|\bar{z}\|}$

The maximum  
 $= \bar{z}^T \cdot \frac{\bar{z}}{\|\bar{z}\|}$   
 $= \|\bar{z}\|_2$   
 Default  $= \|\cdot\|_2$

So the maximum is  $\bar{z}^T \frac{\bar{z}}{\|\bar{z}\|} = \|\bar{z}\|_2$ .

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The maximum  
 $= \bar{z}^T \cdot \frac{\bar{z}}{\|\bar{z}\|}$   
 $= \|\bar{z}\|_2$   
 Default  $= \|\cdot\|_2$

$\|\bar{z}\|_{2^*} = \|\bar{z}\|_2$   
 dual Norm of  $l_2$  norm is  $l_2$  Norm itself

Therefore the dual norm of the  $l_2$  norm is  $l_2$  norm itself.

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Handwritten notes on a whiteboard:

- $\text{Dual Norm} = ?$
- $\|u\|_{\infty} = \max\{|u_1|, |u_2|, \dots, |u_n|\}$
- $= \max\{|u_i|\}$
- $\|z\|_{\infty}^* = \max\left\{\frac{z^T u}{\|u\|_{\infty}} \mid \|u\|_{\infty} \leq 1\right\}$
- $\Rightarrow \max\{|u_i|\} \leq 1$

Now, we want to find the dual norm of the  $l_{\infty}$  norm. This is simply

$$\|z\|_{\infty}^* = \max\left\{\frac{z^T u}{\|u\|_{\infty}} \mid \|u\|_{\infty} \leq 1\right\}. \text{ Now, } \|u\|_{\infty} \leq 1 \Rightarrow \max\{|u_i|\} \leq 1.$$

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Handwritten notes on a whiteboard:

- $= \max\{|u_i|\}$
- $\|z\|_{\infty}^* = \max\left\{\frac{z^T u}{\|u\|_{\infty}} \mid \|u\|_{\infty} \leq 1\right\}$
- $\Rightarrow \max\{|u_i|\} \leq 1$
- $z^T u = \sum_{i=1}^n z_i u_i$
- $\leq \sum_{i=1}^n |z_i u_i|$

Now, assume  $\bar{z}$  and  $\bar{u}$  to be  $n$  dimensional vectors. Now  $\bar{z}^T \bar{u} = \sum_{i=1}^n |z_i u_i|$  is simply the dot product between these two. This is as shown in slide.

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$$\begin{aligned}
 z^T u &= \sum_{i=1}^n z_i u_i \\
 &\leq \sum_{i=1}^n |z_i u_i| \\
 &= \sum_{i=1}^n |z_i| |u_i| \leq \sum_{i=1}^n |z_i| \quad (\text{since } |u_i| \leq 1) \\
 &= \sum_{i=1}^n |z_i|
 \end{aligned}$$

Now we have  $z^T u \leq \sum_{i=1}^n |z_i|$ .

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$z^T u \leq \sum_{i=1}^n |z_i| = \|z\|_1$   
 maximum occurs when  
 $|u_i| = 1, i=1, 2, \dots, n$   
 $\text{sgn}(u_i) = \text{sgn}(z_i)$   
 $u_i = \begin{cases} +1 & \text{if } z_i \geq 0 \\ -1 & \text{if } z_i < 0 \end{cases}$   
 $= \text{sgn}(z_i)$

So the maximum occurs when  $|u_i| = 1$  for each  $i$  and  $\text{sgn}(u_i) = \text{sgn}(z_i)$  as shown in slide.

The maximum value is nothing but the  $l_1$  norm and therefore the dual norm of the infinity norm is the  $l_1$  norm.

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Handwritten slide content:

$$\boxed{\|z\|_{\infty}^* = \|z\|_1}$$

Dual Norm of  $l_1$  norm =  $l_{\infty}$  norm

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#8. DUAL of General Linear Program.

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Gx \leq h \\ & Ax = b \end{array}$$

inequality (pointing to  $Gx \leq h$ )

Equality Constraints (pointing to  $Ax = b$ )

Let us look at another problem to derive the dual optimal problem corresponding to

$$\min c^T x$$

general LP. So consider the general linear program, that is  $s.t. Gx \leq h$ . Now this is a

$$Ax = b$$

general LP implies it has inequality constraints and equality constraints.

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Linear Program

$$\begin{aligned} \min_{\bar{x}} \quad & \bar{c}^T \bar{x} \\ \text{s.t.} \quad & G \bar{x} \leq \bar{h} \\ & A \bar{x} = \bar{b} \end{aligned}$$

inequality constraints      Equality constraints

DUAL Problem

$$L(\bar{x}, \bar{\lambda}, \bar{\mu}) = \bar{c}^T \bar{x} + \bar{\lambda}^T (G \bar{x} - \bar{h}) + \bar{\nu}^T (A \bar{x} - \bar{b})$$

$\bar{\lambda} \geq 0$  (circled in red)

Lagrange multipliers for inequality constraints

So we want to find the dual problem for this, so we have  $L(\bar{x}, \bar{\lambda}, \bar{\mu}) = \bar{c}^T \bar{x} + \bar{\lambda}^T (G \bar{x} - \bar{h}) + \bar{\nu}^T (A \bar{x} - \bar{b})$ . These are the Lagrange multipliers for the both the inequality constraints and the equality constraints.

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$$\begin{aligned} L(\bar{x}, \bar{\lambda}, \bar{\mu}) &= \bar{c}^T \bar{x} + \bar{\lambda}^T (G \bar{x} - \bar{h}) + \bar{\nu}^T (A \bar{x} - \bar{b}) \\ &= \bar{x}^T \bar{c} + (\bar{x}^T G - \bar{h}^T) \bar{\lambda} + (\bar{x}^T A - \bar{b}^T) \bar{\nu} \\ &= \bar{x}^T (\bar{c} + G^T \bar{\lambda} + A^T \bar{\nu}) - (\bar{h}^T \bar{\lambda} + \bar{b}^T \bar{\nu}) \end{aligned}$$

For each inequality constraint      one for each equality constraint

Lagrange multipliers for inequality constraints

Now this is simplified as shown in the slide. So this can be simply rewritten as

$$L(\bar{x}, \bar{\lambda}, \bar{\mu}) = \bar{x}^T (\bar{c} + G^T \bar{\lambda} + A^T \bar{\nu}) - (\bar{h}^T \bar{\lambda} + \bar{b}^T \bar{\nu})$$

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Handwritten derivation on a whiteboard:

$$L(\bar{x}, \bar{\lambda}, \bar{\mu}) = \bar{x}^T (\bar{c} + G^T \bar{\lambda} + A^T \bar{\mu}) - (\bar{h}^T \bar{x} + \bar{b}^T \bar{d}) + (\bar{x}^T A^T - \bar{b}^T) \bar{d}$$

The term  $-\bar{h}^T \bar{x} - \bar{b}^T \bar{d}$  is labeled "Affine in  $\bar{x}$ ".

$$g(\bar{\lambda}, \bar{\mu}) = \min_{\bar{x}} L(\bar{x}, \bar{\lambda}, \bar{\mu})$$

$$\Rightarrow \min_{\bar{x}} \bar{c} + G^T \bar{\lambda} + A^T \bar{\mu} \quad \text{if } \bar{c} + G^T \bar{\lambda} + A^T \bar{\mu} \neq 0$$

And now we have to find  $g(\bar{\lambda}, \bar{\mu}) = \min_x L(\bar{x}, \bar{\lambda}, \bar{\mu})$ . This implies that the minimum is

$-\infty$  if  $\bar{c} + G^T \bar{\lambda} + A^T \bar{\mu} \neq 0$ .

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Handwritten notes on a whiteboard:

$$g(\bar{\lambda}, \bar{\mu}) = \min_{\bar{x}} L(\bar{x}, \bar{\lambda}, \bar{\mu})$$

$$\Rightarrow \min = -\infty \quad \text{if } \bar{c} + G^T \bar{\lambda} + A^T \bar{\mu} \neq 0$$

Also LB for original problem

But NOT very useful!

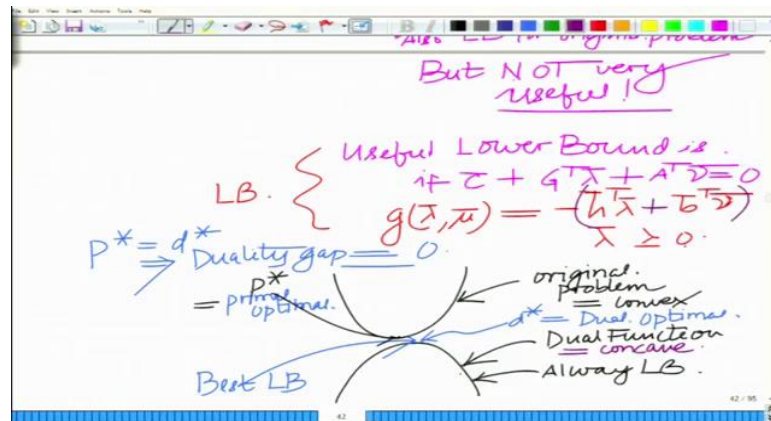
Useful Lower Bound is:  
if  $\bar{c} + G^T \bar{\lambda} + A^T \bar{\mu} = 0$

Now this is also a lower bound for the original problem, but it is not very useful. So instead we want a certain lower bound, which is more useful and that will be obtained if

$\bar{c} + G^T \bar{\lambda} + A^T \bar{\mu} = 0$ .

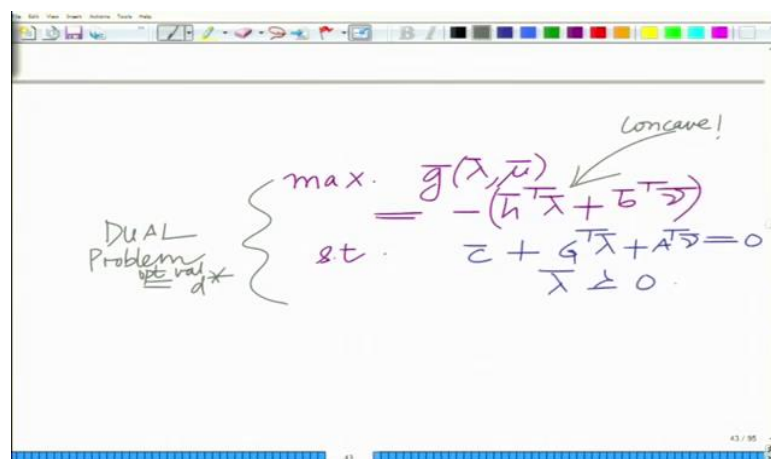


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Therefore, in this case  $g(\bar{\lambda}, \bar{\mu})$  reduces to the constant, which is  $-(\bar{h}^T \bar{\lambda} + \bar{b}^T \bar{\nu})$  and therefore this is a lower bound. This means that all the Lagrange multipliers associated with the inequality constraint have to be greater than or equal to 0 and this is a lower bound for the original optimization. So the best lower bound is the maximum value. So this is the primal optimal and this  $d^*$ , which is the dual optimal and this is what we call as the best lower bound because, it is the one that is closest to the optimum value  $p^*$  of the primal optimization problem and if  $d^* = p^*$  implies that the duality gap is 0.

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Therefore, the dual problem is basically the best lower bound, which is

$$\max g(\bar{\lambda}, \bar{\mu}) = -(\bar{h}^T \bar{\lambda} + \bar{b}^T \bar{\nu})$$

s.t.  $\bar{c} + G^T \bar{\lambda} + A^T \bar{\nu} = 0$  . So this is concave and therefore, you can find the solution  $\bar{\lambda} \geq 0$

which is the optimum value  $d^*$ , where  $d^* \leq p^*$ . But in this case  $d^*$  will be exactly equal to  $p^*$  because this is a linear program. So in general for a convex optimization problem strong duality holds implies that  $d^* = p^*$ . So we will stop here and continue with other examples in the subsequent modules. Thank you very much.