Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture – 18
Introduction to Affine functions and examples: Norm cones l2, l_P, l1, norm balls

Hello. Welcome to another module in this massive open online course. Let us discuss another important operation that preserves convexity which is known as an affine function.

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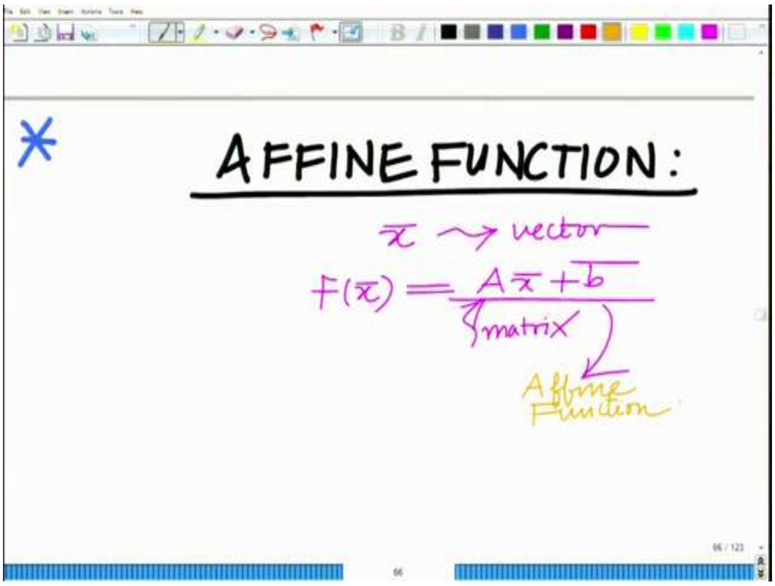

So, the next transformation that preserves convexity is known as an Affine Function. To define an affine function; take a vector $\bar{x}$. So an affine function is a function that is of the form given below.

$$F(\bar{x}) = A\bar{x} + \bar{b}$$

Here A is a matrix, $\bar{b}$ is a vector.

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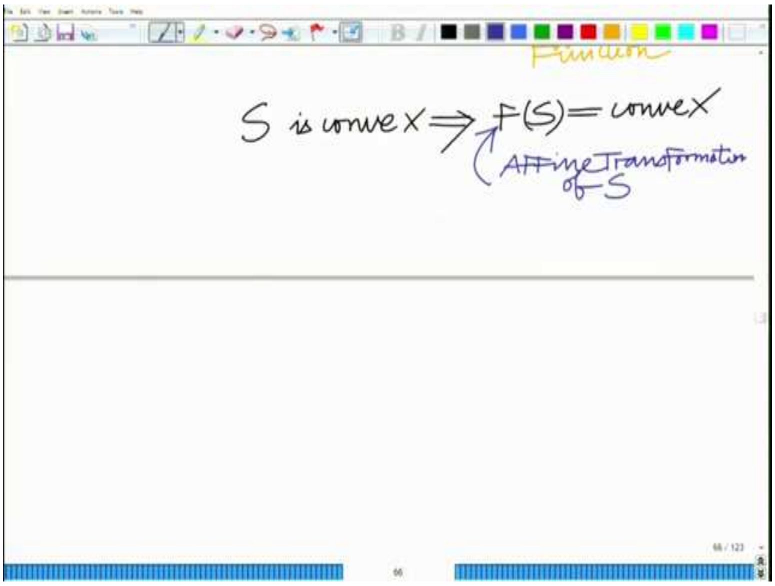

Therefore, according to the second property of convex set, if S is convex, then affine transformation of all elements in S that is F(S) is also convex. This property also includes that an affine pre composition that is $F^{-1}(S)$ also results in a convex set.

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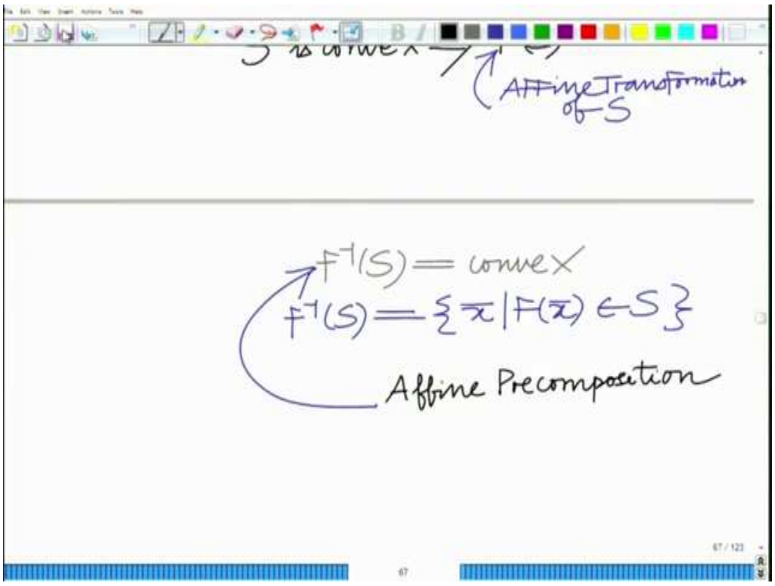

An affined pre composition is defined as follows.

$$F^{-1}(S)=\{\overline{x} \mid F(\overline{x})\in S\}$$

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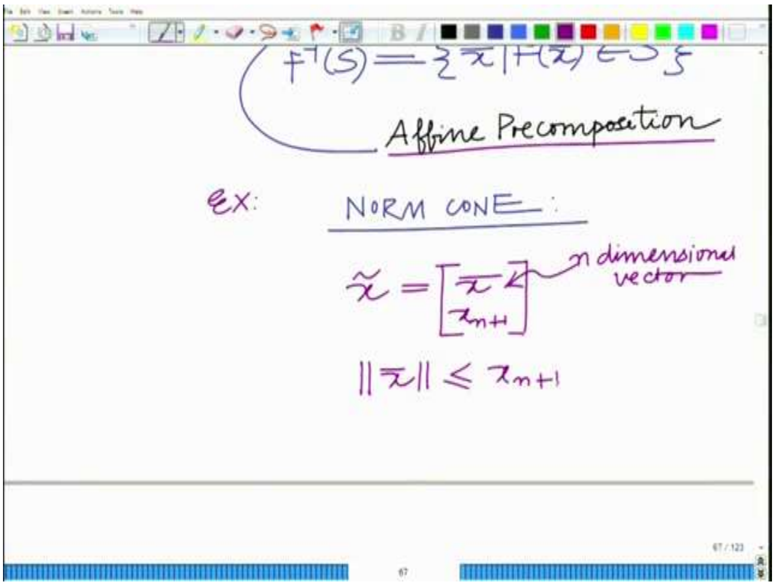

For instance, an application can be demonstrated as follows. Consider a Norm Cone $\tilde{x}$ containing n-dimensional vector $\bar{x}$ such that

$$\tilde{x} = \begin{bmatrix} \bar{x} \\ x_{n+1} \end{bmatrix}$$

As this is a norm cone, therefore

$$\|\bar{x}\| \leq x_{n+1}$$

This basically implies that

$$\bar{x}^T \bar{x} \leq x_{n+1}^2$$

This is an alternative representation of the Norm Core.

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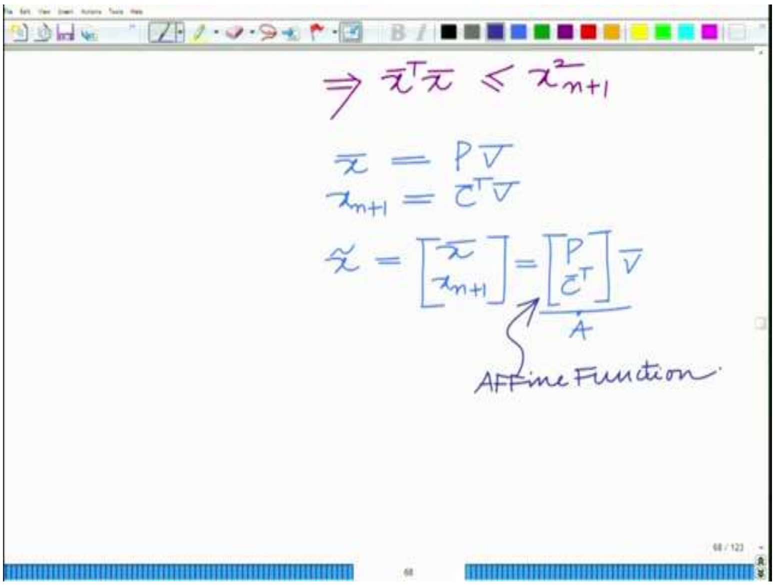

Now, let us see affine pre composition of this vector $\tilde{x}$ where $\bar{x}$ and x_{n+1} are defined as follows.

$$\bar{x} = P\bar{V}$$
$$x_{n+1} = \bar{C}^T\bar{V}$$

Therefore, $\tilde{x}$ would be defined asfollows.

$$\tilde{x} = \begin{bmatrix} P\bar{V} \\ \bar{C}^T\bar{V} \end{bmatrix} = \begin{bmatrix} P \\ \bar{C}^T \end{bmatrix}\bar{V}$$

By the definition of an affine set, the vector A is

$$A = \begin{bmatrix} P \\ \bar{C}^T \end{bmatrix}$$

And vector $\bar{b}$ is a zero vector.

So, this vector $\tilde{x}$ is an affine Function

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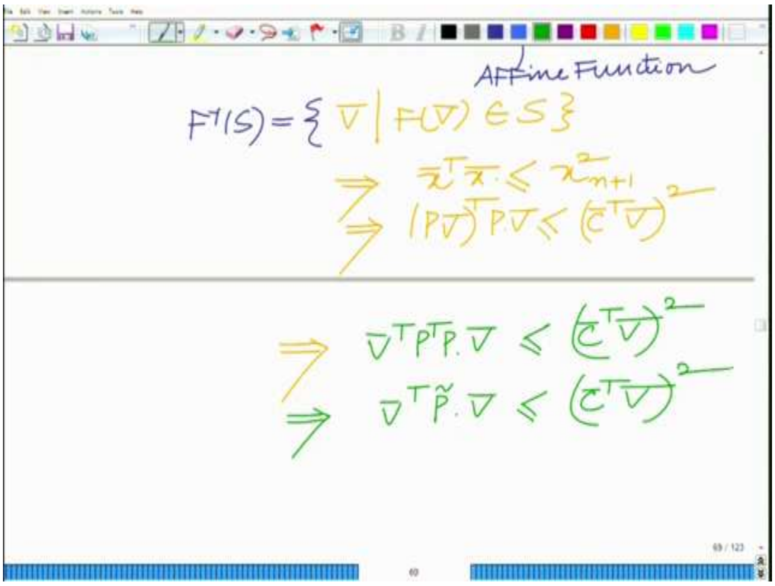

Now, to find affine pre composition of this vector $\tilde{x}$, let us start by its definition.

$$F^{-1}(S) = \{\bar{V} \mid F(\bar{V}) \in S\}$$

This simply implies that

$$\bar{x}^T \bar{x} \le x_{n+1}^2$$

$$(P\bar{V})^T P\bar{V} \le (\bar{C}^T \bar{V})^2$$

$$\bar{V}^T P^T P\bar{V} \le (\bar{C}^T \bar{V})^2$$

$$\bar{V}^T \tilde{P}\bar{V} \le (\bar{C}^T \bar{V})^2$$

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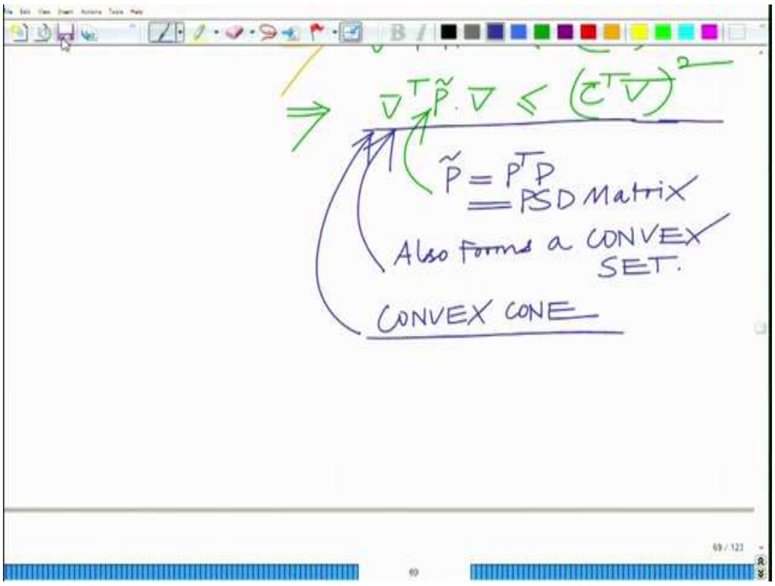

And the matrix $\tilde{P}$ is defined as

$$\tilde{P} = P^T P$$

Which means that $\tilde{P}$ is a positive semi definite matrix. Now, since $F(\bar{V})$ is the norm cone therefore vector $\bar{V}$ is the affine pre composition, and this also forms a convex set or more accurately convex cone.

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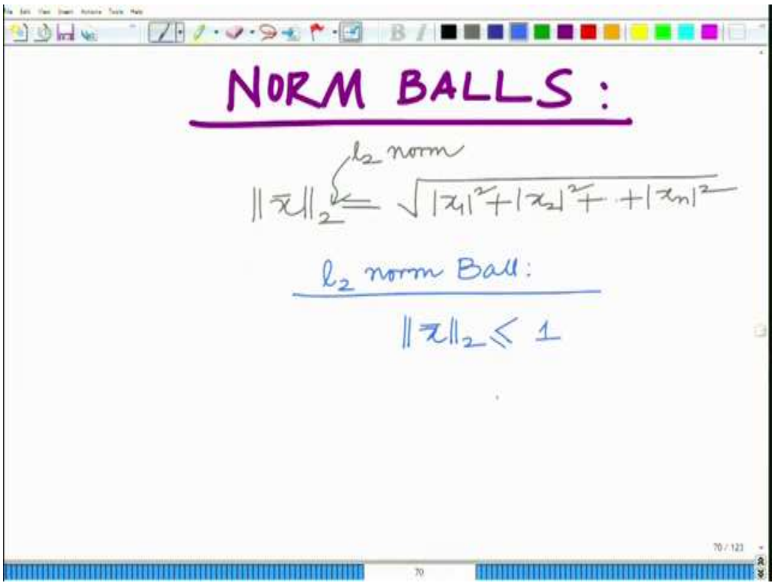

Let us move on to another interesting aspect that is the concept of Norm Ball. Remember the norm ball was defined as follows. If l_2 norm is defined as

$$\|\bar{x}\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}$$

Then the l_2 norm ball is defined as

$$\|\bar{x}\|_2 \le r$$

Where r is the radius of this norm ball. Let us say r equal to 1. Hence this norm ball is basically a circle in 2-dimensions or in n-dimensions it is a sphere.

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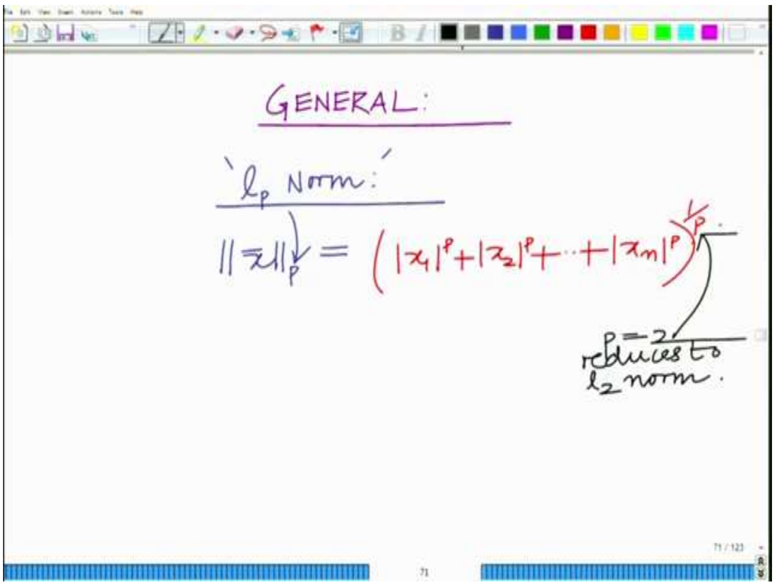

In general, one can define an l_P norm as

$$\|\bar{x}\|_P = \left(\sqrt{|x_1|^P + |x_2|^P + \cdots + |x_n|^P}\right)^{\frac{1}{P}}$$

This is the general form of norm. If P is set as 2 then it will become l_2 norm. This l_P norm can be used to construct other very interesting norm which is l_1 norm.

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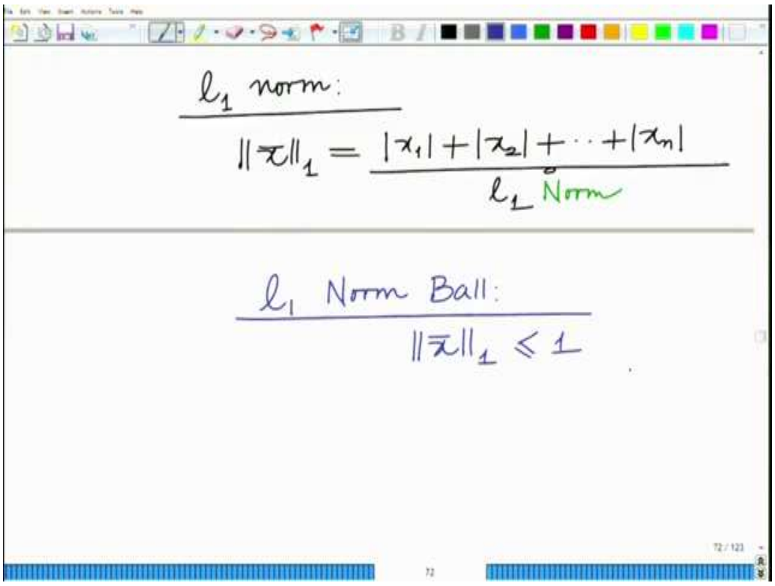

The l_1 norm is one of the most fundamental and widely applied norm. For l_1 norm, set P=1 in the above l_P norm expression.

$$\|\bar{x}\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$

And similarly, l_1 norm ball is given by

$$\|\bar{x}\|_1 \leq 1$$

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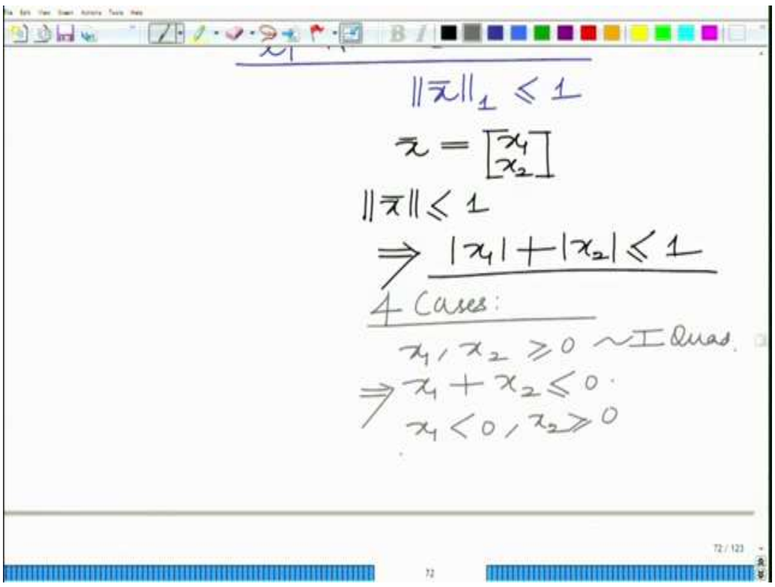

And for instance to look at this, let us consider a 2-dimensional case.

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Therefore the l_1 norm ball of this $\bar{x}$ is

$$\|\bar{x}\| \leq 1$$
$$|x_1| + |x_2| \leq 1$$

For this norm ball, one can consider four cases as;

1. $x_1 \geq 0, \;\; x_2 \geq 0$ Ist Quadrant
2. $x_1 \leq 0, \;\; x_2 \geq 0$ IInd Quadrant
3. $x_1 \leq 0, \;\; x_2 \leq 0$ IIIrd Quadrant
4. $x_1 \geq 0, \;\; x_2 \leq 0$ IVth Quadrant

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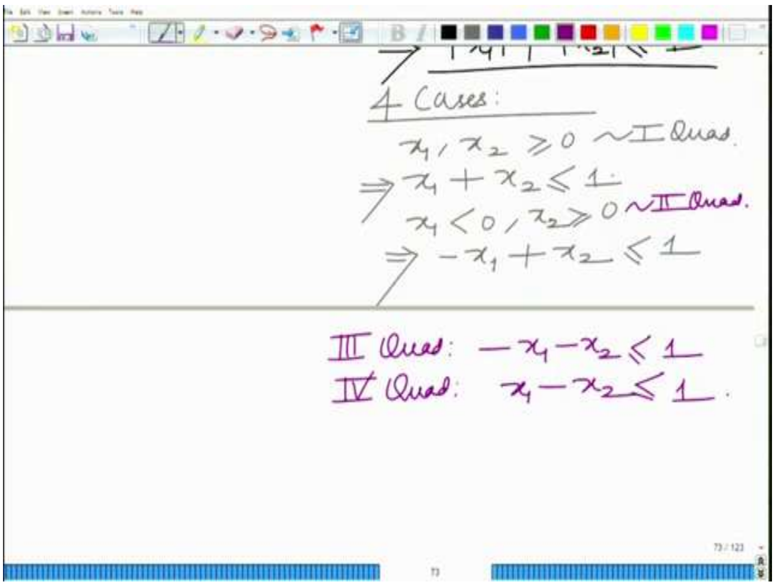

So for first quadrant,

$$x_1 + x_2 \leq 1$$

For second quadrant,

$-x_1 + x_2 \leq 1$

Then in the third quadrant,

$-x_1 - x_2 \leq 1$

And in the fourth quadrant,

$x_1 - x_2 \leq 1$

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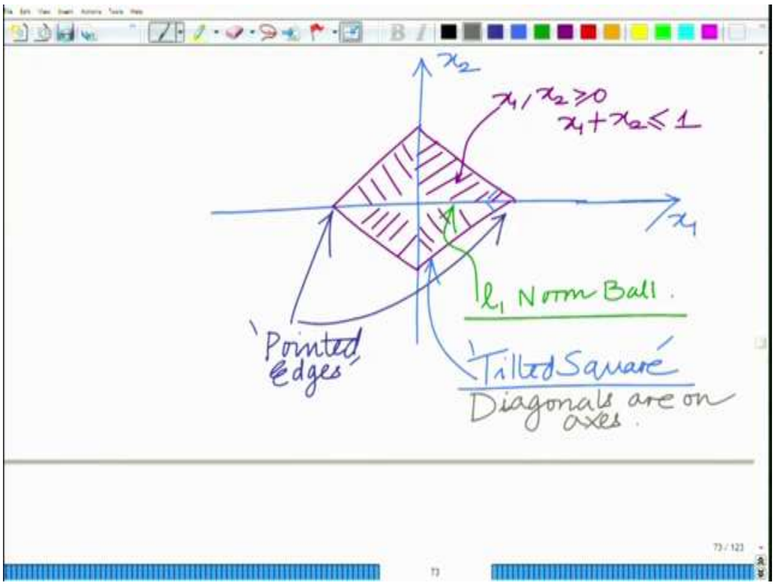

So, these are the four cases and if all these four cases of this norm ball are plotted on a graph with x_1 as x-axis and x_2 as y-axis, then one will observe that this region is a tilted square with the diagonals along the axis. The l_2 norm ball is a circle which means l_1 norm ball is very different from the l_2 norm ball in the sense that l_1 norm ball has pointed edges. This simple observation leads to the profound implications that l_1 norm ball is non-differentiable.

So, the l_2 norm is very amenable for analysis because it can be easily differentiated.

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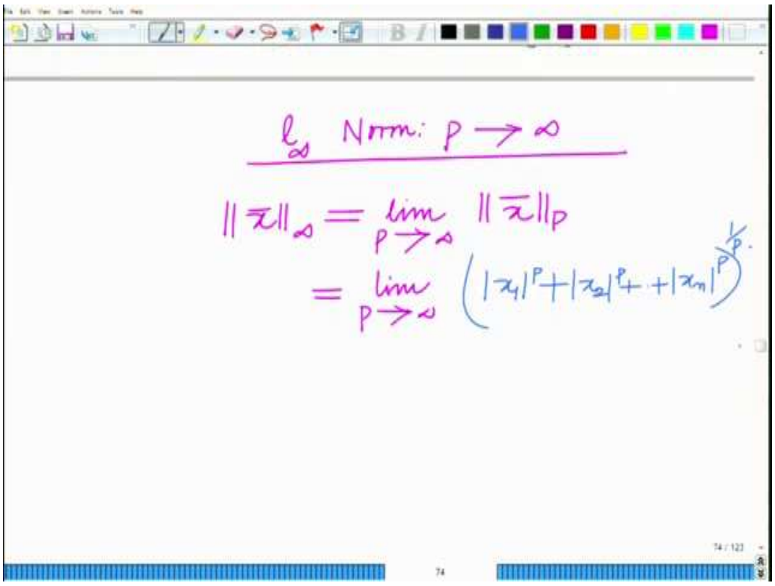

Now, if $P \to \infty$ in l_P norm then l_P norm becomes l_∞ norm which is another class of norm. Therefore

$$\|\bar{x}\|_\infty = \lim_{P \to \infty} \|\bar{x}\|_P$$

$$= \lim_{P \to \infty} \left(\sqrt{|x_1|^P + |x_2|^P + \cdots + |x_n|^P} \right)^{\frac{1}{P}}$$

$$= \max \left\{ |x_1|, |x_2|, \ldots, |x_n| \right\}$$

$$= \max \left\{ |x_i| \mid 1 \leq i \leq n \right\}$$

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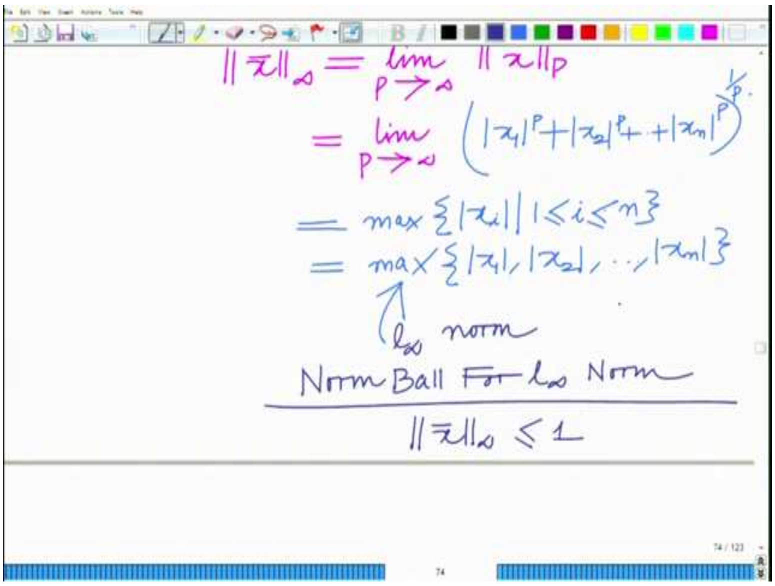

So, l_∞ norm is defined as the maximum of the absolute values of the components of that vector.

And corresponding to this, the l_∞ norm ball will be defined as

$$\|\bar{x}\|_\infty \leq r$$

The l_∞ norm ball is basically the region corresponding to the l_∞ norm of a vector being less than or equal to any radius. In a particular case, discussed above this radius is 1 so

$$\|\bar{x}\|_\infty \leq 1$$

Let us continue this discussion in the subsequent module.