Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture – 68
Optimal MIMO Power allocation(Waterfilling)-II

Keywords: *Optimal MIMO Power allocation, Waterfilling Algorithm, Singular Value Decomposition (SVD)*

Hello welcome to another module in this massive open online course. So we are looking at KKT conditions to solve an optimization problem, we have looked at a specific application of KKT conditions. Now, let us look at an example that is the application in MIMO optimal power allocation.

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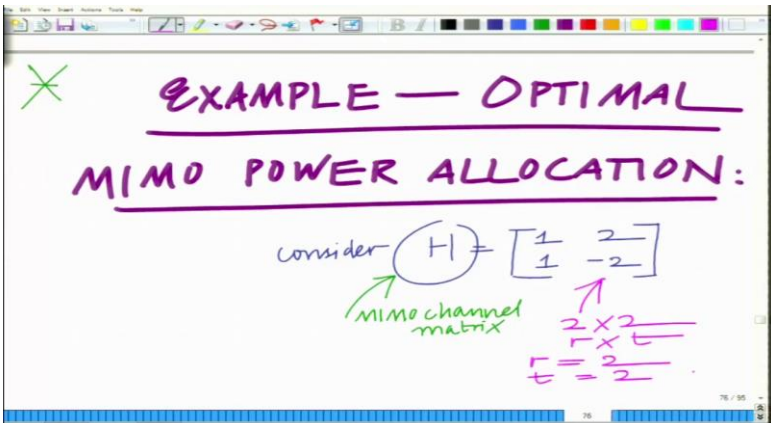

So let us consider the following MIMO system, each MIMO channel can be represented by the equivalent channel matrix. So we have the MIMO channel matrix as $H = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$, so this an $r \times t$, this implies r = 2, t = 2. So the number of receive antennas as well as the number of transmit antennas equals 2.

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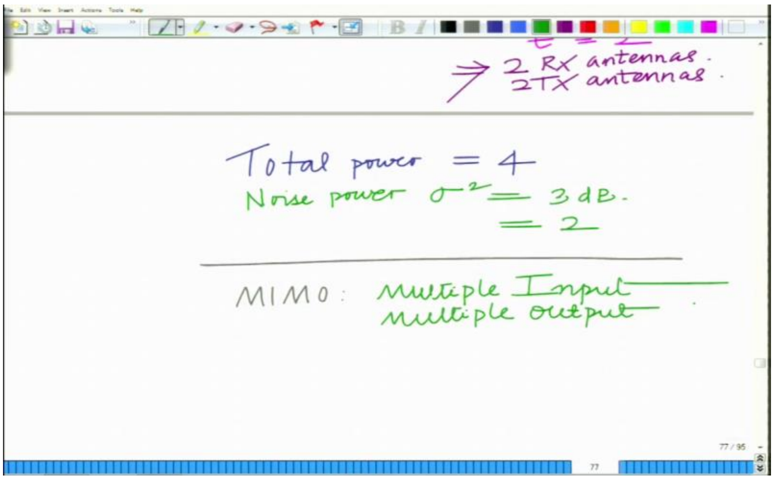

So we want to allocate power optimally to the various modes of this MIMO channel. The modes are given by the singular value decomposition. Let us consider total power = 4 and the noise power $\sigma^2 = 3dB = 2$.

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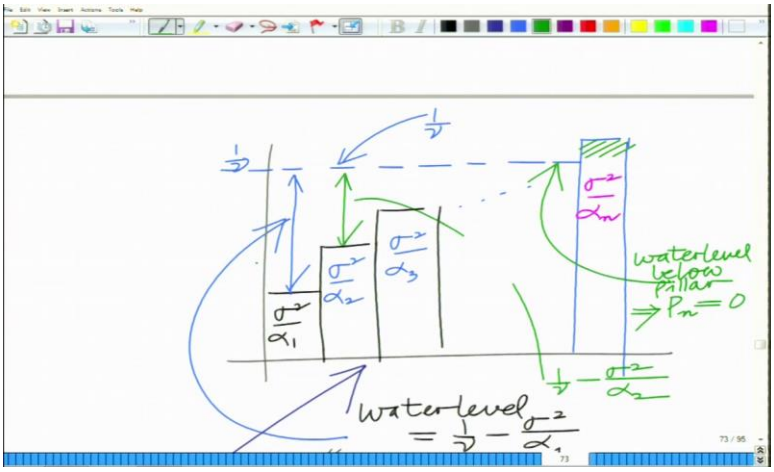

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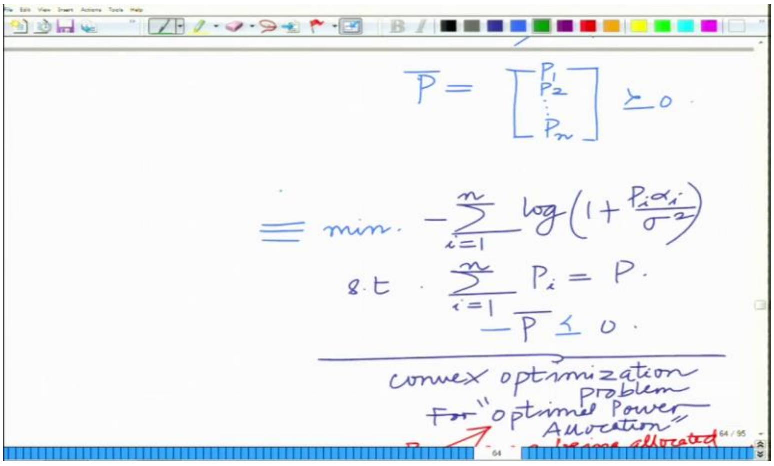

So we have
$$\max -\sum_{i=1}^{n} \log\left(1+\frac{\alpha_i P_i}{\sigma^2}\right)$$
$$s.t \sum_{i=1}^{n} P_i = P$$
$$-P_i \leq 0$$
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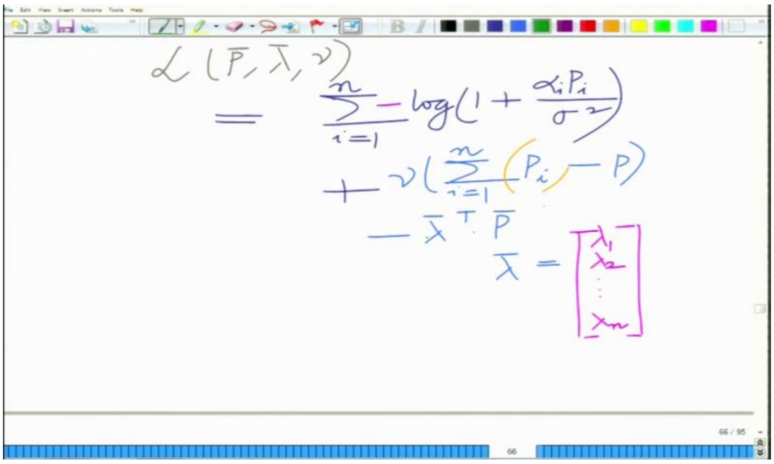

So MIMO stands for Multiple Input Multiple Output system, which means that basically you have a wireless communication system with multiple transmit antennas and multiple receive antennas.

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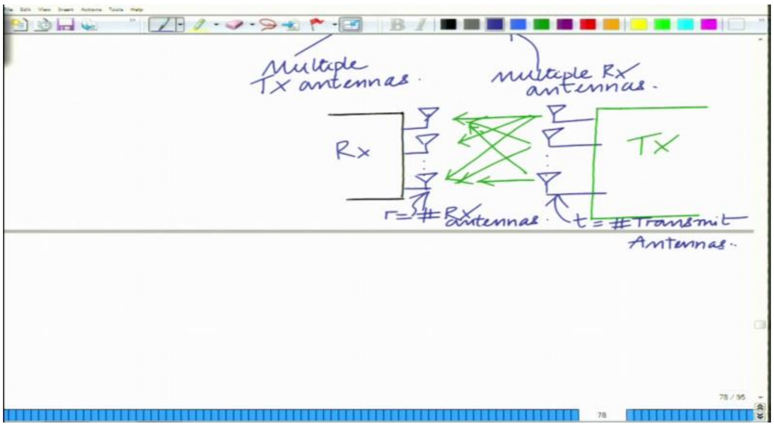

So as shown in slide, we have r receive antennas and t transmit antennas. Now as we have seen the optimal power allocation problem, we have a total transmit power P. We have a set of parallel channels and we want to allocate the power optimally amongst these parallel channels so as to maximize the total bit rate that can be transmitted across this channel. Now, for that first we have to see how this MIMO channel can be decomposed into a set of parallel channels, because only then one can talk about optimal power allocation.

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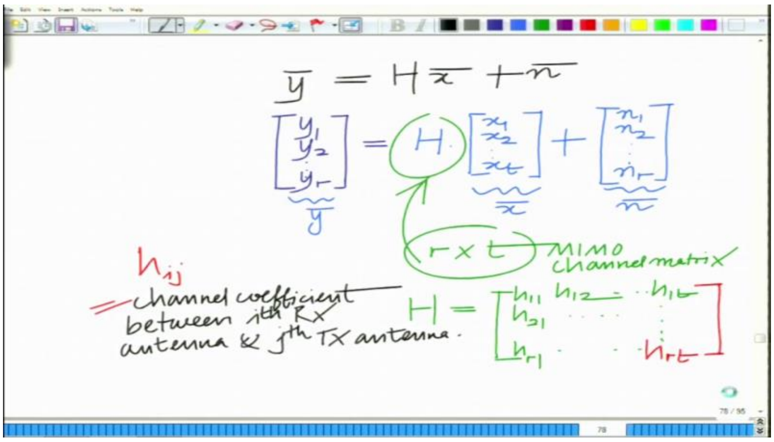

For that we have the received symbol vector $\bar{y} = H\bar{x} + \bar{n}$ for this MIMO system, where $\bar{y}$ has $y_1\ y_2 \ldots y_r$ and these are the r received symbols across the r receive antennas and $\bar{n}$ is the additive white Gaussian noise samples at the r receive antennas. This MIMO channel matrix H is of dimension $r \times t$. For this MIMO channel matrix you have the coefficients h_{11} h_{12} h_{21} so on and finally, the last row last column will be h_{rt}. So h_{ij} is the channel coefficient between i^th^ receive antenna and j^th^ transmit antenna. And now the key to understand this decomposition of MIMO channel into a set of parallel channels is the singular value decomposition.

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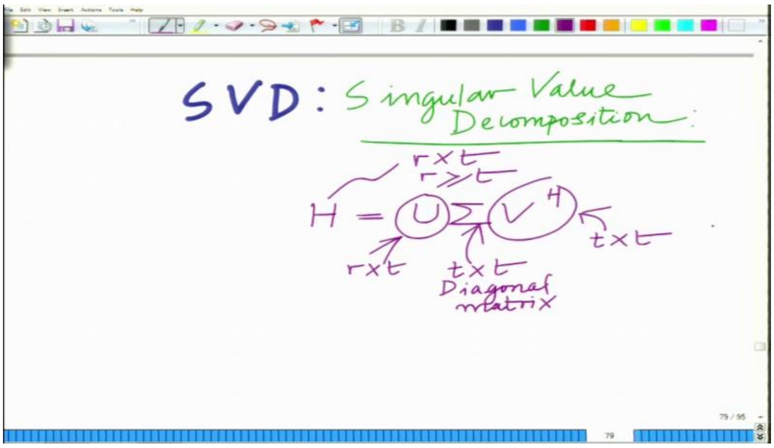

Now, in singular value decomposition given this channel matrix H, you decompose this as a product of three matrices, that is $H = U \Sigma V^H$. Consider now for the sake of simplicity this H to be an r $\times$ t matrix with $r \geq t$. Now, this matrix U is an r $\times$ t matrix, Σ is a t $\times$ t diagonal matrix and V^H is again a t $\times$ t unitary matrix.

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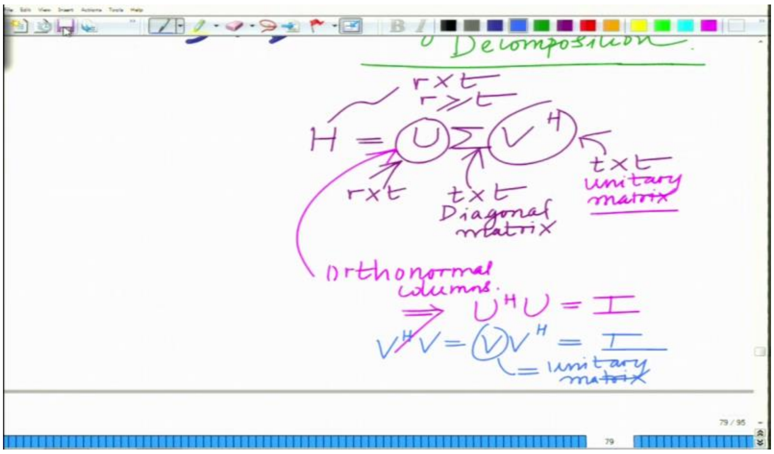

Now, U has orthonormal columns which implies that the columns are orthogonal to each other and are unit norm and therefore, if you perform $U^H U = I$. Now, V is a unitary matrix implies that $V^H V = V V^H = I$, so V is a unitary matrix.

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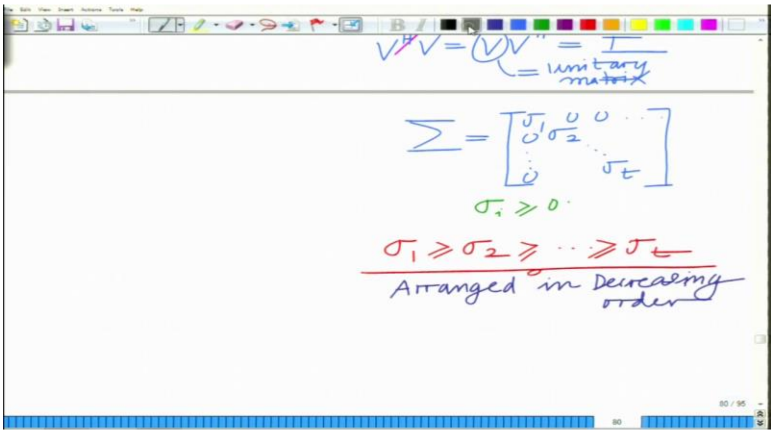

This Σ is a diagonal matrix of what are known as singular values, $\sigma_1 \geq \sigma_2 \geq .... \geq \sigma_t$ with each of this σ_i is non-negative and these are arranged in decreasing order.

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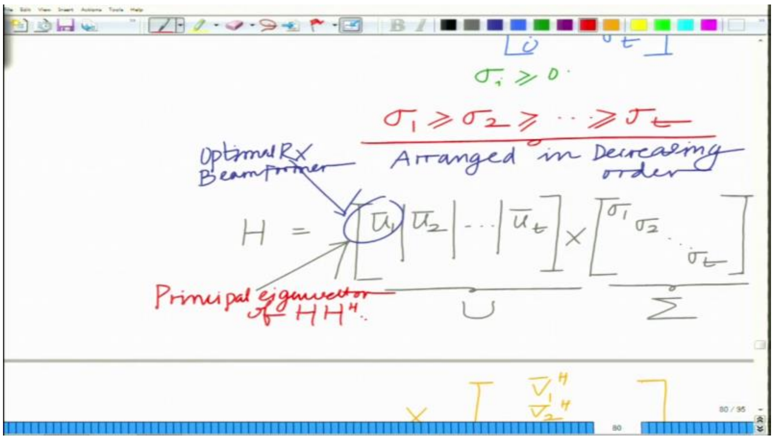

So this matrix can be written as shown in slides.

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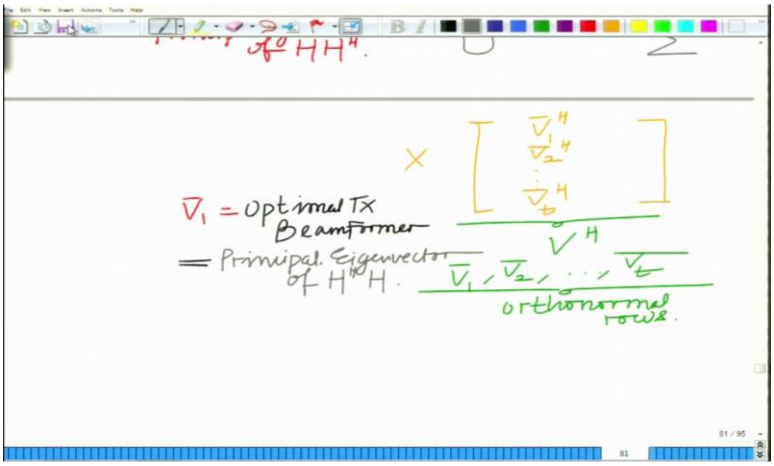

Now, we have seen that $\bar{u}_1$ is the optimal receive beam former for the MIMO system and in fact, it is the principle eigenvector that is an eigenvector corresponding to the largest singular value of HH^H and $\bar{v}_1$ is the optimal transmit beam former, that is also equal to principle eigenvector of $H^H H$.

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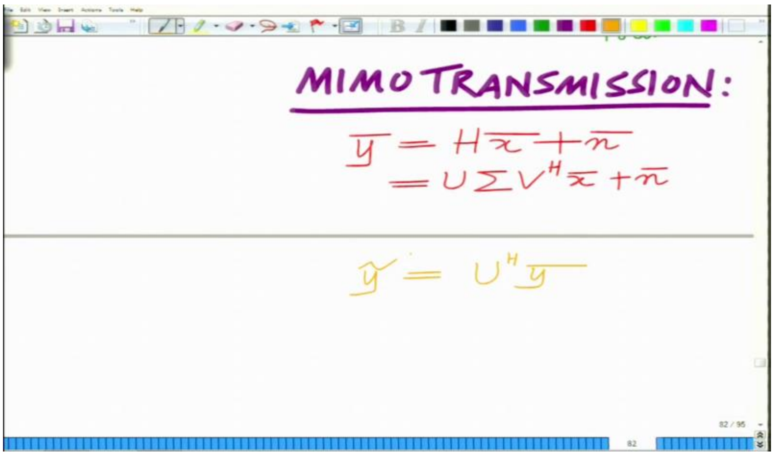

Now let us look at the following MIMO transmission scheme where you have $\bar{y} = H\bar{x} + \bar{n}$ and we have the singular value decomposition as $\bar{y} = U\Sigma V^H \bar{x} + \bar{n}$. Now the first step is, at the receiver I am going to process with U^H . So we have $y = U^H \bar{y}$ and this is the receive processing, so we have $y = U^H \left(U\Sigma V^H \bar{x} + \bar{n}\right) = \Sigma V^H \bar{x} + U^H \bar{n} = \Sigma V^H \bar{x} + n$.

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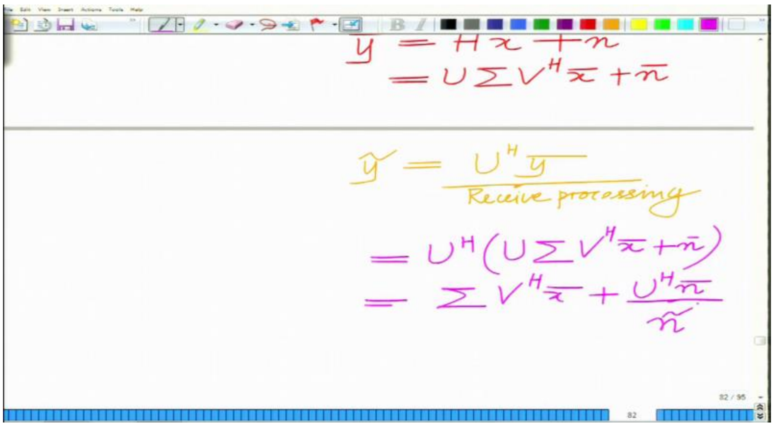

Now, we are going to do similarly at the transmitter even before transmission of $\bar{x}$, we are going to employ a preprocessing operation or this is also known as a pre coding operation. So in case of MIMO, processing can be done at both ends, one is at the transmitter and the other at the receiver.

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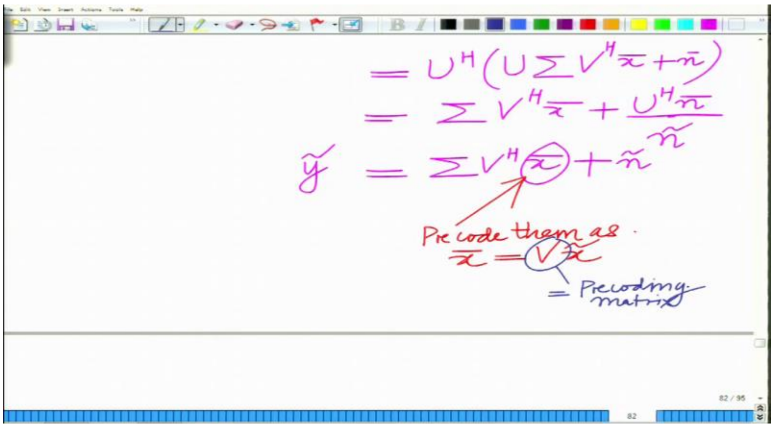

So we have $\bar{x}$ which is to be pre-coded, $\bar{x} = V x$, this is the pre coding operation.

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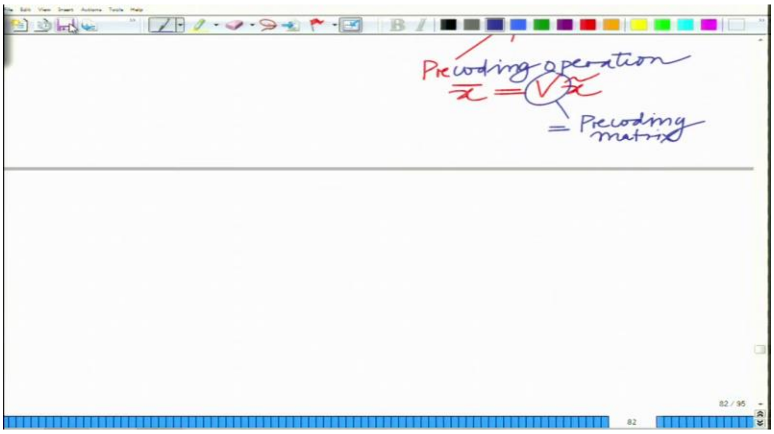

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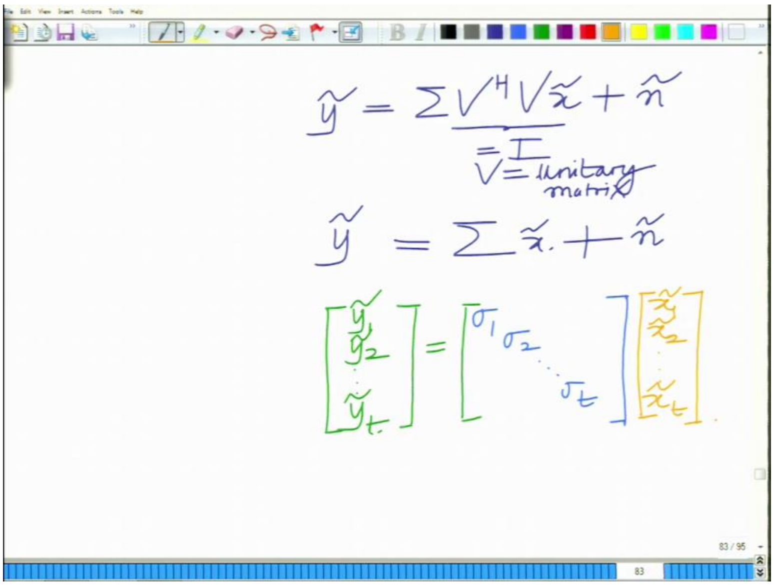

Now we have $y = \Sigma V^H V x + n = \Sigma x + n$. So this is as shown in slide.

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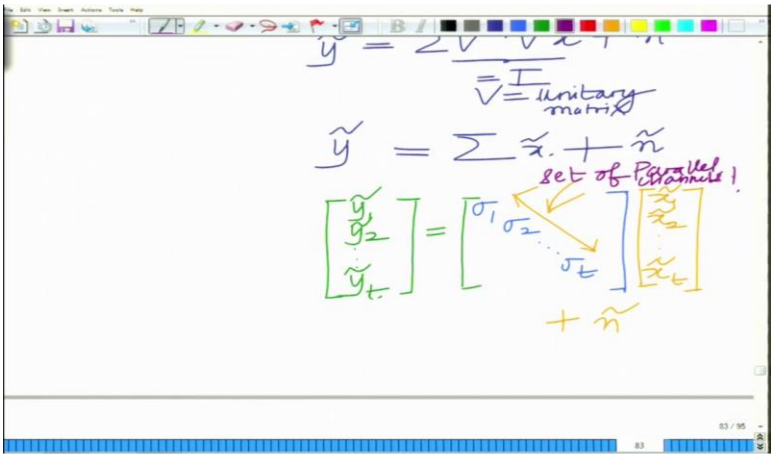

So the singular value decomposition is what gives us our set of parallel channels and this is shown in slides above.

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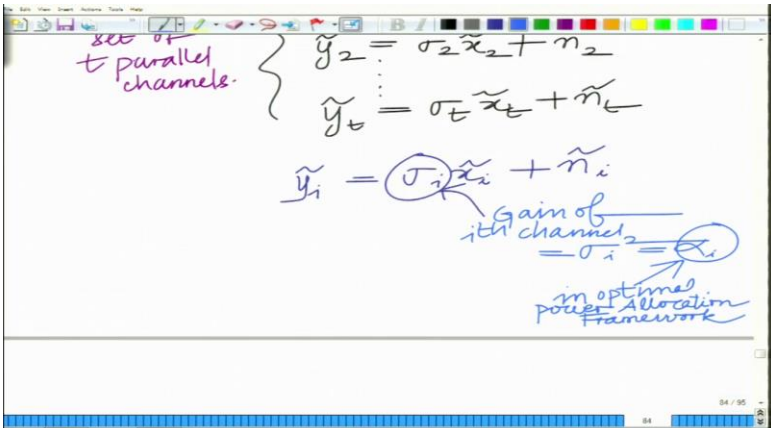

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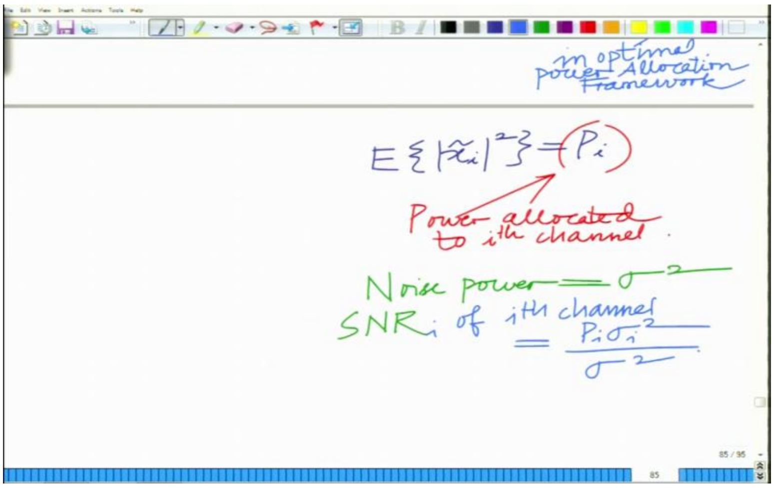

And therefore, optical power allocation can be done as shown. Now power allocated to i^{th} channel is P_i and noise power is σ^2. So SNR of i^{th} channel is $\frac{P_i \sigma_i^2}{\sigma^2}$.

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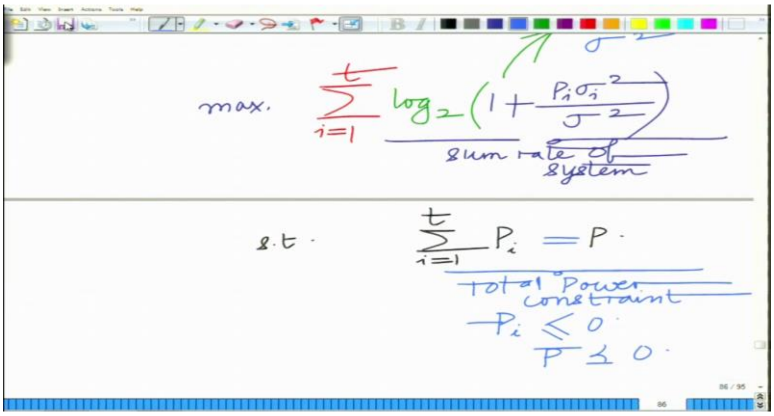

$$\max \sum_{i=1}^{t} \log\left(1 + \frac{P_i \sigma_i^2}{\sigma^2}\right)$$

The sum rate will be $s.t \sum_{i=1}^{t} P_i = P$.

$$-P_i \le 0$$

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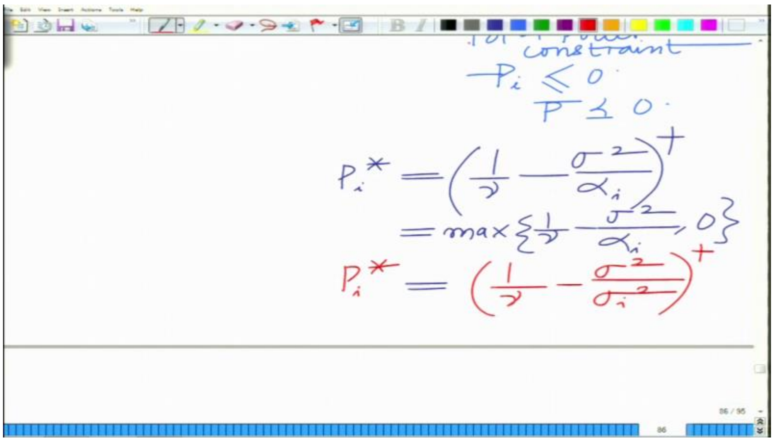

And we have already solved this problem and $P_i^* = \left(\frac{1}{\nu} - \frac{\sigma^2}{\sigma_i^2}\right)^+$, this is given by the water filling power allocation. This is the optimal power allocated to the i^{th} mode of the MIMO channel.

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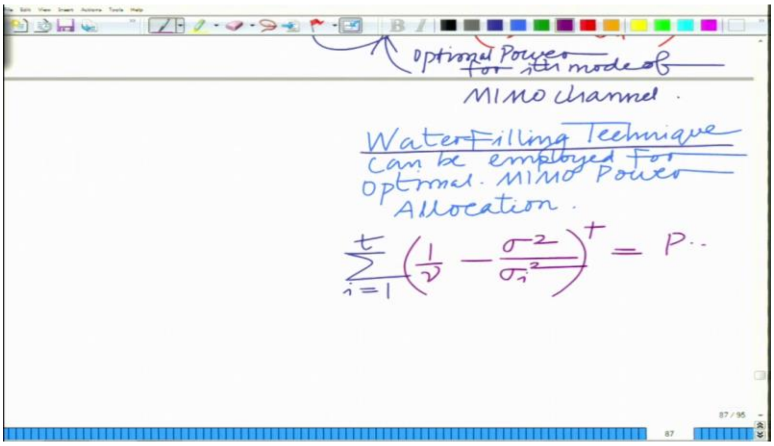

So water filling technique can be used for optimal MIMO power allocation and these parallel channels are already arranged in the decreasing order. So naturally first channel is allocated a larger fraction of the power compared to the next channels. And there might be some channels which are below the water level and which are not allocated any power.

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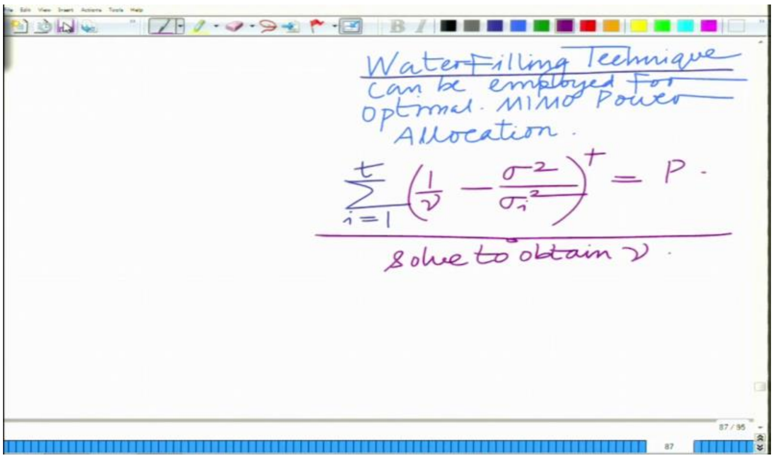

So we will stop here and continue in the next lecture. Thank you very much.