

Applied Optimization for Wireless, Machine Learning, Big data
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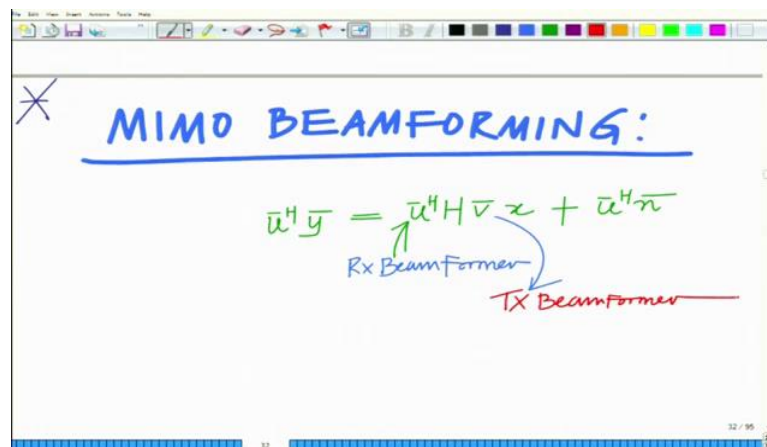
Lecture - 51

Practical Application: Multiple Input Multiple Output (MIMO) Beamformer Design

Keywords: MIMO Beamformer Design, Transmit beamformer, Receive beamformer, Principal Eigenvector

Hello welcome to another module in this massive open online course, so we are looking at MIMO beam forming, how to design the optimal transmit and receive Beamformers in a multiple input multiple output wireless communication system.

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MIMO BEAMFORMING:

$$\bar{u}^H \bar{y} = \bar{u}^H H \bar{v} x + \bar{u}^H \bar{n}$$

Rx Beamformer Tx Beamformer

So at receive beamformer we have $\bar{u}^H \bar{y} = \bar{u}^H H \bar{v} x + \bar{u}^H \bar{n}$ where \bar{u} is the receive beamformer and \bar{v} is the transmit beam former and we want to jointly design the transmit and the receive beamformers to maximize the SNR. Now we set $H \bar{v} = \bar{h}$.

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$$H \bar{v} = \bar{h}$$

$$= \bar{u}^H \bar{h} x + \bar{u}^H \bar{n}$$

effectively becomes multiple Rx antenna system

Optimal RX Beamformer

$$\bar{u} = \frac{\bar{h}}{\|\bar{h}\|}$$

Maximal Ratio combiner

So this becomes $\bar{u}^H \bar{h} x + \bar{u}^H \bar{n}$. Now this effectively becomes a single input multiple output system or simply a multiple receiver antenna system for which we already know the optimal beamformer, $\bar{u} = \frac{\bar{h}}{\|\bar{h}\|}$ which is the maximal ratio combiner.

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Maximal Ratio combiner

output SNR = $\|\bar{h}\|^2 \left(\frac{P}{\sigma^2} \right)$

constant

In order to maximize output SNR, maximize $\|\bar{h}\|^2$

$\bar{h} = H \bar{v}$

\Rightarrow maximize $\|H \bar{v}\|^2$

And the output SNR of the maximal ratio combiner is given as $\|\bar{h}\|^2 \frac{P}{\sigma^2}$ where P is the transmit power and σ^2 is the noise power. So to maximize the SNR we have to maximize $\|\bar{h}\|^2$. But $\bar{h} = H \bar{v}$. So substituting this we have to maximize $\|H \bar{v}\|^2$. Now $\|H \bar{v}\|^2 = \bar{v}^H H^H H \bar{v} = \bar{v}^H G \bar{v}$ where $G = H^H H$.

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$$= \mathbf{v}^H \mathbf{H}^H \mathbf{H} \mathbf{v}$$

$$= \mathbf{v}^H \mathbf{G} \mathbf{v}$$

maximize.

$$\mathbf{G} = \mathbf{H}^H \mathbf{H}$$

$$\max. \mathbf{v}^H \mathbf{G} \mathbf{v}$$

$$\text{s.t. } \|\mathbf{v}\| \leq 1$$

To simplify, consider real vectors.

$$\max. \mathbf{v}^T \mathbf{G} \mathbf{v}$$

$$\text{s.t. } \|\mathbf{v}\| \leq 1$$

So the resulting problem for optimal beam forming becomes $\max_{\mathbf{v}} \mathbf{v}^H \mathbf{G} \mathbf{v}$. And just to $\text{s.t. } \|\mathbf{v}\| \leq 1$

simplify it we are going to assume real vectors and thus it becomes $\max_{\mathbf{v}} \mathbf{v}^T \mathbf{G} \mathbf{v}$ and this is $\text{s.t. } \|\mathbf{v}\| \leq 1$

a non-convex problem.

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To simplify, consider real vectors.

$$\max. \mathbf{v}^T \mathbf{G} \mathbf{v}$$

$$\text{s.t. } \|\mathbf{v}\| \leq 1$$

non-convex

$\mathbf{G} = \text{PSD matrix}$

$\mathbf{v}^T \mathbf{G} \mathbf{v} = \text{convex}$

However,

min. convex objective

⇒ non-convex

In case of a standard form convex optimisation problem, we have a convex objective, but we are minimising it, here we are maximizing it and hence it becomes non-convex. In fact you are minimising over a convex objective function implies that this is non-convex and therefore we will form the Lagrangian in the same way that we have done before.

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Handwritten derivation on a whiteboard:

$$\begin{aligned}
 &\Rightarrow \text{non-convex} \\
 &f(\bar{v}, \lambda) \\
 &= \bar{v}^T G \bar{v} + \lambda(1 - \|\bar{v}\|^2) \\
 &= \bar{v}^T G \bar{v} + \lambda(1 - \bar{v}^T \bar{v}) \\
 &\nabla_{\bar{v}} f = 2G\bar{v} + 0 - \lambda \cdot 2\bar{v} = 0
 \end{aligned}$$

So this is $f(\bar{v}, \lambda) = \bar{v}^T G \bar{v} + \lambda(1 - \|\bar{v}\|^2)$ and this can be solved as shown in slide.

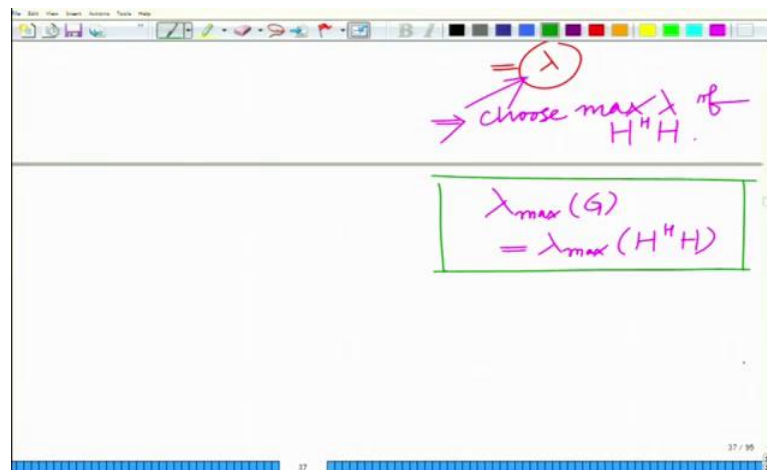
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Handwritten derivation on a whiteboard:

$$\begin{aligned}
 &\Rightarrow \boxed{G\bar{v} = \lambda \bar{v}} \\
 &\bar{v} \text{ — eigenvalue of } G = H^T H \\
 &\text{How to find } \lambda? \\
 &\bar{v}^T G \bar{v} \\
 &= \bar{v}^T \lambda (\bar{v}) \\
 &= \lambda \cdot \|\bar{v}\|^2
 \end{aligned}$$

Now we get $G\bar{v} = \lambda \bar{v}$ and this equation is nothing but the definition of the eigenvector of G . So this has many eigenvalues and we need to find the Lagrange multiplier as shown.

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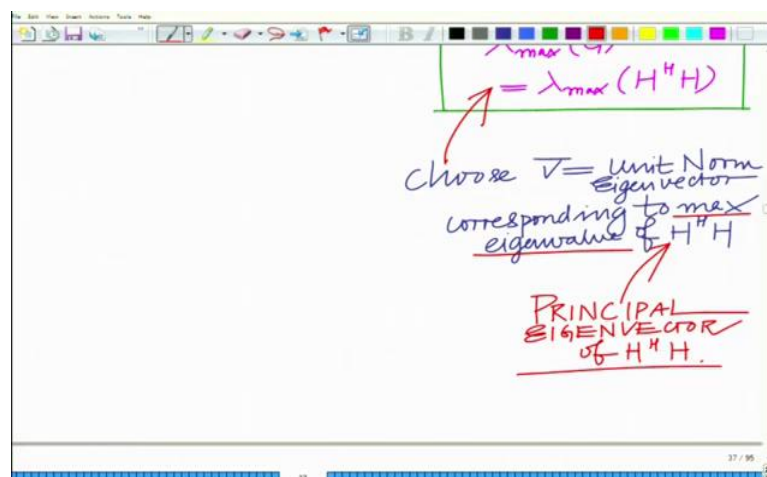


\Rightarrow choose $\max \lambda$ of $H^H H$.

$\lambda_{\max}(G)$
 $= \lambda_{\max}(H^H H)$

So we want to maximize this, implies choose the eigenvector corresponding to maximum eigenvalue of $H^H H$. So the transmit beamformer corresponds to the maximum eigenvalue of $H^H H$ and this maximizes the SNR at the receiver.

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$\lambda_{\max}(G)$
 $= \lambda_{\max}(H^H H)$

Choose $V =$ Unit Norm Eigenvector corresponding to max eigenvalue of $H^H H$

PRINCIPAL EIGENVECTOR of $H^H H$.

So this Eigen vector corresponding to the largest eigenvalue is termed as the principal eigenvector.

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Handwritten derivation on a whiteboard:

$$\alpha = \frac{H^H H \alpha}{\|H^H H \alpha\|} \leftarrow \text{ignore}$$
$$\tilde{\alpha} = H^H H \alpha$$
$$H H^H \tilde{\alpha} = H (H^H H \alpha)$$
$$= H \lambda \alpha$$
$$= \lambda H \alpha$$
$$\boxed{H H^H \tilde{\alpha} = \lambda \tilde{\alpha}}$$

$\Rightarrow \tilde{\alpha} = \text{Principal Eigenvector of } H H^H$

Now the receive beam former is also found as shown in slide and we can see that the receive beam former is the eigenvector corresponding to largest eigenvalue of $H H^H$.

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Handwritten derivation on a whiteboard:

$$\boxed{H H^H \tilde{\alpha} = \lambda \tilde{\alpha}}$$

$\Rightarrow \tilde{\alpha} = \text{Principal Eigenvector of } H H^H$

$$\bar{\alpha} = \frac{\tilde{\alpha}}{\|\tilde{\alpha}\|}$$

\downarrow

$= \text{Principal eigenvector of } H H^H \text{ with unit-norm}$

And that basically gives us both the transmit and receive beamformers.

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Handwritten equations and diagram illustrating beamforming in a MIMO system:

$$\begin{aligned}\bar{y} &= H\bar{x} + \bar{n} \\ \bar{y} &= H \cdot \bar{V}x + \bar{n} \\ \bar{u}^H \bar{y} &= \bar{u}^H H \bar{V}x + \bar{u}^H \bar{n}\end{aligned}$$

The diagram shows a transmitter (Tx) and a receiver (Rx) antenna arrays. A signal path is shown from Tx to Rx, with arrows indicating 'Transmit steering' at the Tx and 'Steering' at the Rx.

And we have to perform beamforming at both the ends in the MIMO system. So you have the transmitter, you have the receiver, you are transmitting from the transmitter in a particular direction, at the receiver you are also collecting or processing the signal, you are steering the receiver antenna array in a particular direction. All these are electronic steering, so you do not need to physically steer.

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Handwritten definitions and relationships:

- \bar{u} = Principal Eigenvector of $H H^H$.
- \bar{v} = Principal Eigenvector of $H^H H$.
- \bar{u}, \bar{v} are Dominant singular vectors of H . From SVD = Singular value decomposition.

So \bar{u} is the principal eigenvector of $H H^H$ and \bar{v} is the principal eigenvector of $H^H H$. Now later when we do the singular value decomposition of the channel matrix it will turn out that \bar{u} and \bar{v} are in fact the dominant left and right singular vectors corresponding to the larger singular value.

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Corresponding to
Largest Singular value

$$\begin{aligned} \max. \quad & \bar{x}^T A \bar{x} \\ \text{s.t.} \quad & \|\bar{x}\| \leq 1 \end{aligned}$$

$A = \text{PSD}$
Positive Semi-Definite matrix

$\bar{x} = \text{Principal Eigenvector of } A$

$$\begin{aligned} \min \quad & \bar{x}^T A \bar{x} \\ & \|\bar{x}\| \geq 1 \\ & \bar{x} = \text{Eigenvector} \end{aligned}$$

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$\max. \quad \bar{x}^T A \bar{x}$
 $\text{s.t.} \quad \|\bar{x}\| \leq 1$

$\bar{x} = \text{Principal Eigenvector of } A$

$$\begin{aligned} \min \quad & \bar{x}^T A \bar{x} \\ & \|\bar{x}\| \geq 1 \\ & \bar{x} = \text{Eigenvector} \\ & \text{Corresponding to smallest Eigenvalue.} \end{aligned}$$

So this is a very interesting application with respect to beam forming in MIMO systems.

So we will stop here and continue in the subsequent modules. Thank you very much.