Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture - 72
Examples on Quadratic Optimization

Keywords: *Quadratic Optimization*

Hello, welcome to another module in this massive open online course. So we are looking at example problems for Convex Optimization. Let us look at another problem that is Quadratic Optimization.

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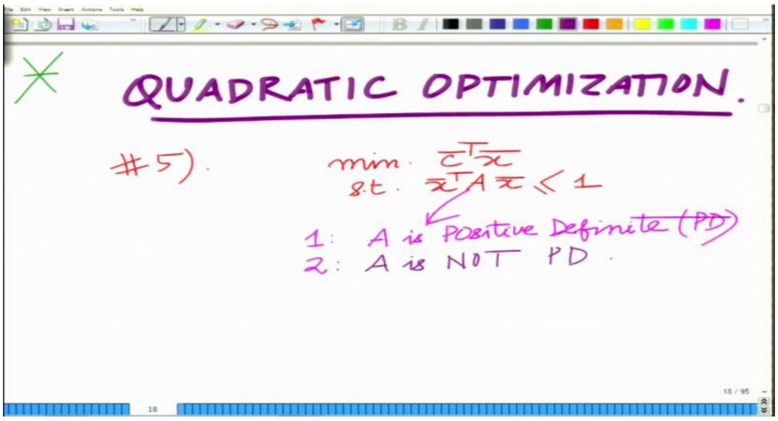

So the quadratic optimization objective is as follows $\begin{array}{l} \min \bar{c}^T \bar{x} \\ s.t\ \bar{x}^T A \bar{x} \le 1 \end{array}$ and we will consider two cases for this, that is when A is positive definite and when A is not positive definite. Let us start with case 1, A is positive definite.

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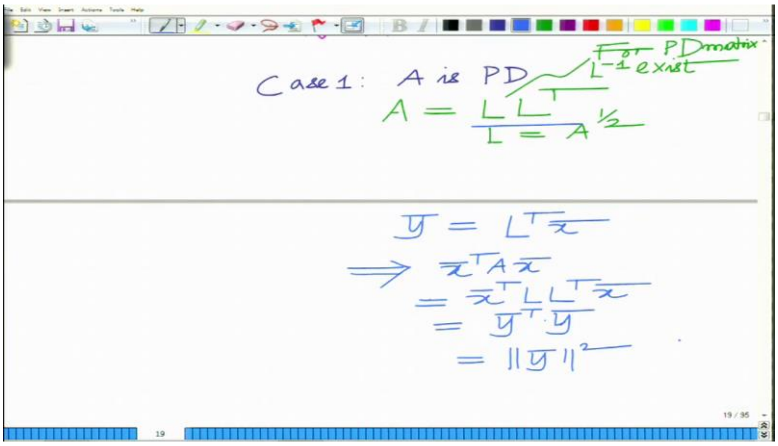

When A is positive definite, you can write $A = LL^T$ where $L = A^{\frac{1}{2}}$, this is obtained by the Cholesky decomposition. So for a positive definite matrix in addition, this L is invertible. So we have $\bar{y} = L^T \bar{x}$ and $\bar{x}^T A \bar{x} = \bar{x}^T L L^T \bar{x} = \bar{y}^T \bar{y} = \left\|\bar{y}\right\|^2$.

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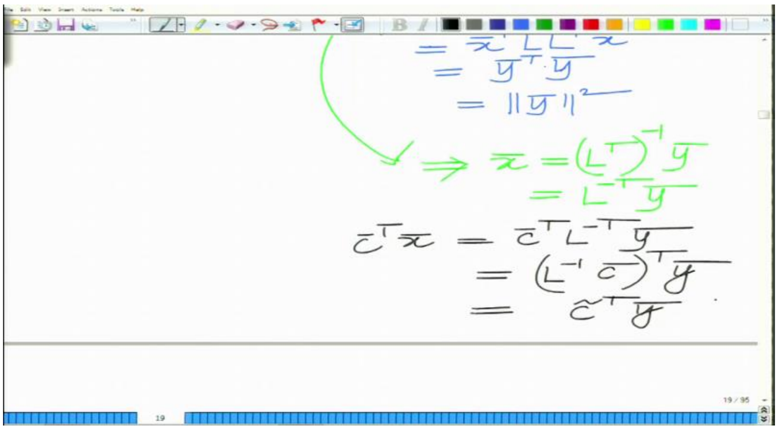

Now as shown in slide we have $\bar{x} = \left(L^T\right)^{-1} \bar{y} = L^{-T} \bar{y}$. Therefore, $\bar{c}^T \bar{x} = \bar{c}^T L^{-T} \bar{y} = \left(L^{-1} \bar{c}\right)^T \bar{y} = \tilde{c}^T \bar{y}$.

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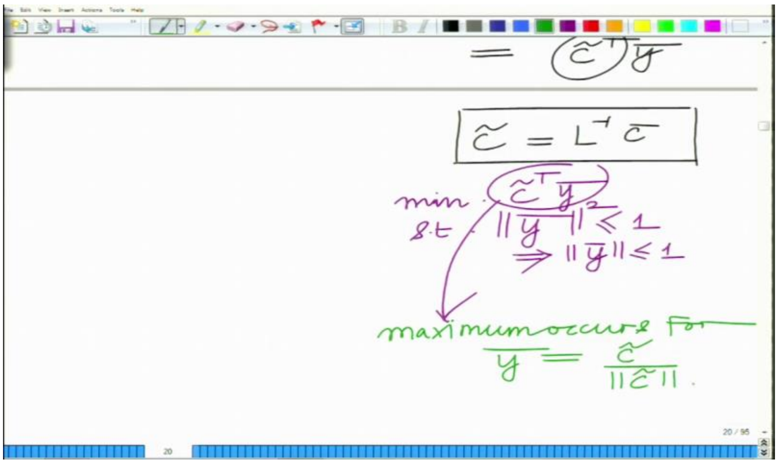

So we can write the optimization problem as $\begin{array}{l} \min \tilde{c}^T \bar{y} \\ s.t \|\bar{y}\| \le 1 \end{array}$. Now the maximum occurs for $\bar{y} = \dfrac{\tilde{c}}{\|\tilde{c}\|}$ which is given as shown in the following slides.

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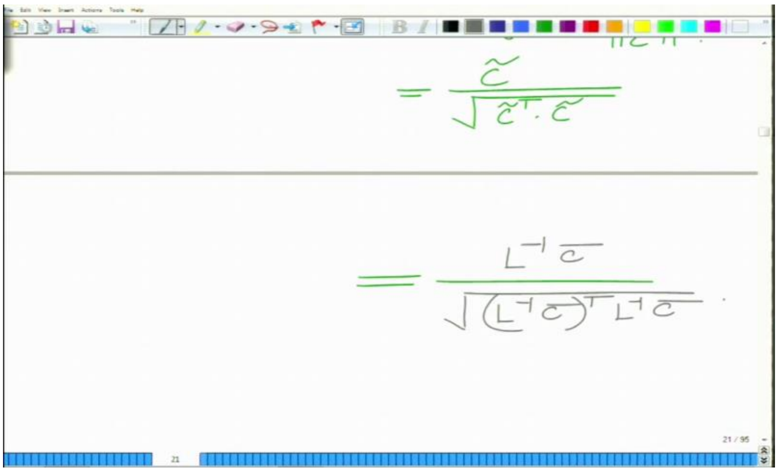

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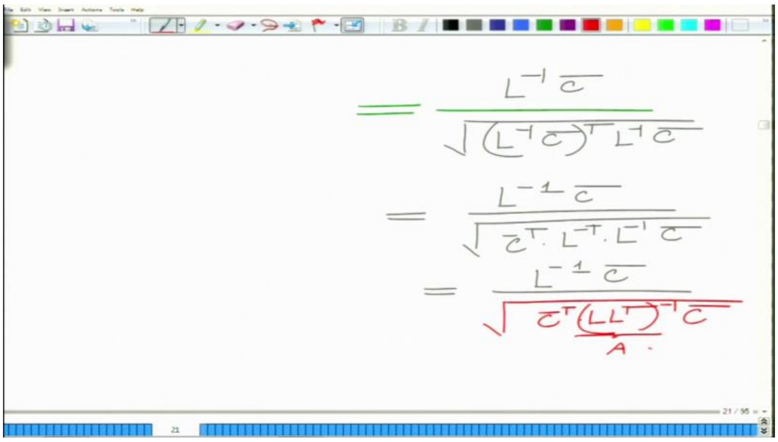

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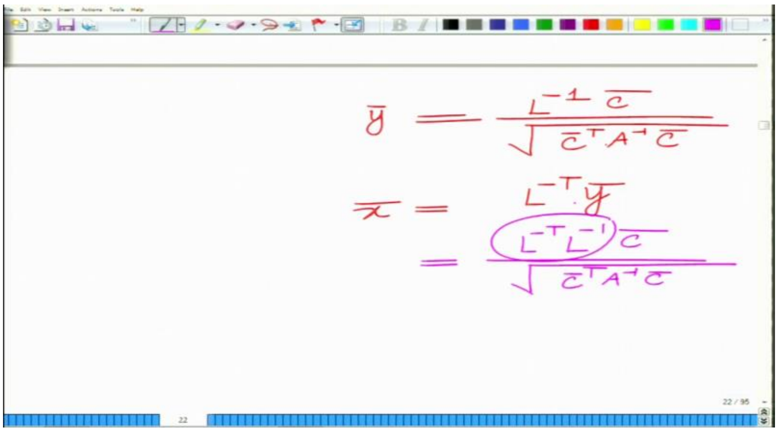

So you have the optimal $\bar{y} = \dfrac{L^{-1}\bar{c}}{\sqrt{\bar{c}^{T} A^{-1}\bar{c}}}$ and the optimal $\bar{x} = \dfrac{A^{-1}\bar{c}}{\sqrt{\bar{c}^{T} A^{-1}\bar{c}}}$ as shown in slides.

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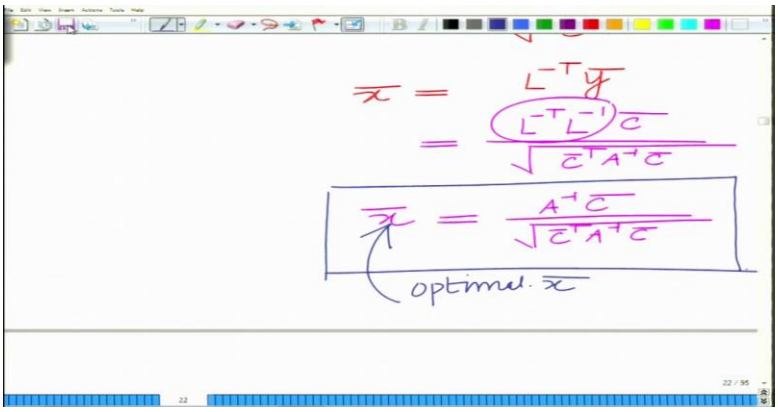

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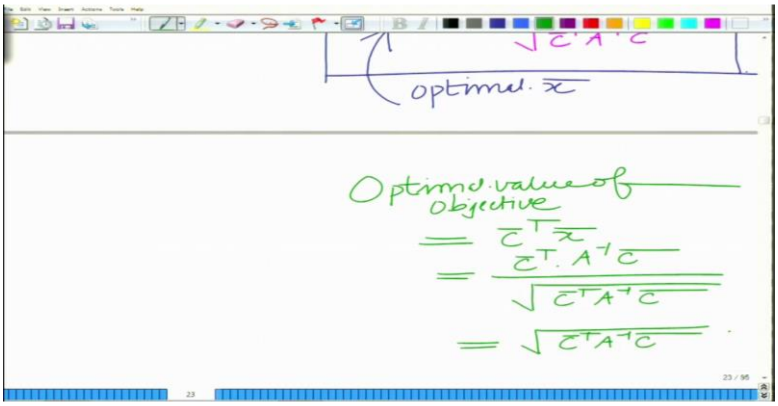

So the optimal value of objective equals $\sqrt{\bar{c}^{T} A^{-1} \bar{c}}$ as shown in slide.

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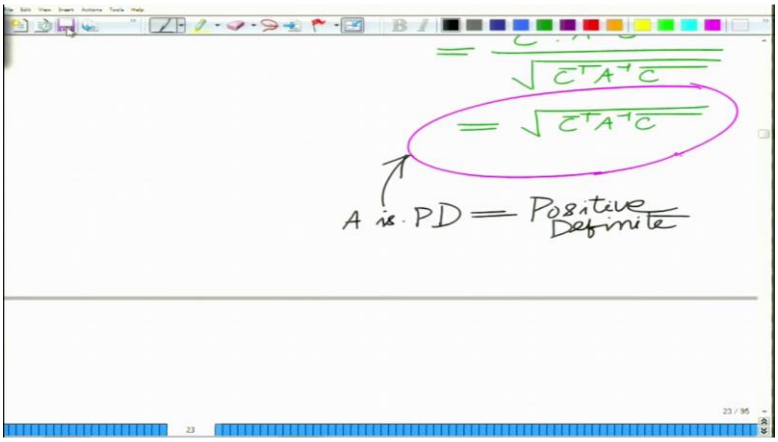

But remember that this entire case is when A is PD. Now when A is not positive definite, then it cannot be decomposed as LL^{T}.

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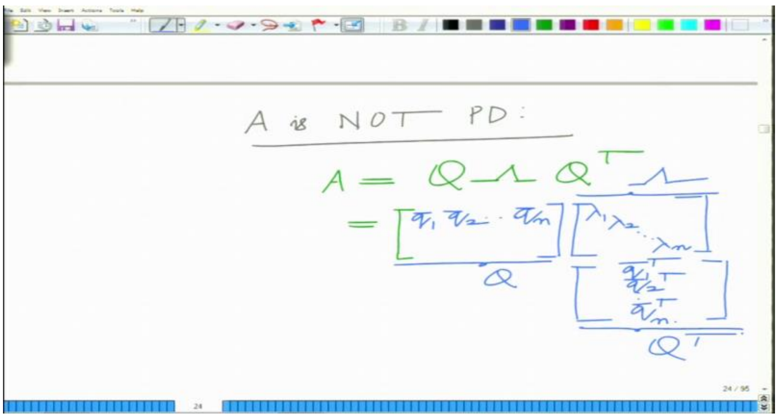

So in that scenario let us say you have an eigenvalue decomposition of A which is $A = Q \Lambda Q^T$. So this can be written as a matrix of eigenvectors as shown in slide.

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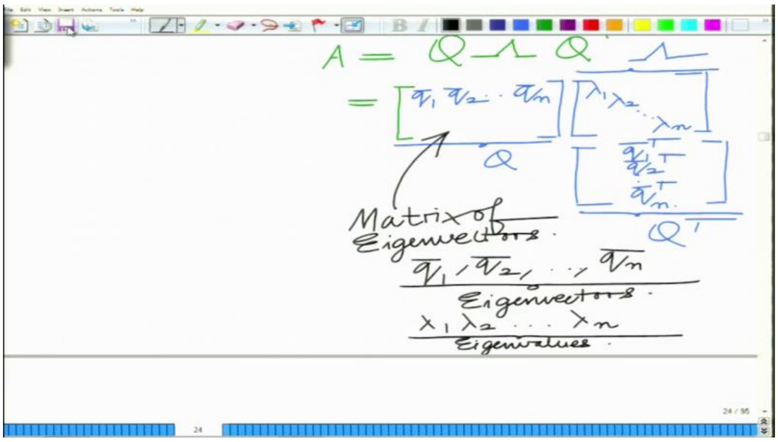

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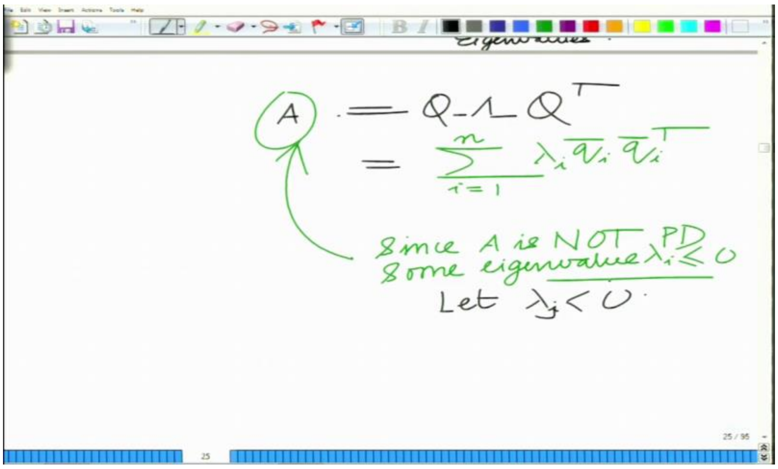

Now you can multiply it out and you can write this as $A = \sum_{i=1}^{n} \lambda_i \bar{q}_i \bar{q}_i^T$. Now, since A is not PD, you have some eigenvalue $\lambda_j < 0$. In this case let us set $\bar{y} = Q^T \bar{x} \Rightarrow \bar{x} = Q^{-T} \bar{y}$.

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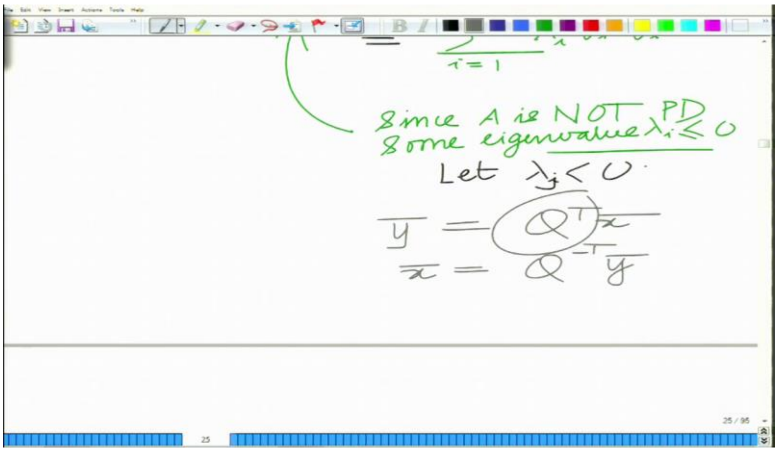

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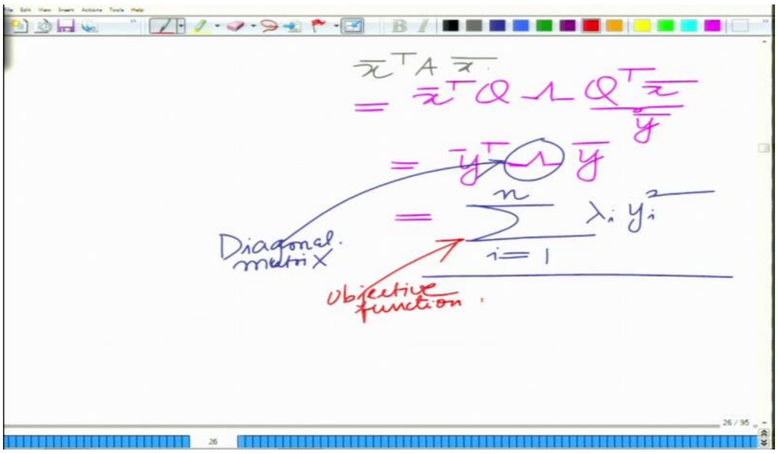

And therefore, now if we look at $\bar{x}^T A \bar{x} = \bar{x}^T Q \Lambda Q^T \bar{x} = \bar{y}^T \Lambda \bar{y} = \sum_{i=1}^{n} \lambda_i y_i^2$. So this is the objective function.

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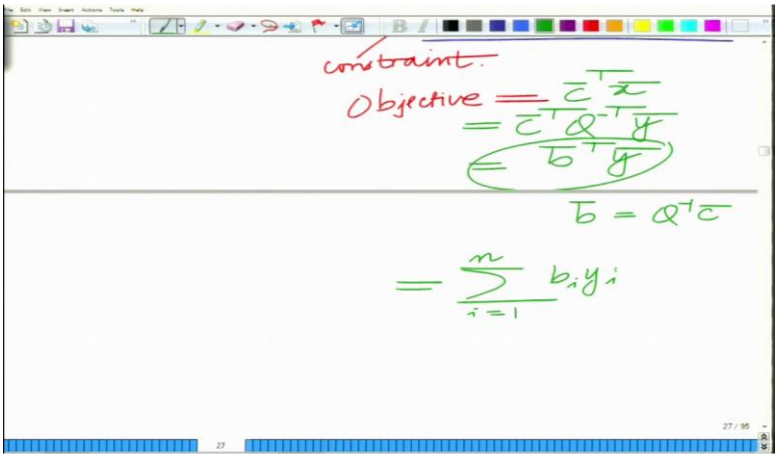

Now we have $\bar{c}^T \bar{x} = \bar{c}^T Q^{-T} \bar{y} = \bar{b}^T \bar{y} = \sum_{i=1}^{n} b_i y_i$. So I can recast this optimization problem.

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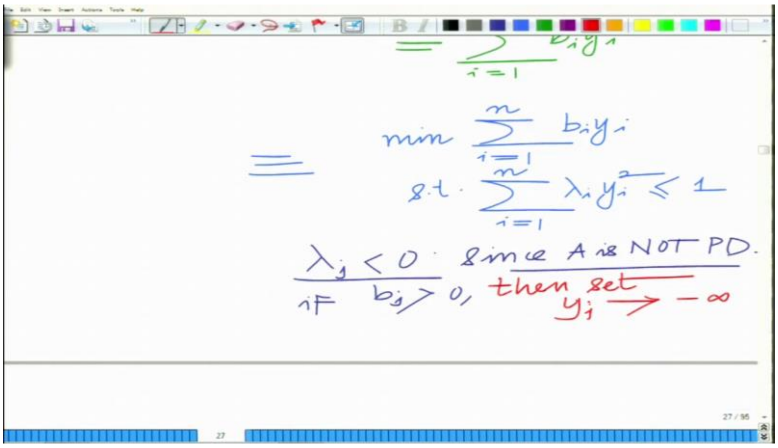

So we have $\begin{aligned} &\min \sum_{i=1}^{n} b_i y_i \\ &s.t \sum_{i=1}^{n} \lambda_i y_i^2 \le 1 \end{aligned}$. Now we are assuming that one particular $\lambda_j < 0$, since A is not positive definite. Now if $b_j > 0$, then set y_i to be a very large negative value.

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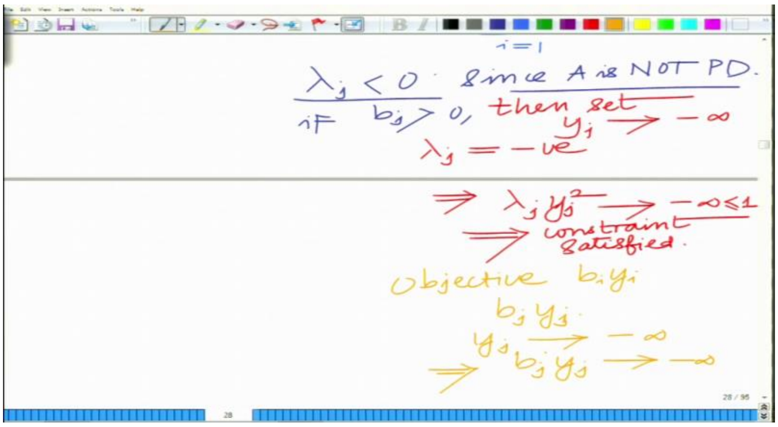

Now, since λ_j is negative implies $\lambda_j y^2_j$ equals negative implies the constraint is satisfied.

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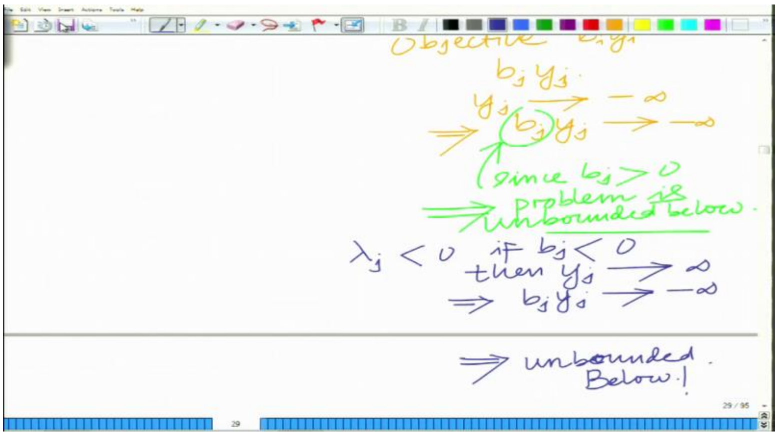

So basically by setting $y_j = -\infty$ we can make the optimization objective as small as possible. Now, consider another scenario, when $\lambda_j < 0$ and if $b_j < 0$, then set y_j to a large positive value. This again implies that the constraint is always satisfied as shown in slide. Let us look at another example, example number 6.

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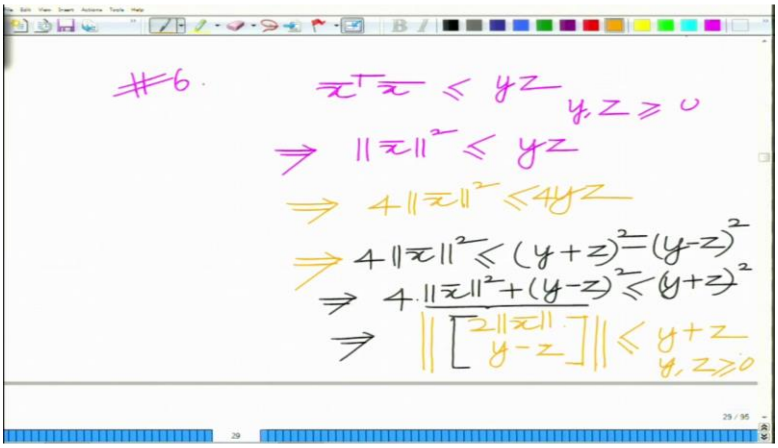

So show that $\bar{x}^T \bar{x} \le yz, y, z \ge 0$. This implies that $\left\|\bar{x}\right\|^2 \le yz \Rightarrow 4\left\|\bar{x}\right\|^2 \le 4yz \Rightarrow 4\left\|\bar{x}\right\|^2 \le (y+z)^2 - (y-z)^2$. So this implies $4\left\|\bar{x}\right\|^2 + (y-z)^2 \le (y+z)^2 \Rightarrow \left\|\begin{bmatrix} 2\left\|\bar{x}\right\| \\ y-z \end{bmatrix}\right\| \le y+z \; y, z \ge 0$.

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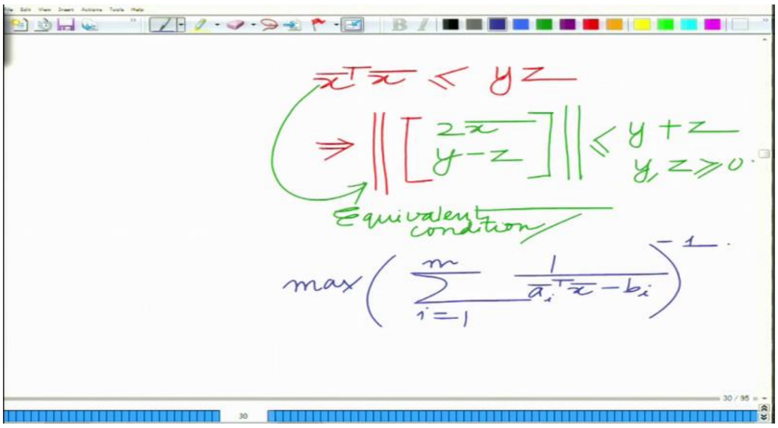

The condition that $\bar{x}^T \bar{x} \le yz$, can be equivalently written as shown in the above slide.

Now, let us say we want to maximize the harmonic mean that is $\max \left(\sum_{i=1}^{m} \frac{1}{\bar{a}_i^T \bar{x} - b_i} \right)^{-1}$.

$s.t \; \bar{a}_i^T \bar{x} - b_i > 0$

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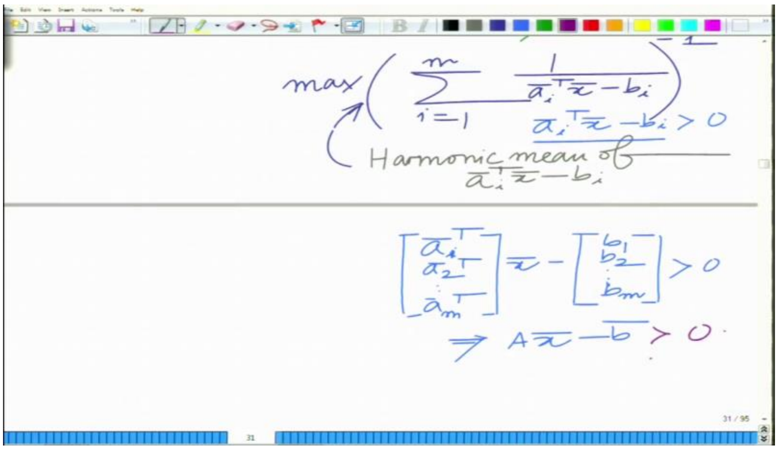

This can be stacked in the form of a matrix which implies that $A\bar{x} - \bar{b} > 0$.

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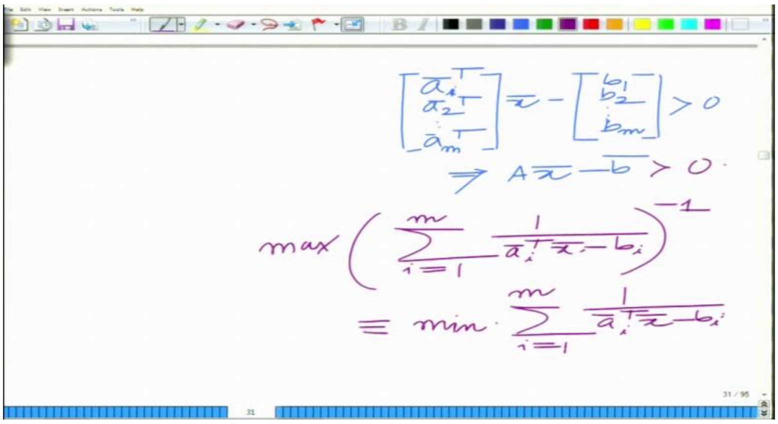

Now this is equivalent to $\min\left(\sum_{i=1}^{m} \frac{1}{\bar{a}_i^T \bar{x} - b_i}\right)$ because everything is non-negative. Now, let us write this in an epigraph form.

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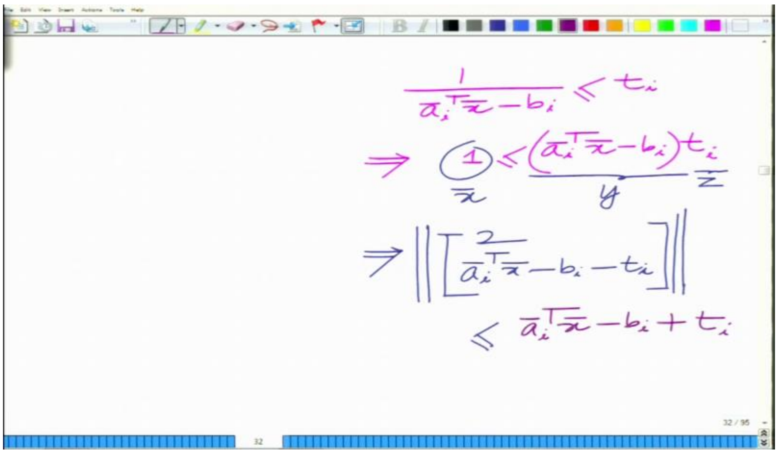

So we have $\frac{1}{\bar{a}_i^T \bar{x} - b_i} \le t_i \Rightarrow 1 \le \left(\bar{a}_i^T \bar{x} - b_i\right) t_i \Rightarrow \left\| \begin{bmatrix} 2 \\ \bar{a}_i^T \bar{x} - b_i - t_i \end{bmatrix} \right\| \le \bar{a}_i^T \bar{x} - b_i + t_i$.

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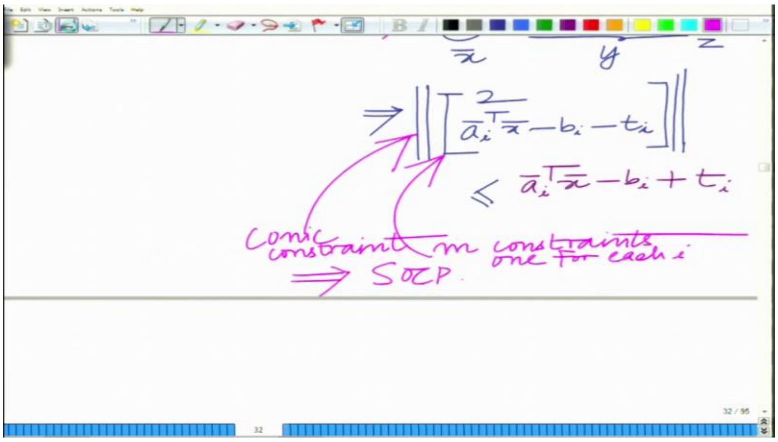

So we will have m constraints, one for each i and this is a second order conic constraint. So the resulting optimization problem will be a second order cone program.

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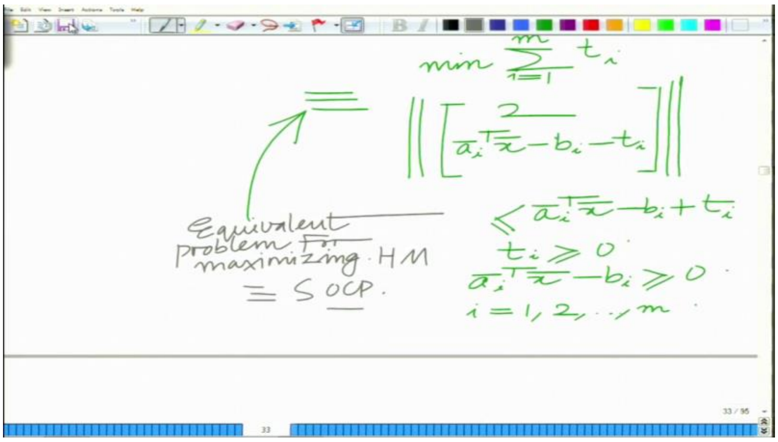

So the equivalent optimization problem can be written as

$$\min \sum_{i=1}^{m} t_i$$

$$s.t \left\| \begin{bmatrix} 2 \\ \bar{a}_i^T \bar{x} - b_i - t_i \end{bmatrix} \right\| \le \bar{a}_i^T \bar{x} - b_i + t_i$$

$$t_i \ge 0$$

$$\bar{a}_i^T \bar{x} - b_i \ge 0$$

$$i = 1, 2, .., m$$

. So once you write this as a second order cone program, you can use the convex solvers readily available to solve this. So let us stop here and continue in the subsequent module. Thank you very much.