Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture – 31
Example Problems: Operations preserving Convexity (log-sum-exp, average) and Quasi-Convexity

Hello. Welcome to another module in this massive open online course. Let us continue the discussion on example problems for convex function.

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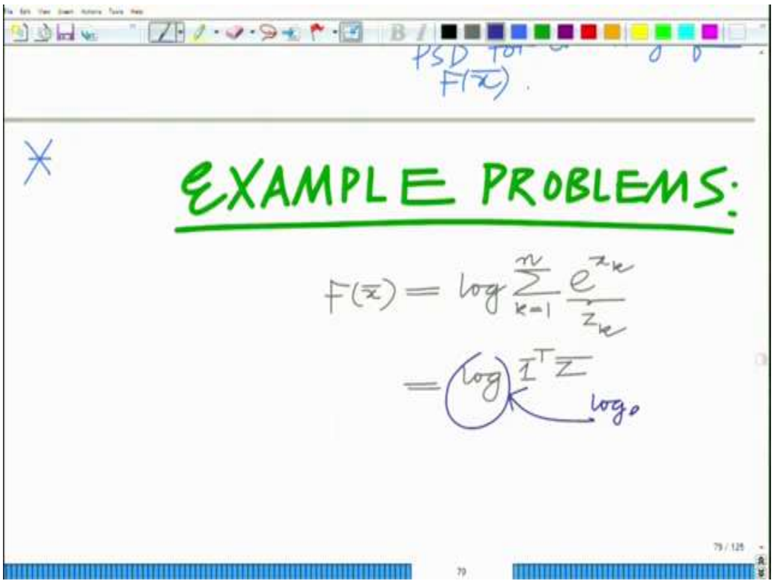

Consider a function $F(\bar{x})$ as follows.

$$F(\bar{x}) = \log_e \left| \sum_{k=1}^{n} e^{x_k} \right|$$

And if e^{x_k} is denoted as z_k i.e.

$$z_k = e^{x_k}$$

Then function $F(\bar{x})$ is

$$F(\bar{x}) = \log_e \left| \bar{1}^T \bar{z} \right|$$

Here similar to the previous modules, $\overline{1}$ is an all ones vector having same dimensions as of $\overline{z}$. So, the hessian of this function $F\left(\overline{x}\right)$ is computed as

$$\nabla^2 F\left(\overline{x}\right) = \frac{\operatorname{diag}\left(\overline{z}\right)}{\overline{1}^T \overline{z}} - \frac{\overline{z}\overline{z}^T}{\left(\overline{1}^T \overline{z}\right)^2}$$

If this hessian is positive semi definite, it will indicate that function $F\left(\overline{x}\right)$ is convex.

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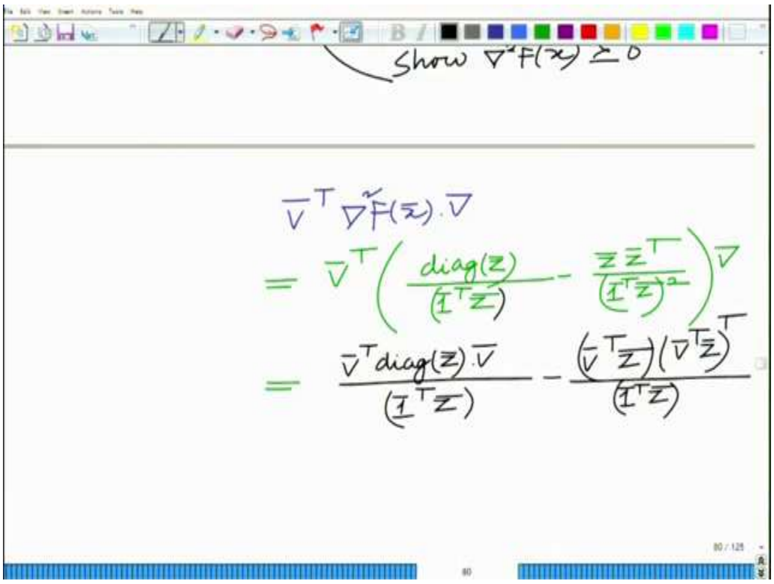

Therefore, substitute the hessian in the definition of positive semi definite.

$$\begin{aligned}
&\overline{V}^T \left(\nabla^2 F\left(\overline{x}\right)\right)\overline{V} \\
&= \overline{V}^T \left(\frac{\operatorname{diag}\left(\overline{z}\right)}{\overline{1}^T \overline{z}} - \frac{\overline{z}\overline{z}^T}{\left(\overline{1}^T \overline{z}\right)^2}\right)\overline{V} \\
&= \frac{\overline{V}^T \operatorname{diag}\left(\overline{z}\right)\overline{V}}{\overline{1}^T \overline{z}} - \frac{\overline{V}^T \overline{z}\overline{z}^T \overline{V}}{\left(\overline{1}^T \overline{z}\right)^2} \\
&= \frac{\overline{V}^T \operatorname{diag}\left(\overline{z}\right)\overline{V}}{\overline{1}^T \overline{z}} - \frac{\left(\overline{V}^T \overline{z}\right)\left(\overline{V}^T \overline{z}\right)^T}{\left(\overline{1}^T \overline{z}\right)^2}
\end{aligned}$$

But, $\overline{V}^T \overline{z}$ is a scalar quantity. Also the above expression can be simplified as

$$\bar{V}^T\left(\nabla^2 F\left(\bar{x}\right)\right)\bar{V} = \frac{\sum_k v_k^2 z_k}{\bar{1}^T \bar{z}} - \frac{\left(\sum_k v_k z_k\right)^2}{\left(\bar{1}^T \bar{z}\right)^2}$$

$$= \frac{\left(\sum_k v_k^2 z_k \sum_k z_k\right) - \left(\sum_k v_k z_k\right)^2}{\left(\bar{1}^T \bar{z}\right)^2}$$

And to demonstrate that this is positive semi definite, let us show that the numerator quantity of above expression is greater than equal to 0.

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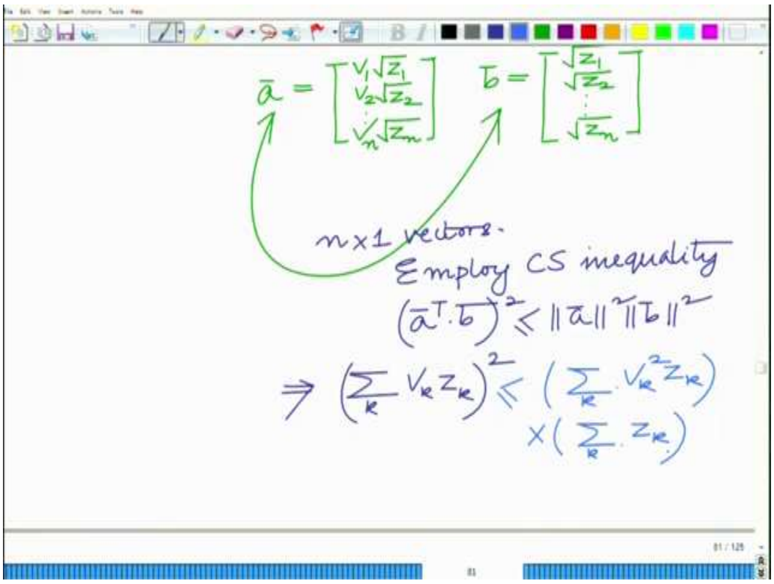

So define two $n\times 1$ vectors $\bar{a}$ and $\bar{b}$ such that

$$\bar{a} = \begin{bmatrix} v_1\sqrt{z_1} \\ v_2\sqrt{z_2} \\ \vdots \\ v_n\sqrt{z_n} \end{bmatrix} \text{ and } \bar{b} = \begin{bmatrix} \sqrt{z_1} \\ \sqrt{z_2} \\ \vdots \\ \sqrt{z_n} \end{bmatrix}$$

Now, employ the Cauchy Schwarz inequality.

$$\left(\bar{a}^T \bar{b}\right)^2 \leq \|\bar{a}\|^2 \|\bar{b}\|^2$$

$$\left(\sum_k v_k z_k\right)^2 \leq \left(\sum_k v_k^2 z_k\right) \sum_k z_k$$

$$\left(\sum_k v_k^2 z_k\right) \sum_k z_k - \left(\sum_k v_k z_k\right)^2 \geq 0$$

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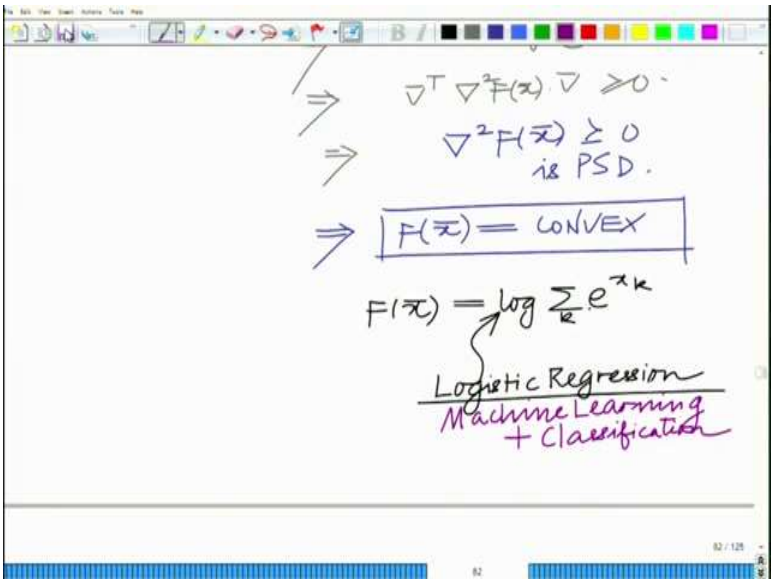

So as the numerator of hessian of function $F(\bar{x})$ is greater than equal to 0, this means

$$\bar{V}^T \left(\nabla^2 F(\bar{x})\right) \bar{V} \geq 0$$

Therefore hessian of function $F(\bar{x})$ is positive semi definite which further implies that function $F(\bar{x})$ is convex. Hence the logarithmic function of summation of exponentials is convex. This can be used to logistic regression that is to fit a curve to a given set of points. This has applications in machine learning and also in classification where a set of data points is classified into two sets.

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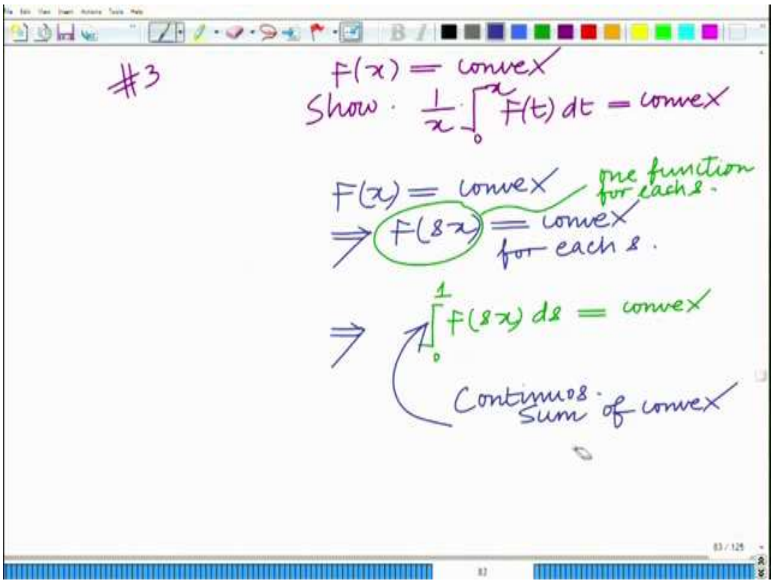

Let us move to the next example which is that if function $F(x)$ is convex, then let us show that $\frac{1}{x}\int_0^x F(t)\,dt$ is also convex. So if $t = sx$ then $dt = x ds$. Thus

$$\frac{1}{x}\int_0^x F(t)\,dt = \int_0^1 F(sx)\,dx$$

Also it is already known that if function $F(x)$ is convex then one function $F(sx)$ for each s is also convex. Therefore the continuous sum $\frac{1}{x}\int_0^x F(t)\,dt$ is also convex.

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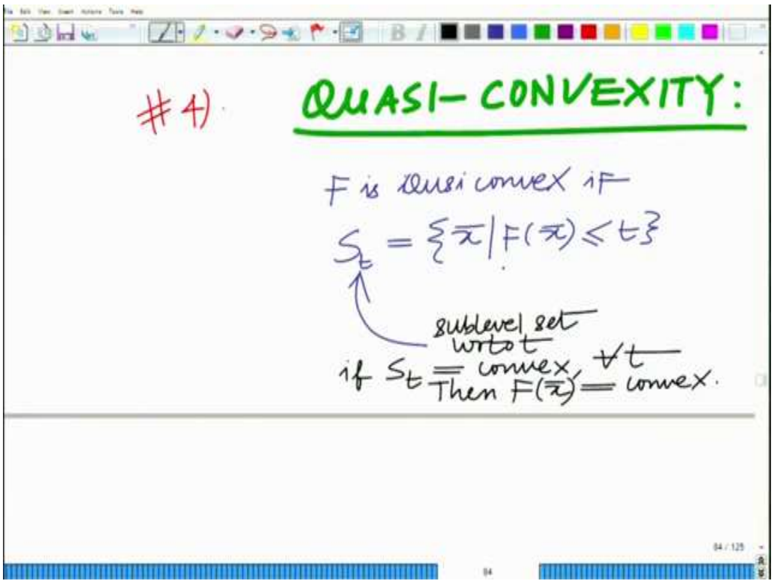

Another interesting concept is the Quasi Convexity. It is defined as follows. A function $F(\bar{x})$ is quasi convex if its sublevel set S_t with respect to t is convex for all t where sublevel set S_t is defined as follows.

$$S_t = \{\bar{x} \mid F(\bar{x}) \leq t\}$$

This is important because there are several functions which are not necessarily convex but, these are qualified as quasi convex.

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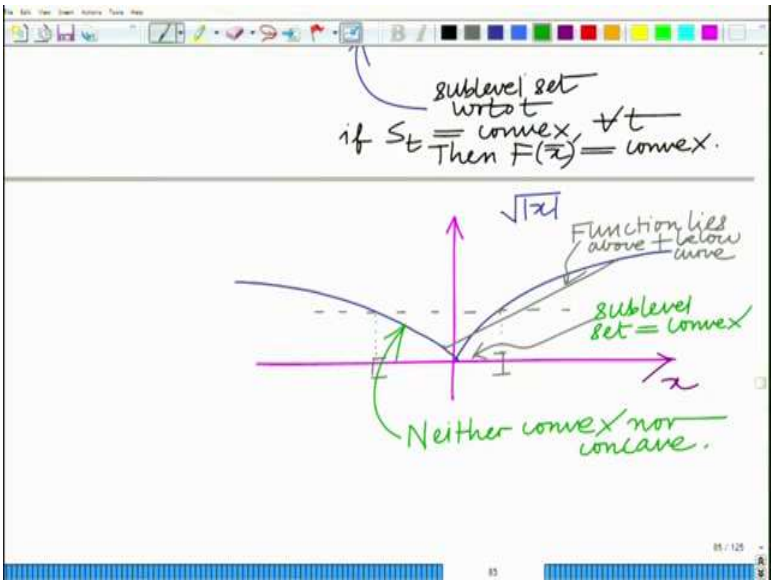

Let us take a simple example. Take a function as

$$F(x) = \sqrt{|x|}$$

It is clearly seen that this function $F(x)$ is neither convex nor concave. However, for any value of *t*, the set of all the points x such that $F(x) \leq t$ is the sublevel set and it is convex.

$$\begin{aligned} S_t &= \{x \mid F(x) \leq t\} \\ &= \{x \mid \sqrt{|x|} \leq t\} \\ &= \{x \mid -t^2 \leq x \leq t^2\} \end{aligned}$$

Therefore this sublevel set is the convex between $[-t^2, t^2]$. Thus this function $\sqrt{|x|}$, which is neither convex nor concave, is Quasi-Convex.

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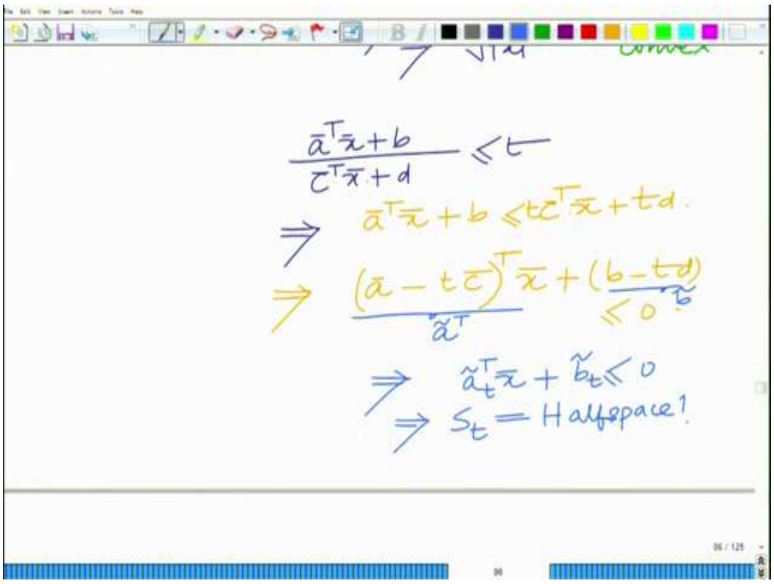

The quasi convex function is not strictly convex function, but it has some properties that are similar to that of a convex function. Let us look at another function.

$$\frac{\bar{a}^T\bar{x}+b}{\bar{c}^T\bar{x}+d} \leq t$$

This is not a convex function. Let us simplify this function.

$$\bar{a}^T \bar{x} + b \le t\bar{c}^T \bar{x} + td$$
$$\left(\bar{a} - t\bar{c}\right)^T \bar{x} + \left(b - td\right) \le 0$$

Consider

$$\left(\bar{a} - t\bar{c}\right) = \tilde{a}_t$$
$$\left(b - td\right) = \tilde{b}_t$$

So the function is now

$$\tilde{a}_t^T \bar{x} + \tilde{b}_t \le 0$$

This now shows that this function's sublevel set is a half space and therefore this sublevel set is convex. This implies that above function is a quasi-convex function.