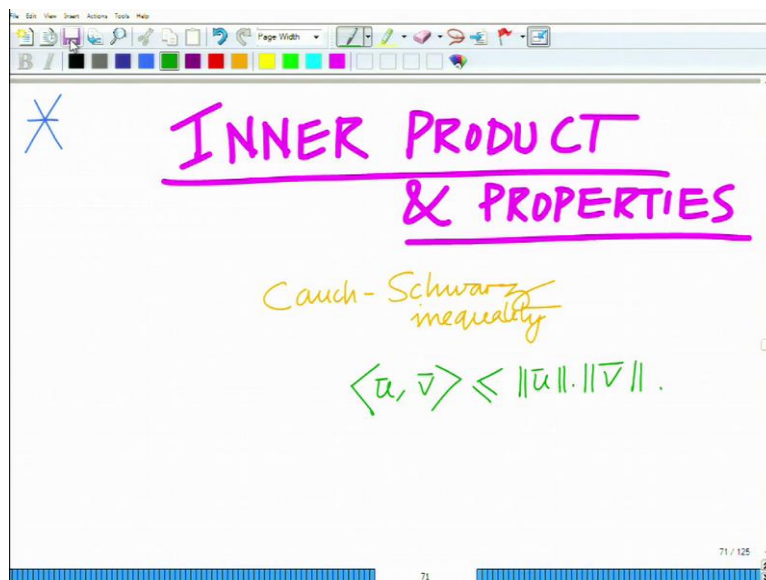


Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture – 06
Properties of Norm, Gaussian Elimination, Echelon form of matrix

Hello. Welcome to another module in this massive open online course. So, we are looking at the concept of inner product and its various properties. In particular, we have also looked at the Cauchy-Schwarz inequality.

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The Cauchy-Schwarz inequality states that

$$\langle \bar{u}, \bar{v} \rangle \leq \|\bar{u}\| \cdot \|\bar{v}\|$$

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PROPERTIES

Cauchy-Schwarz inequality

$$\langle \bar{u}, \bar{v} \rangle \leq \|\bar{u}\| \cdot \|\bar{v}\|$$

$$\langle \bar{u} + t\bar{v}, \bar{u} + t\bar{v} \rangle \geq 0$$

$$\Rightarrow b^2 - 4ac \leq 0$$

$$\Rightarrow 4\langle \bar{u}, \bar{v} \rangle^2 - 4\langle \bar{u}, \bar{u} \rangle \langle \bar{v}, \bar{v} \rangle \leq 0$$

And in fact, in the derivation of the Cauchy-Schwarz inequality, we have seen that

$$\langle \bar{u} + t\bar{v}, \bar{u} + t\bar{v} \rangle \geq 0$$

And this implies that quadratic in

$$b^2 - 4ac \leq 0$$

$$4\langle \bar{u}, \bar{v} \rangle^2 - 4\langle \bar{v}, \bar{v} \rangle \langle \bar{u}, \bar{u} \rangle \leq 0$$

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If $b^2 - 4ac = 0$

$$\Rightarrow \text{Quadratic equation in } t \text{ has a unique root } \tilde{t}$$

$$\Rightarrow \langle \bar{u} + \tilde{t}\bar{v}, \bar{u} + \tilde{t}\bar{v} \rangle = 0$$

$$\Rightarrow \bar{u} + \tilde{t}\bar{v} = 0$$

$$\Rightarrow \bar{u} = -\frac{\tilde{t}}{k}\bar{v}$$

In fact, if quadratic equation has a unique root then it means

$$b^2 - 4ac = 0$$

And now,

$$\langle \bar{u} + \tilde{t}\bar{v}, \bar{u} + \tilde{t}\bar{v} \rangle = 0$$

$$\bar{u} + \tilde{t}\bar{v} = 0$$

$$\bar{u} = -\tilde{t}\bar{v} = k\bar{v}$$

And we can denote \tilde{t} as some constant k .

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has a unique root

$$\Rightarrow \langle \bar{u} + \tilde{t}\bar{v}, \bar{u} + \tilde{t}\bar{v} \rangle = 0$$

$$\Rightarrow \bar{u} + \tilde{t}\bar{v} = 0$$

$$\Rightarrow \bar{u} = -\tilde{t}\bar{v} = k\bar{v}$$

$$\Rightarrow b^2 = 4ac$$

$$\Rightarrow \langle \bar{u}, \bar{v} \rangle = \|\bar{u}\| \cdot \|\bar{v}\|$$

$$\Rightarrow |\langle \bar{u}, \bar{v} \rangle| = \|\bar{u}\| \cdot \|\bar{v}\|$$

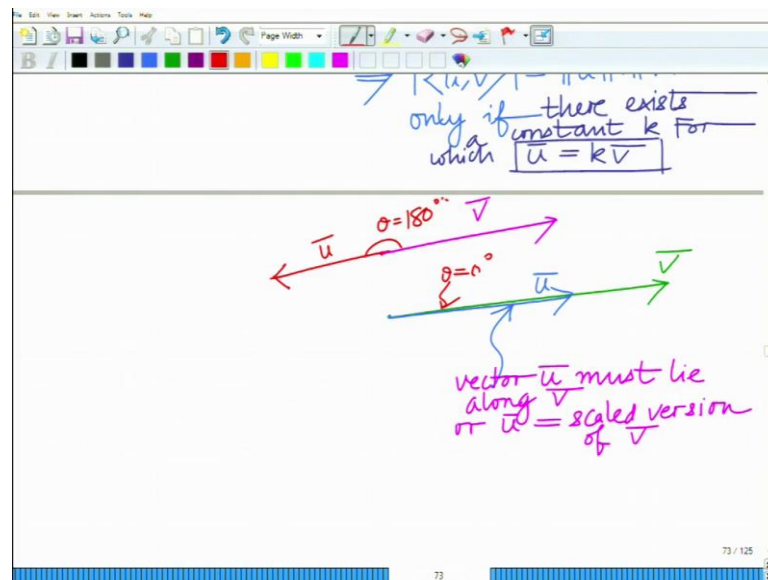
only if there exists a constant k for which $\bar{u} = k\bar{v}$

So, this implies that,

$$\langle \bar{u}, \bar{v} \rangle^2 = \|\bar{u}\|^2 \cdot \|\bar{v}\|^2$$

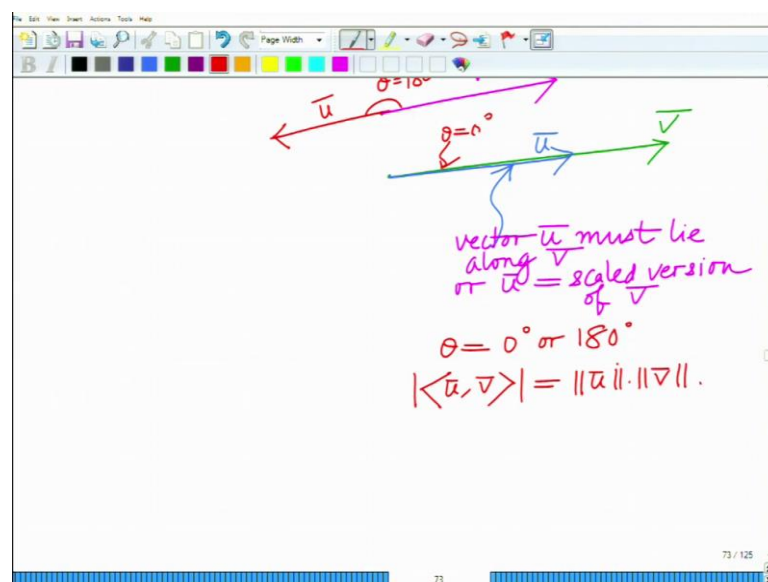
$|\langle \bar{u}, \bar{v} \rangle| = \|\bar{u}\| \cdot \|\bar{v}\|$ is true only if there exists a constant k for which $\bar{u} = k\bar{v}$.

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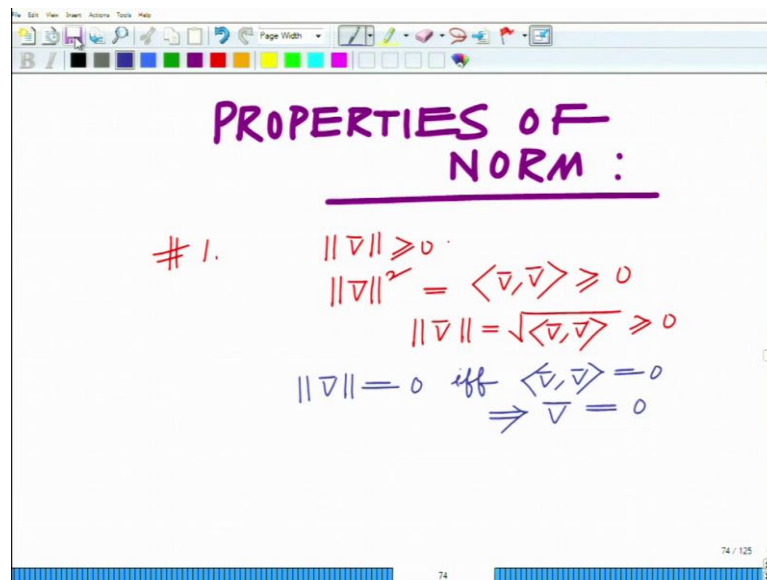
This means, that the vector \underline{u} must also lie along \underline{v} . Which also implies that \underline{u} is simply a scaled version of \underline{v} . Also, both the vectors are in the same direction or in exactly opposite direction (for the negative values of k).

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Also, in both the cases $\theta = 0^\circ$ or 180° the magnitude of inner product of two vectors is same. So conclusively we can say that magnitude of the inner product equal to the product of the norms but these vectors have to be either aligned or exactly anti aligned.

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This is an interesting property that is related to the inner product which we will frequently invoke at instances during our discussion on optimization.

Let us now look at other various properties of the norm and these are as follows. The first property is that

$$\|\bar{v}\| \geq 0$$

So further we can write it as

$$\|\bar{v}\|^2 = \langle \bar{v}, \bar{v} \rangle \geq 0$$

$$\|\bar{v}\| = \sqrt{\langle \bar{v}, \bar{v} \rangle} \geq 0$$

Therefore, we can infer that

$$\|\bar{v}\| = 0 \text{ if and only if } \langle \bar{v}, \bar{v} \rangle = 0.$$

This implies that the inner product of a vector with itself is zero only when the vector is a zero vector.

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The image shows a handwritten derivation on a digital whiteboard. At the top, there is a small toolbar with various drawing tools. The text is written in blue and green ink. The derivation starts with the equation $\|c\vec{v}\| = |c| \cdot \|\vec{v}\|$. Below this, the squared norm is calculated: $\|c\vec{v}\|^2 = \langle c\vec{v}, c\vec{v} \rangle = c \langle \vec{v}, c\vec{v} \rangle = c^2 \langle \vec{v}, \vec{v} \rangle = c^2 \|\vec{v}\|^2$. The final result is boxed: $\Rightarrow \|c\vec{v}\| = |c| \cdot \|\vec{v}\|$. The slide number 75 / 125 is visible in the bottom right corner.

$$\|c\vec{v}\| = |c| \cdot \|\vec{v}\|$$
$$\|c\vec{v}\|^2 = \langle c\vec{v}, c\vec{v} \rangle = c \langle \vec{v}, c\vec{v} \rangle = c^2 \langle \vec{v}, \vec{v} \rangle = c^2 \|\vec{v}\|^2$$
$$\Rightarrow \|c\vec{v}\| = |c| \cdot \|\vec{v}\|$$

Now, the second aspect is that

$$\|c\vec{v}\| = |c| \cdot \|\vec{v}\|$$

In fact, we can also see that

$$\begin{aligned} \|c\vec{v}\|^2 &= \langle c\vec{v}, c\vec{v} \rangle \\ &= c \langle \vec{v}, c\vec{v} \rangle \\ &= c^2 \langle \vec{v}, \vec{v} \rangle \\ &= c^2 \|\vec{v}\|^2 \end{aligned}$$

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Handwritten derivation of the triangle inequality for vector norms. The text is written on a whiteboard with a toolbar at the top. The derivation starts with the expression $\Rightarrow \|u+v\| - \|u\| \leq \|v\|$ and then proceeds to prove the triangle inequality $\|u+v\| \leq \|u\| + \|v\|$ using the Cauchy-Schwarz inequality.

$$\Rightarrow \|u+v\| - \|u\| \leq \|v\|$$

#3. TRIANGLE INEQUALITY:

$$\|u+v\| \leq \|u\| + \|v\|$$

$$\|u+v\|^2 = \langle u+v, u+v \rangle$$

$$= \langle u, u+v \rangle + \langle v, u+v \rangle$$

$$= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle$$

Now the third property is an important property. This is known as the triangle inequality.

This states that

$$\|\bar{u} + \bar{v}\| \leq \|\bar{u}\| + \|\bar{v}\|$$

This follows as

$$\begin{aligned} \|\bar{u} + \bar{v}\|^2 &= \langle \bar{u} + \bar{v}, \bar{u} + \bar{v} \rangle \\ &= \langle \bar{u}, \bar{u} + \bar{v} \rangle + \langle \bar{v}, \bar{u} + \bar{v} \rangle \\ &= \langle \bar{u}, \bar{u} \rangle + \langle \bar{u}, \bar{v} \rangle + \langle \bar{v}, \bar{u} \rangle + \langle \bar{v}, \bar{v} \rangle \\ &= \|\bar{u}\|^2 + 2\langle \bar{u}, \bar{v} \rangle + \|\bar{v}\|^2 \end{aligned}$$

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The image shows a handwritten derivation of the triangle inequality for vector norms. The derivation is written on a whiteboard with a toolbar at the top. The steps are as follows:

$$\begin{aligned} & + \langle \vec{v}, \vec{u} \rangle + \langle \vec{v}, \vec{v} \rangle \\ = & \|\vec{u}\|^2 + 2\langle \vec{u}, \vec{v} \rangle + \|\vec{v}\|^2 \\ \leq & \|\vec{u}\|^2 + 2\|\vec{u}\| \cdot \|\vec{v}\| + \|\vec{v}\|^2 \\ = & (\|\vec{u}\| + \|\vec{v}\|)^2 \\ \Rightarrow & \boxed{\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|} \end{aligned}$$

The final result is boxed in purple. The slide number 76 is visible in the bottom right corner.

Now, applying the Cauchy-Schwarz inequality, we know that

$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 & \leq \|\vec{u}\|^2 + 2\|\vec{u}\| \cdot \|\vec{v}\| + \|\vec{v}\|^2 \\ & \leq (\|\vec{u}\| + \|\vec{v}\|)^2 \end{aligned}$$

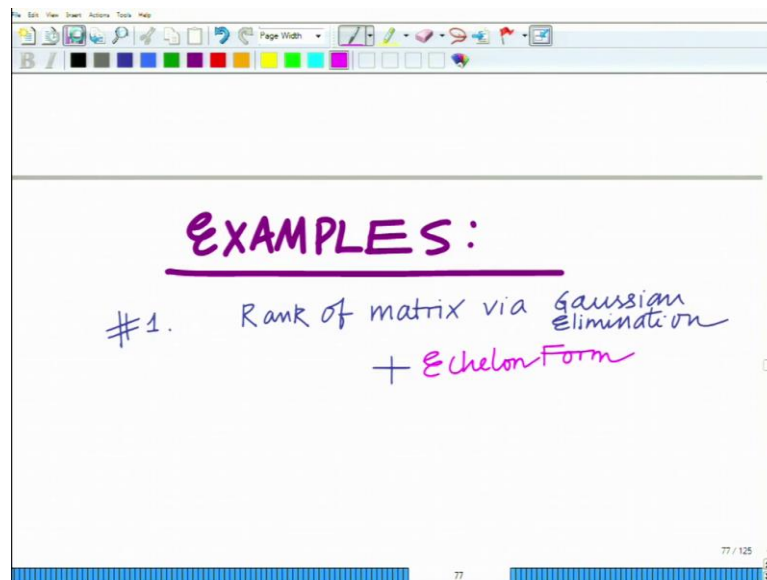
And therefore

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

This is basically the triangle inequality of the norm. This is valid for any norm that is induced by the inner product.

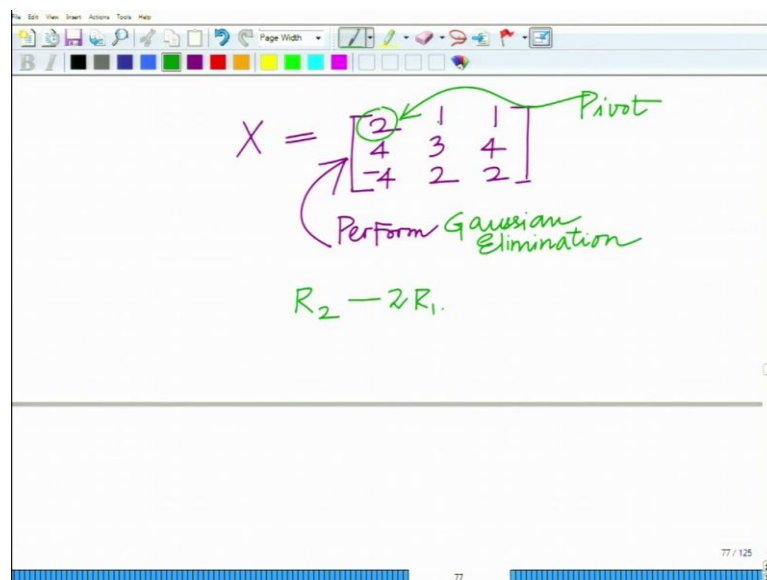
So, that basically concludes our discussion on the various aspects of the norm and the properties of the norm that is induced by the inner product. So, now, let us start doing some examples on these mathematical preliminaries.

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Consider a simple example to discuss the row elimination via Gaussian Elimination to obtain the row echelon form.

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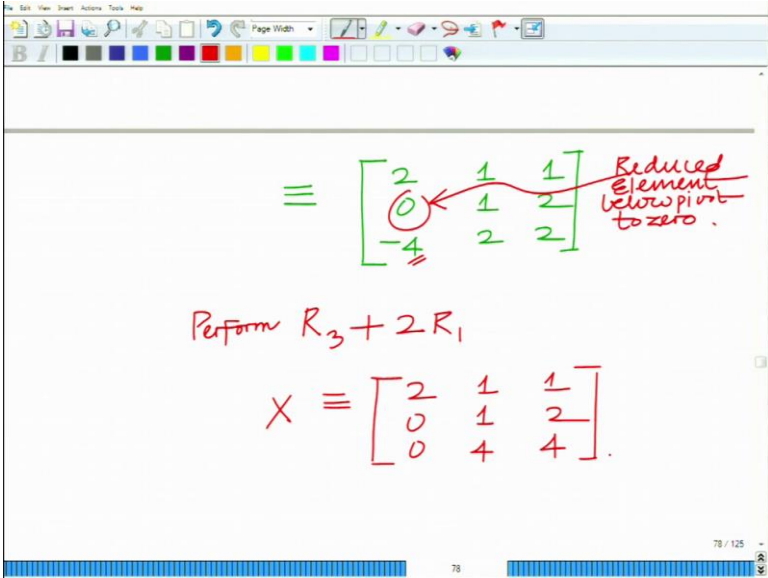
Take a matrix X as

$$X = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 3 & 4 \\ -4 & 2 & 2 \end{pmatrix}$$

In Gaussian elimination, we use a pivot which is going to be subtracted in an operation to reduce an element to zero and hence this pivot helps in converting a matrix into upper triangular matrix. This pivot is chosen for an operation according to element that needs to be zero in this operation. So, first perform $R_2 - 2R_1 \rightarrow R_2$, here R_1 and R_2 are denoting row 1 and row 2 respectively. Also $2R_1$ is used as the pivot to reduce x_{21} to zero. Therefore,

$$X \equiv \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ -4 & 2 & 2 \end{pmatrix}$$

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Handwritten notes on a digital whiteboard showing matrix operations. The matrix is shown with a pivot element circled and an arrow pointing to it from the text "Reduced Element below pivot to zero." Below this, the operation "Perform $R_3 + 2R_1$ " is written, followed by the resulting matrix.

$$\equiv \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ -4 & 2 & 2 \end{bmatrix}$$

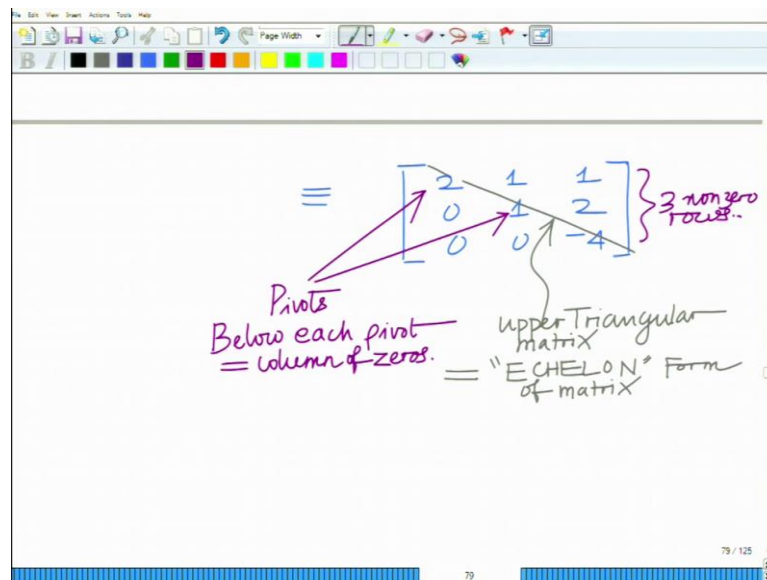
Perform $R_3 + 2R_1$

$$X \equiv \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 4 & 4 \end{bmatrix}$$

Now perform $R_3 + 2R_1 \rightarrow R_3$.

$$X \equiv \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 4 & 4 \end{pmatrix}$$

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Now perform $R_3 - 4R_1 \rightarrow R_3$.

$$X \equiv \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -4 \end{pmatrix}$$

X is now an upper triangular matrix and this is termed as the Echelon form of the matrix.

We can also observe that below each pivot, we have a column of zeros. And here the number of nonzero rows is equal to the rank of the matrix.

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Number of non-zero rows
= Rank of matrix

$$X = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 3 & 6 & 1 \end{bmatrix}$$

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Let us now consider another matrix.

$$X \equiv \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 3 & 6 & 1 \end{pmatrix}$$

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$$X = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 3 & 6 & 1 \end{bmatrix}$$
$$R_2 - 2R_1$$
$$\equiv \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 3 & 6 & 1 \end{bmatrix}$$

Pivot

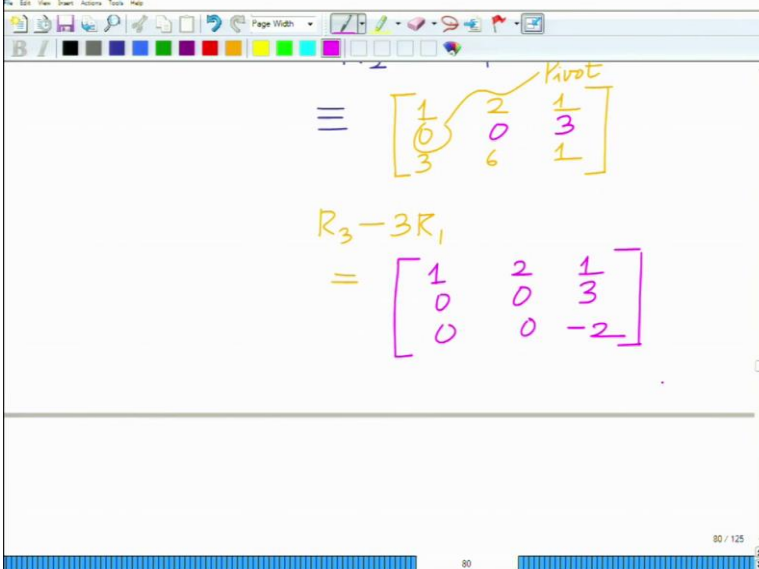
$$R_3 - 3R_1$$
$$=$$

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So now perform $R_2 - 2R_1 \rightarrow R_2$,

$$X \equiv \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 3 & 6 & 1 \end{pmatrix}$$

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The image shows a digital whiteboard interface with a toolbar at the top. Handwritten in yellow and purple ink are the following steps:

$$\equiv \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 3 & 6 & 1 \end{bmatrix}$$

A yellow arrow points from the word "Pivot" to the element '1' in the first row, first column of the matrix.

$$R_3 - 3R_1$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & -2 \end{bmatrix}$$

The bottom right corner of the whiteboard shows the page number "80 / 125".

So now perform $R_3 - 3R_1 \rightarrow R_3$,

$$X \equiv \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & -2 \end{pmatrix}$$

And now, you can interestingly see that all the possible pivots in the second column are 0.

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The whiteboard shows the following handwritten work:

$$= \begin{bmatrix} 1 & 2 & \frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Arrows point to the zeros in the second row with the label "zeros".

$$R_3 + \frac{2}{3}R_2$$
$$\equiv \begin{bmatrix} 1 & 2 & \frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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So, these entries which can be used as possible pivots, these are zeros. Therefore use R_2 as pivot which gives us something interesting. So now perform $R_3 + \frac{2}{3}R_1 \rightarrow R_3$,

$$X \equiv \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

Now we finally have a row of all 0s.

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The whiteboard shows the following handwritten work:

$$R_3 + \frac{2}{3}R_2$$
$$\equiv \begin{bmatrix} 1 & 2 & \frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Annotations include:

- Green circles around the 1 in row 1 and the 3 in row 2, with an arrow pointing to them labeled "Pivots".
- A red bracket on the right side of the matrix with the text "Num of non zero rows = 2".
- A blue arrow pointing to the second row with the text "Echelon Form of matrix".
- A red arrow pointing to the third row with the text "All zero row".
- A red arrow pointing to the text "Rank of matrix = 2".
- A blue arrow pointing to the text "Obtained via Gaussian Elimination".
- A blue arrow pointing to the text "Each pivot lies to right of pivot in row above".

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So in this Echelon form there is an all zero row. Therefore, number of nonzero rows is 2 and this implies that rank of this matrix is 2.

Also note that each chosen pivot for the current operation must have to lie to the right of the pivot in the above row.

This is the simple procedure of Gaussian elimination using pivoting and it can be used to reduce a matrix to the Echelon form and also to determine the rank of the matrix and this makes it much easier to solve a system of linear equations.

Thank you very much.