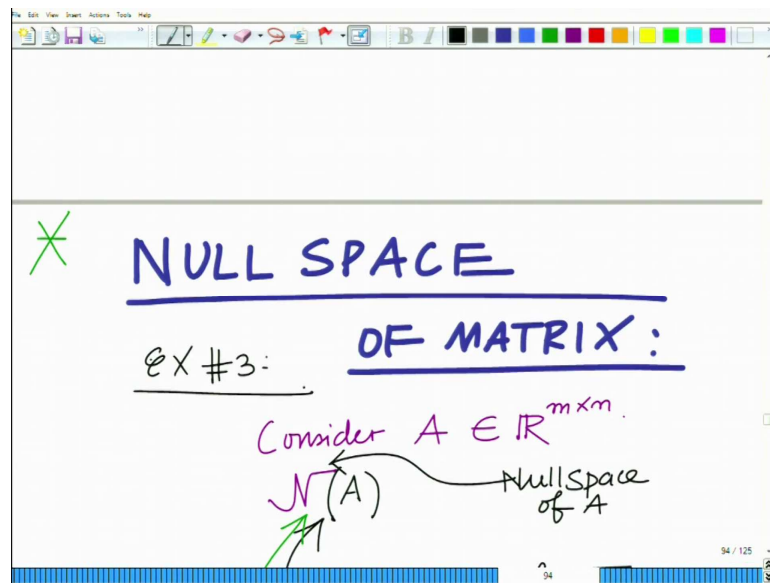


**Applied Optimization for Wireless, Machine Learning, Big Data**  
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**Lecture - 08**  
**Null Space and Trace of Matrices**

Hello, welcome to another module in this massive open online course. In this module, let us discuss another important concept known as the Null Space of a Matrix.

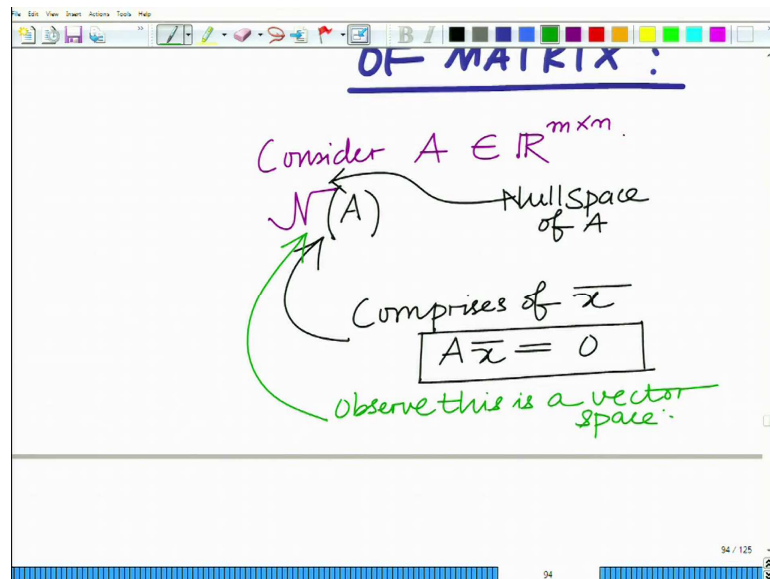
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Consider a  $m \times n$  matrix  $A$  such that  $A \in \mathbb{R}^{m \times n}$ .

Then the null space of  $A$  is denoted as  $N(A)$ .

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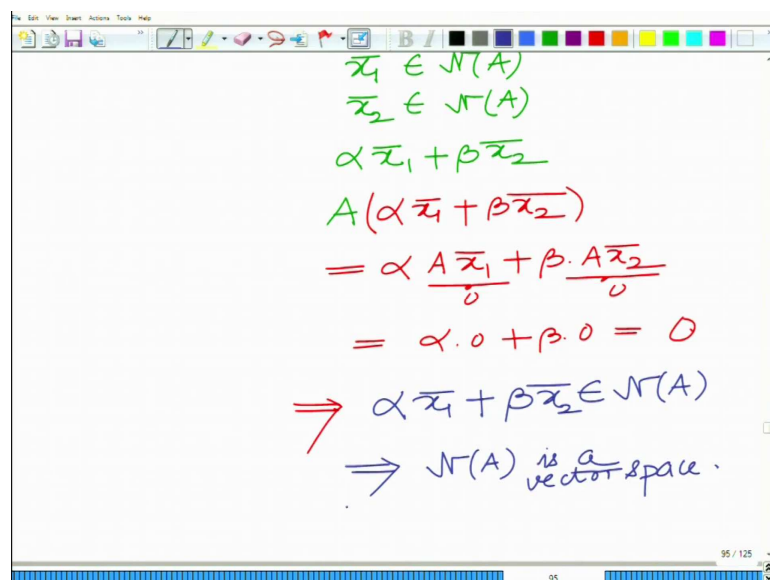


The null space of  $A$ ;  $N(A)$  comprises of all vectors  $\bar{x}$  such that

$$A\bar{x} = 0$$

So the vector space of all vectors  $\bar{x}$  is called the null space of the matrix  $A$ .

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So, observe that the null space as a vector space can be seen as follows. If there are two vectors  $\bar{x}_1$  and  $\bar{x}_2$  have same null space  $N(A)$ .

$$\bar{x}_1 \in N(A) \text{ and } \bar{x}_2 \in N(A)$$

This also means that  $A\bar{x}_1 = 0$  and  $A\bar{x}_2 = 0$ . Consider a linear combination of these two vectors is  $\alpha\bar{x}_1 + \beta\bar{x}_2$  such that  $\bar{x}_1$  and  $\bar{x}_2$  both are the elements of the null space of matrix A.

Therefore

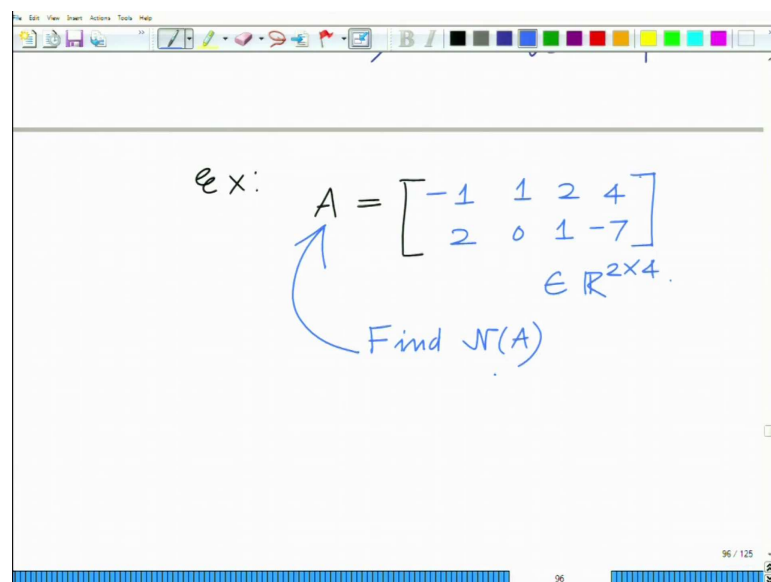
$$\begin{aligned} A(\alpha\bar{x}_1 + \beta\bar{x}_2) &= \alpha A\bar{x}_1 + \beta A\bar{x}_2 \\ &= \alpha \cdot 0 + \beta \cdot 0 \\ &= 0 \end{aligned}$$

This means that

$$\alpha\bar{x}_1 + \beta\bar{x}_2 \in N(A)$$

If  $\bar{x}_1$  and  $\bar{x}_2$  both belongs to the vector space then any linear combination of these vectors also belongs to the set known as a space or a subspace a vector space or a vector subspace. Therefore, the null space of A is a vector space.

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Take an example. Consider a matrix A as

$$A = \begin{bmatrix} -1 & 1 & 2 & 4 \\ 2 & 0 & 1 & -7 \end{bmatrix} \in \mathbb{R}^{2 \times 4}$$

And for the given matrix A, find the null space of matrix A.

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The image shows a digital whiteboard with handwritten notes. At the top, it says 'c x.' followed by the matrix  $A = \begin{bmatrix} -1 & 1 & 2 & 4 \\ 2 & 0 & 1 & -7 \end{bmatrix}$  and  $\in \mathbb{R}^{2 \times 4}$ . Below this, it says 'Find  $\mathcal{N}(A)$ ' and  $A\bar{x} = 0$ . The equation  $\begin{bmatrix} -1 & 1 & 2 & 4 \\ 2 & 0 & 1 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$  is written in green and pink. The vector  $\bar{x}$  is underlined in pink.

$$A = \begin{bmatrix} -1 & 1 & 2 & 4 \\ 2 & 0 & 1 & -7 \end{bmatrix} \in \mathbb{R}^{2 \times 4}$$

So multiply matrix A by a 4 dimensional vector and put it equal to 0. Therefore consider a matrix  $\bar{x}$  as

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

And thus

$$A\bar{x} = 0$$
$$\begin{bmatrix} -1 & 1 & 2 & 4 \\ 2 & 0 & 1 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

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$$A x = 0$$

$$\begin{bmatrix} -1 & 1 & 2 & 4 \\ 2 & 0 & 1 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

Perform row operations on A

$$\begin{bmatrix} -1 & 1 & 2 & 4 \\ 2 & 0 & 1 & -7 \end{bmatrix}$$

To solve the above equation, first perform row operations on the matrix A.

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$$R_1 \rightarrow R_1 / (-1)$$

$$= \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 0 & 1 & -7 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$= \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 2 & 5 & 1 \end{bmatrix}$$

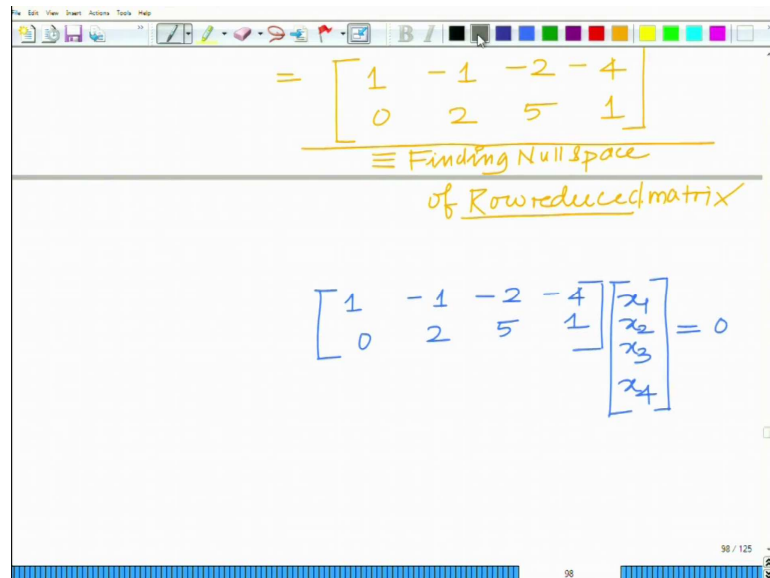
So, first perform we will divide  $R_1 \rightarrow \frac{R_1}{-1}$ .

$$A \equiv \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 0 & 1 & -7 \end{bmatrix}$$

Perform  $R_2 \rightarrow R_2 - 2R_1$  and the matrix becomes

$$A \equiv \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 2 & 5 & 1 \end{bmatrix}$$

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Handwritten notes on a digital whiteboard:

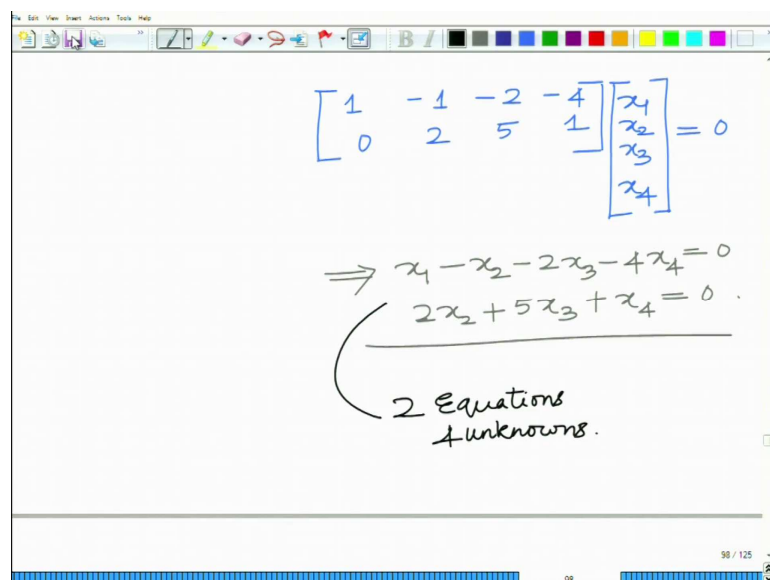
$$= \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 2 & 5 & 1 \end{bmatrix}$$

$\equiv$  Finding Null space of Row reduced matrix

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 2 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

So now let's find the null space of this row reduced matrix of A, because it is equivalent to finding null space of matrix A.

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Handwritten notes on a digital whiteboard:

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 2 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$$\Rightarrow \begin{aligned} x_1 - x_2 - 2x_3 - 4x_4 &= 0 \\ 2x_2 + 5x_3 + x_4 &= 0 \end{aligned}$$

2 Equations & unknowns.

Therefore,

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 2 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

So on solving this, the two equations that came out are

$$\begin{aligned} x_1 - x_2 - 2x_3 - 4x_4 &= 0, \\ 2x_2 + 5x_3 + x_4 &= 0 \end{aligned}$$

This implies that there are two equations and four unknowns. So, set two unknown parameters as free variables. Thus set  $x_3$  and  $x_4$  as free variables and express  $x_1$  and  $x_2$  in terms of  $x_3$  and  $x_4$ .

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Set  $x_3, x_4$  as Free variables.

$$2x_2 + 5x_3 + x_4 = 0$$

$$\Rightarrow x_2 = -\frac{5}{2}x_3 - \frac{1}{2}x_4$$

$$x_1 - x_2 - 2x_3 - 4x_4 = 0$$

$$\Rightarrow x_1 = x_2 + 2x_3 + 4x_4$$

$$= -\frac{5}{2}x_3 - \frac{1}{2}x_4 + 2x_3 + 4x_4$$

Therefore,

$$\begin{aligned} 2x_2 + 5x_3 + x_4 &= 0 \\ x_2 &= -\frac{5}{2}x_3 - \frac{1}{2}x_4 \end{aligned}$$

And also,

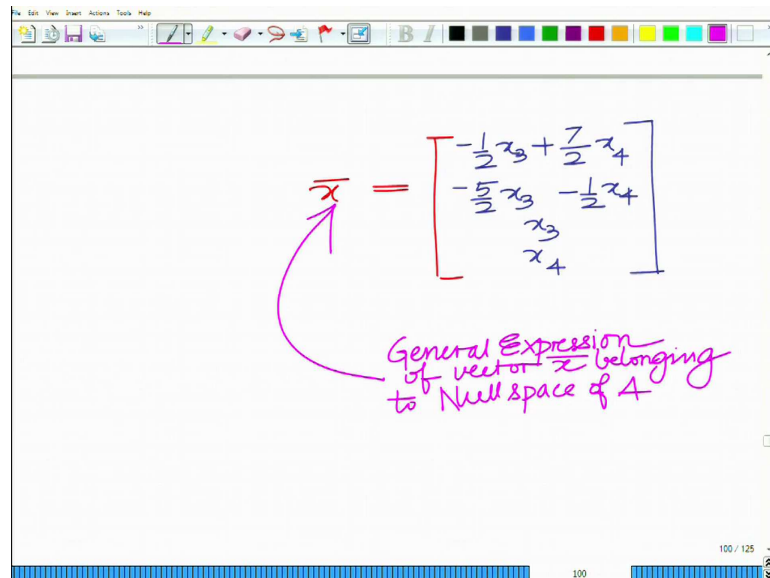
$$x_1 - x_2 - 2x_3 - 4x_4 = 0$$

$$x_1 = x_2 + 2x_3 + 4x_4$$

$$x_1 = -\frac{5}{2}x_3 - \frac{1}{2}x_4 + 2x_3 + 4x_4$$

$$x_1 = -\frac{1}{2}x_3 + \frac{7}{2}x_4$$

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A screenshot of a presentation slide with a white background and a toolbar at the top. The slide contains handwritten mathematical expressions in blue and red ink. A red vector  $\vec{x}$  is shown on the left, with a red arrow pointing to a blue matrix expression. The matrix is a column vector with four entries:  $-\frac{1}{2}x_3 + \frac{7}{2}x_4$ ,  $-\frac{5}{2}x_3 - \frac{1}{2}x_4$ ,  $x_3$ , and  $x_4$ . Below the matrix, there is a handwritten note in blue ink that says "General Expression of vector  $\vec{x}$  belonging to Null space of A". The slide number "100 / 125" is visible in the bottom right corner.

$$\vec{x} = \begin{bmatrix} -\frac{1}{2}x_3 + \frac{7}{2}x_4 \\ -\frac{5}{2}x_3 - \frac{1}{2}x_4 \\ x_3 \\ x_4 \end{bmatrix}$$

General Expression of vector  $\vec{x}$  belonging to Null space of A

And therefore, the general structure of a null vector that belongs to null space of A will be as

$$\vec{x} = \begin{bmatrix} -\frac{1}{2}x_3 + \frac{7}{2}x_4 \\ -\frac{5}{2}x_3 - \frac{1}{2}x_4 \\ x_3 \\ x_4 \end{bmatrix}$$

(Refer Slide Time: 14:52)



General Expression of vector  $\bar{x}$  belonging to Null space of  $A$

$$\bar{x} = \begin{bmatrix} -\frac{5}{2}x_3 & -\frac{1}{2}x_4 \\ x_3 & x_4 \end{bmatrix}$$

$$\bar{x} = x_3 \begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

$$= x_3 \bar{u}_1 + x_4 \bar{u}_2$$

On simplifying this further;

$$\bar{x} = x_3 \begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

$$\bar{x} = x_3 \bar{u}_1 + x_4 \bar{u}_2$$

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$$= x_3 \bar{u}_1 + x_4 \bar{u}_2$$

Linear Combination of  $\bar{u}_1, \bar{u}_2$

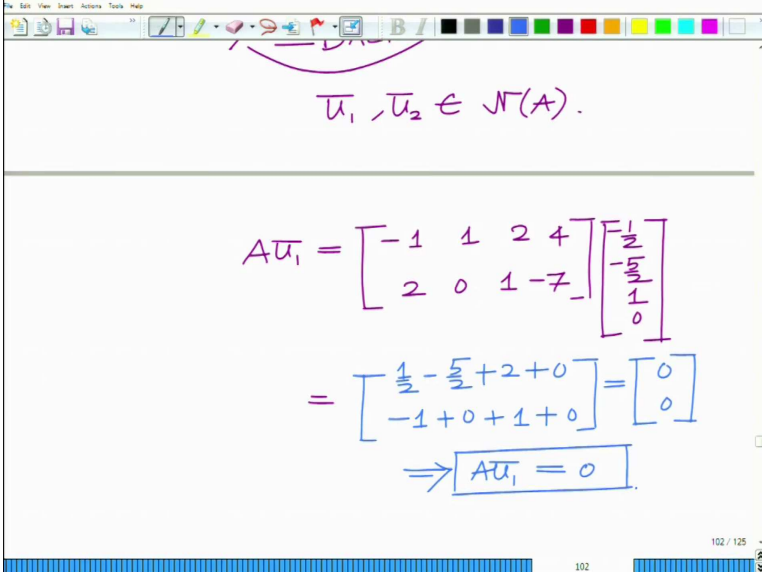
$$\bar{u}_1 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} \quad \bar{u}_2 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow$  BASIS VECTORS FOR  $N(A)$

Observe this is a linear combination of two vectors  $\bar{u}_1$  and  $\bar{u}_2$ . Therefore, this null space of A is formed by all linear combinations of these vectors  $\bar{u}_1$  and  $\bar{u}_2$  and therefore,  $\bar{u}_1$  and  $\bar{u}_2$  are the basis vectors for the null space of the matrix A. Therefore the basis vectors the null space of the matrix A are and

$$\bar{u}_1 = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ \frac{5}{2} \\ 1 \\ 0 \end{bmatrix} \in N(A), \quad \bar{u}_2 = \begin{bmatrix} \frac{7}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \in N(A)$$

(Refer Slide Time: 17:47)



The image shows a digital whiteboard with a toolbar at the top. The text is handwritten in purple and blue ink. A purple bracket underlines the text  $u_1, u_2 \in N(A)$ . Below this, the calculation for  $Au_1$  is shown in blue ink, resulting in the boxed equation  $Au_1 = 0$ .

$$\begin{aligned}
 Au_1 &= \begin{bmatrix} -1 & 1 & 2 & 4 \\ 2 & 0 & 1 & -7 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{5}{2} \\ 1 \\ 0 \end{bmatrix} \\
 &= \begin{bmatrix} \frac{1}{2} - \frac{5}{2} + 2 + 0 \\ -1 + 0 + 1 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
 &\Rightarrow \boxed{Au_1 = 0}
 \end{aligned}$$

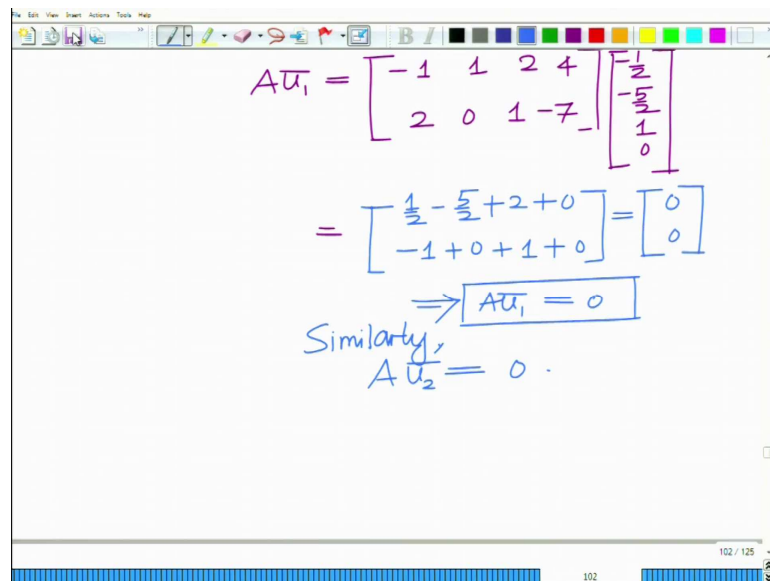
For instance consider  $A\bar{u}_1$ ,

$$\begin{aligned}
 A\bar{u}_1 &= \begin{bmatrix} -1 & 1 & 2 & 4 \\ 2 & 0 & 1 & -7 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ \frac{5}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} \\
 &= \begin{bmatrix} \frac{1}{2} - \frac{5}{2} + 2 + 0 \\ -1 + 0 + 1 + 0 \end{bmatrix} \\
 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
 \end{aligned}$$

So, this implies

$$A\bar{u}_1 = 0$$

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Handwritten derivation on a digital whiteboard:

$$\begin{aligned}
 A\bar{u}_1 &= \begin{bmatrix} -1 & 1 & 2 & 4 \\ 2 & 0 & 1 & -7 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ \frac{5}{2} \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} \\
 &= \begin{bmatrix} \frac{1}{2} - \frac{5}{2} + 2 + 0 \\ -1 + 0 + 1 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
 &\Rightarrow \boxed{A\bar{u}_1 = 0} \\
 \text{Similarly,} \\
 A\bar{u}_2 &= 0.
 \end{aligned}$$

Similarly, consider  $A\bar{u}_2$ ,

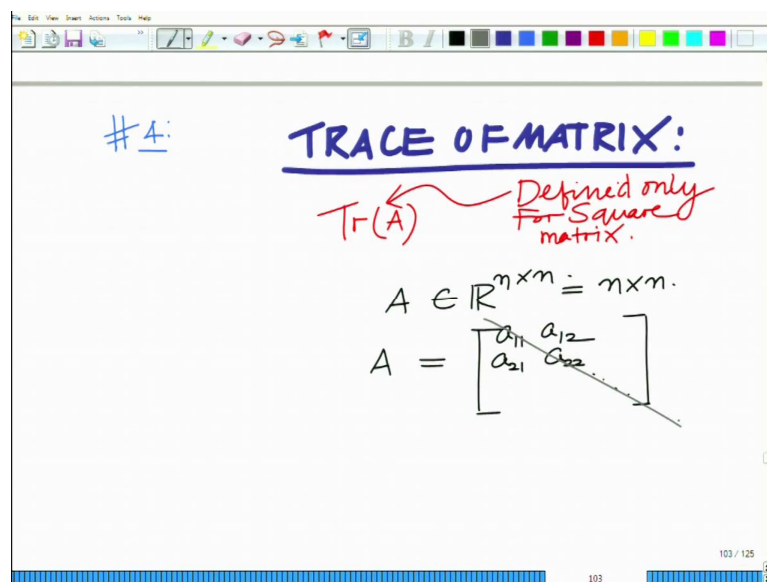
$$\begin{aligned}
 A\bar{u}_2 &= \begin{bmatrix} -1 & 1 & 2 & 4 \\ 2 & 0 & 1 & -7 \end{bmatrix} \begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \\
 &= \begin{bmatrix} -\frac{7}{2} - \frac{1}{2} + 0 + 4 \\ 7 + 0 + 0 - 7 \end{bmatrix} \\
 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
 \end{aligned}$$

So, this implies

$$A\bar{u}_2 = 0$$

So this justifies that  $\bar{u}_1$  and  $\bar{u}_2$  are the null space of matrix A and together it is a vector space because the linear combination of these vectors also lies in this space.

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Let us look at another example regarding the trace of a matrix. The trace of a matrix is defined only for a square matrix and is the sum of its diagonal elements. So, take a  $n \times n$  square matrix  $A \in \mathbb{R}^{n \times n}$  as

$$a = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

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The image shows a digital whiteboard with a toolbar at the top. The handwritten text on the whiteboard is as follows:

$$\begin{aligned} \text{Tr}(A) &= a_{11} + a_{22} + \dots \\ &= \sum_{i=1}^n a_{ii} \end{aligned}$$

An arrow points from the text "= sum of Diagonal Elements" to the diagonal elements in the matrix. Below this, the trace is also expressed as:

$$= \sum_{i=1}^n [A]_{ii}$$

An arrow points from the text " $i, i$  element of A" to the  $[A]_{ii}$  term in the summation.

So, trace of matrix A is

$$\begin{aligned} \text{Tr}(A) &= a_{11} + a_{22} + \dots + a_{nn} \\ &= \sum_{i=1}^n a_{ii} \\ &= \sum_{i=1}^n [A]_{ii} \end{aligned}$$

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$A \in \mathbb{R}^{m \times n}$      $B \in \mathbb{R}^{n \times m}$   
 $m \times n$      $n \times m$

$\boxed{\text{Tr}(AB) = \text{Tr}(BA)}$

$\nRightarrow AB = BA$   
 $m \times m$      $n \times n$

Now, one important property of trace of a matrix is that for two matrix  $A$  and  $B$  such that  $A \in \mathbb{R}^{m \times n}$  and  $B \in \mathbb{R}^{n \times m}$ ; it is true that

$$\text{Tr}(AB) = \text{Tr}(BA)$$

Also it does not imply that  $AB = BA$ . In fact, the sizes of  $AB$  and  $BA$  are not typically equal until  $m = n$ .

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$[AB]_{ij} = \sum_{k=1}^n a_{ik} \cdot b_{kj}$

$i^{\text{th}} \text{ row of } A$      $j^{\text{th}} \text{ column of } B$

$[AB]_{ii} = \sum_{k=1}^n a_{ik} \cdot b_{ki}$

To prove the above property, write down matrix  $AB$  in terms of its elements and this is as

$$[AB]_{ij} = \sum_{k=1}^n a_{ik} \cdot b_{kj}$$

Here  $[AB]_{ij}$  is the  $ij^{\text{th}}$  element of the matrix product  $AB$ ,  $\sum_{k=1}^n a_{ik} \cdot b_{kj}$  is the summation of product of  $k^{\text{th}}$  row elements of matrix A and  $j^{\text{th}}$  column elements of matrix B.

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The image shows a digital whiteboard with a toolbar at the top. The handwritten text on the whiteboard is as follows:

$$[AB]_{ii} = \sum_{k=1}^n a_{ik} \cdot b_{ki}$$

Below this, there is a note: "sum of Diagonal Elements" with an arrow pointing to the  $[AB]_{ii}$  term in the equation above.

$$\text{Tr}(AB) = \sum_{i=1}^m [AB]_{ii}$$

$$= \sum_{i=1}^m \sum_{k=1}^n a_{ik} b_{ki}$$


---


$$= \sum_{i=1}^m \sum_{k=1}^n b_{ki} a_{ik}$$

At the bottom right of the whiteboard, there is a small status bar showing "106 / 125".

So now, the trace of matrix  $AB$  is the summation of all the diagonal elements of matrix  $AB$  and hence

$$\begin{aligned} \text{Tr}(AB) &= \sum_{i=1}^m [AB]_{ii} \\ &= \sum_{i=1}^m \sum_{k=1}^n a_{ik} \cdot b_{ki} \\ &= \sum_{i=1}^m \sum_{k=1}^n b_{ki} \cdot a_{ik} \\ &= \sum_{k=1}^n \sum_{i=1}^m b_{ki} \cdot a_{ik} \end{aligned}$$

Here  $b_{ki}$  is the  $k^{\text{th}}$  row of B and  $a_{ik}$  is the  $k^{\text{th}}$  column of A.

(Refer Slide Time: 28:30)

Handwritten derivation on a digital whiteboard:

$$= \sum_{k=1}^n \sum_{i=1}^m b_{ki} a_{ik}$$

Annotations: An arrow points from  $b_{ki}$  to "k<sup>th</sup> row of B" and another arrow points from  $a_{ik}$  to "i<sup>th</sup> column of A".

$$= \sum_{k=1}^n [BA]_{kk}$$

$$= \text{Tr}(BA)$$

And therefore it can also be written as

$$\begin{aligned} \text{Tr}(AB) &= \sum_{k=1}^n \sum_{i=1}^m b_{ki} a_{ik} \\ &= \sum_{k=1}^n [BA]_{kk} \\ &= \text{Tr}(BA) \end{aligned}$$

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Handwritten final result on a digital whiteboard:

$$= \text{Tr}(BA)$$

$$\boxed{\text{Tr}(AB) = \text{Tr}(BA)}$$

And therefore, the above property is verified. This means that in general the matrices do not have a commutative property, but trace of matrix AB equals the trace of the matrix



BA. This is an interesting property of matrices, which will come handy in several problems or several optimization problems, where matrices or the product of matrices will be needed to manipulate. So, let us stop here and we will continue with some other problems in the subsequent modules.

Thank you very much.