Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture – 24
Properties of Convex Functions with examples

Hello, welcome to another module in this massive open online course. Let us discuss the properties of complex functions.

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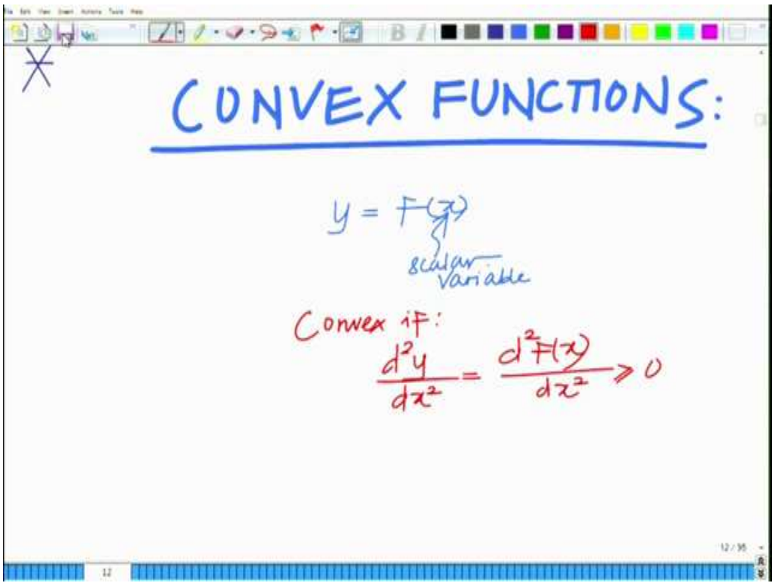

Consider a function $F(x)$ of a scalar variable *x*;

$$y = F(x)$$

Another way to verify the convexity of a function is that a function is convex if the second derivative of the function is greater than or equal to 0. Therefore if $F(x)$ is convex it must satisfies

$$\frac{d^2y}{dx^2} = \frac{d^2F(x)}{dx^2} \geq 0$$

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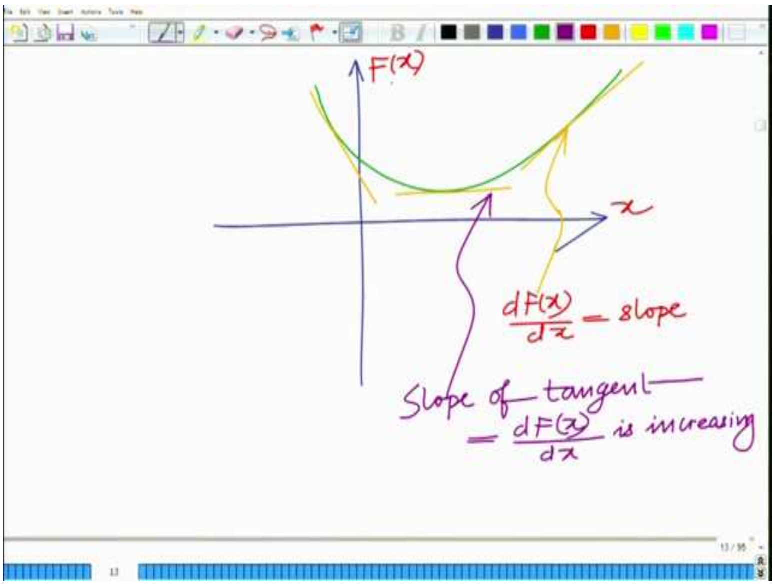

This can be understood as follows. The derivative of a function at some point on the function shows the slope of the tangent to the function drawn on that point. Therefore, in a plot of a convex function, one can easily see that the magnitude of the slope of the tangent is monotonically increasing in both the sides of the lowest point of the function. This means that the second derivative of such functions is greater than or equal to 0. Thus

$$\frac{d^2F(x)}{dx^2} \geq 0$$

Therefore slope of tangent is monotonically increasing for a convex function.

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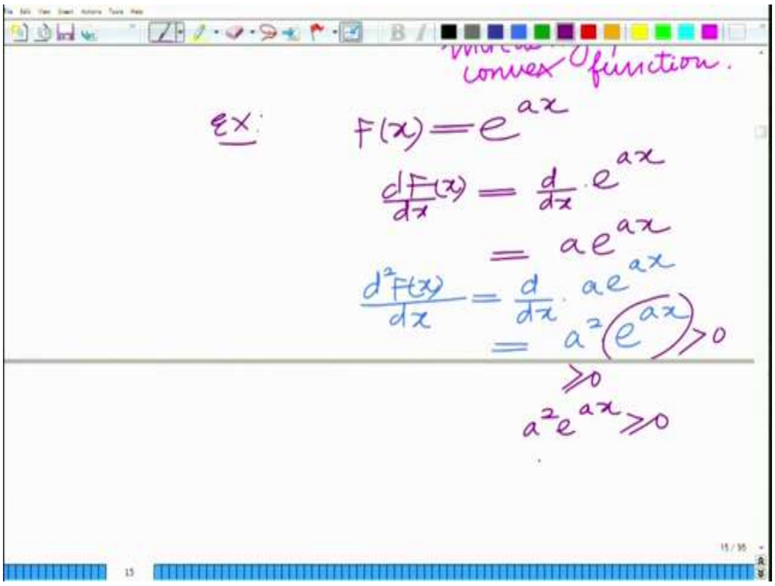

Let us take an example. Consider a convex function

$$F(x) = e^{ax}$$

So the first derivative of this function is

$$\frac{dF(x)}{dx} = \frac{d}{dx} e^{ax}$$
$$= ae^{ax}$$

And the second derivative of this function is

$$\frac{d^2F(x)}{dx^2} = \frac{d}{dx} ae^{ax}$$
$$= a^2e^{ax}$$

As for any value of a; $e^{ax} > 0$. Therefore

$$a^2e^{ax} \geq 0$$
$$\frac{d^2F(x)}{dx^2} \geq 0$$

This implies that this function is a convex function for any value of a. If a is negative, the function is a decreasing exponential and if a is positive, this function is an increasing exponential and both of them are convex. So, it confirms the second order derivative test of convexity.

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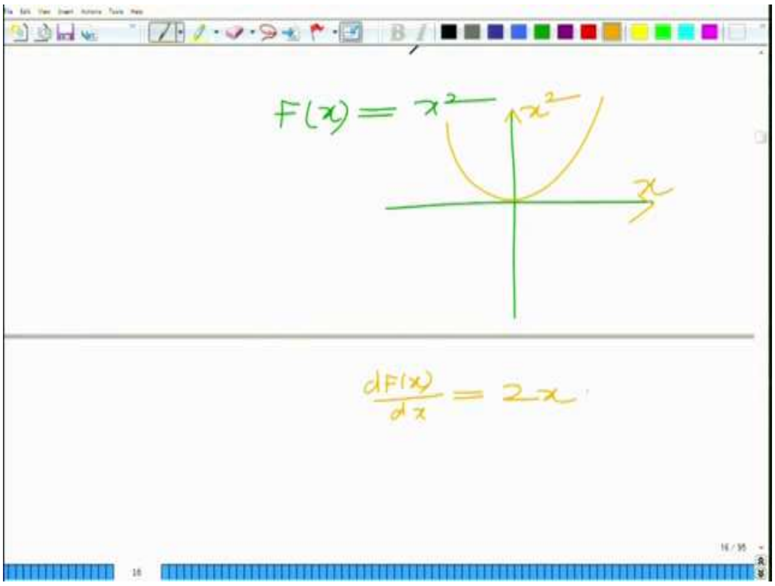

Let us look at another function.

$$F(x) = x^2$$

So the first derivative of this function is

$$\frac{dF(x)}{dx} = \frac{d}{dx}x^2$$
$$= 2x$$

And the second derivative of this function is

$$\frac{d^2F(x)}{dx^2} = \frac{d}{dx}\,2x$$
$$= 2$$

Therefore

$$\frac{d^2F(x)}{dx^2} \geq 0$$

This implies that this function is a convex function.

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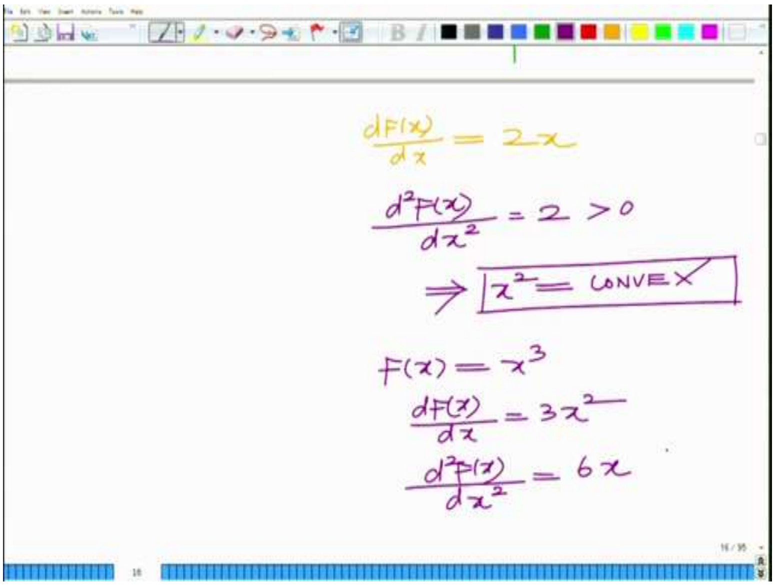

On the other hand another function is

$$F(x) = x^3$$

So the first derivative of this function is

$$\frac{dF(x)}{dx} = \frac{d}{dx}x^3$$
$$= 3x^2$$

And the second derivative of this function is

$$\frac{d^2F(x)}{dx^2} = \frac{d}{dx}\,3x^2$$
$$= 6x$$

Therefore, only for $x \geq 0$;

$$\frac{d^2F(x)}{dx^2} \geq 0$$

This implies that this function is a convex function only for $x \geq 0$.

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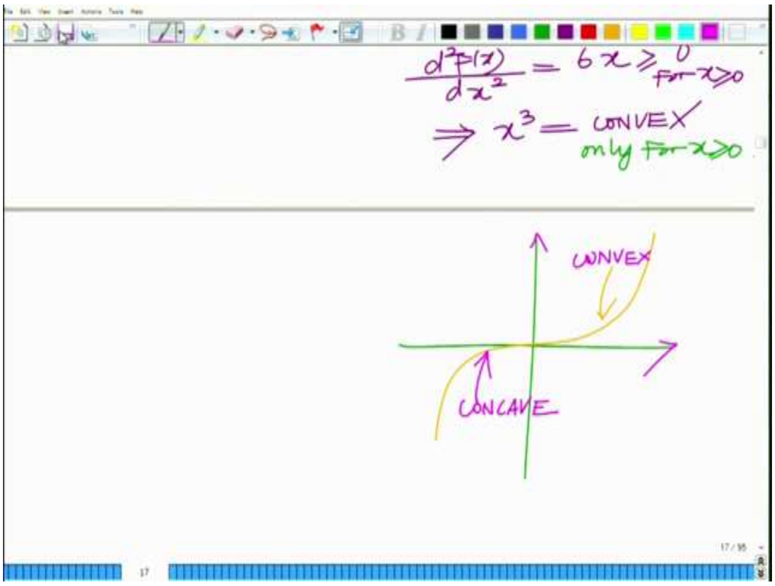

This had already been discussed in the previous module that for $x \geq 0$, this function is convex and for $x \geq 0$, it is concave.

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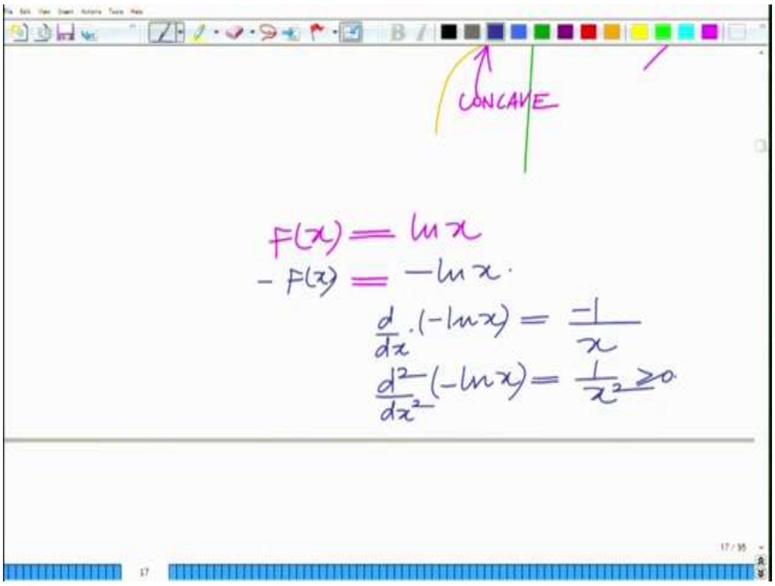

Let us consider a classic example of concave function. The function is

$$F(x) = \ln x$$

So

$$-F(x) = -\ln x$$

So the first derivative of minus of this function is

$$\frac{d(-F(x))}{dx} = \frac{d}{dx}(-\ln x)$$
$$= -\frac{1}{x}$$

And the second derivative of minus of this function is

$$\frac{d^2(-F(x))}{dx^2} = \frac{d}{dx}\left(-\frac{1}{x}\right)$$
$$= \frac{1}{x^2}$$

Therefore, for all values of *x*;

$$\frac{d^2(-F(x))}{dx^2} \geq 0$$

This implies that $-F(x)$ is a convex function and hence above function $F(x)$ is a concave function.

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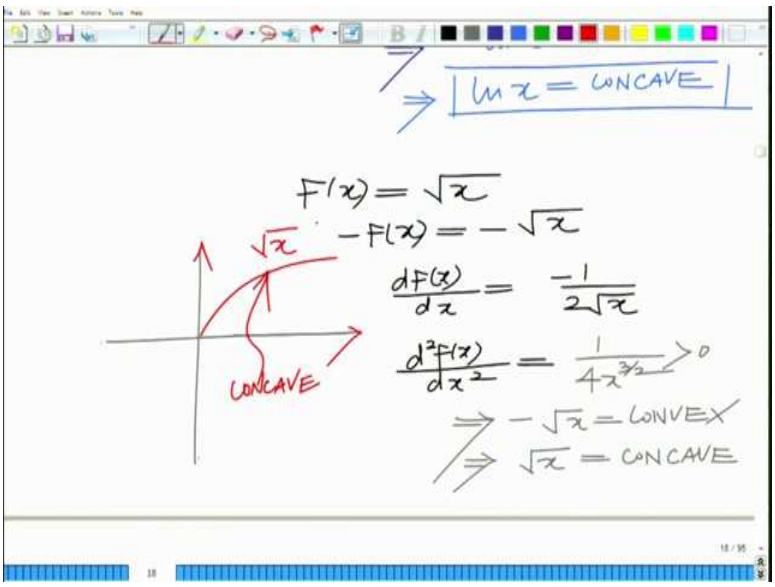

Again the function is

$$F(x) = \sqrt{x}$$

So the first derivative of minus of this function is

$$\frac{d(-F(x))}{dx} = \frac{d}{dx}\left(-\sqrt{x}\right)$$
$$= -\frac{1}{2\sqrt{x}}$$

And the second derivative of minus of this function is

$$\frac{d^2(-F(x))}{dx^2} = \frac{d}{dx}\left(-\frac{1}{2\sqrt{x}}\right)$$
$$= \frac{1}{4x^{\frac{3}{2}}}$$

Therefore, for all values of *x*;

$$\frac{d^2(-F(x))}{dx^2} \geq 0$$

This implies that $-\sqrt{x}$ is a convex function and hence $\sqrt{x}$ is a concave function.

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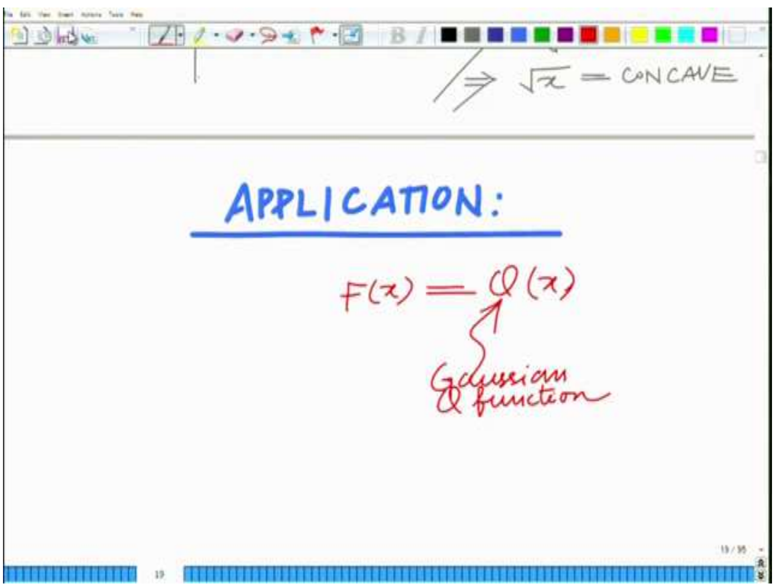

Let us look at a practical application of this. Consider the Gaussian Q function.

$F(x) = Q(x)$

This is the complementary cumulative distribution function (CCDF) of the standard Gaussian random variable with mean 0 and variance 1 and this is also known as the tail probability of the standard Gaussian random variable.

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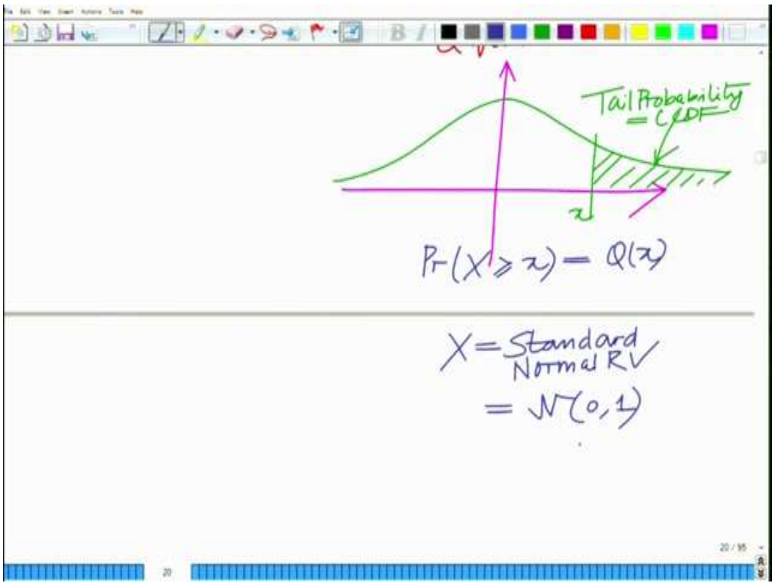

Therefore, $Q(x)$ is basically the probability that X is greater than or equal to x.

$Q(x) = \Pr(X \geq x)$

Where X is the standard normal random as $N(0,1)$. This function is convex.

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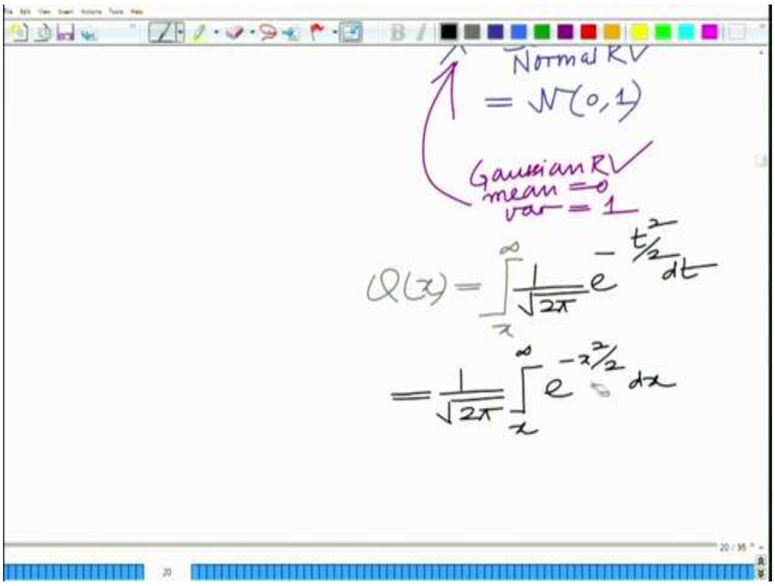

The expression for the Q function is given as follows.

$$Q(x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\frac{t^2}{2}} dt$$

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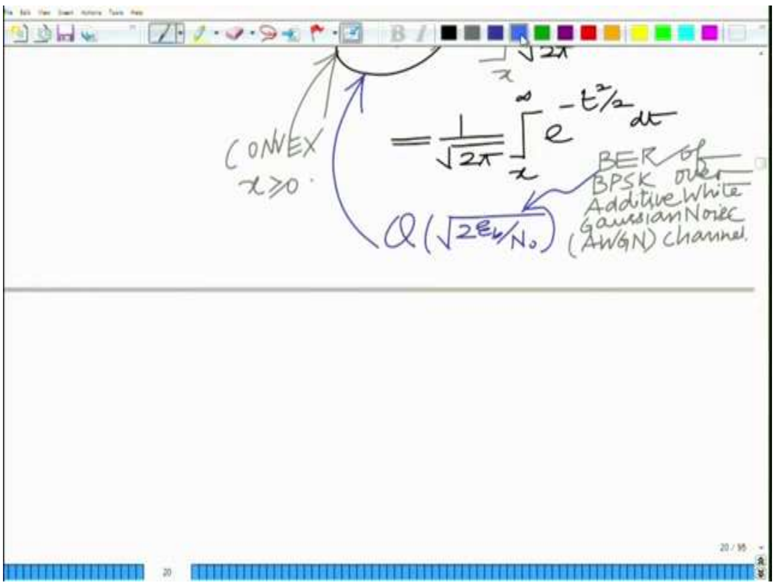

This Q function represents the bit error rate in the wireless communication. The BER of binary phase shift key (BPSK) signal over Additive White Gaussian Noise (AWGN) channel is denotes as $Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$. It has a lot of applications in communications as well as in signal processing.

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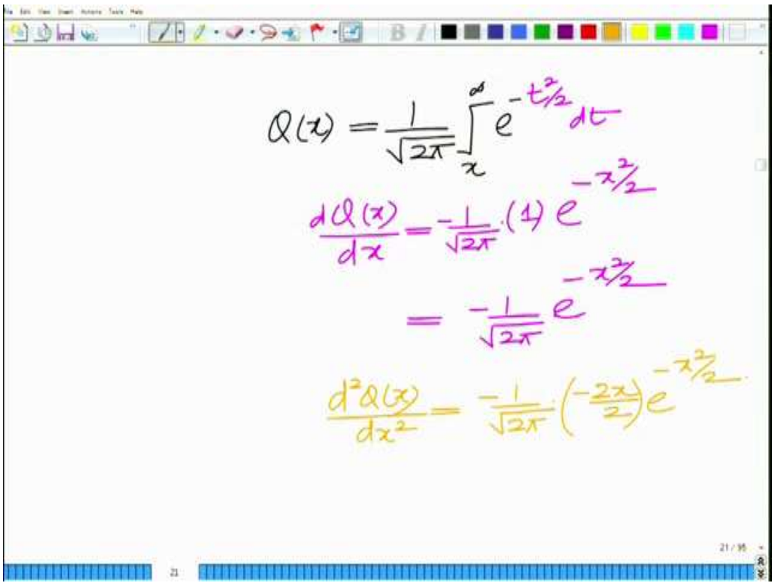

So, to confirm its convexity, let us take first derivative of this function

$$\frac{dQ(x)}{dx} = -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

The second derivative of this function is

$$\frac{d^2}{dx^2} Q(x) = -\frac{1}{\sqrt{2\pi}} \left(\frac{-2x}{2}\right) e^{-\frac{x^2}{2}}$$

$$= \frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

This means that $\frac{d^2}{dx^2} Q(x) \geq 0$ for $x \geq 0$. This implies that the Q function a convex function for $x \geq 0$. This is a very interesting property.

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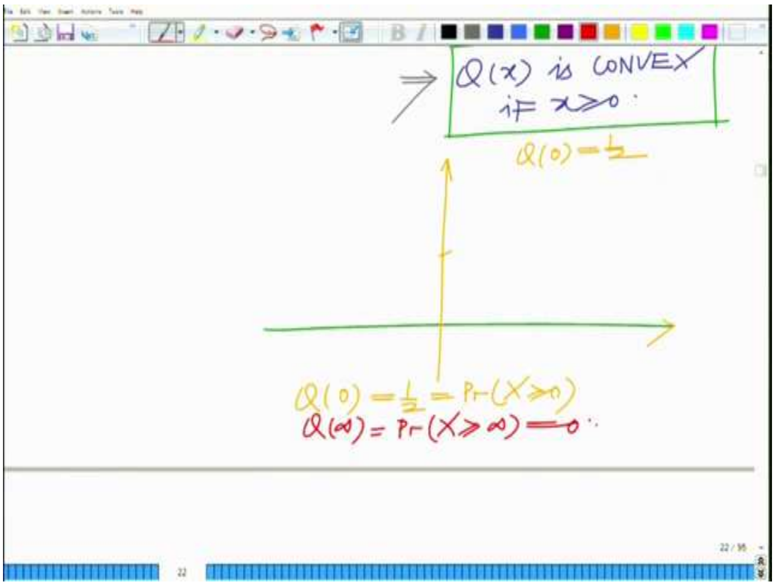

So at $x = 0$,

$$Q(0) = \Pr(X \geq 0) = \frac{1}{2}$$

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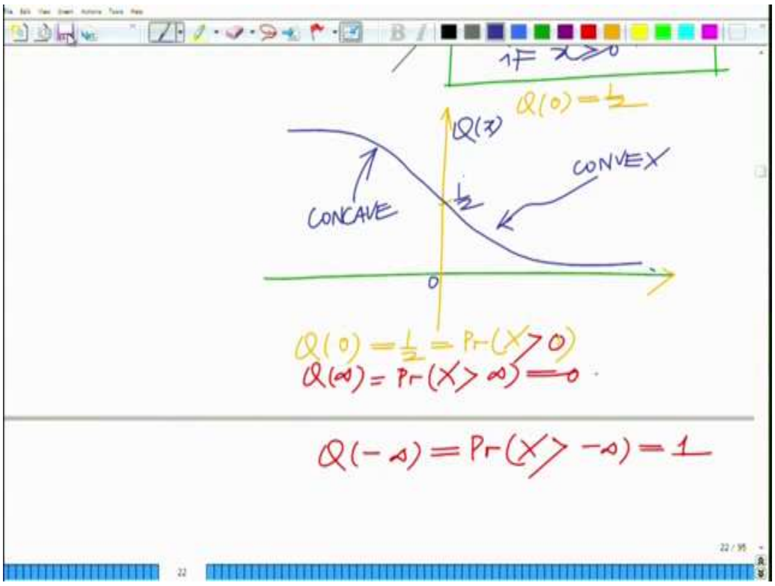

Also at $x = \infty$,

$$Q(\infty) = \Pr(X > \infty) = 0$$

And at $x = -\infty$,

$$Q(-\infty) = \Pr(X > -\infty) = 1$$

This means that this CCDF function decreasing on right hand side of $x = 0$, increasing on left hand side of $x = 0$ and have $\frac{1}{2}$ value at $x = 0$. Therefore this function is convex for $x \geq 0$ and concave for $x < 0$.

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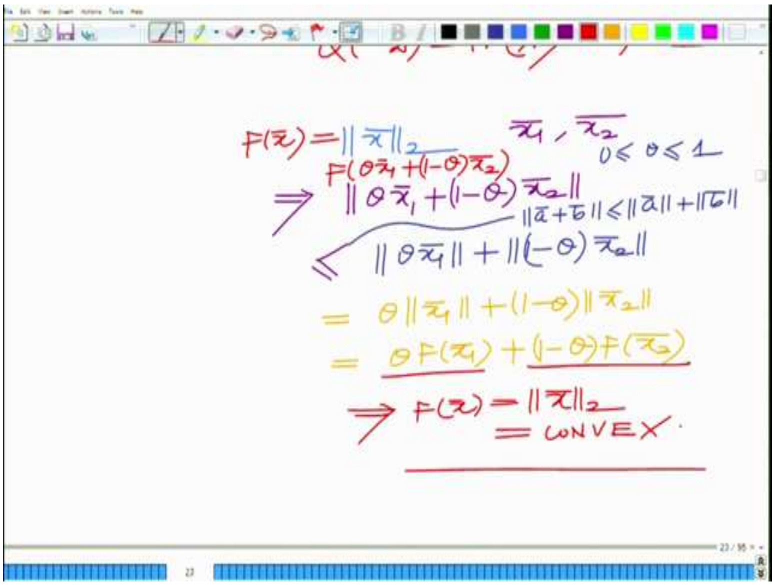

Let us discuss about the norm. Consider any two vectors $\bar{x}_1$ and $\bar{x}_2$ such that $\bar{x} \in \bar{x}_1, \bar{x}_2$. And the function is

$$F(\bar{x}) = \|\bar{x}\|_2$$

So for $0 \leq \theta \leq 1$, the convex combination of $\bar{x}_1$ and $\bar{x}_2$ is

$$F(\theta\bar{x}_1 + (1-\theta)\bar{x}_2) = \|\theta\bar{x}_1 + (1-\theta)\bar{x}_2\|$$

Now use the following triangle inequality.

$$\|\bar{a} + \bar{b}\| \leq \|\bar{a}\| + \|\bar{b}\|$$

Thus

$$
\begin{aligned}
F\left(\theta\overline{x}_1 + (1-\theta)\,\overline{x}_2\right) &\le \left\|\theta\overline{x}_1\right\| + \left\|(1-\theta)\,\overline{x}_2\right\| \\
&\le \theta\left\|\overline{x}_1\right\| + (1-\theta)\left\|\overline{x}_2\right\| \\
&\le \theta F\left(\overline{x}_1\right) + (1-\theta)\,F\left(\overline{x}_2\right)
\end{aligned}
$$

This shows that the l_2 norm is a convex function.