Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture-64
Relation between optimal value of Primal and Dual problems, concepts of Duality gap and Strong Duality

Keywords: *Primal problem, Dual problem, Strong Duality, Duality Gap*

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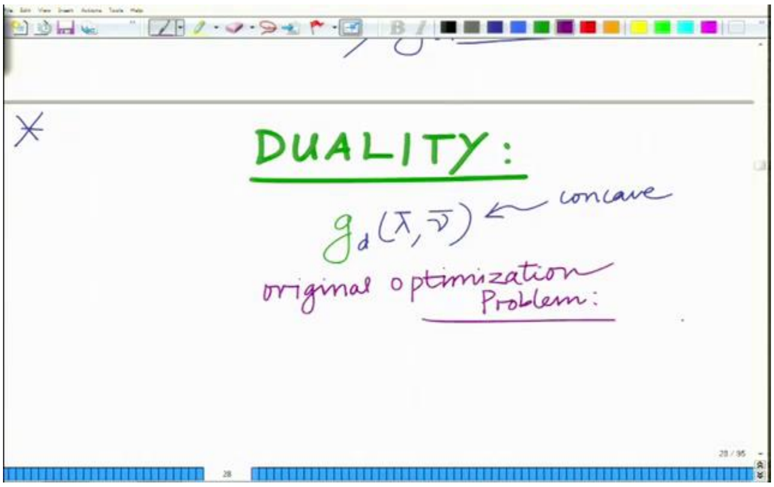

Hello, welcome to another module in this massive open online course and we are looking at the concept of duality for optimization. So let us go back to the original possibly not necessarily convex optimization problem.

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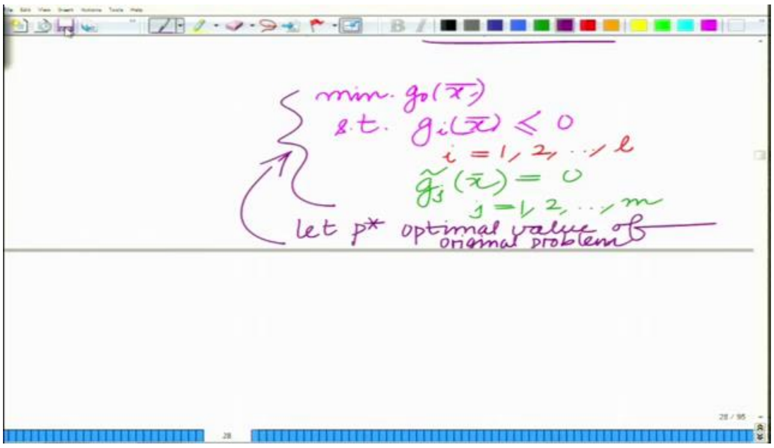

So we have

$$\begin{aligned} \min \quad & g_0(\bar{x}) \\ \text{s.t} \quad & g_i(\bar{x}) \le 0 \\ & i = 1, 2, .., l \\ & g_j(\bar{x}) = 0 \\ & j = 1, 2, .., m \end{aligned}$$

. Now, let P^* denote the optimal value of this original optimization problem.

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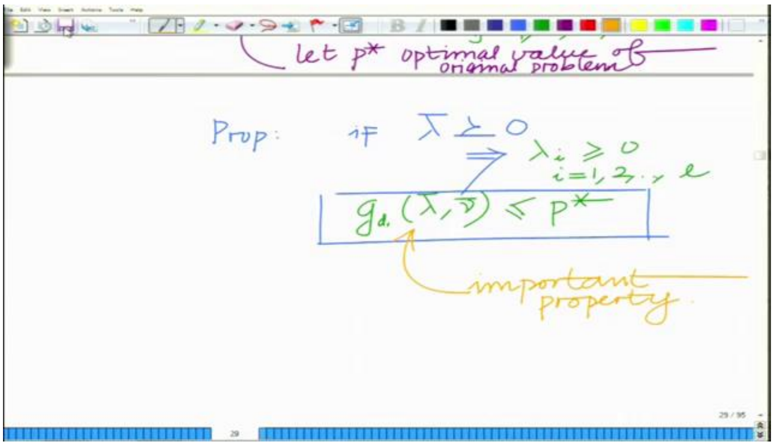

Now we want to show that if this vector of Lagrange multipliers associated with the inequality constraints that is if $\bar{\lambda} \ge 0 \Rightarrow \underset{i=1,2..,l}{\lambda_i \ge 0}$ then the dual that is $g_d(\bar{\lambda}, \bar{\nu}) \le P^*$. This is a very important property of the dual function.

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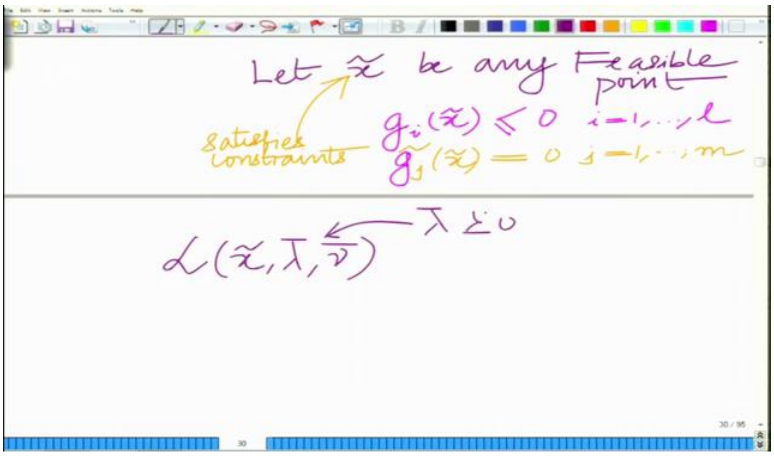

So let us try to demonstrate this, so let us start with the following. Let x be any feasible point, feasible point means, it satisfies the constraint in the sense that you have $g_i(x) \le 0$ i = 1, 2,.., l and $g_j(x) = 0 \; j = 1, 2, .., m$.

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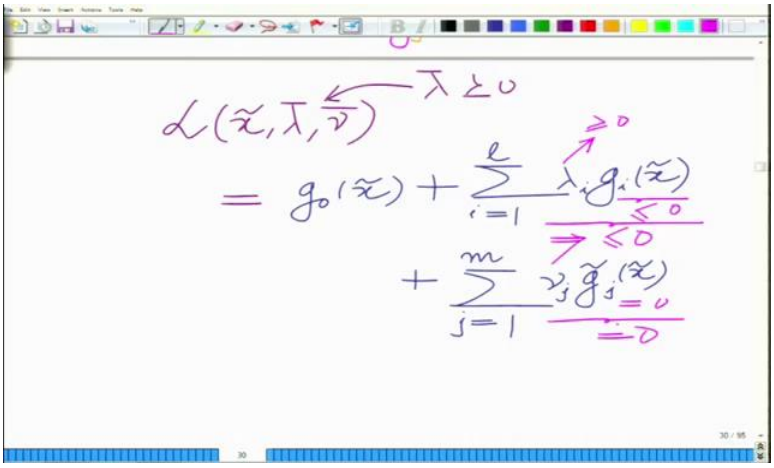

So the dual function is $L\left(x,\bar{\lambda},\bar{\nu}\right) = g_0(x) + \sum_{i=1}^{l} \lambda_i g_i(x) + \sum_{j=1}^{m} \nu_j g_j(x)$ and since each $\lambda_i \ge 0$ and this is shown in slide.

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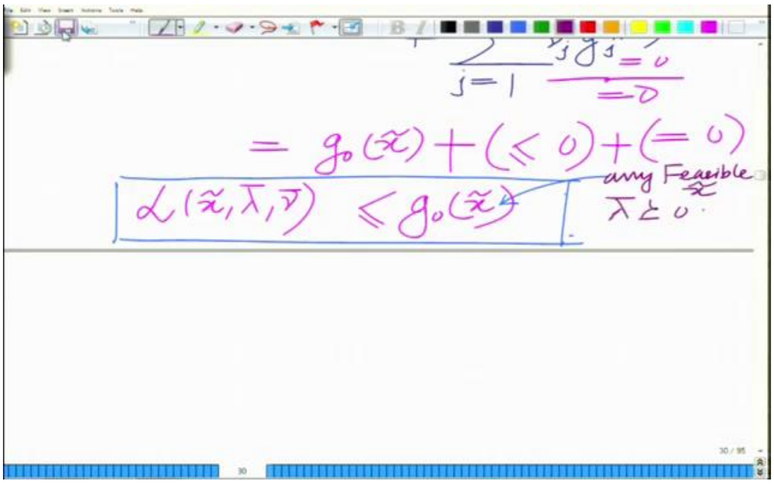

So we have finally $L\left(x,\bar{\lambda},\bar{\nu}\right) \le g_0(x)$. So this holds for any feasible point and as long as all the $\lambda_i \ge 0$.

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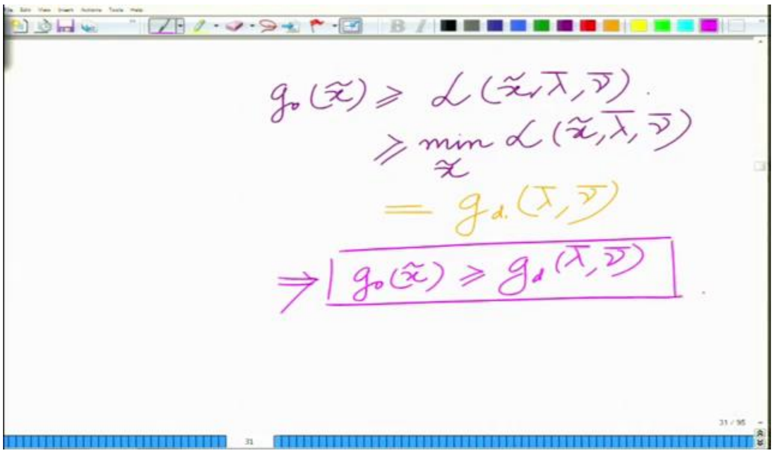

Now this can be rewritten as $g_0(x) \geq g_d(\bar{\lambda}, \bar{\nu})$ as shown in slide.

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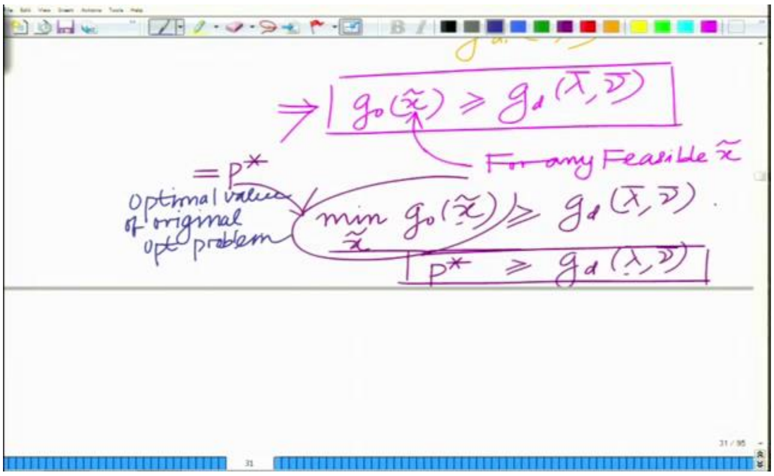

Now, if you take the minimum of this for any feasible x that is nothing but P^* which is the optimal value of the original optimization problem. So we have $P^* \geq g_d(\bar{\lambda}, \bar{\nu})$.

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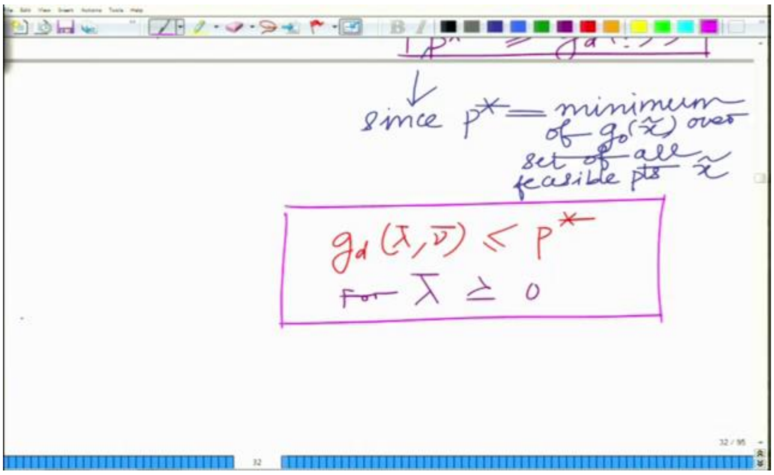

So we have $g_d(\bar{\lambda},\bar{\nu}) \leq P^*$ $for\ \bar{\lambda} \geq 0$. So this means that this Lagrange dual function forms a lower bound for this P^* which is the optimal value of the original optimization problem.

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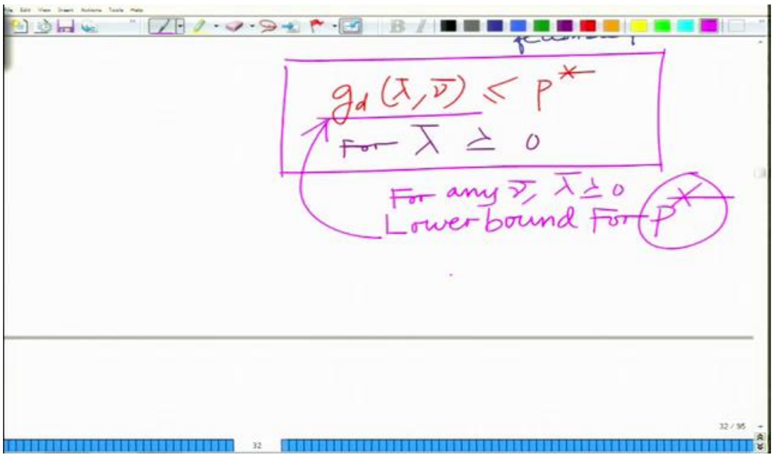

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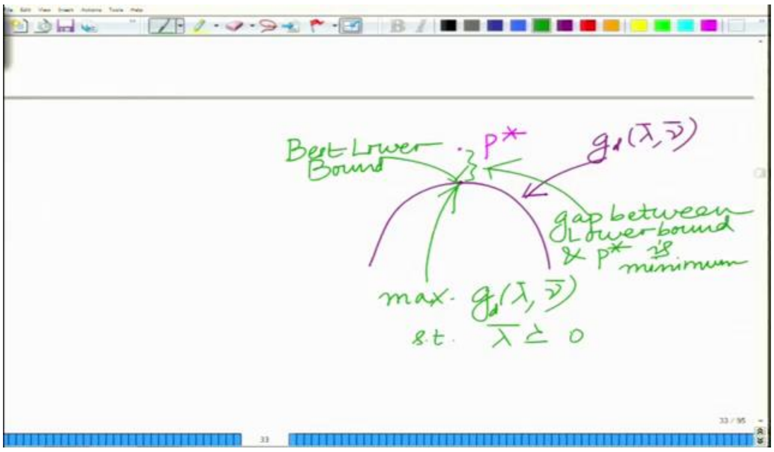

Now the best lower bound is the maximum value of this lower bound which is as close as the optimal value P^*, so that this gap between the lower bound and P^* is minimized. And that is basically given as $\begin{array}{ll} \max & g_d(\bar{\lambda}, \bar{\nu}) \\ s.t & \bar{\lambda} \geq 0 \end{array}$.

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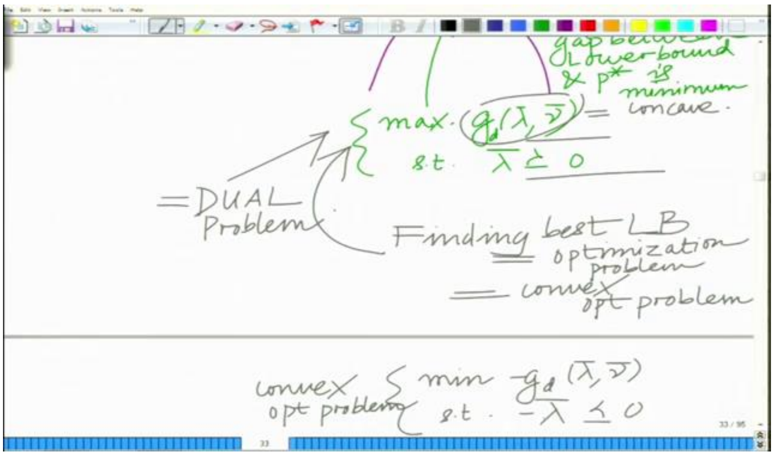

Now we can see that the best lower bound is also an optimization problem, although the original problem need not be convex. So I can equivalently write this as $\begin{array}{ll} \min & -g_d(\bar{\lambda}, \bar{\nu}) \\ s.t & -\bar{\lambda} \geq 0 \end{array}$.

So you can use all the techniques of convex optimization to conveniently solve the dual problem.

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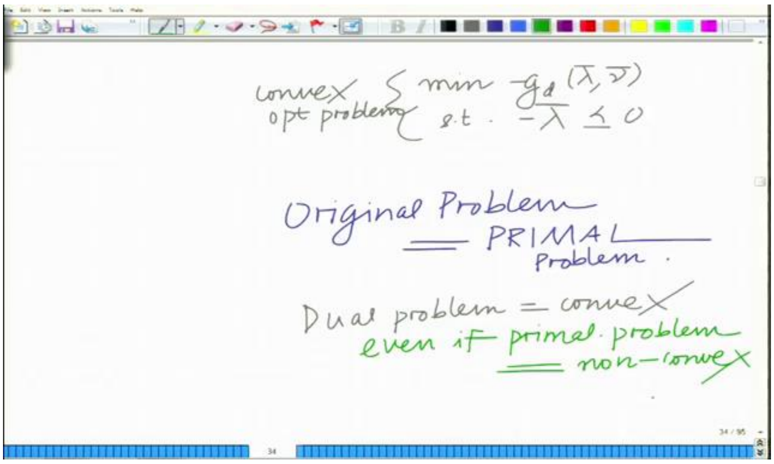

The original problem is termed as the primal problem and even if the primal problem is non-convex, the equivalent dual problem that is derived from the primal problem is convex. And therefore, one can conveniently use all the techniques of convex optimization to solve the Lagrange dual problem.

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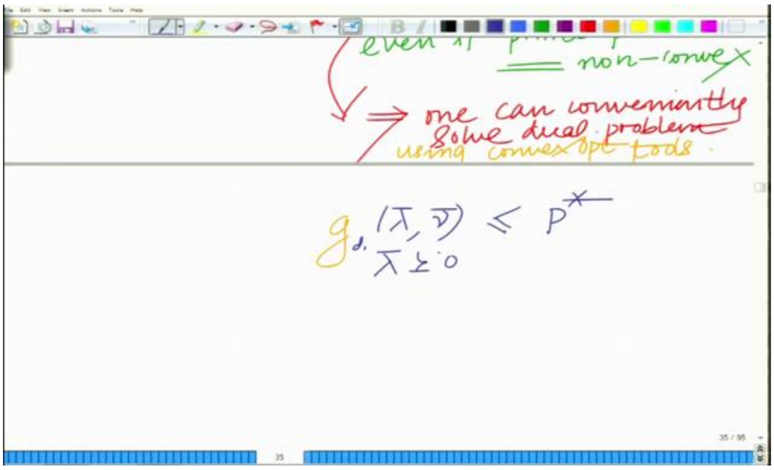

Since you are taking the best lower bound that is going to give you something that is as close as possible to the optimal value P^*, but still it is going to be lower than P^*. So what you get by solving the dual optimization problem is always going to be the best lower bound, but still it is lower than P^*.

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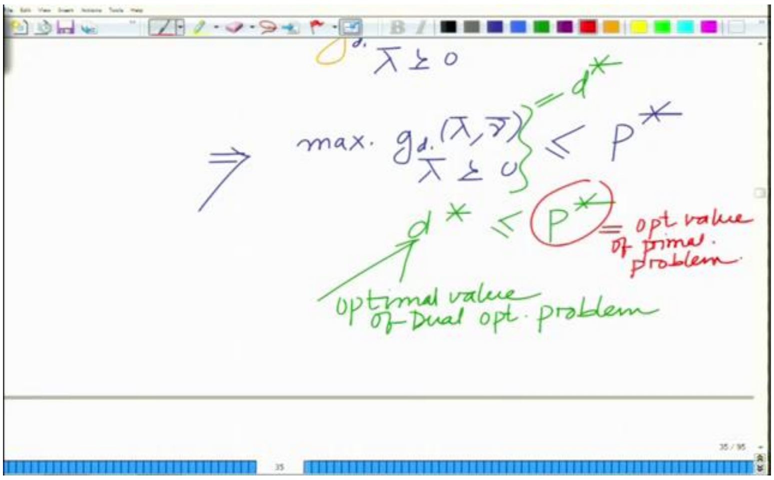

So if we take the maximum for some optimal value of $\bar{\lambda}, \bar{\nu}$, this is still going to be less than or equal to P*. And therefore if you call this optimal value as d*, we have $d^* \leq P^*$. So d* is the optimal value of the dual optimization problem and P* is the optimal value of the primal problem

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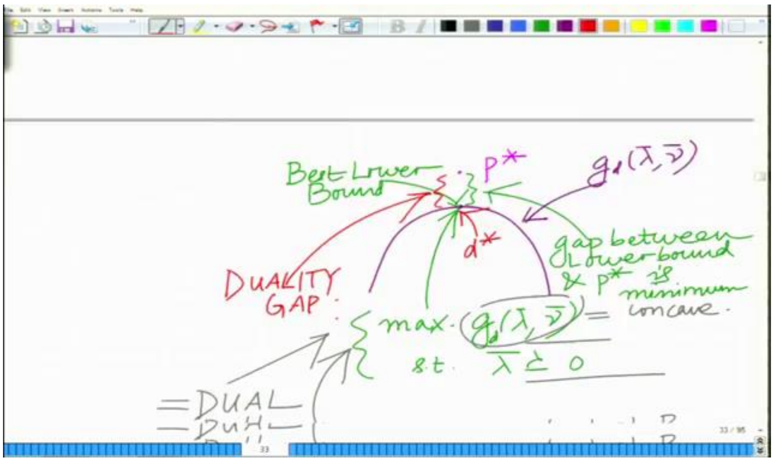

And this gap between d* and P* is the duality gap.

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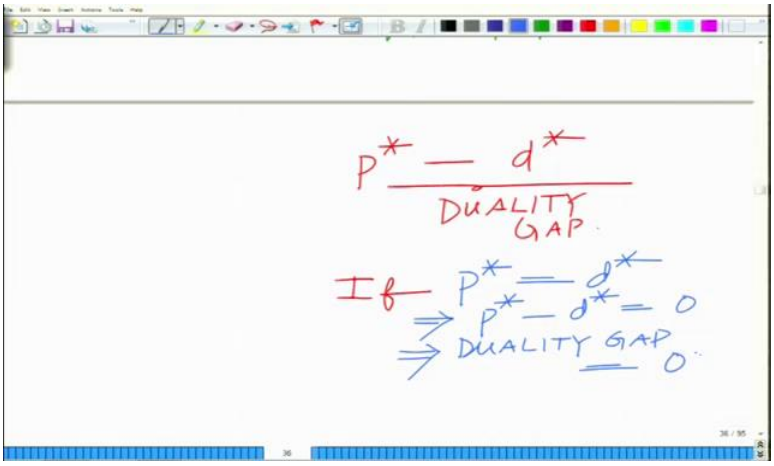

So if $P^* = d^*$, that implies $P^* - d^* = 0$, that implies duality gap is 0.

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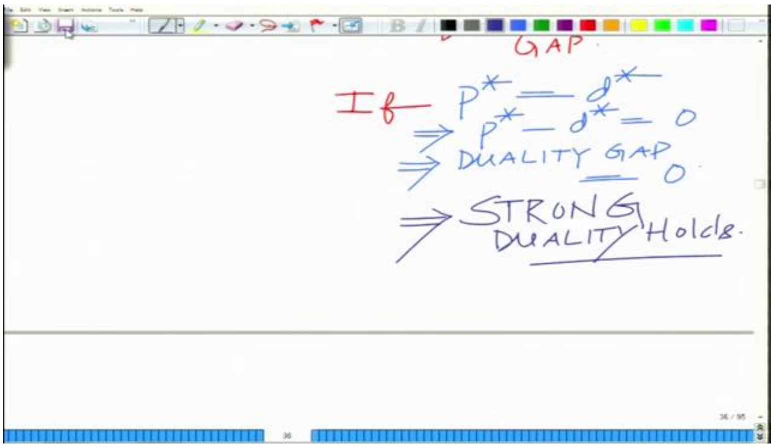

When this happens, it is said that strong duality holds for the problem.

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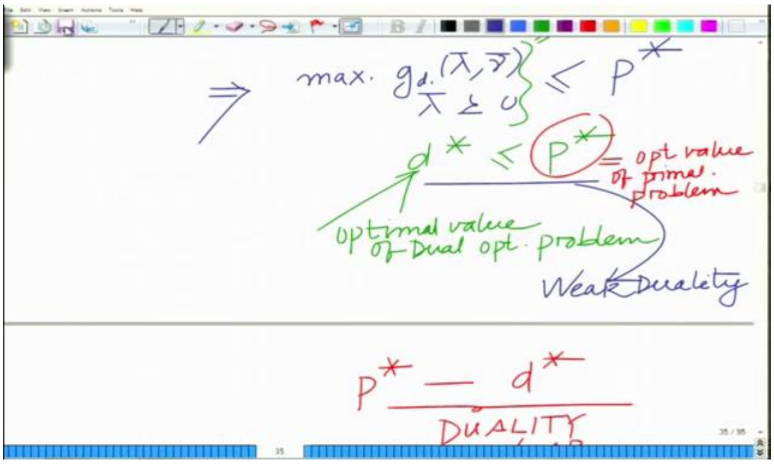

Otherwise, if $d^* \leq P^*$, this is weak duality, it always holds.

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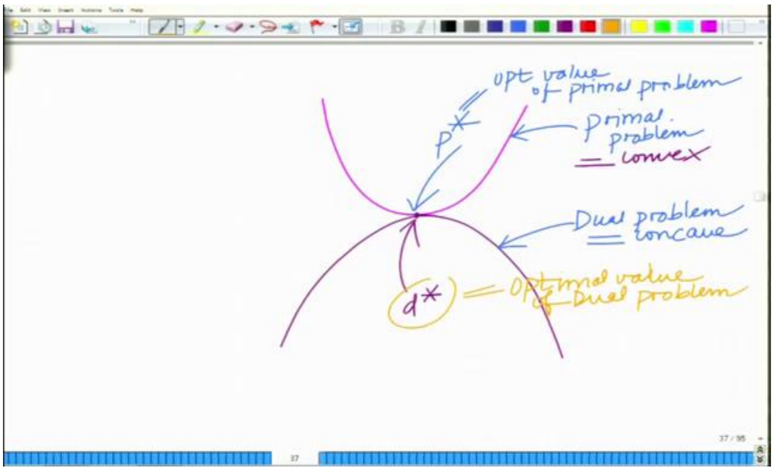

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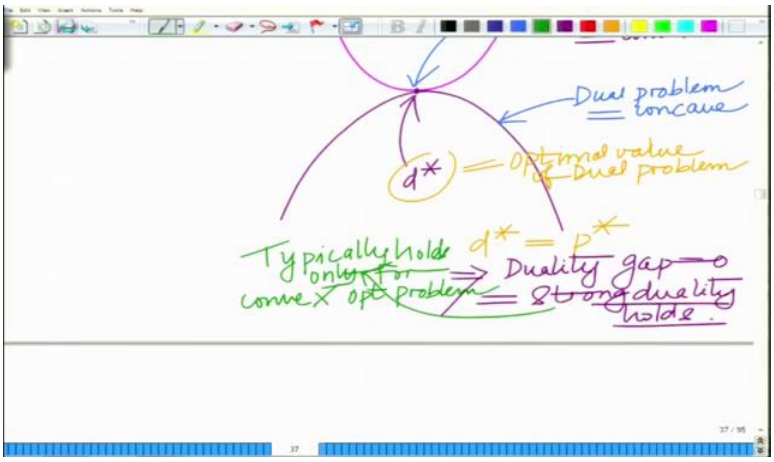

And typically the strong duality holds for any convex problem, although one can form the dual optimization problem, solve the dual optimization problem for any possibly non-convex problem. So the primal or dual, they always go hand in hand for any optimization problem in particular for a convex optimization problem, because the duality gap is 0. So we will stop here and continue in the subsequent modules. Thank you very much.