Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture - 29
Properties of Convex Functions: Operations that preserve Convexity

Hello. Welcome to another module in this massive open online course. In this module, let us start looking at the operations on convex functions that preserve convexity.

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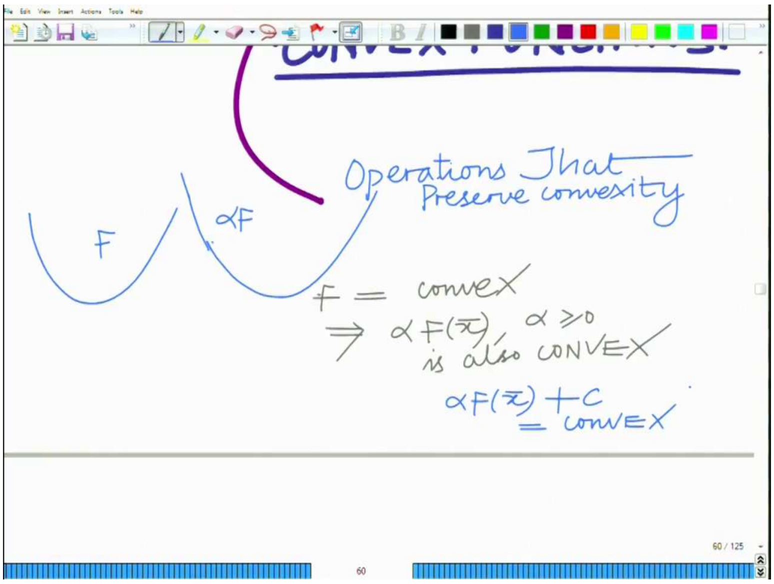

Let function $F(\bar{x})$ is a convex function to discuss some properties of convex function. So for a convex function $F(\bar{x})$; $\alpha F(\bar{x})$ is also convex for $\alpha \geq 0$. This implies that scaling by a non-negative number preserves convexity. Similarly, translation does not change convexity. Thus, for any constant c and $\alpha \geq 0$; it can be concluded $\alpha F(\bar{x})+c$ is also convex.

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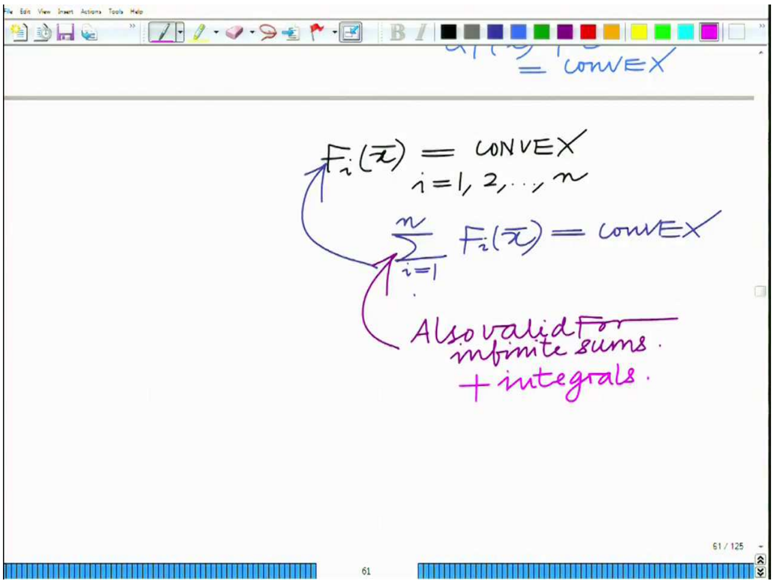

Also, if the several functions of $\bar{x}$ i.e. $F_i(\bar{x})$ are all convex for $i = 1, 2, \ldots, n$, then their sum $\sum_{i=1}^{n} F_i(\bar{x})$ is also convex. This property extends to infinite sum and integrals which is a continuous sum.

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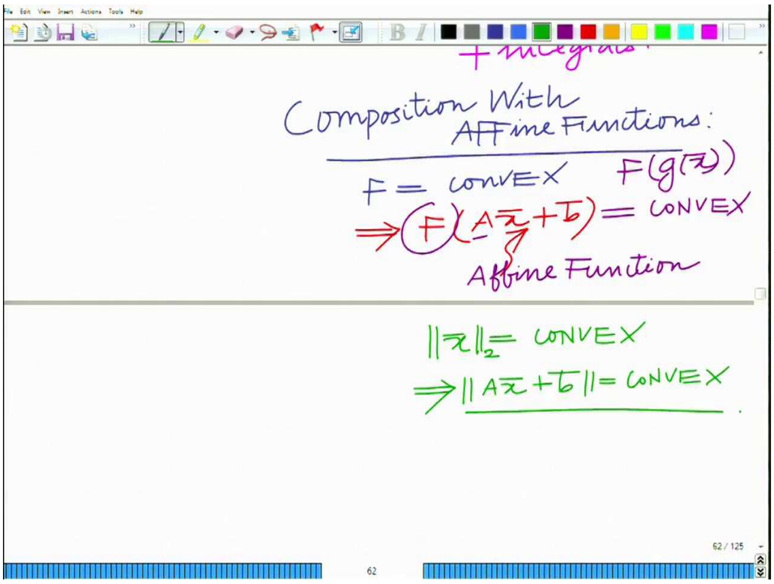

Another property is that the composition of a convex function with affine functions is also a convex function. Thus, for an affine function $A\bar{x}+\bar{b}$; $F\left(A\bar{x}+\bar{b}\right)$ is also a convex function. So for example, as l_2 norm $\|\bar{x}\|_2$ is a convex function so $\|A\bar{x}+\bar{b}\|$ is also convex.

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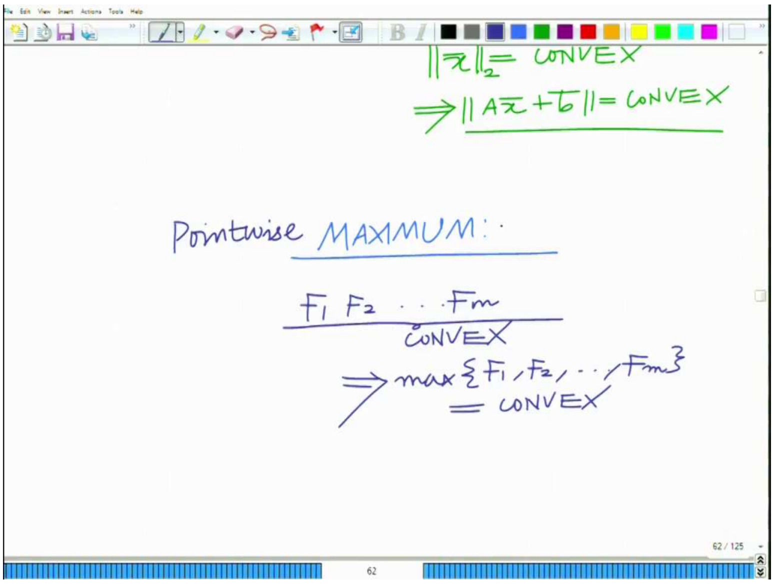

Another interesting property is that the point wise maximum of several convex functions is also a convex function. This has a lot of interesting applications. So if $F_1(\bar{x}), F_2(\bar{x}), \ldots, F_m(\bar{x})$ are convex, then $\max\{F_1(\bar{x}), F_2(\bar{x}), \ldots, F_m(\bar{x})\}$ is also convex where it is point wise maximum.

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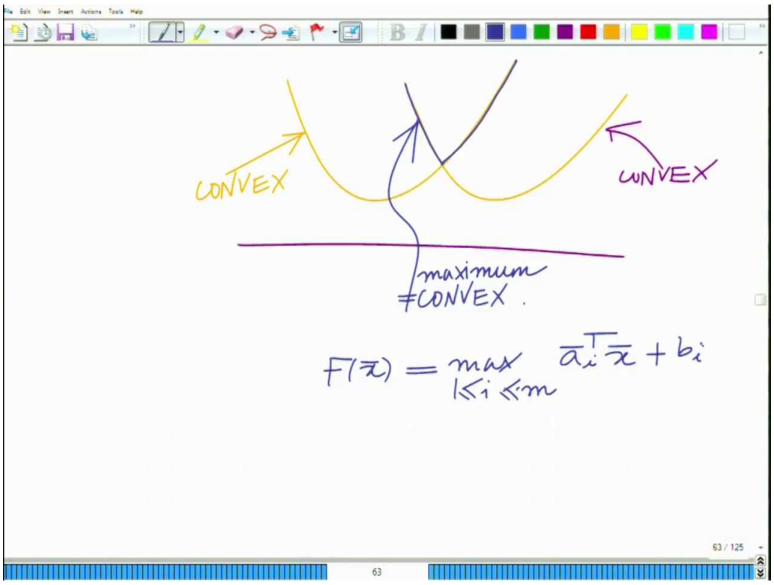

For instance, let us take the maximum of a set of linear functions, that is

$$F\left(\overline{x}\right)=\max_{1\le i\le m}\overline{a}_i^T\overline{x}+b_i$$

This is known as a piecewise linear function.

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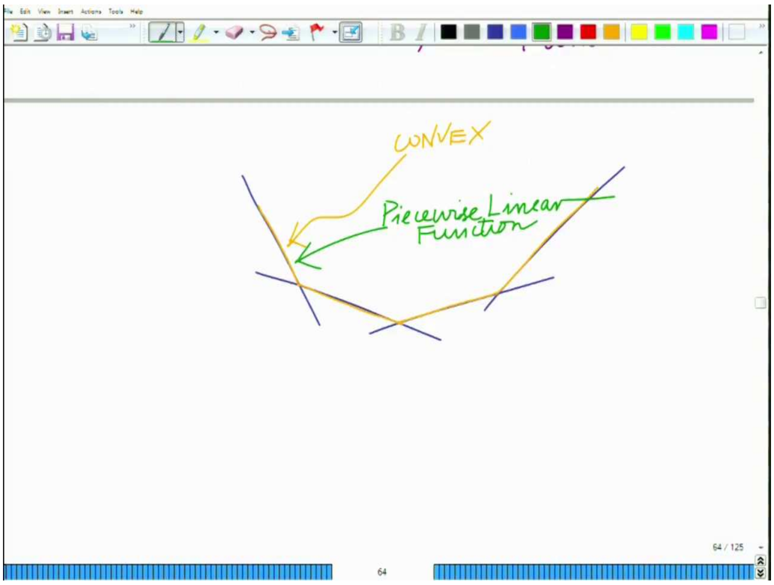

So, the maximum of several linear functions is a piecewise linear function which is also convex. This is because linear functions are basically hyper planes. This means these are convex functions and the maximum of convex functions is also convex. Therefore, the piecewise linear function is also convex.

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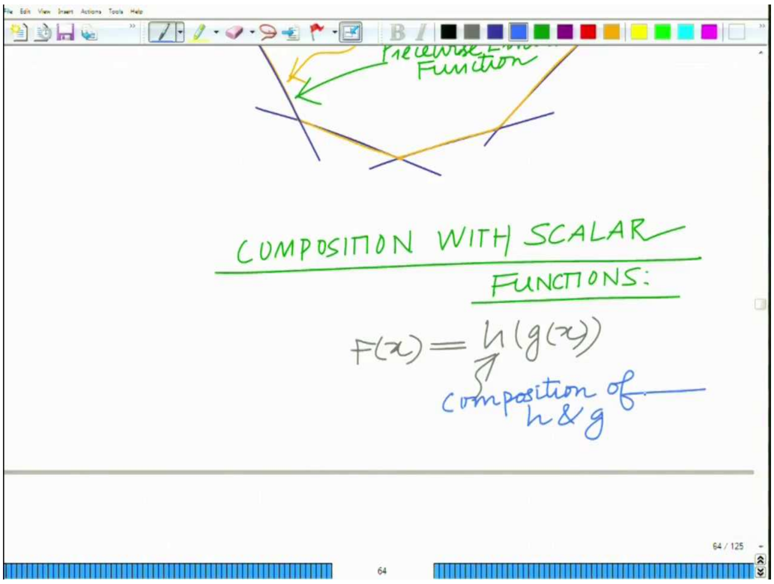

Another concept is the composition of functions. Let us consider a simple case of composition of scalar functions.

$$F(x) = h(g(x))$$

So this function $F(x)$ is a composition of function *h* with function g and this is a convex function for two conditions. First condition is that if function *g* and *h* both are convex functions and function *h* is a non-decreasing function, then their above mentioned composition function $F(x)$ is also convex. Second condition is that if function *g* is a concave function and function *h* is a non-increasing convex function, then their above mentioned composition function $F(x)$ is also convex.

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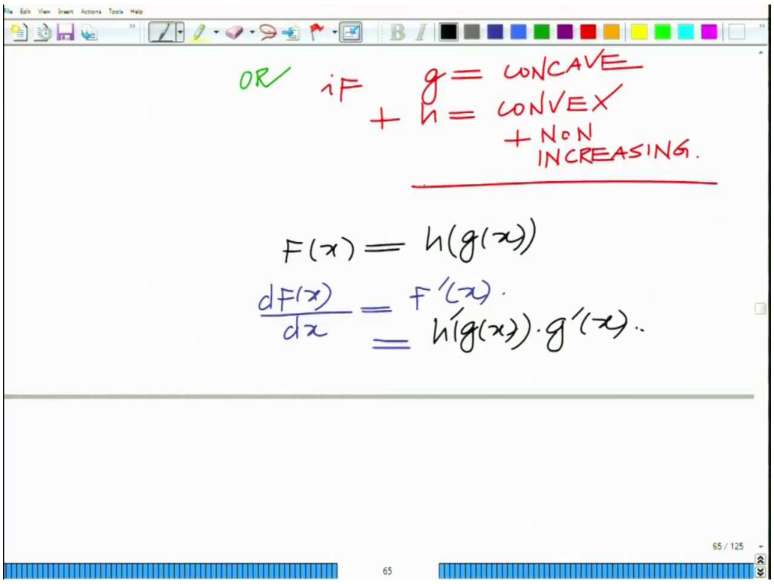

Let us use the derivative test to demonstrate this convexity assuming the functions are differentiable. So take the first order derivative of $F(x)$ using the chain rule.

$$F'(x) = h(g(x)).g'(x)$$

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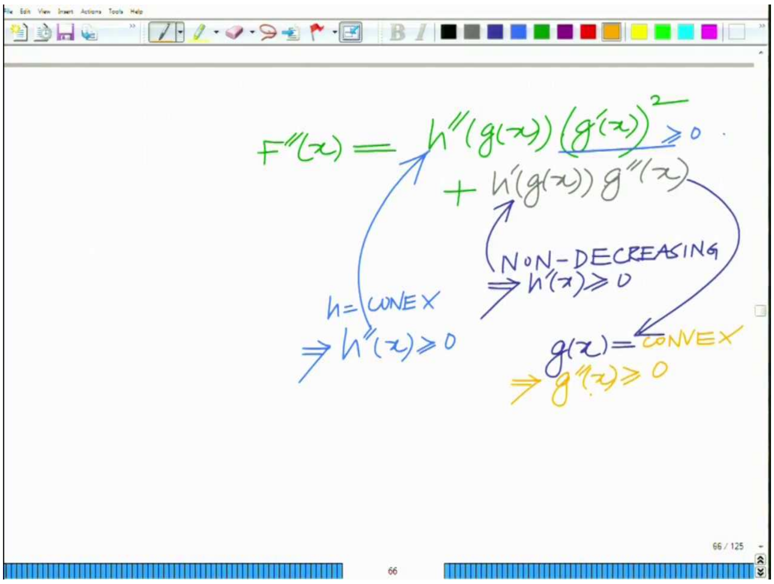

Further, the second order derivative of $F(x)$ is

$$F''(x) = h''(g(x)).(g'(x))^2 + h'(g(x)).g''(x)$$

First consider that $h(x)$ and $g(x)$ are convex. This means

$$h''(x) \geq 0$$

And

$$g''(x) \geq 0$$

Also if $h(x)$ is a non-decreasing function this means

$$h'(x) \geq 0$$

As all four components of $F''(x)$ is greater than equal to 0. This means that $F''(x) \geq 0$ which implies that $F(x)$ is convex. This proves the first condition.

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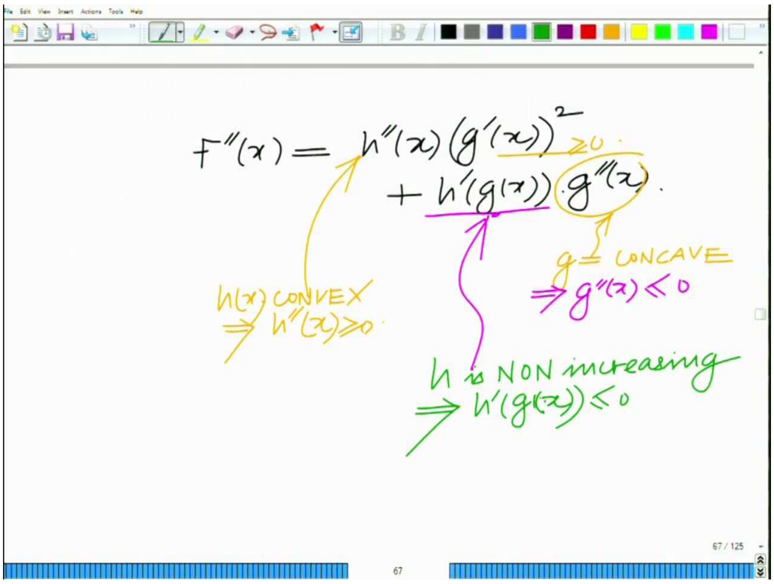

Similarly; for the other condition, consider that $h(x)$ is convex and $g(x)$ is concave. This means

$$h''(x) \geq 0$$

And

$g''(x) \leq 0$

Also if $h(x)$ is a non-increasing function this means

$h'(g(x)) \leq 0$

The product of two negatives makes a positive, so

$h'(g(x)).g''(x) \geq 0$

Thus, as all four components of $F''(x)$ is greater than equal to 0. This means that $F''(x) \geq 0$ which implies that $F(x)$ is convex. This proves the second condition.

Therefore these are the two conditions that ensure that the composition $h(g(x))$ is also convex.

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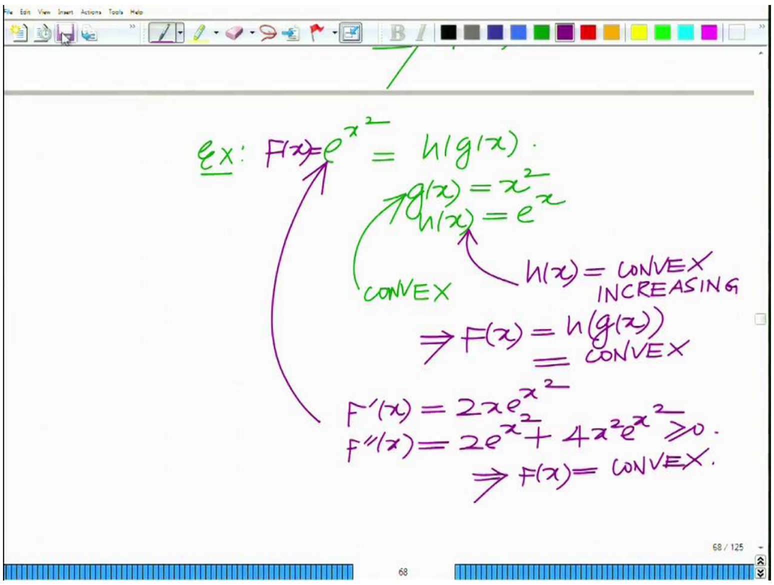

Let us now look at a simple example.

$F(x) = h(g(x)) = e^{x^2}$

For

$$g(x) = x^2$$
$$h(x) = e^x$$

So, $g(x)$ is a convex function and $h(x)$ is a non-decreasing convex function this implies that $F(x)$ is also convex. To show this, first order derivative of $F(x)$ is

$$F'(x) = 2xe^{x^2}$$

And second order derivative of $F(x)$ is

$$F''(x) = 2e^{x^2} + 4x^2e^{x^2}$$

It is clear that

$$F''(x) \geq 0$$

This implies that $F(x)$ is convex.

Similarly, one can derive these conditions for the concavity of composition of functions.