

Applied Optimization for Wireless, Machine Learning, Big Data
Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

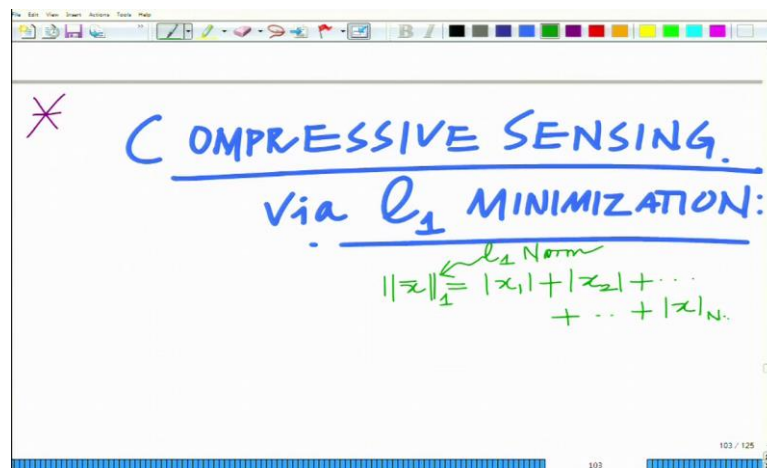
Lecture – 59

Practical Application: L_1 norm minimization and Regularization approach for Compressive Sensing Optimization Problem

Keywords: L_1 norm minimization, Regularization

Hello, welcome to another module in this massive open online course. We were looking at compressive sensing and we have discussed the orthogonal matching pursuit. Let us look at another completely different and radical approach to tackle this compressive sensing problem.

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So we want to look at compressive sensing via L_1 norm minimization. We have seen this L_1 norm of a vector that is if you have an N-dimensional vector \bar{x} the L_1 norm is simply the sum of the magnitudes of the components of \bar{x} . That is we have

$$\|\bar{x}\|_1 = |x_1| + |x_2| + \dots + |x_N|.$$

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Via L_1 MINIMIZATION:

$\|x\|_1 = |x_1| + |x_2| + \dots + |x_N|$ (labeled L_1 Norm)

$\min \|x\|_0$ (labeled L_0 Norm)

s.t. $y = \Phi x$

Φ is $M \times N$, $M \leq N$.

The compressive sensing problem is the following thing that is $\min_{s.t. y = \Phi x} \|x\|_0$. One of the fundamental results in compressive sensing is that this l_0 norm minimization can be replaced by l_1 norm minimization and still you can recover the sparse vector \bar{x} .

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Fundamental Result in CS.

Objective = convex

$\min_{s.t. y = \Phi x} \|x\|_1$

Replacing L_0 Norm By L_1 norm

Yield identical sparse vector \bar{x}

L_1 norm enforces sparsity!

L_1 Norm = convex!

L_0 Norm = Non-convex

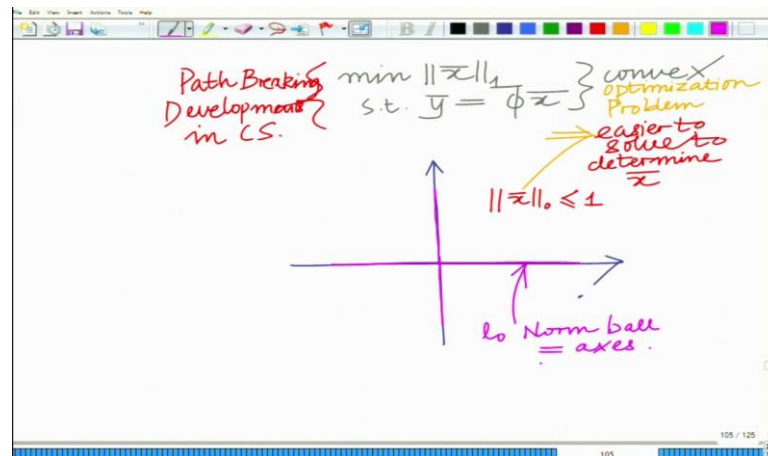
Affine \Rightarrow Problem is convex!

So we have $\min_{s.t. y = \Phi x} \|x\|_1$ what this says is that the l_1 norm also enforces sparsity. And it can

be shown that for a large number of scenarios or with very high probability \bar{x} that is obtained as a solution of both these above optimization problems is the same. The significant advantage is, the l_1 norm is convex in nature and the l_0 norm is highly non-convex. So therefore if you look at this optimization problem, the objective is convex

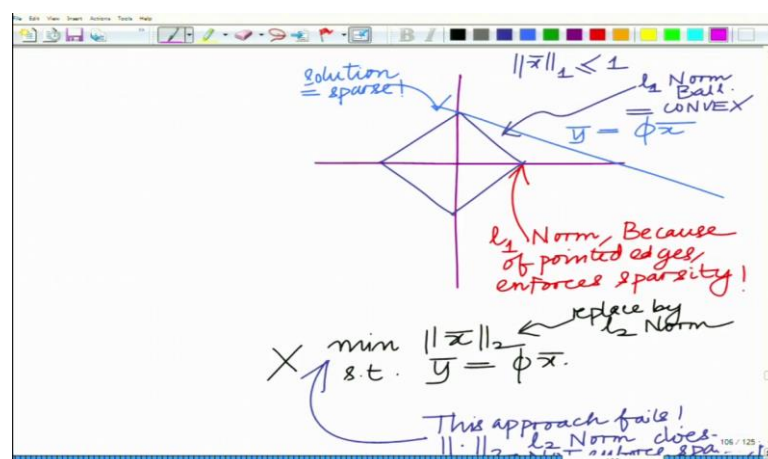
and the constraint is an affine constraint. So we are converting a problem which was previously highly non-convex into something that is convex. So this is much easier to solve and determine the sparse vector \bar{x} .

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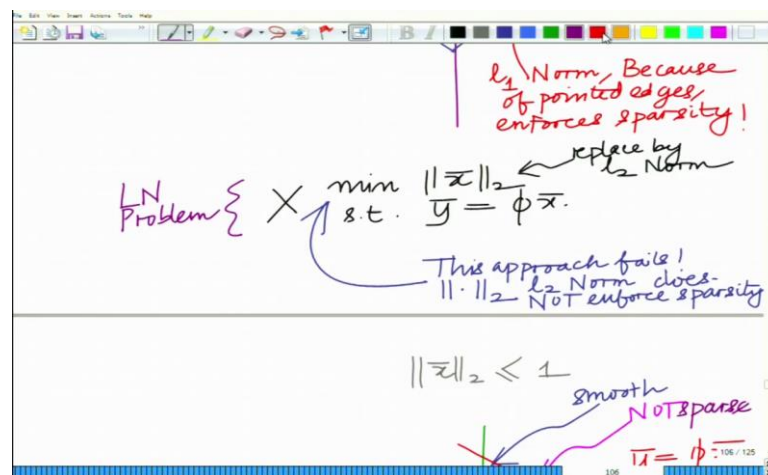
So this is one of the path breaking developments in compressive sensing that is demonstrating that the l_0 norm minimization is equivalent in a large number of scenarios to the l_1 norm minimization. If you look at the l_0 norm ball that is $\|\bar{x}\|_0 \leq 1$, it is simply along the axis, so this is highly non-convex.

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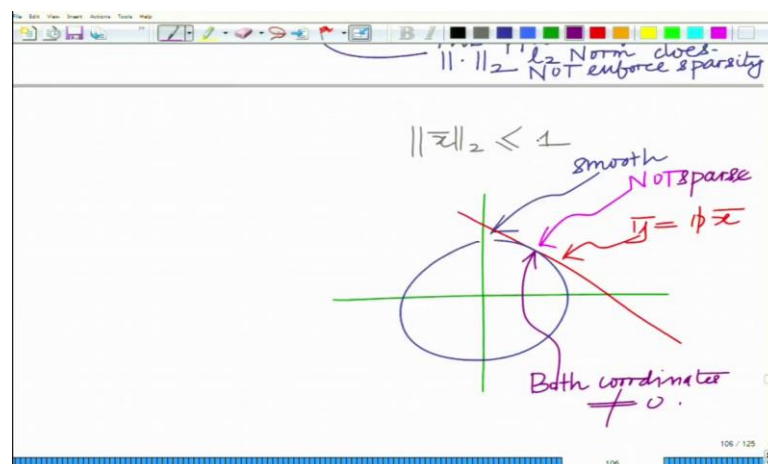
However the l_1 norm ball looks like a diamond shaped object which is a convex shape. And now if you enforce this affine constraint, which is nothing but a line, it intersects at one of these pointed edges, implies the solution is sparse.

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Now, if you replace it by the l_2 norm this approach fails because the l_2 norm does not enforce sparsity.

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The l_2 norm is smooth with no pointed edges and if you look at this affine constraint it intersects at a point which is not sparse.

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The image shows a digital whiteboard with a toolbar at the top. The handwritten text is as follows:

$$\begin{aligned} \min \quad & \|\bar{x}\|_1 \\ \text{s.t.} \quad & \bar{y} = \phi \bar{x} \\ & \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \end{aligned}$$
$$= \min \quad |x_1| + |x_2| + \dots + |x_N|$$
$$\text{s.t.} \quad \bar{y} = \phi \bar{x}$$

At the bottom right of the whiteboard, the text "108 / 125" and the number "108" are visible.

The l_1 norm minimization problem can be further simplified using the epigraph form as shown in the slides below.

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The image shows a digital whiteboard with a toolbar at the top. The handwritten text is as follows:

$$\begin{aligned} \min \quad & \bar{y} = \phi \bar{x} \\ \text{s.t.} \quad & |x_1| \leq t_1 \\ & |x_2| \leq t_2 \\ & \vdots \\ & |x_N| \leq t_N \end{aligned}$$

A green arrow points from the text "Epigraph Form" to the constraints. Below a horizontal line, the following inequalities are written:

$$\begin{aligned} \Rightarrow \quad & -t_1 \leq x_1 \leq t_1 \\ & -t_2 \leq x_2 \leq t_2 \\ & \vdots \\ & -t_N \leq x_N \leq t_N \end{aligned}$$

At the bottom right of the whiteboard, the text "109 / 125" and the number "109" are visible.

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$$E = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_N \end{bmatrix} \quad \min \quad t_1 + t_2 + \dots + t_N$$

$$s.t. \quad \begin{cases} -t_1 \leq x_1 \leq t_1 \\ -t_2 \leq x_2 \leq t_2 \\ \vdots \\ -t_N \leq x_N \leq t_N \end{cases}$$

Linear inequality

$$1^T E = 1^T x$$

$$-E \leq x \leq E$$

$$y = \phi x$$

Affine constraints

$$\min t_1 + t_2 + \dots + t_N$$

$$s.t. \quad \begin{cases} -t_1 \leq x_1 \leq t_1 \\ -t_2 \leq x_2 \leq t_2 \\ \vdots \\ -t_N \leq x_N \leq t_N \end{cases}$$

Therefore, this optimization problem can be written as and this set of

$$\begin{cases} -t_N \leq x_N \leq t_N \\ y = \phi x \end{cases}$$

linear inequalities is also known as box constraint. So these are your linear inequality constraints.

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$$1 = [1 \ 1 \ \dots \ 1]^T$$

$$\min \quad 1^T E$$

$$s.t. \quad -E \leq x \leq E$$

Component-wise inequality

$$y = \phi x$$

Affine constraints

Linear Program! (LP)

$$\min \|\bar{x}\|_1$$

So this can also be written as $\begin{matrix} \text{s.t.} \\ -t \leq \bar{x} \leq t \\ y = \phi \bar{x} \end{matrix}$. So this is the component wise inequality and

you can see these are linear inequalities, affine constraint and linear objective. So the compressive sensing problem to estimate the sparse vector \bar{x} reduces to a linear program for which there are efficient techniques to solve.

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Handwritten notes on a whiteboard:

- min $\|\bar{x}\|_1$
- s.t. $-\bar{t} \leq \bar{x} \leq \bar{t}$
- $y = \phi \bar{x}$
- Component wise inequality
- \Rightarrow Linear Program! (LP)
- Can be solved very efficiently!
- \Rightarrow Revolutionized Compressive sensing

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Handwritten notes on a whiteboard:

- In presence of Noise
- $\bar{y} = \phi \bar{x} + \bar{n}$
- $\bar{x} = \text{sparse}$
- min $\|\bar{y} - \phi \bar{x}\|_2 + \lambda \|\bar{x}\|_1$
- minimizes Approximation error
- enforces sparsity

Now so far we have considered a noiseless observation model. Now in the presence of noise you have the observation model $\bar{y} = \phi \bar{x} + \bar{n}$ and the vector \bar{x} is sparse. Now previously you minimize the least squares. Now here you have $\min \|\bar{y} - \phi \bar{x}\|_2 + \lambda \|\bar{x}\|_1$ that

is the approximation or fit error or the observation model error and this additional term enforces sparsity.

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The image shows a whiteboard with handwritten notes in purple and red ink. At the top, it says $\bar{x} = \text{sparse}$. Below this, the minimization problem is written as $\min \|y - \phi \bar{x}\|_2 + \lambda \|\bar{x}\|_1$. A purple arrow points from the text "minimizes Approximation error" to the first term $\|y - \phi \bar{x}\|_2$. Another purple arrow points from the text "enforces sparsity" to the second term $\lambda \|\bar{x}\|_1$. A red arrow points from the text " λ regularization component" to the λ in the second term. Below that, it says "Regularization Parameter λ " with a circled λ and a red arrow pointing to it. At the bottom, it says "Has to be determined for problem under consideration" in yellow ink.

$$\bar{x} = \text{sparse}$$
$$\min \|y - \phi \bar{x}\|_2 + \lambda \|\bar{x}\|_1$$

minimizes Approximation error

enforces sparsity

λ regularization component

Regularization Parameter λ

Has to be determined for problem under consideration

So basically you are adding an l_1 regularization component and this λ is termed as the regularization parameter. This has to be determined for the problem under consideration. So here you are recovering the sparse vector as well as at the same time minimizing the approximation error. So basically this is known as the regularized version of the previous problem. So we will stop here. Thank you very much.