

Applied Optimization for Wireless, Machine Learning, Big Data  
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Lecture – 69

Example problem on Optimal MIMO Power allocation (Waterfilling)

**Keywords:** Optimal MIMO Power allocation, Waterfilling Algorithm, Singular Value Decomposition(SVD)

Hello, welcome to another module in this massive open online course. So we are looking at Optimal MIMO Power Allocation. Now let us do an example to understand this better.

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EXAMPLE: OPTIMAL MIMO POWER ALLOCATION:

Consider  $H = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}$  (channel matrix)

$r \times t$   
 $2 \times 2$  MIMO channel.  
 $r = t = 2$

Total power  $P = 4$   
Noise power  $\sigma^2 = 2 = 3\text{dB}$

First Find SVD of  $H$ .

So now consider the MIMO channel matrix  $H = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$ .

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Consider  $H = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}$

Total power  $P = 4$   
Noise power  $\sigma^2 = 2 = 3\text{dB}$

First Find SVD of  $H$ .

We have a total power  $P = 4$  and noise power  $\sigma^2 = 3dB = 2$ . And now we have to optimally allocate this total power. So for optimal power allocation which maximizes the sum rate we have to first start with the singular value decomposition.

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$$H = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} = U \Sigma V^H$$

Orthogonal. Norm =  $\sqrt{1+1} = \sqrt{2}$

Orthonormal columns. Norm =  $\sqrt{4+4} = 2\sqrt{2}$

$$H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-2}{2\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix}$$

Now this can be written as  $H = U \Sigma V^H$  where U contains orthonormal columns and these columns are orthogonal to each other. So all we have to do is we have to simply normalize them and this is as shown in slide. So we will get

$$H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{2\sqrt{2}} \\ \frac{2}{\sqrt{2}} & \frac{-2}{2\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix}$$

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$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix}$$

Orthonormal columns.  $U^H U = I$

Possibly  $\Sigma$

Now we have orthonormal columns and this satisfies the property of U. And in fact this matrix can be possibly  $\Sigma$  because this is a diagonal matrix and these are non-negative, so

these are the possible singular values. And now we need the V matrix which is a unitary matrix. So I can simply use the identity matrix in this case as unitary matrix.

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Handwritten slide showing the SVD decomposition of a matrix. The matrix is decomposed into  $U$ ,  $\Sigma$ , and  $V$ .  $U$  is a  $2 \times 2$  matrix with columns  $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$ .  $\Sigma$  is a  $2 \times 2$  diagonal matrix with entries  $\sqrt{2}$  and  $2\sqrt{2}$ .  $V$  is a  $2 \times 2$  identity matrix. The decomposition is written as  $U \Sigma V^H$ . Annotations include "orthonormal columns", " $U^H U = I$ ", "Possibly  $\Sigma$ ", "Valid for  $\Sigma$ ?", " $\sigma_2 = ?$ ", "2x2 identity matrix", and " $V V^H = V^H V = I$ ".

Now the only problem is the that the singular values should be arranged in decreasing order which is not possible in this obtained matrix.

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Handwritten slide showing the SVD decomposition of a matrix. The matrix is decomposed into  $U$ ,  $\Sigma$ , and  $V$ .  $U$  is a  $2 \times 2$  matrix with columns  $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$ .  $\Sigma$  is a  $2 \times 2$  diagonal matrix with entries  $\sqrt{2}$  and  $2\sqrt{2}$ .  $V$  is a  $2 \times 2$  identity matrix. The decomposition is written as  $U \Sigma V^H$ . Annotations include "orthonormal columns", " $U^H U = I$ ", "Possibly  $\Sigma$ ", "Valid for  $\Sigma$ ?", " $\sigma_2 = ?$ ", "2x2 identity matrix", " $V V^H = V^H V = I$ ", "interchange columns of U", " $\sigma_1 < \sigma_2$ ", and "NOT valid SVD! We need  $\sigma_1 \geq \sigma_2$ ".

So this is not a valid SVD. So we need the singular values to be ordered in decreasing order. So we have to somehow switch these values and this is as shown in slide. So this is possible if I basically interchange the columns of  $U$  and the rows of  $V^H$  and then I can flip the singular values.

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We need  $\sigma_1 \geq \sigma_2$

$$= \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}_U \underbrace{\begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{V^H}$$

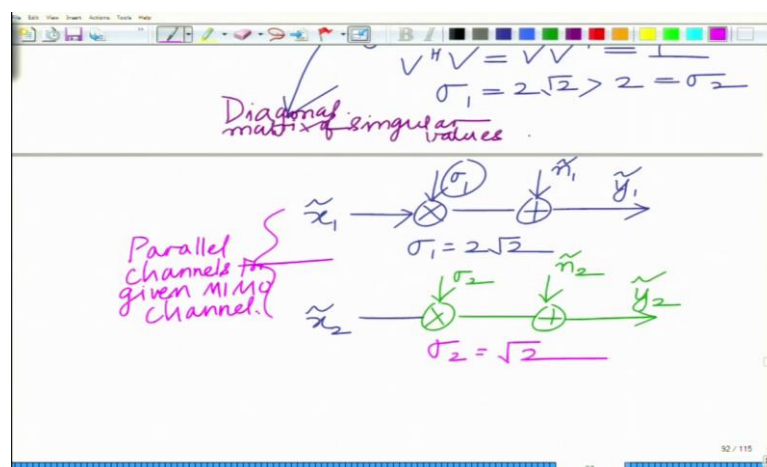
$U^H U = I$   
 $V^H V = V V^H = I$   
 $\sigma_1 = 2\sqrt{2} > 2 = \sigma_2$

Diagonal matrix of singular values.

So finally we have  $H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  and now we have to do optimal

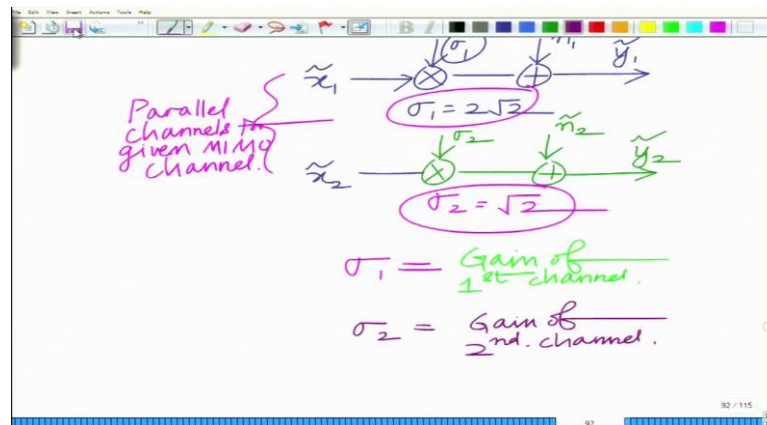
power allocation that is you can decompose this using pre coding as the combination of two parallel channels.

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So you are transmitting  $x_1$  through the first channel that has gain  $\sigma_1$  and noise  $n_1$  to give  $y_1$ . And similarly  $x_2$  through the second channel that has gain  $\sigma_2$  and noise  $n_2$  to give  $y_2$ , so these are the parallel channels for the given MIMO channel. So this is a  $2 \times 2$  MIMO channel.

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$$P_i = \left( \frac{1}{2} - \frac{\sigma^2}{\sigma_i^2} \right)^+$$

Optimal power for channel i

$\sigma^2 = 2$

$\sigma_1 = 2\sqrt{2} \quad \sigma_2 = \sqrt{2}$

$$P_1 = \left( \frac{1}{2} - \frac{\sigma^2}{\sigma_1^2} \right)^+$$

$$= \left( \frac{1}{2} - \frac{2}{8} \right)^+$$

Now for the optimal power allocation we substitute the required quantities as shown in slide.

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$$\sigma_1 = 2\sqrt{2} \quad \sigma_2 = \sqrt{2}$$

Optimal power for channel 1:

$$P_1 = \left( \frac{1}{2} - \frac{\sigma^2}{\sigma_1^2} \right)^+$$

$$= \left( \frac{1}{2} - \frac{2}{8} \right)^+$$

$$= \left( \frac{1}{2} - \frac{1}{4} \right)^+$$

Optimal power for channel 2:

$$P_2 = \left( \frac{1}{2} - \frac{\sigma^2}{\sigma_2^2} \right)^+$$

$$= \left( \frac{1}{2} - \frac{2}{2} \right)^+$$

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Handwritten slide content showing a non-linear equation for  $x^+$ . The equation is:

$$\left(\frac{1}{2} - \frac{1}{4}\right)^+ + \left(\frac{1}{2} - 1\right)^+ = 4$$

Annotations include:

- Arrows pointing to the terms in the equation with the text: "To find  $\frac{1}{2}$  use Total Power constraint".
- The text "Non Linear Equation" written below the equation.
- A partial equation at the top:  $= \left(\frac{1}{2} - 1\right)^+$ .

So finally to find  $v$  we have to use the total power constraint. So this is basically a non-linear equation and this is proceeded as shown in slide.

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Handwritten slide content showing assumptions and the non-linear equation. The equation is:

$$\left(\frac{1}{2} - \frac{1}{4}\right)^+ + \left(\frac{1}{2} - 1\right)^+ = 4$$

Annotations include:

- The text "Non Linear Equation" written below the equation.
- A series of assumptions:
  - Assume  $\frac{1}{2} \geq 1$
  - $\Rightarrow \frac{1}{2} \geq \frac{1}{4}$
  - $P_1 \geq 0, P_2 \geq 0$
  - $\Rightarrow \left(\frac{1}{2} - \frac{1}{4}\right)^+ = \frac{1}{2} - \frac{1}{4}$
  - $\left(\frac{1}{2} - 1\right)^+ = \frac{1}{2} - 1$

So start with the assumption,  $\frac{1}{v} \geq 1$ .

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$$\begin{aligned} & \Rightarrow \frac{1}{2} - \frac{1}{4} + \frac{1}{2} - 1 = 4 \\ & \Rightarrow \frac{3}{2} = 4 + 1 + \frac{1}{4} \\ & \Rightarrow \frac{3}{2} = \frac{21}{4} \\ & \Rightarrow \boxed{\frac{1}{2} = \frac{21}{8}} \end{aligned}$$


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$$\begin{aligned} P_2 &= \frac{1}{2} - 1 = \frac{21}{8} - 1 \\ &= \frac{13}{8} \\ P_1 &= \frac{1}{2} - \frac{1}{4} = \frac{21}{8} - \frac{1}{4} \end{aligned}$$

So finally we get  $P_1 = \frac{19}{8} > 0, P_2 = \frac{13}{8} > 0$ .

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$$\begin{aligned} P_1 &= \frac{1}{2} - \frac{1}{4} = \frac{21}{8} - \frac{1}{4} \\ &= \frac{19}{8} > 0 \\ P_2 &= \frac{13}{8} > 0 \\ \hline &P_1 > 0, P_2 > 0 \\ \Rightarrow &\text{Optimal Powers are} \\ &P_1^* = \frac{19}{8} \\ &P_2^* = \frac{13}{8} \end{aligned}$$

So the original assumption holds and we get the optimal powers.

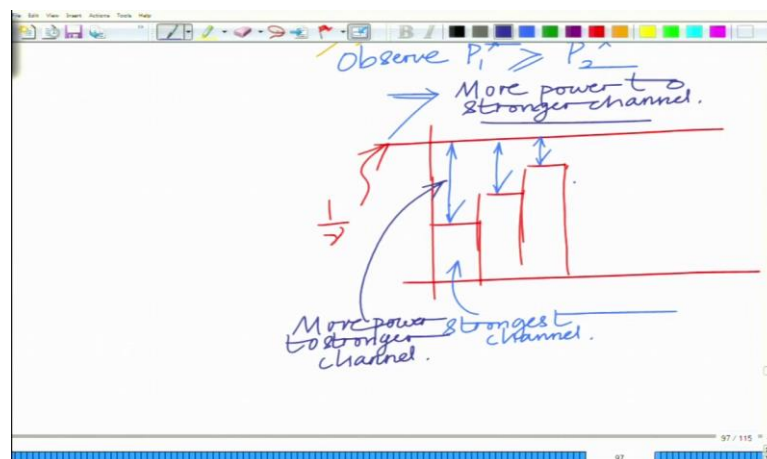


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Handwritten mathematical derivation for optimal power allocation. At the top,  $P_2 = \frac{13}{8} > 0$  is written. Below it,  $P_1 > 0, P_2 > 0$  is written. An arrow points to a box containing the optimal powers:  $P_1^* = \frac{19}{8}$  and  $P_2^* = \frac{13}{8}$ . Below the box, it says "Observe  $P_1^* > P_2^*$ " and "More power to stronger channel." with an arrow pointing to the box.

Now, if one of the powers would have been negative that implies our original assumption is incorrect. So the power is negative implies that that corresponding channel is above the water level. So power is not allocated. So in the corresponding channel the power has to be set to 0 and the problem has to be repeated with the total power constraint. So this is the procedure alright and now you also observe that more power is allocated to the stronger channel.

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So what this shows is that to maximize capacity more power is allocated to the stronger channel that is the one with the largest singular value.. So we will stop here and continue in the subsequent modules. Thank you very much.