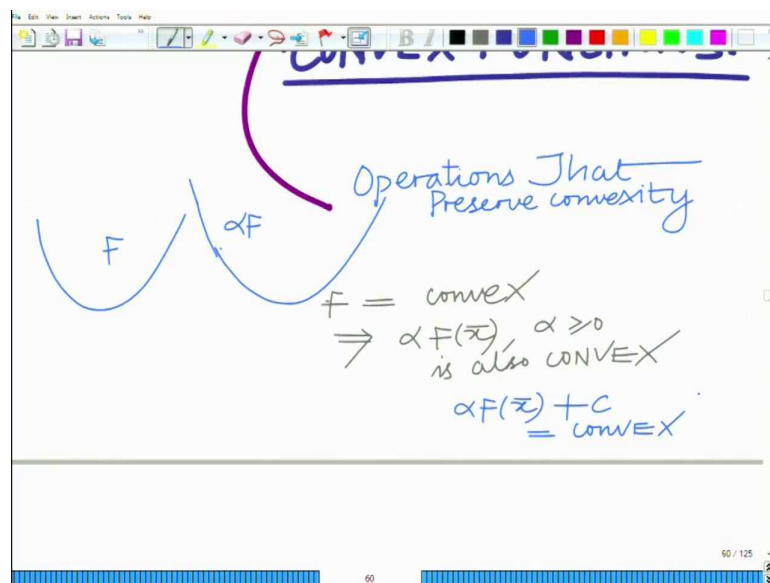


Applied Optimization for Wireless, Machine Learning, Big Data
Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture - 29
Properties of Convex Functions: Operations that preserve Convexity

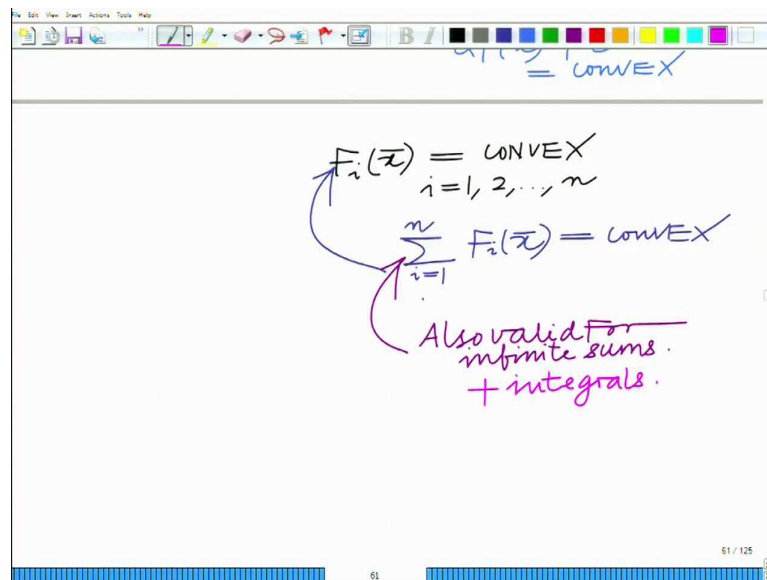
Hello. Welcome to another module in this massive open online course. In this module, let us start looking at the operations on convex functions that preserve convexity.

(Refer Slide Time: 01:24)



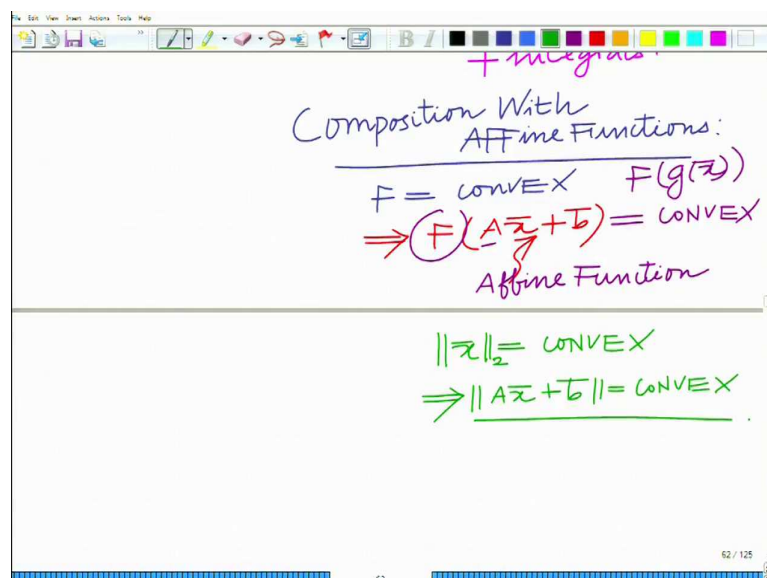
Let function $F(\bar{x})$ is a convex function to discuss some properties of convex function. So for a convex function $F(\bar{x})$; $\alpha F(\bar{x})$ is also convex for $\alpha \geq 0$. This implies that scaling by a non-negative number preserves convexity. Similarly, translation does not change convexity. Thus, for any constant c and $\alpha \geq 0$; it can be concluded $\alpha F(\bar{x}) + c$ is also convex.

(Refer Slide Time: 02:51)



Also, if the several functions of \bar{x} i.e. $F_i(\bar{x})$ are all convex for $i = 1, 2, \dots, n$, then their sum $\sum_{i=1}^n F_i(\bar{x})$ is also convex. This property extends to infinite sum and integrals which is a continuous sum.

(Refer Slide Time: 04:39)



Another property is that the composition of a convex function with affine functions is also a convex function. Thus, for an affine function $A\bar{x} + \bar{b}$; $F(A\bar{x} + \bar{b})$ is also a convex

function. So for example, as l_2 norm $\|\bar{x}\|_2$ is a convex function so $\|A\bar{x} + \bar{b}\|$ is also convex.

(Refer Slide Time: 06:36)

Handwritten notes on a whiteboard:

- $\|\bar{x}\|_2 = \text{CONVEX}$
- $\Rightarrow \|A\bar{x} + \bar{b}\| = \text{CONVEX}$
- Pointwise MAXIMUM:
- F_1, F_2, \dots, F_m
- CONVEX
- $\Rightarrow \max\{F_1, F_2, \dots, F_m\} = \text{CONVEX}$

Another interesting property is that the point wise maximum of several convex functions is also a convex function. This has a lot of interesting applications. So if $F_1(\bar{x}), F_2(\bar{x}), \dots, F_m(\bar{x})$ are convex, then $\max\{F_1(\bar{x}), F_2(\bar{x}), \dots, F_m(\bar{x})\}$ is also convex where it is point wise maximum.

(Refer Slide Time: 07:37)

Handwritten notes on a whiteboard:

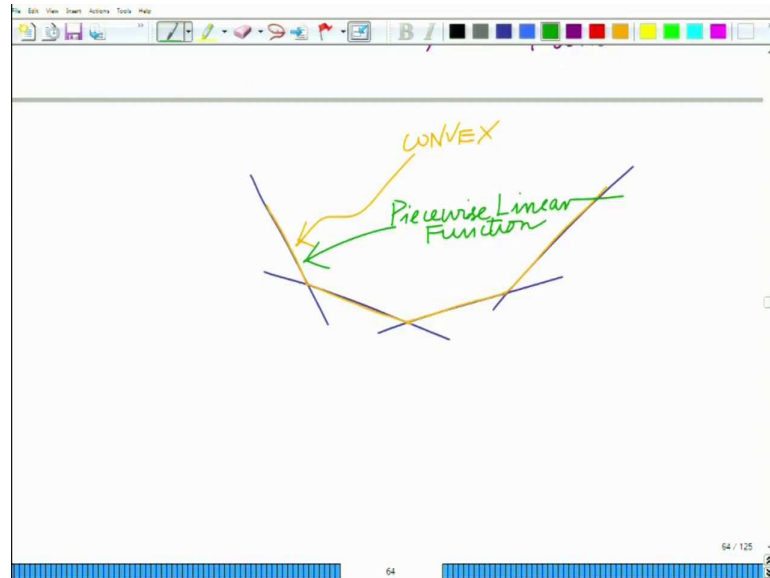
- Two convex functions are shown, each labeled CONVEX .
- The maximum of these two functions is shown as a piecewise linear function, labeled $\text{maximum} \neq \text{CONVEX}$.
- The formula $F(\bar{x}) = \max_{1 \leq i \leq m} \bar{a}_i^T \bar{x} + b_i$ is written.

For instance, let us take the maximum of a set of linear functions, that is

$$F(\bar{x}) = \max_{1 \leq i \leq m} \bar{a}_i^T \bar{x} + b_i$$

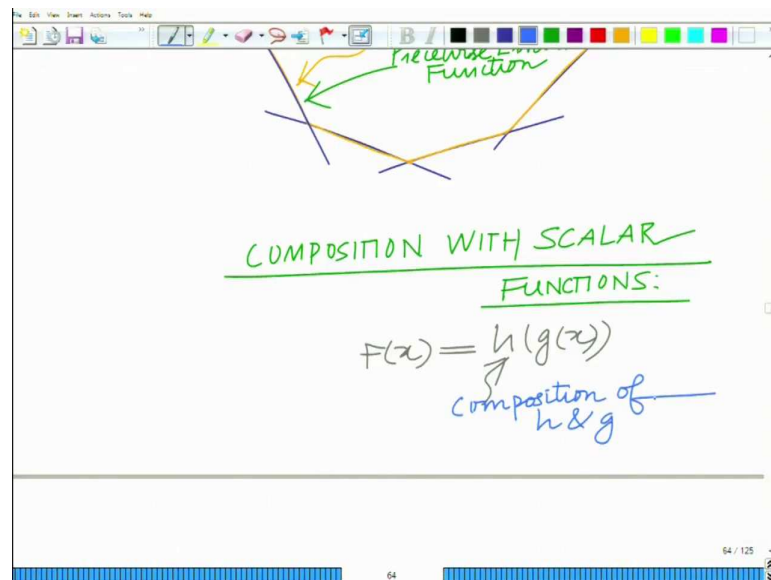
This is known as a piecewise linear function.

(Refer Slide Time: 09:29)



So, the maximum of several linear functions is a piecewise linear function which is also convex. This is because linear functions are basically hyper planes. This means these are convex functions and the maximum of convex functions is also convex. Therefore, the piecewise linear function is also convex.

(Refer Slide Time: 10:48)



Another concept is the composition of functions. Let us consider a simple case of composition of scalar functions.

$$F(x) = h(g(x))$$

So this function $F(x)$ is a composition of function h with function g and this is a convex function for two conditions. First condition is that if function g and h both are convex functions and function h is a non-decreasing function, then their above mentioned composition function $F(x)$ is also convex. Second condition is that if function g is a concave function and function h is a non-increasing convex function, then their above mentioned composition function $F(x)$ is also convex.

(Refer Slide Time: 13:41)

OR if $g = \text{CONCAVE}$
 $+ h = \text{CONVEX}$
 $+ \text{NON INCREASING.}$

$$F(x) = h(g(x))$$

$$\frac{dF(x)}{dx} = F'(x) = h'(g(x)) \cdot g'(x)$$

Let us use the derivative test to demonstrate this convexity assuming the functions are differentiable. So take the first order derivative of $F(x)$ using the chain rule.

$$F'(x) = h(g(x)) \cdot g'(x)$$

(Refer Slide Time: 14:53)

$$F''(x) = h''(g(x))(g'(x))^2 + h'(g(x))g''(x)$$

$h = \text{CONVEX} \Rightarrow h''(x) \geq 0$
 $\text{NON-DECREASING} \Rightarrow h'(x) \geq 0$
 $g(x) = \text{CONVEX} \Rightarrow g''(x) \geq 0$

Further, the second order derivative of $F(x)$ is

$$F''(x) = h''(g(x)) \cdot (g'(x))^2 + h'(g(x)) \cdot g''(x)$$

First consider that $h(x)$ and $g(x)$ are convex. This means

$$h''(x) \geq 0$$

And

$$g''(x) \geq 0$$

Also if $h(x)$ is a non-decreasing function this means

$$h'(x) \geq 0$$

As all four components of $F''(x)$ is greater than equal to 0. This means that $F''(x) \geq 0$ which implies that $F(x)$ is convex. This proves the first condition.

(Refer Slide Time: 17:36)

The image shows a handwritten derivation of the second derivative of a function $F(x)$. The main equation is $F''(x) = h''(x)(g'(x))^2 + h'(g(x))g''(x)$. Annotations include:
 - A yellow arrow pointing from $h''(x) \geq 0$ to the first term, with the note " $h(x)$ CONVEX $\Rightarrow h''(x) \geq 0$ ".
 - A purple arrow pointing from $g''(x) \leq 0$ to the second term, with the note " g = CONCAVE $\Rightarrow g''(x) \leq 0$ ".
 - A green arrow pointing from $h'(g(x)) \leq 0$ to the second term, with the note " h is NON increasing $\Rightarrow h'(g(x)) \leq 0$ ".
 The slide number 67 is visible at the bottom.

Similarly; for the other condition, consider that $h(x)$ is convex and $g(x)$ is concave.

This means

$$h''(x) \geq 0$$

And

$$g''(x) \leq 0$$

Also if $h(x)$ is a non-increasing function this means

$$h'(g(x)) \leq 0$$

The product of two negatives makes a positive, so

$$h'(g(x)) \cdot g''(x) \geq 0$$

Thus, as all four components of $F''(x)$ is greater than equal to 0. This means that $F''(x) \geq 0$ which implies that $F(x)$ is convex. This proves the second condition.

Therefore these are the two conditions that ensure that the composition $h(g(x))$ is also convex.

(Refer Slide Time: 20:52)

Handwritten notes on a whiteboard illustrating the proof of convexity for the composition $F(x) = h(g(x)) = e^{x^2}$.

Example: $F(x) = e^{x^2} = h(g(x))$.

Define $g(x) = x^2$ and $h(x) = e^x$.

$g(x)$ is CONVEX.

$h(x) = e^x$ is CONVEX INCREASING.

$\Rightarrow F(x) = h(g(x))$ is CONVEX.

Derivatives:

$$F'(x) = 2x e^{x^2}$$

$$F''(x) = 2e^{x^2} + 4x^2 e^{x^2} \geq 0.$$

$\Rightarrow F(x) = e^{x^2}$ is CONVEX.

Let us now look at a simple example.

$$F(x) = h(g(x)) = e^{x^2}$$

For

$$g(x) = x^2$$

$$h(x) = e^x$$

So, $g(x)$ is a convex function and $h(x)$ is a non-decreasing convex function this implies that $F(x)$ is also convex. To show this, first order derivative of $F(x)$ is

$$F'(x) = 2xe^{x^2}$$

And second order derivative of $F(x)$ is

$$F''(x) = 2e^{x^2} + 4x^2e^{x^2}$$

It is clear that

$$F''(x) \geq 0$$

This implies that $F(x)$ is convex.

Similarly, one can derive these conditions for the concavity of composition of functions.