

Applied Optimization for Wireless, Machine Learning Big Data
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Lecture - 56
Practical Application

Keywords: *Compressive Sensing, Sparsity*

Hello welcome to another module in this Massive Open Online Course. So we are looking at compressive sensing where we try to compress during the sensing process itself by making much fewer number of measurements in comparison to the dimension of the signal and then later try to reconstruct the signal from the very few measurements made.

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COMPRESSIVE SENSING:

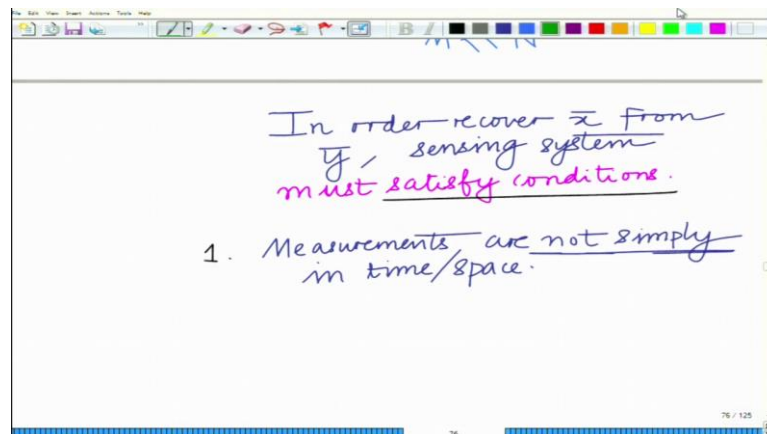
$$\bar{y} = \Phi \bar{x}$$

Measurement vector $\bar{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$ Sensing matrix Φ ($M \times N$) Signal vector $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$

$M = \# \text{ Equations}$
 $N = \# \text{ Unknowns}$
 $M \ll N$

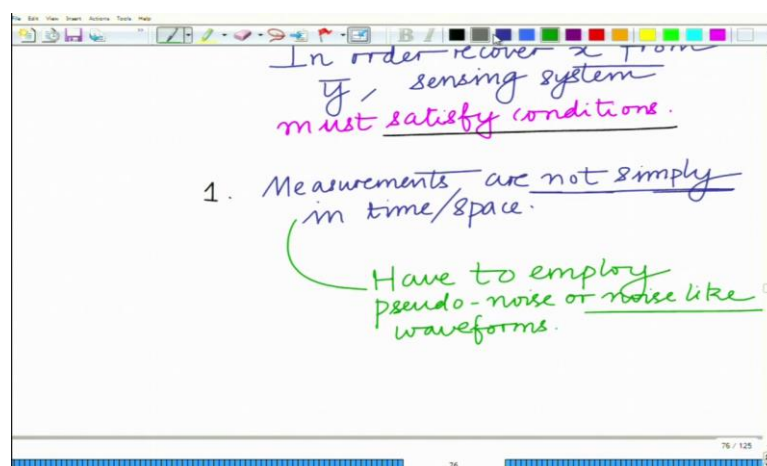
So we have a measurement vector $\bar{y} = \Phi \bar{x}$ and there are M measurements as shown in slide. The sensing matrix is $M \times N$ and we make significantly fewer measurements that is $M \ll N$. Now if you view this as a system of equation then we have M number of equations and N number of unknowns. So simple linear algebra tells us that one cannot reconstruct the vector \bar{x} of length N from M equations, since the number of equations is much lower than the number of unknowns. So this is an underdetermined system. Therefore, this sensing system has to satisfy certain special properties in order to recover \bar{x} .

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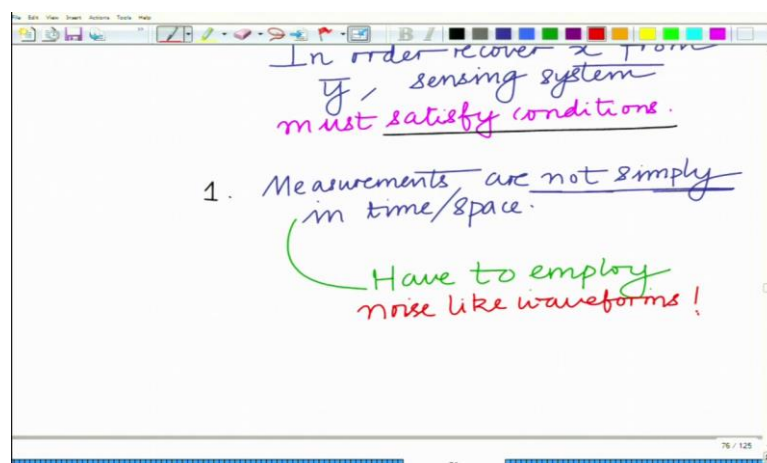


Now the first condition states that the measurements are not simply in the time or space. Rather, they have to employ noise like or one can say pseudo noise like waveform.

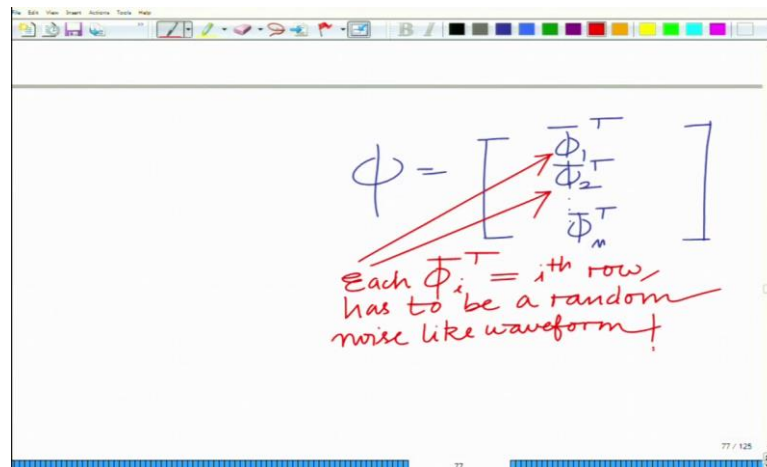
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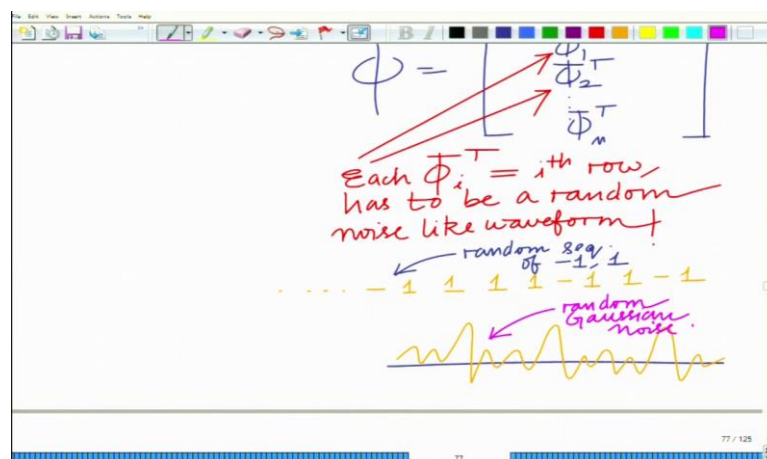
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A handwritten diagram on a whiteboard showing a matrix Φ enclosed in large square brackets. Inside the brackets, there are n rows, each labeled $\phi_1^T, \phi_2^T, \dots, \phi_n^T$. Red arrows point from the text below to the rows of the matrix. The text reads: "Each $\phi_i^T = i^{\text{th}}$ row, has to be a random noise like waveform".

If you look at each row of the sensing matrix which we are denoting by $\phi_1^T, \phi_2^T, \phi_M^T$ this has to be a noise like waveform which means that it has to be something very random, either can be a random sequence of - 1, 1 or it has to be some random noise like waveform such as Gaussian noise.

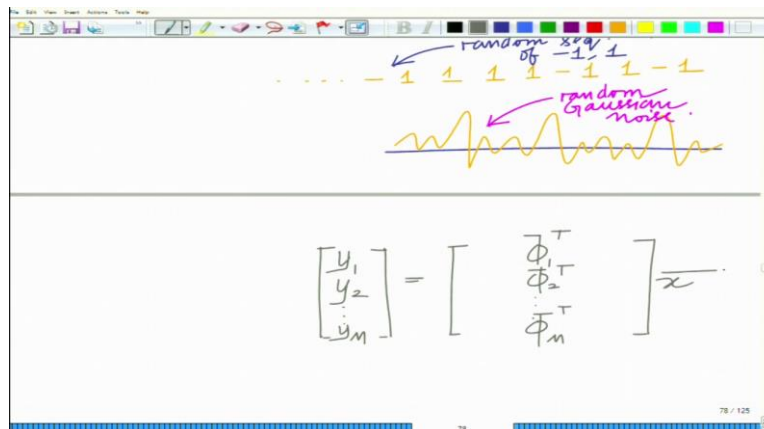
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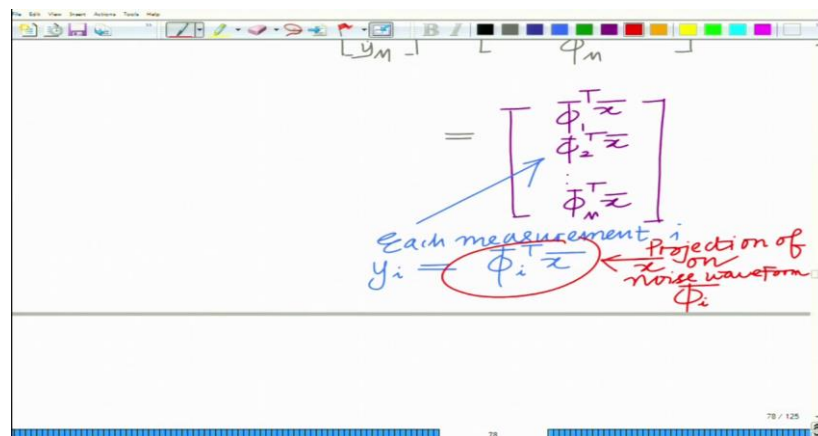
A handwritten diagram on a whiteboard, similar to the previous one, showing the matrix Φ with rows $\phi_1^T, \phi_2^T, \dots, \phi_n^T$. Below the matrix, there are two examples of random waveforms. The first is a sequence of numbers: "... -1 1 1 1 -1 1 -1", with a blue arrow pointing to it from the text "random seq. of -1, 1". The second is a yellow wavy line representing a signal, with a purple arrow pointing to it from the text "random Gaussian noise".

So these rows have to look like independent realization of the noise waveforms. And when you are making the measurement you are taking the projections of the signal on this noise like waveform.

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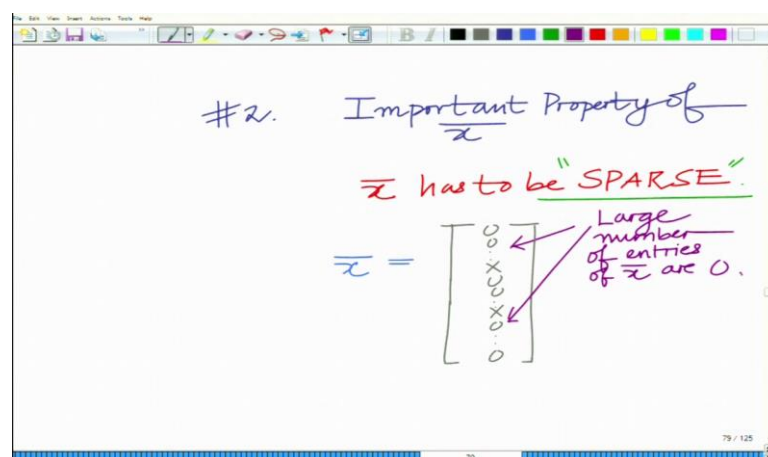


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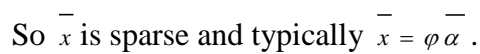


So each $\phi_i^T \bar{x}$ is a projection of \bar{x} and we are taking the linear combination of \bar{x} using this noise like waveform.

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So \bar{x} is $N \times 1$, this $\bar{\alpha}$ is $N \times 1$ and φ is an $N \times N$ basis such that $\bar{\alpha}$ is a sparse vector. For instance, we take an image and if you look at the wavelet coefficients of an image then they are sparse.

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$$\bar{x} = \Psi \bar{\alpha}$$

image wavelet coeffs. Wavelet transform matrix

$$\bar{y} = \Phi \bar{x}$$

$$= \Phi \Psi \bar{\alpha}$$

$$= \tilde{\Phi} \bar{\alpha}$$

So \bar{x} is sparse can be expressed in terms of $\bar{\alpha}$ which is sparse and therefore, now if you substitute this, the sensing model becomes $\bar{y} = \Phi \bar{x} = \Phi \Psi \bar{\alpha} = \tilde{\Phi} \bar{\alpha}$.

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$$\bar{y} = \Phi \bar{x}$$

$$= \Phi \Psi \bar{\alpha}$$

$$= \tilde{\Phi} \bar{\alpha}$$

$N \times 1$ $N \times N$ $N \times L$ $L \times 1$

"effective" sensing matrix sparse

Now Φ becomes your effective sensing matrix. Now, it is as if you are trying to sense the vector $\bar{\alpha}$ which is in the wavelet domain. Now once you get the wavelet coefficients you can reconstruct the image because image and wavelet have a 1 to 1 correspondence. But the wavelet coefficient is sparse and that is very amenable to compressive sensing.

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Handwritten slide content showing matrix equations and annotations:

$$\begin{aligned} \underline{y} &= \phi \bar{x} \\ \underline{y} &= \phi \psi \alpha \end{aligned}$$

Annotations:

- \underline{y} is $N \times 1$.
- ϕ is $N \times N$.
- ψ is $N \times 1$.
- α is 1×1 .
- ϕ is annotated as "effective sensing matrix" and "sparse".
- ψ is annotated as "has to comprise of noise like waveforms".
- Equation: $\hat{x} = \psi \hat{\alpha}$ to obtain \hat{x} from $\hat{\alpha}$.

So once you get $\bar{\alpha}$ use $x = \phi \alpha$ to obtain the estimate x from α and this ϕ matrix has to contain noise like waveforms.

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Handwritten slide content discussing sparsity and reconstruction:

$\bar{x} = \text{sparse}$.

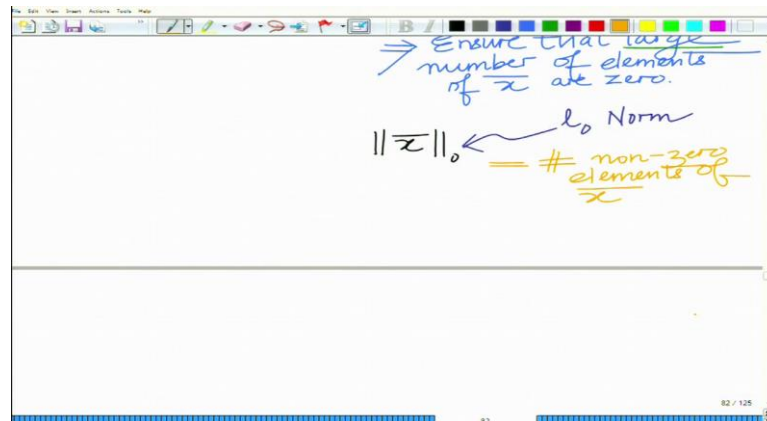
How to reconstruct \bar{x} ?

Enforce sparsity!

\Rightarrow Ensure that large number of elements of \bar{x} are zero.

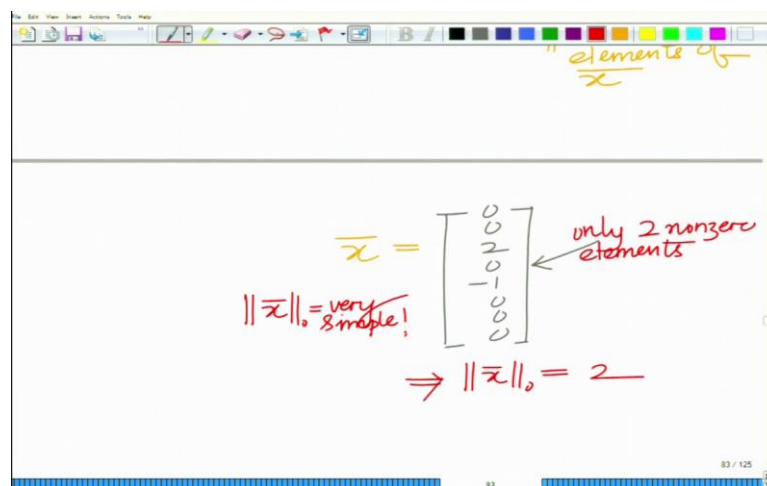
Now to reconstruct \bar{x} enforce sparsity which implies that we have to ensure that the reconstructed vector \bar{x} is such that a large number of elements are 0's and only some elements are non-zero. This is precisely what we call the l_0 norm that is if you denote the l_0 norm of a vector this equals the number of non-zero elements of \bar{x} .

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So we want to minimize the number of non-zero elements of \bar{x} which is l_0 norm.

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For instance, let us say you have the vector \bar{x} as shown in slide and there are 8 elements, but only 2 non-zero elements, which implies the l_0 norm of \bar{x} is 2.

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Handwritten slide content showing a vector \tilde{x} and its L_1 norm. The vector is $\tilde{x} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. The L_1 norm is calculated as $\|\tilde{x}\|_1 = 2 + 1 = 3$. The slide also notes that only 2 non-zero elements are present (2 and -1) and that the norm is very simple to calculate.

We are not concerned with the values of the non-zero elements we just have to consider the number of non-zero elements.

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Handwritten slide content showing an optimization problem for reconstruction. The problem is formulated as:

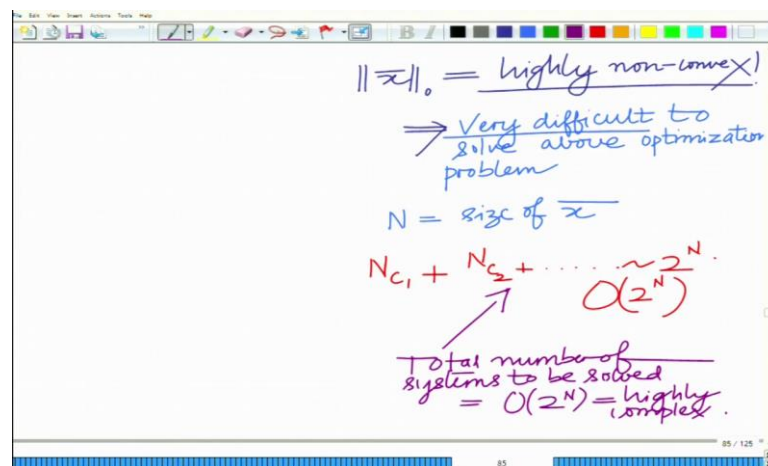
$$\begin{aligned} \text{minimize } & \|\tilde{x}\|_1 \\ \text{s.t. } & y = \phi \tilde{x} \end{aligned}$$

The slide also notes that $M \ll N$ and that the number of equations is less than the number of unknowns. The goal is to find the sparsest vector \tilde{x} that satisfies the observation model $y = \phi \tilde{x}$. The slide emphasizes that reconstruction is only possible via exploiting sparsity.

Therefore, the optimization problem for reconstruction of \tilde{x} can be given as
$$\begin{aligned} \min_{\tilde{x}} & \|\tilde{x}\|_1 \\ \text{s.t. } & y = \phi \tilde{x} \end{aligned}$$

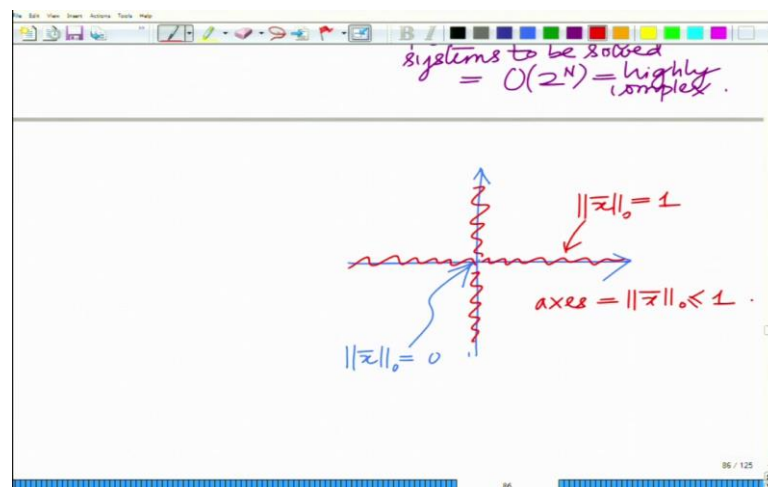
And we need this because $M \ll N$ and therefore, one has to exploit sparsity. So you are trying to find the sparsest vector which satisfies this observation model. Now the problem with this optimization problem is that not only the objective is non-differentiable, this optimization problem is highly non convex.

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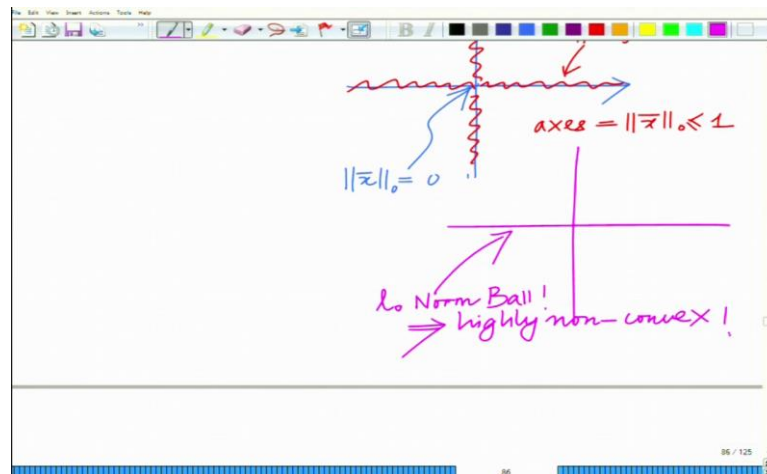
So this implies it is very difficult to solve the optimization problem. So the point is that although it is very simple to state the optimization problem it is an extremely complicated one.

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Now, if you look at l_0 norm $\|\bar{x}\|_0 \leq 1$ that comprises only of the axis, so it is highly non convex.

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Therefore, one has to come up with other intelligent techniques to solve this optimization problem and that forms the basis for compressive sensing. So let us stop here and continue in the subsequent modules. Thank you very much.