Applied Optimization for Wireless, Machine Learning, Big Data
Prof. Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology, Kanpur

Lecture - 23
Introduction to Convex and Concave Functions

Hello, welcome to another module in this Massive Open Online Course. In this module, let us discuss about the Convex Functions.

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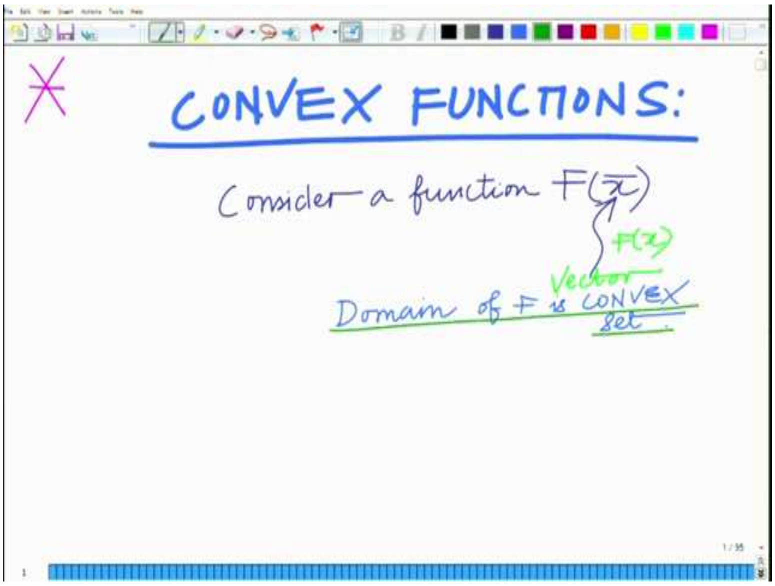

A convex function can be defined as follows. Consider a function $F(\bar{x})$ where $\bar{x}$ is a vector. Let the domain of this function $F(\bar{x})$ is a convex set. Thus function $F(\bar{x})$ is also convex, if it satisfies the following property.

$$F\left(\theta\bar{x}_1 + (1-\theta)\bar{x}_2\right) \le \theta F\left(\bar{x}_1\right) + (1-\theta)F\left(\bar{x}_2\right)$$

Where the two points $\bar{x}_1$ and $\bar{x}_2$ belongs to the domain of function $F(\bar{x})$ and θ is a scalar quantity such that $0 \le \theta \le 1$. This can be pictorially represented in the following figure.

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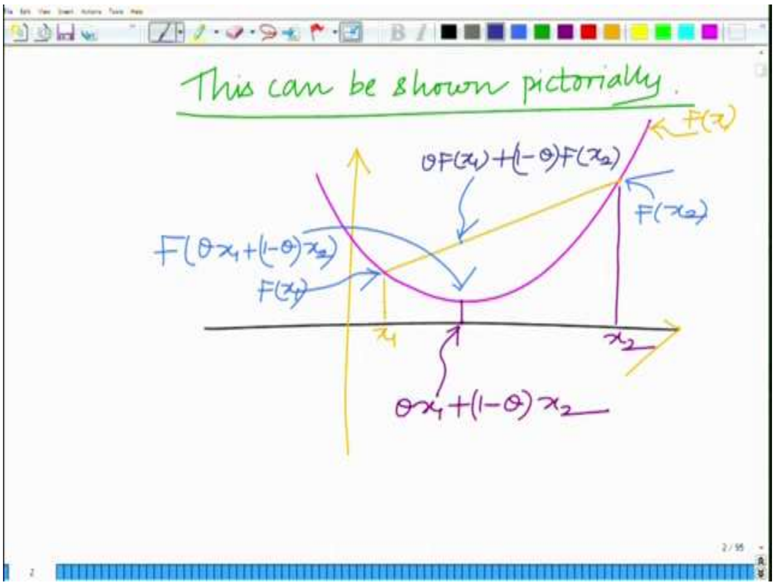

This convex function $F(\bar{x})$ can be pictorially represented as shown in the above figure. Here $F(\bar{x}_1)$ is the value of function $F(\bar{x})$ at $\bar{x}=\bar{x}_1$ and $F(\bar{x}_2)$ is the value of function $F(\bar{x})$ at $\bar{x}=\bar{x}_2$. Thus the function value at the convex combination of points $\bar{x}_1$ and $\bar{x}_2$; $F(\theta\bar{x}_1+(1-\theta)\bar{x}_1)$ shows the function curve in between $F(\bar{x}_1)$ and $F(\bar{x}_2)$. On the other hand, $\theta F(\bar{x}_1)+(1-\theta)F(\bar{x}_1)$ is the convex combination of two function value points $F(\bar{x}_1)$ and $F(\bar{x}_2)$ and it represents a line segment joining $F(\bar{x}_1)$ and $F(\bar{x}_2)$. In the above figure it is shown that this line segment lies above the function curve in between $F(\bar{x}_1)$ and $F(\bar{x}_2)$. Therefore, a convex function curve can be considered as a U- shaped curve which open upward; although it is not always true (as in the case of a line).

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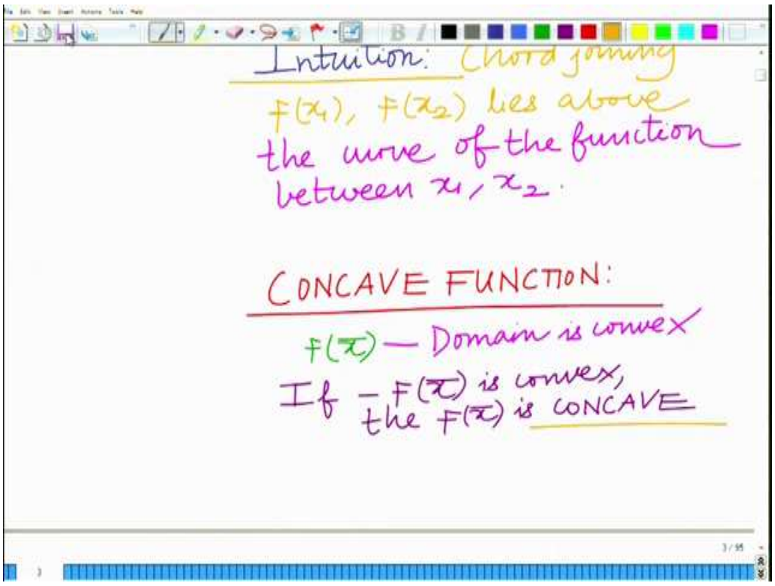

Most of the optimization problems will be built on convex functions; therefore it is a very important class of functions.

On the other hand; a concave function is defined as the minus of the convex function. This means that if $-F(\bar{x})$ is a convex function, then $F(\bar{x})$ is a concave function. Concave function has a convex domain. Therefore a concave function $F(\bar{x})$ satisfies the following.

$$F(\theta\bar{x}_1 + (1-\theta)\bar{x}_2) \geq \theta F(\bar{x}_1) + (1-\theta)F(\bar{x}_2)$$

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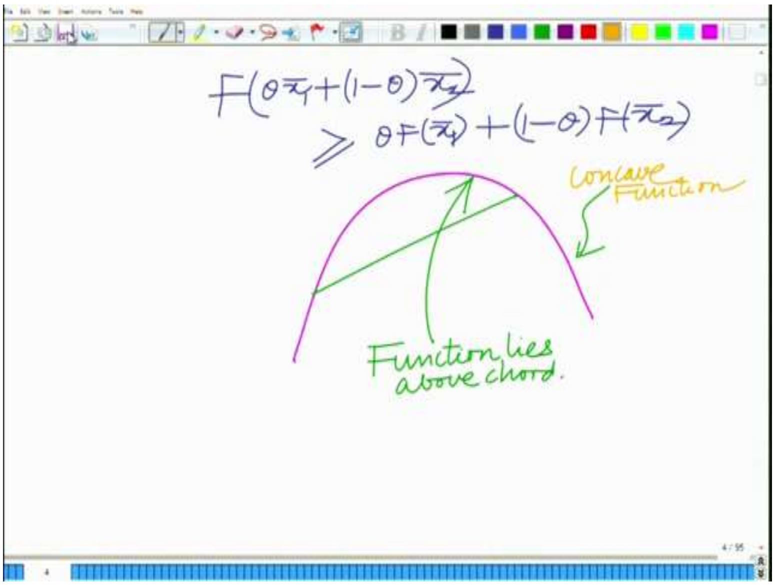

Naturally a concave function curves downwards and the values that belong to the concave function lies above of this curvature.

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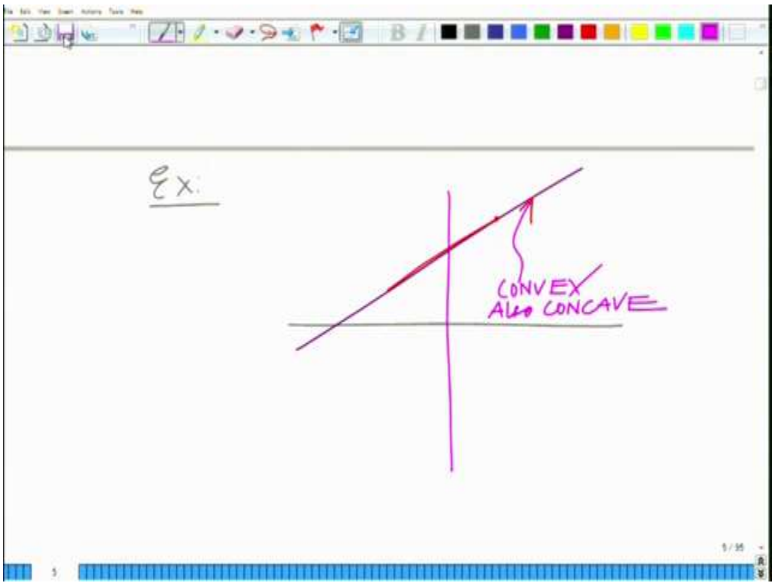

The simplest example of these functions is a straight line. The line segment between any two points on the straight line itself lies on the straight line. Therefore, the straight line is convex. It is also a concave function because the function curve coincides with the chord joining any two points. Thus any straight line is convex and also concave.

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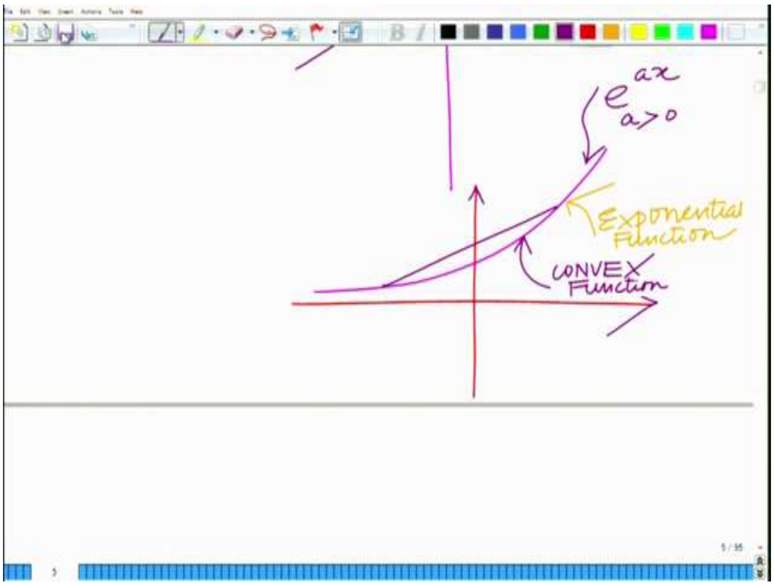

A very important class of convex functions is the exponential function. This function is defined as

$$F(\bar{x}) = e^{ax} \qquad \text{for } a > 0$$

This is an increasing exponential which is a convex function.

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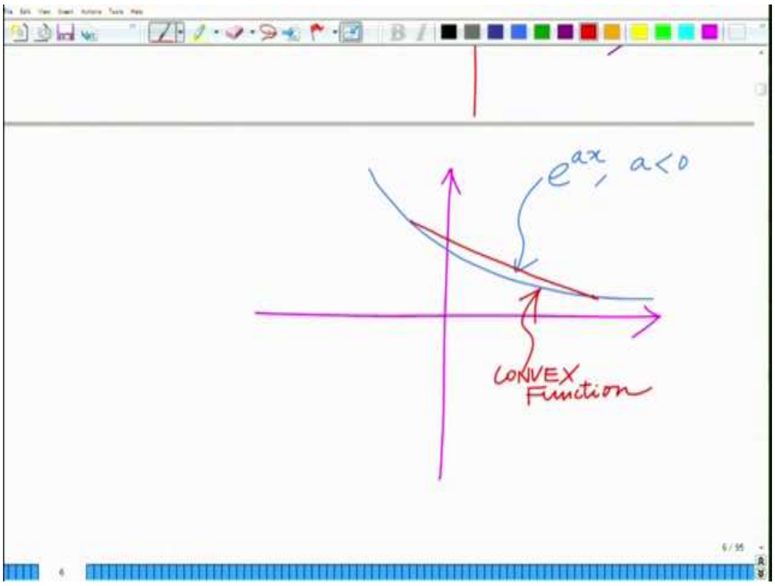

In fact, the decreasing exponential is also a convex function.

$F(\bar{x}) = e^{ax}$ for $a < 0$

The convexity of these functions can easily be observed from the plot of both increasing and decreasing exponential function. If one takes two points on this function, then the chord joining these two points always lies above the function and therefore exponential function with any real value of a is a convex function.

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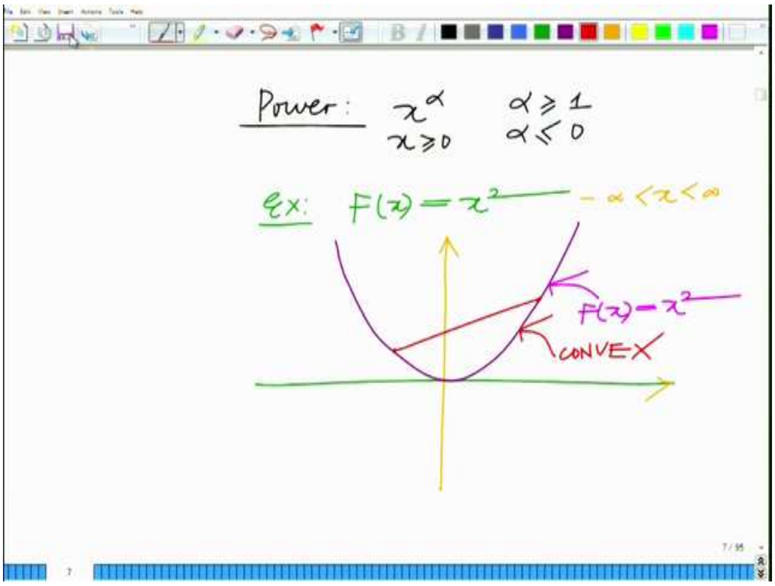

Let us look at the another function

$F(x) = x^{\alpha}$ for $x \geq 0$

There are two cases for this function.

Case 1: $\alpha \geq 1$
Case 2: $\alpha \leq 0$

In both cases this function $F(x)$ is convex.

For example; function is given as

$F(x) = x^2$ for $-\infty < x < \infty$

Then this function is a convex function. This can easily be observed from its plot. This function curve has classic bowl shape. Therefore the chord joining any two points on this function lies above the function. So, this is a classic example of a convex function.

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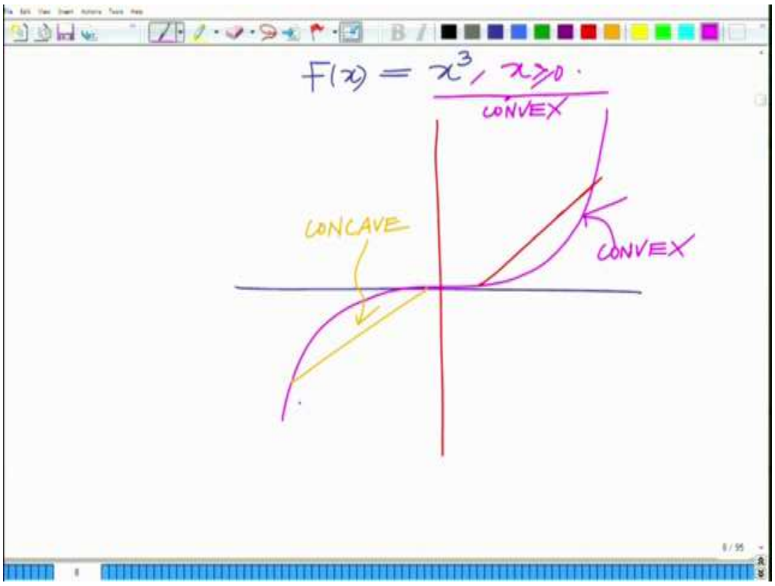

Now if the function is given as

$$F(x) = x^3$$

The function is convex for $x \geq 0$. On the other hand, this function is concave for $x < 0$. Therefore, it can be concluded that the convexity of the power function depends on the value of both x and α. This means that it is important to consider the suitable domain of a function for its convexity.

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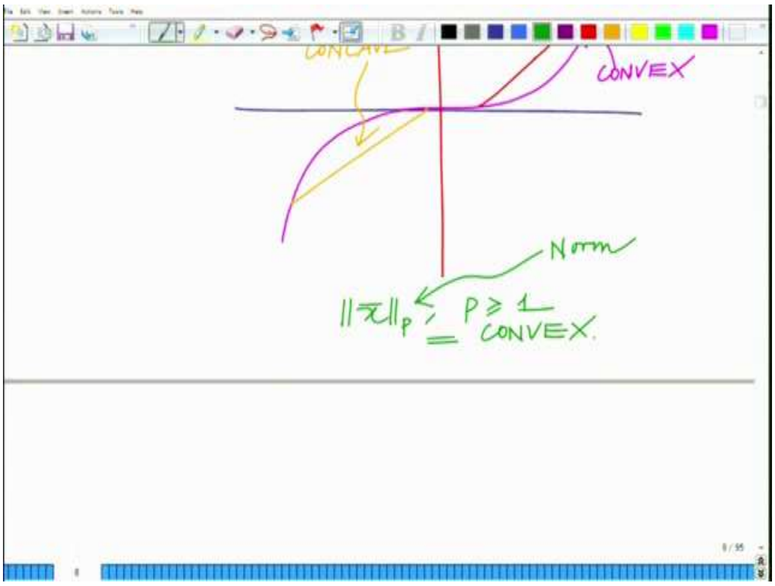

Another function is the norm function which is given as

$$F(x) = \|x\|_p \qquad \text{for } p \geq 1$$

Here x is a vector and this function is a convex field.

So, these were some examples of convex functions which are mostly one dimensional function. Let us look at the examples of concave functions.

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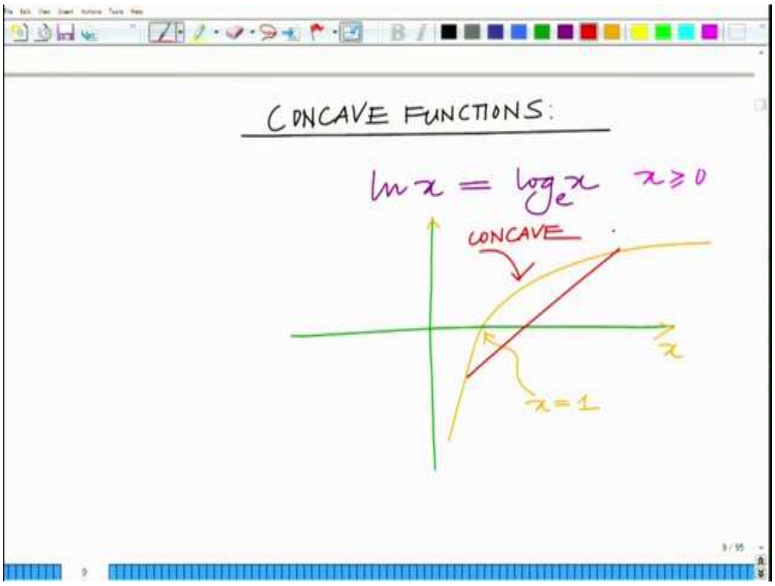

The classical example of concave function is the logarithmic function. This function $F(x)$ is given as

$$F(x) = \log_e x \qquad \text{for } x \geq 0$$

Simply from the plot of this function, a chord joining any two points of this function always lies below the function curve. Hence this function is a concave function.

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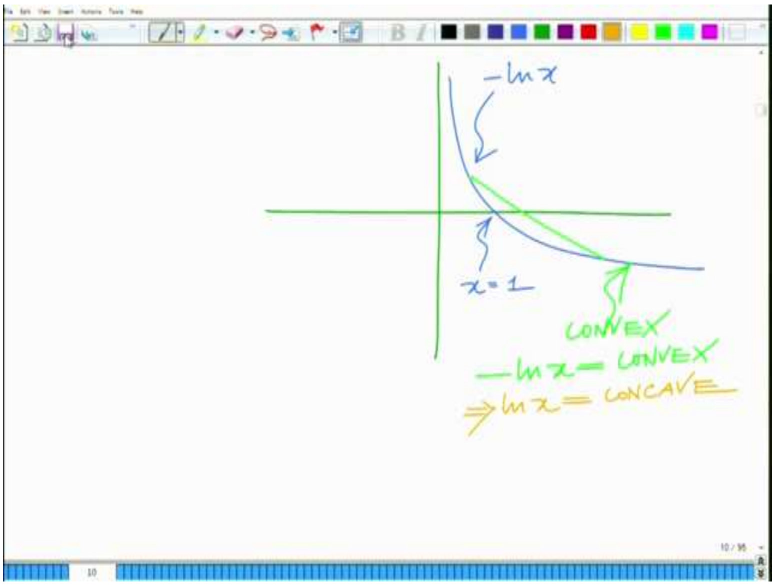

In fact, the correct way to demonstrate concavity of tis function is to look upon the minus of this function; which is

$$-F(x) = -\log_e x$$

The curve of $-F(x)$ is a convex curve. This implies that the natural logarithm of x is a concave function.

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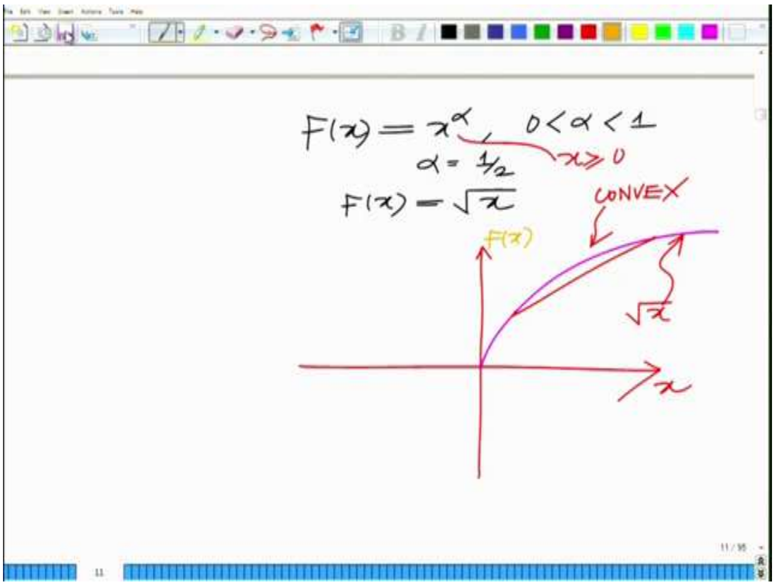

Another important class of concave functions is

$$F(x) = x^{\alpha} \qquad \text{for } 0 < \alpha < 1,\ x \geq 0$$

Therefore for $\alpha = \frac{1}{2}$, this function is

$$F(x) = \sqrt{x}$$

So this function is a concave function for $x \geq 0$ which can easily be seen from its curvature. Here the domain of the function is restricted as $x \geq 0$ which majorly affects the concavity of this function. Therefore the domain of a function plays major role in its convexity.