

**Applied Optimization for Wireless, Machine Learning, Big Data**  
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**Lecture - 58**

**Example Problem: Orthogonal Matching Pursuit (OMP) algorithm**

**Keywords:** *Orthogonal Matching Pursuit (OMP) algorithm*

Hello, welcome to another module in this massive open online course. So we are looking at techniques for compressive sensing and we have seen that orthogonal matching pursuit can be used for sparse signal recovery, so let us now look at an example to understand this better.

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**OMP: EXAMPLE:**

$$\begin{bmatrix} 0 \\ 2 \\ 3 \\ 5 \end{bmatrix}_{4 \times 1} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}_{4 \times 6} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}_{6 \times 1}$$

rows of  $\phi$  are random 0, 1

$M = 4$  # Equations  
 $N = 6$  # unknowns  
 $M < N \Rightarrow$  # Equations < # unknowns

So let us consider the following example, we have  $\bar{y} = \phi \bar{x}$  and we have to estimate the

vector  $\bar{x}$ . So let  $\bar{y} = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 5 \end{bmatrix}$ , the matrix  $\phi = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$  and  $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$ .

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Handwritten notes on a whiteboard:

- $4 \times 1$  (in blue)
- $4 \times 6$  (in blue)
- $6 \times 1$  (in blue)
- $M = 4$  # Equations.
- $N = 6$  # unknowns.
- $M < N \Rightarrow$  # Equations < # unknowns.
- $\bar{x} = \text{sparse}$ .
- Estimate sparse  $\bar{x}$
- $\Rightarrow$  Sparse signal Recovery.

In this problem we have  $M = 4$  which is basically the number of equations and  $N = 6$  which is the number of unknowns and  $M < N$  implies number of equations is less than number of unknowns. Therefore, to estimate  $\bar{x}$  you cannot use conventional linear algebra, because in linear algebra you need the number of equations at least equal to the number of unknowns to uniquely determine the unknown vector  $\bar{x}$ . And therefore, one has to enforce sparsity, so we assume that  $\bar{x}$  is sparse and then we want to estimate this sparse vector. This is basically termed as sparse signal recovery.

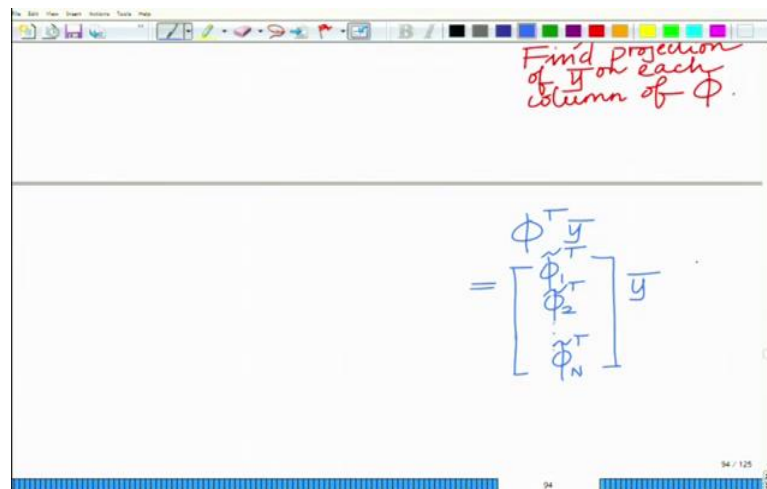
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Handwritten notes on a whiteboard:

- OMP:
- 1<sup>st</sup> iteration:
- $i(1) = \arg \max_{1 \leq j \leq N} |\Phi_j^T \bar{x}|$
- Find projection of  $\bar{x}$  on each column of  $\Phi$ .

The algorithm for sparse signal recovery is OMP and this can be done as shown in slide.

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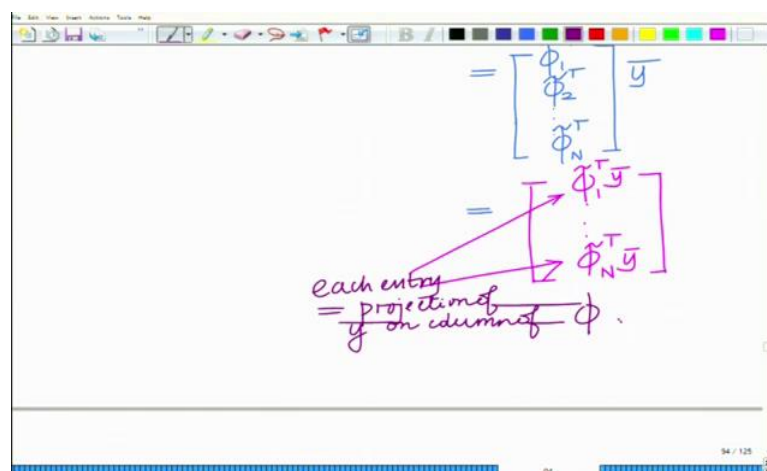


Find projection of  $\bar{y}$  on each column of  $\Phi$ .

$$= \begin{bmatrix} \phi_1^T \bar{y} \\ \phi_2^T \bar{y} \\ \vdots \\ \phi_N^T \bar{y} \end{bmatrix}$$

So we perform  $\phi^T \bar{y}$  and this is as shown in slide.

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$$= \begin{bmatrix} \phi_1^T \\ \phi_2^T \\ \vdots \\ \phi_N^T \end{bmatrix} \bar{y}$$

$$= \begin{bmatrix} \phi_1^T \bar{y} \\ \phi_2^T \bar{y} \\ \vdots \\ \phi_N^T \bar{y} \end{bmatrix}$$

each entry = projection of  $\bar{y}$  on column of  $\Phi$ .

So each of these entries equals the projection of  $\bar{y}$  on each column of  $\phi$ . Now the other thing that you must have observed is if you look at these rows, you can see that these rows are random 0's and 1. So these are noise like waveforms. So each measurement is a projection of  $\bar{y}$  on this noise like waveform.

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$$\Phi^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

$$\Phi_1^T y = 3, \quad \Phi_2^T y = \frac{7}{2}$$

$$\Phi_5^T y = 8 \quad \leftarrow \text{maximum} = 5^{\text{th}} \text{ entry} = \Phi_5^T y$$

So when we compute this  $\phi^T y$  we will get the vector  $\begin{bmatrix} 3 \\ 7 \\ 2 \\ 5 \\ 8 \\ 5 \end{bmatrix}$  and the maximum is at the 5<sup>th</sup>

entry which is equal to  $\phi_5^T y$ .

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$$A_{(1)} = \begin{bmatrix} \Phi_{i(1)} \end{bmatrix} = \begin{bmatrix} \Phi_5 \end{bmatrix}$$

Therefore we form the basis matrix using this column, so we have  $A_{(1)} = \begin{bmatrix} \phi_5 \end{bmatrix}$ .

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$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\min \| \bar{y} - A_{(4)} \bar{x}^{(4)} \|^2$$

Least Squares Problem

$$\hat{x}^{(4)} = (A_{(4)}^T A_{(4)})^{-1} A_{(4)}^T \bar{y}$$

$$= \left( \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^T \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)^{-1}$$

This is nothing but  $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$  and now you solve the least squares problem  $\| \bar{y} - A_{(1)} \bar{x}^{(1)} \|^2$ , this

is the first iteration. So the solution to this is  $x^{(1)} = (A_{(1)}^T A_{(1)})^{-1} A_{(1)}^T \bar{y}$ . So on evaluating this as shown in slide, we get  $x^{(1)} = 4$ .

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$$\frac{1}{2} \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 8 \\ 8 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 8 \\ 8 \end{bmatrix}$$

$$\hat{x}^{(1)} = \frac{1}{2} \times 8 = 4$$

Estimate in 1<sup>st</sup> iteration

This corresponds to the index of the column that is chosen in the first iteration that is column number 5. So this corresponds to the 5<sup>th</sup> column or the 5<sup>th</sup> entry of the vector  $\bar{x}$ . Now we find the residue for the first iteration.

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Estimate in 1<sup>st</sup> iteration

Residue:

$$\begin{aligned} r(1) &= \bar{y} - A_{(1)} x^{(1)} \\ &= \begin{bmatrix} 0 \\ 2 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} 4 \\ &= \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix} \end{aligned}$$

Residue in 1<sup>st</sup> iteration

The residue is  $\bar{r}(1) = \bar{y} - A_{(1)} x^{(1)}$  which will basically be  $\begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}$  and this is what we carry

over to the second iteration. So we subsequently find the projections of the columns of  $\phi$  on this residue, choose the one that has the maximum and perform the least square solution, find the residue and repeat the process.

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2<sup>nd</sup> iteration:

Find projection of  $\bar{r}(1)$  i.e. residue from 1<sup>st</sup> iteration on each column of  $\phi$ .

$$\begin{aligned} &\phi^T \bar{r}(1) \\ &= \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix} \end{aligned}$$

So proceeding the same way, we get the residue as shown in the slides below.

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$$\Phi^T \bar{r}(4) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad \Phi_2^T \bar{r}(4)$$

maximum corresponds to 2nd column  $i(2) = 2$

$$A_{(2)} = \begin{bmatrix} \Phi_2 & \Phi_5 \end{bmatrix}$$

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Orthogonal to 2nd column  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$A_{(2)} = \begin{bmatrix} \Phi_2 & \Phi_5 \end{bmatrix}$$

Augmented Basis matrix

$$= \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\min. \| \bar{y} - A_{(2)} \bar{x}^{(2)} \|^2$$

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$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\min. \| \bar{y} - A_{(2)} \bar{x}^{(2)} \|^2$$

$$\hat{x}^{(2)} = (A_{(2)}^T A_{(2)})^{-1} A_{(2)}^T \bar{y}$$

$$= \left( \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \right)^{-1}$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

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$$\begin{aligned}
 & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 3 \\ 5 \end{bmatrix} \\
 &= \left( \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right)^{-1} \times \begin{bmatrix} 7 \\ 8 \end{bmatrix} \\
 &= \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix} \\
 \hat{x}^{(2)} &= \frac{1}{3} \begin{bmatrix} 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}
 \end{aligned}$$

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$$\begin{aligned}
 &= \left( \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right)^{-1} \times \begin{bmatrix} 7 \\ 8 \end{bmatrix} \\
 &= \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix} \\
 \hat{x}^{(2)} &= \frac{1}{3} \begin{bmatrix} 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\
 &\text{Estimate in 2nd iteration} \\
 r(2) &= y - A_{(2)} \hat{x}^{(2)}
 \end{aligned}$$

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$$\begin{aligned}
 &= \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\
 r(2) &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \leftarrow \text{residue} \\
 \hat{x}^{(2)} &= \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\
 y &= A_{(2)} \hat{x}^{(2)} \\
 &= \begin{bmatrix} \phi_2 & \phi_5 \end{bmatrix} \hat{x}^{(2)}
 \end{aligned}$$



So here the residue is exactly 0 which basically means that you are exactly able to approximate  $\bar{y}$  in the second iteration.

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Handwritten slide content showing the equation  $\bar{y} = A^{(2)} x$  and its approximation using columns of matrix  $\phi$ . The equation is written as  $\bar{y} = [\tilde{\phi}_2 \tilde{\phi}_5] \hat{x}^{(2)}$ . An arrow points to the matrix  $[\tilde{\phi}_2 \tilde{\phi}_5]$  with the text "Exactly approximates  $\bar{y}$ ". Below this, the vector  $\hat{x}^{(2)} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$  is shown. Arrows point from the components of  $\hat{x}^{(2)}$  to the columns of  $\phi$ :  $\tilde{\phi}_2 = 2^{\text{nd}} \text{ column of } \phi$  and  $\tilde{\phi}_5 = 5^{\text{th}} \text{ column of } \phi$ .

So no further iterations are needed which means  $x^{(2)} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$  and the components 2 and 3 corresponds to  $\phi_2$  and  $\phi_5$  which are basically the second and fifth columns of the matrix  $\phi$  respectively.

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Handwritten slide content showing the reconstruction of the sparse vector  $\bar{x}$  using the columns of  $\phi$  and the components of  $\hat{x}^{(2)}$ . The equation is written as  $\bar{x} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}$ . Arrows point from the components of  $\bar{x}$  to the columns of  $\phi$ :  $\tilde{\phi}_2 = 2^{\text{nd}} \text{ column of } \phi$  and  $\tilde{\phi}_5 = 5^{\text{th}} \text{ column of } \phi$ . The text "Estimate of sparse vector  $\bar{x}$ " is written above the vector. The text "Rest of entries 0" is written below the vector.

And therefore, now you can reconstruct the sparse vector  $\bar{x}$  as follows, only the second entry and the 5<sup>th</sup> entry will be 2 and 3 respectively and the rest of the entries are 0. This is a simple example, but problems in practice are frequently more complex. But you can

use this OMP algorithm for similar scenarios. So let us stop here and continue in the subsequent modules. Thank you very much.