**Applied Optimization for Wireless, Machine Learning, Big Data**

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**Lecture - 35**
**Practical Application: Maximal Ratio Combiner for Wireless Systems**

Hello, welcome to another module in this massive open online course.

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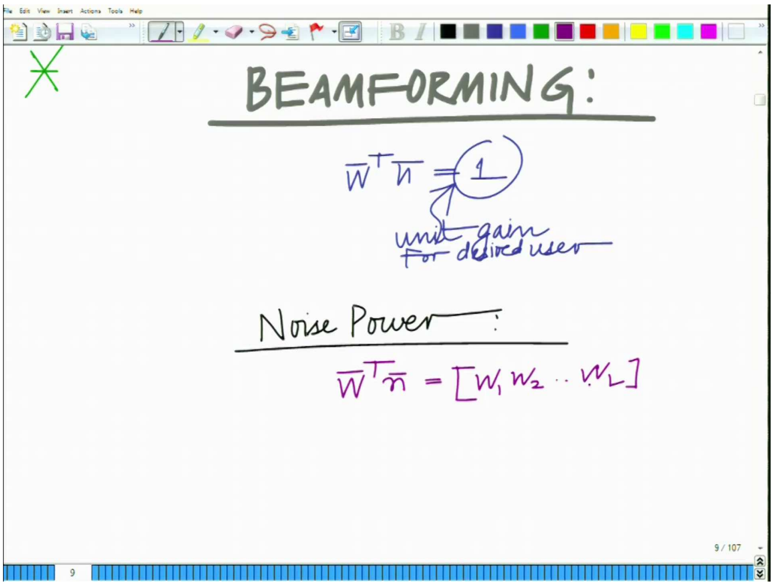


Let us continue the discussion on the beam forming problem. The constraint for the signal gain is

$$\bar{w}^T\bar{h} = 1$$

Where $\bar{w}$ is the beam forming vector and $\bar{h}$ is the channel coefficient vector of a SIMO system. Above expression shows the unit gain for desired user to minimize the noise power. Now, the noise power $\bar{w}^T\bar{n}$ can be calculated as follows.

$$\bar{w}^T\bar{n} = \begin{bmatrix} w_1 & w_2 & \cdots & w_L \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_L \end{bmatrix}$$

$$= \sum_{i=1}^{L} w_i n_i$$

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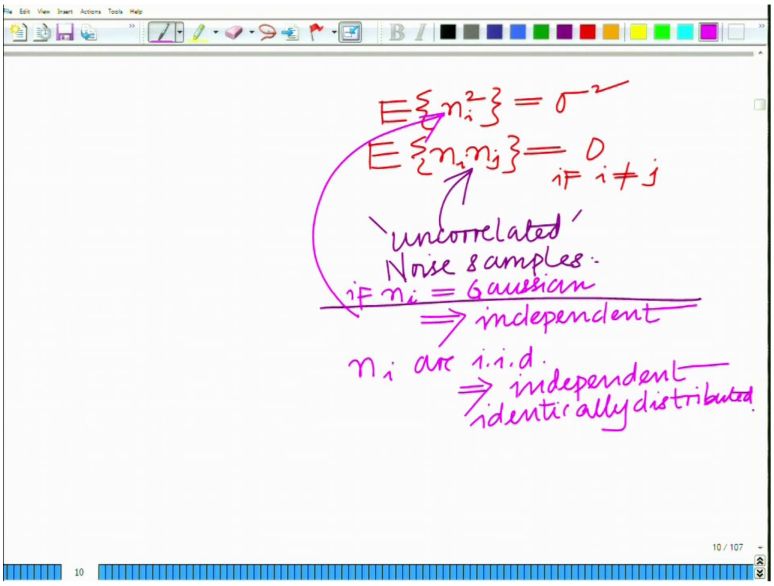


Typically the noise samples are assumed to be additive white Gaussian which means that the different noise samples at the different antennas are independent and identically distributed which means they have zero mean and its variance is $\sigma^2$. Also these noise samples are uncorrelated. Thus the properties of this noise samples are as follows.

$$E\left\{n_i^2\right\} = \sigma^2$$

$$E\left\{n_i n_j\right\} = 0 \qquad \text{for } i \neq j$$

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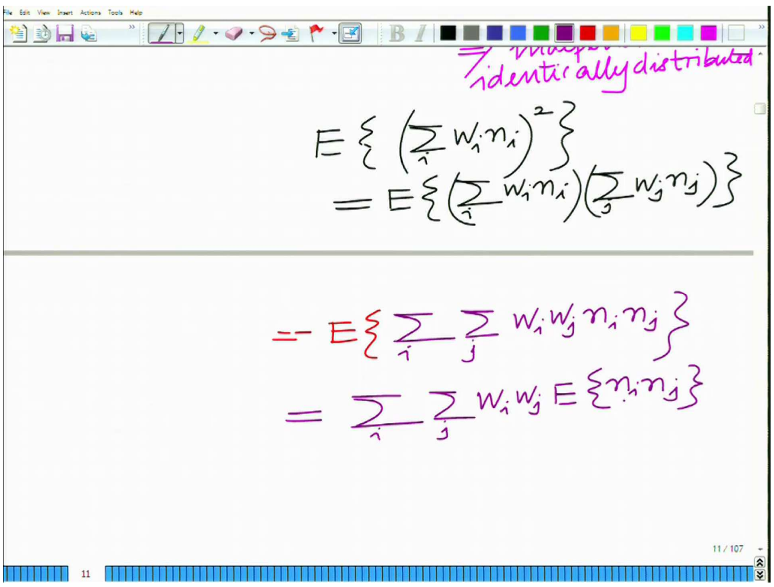

Therefore the noise power is

$$E\left\{\left(\sum_i w_i n_i\right)^2\right\} = E\left\{\left(\sum_i w_i n_i\right)\left(\sum_j w_j n_j\right)\right\}$$
$$= E\left\{\sum_i \sum_j w_i w_j n_i n_j\right\}$$
$$= \sum_i \sum_j w_i w_j E\left\{n_i n_j\right\}$$

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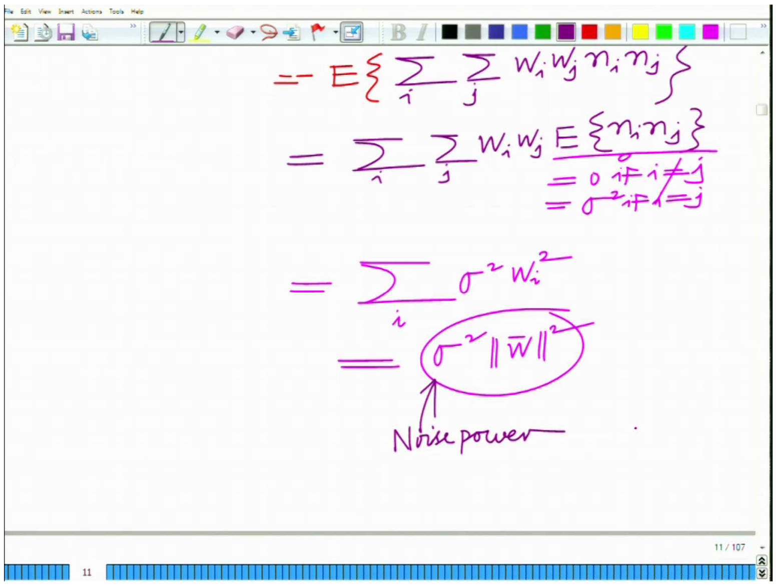


There are samples of the two different antennas are uncorrelated. So for that the noise power is zero. But for $i = j$;

$$E\left\{\left(\sum_i w_i n_i\right)^2\right\} = \sum_i \sigma^2 w_i^2$$
$$= \sigma^2 \|\bar{w}\|^2$$

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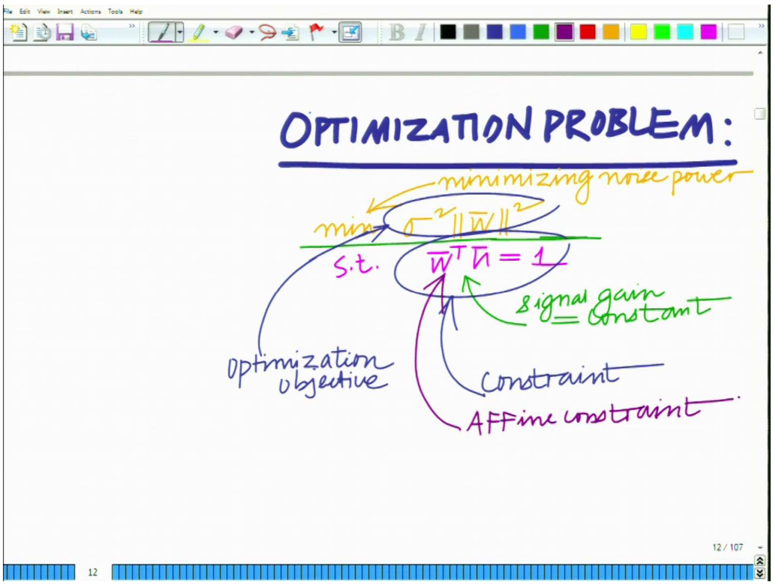


Therefore, the optimization problem for beamforming will be formulated as follows.

$$\min \ \sigma^2 \|\bar{w}\|^2$$
$$\text{such that } \ \bar{w}^T \bar{h} = 1$$

This optimization problem is to minimizing the noise power such that the signal gain is unity in the desired direction. This maximizes the signal to noise power ratio. This optimization problem can be looked in two parts. First is the minimization which is termed as the optimization objective or the objective function. In this case, it is the sum of convex functions and hence it is convex. Second part is the constraint which is here an affine constraint. So in beam forming, objective and constraint both are convex. Therefore the above optimization problem is a constrained convex optimization problem.

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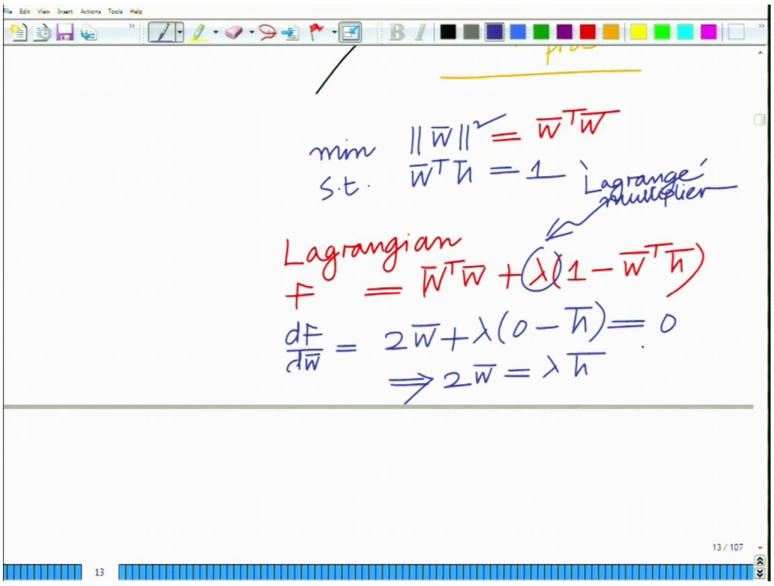


Let us formulate the objective power. As $\sigma^2$ is a constant so it can be avoided from the objective. Also

$$\|\bar{w}\|^2 = \bar{w}^T\bar{w}$$

Therefore objective can be rewritten as follows.

$$\min \ \bar{w}^T\bar{w}$$
$$\text{such that } \ \bar{w}^T\bar{h} = 1$$

With reference to the early course on calculus, to solve a constraint or constrained optimization problem, one needs to use Lagrange multipliers. In the above optimization problem with $\lambda$ as the Lagrange multiplier, the Lagrangian function is as follows.

$$F = \bar{w}^T\bar{w} + \lambda\left(1 - \bar{w}^T\bar{h}\right)$$

This Lagrange multiplier $\lambda$ has to be determined as the solution to the optimization problem. Therefore for minimization, set the differentiation of Lagrangian function equal to zero.

$$\frac{dF}{d\bar{w}} = 0$$

$$2\bar{w} + \lambda\left(0 - \bar{h}\right) = 0$$

$$\bar{w} = \frac{\lambda}{2}\bar{h}$$

This is the optimal beam former because it maximizes the signal to noise power ratio. It is evident here that the optimal beam former is proportional to channel coefficient vector. So, this is similar to the spatially matched filter, because typically matched filter is in time. Hence it is a matched filter across the antennas. So, this optimal beamformer is analogues to a matched filter that is employed in a digital communication system.

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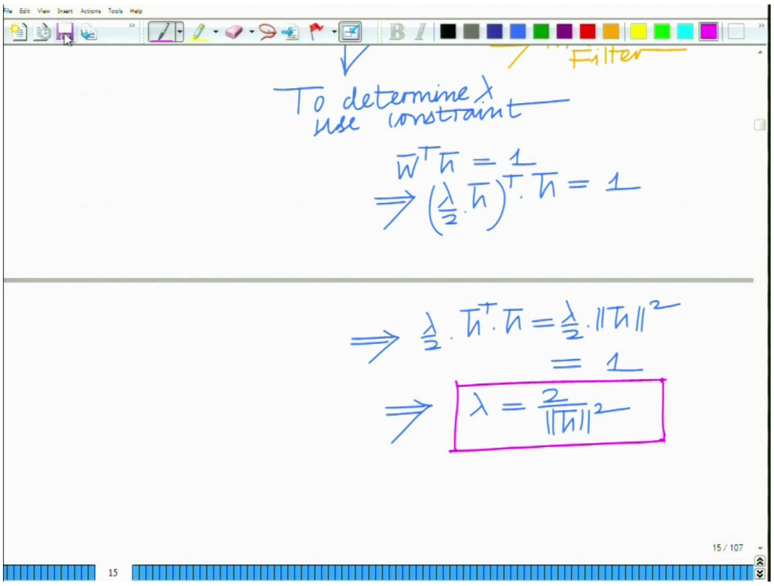


Also to determine this $\lambda$, one has to use the constraint. Therefore on substituting the value of $\bar{w}$ from the above expression in the constraint, the value of this Lagrange multiplier is

$$\bar{w}^T\bar{h} = 1$$

$$\left(\frac{\lambda}{2}\bar{h}\right)^T\bar{h} = 1$$

$$\frac{\lambda}{2}\|\bar{h}\|^2 = 1$$

$$\lambda = \frac{2}{\|\bar{h}\|^2}$$

Therefore, it implies that the optimal beam former is

$$\bar{w}^* = \frac{1}{2}\frac{2}{\left\|\bar{h}\right\|^2}\bar{h}$$

$$\bar{w}^* = \frac{\bar{h}}{\left\|\bar{h}\right\|^2}$$

This is also termed as the maximal ratio combiner because it maximizes the signal to noise power ratio.

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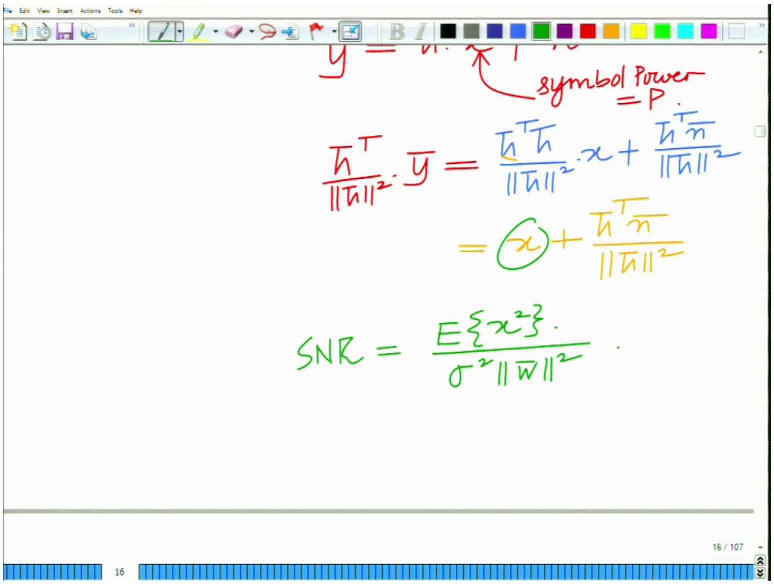


Let us beam form the above system with the maximal ratio combiner.

$$\bar{w}^*.\bar{y} = \frac{\bar{h}}{\left\|\bar{h}\right\|^2}.\left(\bar{h}x + \bar{n}\right)$$

$$= \frac{\bar{h}^T\bar{h}}{\left\|\bar{h}\right\|^2}x + \frac{\bar{h}^T\bar{n}}{\left\|\bar{h}\right\|^2}$$

$$= x + \frac{\bar{h}^T\bar{n}}{\left\|\bar{h}\right\|^2}$$

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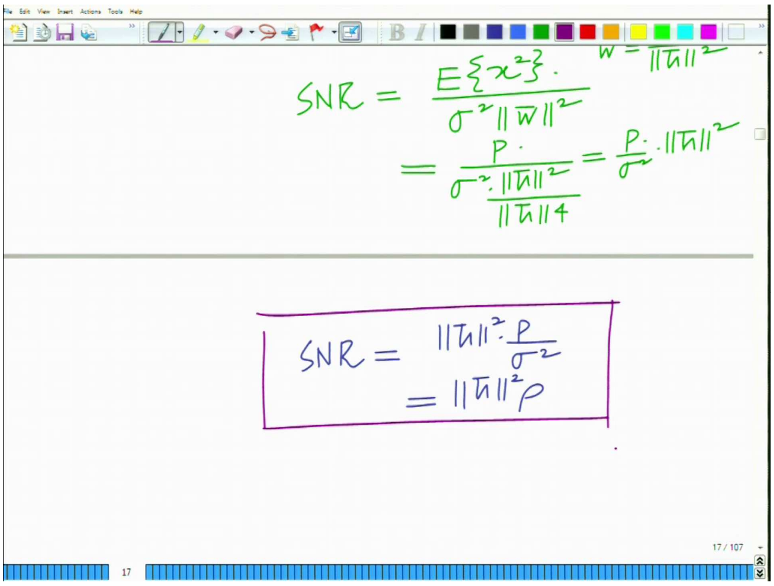


Therefore, the SNR at the output of the maximal ratio combiner is derived as follows.

$$\mathrm{SNR} = \frac{E\left\{x^2\right\}}{\sigma^2 \frac{\left\|\bar{h}\right\|^2}{\left\|\bar{h}\right\|^4}}$$

$$= \frac{P}{\sigma^2}\left\|\bar{h}\right\|^2$$

Here $P$ is the symbol power and $\sigma^2$ is the noise power. This can also be written as follows.

$$\mathrm{SNR} = \left\|\bar{h}\right\|^2 \rho$$

Where $\rho$ is the transmit SNR equal to $\frac{P}{\sigma^2}$.

So, this designing of the optimal beam former is the first convex optimization problem which is a simple application of the optimization framework. Here the beam forming gain in the desired direction and the direction of the signal is set to unity as the constraint and the noise power is minimized resulting in the maximization of the signal to noise power ratio. The Lagrange multiplier framework is used to formulate the Lagrangian function. This is also known as the KKT frame.