Applied Optimization for Wireless, Machine Learning, Big Data
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Lecture – 65
Example problem on Strong Duality

***Keywords**: Strong Duality*

Hello welcome to another module in this massive open online course. So we are looking at duality and we have seen the concept of strong duality that is for any optimization problem written in the standard form, one can come up with an equivalent dual optimization problem which is convex, you can solve that and to obtain the optimal point d^* and usually $d^* \leq P^*$ where P^* is the optimal value of the original primal problem, but when strong duality holds which is usually true for a convex optimization problem we have $d^* = P^*$. And now let us understand that through an example.

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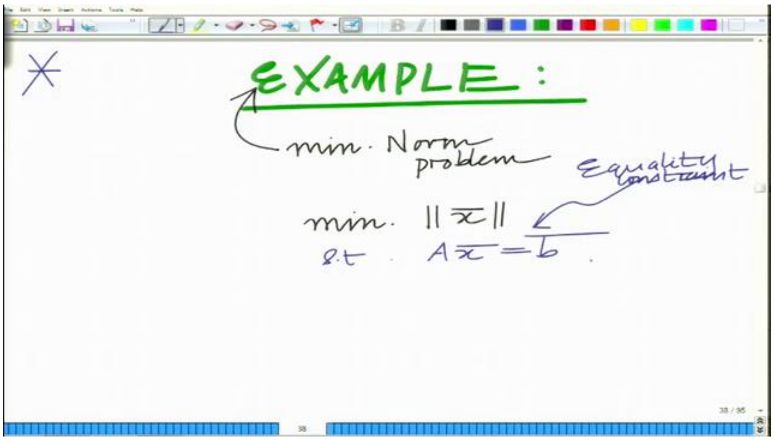

So let us look at the minimum norm problem. So we have $\begin{matrix} \min \|\bar{x}\| \\ s.t\ A\bar{x} = \bar{b} \end{matrix}$. Here there are only equality constraints, there is no inequality constraint.

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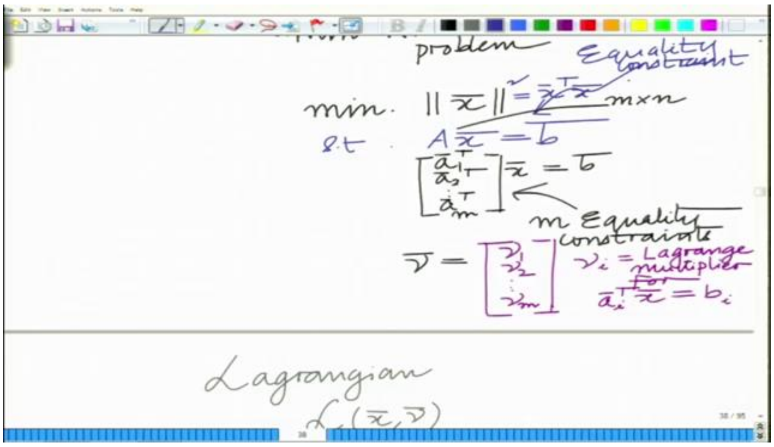

Here this matrix A has m rows as shown in slide and therefore there are m equality constraints, one for each row of the matrix A. Therefore you need to have one Lagrange multiplier for each equality constraint, so you have a vector $\bar{\nu}$ where each ν_i is for $\bar{a}_i^T \bar{x} = b_i$.

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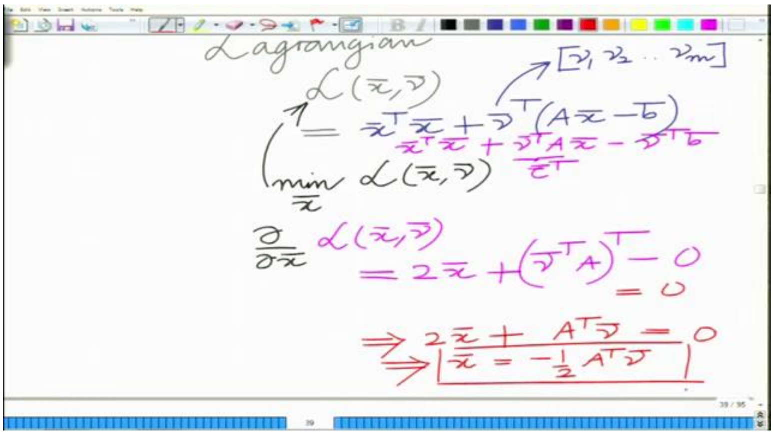

Now, the Lagrangian can be formulated as shown in slide and on solving it as shown we get $\bar{x} = -\frac{1}{2} A^T \bar{\nu}$. So this is the $\bar{x}$ for which the minimum is achieved for the Lagrangian corresponding to the original optimization problem. Now to get the dual optimization problem we substitute this.

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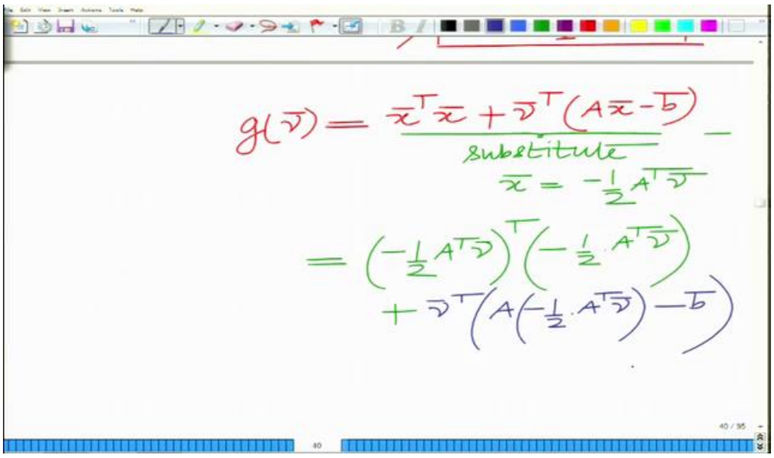

So after substitution and further simplification as shown in the slides we get the Lagrange Dual function as $g(\bar{v}) = -\frac{1}{4}\bar{v}^T A A^T \bar{v} - \bar{v}^T \bar{b}$.

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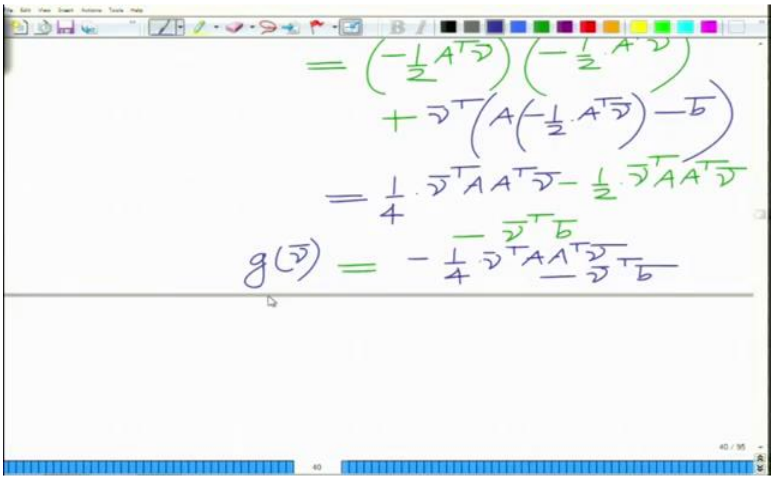

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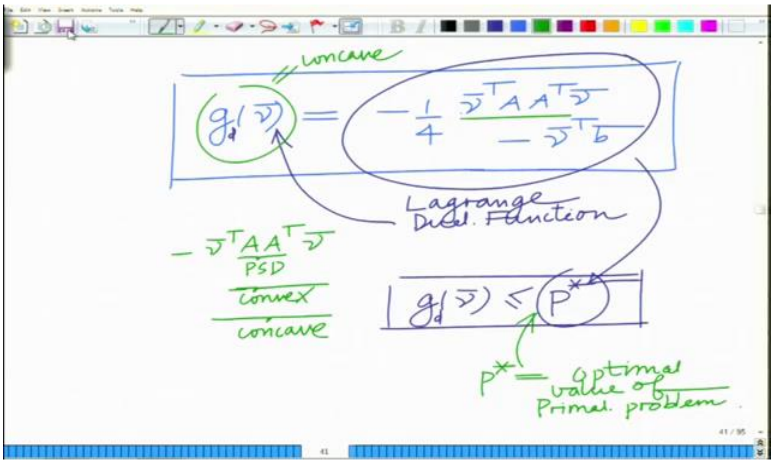

So this will always give a lower bound that is $g_d(\bar{\nu}) \leq P^*$. Now this is a concave function.

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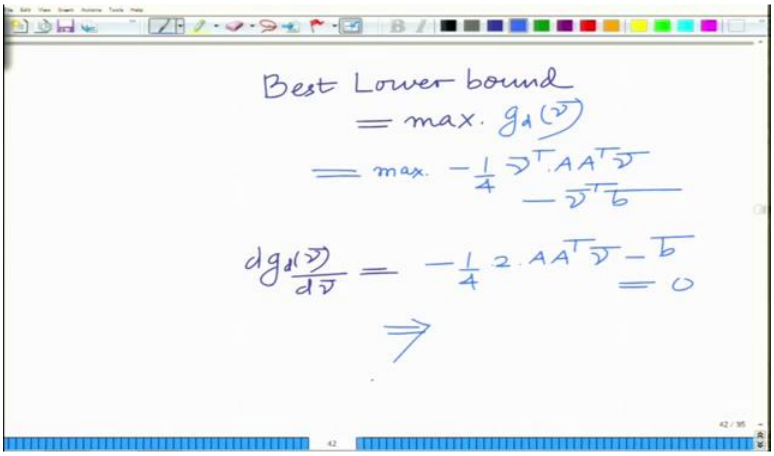

Now the best lower bound is given by the maximum value. So we have $\max\ g_d(\bar{\nu})$.

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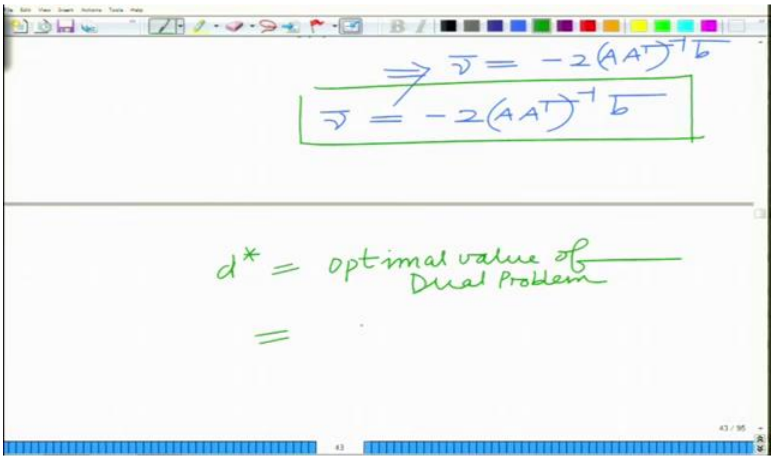

So on solving this we get $\bar{\nu} = -2(AA^T)^{1}\bar{b}$ and for this value the Lagrange dual function is maximized. Now to find the optimal value d*, simply substitute $\bar{\nu}$ in the dual problem,.

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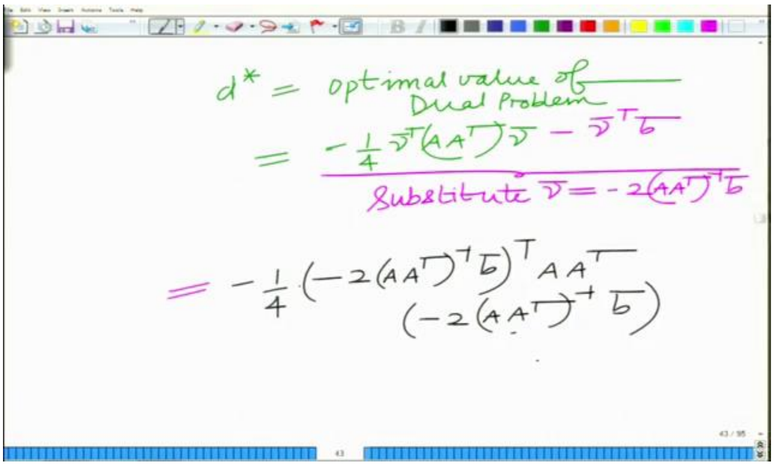

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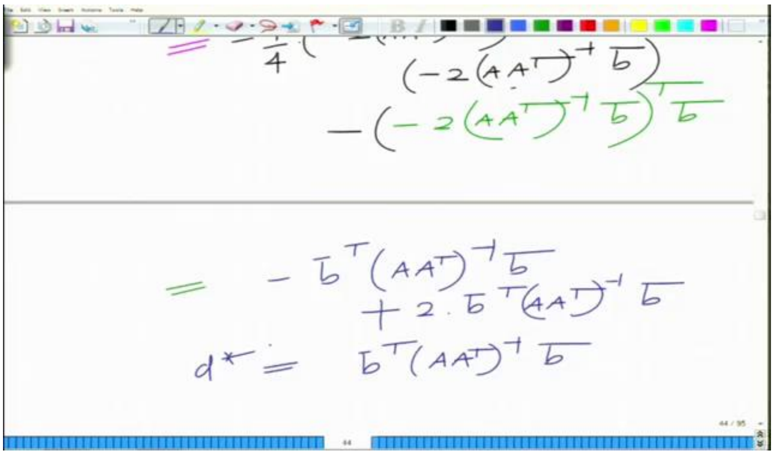

So d^* that is the optimal value of the dual problem is obtained as $d^* = \bar{b}^T \left(AA^T \right)^{-1} \bar{b}$.

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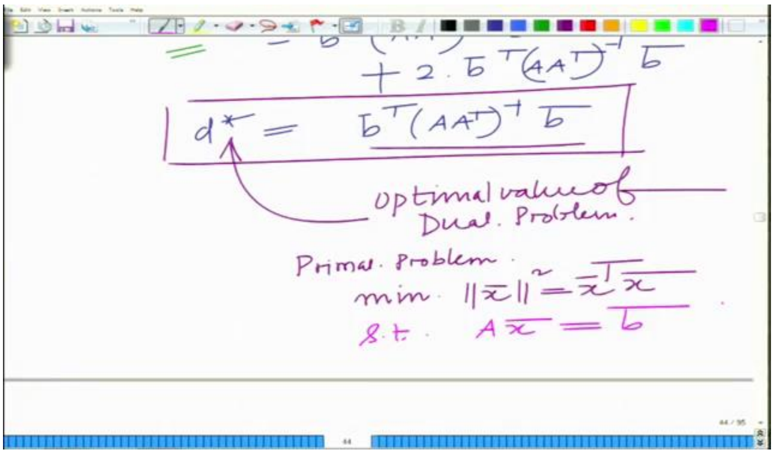

So this is d^* is always less than or equal to P^*. Now we need to find P^* that is optimal value of the primal problem. So we have $\begin{array}{l} \min \left\| \bar{x} \right\|^2 = \bar{x}^T \bar{x} \\ s.t\ A\bar{x} = \bar{b} \end{array}$.

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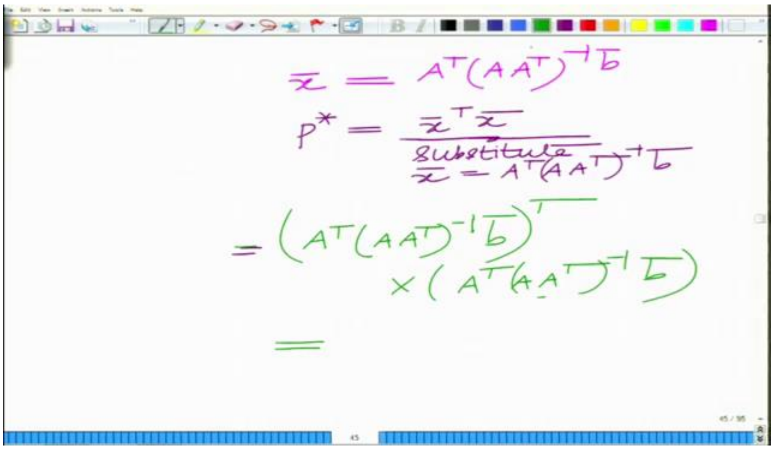

And we already know that the optimal solution for this is $\bar{x} = A^T\left(AA^T\right)^{-1}\bar{b}$ from the previous modules. And now $P^* = \bar{x}^T\bar{x}$ and we substitute $\bar{x}$ in P^* and if you simplify it we get $P^* = \bar{b}^T\left(AA^T\right)^{-1}\bar{b} = d^*$.

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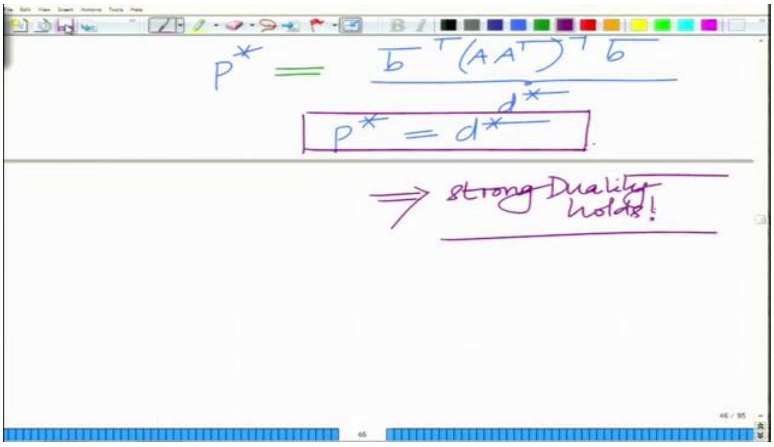

Therefore, strong duality holds and the dual objective and the primal objective are coinciding at the same point which is the maximum value of the dual objective function as well as the optimal value of the primal objective. So this is one of the simplest and most elegant optimization problems.

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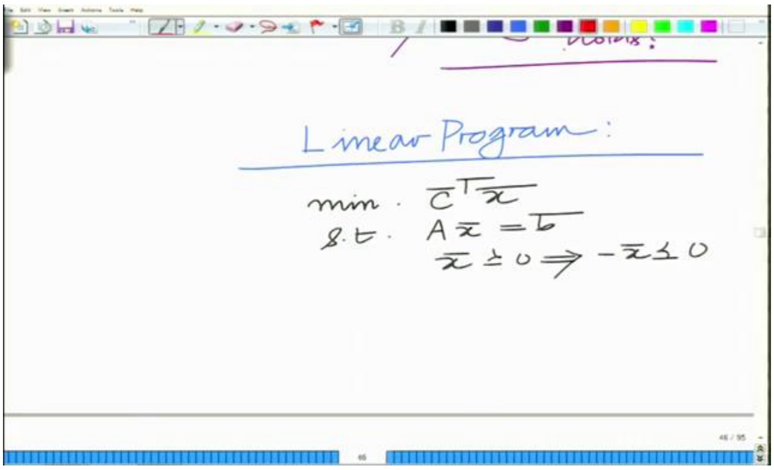

Let us look at another interesting problem and that is a linear program. So we have

$$\begin{aligned} &\min \ \bar{c}^T \bar{x} \\ &s.t \ \ A\bar{x} = \bar{b} \\ &\qquad \bar{x} \geq 0 \end{aligned}$$

. This can be written as a standard form convex optimization problem as

$$\begin{aligned} &\min \ \bar{c}^T \bar{x} \\ &s.t \ \ A\bar{x} = \bar{b} \\ &\quad -\bar{x} \leq 0 \end{aligned}$$

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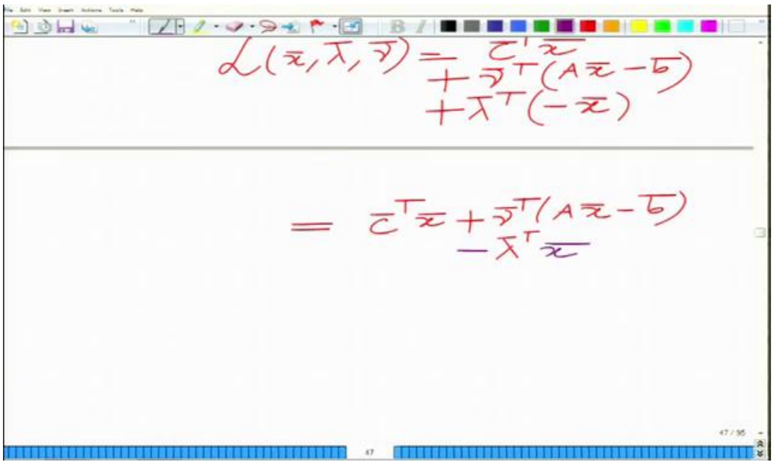

Now, the Lagrangian of this can be formulated as $L(\bar{x},\bar{\lambda},\bar{\nu}) = \bar{c}^T \bar{x} + \bar{\nu}^T \left(A\bar{x} - \bar{b}\right) + \bar{\lambda}^T \left(-\bar{x}\right)$. So this comprises of the Lagrange multiplier for the equality constraint and one Lagrange multiplier for each inequality constraint. Now, we have to take the minimum of the Lagrangian and typically for that we differentiate it with respect to the vector $\bar{x}$, but since this is an affine function we will follow a slightly different approach.

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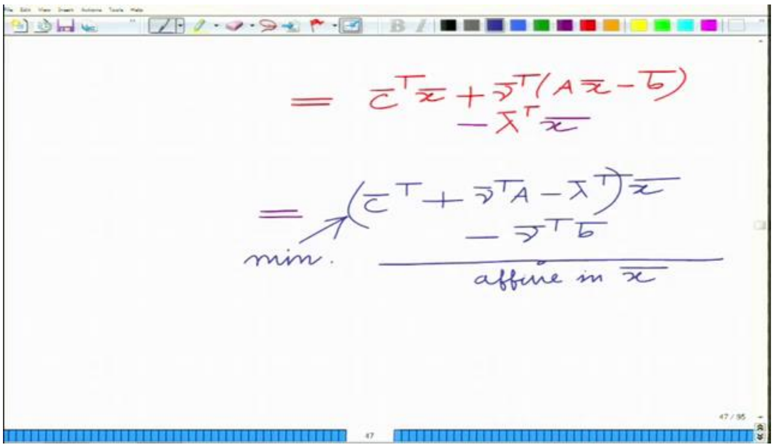

If you separate the terms, you can see this is the equation of a hyperplane. Now this is an affine function, it is like a line.

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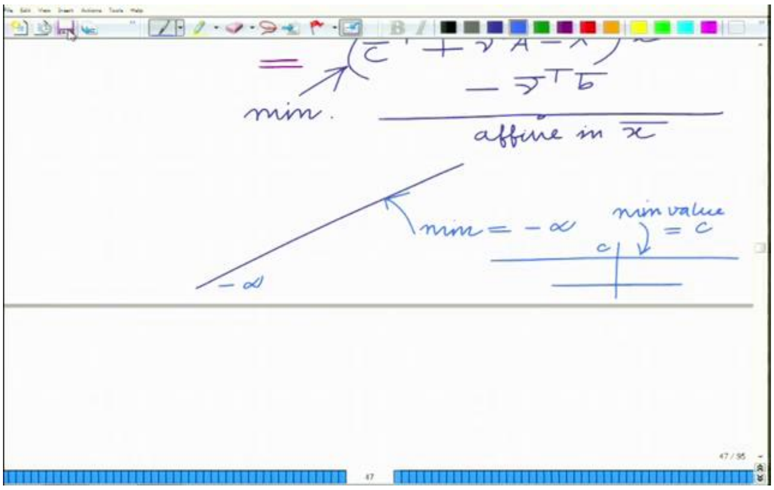

So if this line has a slope, then the minimum value of this will always be equal to $-\infty$, only if the line is parallel, then the minimum value is a constant.

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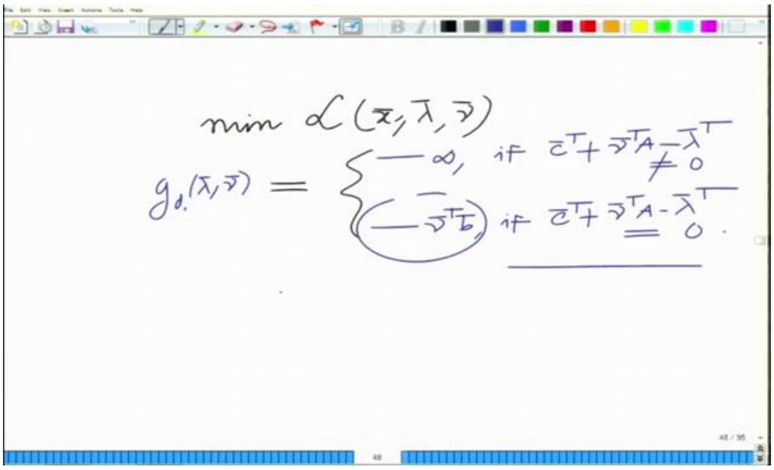

$$\min L(\bar{x}, \bar{\lambda}, \bar{\nu})$$

So with that observation we have $g_d(\bar{\lambda}, \bar{\nu}) = \begin{cases} -\infty \; if \; \bar{c}^T + \bar{\nu}^T A - \bar{\lambda}^T \neq 0 \\ -\bar{\nu}^T \bar{b} \; if \; \bar{c}^T + \bar{\nu}^T A - \bar{\lambda}^T = 0 \end{cases}$. So this is the Lagrange dual function and the best lower bound is available, when you maximize this.

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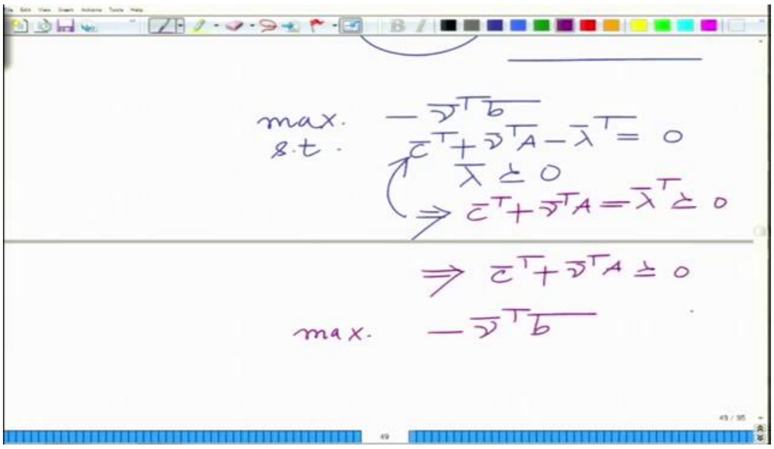

So the dual optimization problem can be equivalently written as $\begin{array}{l} \max -\bar{\nu}^T \bar{b} \\ s.t \; A^T \bar{\nu} + \bar{c} \geq 0 \end{array}$ as shown in the slides.

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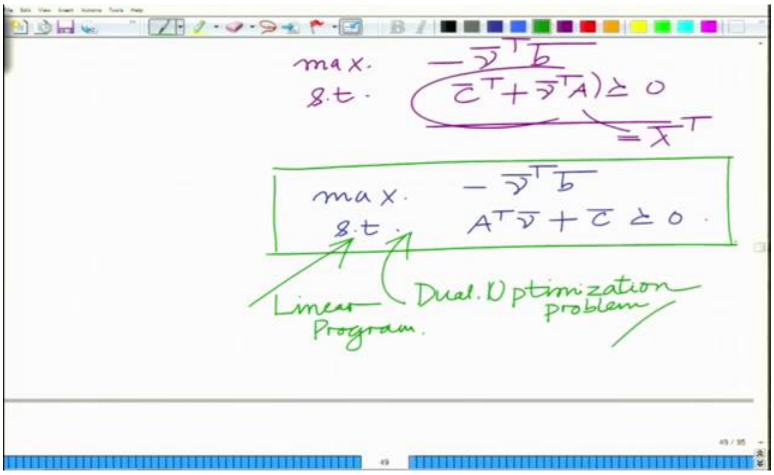

And since the original problem is a linear program, the dual optimization problem is also a linear program. Therefore, strong duality holds. So we will stop here and continue in the subsequent modules. Thank you very much.