

# 电动力学-第十次作业

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2023 年 4 月 22 日

Problem 10.16

Answer: The potential of the particle is equal to:

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{\sqrt{(c^2t - \vec{r} \cdot \vec{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}}$$

We need to somehow "pull out" the factor of  $Rc$  from the denominator. The expression under the root is:

$$\begin{aligned} & (c^2t - \vec{r} \cdot \vec{v})^2 + (c^2 - v^2)(r^2 - c^2t^2) = \\ & = c^4t^2 - 2c^2t\vec{r} \cdot \vec{v} + (\vec{r} \cdot \vec{v})^2 + c^2r^2 - c^4t^2 - v^2r^2 + v^2c^2t^2 \\ & = -2c^2t\vec{r} \cdot \vec{v} + (\vec{r} \cdot \vec{v})^2 + c^2r^2 - v^2r^2 + v^2c^2t^2 \end{aligned}$$

$$\begin{aligned} \vec{R} &= \vec{r} - \vec{v}t \implies R^2 = r^2 + v^2t^2 - 2t\vec{v} \cdot \vec{r} \\ & (c^2t - \vec{r} \cdot \vec{v})^2 + (c^2 - v^2)(r^2 - c^2t^2) = \\ & = c^2(r^2 + v^2t^2 - 2t\vec{v} \cdot \vec{r}) + (\vec{r} \cdot \vec{v})^2 - v^2r^2 \\ & = c^2R^2 + (\vec{r} \cdot \vec{v})^2 - v^2r^2 \end{aligned}$$

Now, the two other two terms are equal to:

$$\begin{aligned} (\vec{r} \cdot \vec{v})^2 - v^2r^2 &= ((\vec{R} + \vec{v}t) \cdot \vec{v})^2 - v^2(R^2 + 2t\vec{r} \cdot \vec{v} - v^2t^2) \\ &= v^2R^2 \cos^2 \theta + 2(\vec{R} \cdot \vec{v})v^2t + v^4t^2 - v^2R^2 + 2v^2t\vec{r} \cdot \vec{v} + v^4t^2 \\ &= v^2R^2 (\cos^2 \theta - 1) + 2((\vec{r} - \vec{v}t) \cdot \vec{v})v^2t + 2v^2t\vec{r} \cdot \vec{v} + 2v^4t^2 \\ &= -v^2R^2 \sin^2 \theta + 2v^2t\vec{r} \cdot \vec{v} - 2v^4t^2 - 2v^2t\vec{r} \cdot \vec{v} + 2v^4t^2 \\ &= -v^2R^2 \sin^2 \theta \end{aligned}$$

where  $\theta$  is the angle the velocity makes with the vector  $\vec{R}$ . The expression under the root is then:

$$\begin{aligned} (c^2t - \vec{r} \cdot \vec{v})^2 + (c^2 - v^2)(r^2 - c^2t^2) &= c^2R^2 - v^2R^2 \sin^2 \theta \\ &= c^2R^2 (1 - \beta^2 \sin^2 \theta) \end{aligned}$$

So, the potential is:

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon_0} \frac{qc}{\sqrt{(c^2t - \vec{r} \cdot \vec{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}} \\ &= \frac{1}{4\pi\epsilon_0} \frac{qc}{Rc\sqrt{1 - \beta^2 \sin^2 \theta}} = \sqrt{\frac{1}{4\pi\epsilon_0}} \frac{q}{R\sqrt{1 - \beta^2 \sin^2 \theta}} \end{aligned}$$

Q.E.D

#### Problem 10.19

Answer:

Due to the limitations of this site I will be using letter  $l$  as the distance from the charge to the point of interest.

The Lienard-Wiechert vector potential of the charge in motion is:

$$\vec{A} = \frac{\mu_0}{4\pi} \frac{qc\vec{v}}{l_c - \vec{l} \cdot \vec{v}} = \frac{\mu_0}{4\pi} \frac{qc\vec{v}}{\vec{l} \cdot \vec{u}}$$

all quantities evaluated at retarded time. We want the partial time derivative of this, but first:

$$l = c(t - t_r)$$

$$l^2 = c^2(t - t_r)^2$$

$$2\vec{l} \cdot \frac{\partial \vec{l}}{\partial t} = 2c^2(t - t_r) \left(1 - \frac{\partial t_r}{\partial t}\right) \Rightarrow \frac{\partial t_r}{\partial t} = 1 - \frac{\hat{l}}{c} \cdot \frac{\partial \vec{l}}{\partial t}$$

But:

$$\vec{l} = \vec{r} - \vec{w}(t_r)$$

$$\frac{\partial \vec{l}}{\partial t} = -\frac{\partial \vec{w}}{\partial t_r} \frac{\partial t_r}{\partial t} = -\vec{v}(t_r) \frac{\partial t_r}{\partial t}$$

$$\frac{\partial t_r}{\partial t} = 1 + \frac{\hat{l}}{c} \cdot \vec{v}(t_r) \frac{\partial t_r}{\partial t}$$

So, as we need to prove first:

$$\frac{\partial t_r}{\partial t} = \frac{cl}{\vec{l} \cdot (c\hat{l} - \vec{v})} = \frac{cl}{\vec{l} \cdot \vec{u}}$$

With this we can get the partial time derivative of the vector potential:

$$\begin{aligned} \frac{\partial \vec{A}}{\partial t} &= \frac{\mu_0 qc}{4\pi} \left[ \frac{1}{\vec{l} \cdot \vec{u}} \frac{\partial \vec{v}}{\partial t} - \frac{\vec{v}}{(\vec{l} \cdot \vec{u})^2} \frac{\partial}{\partial t} (\vec{l} \cdot \vec{u}) \right] \\ \frac{\partial \vec{v}}{\partial t} &= \frac{\partial \vec{v}}{\partial t_r} \frac{\partial t_r}{\partial t} = \vec{a} \frac{cl}{\vec{l} \cdot \vec{u}} \\ \frac{\partial}{\partial t} (\vec{l} \cdot \vec{u}) &= c \frac{\partial l}{\partial t} - \frac{\partial \vec{l}}{\partial t} \cdot \vec{v} - \frac{\partial \vec{v}}{\partial t} \cdot \vec{l} = c \frac{\partial l}{\partial t} + v^2 \frac{cl}{\vec{l} \cdot \vec{u}} - (\vec{a} \cdot \vec{l}) \frac{cl}{\vec{l} \cdot \vec{u}} \end{aligned}$$

Hence:

$$\begin{aligned} \frac{\partial \vec{A}}{\partial t} &= \frac{\mu_0 qc}{4\pi} \frac{1}{(\vec{l} \cdot \vec{u})^3} \left[ lc\vec{a}(\vec{l} \cdot \vec{u}) + \vec{v} \left( cl \left( c^2 - v^2 + \vec{l} \cdot \vec{a} \right) + c^2 (\vec{l} \cdot \vec{u}) \right) \right] \\ &= \frac{qc}{4\pi\epsilon_0} \left[ (\vec{l} \cdot \vec{u}) \left( \frac{l}{c} \vec{a} - \vec{v} \right) + \frac{l}{c} \vec{v} (c^2 - v^2 + \vec{r} \cdot \vec{a}) \right] \end{aligned}$$

Q.E.D

### Problem 10.22

Answer:

The electric field due to the leght of the wire of lenght  $dx$  is the field of the uniformly moving point charge:

$$d\vec{E} = \frac{\lambda dx}{4\pi\epsilon_0} \frac{1 - \beta^2}{(1 - \beta^2 \sin^2 \theta)^{3/2}} \frac{\hat{R}}{R^2}$$

Only the vertical component of the field will survive the integration, so:

$$\begin{aligned} dE &= \frac{\lambda dx}{4\pi\epsilon_0} \frac{1 - \beta^2}{(1 - \beta^2 \sin^2 \theta)^{3/2}} \frac{\sin \theta}{R^2} \\ R &= \frac{d}{\sin \theta} \quad - x = R \cos \theta = d \cot \theta \quad dx = d \frac{d\theta}{\sin^2 \theta} \\ &= \frac{\lambda d}{4\pi\epsilon_0} \frac{1 - \beta^2}{(1 - \beta^2 \sin^2 \theta)^{3/2}} \frac{\sin^3 \theta}{d^2} \frac{d\theta}{\sin^2 \theta} \\ &= \frac{\lambda}{4\pi\epsilon_0 d} (1 - \beta^2) \frac{\sin \theta d\theta}{(1 - \beta^2 \sin^2 \theta)^{3/2}} \end{aligned}$$

The total electric field is then the integral:

$$\begin{aligned} E &= \frac{\lambda}{4\pi\epsilon_0 d} (1 - \beta^2) \int_0^\pi \frac{\sin \theta d\theta}{(1 - \beta^2 \sin^2 \theta)^{3/2}} \\ &= \frac{\lambda}{2\pi\epsilon_0 d} (1 - \beta^2) \int_0^{\pi/2} \frac{\sin \theta d\theta}{(1 - \beta^2 \sin^2 \theta)^{3/2}} \end{aligned}$$

since the integrand is symmetrical about  $\theta = \pi/2$ . Make the substitution:

$$\begin{aligned} u &= \cos \theta \quad du = -\sin \theta d\theta \\ E &= \frac{\lambda}{2\pi\epsilon_0 d} (1 - \beta^2) \int_0^1 \frac{du}{(1 - \beta^2(1 - u^2))^{3/2}} \end{aligned}$$

The integral is of the table variety:

$$\begin{aligned} \int_0^1 \frac{du}{(1 - \beta^2(1 - u^2))^{3/2}} &= \frac{1}{\beta^3} \int_0^1 \frac{du}{(\beta^{-2} - 1 + u^2)^{3/2}} \\ &= \frac{1}{\beta^3} \int_0^1 \frac{du}{(A^2 + u^2)^{3/2}} \\ A^2 &= \beta^{-2} - 1 \\ \frac{1}{\beta^3} \int_0^1 \frac{du}{(A^2 + u^2)^{3/2}} &= \frac{1}{\beta^3} \frac{u}{A^2 \sqrt{A^2 + u^2}} \Big|_0^1 \\ &= \frac{1}{\beta^3} \frac{1}{A^2 \sqrt{A^2 + 1}} = \frac{1}{\beta^3} \frac{1}{(\beta^{-2} - 1) \beta - 1} = \frac{1}{1 - \beta^2} \end{aligned}$$

The electric field is thus:

$$\begin{aligned} E &= \frac{\lambda}{2\pi\epsilon_0 d} (1 - \beta^2) \frac{1}{1 - \beta^2} = \frac{\lambda}{2\pi\epsilon_0 d} \\ \vec{E} &= \frac{\lambda}{2\pi\epsilon_0 d} \hat{s} \end{aligned}$$