

电动力学-第十一次作业

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Problem 11.12

Answer:

The acceleration of the particle is $a = g$ (downwards), so the power radiated is:

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} = \frac{\mu_0 q^2 g^2}{6\pi c}$$

The total energy radiated in the time needed to traverse 1cm is:

$$U = \int_0^{t_f} P dt = P t_f$$

$$y = \frac{1}{2} g t_f^2 \implies t_f = \sqrt{\frac{2y}{g}}$$

$$U = P \sqrt{\frac{2y}{g}} = \frac{\mu_0 q^2 g^2}{6\pi c} \sqrt{\frac{2y}{g}}$$

The ratio of this energy versus the potential energy lost by traversing that distance is:

$$\eta = \frac{U}{U_{pot}} = \frac{\frac{\mu_0 q^2 g^2}{6\pi c} \sqrt{\frac{2y}{g}}}{mgy} = \frac{\mu_0 q^2}{6\pi c m} \sqrt{\frac{2g}{y}}$$

Numerically this ratio is equal to, with $m = 9.11 \cdot 10^{-31} \text{kg}$ and $q = 1.609 \cdot 10^{-19} \text{C}$:

$$\eta \approx 2.8 \cdot 10^{-22}$$

Problem 11.20

Answer:

(c)

We want to sum all of the interaction terms, and hopefully that will reproduce the 11.100). Now:

$$\begin{aligned} dF^{int} &= \frac{\mu_0 \dot{a}}{12\pi c} dq_1 dq_2 \\ dq_1 &= 2\lambda dy_1 \quad dq_2 = 2\lambda dy_2 \\ dF^{int} &= \frac{\mu_0 \dot{a} \lambda^2}{3\pi c} dy_1 dy_2 \end{aligned}$$

Let $y_1 > y_2$ and let the bar run from $y = 0$ to $y = L$. First we integrate from $y_2 = 0$ to $y_2 = y_1$ to find the interaction of everything below y_1 with the dq_1 at y_1

$$dF^{int} = \frac{\mu_0 \dot{a} \lambda^2}{3\pi c} \left(\int_0^{y_1} dy_2 \right) dy_1 = \frac{\mu_0 \dot{a} \lambda^2}{3\pi c} y_1 dy_1$$

Then just integrate over all y_1 over the entire bar:

$$F^{int} = \frac{\mu_0 \dot{a} \lambda^2}{3\pi c} \int_0^L y_1 dy_1 = \frac{\mu_0 \dot{a} \lambda^2}{6\pi c} L^2 = \frac{\mu_0 \dot{a} q^2}{6\pi c}$$

Problem 11.24

Answer:

Refer to the Figure 11.19). The potentials of a dipole are:

$$V = -\frac{p_0 \omega}{4\pi \epsilon_0 c} \frac{\cos \theta}{r} \sin[\omega(t - r/c)] \quad \vec{A} = -\frac{\mu_0 p_0 \omega}{4\pi r} \sin[\omega(t - r/c)]$$

Now, the dipoles are not at the origin, so:

$$l_{\pm} = \sqrt{r^2 + d^2/4 \mp r d \cos \theta} \approx r \left(1 \mp \frac{d}{2r} \cos \theta \right) \Rightarrow \frac{1}{l_{\pm}} \approx \frac{1}{r} \left(1 \pm \frac{d}{2r} \cos \theta \right)$$

The angles θ_{\pm} are different, as well:

$$\begin{aligned} r \cos \theta &= \frac{d}{2} + l_+ \cos \theta_+ \quad l_- \cos \theta_- = r \cos \theta + \frac{d}{2} \\ l_{\pm} \cos \theta_{\pm} &= r \cos \theta \mp \frac{d}{2} \end{aligned}$$

$$\begin{aligned} \Rightarrow \cos \theta_{\pm} &= \frac{1}{l_{\pm}} (r \cos \theta \mp d/2) \approx \cos \theta \pm \frac{d}{2r} \cos^2 \theta \mp \frac{d}{2r} \\ &= \cos \theta \mp \frac{d}{2r} (1 - \cos^2 \theta) = \cos \theta \mp \frac{d}{2r} \sin^2 \theta \end{aligned}$$

With these results the potentials are (via a lengthy process) easy to get. Let us start with the vector potential:

$$\begin{aligned} \vec{A} &= -\frac{\mu_0 p_0 \omega}{4\pi} \left[\frac{\sin[\omega(t - l_+/c)]}{l_+} - \frac{\sin[\omega(t - l_-/c)]}{l_-} \right] \hat{z} \\ \frac{\sin[\omega(t - l_{\pm}/c)]}{l_{\pm}} &\approx \frac{1}{r} \left(1 \pm \frac{d}{2r} \cos \theta \right) \sin \left[t_0 \pm \frac{d}{2c} \cos \theta \right] \end{aligned}$$

Taylor-expand the sine term:

$$\begin{aligned} \sin \left[\omega t_0 \pm \frac{\omega d}{2c} \cos \theta \right] &\approx \sin (\omega t_0) \pm \cos (\omega t_0) \frac{\omega d}{2c} \cos \theta \\ \vec{A} &= -\frac{\mu_0 p_0 \omega}{4\pi} \left\{ \frac{1}{r} \left(1 + \frac{d}{2r} \cos \theta \right) \left[\sin (\omega t_0) + \cos (\omega t_0) \frac{\omega d}{2c} \cos \theta \right] \right. \\ &\quad \left. - \frac{1}{r} \left(1 - \frac{d}{2r} \cos \theta \right) \left[\sin (\omega t_0) - \cos (\omega t_0) \frac{\omega d}{2c} \cos \theta \right] \right\} \hat{z} \end{aligned}$$

Add up all the terms and neglect all $\mathcal{O}(d^2)$ terms:

$$\begin{aligned} \vec{A} &= -\frac{\mu_0 p_0 \omega^2 d}{4\pi r c} \left[\cos \theta \cos (\omega t_0) + \frac{\omega}{rc} \cos \theta \sin (\omega t_0) \right] \hat{z} \\ &\approx -\frac{\mu_0 p_0 \omega^2 d}{4\pi r c} \cos \theta \cos (\omega t_0) \hat{z} \\ &= -\frac{\mu_0 p_0 \omega^2 d}{4\pi r c} \cos \theta \cos (\omega t_0) (\cos \theta \hat{r} - \sin \theta \hat{\theta}) \end{aligned}$$

Now, the scalar potential:

$$\begin{aligned} V &= -\frac{p_0 \omega}{4\pi \epsilon_0 c} \left\{ \frac{\cos_+}{l_+} \sin [\omega (t - l_+/c)] - \frac{\cos_-}{l_-} \sin [\omega (t - l_-/c)] \right\} \\ &= -\frac{p_0 \omega}{4\pi \epsilon_0 c} \left\{ \frac{1}{r} \left(1 + \frac{d}{2r} \cos \theta \right) \left[\sin (\omega t_0) + \cos (\omega t_0) \cos \theta \frac{\omega d}{2c} \right] \left(\cos - \frac{d}{2r} \sin^2 \theta \right) \right. \\ &\quad \left. - \frac{1}{r} \left(1 - \frac{d}{2r} \cos \theta \right) \left[\sin (\omega t_0) - \cos (\omega t_0) \cos \theta \frac{\omega d}{2c} \right] \left(\cos + \frac{d}{2r} \sin^2 \theta \right) \right\} \end{aligned}$$

Again, by collecting all the terms and neglecting $\mathcal{O}(d^2)$ terms:

$$\begin{aligned} V &= -\frac{\mu_0 p_0 \omega^2 d}{4\pi r} \left[\cos^2 \theta \cos (\omega t_0) + \frac{c}{r\omega} (\cos^2 \theta - \sin^2 \theta) \sin (\omega t_0) \right] \\ &\approx -\frac{\mu_0 p_0 \omega^2 d}{4\pi r} \cos^2 \theta \cos (\omega t_0) \end{aligned}$$

(b)

We have the potentials. The fields are then:

$$\begin{aligned} \vec{B} &= \nabla \times \vec{A} = \frac{1}{r} \left(\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right) \hat{\phi} \\ &= \frac{\mu_0 p_0 \omega^2 d}{4\pi c r} \cos \theta \sin \theta \sin (\omega t_0) \frac{\omega}{c} \hat{\phi} - \frac{\mu_0 p_0 \omega^2 d}{2\pi c r^2} \sin \theta \cos \theta \cos (\omega t_0) \hat{\phi} \\ &\approx \frac{\mu_0 p_0 \omega^3 d}{4\pi c^2 r} \cos \theta \sin \theta \sin (\omega t_0) \hat{\phi} \\ \vec{E} &= -\nabla V - \frac{\partial \vec{A}}{\partial t} \\ \nabla V &= -\frac{\mu_0 p_0 \omega^2 d}{4\pi} \cos^2 \theta \left(-\frac{1}{r^2} \cos (\omega t_0) + \frac{\omega}{rc} \sin (\omega t_0) \right) \hat{r} \\ &\quad + \frac{\mu_0 p_0 \omega^2 d}{4\pi r^2} 2 \cos \theta \sin \theta \cos (\omega t_0) \hat{\theta} \end{aligned}$$

$$\begin{aligned}
&\approx -\frac{\mu_0 p_0 \omega^3 d}{4\pi r c} \cos^2 \theta \sin(\omega t_0) \hat{r} \\
\frac{\partial \vec{A}}{\partial t} &= \frac{\mu_0 p_0 \omega^3 d}{4\pi r c} \cos \theta \sin(\omega t_0) (\cos \theta \hat{r} - \sin \theta \hat{\theta}) \\
\vec{E} &= \frac{\mu_0 p_0 \omega^3 d}{4\pi r c} \cos \theta \sin \theta \sin(\omega t_0) \hat{\theta}
\end{aligned}$$

(c)

The Poynting vector, the intensity and the total power radiated are:

$$\begin{aligned}
\vec{S} &= \frac{1}{\mu_0} \vec{E} \times \vec{B} \\
&= \frac{\mu_0}{c} \left(\frac{p_0 \omega^3 d}{4\pi c r} \right)^2 \sin^2 \theta \cos^2 \theta \sin^2(\omega t_0) \hat{r} \\
\vec{I} &= \langle \vec{S} \rangle \\
&= \frac{\mu_0}{2c} \left(\frac{p_0 \omega^3 d}{4\pi c r} \right)^2 \sin^2 \theta \cos^2 \theta \hat{r} = \sqrt{\frac{\mu_0}{8c} \left(\frac{p_0 \omega^3 d}{4\pi c r} \right)^2 \sin^2(2\theta)} \hat{r} \\
P &= \frac{\mu_0}{2c} \left(\frac{p_0 \omega^3 d}{4\pi c} \right)^2 \int_0^{2\pi} \int_0^\pi \sin^2 \theta \cos^2 \theta \sin \theta d\theta d\phi \\
&\quad \cos \theta = x \quad dx = -\sin \theta d\theta \\
P &= \frac{\mu_0}{2c} \left(\frac{p_0 \omega^3 d}{4\pi c} \right)^2 \int_0^{2\pi} \int_0^\pi \sin^2 \theta \cos^2 \theta \sin \theta d\theta d\phi \\
&\quad \cos \theta = x \quad dx = -\sin \theta d\theta \\
&= \frac{\mu_0 \pi}{c} \left(\frac{p_0 \omega^3 d}{4\pi c} \right)^2 \int_{-1}^1 (1-x^2) x^2 dx = \frac{\mu_0 \pi}{c} \left(\frac{p_0 \omega^3 d}{4\pi c} \right)^2 \frac{4}{15} \\
&= \frac{\mu_0}{60\pi} \frac{p_0^2 \omega^6 d^2}{c^3}
\end{aligned}$$