

# 电动力学-第九次作业

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Problem 9.19

Answer:

(a)

For glass:

$$\epsilon = 4.7\epsilon_0 \quad \sigma = 1 \cdot 10^{-13} \quad (1.1)$$

With this the relaxation time  $\tau$  is:

$$\tau = \frac{\epsilon}{\sigma} \approx 416.138\text{s} \quad (1.2)$$

(b)

The skin depth  $d$  is given by:

$$d = \frac{1}{\kappa} = \sqrt{\frac{2}{\epsilon\mu}} [\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1]^{-1/2} \quad (1.3)$$

Using the values of the constants for the silver (again, in SI units), and the provided angular frequency:

$$\epsilon \approx \epsilon_0 \quad \mu \approx \mu_0 \quad \sigma = 6.29 \cdot 10^7 \quad (1.4)$$

The skin depth has the value:

$$d \approx 6.35 \cdot 10^{-7}\text{m} = 0.64\mu\text{m} \quad (1.5)$$

(c)

The frequency is  $1\text{MHz} = 10^6\text{Hz}$ . To get the wave length we calculate the wave number  $k$ :

$$k = \sqrt{\frac{2}{\epsilon\mu}} [\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} + 1]^{1/2} \quad (1.6)$$

Similiarly with (b), we can get:

$$k \approx 15299.807 \text{m}^{-1} \quad (1.7)$$

So:

$$\lambda = \frac{2\pi}{k} \approx 4.107 \cdot 10^{-4} \text{m} \quad (1.8)$$

$$v = \frac{\omega}{k} = f\lambda \approx 410.7 \text{m/s} \quad (1.9)$$

Problem 9.31

Answer:

For the TM mode  $B_z = 0$ , so:

$$\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (\omega/c)^2 - k^2 \right] E_z = 0 \quad (2.1)$$

with the boundary condition  $\vec{E}_{\parallel} = 0$ , meaning that  $E_z = 0$  at the boundary of the waveguide. This translates into:

$$E_z(x, b) = E_z(x, 0) = E_z(0, y) = E_z(a, y) = 0 \quad (2.2)$$

Now, let us assume that  $E_z = X(x)Y(y)$ , so:

$$\frac{X''}{X} + \frac{Y''}{Y} = -(\omega/c)^2 + k^2 \quad (2.3)$$

$$-k_x^2 - k_y^2 = -(\omega/c)^2 + k^2 \quad (2.4)$$

Then:

$$X(x) = A \cos(k_x x) + B \sin(k_x x) \quad (2.5)$$

$$Y(y) = C \cos(k_y y) + D \sin(k_y y) \quad (2.6)$$

Apply the boundary conditions, which yield:

$$\begin{aligned} E_z(x, 0) = 0 &\rightarrow C = 0 \\ E_z(0, y) = 0 &\rightarrow A = 0 \\ E_z(x, b) = 0 &\rightarrow k_y = \frac{m\pi}{b} \\ E_z(a, y) = 0 &\rightarrow k_x = \frac{n\pi}{a} \end{aligned} \quad (2.7)$$

where  $m$  and  $n$  are integers larger than zero, otherwise the solution is trivial (zero).

Thus, the  $z$  -component of the field is:

$$E_z = XY = E_0 \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \quad (2.8)$$

The cutoff frequencies:

$$(\omega/c)^2 - k^2 - k_x^2 - k_y^2 = 0 \quad (2.9)$$

$$\omega = c\sqrt{k^2 + \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2} = c\sqrt{k^2 + \frac{\omega_{mn}^2}{c^2}} \quad (2.10)$$

So:

$$\omega_{mn} = c\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}} \quad (2.11)$$

The lowest cutoff frequency is, as we mentioned, the (1,1) mode, so:

$$\omega_{11} = c\pi\sqrt{\frac{1}{a^2} + \frac{1}{b^2}} \quad (2.12)$$

The wave velocity and the group velocity are as follows:

$$v = \frac{\omega}{k} = \frac{\omega}{\frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}} = \frac{c}{\sqrt{1 - (\omega_{mn}/\omega)^2}} \quad (2.13)$$

$$\begin{aligned} v_g = \frac{d\omega}{dk} &= \frac{d}{dk} c\sqrt{k^2 + (\omega_{mn}/c)^2} = \frac{ck}{\sqrt{k^2 + (\omega_{mn}/c)^2}} = \frac{ck}{\omega/c} \\ &= \frac{c^2}{v} = c\sqrt{1 - (\omega_{mn}/\omega)^2} \end{aligned} \quad (2.14)$$

Finally, the ratio of lowest TE cutoff frequency to the lowest TM cutoff frequency is:

$$\frac{\omega_{11}}{\omega_{10}} = \frac{c\pi\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}{\frac{c\pi}{a}} = \sqrt{1 + (a/b)^2} \quad (2.15)$$

### Problem 3.36

Answer:

Let the two planes be at  $z = 0$  and  $z = d$ , let the electric field travel along the  $z$  -axis and be polarized along the  $x$  -axis, then the fields are, in material 1:

$$\vec{E}_I = E_{0I} e^{i(k_1 z - \omega t)} \hat{x} \quad (3.1)$$

$$\vec{B}_I = \frac{E_{0I}}{v_1} e^{i(k_1 z - \omega t)} \hat{z} \times \hat{x} = \frac{E_{0I}}{v_1} e^{i(k_1 z - \omega t)} \hat{y} \quad (3.2)$$

$$\vec{E}_R = E_{0R} e^{i(-k_{12} - \omega t)} \hat{x} \quad (3.3)$$

$$\vec{B}_R = \frac{E_{0R}}{v_1} e^{i(-k_1 z - \omega t)} (-\hat{z}) \times \hat{x} = -\frac{E_{0R}}{v_1} e^{i(-k_1 z - \omega t)} \hat{y} \quad (3.4)$$

For material 2:

$$\vec{E}_r = E_{0r} e^{i(k_2 z - \omega t)} \hat{x} \quad (3.5)$$

$$\vec{B}_r = \frac{E_{0r}}{v_2} e^{i(k_2 z - \omega t)} \hat{y} \quad (3.6)$$

$$\vec{E}_l = E_{0l} e^{i(-k_2 z - \omega t)} \hat{x} \quad (3.7)$$

$$\vec{B}_l = -\frac{E_{0l}}{v_2} e^{i(-k_2 z - \omega t)} \hat{y} \quad (3.8)$$

where  $r$  stands for wave going to the right, and  $l$  for one going to the left.

Material 3 only has the transmitted wave:

$$\vec{E}_T = E_{0T} e^{i(k_3 z - \omega t)} \hat{x} \quad (3.9)$$

$$\vec{B}_T = \frac{E_{0T}}{v_3} e^{i(k_3 z - \omega t)} \hat{y} \quad (3.10)$$

On both of the planes we impose boundary conditions. The first one,  $\vec{E}_{\parallel,1} = \vec{E}_{\parallel,2}$ , gives:

$$E_{0I} + E_{0R} = E_{0r} + E_{0l} \quad (3.11)$$

$$E_{0T} e^{ik_3 d} = E_{0r} e^{ik_2 d} + E_{0l} e^{-ik_2 d} \quad (3.12)$$

and the second one,  $\vec{B}_{\parallel,1}/\mu_1 = \vec{B}_{\parallel,2}/\mu_2$ , gives:

$$\frac{1}{\mu_1 v_1} (E_{0I} - E_{0R}) = \frac{1}{\mu_2 v_2} (E_{0r} - E_{0l}) \quad (3.13)$$

$$\frac{1}{\mu_3 v_3} E_{0T} e^{ik_3 d} = \frac{1}{\mu_2 v_2} (E_{0r} e^{ik_2 d} - E_{0l} e^{-ik_2 d}) \quad (3.14)$$

Combine (3.11) (3.12) (3.13) (3.14) to solve simultaneous equations.

So:

$$\begin{aligned}
T &= \frac{n_3^2}{n_1^2} \frac{1}{\beta_{13} (1 + \beta_{13})^2 + \sin^2(k_2 d) (\beta_{12}^2 + \beta_{23}^2 + 2\beta_{12}\beta_{23} - 1 - 2\beta_{13} - \beta_{13}^2)} \\
&= \frac{n_3}{n_1} \frac{4}{(1 + n_3/n_1)^2 + \sin^2 k_2 d \left( (n_3/n_2)^2 + (n_2/n_1)^2 - (n_3/n_1)^2 - 1 \right)} \\
&= \frac{4n_1 n_3}{(n_1 + n_3)^2 + \sin^2 k_2 d \left( n_1^2 (n_3/n_2)^2 + n_2^2 - n_3^2 - n_1^2 \right)} \\
&= \frac{4n_1 n_3}{(n_1 + n_3)^2 + \sin^2 \left( \frac{\omega n_2 d}{c} \right) \frac{(n_1^2 - n_2^2)(n_3^2 - n_2^2)}{n_2^2}}
\end{aligned} \tag{3.15}$$