Trajectory Design for Space Systems

EN.530.626 (Fall 2025)

Lecture 8

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Course Assistant 1: Arnab Chatterjee
Course Assistant 2: Mark Gonzales

Class review

- So far: smooth constrained optimization

$$\begin{aligned} &\min_x f(x) && && f \in \mathscr{C}^2 \\ \texttt{subject to:}\quad & h_i(x) = 0, && i = 1,\ldots,m && h_i \in \mathscr{C}^2 \\ & g_i(x) = 0, && i = 0,\ldots,p && g_i \in \mathscr{C}^2 \end{aligned}$$

- Allows us to leverage Newton method-style approaches for solving problem

Class review

Quadratic programs

- One particular formulation we have returned to has been *convex* quadratic programs

$$\min_x \frac{1}{2}x^T P x + p^T x \qquad P \in \mathbb{S}^n_+$$

`subject to:`

$$Ax = 0 \qquad A \in \mathbb{R}^{m\times n}$$

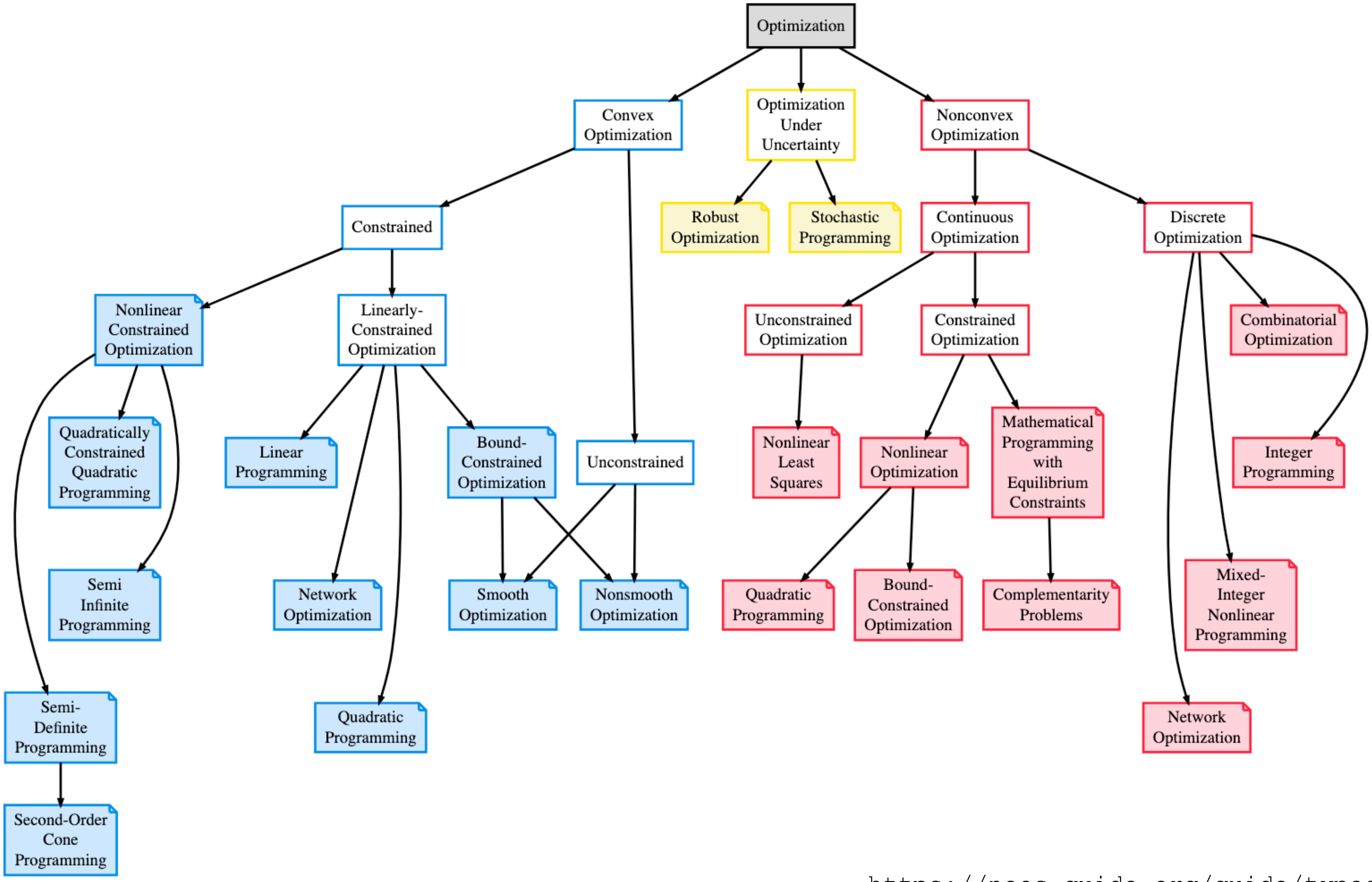
$$Gx \leq h \qquad G \in \mathbb{R}^{p\times n}$$

- Applied primal-dual interior point methods to find solutions for these

What's next?

Recall optimization classes:

- Unconstrained vs. discrete
- Smooth vs. non smooth
- Convex vs. non-convex
- Continuous vs. discrete



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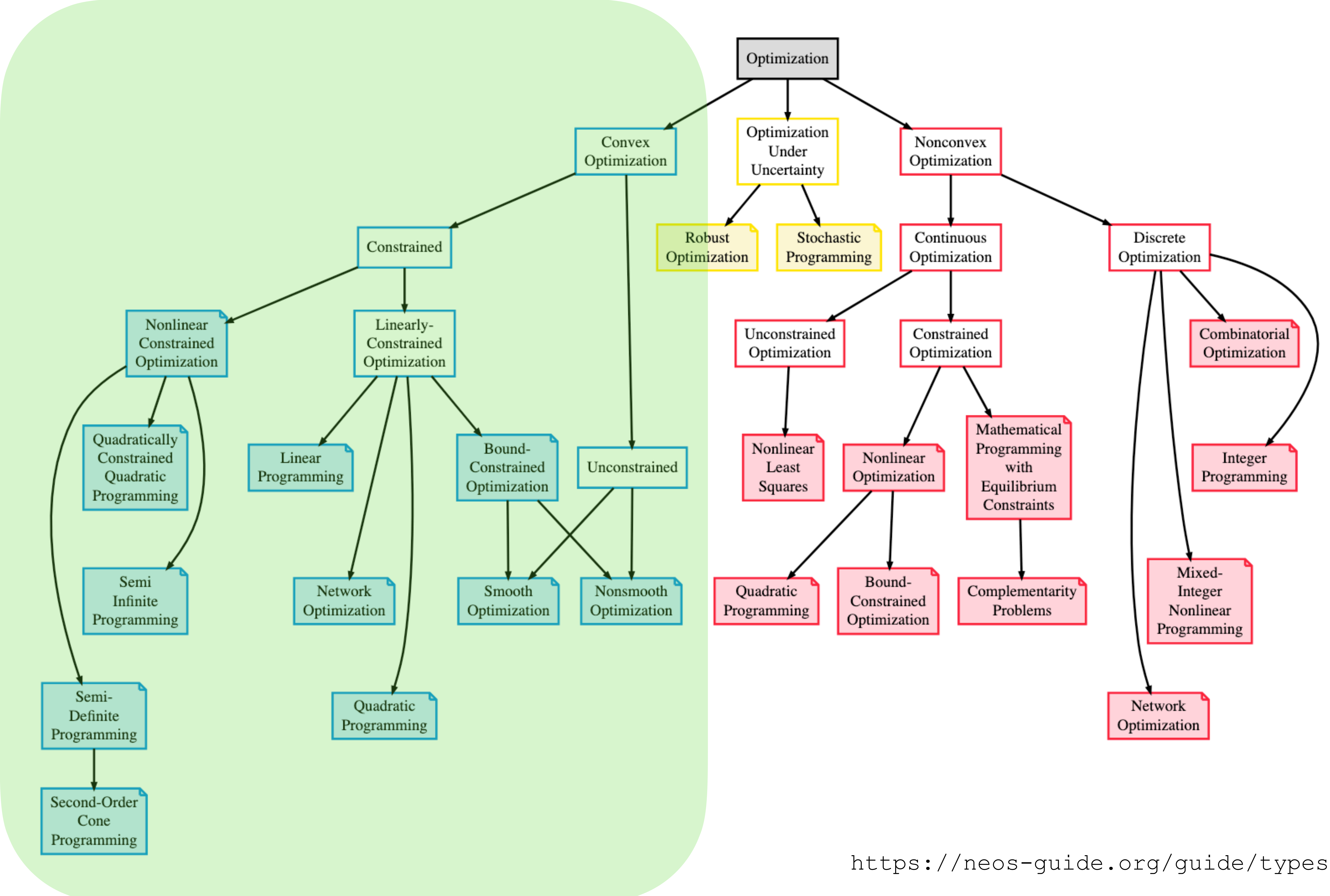

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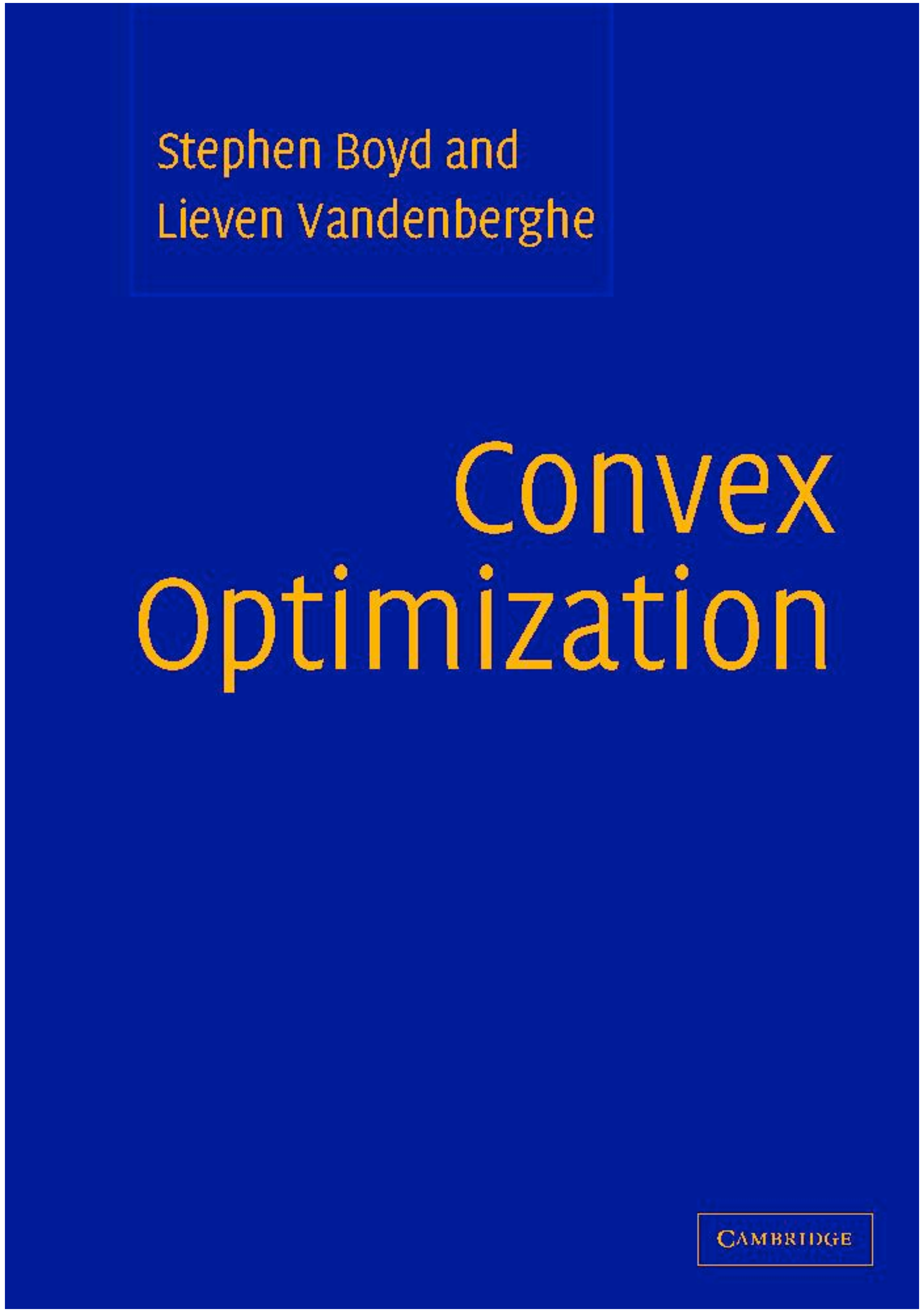
Convexity

- A whole field of optimization on its own
- Convex $\neq$ easy
- Advantages
  - (1) Enjoys strong convergence guarantees
  - (2) Local optimizer is global one



Convexity

Convex sets

- A set $\mathscr{X}$ is convex if $\forall x_1, x_2 \in \mathscr{X}$ and $\theta \in [0,1]$

$$\theta x_1 + (1-\theta)x_2 \in \mathscr{X}$$

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$\|x\|_2 \leq r_{\text{dist}}$

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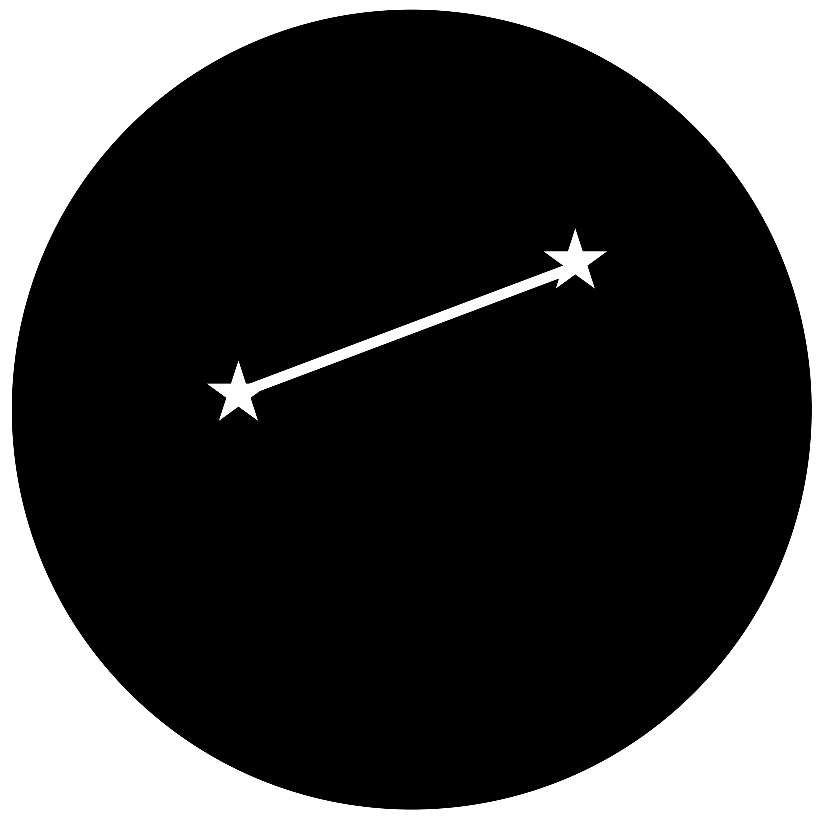
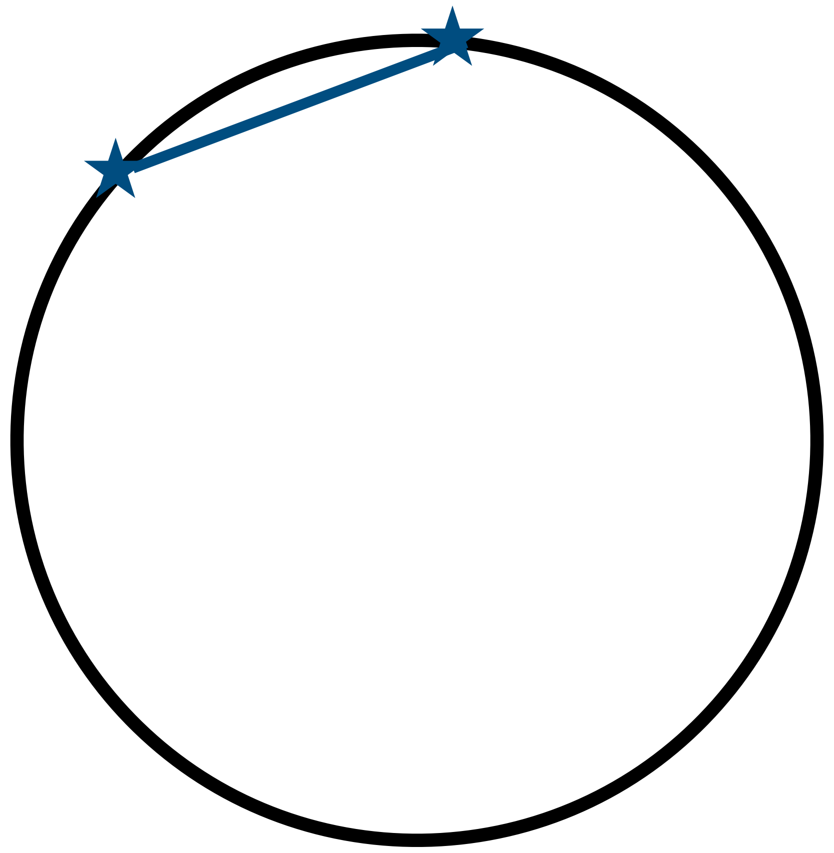


$\|x\|_2 = r_{\text{dist}}$



Convexity

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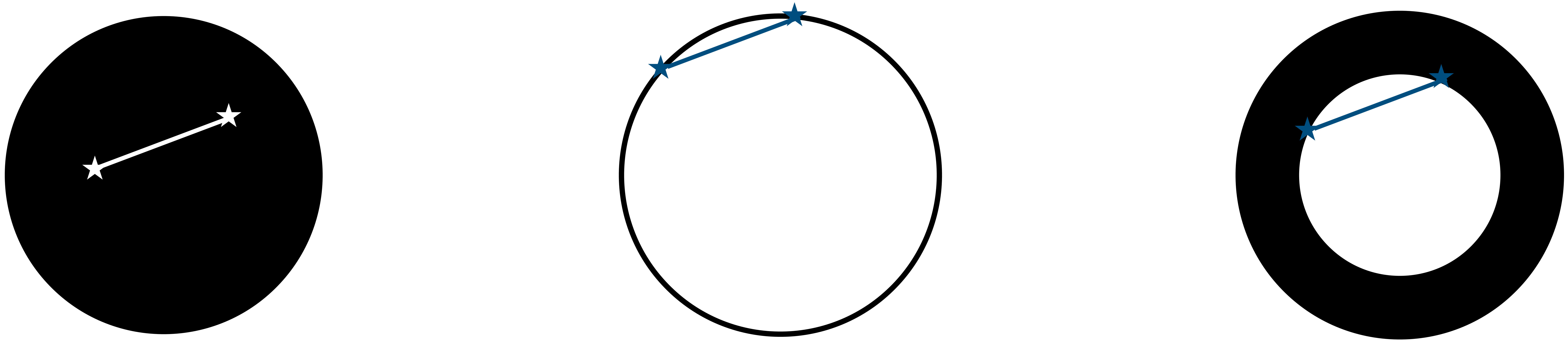
$\|x\|_2 \leq r_{\text{dist}}$ $\quad$ $\|x\|_2 = r_{\text{dist}}$ $\quad$ $r_{\text{min}} \leq \|x\|_2 \leq r_{\text{max}}$

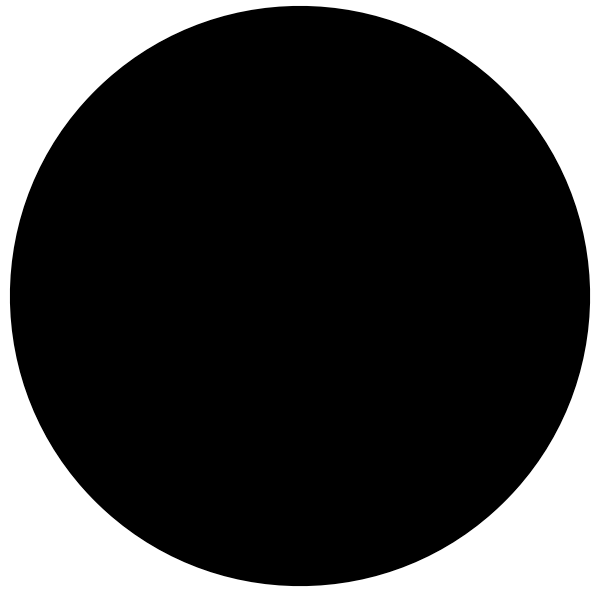


Convexity

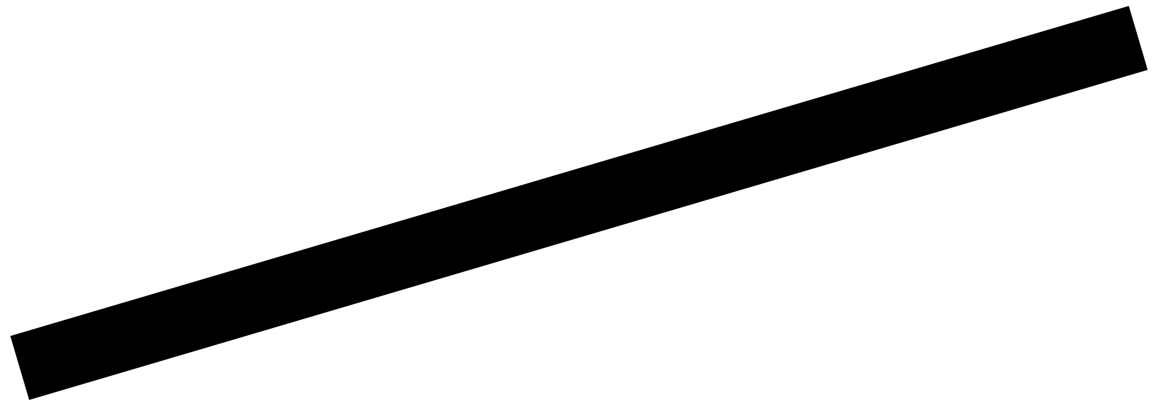
Convex sets

- The intersection of convex sets is convex

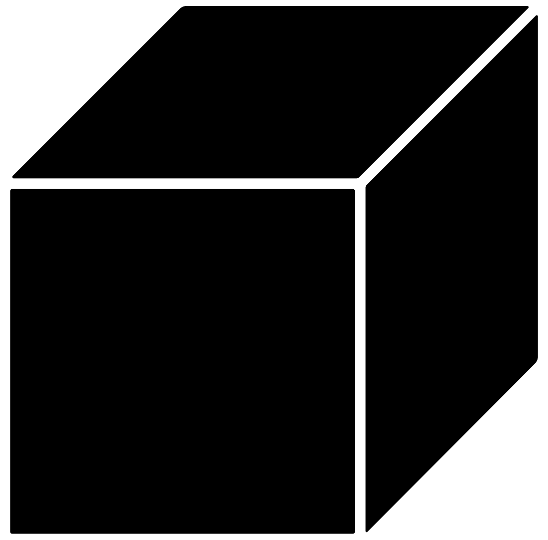
$\|x\|_2 \leq r_{\text{dist}}$



$y = c^T x + b$



$\|u\|_1 \leq u_{\text{max}}$



Convexity

Convex functions

- A function $f(x)$ is convex if $\forall x_1, x_2 \in \mathrm{dom}\, f$ and $\theta \in [0,1]$

$$f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta)f(x_2)$$

Convexity

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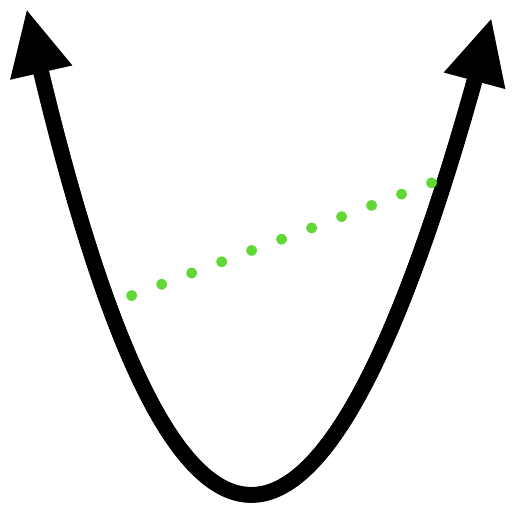
$f(x) = x^2$



Convex

Convexity

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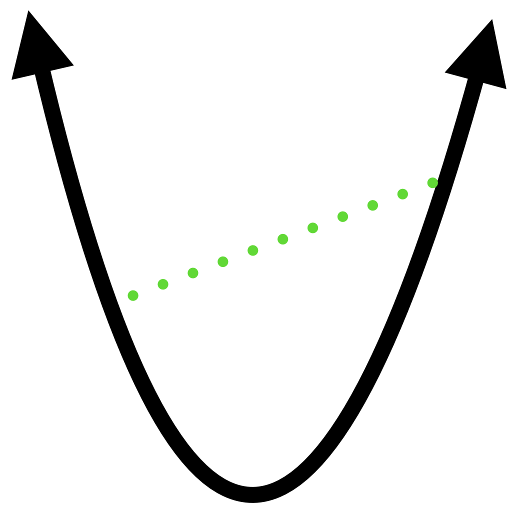
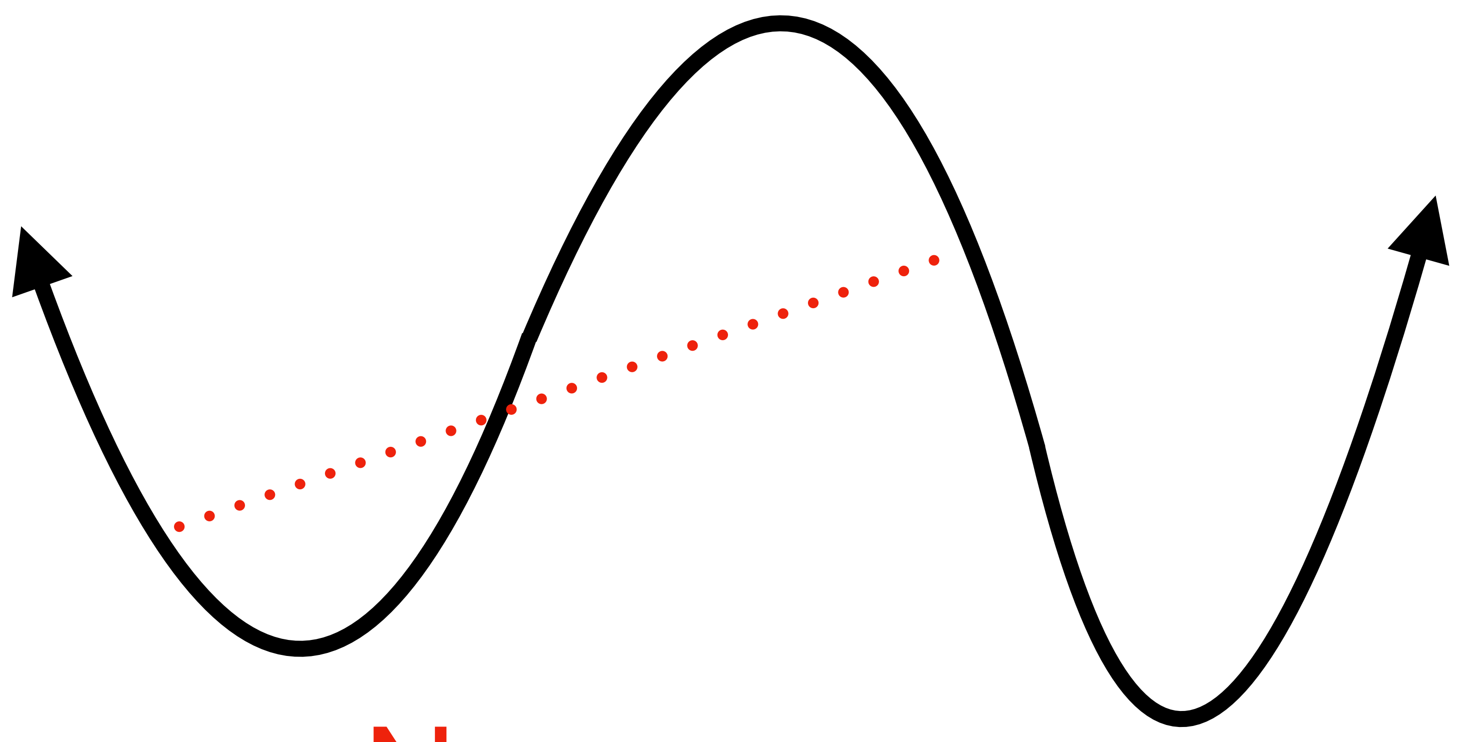


Convex

$f(x) = x^4 - 2 * x^3 - 10x^2 + 7x - 17$



Non-convex

Convexity

Convex functions

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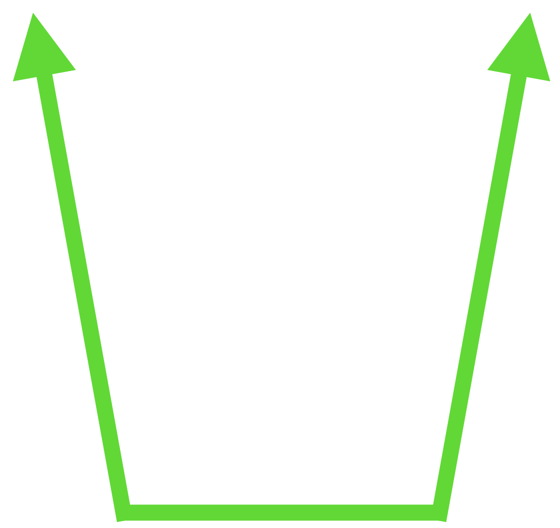
$$f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta)f(x_2)$$

- Note: f need not be smooth (i.e., $\nabla_x f(x)$ might not exist)



Convex

Non-convex

Convexity

Convex functions and sets

- The epigraph of a function $f : \mathbb{R}^n \to \mathbb{R}$ is:

$$\text{epi}\, f = \{(x, t) \,|\, x \in \text{dom}\, f, f(x) \leq t\}$$

Convexity

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$$\text{epi}\, f = \{(x, t) \,|\, x \in \text{dom}\, f, f(x) \leq t\}$$

- f is a convex function if and only if epi f is a convex set

$$f \text{ is convex function} \Leftrightarrow \text{epi}\, f \text{ is convex set}$$

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Convexity

Why does this matter?

- A *local* solution to a convex problem is also the *global* solution
  - For smooth unconstrained optimization, if $f \in \mathscr{C}^2$ and convex:

$$x^* \text{ is global optimizer} \Leftrightarrow \nabla_x f(x^*) = 0$$

  - For smooth constrained optimization, if $f, h, g \in \mathscr{C}^2$ and convex:

$$x^* \text{ is global optimizer} \Leftrightarrow \nabla_x \mathscr{L}(x^*, y^*, z^*) = 0$$

Convexity

Completeness guarantees

- A function $f : \mathbb{R} \to \mathbb{R}$ is called *self-concordant* if f is convex and:

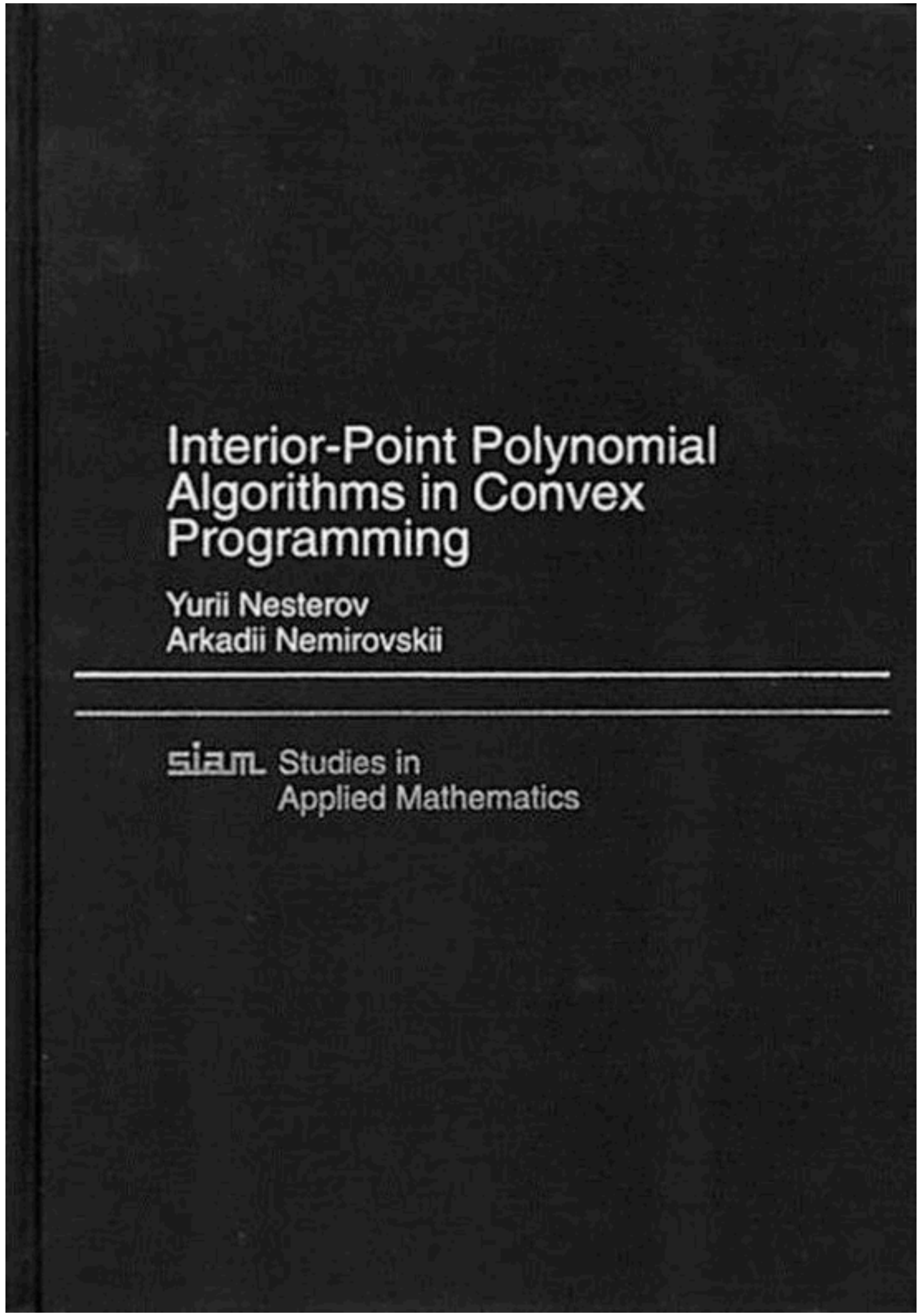
$$f'''(x) \leq 2f''(x)^{\frac{3}{2}}$$

- Loosely, “third derivative doesn’t change too quickly”

- Interior point methods solve convex optimization problems in polynomial time

$$N = \mathcal{O}\left(\sqrt{n}\log\left(\frac{1}{\varepsilon}\right)\right)$$

Convexity


Interior-Point Polynomial Algorithms in Convex Programming
Yurii Nesterov
Arkadii Nemirovskii
siam Studies in Applied Mathematics

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Article electronically published on September 21, 2004

THE INTERIOR-POINT REVOLUTION IN OPTIMIZATION: HISTORY, RECENT DEVELOPMENTS, AND LASTING CONSEQUENCES

MARGARET H. WRIGHT

1. Overview

REVOLUTION:
(i) a sudden, radical, or complete change;
(ii) a fundamental change in political organization, especially the overthrow or renunciation of one government or ruler and the substitution of another.[1]

It can be asserted with a straight face that the field of continuous optimization has undergone a revolution since 1984 in the sense of the first definition and that the second definition applies in a philosophical sense: Because the interior-point presence in optimization today is ubiquitous, it is easy to lose sight of the magnitude and depth of the shifts that have occurred during the past twenty years. Building on the implicit political metaphor of our title, successful revolutions eventually become the status quo.

M. H. Wright, "The interior-point revolution in optimization: History, recent developments, and lasting consequences," *Bulletin of the American Mathematical Society*, vol. 42, no. 1, pp. 39 — 56, 2004.

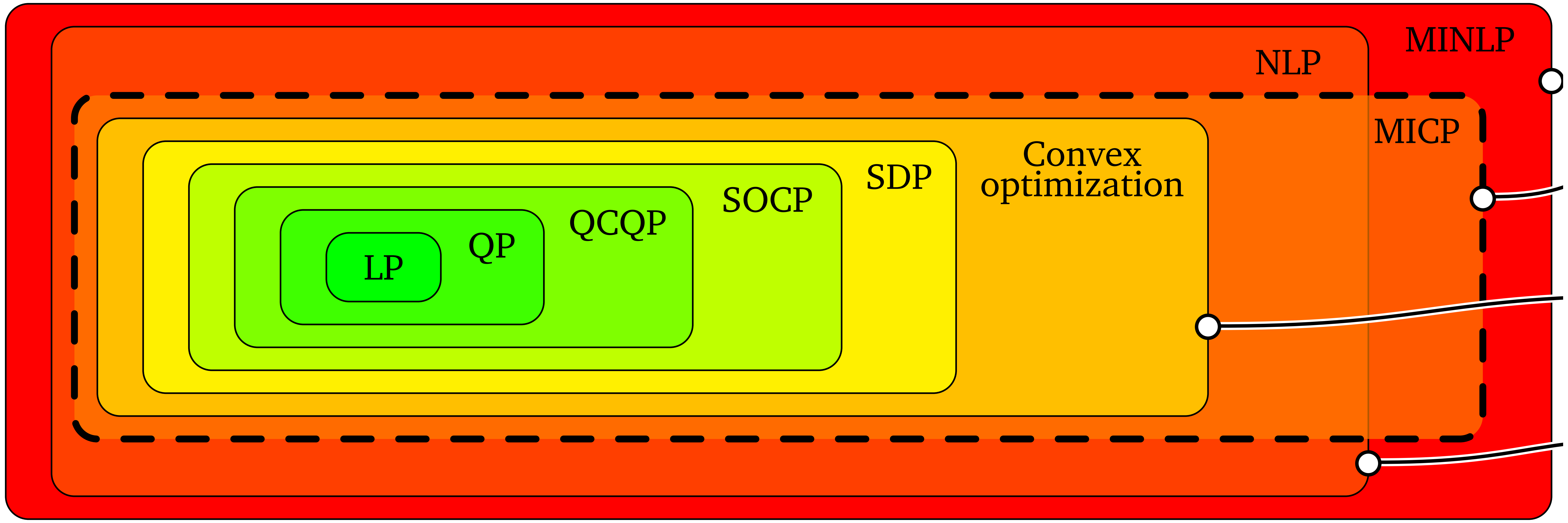
Convexity

In summary

Convexity

In summary

- Convexity does not mean the problem is *easy*

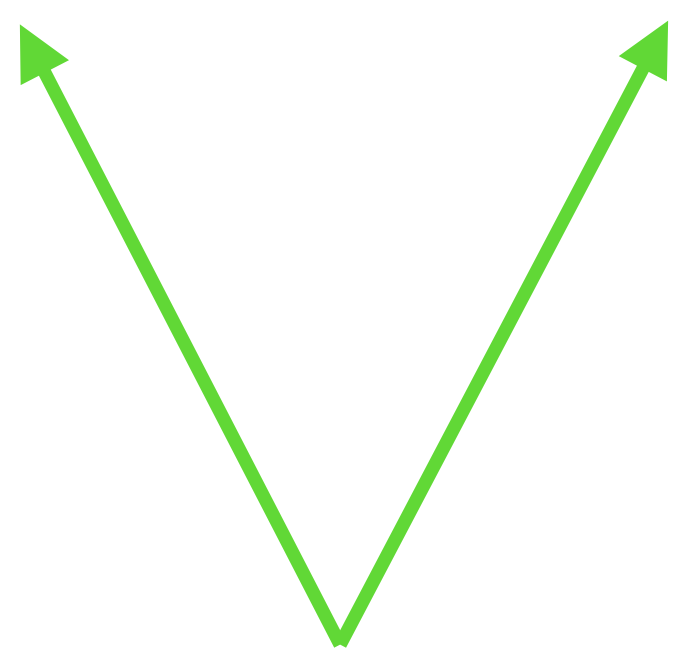


D. Malyuta, Y. Yu, P. Elango, and B. Acikmese, "Advances in trajectory optimization for space vehicle control," *Annual Reviews in Control*, vol. 52, pp. 282 — 315, 2021.

Convexity

In summary

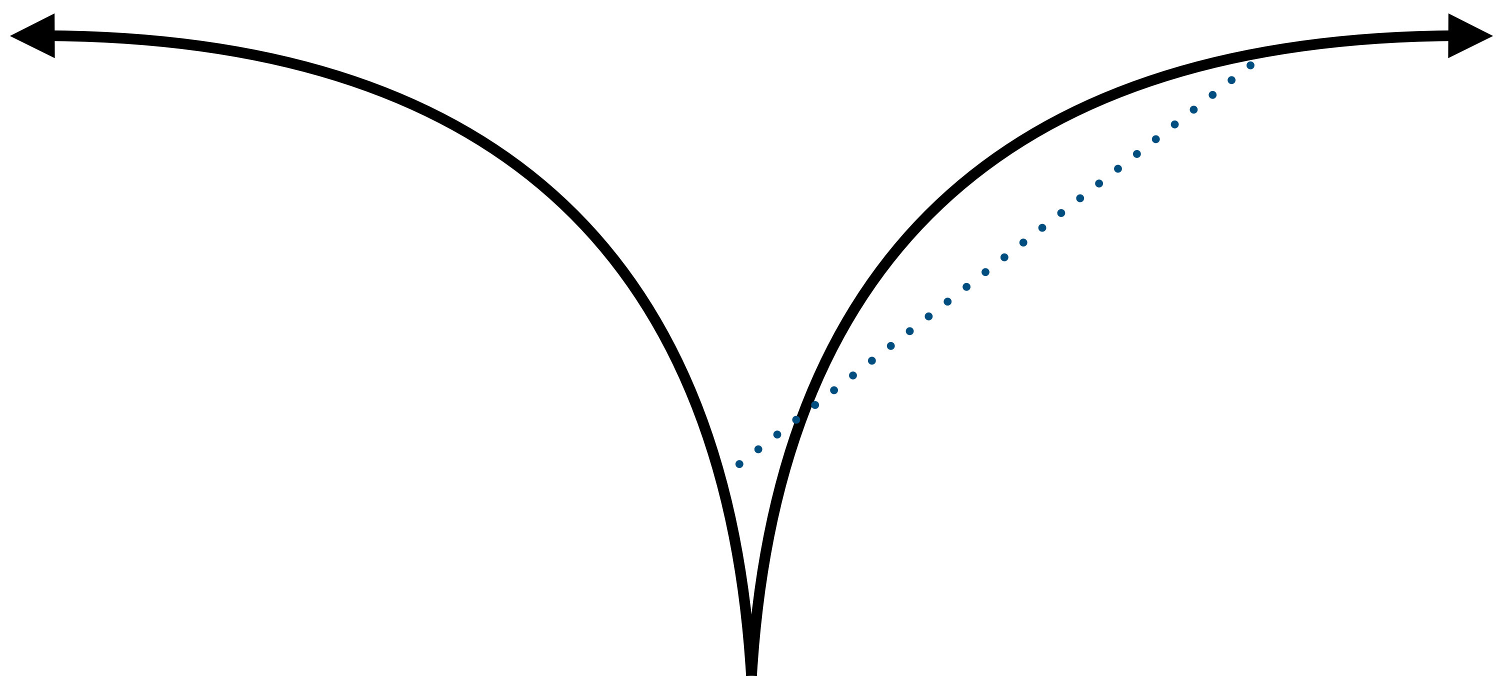
- Convexity does not mean the problem is *easy*
- Convex programs do not have to be smooth



Convexity

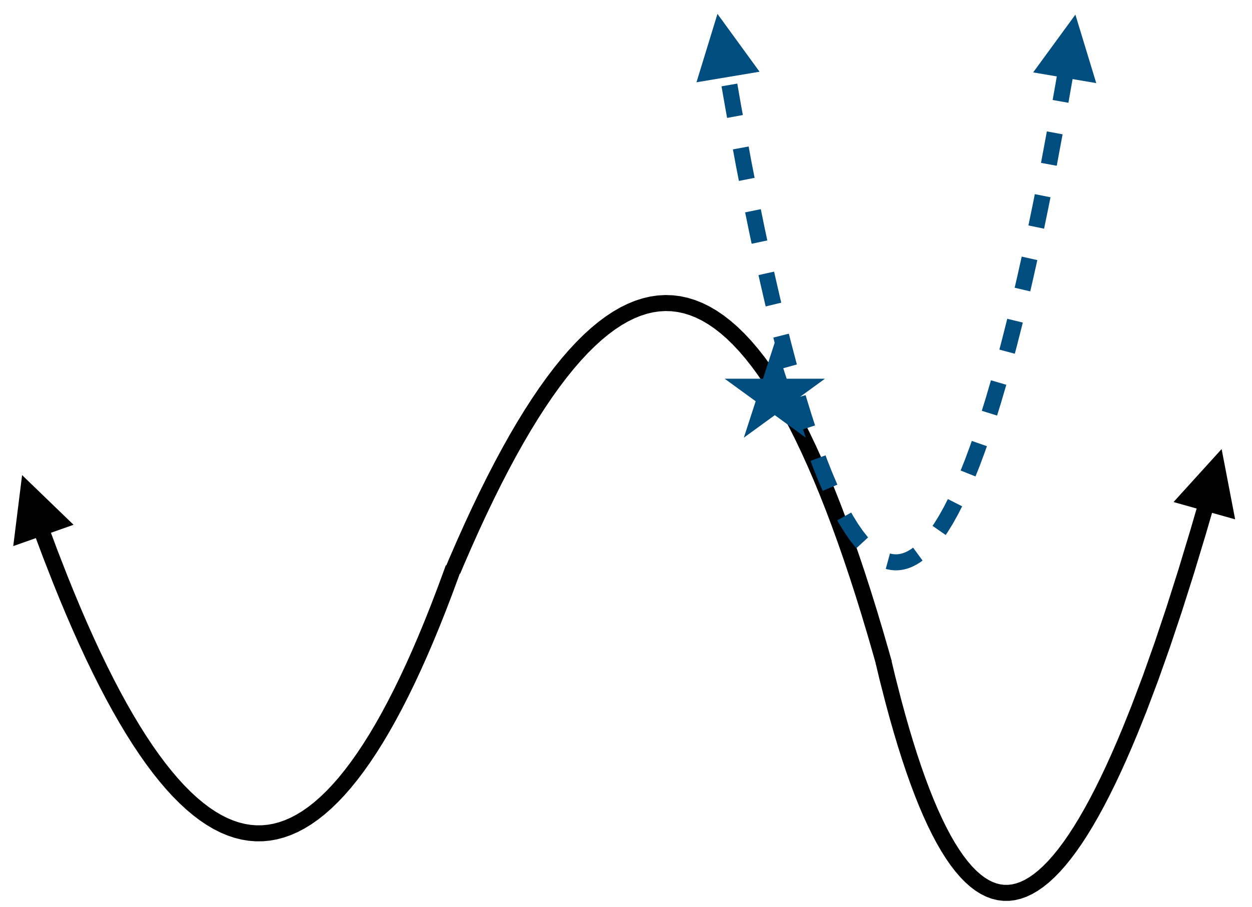
In summary

- Convexity does not mean the problem is *easy*
- Convex programs do not have to be smooth
- Convex programs are not the only ones with global optima



Relevance to trajectory design

- Most practical systems of interest are non-convex
- For *smooth* non-convex optimization problems, most approaches rely on solving a sequence of convex approximations to the problem



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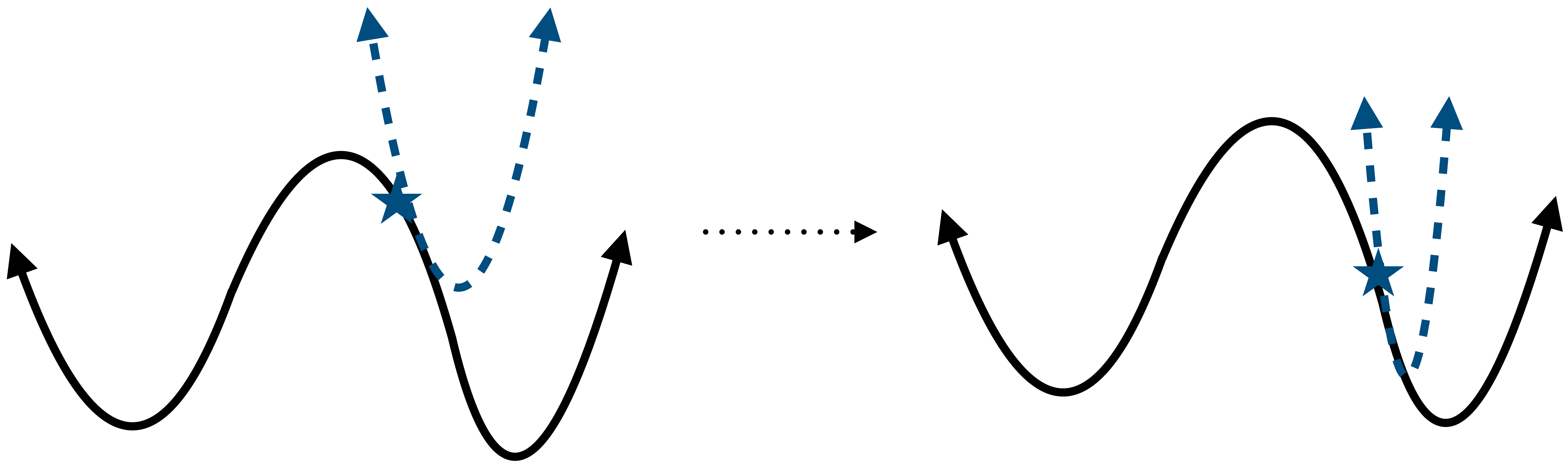

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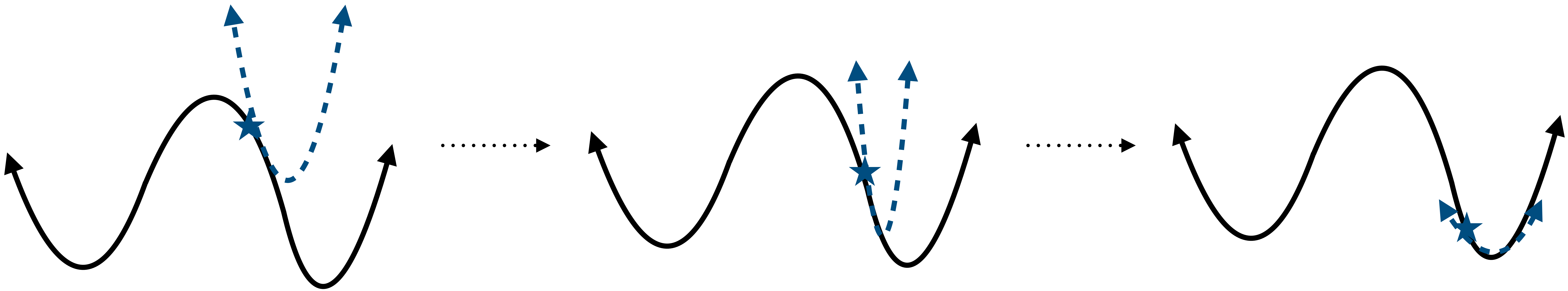


Relevance to trajectory design

Sequential quadratic programming

$$\min_x f(x)$$

subj. to:

$$h(x) = 0$$

$$g(x) \leq 0$$

$$f, h, g \in \mathscr{C}^2$$

Relevance to trajectory design

Sequential quadratic programming

$$\min_x f(x)$$

`subj. to:`
$$h(x) = 0$$
$$g(x) \leq 0$$
$$f, h, g \in \mathscr{C}^2$$

$\longrightarrow$

1. Given $w^0 = (x^0, y^0, z^0)$

2. Construct Lagrangian using w

`for` $k \in [1, N_{\mathtt{max}}]$

3. Solve local QP approximation

$$\min_{\Delta x} \frac{1}{2} \Delta x^T \nabla_x \mathscr{L}(\bar{x}) \Delta x + \nabla_x \mathscr{L}(\bar{x})$$

`subj. to:`
$$h(\bar{x}) + \nabla_x h(\bar{x}) \Delta x = 0$$
$$g(\bar{x}) + \nabla_x g(\bar{x}) \Delta x \leq 0$$

4. Line search to find α

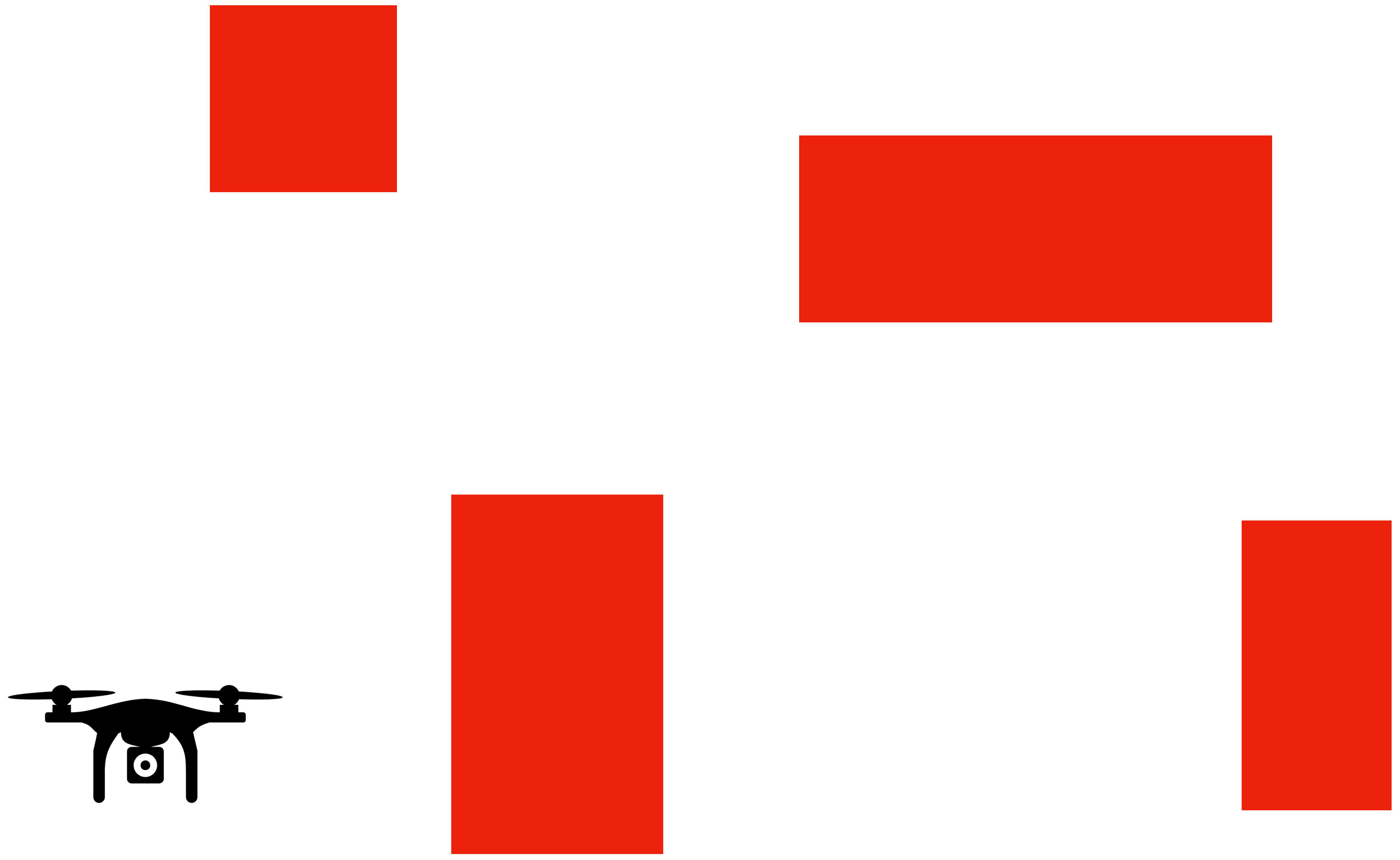
5. $w^k = w^{k-1} + \alpha \Delta w^k$

5. Terminate when necessary conditions satisfied

Convexity of practical constraints

Convexity of practical constraints

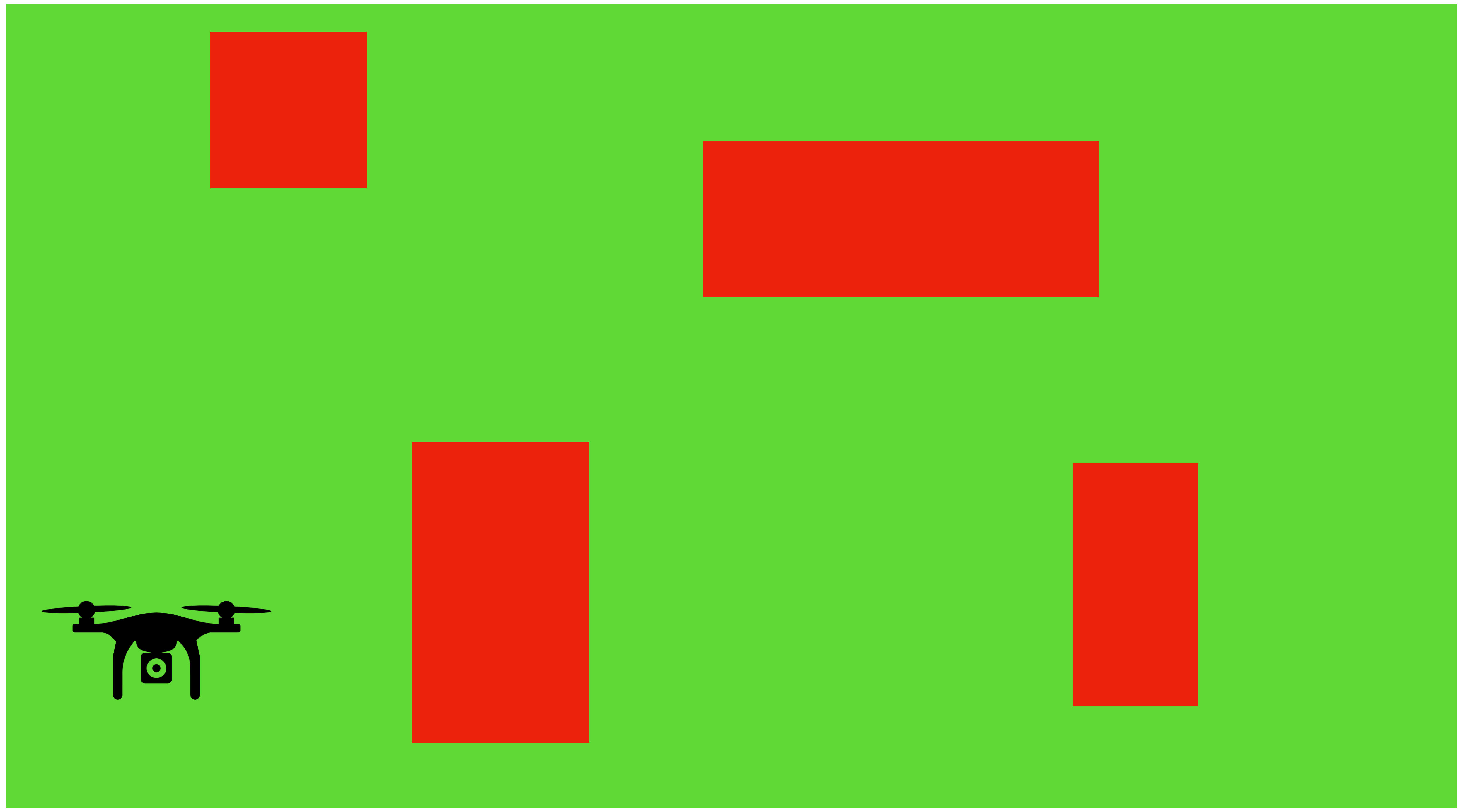
Collision avoidance



$$x_k \notin \mathcal{X}_{\text{obs}}$$

Convexity of practical constraints

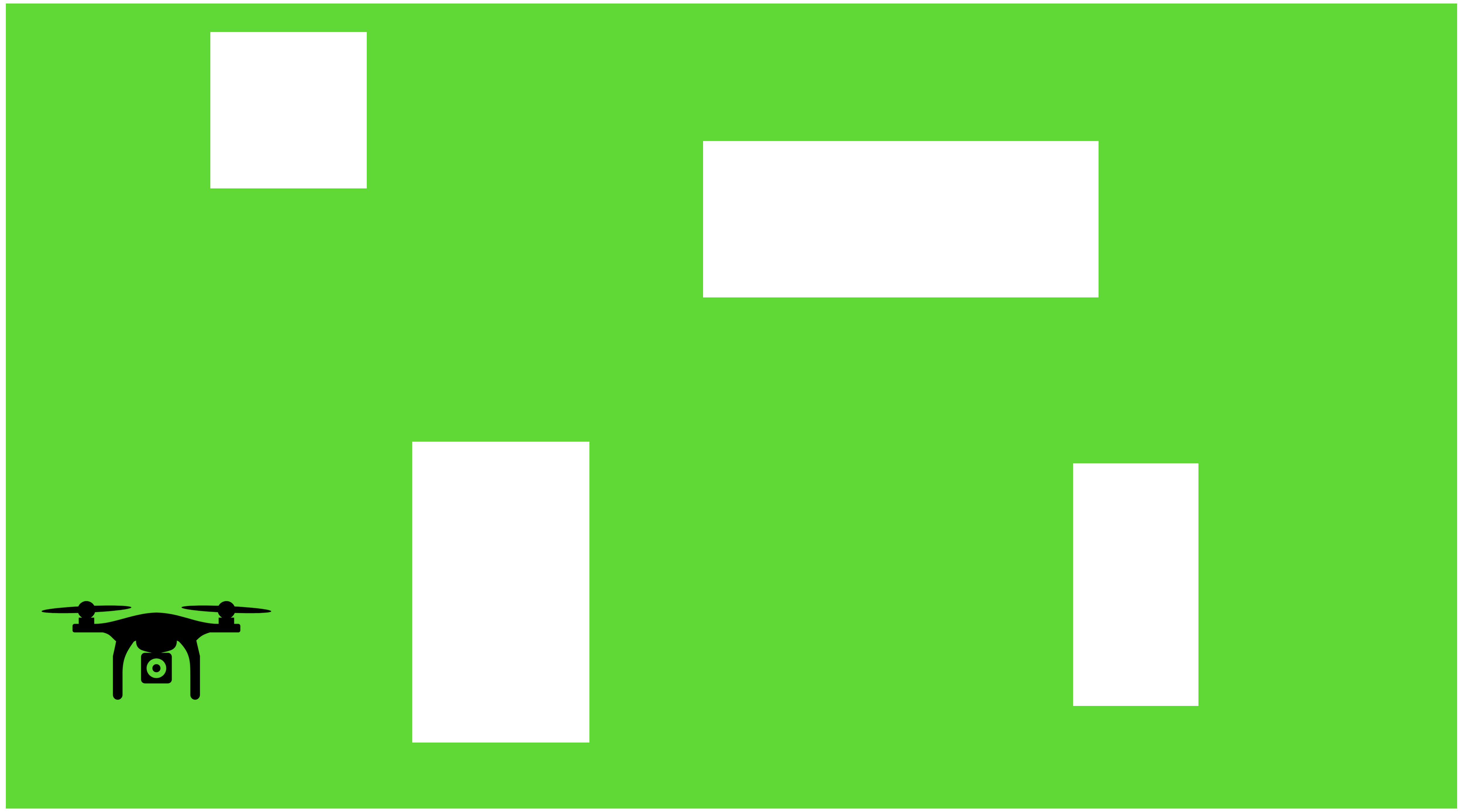
Collision avoidance



$$x_k \in \mathcal{X}_{\text{safe}}$$

Convexity of practical constraints

Collision avoidance



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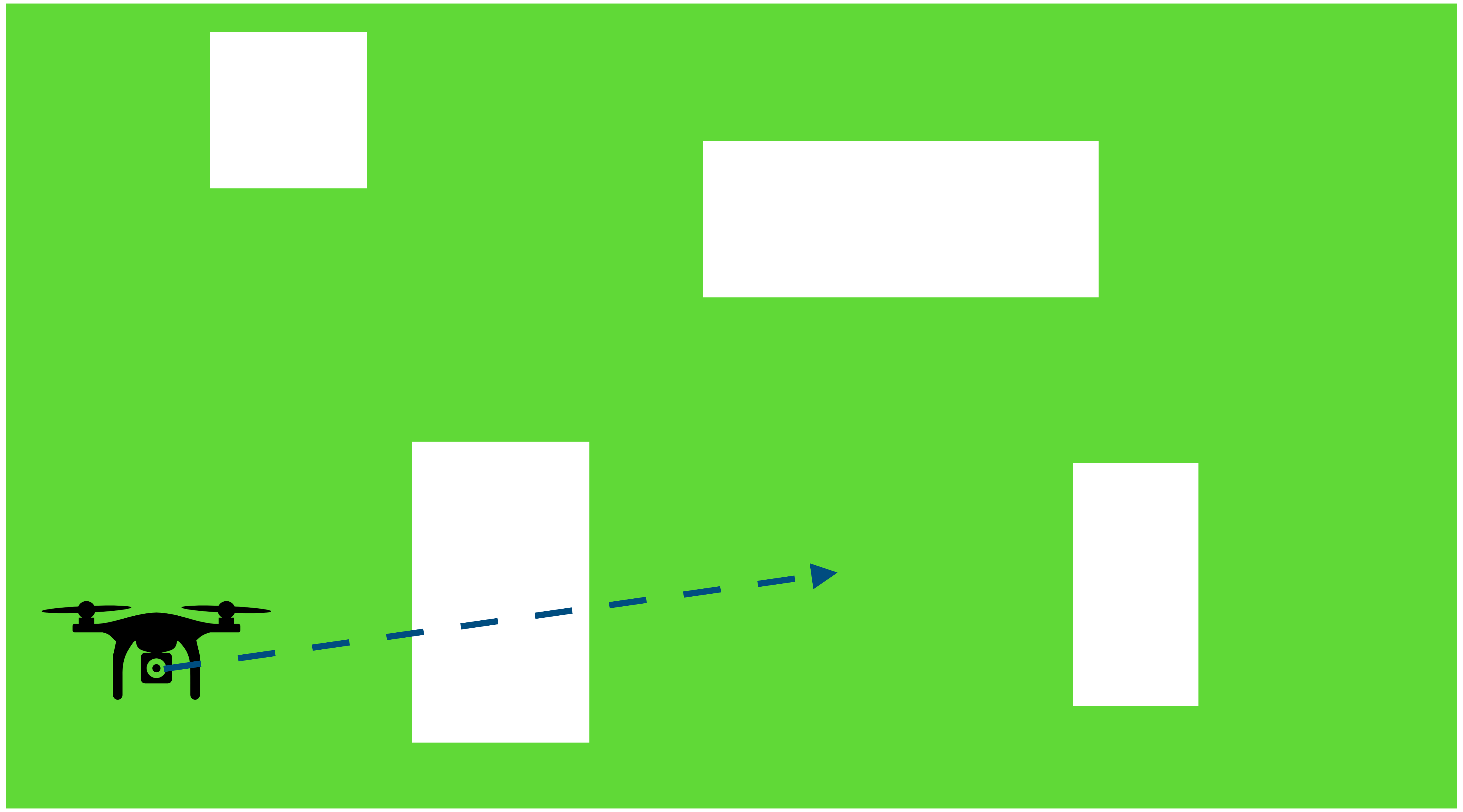

$x_k \in \mathcal{X}_{\text{safe}}$ Non-convex

Convexity of practical constraints

Dynamics constraints

- An equality constraint can be written as a two-sided inequality

$$x_{k+1} = f(x_k, u_k) \quad \longleftrightarrow \quad \begin{aligned} x_{k+1} &\leq f(x_k, u_k) \\ x_{k+1} &\geq f(x_k, u_k) \end{aligned}$$

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- For both a function $f(x_k, u_k)$ and its negative $-f(x_k, u_k)$ to be convex with respect to x_k and u_k, $f(\cdot)$ must be an affine function

$$x_{k+1} = Ax_k + Bu_k + c$$

Convexity of practical constraints

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$$x_{k+1} = Ax_k + Bu_k + c$$

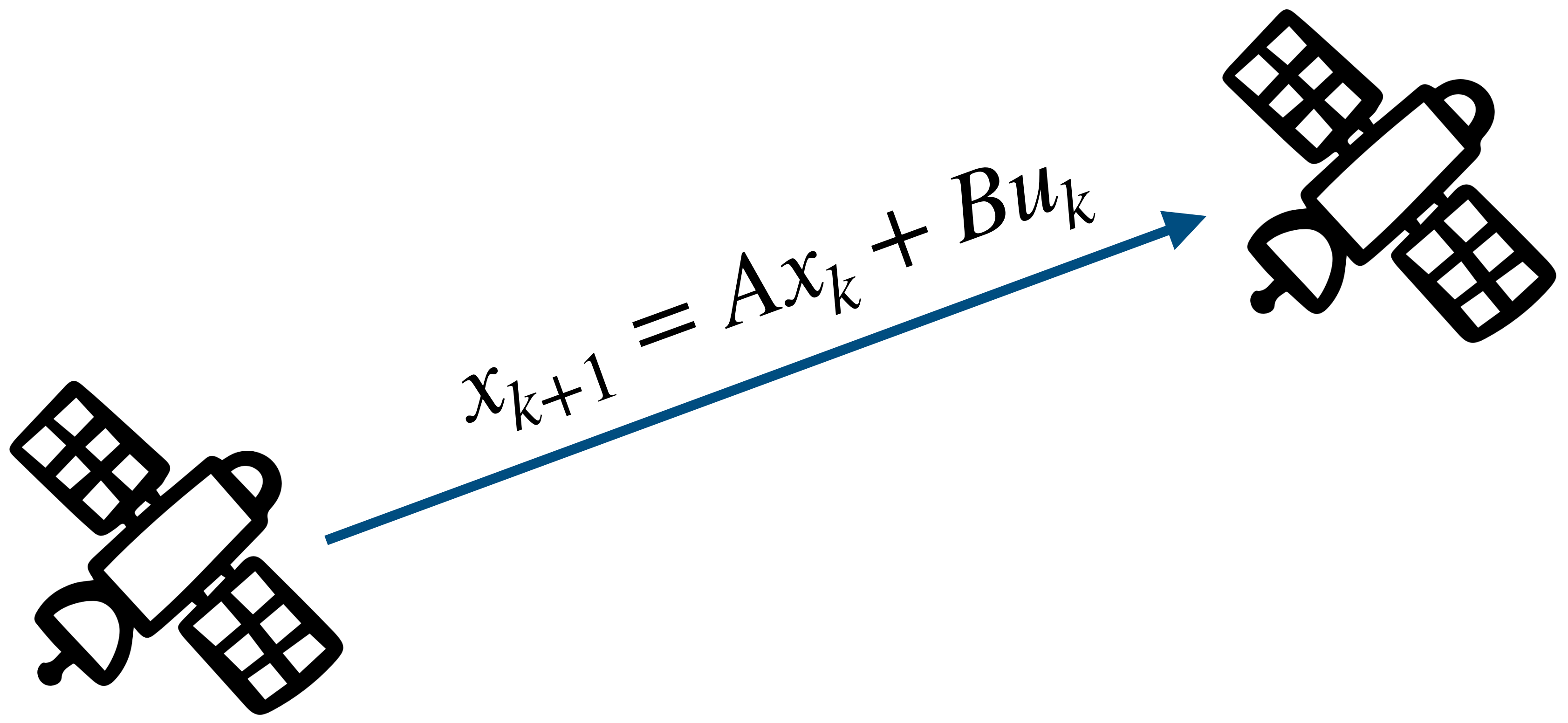
- Any system with nonlinear dynamics is non-convex

Convexity of practical constraints

Dynamics constraints

Spacecraft double integrator (convex)

$$x_{k+1} = \begin{pmatrix} I & \Delta t I \\ 0 & I \end{pmatrix} x_k + \begin{pmatrix} \Delta t^2 I \\ \Delta t I \end{pmatrix} u_k$$



Convexity of practical constraints

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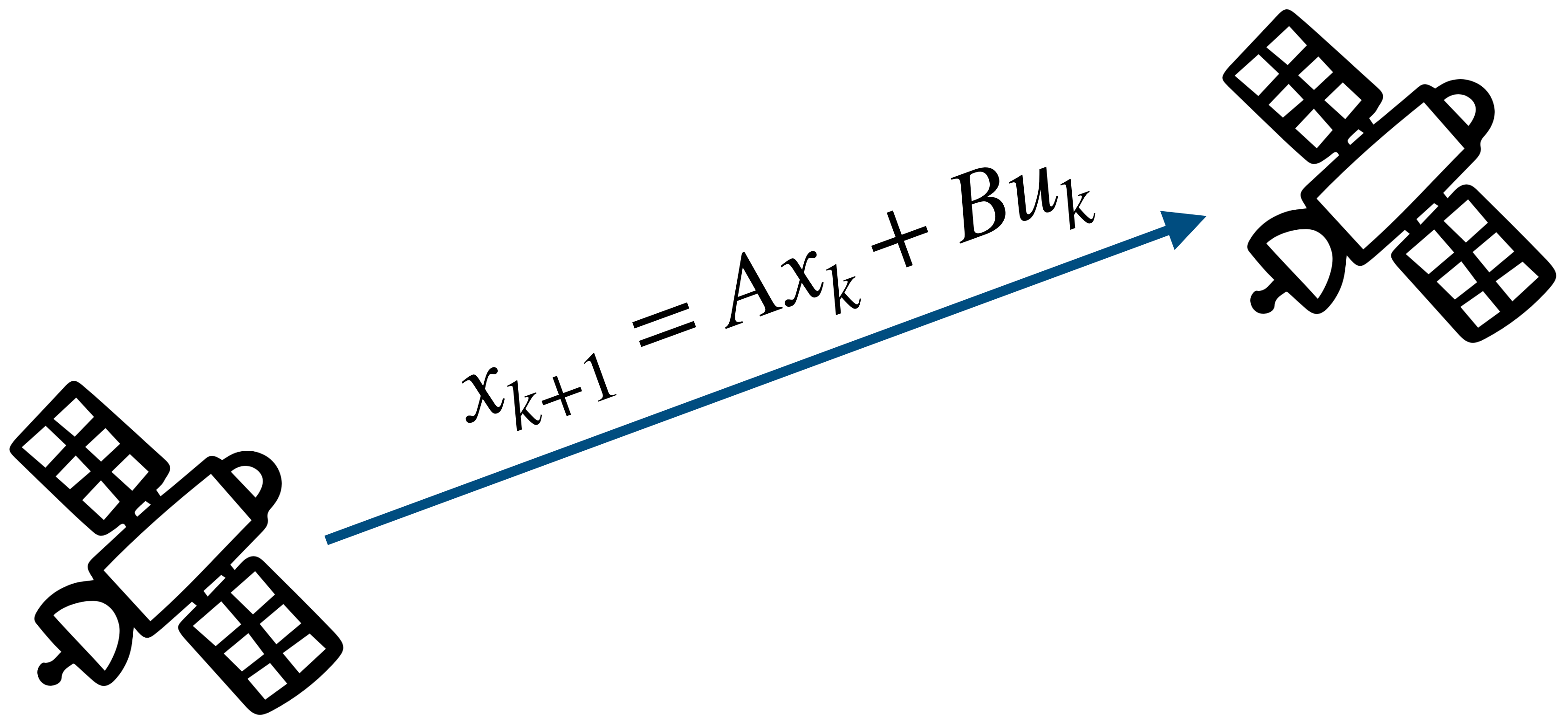
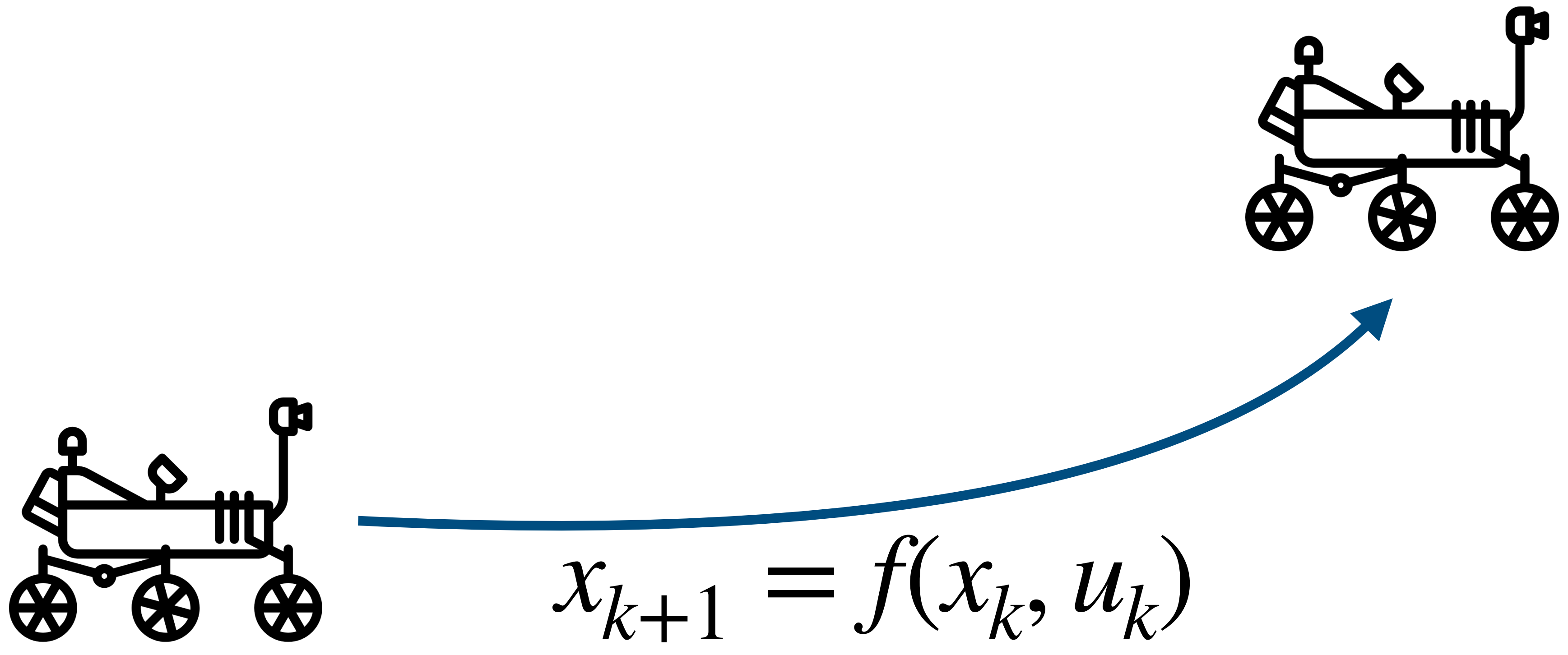


Mars rover (non-convex)

$$x_{k+1} = x_k + \Delta t \begin{pmatrix} v_k \cos \theta_k \\ v_k \sin \theta_k \\ \omega_k \end{pmatrix}$$



Optimal control as nonlinear programs

How to convert a trajectory generation problem into standard optimization form?

Solve for trajectory $(x_{0:N}, u_{0:N})$

$$\min_{x_{0:N}, u_{0:N}} g_T(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k)$$

`subject to:`

$$x_0 = x_{\text{init}}$$

$$x_{k+1} = f(x_k, u_k)$$

$$x_k \in \mathcal{X}_{\text{safe}}$$

$$u_k \in \mathcal{U}$$

Optimal control as nonlinear programs

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```
subject to:
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$$x_0 = x_{\text{init}}$$

$$x_{k+1} = f(x_k, u_k)$$

$$x_k \in \mathcal{X}_{\text{safe}}$$

$$u_k \in \mathcal{U}$$

1. Convert to standard form with inequality constraints written as $g_i(x_k, u_k) \leq 0$ and equality constraints as $h_i(x_{k+1}, x_k, u_k) = 0$

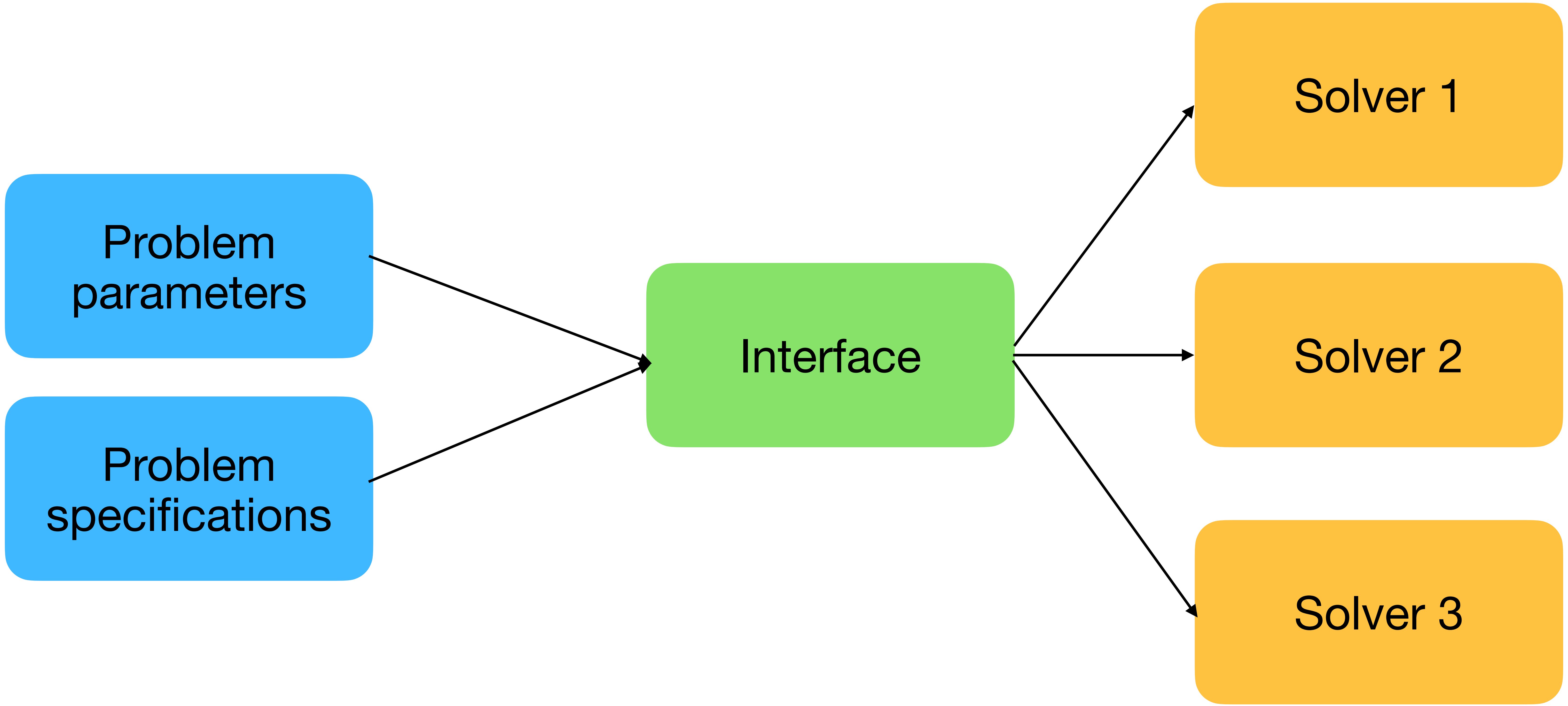
2. Provide gradient and Hessian for each constraint and cost

Converting an optimal control problem into standard form can be <u>cumbersome</u>!

Off-the-shelf trajectory optimization

Interface vs. solver

- An interface allows for solving the same optimization problem with different backend solvers
- Allows for rapid prototyping and testing



Off-the-shelf trajectory optimization

Interface vs. solver

```python
X = cp.Variable(shape=(nx, N+1))
U = cp.Variable(shape=(nu, N))

cost = 0.0
constraints = []

# Initial condition for system
constraints += [X[:,0] == x_init]

for cp_idx in range(N):
    # Add stage-wise cost
    cost += cp.quad_form(X[:, cp_idx+1] - x_goal, Q) + cp.quad_form(U[:, cp_idx], R)

    # Dynamics constraint
    constraints += [X[:, cp_idx+1] == Ak @ X[:,cp_idx] + Bk @ U[:,cp_idx]]

    # State upper and lower bounds
    constraints += [X[:, cp_idx+1] <= x_max]
    constraints += [X[:, cp_idx+1] >= x_min]

    # Control upper and lower bounds
    constraints += [U[:, cp_idx] <= u_max]
    constraints += [U[:, cp_idx] >= u_min]

prob = cp.Problem(cp.Minimize(cost), constraints)
prob.solve()
print(prob.value)
```

```
4677.378851800433
```

CVXPY

Off-the-shelf trajectory optimization

Engineering considerations

- Linear algebra routines
- Problem scaling
- Presolve techniques
- Infeasibility detection
- Parameter tuning

Solver interfaces

Python

acados
PYOMO
CVXPY
CasADi

Julia

JuMP

jump-dev/**Convex.jl**

A Julia package for disciplined convex programming

C++

CasADi
acados

MATLAB

yalmip/**YALMIP**

MATLAB toolbox for optimization modeling

cvx

Convex solvers

Linear programs: GLPK, SCIP

Quadratic programs: OSQP, HPIPM, QPOASES

Second-order cone programs: SCS, ECOS, Clarabel

Semidefinite programs: SEDUMI, SDPT3

General nonlinear programs: IPOPT, SNOPT, KNITRO

Mixed integer linear programs: CPLEX, KNITRO, MOSEK, Gurobi

Mixed integer quadratic programs: MOSEK, Gurobi

Mixed integer nonlinear programming

NLLS solvers

Popular for SLAM

- Many problems in SLAM yield nonlinear least squares (NLLS) problems

$$\min_{x} ||f(x) - y||_2^2$$

- Constraints are managed by “penalizing” violations
- Weaker guarantees and recent work in SLAM has sought to connect convex optimization with such problems
- Solvers: g2o, GTSAM, ceres, symforce, theseus

References

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