Trajectory Design for Space Systems

EN.530.626 (Fall 2025)

Lecture 8

Instructor: Prof. Abhishek Cauligi
Course Assistant 1: Arnab Chatterjee
Course Assistant 2: Mark Gonzales

Class review

- So far: smooth constrained optimization

$$\min_x f(x) \qquad\qquad f \in \mathscr{C}^2$$

$$\text{subject to:} \quad h_i(x) = 0, \quad i = 1,\ldots,m \quad h_i \in \mathscr{C}^2$$

$$g_i(x) = 0, \quad i = 0,\ldots,p \quad g_i \in \mathscr{C}^2$$

- Allows us to leverage Newton method-style approaches for solving problem

Class review
Quadratic programs

- One particular formulation we have returned to has been *convex* quadratic programs
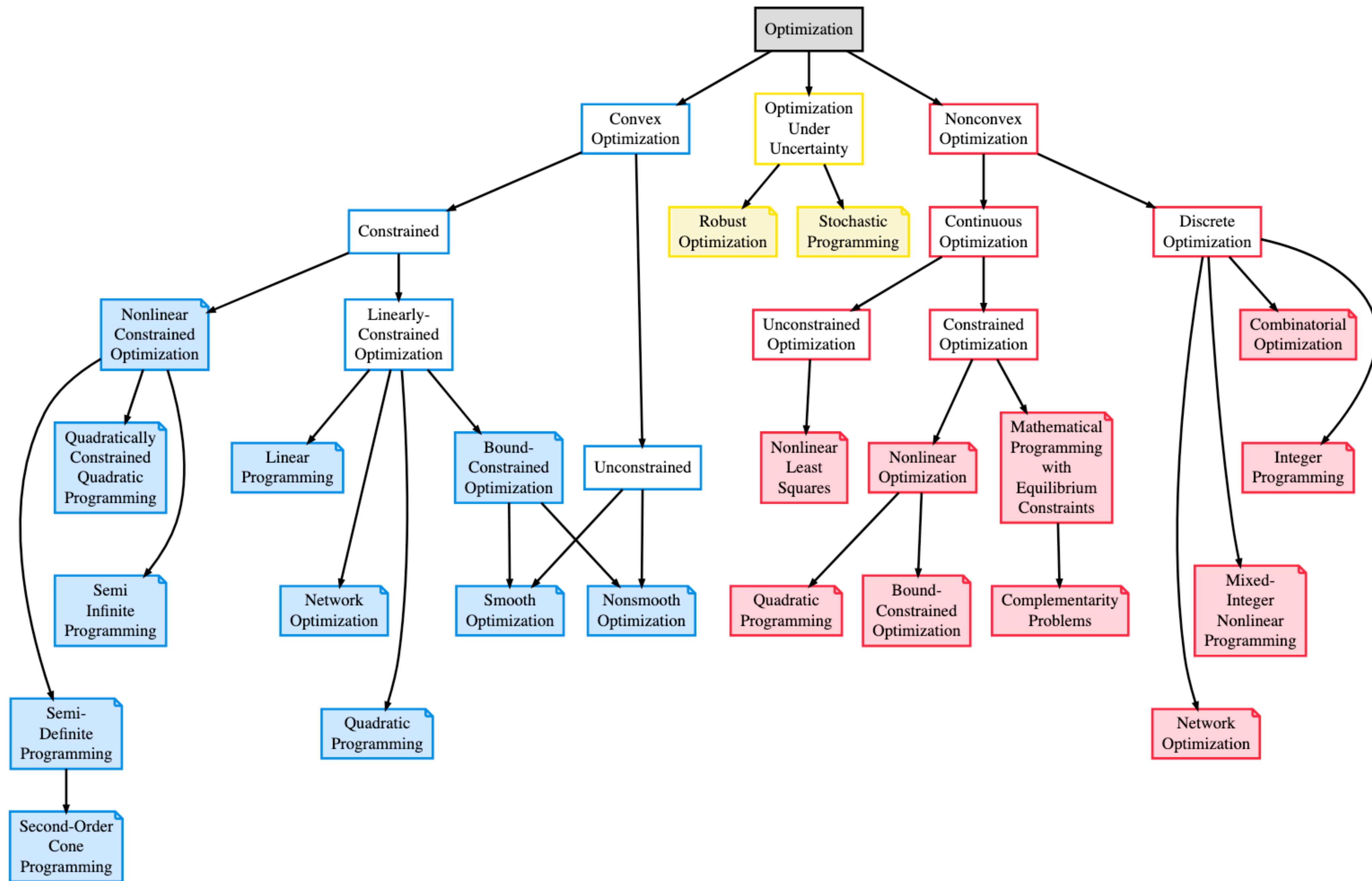
$$\min_x \frac{1}{2} x^T P x + p^T x \qquad\qquad P \in \mathbb{S}^n_+$$

$$\text{subject to:} \qquad Ax = 0 \qquad\qquad A \in \mathbb{R}^{m \times n}$$

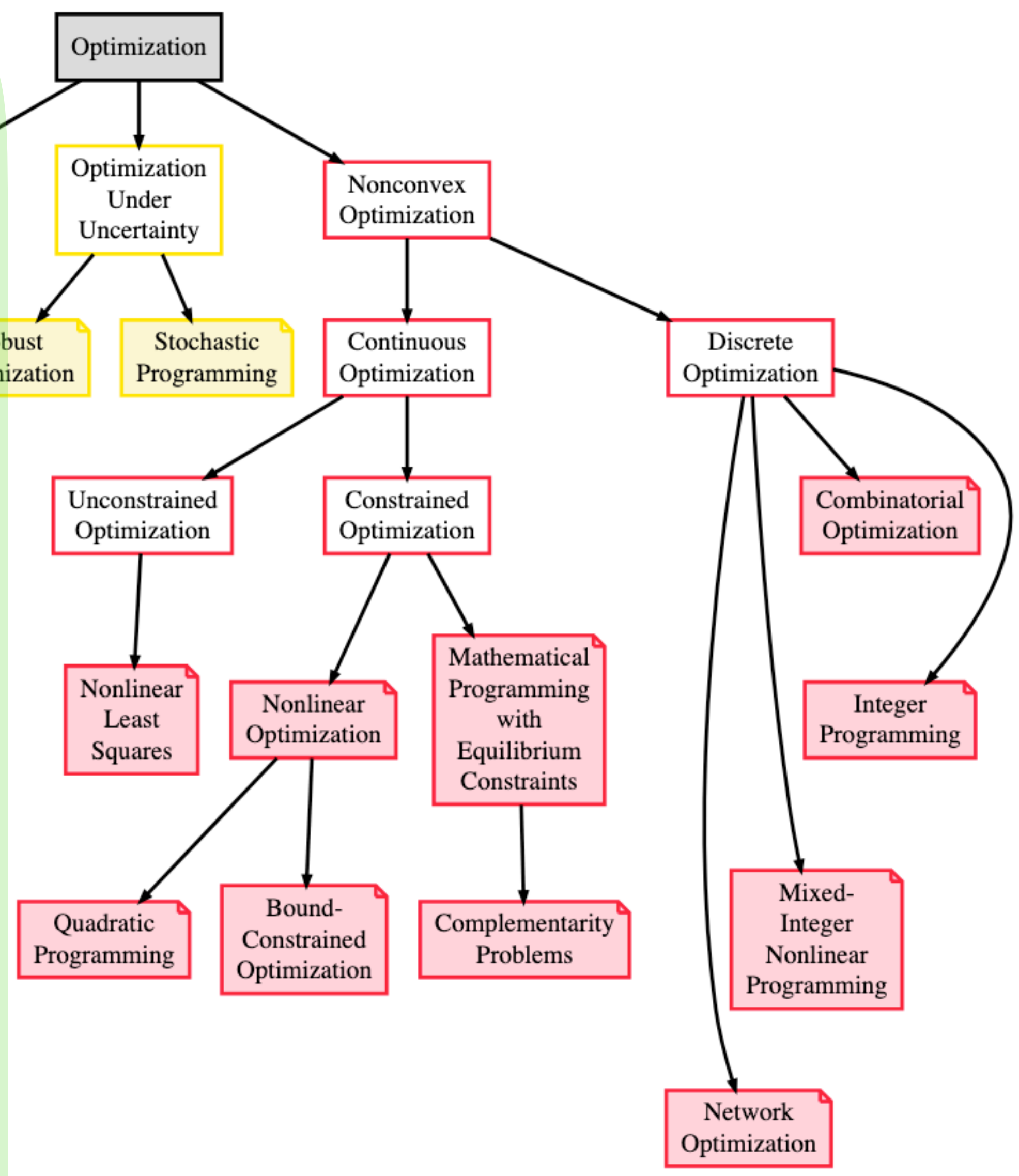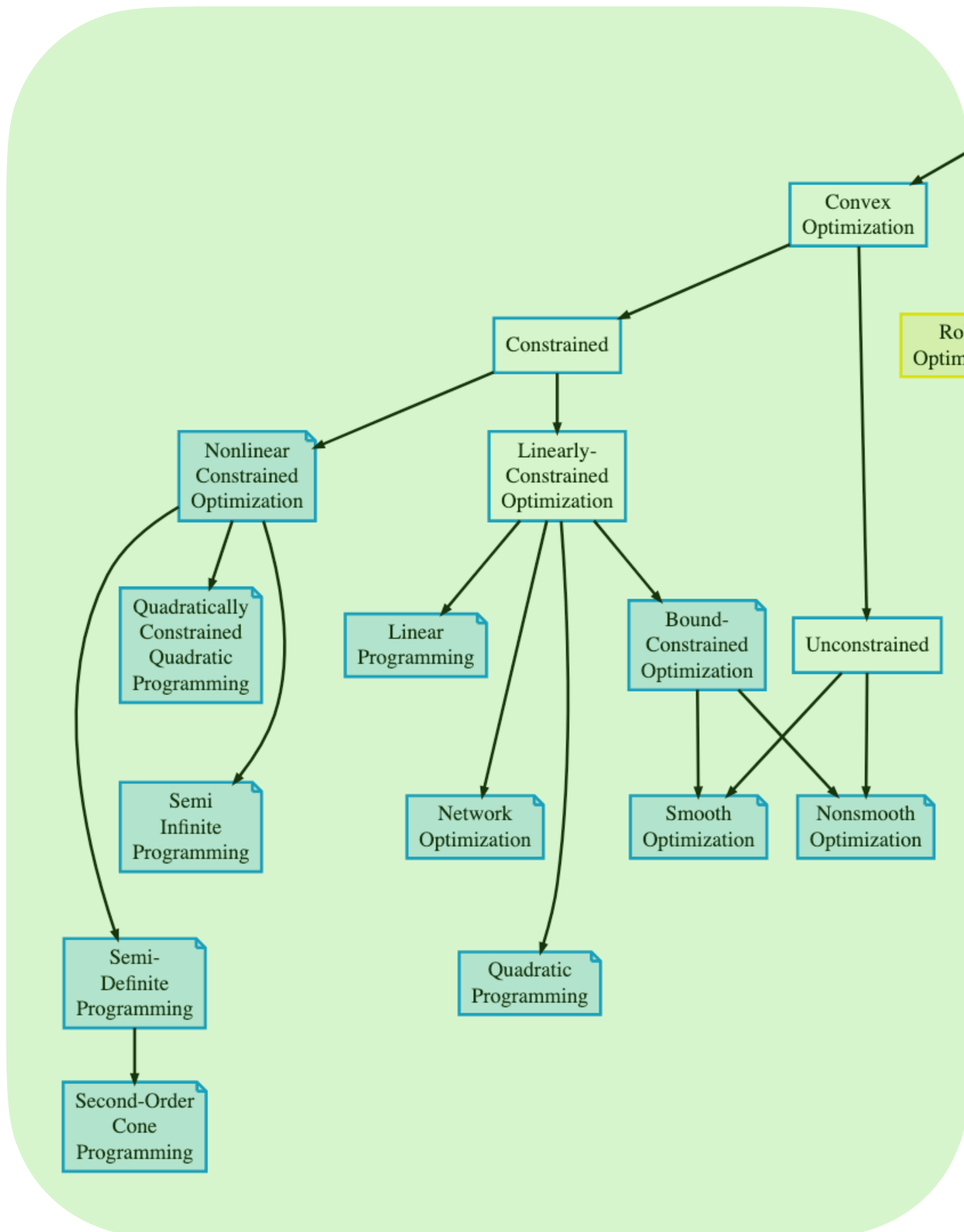$$Gx \leq h \qquad\qquad G \in \mathbb{R}^{p \times n}$$

- Applied primal-dual interior point methods to find solutions for these

What's next?

Recall optimization classes:

- Unconstrained vs. constrained

- Smooth vs. non smooth

- Convex vs. non-convex

- Continuous vs. discrete

https://neos-guide.org/guide/types/

```
Optimization
├── Convex Optimization
│   ├── Constrained
│   │   ├── Nonlinear Constrained Optimization
│   │   │   ├── Quadratically Constrained Quadratic Programming
│   │   │   ├── Semi Infinite Programming
│   │   │   ├── Semi-Definite Programming
│   │   │   │   └── Second-Order Cone Programming
│   │   └── Linearly-Constrained Optimization
│   │       ├── Linear Programming
│   │       ├── Network Optimization
│   │       ├── Quadratic Programming
│   │       └── Bound-Constrained Optimization
│   │           ├── Smooth Optimization
│   │           └── Nonsmooth Optimization
│   └── Unconstrained
│       ├── Smooth Optimization
│       └── Nonsmooth Optimization
├── Optimization Under Uncertainty
│   ├── Robust Optimization
│   └── Stochastic Programming
└── Nonconvex Optimization
    ├── Continuous Optimization
    │   ├── Unconstrained Optimization
    │   │   └── Nonlinear Least Squares
    │   └── Constrained Optimization
    │       ├── Nonlinear Optimization
    │       │   ├── Quadratic Programming
    │       │   └── Bound-Constrained Optimization
    │       └── Mathematical Programming with Equilibrium Constraints
    │           └── Complementarity Problems
    └── Discrete Optimization
        ├── Combinatorial Optimization
        ├── Integer Programming
        ├── Mixed-Integer Nonlinear Programming
        └── Network Optimization
```

https://neos-guide.org/guide/types/

Convexity

- A whole field of optimization on its own

- Convex ≠ easy

- Advantages

  - (1) Enjoys strong convergence guarantees

  - (2) Local optimizer is global one

Stephen Boyd and
Lieven Vandenberghe

Convex Optimization

CAMBRIDGE
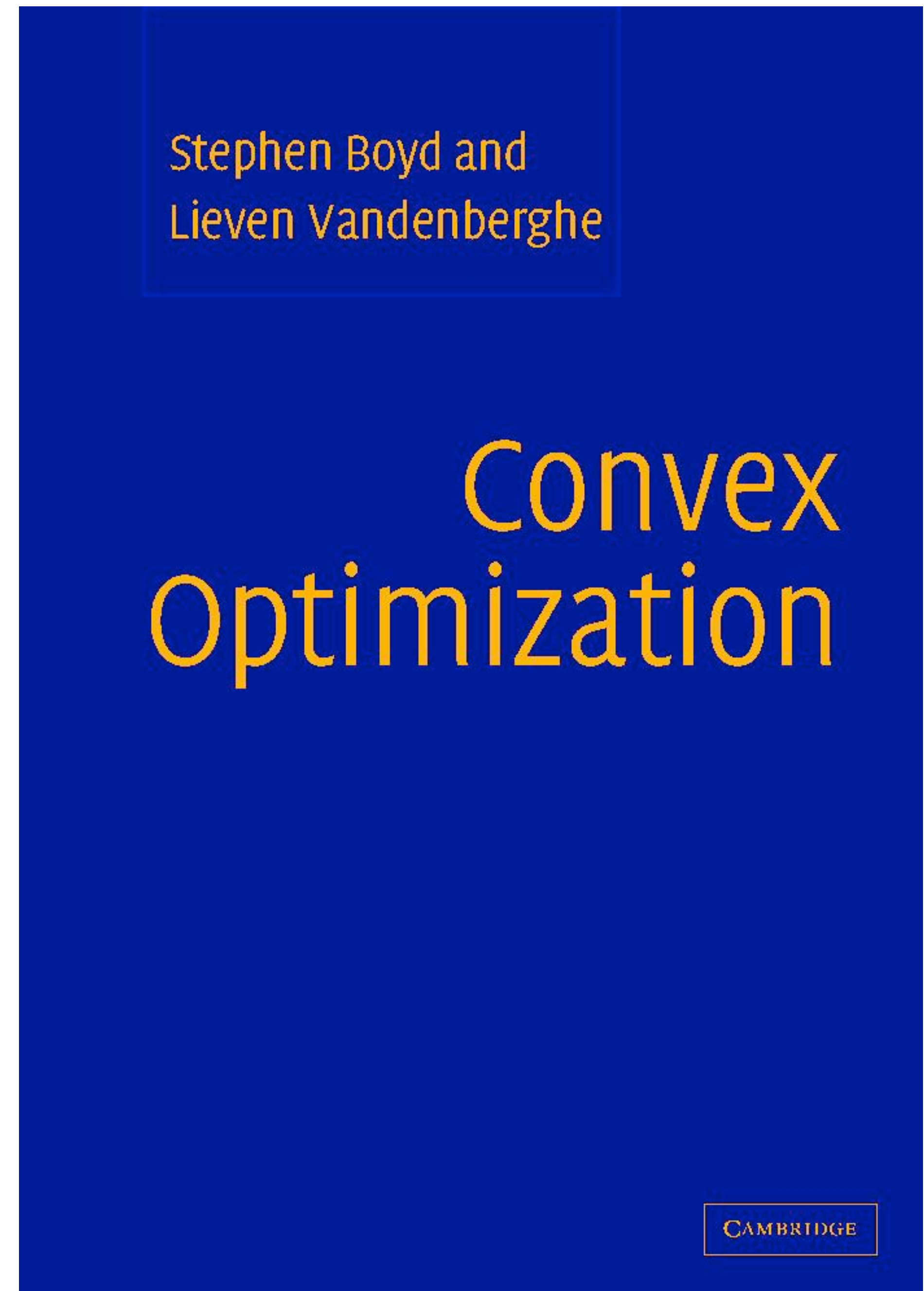
Convexity
Convex sets

- A set $\mathcal{X}$ is convex if $\forall x_1, x_2 \in \mathcal{X}$ and $\theta \in [0,1]$

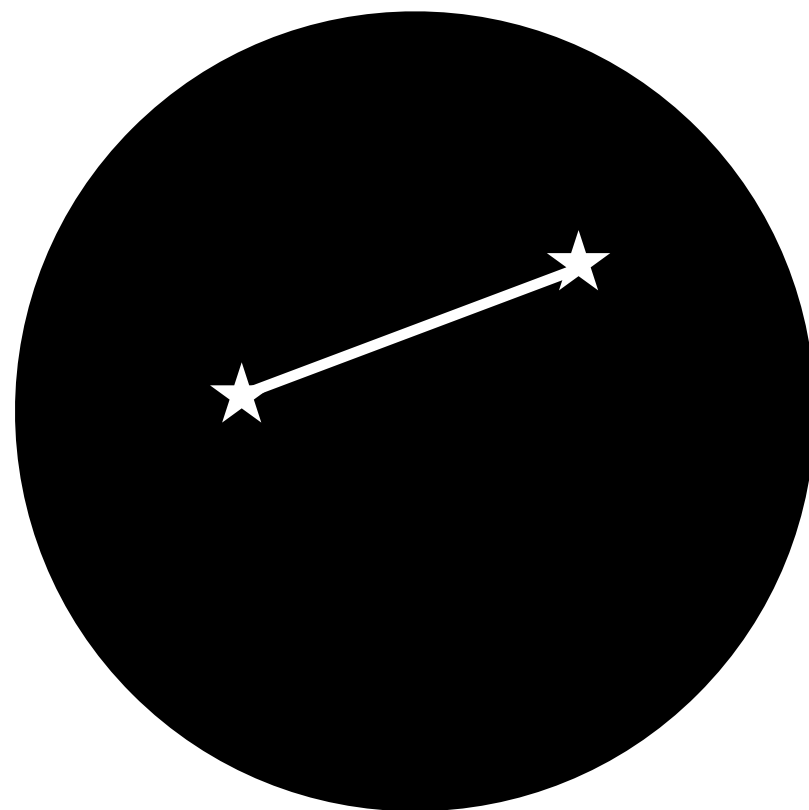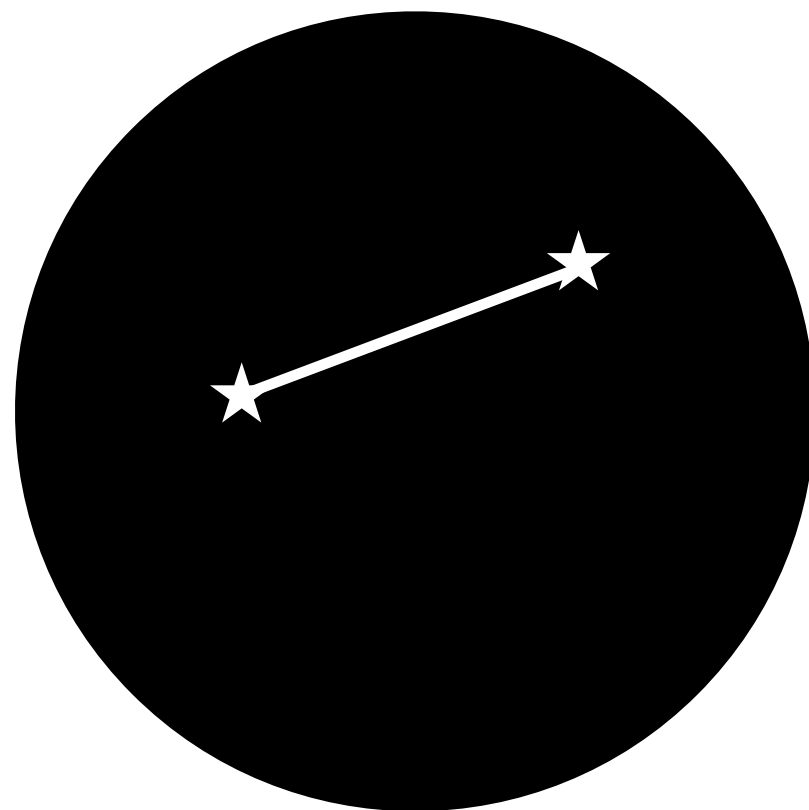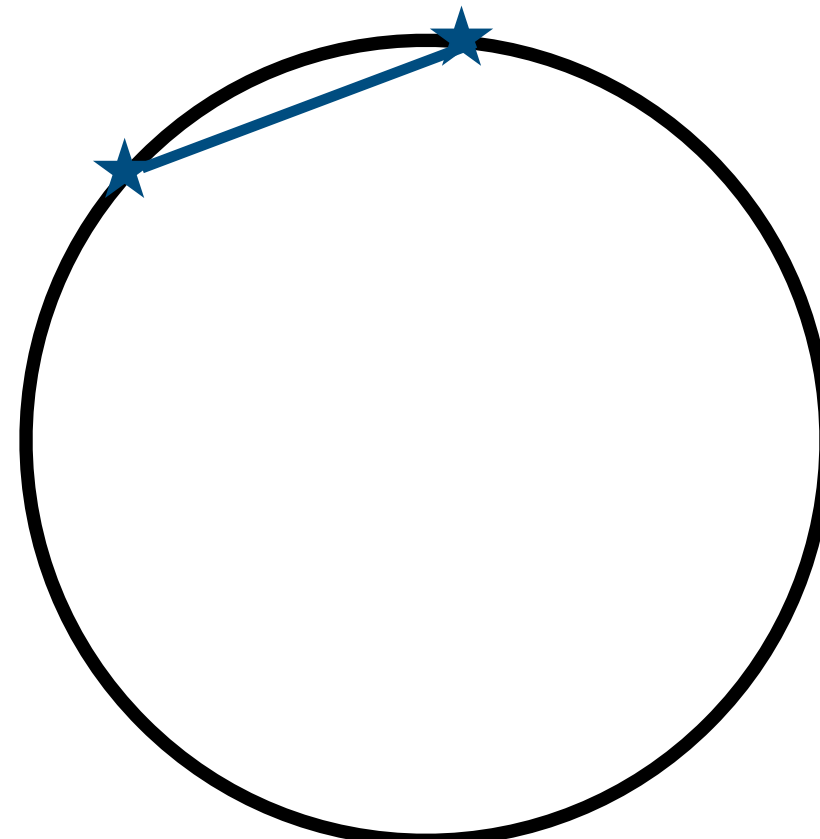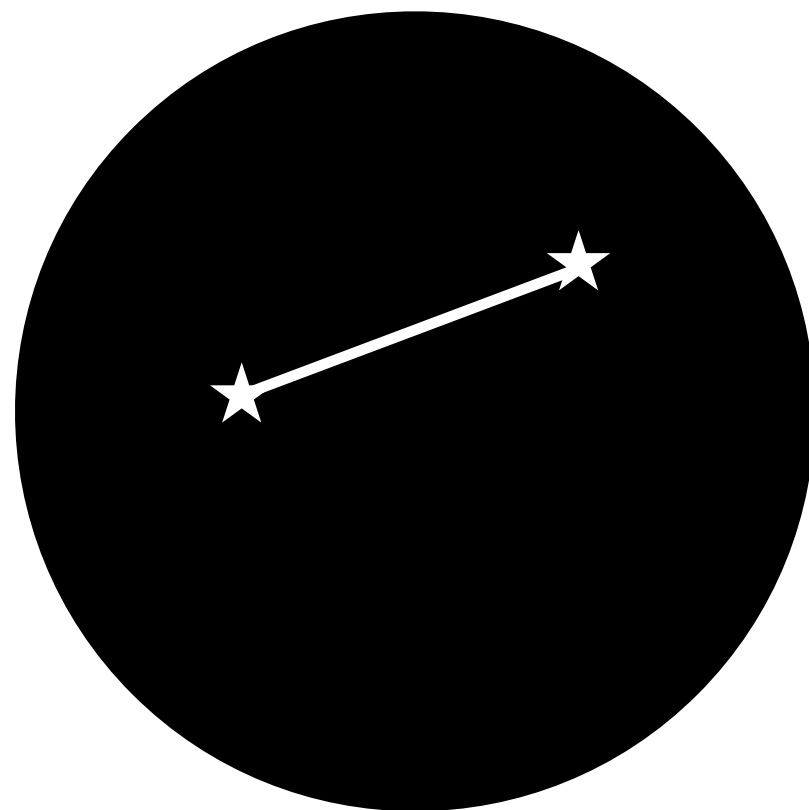$$\theta x_1 + (1 - \theta)x_2 \in \mathcal{X}$$
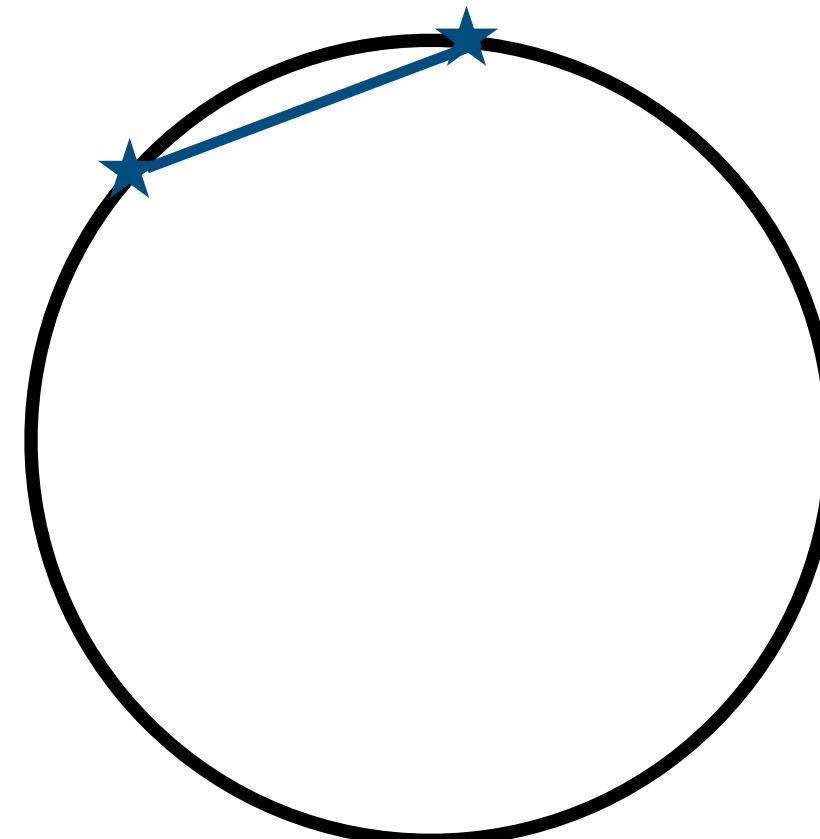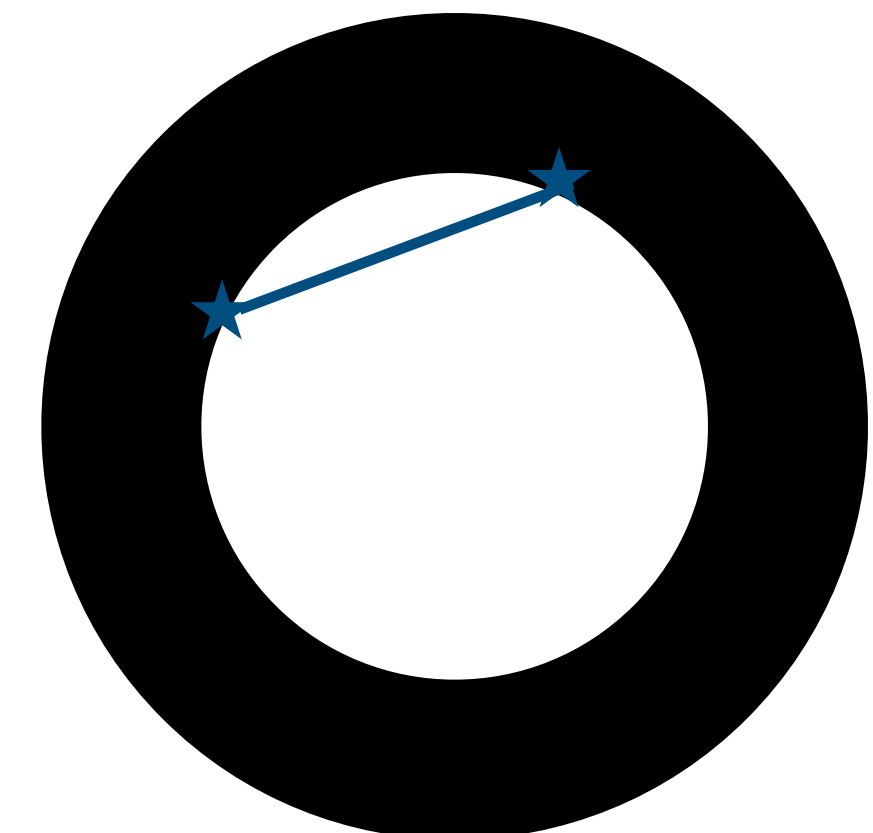
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$$\theta x_1 + (1 - \theta)x_2 \in \mathcal{X}$$

$$\|x\|_2 \leq r_{\text{dist}}$$

Convexity
Convex sets

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$\|x\|_2 \leq r_{\text{dist}}$

$\|x\|_2 = r_{\text{dist}}$

Convexity
Convex sets

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$\|x\|_2 \leq r_{\text{dist}}$ $\|x\|_2 = r_{\text{dist}}$ $r_{\text{min}} \leq \|x\|_2 \leq r_{\text{max}}$
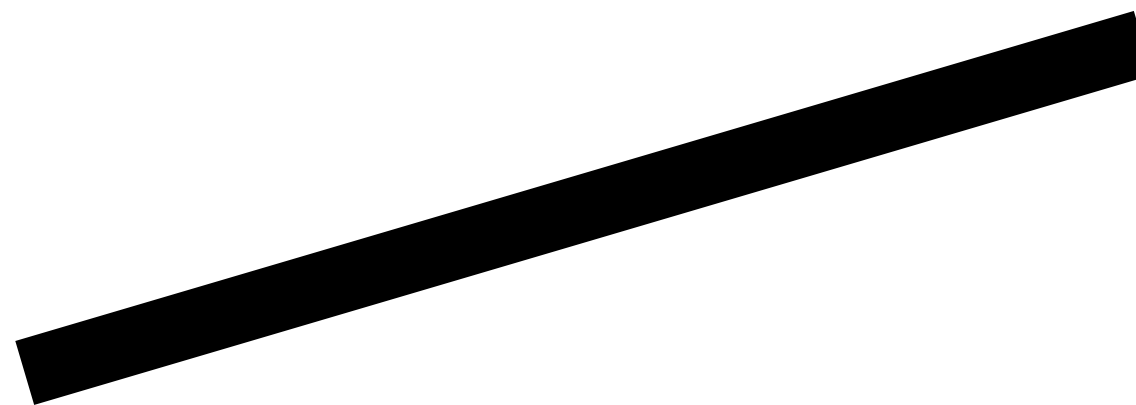
Convexity
Convex sets

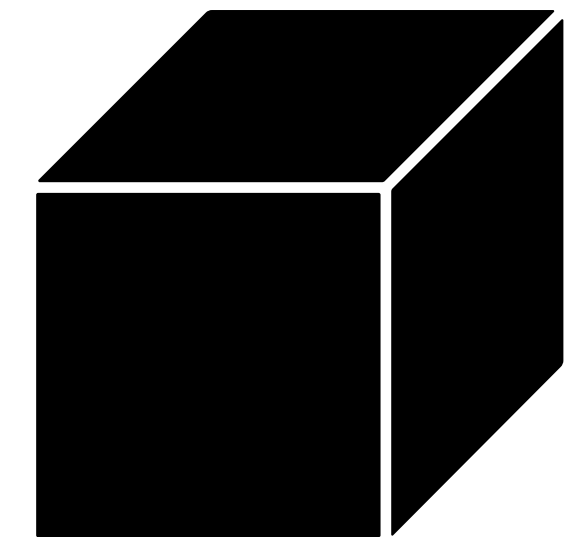- The intersection of convex sets is convex

$$\|x\|_2 \leq r_{\text{dist}}$$
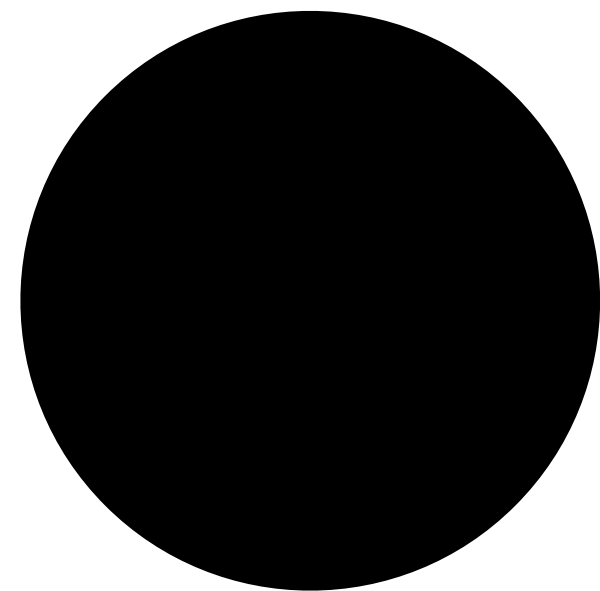
$$y = c^T x + b$$

$$\|u\|_1 \leq u_{\text{max}}$$

Convexity
Convex functions

- A function $f(x)$ is convex if $\forall x_1, x_2 \in \text{dom } f$ and $\theta \in [0,1]$

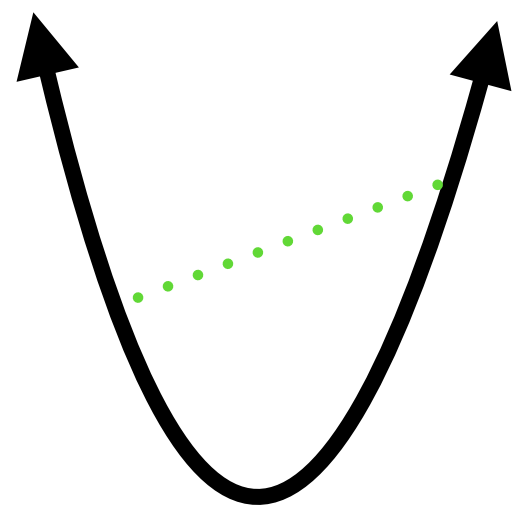$$f(\theta x_1 + (1 - \theta)x_2) \leq \theta f(x_1) + (1 - \theta)f(x_2)$$

Convexity
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$$f(\theta x_1 + (1 - \theta)x_2) \leq \theta f(x_1) + (1 - \theta)f(x_2)$$

$f(x) = x^2$

Convex

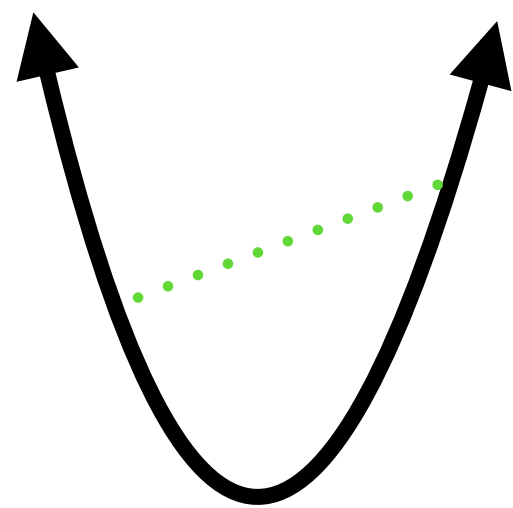# Convexity
Convex functions

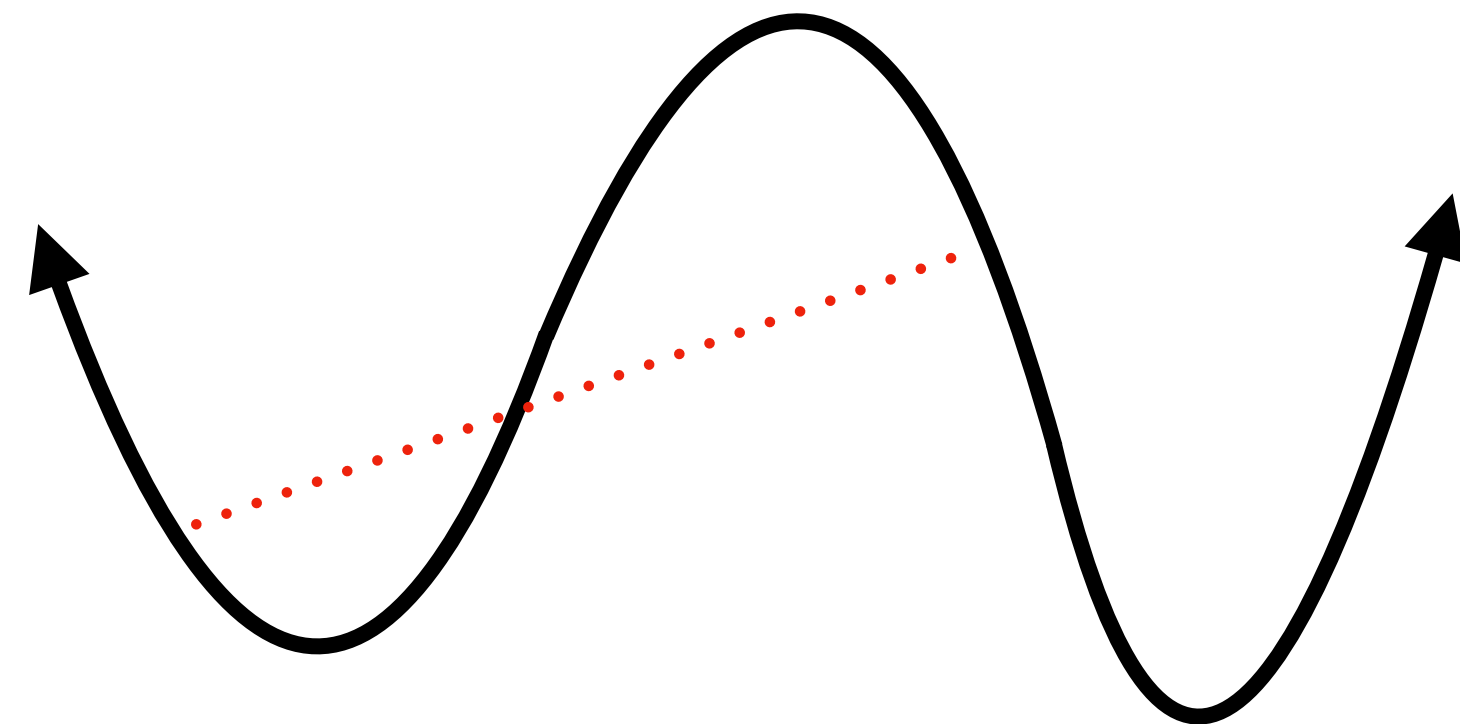- A function $f(x)$ is convex if $\forall x_1, x_2 \in$ dom f and $\theta \in [0,1]$

$$f(\theta x_1 + (1 - \theta)x_2) \leq \theta f(x_1) + (1 - \theta)f(x_2)$$

$f(x) = x^2$

$f(x) = x^4 - 2 * x^3 - 10x^2 + 7x - 17$
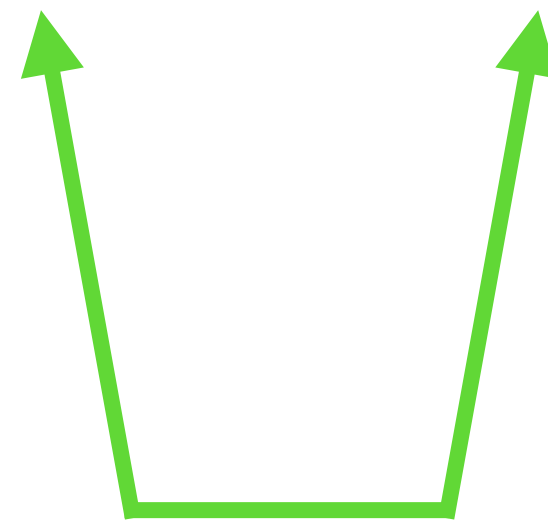
Convex

Non-convex

Convexity
Convex functions

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$$f(\theta x_1 + (1 - \theta)x_2) \leq \theta f(x_1) + (1 - \theta)f(x_2)$$

- Note: f need not be smooth (i.e., $\nabla_x f(x)$ might not exist)

Convex

Non-convex

Convexity
Convex functions and sets

- The epigraph of a function $f : \mathbb{R}^n \to \mathbb{R}$ is:

$$\operatorname{epi} f = \{(x, t) \mid x \in \operatorname{dom} f, f(x) \leq t\}$$

Convexity
Convex functions and sets

- The epigraph of a function $f : \mathbb{R}^n \to \mathbb{R}$ is:

$$\text{epi } f = \{(x, t) \mid x \in \text{dom } f, f(x) \leq t\}$$

- f is a convex function if and only if epi f is a convex set

$$f \text{ is convex function} \Leftrightarrow \text{epi } f \text{ is convex set}$$

Convexity
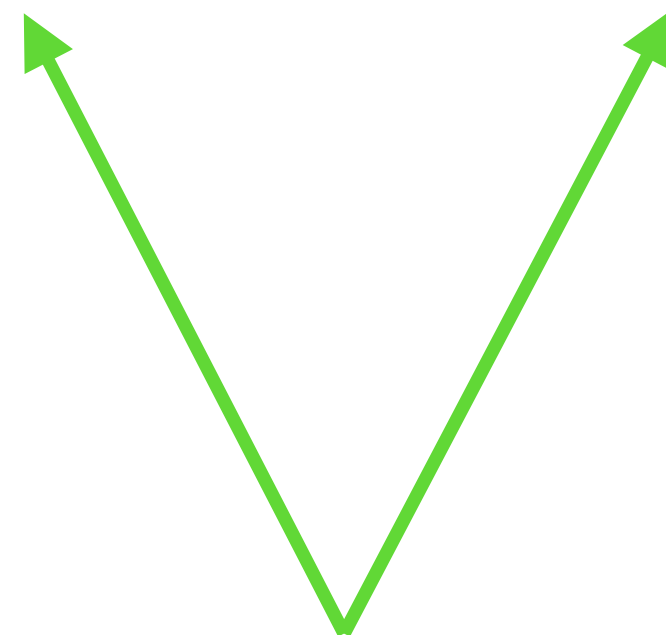## Convex functions and sets

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Convexity
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$$f \text{ is convex function} \Leftrightarrow \text{epi } f \text{ is convex set}$$

epi f

Convexity
Why does this matter?

- A *local* solution to a convex problem is also the *global* solution

  - For smooth unconstrained optimization, if $f \in \mathscr{C}^2$ and convex:

  $$x^* \text{ is global optimizer} \Leftrightarrow \nabla_x f(x^*) = 0$$

  - For smooth constrained optimization, if $f, h, g \in \mathscr{C}^2$ and convex:

  $$x^* \text{ is global optimizer} \Leftrightarrow \nabla_x \mathscr{L}(x^*, y^*, z^*) = 0$$

Convexity
Completeness guarantees

- A function $f : \mathbb{R} \to \mathbb{R}$ is called *self-concordant* if f is convex and:

$$f'''(x) \leq 2f''(x)^{\frac{3}{2}}$$

- Loosely, "third derivative doesn't change too quickly"

- Interior point methods solve convex optimization problems in polynomial time

$$N = \mathcal{O}\left(\sqrt{n} \log\left(\frac{1}{\varepsilon} \right) \right)$$

Convexity

Interior-Point Polynomial
Algorithms in Convex
Programming

Yurii Nesterov
Arkadii Nemirovskii

siam Studies in
Applied Mathematics

THE INTERIOR-POINT REVOLUTION IN OPTIMIZATION: HISTORY, RECENT DEVELOPMENTS, AND LASTING CONSEQUENCES

MARGARET H. WRIGHT

1. OVERVIEW

REVOLUTION:
(i) a sudden, radical, or complete change;
(ii) a fundamental change in political organization, especially the overthrow or renunciation of one government or ruler and the substitution of another.[1]

It can be asserted with a straight face that the field of continuous optimization has undergone a revolution since 1984 in the sense of the first definition and that the second definition applies in a philosophical sense: Because the interior-point presence in optimization today is ubiquitous, it is easy to lose sight of the magnitude and depth of the shifts that have occurred during the past twenty years. Building on the implicit political metaphor of our title, successful revolutions eventually become the status quo.
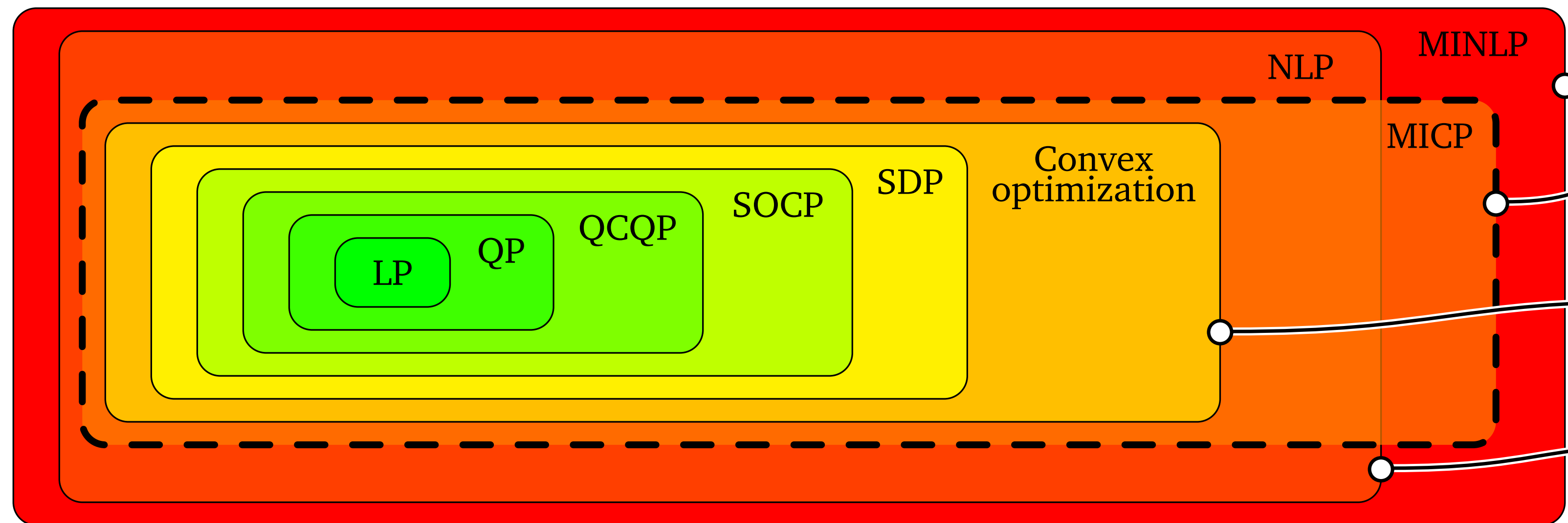
Convexity
In summary

Convexity

In summary

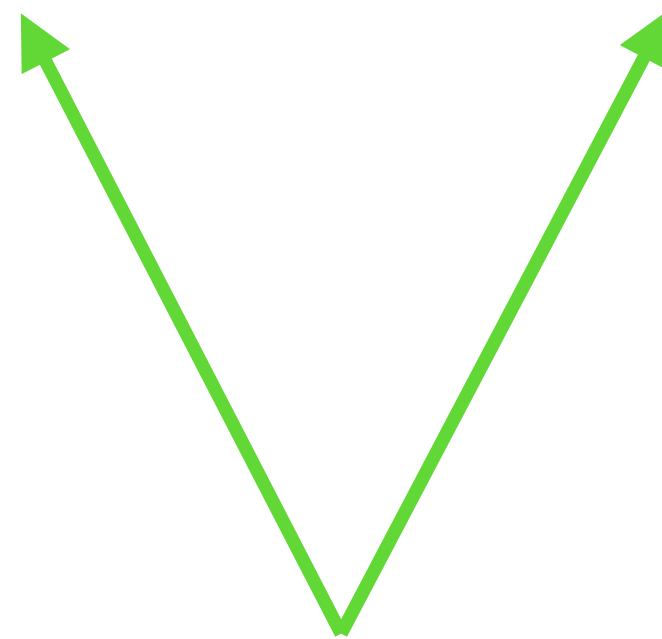- Convexity does not mean the problem is *easy*

D. Malyuta, Y. Yu, P. Elango, and B. Acikmese, "Advances in trajectory optimization for space vehicle control," *Annual Reviews in Control*, vol. 52, pp. 282 — 315, 2021.
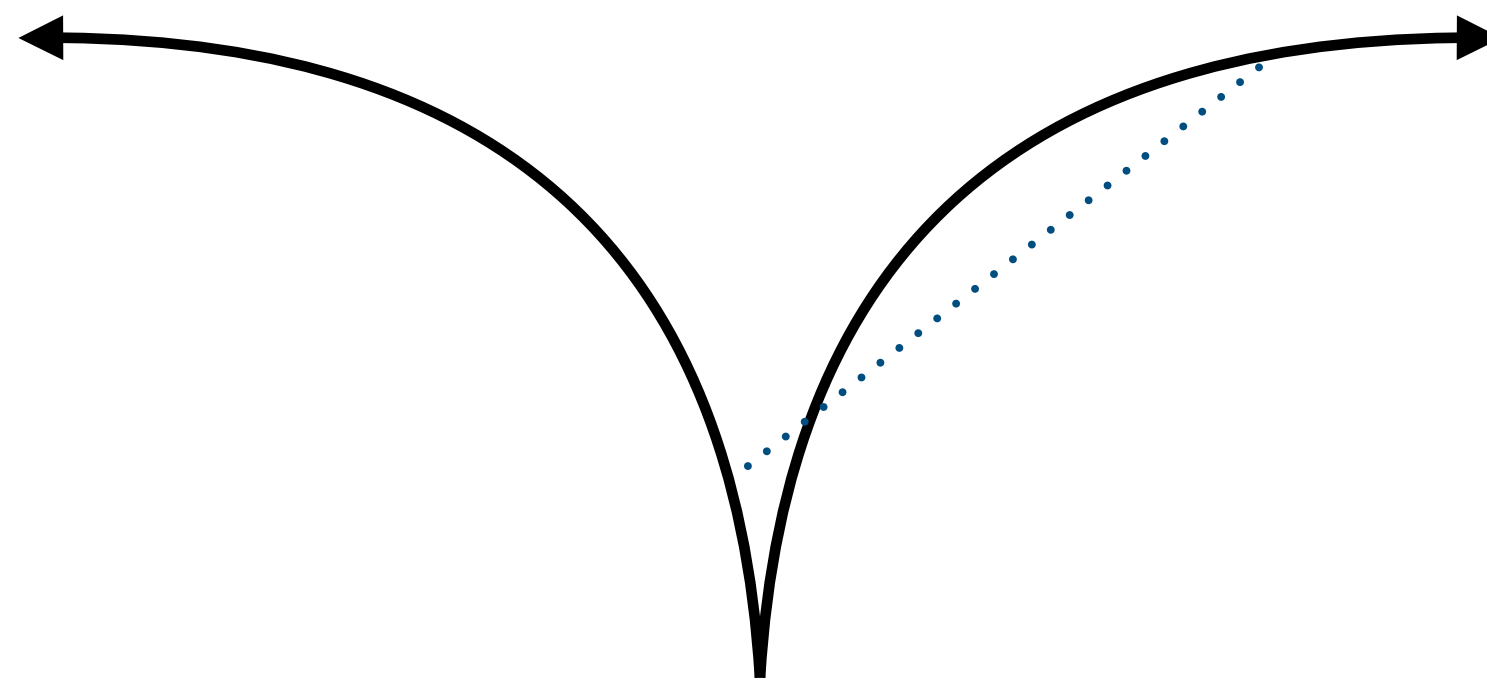
Convexity
In summary

- Convexity does not mean the problem is *easy*
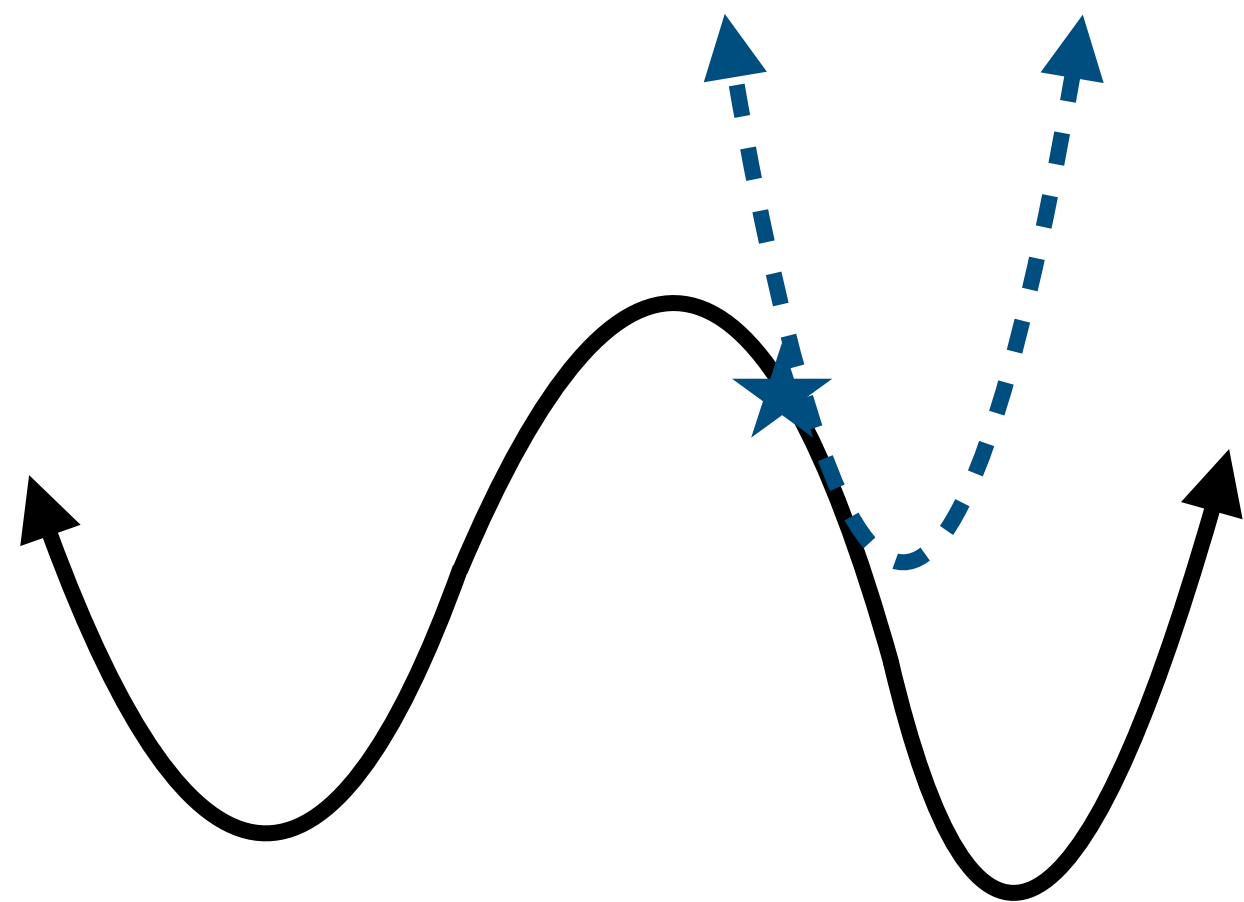- Convex programs do not have to be smooth

Convexity

In summary

- Convexity does not mean the problem is *easy*

- Convex programs do not have to be smooth

- Convex programs are not the only ones with global optima

Relevance to trajectory design

- Most practical systems of interest are non-convex

- For *smooth* non-convex optimization problems, most approaches rely on solving a sequence of convex approximations to the problem
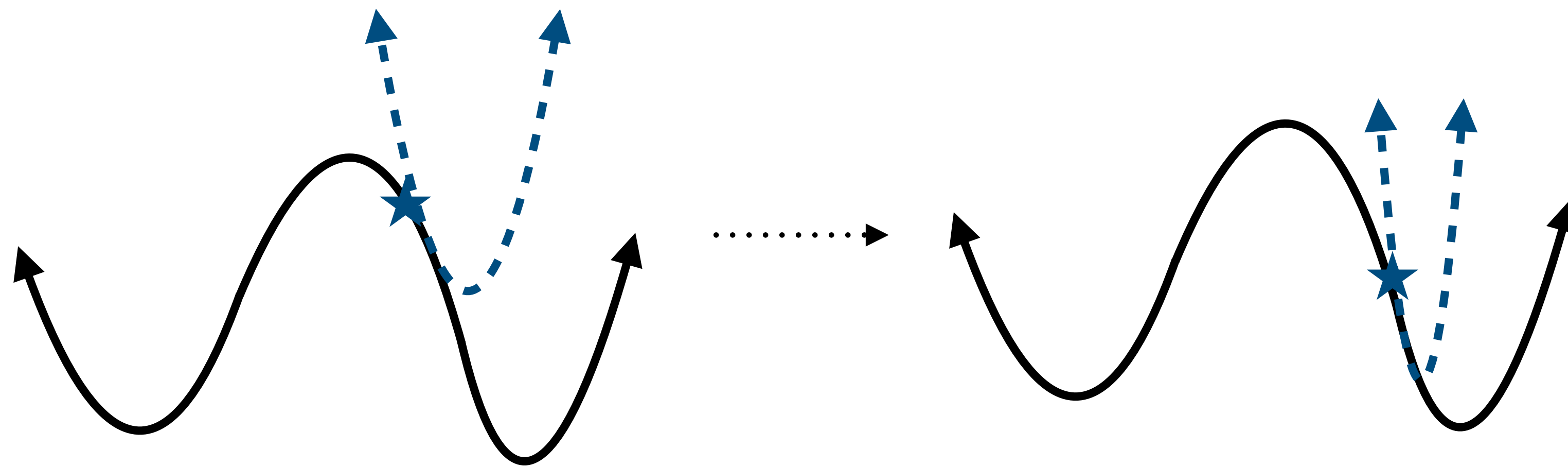
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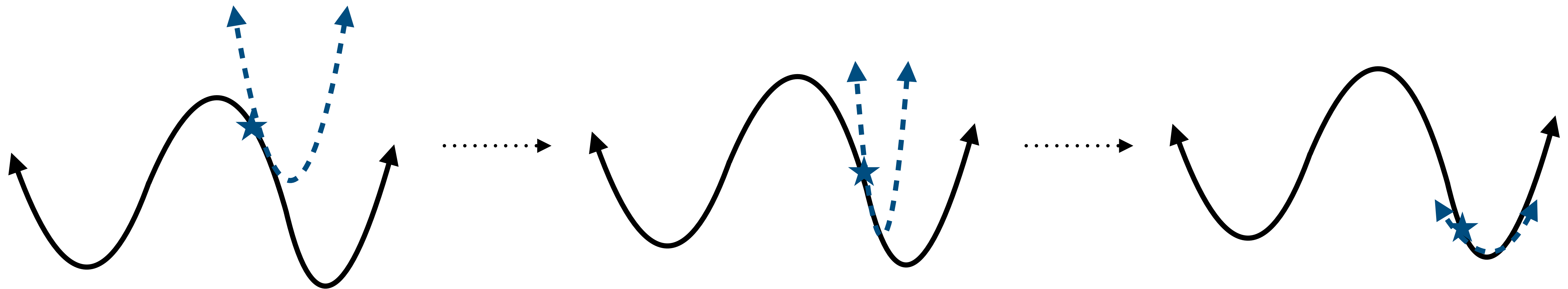
Relevance to trajectory design

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Relevance to trajectory design
Sequential quadratic programming

$$\min_x f(x)$$

subj. to:
$$h(x) = 0$$
$$g(x) \leq 0$$
$$f, h, g \in \mathscr{C}^2$$

Relevance to trajectory design
Sequential quadratic programming

1. Given $w^0 = (x^0, y^0, z^0)$

2. Construct Lagrangian using w

`for` $k \in [1, N_{\max}]$

   3. Solve local QP approximation for Δw^k

$$\min_{\Delta x} \frac{1}{2} \Delta x^T \nabla_{xx}^2 \mathscr{L}(\bar{x}) \Delta x + \nabla_x \mathscr{L}(\bar{x}) \Delta x$$

$$\text{subj. to:} \quad \begin{aligned} h(\bar{x}) + \nabla_x h(\bar{x}) \Delta x = 0 \\ g(\bar{x}) + \nabla_x g(\bar{x}) \Delta x \leq 0 \end{aligned}$$

4. Line search to find α

5. $w^k = w^{k-1} + \alpha \Delta w^k$

6. Terminate when necessary conditions satisfied

$$\min_x f(x)$$

$$\text{subj. to:} \quad \begin{aligned} h(x) = 0 \\ g(x) \leq 0 \\ f, h, g \in \mathscr{C}^2 \end{aligned}$$

$\longrightarrow$

Convexity of practical constraints
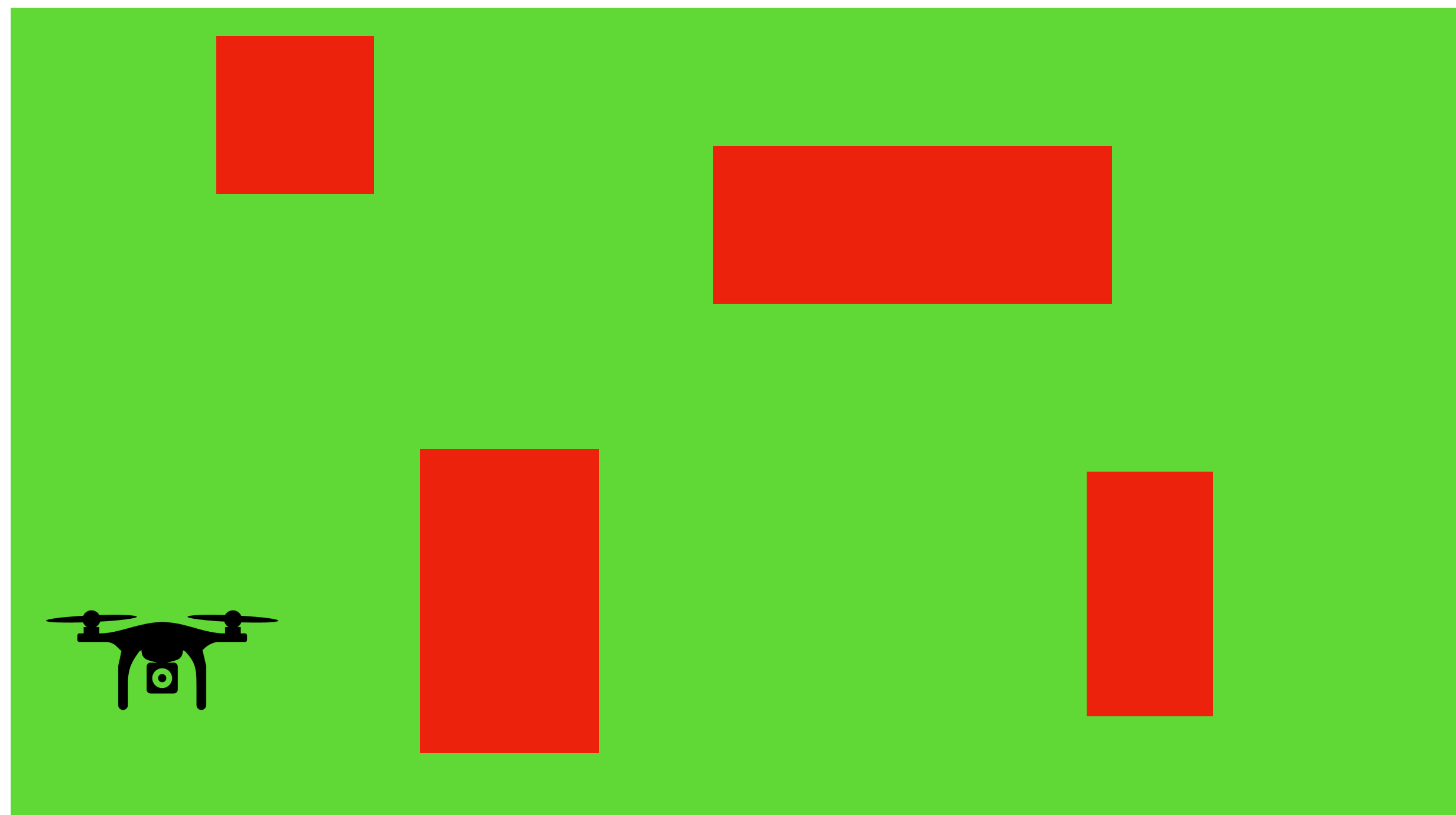
Convexity of practical constraints
Collision avoidance

$$x_k \notin \mathcal{X}_{\text{obs}}$$

Convexity of practical constraints
Collision avoidance

$$x_k \in \mathscr{X}_{\text{safe}}$$
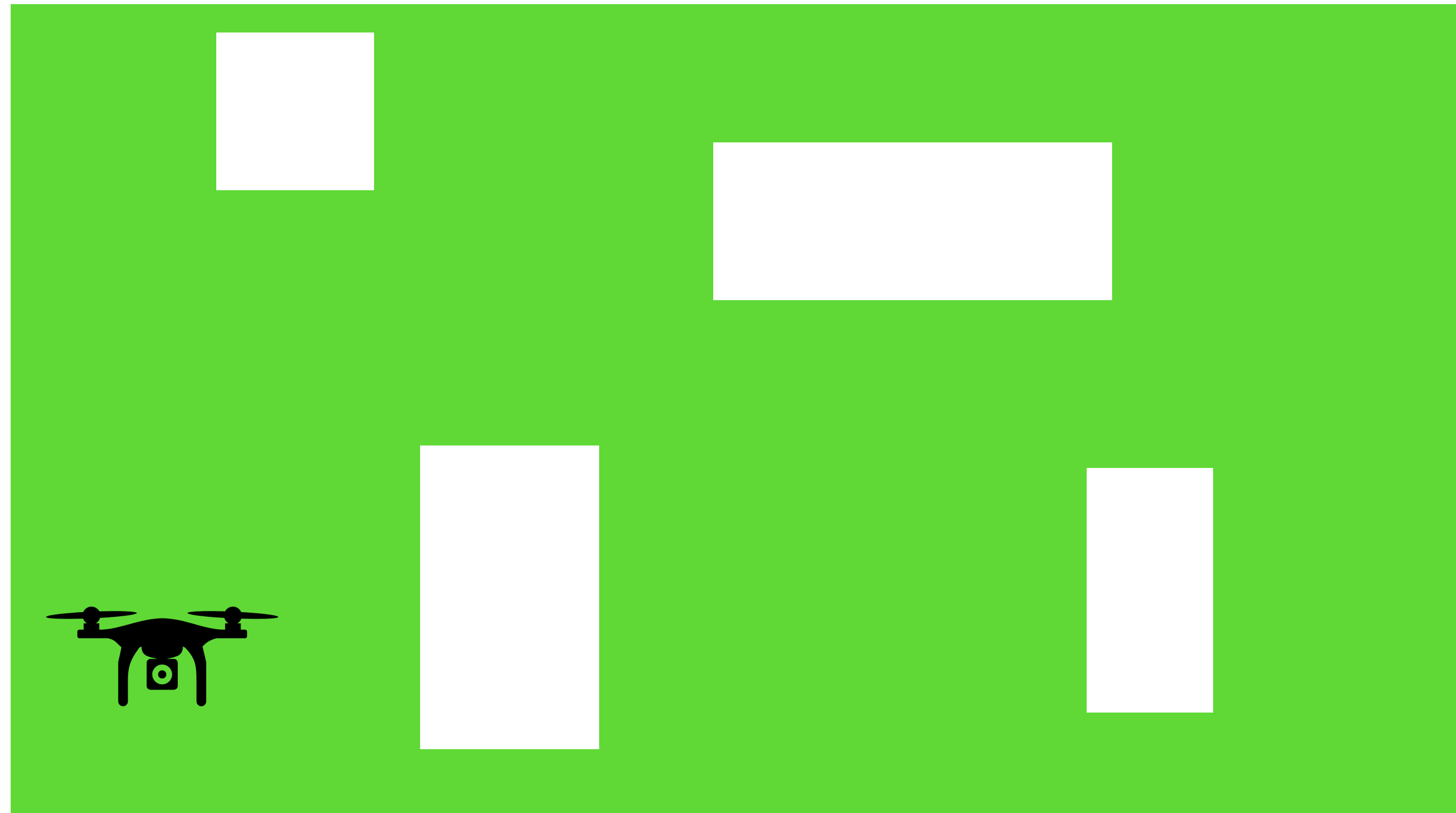
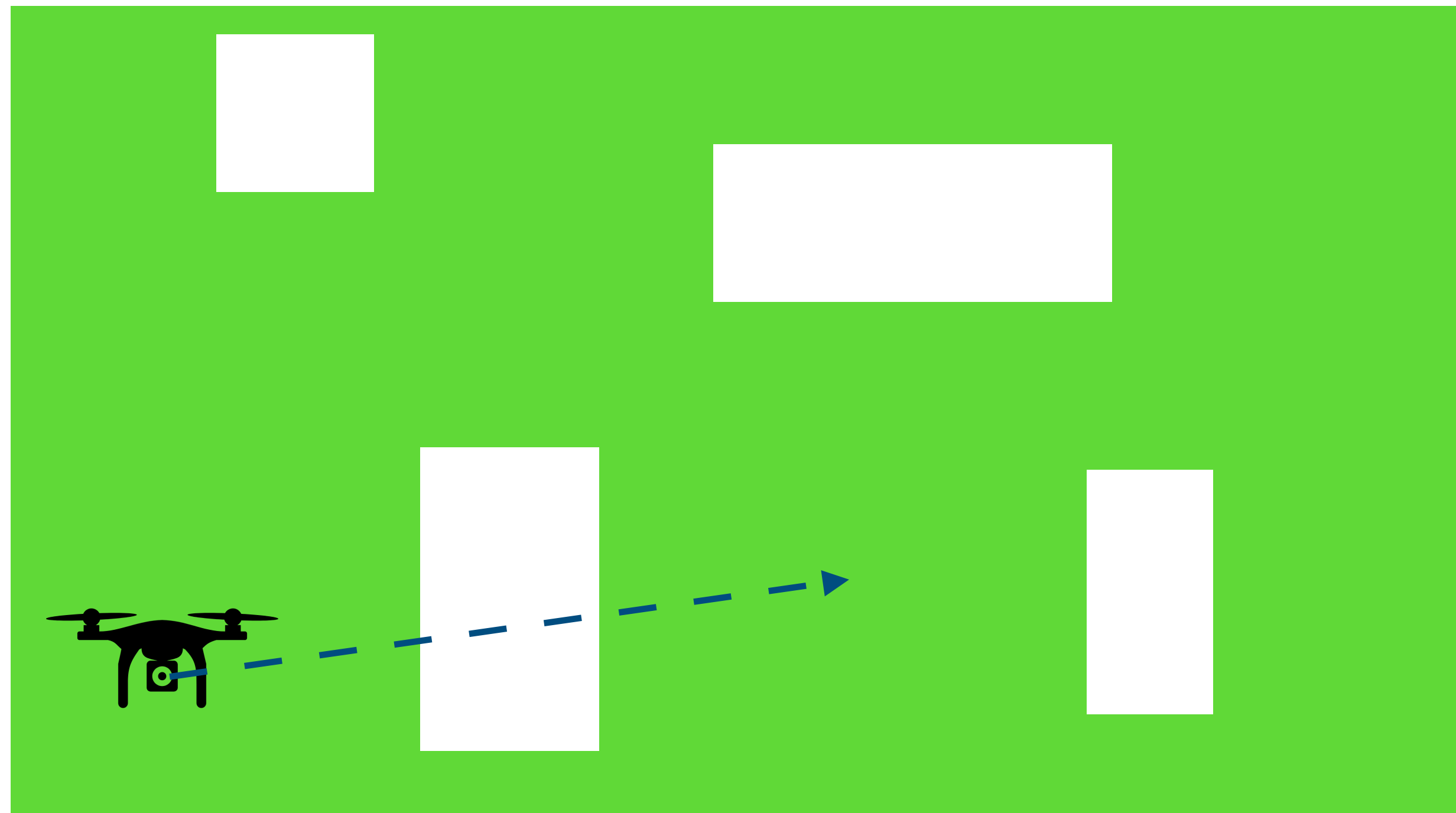# Convexity of practical constraints
Collision avoidance

$$x_k \in \mathcal{X}_{\text{safe}}$$

Convexity of practical constraints
Collision avoidance

$$x_k \in \mathcal{X}_{\text{safe}} \qquad \text{Non-convex}$$

Convexity of practical constraints
Dynamics constraints

- An equality constraint can be written as a two-sided inequality

$$x_{k+1} = f(x_k, u_k)$$

$$\longleftrightarrow$$

$$x_{k+1} \leq f(x_k, u_k)$$
$$x_{k+1} \geq f(x_k, u_k)$$

Convexity of practical constraints
Dynamics constraints

- An equality constraint can be written as a two-sided inequality

$$x_{k+1} = f(x_k, u_k)$$

$$\begin{aligned} x_{k+1} &\leq f(x_k, u_k) \\ -x_{k+1} &\leq -f(x_k, u_k) \end{aligned}$$

Convexity of practical constraints
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- For both a function $f(x_k, u_k)$ and its negative $-f(x_k, u_k)$ to be convex with respect to x_k and u_k, $f(\cdot)$ must be an affine function

$$x_{k+1} = Ax_k + Bu_k + c$$

Convexity of practical constraints
Dynamics constraints

- An equality constraint can be written as a two-sided inequality

$$x_{k+1} = f(x_k, u_k) \quad \longleftrightarrow \quad \begin{aligned} x_{k+1} &\leq f(x_k, u_k) \\ -x_{k+1} &\leq -f(x_k, u_k) \end{aligned}$$

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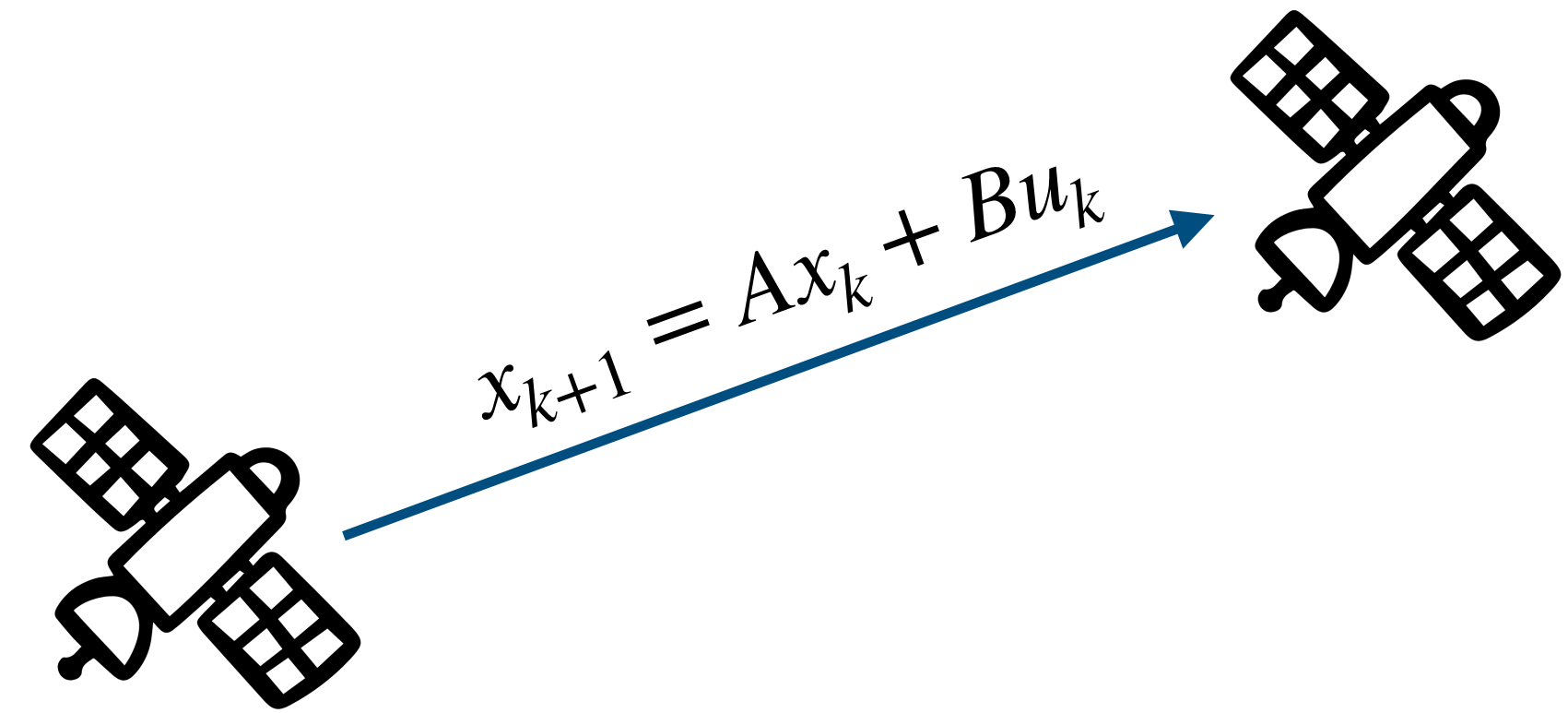
$$x_{k+1} = Ax_k + Bu_k + c$$

- Any system with nonlinear dynamics is non-convex

Convexity of practical constraints
Dynamics constraints

Spacecraft double integrator (convex)

$$x_{k+1} = \begin{pmatrix} I & \Delta t I \\ 0 & I \end{pmatrix} x_k + \begin{pmatrix} \Delta t^2 I \\ \Delta t I \end{pmatrix} u_k$$

$x_{k+1} = A x_k + B u_k$

Convexity of practical constraints
Dynamics constraints

Spacecraft double integrator (convex)

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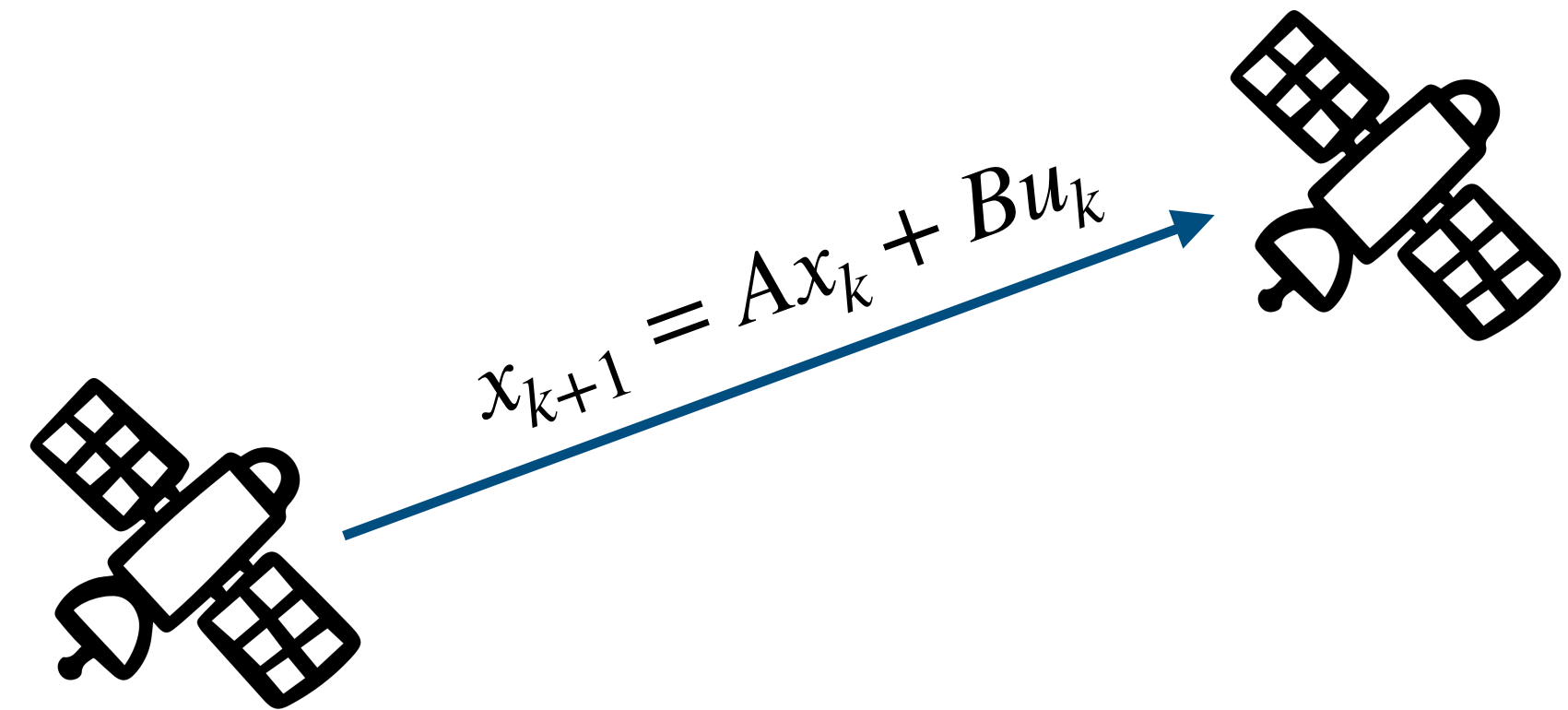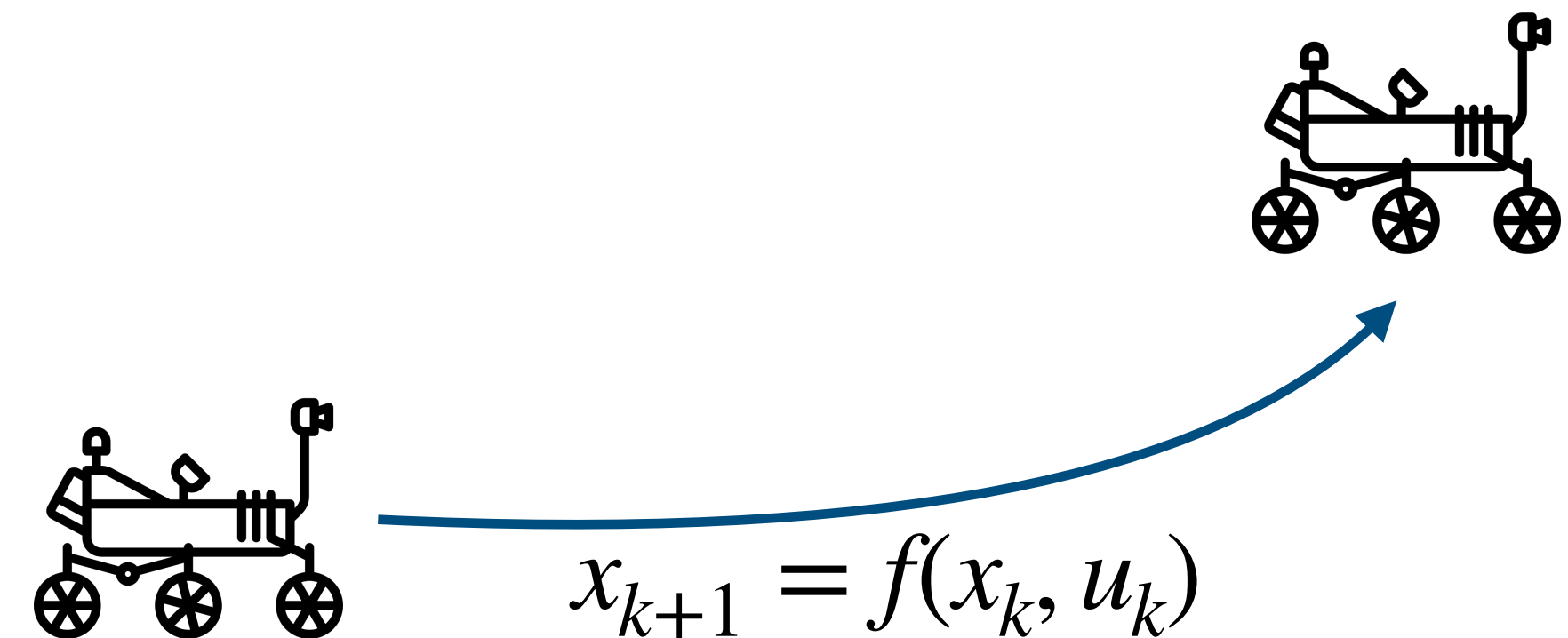
$$x_{k+1} = A x_k + B u_k$$

Mars rover (non-convex)

$$x_{k+1} = x_k + \Delta t \begin{pmatrix} v_k \cos \theta_k \\ v_k \sin \theta_k \\ \omega_k \end{pmatrix}$$

$$x_{k+1} = f(x_k, u_k)$$

Optimal control as nonlinear programs

How to convert a trajectory generation problem into standard optimization form?

Solve for trajectory $(x_{0:N}, u_{0:N})$

$$\min_{x_{0:N}, u_{0:N}} g_T(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k)$$

subject to:
$$x_0 = x_{\text{init}}$$

$$x_{k+1} = f(x_k, u_k)$$

$$x_k \in \mathcal{X}_{\text{safe}}$$

$$u_k \in \mathcal{U}$$

Optimal control as nonlinear programs

How to convert a trajectory generation problem into standard optimization form?

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$$\min_{x_{0:N}, u_{0:N}} \; g_T(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k)$$

subject to:

$$x_0 = x_{\text{init}}$$

$$x_{k+1} = f(x_k, u_k)$$

$$x_k \in \mathcal{X}_{\text{safe}}$$

$$u_k \in \mathcal{U}$$

1. Convert to standard form with inequality constraints written as $g_i(x_k, u_k) \leq 0$ and equality constraints as $h_i(x_{k+1}, x_k, u_k) = 0$
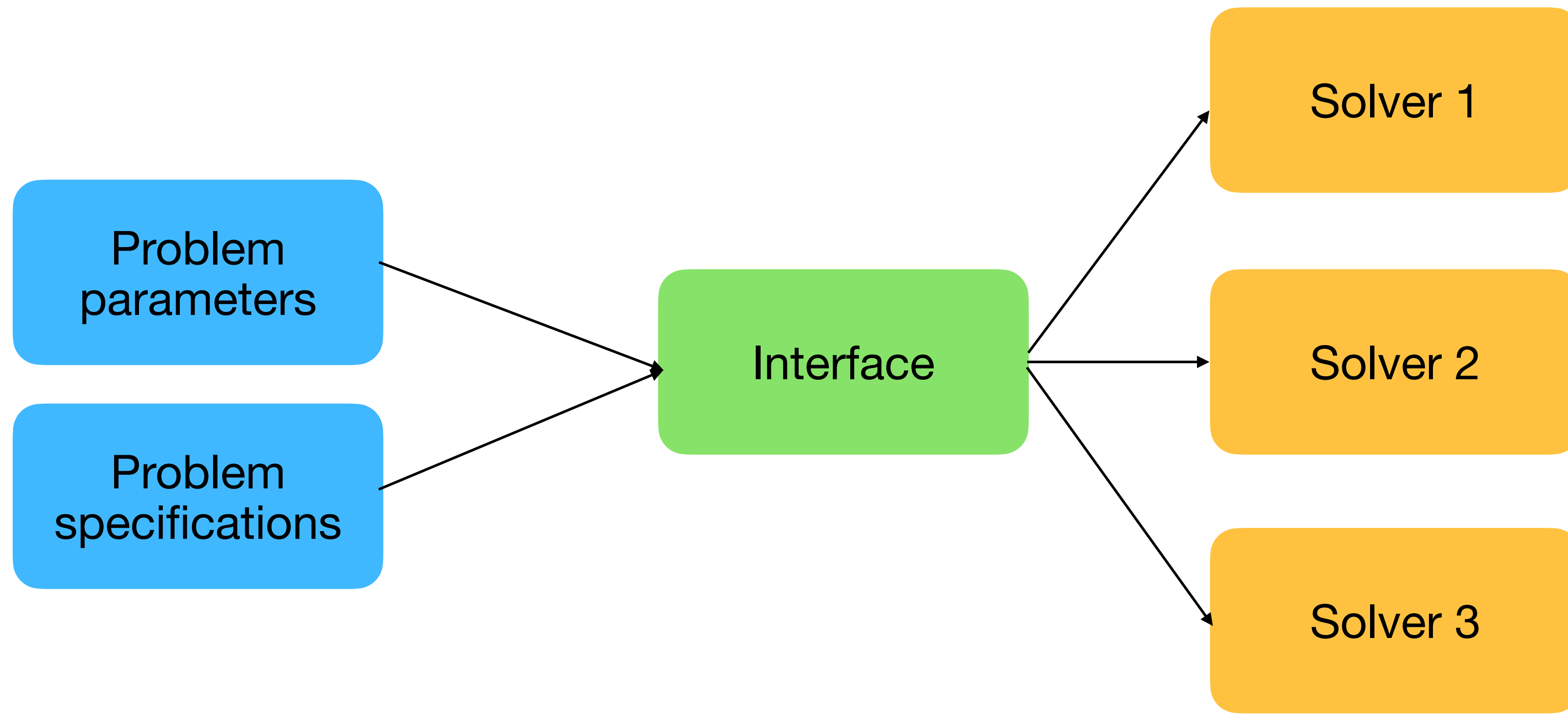
2. Provide gradient and Hessian for each constraint and cost

Converting an optimal control problem into standard form can be cumbersome!

Off-the-shelf trajectory optimization
Interface vs. solver

- An interface allows for solving the same optimization problem with different backend solvers

- Allows for rapid prototyping and testing

Off-the-shelf trajectory optimization
Interface vs. solver

```python
1   X = cp.Variable(shape=(nx, N+1))
2   U = cp.Variable(shape=(nu, N))
3
4   cost = 0.0
5   constraints = []
6
7   # Initial condition for system
8   constraints += [X[:,0] == x_init]
9
10  for cp_idx in range(N):
11      # Add stage-wise cost
12      cost += cp.quad_form(X[:, cp_idx+1] - x_goal, Q) + cp.quad_form(U[:, cp_idx], R)
13
14      # Dynamics constraint
15      constraints += [X[:, cp_idx+1] == Ak @ X[:,cp_idx] + Bk @ U[:,cp_idx]]
16
17      # State upper and lower bounds
18      constraints += [X[:, cp_idx+1] <= x_max]
19      constraints += [X[:, cp_idx+1] >= x_min]
20
21      # Control upper and lower bounds
22      constraints += [U[:, cp_idx] <= u_max]
23      constraints += [U[:, cp_idx] >= u_min]
24
25  prob = cp.Problem(cp.Minimize(cost), constraints)
26  prob.solve()
27  print(prob.value)
```

4677.378851800433

Off-the-shelf trajectory optimization
Engineering considerations

- Linear algebra routines

- Problem scaling

- Presolve techniques

- Infeasibility detection

- Parameter tuning

Solver interfaces
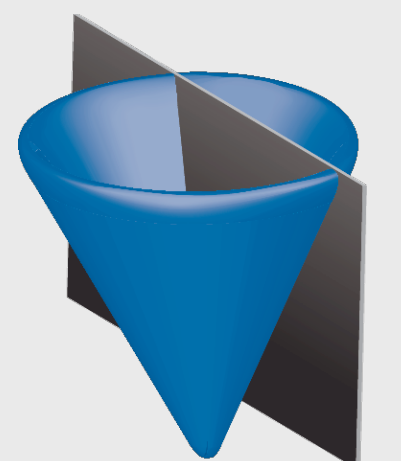
Convex solvers

Linear programs: GLPK, SCIP

Quadratic programs: OSQP, HPIPM, QPOASES

Second-order cone programs: SCS, ECOS, Clarabel

Semidefinite programs: SEDUMI, SDPT3

General nonlinear programs: IPOPT, SNOPT, KNITRO

Mixed integer linear programs: CPLEX, KNITRO, MOSEK, Gurobi

Mixed integer quadratic programs: MOSEK, Gurobi

Mixed integer nonlinear programming

NLLS solvers
Popular for SLAM

- Many problems in SLAM yield nonlinear least squares (NLLS) problems

$$\min_{x} \|f(x) - y\|_2^2$$

- Constraints are managed by "penalizing" violations

- Weaker guarantees and recent work in SLAM has sought to connect convex optimization with such problems

- Solvers: g2o, GTSAM, ceres, symforce, theseus

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