Lecture 2 - 2023-08-28

Recap: GNC vs. sense-plan-act

Notation for vectors

Today: Propagating diff. eq. (i.e., trajectories.

$x \in \mathbb{R}^{n_x}$: state

$u \in \mathbb{R}^{n_u}$: control

LTI vs. LTV, vs. nonlinear

time invariant — time varying

↳ continuous $\dot{x} = Ax + Bu$ $\quad \dot{x} = A(t)x + B(t)u$

discrete $x_{k+1} = A_k x_k + B_k u_k$

$$x_{k+1} = Ax_k + Bu_k$$

converting LTI from continuous → discrete

$$\dot{x} = Ax + Bu \to \text{duration } \Delta t$$

$$\dot{x} \approx \frac{x_{k+1} - x_k}{\Delta t} = Ax_k + Bu_k \to x_{k+1} = (I + \Delta t A)x_k + (\Delta t B)\, u_k$$

Δt ← reasonably small

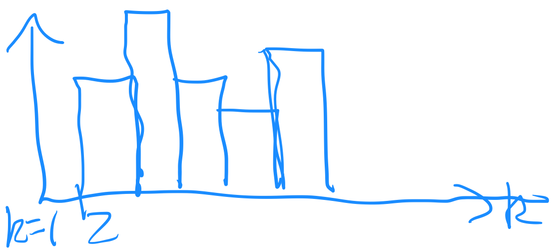
$$I = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \to Ix_k = x_k$$

$$x_{k+1} - x_k = \Delta t(Ax_k + Bu_k) = \Delta t A x_k + \Delta t B u_k$$

$$\to x_{k+1} = (I + \Delta t A)x_k + \Delta t B \underline{u_k}$$

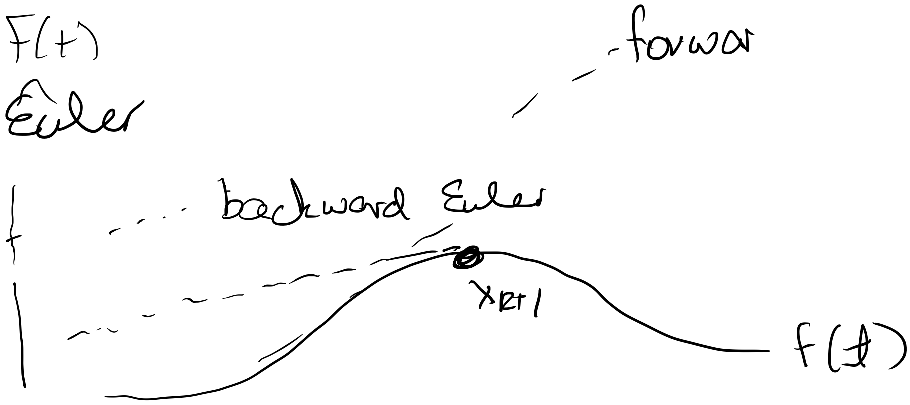
zero-order hold

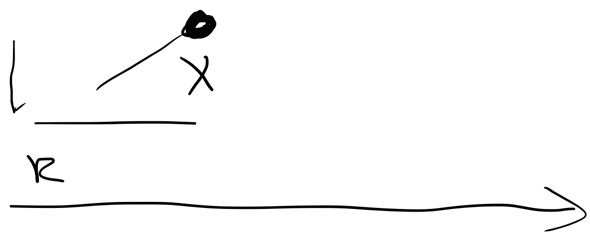
u



Can we do better for linear systems?

$x_{k+1} = (I + \Delta t\, A)x_k + \Delta t B u_k$ forward Euler





backward Euler: $\dot{x} \approx \dfrac{x_{k+1} - x_k}{\Delta t} = A x_{k+1} + B u_{k+1}$

evaluating at k+1

$\rightarrow (I - \Delta t A) x_{k+1} = x_k + \Delta t B u_{k+1}$ (backward Euler)

are there better ways? → Runge-Kutta (RK4)

Trapezoidal.

Hermite-Simpson.

Verlet (?) Jacobi(?)

Newton(?)

→ pseudo spectral methods

(TODO(ac))

analytical solutions via exponential matrix:

$$\dot{x} = Ax + Bu \longleftrightarrow \dot{x} = ax + bu$$

$x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_x}$ $\quad x \in \mathbb{R}, u \in \mathbb{R}$

$\dot{x} = ax$ $\leftarrow$ how to solve this?

re-writing as characteristic equation:

$\dot{x} - ax = 0$

$s - a =$

$\rightarrow x(t) = x(t_0)\, e^{at}$

$\dot{x} = ax + bu \rightarrow \cancel{\dot{x}} - ax = bu$

$x(t)$ $\quad u(t)$

$\dot{x} - ax = bu$

$\Big\downarrow$ multiply by e^{-at}

$e^{-at}(\dot{x} - ax) = e^{-at}\dot{x} - a e^{-at} x = e^{-at} bu$

$$\frac{d}{dt}\left[x(t)\, e^{-at}\right]$$

$$\frac{d}{dt}[ab] = \dot{a}b + a\dot{b}$$

$$\int_{T_0}^{T_f} \frac{d}{dt}\left[x(t)\, e^{-at}\right] dt = \int_{T_0}^{T_f} e^{-at}\, bu\, dt$$

$$\rightarrow x(t)\, e^{-at} \Big|_{T_0}^{T_f} = \int_{T_0}^{T_f} e^{-at}\, bu(t)\, dt$$

$$= x(T_f)\, e^{-aT_f} - x(T_0)\, e^{-aT_0} = \int_{T_0}^{T_f} e^{-at}\, bu\, dt$$

$$\downarrow \rightarrow x(T_f) = e^{-(T_f - T_0)a}\, x(T_0) + \int_{T_0}^{T_f} e^{-a(T_f - T_0)}\, bu\, dt$$

multiply by e^{aT_f} on both sides

$\dot{x} = Ax + Bu \leftarrow$ can we apply this when

$x \in \mathbb{R}^{n_x}$ and $u \in \mathbb{R}^{n_u}$?

1. solution takes exponential form
2. multiply by e^{-at} on L.H.S. and R.H.S.

exponential matrix:

$$e^{At} = \sum_{i=0}^{\infty} (A)^i \frac{t^i}{i!} \text{ here } A \in \mathbb{R}^{n \times n}$$

$$= A^0 \frac{t^0}{0!} + A^1 \frac{t^1}{1!} + A^2 \frac{t^2}{2!} + \dots$$

$$= I + At + A^2 \frac{t^2}{2!} + \dots$$

$$\frac{d}{dt} e^{At} = \frac{d}{dt}\left[\cancel{I} + At + A^2 \frac{t^2}{2!} + \dots\right]$$

$$= A + A^2 t + A^3 \frac{t^2}{2!} + \dots$$

$$= A\left(I + At + A^2 \frac{t^2}{2!} + \dots\right) = Ae^{At}$$

$$\underbrace{\hspace{6cm}}_{e^{At}}$$

$$\Rightarrow \frac{d}{dt} e^{At} = A e^{At} = e^{At} A$$

e^{at} vs. e^{At}

1. $e^{A(t_1+t_2)} = e^{At_1} e^{At_2}$

2. $e^{A0} = I$

3. $e^{A+B} \neq e^{A} e^{B}$

$\Downarrow$ apply e^{At} to LTI system

$$\dot{x} = Ax + Bu \rightarrow \dot{x} - Ax = Bu$$

left

→ multiply by e^{-At}

→ $e^{-At}\dot{x} - e^{-At}Ax = e^{-At}Bu$

$$\frac{d}{dt}\left[e^{-At}x(t)\right] = e^{-At}Bu$$

→ integrate: $\int_{T_0}^{T_f} \frac{d}{dt}\left[e^{-At}x(t)\right]dt = \int_{T_0}^{T_f} e^{-At}Bu\,dt$

→ $e^{-AT_f}x(T_f) - e^{-AT_0}x(T_0) = \int_{T_0}^{T_f} e^{-At}Bu\,dt$

→ $e^{-AT_f}x(T_f) = e^{-AT_0}x(T_0) + \int_{T_0}^{T_f} e^{-At}Bu\,dt$

left multiply by e^{AT_f}

$$\rightarrow x(T_f) = e^{A(T_f - T_0)} x(T_0) + \int_{T_0}^{T_f} e^{A(T_f - t)} Bu\,dt$$

swap $T_f \to t$ to be more useful
and $T_0 = 0$ & $t \to \tau$

$$\rightarrow x(t) = \underline{e^{At}} x(0) + \int_0^t e^{A(t-\tau)} Bu\,d\tau$$

state transition matrix $\rightarrow \underline{\Phi}(t, t_0)$

double integrator: $\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$

$\uparrow A \quad \uparrow B$

say: robot arm joint angles q

joint velocities $\dot{q}$

$$x = \begin{pmatrix} q \\ \dot{q} \end{pmatrix} \qquad \dot{x} = \begin{pmatrix} \dot{q} \\ \ddot{q} \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \dot{q} \\ \ddot{q} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ M\ddot{q} & 0 \end{pmatrix} \begin{pmatrix} q \\ \dot{q} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$r_k \qquad r_{k+1} = r_k + v_k \Delta t + \frac{1}{2} \Delta t^2 u_k$

$v_k \rightarrow v_{k+1} = v_k + v_k \Delta t$

NOTE: computing $e^{At} = I + At + A^2 \frac{t^2}{2!} + \ldots$

Cayley-Hamilton Theorem: if $A \in$

$\mathbb{R}^{n\times n}$,

then it also satisfies its characteristic equaton.

$$p_A(\lambda) = \det(\lambda I - A) = 0$$

e.g. $A = \begin{pmatrix} 3 & 4 \\ 5 & 8 \end{pmatrix} \to p_A(\lambda) = \det$ $(\lambda I - A)$

$\det|\lambda I - A| = \det \begin{vmatrix} \lambda-3 & -4 \\ -5 & \lambda-8 \end{vmatrix}$

$= (\lambda-3)(\lambda-8) - (-4)(-5)$

$= \lambda^2 - 11\lambda + 24 - 20 = \lambda^2 - 11\lambda + 4$

$\to A^2 - 11A + 4 = 0 \Rightarrow A^2 = 11A - 4$

$A^3 = AA^2 = A(11A-4) = 11A^2 - 4A =$

$117A - 44$

defining linearity:

$y = Ax \rightarrow y \in \mathbb{R}^m \quad x \in \mathbb{R}^n \quad A \in \mathbb{R}^{m \times n}$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & a_{22} & \vdots \\ \vdots & & \ddots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

a function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$

1. $f(x+y) = f(x) + f(y)$ $\forall x, y \in \mathbb{R}^n$

2. $f(\alpha x) = \alpha f(x) \; \forall \alpha \in \mathbb{R}, x$

$\in \mathbb{R}^n$

e.g. $y = Ax$ is linear

e.g., $y = Ax + b$ is linear?

$$\underbrace{\begin{pmatrix} y \\ 1 \end{pmatrix}}_{\bar{y}} = \underbrace{\begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix}}_{\bar{A}} \underbrace{\begin{pmatrix} x \\ 1 \end{pmatrix}}_{\bar{x}}$$

1. $\bar{y} = \bar{A}(\bar{x}_1 + \bar{x}_2) = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 + x_2 \\ 1+1 \end{pmatrix}$

$$= \begin{pmatrix} Ax_1 + b + Ax_2 + b \end{pmatrix} = \begin{pmatrix} Ax_1 + b \\ 1 \end{pmatrix} + \begin{pmatrix} Ax_2 + b \\ 1 \end{pmatrix}$$

$$= \bar{A}\bar{x}_1 + \bar{A} x_2$$

complexity: $\underline{Ax}$ where $A \in \mathbb{R}^{m \times n}$ $x \in \mathbb{R}^n$

pseudo code:

$$\begin{pmatrix} a_1 & — \\ & \vdots \\ a_m & — \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

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for i = 1 -> m
    y_i = 0
    for j = 1 -> n
        y_i += a_ij · x_j
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$\rightarrow \Theta(mn)$