HW 2. Mingzuo Sun

1. $\max\ u(C_{t,1}) + \beta u(C_{t+1,2})$ s.t $\begin{cases} C_{t,1} + a_t = w_t \\ N_t a_t = N_{t+1} k_{t+1} \\ C_{t+1,2} = a_t R_{t+1} \end{cases}$

$= \ln(C_{t,1}) + \beta \ln(C_{t+1,2})$

From the FOC with respective to $C_{t,1}$, we can easily get the Euler Equation:

$u'(C_{t,1}) = \beta R\, u'(C_{t+1,2})$ $u'(\cdot) = \frac{1}{\cdot}$. then

$\frac{C_{t+1,2}}{C_{t,1}} = \beta R$ From the market clear condition. we have

$\begin{cases} w_t = f(k_t) - f'(k_t)\cdot k_t = (1-\varepsilon)\cdot k_t^{\varepsilon} \\ r_{t+1} = f'(k_{t+1}) = \varepsilon \cdot k_t^{\varepsilon-1} \end{cases}$ From intertemporal budget constraint,

we can solve $C_{t,1}$ $C_{t,1} = \frac{w_t}{1+\beta}$, then $a_t = w_t - C_{t,1} = \frac{\beta w_t}{1+\beta}$

then $k_{t+1} = \frac{N_t a_t}{N_{t+1}} = \frac{a_t}{\frac{N_{t+1}}{N_t}} = \frac{a_t}{\Xi_{t+1}} = \frac{\beta w_t}{(1+\beta)\Xi_{t+1}}$

From $w_t = (1-\varepsilon) k_t^{\varepsilon}$, we derive the equation for the evolution capital per worker. as below:

$$k_{t+1} = \frac{\beta(1-\varepsilon) k_t^{\varepsilon}}{(1+\beta)\Xi_{t+1}}$$

(b). If $\Xi_t = \Xi \ \forall t$. then we have

$$\bar{k} = \bar{k}^{\varepsilon} \cdot \frac{\beta(1-\varepsilon)}{(1+\beta)\Xi} \Rightarrow \bar{k} = \left[\frac{\beta(1-\varepsilon)}{(1+\beta)\Xi}\right]^{\frac{1}{1-\varepsilon}}$$

(c) At new steady state, $\bar{k}' = \left[\frac{\beta(1-\varepsilon)}{(1+\beta)\hat{\Xi}}\right]^{\frac{1}{1-\varepsilon}}$, because $\hat{\Xi} > \Xi$

we have $\bar{k}' < \bar{k}$. This is easy to understand, at the very begin when Ξ turns into $\hat{\Xi}$ in $t+1$, the generation born in t have no expectancy of it and leave $a_t = \Xi \bar{k}$ to next period $t+1$. then $k_{t+1} = \frac{N_t a_t}{N_{t+1}} = \frac{\Xi \bar{k}}{\hat{\Xi}} < \bar{k}$. Even though in the later periods. k_t will increase for a higher return of capital. the level of k_t cannot come back to the original level.

(e). $X_t = C_{1,t} + \frac{C_{2,t}}{\Xi}$ is a good measure of consumption per person for the whole society. because it gives different weight to different generations. If $\Xi > 1$, the the young's population will be larger than the old thus they share a large proportion. For sustainable level of X. we should have

$\forall k_t$, $f(k_t) + k_t = X_t + k_{t+1}\Xi$ where $k_t = k_{t+1}$.

then $X_t = f(k_t) + k_t(1-\Xi)$

(7) From F.O.C we know $C_{t+1,2} = \beta R C_{t,1}$ for steady state, $C_{t+1,2} = C_{t,2}$ then

$X_t = C_{t,1} + \frac{\beta R C_{t,1}}{\Xi} = C_{t,1}\left[1 + \frac{\beta}{\Xi/R}\right]$ It is easy to prove that $\frac{\partial C_{t,1}}{\partial \Xi} < 0$ then if we can ~~prove~~ prove Ξ/R increases with Ξ, we can finish the whole proof. $\frac{\partial \frac{\Xi}{R}}{\partial \Xi} = \frac{1}{R} \cdot \left[1 - \frac{\partial R}{\partial \Xi} \cdot \frac{\Xi}{R}\right]$

then we need to prove $\left|\frac{\partial R}{\partial \Xi} \cdot \frac{\Xi}{R}\right| < 1$ From the fact $R = R(k) = R(k(\Xi))$. we have

$$\left|\frac{\partial R}{\partial \Xi} \cdot \frac{\Xi}{R}\right| = \left|\frac{\frac{\partial R}{R}}{\frac{\partial k}{k}} \cdot \frac{\frac{\partial k}{k}}{\frac{\partial \Xi}{\Xi}}\right| = \left|\frac{\frac{\partial R}{R}}{\frac{\partial k}{k}}\right| \cdot \left|\frac{\frac{\partial k}{k}}{\frac{\partial \Xi}{\Xi}}\right|$$ It is the product of two elasticities.

It is easy to know $\left|\frac{\frac{\partial k}{k}}{\frac{\partial \Xi}{\Xi}}\right| = \frac{1}{1-\varepsilon}$ then $\left|\frac{\frac{\partial R}{R}}{\frac{\partial k}{k}}\right| = \left|\frac{\frac{\partial r}{R}}{\frac{\partial k}{k}}\right| < \left|\frac{\frac{\partial r}{r}}{\frac{\partial k}{k}}\right| = 1-\varepsilon$

then $\left|\frac{\partial R}{\partial \Xi} \cdot \frac{\Xi}{R}\right| < (1-\varepsilon) \cdot \frac{1}{1-\varepsilon} = 1$. then $\frac{\partial \frac{\Xi}{R}}{\partial \Xi} > 0$. Ξ/R goes larger with Ξ

If $C_{t,1}$ decreases when Ξ increases, $1 + \frac{\beta}{\Xi/R}$ decreases when Ξ increases.

X_t decreases when Ξ increases. Q.E.D

~~steady state~~ associated with (CE). please see the end of the homework.

2. Let's first consider the case of constant per capita benefits ($Z_{2,t} = Z, \forall t$)

For generation ~~boa~~ born at $t-1$, ~~their~~ their budget constraint is given below

$$C_{1,t-1} = W_{t-1} - Z_{1,t-1} - a_{t-1}$$

$$C_{2,t} = a_{t-1} R_t + Z_{2,t}$$

For generation born at t, their budget constraint is:

$$C_{1,t} = W_t - Z_{1,t} - a_t$$

$$C_{2,t+1} = a_t R_{t+1} + Z_{2,t+1}$$

For generation born in t+1, their budget constraint is:

$$C_{1,t+1} = W_{t+1} - Z_{1,t+1} - a_{t+1}$$

$$C_{2,t+2} = a_{t+1} R_{t+2} + Z_{2,t+2}$$

Let's assume that agents have logarithmic utility, and the economy is a small open economy with no cost of the flow of labors and capitals, which means $R_t = R$ and $W_t = W$, $\forall t$. However, though the agents could work for other countries, they are still in their own country's pension system. With above assumption and the equation $N_t Z_{1,t} = N_{t-1} \cdot Z_{2,t}$ we can easily solve the optimal problem:

$$C_{1,t-1} = \left(W - \frac{Z}{\Xi_{t-1}} + \frac{Z}{R} \right) \cdot \frac{1}{1+\beta} \qquad C_{2,t} = \beta R C_{1,t-1}$$

$$C_{1,t} = \left(W - \frac{Z}{\Xi_t} + \frac{Z}{R} \right) \cdot \frac{1}{1+\beta} \qquad C_{2,t+1} = \beta R C_{1,t}$$

$$C_{1,t+1} = \left(W - \frac{Z}{\Xi_{t+1}} + \frac{Z}{R} \right) \frac{1}{1+\beta} \qquad C_{2,t+2} = \beta R C_{1,t+1}$$

We denote $\Xi_t = \frac{N_t}{N_{t-1}}$ here. So $\Xi_t > \Xi_{t-1} > \Xi_{t+1}$

then $C_{1,t} > C_{1,t-1} > C_{1,t+1}$, $C_{2,t+1} > C_{2,t} > C_{2,t+2}$

Let's denote the welfare of each generation born in t,

W_t then, $W_t > W_{t-1} > W_{t+1}$.

Though they share the same benefits in their old age, their tax burden in their young age are different. Due to the baby boom, the t generation have the least burden and the $t+1$ generation have the heaviest tax burden.

Now, let's turn to the case of same tax per capita. $(\bar{z}_{1,t} = \bar{z}, \forall t)$

From the similar argument, we have:

$$C_{1,t-1} = \frac{\left(w - \bar{z} + \frac{\bar{z}\,\Xi_{t}}{R}\right)}{1+\beta}, \quad C_{2,t} = \beta R\, C_{1,t-1}$$

$$C_{1,t} = \frac{\left(w - \bar{z} + \frac{\bar{z}\,\Xi_{t+1}}{R}\right)}{1+\beta}, \quad C_{2,t+1} = \beta R\, C_{1,t}$$

$$C_{1,t+1} = \frac{\left(w - \bar{z} + \frac{\bar{z}\,\Xi_{t+2}}{R}\right)}{1+\beta}, \quad C_{2,t+2} = \beta R\, C_{1,t+1}$$

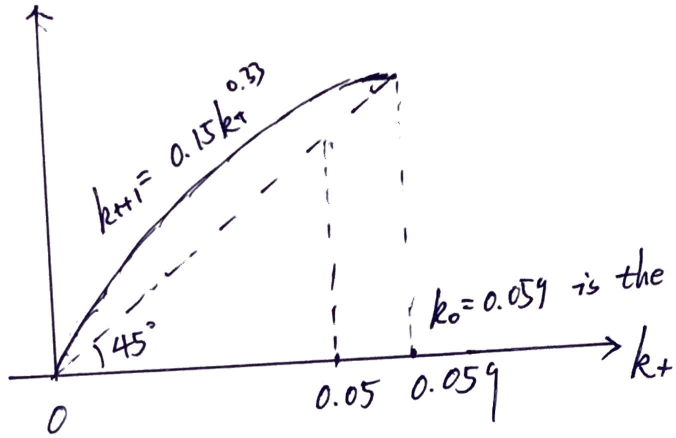
Now we have $C_{1,t-1} > C_{1,t+1} > C_{1,t}$, $C_{2,t} > C_{2,t+2} > C_{2,t+1}$

$W_{t-1} > W_{t+1} > W_t$. Though they share the same tax, their benefits are different. $t-1$ generation benefit most, and t generation benefit least.

(d). (i) If $\beta = 0.96^{30} = 0.294$. $\varepsilon = 0.33$. (I use ε instead of α). $\Xi = 1.01$

then $k_{t+1} = \dfrac{\beta(1-\varepsilon) k_t^{\varepsilon}}{(1+\beta)\Xi} = 0.15 k_t^{0.33}$

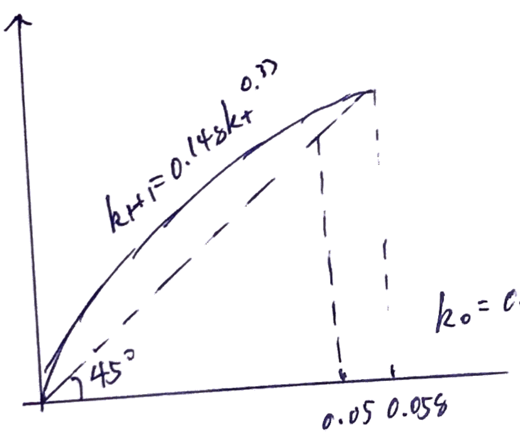
The $k_{t+1}(k_t)$ curve looks like below.



$k_0 = 0.059$ is the level of k in steady state.

(ii). If $\Xi = 1.03$. then $k_{t+1} = 0.148 k_t^{0.33}$

The new $k_{t+1}(k_t)$ curve looks like below



$k_0 = 0.058$ is the level of k in the new steady state.