1. Population growth and dynamic inefficiency in OLG

a)

By Euler's Theorem, $F(K,L) = F_L \cdot L + F_K \cdot K$

Then $f(k_t) = F(K_t, L_t)/L_t = F(K_t/L_t, 1)$

By perfect competition competition,

$$r_t = F_K(K_t, L_t) = L_t \cdot \frac{dF(K_t/L_t, 1)}{dK_t} = L_t \cdot \frac{dF(k_t,1)}{dk_t} \cdot \frac{d\,K_t/L_t}{dK_t}$$
$$= f'(k_t)$$

$$W_t = F_L(K_t, L_t) = \frac{1}{L} \cdot [F(K,L) - F_K \cdot K]$$
$$= f(k_t) - k_t \cdot f'(k_t)$$

Thus we get $\begin{cases} W_t = f(k_t) - k_t \cdot f'(k_t) & ① \\ r_t = f'(k_t) & ② \end{cases}$

Since we have a Cobb-Douglas aggregate production function $F(K,L) = K^{\varepsilon} \cdot L^{1-\varepsilon}$, then $f(k_t) = k_t^{\varepsilon}$.

plug into ①, ②, we get:

$$\begin{cases} W_t = k_t^{\varepsilon} - k_t \cdot \varepsilon \cdot k_t^{\varepsilon-1} = (1-\varepsilon) \cdot k_t^{\varepsilon} & ③ \\ r_t = \varepsilon \cdot k_t^{\varepsilon-1} \end{cases}$$

The individual's maximization problem yields:

$$u'(C_{1,t}) = \beta \cdot R_{t+1} \cdot u'(C_{2,t+1})$$

Assuming logarithmic utility: $u(c) = \log c$, we have:

$C_t = \frac{b_t + h_t}{1+\beta}$, $b_t = 0$, $h_t = W_{1,t} + \frac{1}{R_{t+1}} \cdot W_{2,t+1}$.

Since $W_{2,t+1} = 0$, we get: $C_t = \frac{W_{1,t}}{1+\beta}$

Then we have:

$$a_{1,t} = W_{1,t} - C_{1,t} = W_{1,t} - \frac{W_{1,t}}{1+\beta} = \frac{\beta}{1+\beta} W_{1,t}$$

plug ③ into the equation above:

$$a_{1,t} = \frac{\beta}{1+\beta} \cdot (1-\varepsilon) \cdot k_t^{\varepsilon} = k_t^{\varepsilon} \cdot \left[\frac{(1-\varepsilon) \cdot \beta}{1+\beta}\right]$$

$$k_{t+1} = \frac{L_t \cdot a_{1,t}}{L_{t+1}} = \frac{L_t \cdot a_{1,t}}{\Xi_{t+1} \cdot L_t} = \frac{a_{1,t}}{\Xi_{t+1}} = k_t^{\varepsilon} \cdot \left[\frac{(1-\varepsilon) \cdot \beta}{(1+\beta) \cdot \Xi_{t+1}} \right]$$

b)

When $\Xi_t = \Xi$ at any t, we have:

$$k_{t+1} = k_t^{\varepsilon} \cdot \left[\frac{(1-\varepsilon) \cdot \beta}{(1+\beta)\Xi} \right]$$

When reaching the steady state, we get:

$k_{t+1} = k_t = \bar{k}$, and thus:

$$\bar{k} = \bar{k}^{\varepsilon} \cdot \left[\frac{(1-\varepsilon) \cdot \beta}{(1+\beta)\Xi} \right] \Rightarrow \bar{k} = \left[\frac{(1-\varepsilon) \cdot \beta}{(1+\beta)\Xi} \right]^{\frac{1}{1-\varepsilon}}$$

c)

The new steady-state level of capital stock per capita is:

$$\hat{k} = \left[\frac{(1-\varepsilon) \cdot \beta}{(1+\beta)\hat{\Xi}} \right]^{\frac{1}{1-\varepsilon}}$$, since $\hat{\Xi} > \Xi$, we have:

$\hat{k} < \bar{k}$. Thus the new stead-state of k would be smaller than before.

d)

please find the graphs on jupyter notebook.

e)

$X_t = c_{1,t} + c_{2,t} / \Xi$. Since $c_{1,t} = \frac{C_{1,t}}{L_t}$, $c_{2,t} = \frac{C_{2,t}}{L_{t-1}}$,

$X_t = \frac{C_{1,t}}{L_t} + \frac{C_{2,t}}{L_{t-1} \cdot \Xi} \Rightarrow X_t = \frac{C_{1,t} + C_{2,t}}{L_t}$ this is appropriate because it uses a weighted average for two generation's consumptions to take into consideration growth of population.

We have $k_{t+1} = \frac{a_{1,t}}{\Xi} = \frac{w_{1,t} - c_{1,t}}{\Xi} = \frac{(1-\varepsilon) \cdot k_t^{\varepsilon} - c_{1,t}}{\Xi}$

For a given level of k_t, and to make k_t unchanged over time, we need:

$$k_{t+1} = k_t = \frac{(1-\varepsilon) \cdot k_t^{\varepsilon} - c_{1,t}}{\Xi} \Rightarrow c_{1,t} = (1-\varepsilon) \cdot k_t^{\varepsilon} - k_t \cdot \Xi \quad ①$$

Notice that:

$C_{2,t} = a_{1,t-1} \cdot R = (W_{1,t-1} - C_{1,t-1}) \cdot R$

Since k_t is unchanged over time, by ① we have: $C_{1,t-1} = C_{1,t}$

Then $C_{2,t} = (W_{1,t-1} - C_{1,t}) \cdot R$

$$= [(1-\varepsilon) \cdot k_{t-1}^{\varepsilon} - (1-\varepsilon) \cdot k_t^{\varepsilon} + k_t \cdot \Xi] \cdot R$$

$$= [(1-\varepsilon) \cdot k_t^{\varepsilon} - (1-\varepsilon) \cdot k_t^{\varepsilon} + k_t \cdot \Xi] \cdot R$$

$$= k_t \cdot \Xi \cdot R$$

Thus $X_t = C_{1,t} + \frac{C_{2,t}}{\Xi} = (1-\varepsilon) \cdot k_t^{\varepsilon} - k_t \cdot \Xi + \frac{k_t \cdot \Xi \cdot R}{\Xi}$

$$= (1-\varepsilon) \cdot k_t^{\varepsilon} + (R - \Xi) \cdot k_t$$

Let $k_t = \bar{k}$ for any t, we have:

$X = (1-\varepsilon) \cdot \bar{k}^{\varepsilon} + (R - \Xi) \cdot \bar{k}$. Because $R = 1 + r$, where $r = \varepsilon \cdot k^{\varepsilon - 1}$. plug in we get: $\boxed{X = \bar{k}^{\varepsilon} + (1 - \Xi)\bar{k}}$

f).

We have $X = \bar{k}^{\varepsilon} - \xi \cdot \bar{k}$, where $\xi = \Xi - 1$. Since $\bar{k} = \left[\frac{\beta \cdot (1-\varepsilon)}{(1+\beta)(1+\xi)}\right]^{\frac{1}{1-\varepsilon}}$

Then $X = \left[\frac{\beta(1-\varepsilon)}{(1+\beta)(1+\xi)}\right]^{\frac{\varepsilon}{1-\varepsilon}} - \xi \cdot \left[\frac{\beta(1-\varepsilon)}{(1+\beta)(1+\xi)}\right]^{\frac{1}{1-\varepsilon}}$

$= \left[\frac{\beta(1-\varepsilon)}{(1+\beta)}\right]^{\frac{\varepsilon}{1-\varepsilon}} \cdot (1+\xi)^{\frac{1-\varepsilon}{\varepsilon}} - \left[\frac{\beta(1-\varepsilon)}{(1+\beta)}\right]^{\frac{1}{1-\varepsilon}} \cdot \xi \cdot (1+\xi)^{\frac{1}{\varepsilon-1}}$

Thus $\frac{\partial X}{\partial \xi} = (1+\xi)^{\frac{1}{\varepsilon-1}} \cdot \left[A \cdot \frac{\varepsilon}{\varepsilon-1} - B - B \cdot g^{\frac{1}{\varepsilon-1}} \cdot (1+g)^{-1}\right]$, where

$A = \left[\frac{\beta(1-\varepsilon)}{1+\beta}\right]^{\frac{\varepsilon}{1-\varepsilon}}$, $B = \left[\frac{\beta(1-\varepsilon)}{1+\beta}\right]^{\frac{1}{1-\varepsilon}}$. If $\frac{\partial X}{\partial \xi} < 0$, need $A \cdot \frac{\varepsilon}{\varepsilon-1} - B - B \cdot g^{\frac{1}{\varepsilon-1}} \cdot (1+g)^{-1} < 0$ $\Rightarrow$ need $\frac{(A \cdot \frac{\varepsilon}{\varepsilon-1} - B) \cdot (\varepsilon - 1)}{B} > \frac{g}{1+g}$ ①. plug in A and B,

$[(A \cdot \frac{\varepsilon}{\varepsilon-1} - B)(\varepsilon-1)]/B = \varepsilon \cdot \left[\underbrace{\frac{1+\beta}{\beta \cdot (1-\varepsilon)}}_{>1} - 1\right] + 1 > 1$, $\frac{g}{1+g} < 1$, thus ① holds. ■

2.

Assume "baby boom" happens at time t, then we have:

$L_t = L_{t-1} \cdot \Xi_1$, $L_t = L_{t+1} \cdot \Xi_2$, $\Xi_1, \Xi_2 > 1$.

① assume per-capita taxes constant:

i. Consider the "baby boom" generation at time:

$\tilde{z}_t = z_{1,t} + z_{2,t+1} / R_{t+1}$

$z_{2,t+1} = \dfrac{Z_{2,t+1}}{L_t} = \dfrac{-Z_{1,t+1}}{L_t} = \dfrac{-Z_{1,t+1}}{L_{t+1}} \cdot \dfrac{L_{t+1}}{L_t}$

$= -z_{1,t+1} / \Xi_2 = -z_{1,t} / \Xi_2$

Thus $\tilde{z}_t = z_{1,t} - z_{1,t} / (\Xi_2 \cdot R_{t+1}) = \dfrac{\Xi_2 \cdot R_{t+1} - 1}{\Xi_2 \cdot R_{t+1}} \cdot z_{1,t} = \left(1 - \dfrac{1}{\Xi_2 \cdot R_{t+1}}\right) \cdot z_{1,t}$

Since $\Xi_2 > 1$, $R_{t+1} > 1$, $z_{1,t} < 0$, $1 - \dfrac{1}{\Xi_2 \cdot R_{t+1}} > 1 - \dfrac{1}{R_{t+1}}$

Thus the "baby boom" generation is worse off by this amount.

ii. consider the generation before "baby boom", at time t-1:

$\tilde{z}_{t-1} = z_{1,t-1} + z_{2,t} / R_t$

$z_{2,t} = \dfrac{Z_{2,t}}{L_{t-1}} = \dfrac{-Z_{1,t}}{L_t} \cdot \dfrac{L_t}{L_{t-1}} = -z_{1,t} \cdot \Xi_1 = -z_{1,t-1} \cdot \Xi_1$

$\tilde{z}_{t-1} = z_{1,t-1} - z_{1,t-1} \cdot \Xi_1 / R_t = \dfrac{R_t - \Xi_1}{R_t} \cdot z_{1,t-1} = \left(1 - \dfrac{\Xi_1}{R_t}\right) z_{1,t-1}$

Since $\Xi_1 > 1$, $1 - \dfrac{\Xi_1}{R_t} < 1 - \dfrac{1}{R_t}$. Thus the generation at time t-1 is better off.

iii. consider the generation after "baby boom", at time t+1:

$\tilde{z}_{t+1} = z_{1,t+1} + z_{2,t+2} / R_{t+2}$

$z_{2,t+2} = \dfrac{Z_{2,t+2}}{L_{t+1}} = \dfrac{-Z_{1,t+2}}{L_{t+2}} \cdot \dfrac{L_{t+2}}{L_{t+1}} = -z_{1,t+2} \cdot \dfrac{L_{t+2}}{L_{t+1}} = -z_{1,t+1} \cdot \dfrac{L_{t+2}}{L_{t+1}}$

Thus $\tilde{z}_{t+1} = z_{1,t+1} - z_{1,t+1} \cdot \dfrac{L_{t+2}}{L_{t+1}} \cdot \dfrac{1}{R_{t+2}}$

$= \left(1 - \dfrac{L_{t+2}}{L_{t+1} \cdot R_{t+2}}\right) \cdot z_{1,t+1} \Rightarrow$ unchanged by the "baby boom" generation.

② assume per-capita benefits constant.

i. consider the "baby boom" generation at time t:

$\tilde{z}_t = z_{1,t} + z_{2,t+1} / R_{t+1}$

$z_{2,t+1} = z_{2,t} = \dfrac{Z_{2,t}}{L_{t-1}} = \dfrac{-Z_{1,t}}{L_{t-1}} = \dfrac{-Z_{1,t}}{L_t} \cdot \dfrac{L_t}{L_{t-1}}$

$= -z_{1,t} \cdot \Xi_1$

Thus $\tilde{z}_t = z_{1,t} - \dfrac{z_{1,t} \cdot \Xi_1}{R_{t+1}} = \dfrac{R_{t+1} - \Xi_1}{R_{t+1}} \cdot z_{1,t} = \left(1 - \dfrac{\Xi_1}{R_{t+1}}\right) \cdot z_{1,t}$

Since $\text{日}_1 > 1$, $1 - \frac{\text{日}_1}{R_{t+1}} < 1 - \frac{1}{R_{t+1}}$, then the "baby boom" generation at time t is better off.

ii. consider the generation before "baby boom", at time $t-1$:

$\tilde{z}_{t-1} = z_{1,t-1} + z_{2,t} / R_t$

$z_{2,t} = z_{2,t-1} = \frac{Z_{2,t-1}}{L_{t-2}} = \frac{-Z_{1,t-1}}{L_{t-1}} \cdot \frac{L_{t-1}}{L_{t-2}} = -Z_{1,t-1} \cdot \frac{L_{t-1}}{L_{t-2}}$

Then $\tilde{z}_{t-1} = z_{1,t-1} - z_{1,t-1} \cdot \frac{L_{t-1}}{L_{t-2}} \cdot \frac{1}{R_t}$

$= z_{1,t-1} \cdot \left(1 - \frac{L_{t-1}}{L_{t-2} \cdot R_t}\right) \Rightarrow$ has nothing to do with the "baby boom" generation.

iii. Consider the generation after "baby boom", at time $t+1$:

$\tilde{z}_{t+1} = z_{1,t+1} + z_{2,t+2} / R_{t+2}$

$z_{2,t+2} = z_{2,t+1} = \frac{Z_{2,t+1}}{L_t} = \frac{-Z_{1,t+1}}{L_{t+1}} \cdot \frac{L_{t+1}}{L_t} = -z_{1,t+1} \cdot \frac{1}{\text{日}_2}$

Thus $\tilde{z}_{t+1} = z_{1,t+1} - z_{1,t+1} / (R_{t+2} \cdot \text{日}_2)$

$= \frac{R_{t+2} \cdot \text{日}_2 - 1}{R_{t+2} \cdot \text{日}_2} \cdot z_{1,t+1} = \left(1 - \frac{1}{R_{t+2} \cdot \text{日}_2}\right) \cdot z_{1,t+1}$

Since $R_{t+2} > 1$, $\text{日}_2 > 1$, $1 - \frac{1}{R_{t+2} \cdot \text{日}_2} > \left(1 - \frac{1}{R_{t+2}}\right) \cdot z_{1,t+1}$

Thus the generation after "baby boom" is worse off.