Macro Problem Set 2

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1. Population Growth and Dynamic Inefficiency

$$u(c) = \log c$$

$$F(K, L) = K^{\varepsilon} L^{1-\varepsilon}$$

$$f(k) = k^{\varepsilon}, \quad \text{where } k = \frac{K}{L}.$$

The wage $w = (1-\varepsilon) k^{\varepsilon}$

The rent for capital

$$r = \varepsilon k^{\varepsilon - 1}$$

The consumer's problem:

$$V_t = \log(C_{t,1}) + \beta \log(C_{t+1,2})$$

s.t.

$$C_{t+1,2} = (W_{t,1} - C_{t,1}) R_{t+1}$$

The Euler's Equation:

$$\frac{1}{C_{t,1}} = \beta R_{t+1} \frac{1}{C_{t+1,2}}$$

$$\Rightarrow \quad C_{t+1,2} = \beta R_{t+1} C_{t,1} = W_{t,1} R_{t+1} - C_{t,1} R_{t+1}$$

$$\Rightarrow \quad c_{t,1} = \frac{w_{t,1}}{1+\beta} =$$

$$a_{t,1} = w_{t,1} - c_{t,1}$$

$$= \frac{\beta w_{t,1}}{1+\beta}$$

$$K_{t+1} = L_t \, a_{t,1}$$

$$k_{t+1} = \frac{L_t}{L_{t+1}} a_{t,1} = \frac{\beta w_{t,1}}{\Xi_{t+1}(1+\beta)} = \frac{\beta(1-\varepsilon)k_t^{\varepsilon}}{\Xi_{t+1}(1+\beta)}$$

(b) If $\Xi_t = \Xi$ for all t,

$$k_{t+1}(k_t) = \frac{\beta(1-\varepsilon)k_t^{\varepsilon}}{\Xi(1+\beta)}$$

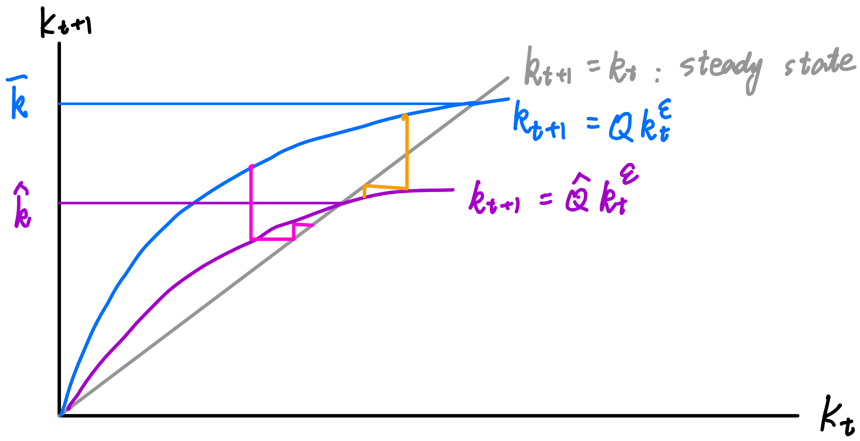
In steady state, $k_{t+1} = k_t = \bar{k}$

$$\bar{k} = \left[\frac{\beta(1-\varepsilon)}{\Xi(1+\beta)}\right]^{\frac{1}{1-\varepsilon}}$$

(c) Again denote $Q = \dfrac{\beta(1-\varepsilon)}{\Xi(1+\beta)}$

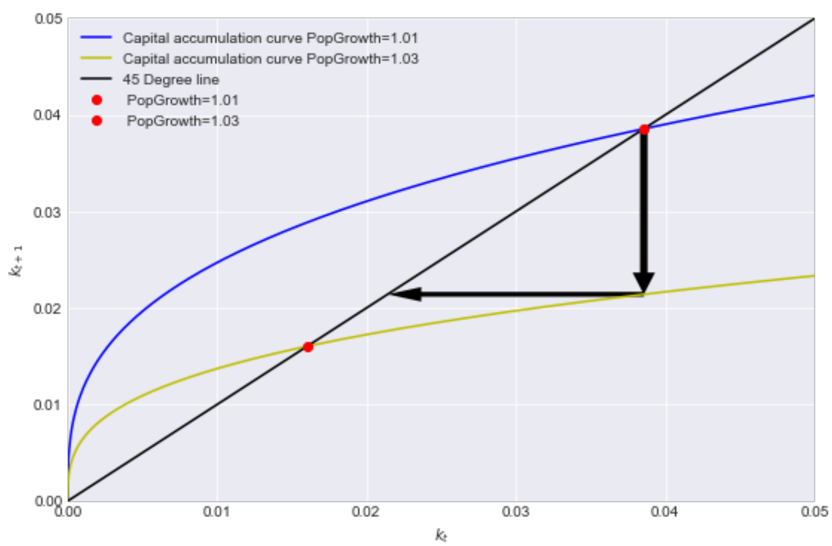
If $\hat{\Xi} > \Xi$, the corresponding $\hat{Q} < Q$

This will make $\hat{k}$ smaller than $\bar{k}$.



No matter $k_t > \hat{k}$ or $k_t < \hat{k}$, the steady state capital per worker will converge to $\hat{k}$.

(d)



Original code in separate file.

(e) ① $X_t = c_{1,t} + c_{2,t}/\Xi$. the average consumption per person is

$$\frac{C_{1,t} + C_{2,t}}{L_t + L_{t-1}}$$

$$= \frac{L_t c_{1,t} + L_{t-1} c_{2,t}}{L_{t-1}(1+\Xi)} = \frac{c_{1,t}\,\Xi + c_{2,t}}{1+\Xi}$$

$$= \frac{c_{1,t} + c_{2,t}/\Xi}{\Xi(1+\Xi)} = \frac{1}{\Xi(1+\Xi)} X_t$$

② Sustainable level of X should be such that $k_t = k_{t+1}$

$$k_{t+1} = \frac{a_{1,t}}{\Xi} = \frac{w_{1,t} - c_{1,t}}{\Xi} = \frac{(1-\varepsilon)k_t^{\varepsilon} - c_{1,t}}{\Xi};$$

If $k_{t+1} = k_t$, $\quad c_{1,t} = (1-\varepsilon)k_t^{\varepsilon} - \Xi k_t$

Since every generation faces the same budget constraint

and same utility function, $c_{2,t} = c_{2,t+1} = (w_{1,t} - c_{1,t})R$

$$= \Xi R k_t$$

$$X_t = c_{1,t} + c_{2,t}/\Xi$$

$$= (1-\varepsilon)k_t^{\varepsilon} + (R-\Xi)k_t$$

$$= k_t^{\varepsilon} + (1-\Xi)k_t$$

(f) In steady state, $k_t = \bar{k} = \left[\frac{(1-\varepsilon)\beta}{\Xi(1+\beta)} \right]^{\frac{1}{1-\varepsilon}} = \bar{k}(\Xi)$

$$\frac{\partial \bar{x}}{\partial \Xi} = -\bar{k} + \frac{\partial \bar{x}}{\partial \bar{k}} \cdot \underbrace{\frac{\partial \bar{k}}{\partial \Xi}}_{<0}$$

$\frac{\partial \bar{x}}{\partial \Xi}$ is positive only if $\frac{\partial \bar{x}}{\partial \bar{k}} < 0$, i.e. $f'(\bar{k}) - \xi < 0$ (*).

where $\xi = \Xi - 1$ = the population growth rate.

(*) means dynamically inefficient.

2. Pay-as-you-go vs. baby boom.

① constant per-capita tax.

$$z_{1,t} = z_{1,t+1} = z_1$$

$$Z_{2,t} = -z_{1,t} L_t.$$

$$z_{2,t} = -z_{1,t}\,\Xi_t$$

$$z_{2,t+1} = \frac{-z_{1,t+1} L_{t+1}}{L_t} = -z_{1,t+1}\,\Xi_{t+1} = -z_{1,t}\,\Xi_{t+1}$$

$$\bar{z}_t = z_{1,t}\left(1 - \Xi/R\right). = z_1\left(1 - \Xi_{t+1}/R\right).$$

Suppose that $\Xi_t = \begin{cases} \Xi > 1 & \text{if } t = \tau \\ 1/\Xi < 1 & \text{if } t = \tau+1 \\ 1 & \text{else.} \end{cases}$

which means there's a baby boom in time τ.

i.e. $L_{\tau-1} = L$

$L_\tau = L\Xi$

$L_{\tau+1} = L$

$$\bar{z}_{\tau-1} = z_1\left(1 - \Xi_\tau/R\right) < 0 : \text{ generation } \tau-1 \text{ is better off}$$

$$\bar{z}_\tau = z_1\left(1 - \Xi_{\tau+1}/R\right) > z_1\left(1 - 1/R\right) > 0.$$

generation τ is worse than other generations.

For $t \geq \tau+1$, $\bar{z}_t = z_1\left(1 - \frac{1}{R}\right)$

② constant per-capita benefit.

$$z_{2,t} = z_{2,t+1}$$

$$z_{2,t} = -z_{1,t}\,\Xi_t$$

So: $z_{2,t+1} = -z_{1,t}\,\Xi_t$.

$$\bar{z}_t = z_{1,t}\left(1 - \Xi_t/R\right)$$

Given the same baby boom; i.e. $\Xi_t = \begin{cases} \Xi > 1 & \text{if } t = \tau \\ 1/\Xi < 1 & \text{if } t = \tau+1 \\ 1 & \text{otherwise} \end{cases}$

$\bar{z}_\tau = z_{1,t}\left(1 - \Xi/R\right)$ better off.

$\bar{z}_{\tau+1} = z_{1,t}\left(1 - \frac{1}{R\Xi}\right)$ worse off.

other generations are the same.

Difference:

1. Under the fixed per-capita tax scheme,

the baby-boom generation paid the same but receives less when they are old, because the next generation has smaller population.

2. Under the fixed per-capita benefit scheme, the latter generation has to pay more for the baby-boom generation.