Problem Set 6: Solutions

Differential Equations

Fall 2024

1. (Reduction of Order or Integrating Method). Let a differential equation be:

$$y''(t) + \frac{2}{t}y'(t) = 0.$$

   (a) Verify that $y(t) = 1/t$ is one solution, then find a full set of solution.

   (b) Consider $\omega(t) = y'(t)$, solve the differential equation by using integrating factor.

   (c) Verify that the two methods give you the same set of the solutions.

Solution:

(a) The verification should be easy, we have the derivatives as:

$$\frac{d}{dt}\left[\frac{1}{t}\right] = -\frac{1}{t^2} \text{ and } \frac{d^2}{dt^2}\left[\frac{1}{t}\right] = \frac{2}{t^3},$$

and hence by plugging in, we have:

$$y'' + \frac{1}{t}y' = \frac{2}{t^3} + \frac{2}{t}\cdot\left(-\frac{1}{t^2}\right) = \frac{2}{t^3} - \frac{2}{t^3} = 0,$$

hence we have verified that $1/t$ is a solution. To obtain the other solution, we let $y_2 = u(t)/t$, and we take the derivatives as:

$$\left(\frac{1}{t}u(t)\right)'' + \frac{2}{t}\left(\frac{1}{t}u(t)\right)' = \frac{u''(t)}{t} - \frac{2u'(t)}{t^2} + \frac{2u(t)}{t^3} + \frac{2}{t}\left(\frac{u'(t)}{t} - \frac{u(t)}{t^2}\right) = 0.$$

Hence, we can reduce the ODE into:

$$t^2u''(t) - 2tu'(t) + 2u(t) + 2tu'(t) - 2u(t) = t^2u''(t) = 0 \implies u''(t) = 0,$$

therefore, the solution is $u(t) = at + b$, and by multiplying $1/t$, we have the set as $\boxed{\{1, 1/t\}}$.

(b) Consider $\omega(t) = y'(t)$, we have $\omega'(t) + \frac{2}{t}\omega(t) = 0$, so the integrating factor is:

$$\mu(t) = \exp\left(\int_0^1 \frac{2}{s}ds\right) = e^{2\ln|t|} = t^2,$$

and hence by multiplying it, we have:

$$t^2\omega'(t) + 2t\omega(t) = 0,$$

which can be solved as:

$$\frac{d}{dt}[t^2\omega(t)] = 0 \implies t^2\omega(t) = C \implies \omega(t) = Ct^{-2}.$$

Hence, by integrating ω, we have:

$$y(t) = \int \omega dt = \boxed{C_1t^{-1} + C_2}.$$

(c) The two methods give the same set of solutions, *and it is better that they do.*

2. (Complex Characteristics, Again). Find a full set of real solutions to the differential equation:

$$\frac{d^3y}{dx^3} = -y.$$

Solution:

Clearly, the characteristic equation is $r^3 = -1$. For this part, you will still have two options to proceed:

- By observing that -1 is a result, you may induct a long division of $(r^3+1)/(r+1)$, and factor as of how you factor quadratics, or
- by Euler's method's heuristics, namely finding the roots for $x^6 = 1$, that is $\zeta_6 = e^{2\pi \mathrm{i}/6} = e^{\pi \mathrm{i}/3}$, and take its odd powers, that is ζ_6^1, ζ_6^3, and ζ_6^5.

Whatever your choice is, you should obtain your three roots as:

$$r = -1, \frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}, \text{ and } \frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2},$$

hence inducing the solution set as:

$$\left\{e^{-t}, e^{(1/2+\mathrm{i}\sqrt{3}/2)t}, e^{(1/2-\mathrm{i}\sqrt{3}/2)t}\right\}.$$

By some simply arithmetics of linear combinations, we have:

$$\boxed{\left\{e^{-t}, e^{t/2}\cos\left(\frac{\sqrt{3}}{2}t\right), e^{t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)\right\}}.$$

*At this moment, readers should be utterly clear with Euler's identity and the method of transforming from complex-valued solutions to real-valued solution. If you are having trouble on this question, we suggest you to review on this part and check on **Question 3** in **Problem Set 5**.*

3. (Non-homogeneous Differential Equations). Solve the following differential equations.

(a)
$$y'' + 4y = t^2 + 3e^t.$$

(b)
$$y'' + 2y' + y = \frac{e^{-x}}{x}.$$

Solution:

(a) For the first part, we first find the solution to the homogeneous case, that is $y'' + 4y = 0$, whose characteristic equation is $r^2 + 4 = 0$, with roots $r = \pm 2\mathrm{i}$. Hence, the homogeneous solution is:

$$y = C_1 \cos(2t) + C_2 \sin(2t).$$

Based on the non-homogeneous part, our guess of the solution should be:

$$y_p(t) = \underbrace{At^2 + Bt + C}_{\text{Guess for } t^2} + \underbrace{De^t}_{\text{Guess for } 3e^t}.$$

Of course, readers can make separated guess since differentiation is linear operator, and solve for a, b, c and d separately. However, we will provide the whole derivatives as:

$$y_p' = 2At + B + De^t,$$
$$y_p'' = 2A + De^t.$$

Therefore, as we plug in the particular solution, we have:

$$(2A + De^t) + 4(At^2 + Bt + C + De^t) = t^2 + 3e^t,$$
$$4At^2 + Bt + (4C + 2A) + 5De^t = t^2 + 3e^t,$$

so the solutions are $A = 1/4$, $B = 0$, $C = -1/8$, and $D = 3/5$, so we have:

$$y(t) = \boxed{C_1 \cos(2t) + C_2 \sin(2t) + \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t}.$$

(b) Still, we first look for the homogeneous solution, for $y'' + 2y' + y = 0$, with characteristic equation as $r^2 + 2r + 1 = 0$, the roots is $r = -1$ with multiplicity 2, that is:

$$y = C_1 e^{-x} + C_2 x e^{-x}.$$

Here, we use the variation of parameter that we first take the Wronskian:

$$W[e^{-x}, xe^{-x}] = \det \begin{pmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{pmatrix} = -xe^{2x} + e^{-2x} + xe^{-2x} = e^{-2x}.$$

Therefore, we have the particular solution as:

$$\begin{aligned} y_p(x) &= -y_1(x) \int \frac{y_2(x) g(x)}{W} dx + y_2 \int \frac{y_1(x) g(x)}{W} dx \\ &= -e^{-x} \int \frac{xe^{-x}}{e^{-2x}} \frac{e^{-x}}{x} dx + xe^{-x} \int \frac{e^{-x}}{e^{-2x}} \frac{e^{-x}}{x} dx \\ &= -e^{-x} \int dx + xe^{-x} \int \frac{dx}{x} = -xe^{-x} + K_1 e^{-x} + K_2 x e^{-x} + xe^{-x} \log|x| \\ &= xe^{-x} \log|x|. \end{aligned}$$

Hence, the solution would be:

$$y(x) = \boxed{C_1 e^{-x} + C_2 x e^{-x} + xe^{-x} \log|x|}.$$

4. (Warm up in Linear Algebra). This problem reviews the basic concepts linear algebra concepts.

   (a) Which of the following set of vectors are linearly independent in $\mathbb{R}$-vector space, what about $\mathbb{C}$-vector space? Justify your answer.

   (i) $$\alpha = \{(1,1,0),(0,1,1),(1,0,1)\},$$

   (ii) $$\beta = \{(0,1),(2,3),(4,5)\},$$

   (iii) $$\gamma = \{1, \mathrm{i}\}.$$

   (b) Let $A = \begin{pmatrix} 1+\mathrm{i} & -1+2\mathrm{i} \\ 3+2\mathrm{i} & 2-\mathrm{i} \end{pmatrix}$ and $B = \begin{pmatrix} \mathrm{i} & 3 \\ 2 & -2\mathrm{i} \end{pmatrix}$, compute the following:

   (i) $$A - 2B,$$

   (ii) $$BA,$$

   (iii) $$B^{-1}.$$

Solution:

(a) For the first part, the $\mathbb{R}$ and $\mathbb{C}$-vector spaces should generally be the same:

i. Consider the determinant of vertically concatenating the vectors that:
$$\det \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 1+0+1-0-0-0 = 2 \neq 0,$$
hence it is $\boxed{\text{linearly independent}}$.

ii. Here, the vector space is $\mathbb{R}^2$ or $\mathbb{C}^2$, which has determinant 2, but since there are three vectors, it is $\boxed{\text{not linearly independent}}$.

iii. This case is interesting, consider the $\mathbb{R}$-vector space, for any $\lambda_1, \lambda_2 \in \mathbb{R}$, we have $\lambda_1 \cdot 1 + \lambda_2 \cdot \mathrm{i} = 0$ if and only if $\lambda_1 = \lambda_2 = 0$, so it is $\boxed{\text{linearly independent in } \mathbb{R}\text{-vector space}}$. Consider the $\mathbb{C}$-vector space, we have $1 \cdot 1 + \mathrm{i} \cdot \mathrm{i} = 1 - 1 = 0$, so it is $\boxed{\text{not linearly independent in } \mathbb{C}\text{-vector space}}$.

(b) For the second part, we do the computation on the matrix operations:

i. Consider $A - 2B$, we have:
$$\begin{aligned} A - 2B &= \begin{pmatrix} 1+\mathrm{i} & -1+2\mathrm{i} \\ 3+2\mathrm{i} & 2-\mathrm{i} \end{pmatrix} - 2\begin{pmatrix} \mathrm{i} & 3 \\ 2 & -2\mathrm{i} \end{pmatrix} \\ &= \begin{pmatrix} 1+\mathrm{i}-2\mathrm{i} & -1+2\mathrm{i}-6 \\ 3+2\mathrm{i}-4 & 2-\mathrm{i}+4\mathrm{i} \end{pmatrix} = \boxed{\begin{pmatrix} 1-\mathrm{i} & -7+2\mathrm{i} \\ -1+2\mathrm{i} & 2+3\mathrm{i} \end{pmatrix}}. \end{aligned}$$

ii. For BA, we do the matrix multiplication entry wise, that is:
$$BA = \begin{pmatrix} (1+\mathrm{i})\cdot\mathrm{i} + (-1+2\mathrm{i})\cdot 2 & \mathrm{i}(-1+2\mathrm{i}) + 3(2-\mathrm{i}) \\ (3+2\mathrm{i})\cdot\mathrm{i} + (2-\mathrm{i})\cdot 2 & (3+2\mathrm{i})\cdot 3 + (2-\mathrm{i})\cdot(-2\mathrm{i}) \end{pmatrix} = \boxed{\begin{pmatrix} 8+7\mathrm{i} & 4-4\mathrm{i} \\ 6-4\mathrm{i} & -4 \end{pmatrix}}.$$

iii. To find the inverse, we can use the formula that:
$$B^{-1} = \frac{1}{\det B}\begin{pmatrix} -2\mathrm{i} & -3 \\ -2 & \mathrm{i} \end{pmatrix} = \frac{1}{-4}\begin{pmatrix} -2\mathrm{i} & -3 \\ -2 & \mathrm{i} \end{pmatrix} = \boxed{\begin{pmatrix} \mathrm{i}/2 & 3/4 \\ 1/2 & -\mathrm{i}/4 \end{pmatrix}}.$$