Problem Set 1: Solutions

Differential Equations

Fall 2024

1. (Review: Integration.) As one of the most important skills of differential equations, the study requires proficiency in integration. By the *Fundamental Theorem of Calculus*, the basics of most calculations are on finding antiderivatives. Please evaluate the following indefinite integrals:

   (a) $$\int e^{1/x} \cdot \frac{1}{x^2} dx.$$

   (b) $$\int \sin(5x) e^{-x} dx.$$

   (c) $$\int \cos(2t) \tan(t) dt.$$

Solution:

(a) For the first integration, readers should observe that we may use integration by substitution (or *u-substitution*) with $u = 1/x$ where $du/dx = -1/x^2$, hence giving us:

$$\int e^{1/x} \cdot \frac{1}{x^2} dx = -\int e^u du = -e^u + C = \boxed{-e^{1/x} + C}.$$

(b) Here, we shall introduce a basic technique that you will see a lot over the course, we may integrate by parts first.

$$\begin{aligned}
\int \sin(5x) e^{-x} dx &= -\sin(5x) e^{-x} + \int 5\cos(5x) e^{-x} dx \\
&= -\sin(5x) e^{-x} + 5\left[-\cos(5x) e^{-x} - \int 5\sin(5x) e^{-x} dx\right] \\
26 \int \sin(5x) e^{-x} dx &= -\sin(5x) e^{-x} - 5\cos(5x) e^{-x} + C \\
\int \sin(5x) e^{-x} dx &= \boxed{-\frac{1}{26}\sin(5x) e^{-x} - \frac{5}{26}\cos(5x) e^{-x} + \tilde{C}}.
\end{aligned}$$

(c) Evaluating the last integral inhibits trigonometric identities, that is:

$$\begin{aligned}
\int \cos(2t) \tan(t) dt &= \int (2\cos^2(t) - 1) \frac{\sin(t)}{\cos(t)} dt = \int \big(2\sin(t)\cos(t) - \tan(t)\big) dt \\
&= \int \big(\sin(2t) - \tan(t)\big) dt = \boxed{-\frac{1}{2}\cos(2t) + \log|\cos t| + C}.
\end{aligned}$$

2. (Separable ODE.) Solve the following initial value problem (IVP) on $y = y(x)$, and specify the domain for your solution:

$$\begin{cases} y' = (x \log x)^{-1}, \\ y(e) = -6. \end{cases}$$

Solution:

Here, we notice that this problem is separable, hence we can write:

$$\begin{aligned} y' = \frac{dy}{dx} &= (x \log x)^{-1}, \\ dy &= \frac{1}{x \log x} dx, \\ \int dy &= \int \frac{1}{x \log x} dx. \end{aligned}$$

Now, we evaluate the integral by substitution, *i.e.*, $u = \log x$ and $du = dx/x$, which give that:

$$y = \int \frac{1}{u} du = \log |u| + C = \log |\log x| + C.$$

Eventually, we plug in the initial condition, that is $y(e) = -6$, giving us that:

$$\begin{aligned} -6 &= \log |\log e| + C, \\ C &= 6. \end{aligned}$$

Therefore, the solution is:

$$y = \boxed{\log |\log x| - 6}.$$

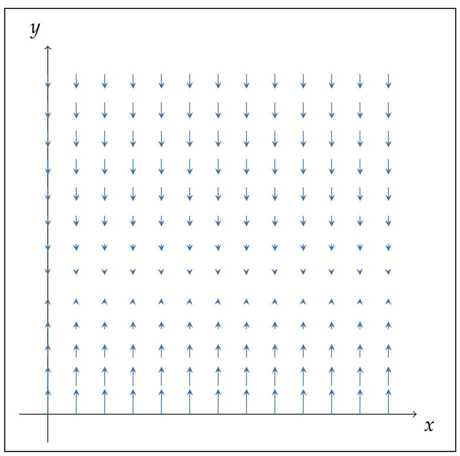
Here, we note that $\log(*)$ has a valid domain over positive numbers, and the double $\log(*)$ functions enforces that x must be greater than 1. Since our initial condition is e, and $e \in (1, \infty)$, the domain of the solution is $\boxed{(1, \infty)}$.

3. (Direction Field.) Let a differential equation be defined as follows:
$$\frac{dy}{dx} = y^3 - 7y^2 + 16y - 12 \text{ where } x \geq 0 \text{ and } y \geq 0.$$
   (a) Classify the above differential equation.
   (b) Sketch a direction field on the differential equation, then state the equilibriums of y, interpret their stability.

Solution:

(a) Note that we can rewrite the equation as:
$$F[y', y] = y' - y^3 + 7y^2 - 16y + 12 = 0,$$
and clearly it is $\boxed{\text{non-linearly}}$ (or you can explicitly show that $F[(y+1)', (y+1)] \neq 0$).
Note that the highest derivative is of degree 1, hence it is $\boxed{\text{first order}}$.

(b) Recall from Pre-Calculus (or Algebra) the following *Rational root test*:
Theorem 1: Rational Root Test. Let the polynomial:
$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0$$
have integer coefficients $a_i \in \mathbb{Z}$ and $a_0, a_n \neq 0$, then any rational root $r = p/q$ such that $p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$ satisfies that $p | a_0$ and $q | a_n$. ⌟
From the theorem, we can note that if the equation has a rational root, it must be one of:
$$r = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \text{ and } \pm 12.$$
By plugging in, one should notice that $y = 2$ is a root (one might also notice 3 is a root, but we will get the step slowly), so we can apply the long division (dividing $y - 2$) to obtain that:
$$\frac{y^3 - 7y^2 + 16y - 12}{y - 2} = y^2 - 5y + 6,$$
Clear, we notice that the right hand side is $(y-2)(y-3)$, so we now know that the roots (or equilibrium) are $\boxed{\text{2 (multiplicity 2) and 3}}$, and the direction field looks like:



Note that **Theorem 1** can also be generalized in ring theory (particularly, in UFDs), please check on it if you are interested in it. *Moreover, capable readers should attempt to prove that a polynomial of degree 3 with integer coefficients must have at least one rational root.*

4. (Constructing Solutions.) Let $x(t) = t^2 e^t$. Construct a second order ODE that has $x(t)$ as a solution and includes all of $x(t)$, $x'(t)$ and $x''(t)$, along with maybe some leftover stuff.
Hint: Take the first and second derivative of $x(t)$ and fit them together into some linear combinations.

Solution:
Here, we first take the derivatives as:

$$\begin{aligned} x(t) &= t^2 e^t, \\ x'(t) &= 2te^t + t^2 e^t, \\ x''(t) &= 2e^t + 4te^t. \end{aligned}$$

Here, we simply want to put the derivatives as linear combinations, *i.e.*, $f[t, x, x', x''] = 0$, in which one straightforward example could be:

$$\boxed{x(t) + x'(t) + x''(t) - t^2 e^t - (2te^t + t^2 e^t) - (2e^t + 4te^t + t^2 e^t) = 0}.$$

In general, you may have any non-zero linear combinations of $x(t)$, $x'(t)$, $x''(t)$, and a function of t.