Problem Set 3: Solutions

Differential Equations

Fall 2024

1. (Stability.) Draw the phase line and determine the stability of each equilibrium:

$$y' = y^2(y-1)(y-2).$$

Solution:

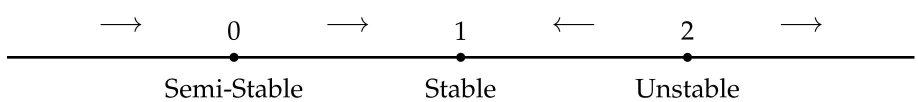
For this question, we can trivially note the roots as:

$$y = 0 \text{ with multiplicity } 2, y = 1, \text{ and } y = 2.$$

Sophisticated readers shall notice that this polynomial has a positive leading coefficient, hence it approaches $+\infty$ when $y \to \infty$, hence the arrows can be easily determined.

Otherwise, readers can plug in a value within each intervals, such as $y = 3$ for $y > 2$, $y = 3/2$ for $1 < y < 2$, etc., which should work equivalently.

Hence, we should expect a graph as follows:



The stability is obvious given the directions of the arrows.

2. (Ending Behavior.) Given an IVP as follows:
$$\begin{cases} y' + \frac{1}{2}y = \sin t; \\ y(0) = 1. \end{cases}$$
Find the specific solution of the equation and describe its end behavior

Solution:

(a) Here, we solve the IVP by first finding the general solution. Notice that we can apply the integrating factor method, that is:
$$\mu(t) = \exp\left(\int_0^t \frac{1}{2}ds\right) = \exp\left(\frac{t}{2}\right).$$
Then, by multiplying both sides of the differential equation with the integrating factor, we shall obtain the following (*details left as exercise to readers*):
$$e^{t/2}y = \int e^{t/2}\sin t dt.$$
Recall from **Problem set 1 Question 1 (ii)**, we have observed this anti-derivative case in prior, which requires two integration by parts to cancel out corresponding terms. In particular, we have:
$$\begin{aligned} \int e^{t/2}\sin t dt &= 2e^{t/2}\sin t - \int 2e^{t/2}\cos t dt \\ &= 2e^{t/2}\sin t - 2\left[2e^{t/2}\cos t - \int 2e^{t/2}(-\sin t)dt\right] \\ &= 2e^{t/2}\sin t - 4e^{t/2}\cos t - 4\int e^{t/2}\sin t dt. \\ 5\int e^{t/2}\sin t dt &= 2e^{t/2}\sin t - 4e^{t/2}\cos t + C. \\ \int e^{t/2}\sin t dt &= \frac{2}{5}e^{t/2}\sin t - \frac{4}{5}e^{t/2}\cos t + \tilde{C}. \end{aligned}$$
Hence, by dividing by sides by $e^{t/2}$, we obtain:
$$y = \frac{2}{5}\sin t - \frac{4}{5}\cos t + \tilde{C}e^{-t/2}.$$
Then, we use the initial condition that $y(0) = 1$ to get that:
$$1 = \frac{2}{5}\cdot 0 - \frac{4}{5}\cdot 1 + \tilde{C},$$
which enforces $\tilde{C} = 9/5$, and our solution is:
$$\boxed{y(t) = \frac{2}{5}\sin t - \frac{4}{5}\cos t + \frac{9}{5}e^{-t/2}}.$$

(b) Note that although $e^{-t/2}$ converges to 0 as $t \to \infty$. However, the sine and cosine functions has an oscillating behavior, *i.e.*, moving above and below 0 but not converging to it, hence the solution $\boxed{\text{oscillates around } y = 0}$.

3. (Bifurcation.) For the first-order autonomous ODE:

$$\frac{dx}{dt} = x^2 - 2x + c,$$

with parameter $c \in \mathbb{R}$, do the following:

(a) Sketch all of the qualitatively different graphs of $f(x) = x^2 - 2x + c$, as c is varied.

(b) Determine any and all bifurcation values for the parameter c.

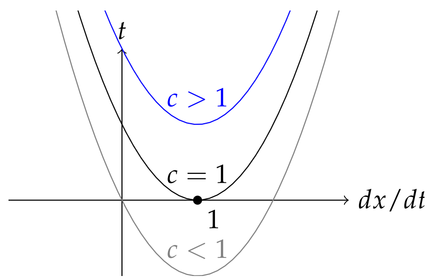
(c) Sketch a bifurcation diagram for this ODE.

Solutions:

(a) Given the right hand side, we want to find its critical point, *i.e.*, $x^2 - 2x + c = 0$, that is:

$$x = \frac{2 \pm \sqrt{4-4c}}{2} = 1 \pm \sqrt{1-c}.$$

Graphically, we may draw the diagram as:

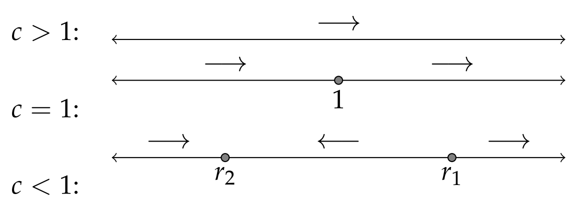


(b) Then, we find the bifurcation value, that is the critical points equivalent to each other, *i.e.*:

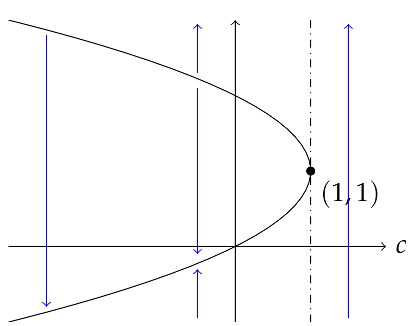
$$1 + \sqrt{1-c} = 1 - \sqrt{1-c}$$
$$c = \boxed{1}.$$

Diligent readers might also notice that this is the value c such that $\Delta = 0$.

(c) When $c > 1$, $dx/dt > 0$. When $c = 1$, the two roots are both 1. When $c < 1$, the $dx/dt > 0$ when larger than the larger root or smaller than the smaller roots, hence the phase diagrams are:



Thus, the bifurcation diagram is:



4. (Existence of Higher Order ODEs.) Determine intervals that the initial condition must be in so that the solutions are sure to exist:

   (a) $y^{(4)} + 4y''' + 3y = t$,

   (b) $y''' + ty'' + t^2y' + t^3y = \ln t$.

Solutions: This question concerns the existence and uniqueness for higher order ODEs, *i.e.*, we want to write them in the form of:

$$y^{(q)} + p_{q-1}(t)y^{(q-1)} + \cdots + p_1(t)y' + p_0(t)y + p(t) = 0.$$

Luckily, equation in this question is already given in this form, hence we want to evaluate them case by case:

(a) For this equation, we have:

$$\begin{aligned} p_3(t) &= 4, \\ p_2(t) &= 0, \\ p_1(t) &= 0, \\ p_0(t) &= 3, \\ p(t) &= t. \end{aligned}$$

We want all above to be continuous, which is valid for all $x \in \mathbb{R}$, so the interval would be $\boxed{\mathbb{R} = (-\infty, \infty)}$.

(b) For this equation, we have:

$$\begin{aligned} p_2(t) &= t, \\ p_1(t) &= t^2, \\ p_0(t) &= t^3, \\ p(t) &= \ln t. \end{aligned}$$

We want all above to be continuous, we must concern on $\ln t$, which is continuous only for $t > 0$, so the interval would be $\boxed{(0, \infty)}$.

Note that since we do not have an initial condition, the initial condition must be in the given interval so that existence is guaranteed for that interval.