PILOT

Problem Set 9: Solutions

Differential Equations

Fall 2024

1. (Repeated Eigenvalue). This problem investigates the case for repeated eigenvalues. First, we let the matrix $A \in \mathbb{R}^{2\times 2}$ be:
$$A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$
Here, we define the *algebraic multiplicity* of an eigenvalue as its multiplicity as a root to the characteristic polynomial, and the *geometric multiplicity* is the dimension of the eigenspace.

   (a) Find the eigenvalue and its corresponding eigenvector. State its algebraic and geometric multiplicity.

   (b) Find a the general solution to $\mathbf{x}' = A.\mathbf{x}$, where $\mathbf{x}' \in \mathbb{R}^2$.

   Then, we consider the diagonal n-by-n matrices, that is matrices with entries only on the diagonal, which can be characterized as:
$$D = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

   (c) Show that the eigenvalues are exactly $a_1, \cdots, a_n$, and the algebraic multiplicity is exactly the same as geometric multiplicity for all eigenvalues.

   (d) Consider the linear system $\mathbf{x} = D.\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^n$, solve for the general solution for $\mathbf{x} = (x_1, \cdots, x_n)$. Explain why do not have to find the eigenvalues in this case.

Solution:

(a) For the eigenvalue and eigenvector, we set:
$$0 = \det \begin{pmatrix} 2-\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = (2-\lambda)(-\lambda) + 1 = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2.$$
Hence, the eigenvalue is $\boxed{\text{1 with algebraic multiplicity 2}}$.

Then, we consider the eigenvector, that is $\boldsymbol{\xi}$ such that $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.\boldsymbol{\xi} = \mathbf{0}$, hence $\xi_1 + \xi_2 = 0$, so we have $\xi_2 = -\xi_1$, so the eigenvalue is $\boxed{(1,-1) \text{ with geometric multiplicity 1}}$.

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(b) To find the solution to the differential equations, we first obtain a solution as:

$$\mathbf{x}^{(1)}(t) = \boxed{C_1 e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}}.$$

However, for the linear system, we want another solution with the repeated roots. Here, we think of the second root by having the vector $\boldsymbol{\eta}$ such that $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} .\boldsymbol{\eta} = \boldsymbol{\xi}$, so we have:

$$\begin{cases} \eta_1 + \eta_2 = \xi_1 = 1, \\ -\eta_1 - \eta_2 = \xi_2 = -1. \end{cases}$$

This solves to $\eta_1 + \eta_2 = 1$, so we have $\boldsymbol{\eta} = (\eta_1, 1 - \eta_1)$, and here, we consider $(0, 1)$ since $(1, -1)$ is the eigenvector already, so the second solution is:

$$\mathbf{x}^{(2)}(t) = \boxed{C_2 \left(te^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)}.$$

(c) *Proof.* Note that as we subtract λ on the diagonal, we have:

$$D - \lambda\,\mathrm{Id} = \begin{pmatrix} a_1 - \lambda & 0 & \cdots & 0 \\ 0 & a_2 - \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n - \lambda \end{pmatrix}.$$

Here, by the recursive definition of determinant, we disregard the zeros in the first row and entry, we have:

$$\begin{aligned} 0 &= \det \begin{pmatrix} a_1 - \lambda & 0 & \cdots & 0 \\ 0 & a_2 - \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n - \lambda \end{pmatrix} = (a_1 - \lambda) \det \begin{pmatrix} a_2 - \lambda & 0 & \cdots & 0 \\ 0 & a_3 - \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n - \lambda \end{pmatrix} \\ &= (a_1 - \lambda)(a_2 - \lambda) \det \begin{pmatrix} a_3 - \lambda & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n - \lambda \end{pmatrix} = (a_1 - \lambda)(a_2 - \lambda) \cdots (a_n - \lambda). \end{aligned}$$

Therefore, we know that the eigenvalues are exactly the diagonal matrices, with algebraic multiplicity as how many times they appear. Now, when we shift to the geometric multiplicity, it becomes the *kernel* (or null space) for $D - a_i\,\mathrm{Id}$, which we have:

$$D - a_i\,\mathrm{Id} = \begin{pmatrix} a_1 - a_i & 0 & \cdots & 0 \\ 0 & a_2 - a_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n - a_i \end{pmatrix}.$$

Here, we consider the matrix as column vectors and only the columns with zero is increasing the dimension of the kernel by 1, so the dimension of the eigenspace is exactly the number of times that a_i appears, hence the geometric multiplicity is exactly the algebraic multiplicity. □

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(d) With a diagonal system, our linear system degenerates into:

$$\begin{cases} x_1' = a_1 x_1, \\ x_2' = a_2 x_2, \\ \vdots \\ x_n' = a_n x_n. \end{cases}$$

Hence, the solution is exactly:

$$\boxed{\begin{cases} x_1(t) = C_1 e^{a_1 t}, \\ x_2(t) = C_2 e^{a_2 t}, \\ \vdots \\ x_n(t) = C_n e^{a_n t}. \end{cases}}$$

Note that this is really a simple case of single first order linear differential equations. *We suggest diligent readers to also think about how to solve if this is a linear system.*

2. (Complex Eigenvalue and Phase Portraits). Find the general solution and sketch a few phase portraits for:

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} .\mathbf{x}.$$

Solution:

It is standard process that we find the eigenvalues and eigenvector of $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, that is:

$$0 = \det \begin{pmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 + 1.$$

Hence the eigenvalues are $\lambda = 1 \pm \mathrm{i}$. Note that for solving complex system, we simply solve with one complex eigenvalue for the conjugate pair. We pick $\lambda = 1 - \mathrm{i}$ here, so we want eigenvector $\boldsymbol{\xi}$ such that $\begin{pmatrix} \mathrm{i} & 1 \\ -1 & \mathrm{i} \end{pmatrix} .\boldsymbol{\xi} = \mathbf{0}$. There, we have $\xi_1 = \mathrm{i}\xi_2$, so we have $\boldsymbol{\xi} = (\mathrm{i}, 1)$.

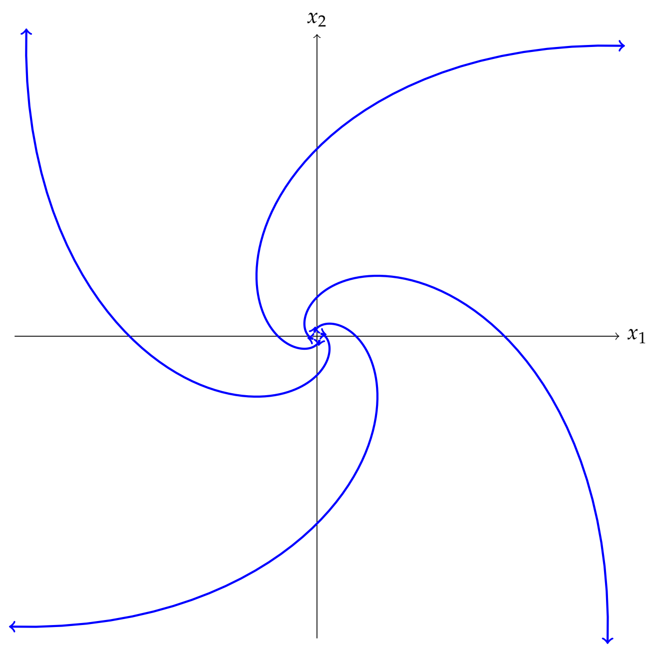
Therefore, we form a solution as:

$$\mathbf{x}(t) = e^{(1-\mathrm{i})t} \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix} = e^t(\cos t - \mathrm{i}\sin t) \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix} = e^t \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + \mathrm{i}e^t \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}.$$

Since the real and imaginary parts forms a linearly independent set, we have the solution as:

$$\mathbf{x}(t) = \boxed{C_1 e^t \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + C_2 e^t \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}}.$$

The phase portrait looks as follows:



3. (Stability). Complete the following table for stability of dimension 2 linear systems.

Solution:

Eigenvalues	**Type**	**Stability**
Eigenvalues are λ_1 and λ_2		
$0 < \lambda_1 < \lambda_2$	Node	Unstable
$\lambda_1 < \lambda_2 < 0$	Node	Asymptotically Stable
$\lambda_1 < 0 < \lambda_2$	Saddle Point	Unstable
$\lambda_1 = \lambda_2 > 0$	Node	Unstable
$\lambda_1 = \lambda_2 < 0$	Node	Asymptotically Stable
Eigenvalues are $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$		
$\alpha > 0$	Spiral Point	Unstable
$\alpha = 0$	Center	Stable
$\alpha < 0$	Spiral Point	Asymptotically Stable

We encourage diligent readers to sketch some *phase portraits* if possible for each case.

4. (Fundamental Matrix). Let a system be defined as:

$$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}.$$

Find the general solution using a fundamental matrix.

Solution:

We first solve for the eigenvalues and eigenvectors:

$$0 = \det \begin{pmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{pmatrix} = (3-\lambda)(-1-\lambda) + 8 = \lambda^2 - 2\lambda + 5.$$

Hence, the eigenvalues are $\lambda = 1 \pm 2\mathrm{i}$.

Since the two eigenvalues are complex conjugates, they will cancel out eventually. Then, we can look for the eigenvector for $\lambda_1 = 1 - 2i$, (you may also choose the other eigenvalue, this choice is discretionary to the reader), which is $\boldsymbol{\xi}^{(1)}$ such that $\begin{pmatrix} 2+2\mathrm{i} & -2 \\ 4 & -2+2\mathrm{i} \end{pmatrix} . \boldsymbol{\xi}^{(1)} = \mathbf{0}$, so we have $(2+2i)\xi_1^{(1)} = 2\xi_2^{(1)}$, so we the eigenvector is $\boldsymbol{\xi}^{(1)} = (1, 1+\mathrm{i})$. Therefore, we can get our solution as:

$$\begin{aligned} \mathbf{x} = e^{(1-2i)t} \begin{pmatrix} 1+\mathrm{i} \\ 1 \end{pmatrix} &= e^t \big(\cos(2t) - \mathrm{i} \sin(2t) \big) \begin{pmatrix} 1 \\ 1+\mathrm{i} \end{pmatrix} \\ &= e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + \mathrm{i} e^t \begin{pmatrix} -\sin 2t \\ \cos 2t - \sin 2t \end{pmatrix}. \end{aligned}$$

Notice that $\begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix}$ and $\begin{pmatrix} -\sin 2t \\ \cos 2t - \sin 2t \end{pmatrix}$ are linearly independent, it gives the fundamental matrix as:

$$\Psi = \boxed{e^t \begin{pmatrix} \cos 2t & -\sin 2t \\ \cos 2t + \sin 2t & \cos 2t - \sin 2t \end{pmatrix}}.$$