PILOT

Problem Set 1: Solutions

Differential Equations

Fall 2024

1. (Review: Integration.) As one of the most important skills of differential equations, the study requires proficiency in integration. By the *Fundamental Theorem of Calculus*, the basics of most calculations are on finding antiderivatives. Please evaluate the following indefinite integrals:

   (a) $$\int e^{1/x} \cdot \frac{1}{x^2} dx.$$

   (b) $$\int \sin(5x) e^{-x} dx.$$

   (c) $$\int \cos(2t)\tan(t) dt.$$

Solution:

(a) For the first integration, readers should observe that we may use integration by substitution (or *u-substitution*) with $u = 1/x$ where $du/dx = -1/x^2$, hence giving us:

$$\int e^{1/x} \cdot \frac{1}{x^2} dx = -\int e^u du = -e^u + C = \boxed{-e^{1/x} + C}.$$

(b) Here, we shall introduce a basic technique that you will see a lot over the course, we may integrate by parts first.

$$\begin{aligned} \int \sin(5x)e^{-x} dx &= -\sin(5x)e^{-x} + \int 5\cos(5x)e^{-x} dx \\ &= -\sin(5x)e^{-x} + 5\left[-\cos(5x)e^{-x} - \int 5\sin(5x)e^{-x} dx\right] \\ 26\int \sin(5x)e^{-x} dx &= -\sin(5x)e^{-x} - 5\cos(5x)e^{-x} + C \\ \int \sin(5x)e^{-x} dx &= \boxed{-\frac{1}{26}\sin(5x)e^{-x} - \frac{5}{26}\cos(5x)e^{-x} + \tilde{C}}. \end{aligned}$$

(c) Evaluating the last integral inhibits trigonometric identities, that is:

$$\begin{aligned} \int \cos(2t)\tan(t) dt &= \int (2\cos^2(t) - 1)\frac{\sin(t)}{\cos(t)} dt = \int \big(2\sin(t)\cos(t) - \tan(t)\big) dt \\ &= \int \big(\sin(2t) - \tan(t)\big) dt = \boxed{-\frac{1}{2}\cos(2t) + \log|\cos t| + C}. \end{aligned}$$

2. (Separable ODE.) Solve the following initial value problem (IVP) on $y = y(x)$, and specify the domain for your solution:

$$\begin{cases} y' = (x \log x)^{-1}, \\ y(e) = -6. \end{cases}$$

Note: Here $\log(x) := \log_e(x)$ is the natural logarithm function, which may be written as $\ln(x)$.

Solution:

Here, we notice that this problem is separable, hence we can write:

$$y' = \frac{dy}{dx} = (x \log x)^{-1},$$

$$dy = \frac{1}{x \log x} dx,$$

$$\int dy = \int \frac{1}{x \log x} dx.$$

Now, we evaluate the integral by substitution, *i.e.*, $u = \log x$ and $du = dx/x$, which give that:

$$y = \int \frac{1}{u} du = \log |u| + C = \log |\log x| + C.$$

Eventually, we plug in the initial condition, that is $y(e) = -6$, giving us that:

$$-6 = \log |\log e| + C,$$

$$C = -6.$$

Therefore, the solution is:

$$y = \boxed{\log |\log x| - 6}.$$

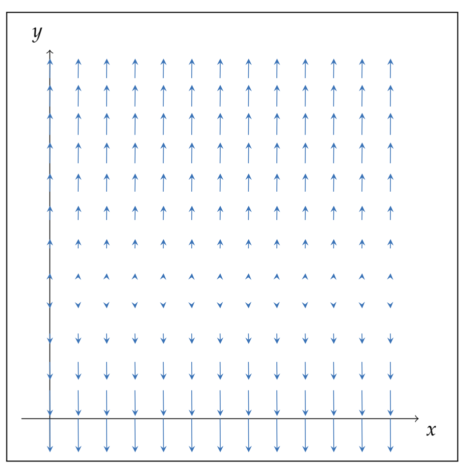
Here, we note that $\log(\bullet)$ has a valid domain over positive numbers, and the double $\log(\bullet)$ functions enforces that x must be greater than 1, as $\log(0)$ is undefined. Since our initial condition is e, and $e \in (1, \infty)$, the domain of the solution is $\boxed{(1, \infty)}$.

3. (Direction Field.) Let a differential equation be defined as follows:
$$\frac{dy}{dx} = y^3 - 7y^2 + 16y - 12 \text{ where } x \geq 0 \text{ and } y \geq 0.$$
   (a) Classify the above differential equation.
   (b) Sketch a direction field on the differential equation, then state the equilibriums of y, interpret their stability.

Solution:

(a) Note that we can rewrite the equation as:
$$F[y', y] = y' - y^3 + 7y^2 - 16y + 12 = 0,$$
and clearly it is $\boxed{\text{non-linearly}}$ (or you can explicitly show that $F[(y+1)', (y+1)] \neq 0$).
Note that the highest derivative is of degree 1, hence it is $\boxed{\text{first order}}$.

(b) Recall from Pre-Calculus (or Algebra) the following *Rational root test*:
Theorem 1: Rational Root Test. Let the polynomial:
$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0$$
have integer coefficients $a_i \in \mathbb{Z}$ and $a_0, a_n \neq 0$, then any rational root $r = p/q$ such that $p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$ satisfies that $p | a_0$ and $q | a_n$. ⌟
From the theorem, we can note that if the equation has a rational root, it must be one of:
$$r = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \text{ and } \pm 12.$$
By plugging in, one should notice that $y = 2$ is a root (one might also notice 3 is a root, but we will get the step slowly), so we can apply the long division (dividing $y - 2$) to obtain that:
$$\frac{y^3 - 7y^2 + 16y - 12}{y - 2} = y^2 - 5y + 6,$$
Clear, we notice that the right hand side is $(y-2)(y-3)$, so we now know that the roots (or equilibrium) are $\boxed{\text{2 (multiplicity 2) and 3}}$, and the direction field looks like:



Note that **Theorem 1** can also be generalized in ring theory (particularly, in UFDs), please check on it if you are interested in it. *Moreover, capable readers should attempt to prove that a polynomial of degree 3 with integer coefficients must have at least one rational root.*

4. (Constructing Solutions.) Let $x(t) = t^2 e^t$. Construct a second order ODE that has $x(t)$ as a solution and includes all of $x(t)$, $x'(t)$ and $x''(t)$, along with maybe some leftover stuff.
   Hint: Take the first and second derivative of $x(t)$ and fit them together into some linear combinations.

Solution:

Here, we first take the derivatives as:

$$\begin{aligned} x(t) &= t^2 e^t, \\ x'(t) &= 2te^t + t^2 e^t, \\ x''(t) &= 2e^t + 4te^t + t^2 e^t. \end{aligned}$$

Here, we simply want to put the derivatives as linear combinations, *i.e.*, $f[t, x, x', x''] = 0$, in which one straightforward example could be:

$$\boxed{x(t) + x'(t) + x''(t) - t^2 e^t - (2te^t + t^2 e^t) - (2e^t + 4te^t + t^2 e^t) = 0}.$$

In general, you may have any non-zero linear combinations of $x(t)$, $x'(t)$, $x''(t)$, and a function of t.

Problem Set 2: Solutions

Differential Equations

Fall 2024

1. (Linearity of Solutions.) Let $y = y_1(t)$ be a solution to $y' + p(t)y = 0$, and let $y = y_2(t)$ be a solution to $y' + p(t)y = q(t)$. Show that $y = y_1(t) + y_2(t)$ is then also a solution to $y' + p(t)y = q(t)$.

Proof:
Here, we note that $y_1(t)$ and $y_2(t)$ satisfies that:

$$\begin{cases} y_1'(t) + p(t)y_1(t) = 0, \\ y_2'(t) + p(t)y_2(t) = q(t). \end{cases}$$

Thus, by added both the left hand side and the right hand side, we obtain:

$$\begin{aligned} \big(y_1'(t) + p(t)y_1(t)\big) + \big(y_2'(t) + p(t)y_2(t)\big) &= 0 + q(t) \\ \big(y_1'(t) + y_2'(t)\big) + p(t)\big(y_1(t) + y_2(t)\big) &= q(t) \\ \big(y_1(t) + y_2(t)\big)' + p(t)\big(y_1(t) + y_2(t)\big) &= q(t), \end{aligned}$$

implying that $y = y_1(t) + y_2(t)$ is the solution to $y' + p(t)y = q(t)$, as desired. $\square$

2. (Integrating Factor.) Solve for the general solution to the following ODE with $y = y(t)$:

$$2y' + y = 3t.$$

Solution:
Here, we first convert the equation to standard form, *i.e.*:

$$y' + \frac{1}{2}y = \frac{3}{2}t.$$

Hence, with $p(t) = 1/2$, the integration factor must be:

$$\mu(t) = \exp\left(\int_0^t p(s)ds\right) = \exp\left(\int_0^t \frac{1}{2}ds\right) = \exp\left(\frac{1}{2}t\right).$$

Now, we multiply the integration factor on both sides, giving that:

$$\begin{aligned}
y'e^{t/2} + \frac{1}{2}ye^{t/2} &= \frac{3}{2}te^{t/2}, \\
\frac{d}{dt}\left[e^{t/2}y\right] &= \frac{3}{2}te^{t/2}, \\
e^{t/2}y &= \frac{3}{2}\int te^{t/2}dt \\
&= \frac{3}{2}\left[2te^{t/2} - \int 2e^{t/2}\right] \\
&= \frac{3}{2}\left[2te^{t/2} - 4e^{t/2} + C\right] \\
&= 3te^{t/2} - 6e^{t/2} + \tilde{C}, \\
y &= \boxed{\tilde{C}e^{-t/2} + 3t - 6}.
\end{aligned}$$

3. (Integrating Factor or Exactness?) Let a differential equation be defined as follows:
$$\frac{dy}{dx} = e^{2x} + y - 1.$$
   (a) What is the integrating factor ($\mu(x)$) for the equation? Solve for the general solution.
   (b) Is the equation *exact*? If not, make it exact, then find the general solution.
   (c) Do solutions from part (a) and (b) agree?

Solutions:

(a) First, we write the equation in standard form, that is:
$$y' - y = e^{2x} - 1.$$
Hence, with $p(x) = -1$, the integrating factor is:
$$\mu(x) = \exp\left(\int_0^x p(s)ds\right) = \exp\left(\int_0^x (-1)ds\right) = \boxed{\exp(-x)}.$$
Then, we multiply the integrating factors on both ends to obtain:
$$\begin{aligned} y'e^{-x} - ye^{-x} &= e^x - e^{-x}, \\ \frac{d}{dx}\left[ye^{-x}\right] &= e^x - e^{-x}, \\ ye^{-x} &= \int \left(e^x - e^{-x}\right) dx = e^x + e^{-x} + C, \\ y &= \boxed{Ce^x + e^{2x} + 1}. \end{aligned}$$

(b) Note that for exactness, we write the equation as:
$$\underbrace{(-e^{2x} - y + 1)}_{M(x,y)} + \underbrace{(1)}_{N(x,y)} \frac{dy}{dx} = 0,$$
meaning that their partial derivatives are, respectively:
$$\partial_y M(x,y) = -1 \text{ and } \partial_x N(x,y) = 0,$$
and since they are different, the equation is $\boxed{\text{not exact}}$.
Thus, we look for the integrating factor, *i.e.*:
$$\mu(t) = \exp\left(\int_0^x \frac{\partial_y M(x,y) - \partial_x N(x,y)}{N(x,y)}\right) = \exp\left(\int_0^x \frac{-1-0}{1} ds\right) = \exp(-x).$$
Now, we multiply e^{-x} on both sides, giving us that:
$$\underbrace{(-e^x - ye^{-x} + e^{-x})}_{\tilde{M}(x,y)} + \underbrace{(e^{-x})}_{\tilde{N}(x,y)} \frac{dy}{dx} = 0.$$
Now, the equation is exact. *We leave the check to the readers as an exercise.*
To get the solution, we first integrate $\tilde{M}(x,y)$ with respect to x, that is:
$$\varphi(x,y) = \int (-e^x - ye^{-x} + e^{-x})dx = -e^x + ye^{-x} - e^{-x} + h(y).$$
Now, taking the derivative with respect to y gives:
$$\partial_y \varphi(x,y) = e^{-x} + h'(y) = e^{-x},$$
which pushes $h(y)$ to be constant, hence we have solution:
$$\varphi(x,y) = \boxed{-e^x + ye^{-x} - e^{-x} = C}.$$

(c) The solutions $\boxed{\text{agree}}$ by simple arithmetic deductions.

4. (Decay and Dating.) Carbon-14, a radioactive isotope of carbon, is an effective tool in dating the age of organic compounds, as it decays with a relatively long period. Let $Q(t)$ denote the amount of carbon-14 at time t, we suppose that the decay of $Q(t)$ satisfies the following differential equation:

$$\frac{dQ}{dt} = -\lambda Q \text{ where } \lambda \text{ is the rate of decay constant.}$$

   (a) Let the half-life of carbon-14 be τ, find the rate of decay, λ.

   (b) Suppose that a piece of remain is discovered to have 10% of the original amount of carbon-14, find the age of the remain in terms of τ.

Solutions:

(a) Note that the differential equation is separable, hence:

$$\begin{aligned}\frac{dQ}{Q} &= -\lambda dt,\\ \int \frac{dQ}{Q} &= -\int \lambda dt,\\ \log|Q| &= -\lambda t + C,\\ Q &= \tilde{C}e^{-\lambda t}.\end{aligned}$$

Here, we assume $Q = Q_0$ at $t = t_0$, then we have $Q = Q_0/2$ when $t = t_0 + \tau$, so:

$$\frac{1}{2} = e^{-\lambda\tau},$$

which deduces to:

$$\lambda = -\frac{1}{\tau}\log\left(\frac{1}{2}\right) = \boxed{\frac{\log 2}{\tau}}.$$

(b) If there are only 10% of remain, we suppose that we have $Q = Q_0$ at $t = t_0$, and have $Q = Q_0/10$ at $t = t_0 + s$, hence giving that:

$$\frac{Q_0}{10} = Q_0 \exp\left(-\lambda t_0\right) = Q_0 \exp\left(-\frac{\log(2)t_0}{\tau}\right).$$

Thus, we obtain that:

$$\frac{1}{10} = \exp\left(-\frac{\log(2)t_0}{\tau}\right),$$

and by solving for t_0, we obtain:

$$t_0 = -\frac{\tau}{\log 2}\log\left(\frac{1}{10}\right) = \boxed{\frac{\log 10}{\log 2}\tau}.$$

Problem Set 3: Solutions

Differential Equations

Fall 2024

1. (Stability.) Draw the phase line and determine the stability of each equilibrium:

$$y' = y^2(y-1)(y-2).$$

Solution:

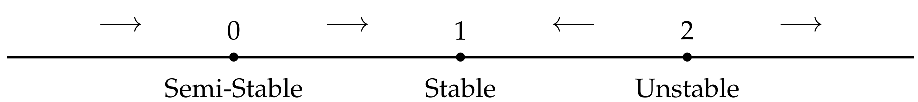
For this question, we can trivially note the roots as:

$$y = 0 \text{ with multiplicity } 2, y = 1, \text{ and } y = 2.$$

Sophisticated readers shall notice that this polynomial has a positive leading coefficient, hence it approaches $+\infty$ when $y \to \infty$, hence the arrows can be easily determined.

Otherwise, readers can plug in a value within each intervals, such as $y = 3$ for $y > 2$, $y = 3/2$ for $1 < y < 2$, etc., which should work equivalently.

Hence, we should expect a graph as follows:



The stability is obvious given the directions of the arrows.

2. (Ending Behavior.) Given an IVP as follows:

$$\begin{cases} y' + \frac{1}{2}y = \sin t; \\ y(0) = 1. \end{cases}$$

Find the specific solution of the equation and describe its end behavior

Solution:

(a) Here, we solve the IVP by first finding the general solution. Notice that we can apply the integrating factor method, that is:

$$\mu(t) = \exp\left(\int_0^t \frac{1}{2} ds\right) = \exp\left(\frac{t}{2}\right).$$

Then, by multiplying both sides of the differential equation with the integrating factor, we shall obtain the following (*details left as exercise to readers*):

$$e^{t/2}y = \int e^{t/2} \sin t dt.$$

Recall from **Problem set 1 Question 1 (ii)**, we have observed this anti-derivative case in prior, which requires two integration by parts to cancel out corresponding terms. In particular, we have:

$$\begin{aligned} \int e^{t/2} \sin t dt &= 2e^{t/2} \sin t - \int 2e^{t/2} \cos t dt \\ &= 2e^{t/2} \sin t - 2\left[2e^{t/2} \cos t - \int 2e^{t/2}(-\sin t) dt\right] \\ &= 2e^{t/2} \sin t - 4e^{t/2} \cos t - 4\int e^{t/2} \sin t dt. \\ 5\int e^{t/2} \sin t dt &= 2e^{t/2} \sin t - 4e^{t/2} \cos t + C. \\ \int e^{t/2} \sin t dt &= \frac{2}{5}e^{t/2} \sin t - \frac{4}{5}e^{t/2} \cos t + \tilde{C}. \end{aligned}$$

Hence, by dividing by sides by $e^{t/2}$, we obtain:

$$y = \frac{2}{5} \sin t - \frac{4}{5} \cos t + \tilde{C}e^{-t/2}.$$

Then, we use the initial condition that $y(0) = 1$ to get that:

$$1 = \frac{2}{5} \cdot 0 - \frac{4}{5} \cdot 1 + \tilde{C},$$

which enforces $\tilde{C} = 9/5$, and our solution is:

$$\boxed{y(t) = \frac{2}{5} \sin t - \frac{4}{5} \cos t + \frac{9}{5}e^{-t/2}}.$$

(b) Note that although $e^{-t/2}$ converges to 0 as $t \to \infty$. However, the sine and cosine functions has an oscillating behavior, *i.e.*, moving above and below 0 but not converging to it, hence the solution $\boxed{\text{oscillates around } y = 0}$.

3. (Bifurcation.) For the first-order autonomous ODE:

$$\frac{dx}{dt} = x^2 - 2x + c,$$

with parameter $c \in \mathbb{R}$, do the following:

(a) Sketch all of the qualitatively different graphs of $f(x) = x^2 - 2x + c$, as c is varied.

(b) Determine any and all bifurcation values for the parameter c.

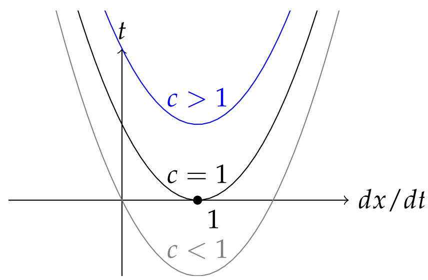
(c) Sketch a bifurcation diagram for this ODE.

Solutions:

(a) Given the right hand side, we want to find its critical point, *i.e.*, $x^2 - 2x + c = 0$, that is:

$$x = \frac{2 \pm \sqrt{4 - 4c}}{2} = 1 \pm \sqrt{1 - c}.$$

Graphically, we may draw the diagram as:



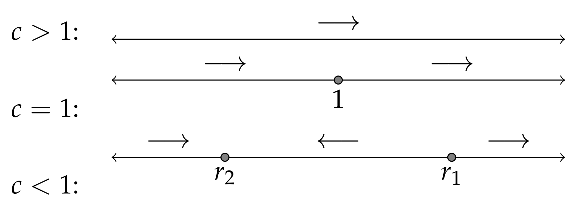
(b) Then, we find the bifurcation value, that is the critical points equivalent to each other, *i.e.*:

$$1 + \sqrt{1 - c} = 1 - \sqrt{1 - c}$$

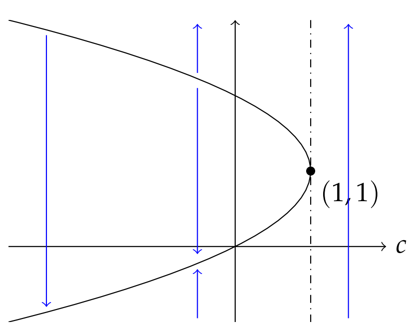
$$c = \boxed{1}.$$

Diligent readers might also notice that this is the value c such that $\Delta = 0$.

(c) When $c > 1$, $dx/dt > 0$. When $c = 1$, the two roots are both 1. When $c < 1$, the $dx/dt > 0$ when larger than the larger root or smaller than the smaller roots, hence the phase diagrams are:



Thus, the bifurcation diagram is:



4. (Existence of Higher Order ODEs.) Determine intervals that the initial condition must be in so that the solutions are sure to exist:

   (a) $y^{(4)} + 4y''' + 3y = t$,

   (b) $y''' + ty'' + t^2y' + t^3y = \ln t$.

Solutions: This question concerns the existence and uniqueness for higher order ODEs, *i.e.*, we want to write them in the form of:

$$y^{(q)} + p_{q-1}(t)y^{(q-1)} + \cdots + p_1(t)y' + p_0(t)y + p(t) = 0.$$

Luckily, equation in this question is already given in this form, hence we want to evaluate them case by case:

(a) For this equation, we have:

$$\begin{aligned} p_3(t) &= 4, \\ p_2(t) &= 0, \\ p_1(t) &= 0, \\ p_0(t) &= 3, \\ p(t) &= t. \end{aligned}$$

We want all above to be continuous, which is valid for all $x \in \mathbb{R}$, so the interval would be $\boxed{\mathbb{R} = (-\infty, \infty)}$.

(b) For this equation, we have:

$$\begin{aligned} p_2(t) &= t, \\ p_1(t) &= t^2, \\ p_0(t) &= t^3, \\ p(t) &= \ln t. \end{aligned}$$

We want all above to be continuous, we must concern on $\ln t$, which is continuous only for $t > 0$, so the interval would be $\boxed{(0, \infty)}$.

Note that since we do not have an initial condition, the initial condition must be in the given interval so that existence is guaranteed for that interval.

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Problem Set 4: Solutions

Differential Equations

Fall 2024

1. (Second Order Differential Equation.) Let an initial value problem for $y = y(t)$ be defined as follows:

$$\begin{cases} 4y'' - y = 0, \\ y(0) = 2,\ y'(0) = \beta, \end{cases}$$

where β is a real constant.

   (a) Find the specific solution to the initial value problem. Express your solution with constant β.

   (b) Find the value of β such that the solution *converges* to 0 as t tends to infinity.

Solution:

(a) First, we note that the characteristic equation is $4r^2 - 1 = 0$, whose roots are $\pm 1/2$, hence the general solution to the differential equation is:

$$y(t) = C_1 e^{t/2} + C_2 e^{-t/2}, \text{ where } C_1, C_2 \text{ are constants.}$$

To find the specific solution, we input initial conditions, namely we find the derivative:

$$y'(t) = \frac{C_1}{2} e^{t/2} - \frac{C_2}{2} e^{-t/2}.$$

Hence, the initial data tells us that:

$$y(0) = C_1 + C_2 = 2 \text{ and } y'(0) = \frac{C_1}{2} - \frac{C_2}{2} = \beta.$$

By algebraically manipulating the equations, we find:

$$C_1 = 1 + \beta \text{ and } C_2 = 1 - \beta.$$

Hence, the solution is:

$$y(t) = \boxed{(1+\beta)e^{t/2} + (1-\beta)e^{-t/2}}.$$

(b) Considering $t \to \infty$, we note that $e^{t/2} \to \infty$ and $e^{-t/2} \to 0$, hence we only need to consider about the $(1+\beta)e^{t/2}$ part.

In order for a convergence to 0, we want this part to vanish, *i.e.*:

$$1 + \beta = 0 \text{ or } \beta = \boxed{-1}.$$

2. (LI Set of Solutions.) Find the general solution to the following differential equation, and verify that your solution is a linearly independent set of solutions.

$$y^{(3)}(x) - 6y''(x) + 11y'(x) - 6y(x) = 0.$$

Solution: As usual, we first find the characteristic equation, that is:

$$r^3 - 6r^2 + 11r - 6 = 0.$$

Up to this point, readers should be quite familiar with the *rational root theorem*, so we know that if the polynomial has a rational root, it must be one of the following:

$$\pm 1, \pm 2, \pm 3, \text{ and } \pm 6.$$

In fact, for degree 3 polynomials of integer/rational coefficients, it must have at least one rational root. *We leave the check of this claim to the readers, as an exercise to get more familiar with polynomials.*
By easy checking, we note that 1 is a root, as $1^3 - 6 \times 1^2 + 11 \times 1 - 6 = 0$, then we can eliminate the polynomial to:

$$r^3 - 6r^2 + 11r - 6 = (r-1)(r^2 - 5r + 6) = (r-1)(r-2)(r-3),$$

hence the roots are $r = 1, 2, 3$, each with multiplicity 1.
Therefore, the general solution should be:

$$y(x) = \boxed{C_1 e^x + C_2 e^{2x} + C_3 e^{3x}}.$$

Note that since we are asked to verify linear independence, we use the Wronskian, that is:

$$\mathrm{W}[e^x, e^{2x}, e^{3x}] = \det \begin{pmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{pmatrix} = 18e^{6x} + 3e^{6x} + 4e^{6x} - 2e^{6x} - 12e^{6x} - 9e^{6x} = 2e^{6x} \neq 0,$$

hence the set $\{e^x, e^{2x}, e^{3x}\}$ is linearly independent.

3. (A Missing Solution.) Let a third order differential equation of $y = y(x)$ be defined as below:

$$y''' - y'' + y' - y = 0.$$

(a) Verify that $\sin(x)$ and $\cos(x)$ are two solutions to the above differential equation. Can you explain how we can find these two solutions?

(b) Is the set $\{\sin(x), \cos(x)\}$ linearly independent?

(c) Does $\{\sin(x), \cos(x)\}$ constitute a full set of solution to the differential equation?

(d) Give the general solution to the differential equation.

Solution:

(a) *Proof.* The verification is trivial. Since:

$$(\sin x)''' - (\sin x)'' + (\sin x)' - (\sin x) = -\cos x - \sin x + \cos x - \sin x = 0,$$
$$(\cos x)''' - (\cos x)'' + (\cos x)' - (\cos x) = \sin x + \cos x - \sin x - \cos x = 0.$$

Hence, they are solutions to the differential equation. □

Moreover, this is the case, since the characteristic equation has roots being $\pm\mathrm{i}$, which by Euler's identity, can be changed to real-valued functions $\cos x$ and $\sin x$.

(b) To verify linear independence, we have:

$$\mathrm{W}[\sin x, \cos x] = \det \begin{pmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{pmatrix} = -\sin^2 x - \cos^2 x = 1 \neq 0.$$

Hence the set is $\boxed{\text{linearly independent}}$.

(c) Note that we have a degree 3 differential equations, hence we need at least 3 linearly independent solution. Hence it does $\boxed{\text{not}}$ constitute a full set of solutions.

(d) To find the general solution, we want to first find the third solution. Recall the characteristic polynomial $r^3 - r^2 + r - 1$, we find roots $r = 1, \pm\mathrm{i}$, hence the third solution should be:

$$e^x,$$

and hence, by the principle of superposition, the general solution is:

$$y(x) = \boxed{C_1 \sin x + C_2 \cos x + C_3 e^x}.$$

4. (A Symmetric Solution.) Given the following second order initial value problem:

$$\begin{cases} \dfrac{d^2y}{dx^2} + \sin^2(1-x)y = \cosh(x-1), \\ y(1) = e, \ \dfrac{dy}{dx}(1) = 0. \end{cases}$$

Prove that the solution $y(x)$ is symmetric about $x = 1$, *i.e.*, satisfying that $y(x) = y(2-x)$.
Hint: Consider the interval in which the solution is unique.
Also, note that $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$.

Solution:
Note that I deliberately messed up with all the messy functions. Not only haven't I found a solution to the system, Wolfram cannot have an elementary solution as well. Hence, we need to think, alternatively, on some theorems.

Proof. Here, we suppose that $y(x)$ is a solution, and we want to show that $y(2-x)$ is also a solution. First we note that we can think of taking the derivatives of $y(2-x)$, by the chain rule:

$$\frac{d}{dx}[y(2-x)] = -y'(2-x),$$
$$\frac{d^2}{dx^2}[y(2-x)] = y''(2-x).$$

Now, if we plug in $y(2-x)$ into the system of equations, we have:

- First, for the differential equation, we have:

$$\begin{aligned} \frac{d^2}{dx^2}[y(2-x)] + \sin^2(1-x)y(2-x) &= y''(2-x) + \sin^2(x-1)y(2-x) \\ &= y''(2-x) + \sin^2\big(1-(2-x)\big)y(2-x) \\ &= y''(z) + \sin^2(1-z)y(z) \\ &= \cosh(z-1) = \frac{e^{z-1} + e^{-z+1}}{2} = \frac{e^{-(2-z)+1} + e^{(2-z)-1}}{2} \\ &= \cosh\big(x-1\big). \end{aligned}$$

- For the initial conditions, we trivially have that:

$$y(1) = y(2-1) = e \text{ and } y'(1) = y'(2-1) = 0.$$

Hence, we have shown that $y(2-x)$ is a solution if $y(x)$ is a solution.
Again, we observe the original initial value problem that:

$$\sin^2(1-x) \text{ and } \cosh(x-1) \text{ are continuous on } \mathbb{R}.$$

Therefore, by the *existence and uniqueness theorem for second order linear case*, there could be only one solution, which forces that:

$$y(x) = y(2-x),$$

so the solution is symmetric about $x = 1$, as desired. □

Problem Set 5: Solutions

Differential Equations

Fall 2024

1. (Constructing Solutions, Again.) Construct an initial value problem for the following solutions:

   (a) $$y(t) = 4e^{3t} - e^{-2t}.$$

   (b) $$y(t) = e^{2t}\cos t + e^{2t}\sin t + e^{2t}.$$

Solution:

(a) Here, we trivially note that the two solutions are e^{3t} and e^{-2t}, so the roots of the characteristic equation is $r = 3$ and $r = -2$, so $(r-3)(r+2) = r^2 - r - 6 = 0$, hence the differential equation should be $y'' - y' - 6y = 0$.

For the initial value, since the function is well defined over $\mathbb{R}$, we pick any initial value, and we pick 0 for simplicity here, so $y(0) = 4 - 1 = 3$, and while for the derivative being:

$$y'(t) = 12e^{3t} + 2e^{-2t},$$

so we have $y'(0) = 12 + 2 = 14$. Hence, one example of the IVP is:

$$\boxed{\begin{cases} y'' - y' - 6y = 0, \\ y(0) = 3,\ y'(0) = 14. \end{cases}}$$

(b) For this part, we reverse engineer the solution as:

$$e^{2t}(\cos t + \mathrm{i}\sin t), e^{2t}(\cos t - \mathrm{i}\sin t), \text{ and } e^{2t},$$

which is e^{2t} with $e^{2t \pm it}$, so the roots are 2 and $2 \pm \mathrm{i}$, thus, we reconstruct the characteristic polynomial that:

$$(r-2)\big((r-2-\mathrm{i})(r-2+\mathrm{i})\big) = (r-2)(r^2 - 4r + 5) = r^3 - 6r^2 + 13r - 10.$$

Thus, we have the differential equation as:

$$y''' - 6y'' + 13y' - 10y = 0.$$

Now, we consider that the initial condition, we take the first and second derivatives as:

$$y'(t) = 2e^{2t}\cos t + 2e^{2t}\sin t + 2e^{2t} - e^{2t}\sin t + e^{2t}\cos t = 2e^{2t}\cos t + e^{2t}\sin t + 2e^{2t},$$
$$y''(t) = 4e^{2t}\cos t + 2e^{2t}\sin t + 4e^{2t} - 2e^{2t}\sin t + e^{2t}\cos t = 5e^{2t}\cos t + 4e^{2t}.$$

Hence, we choose 0 as initial point and obtain $y(0) = 1 + 1 = 2$, $y'(0) = 2 + 2 = 4$, and $y''(0) = 5 + 4 = 9$, so one example of initial value problem is:

$$\boxed{\begin{cases} y''' - 6y'' + 13y' - 10y = 0, \\ y(0) = 2,\ y'(0) = 4,\ y''(0) = 9. \end{cases}}$$

2. ($L^2([0,2\pi])$ Space.) Recall that we have defined linear independence of functions, we define *orthogonality* of two real-valued, "square-integrable" functions over $[0,2\pi]$, f and g, as:

$$\int_0^{2\pi} f(x)g(x)dx = 0.$$

(a) Show that the set $\{\sin x, \cos x\}$ is linearly independent and orthogonal.

(b) Show that if $\{f(x), g(x)\}$ is orthogonal, then $C_1 f(x)$ and $C_2 g(x)$ is orthogonal.

(c) Note that $\{x, x^2\}$ are linearly independent, construct a basis that is orthogonal.

Solution:

(a) *Proof.* To show linear independence, we compute the Wronskian as:

$$\mathrm{W}[\sin x, \cos x] = \det\begin{pmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{pmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0.$$

Then, to show orthogonality, we have:

$$\int_0^{2\pi} \sin x \cos x dx = \frac{1}{2}\int_0^{2\pi} \sin(2x)dx = \frac{1}{2}\left[-\frac{1}{2}\cos(2x)\right]_0^{2\pi} = \frac{1}{4}\big(\cos 0 - \cos(4\pi)\big) = 0,$$

hence we have shown linear independence and orthogonality. □

(b) *Proof.* By orthogonality, we have $\int_0^{2\pi} f(x)g(x)dx = 0$, so we have:

$$\int_0^{2\pi} C_1 f(x) \cdot C_2 g(x)dx = C_1 C_2 \int_0^{2\pi} f(x)g(x)dx = C_1 C_2 \cdot 0 = 0.$$

Hence orthogonality is preserved with scalar multiplications. □

(c) The check of x and x^2 being linearly independent can be verified by Wronskian, and we leave this check to the readers. By the principle of superposition, we want to construct the second argument as $x^2 - Ax$, where A is a constant, now we take the inner product as:

$$\int_0^{2\pi} x(x^2 - Ax)dx = \int_0^{2\pi} (x^3 - Ax^2)dx = \frac{x^4}{4} - \frac{Ax^3}{3}\bigg|_0^{2\pi} = 4\pi^4 - \frac{8A\pi^3}{3} = 0,$$

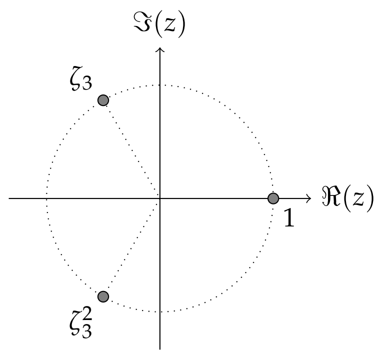
which forces A to be $3\pi/2$, so the orthogonal basis is now:

$$\boxed{\left\{x, x^2 - \frac{3\pi x}{2}\right\}}.$$

Diligent should notice that we have somehow constructed a "vector space" with a proper inner product. In fact, this space $L^2([0,2\pi])$ is considered a Hilbert Space, that is a infinite dimensional vector space with completeness and denseness. The $L^2([0,2\pi])$ is closely related to Fourier series, that has inarguable impacts on mathematics as well as science and engineering disciplines.

3. (Repeated and Complex Root.) Let a six order differential equation of $y = y(t)$ be defined as follows:
$$y^{(6)} - 2y^{(3)} + y = 0.$$
Find a set of real-valued function being the general solution to the above differential equation.

Solution:
For this questions, we first find the characteristic equation, which should be a fairly easy perfect square:
$$r^6 - 2r^3 + 1 = (r^3 - 1)^2 = 0.$$
Hence, our concern follows to r being the solution to $r^3 = 1$, with double multiplicity. In particular, we have the roots being on the unit circle, with ζ_3 being the 3rd root of unity, as:



Hence, the roots of the polynomial is:
$$r = 1, \zeta_3, \zeta_3^2,$$
each with multiplicity 2, where ζ_3 and ζ_3^2 can be expressed as:
$$\zeta_3 = \cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2},$$
$$\zeta_3^2 = \cos\left(\frac{4\pi}{3}\right) + \mathrm{i}\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}.$$
Hence, one set of solution is:
$$y_1 = e^t,$$
$$y_2 = e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right),$$
$$y_3 = e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right),$$
where this set is already manipulated by Euler's identity. By multiplicity of roots, we have repeated roots, leading to solutions:
$$y_4 = te^t,$$
$$y_5 = te^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right),$$
$$y_6 = te^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right).$$
And the set is $\boxed{\{y_1, y_2, y_3, y_4, y_5, y_6\}}$.

4. (Reduction of Order.) Let a ODE be defined as follows:

$$t^2y'' + 2ty' = 2y,\ t > 0.$$

Given a solution is $y_1(t) = t$, find the full fundamental set of solution.

Solution:

Here, we let the second solution to be $y_2(t) = u(t)y_1(t) = tu(t)$, thus we take its derivatives as:

$$\begin{aligned} y_2'(t) &= u'(t)y_1(t) + u(t)y_1'(t) = tu'(t) + u(t), \\ y_2''(t) &= u''(t)y_1(t) + 2u'(t)y_1'(t) + u(t)y_1''(t) = tu''(t) + 2u'(t). \end{aligned}$$

Now, as we plug in the above results back to the ODE, we have:

$$t^2\big(tu''(t) + 2u'(t)\big) + 2t\big(tu'(t) + u(t)\big) = 2\big(tu(t)\big).$$

By some arithmetic expansions and rearranging, we have:

$$t^3u''(t) + 4t^2u'(t) = tu'' + 4u' = 0.$$

Here, we denote $\omega = u'$ to have $t\omega' + 4\omega = 0$, which is separable, so we have:

$$\begin{aligned} \frac{d\omega}{\omega} &= -\frac{4dt}{t}, \\ \int \frac{d\omega}{\omega} &= -4\int \frac{dt}{t}, \\ \log|\omega| &= -4\log|t| + C, \\ \omega &= \tilde{C}t^{-4} \end{aligned}$$

Now, we integrate ω to obtain that:

$$\int \omega dt = \int \tilde{C}t^{-4}dt = -\frac{\tilde{C}}{3}t^{-3} + D_2 = D_1t^{-3} + D_2.$$

By multiplying back with $y_1(t)$, we have the fundamental set of solution as:

$$y(t) = (D_1t^{-3} + D_2)t = \boxed{\frac{D_1}{t^2} + D_2t}.$$

Problem Set 6: Solutions

Differential Equations

Fall 2024

1. (Reduction of Order or Integrating Method). Let a differential equation be:

$$y''(t) + \frac{2}{t}y'(t) = 0.$$

   (a) Verify that $y(t) = 1/t$ is one solution, then find a full set of solution.

   (b) Consider $\omega(t) = y'(t)$, solve the differential equation by using integrating factor.

   (c) Verify that the two methods give you the same set of the solutions.

Solution:

(a) The verification should be easy, we have the derivatives as:

$$\frac{d}{dt}\left[\frac{1}{t}\right] = -\frac{1}{t^2} \text{ and } \frac{d^2}{dt^2}\left[\frac{1}{t}\right] = \frac{2}{t^3},$$

and hence by plugging in, we have:

$$y'' + \frac{1}{t}y' = \frac{2}{t^3} + \frac{2}{t}\cdot\left(-\frac{1}{t^2}\right) = \frac{2}{t^3} - \frac{2}{t^3} = 0,$$

hence we have verified that $1/t$ is a solution. To obtain the other solution, we let $y_2 = u(t)/t$, and we take the derivatives as:

$$\left(\frac{1}{t}u(t)\right)'' + \frac{2}{t}\left(\frac{1}{t}u(t)\right)' = \frac{u''(t)}{t} - \frac{2u'(t)}{t^2} + \frac{2u(t)}{t^3} + \frac{2}{t}\left(\frac{u'(t)}{t} - \frac{u(t)}{t^2}\right) = 0.$$

Hence, we can reduce the ODE into:

$$t^2u''(t) - 2tu'(t) + 2u(t) + 2tu'(t) - 2u(t) = t^2u''(t) = 0 \implies u''(t) = 0,$$

therefore, the solution is $u(t) = at + b$, and by multiplying $1/t$, we have the set as $\boxed{\{1, 1/t\}}$.

(b) Consider $\omega(t) = y'(t)$, we have $\omega'(t) + \frac{2}{t}\omega(t) = 0$, so the integrating factor is:

$$\mu(t) = \exp\left(\int_0^1 \frac{2}{s}ds\right) = e^{2\ln|t|} = t^2,$$

and hence by multiplying it, we have:

$$t^2\omega'(t) + 2t\omega(t) = 0,$$

which can be solved as:

$$\frac{d}{dt}[t^2\omega(t)] = 0 \implies t^2\omega(t) = C \implies \omega(t) = Ct^{-2}.$$

Hence, by integrating ω, we have:

$$y(t) = \int \omega dt = \boxed{C_1t^{-1} + C_2}.$$

(c) The two methods give the same set of solutions, *and it is better that they do.*

2. (Complex Characteristics, Again). Find a full set of real solutions to the differential equation:
$$\frac{d^3y}{dx^3} = -y.$$

Solution:
Clearly, the characteristic equation is $r^3 = -1$. For this part, you will still have two options to proceed:

- By observing that -1 is a result, you may induct a long division of $(r^3+1)/(r+1)$, and factor as of how you factor quadratics, or
- by Euler's method's heuristics, namely finding the roots for $x^6 = 1$, that is $\zeta_6 = e^{2\pi \mathrm{i}/6} = e^{\pi \mathrm{i}/3}$, and take its odd powers, that is ζ_6^1, ζ_6^3, and ζ_6^5.

Whatever your choice is, you should obtain your three roots as:
$$r = -1, \frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}, \text{ and } \frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2},$$
hence inducing the solution set as:
$$\left\{e^{-t}, e^{(1/2+\mathrm{i}\sqrt{3}/2)t}, e^{(1/2-\mathrm{i}\sqrt{3}/2)t}\right\}.$$
By some simply arithmetics of linear combinations, we have:
$$\boxed{\left\{e^{-t}, e^{t/2}\cos\left(\frac{\sqrt{3}}{2}t\right), e^{t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)\right\}}.$$
At this moment, readers should be utterly clear with Euler's identity and the method of transforming from complex-valued solutions to real-valued solution. If you are having trouble on this question, we suggest you to review on this part and check on ***Question 3*** *in* ***Problem Set 5***.

3. (Non-homogeneous Differential Equations). Solve the following differential equations.

   (a)
$$y'' + 4y = t^2 + 3e^t.$$
   (b)
$$y'' + 2y' + y = \frac{e^{-x}}{x}.$$

Solution:

(a) For the first part, we first find the solution to the homogeneous case, that is $y'' + 4y = 0$, whose characteristic equation is $r^2 + 4 = 0$, with roots $r = \pm 2\mathrm{i}$. Hence, the homogeneous solution is:
$$y = C_1 \cos(2t) + C_2 \sin(2t).$$
Based on the non-homogeneous part, our guess of the solution should be:
$$y_p(t) = \underbrace{At^2 + Bt + C}_{\text{Guess for } t^2} + \underbrace{De^t}_{\text{Guess for } 3e^t}.$$
Of course, readers can make separated guess since differentiation is linear operator, and solve for a, b, c and d separately. However, we will provide the whole derivatives as:
$$y_p' = 2At + B + De^t,$$
$$y_p'' = 2A + De^t.$$
Therefore, as we plug in the particular solution, we have:
$$(2A + De^t) + 4(At^2 + Bt + C + De^t) = t^2 + 3e^t,$$
$$4At^2 + Bt + (4C + 2A) + 5De^t = t^2 + 3e^t,$$
so the solutions are $A = 1/4$, $B = 0$, $C = -1/8$, and $D = 3/5$, so we have:
$$y(t) = \boxed{C_1 \cos(2t) + C_2 \sin(2t) + \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t}.$$

(b) Still, we first look for the homogeneous solution, for $y'' + 2y' + y = 0$, with characteristic equation as $r^2 + 2r + 1 = 0$, the roots is $r = -1$ with multiplicity 2, that is:
$$y = C_1 e^{-x} + C_2 x e^{-x}.$$
Here, we use the variation of parameter that we first take the Wronskian:
$$W[e^{-x}, xe^{-x}] = \det \begin{pmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{pmatrix} = -xe^{2x} + e^{-2x} + xe^{-2x} = e^{-2x}.$$
Therefore, we have the particular solution as:
$$\begin{aligned} y_p(x) &= -y_1(x) \int \frac{y_2(x)g(x)}{W} dx + y_2 \int \frac{y_1(x)g(x)}{W} dx \\ &= -e^{-x} \int \frac{xe^{-x}}{e^{-2x}} \frac{e^{-x}}{x} dx + xe^{-x} \int \frac{e^{-x}}{e^{-2x}} \frac{e^{-x}}{x} dx \\ &= -e^{-x} \int dx + xe^{-x} \int \frac{dx}{x} = -xe^{-x} + K_1 e^{-x} + K_2 x e^{-x} + xe^{-x} \log|x| \\ &= xe^{-x} \log|x|. \end{aligned}$$
Hence, the solution would be:
$$y(x) = \boxed{C_1 e^{-x} + C_2 x e^{-x} + xe^{-x} \log|x|}.$$

4. (Warm up in Linear Algebra). This problem reviews the basic concepts linear algebra concepts.

   (a) Which of the following set of vectors are linearly independent in $\mathbb{R}$-vector space, what about $\mathbb{C}$-vector space? Justify your answer.

   (i) $\alpha = \{(1,1,0),(0,1,1),(1,0,1)\}$,

   (ii) $\beta = \{(0,1),(2,3),(4,5)\}$,

   (iii) $\gamma = \{1, \mathrm{i}\}$.

   (b) Let $A = \begin{pmatrix} 1+\mathrm{i} & -1+2\mathrm{i} \\ 3+2\mathrm{i} & 2-\mathrm{i} \end{pmatrix}$ and $B = \begin{pmatrix} \mathrm{i} & 3 \\ 2 & -2\mathrm{i} \end{pmatrix}$, compute the following:

   (i) $A - 2B$,

   (ii) BA,

   (iii) B^{-1}.

Solution:

(a) For the first part, the $\mathbb{R}$ and $\mathbb{C}$-vector spaces should generally be the same:

i. Consider the determinant of vertically concatenating the vectors that:

$$\det \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 1+0+1-0-0-0 = 2 \neq 0,$$

hence it is $\boxed{\text{linearly independent}}$.

ii. Here, the vector space is $\mathbb{R}^2$ or $\mathbb{C}^2$, which has determinant 2, but since there are three vectors, it is $\boxed{\text{not linearly independent}}$.

iii. This case is interesting, consider the $\mathbb{R}$-vector space, for any $\lambda_1, \lambda_2 \in \mathbb{R}$, we have $\lambda_1 \cdot 1 + \lambda_2 \cdot \mathrm{i} = 0$ if and only if $\lambda_1 = \lambda_2 = 0$, so it is $\boxed{\text{linearly independent in } \mathbb{R}\text{-vector space}}$. Consider the $\mathbb{C}$-vector space, we have $1 \cdot 1 + \mathrm{i} \cdot \mathrm{i} = 1 - 1 = 0$, so it is $\boxed{\text{not linearly independent in } \mathbb{C}\text{-vector space}}$.

(b) For the second part, we do the computation on the matrix operations:

i. Consider $A - 2B$, we have:

$$A - 2B = \begin{pmatrix} 1+\mathrm{i} & -1+2\mathrm{i} \\ 3+2\mathrm{i} & 2-\mathrm{i} \end{pmatrix} - 2\begin{pmatrix} \mathrm{i} & 3 \\ 2 & -2\mathrm{i} \end{pmatrix}$$
$$= \begin{pmatrix} 1+\mathrm{i}-2\mathrm{i} & -1+2\mathrm{i}-6 \\ 3+2\mathrm{i}-4 & 2-\mathrm{i}+4\mathrm{i} \end{pmatrix} = \boxed{\begin{pmatrix} 1-\mathrm{i} & -7+2\mathrm{i} \\ -1+2\mathrm{i} & 2+3\mathrm{i} \end{pmatrix}}.$$

ii. For BA, we do the matrix multiplication entry wise, that is:

$$BA = \begin{pmatrix} (1+\mathrm{i})\cdot\mathrm{i} + (-1+2\mathrm{i})\cdot 2 & \mathrm{i}(-1+2\mathrm{i}) + 3(2-\mathrm{i}) \\ (3+2\mathrm{i})\cdot\mathrm{i} + (2-\mathrm{i})\cdot 2 & (3+2\mathrm{i})\cdot 3 + (2-\mathrm{i})\cdot(-2\mathrm{i}) \end{pmatrix} = \boxed{\begin{pmatrix} 8+7\mathrm{i} & 4-4\mathrm{i} \\ 6-4\mathrm{i} & -4 \end{pmatrix}}.$$

iii. To find the inverse, we can use the formula that:

$$B^{-1} = \frac{1}{\det B}\begin{pmatrix} -2\mathrm{i} & -3 \\ -2 & \mathrm{i} \end{pmatrix} = \frac{1}{-4}\begin{pmatrix} -2\mathrm{i} & -3 \\ -2 & \mathrm{i} \end{pmatrix} = \boxed{\begin{pmatrix} \mathrm{i}/2 & 3/4 \\ 1/2 & -\mathrm{i}/4 \end{pmatrix}}.$$

Problem Set 7: Solutions (Fall Break Special)

Differential Equations

Fall 2024

Regular Questions: Review of course contents.

1. (Third-Order Nonhomogeneous). Solve the following third order differential equation of $y = y(t)$:
$$y''' - 4y' = e^{-2t}.$$

Solution:

First, we find the homogeneous case, that is:
$$y''' - 4y' = 0,$$
whose characteristic equation is $r^3 - 4r = 0$, so the roots are $r = 0, 2, -2$, hence the homogeneous solution is:
$$y(t) = C_1 + C_2 e^{2t} + C_3 e^{-2t}.$$
Given that the non-homogeneous part already exists in the equation, then our guess should be $y_p(t) = Ate^{-2t}$, which the derivatives as:
$$y_p'(t) = Ae^{-2t} - 2Ate^{-2t},$$
$$y_p''(t) = -4Ae^{-2t} + 4Ate^{-2t},$$
$$y_p'''(t) = 12Ae^{-2t} - 8Ate^{-2t}.$$
Note that when we plug into our equation, we have:
$$(12Ae^{-2t} - 8Ate^{-2t}) - 4(Ae^{-2t} - 2Ate^{-2t}) = e^{-2t}.$$
Note that the te^{-2t} term vanishes (why?), we now have:
$$8Ae^{-2t} = e^{-2t},$$
so we have that $A = 1/8$, so our general solution is:
$$y(t) = \boxed{C_1 + C_2 e^{2t} + C_3 e^{-2t} + \frac{1}{8} te^{-2t}}.$$
We invite diligent readers to attempt solving this problem using variation of parameters, as well, namely:
$$y_p(t) = y_1(t) \int_0^t \frac{W_1(s)g(s)}{W(s)} ds + y_2(t) \int_0^t \frac{W_2(s)g(s)}{W(s)} ds + y_3(t) \int_0^t \frac{W_3(s)g(s)}{W(s)} ds.$$

2. (Eigenvalues & Eigenvectors). Find all eigenvectors and eigenvalues of the following matrix:

   (a) $$A = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix},$$

   (b) $$B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}.$$

Solution:

(a) Here, we find the characteristic equation as:

$$0 = \det \begin{pmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{pmatrix} = (5-\lambda)(1-\lambda) - (-1)\cdot 3 = 8 - 6\lambda + \lambda^2 = (\lambda-2)(\lambda-4).$$

Hence, the eigenvalues are 2 and 4, and the eigenvalues, respectively, are:

i. For $\lambda_1 = \boxed{2}$, we have $A - 2\,\mathrm{Id}$ as $\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}$, we want find $\boldsymbol{\xi}^{(1)}$ such that $\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}.\boldsymbol{\xi}^{(1)} = \mathbf{0}$, that is $3\xi_1^{(1)} - \xi_2^{(1)} = 0$, so we have $\xi_2^{(1)} = 3\xi_1^{(1)}$, so we have $\boldsymbol{\xi}^{(1)} = \boxed{\begin{pmatrix} 1 \\ 3 \end{pmatrix}}$.

ii. For $\lambda_2 = \boxed{4}$, we have $A - 4\,\mathrm{Id}$ as $\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}$, we want find $\boldsymbol{\xi}^{(2)}$ such that $\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}.\boldsymbol{\xi}^{(2)} = \mathbf{0}$, that is $\xi_1^{(2)} - \xi_2^{(2)} = 0$, so we have $\xi_2^{(2)} = \xi_1^{(2)}$, so we have $\boldsymbol{\xi}^{(2)} = \boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$.

(b) Here, we also find the characteristic equation as:

$$0 = \det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{pmatrix} = (1-\lambda)^3 - (1-\lambda)(-4) = (1-\lambda)\big((1-\lambda)^2 + 4\big).$$

Hence, the eigenvalues are 1, $1+2i$ and $1-2i$, and the eigenvalues, respectively, are:

i. For $\lambda_1 = \boxed{1}$, we have $B - \mathrm{Id}$ as $\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{pmatrix}$, we want find $\boldsymbol{\xi}^{(1)}$ such that $\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{pmatrix}.\boldsymbol{\xi}^{(1)} = \mathbf{0}$, that is $\xi_1^{(1)} - \xi_3^{(1)} = 0$ and $3\xi_1^{(1)} + 2\xi_2^{(1)} = 0$, so we have $\boldsymbol{\xi}^{(1)} = \boxed{\begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}}$.

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ii. For $\lambda_2 = \boxed{1+2\mathrm{i}}$, we have $B-(1+2\mathrm{i})\,\mathrm{Id}$ as $\begin{pmatrix} -2\mathrm{i} & 0 & 0 \\ 2 & -2\mathrm{i} & -2 \\ 3 & 2 & -2\mathrm{i} \end{pmatrix}$, we want find $\boldsymbol{\xi}^{(2)}$ such that $\begin{pmatrix} -2\mathrm{i} & 0 & 0 \\ 2 & -2\mathrm{i} & -2 \\ 3 & 2 & -2\mathrm{i} \end{pmatrix}.\boldsymbol{\xi}^{(2)} = \mathbf{0}$, that is $\xi_1^{(2)} = 0$ and $\mathrm{i}\xi_2^{(2)} + \xi_3^{(2)} = 0$, so we have $\boldsymbol{\xi}^{(2)} = \boxed{\begin{pmatrix} 0 \\ 1 \\ -\mathrm{i} \end{pmatrix}}$.

iii. For $\lambda_3 = \boxed{1-2\mathrm{i}}$, we have $B-(1-2\mathrm{i})\,\mathrm{Id}$ as $\begin{pmatrix} 2\mathrm{i} & 0 & 0 \\ 2 & 2\mathrm{i} & -2 \\ 3 & 2 & 2\mathrm{i} \end{pmatrix}$, we want find $\boldsymbol{\xi}^{(3)}$ such that $\begin{pmatrix} 2\mathrm{i} & 0 & 0 \\ 2 & 2\mathrm{i} & -2 \\ 3 & 2 & 2\mathrm{i} \end{pmatrix}.\boldsymbol{\xi}^{(3)} = \mathbf{0}$, that is $\xi_1^{(3)} = 0$ and $\mathrm{i}\xi_2^{(3)} - \xi_3^{(3)} = 0$, so we have $\boldsymbol{\xi}^{(3)} = \boxed{\begin{pmatrix} 0 \\ 1 \\ \mathrm{i} \end{pmatrix}}$.

3. (Linear Systems). Let $\mathbf{x} \in \mathbb{R}^2$, find the general solution of $\mathbf{x}$ if $\mathbf{x}$ satisfies:

$$\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} .\mathbf{x}.$$

Solution:

Some readers might notice that this is the same matrix in the previous problem, we recall that the eigenvalues and eigenvectors, respectively, are:

$$\lambda_1 = 2, \qquad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix},$$

$$\lambda_2 = 4, \qquad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Hence, the solution to the linear system is:

$$\mathbf{x} = \boxed{C_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}}.$$

Additional Questions: More challenging and fun problems related with the course.

1. (PDEs: Wave Equation). The following system of partial differential equations portraits the propagation of waves on a segment of the 1-dimensional string of length L, the displacement of string at $x \in [0, L]$ at time $t \in [0, \infty)$ is described as the function $u = u(x, t)$:

$$\begin{cases} \text{Differential Equation:} & \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0, & \text{where } x \in (0, L) \text{ and } t \in [0, \infty); \\ \text{Initial Conditions:} & u(x, 0) = \sin\left(\dfrac{2\pi x}{L}\right), & \\ & \dfrac{\partial u}{\partial t}(x, 0) = \sin\left(\dfrac{5\pi x}{L}\right), & \text{where } x \in [0, L]; \\ \text{Boundary Conditions:} & u(0, t) = u(L, t) = 0, & \text{where } t \in [0, \infty); \end{cases}$$

where c is a constant and $g(x)$ has "good" behavior. Apply the method of separation, *i.e.*, $u(x, t) = v(x) \cdot w(t)$, and attempt to obtain a general solution that is *non-trivial*.
Hint: Use the fact that $\{\sin(n\pi x/L), \cos(n\pi x/L)\}_{n \in \mathbb{Z}^+}$ forms an orthonormal basis (cf. §5.2).

Solution:
With the method of separation, we insert the separations back to the system of equation to obtain:

$$v(x)w''(t) = c^2 v''(x) w(t).$$

Now, we apply the separation and set the common ratio to be λ:

$$\frac{v''(x)}{v(x)} = \frac{1}{c^2} \cdot \frac{w''(t)}{w(t)} = \lambda.$$

Reformatting the boundary condition gives use the following initial value problem:

$$\begin{cases} v''(x) - \lambda v(x) = 0, \\ v(0) = v(L) = 0. \end{cases}$$

As a second order linear ordinary differential equation, we discuss all following cases:

- If $\lambda = 0$, then $v(x) = a + Bx$ and by the initial condition, $A = B = 0$, which gives the trivial solution, *i.e.*, $v(x) = 0$;
- If $\lambda = \mu^2 > 0$, then we have $v(x) = Ae^{-\mu x} + Be^{\mu x}$ and again giving that $A = B = 0$, or the trivial solution;
- Eventually, if $\lambda = -\mu^2 < 0$, then we have the solution as:

$$v(x) = A \sin(\mu x) + B \cos(\mu x),$$

and the initial conditions gives us that:

$$\begin{cases} v(0) = B = 0, \\ v(L) = A \sin(\mu L) + B \cos(\mu L) = 0, \end{cases}$$

where A is arbitrary, $B = 0$, and $\mu L = m\pi$ positive integer m.
Overall, the only non-trivial solution would be:

$$v_m(x) = A \sin(\mu_m x), \text{ where } \mu_m = \frac{m\pi}{L}.$$

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Eventually, by inserting back $\lambda = -\mu_m^2$, we have $\lambda = -m^2\pi^2/L^2$, giving the solution to $w_m(t)$, another second order linear ordinary differential equation, as:

$$w_m(t) = C\cos(\mu_m ct) + D\sin(\mu_m ct), \text{ with } C, D \in \mathbb{R}.$$

By the *principle of superposition*, we can have our solution in the form:

$$u(x,t) = \sum_{m=1}^{\infty} [a_m \cos(\mu_m ct) + b_m \sin(\mu_m ct)] \sin(\mu_m x),$$

where our coefficients a_m and b_m have to be chosen to satisfy the initial conditions for $x \in [0, L]$:

$$u(x,0) = \sum_{m=1}^{\infty} a_m \sin(\mu_m x) = \sin\left(\frac{2\pi x}{L}\right),$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{m=1}^{\infty} c\mu_m b_m \sin(\mu_m x) = \sin\left(\frac{5\pi x}{L}\right).$$

Since we are hinted that $\{\sin(n\pi x/L), \cos(n\pi x/L)\}_{n\in\mathbb{Z}^+}$ forms an orthonormal basis, we now know that except for the following:

$$a_2 = 1 \text{ and } c\mu_5 b_5 = 1,$$

all the other coefficients are zero, so we have:

$$u(x,t) = \boxed{\cos\left(\frac{2\pi ct}{L}\right)\sin\left(\frac{2\pi x}{L}\right) + \frac{L}{5\pi c}\sin\left(\frac{5\pi ct}{L}\right)\sin\left(\frac{5\pi x}{L}\right)}.$$

2. (*Putnam 2023*: First Positive Root). Determine the smallest positive real number r such that there exists differentiable functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ satisfying:

   - $f(0) > 0$,
   - $g(0) = 0$,
   - $|f'(x)| \le |g(x)|$ for all x,
   - $|g'(x)| \le |f(x)|$ for all x, and
   - $f(r) = 0$.

   You may give an answer *without* a rigorous proof, as the proof is out of scope of the course.
   Hint: Assume that the function "moves" the fastest when the cap of the derivatives are "moving" the fastest, then think of constructing a dynamical system relating f and g.

Solution:
Here, we first provide a "simplified" case, *i.e.*, we are constructing a dynamical system in which we pick equality for the inequality, that is:

$$\begin{cases} |f'(x)| = |g(x)|, \text{ and} \\ |g'(x)| = |f(x)|. \end{cases}$$

Without loss of generality, we may assume that f and g are non-negative before r, so the system becomes:

$$\begin{cases} f' = -g \\ g' = f \end{cases},$$

or equivalently, $\mathbf{y} = \begin{pmatrix} f \\ g \end{pmatrix}$ that $\mathbf{y} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{y}$. Clearly, we observe the eigenvalues are $\pm \mathrm{i}$ as the polynomial is $\lambda^2 + 1 = 0$. Moreover, the eigenvectors for $\lambda_1 = \mathrm{i}$ is when $\begin{pmatrix} -\mathrm{i} & -1 \\ 1 & -\mathrm{i} \end{pmatrix} \boldsymbol{\xi} = \mathbf{0}$, in which we have $\boldsymbol{\xi} = y \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix}$, and that solution is:

$$\mathbf{y} = \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix} e^{\mathrm{i}x} = \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix} (\cos x + \mathrm{i} \sin x) = \begin{pmatrix} -\sin x \\ \cos x \end{pmatrix} + \mathrm{i} \begin{pmatrix} \cos x \\ \sin x, \end{pmatrix}$$

and by conjugation, the solution should be:

$$\begin{pmatrix} f \\ g \end{pmatrix} = C_1 \begin{pmatrix} -\sin x \\ \cos x \end{pmatrix} + C_2 \begin{pmatrix} \cos x \\ \sin x \end{pmatrix}.$$

Note that with the given initial condition that $g(0) = 0$, this enforces $C_1 = 0$, thus $f(x) = C \cos x$ and $g(x) = C \sin x$, and we know that $f(r)$ is zero first at $r = \boxed{\pi/2}$.
The above version has some reasoning, but is not a rigorous proof at all, since this does not consider if r could be smaller than $\pi/2$. For students with interests, we provide the complete proof from the Putnam competition from Victor Lie, as follows.

Proof. Without loss of generality, we assume $f(x) > 0$ for all $x \in [0, r)$ as it is the first positive zero. By the fundamental theorem of calculus, we have:

$$|f'(x)| \le |g(x)| \le \left| \int_0^x g(s) ds \right| \le \int_0^x |g(s)| ds \le \int_0^t |f(s)| ds.$$

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Now, as we denote $F(x) = \int_0^x f(s)ds$, we have:

$$f'(x) + F(x) \geq 0 \text{ for } x \in [0, r].$$

For the sake of contradiction, we suppose $r < \pi/2$, then we have:

$$f'(x)\cos x + F(x)\cos x \geq 0 \text{ for } x \in [0, r].$$

Notice that the left hand side is the derivative of $f(x)\cos x + F(x)\sin x$, so an integration on $[y, r]$ gives:

$$F(r)\sin r \geq f(y)\cos y + F(y)\sin(y).$$

With some rearranging, we can have:

$$F(r)\sin r\sec^2 y \geq f(y)\sec y + F(y)\sin y\sec^2 y$$

Again, we integrate both sides with respect to y on $[0, r]$, which gives:

$$F(r)\sin^2 r \geq F(r),$$

and this is impossible, so we have a contradiction.
Hence we must have $r \geq \pi/2$, and since we have noted the solution $f(x) = C\cos x$ and $g(x) = C\sin x$, we have proven that $r = \pi/2$ is the smallest case. □

3. (Nilpotent Operator). Let M be a square matrix, M is defined to be *nilpotent* if:

$$M^k = 0 \text{ for some positive integer } k.$$

Similar to how we defined the exponential function analytically, the exponential function is also defined for matrices, let A be a square matrix, we define:

$$\exp(A) = \sum_{i=0}^{\infty} \frac{1}{i!} A^i.$$

(a) Show that $N = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is nilpotent, then write down the result of $\exp(N)$.

Now, suppose that $N \in \mathcal{L}(\mathbb{R}^n)$ is a square matrix and is *nilpotent*.

(b) Suppose that $\mathrm{Id}_n \in \mathcal{L}(\mathbb{R}^n)$ is the identity matrix, prove that $\mathrm{Id}_n + N$ is invertible.
Hint: Use the differences of squares for matrices.

(c) If all the entries in N are rational, show that $\exp(N)$ has rational entries.

Solution:

(a) *proof of N is nilpotent.* Here, we want to do the matrix multiplication:

$$N^2 = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$N^3 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now, we have shown that $N^3 = 0$, or the zero matrix, hence N is nilpotent. □

Then, we want to calculate the matrix exponential, that is:

$$\exp(N) = \sum_{k=0}^{\infty} \frac{1}{k!} N^k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}.$$

(b) *Proof.* Here, we recall the differences of squares still works when commutativity for multiplications fails, hence the we can still use it for matrix multiplication, namely, for all $m \in \mathbb{Z}^+$:

$$(\mathrm{Id}_n + N) \cdot (\mathrm{Id}_n - N) \cdot (\mathrm{Id}_n + N^2) \cdots (\mathrm{Id}_n + N^{2^m}) = \mathrm{Id}_n - N^{2^{m+1}}$$

Since N is *nilpotent*, this implies that we have some k such that $N^\ell = 0$ for all $\ell \geq k$. Meanwhile, note that $2^\ell \geq \ell$ for all positive integer ℓ. (This can be proven by induction.) Therefore, we select $m + 1 \geq k$ so that $N^{2m+1} = 0$, and we have:

$$(\mathrm{Id}_n + N) \cdot \left[(\mathrm{Id}_n - N) \cdot (\mathrm{Id}_n + N^2) \cdots (\mathrm{Id}_n + N^{2^m})\right] = \mathrm{Id}_n,$$

thus $\mathrm{Id}_n + N$ is invertible. □

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(c) *Proof.* By the definition that N is nilpotent, we know that $N^m = 0$ for some finite positive integer m, hence, we can make the (countable) infinite sum into a finite sum:

$$\exp(N) = \sum_{k=0}^{\infty} \frac{1}{k!} N^k = \sum_{k=0}^{m} \frac{1}{k!} N^k,$$

thus all the entries are sum and non-zero divisions of rational numbers, while rational numbers are closed under addition and non-zero divisions, hence, all entries of $\exp(N)$ is rational. □

Note that the elements of all n-by-n matrices can be considered as a *ring*, while *nilpotent* can be defined more generally for *rings*. We invite capable readers to investigate more properties of *nilpotent* elements of *rings* in the discipline of *Modern Algebra*.

4. (Convergence of Series.) As we dive into fundamentals of mathematics, it is inevitable to encounter *sequences* and their sums. Discuss about the following sequences if they converge or not. If they converge, find the explicit sum.

   (a) $$\sum_{k=0}^{\infty} \frac{1}{k}.$$

   (b) $$\sum_{k=0}^{\infty} \frac{1}{k!}.$$

   (c) $$\sum_{k=0}^{\infty} \frac{1}{(4k+1)!}.$$

Solution:

(a) Diligent readers should observe that $\sum_{k=0}^{\infty} 1/k$ is a harmonic series, hence it $\boxed{\text{diverges}}$. Otherwise, we can simply notice that:

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{1}{k} &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} \cdots \\ &\geq \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \cdots \\ &= \frac{1}{1} + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \cdots\right) + \cdots \\ &= \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots,\end{aligned}$$

which diverges, hence our sequence $\sum_{k=0}^{\infty} 1/k$ must diverge.

(b) Here, we recall that the Taylor expansion of e^x at 0 is:

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} e^0 (x-0)^k = \sum_{k=0}^{\infty} \frac{1}{k!} x^k.$$

Evaluating the above equation at 1 gives that:

$$\sum_{k=0}^{\infty} \frac{1}{k!} = e^1 = \boxed{e},$$

in which the sequence $\boxed{\text{converges}}$.

(c) For this part, we want to note the Taylor series of e^x, e^{-x}, $\sin x$ and $\cos x$ at 0 evaluated at $x = 1$ are, respectively:

$$\begin{array}{rcccccccc} e^1 & = & +\frac{1}{0!} & +\frac{1}{1!} & +\frac{1}{2!} & +\frac{1}{3!} & +\frac{1}{4!} & +\frac{1}{5!} & +\cdots \\ e^{-1} & = & +\frac{1}{0!} & -\frac{1}{1!} & +\frac{1}{2!} & -\frac{1}{3!} & +\frac{1}{4!} & -\frac{1}{5!} & +\cdots \\ \sin 1 & = & & +\frac{1}{1!} & & -\frac{1}{3!} & & +\frac{1}{5!} & -\cdots \\ \cos 1 & = & +\frac{1}{0!} & & -\frac{1}{2!} & & +\frac{1}{4!} & & -\cdots \end{array}$$

Since the first series converges, we know that the later three series converges *absolutely*, so we are free to move around terms. Thus comparing vertically gives us that:

$$\sum_{k=0}^{\infty} \frac{1}{(4k+1)!} = \boxed{\frac{e^1 - e^{-1}}{4} + \frac{\sin 1}{2}}.$$

Problem Set 8: Solutions

Differential Equations

Fall 2024

1. (Linear System versus Second Order). Let an initial value problem for linear system on $x_1 := x_1(t)$ and $x_2 := x_2(t)$ be defined as follows:

$$\begin{cases} x_1' = 3x_1 - 2x_2, & x_1(0) = 3, \\ x_2' = 2x_1 - 2x_2, & x_2(0) = \dfrac{1}{2}. \end{cases}$$

   (a) Solve for the *general solution* for the linear system by considering $\mathbf{x} = (x_1, x_2)$.

   (b) Transform the *general system* into a single equation of second order. Then solve the second-order equation. Eventually, convert your solution of one variable back to the *general solution* to $x_1(t)$ and $x_2(t)$.

   (c) Find the particular solution using the initial conditions, then graph the parameterized curve on a x_1x_2-plane with $t \geq 0$.

Solution:

(a) Here, when we consider the general linear system, we have the matrix as:

$$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} .\mathbf{x}.$$

Hence, we look for the eigenvalues for the matrix, that is when:

$$0 = \det \begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix} = (3-\lambda)(-2-\lambda) - (-2) \times 2 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1).$$

Hence the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$, we solve for the eigenvectors:

- For $\lambda_1 = 2$, we want $\boldsymbol{\xi}^{(1)}$ satisfy that $\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} .\boldsymbol{\xi}^{(1)} = \mathbf{0}$, that is $\xi_1^{(1)} = 2\xi_2^{(1)}$, so the eigenvector is $(2, 1)$.
- For $\lambda_2 = -1$, we want $\boldsymbol{\xi}^{(2)}$ satisfy that $\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} .\boldsymbol{\xi}^{(2)} = \mathbf{0}$, that is $2\xi_1^{(2)} = \xi_2^{(2)}$, so the eigenvector is $(1, 2)$.

Therefore, the general solution for the system of differential equation is:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{x} = C_1 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \boxed{\begin{pmatrix} 2C_1 e^{2t} + C_2 e^{-t} \\ C_1 e^{2t} + 2C_2 e^{-t}. \end{pmatrix}}.$$

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(b) By manipulating the first differential equation, we are able to obtain that:

$$x_2 = \frac{3}{2}x_1 - \frac{1}{2}x_1',$$

whose derivative is correspondingly:

$$x_2' = \frac{3}{2}x_1' - \frac{1}{2}x_1''.$$

Then, we substitute into the second differential equation to obtain that:

$$\frac{3}{2}x_1' - \frac{1}{2}x_1'' = 2x_1 - 2\left(\frac{3}{2}x_1 - \frac{1}{2}x_1'\right),$$

$$x_1'' - x_1' - 2x_1 = 0.$$

By solving the second order linear differential equation on $x_1 = x_1(t)$, its characteristic equation is $r^2 - r - 2 = 0$, which factors to $(r-2)(r+1)$, so the solution is:

$$x_1 = \boxed{D_1e^{2t} + D_2e^{-t}}.$$

Now, we take the derivative of x_1 to obtain that:

$$x_1' = 2D_1e^{2t} - D_2e^{-t},$$

so we can plug it into equation for x_2, that is:

$$x_2 = \frac{3}{2}(D_1e^{2t} + D_2e^{-t}) - \frac{1}{2}(2D_1e^{2t} - D_2e^{-t}) = \boxed{\frac{1}{2}D_1e^{2t} + 2D_2e^{-t}}.$$

Note: We hope that the readers have already realized that this is equivalent to the solution in part (a), simply having $D_1 = 2C_1$ and $D_2 = C_2$.

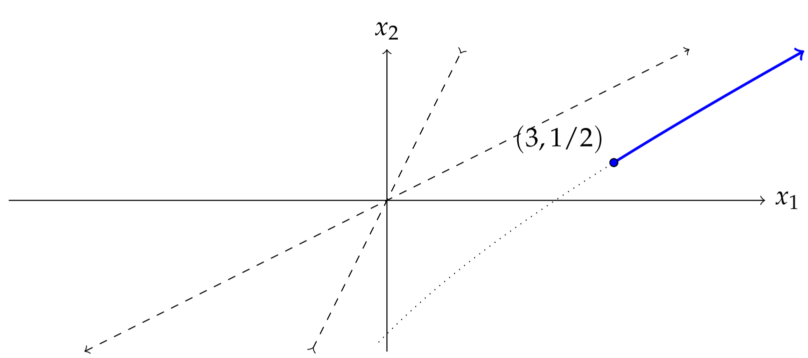
(c) Since the results are equal up to multiplying constants, we use the results from part (a). By plugging in the solutions, we have:

$$\begin{cases} 3 = x_1(0) = 2C_1 + C_2, \\ \frac{1}{2} = x_2(0) = C_1 + 2C_2. \end{cases}$$

This induces the solution that $3C_1 + 3C_2 = 7/2$, so $C_1 + C_2 = 7/6$, and thus $C_1 = 11/6$ and $C_2 = -2/3$, so the particular solution is:

$$x_1 = \boxed{\frac{11}{3}e^{2t} - \frac{2}{3}e^{-t}} \qquad x_2 = \boxed{\frac{11}{6}e^{2t} - \frac{4}{3}e^{-t}}.$$

The graph on the x_1x_2-plane can be visualized as follows:



2. (Deal with Complex Eigenvalues). Let $\mathbf{x} = (x_1, x_2, x_3)$ be in dimension 3, we define a linear system as follows:

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} .\mathbf{x}.$$

Find the general solution to the above system in terms of real valued functions.

Solution:

Diligent readers should notice that this is the exact same matrix with **Question 2, Problem Set 7**. Here, we omit the lengthy calculation for obtaining the eigenvalues and eigenvectors. Thus, the eigenvalues and eigenvectors are:

- For $\lambda_1 = 1$, we have eigenvector $\boldsymbol{\xi}^{(\mathbf{1})} = (2, -3, 2)$.
- For $\lambda_2 = 1 + 2\mathrm{i}$, we have eigenvector $\boldsymbol{\xi}^{(\mathbf{2})} = (0, 1, -\mathrm{i})$.
- For $\lambda_3 = 1 - 2\mathrm{i}$, we have eigenvector $\boldsymbol{\xi}^{(\mathbf{3})} = (0, 1, \mathrm{i})$.

Technically, we only need one complex eigenvalue for the conjugate pair, so you do not need to calculate the last eigenvector.

For λ_1, we trivially obtain one solution that $\mathbf{x} = C_1 e^t \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$.

For λ_2, we form a solution as:

$$\mathbf{x} = e^{(1+2\mathrm{i})t} \begin{pmatrix} 0 \\ 1 \\ -\mathrm{i} \end{pmatrix} = e^t \big(\cos(2t) + \mathrm{i} \sin(2t) \big) \begin{pmatrix} 0 \\ 1 \\ -\mathrm{i} \end{pmatrix} = \begin{pmatrix} 0 \\ e^t \cos(2t) \\ e^t \sin(2t) \end{pmatrix} + \mathrm{i} \begin{pmatrix} 0 \\ e^t \sin(2t) \\ -e^t \cos(2t) \end{pmatrix}.$$

Note that we have proven that $\{1, \mathrm{i}\}$ is a linearly independent set in a $\mathbb{R}$-vector space from **Question 4(a)(iii), Problem Set 6**, we obtain the other two solutions as:

$$\mathbf{x} = C_2 e^t \begin{pmatrix} 0 \\ \cos(2t) \\ \sin(2t) \end{pmatrix} \text{ and } \mathbf{x} = C_3 e^t \begin{pmatrix} 0 \\ \sin(2t) \\ -\cos(2t). \end{pmatrix}$$

Now, by the principle of superposition, we have the full set of solution as:

$$\mathbf{x} = \boxed{C_1 e^t \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} + C_2 e^t \begin{pmatrix} 0 \\ \cos(2t) \\ \sin(2t) \end{pmatrix} + C_3 e^t \begin{pmatrix} 0 \\ \sin(2t) \\ -\cos(2t). \end{pmatrix}}.$$

3. (Directional Field for Linear System). For the following systems with $\mathbf{x} = (x_1, x_2)$, draw a direction field and plot some trajectories to characterize the solutions.

   (a)
$$\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ -2 & 2 \end{pmatrix} .\mathbf{x}.$$

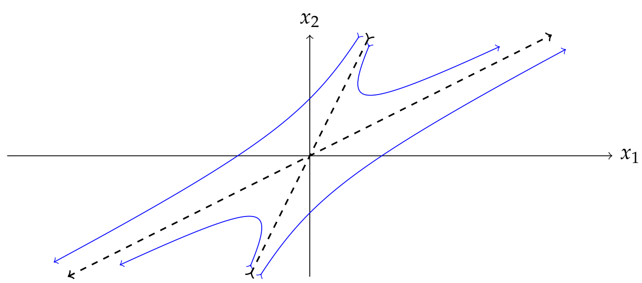
   (b)
$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} .\mathbf{x}.$$

Solution:

(a) Here, we note that this matrix is identical with **Question 1**, so we obtain the eigenvalues and eigenvectors as:

- For $\lambda_1 = 2$, the eigenvector is $(2, 1)$.
- For $\lambda_2 = -1$, the eigenvector is $(1, 2)$.

Hence, the plot of some trajectories can be visualized as:



The system is unstable.

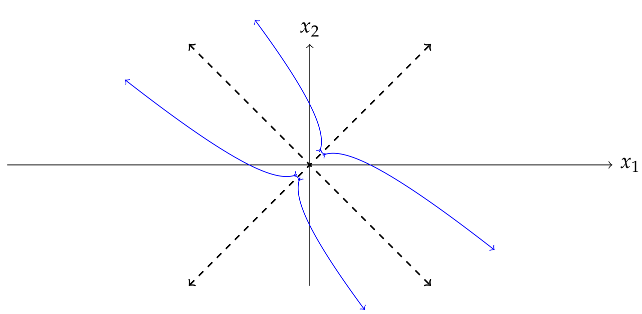
(b) Here, we actually want to solve for the eigenvalues and eigenvectors of the matrix, that is:

$$0 = \det \begin{pmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{pmatrix} = (2-\lambda)(2-\lambda) - 1 = (2-\lambda)^2 - 1 = 0.$$

Therefore, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 3$. Then, we look for the eigenvectors as:

- For $\lambda_1 = 1$, we want $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} .\boldsymbol{\xi}^{(1)} = \mathbf{0}$, hence $\xi_1^{(1)} = \xi_2^{(1)}$, so the eigenvector is $(1, 1)$.
- For $\lambda_2 = 3$, we want $\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} .\boldsymbol{\xi}^{(2)} = \mathbf{0}$, hence $\xi_1^{(2)} = -\xi_2^{(1)}$, so the eigenvector is $(-1, 1)$.

Hence, the plot of some trajectories can be visualized as:



The system is unstable.

4. (Zero Eigenvalue). Let a system of $\mathbf{x} = (x_1, x_2)$ be defined as:

$$\mathbf{x}' = \begin{pmatrix} -3 & -6 \\ 1 & 2 \end{pmatrix} .\mathbf{x}.$$

(a) Find the eigenvalues and eigenvectors for $\begin{pmatrix} -3 & -6 \\ 1 & 2 \end{pmatrix}$.

(b) Give a full set of solutions to the differential equation. Plot some trajectory on the x_1x_2-plane.

(c)* Let A be an arbitrary square matrix. Show that A is non-invertible if and only if A has zero as an eigenvalue.

Note: Please avoid using the definition that the determinant is the product of all eigenvalues. Moreover, consider the geometric implication of eigenvalue to account for invertibility.

Solution:

(a) Similarly, we find eigenvalue by:

$$0 = \det \begin{pmatrix} -3-\lambda & -6 \\ 1 & 2-\lambda \end{pmatrix} = (-3-\lambda)(2-\lambda) + 6 = \lambda^2 + \lambda = \lambda(\lambda+1),$$

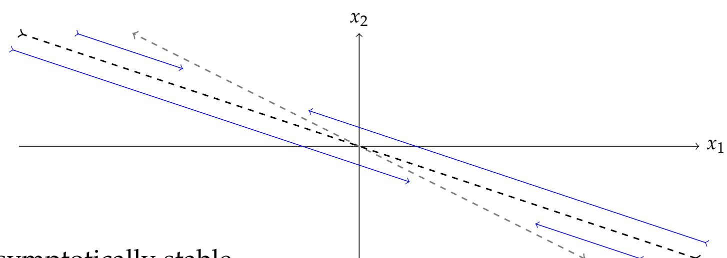
so the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -1$. Now, we look for the eigenvectors, as follows:

- For $\lambda_1 = 0$, we want $\begin{pmatrix} -3 & -6 \\ 1 & 2 \end{pmatrix} .\boldsymbol{\xi}^{(1)} = \mathbf{0}$, so $\xi_1^{(1)} = -2\xi_2^{(1)}$, so the eigenvector is $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.
- For $\lambda_2 = -1$, we want $\begin{pmatrix} -2 & -6 \\ 1 & 3 \end{pmatrix} .\boldsymbol{\xi}^{(2)} = \mathbf{0}$, so $\xi_1^{(2)} = -3\xi_2^{(2)}$, so the eigenvector is $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

(b) With the eigenvectors and eigenvalues, we trivially obtain the full set of solutions as:

$$\boxed{\left\{ C_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix}, C_2 e^{-t} \begin{pmatrix} 3 \\ -1 \end{pmatrix} \right\}}.$$

Graphically, the plot of some trajectories can be visualized as:



The system is asymptotically stable.

(c) Here, we advised readers not to consider the definition of determinant being the product of all eigenvalues, since this equivalence is trivial by that definition. Alternatively, we want readers to think of the definition of determinants in the scope of this class, *i.e.*, the root of $\det(A - \lambda\,\mathrm{Id})$.

Proof. Here, we have the following equivalences:

$$A \text{ is non-invertible} \iff \det(A) = 0 \iff \det(A - 0\,\mathrm{Id}) = 0 \iff 0 \text{ is an eigenvalue of } A.$$

As desired. □

Geometrically, think of eigenvalue monitoring the action in terms of a scalar multiplication, so having a zero eigenvalue collapse a dimension, making the matrix unable to be bijective. Conversely, if the map is not injective, we can easily form the kernel as a subspace whose eigenvalue is zero.

Problem Set 9: Solutions

Differential Equations

Fall 2024

1. (Repeated Eigenvalue). This problem investigates the case for repeated eigenvalues. First, we let the matrix $A \in \mathbb{R}^{2\times 2}$ be:
$$A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$
Here, we define the *algebraic multiplicity* of an eigenvalue as its multiplicity as a root to the characteristic polynomial, and the *geometric multiplicity* is the dimension of the eigenspace.

   (a) Find the eigenvalue and its corresponding eigenvector. State its algebraic and geometric multiplicity.

   (b) Find a the general solution to $\mathbf{x}' = A.\mathbf{x}$, where $\mathbf{x}' \in \mathbb{R}^2$.

   Then, we consider the diagonal n-by-n matrices, that is matrices with entries only on the diagonal, which can be characterized as:
$$D = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

   (c) Show that the eigenvalues are exactly $a_1, \cdots, a_n$, and the algebraic multiplicity is exactly the same as geometric multiplicity for all eigenvalues.

   (d) Consider the linear system $\mathbf{x} = D.\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^n$, solve for the general solution for $\mathbf{x} = (x_1, \cdots, x_n)$. Explain why do not have to find the eigenvalues in this case.

Solution:

(a) For the eigenvalue and eigenvector, we set:
$$0 = \det \begin{pmatrix} 2-\lambda & 1 \\ -1 & -\lambda \end{pmatrix} = (2-\lambda)(-\lambda) + 1 = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2.$$
Hence, the eigenvalue is $\boxed{\text{1 with algebraic multiplicity 2}}$.

Then, we consider the eigenvector, that is $\boldsymbol{\xi}$ such that $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.\boldsymbol{\xi} = \mathbf{0}$, hence $\xi_1 + \xi_2 = 0$, so we have $\xi_2 = -\xi_1$, so the eigenvalue is $\boxed{(1, -1) \text{ with geometric multiplicity 1}}$.

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(b) To find the solution to the differential equations, we first obtain a solution as:

$$\mathbf{x}^{(1)}(t) = \boxed{C_1 e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}}.$$

However, for the linear system, we want another solution with the repeated roots. Here, we think of the second root by having the vector $\boldsymbol{\eta}$ such that $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} .\boldsymbol{\eta} = \boldsymbol{\xi}$, so we have:

$$\begin{cases} \eta_1 + \eta_2 = \xi_1 = 1, \\ -\eta_1 - \eta_2 = \xi_2 = -1. \end{cases}$$

This solves to $\eta_1 + \eta_2 = 1$, so we have $\boldsymbol{\eta} = (\eta_1, 1 - \eta_1)$, and here, we consider $(0, 1)$ since $(1, -1)$ is the eigenvector already, so the second solution is:

$$\mathbf{x}^{(2)}(t) = \boxed{C_2 \left(te^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)}.$$

(c) *Proof*. Note that as we subtract λ on the diagonal, we have:

$$D - \lambda \operatorname{Id} = \begin{pmatrix} a_1 - \lambda & 0 & \cdots & 0 \\ 0 & a_2 - \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n - \lambda \end{pmatrix}.$$

Here, by the recursive definition of determinant, we disregard the zeros in the first row and entry, we have:

$$\begin{aligned} 0 &= \det \begin{pmatrix} a_1 - \lambda & 0 & \cdots & 0 \\ 0 & a_2 - \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n - \lambda \end{pmatrix} = (a_1 - \lambda) \det \begin{pmatrix} a_2 - \lambda & 0 & \cdots & 0 \\ 0 & a_3 - \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n - \lambda \end{pmatrix} \\ &= (a_1 - \lambda)(a_2 - \lambda) \det \begin{pmatrix} a_3 - \lambda & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n - \lambda \end{pmatrix} = (a_1 - \lambda)(a_2 - \lambda) \cdots (a_n - \lambda). \end{aligned}$$

Therefore, we know that the eigenvalues are exactly the diagonal matrices, with algebraic multiplicity as how many times they appear. Now, when we shift to the geometric multiplicity, it becomes the *kernel* (or null space) for $D - a_i \operatorname{Id}$, which we have:

$$D - a_i \operatorname{Id} = \begin{pmatrix} a_1 - a_i & 0 & \cdots & 0 \\ 0 & a_2 - a_i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n - a_i \end{pmatrix}.$$

Here, we consider the matrix as column vectors and only the columns with zero is increasing the dimension of the kernel by 1, so the dimension of the eigenspace is exactly the number of times that a_i appears, hence the geometric multiplicity is exactly the algebraic multiplicity. $\square$

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(d) With a diagonal system, our linear system degenerates into:

$$\begin{cases} x_1' = a_1 x_1, \\ x_2' = a_2 x_2, \\ \vdots \\ x_n' = a_n x_n. \end{cases}$$

Hence, the solution is exactly:

$$\boxed{\begin{cases} x_1(t) = C_1 e^{a_1 t}, \\ x_2(t) = C_2 e^{a_2 t}, \\ \vdots \\ x_n(t) = C_n e^{a_n t}. \end{cases}}$$

Note that this is really a simple case of single first order linear differential equations. *We suggest diligent readers to also think about how to solve if this is a linear system.*

2. (Complex Eigenvalue and Phase Portraits). Find the general solution and sketch a few phase portraits for:

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} .\mathbf{x}.$$

Solution:

It is standard process that we find the eigenvalues and eigenvector of $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, that is:

$$0 = \det \begin{pmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 + 1.$$

Hence the eigenvalues are $\lambda = 1 \pm \mathrm{i}$. Note that for solving complex system, we simply solve with one complex eigenvalue for the conjugate pair. We pick $\lambda = 1 - \mathrm{i}$ here, so we want eigenvector $\boldsymbol{\xi}$ such that $\begin{pmatrix} \mathrm{i} & 1 \\ -1 & \mathrm{i} \end{pmatrix} .\boldsymbol{\xi} = \mathbf{0}$. There, we have $\xi_1 = \mathrm{i}\xi_2$, so we have $\boldsymbol{\xi} = (\mathrm{i}, 1)$.

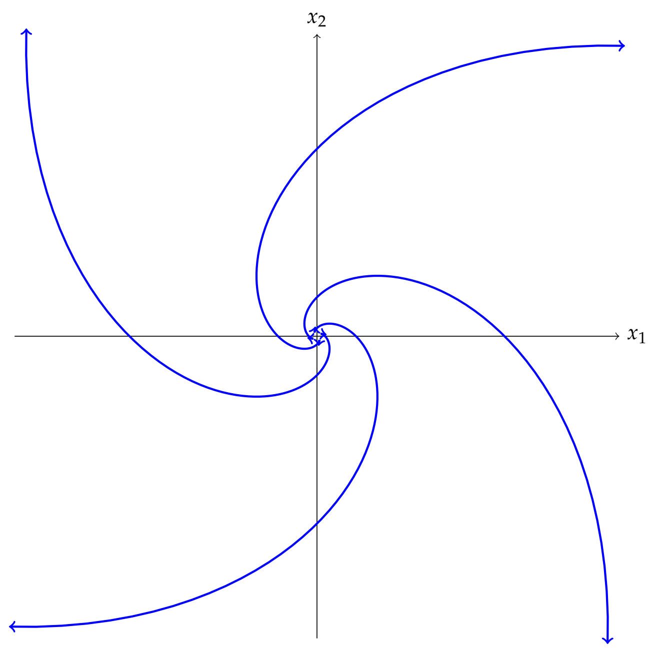
Therefore, we form a solution as:

$$\mathbf{x}(t) = e^{(1-\mathrm{i})t} \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix} = e^t(\cos t - \mathrm{i} \sin t) \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix} = e^t \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + \mathrm{i}e^t \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}.$$

Since the real and imaginary parts forms a linearly independent set, we have the solution as:

$$\mathbf{x}(t) = \boxed{C_1 e^t \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + C_2 e^t \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}}.$$

The phase portrait looks as follows:



3. (Stability). Complete the following table for stability of dimension 2 linear systems.

Solution:

Eigenvalues	Type	Stability
Eigenvalues are λ_1 and λ_2		
$0 < \lambda_1 < \lambda_2$	Node	Unstable
$\lambda_1 < \lambda_2 < 0$	Node	Asymptotically Stable
$\lambda_1 < 0 < \lambda_2$	Saddle Point	Unstable
$\lambda_1 = \lambda_2 > 0$	Node	Unstable
$\lambda_1 = \lambda_2 < 0$	Node	Asymptotically Stable
Eigenvalues are $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$		
$\alpha > 0$	Spiral Point	Unstable
$\alpha = 0$	Center	Stable
$\alpha < 0$	Spiral Point	Asymptotically Stable

We encourage diligent readers to sketch some *phase portraits* if possible for each case.

4. (Fundamental Matrix). Let a system be defined as:

$$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}.$$

Find the general solution using a fundamental matrix.

Solution:

We first solve for the eigenvalues and eigenvectors:

$$0 = \det \begin{pmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{pmatrix} = (3-\lambda)(-1-\lambda) + 8 = \lambda^2 - 2\lambda + 5.$$

Hence, the eigenvalues are $\lambda = 1 \pm 2\mathrm{i}$.

Since the two eigenvalues are complex conjugates, they will cancel out eventually. Then, we can look for the eigenvector for $\lambda_1 = 1 - 2i$, (you may also choose the other eigenvalue, this choice is discretionary to the reader), which is $\boldsymbol{\xi}^{(1)}$ such that $\begin{pmatrix} 2+2\mathrm{i} & -2 \\ 4 & -2+2\mathrm{i} \end{pmatrix} . \boldsymbol{\xi}^{(1)} = \mathbf{0}$, so we have $(2+2i)\xi_1^{(1)} = 2\xi_2^{(1)}$, so we the eigenvector is $\boldsymbol{\xi}^{(1)} = (1, 1+\mathrm{i})$. Therefore, we can get our solution as:

$$\begin{aligned} \mathbf{x} = e^{(1-2i)t} \begin{pmatrix} 1+\mathrm{i} \\ 1 \end{pmatrix} &= e^t \big(\cos(2t) - \mathrm{i}\sin(2t) \big) \begin{pmatrix} 1 \\ 1+\mathrm{i} \end{pmatrix} \\ &= e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + \mathrm{i}e^t \begin{pmatrix} -\sin 2t \\ \cos 2t - \sin 2t \end{pmatrix}. \end{aligned}$$

Notice that $\begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix}$ and $\begin{pmatrix} -\sin 2t \\ \cos 2t - \sin 2t \end{pmatrix}$ are linearly independent, it gives the fundamental matrix as:

$$\Psi = \boxed{e^t \begin{pmatrix} \cos 2t & -\sin 2t \\ \cos 2t + \sin 2t & \cos 2t - \sin 2t \end{pmatrix}}.$$

Problem Set 10: Solutions

Differential Equations

Fall 2024

1. (Stability for Nonlinear System). Complete the following table for stability of dimension 2 linear and nonlinear systems.

Solution:

Eigenvalues	**Linear System**		**Nonlinear System**	
	Type	**Stability**	**Type**	**Stability**
Eigenvalues are λ_1 and λ_2				
$0 < \lambda_1 < \lambda_2$	Node	Unstable	Node	Unstable
$\lambda_1 < \lambda_2 < 0$	Node	Asymptotically Stable	Node	Asymptotically Stable
$\lambda_1 < 0 < \lambda_2$	Saddle Point	Unstable	Saddle Point	Unstable
$\lambda_1 = \lambda_2 > 0$	Node	Unstable	Node or Spiral Point	Unstable
$\lambda_1 = \lambda_2 < 0$	Node	Asymptotically Stable	Node or Spiral Points	Asymptotically Stable
Eigenvalues are $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$				
$\alpha > 0$	Spiral Point	Unstable	Spiral Point	Unstable
$\alpha = 0$	Center	Stable	Center or Spiral Point	Indeterminate
$\alpha < 0$	Spiral Point	Asymptotically Stable	Spiral Point	Asymptotically Stable

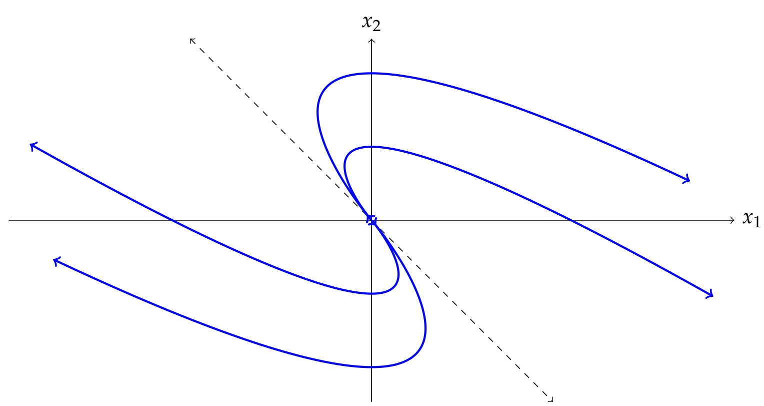
2. (Phase Portraits for Repeated Roots). Find the solutions to the following linear system differential equation, sketch a few phase portraits, and classify its type and stability.

(a)
$$\mathbf{x}' = \begin{pmatrix} 4 & 2 \\ -2 & 0 \end{pmatrix} \cdot \mathbf{x}.$$

(b)
$$\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \cdot \mathbf{x}.$$

Solution:

(a) Again, we first find the eigenvalues as:
$$0 = \det \begin{pmatrix} 4-\lambda & 2 \\ -2 & 0-\lambda \end{pmatrix} = (4-\lambda)(-\lambda) + 4 = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2.$$
Hence, the eigenvalue is 2 (with *algebraic* multiplicity 2, c.f. §9.1). Then, we look for the eigenvalue, that is $\boldsymbol{\xi}$ such that :
$$\begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix} \cdot \boldsymbol{\xi} = \mathbf{0}.$$
Hence, we have $2\xi_1 + 2\xi_2 = 0$, so $-\xi_1 = \xi_2$, which is $\boldsymbol{\xi} = (1, -1)$.
Then, we need to find the other solution with vector $\boldsymbol{\eta}$ such that:
$$\begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix} \cdot \boldsymbol{\eta} = \boldsymbol{\xi}.$$
Hence, we have $2\eta_1 + 2\eta_2 = 1$, so $\boldsymbol{\eta} = (1/2, 0)$. Hence, the solution is:
$$\mathbf{x} = \boxed{C_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 \left(t e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t} \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \right)}.$$
Here, we may graph the phase portraits as:



Here, the graph has a $\boxed{\text{unstable improper node}}$.

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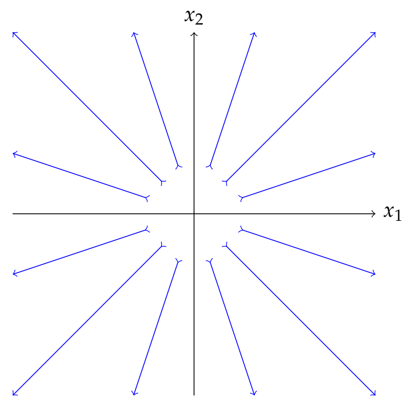
(b) Here, we find that the eigenvalue is 3 with *algebraic* multiplicity 2, and since it is diagonal, the *geometric* multiplicity is also 2. Here, we make this into a degenerated case:

$$\begin{cases} x_1' = 3x_1, \\ x_2' = 3x_2. \end{cases}$$

Hence, the solution is:

$$\begin{cases} x_1 = \boxed{C_1 e^{3t}}, \\ x_2 = \boxed{C_2 e^{3t}}. \end{cases}$$

Here, we may graph the phase portraits as:



Here, the graph has a $\boxed{\text{unstable proper node}}$.

3. (Critical Point). Find all the critical point in the following first order system:

$$\begin{cases} x' = 2x^3 - x^2 - 4x + 3 - y^2, \\ y' = 2x - y. \end{cases}$$

Solution:
Here, we set both equations as 0, so we have:

$$\begin{cases} 0 = 2x^3 - x^2 - 4x + 3 - y^2, \\ 0 = 2x - y. \end{cases}$$

Hence, we let the second equation be:

$$y = 2x,$$

and we plug it into the first equation to be:

$$0 = 2x^3 - x^2 - 4x + 3 - 4x^2 = 2x^3 - 5x^2 - 4x + 3.$$

Here, we attempt to factor the above by the rational root test (c.f. §1.3), the rational roots has to be one of $\pm 1, \pm 3, \pm 1/2, \pm 3/2$. We first note that $x = -1$ is a root, then we factor it into:

$$(2x^3 - 5x^2 - 4x + 3)/(x + 1) = 2x^2 - 7x + 3.$$

Thus, the last two roots are $1/2$ and 3. Hence, the critical points are:

$$\boxed{(-1, -2), (1/2, 1), \text{ and } (3, 6)}.$$

4. (Locally Linear System). Let a linear system be defined as follows:

$$\begin{cases} x' = x - y, \\ y' = x - 2y + x^2. \end{cases}$$

(a) Verify that $(0,0)$ is an equilibrium.

(b) Verify that the system is locally linear at $(0,0)$.

(c) Classify the type and stability of $(0,0)$ locally.

Solution:

(a) *Proof.* The verification is trivial. We evaluate x and y both at 0 for the differential equation, hence:

$$\begin{cases} x' = 0 - 0 = 0, \\ y' = 0 - 0 + 0 = 0. \end{cases}$$

Hence, $(0,0)$ is a equilibrium. □

(b) *Proof.* Here, we consider the Jacobian matrix as:

$$\mathrm{J}[x'(x,y), y'(x,y)] = \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2x+1 & -2 \end{pmatrix}.$$

Now, we evaluate the matrix at $(0,0)$, which gives:

$$\mathrm{ev}_0 \begin{pmatrix} 1 & -1 \\ 2x+1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0+1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix}.$$

Note that the determinant is $-2+1 = -1 \neq 0$, so the system is locally linear. □

(c) Here, the linear system locally at $(0,0)$ should be:

$$\begin{pmatrix} x \\ y \end{pmatrix}' \sim \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

We find its eigenvalue as:

$$0 = \det \begin{pmatrix} 1-\lambda & -1 \\ 1 & -2-\lambda \end{pmatrix} = (1-\lambda)(-2-\lambda) + 1 = \lambda^2 + \lambda - 1.$$

By using the quadratic formula, we have the eigenvalues as $\lambda = \frac{-1 \pm \sqrt{5}}{2}$.
Thus, we have $\lambda_1 < 0 < \lambda_2$, so we have a $\boxed{\text{unstable saddle point}}$.

Problem Set 11: Solutions

Differential Equations

Fall 2024

1. (System with Unknown Coefficients). Let a non-linear system for $x = x(t)$ and $y = y(t)$ be:

$$\begin{cases} x' = \alpha x - y + y^2, \\ y' = x + \alpha y. \end{cases}$$

   (a) Show that $(0,0)$ is a critical point, and show system is locally linear at $(0,0)$ for all $\alpha \in \mathbb{R}$.

   (b) Classify the critical point $(0,0)$ and sketch a few phase portraits of the linearized system.

Solution:

(a) *Proof*. To show that $(0,0)$ is a critical point, we just plug it into the system as:

$$\begin{cases} x'(0,0) = 0 - 0 + 0 = 0, \\ y'(0,0) = 0 + 0 = 0. \end{cases}$$

Hence $(0,0)$ is a critical point.

To consider the local linearity, we compute the Jacobian matrix as:

$$\mathrm{J}[x', y'] = \begin{pmatrix} \partial_x x' & \partial_y x' \\ \partial_x y' & \partial_y y' \end{pmatrix} = \begin{pmatrix} \alpha & -1 + 2y \\ 1 & \alpha \end{pmatrix}.$$

When we evaluate it at $(0,0)$, we have:

$$\mathrm{ev}_{(0,0)}\,\mathrm{J}[x', y'] = \begin{pmatrix} \alpha & -1 \\ 1 & \alpha \end{pmatrix}.$$

We see that the determinant is $\alpha^2 + 1 > 0$, hence it is locally linear for all $\alpha \in \mathbb{R}$. □

(b) For the locally linear system, we classify the linear approximation as:

$$\begin{pmatrix} x \\ y \end{pmatrix}' \sim \begin{pmatrix} \alpha & -1 \\ 1 & \alpha \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

Hence, the characteristic equation is $(\alpha - \lambda)^2 + 1 = 0$, that is $\lambda = \alpha \pm \mathrm{i}$.

Depending on different cases for α, we have different results:

- When $\alpha = 0$, it is a stable center.
- When $\alpha > 0$, it is unstable spiral.
- When $\alpha < 0$, it is asymptotically stable spiral.

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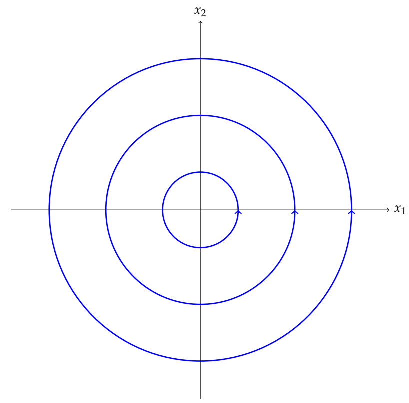
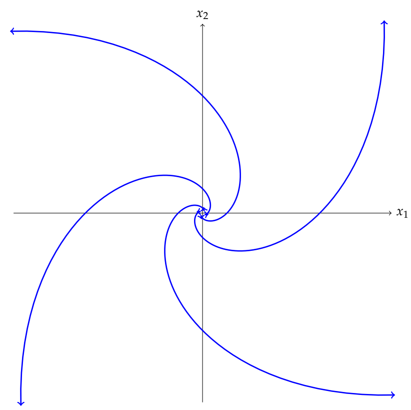
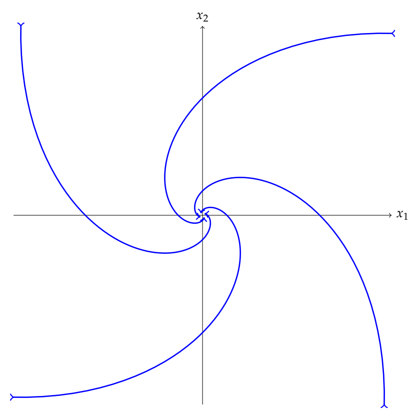
- For $\alpha = 0$:



- For $\alpha > 0$:



- For $\alpha < 0$:



2. (Nonlinear at origin). Let the linear system be:

$$\begin{cases} x' = y, \\ y' = x + 2x^3. \end{cases}$$

(a) Show that the origin is a saddle point.

(b) Sketch a phase portrait for the linearized system. Note that where all the trajectories of the linear system tend to the origin.

Solution:

(a) *Proof.* Here, we first verify that it is critical point, that is $x'(0,0) = 0$ and $y'(0,0) = 0 + 0 = 0$. Then, we check the Jacobian matrix:

$$\mathrm{ev}_{(0,0)}\,\mathrm{J}[x', y'] = \mathrm{ev}_{(0,0)} \begin{pmatrix} 0 & 1 \\ 1 + 6x^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

whose determinant is -1, so the system is locally linear, so the linear system is:

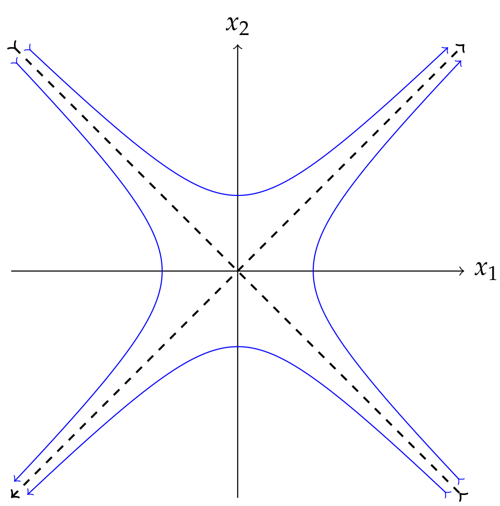
$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

Hence, we have the characteristic equation as $\lambda^2 - 1 = 0$, so the roots are $\lambda_1 = 1$ and $\lambda_2 = -1$. Thus, it is a unstable saddle point. $\square$

(b) To sketch the diagram, we would want the eigenvectors.

- When $\lambda_1 = 1$, then we have $\boldsymbol{\xi}^{(1)}$ such that $-\xi_1^{(1)} + \xi_2^{(1)} = 0$, so the eigenvector is $(1, 1)$.
- When $\lambda_2 = -1$, then we have $\boldsymbol{\xi}^{(2)}$ such that $\xi_1^{(1)} + \xi_2^{(1)} = 0$, so the eigenvector is $(1, -1)$.

Hence, the diagram is as:



3. (Modeling Politics). Suppose D and R are two parties on a non-existing country on the center of Mars. For the simplicity of this problem, they, *unfortunately*, have no elections. Therefore, we can model the amount of the supporter for each party (in millions), denoted x_D and x_R with the following relationship:
$$\begin{cases} \dfrac{dx_D}{dt} = x_D(1 - x_D - x_R), \\ \dfrac{dx_R}{dt} = x_R(3 - 2x_D - 4x_R). \end{cases}$$
Find all possible endings (say arbitrarily long after, that is $t \to \infty$) of the number of supporters (in millions) for the two parties.

Solution:
Alright, we do assume that these parts are just having random letters, so we do not dive into actual politics. This is a typical "Competing Species" models, in which the two parties are competing for the limited resources.
First find the critical points, that is:
$$\begin{cases} x_D(1 - x_D - x_R) = 0, \\ x_R(3 - 2x_D - 4x_R) = 0. \end{cases}$$
We know that we could have many cases:

- When $x_D = x_R = 0$, we have both equations being 0.
- When $x_D = 0$, we can also have $x_R = 3/4$.
- When $x_R = 0$, we can also have $x_D = 1$.
- Or, we can have $1 - x_D - x_R = 0$ and $3 - 2x_D - 4x_R = 0$, and in this case, we have $x_D = x_R = 1/2$.

Here, we note that the Jacobian matrix is:
$$J[x_D', x_R'] = \begin{pmatrix} 1 - 2x_D & -x_D \\ -2x_R & 3 - 8x_R \end{pmatrix}.$$
It can be easily verified that the system are locally linear at all of the critical points, and we leave this as an exercise to the readers.
Here, as we inspect these points, we could conclude that:

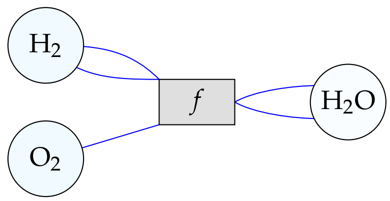
- Case 1: $x_D = x_R = 0$. This is when two party has no supporting population, it they remain uncared.
- Case 2: $x_D = 0$ and $x_R = 3/4$. This is when the R party got the support initially, and they keep absolute advantage over the D party.
- Case 3: $x_R = 0$ and $x_D = 1$. This is when the D party got the support initially, and they keep absolute advantage over the R party.
- Case 4: $x_D = x_R = 1/2$. This is when two party all got some initial support. While after some political campaigns (maybe also fightings), they got to a balanced equilibrium.

If the readers are still confused, we recommend them taking a look at the *directional field*. A website of the directional field can be found here.

4. (Chemical Reaction). Consider the following chemical equation of hydrogen gas combustion in oxygen gas:

$$2H_2 + O_2 \xrightarrow{f} 2H_2O.$$

We may represent it from a graphical representation.

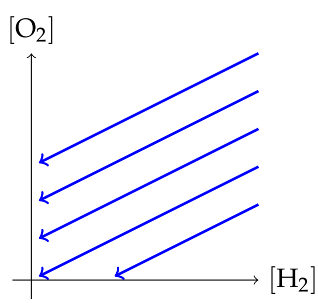


Assume that the reaction rate is constant $\kappa := \text{rate}(f)$. Construct the nonlinear system of the concentration of H_2 and O_2, sketch a few trajectories for different initial conditions for different starting concentrations.

Solution:
For the model, we recall that the reaction rate is proportional to the concentration of the current amount of reactants. Hence, we obtain the system as:

$$\boxed{\begin{cases} \dfrac{d[H_2]}{dt} = -2\kappa[H_2][O_2], \\ \dfrac{d[O_2]}{dt} = -\kappa[H_2][O_2]. \end{cases}}$$

This is system is of course not linear, but we may notice that the rates are exactly the same, except for the linear factor $1/2$, so that means that the concentration of H_2 will be consuming twice as fast as the concentration of O_2 being consumed.
If we are to plot this, we should have the line with a slope of $1/2$, namely:



If you are interested in graphical (or more precisely *categorical*) representation of chemical equations, please refer to this slide deck.

Problem Set 12: Solutions

Differential Equations

Fall 2024

1. (Limit Cycles). Determine the periodic solution, if there are any, of the following system:

$$\begin{cases} x' = y + \dfrac{x}{\sqrt{x^2+y^2}}(x^2+y^2-2), \\ y' = -x + \dfrac{y}{\sqrt{x^2+y^2}}(x^2+y^2-2). \end{cases}$$

Solution:

Here, we recall the formula converting between polar coordinates and Cartesian coordinates:

$$\begin{cases} x = r\cos\theta, & y = r\sin\theta, \\ rr' = xx' + yy', & r^2\theta' = xy' - yx'. \end{cases}$$

Now, we are able to convert the system as:

$$\begin{cases} rr' = x\left[y + \dfrac{x}{\sqrt{x^2+y^2}}(x^2+y^2-2)\right] + y\left[-x + \dfrac{y}{\sqrt{x^2+y^2}}(x^2+y^2-2)\right], \\ r^2\theta' = x\left[-x + \dfrac{y}{\sqrt{x^2+y^2}}(x^2+y^2-2)\right] - y\left[y + \dfrac{x}{\sqrt{x^2+y^2}}(x^2+y^2-2)\right]. \end{cases}$$

Here, by simple deductions, we trivially have:

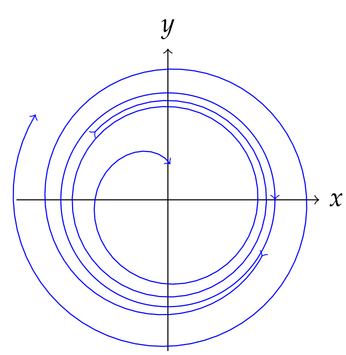
$$rr' = \frac{x^2+y^2}{\sqrt{x^2+y^2}}(x^2+y^2-2) = \frac{r^2}{r}(r^2-2) \rightsquigarrow r' = r^2-2.$$

$$r^2\theta' = -x^2-y^2 = -r^2 \rightsquigarrow \theta' = -1.$$

Thereby, we consider the radius as:

$$r' = r^2 - 2 = (r-\sqrt{2})(r+\sqrt{2}).$$

Hence, we note that the critical point is $\boxed{r = \sqrt{2}}$ (since r must be positive). Note that $r' < 0$ for $0 < r < \sqrt{2}$ and $r' > 0$ for $r > \sqrt{2}$. Hence, this is an unstable limit cycle.



2. (Converging Sequences). In this question, we will review some common power series.

   (a) Construct the power series of e^x, $\sin x$, and $\cos x$ centered at 0.

   (b) Consider the following power series:
$$\sum_{k=0}^{\infty} \frac{x^{4k+3}}{(4k+3)!}.$$
   Identify if such series converges. Compute the limit if the series converges.

Solution:

(a) Recall that we have the power series of Taylor's polynomial centered at $x_0 = 0$ being:
$$f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$
Consider the derivatives of e^x being always e^x, so we have $\frac{d^n e^x}{dx^n}(0) = 1$, thus:
$$e^x \sim \boxed{\sum_{n=0}^{\infty} \frac{1}{n!} x^n}.$$
Then, we consider the derivatives of sine and cosine that cycles every 4 times, so we have:
$$(\sin x)' = \cos x, \qquad (\sin x)'' = -\sin x, \qquad (\sin x)''' = -\cos x, \qquad (\sin x)'''' = \sin x.$$
$$(\cos x)' = -\sin x, \qquad (\cos x)'' = -\cos x, \qquad (\cos x)''' = \sin x, \qquad (\cos x)'''' = \cos x.$$
Hence, it is trivial to conclude that:
$$\sin x \sim \boxed{\sum_{n=0}^{\infty} \left(\frac{x^{4n+1}}{(4n+1)!} - \frac{x^{4n+3}}{(4n+3)!} \right)},$$
$$\cos x \sim \boxed{\sum_{n=0}^{\infty} \left(\frac{x^{4n}}{(4n)!} - \frac{x^{4n+2}}{(4n+2)!} \right)}.$$

(b) Here, we also consider the power series of e^{-x}, namely:
$$e^{-x} \sim \sum_{n=0}^{\infty} \left(\frac{1}{(2n)!} x^{2n} - \frac{1}{(2n+1)!} x^{2n+1} \right).$$
For simplicity, we expand all the terms out, explicitly as:
$$\begin{array}{lcccccccc}
e^x & \sim & +\frac{x^0}{0!} & +\frac{x^1}{1!} & +\frac{x^2}{2!} & +\frac{x^3}{3!} & +\frac{x^4}{4!} & +\frac{x^5}{5!} & +\cdots \\
e^{-x} & \sim & +\frac{x^0}{0!} & -\frac{x^1}{1!} & +\frac{x^2}{2!} & -\frac{x^3}{3!} & +\frac{x^4}{4!} & -\frac{x^5}{5!} & +\cdots \\
\sin x & \sim & & +\frac{x^1}{1!} & & -\frac{x^3}{3!} & & +\frac{x^5}{5!} & -\cdots \\
\cos x & \sim & +\frac{x^0}{0!} & & -\frac{x^2}{2!} & & +\frac{x^4}{4!} & & -\cdots
\end{array}$$
By some arithmetics, one should notice that:
$$\sum_{k=0}^{\infty} \frac{x^{4k+3}}{(4k+3)!} = \boxed{\frac{e^x - e^{-x}}{4} - \frac{\sin x}{2}}.$$
Hence, the power series converges.

3. (Analytic Function). Recall the definition of analytic function in class defined for $f : \mathbb{R} \to \mathbb{R}$ at a fixed point $x \in \mathbb{R}$.

   (a) Write down the power series of $f(x) = \frac{1}{1-x}$ around 0. Show that $f(x)$ is analytic at 0.

   Here, we extend the definition to function with complex input and complex output, say $f : \mathbb{C} \to \mathbb{C}$. We can similarly define f being analytic at $x_0 \in \mathbb{C}$ when the function has a positive radius of convergence at x_0.

   (b) Convince yourself that if $f : \mathbb{C} \to \mathbb{C}$ is analytical over $\mathbb{C}$ and $f(\mathbb{R}) \subset \mathbb{R}$, then $f\mid_{\mathbb{R}}$ is analytical $\mathbb{R}$.

   By Cauchy-Riemann equations, $f(z) := f(x + \mathrm{i}y)$ being analytical is identical with:
$$\frac{\partial f}{\partial x} = -\mathrm{i}\frac{\partial f}{\partial y}.$$

   (c) Show that the Möbius transform $\psi : \mathbb{C} \to \mathbb{C}$ such that $\psi_a(z) := \frac{a-z}{1-\bar{a}z}$ is analytic on $\mathbb{C} \setminus \{1/\bar{a}\}$. Conclude with a condition for which $\psi_a(z)\mid_{\mathbb{R}}$ is analytic on some set A.

Solution:

(a) Consider $f(x) = \frac{1}{1-x}$, when we are around zero, consider $|x| < 1$, we can think of the function as the sum of the geometric series, so we have:
$$f(x) \sim 1 + x + x^2 + \cdots = \boxed{\sum_{n=0}^{\infty} x^n}.$$
Since for all $|x| < 1$, we can think of it as a geometric series, that is:
$$1 + x + x^2 + \cdots = \frac{1}{1-x},$$
so we have pointwise convergence for all $|x| < 1$, so it is analytic at 0.

(b) *Proof.* Suppose $f : \mathbb{C} \to \mathbb{C}$ is analytic over $\mathbb{C}$, there exists complex power series around 0:
$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$
By assumption that $f(\mathbb{R}) = \mathbb{R}$, that means that for any $x \in \mathbb{R}$, we have $f(x) \in \mathbb{R}$. Hence, $f\mid_{\mathbb{R}}$ has real output, then we can write it still as the power series expansions. Since at every point $z \in \mathbb{C}$, we have a radius of convergence of ρ, that is, for all $z_0 \in \mathbb{C}$ such that $|z - z_0| < \rho$ in which the power sequence converges. Hence for any $x \in \mathbb{R}$, we have all $x_0 \in \mathbb{R}$ such that $|x - x_0| < \rho$ in which the power sequence converges at x_0, thus $f\mid_{\mathbb{R}}$ has a positive radius of convergence for all $x \in \mathbb{R}$, so it is analytic. □

(c) *Proof.* Given the Cauchy-Riemann equation, we can write the Möbius transformation as:
$$\psi_a(z) = \frac{a - x - \mathrm{i}y}{1 - \bar{a}x - \bar{a}\mathrm{i}y}.$$
Then, we may take the partial derivatives as:
$$\frac{\partial \psi_a}{\partial x} = \frac{-(1 - \bar{a}x - \bar{a}\mathrm{i}y) - (a - x - \mathrm{i}y)(-\bar{a})}{(1 - \bar{a}x - \bar{a}\mathrm{i}y)^2},$$
$$\frac{\partial \psi_a}{\partial y} = \frac{-\mathrm{i}(1 - \bar{a}x - \bar{a}\mathrm{i}y) - (a - x - \mathrm{i}y)(-\mathrm{i}\bar{a})}{(1 - \bar{a}x - \bar{a}\mathrm{i}y)^2}.$$

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Here, we note that:

$$-\mathrm{i}\frac{\partial\psi_a}{\partial y} = \frac{-(1-\overline{a}x-\overline{a}\mathrm{i}y)-(a-x-\mathrm{i}y)(-\overline{a})}{(1-\overline{a}x-\overline{a}\mathrm{i}y)^2}.$$

Hence, we observe that the two expressions are the same when $\overline{a}(x+\mathrm{i}y) \neq 1$, that is $(x+\mathrm{i}y) \neq 1/\overline{a}$. Thus, we have shown that $\psi_a(z)$ is in fact analytic at all $\mathbb{C} \setminus \{1/\overline{a}\}$. $\square$

Then, recall from the precious part, as long as $a \in \mathbb{R}$, we can guarantee that $\psi_a(\mathbb{R}) \subset \mathbb{R}$ (since addition, subtraction, multiplication, and nonzero divisions of real numbers are closed).

Thus, when $a \in \mathbb{R}$, $\psi_a(z)\mid_{\mathbb{R}}$ is analytic on $\mathbb{R} \setminus \{1/a\}$.

Note that the Cauchy-Riemann equation is an important consequence in *complex analysis*. In complex analysis, there has been an equivalence established between holomorphic functions and analytic functions. It is noteworthy for diligent readers to think about the relationship in (real) power series between differentiability and analytic properties.

4. (Recurrence Relation). Solve the following differential equation using power series method. Include the recurrence relation.

$$y'' + y = 0.$$

Solution:
Here, we note that we have constant coefficients, so they are automatically analytic. Now, we take $x_0 = 0$, and assume that our solution is in the form that:

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

Now, by the assumption that the series converges absolute, we take differentiate the terms twice, which gives that:

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1},$$

and:

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

With the derivative, we plug it back into the differential equations, that is:

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$

By the term-wise addition, we have:

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n \right] x^n = 0.$$

Given that the sequence is equivalently zero, then we have the recurrence relation as:

$$\boxed{a_{n+2} = -\frac{a_n}{(n+2)(n+1)} \text{ for all } n \geq 0}.$$

Given that this differential equation has order 2, we let the first two coefficients fixed, that is a_0 and a_1, then we can form the rest of the coefficients as:

$$\begin{aligned} a_2 &= -\frac{a_0}{2 \cdot 1}, & a_4 &= -\frac{a_2}{4 \cdot 3} = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1}, & &\cdots, & a_{2n} &= \frac{(-1)^n a_0}{(2n)!}, \\ a_3 &= -\frac{a_1}{3 \cdot 2}, & a_5 &= -\frac{a_3}{5 \cdot 4} = \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2}, & &\cdots, & a_{2n+1} &= \frac{(-1)^n a_1}{(2n+1)!}. \end{aligned}$$

When we sum all the terms, we have that:

$$\begin{aligned} y(x) &= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + a_1 \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} \\ &= a_0 \sum_{n=0}^{\infty} \left(\frac{x^{4n}}{(4n)!} - \frac{x^{4n+2}}{(4n+2)!} \right) + a_1 \sum_{m=0}^{\infty} \left(\frac{x^{4m+1}}{(4m+1)!} - \frac{x^{4m+3}}{(4m+3)!} \right) \\ &= \boxed{a_0 \cos x + a_1 \sin x}. \end{aligned}$$

We encourage diligent readers to also attempt this particular question with the characteristic polynomial method, and *hopefully* obtain the same set of linearly independent solution (unique up to linear combinations).

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Problem Set 13: Solutions

Differential Equations

Fall 2024

1. (Logarithms and Recurrence Relations). The following problem aims to solve the differential equation for $y := y(x)$:
$$(x+1)^2 \frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} = 0.$$
using recurrence relationship.

   (a) Write down the power series of $\log(x+1)$.

   (b) Find the *recurrence relationship* for the differential equation.

   (c)* Find the fundamental set of solutions for the differential equation.
   Hint: Make a conjecture from a pattern of the first few terms.

Solution:

(a) Here, we note that:
$$\frac{d}{dx}\big(\log(x+1)\big) = \frac{1}{x+1} = \frac{1}{1+x},$$
hence, as how we considered for geometric sequence, we have:
$$\frac{1}{1+x} = 1 + (-x) + (-x)^2 + \cdots = 1 - x + x^2 - x^3 + \cdots$$
Hence, we learned that the coefficients are just simply:
$$\mathrm{ev}_0\left(\frac{d^n}{dx^n}\frac{1}{1+x}\right) = (-1)^n n!.$$
Hence, we can easily deduce the coefficients for $\log(x+1)$, that is:
$$\mathrm{ev}_0\left(\frac{d^n}{dx^n}\big(\log(x+1)\big)\right) = (-1)^{n-1}(n-1)! \text{ for } n \geq 1.$$
Now, we can form the power series of $\log(x+1)$ as:
$$\log(x+1) \sim \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}}.$$

(b) Now, we are handling the recurrence relationship. First, we assume that the solution is:
$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$
by the assumption that the series converges absolute, we take differentiate the terms twice, which gives that:
$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n,$$

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and:

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n.$$

With the derivative, we plug it back into the differential equations, that is:

$$\begin{aligned}
0 &= (x+1)^2 \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + (x+1)\sum_{n=1}^{\infty} na_n x^{n-1} \\
&= x^2 \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + 2x\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n \\
&\quad + x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n \\
&= \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} 2(n+1)na_{n+1}x^n + \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n \\
&\quad + \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n \\
&= \sum_{n=2}^{\infty}\left[n(n-1)a_n + 2(n+1)na_{n+1} + (n+2)(n+1)a_{n+2} + na_n + (n+1)a_{n+1}\right]x^n \\
&\quad + a_1 + 2a_2 + 4a_2x + 6a_3x + a_1x + 2a_2x.
\end{aligned}$$

Now, we suppose the initial conditions a_0 and a_1, we have the relationship:

$$\begin{cases} a_1 + 2a_2 = 0, \\ 6a_3 + 6a_2 + a_1 = 0, \\ n(n-1)a_n + 2(n+1)na_{n+1} + (n+2)(n+1)a_{n+2} + na_n + (n+1)a_{n+1} = 0 \text{ for all } n \geq 2. \end{cases}$$

In particular, we can find the recurrence relationship as:

$$a_{n+2} = \boxed{-\frac{n^2 a_n + (n+1)(2n+1)a_{n+1}}{(n+2)(n+1)}} \text{ for } n \geq 2.$$

(c) Now, since we have a second order differential equation, we set:

$$a_0 = a_0, \qquad a_1 = a_1, \qquad a_2 = -\frac{1}{2}a_1, \qquad a_3 = \frac{-a_1 - 6a_2}{6} = \frac{a_1}{3},$$
$$a_4 = \frac{-15a_3 + 4a_2}{12} = -\frac{a_1}{4}, \qquad a_5 = \frac{-9a_3 - 28a_4}{20} = \frac{a_1}{5}, \qquad \cdots.$$

Here, it is fair to conjecture that $a_n = \frac{(-1)^{n-1}a_1}{n}$. In fact, we can prove this by strong induction.

Proof. The base case is already check in the previous argument, so we check on the inductive step, suppose that:

$$a_k = \frac{(-1)^{k-1}a_1}{k} \text{ and } a_{k+1} = \frac{(-1)^k a_1}{k+1}.$$

Then, using the relationship above, we have:

$$\begin{aligned}
a_{k+2} &= -\frac{\frac{k^2(-1)^{k-1}a_1}{k} + \frac{(-1)^k a_1(k+1)(2k+1)}{k+1}}{(k+2)(k+1)} \\
&= -\frac{a_1\big(2k+1-k\big)(-1)^k}{(k+2)(k+1)} = -\frac{a_1(-1)^k}{k+2} = \frac{a_1(-1)^{k+2-1}}{k+2},
\end{aligned}$$

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which completes the proof. □

Therefore, we can write our solution, by part (a), as:

$$y(x) := a_0 + a_1 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = a_0 + a_1 \log(1+x),$$

so the fundamental set of solutions are:

$$\boxed{1 \quad \text{and} \quad \log(1+x)}.$$

2. (Euler's Equations). Let a differential equation of $y := y(x)$ defined as:

$$x^2y'' + xy' + cy = 0,$$

where $c \in \mathbb{R}$ is a fixed constant, we want to solve the differential equation using *Euler's equations*.

(a) Assume $c = -4$, solve the solution to the differential equation.

(b) Assume $c = 9$, solve the solution to the differential equation.

(c)* Find the critical point to this system where the behavior of the solution changes.

Solution:

(a) For $c = -4$, we have:

$$x^2y'' + xy' - 4y = 0,$$

and our characteristic equation is:

$$0 = r(r-1) + r - 4 = r^2 - 4,$$

whose roots are ± 2, so the solution is:

$$y(x) = c_1|x|^2 + c_2|x|^{-2} = \boxed{c_1x^2 + c_2x^{-2}}.$$

(b) For $c = 9$, we have:

$$x^2y'' + xy' + 9y = 0,$$

and our characteristic equation is:

$$0 = r(r-1) + r + 9 = r^2 + 9,$$

whose roots are $\pm 3\mathrm{i}$, so the solution is:

$$y(x) = c_1|x|^0 \cos\left(3\log|x|\right) + c_2|x|^0 \sin\left(3\log|x|\right) = \boxed{c_1\cos(3\log|x|) + c_2\sin(3\log|x|)}.$$

(c) Consider that we have the characteristic equation:

$$0 = r(r+1) + r + c = r^2 + c.$$

Hence, we know that the system has two distinct real roots when $c < 0$, it has repeated zero roots when $c = 0$, and complex roots when $c > 0$, so the critical point is $\boxed{c = 0}$.

3. (Singularities, Zeros, and Poles). For any function $f : \mathbb{C} \to \mathbb{C}$, and $z_0 \in \mathbb{C}$, we have the following:

   - It has **a zero of order *m*** at z_0 if $f(z_0) = 0$, and m is the smallest positive integer such that $f(z) = (z - z_0)^m g(z)$, where g is analytic at z_0 and $g(z_0) \neq 0$.
   - It has **a pole of order *n*** at z_0 if $f(z_0)$ is not defined, and n is the smallest integer such that $g(z) = (z - z_0)^n f(z)$, where g is analytic at z_0 and $g(z_0) \neq 0$.
   - If a zero/pole has order 1, it is **simple**.

   As a side note, such definition applies for any real valued functions, *i.e.*, $f : \mathbb{R} \to \mathbb{R}$.
   Here, we define a differential equation for $y := y(x)$ as:

   $$\sin(x)y'' + \sin(x)\big(\cos(x) - e^x + x\big)y' + \big(\csc(x)\big)y = 0$$

   (a) Write the differential equation in the form of:

   $$y'' + p(x)y' + q(x)y = 0.$$

   (b)* Identify all zeros and poles of $p(x)$ and $q(x)$ as real functions, *i.e.*, $p, q : \mathbb{R} \to \mathbb{R}$. Find the order of the zeros and poles.

   (c) Identify all the points $x_0 \in \mathbb{R}$ such that the differential equation has a *regular singular point*.

Solution:

(a) Here, we easily write this as $\boxed{y'' + \big(\cos(x) - e^x + x\big)y' + \dfrac{1}{\sin^2(x)}y = 0}$.

(b) Here, we have that:

$$p(x) = \cos(x) - e^x + x \text{ and } q(x) = \frac{1}{\sin^2(x)}.$$

Note that $p(x)$ is the sum of analytic functions, so it has no poles, we note that $p(x)$ has zero when $0 = \cos(x) - e^x + x$. We note that the derivative of $p(x)$ is:

$$p'(x) = -\sin(x) - e^x + 1.$$

It is not hard to observe that $p'(x) \leq 0$ for all $x > 0$ and $p(x) \geq 0$ for all $-2 < x < 0$, and we note that $p(0) = 0$, so it must be the unique zero. Now, we determine its order, we note that:

$$\lim_{x\to 0} \frac{p(x)}{x} = \lim_{x\to 0} \frac{\cos(x) - e^x + x}{x} \overset{\text{L'Hôpital}}{=\!=} \lim_{x\to 0} \frac{-\sin(x) - e^x + 1}{1} = 0.$$

Then, we need to take second order again:

$$\lim_{x\to 0} \frac{p(x)}{x^2} = \lim_{x\to 0} \frac{\cos(x) - e^x + x}{x^2} \overset{\text{L'Hôpital}}{=\!=} \lim_{x\to 0} \frac{-\sin(x) - e^x + 1}{2x} \overset{\text{L'Hôpital}}{=\!=} \lim_{x\to 0} \frac{-\cos x - e^x}{2} = -1.$$

Hence, $p(x)$ has $\boxed{\text{a zero of order 2 at } x = 0}$.
For $q(x)$, we note that it is nonzero, but it is undefined when $\sin(x) = 0$, that is all $k\pi$ for $k \in \mathbb{Z}$.
For $\sin x$, we note that it has zeros there, and each of them has order 1, since we have:

$$\text{ev}_{k\pi}\left[\frac{d}{dx}(\sin x)\right] = \cos(k\pi) = 1 \neq 0,$$

thus we have shown that $\sin x$ has zeros of order 1 at all $k\pi$, so $\sin^2(x)$ has zero of order 2 at all $k\pi$, and thus $\frac{1}{\sin^2(x)}$ has $\boxed{\text{poles of order 2 at all } k\pi}$.

(c) Hence, all the singular points are $k\pi$ for $k \in \mathbb{Z}$, and since they have poles of order 2, all of them are regular singular points. Thus, all the regular singular points are $\boxed{k\pi \text{ for } k \in \mathbb{Z}}$.

4. (Dispersion of Heat). For this problem, we consider the dispersion of heat for an object in an environment with fixed temperature. Here, let $\theta := \theta(t)$ be the temperature of the object and θ_0 denote the fixed temperature of the environment, we may model the temperature of the object by:
$$\frac{d\theta}{dt} = -\frac{1}{\kappa}(\theta - \theta_0),$$
where κ is a fixed positive constant, representing the rate of heat dispersion.
Suppose that we have a rigid body of 100°C (equivalently 212°F), and the room temperature is fixed as 20°C (equivalently 68°F, and this is also condition for STP, standard temperature and pressure). Also, we assume that $\kappa = 2$.

   (a) Construct the differential equation for the above system.

   (b) Use *Euler's method* with step size of 1 to approximate the temperature at $t = 3$.

   (c)* Identify if the approximation of temperature is an underestimate or an overestimate.

Solution:

(a) The system can be easily constructed, namely:
$$\boxed{\frac{d\theta}{dt} = -\frac{1}{2}(\theta - 20)}.$$

(b) For Euler's method of step size 1, with $\theta(0) = 100$, we have $\theta'(0) = -80/2 = -40$. We do the following steps:

- We approximate $\theta(1) \approx \theta(0) + \theta'(0) = 100 - 40 = 60$, then we have $\theta'(1) \approx -40/2 = -20$.
- We approximate $\theta(2) \approx \theta(1) + \theta'(1) \approx 60 - 20 = 40$, then we have $\theta'(2) \approx -20/2 = -10$.
- We approximate $\theta(3) \approx \theta(2) + \theta'(2) \approx 40 - 10 = 30$.

Then, we have approximated that:
$$\theta(3) \approx \boxed{30(°\text{C})}.$$

(c) To show underestimate or overestimate, we note that:
$$\frac{d^2\theta}{dt^2} = -\frac{1}{2} < 0,$$
so the function is concave (typically called *concave down* in high school). Hence, we know that the tangent line is having a smaller slope compared to the actual curve, hence it is decreasing slower than actual, so the temperature is an $\boxed{\text{overestimate}}$.