# Problem Set 13

Differential Equations

Fall 2024

Welcome back from the Fall Recess. We hope that you have gained some nice rests and got ready to end the semester strong! While we finish up the digression into series solutions for second order differential equation, make sure that you are familiar with the following concepts:

- Concepts:
  - Analytic at a Point
  - Pointwise Convergence
  - Absolute Convergence
  - Ordinary/Singular Point
  - Regular/Irregular Singularity
- Methodologies:
  - Recurrence Relationship
  - Euler's Equations
  - Regular Singularity

As you are familiar with the above models, we are also moving towards of utilizing series to solve second order linear differential equations.

1. (Logarithms and Recurrence Relations). The following problem aims to solve the differential equation for $y := y(x)$:
$$(x+1)^2\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} = 0.$$
using recurrence relationship.
   (a) Write down the power series of $\log(x+1)$.
   (b) Find the *recurrence relationship* for the differential equation.
   (c)* Find the fundamental set of solutions for the differential equation.
   *Note: Make a conjecture from a pattern of the first few terms.*

2. (Euler's Equations). Let a differential equation of $y := y(x)$ defined as:
$$x^2y'' + xy' + cy = 0,$$
where $c \in \mathbb{R}$ is a fixed constant, we want to solve the differential equation using *Euler's equations*.
   (a) Assume $c = -4$, solve the solution to the differential equation.
   (b) Assume $c = 9$, solve the solution to the differential equation.
   (c)* Find the critical point to this system where the behavior of the solution changes.

3. (Singularities, Zeros, and Poles). For any function $f : \mathbb{C} \to \mathbb{C}$, and $z_0 \in \mathbb{C}$, we have the following:

   - It has **a zero of order $m$** at $z_0$ if $f(z_0) = 0$, and $m$ is the smallest positive integer such that $f(z) = (z - z_0)^m g(z)$, where $g$ is analytic at $z_0$ and $g(z_0) \neq 0$.
   - It has **a pole of order $n$** at $z_0$ if $f(z_0)$ is not defined, and $n$ is the smallest integer such that $g(z) = (z - z_0)^n f(z)$, where $g$ is analytic at $z_0$ and $g(z_0) \neq 0$.
   - If a zero/pole has order 1, it is **simple**.

   As a side note, such definition applies for any real valued functions, *i.e.*, $f : \mathbb{R} \to \mathbb{R}$.
   Here, we define a differential equation for $y := y(x)$ as:
   $$\sin(x)y'' + \sin(x)\big(\cos(x) + e^x + x\big)y' + \big(\csc(x)\big)y = 0$$

   (a) Write the differential equation in the form of:
   $$y'' + p(x)y' + q(x)y = 0.$$

   (b)* Identify all zeros and poles of $p(x)$ and $q(x)$ as real functions, *i.e.*, $p, q : \mathbb{R} \to \mathbb{R}$. Find the order of the zeros and poles.

   (c) Identify all the points $x_0 \in \mathbb{R}$ such that the differential equation has a *regular singular point*.

4. (Dispersion of Heat). For this problem, we consider the dispersion of heat for an object in an environment with fixed temperature. Here, let $\theta := \theta(t)$ be the temperature of the object and $\theta_0$ denote the fixed temperature of the environment, we may model the temperature of the object by:
   $$\frac{d\theta}{dt} = -\frac{1}{\kappa}(\theta - \theta_0),$$
   where $\kappa$ is a fixed positive constant, representing the rate of heat dispersion.
   Suppose that we have a rigid body of 100°C (equivalently 212°F), and the room temperature is fixed as 20°C (equivalently 68°F, and this is also condition for STP, standard temperature and pressure). Also, we assume that $\kappa = 2$.

   (a) Construct the differential equation for the above system.

   (b) Use *Euler's method* with step size of 1 to approximate the temperature at $t = 3$.

   (c)* Identify if the approximation of temperature is an underestimate or an overestimate.

**Congrats on finishing the last weekly PILOT Problem Set!**

As you prepare for the final examination, please consider the following resources:

- PILOT webpage for ODEs: `https://jhu-ode-pilot.github.io/FA24/`
  - Check on the weekly problem sets.
  - Find the review problem set for this semester.
- *Homework sets*, *quizzes*, and *practice final set* provided by the instructor.
- PILOT Review Session for `AS.110.302` Differential Equations and Application.
  - Date & Time: December 8th, 4pm to 6pm.
  - Location: Maryland 110.

**Clubs & Orgs Bulletin**

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TBA

**Tip of the Week**

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