PILOT

Problem Set 4: Solutions

Differential Equations

Fall 2024

1. (Second Order Differential Equation.) Let an initial value problem for $y = y(t)$ be defined as follows:

$$\begin{cases} 4y'' - y = 0, \\ y(0) = 2,\ y'(0) = \beta, \end{cases}$$

where β is a real constant.

(a) Find the specific solution to the initial value problem. Express your solution with constant β.

(b) Find the value of β such that the solution *converges* to 0 as t tends to infinity.

Solution:

(a) First, we note that the characteristic equation is $4r^2 - 1 = 0$, whose roots are $\pm 1/2$, hence the general solution to the differential equation is:

$$y(t) = C_1 e^{t/2} + C_2 e^{-t/2}, \text{ where } C_1, C_2 \text{ are constants.}$$

To find the specific solution, we input initial conditions, namely we find the derivative:

$$y'(t) = \frac{C_1}{2} e^{t/2} - \frac{C_2}{2} e^{-t/2}.$$

Hence, the initial data tells us that:

$$y(0) = C_1 + C_2 = 2 \text{ and } y'(0) = \frac{C_1}{2} - \frac{C_2}{2} = \beta.$$

By algebraically manipulating the equations, we find:

$$C_1 = 1 + \beta \text{ and } C_2 = 1 - \beta.$$

Hence, the solution is:

$$y(t) = \boxed{(1+\beta)e^{t/2} + (1-\beta)e^{-t/2}}.$$

(b) Considering $t \to \infty$, we note that $e^{t/2} \to \infty$ and $e^{-t/2} \to 0$, hence we only need to consider about the $(1+\beta)e^{t/2}$ part.

In order for a convergence to 0, we want this part to vanish, *i.e.*:

$$1 + \beta = 0 \text{ or } \beta = \boxed{-1}.$$

2. (LI Set of Solutions.) Find the general solution to the following differential equation, and verify that your solution is a linearly independent set of solutions.

$$y^{(3)}(x) - 6y''(x) + 11y'(x) - 6y(x) = 0.$$

Solution: As usual, we first find the characteristic equation, that is:

$$r^3 - 6r^2 + 11r - 6 = 0.$$

Up to this point, readers should be quite familiar with the *rational root theorem*, so we know that if the polynomial has a rational root, it must be one of the following:

$$\pm 1, \pm 2, \pm 3, \text{ and } \pm 6.$$

In fact, for degree 3 polynomials of integer/rational coefficients, it must have at least one rational root. *We leave the check of this claim to the readers, as an exercise to get more familiar with polynomials.*
By easy checking, we note that 1 is a root, as $1^3 - 6 \times 1^2 + 11 \times 1 - 6 = 0$, then we can eliminate the polynomial to:

$$r^3 - 6r^2 + 11r - 6 = (r-1)(r^2 - 5r + 6) = (r-1)(r-2)(r-3),$$

hence the roots are $r = 1, 2, 3$, each with multiplicity 1.
Therefore, the general solution should be:

$$y(x) = \boxed{C_1 e^x + C_2 e^{2x} + C_3 e^{3x}}.$$

Note that since we are asked to verify linear independence, we use the Wronskian, that is:

$$\mathrm{W}[e^x, e^{2x}, e^{3x}] = \det \begin{pmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{pmatrix} = 18e^{6x} + 3e^{6x} + 4e^{6x} - 2e^{6x} - 12e^{6x} - 9e^{6x} = 2e^{6x} \neq 0,$$

hence the set $\{e^x, e^{2x}, e^{3x}\}$ is linearly independent.

3. (A Missing Solution.) Let a third order differential equation of $y = y(x)$ be defined as below:

$$y''' - y'' + y' - y = 0.$$

   (a) Verify that $\sin(x)$ and $\cos(x)$ are two solutions to the above differential equation. Can you explain how we can find these two solutions?
   (b) Is the set $\{\sin(x), \cos(x)\}$ linearly independent?
   (c) Does $\{\sin(x), \cos(x)\}$ constitute a full set of solution to the differential equation?
   (d) Give the general solution to the differential equation.

Solution:

(a) *Proof.* The verification is trivial. Since:

$$(\sin x)''' - (\sin x)'' + (\sin x)' - (\sin x) = -\cos x - \sin x + \cos x - \sin x = 0,$$
$$(\cos x)''' - (\cos x)'' + (\cos x)' - (\cos x) = \sin x + \cos x - \sin x - \cos x = 0.$$

Hence, they are solutions to the differential equation. □

Moreover, this is the case, since the characteristic equation has roots being $\pm \mathrm{i}$, which by Euler's identity, can be changed to real-valued functions $\cos x$ and $\sin x$.

(b) To verify linear independence, we have:

$$\mathrm{W}[\sin x, \cos x] = \det \begin{pmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{pmatrix} = -\sin^2 x - \cos^2 x = 1 \neq 0.$$

Hence the set is $\boxed{\text{linearly independent}}$.

(c) Note that we have a degree 3 differential equations, hence we need at least 3 linearly independent solution. Hence it does $\boxed{\text{not}}$ constitute a full set of solutions.

(d) To find the general solution, we want to first find the third solution. Recall the characteristic polynomial $r^3 - r^2 + r - 1$, we find roots $r = 1, \pm \mathrm{i}$, hence the third solution should be:

$$e^x,$$

and hence, by the principle of superposition, the general solution is:

$$y(x) = \boxed{C_1 \sin x + C_2 \cos x + C_3 e^x}.$$

4. (A Symmetric Solution.) Given the following second order initial value problem:

$$\begin{cases} \dfrac{d^2y}{dx^2} + \sin^2(1-x)y = \cosh(x-1), \\ y(1) = e,\ \dfrac{dy}{dx}(1) = 0. \end{cases}$$

Prove that the solution $y(x)$ is symmetric about $x = 1$, *i.e.*, satisfying that $y(x) = y(2-x)$.
Hint: Consider the interval in which the solution is unique.
Also, note that $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$.

Solution:
Note that I deliberately messed up with all the messy functions. Not only haven't I found a solution to the system, Wolfram cannot have an elementary solution as well. Hence, we need to think, alternatively, on some theorems.

Proof. Here, we suppose that $y(x)$ is a solution, and we want to show that $y(2-x)$ is also a solution. First we note that we can think of taking the derivatives of $y(2-x)$, by the chain rule:

$$\frac{d}{dx}[y(2-x)] = -y'(2-x),$$
$$\frac{d^2}{dx^2}[y(2-x)] = y''(2-x).$$

Now, if we plug in $y(2-x)$ into the system of equations, we have:

- First, for the differential equation, we have:
$$\begin{aligned} \frac{d^2}{dx^2}[y(2-x)] + \sin^2(1-x)y(2-x) &= y''(2-x) + \sin^2(x-1)y(2-x) \\ &= y''(2-x) + \sin^2\big(1-(2-x)\big)y(2-x) \\ &= y''(z) + \sin^2(1-z)y(z) \\ &= \cosh(z-1) = \frac{e^{z-1} + e^{-z+1}}{2} = \frac{e^{-(2-z)+1} + e^{(2-z)-1}}{2} \\ &= \cosh\big(x-1\big). \end{aligned}$$
- For the initial conditions, we trivially have that:
$$y(1) = y(2-1) = e \text{ and } y'(1) = y'(2-1) = 0.$$

Hence, we have shown that $y(2-x)$ is a solution if $y(x)$ is a solution.
Again, we observe the original initial value problem that:

$$\sin^2(1-x) \text{ and } \cosh(x-1) \text{ are continuous on } \mathbb{R}.$$

Therefore, by the *existence and uniqueness theorem for second order linear case*, there could be only one solution, which forces that:

$$y(x) = y(2-x),$$

so the solution is symmetric about $x = 1$, as desired. $\square$