Problem Set 11: Solutions

Differential Equations

Fall 2024

1. (System with Unknown Coefficients). Let a non-linear system for $x = x(t)$ and $y = y(t)$ be:
$$\begin{cases} x' = \alpha x - y + y^2, \\ y' = x + \alpha y. \end{cases}$$

   (a) Show that $(0,0)$ is a critical point, and show system is locally linear at $(0,0)$ for all $\alpha \in \mathbb{R}$.

   (b) Classify the critical point $(0,0)$ and sketch a few phase portraits of the linearized system.

Solution:

(a) *Proof.* To show that $(0,0)$ is a critical point, we just plug it into the system as:
$$\begin{cases} x'(0,0) = 0 - 0 + 0 = 0, \\ y'(0,0) = 0 + 0 = 0. \end{cases}$$
Hence $(0,0)$ is a critical point.
To consider the local linearity, we compute the Jacobian matrix as:
$$\mathrm{J}[x', y'] = \begin{pmatrix} \partial_x x' & \partial_y x' \\ \partial_x y' & \partial_y y' \end{pmatrix} = \begin{pmatrix} \alpha & -1 + 2y \\ 1 & \alpha \end{pmatrix}.$$
When we evaluate it at $(0,0)$, we have:
$$\mathrm{ev}_{(0,0)}\, \mathrm{J}[x', y'] = \begin{pmatrix} \alpha & -1 \\ 1 & \alpha \end{pmatrix}.$$
We see that the determinant is $\alpha^2 + 1 > 0$, hence it is locally linear for all $\alpha \in \mathbb{R}$. $\square$

(b) For the locally linear system, we classify the linear approximation as:
$$\begin{pmatrix} x \\ y \end{pmatrix}' \sim \begin{pmatrix} \alpha & -1 \\ 1 & \alpha \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$
Hence, the characteristic equation is $(\alpha - \lambda)^2 + 1 = 0$, that is $\lambda = \alpha \pm \mathrm{i}$.
Depending on different cases for α, we have different results:

- When $\alpha = 0$, it is a stable center.
- When $\alpha > 0$, it is unstable spiral.
- When $\alpha < 0$, it is asymptotically stable spiral.

The graphs are on the next page.

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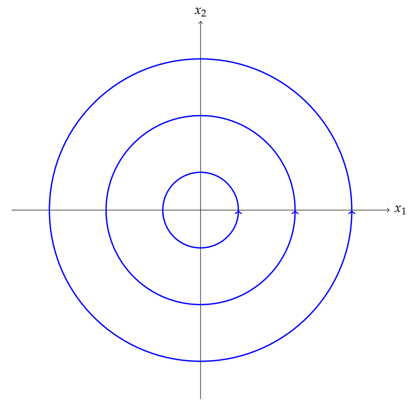
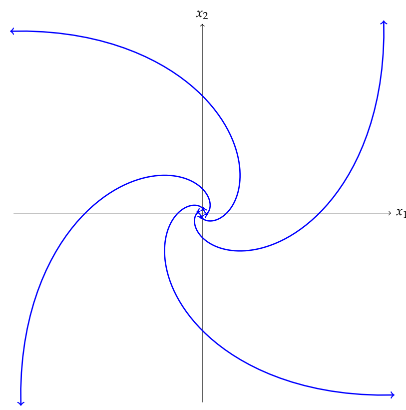
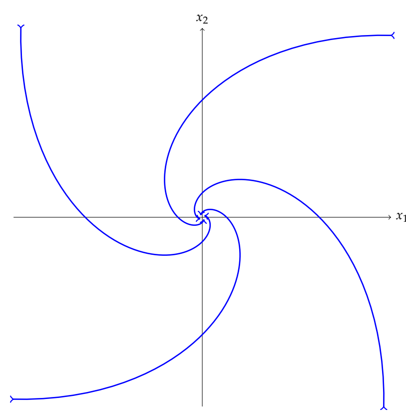
- For $\alpha = 0$:



- For $\alpha > 0$:



- For $\alpha < 0$:



2. (Nonlinear at origin). Let the linear system be:

$$\begin{cases} x' = y, \\ y' = x + 2x^3. \end{cases}$$

   (a) Show that the origin is a saddle point.

   (b) Sketch a phase portrait for the linearized system. Note that where all the trajectories of the linear system tend to the origin.

Solution:

(a) *Proof.* Here, we first verify that it is critical point, that is $x'(0,0) = 0$ and $y'(0,0) = 0 + 0 = 0$. Then, we check the Jacobian matrix:

$$\mathrm{ev}_{(0,0)}\,\mathrm{J}[x', y'] = \mathrm{ev}_{(0,0)} \begin{pmatrix} 0 & 1 \\ 1 + 6x^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

whose determinant is -1, so the system is locally linear, so the linear system is:

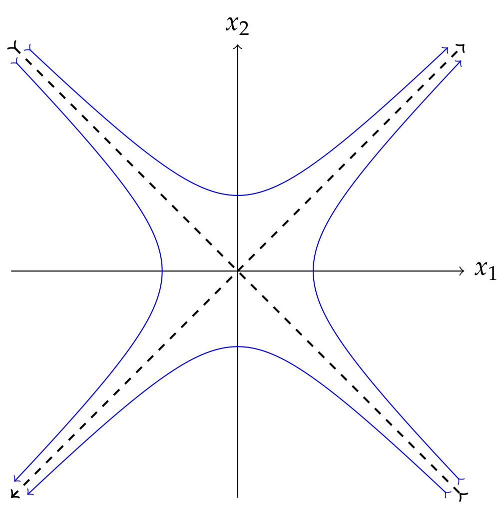
$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

Hence, we have the characteristic equation as $\lambda^2 - 1 = 0$, so the roots are $\lambda_1 = 1$ and $\lambda_2 = -1$. Thus, it is a unstable saddle point. □

(b) To sketch the diagram, we would want the eigenvectors.

- When $\lambda_1 = 1$, then we have $\boldsymbol{\xi}^{(1)}$ such that $-\xi_1^{(1)} + \xi_2^{(1)} = 0$, so the eigenvector is $(1, 1)$.
- When $\lambda_2 = -1$, then we have $\boldsymbol{\xi}^{(2)}$ such that $\xi_1^{(1)} + \xi_2^{(1)} = 0$, so the eigenvector is $(1, -1)$.

Hence, the diagram is as:



3. (Modeling Politics). Suppose D and R are two parties on a non-existing country on the center of Mars. For the simplicity of this problem, they, *unfortunately*, have no elections. Therefore, we can model the amount of the supporter for each party (in millions), denoted x_D and x_R with the following relationship:

$$\begin{cases} \dfrac{dx_D}{dt} = x_D(1 - x_D - x_R), \\ \dfrac{dx_R}{dt} = x_R(3 - 2x_D - 4x_R). \end{cases}$$

Find all possible endings (say arbitrarily long after, that is $t \to \infty$) of the number of supporters (in millions) for the two parties.

Solution:

Alright, we do assume that these parts are just having random letters, so we do not dive into actual politics. This is a typical "Competing Species" models, in which the two parties are competing for the limited resources.

First find the critical points, that is:

$$\begin{cases} x_D(1 - x_D - x_R) = 0, \\ x_R(3 - 2x_D - 4x_R) = 0. \end{cases}$$

We know that we could have many cases:

- When $x_D = x_R = 0$, we have both equations being 0.
- When $x_D = 0$, we can also have $x_R = 3/4$.
- When $x_R = 0$, we can also have $x_D = 1$.
- Or, we can have $1 - x_D - x_R = 0$ and $3 - 2x_D - 4x_R = 0$, and in this case, we have $x_D = x_R = 1/2$.

Here, we note that the Jacobian matrix is:

$$J = \begin{pmatrix} 1 - 2x_D & -x_D \\ -2x_R & 3 - 8x_R \end{pmatrix}.$$

It can be easily verified that the system are locally linear at all of the critical points, and we leave this as an exercise to the readers.

Here, as we inspect these points, we could conclude that:

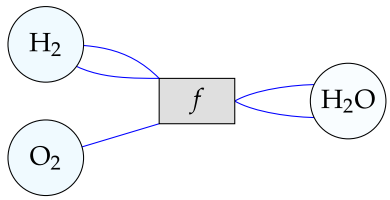
- Case 1: $x_D = x_R = 0$. This is when two party has no supporting population, it they remain uncared.
- Case 2: $x_D = 0$ and $x_R = 3/4$. This is when the R party got the support initially, and they keep absolute advantage over the D party.
- Case 3: $x_R = 0$ and $x_D = 1$. This is when the D party got the support initially, and they keep absolute advantage over the R party.
- Case 4: $x_D = x_R = 1/2$. This is when two party all got some initial support. While after some political campaigns (maybe also fightings), they got to a balanced equilibrium.

If the readers are still confused, we recommend them taking a look at the *directional field*.

4. (Chemical Reaction). Consider the following chemical equation of hydrogen gas combustion in oxygen gas:

$$2H_2 + O_2 \xrightarrow{f} 2H_2O.$$

We may represent it from a graphical representation.

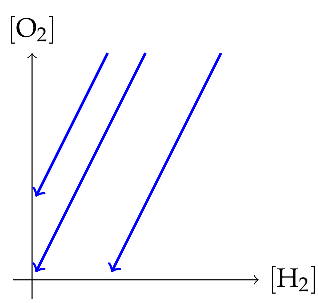


Assume that the reaction rate is constant $\kappa := \mathrm{rate}(f)$. Construct the nonlinear system of the concentration of H_2 and O_2, sketch a few trajectories for different initial conditions for different starting concentrations.

Solution:
For the model, we recall that the reaction rate is proportional to the concentration of the current amount of reactants. Hence, we obtain the system as:

$$\boxed{\begin{cases} \dfrac{d[H_2]}{dt} = -2\kappa[H_2][O_2], \\ \dfrac{d[O_2]}{dt} = -\kappa[H_2][O_2]. \end{cases}}$$

This is system is of course not linear, but we may notice that the rates are exactly the same, except for the linear factor 2, so that means that the concentration of H_2 will be consuming twice as fast as the concentration of O_2 being consumed.
If we are to plot this, we should have the line with a slope of 2, namely:



If you are interested in graphical (or more precisely *categorical*) representation of chemical equations, please refer to this slide deck.