Problem Set 5: Solutions

Differential Equations

Fall 2024

1. (Constructing Solutions, Again.) Construct an initial value problem for the following solutions:

   (a) $$y(t) = 4e^{3t} - e^{-2t}.$$

   (b) $$y(t) = e^{2t}\cos t + e^{2t}\sin t + e^{2t}.$$

Solution:

(a) Here, we trivially note that the two solutions are e^{3t} and e^{-2t}, so the roots of the characteristic equation is $r = 3$ and $r = -2$, so $(r-3)(r+2) = r^2 - r - 6 = 0$, hence the differential equation should be $y'' - y' - 6y = 0$.
For the initial value, since the function is well defined over $\mathbb{R}$, we pick any initial value, and we pick 0 for simplicity here, so $y(0) = 4 - 1 = 3$, and while for the derivative being:
$$y'(t) = 12e^{3t} + 2e^{-2t},$$
so we have $y'(0) = 12 + 2 = 14$. Hence, one example of the IVP is:
$$\boxed{\begin{cases} y'' - y' - 6y = 0, \\ y(0) = 3,\ y'(0) = 14. \end{cases}}$$

(b) For this part, we reverse engineer the solution as:
$$e^{2t}(\cos t + \mathrm{i}\sin t), e^{2t}(\cos t - \mathrm{i}\sin t), \text{ and } e^{2t},$$
which is e^{2t} with $e^{2t\pm it}$, so the roots are 2 and $2 \pm \mathrm{i}$, thus, we reconstruct the characteristic polynomial that:
$$(r-2)\big((r-2-\mathrm{i})(r-2+\mathrm{i})\big) = (r-2)(r^2-4r+5) = r^3 - 6r^2 + 13r - 10.$$
Thus, we have the differential equation as:
$$y''' - 6y'' + 13y' - 10y = 0.$$
Now, we consider that the initial condition, we take the first and second derivatives as:
$$\begin{aligned} y'(t) &= 2e^{2t}\cos t + 2e^{2t}\sin t + 2e^{2t} - e^{2t}\sin t + e^{2t}\cos t = 2e^{2t}\cos t + e^{2t}\sin t + 2e^{2t}, \\ y''(t) &= 4e^{2t}\cos t + 2e^{2t}\sin t + 4e^{2t} - 2e^{2t}\sin t + e^{2t}\cos t = 5e^{2t}\cos t + 4e^{2t}. \end{aligned}$$
Hence, we choose 0 as initial point and obtain $y(0) = 1 + 1 = 2$, $y'(0) = 2 + 2 = 4$, and $y''(0) = 5 + 4 = 9$, so one example of initial value problem is:
$$\boxed{\begin{cases} y''' - 6y'' + 13y' - 10y = 0, \\ y(0) = 2,\ y'(0) = 4,\ y''(0) = 9. \end{cases}}$$

2. ($L^2([0,2\pi])$ Space.) Recall that we have defined linear independence of functions, we define *orthogonality* of two real-valued, "square-integrable" functions over $[0,2\pi]$, f and g, as:

$$\int_0^{2\pi} f(x)g(x)dx = 0.$$

(a) Show that the set $\{\sin x, \cos x\}$ is linearly independent and orthogonal.

(b) Show that if $\{f(x), g(x)\}$ is orthogonal, then $C_1 f(x)$ and $C_2 g(x)$ is orthogonal.

(c) Note that $\{x, x^2\}$ are linearly independent, construct a basis that is orthogonal.

Solution:

(a) *Proof.* To show linear independence, we compute the Wronskian as:

$$\mathrm{W}[\sin x, \cos x] = \det \begin{pmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{pmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0.$$

Then, to show orthogonality, we have:

$$\int_0^{2\pi} \sin x \cos x dx = \frac{1}{2}\int_0^{2\pi} \sin(2x) dx = \frac{1}{2}\left[-\frac{1}{2}\cos(2x)\right]_0^{2\pi} = \frac{1}{4}\big(\cos 0 - \cos(4\pi)\big) = 0,$$

hence we have shown linear independence and orthogonality. $\square$

(b) *Proof.* By orthogonality, we have $\int_0^{2\pi} f(x)g(x)dx = 0$, so we have:

$$\int_0^{2\pi} C_1 f(x) \cdot C_2 g(x) dx = C_1 C_2 \int_0^{2\pi} f(x)g(x)dx = C_1 C_2 \cdot 0 = 0.$$

Hence orthogonality is preserved with scalar multiplications. $\square$

(c) The check of x and x^2 being linearly independent can be verified by Wronskian, and we leave this check to the readers. By the principle of superposition, we want to construct the second argument as $x^2 - Ax$, where A is a constant, now we take the inner product as:

$$\int_0^{2\pi} x(x^2 - Ax)dx = \int_0^{2\pi} (x^3 - Ax^2)dx = \frac{x^4}{4} - \frac{Ax^3}{3}\bigg|_0^{2\pi} = 4\pi^4 - \frac{8A\pi^3}{3} = 0,$$

which forces A to be $3\pi/2$, so the orthogonal basis is now:

$$\boxed{\left\{x, x^2 - \frac{3\pi x}{2}\right\}}.$$

Diligent should notice that we have somehow constructed a "vector space" with a proper inner product. In fact, this space $L^2([0,2\pi])$ is considered a Hilbert Space, that is a infinite dimensional vector space with completeness and denseness. The $L^2([0,2\pi])$ is closely related to Fourier series, that has inarguable impacts on mathematics as well as science and engineering disciplines.

3. (Repeated and Complex Root.) Let a six order differential equation of $y = y(t)$ be defined as follows:

$$y^{(6)} - 2y^{(3)} + y = 0.$$

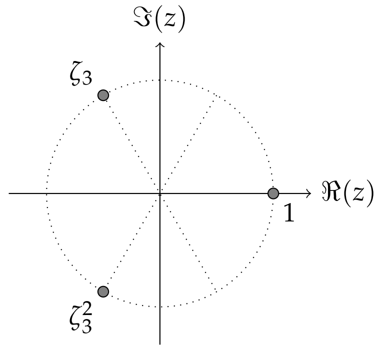
Find a set of real-valued function being the general solution to the above differential equation.

Solution:

For this questions, we first find the characteristic equation, which should be a fairly easy perfect square:

$$r^6 - 2r^3 + 1 = (r^3 - 1)^2 = 0.$$

Hence, our concern follows to r being the solution to $r^3 = 1$, with double multiplicity. In particular, we have the roots being on the unit circle, with ζ_3 being the 3rd root of unity, as:



Hence, the roots of the polynomial is:

$$r = 1, \zeta_3, \zeta_3^2,$$

each with multiplicity 2, where ζ_3 and ζ_3^2 can be expressed as:

$$\zeta_3 = \cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2},$$
$$\zeta_3^2 = \cos\left(\frac{4\pi}{3}\right) + \mathrm{i}\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}.$$

Hence, one set of solution is:

$$y_1 = e^t,$$
$$y_2 = e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right),$$
$$y_3 = e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right),$$

where this set is already manipulated by Euler's identity. By multiplicity of roots, we have repeated roots, leading to solutions:

$$y_4 = te^t,$$
$$y_5 = te^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right),$$
$$y_6 = te^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right).$$

And the set is $\boxed{\{y_1, y_2, y_3, y_4, y_5, y_6\}}$.

4. (Reduction of Order.) Let a ODE be defined as follows:

$$t^2y'' + 2ty' = 2y,\ t > 0.$$

Given a solution is $y_1(t) = t$, find the full fundamental set of solution.

Solution:

Here, we let the second solution to be $y_2(t) = u(t)y_1(t) = tu(t)$, thus we take its derivatives as:

$$\begin{aligned} y_2'(t) &= u'(t)y_1(t) + u(t)y_1'(t) = tu'(t) + u(t), \\ y_2''(t) &= u''(t)y_1(t) + 2u'(t)y_1'(t) + u(t)y_1''(t) = tu''(t) + 2u'(t). \end{aligned}$$

Now, as we plug in the above results back to the ODE, we have:

$$t^2\big(tu''(t) + 2u'(t)\big) + 2t\big(tu'(t) + u(t)\big) = 2\big(tu(t)\big).$$

By some arithmetic expansions and rearranging, we have:

$$t^3u''(t) + 4t^2u'(t) = tu'' + 4u' = 0.$$

Here, we denote $\omega = u'$ to have $t\omega' + 4\omega = 0$, which is separable, so we have:

$$\begin{aligned} \frac{d\omega}{\omega} &= -\frac{4dt}{t}, \\ \int \frac{d\omega}{\omega} &= -4\int \frac{dt}{t}, \\ \log|\omega| &= -4\log|t| + C, \\ \omega &= \tilde{C}t^{-4} \end{aligned}$$

Now, we integrate ω to obtain that:

$$\int \omega dt = \int \tilde{C}t^{-4}dt = -\frac{\tilde{C}}{3}t^{-3} + D_2 = D_1t^{-3} + D_2.$$

By multiplying back with $y_1(t)$, we have the fundamental set of solution as:

$$y(t) = (D_1t^{-3} + D_2)t = \boxed{\frac{D_1}{t^2} + D_2t}.$$