# Problem Set 7 (Fall Break Special)

Differential Equations

Fall 2024

**Zoom Session Information:**

| Date: October 16, 2024 Time: 7:30pm-9:00pm ET | Meeting ID: 959 8028 3654 Passcode: 781316 |
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| Link: `https://JHUBlueJays.zoom.us/j/95980283654?pwd=KdY2jBJxkuIjYL7hmu5NVLzuIwydJG.1` | |

Fall Break is now coming! We hope that you are ready to enjoy a short break from the hard work in the past weeks. As of right now, you should have seen various types of differential equations, from higher order linear equation, or a system of linear differential equations. In the set this week, we are aiming to solidify our understanding on eigenspace, while exploring on certain structures of our materials and some unanticipated applications.

Since certain sections will not meeting this week, and this problem set will be composed of two parts. The first part is the basic reviews on class materials, whereas the second part is more challenging and for fun. *It is fine if you have no ideas on the second part of the problem set, but please attempt to give them a try.*

**Regular Questions:** Review of course contents.

1. (Variation of Parameters). Solve the following third order differential equation of $y = y(t)$:
$$y''' - 4y' = e^{-2t}.$$

2. (Eigenvalues & Eigenvectors). Find all eigenvectors and eigenvalues of the following matrix:
   (a) $$A = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix},$$
   (b) $$B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}.$$

3. (Linear Systems). Let $\mathbf{x} \in \mathbb{R}^2$, find the general solution of $\mathbf{x}$ if $\mathbf{x}$ satisfies:
$$\mathbf{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}.\mathbf{x}.$$

**Additional Questions:** More challenging and fun problems related with the course.

1. (PDEs: Wave Equation). The following system of partial differential equations portraits the propagation of waves on a segment of the 1-dimensional string of length $L$, the displacement of string at $x \in [0, L]$ at time $t \in [0, \infty)$ is described as the function $u = u(x, t)$:

$$\begin{cases} \text{Differential Equation:} & \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0, & \text{where } x \in (0, L) \text{ and } t \in [0, \infty); \\ \text{Initial Conditions:} & u(x, 0) = \sin\left(\dfrac{2\pi x}{L}\right), & \\ & \dfrac{\partial u}{\partial t}(x, 0) = \sin\left(\dfrac{5\pi x}{L}\right), & \text{where } x \in [0, L]; \\ \text{Boundary Conditions:} & u(0, t) = u(L, t) = 0, & \text{where } t \in [0, \infty); \end{cases}$$

where $c$ is a constant and $g(x)$ has "good" behavior. Apply the method of separation, *i.e.*, $u(x, t) = v(x) \cdot w(t)$, and attempt to obtain a general solution that is *non-trivial*.
*Hint:* Use the fact that $\{\sin(n\pi x/L), \cos(n\pi x/L)\}_{n \in \mathbb{Z}^+}$ forms an orthonormal basis (cf. §5.2).

2. (*Putnam 2023*: First Positive Root). Determine the smallest positive real number $r$ such that there exists differentiable functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ satisfying:

   - $f(0) > 0$,
   - $g(0) = 0$,
   - $|f'(x)| \leq |g(x)|$ for all $x$,
   - $|g'(x)| \leq |f(x)|$ for all $x$, and
   - $f(r) = 0$.

   You may give an answer *without* a rigorous proof, as the proof is out of scope of the course.
   *Hint:* Assume that the function "moves" the fastest when the cap of the derivatives are "moving" the fastest, then think of constructing a dynamical system relating $f$ and $g$.

3. (Nilpotent Operator). Let $M$ be a square matrix, $M$ is *nilpotent* if $M^k = 0$ for some positive integer $k$. Similar to how we defined the exponential function analytically, the exponential function is also defined for matrices, let $A$ be a square matrix, we define:

$$\exp(A) = \sum_{i=0}^{\infty} \frac{1}{i!} A^i.$$

   (a) Show that $N = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is nilpotent, then write down the result of $\exp(N)$.

   Now, suppose that $N \in \mathcal{L}(\mathbb{R}^n)$ is a square matrix and is *nilpotent*.

   (b) Suppose that $\mathrm{Id}_n \in \mathcal{L}(\mathbb{R}^n)$ is the identity matrix, prove that $\mathrm{Id}_n + N$ is invertible.
   *Hint:* Use the differences of squares for matrices.

   (c) If all the entries in $N$ are rational, show that $\exp(N)$ has rational entries.

4. (Convergence of Series.) As we dive into fundamentals of mathematics, it is inevitable to encounter *sequences* and their sums. Discuss about the following sequences if they converge or not. If they converge, find the explicit sum.

   (a) $$\sum_{k=0}^{\infty} \frac{1}{k}.$$

   (b) $$\sum_{k=0}^{\infty} \frac{1}{k!}.$$

   (c) $$\sum_{k=0}^{\infty} \frac{1}{(4k+1)!}.$$

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**Tip of the Week**

To be updated.