Problem Set 12: Solutions

Differential Equations

Fall 2024

1. (Limit Cycles). Determine the periodic solution, if there are any, of the following system:

$$\begin{cases} x' = y + \dfrac{x}{\sqrt{x^2+y^2}}(x^2+y^2-2), \\ y' = -x + \dfrac{y}{\sqrt{x^2+y^2}}(x^2+y^2-2). \end{cases}$$

Solution:

Here, we recall the formula converting between polar coordinates and Cartesian coordinates:

$$\begin{cases} x = r\cos\theta, & y = r\sin\theta, \\ rr' = xx' + yy', & r^2\theta' = xy' - yx'. \end{cases}$$

Now, we are able to convert the system as:

$$\begin{cases} rr' = x\left[y + \dfrac{x}{\sqrt{x^2+y^2}}(x^2+y^2-2)\right] + y\left[-x + \dfrac{y}{\sqrt{x^2+y^2}}(x^2+y^2-2)\right], \\ r^2\theta' = x\left[-x + \dfrac{y}{\sqrt{x^2+y^2}}(x^2+y^2-2)\right] - y\left[y + \dfrac{x}{\sqrt{x^2+y^2}}(x^2+y^2-2)\right]. \end{cases}$$

Here, by simple deductions, we trivially have:

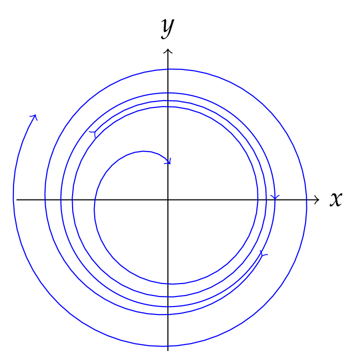
$$rr' = \frac{x^2+y^2}{\sqrt{x^2+y^2}}(x^2+y^2-2) = \frac{r^2}{r}(r^2-2) \rightsquigarrow r' = r^2 - 2.$$

$$r^2\theta' = -x^2 - y^2 = -r^2 \rightsquigarrow \theta' = -1.$$

Thereby, we consider the radius as:

$$r' = r^2 - 2 = (r - \sqrt{2})(r + \sqrt{2}).$$

Hence, we note that the critical point is $\boxed{r = \sqrt{2}}$ (since r must be positive). Note that $r' < 0$ for $0 < r < \sqrt{2}$ and $r' > 0$ for $r > \sqrt{2}$. Hence, this is an unstable limit cycle.



2. (Converging Sequences). In this question, we will review some common power series.

   (a) Construct the power series of e^x, $\sin x$, and $\cos x$ centered at 0.

   (b) Consider the following power series:
   $$\sum_{k=0}^{\infty} \frac{x^{4k+3}}{(4k+3)!}.$$
   Identify if such series converges. Compute the limit if the series converges.

Solution:

(a) Recall that we have the power series of Taylor's polynomial centered at $x_0 = 0$ being:
$$f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$
Consider the derivatives of e^x being always e^x, so we have $\frac{d^n e^x}{dx^n}(0) = 1$, thus:
$$e^x \sim \boxed{\sum_{n=0}^{\infty} \frac{1}{n!} x^n}.$$
Then, we consider the derivatives of sine and cosine that cycles every 4 times, so we have:
$$(\sin x)' = \cos x, \quad (\sin x)'' = -\sin x, \quad (\sin x)''' = -\cos x, \quad (\sin x)'''' = \sin x.$$
$$(\cos x)' = -\sin x, \quad (\cos x)'' = -\cos x, \quad (\cos x)''' = \sin x, \quad (\cos x)'''' = \cos x.$$
Hence, it is trivial to conclude that:
$$\sin x \sim \boxed{\sum_{n=0}^{\infty} \left(\frac{x^{4n+1}}{(4n+1)!} - \frac{x^{4n+3}}{(4n+3)!} \right)},$$
$$\cos x \sim \boxed{\sum_{n=0}^{\infty} \left(\frac{x^{4n}}{(4n)!} - \frac{x^{4n+2}}{(4n+2)!} \right)}.$$

(b) Here, we also consider the power series of e^{-x}, namely:
$$e^{-x} \sim \sum_{n=0}^{\infty} \left(\frac{1}{(2n)!} x^{2n} - \frac{1}{(2n+1)!} x^{2n+1} \right).$$
For simplicity, we expand all the terms out, explicitly as:
$$\begin{array}{lcccccccc}
e^x & \sim & +\frac{x^0}{0!} & +\frac{x^1}{1!} & +\frac{x^2}{2!} & +\frac{x^3}{3!} & +\frac{x^4}{4!} & +\frac{x^5}{5!} & +\cdots \\
e^{-x} & \sim & +\frac{x^0}{0!} & -\frac{x^1}{1!} & +\frac{x^2}{2!} & -\frac{x^3}{3!} & +\frac{x^4}{4!} & -\frac{x^5}{5!} & +\cdots \\
\sin x & \sim & & +\frac{x^1}{1!} & & -\frac{x^3}{3!} & & +\frac{x^5}{5!} & -\cdots \\
\cos x & \sim & +\frac{x^0}{0!} & & -\frac{x^2}{2!} & & +\frac{x^4}{4!} & & -\cdots
\end{array}$$
By some arithmetics, one should notice that:
$$\sum_{k=0}^{\infty} \frac{x^{4k+3}}{(4k+3)!} = \boxed{\frac{e^x - e^{-x}}{4} - \frac{\sin x}{2}}.$$
Hence, the power series converges.

3. (Analytic Function). Recall the definition of analytic function in class defined for $f : \mathbb{R} \to \mathbb{R}$ at a fixed point $x \in \mathbb{R}$.

   (a) Write down the power series of $f(x) = \frac{1}{1-x}$ around 0. Show that $f(x)$ is analytic at 0.

   Here, we extend the definition to function with complex input and complex output, say $f : \mathbb{C} \to \mathbb{C}$. We can similarly define f being analytic at $x_0 \in \mathbb{C}$ when the function has a positive radius of convergence at x_0.

   (b) Convince yourself that if $f : \mathbb{C} \to \mathbb{C}$ is analytical over $\mathbb{C}$ and $f(\mathbb{R}) \subset \mathbb{R}$, then $f\mid_{\mathbb{R}}$ is analytical $\mathbb{R}$.

   By Cauchy-Riemann equations, $f(z) := f(x + \mathrm{i}y)$ being analytical is identical with:
$$\frac{\partial f}{\partial x} = -\mathrm{i}\frac{\partial f}{\partial y}.$$

   (c) Show that the Möbius transform $\psi : \mathbb{C} \to \mathbb{C}$ such that $\psi_a(z) := \frac{a-z}{1-\bar{a}z}$ is analytic on $\mathbb{C} \setminus \{1/\bar{a}\}$. Conclude with a condition for which $\psi_a(z)\mid_{\mathbb{R}}$ is analytic on some set A.

Solution:

(a) Consider $f(x) = \frac{1}{1-x}$, when we are around zero, consider $|x| < 1$, we can think of the function as the sum of the geometric series, so we have:
$$f(x) \sim 1 + x + x^2 + \cdots = \boxed{\sum_{n=0}^{\infty} x^n}.$$
Since for all $|x| < 1$, we can think of it as a geometric series, that is:
$$1 + x + x^2 + \cdots = \frac{1}{1-x},$$
so we have pointwise convergence for all $|x| < 1$, so it is analytic at 0.

(b) *Proof.* Suppose $f : \mathbb{C} \to \mathbb{C}$ is analytic over $\mathbb{C}$, there exists complex power series around 0:
$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$
By assumption that $f(\mathbb{R}) = \mathbb{R}$, that means that for any $x \in \mathbb{R}$, we have $f(x) \in \mathbb{R}$. Hence, $f\mid_{\mathbb{R}}$ has real output, then we can write it still as the power series expansions. Since at every point $z \in \mathbb{C}$, we have a radius of convergence of ρ, that is, for all $z_0 \in \mathbb{C}$ such that $|z - z_0| < \rho$ in which the power sequence converges. Hence for any $x \in \mathbb{R}$, we have all $x_0 \in \mathbb{R}$ such that $|x - x_0| < \rho$ in which the power sequence converges at x_0, thus $f\mid_{\mathbb{R}}$ has a positive radius of convergence for all $x \in \mathbb{R}$, so it is analytic. □

(c) *Proof.* Given the Cauchy-Riemann equation, we can write the Möbius transformation as:
$$\psi_a(z) = \frac{a - x - \mathrm{i}y}{1 - \bar{a}x - \bar{a}\mathrm{i}y}.$$
Then, we may take the partial derivatives as:
$$\frac{\partial \psi_a}{\partial x} = \frac{-(1 - \bar{a}x - \bar{a}\mathrm{i}y) - (a - x - \mathrm{i}y)(-\bar{a})}{(1 - \bar{a}x - \bar{a}\mathrm{i}y)^2},$$
$$\frac{\partial \psi_a}{\partial y} = \frac{-\mathrm{i}(1 - \bar{a}x - \bar{a}\mathrm{i}y) - (a - x - \mathrm{i}y)(-\mathrm{i}\bar{a})}{(1 - \bar{a}x - \bar{a}\mathrm{i}y)^2}.$$

Continues on the next page...

Continued from last page.

Here, we note that:

$$-\mathrm{i}\frac{\partial \psi_a}{\partial y} = \frac{-(1-\overline{a}x-\overline{a}\mathrm{i}y)-(a-x-\mathrm{i}y)(-\overline{a})}{(1-\overline{a}x-\overline{a}\mathrm{i}y)^2}.$$

Hence, we observe that the two expressions are the same when $\overline{a}(x+\mathrm{i}y) \neq 1$, that is $(x+\mathrm{i}y) \neq 1/\overline{a}$. Thus, we have shown that $\psi_a(z)$ is in fact analytic at all $\mathbb{C} \setminus \{1/\overline{a}\}$. $\square$

Then, recall from the precious part, as long as $a \in \mathbb{R}$, we can guarantee that $\psi_a(\mathbb{R}) = \mathbb{R}$ (since addition, subtraction, multiplication, and nonzero divisions of real numbers are closed).

Thus, when $a \in \mathbb{R}$, $\psi_a(z)\mid_{\mathbb{R}}$ is analytic on $\mathbb{R} \setminus \{1/a\}$.

Note that the Cauchy-Riemann equation is an important consequence in *complex analysis*. In complex analysis, there has been an equivalence established between holomorphic functions and analytic functions. It is noteworthy for diligent readers to think about the relationship in (real) power series between differentiability and analytic properties.

4. (Recurrence Relation). Solve the following differential equation using power series method. Include the recurrence relation.
$$y'' + y = 0.$$

Solution:
Here, we note that we have constant coefficients, so they are automatically analytic. Now, we take $x_0 = 0$, and assume that our solution is in the form that:
$$y = \sum_{n=0}^{\infty} a_n x^n.$$
Now, by the assumption that the series converges absolute, we take differentiate the terms twice, which gives that:
$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1},$$
and:
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$
With the derivative, we plug it back into the differential equations, that is:
$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$
By the term-wise addition, we have:
$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n \right] x^n = 0.$$
Given that the sequence is equivalently zero, then we have the recurrence relation as:
$$\boxed{a_{n+2} = -\frac{a_n}{(n+2)(n+1)} \text{ for all } n \geq 0}.$$
Given that this differential equation has order 2, we let the first two coefficients fixed, that is a_0 and a_1, then we can form the rest of the coefficients as:
$$a_2 = -\frac{a_0}{2 \cdot 1}, \qquad a_4 = -\frac{a_2}{4 \cdot 3} = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1}, \qquad \cdots, \qquad a_{2n} = \frac{(-1)^n a_0}{(2n)!},$$
$$a_3 = -\frac{a_1}{3 \cdot 2}, \qquad a_5 = -\frac{a_3}{5 \cdot 4} = \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2}, \qquad \cdots, \qquad a_{2n+1} = \frac{(-1)^n a_1}{(2n+1)!}.$$
When we sum all the terms, we have that:
$$\begin{aligned} y(x) &= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + a_1 \sum_{m=1}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ &= a_0 \sum_{n=0}^{\infty} \left(\frac{x^{4n}}{(4n)!} - \frac{x^{4n+2}}{(4n+2)!} \right) + a_1 \sum_{n=0}^{\infty} \left(\frac{x^{4n+1}}{(4n+1)!} - \frac{x^{4n+3}}{(4n+3)!} \right) \\ &= \boxed{a_0 \cos x + a_1 \sin x}. \end{aligned}$$
We encourage diligent readers to also attempt this particular question with the characteristic polynomial method, and *hopefully* obtain the same set of linearly independent solution (unique up to linear combinations).