Problem Set 8: Solutions

Differential Equations

Fall 2024

1. (Linear System versus Second Order). Let an initial value problem for linear system on $x_1 := x_1(t)$ and $x_2 := x_2(t)$ be defined as follows:

$$\begin{cases} x_1' = 3x_1 - 2x_2, & x_1(0) = 3, \\ x_2' = 2x_1 - 2x_2, & x_2(0) = \dfrac{1}{2}. \end{cases}$$

   (a) Solve for the *general solution* for the linear system by considering $\mathbf{x} = (x_1, x_2)$.

   (b) Transform the *general system* into a single equation of second order. Then solve the second-order equation. Eventually, convert your solution of one variable back to the *general solution* to $x_1(t)$ and $x_2(t)$.

   (c) Find the particular solution using the initial conditions, then graph the parameterized curve on a x_1x_2-plane with $t \geq 0$.

Solution:

(a) Here, when we consider the general linear system, we have the matrix as:

$$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} .\mathbf{x}.$$

Hence, we look for the eigenvalues for the matrix, that is when:

$$0 = \det \begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix} = (3-\lambda)(-2-\lambda) - (-2)\times 2 = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1).$$

Hence the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$, we solve for the eigenvectors:

- For $\lambda_1 = 2$, we want $\boldsymbol{\xi}^{(1)}$ satisfy that $\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} .\boldsymbol{\xi}^{(1)} = \mathbf{0}$, that is $\xi_1^{(1)} = 2\xi_2^{(1)}$, so the eigenvector is $(2,1)$.
- For $\lambda_2 = -1$, we want $\boldsymbol{\xi}^{(2)}$ satisfy that $\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} .\boldsymbol{\xi}^{(2)} = \mathbf{0}$, that is $2\xi_1^{(2)} = \xi_2^{(2)}$, so the eigenvector is $(1,2)$.

Therefore, the general solution for the system of differential equation is:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{x} = C_1 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \boxed{\begin{pmatrix} 2C_1 e^{2t} + C_2 e^{-t} \\ C_1 e^{2t} + 2C_2 e^{-t}. \end{pmatrix}}.$$

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(b) By manipulating the first differential equation, we are able to obtain that:

$$x_2 = \frac{3}{2}x_1 - \frac{1}{2}x_1',$$

whose derivative is correspondingly:

$$x_2' = \frac{3}{2}x_1' - \frac{1}{2}x_1''.$$

Then, we substitute into the second differential equation to obtain that:

$$\frac{3}{2}x_1' - \frac{1}{2}x_1'' = 2x_1 - 2\left(\frac{3}{2}x_1 - \frac{1}{2}x_1'\right),$$

$$x_1'' - x_1' - 2x_1 = 0.$$

By solving the second order linear differential equation on $x_1 = x_1(t)$, its characteristic equation is $r^2 - r - 2 = 0$, which factors to $(r-2)(r+1)$, so the solution is:

$$x_1 = \boxed{D_1e^{2t} + D_2e^{-t}}.$$

Now, we take the derivative of x_1 to obtain that:

$$x_1' = 2D_1e^{2t} - D_2e^{-t},$$

so we can plug it into equation for x_2, that is:

$$x_2 = \frac{3}{2}(D_1e^{2t} + D_2e^{-t}) - \frac{1}{2}(2D_1e^{2t} - D_2e^{-t}) = \boxed{\frac{1}{2}D_1e^{2t} + 2D_2e^{-t}}.$$

Note: We hope that the readers have already realized that this is equivalent to the solution in part (a), simply having $D_1 = 2C_1$ and $D_2 = C_2$.

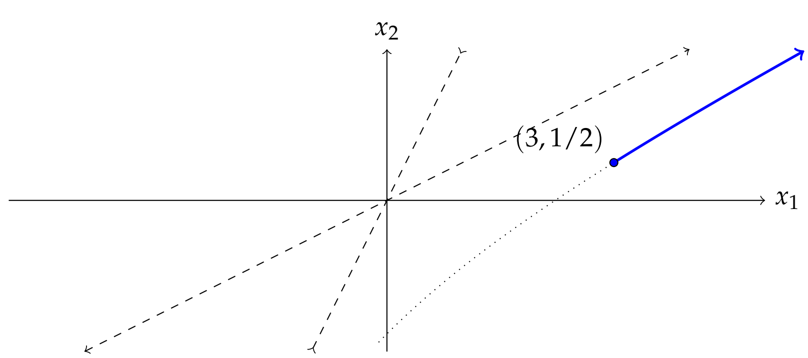
(c) Since the results are equal up to multiplying constants, we use the results from part (a). By plugging in the solutions, we have:

$$\begin{cases} 3 = x_1(0) = 2C_1 + C_2, \\ \frac{1}{2} = C_1 + 2C_2. \end{cases}$$

This induces the solution that $3C_1 + 3C_2 = 7/2$, so $C_1 + C_2 = 7/6$, and thus $C_1 = 11/6$ and $C_2 = -2/3$, so the particular solution is:

$$x_1 = \boxed{\frac{11}{3}e^{2t} - \frac{2}{3}e^{-t}} \qquad x_2 = \boxed{\frac{11}{6}e^{2t} - \frac{4}{3}e^{-t}}.$$

The graph on the x_1x_2-plane can be visualized as follows:



2. (Deal with Complex Eigenvalues). Let $\mathbf{x} = (x_1, x_2, x_3)$ be in dimension 3, we define a linear system as follows:

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}.\mathbf{x}.$$

Find the general solution to the above system in terms of real valued functions.

Solution:

Diligent readers should notice that this is the exact same matrix with **Question 2, Problem Set 7**. Here, we omit the lengthy calculation for obtaining the eigenvalues and eigenvectors. Thus, the eigenvalues and eigenvectors are:

- For $\lambda_1 = 1$, we have eigenvector $\boldsymbol{\xi}^{(\mathbf{1})} = (2, -3, 2)$.
- For $\lambda_2 = 1 + 2\mathrm{i}$, we have eigenvector $\boldsymbol{\xi}^{(\mathbf{2})} = (0, 1, -\mathrm{i})$.
- For $\lambda_3 = 1 - 2\mathrm{i}$, we have eigenvector $\boldsymbol{\xi}^{(\mathbf{3})} = (0, 1, \mathrm{i})$.

Technically, we only need one complex eigenvalue for the conjugate pair, so you do not need to calculate the last eigenvector.

For λ_1, we trivially obtain one solution that $\mathbf{x} = C_1 e^t \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix}$.

For λ_2, we form a solution as:

$$\mathbf{x} = e^{(1+2\mathrm{i})t} \begin{pmatrix} 0 \\ 1 \\ -\mathrm{i} \end{pmatrix} = e^t \big(\cos(2t) + \mathrm{i} \sin(2t) \big) \begin{pmatrix} 0 \\ 1 \\ -\mathrm{i} \end{pmatrix} = \begin{pmatrix} 0 \\ e^t \cos(2t) \\ e^t \sin(2t) \end{pmatrix} + \mathrm{i} \begin{pmatrix} 0 \\ e^t \sin(2t) \\ -e^t \cos(2t) \end{pmatrix}.$$

Note that we have proven that $\{1, \mathrm{i}\}$ is a linearly independent set in a $\mathbb{R}$-vector space from **Question 4(a)(iii), Problem Set 6**, we obtain the other two solutions as:

$$\mathbf{x} = C_2 e^t \begin{pmatrix} 0 \\ \cos(2t) \\ \sin(2t) \end{pmatrix} \text{ and } \mathbf{x} = C_3 e^t \begin{pmatrix} 0 \\ \sin(2t) \\ -\cos(2t). \end{pmatrix}$$

Now, by the principle of superposition, we have the full set of solution as:

$$\mathbf{x} = \boxed{C_1 e^t \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} + C_2 e^t \begin{pmatrix} 0 \\ \cos(2t) \\ \sin(2t) \end{pmatrix} + C_3 e^t \begin{pmatrix} 0 \\ \sin(2t) \\ -\cos(2t). \end{pmatrix}}.$$

3. (Directional Field for Linear System). For the following systems with $\mathbf{x} = (x_1, x_2)$, draw a direction field and plot some trajectories to characterize the solutions.

(a)
$$\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ -2 & 2 \end{pmatrix} .\mathbf{x}.$$

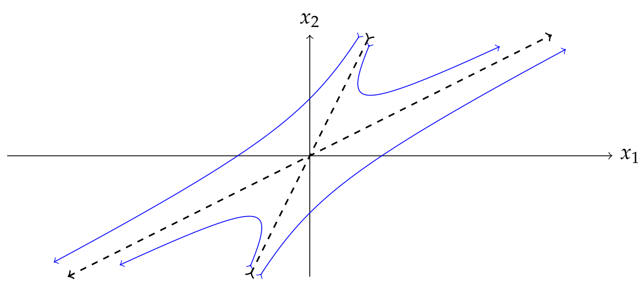
(b)
$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} .\mathbf{x}.$$

Solution:

(a) Here, we note that this matrix is identical with **Question 1**, so we obtain the eigenvalues and eigenvectors as:

- For $\lambda_1 = 2$, the eigenvector is $(2, 1)$.
- For $\lambda_2 = -1$, the eigenvector is $(1, 2)$.

Hence, the plot of some trajectories can be visualized as:



The system is unstable.

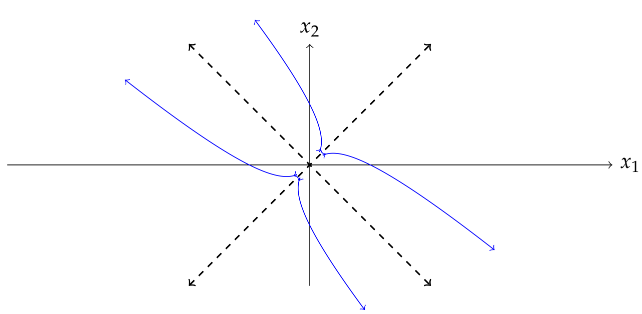
(b) Here, we actually want to solve for the eigenvalues and eigenvectors of the matrix, that is:

$$0 = \det \begin{pmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{pmatrix} = (2-\lambda)(2-\lambda) - 1 = (2-\lambda)^2 - 1 = 0.$$

Therefore, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 3$. Then, we look for the eigenvectors as:

- For $\lambda_1 = 1$, we want $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} .\boldsymbol{\xi}^{(1)} = \mathbf{0}$, hence $\xi_1^{(1)} = \xi_2^{(1)}$, so the eigenvector is $(1, 1)$.
- For $\lambda_2 = 3$, we want $\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} .\boldsymbol{\xi}^{(2)} = \mathbf{0}$, hence $\xi_1^{(2)} = -\xi_2^{(1)}$, so the eigenvector is $(-1, 1)$.

Hence, the plot of some trajectories can be visualized as:



The system is unstable.

4. (Zero Eigenvalue). Let a system of $\mathbf{x} = (x_1, x_2)$ be defined as:

$$\mathbf{x}' = \begin{pmatrix} -3 & -6 \\ 1 & 2 \end{pmatrix} .\mathbf{x}.$$

(a) Find the eigenvalues and eigenvectors for $\begin{pmatrix} -3 & -6 \\ 1 & 2 \end{pmatrix}$.

(b) Give a full set of solutions to the differential equation. Plot some trajectory on the $x_1 x_2$-plane.

(c)* Let A be an arbitrary square matrix. Show that A is non-invertible if and only if A has zero as an eigenvalue.

Note: Please avoid using the definition that the determinant is the product of all eigenvalues. Moreover, consider the geometric implication of eigenvalue to account for invertibility.

Solution:

(a) Similarly, we find eigenvalue by:

$$0 = \det \begin{pmatrix} -3-\lambda & -6 \\ 1 & 2-\lambda \end{pmatrix} = (-3-\lambda)(2-\lambda) + 6 = \lambda^2 + \lambda = \lambda(\lambda+1),$$

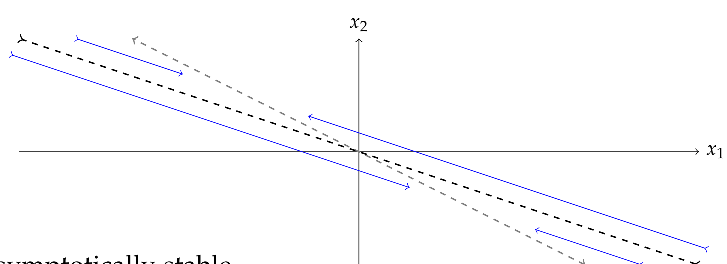
so the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -1$. Now, we look for the eigenvectors, as follows:

- For $\lambda_1 = 0$, we want $\begin{pmatrix} -3 & -6 \\ 1 & 2 \end{pmatrix} .\boldsymbol{\xi}^{(\mathbf{1})} = \mathbf{0}$, so $\xi_1^{(1)} = -2\xi_2^{(1)}$, so the eigenvector is $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.
- For $\lambda_2 = -1$, we want $\begin{pmatrix} -2 & -6 \\ 1 & 3 \end{pmatrix} .\boldsymbol{\xi}^{(\mathbf{2})} = \mathbf{0}$, so $\xi_1^{(2)} = -3\xi_2^{(1)}$, so the eigenvector is $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

(b) With the eigenvectors and eigenvalues, we trivially obtain the full set of solutions as:

$$\boxed{\left\{ C_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix}, C_2 e^{-t} \begin{pmatrix} 3 \\ -1 \end{pmatrix} \right\}}.$$

Graphically, the plot of some trajectories can be visualized as:



The system is asymptotically stable.

(c) Here, we advised readers not to consider the definition of determinant being the product of all eigenvalues, since this equivalence is trivial by that definition. Alternatively, we want readers to think of the definition of determinants in the scope of this class, *i.e.*, the root of $\det(A - \lambda\,\mathrm{Id})$.

Proof. Here, we have the following equivalences:

$$A \text{ is non-invertible} \iff \det(A) = 0 \iff \det(A - 0\,\mathrm{Id}) = 0 \iff 0 \text{ is an eigenvalue of } A.$$

As desired. □

Geometrically, think of eigenvalue monitoring the action in terms of a scalar multiplication, so having a zero eigenvalue collapse a dimension, making the matrix unable to be bijective. Conversely, if the map is not injective, we can easily form the kernel as a subspace whose eigenvalue is zero.