PILOT

# Problem Set 10: Solutions

Differential Equations

Fall 2024

1. (Stability for Nonlinear System). Complete the following table for stability of dimension 2 linear and nonlinear systems.

**Solution:**

| **Eigenvalues** | **Linear System** | | **Nonlinear System** | |
|---|---|---|---|---|
| | **Type** | **Stability** | **Type** | **Stability** |
| Eigenvalues are $\lambda_1$ and $\lambda_2$ | | | | |
| $0 < \lambda_1 < \lambda_2$ | Node | Unstable | Node | Unstable |
| $\lambda_1 < \lambda_2 < 0$ | Node | Asymptotically Stable | Node | Asymptotically Stable |
| $\lambda_1 < 0 < \lambda_2$ | Saddle Point | Unstable | Saddle Point | Unstable |
| $\lambda_1 = \lambda_2 > 0$ | Node | Unstable | Node or Spiral Point | Unstable |
| $\lambda_1 = \lambda_2 < 0$ | Node | Asymptotically Stable | Node or Spiral Points | Asymptotically Stable |
| Eigenvalues are $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$ | | | | |
| $\alpha > 0$ | Spiral Point | Unstable | Spiral Point | Unstable |
| $\alpha = 0$ | Center | Stable | Center or Spiral Point | Indeterminate |
| $\alpha < 0$ | Spiral Point | Asymptotically Stable | Spiral Point | Asymptotically Stable |

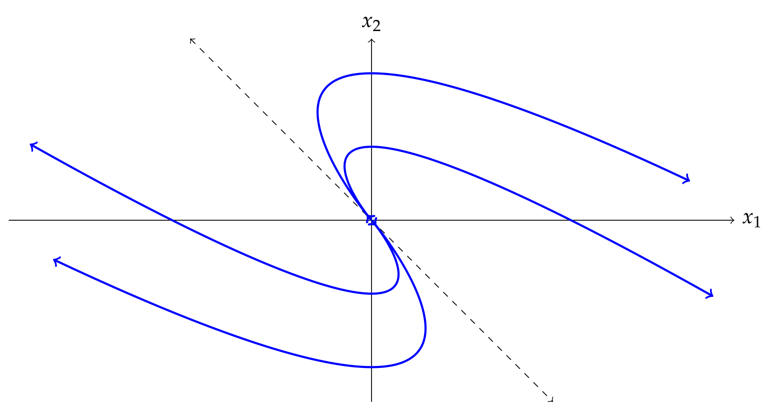
2. (Phase Portraits for Repeated Roots). Find the solutions to the following linear system differential equation, sketch a few phase portraits, and classify its type and stability.

(a)
$$\mathbf{x}' = \begin{pmatrix} 4 & 2 \\ -2 & 0 \end{pmatrix} \cdot \mathbf{x}.$$

(b)
$$\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \cdot \mathbf{x}.$$

**Solution:**

(a) Again, we first find the eigenvalues as:
$$0 = \det \begin{pmatrix} 4-\lambda & 2 \\ -2 & 0-\lambda \end{pmatrix} = (4-\lambda)(-\lambda) + 4 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2.$$
Hence, the eigenvalue is 2 (with *algebraic* multiplicity 2, c.f. §9.1). Then, we look for the eigenvalue, that is $\boldsymbol{\xi}$ such that :
$$\begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix} \cdot \boldsymbol{\xi} = \mathbf{0}.$$
Hence, we have $2\xi_1 + 2\xi_2 = 0$, so $-\xi_1 = \xi_2$, which is $\boldsymbol{\xi} = (1, -1)$.
Then, we need to find the other solution with vector $\boldsymbol{\eta}$ such that:
$$\begin{pmatrix} 2 & 2 \\ -2 & -2 \end{pmatrix} \cdot \boldsymbol{\eta} = \boldsymbol{\xi}.$$
Hence, we have $2\eta_1 + 2\eta_2 = 1$, so $\boldsymbol{\eta} = (1/2, 0)$. Hence, the solution is:
$$\mathbf{x} = \boxed{C_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 \left( t e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t} \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \right)}.$$
Here, we may graph the phase portraits as:




Here, the graph has a $\boxed{\text{unstable improper node}}$.

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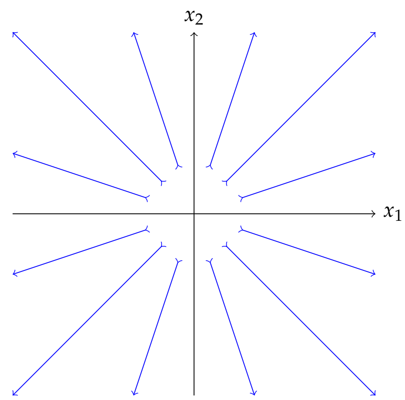
(b) Here, we find that the eigenvalue is 3 with *algebraic* multiplicity 2, and since it is diagonal, the *geometric* multiplicity is also 2. Here, we make this into a degenerated case:

$$\begin{cases} x_1' = 3x_1, \\ x_2' = 3x_2. \end{cases}$$

Hence, the solution is:

$$\begin{cases} x_1 = \boxed{C_1 e^{3t}}, \\ x_2 = \boxed{C_2 e^{3t}}. \end{cases}$$

Here, we may graph the phase portraits as:




Here, the graph has a $\boxed{\text{unstable proper node}}$.

3. (Critical Point). Find all the critical point in the following first order system:

$$\begin{cases} x' = 2x^3 - x^2 - 4x + 3 - y^2, \\ y' = 2x - y. \end{cases}$$

**Solution:**
Here, we set both equations as 0, so we have:

$$\begin{cases} 0 = 2x^3 - x^2 - 4x + 3 - y^2, \\ 0 = 2x - y. \end{cases}$$

Hence, we let the second equation be:

$$y = 2x,$$

and we plug it into the first equation to be:

$$0 = 2x^3 - x^2 - 4x + 3 - 4x^2 = 2x^3 - 5x^2 - 4x + 3.$$

Here, we attempt to factor the above by the rational root test (c.f. §1.3), the rational roots has to be one of $\pm 1, \pm 3, \pm 1/2, \pm 3/2$. We first note that $x = -1$ is a root, then we factor it into:

$$(2x^3 - 5x^2 - 4x + 3)/(x + 1) = 2x^2 - 7x + 3.$$

Thus, the last two roots are $1/2$ and 3. Hence, the critical points are:

$$\boxed{(-1, -2), (1/2, 1), \text{ and } (3, 6)}.$$

4. (Locally Linear System). Let a linear system be defined as follows:

$$\begin{cases} x' = x - y, \\ y' = x - 2y + x^2. \end{cases}$$

(a) Verify that $(0,0)$ is an equilibrium.

(b) Verify that the system is locally linear at $(0,0)$.

(c) Classify the type and stability of $(0,0)$ locally.

**Solution:**

(a) *Proof.* The verification is trivial. We evaluate $x$ and $y$ both at 0 for the differential equation, hence:

$$\begin{cases} x' = 0 - 0 = 0, \\ y' = 0 - 0 + 0 = 0. \end{cases}$$

Hence, $(0,0)$ is a equilibrium. $\square$

(b) *Proof.* Here, we consider the Jacobian matrix as:

$$\mathrm{J}[x'(x,y), y'(x,y)] = \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2x+1 & -2 \end{pmatrix}.$$

Now, we evaluate the matrix at $(0,0)$, which gives:

$$\mathrm{ev}_0 \begin{pmatrix} 1 & -1 \\ 2x+1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0+1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix}.$$

Note that the determinant is $-2+1 = -1 \neq 0$, so the system is locally linear. $\square$

(c) Here, the linear system locally at $(0,0)$ should be:

$$\begin{pmatrix} x \\ y \end{pmatrix}' \sim \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

We find its eigenvalue as:

$$0 = \det \begin{pmatrix} 1-\lambda & -1 \\ 1 & -2-\lambda \end{pmatrix} = (1-\lambda)(-2-\lambda) + 1 = \lambda^2 + \lambda - 1.$$

By using the quadratic formula, we have the eigenvalues as $\lambda = \frac{-1 \pm \sqrt{5}}{2}$.
Thus, we have $\lambda_1 < 0 < \lambda_2$, so we have a $\boxed{\text{unstable saddle point}}$.