Problem Set 13: Solutions

Differential Equations

Fall 2024

1. (Logarithms and Recurrence Relations). The following problem aims to solve the differential equation for $y := y(x)$:

$$(x+1)^2 \frac{d^2 y}{dx^2} + (x+1)\frac{dy}{dx} = 0.$$

using recurrence relationship.

(a) Write down the power series of $\log(x+1)$.

(b) Find the *recurrence relationship* for the differential equation.

(c)* Find the fundamental set of solutions for the differential equation.
Hint: Make a conjecture from a pattern of the first few terms.

Solution:

(a) Here, we note that:

$$\frac{d}{dx}\big(\log(x+1)\big) = \frac{1}{x+1} = \frac{1}{1+x},$$

hence, as how we considered for geometric sequence, we have:

$$\frac{1}{1+x} = 1 + (-x) + (-x)^2 + \cdots = 1 - x + x^2 - x^3 + \cdots$$

Hence, we learned that the coefficients are just simply:

$$\mathrm{ev}_0\left(\frac{d^n}{dx^n}\frac{1}{1+x}\right) = (-1)^n n!.$$

Hence, we can easily deduce the coefficients for $\log(x+1)$, that is:

$$\mathrm{ev}_0\left(\frac{d^n}{dx^n}\big(\log(x+1)\big)\right) = (-1)^{n-1}(n-1)! \text{ for } n \geq 1.$$

Now, we can form the power series of $\log(x+1)$ as:

$$\log(x+1) \sim \boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n}}.$$

(b) Now, we are handling the recurrence relationship. First, we assume that the solution is:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

by the assumption that the series converges absolute, we take differentiate the terms twice, which gives that:

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n,$$

Continues on the next page...

Continued from last page.

and:

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n.$$

With the derivative, we plug it back into the differential equations, that is:

$$\begin{aligned}
0 &= (x+1)^2 \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + (x+1) \sum_{n=1}^{\infty} na_n x^{n-1} \\
&= x^2 \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + 2x \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n \\
&\quad + x \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n \\
&= \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=1}^{\infty} 2(n+1)na_{n+1}x^n + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n \\
&\quad + \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n \\
&= \sum_{n=2}^{\infty} \left[n(n-1)a_n + 2(n+1)na_{n+1} + (n+2)(n+1)a_{n+2} + na_n + (n+1)a_{n+1}\right] x^n \\
&\quad + a_1 + 2a_2 + 4a_2 x + 6a_3 x + a_1 x + 2a_2 x.
\end{aligned}$$

Now, we suppose the initial conditions a_0 and a_1, we have the relationship:

$$\begin{cases}
a_1 + 2a_2 = 0, \\
6a_3 + 6a_2 + a_1 = 0, \\
n(n-1)a_n + 2(n+1)na_{n+1} + (n+2)(n+1)a_{n+2} + na_n + (n+1)a_{n+1} = 0 \text{ for all } n \geq 2.
\end{cases}$$

In particular, we can find the recurrence relationship as:

$$a_{n+2} = \boxed{-\frac{n^2 a_n + (n+1)(2n+1)a_{n+1}}{(n+2)(n+1)}} \text{ for } n \geq 2.$$

(c) Now, since we have a second order differential equation, we set:

$$a_0 = a_0, \qquad a_1 = a_1, \qquad a_2 = -\frac{1}{2}a_1, \qquad a_3 = \frac{-a_1 - 6a_2}{6} = \frac{a_1}{3},$$
$$a_4 = \frac{-15a_3 + 4a_2}{12} = -\frac{a_1}{4}, \qquad a_5 = \frac{-9a_3 - 28a_4}{20} = \frac{a_1}{5}, \qquad \cdots.$$

Here, it is fair to conjecture that $a_n = \frac{(-1)^{n-1}a_1}{n}$. In fact, we can prove this by strong induction.

Proof. The base case is already check in the previous argument, so we check on the inductive step, suppose that:

$$a_k = \frac{(-1)^{k-1}a_1}{k} \text{ and } a_{k+1} = \frac{(-1)^k a_1}{k+1}.$$

Then, using the relationship above, we have:

$$\begin{aligned}
a_{k+2} &= -\frac{\frac{k^2(-1)^{k-1}a_1}{k} + \frac{(-1)^k a_1 (k+1)(2k+1)}{k+1}}{(k+2)(k+1)} \\
&= -\frac{a_1\big(2k+1-k\big)(-1)^k}{(k+2)(k+1)} = -\frac{a_1(-1)^k}{k+2} = \frac{a_1(-1)^{k+2-1}}{k+2},
\end{aligned}$$

Continues on the next page...

Continued from last page.

which completes the proof. □

Therefore, we can write our solution, by part (a), as:

$$y(x) := a_0 + a_1 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = a_0 + a_1 \log(1+x),$$

so the fundamental set of solutions are:

$$\boxed{1 \quad \text{and} \quad \log(1+x)}.$$

2. (Euler's Equations). Let a differential equation of $y := y(x)$ defined as:

$$x^2y'' + xy' + cy = 0,$$

where $c \in \mathbb{R}$ is a fixed constant, we want to solve the differential equation using *Euler's equations*.

(a) Assume $c = -4$, solve the solution to the differential equation.

(b) Assume $c = 9$, solve the solution to the differential equation.

(c)* Find the critical point to this system where the behavior of the solution changes.

Solution:

(a) For $c = -4$, we have:

$$x^2y'' + xy' - 4y = 0,$$

and our characteristic equation is:

$$0 = r(r-1) + r - 4 = r^2 - 4,$$

whose roots are ± 2, so the solution is:

$$y(x) = c_1|x|^2 + c_2|x|^{-2} = \boxed{c_1x^2 + c_2x^{-2}}.$$

(b) For $c = 9$, we have:

$$x^2y'' + xy' + 9y = 0,$$

and our characteristic equation is:

$$0 = r(r-1) + r + 9 = r^2 + 9,$$

whose roots are $\pm 3i$, so the solution is:

$$y(x) = c_1|x|^0 \cos\left(3\log|x|\right) + c_2|x|^0 \sin\left(3\log|x|\right) = \boxed{c_1\cos(3\log|x|) + c_2\sin(3\log|x|)}.$$

(c) Consider that we have the characteristic equation:

$$0 = r(r+1) + r + c = r^2 + c.$$

Hence, we know that the system has two distinct real roots when $c < 0$, it has repeated zero roots when $c = 0$, and complex roots when $c > 0$, so the critical point is $\boxed{c = 0}$.

3. (Singularities, Zeros, and Poles). For any function $f : \mathbb{C} \to \mathbb{C}$, and $z_0 \in \mathbb{C}$, we have the following:

- It has **a zero of order *m*** at z_0 if $f(z_0) = 0$, and m is the smallest positive integer such that $f(z) = (z - z_0)^m g(z)$, where g is analytic at z_0 and $g(z_0) \neq 0$.
- It has **a pole of order *n*** at z_0 if $f(z_0)$ is not defined, and n is the smallest integer such that $g(z) = (z - z_0)^n f(z)$, where g is analytic at z_0 and $g(z_0) \neq 0$.
- If a zero/pole has order 1, it is **simple**.

As a side note, such definition applies for any real valued functions, *i.e.*, $f : \mathbb{R} \to \mathbb{R}$.
Here, we define a differential equation for $y := y(x)$ as:

$$\sin(x)y'' + \sin(x)\big(\cos(x) - e^x + x\big)y' + \big(\csc(x)\big)y = 0$$

(a) Write the differential equation in the form of:

$$y'' + p(x)y' + q(x)y = 0.$$

(b)* Identify all zeros and poles of $p(x)$ and $q(x)$ as real functions, *i.e.*, $p, q : \mathbb{R} \to \mathbb{R}$. Find the order of the zeros and poles.

(c) Identify all the points $x_0 \in \mathbb{R}$ such that the differential equation has a *regular singular point*.

Solution:

(a) Here, we easily write this as $\boxed{y'' + \big(\cos(x) - e^x + x\big)y' + \dfrac{1}{\sin^2(x)}y = 0}$.

(b) Here, we have that:

$$p(x) = \cos(x) - e^x + x \text{ and } q(x) = \frac{1}{\sin^2(x)}.$$

Note that $p(x)$ is the sum of analytic functions, so it has no poles, we note that $p(x)$ has zero when $0 = \cos(x) - e^x + x$. We note that the derivative of $p(x)$ is:

$$p'(x) = -\sin(x) - e^x + 1.$$

It is not hard to observe that $p'(x) \leq 0$ for all $x > 0$ and $p(x) \geq 0$ for all $-2 < x < 0$, and we note that $p(0) = 0$, so it must be the unique zero. Now, we determine its order, we note that:

$$\lim_{x\to 0} \frac{p(x)}{x} = \lim_{x\to 0} \frac{\cos(x) - e^x + x}{x} \overset{\text{L'Hôpital}}{=\!=} \lim_{x\to 0} \frac{-\sin(x) - e^x + 1}{1} = 0.$$

Then, we need to take second order again:

$$\lim_{x\to 0} \frac{p(x)}{x^2} = \lim_{x\to 0} \frac{\cos(x) - e^x + x}{x^2} \overset{\text{L'Hôpital}}{=\!=} \lim_{x\to 0} \frac{-\sin(x) - e^x + 1}{2x} \overset{\text{L'Hôpital}}{=\!=} \lim_{x\to 0} \frac{-\cos x - e^x}{2} = -1.$$

Hence, $p(x)$ has $\boxed{\text{a zero of order 2 at } x = 0}$.
For $q(x)$, we note that it is nonzero, but it is undefined when $\sin(x) = 0$, that is all $k\pi$ for $k \in \mathbb{Z}$.
For $\sin x$, we note that it has zeros there, and each of them has order 1, since we have:

$$\mathrm{ev}_{k\pi}\left[\frac{d}{dx}(\sin x)\right] = \cos(k\pi) = 1 \neq 0,$$

thus we have shown that $\sin x$ has zeros of order 1 at all $k\pi$, so $\sin^2(x)$ has zero of order 2 at all $k\pi$, and thus $\frac{1}{\sin^2(x)}$ has $\boxed{\text{poles of order 2 at all } k\pi}$.

(c) Hence, all the singular points are $k\pi$ for $k \in \mathbb{Z}$, and since they have poles of order 2, all of them are regular singular points. Thus, all the regular singular points are $\boxed{k\pi \text{ for } k \in \mathbb{Z}}$.

4. (Dispersion of Heat). For this problem, we consider the dispersion of heat for an object in an environment with fixed temperature. Here, let $\theta := \theta(t)$ be the temperature of the object and θ_0 denote the fixed temperature of the environment, we may model the temperature of the object by:
$$\frac{d\theta}{dt} = -\frac{1}{\kappa}(\theta - \theta_0),$$
where κ is a fixed positive constant, representing the rate of heat dispersion.
Suppose that we have a rigid body of 100°C (equivalently 212°F), and the room temperature is fixed as 20°C (equivalently 68°F, and this is also condition for STP, standard temperature and pressure). Also, we assume that $\kappa = 2$.

   (a) Construct the differential equation for the above system.

   (b) Use *Euler's method* with step size of 1 to approximate the temperature at $t = 3$.

   (c)* Identify if the approximation of temperature is an underestimate or an overestimate.

Solution:

(a) The system can be easily constructed, namely:
$$\boxed{\frac{d\theta}{dt} = -\frac{1}{2}(\theta - 20)}.$$

(b) For Euler's method of step size 1, with $\theta(0) = 100$, we have $\theta'(0) = -80/2 = -40$. We do the following steps:

- We approximate $\theta(1) \approx \theta(0) + \theta'(0) = 100 - 40 = 60$, then we have $\theta'(1) \approx -40/2 = -20$.
- We approximate $\theta(2) \approx \theta(1) + \theta'(1) \approx 60 - 20 = 40$, then we have $\theta'(2) \approx -20/2 = -10$.
- We approximate $\theta(3) \approx \theta(2) + \theta'(2) \approx 40 - 10 = 30$.

Then, we have approximated that:
$$\theta(3) \approx \boxed{30(^\circ\mathrm{C})}.$$

(c) To show underestimate or overestimate, we note that:
$$\frac{d^2\theta}{dt^2} = -\frac{1}{2} < 0,$$
so the function is concave (typically called *concave down* in high school). Hence, we know that the tangent line is having a smaller slope compared to the actual curve, hence it is decreasing slower than actual, so the temperature is an $\boxed{\text{overestimate}}$.