Problem Set 2: Solutions

Differential Equations

Fall 2024

1. (Linearity of Solutions.) Let $y = y_1(t)$ be a solution to $y' + p(t)y = 0$, and let $y = y_2(t)$ be a solution to $y' + p(t)y = q(t)$. Show that $y = y_1(t) + y_2(t)$ is then also a solution to $y' + p(t)y = q(t)$.

Proof:

Here, we note that $y_1(t)$ and $y_2(t)$ satisfies that:

$$\begin{cases} y_1'(t) + p(t)y_1(t) = 0, \\ y_2'(t) + p(t)y_2(t) = q(t). \end{cases}$$

Thus, by added both the left hand side and the right hand side, we obtain:

$$\begin{aligned} \big(y_1'(t) + p(t)y_1(t)\big) + \big(y_2'(t) + p(t)y_2(t)\big) &= 0 + q(t) \\ \big(y_1'(t) + y_2'(t)\big) + p(t)\big(y_1(t) + y_2(t)\big) &= q(t) \\ \big(y_1(t) + y_2(t)\big)' + p(t)\big(y_1(t) + y_2(t)\big) &= q(t), \end{aligned}$$

implying that $y = y_1(t) + y_2(t)$ is the solution to $y' + p(t)y = q(t)$, as desired. $\square$

2. (Integrating Factor.) Solve for the general solution to the following ODE with $y = y(t)$:

$$2y' + y = 3t.$$

Solution:
Here, we first convert the equation to standard form, *i.e.*:

$$y' + \frac{1}{2}y = \frac{3}{2}t.$$

Hence, with $p(t) = 1/2$, the integration factor must be:

$$\mu(t) = \exp\left(\int_0^t p(s)ds\right) = \exp\left(\int_0^t \frac{1}{2}ds\right) = \exp\left(\frac{1}{2}t\right).$$

Now, we multiply the integration factor on both sides, giving that:

$$\begin{aligned}
y'e^{t/2} + \frac{1}{2}ye^{t/2} &= \frac{3}{2}te^{t/2}, \\
\frac{d}{dt}\left[e^{t/2}y\right] &= \frac{3}{2}te^{t/2}, \\
e^{t/2}y &= \frac{3}{2}\int te^{t/2}dt \\
&= \frac{3}{2}\left[2te^{t/2} - \int 2e^{t/2}\right] \\
&= \frac{3}{2}\left[2te^{t/2} - 4e^{t/2} + C\right] \\
&= 3te^{t/2} - 6e^{t/2} + \tilde{C}, \\
y &= \boxed{\tilde{C}e^{-t/2} + 3t - 6}.
\end{aligned}$$

3. (Integrating Factor or Exactness?) Let a differential equation be defined as follows:
$$\frac{dy}{dx} = e^{2x} + y - 1.$$
   (a) What is the integrating factor ($\mu(x)$) for the equation? Solve for the general solution.
   (b) Is the equation *exact*? If not, make it exact, then find the general solution.
   (c) Do solutions from part (a) and (b) agree?

Solutions:

(a) First, we write the equation in standard form, that is:
$$y' - y = e^{2x} - 1.$$
Hence, with $p(x) = -1$, the integrating factor is:
$$\mu(x) = \exp\left(\int_0^x p(s)ds\right) = \exp\left(\int_0^x (-1)ds\right) = \boxed{\exp(-x)}.$$
Then, we multiply the integrating factors on both ends to obtain:
$$\begin{aligned} y'e^{-x} - ye^{-x} &= e^x - e^{-x}, \\ \frac{d}{dx}\left[ye^{-x}\right] &= e^x - e^{-x}, \\ ye^{-x} &= \int \left(e^x - e^{-x}\right) dx = e^x + e^{-x} + C, \\ y &= \boxed{Ce^x + e^{2x} + 1}. \end{aligned}$$

(b) Note that for exactness, we write the equation as:
$$\underbrace{(-e^{2x} - y + 1)}_{M(x,y)} + \underbrace{(1)}_{N(x,y)} \frac{dy}{dx} = 0,$$
meaning that their partial derivatives are, respectively:
$$\partial_y M(x,y) = -1 \text{ and } \partial_x N(x,y) = 0,$$
and since they are different, the equation is $\boxed{\text{not exact}}$.
Thus, we look for the integrating factor, *i.e.*:
$$\mu(t) = \exp\left(\int_0^x \frac{\partial_y M(x,y) - \partial_x N(x,y)}{N(x,y)}\right) = \exp\left(\int_0^x \frac{-1-0}{1} ds\right) = \exp(-x).$$
Now, we multiply e^{-x} on both sides, giving us that:
$$\underbrace{(-e^x - ye^{-x} + e^{-x})}_{\tilde{M}(x,y)} + \underbrace{(e^{-x})}_{\tilde{N}(x,y)} \frac{dy}{dx} = 0.$$
Now, the equation is exact. *We leave the check to the readers as an exercise.*
To get the solution, we first integrate $\tilde{M}(x,y)$ with respect to x, that is:
$$\varphi(x,y) = \int (-e^x - ye^{-x} + e^{-x})dx = -e^x + ye^{-x} - e^{-x} + h(y).$$
Now, taking the derivative with respect to y gives:
$$\partial_y \varphi(x,y) = e^{-x} + h'(y) = e^{-x},$$
which pushes $h(y)$ to be constant, hence we have solution:
$$\varphi(x,y) = \boxed{-e^x + ye^{-x} - e^{-x} = C}.$$

(c) The solutions $\boxed{\text{agree}}$ by simple arithmetic deductions.

4. (Decay and Dating.) Carbon-14, a radioactive isotope of carbon, is an effective tool in dating the age of organic compounds, as it decays with a relatively long period. Let $Q(t)$ denote the amount of carbon-14 at time t, we suppose that the decay of $Q(t)$ satisfies the following differential equation:
$$\frac{dQ}{dt} = -\lambda Q \text{ where } \lambda \text{ is the rate of decay constant.}$$
   (a) Let the half-life of carbon-14 be τ, find the rate of decay, λ.
   (b) Suppose that a piece of remain is discovered to have 10% of the original amount of carbon-14, find the age of the remain in terms of τ.

Solutions:

(a) Note that the differential equation is separable, hence:
$$\begin{aligned} \frac{dQ}{Q} &= -\lambda dt, \\ \int \frac{dQ}{Q} &= -\int \lambda dt, \\ \log|Q| &= -\lambda t + C, \\ Q &= \tilde{C}e^{-\lambda t}. \end{aligned}$$
Here, we assume $Q = Q_0$ at $t = t_0$, then we have $Q = Q_0/2$ when $t = t_0 + \tau$, so:
$$\frac{1}{2} = e^{-\lambda\tau},$$
which deduces to:
$$\lambda = -\frac{1}{\tau}\log\left(\frac{1}{2}\right) = \boxed{\frac{\log 2}{\tau}}.$$

(b) If there are only 10% of remain, we suppose that we have $Q = Q_0$ at $t = t_0$, and have $Q = Q_0/10$ at $t = t_0 + s$, hence giving that:
$$\frac{Q_0}{10} = Q_0 \exp(-\lambda t_0) = Q_0 \exp\left(-\frac{\log(2)t_0}{\tau}\right).$$
Thus, we obtain that:
$$\frac{1}{10} = \exp\left(-\frac{\log(2)t_0}{\tau}\right),$$
and by solving for t_0, we obtain:
$$t_0 = -\frac{\tau}{\log 2}\log\left(\frac{1}{10}\right) = \boxed{\frac{\log 10}{\log 2}\tau}.$$