



Problem Set 1: Solutions

Differential Equations

Fall 2024

1. (Review: Integration.) As one of the most important skills of differential equations, the study requires proficiency in integration. By the *Fundamental Theorem of Calculus*, the basics of most calculations are on finding antiderivatives. Please evaluate the following indefinite integrals:

(a) $\int e^{1/x} \cdot \frac{1}{x^2} dx.$

(b) $\int \sin(5x)e^{-x} dx.$

(c) $\int \cos(2t) \tan(t) dt.$

Solution:

- (a) For the first integration, readers should observe that we may use integration by substitution (or *u*-substitution) with $u = 1/x$ where $du/dx = -1/x^2$, hence giving us:

$$\int e^{1/x} \cdot \frac{1}{x^2} dx = - \int e^u du = -e^u + C = \boxed{-e^{1/x} + C}.$$

- (b) Here, we shall introduce a basic technique that you will see a lot over the course, we may integrate by parts first.

$$\begin{aligned} \int \sin(5x)e^{-x} dx &= -\sin(5x)e^{-x} + \int 5 \cos(5x)e^{-x} dx \\ &= -\sin(5x)e^{-x} + 5 \left[-\cos(5x)e^{-x} - \int 5 \sin(5x)e^{-x} dx \right] \\ 26 \int \sin(5x)e^{-x} dx &= -\sin(5x)e^{-x} - 5 \cos(5x)e^{-x} + C \\ \int \sin(5x)e^{-x} dx &= \boxed{-\frac{1}{26} \sin(5x)e^{-x} - \frac{5}{26} \cos(5x)e^{-x} + \tilde{C}}. \end{aligned}$$

- (c) Evaluating the last integral inhibits trigonometric identities, that is:

$$\begin{aligned} \int \cos(2t) \tan(t) dt &= \int (2 \cos^2(t) - 1) \frac{\sin(t)}{\cos(t)} dt = \int (2 \sin(t) \cos(t) - \tan(t)) dt \\ &= \int (\sin(2t) - \tan(t)) dt = \boxed{-\frac{1}{2} \cos(2t) + \log |\cos t| + C}. \end{aligned}$$

2. (Separable ODE.) Solve the following initial value problem (IVP) on $y = y(x)$, and specify the domain for your solution:

$$\begin{cases} y' = (x \log x)^{-1}, \\ y(e) = -6. \end{cases}$$

Note: Here $\log(x) := \log_e(x)$ is the natural logarithm function, which may be written as $\ln(x)$.

Solution:

Here, we notice that this problem is separable, hence we can write:

$$\begin{aligned} y' &= \frac{dy}{dx} = (x \log x)^{-1}, \\ dy &= \frac{1}{x \log x} dx, \\ \int dy &= \int \frac{1}{x \log x} dx. \end{aligned}$$

Now, we evaluate the integral by substitution, *i.e.*, $u = \log x$ and $du = dx/x$, which give that:

$$y = \int \frac{1}{u} du = \log |u| + C = \log |\log x| + C.$$

Eventually, we plug in the initial condition, that is $y(e) = -6$, giving us that:

$$\begin{aligned} -6 &= \log |\log e| + C, \\ C &= 6. \end{aligned}$$

Therefore, the solution is:

$$y = \boxed{\log |\log x| - 6}.$$

Here, we note that $\log(\bullet)$ has a valid domain over positive numbers, and the double $\log(\bullet)$ functions enforces that x must be greater than 1, as $\log(0)$ is undefined. Since our initial condition is e , and $e \in (1, \infty)$, the domain of the solution is $\boxed{(1, \infty)}$.

3. (Direction Field.) Let a differential equation be defined as follows:

$$\frac{dy}{dx} = y^3 - 7y^2 + 16y - 12 \text{ where } x \geq 0 \text{ and } y \geq 0.$$

- (a) Classify the above differential equation.
 (b) Sketch a direction field on the differential equation, then state the equilibriums of y , interpret their stability.

Solution:

- (a) Note that we can rewrite the equation as:

$$F[y', y] = y' - y^3 + 7y^2 - 16y + 12 = 0,$$

and clearly it is **non-linearly** (or you can explicitly show that $F[(y+1)', (y+1)] \neq 0$).

Note that the highest derivative is of degree 1, hence it is **first order**.

- (b) Recall from Pre-Calculus (or Algebra) the following *Rational root test*:

Theorem 1: Rational Root Test. Let the polynomial:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0$$

have integer coefficients $a_i \in \mathbb{Z}$ and $a_0, a_n \neq 0$, then any rational root $r = p/q$ such that $p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$ satisfies that $p|a_0$ and $q|a_n$. \square

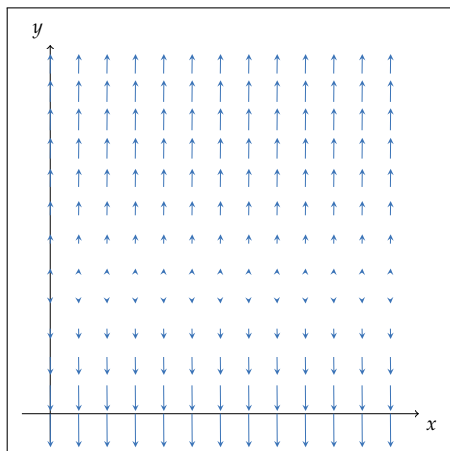
From the theorem, we can note that if the equation has a rational root, it must be one of:

$$r = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \text{ and } \pm 12.$$

By plugging in, one should notice that $y = 2$ is a root (one might also notice 3 is a root, but we will get the step slowly), so we can apply the long division (dividing $y - 2$) to obtain that:

$$\frac{y^3 - 7y^2 + 16y - 12}{y - 2} = y^2 - 5y + 6,$$

Clear, we notice that the right hand side is $(y - 2)(y - 3)$, so we now know that the roots (or equilibrium) are **2 (multiplicity 2) and 3**, and the direction field looks like:



Note that **Theorem 1** can also be generalized in ring theory (particularly, in UFDs), please check on it if you are interested in it. Moreover, capable readers should attempt to prove that a polynomial of degree 3 with integer coefficients must have at least one rational root.

4. (Constructing Solutions.) Let $x(t) = t^2 e^t$. Construct a second order ODE that has $x(t)$ as a solution and includes all of $x(t)$, $x'(t)$ and $x''(t)$, along with maybe some leftover stuff.

Hint: Take the first and second derivative of $x(t)$ and fit them together into some linear combinations.

Solution:

Here, we first take the derivatives as:

$$x(t) = t^2 e^t,$$

$$x'(t) = 2te^t + t^2 e^t,$$

$$x''(t) = 2e^t + 4te^t + t^2 e^t.$$

Here, we simply want to put the derivatives as linear combinations, *i.e.*, $f[t, x, x', x''] = 0$, in which one straightforward example could be:

$$x(t) + x'(t) + x''(t) - t^2 e^t - (2te^t + t^2 e^t) - (2e^t + 4te^t + t^2 e^t) = 0.$$

In general, you may have any non-zero linear combinations of $x(t)$, $x'(t)$, $x''(t)$, and a function of t .