Final Review Set

Differential Equations

Fall 2025

Attention on Different Contents.

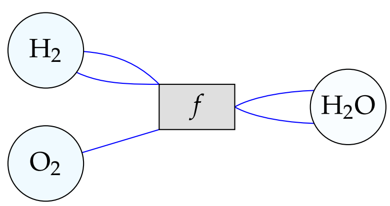
The questions labeled with **S** are the questions designated for Series Solutions for Second Order ODEs (Section 01-03) and the questions labeled with **L** are the questions designated for Laplace Transformation (Section 04-06).

Note that the final exam is cumulative, please refer to the earlier review sets for review as well.

1. Consider the following chemical equation of hydrogen gas combustion in oxygen gas:

$$2\mathrm{H}_2 + \mathrm{O}_2 \xrightarrow{f} 2\mathrm{H}_2\mathrm{O}.$$

We may represent it from a graphical representation.



Assume that the reaction rate is constant $\kappa := \mathrm{rate}(f)$. Construct the nonlinear system of the concentration of H_2 and O_2, sketch a few trajectories for different initial conditions for different starting concentrations.

2. Carbon-14, a radioactive isotope of carbon, is an effective tool in dating the age of organic compounds, as it decays with a relatively long period. Let $Q(t)$ denote the amount of carbon-14 at time t, we suppose that the decay of $Q(t)$ satisfies the following differential equation:

$$\frac{dQ}{dt} = -\lambda Q \text{ where } \lambda \text{ is the rate of decay constant.}$$

   (a) Let the half-life of carbon-14 be τ, find the rate of decay, λ.

   (b) Suppose that a piece of remain is discovered to have 10% of the original amount of carbon-14, find the age of the remain in terms of τ.

3. Let a locally linearly system be defined as:

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix} \mathbf{x} + \mathbf{f}(\mathbf{x}),$$

where $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is a vector-valued function. Find the necessary condition(s) in which the equilibrium(s) have a stable *center* in linear system. Then, state the stability and type (if possible).

Hint: Consider the solution for the linear case.

4. Given the a system of differential equations as follows:

$$\begin{cases} x' = x - y - x(x^2 + y^2), \\ y' = x + y - y(x^2 + y^2). \end{cases}$$

Find the limit cycle of the system, classify the critical points, and sketch a phase portrait of the system.

5. Let a differential equation be defined as:

$$\frac{dy}{dt} = t - y \text{ and } y(0) = 0.$$

Use Euler's Method with step size $h = 1$ to approximate $y(5)$.

S6. Consider the following series. Identify if such series converges. Compute the limit if the series converges.

(a)
$$\sum_{k=0}^{\infty} \frac{n!}{2^n}.$$

(b)
$$\sum_{k=0}^{\infty} \frac{x^{4k+1}}{(4k+1)!}.$$

(c)
$$\sum_{k=0}^{\infty} \frac{x^{4k}}{(4k)!} - \sum_{k=0}^{\infty} \frac{x^{4k+2}}{(4k+2)!}.$$

L6. Given the following the results after Laplace transformation $F(s) = \mathcal{L}\{f(t)\}$, find each $f(t)$ prior to the Laplace transformation.

(a)
$$F(s) = \frac{2s^2 + 4}{s^3 + 4s}.$$

(b)*
$$F(s) = \frac{s^2}{s^2 + 9} - 1.$$

S7. Use the *series expansions* to find the solutions to the following differential equation:

$$y'' + 3y' = 0.$$

L7. Find the solution of $y = y(t)$ to the following IVP using Laplace transformation:

$$\begin{cases} y'' - 2y' + 2y = e^{-t}, \\ y(0) = 0, \qquad y'(0) = 1. \end{cases}$$

S8. Use the *Euler's equation* to find the solution to the following differential equations:

(a)
$$x^2y'' + 5xy' + 4y = 0.$$

(b)
$$5x^2y'' + 3xy' + 7y = 0.$$

L8. Given a piecewise defined function as follows:

$$f(t) = \begin{cases} 0, & \text{when } t < 1, \\ t - 1, & \text{when } t > 1. \end{cases}$$

(a) Find the Laplace transformation of $f(t)$.

(b) Solve the following IVP:

$$\begin{cases} y'' + 3y' + 2y = f(t), \\ y(0) = 1, \qquad y(1) = e^{-2}. \end{cases}$$