Additional Materials: Differential Forms

Differential Equations

Fall 2025

This brief digression to "differential forms" aims for the following goals:

- Legitimize $\frac{\partial y}{\partial x} = \frac{f(x)}{g(y)} \iff g(y)dy = f(x)dx$ via the differential operator d.
- Get the foundation of *exactness* for certain differential equation relationship.

First, consider variables $x_1, x_2, \cdots, x_n$, we may defined the wedge product ($\wedge$) to connect any two variables satisfying that:

$$x_i \wedge x_j = -x_j \wedge x_i \text{ for all } 1 \leq i, j \leq n.$$

(a) Show that $x_i \wedge x_i = 0$ for $1 \leq i \leq n$.

Now, given any smooth function f, we defined the differential operator (d) as:

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

(b) Suppose $y(x) = e^x$, find dy.

(c) Now, suppose that $\frac{\partial y}{\partial x} = \frac{f(x)}{g(y)}$, can you express dy in terms of the differential form of x.
Note: Since we have just one variable, we have $dy/dx = \partial y/\partial x$, leading to our first goal.

Furthermore, we can apply the differential operator over differential forms with wedge products already. Suppose:

$$\omega = \sum_{i_1, \cdots, i_k} f_{i_1, \cdots, i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

we may have the differential of ω as:

$$d\omega = \sum_{i_1, \cdots, i_k} (df_{i_1, \cdots, i_k}) dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

(d) Suppose x, y are the variables, and $\omega = 2xy^2 dx + 2x^2 y dy$, show that $d\omega = 0$.

This then relates to a concept called *exactness* in differential equations. Consider the equation:

$$\frac{dy}{dx} + \frac{F(x,y)}{G(x,y)} = 0,$$

we can rewrite it as $F(x,y)dx + G(x,y)dy = 0$. Exactness enforces that:

$$\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}.$$

Similarly, exactness is considering finding a solution $f(x,y) = c$ such that $F = \frac{\partial f}{\partial x}$ and $G = \frac{\partial f}{\partial y}$.

(e) Show that $df = F(x,y)dx + G(x,y)dy$ and exactness is equivalently $d(df) = 0$.
Note: This implies that the differential equation in part (d) satisfies *exactness*.

Solution:

(a) *Proof.* Since we have:

$$x_i \wedge x_i = -x_i \wedge x_i,$$

we must have $x_i \wedge x_i = 0$. □

(b) By the given differential operator:

$$dy = \frac{\partial y}{\partial x}dx = \boxed{e^x}dx.$$

(c) Then, we have:

$$dy = \frac{\partial y}{\partial x}dx = \boxed{\frac{f(x)}{g(y)}dx}.$$

Hence, we justify the separation of the variables as $g(y)dy = f(x)dx$.

(d) *Proof.* As instructed, we have:

$$\begin{aligned} d\omega &= \frac{\partial}{\partial x}(2xy^2)dx \wedge dx + \frac{\partial}{\partial y}(2xy^2)dy \wedge dx + \frac{\partial}{\partial x}(2x^2y)dx \wedge dy + \frac{\partial}{\partial y}(2x^2y)dy \wedge dy \\ &= 0 + 4xydy \wedge dx + 4xydx \wedge dy + 0 = -4xydx \wedge dy + 4xydx \wedge dy = 0, \end{aligned}$$

as desired. □

(e) *Proof.* First, we have that:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Fdx + Gdy.$$

Then, in terms of the exactness relationship, we have:

$$\begin{aligned} \frac{\partial F}{\partial y} = \frac{\partial G}{\partial x} &\iff \frac{\partial F}{\partial y}dy \wedge dx = -\frac{\partial G}{\partial x}dx \wedge dy \\ &\iff \frac{\partial F}{\partial y}dy \wedge dx + \frac{\partial G}{\partial x}dx \wedge dy = 0 \\ &\iff \frac{\partial F}{\partial x}dx \wedge dx + \frac{\partial F}{\partial y}dy \wedge dx + \frac{\partial G}{\partial x}dx \wedge dy + \frac{\partial G}{\partial y}dy \wedge dy = 0 \\ &\iff d(df) = 0. \end{aligned}$$

Hence, we have shown that the exactness is exactly that the differential form satisfies that $d(df) = 0$. □

In fact, for any smooth function f, we have $d(df) = 0$, which is the equivalent of the conclusion such that mixed partials are equal. We invite capable readers to investigate that $d^2 := d \circ d = 0$ for all smooth function f. Additionally, people with experiences in vector calculus could investigate the following *commutative diagram*.

$$\begin{tikzcd} \Omega^0(\mathbb{R}^3) \arrow[r, "d(-)"] \arrow[d, "\cong"] & \Omega^1(\mathbb{R}^3) \arrow[r, "d(-)"] \arrow[d, "\cong"] & \Omega^2(\mathbb{R}^3) \arrow[r, "d(-)"] \arrow[d, "\cong"] & \Omega^3(\mathbb{R}^3) \arrow[d, "\cong"] \\ \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R}) \arrow[r, "\nabla(-)"] & \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R}^3) \arrow[r, "\nabla\times(-)"] & \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R}^3) \arrow[r, "\nabla\cdot(-)"] & \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R}) \end{tikzcd}$$

The above are respectively 0-form, 1-form, 2-form, and 3-form (with 0, 1, 2, or, 3 $\wedge$'s in the differential form) and the below are smooth functions mapping in respective Euclidean spaces.