PILOT

Conceptual Problem Set

Differential Equations

Summer 2025

Contents

Acknowledgments:

- The CONCEPTUAL PROBLEM Set consists of extra conceptual questions for `AS.110.302` Differential Equations and Applications at *Johns Hopkins University*.
- The problem set contains exercises exploring the conceptual points in the course with most connections with *Linear Algebra* and *Multivariable Calculus*, together with some extensions to *Modern Algebra*, *Real Analysis*, and *Partial Differential Equations*.
- The problem set has been created and modified every term since Fall 2023 for PILOT by James Guo. It might contain minor typos or errors. Please point out any notable error(s).

Best regards, James Guo.
May 2024.

1 Foundations to this Set

Welcome to the ODEs PILOT. Whether you have previously taken any math class or not at Hopkins, it would be interesting to get into the foundations of mathematical components.

This problem set is **entirely voluntary**, you are not expected to know most of these contents for this class. However, if you are looking for better conceptual understanding on the materials, or wishing to see how many of the concepts develop in other areas of mathematics, you are definitely encouraged to attempt on this problem set. In general, the pace of this set should roughly align with the current design of the ODE course. To catch up with the pace, try to attempt around 2 *exercises* each week.

Right now, without many foundations into the class yet, let put our attention to some pre-requisites for this class, namely some pre-calculus and elementary calculus stuffs.

Definition. Linear Maps: For any vector spaces V, W, a linear map $T : V \to W$ satisfies that:

1. $T(u+v) = T(u) + T(v)$ for all $u, v \in V$;
2. $T(\lambda u) = \lambda T(u)$ for all $\lambda \in \mathbb{F}$ (where $\mathbb{F}$ is the field for the vector field) and $u \in V$.

In mathematical logics, we always want to show something works rather than from intuitions. With use of logics, try on the following problem.

Exercise 1. Let $T : V \to W$ be a linear map where V and W are vector fields, prove or disprove the following:

(a) $T(0_V) = 0_W$ where 0_V is the additive identity in V and 0_W is the additive identity in W.

(b) Let $f : V \to W$ be a map, if $f(0_V) = 0_W$, then f is linear.

The above problem is a prove/disprove question. Hopefully, your intuitions are correct for the above question. However, we can no longer believe in our intuitions all the time, for example, accounting the following example.

Example. Definitionine a function $f : (0,1) \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational,} \\ \dfrac{1}{n}, & \text{if } x = \frac{m}{n}, \text{ where } m, n \in \mathbb{Z}^+ \text{ and } \gcd(m, n) = 1. \end{cases}$$

Determine if f is continuous at irrational points, and if f is continuous at rational points.
From our intuition the answers should be not continuous at both scenarios, but this is not the case, since by the definition of continuity, f is continuous at irrational points but not at rational points. This is a great example from Real Analysis and could always trick many people on intuitions.

Nonetheless, if you consider a continuous map f from $\mathbb{R}$ to $\mathbb{R}$, then satisfying condition 2 would sufficient to prove that f is linear.

Back to mathematics in our scope, now let's have a brief review on integration by parts.

Exercise 2. Let f, g be continuous functions from $\mathbb{R}$ to $\mathbb{R}$, show that:

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

using the Fundamental Theorem of Calculus.

Spoiler alert, integration by parts would be one of the *most intensive* calculation in this course.

2 Wronskian, First Encounter

During the class, you might have been introduced for using *Wronskian* to determine if two functions are linearly independent. Let's explore deeper into this concept.

Definition. Linear Independence: Let $f_1, f_2, \cdots, f_n$ be a sequence of functions. The sequence of functions are linearly independent if:

$$k_1 f_1 + k_2 f_2 + \cdots + k_n f_n = 0 \Longrightarrow k_1 = k_2 = \cdots = k_n = 0,$$

while the functions are linearly dependent if they are not linearly independent.

Exercise 3. Explore the role of 0 in linear Independence. Here consider 0 as a function that maps every input to 0.

(a) Show that $f = 0$ is not linearly independent.

(b) Show that if a series of functions contains $f_k = 0$, then the series of functions are not linearly independent.

Diligent readers might have noticed that we have not mentioned if the 0 function is *well-defined*, in particular, we do not know if 0 is contained in the space of the co-domain. However, this problem should be trivial as most predominant algebraic structures in *Modern Algebra* contains 0, whether for groups, rings, and fields, or for R-modules which contains a specific example of *vector space* which most readers should have familiarity with.

Definition. Wronskian for 2 functions: The Wronskian for functions f_1 and f_2 is defined to be:

$$W[f_1, f_2] := \det \begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix} = f_1 f_2' - f_2 f_1'$$

Exercise 4. Suppose f_1 and f_2 are non-zero. Convince (or prove to) yourself with the following statement:

$$f_1 \text{ and } f_2 \text{ are linearly dependent if and only if there exists some } k \in \mathbb{R} \text{ such that } f_1 = kf_2.$$

Now, we have established a new way of expressing linear dependence with another approach.

Exercise 5. With the proceedings, prove that f_1 and f_2 are linearly dependent if and only if $W[f_1, f_2] = 0$.
Remark. For this question, just regard determinant as an operator for calculation, and a non-zero determinant implies the linear map being invertible.

Now have we concluded that Wronskian is valid for 2 functions, however, we want to aim for a more generalized case.

Exercise 6. Let $f_1, f_2, f_3, \cdots, f_n$ be continuous and differentiable functions. If the Wronskian of this set of functions is not identically zero then the set of functions is linearly independent.

For this question, you may assume that a non-zero determinant is equivalent to invertible. We will be exploring that further in the future.

On the other hand, as you explore deeper in *Real Analysis*, you will be familiarized with more classes of functions. For example, for functions that are square integrable ($L^2(\mathbb{R})$), it forms a good structure, or *Hilbert Space*. In particular, in $L^2([-L, L])$ space, a basis could be consisted of sine and cosine curves. If you find this interesting, please refer to *Fourier Series*.

3 Exactness and Mixed Partials

When contacting exactness, you might be wondering why iterated partials are equal, and why we verify the mixed partial when checking the exactness condition. Let's explore such topic now.

Theorem. Equality of Mixed Partials: If $f(x, y)$ is of class C^2 (twice continuously differentiable), then the mixed partial derivatives are equal:

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

Exercise 7. In considering this problem, define a function as follows:

$$S(\Delta x, \Delta y) := f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0) - f(x_0, y_0 + \Delta y) + f(x_0, y_0).$$

Assume that Δx and Δy are positive, draw a diagram representing x and y in $\mathbb{R}^2$.

Then we can hold y_0 and Δy fixed, define a function g as follows:

$$g(x) := f(x, y_0 + \Delta y) - f(x, y_0).$$

Exercise 8. Rewrite $S(\Delta x, \Delta y)$ in terms of the definition of g, and variables x_0 with Δx.

Theorem. Mean Value Theorem: If f is a continuous function on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then there exists a point c in (a,b) such that the tangent at c is parallel to the secant line through the endpoints $(a, f(a))$ and $(b, f(b))$, that is,

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Exercise 9. With the mean-value theorem, prove that there exists some $x_0 < \bar{x} < x_0 + \Delta x$ satisfying that $g(x_0 + \Delta x) - g(x_0)$. Then, express $S(\Delta x, \Delta y)$ in another way.

After showing the case for $\bar{x}$, we have a very similar case for $\bar{y}$ in demonstrating this proof. Convince yourself that it is correct.

Exercise 10. Prove the theorem on equality if mixed partials by using definition of limits.

Now, we have shown that mixed partials should be equal for functions that are "good" enough. For sure, we want to make our partial derivatives good so we can do the reverse process to find our target function.

4 Euler's Formula

With no doubt, Euler's Formula is considered as one of the prettiest formula in mathematics, as it combines integer 1, with π and e as transcendental numbers, and with i as imaginary number to 0, as of follows:

$$e^{\pi i} + 1 = 0.$$

More generally, Euler's formula writes e to an imaginary power as of sin and cos, such as:

$$e^{\theta i} = \cos\theta + i \sin\theta.$$

The conventional way to show Euler's Formula is through the power series. Recall that a Taylor Approximation of $f(x)$ is:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Exercise 11. Consider the power series of $e^{i\theta}$, $\sin\theta$, and $\cos\theta$, show that Euler's Formula is valid.
Remark. Since we have not constructed a clear definition of sequences and justifying the rearrangement of terms, this proof is not entirely rigorous.

Alternatively, there is another proof by using the product rule of derivatives.

Exercise 12. Consider a function f, defined as follows:

$$f(\theta) := \frac{\cos\theta + i\sin\theta}{e^{i\theta}} = e^{-i\theta}(\cos\theta + i\sin\theta),$$

show that f is a constant function, then conclude Euler's Formula.

Now have we explored Euler's formula. One significant remark is that Euler's formula exhibits a periodic patter, which greatly fits the cores of ODEs.

Exercise 13. Given a function $f\colon \mathbb{R} \to \mathbb{C}$ defined as:

$$f(x) := e^{xi}.$$

Decompose $f = i_f \circ \tilde{f} \circ \pi_\sim$, where $\pi_\sim$ is surjective, i_f is injective, and $\tilde{f}$ is bijective, as of follows:

$$\begin{tikzcd} \mathbb{R} \arrow[r, two heads, "\pi_\sim"'] \arrow[rrr, bend left, "f"] & X \arrow[r, hook, two heads, "\sim", "\tilde{f}"'] & Y \arrow[r, hook, "i_f"'] & \mathbb{C}, \end{tikzcd}$$

Further, you can consider $\pi_\sim$ as a projection to an equivalent class, $\tilde{f}$ as a modification of f, and i_f as a map from the image to the co-domain. Such decomposition is called a "Canonical Decomposition."
Remark. For the simplicity of this question, consider X as an interval of $\mathbb{R}$ and Y geometrically in $\mathbb{C}$.

While the above inquiry shows a canonical way of decomposition (namely, the *Canonical Decomposition* in *Modern Algebra*), this reveals some topological relationship, as described follows:

$$\begin{tikzcd} \mathbb{R} \arrow[r, "e^{i\bullet}"] \arrow[d, "\pi_\sim"'] & \mathbb{S}^1 \\ \mathbb{R}/\tau\mathbb{Z} \arrow[ur, dashed, "\sim"'] & \end{tikzcd}$$

Here, $\mathbb{S}^1$ (you can consider this a disk, *i.e.* boundary of ball in $\mathbb{R}^2$) is covering space of $\mathbb{R}$, which can be corresponded to $\mathbb{R}/\tau\mathbb{Z}$, known as the partition of $\mathbb{R}$ such that each equivalence class is consisted of elements in form $x + k\tau$ where $0 \le x < \tau$ and $k \in \mathbb{Z}$.

Exercise 14. Based on the definition of $\mathbb{R}/\tau\mathbb{Z}$.

(a) Find τ, then explain how it corresponds to the interval you have made in the preceding problem.

(b) Consider the equation $x^n = 1$ where $n \in \mathbb{Z}$. Use the concept of partition to find the new periodicity τ, *i.e.*, what are the differences in angle between the solutions.

Now, you can see that the Euler formula can be interpreted as a map (or *morphism*) between some structures, and you can see some sort of equal relationship (known as *isomorphism*). If you find such topic interesting, please refer to some *Topology* contents to find a good definition on what are considered "same" in mathematics.

5 Laplace Transformations

At this moment of class, you should be introduced to Laplace Transformation. However, we are going to explore about it through a different perspective. Namely, to remind you what Laplace Transformation is:

Definition. Laplace Transformation: For a function $f(x)$, its Laplace transformation is defined as:

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

First, let us review about some properties of Laplace Transformation:

Exercise 15. Let $f(x)$ and $g(x)$ be "good" functions, and let a be a real constant, show that:

(a) $\mathcal{L}\{f(x) + g(x)\} = \mathcal{L}\{f(x)\} + \mathcal{L}\{g(x)\}$.

(b) $\mathcal{L}\{af(x)\} = a\mathcal{L}\{f(x)\}$.

In particular, $\mathcal{L}$ is a linear operator if it aligns with the above arguments.

Note that we used "good" here, as we genially want to have $e^{-st}f(t)$ to be Riemann-integrable. Here, the "good" does not imply any sort of continuity, so if we imagine $f(x)$ as a "good" function such that $f(x) \neq 0$ for all $x \in \mathbb{R}$, and we define $g(x)$ by the follows:

$$g(x) = \begin{cases} f(x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

If we consider this function, it is immediate to observe that $\mathcal{L}\{f\} = \mathcal{L}\{g\}$. Intuitively, we know that a variation of a single point does not matter.

Exercise 16. Given an example of $f(x)$ and $g(x)$ from the above argument. Moreover, conclude that for any $F(s)$ that has a Laplace inverse, there must be infinitely (or countably) many inverses.

6 Eigenspace

Most likely in most ODE courses, you might be introduced to eigenvectors and eigenvalues purely by calculation approaches, but most students fails to know that eigenspace means in linear algebra. Alternatively, we will attempt to understand the concept of eigenspace through another approach.

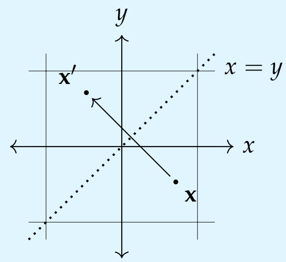
To approach the purest understanding of eigenspace, let's forget everything about determinants.

Definition. Eigenvalue and Eigenvectors: Consider for a linear map T that maps V as a vector space to itself, denoted $T \in \mathcal{L}(V)$, the eigenvalue of T are values $\lambda \in \mathbb{F}$ (where $\mathbb{F}$ denotes the field) such that there exists some $v \in V$ such that:

$$T(v) = \lambda x,$$

and the corresponded elements (not necessarily vector) $v \in V$ are the eigenvectors.

Exercise 17. Consider the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as reflecting each point $(x, y) \in \mathbb{R}^2$ by the axis $x = y$, as of follows:



(a) What are the eigenvalues and eigenvectors of T.

(b) Can you think of a way to represent T as a matrix? Feel free to check the eigenvalues and eigenvectors for the matrix align with what you get in part (a).

Hence, at this moment, you should really realize that **determinant** is **not** a definition of eigenvectors, but rather a tool. Even further, you would notice the exact opposite construction.

However, for now, let's consider another case of eigenvalues.

Exercise 18. Consider a linear map $T \in \mathcal{L}(V)$ with a series of eigenvalues.

(a) Give a specific example of linear map T that has 0 as an eigenvalue.

(b) Now, prove that the operator T is invertible if and only if 0 is not an eigenvalue of T.

From here, the reader might find some connections with determinant being 0 and have 0 as an eigenvalue. Now, we can define determinant as follows:
Definition. Eigenvalue and Eigenvectors: The determinant of a linear map $T \in \mathcal{L}(V)$, denoted $\det T$, is the product of the eigenvalues in the complex field with each eigenvalue repeated according to its multiplicity.

Exercise 19. Prove that invertible is equivalent to nonzero determinant.

Therefore, we have shown that a non-zero determinant implies an operator being invertible. Now, we want to build further connections between the determinant of a linear map T with its eigenvalues by finding the characteristic polynomial.

Exercise 20. Suppose $T \in \mathcal{L}(V)$, show that the characteristic polynomial of T equals $\det(zI - T)$.

Readers should immediately realize that what we have get all the way through is another approach to "redefine" determinants. However, you should see that the determinant is one of the most straightforward way to approach the eigenvalues.

7 Wronskian, Second Encounter

Since we have now had a better view on determinants, it is a great time for us to review on the techniques of Wronskian. It is a great opportunity for us to view determinants of matrices through another construction. *Some readers might notice that I "stole" this title from a famous algebra textbook – Algebra: Chapter 0.* Despite all of these, let's reprise the information of Wronskian to learn the contents again.

Thus, prior to seeing the construction for matrices, we also want to explore the permutations, with a definition of the "sgn" function.

Definition. Permutation: A permutation of n, denoted perm n, is a list $(m_1, m_2, \cdots, m_n)$ that contains each of the numbers $1, 2, \cdots, n$ exactly once. To understand, you can interpret this as mapping each element in the list to the later one, while the last one is mapped to the first.

Exercise 21. For the simplicity of this question, with a permutation written as $\sigma = (a, b)$, we can understand that as $\sigma(a) = b$ and $\sigma(b) = a$.

(a) Prove that $(1, 2, 3) = (1, 2)(1, 3)$ holds.

(b) Given any permutation as $(m_1, m_2, \cdots, m_n) \in \operatorname{perm} n$, write it in the form as only permutation of two elements. Specifically, you should generalize the previous part to any m_i.

The reader shall have noticed a fundamental discrepancy between some permutations. For instance, we can decompose as follows:

$$\begin{aligned}(1,2) &= (1,2)(1,3)(3,1),\\ (1,2,3) &= (1,2)(1,3),\\ (1,2,3,4) &= (1,2)(1,3)(1,4).\end{aligned}$$

Specifically, some permutations can only be "decomposed" into even numbers of single permutations (called **transposition**), while the others can only be "decomposed" into odd numbers of single permutations. One should found the parity here suspicious, thus shall lead to our definition of the sign function.

Definition. Sign Function: If a permutation σ can be decomposed into odd number of transposition, it is odd, written as $\operatorname{sgn}(\sigma) = -1$. If it can be decomposed into even number of transposition, it is even, written as $\operatorname{sgn}(\sigma) = 1$.

Exercise 22. Prove that even permutations are "closed" under composition, that it, if you compose two even permutations, their composition shall still be even.

If fact, you can also show that composition of odd permutations is not closed, and this corresponds to the parity of addition.

After defining our sign function, we shall finally see the construction of determinant for a matrix. Given a matrix $A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix}$, its determinant is $\det(A) = \sum_{\sigma \in \text{perm}\, n} \text{sgn}(\sigma) a_{\sigma(1),1} \cdots a_{\sigma(n)}, n$.

Exercise 23. With this definition, the sign function is determined by the number of transpositions in σ, argue that we may remove the row and column of $a_{\sigma(1),1}$ so it give rise to a recursive definition of determinants.

If you get really interested into mathematics, you could know that perm n is some group structure, known as the cyclic group, in which the elements rotates inside each permutation. The **group theory** definitely brings many nice properties, while you are encouraged to explore this on yourself later on.

Maybe this brings us too much away from Wronskian, which is the topic of this section. However, you might recall the **Variation of Parameter** for higher order linear differential equations, as the particular solution is:

$$y_p = y_1(t) \int \frac{W_1 g}{W} dt + y_2(t) \int \frac{W_2 g}{W} dt + \cdots + y_n(t) \int \frac{W_n g}{W} dt,$$

where W_i is defined to be the Wronskian with the i-th row alternated into $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$.

Exercise 24. With the recursive definition of determinants, we now want to tackle how W_i is valid in terms of its definition.

(a) Prove that interchanging two columns of a matrix makes the determinant into its opposite number (in terms of addition, known as the **additive inverse**).

(b) By applying part (a), conceptually describe the processes of each W_i based on parity.

Also, you should now reconsider the variation of parameter with second order linear differential equations. Hopefully, you should be able to see why there is an extra negative sign for the variation of parameters in that case.

This marks the end of the extra sheets for ODEs PILOT. We hope that you have encountered something fun and might observed some interests in other areas of mathematics.