Additional Materials: Dilemma with Existence & Uniqueness Theorem

Differential Equations

Fall 2025

Let a first order IVP on $y := y(t)$ be defined as follows:

$$\begin{cases} y' = \frac{2}{t}y, \\ y(1) = 1. \end{cases}$$

(a) Find the solution to the above initial value problem.

(b) Recall the theorem on existence and uniqueness, as follows:

> For an IVP in simple form:
>
> $$\begin{cases} \frac{dy}{dt} = a(t)y + b(t), \\ y(t_0) = y_0. \end{cases}$$
>
> For some $I = (\alpha, \beta) \ni t_0$, if $a(t)$ and $b(t)$ are continuous on the interval I. Then, there exists a unique solution to the IVP on the interval I.

Show that the IVP in this problem does not satisfy the condition for the existence and uniqueness theorem for $\mathbb{R}$.

(c) Does the above example violates the existence and uniqueness theorem? Why?

Solution:

1. This problem is clearly separable, we may compute:

$$\frac{dy}{y} = 2\frac{dt}{t}$$

$$\int \frac{dy}{y} = 2\int \frac{dt}{t}$$

$$\log|y| = 2\log|t| + C$$

$$y = \tilde{C}t^2.$$

Note that the initial condition enforces that $y(1) = 1$, so the solution is just:

$$y = \boxed{t^2}.$$

2. Note that $a(t) = 2/t$, which is not continuous over $(-\infty, 0) \cup (0, \infty)$, then the theorem does not guarantee the existence and uniqueness of a solution over $\mathbb{R}$.

3. This is not a violation since the converse of the theorem is not necessarily true. In propositional logic, if A implies B (written as $A \implies B$), the converse (B implies A, written as $B \implies A$) is not necessarily true. Hence, we can still have a solution that is unique over $\mathbb{R}$.