PILOT

Problem Set 6

Differential Equations

Fall 2025

1. (Constructing Solutions, Again.) Construct an initial value problem with Cauchy conditions for the following solutions:

   (a) $$y(t) = 4e^{3t} - e^{-2t}.$$

   (b) $$y(t) = e^{2t}\cos t + e^{2t}\sin t + e^{2t}.$$

2. (Reduction of Order). Let a ODE be defined as follows:
$$t^2y'' + 2ty' = 2y,\ t > 0.$$
Given a solution is $y_1(t) = t$, find the other solution by assuming $y_2(t) = u(t) \cdot y_1(t)$.

3. (Reduction of Order or Integrating Method). Let a differential equation be:

$$y''(t) + \frac{2}{t}y'(t) = 0.$$

   (a) Verify that $y(t) = 1/t$ is one solution, then find a full set of solution.

   (b) Consider $\omega(t) = y'(t)$, solve the differential equation by using integrating factor.

   (c) Verify that the two methods give you the same set of the solutions.

4. (A Criterion on Linearly Independence). Recall that for the complex characteristic roots $x = \lambda \pm \mathrm{i}\mu$, the corresponding solutions are:

$$y_1 = e^{\lambda x}\sin(\mu x) \qquad \text{and} \qquad y_2 = e^{\lambda x}\cos(\mu x).$$

Of course, you may compute the Wronskian of $\mathrm{W}(e^{\lambda x}\sin(\mu x), e^{\lambda x}\cos(\mu x))$, which involves taking derivative with chain rule and product rule, and a lot of computation. Another approach to show linear independence is by returning to its definition.

> **Definition**. (Linearly Independence).
> Two functions f and g are *linearly independent* if $\lambda_1 f + \lambda_2 g = 0$ implies $\lambda_1 = \lambda_2 = 0$.

(a) Show that $y_1 = x$ and $y_2 = x^2$ is linearly independent via both approach.

(b) Show that $y_1 = e^{\lambda x}\sin(\mu x)$ and $y_2 = e^{\lambda x}\cos(\mu x)$ are linearly independent by using the definition.

(c) Let two functions be defined as follows:

$$y_1(x) = \mathbb{1}_{[0,1]}(x) = \begin{cases} 1, & \text{when } 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases} \qquad \text{and} \qquad y_2 = \mathbb{1}_{[2,3]}(x) = \begin{cases} 1, & \text{when } 2 \le x \le 3, \\ 0, & \text{otherwise.} \end{cases}$$

Show that the two functions are linearly independent.