PILOT

Problem Set 3: Solutions

Differential Equations

Fall 2025

1. (Stability of Autonomous ODEs). Draw the phase line and determine the stability of each equilibrium for the following autonomous differential equations:

   (a) $$y' = y^4 - 3y^3 + 2y^2.$$

   (b) $$y' = y^{2025} - 1.$$

   (c) $$y' = y^2 + 2y + C, \text{ where } C \in \mathbb{R} \text{ is a constant.}$$

   For part (c), determine the bifurcation values for the parameter C and sketch a bifurcation diagram.

2. (More IVPs). Given following IVPs:

   (a)
   $$\begin{cases} y' + \dfrac{1}{2}y = \sin t, \\ y(0) = 1. \end{cases}$$

   (b)
   $$\begin{cases} y' = \dfrac{1}{x^4 - 1}, \\ y(0) = 0. \end{cases}$$

   Find the specific solution of the equation, note the domain of the solution, and describe the end behavior of the solution if it appears in the valid domain.

3. (Existence and Uniqueness for IVP). Suppose $f(x)$ is non-zero, let an initial value problem be:

$$\begin{cases} \dfrac{1-y}{x} \cdot \dfrac{dy}{dx} = \dfrac{f(x)}{1+y}, \\ y(0) = 0. \end{cases}$$

   (a) Show that the differential equation is **not** linear.

   For the next two parts, suppose $f(x) = \tan x$.

   (b) State, without justification, the open interval(s) in which $f(x)$ is continuous.

   (c) Show that there exists some $\delta > 0$ such that there exists a unique solution $y(x)$ for $x \in (-\delta, \delta)$.

   Now, suppose that $f(x)$ is some function, **not** necessarily continuous.

   (d) Suppose that the condition in (c) does **not** hold, give three examples in which $f(x)$ could be.

4. (Existence of Largest Interval). For the following IVPs, determine the largest interval in which a solution is guaranteed to exist.

   (a)
   $$\begin{cases} (t-3)y' + (\log t)y = 2t, \\ y(1) = 2. \end{cases}$$

   (b)
   $$\begin{cases} (4-t^2)y' + 2ty = 3t^2, \\ y(1) = -3. \end{cases}$$

   (c)
   $$\begin{cases} y' + (\tan t)y = \sin t, \\ y(\pi) = 0. \end{cases}$$