Problem Set 5

Differential Equations

Fall 2025

1. (Second Order Differential Equations). Find the general solution to the following second order differential equations on $y := y(t)$. Then verify using Wronskian that the solutions are linearly independent.

   (a) $$y'' - 3y' + 2y = 0.$$

   (b) $$y'' + 12y' - 3y = 0.$$

2. (Second Order IVP). Let an initial value problem for $y = y(t)$ be defined as follows:

$$\begin{cases} 4y'' - y = 0, \\ y(0) = 2,\ y'(0) = \beta, \end{cases}$$

where β is a real constant.

(a) Find the specific solution to the initial value problem. Express your solution with constant β.

(b) Find the value of β such that the solution *converges* to 0 as t tends to infinity.

3. (Euler's Theorem). In our study of differential equations, our main focus is on *real-valued functions*. But we are about to see complex numbers in our story. **Euler's theorem** states that for any $z \in \mathbb{C}$, we have:

$$\exp(\mathrm{i}z) = \cos(z) + \mathrm{i}\sin(z).$$

   (a) To review on complex numbers, compute/simplify the following expressions:

$$\text{(i)}(2+5\mathrm{i}) \times (1+2\mathrm{i}), \qquad \text{(ii)}\frac{2-3\mathrm{i}}{1+\mathrm{i}}, \qquad \text{(iii)}\overline{2+5\mathrm{i}}, \qquad \text{(iv)}(20+25\mathrm{i}) \times (\overline{20+25\mathrm{i}}).$$

   (b) Write the following complex exponentials in terms of a sum of the real and imaginary parts:

$$\text{(i)}\exp(\mathrm{i}), \qquad \text{(ii)}\exp\left(\frac{\pi\mathrm{i}}{3}\right), \qquad \text{(iii)}\exp(2+2\mathrm{i}).$$

   (c) Express $\sin(z)$ and $\cos(z)$ in terms of exponential functions, where $z \in \mathbb{C}$ is a complex number.

4. (Second Order IVP with Dirichlet Condition). Given following initial value problem.

$$\begin{cases} y'' = -y, \\ y(0) = 0, \\ y(b) = 0, \end{cases}$$

where $b \in \mathbb{R}$ and $b \neq 0$.

(a) Find the general solution to $y'' = -y$.

(b) Suppose $b = \frac{\pi}{4}$, find the solution.

(c) Give a value of b such that there is not a unique solution.