



Quiz 1 Review Set: Solutions

Differential Equations

Fall 2025

1. Find the general solution for $y = y(t)$:

$$y' + 3y = t + e^{-2t},$$

then, describe the behavior of the solution as $t \rightarrow \infty$.

Solution:

Here, one could note that this differential equation is not separable but in the form of integrating factor problem, then we find the integrating factor as:

$$\mu(t) = \exp\left(\int_0^t 3ds\right) = \exp(3t).$$

By multiplying both sides with $\exp(3t)$, we obtain the equation:

$$y'e^{3t} + 3ye^{3t} = te^{3t} + e^{-2t}e^{3t}.$$

Clearly, we observe that the left hand side is the derivative after product rule for ye^{3t} and the right hand side can be simplified as:

$$\frac{d}{dt}[ye^{3t}] = te^{3t} + e^t.$$

Therefore, we have turned this into an integration problem, so we do the respective integrations, giving us that:

$$\begin{aligned} ye^{3t} &= \int te^{3t} dt + \int e^t dt \\ &= \frac{te^{3t}}{3} - \int \frac{1}{3}e^{3t} dt + e^t + C \\ &= \frac{te^{3t}}{3} - \frac{e^{3t}}{9} + e^t + C. \end{aligned}$$

Eventually, we divide both sides by e^{3t} to obtain that:

$$y(t) = \boxed{\frac{t}{3} - \frac{1}{9} + e^{-2t} + Ce^{-3t}}.$$

As $t \rightarrow \infty$, the solution diverges to ∞ .

2. An autonomous differential equation is given as follows:

$$\frac{dy}{dt} = 4y^3 - 12y^2 + 9y - 2 \quad \text{where } t \geq 0 \text{ and } y \geq 0.$$

Draw a phase portrait and sketch a few solutions with different initial conditions.

Solution:

Recall the *Rational root test* (c.f. §1.3). Let the polynomial with integer coefficients be defined as:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0,$$

then any rational root $r = p/q$ such that $p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$ satisfies that $p|a_0$ and $q|a_n$.

From the theorem, we can note that if the equation has a rational root, it must be one of:

$$r = \pm 1, \pm 2, \pm \frac{1}{2}, \pm \frac{1}{4}.$$

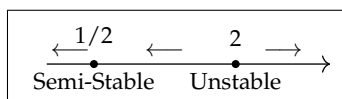
By plugging in, one should notice that $y = 2$ is a root (one might also notice $1/2$ is a root as well, but we will get the step slowly), so we can apply the long division (dividing $y - 2$) to obtain that:

$$\frac{4y^3 - 12y^2 + 9y - 2}{y - 2} = 4y^2 - 4y + 1.$$

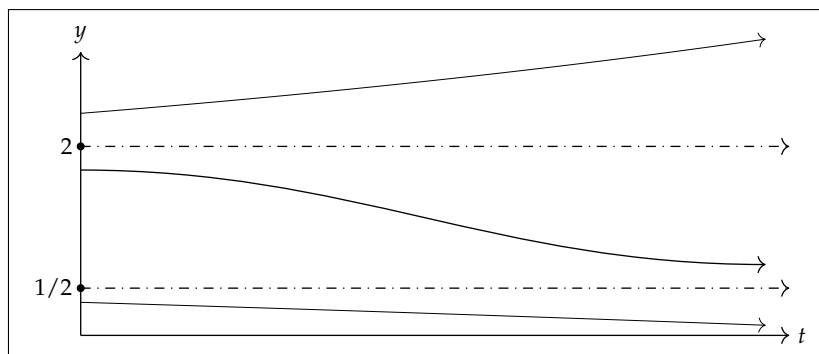
Clear, we can notice that the right hand side is a perfect square (else, you could use the quadratic formula) that:

$$4y^2 - 4y + 1 = (2y - 1)^2.$$

Thus, we now know that the roots are 2 and $1/2$ (multiplicity 2). Hence, the phase portrait is:



Correspondingly, we can sketch a few solutions (not necessarily in scale):



3. For the first-order autonomous ODE:

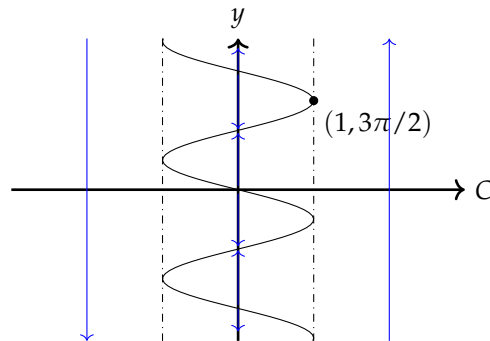
$$\frac{dy}{dt} = \sin y + C,$$

where $C \in \mathbb{R}$ is a parameter. Determine any and all bifurcation values for the parameter C and sketch a bifurcation diagram.

Solution:

It is not hard to observe that $\sin y$ will intersect the axis infinitely many times, while $\sin(\mathbb{R}) = [-1, 1]$, one shall then realize that the bifurcation value would be ± 1 , since when $C > 1$ or $C < -1$, there will be no equilibria at all.

Therefore, the bifurcation diagram can be illustrated as:



4. Let an initial value problem be defined as follows:

$$\begin{cases} (12x^4 + 5x^2 + 6) \frac{dy}{dx} - (x^2 \sin(x) + x^3)y = 0, \\ y(0) = 1. \end{cases}$$

Show that the solution to the above initial value problem is symmetric about $x = 0$.

Solution:

If you were attempting to solve this problem by integrating factor or exactness, you are on the wrong track. The functions are deliberately chosen so that these operations will be hardly possible.

However, this does not necessarily mean that it is not possible to prove without solving the solution out, one shall utilize the existence and uniqueness theorem to proceed.

Proof. Now, we first observe that when we rewrite the problem, we have:

$$y' = \frac{x^2 \sin(x) + x^3}{12x^4 + 5x^2 + 6} y,$$

where we clearly notice that the numerator and denominator are composed of continuous function while the denominator is positive, so we know that it is continuous over \mathbb{R} , so the initial value problem exhibits a unique solution.

Now, suppose $y(x)$ is a solution of the above IVP, we want to show that $\tilde{y}(x) := y(-x)$ is also a solution to the above IVP.

Clearly, we have:

$$\tilde{y}(0) = y(-0) = y(0) = 1,$$

so the initial condition is satisfied, so we are left to check the differential equation. By chain rule, we have:

$$\frac{d\tilde{y}}{dx}(x) = \frac{dy}{dx}(-x) \cdot \frac{d}{dx}[-x] = -\tilde{y}'(x).$$

With the first equation and y being a solution, we can make all x into $-x$ to obtain that:

$$(12(-x)^4 + 5(-x)^2 + 6) \frac{dy(-x)}{dx} - ((-x)^2 \sin(-x) + (-x)^3) \tilde{y} = 0,$$

and if we organize the left hand side, we have:

$$(12x^4 + 5x^2 + 6) \tilde{y}' - (x^2 \sin(x) + x^3) \tilde{y} = 0,$$

so \tilde{y} is clearly another solution to the IVP, so by uniqueness, we must have $\tilde{y}(x) = y(x)$, or namely $y(-x) = y(x)$, so the solution must be symmetric about $x = 0$. \square