Problem Set 6: Solutions

Differential Equations

Spring 2025

1. (Second Order Differential Equations). Find the general solution to the following second order differential equations on $y := y(t)$:

   (a) $$y'' - 3y' + 2y = 0.$$

   (b) $$y'' + 12y' - 3y = 0.$$

Solution:

(a) Here, we note that the characteristic equation is:

$$r^2 - 3r + 2 = (r-1)(r-2) = 0,$$

so the roots are $r = 1$ and $r = 2$, and the solution is correspondingly:

$$y = \boxed{C_1 e^t + C_2 e^{2t}}.$$

(b) For the second differential equation, the characteristic equation is:

$$r^2 + 12r - 3 = 0,$$

we may use the quadratic formula to obtain roots as:

$$r = \frac{-12 \pm \sqrt{144+12}}{2} = -6 \pm \sqrt{52},$$

so the general solution is:

$$y = \boxed{C_1 \exp\left((-6+\sqrt{52})t\right) + C_2 \exp\left((-6-\sqrt{52})t\right)}.$$

2. (Second Order IVP). Let an initial value problem for $y = y(t)$ be defined as follows:

$$\begin{cases} 4y'' - y = 0, \\ y(0) = 2,\ y'(0) = \beta, \end{cases}$$

where β is a real constant.

(a) Find the specific solution to the initial value problem. Express your solution with constant β.

(b) Find the value of β such that the solution *converges* to 0 as t tends to infinity.

Solution:

(a) First, we note that the characteristic equation is $4r^2 - 1 = 0$, whose roots are $\pm 1/2$, hence the general solution to the differential equation is:

$$y(t) = C_1 e^{t/2} + C_2 e^{-t/2}, \text{ where } C_1, C_2 \text{ are constants.}$$

To find the specific solution, we input initial conditions, namely we find the derivative:

$$y'(t) = \frac{C_1}{2} e^{t/2} - \frac{C_2}{2} e^{-t/2}.$$

Hence, the initial data tells us that:

$$y(0) = C_1 + C_2 = 2 \text{ and } y'(0) = \frac{C_1}{2} - \frac{C_2}{2} = \beta.$$

By algebraically manipulating the equations, we find:

$$C_1 = 1 + \beta \text{ and } C_2 = 1 - \beta.$$

Hence, the solution is:

$$y(t) = \boxed{(1+\beta)e^{t/2} + (1-\beta)e^{-t/2}}.$$

(b) Considering $t \to \infty$, we note that $e^{t/2} \to \infty$ and $e^{-t/2} \to 0$, hence we only need to consider about the $(1+\beta)e^{t/2}$ part.

In order for a convergence to 0, we want this part to vanish, *i.e.*:

$$1 + \beta = 0 \text{ or } \beta = \boxed{-1}.$$

3. ($L^2([0, 2\pi])$ Space.) Recall that we have defined linear independence of functions, we define *orthogonality* of two real-valued, "square-integrable" functions over $[0, 2\pi]$, f and g, as:

$$\int_0^{2\pi} f(x)g(x)dx = 0.$$

(a) Show that the set $\{\sin x, \cos x\}$ is linearly independent and orthogonal.

(b) Show that if $\{f(x), g(x)\}$ is orthogonal, then $C_1 f(x)$ and $C_2 g(x)$ is orthogonal.

(c)* Note that $\{x, x^2\}$ are linearly independent, construct a basis that is orthogonal.

Solution:

(a) *Proof.* To show linear independence, we compute the Wronskian as:

$$\mathrm{W}[\sin x, \cos x] = \det \begin{pmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{pmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0.$$

Then, to show orthogonality, we have:

$$\int_0^{2\pi} \sin x \cos x dx = \frac{1}{2}\int_0^{2\pi} \sin(2x)dx = \frac{1}{2}\left[-\frac{1}{2}\cos(2x)\right]_0^{2\pi} = \frac{1}{4}\big(\cos 0 - \cos(4\pi)\big) = 0,$$

hence we have shown linear independence and orthogonality. □

(b) *Proof.* By orthogonality, we have $\int_0^{2\pi} f(x)g(x)dx = 0$, so we have:

$$\int_0^{2\pi} C_1 f(x) \cdot C_2 g(x)dx = C_1 C_2 \int_0^{2\pi} f(x)g(x)dx = C_1 C_2 \cdot 0 = 0.$$

Hence orthogonality is preserved with scalar multiplications. □

(c) The check of x and x^2 being linearly independent can be verified by Wronskian, and we leave this check to the readers. By the principle of superposition, we want to construct the second argument as $x^2 - Ax$, where A is a constant, now we take the inner product as:

$$\int_0^{2\pi} x(x^2 - Ax)dx = \int_0^{2\pi} (x^3 - Ax^2)dx = \frac{x^4}{4} - \frac{Ax^3}{3}\bigg|_0^{2\pi} = 4\pi^4 - \frac{8A\pi^3}{3} = 0,$$

which forces A to be $3\pi/2$, so the orthogonal basis is now:

$$\boxed{\left\{x, x^2 - \frac{3\pi x}{2}\right\}}.$$

Diligent readers should notice that we have somehow constructed a "vector space" with a proper inner product. In fact, this space $L^2([0, 2\pi])$ is considered a Hilbert Space, that is a infinite dimensional vector space with completeness and denseness. The $L^2([0, 2\pi])$ is closely related to Fourier series, that has inarguable impacts on mathematics as well as sciences and engineering disciplines.

4. (Preview on Euler's Theorem). In our study of differential equations, our main focus is on *real-valued functions*. But we are about to see complex numbers in our story. **Euler's theorem** states that for any $z \in \mathbb{C}$, we have:

$$\exp(\mathrm{i}z) = \cos(z) + \mathrm{i}\sin(z).$$

(a) To review on complex numbers, compute/simplify the following expressions:

$$(\mathrm{i})(2+5\mathrm{i}) \times (1+2\mathrm{i}), \qquad (\mathrm{ii})\frac{2-3\mathrm{i}}{1+\mathrm{i}}, \qquad (\mathrm{iii})\overline{2+5\mathrm{i}}, \qquad (\mathrm{iv})(20+25\mathrm{i}) \times (\overline{20+25\mathrm{i}}).$$

(b) Write the following complex exponentials in terms of a sum of the real and imaginary parts:

$$(\mathrm{i})\exp(\mathrm{i}), \qquad (\mathrm{ii})\exp\left(\frac{\pi \mathrm{i}}{3}\right), \qquad (\mathrm{iii})\exp(2+2\mathrm{i}).$$

(c) Express $\sin(z)$ and $\cos(z)$ in terms of exponential functions, where $z \in \mathbb{C}$ is a complex number.

(d)* Given a function $\varphi \colon \mathbb{R} \to \mathbb{C}$ defined as $\varphi(x) = \exp(ix)$. We can decompose $\varphi = i_\varphi \circ \tilde{\varphi} \circ \pi_\sim$, where $\pi_\sim$ is surjective, i_φ is injective, and $\tilde{\varphi}$ is bijective, which can be expressed as follows:

$$\begin{tikzcd}
\mathbb{R} \arrow[r, two heads, "\pi_\sim"'] \arrow[rrr, bend left, "\varphi"] & X \arrow[r, hook, two heads, "\sim", "\tilde{\varphi}"'] & Y \arrow[r, hook, "i_\varphi"'] & \mathbb{C},
\end{tikzcd}$$

Find X and Y in the above commutative diagram.
Hint: Consider $\pi_\sim$ as a projection to an equivalent class, $\tilde{\varphi}$ as a modification of φ, and i_f as a map from the image to the co-domain.

Solution:

(a) These calculations should be trivial:

(i) $$(2+5\mathrm{i}) \times (1+2\mathrm{i}) = 3 - 10 + 5\mathrm{i} + 4\mathrm{i} = \boxed{-7+9\mathrm{i}}.$$

(ii) $$\frac{2-3\mathrm{i}}{1+\mathrm{i}} = \frac{(2-3\mathrm{i})(1-\mathrm{i})}{(1+\mathrm{i})(1-\mathrm{i})} = \frac{2-3-3\mathrm{i}-2\mathrm{i}}{2} = \boxed{\frac{-1-5\mathrm{i}}{2}}.$$

(iii) $$\overline{2+5\mathrm{i}} = \boxed{2-5\mathrm{i}}$$

(iv) $$(20+25\mathrm{i}) \times (\overline{20+25\mathrm{i}}) = 20^2 + 25^2 = 400 + 625 = \boxed{1025}.$$

(b) We simply apply Euler's formula to obtain that:

(i) $$\exp(\mathrm{i}) = \boxed{\cos(1) + \mathrm{i}\sin(1)}.$$

(ii) $$\exp\left(\frac{\pi \mathrm{i}}{3}\right) = \cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right) = \boxed{\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}}.$$

(iii) $$\exp(2+2\mathrm{i}) = \exp(2)\exp(2\mathrm{i}) = \boxed{\exp(2)\cos(2) + \mathrm{i}\exp(2)\sin(2)}.$$

(c) Here, readers shall recall that sine is odd and cosine is even. Now, we shall utilize this property with the exponentials, namely:

$$\exp(-\mathrm{i}z) = \cos(-z) + \mathrm{i}\sin(-z) = \cos(z) - \mathrm{i}\sin(z).$$

Combining this with the $\exp(\mathrm{i}z)$ expression, we may find that:

$$\sin(z) = \boxed{\frac{\exp(\mathrm{i}z) - \exp(-\mathrm{i}z)}{2\mathrm{i}}} \text{ and } \cos(z) = \boxed{\frac{\exp(\mathrm{i}z) + \exp(-\mathrm{i}z)}{2}}.$$

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(d) Here, lets discuss the motivation of this first, we want to decompose our function of several pieces, in which each piece has some unique properties, note that from Euler's formula, we see repetitive patterns in every 2π rotation, so our intention is to quotient out this triviality, so we have:

$$X = \boxed{\mathbb{R}/2\pi} := \big\{\{x + 2k\pi : k \in \mathbb{Z}\} : x \in \mathbb{R}\big\},$$

while the $\pi_\sim$ map will be a simple projection that projects $\mathbb{R}$ to its equivalent classes in $\mathbb{R}/2\pi$. Since i_φ is injective, so we want Y to be the image of φ, so:

$$Y = \boxed{\mathbb{D}_1(0)} := \{x \in \mathbb{C} : |x| = 1\},$$

with the corresponding i_φ being the identity map.

Eventually, we now see $\tilde{\varphi}$ is bijective, since it is mapped to the image of φ and the injective part is enforced by having $\exp(z)$ and $\exp(z')$ being unique up to $z \sim z'$ in the equivalent class.

If you have taken abstract algebra or basic category theory, this is called a canonical decomposition, so we can think of more properties of each respective map and use various universal properties. Please check on some of these text to learn more about such decomposition.