PILOT Final Review

Differential Equations

Johns Hopkins University

Spring 2025

As you prepare for the final, please consider the following resources:

- PILOT webpage for ODEs: `https://jhu-ode-pilot.github.io/SP25/`
  - Since the final is cumulative, all problem sets will be relevant.
  - Find the review problem set for final exam, as well as review problem sets for midterm 1 and midterm 2.
- Review the *homework sets* provided by the instructor.
- Join the PILOT Final Review Session. (You are here.)

Plan for today:

1. Go over all contents that we have covered for this semester so far.
2. In the end, we will open the poll to you. Please indicate which problems from the PSets or Review Set that you want us to go over.

Contents:

Part 1: Contents Review

We will get through all contents over this semester.

- Feel free to download the slide deck from the webpage and annotate on it.
- If you have any questions, ask by the end of each chapter.

For a given first order linear ODE in form:

$$\mathbf{x}' = A\mathbf{x},$$

the eigenvalues can be found as the solutions to the characteristic equation:

$$\det(A - Ir) = 0,$$

and the eigenvectors can be then found by solving the linear system that:

$$(A - Ir) \cdot \boldsymbol{\xi} = \mathbf{0}.$$

The solution to the ODE is:

$$\mathbf{x} = c_1 \boldsymbol{\xi}^{(1)} e^{r_1 t} + c_2 \boldsymbol{\xi}^{(2)} e^{r_2 t} + \cdots + c_n \boldsymbol{\xi}^{(n)} e^{r_n t}.$$

Let the solutions form the fundamental matrix $\Psi(t)$, thus the Wronskian is:

$$\det\left(\Psi(t)\right).$$

The system is linearly independent if the Wronskian is non-zero.
For the linear system in form:

$$\mathbf{x}' = A\mathbf{x},$$

the Wronskian can be found by the trace of A, which is the sum of the diagonals, that is:

$$W = Ce^{\int \operatorname{trace} A dt} = Ce^{\int (A_{1,1}+A_{2,2}+\cdots+A_{n,n})dt}.$$

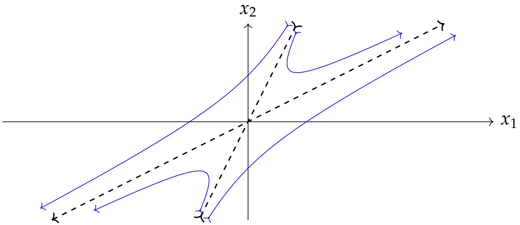
For repeated eigenvalue r with only one eigenvector, if a given a solution is $\mathbf{x}^{(1)} = \boldsymbol{\xi} e^{rt}$, the other solution would be:

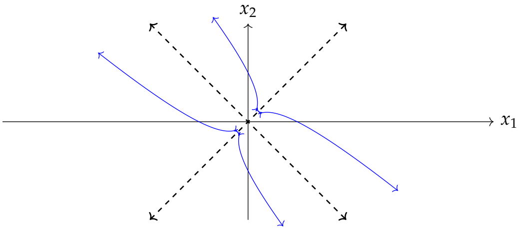
$$\mathbf{x}^{(2)} = \boldsymbol{\xi} t e^{rt} + \boldsymbol{\eta} e^{rt},$$

where $(A - Ir) \cdot \boldsymbol{\eta} = \boldsymbol{\xi}$.

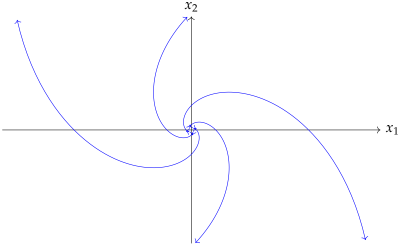
In particular, we can sketch the linear system of $\mathbb{R}^2$ in terms of phase portraits given the eigenvalues and eigenvectors.

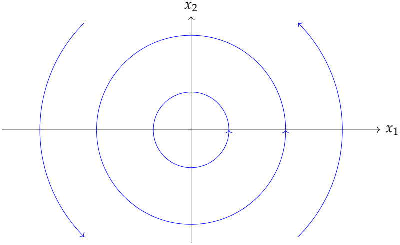
- For a node graph, we have it as (directions might vary):



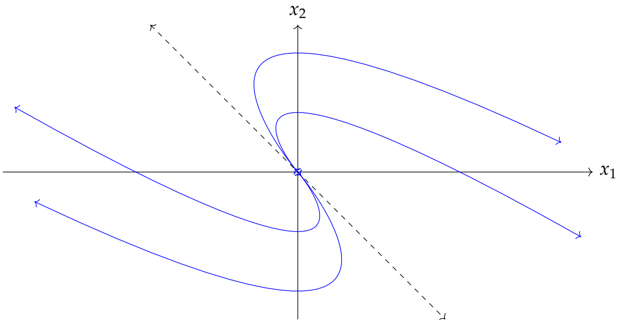


- For a spiral/center graph, we have it as (directions might vary):





- For repeated eigenvalues, the solution depends is (directions might vary):



For non-linear system $\mathbf{x}' = \begin{pmatrix} F \\ G \end{pmatrix} \mathbf{x}$, if $F, G \in C^2$, i.e. locally linear, the approximation at critical point (x_0, y_0) is:

$$\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} = \mathbf{J}(x_0, y_0) \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix},$$

where Jacobian is:

$$\mathbf{J}(x_0, y_0) = \begin{pmatrix} F_x(x_0, y_0) & F_y(x_0, y_0) \\ G_x(x_0, y_0) & G_y(x_0, y_0) \end{pmatrix}.$$

When $\mathbf{x} = \begin{pmatrix} F(y) \\ G(x) \end{pmatrix}$, it can be solved implicitly for:

$$\frac{dy}{dx} = \frac{G(x)}{F(y)}.$$

For linearized system with eigenvalues r_1, r_2:

1. If $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$: $r_1 < r_2 < 0$ indicates an asymptotically stable node, $r_1 < 0 < r_2$ indicates a mostly unstable saddle, and $0 < r_1 < r_2$ indicates an unstable node. Note that these will not change for the non-linear case.
2. If $r_1 = r_2$: $r_1 = r_2 < 0$ indicates a asymptotically stable node and $r_1 = r_2 > 0$ indicates an unstable node. The stability preserves but the shape either node or spiral.
3. If $r_1, r_2 \in \mathbb{C}$ and $\mathrm{Re}(r_1) = \mathrm{Re}(r_2) \neq 0$: $\mathrm{Re}(r_1) = \mathrm{Re}(r_2) > 0$ indicates an unstable spiral and $\mathrm{Re}(r_1) = \mathrm{Re}(r_2) < 0$ indicates an asymptotically stable spiral. Note that these will not change for the non-linear case.
4. If $r_1, r_2 \in \mathbb{C}$ and $\mathrm{Re}(r_1) = \mathrm{Re}(r_2) = 0$: That indicates a stable center. In the non-linear case, the shape is either spiral or center, but the stability is in-determinant.

The stability can be concluded as follows:

Eigenvalues	**Linear System**		**Nonlinear System**	
	Type	**Stability**	**Type**	**Stability**
Eigenvalues are λ_1 and λ_2				
$0 < \lambda_1 < \lambda_2$	Node	Unstable	Node	Unstable
$\lambda_1 < \lambda_2 < 0$	Node	Asymptotically Stable	Node	Asymptotically Stable
$\lambda_1 < 0 < \lambda_2$	Saddle Point	Unstable	Saddle Point	Unstable
$\lambda_1 = \lambda_2 > 0$	Node	Unstable	Node or Spiral Point	Unstable
$\lambda_1 = \lambda_2 < 0$	Node	Asymptotically Stable	Node or Spiral Points	Asymptotically Stable

Eigenvalues are $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$				
$\alpha > 0$	Spiral Point	Unstable	Spiral Point	Unstable
$\alpha = 0$	Center	Stable	Center or Spiral Point	Indeterminate
$\alpha < 0$	Spiral Point	Asymptotically Stable	Spiral Point	Asymptotically Stable

A closed trajectory or periodic solution repeats back to itself with period τ:

$$\begin{pmatrix} x(t+\tau) \\ y(t+\tau) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Closed trajectories with either side converging to/diverging from the solution is a limit cycle.

A Cartesian coordinate can be converted by:

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \\ rr' = xx' + yy', \\ r^2\theta' = xy' - yx'. \end{cases}$$

For a linear system $x = \begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix}$ with $F, G \in C^1$:

1. A closed trajectory of the system must enclose at least 1 critical point.
2. If it only encloses 1 critical point, then that critical point cannot be saddle point.
3. If there are no critical points, there are no closed trajectories.
4. If the unique critical point is saddle, there are no trajectories.
5. For a simple connected domain D in the xy-plane with no holes. If $F_x + G_y$ had the same sign throughout D, then there is no closed trajectories in D.

Part 2: Open Poll

We will work out some sample questions.

- If you have a problem that you are interested with, tell us now.
- Otherwise, we will work through the practice problem set sequentially.
- We are also open to conceptual questions with the course.

1. Let systems of differential equations be defined as follows, find the general solutions to the equations:

$$\text{(a)} \qquad \mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x} = (x_1, x_2).$$

$$\text{(b)} \qquad \mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix}, \qquad \mathbf{x} = (x_1, x_2).$$

$$\text{(c)} \qquad \mathbf{x}' = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 1 & 3 \\ 0 & 4 & 1 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x} = (x_1, x_2, x_3).$$

2. Solve the following initial value problem, represent your solution as a fundamental matrix:

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

3 For the following non-linear systems, find all equilibrium(s) and classify their stability locally.

(a) $$\begin{cases} \dfrac{dx}{dt} = x - y^2, \\ \dfrac{dy}{dt} = x + x^2 - 2y. \end{cases}$$

(b) $$\begin{cases} \dfrac{dx}{dt} = 2x + 3y^2, \\ \dfrac{dy}{dt} = x + 4y^2. \end{cases}$$

4. Let a system of equations for $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ be:

$$\mathbf{x}' = \begin{pmatrix} F(\mathbf{x}) \\ F(\mathbf{x}) \end{pmatrix}$$

Suppose that $F(x_1, x_2) = \sin x_1 + \csc(3x_2)$.

   1. Find the set of all equilibrium(s) for $\mathbf{x}$.
   2. Find the set in which the equilibrium(s) is locally linear.

   Now, $F : \mathbb{R}^2 \to \mathbb{R}$ is not necessarily well-behaved.

   3. Construct a function F such that $\mathbf{x}$ has a equilibrium that is not locally linear.
   Hint: Consider the condition in which a non-linear system is locally linear.

5 Let the following systems of (x, y) be functions of variable t:

$$\text{(a)} \qquad \begin{cases} x' = (1+x)\sin y, \\ y' = 1 - x - \cos y. \end{cases}$$

$$\text{(b)} \qquad \begin{cases} x' = x - y, \\ y' = x - 2y + x^2. \end{cases}$$

Identify the corresponding linear system, then evaluate the stability for the equilibrium at $(0, 0)$ by showing it is locally linear.

6 Determine the periodic solution, if there are any, of the following system:

$$\begin{cases} x' = y + \dfrac{x}{\sqrt{x^2+y^2}}(x^2+y^2-2), \\ y' = -x + \dfrac{y}{\sqrt{x^2+y^2}}(x^2+y^2-2). \end{cases}$$

Good luck on your final exam.

Thank you for being with PILOT for Differential Equations this semester. We wish the best of all for you future academic and career pursuits!