Problem Set 4

Differential Equations

Spring 2025

As we wrap up the first part of the class, let's remark on the key components so far about first order ordinary differential equations:

- Concepts:
  - Existence
  - Uniqueness
  - General Solutions
  - Specific Solutions (IVPs)
- Methods to solve first order ODEs:
  - Separable ODEs
  - Integrating Factor
  - Exact ODEs
  - Autonomous ODEs
- Behavior analyses:
  - Directional Field
  - End Behavior
  - Phase Line
  - Bifurcation Diagram

1. (IVP with Specific Domain). Given following IVP for $y := y(x)$.
$$\begin{cases} y' = \dfrac{1}{x^4 - 1}, \\ y(0) = 0. \end{cases}$$
Find the specific solution and state the domain of the solution.

2. (Bifurcation Diagram). For the first-order autonomous ODE:
$$\frac{dx}{dt} = x^2 - 2x + c,$$
with parameter $c \in \mathbb{R}$, do the following:
   (a) Sketch all of the qualitatively different graphs of $f(x) = x^2 - 2x + c$, as c is varied.
   (b) Determine any and all bifurcation values for the parameter c.
   (c) Sketch a bifurcation diagram for this ODE.

3. (Make an Exact ODE). Let a differential equation on $y := y(x)$ be defined as follows:
$$xy^2 + bx^2y + (x + y)x^2y' = 0.$$
Suppose this differential equation is exact. Find the appropriate value of b and then solve for the solution of the differential equation.

4. (Existence and Uniqueness for IVP). Suppose $f(x)$ is non-zero, let an initial value problem be:

$$\begin{cases} \dfrac{1-y}{x} \cdot \dfrac{dy}{dx} = \dfrac{f(x)}{1+y}, \\ y(0) = 0. \end{cases}$$

   (a) Show that the differential equation is **not** linear.

   For the next two questions, suppose $f(x) = \tan x$.

   (b) State, without justification, the open interval(s) in which $f(x)$ is continuous.

   (c)* Show that there exists some $\delta > 0$ such that there exists a unique solution $y(x)$ for $x \in (-\delta, \delta)$.

   Now, suppose that $f(x)$ is some function, **not** necessarily continuous.

   (d) Suppose that the condition in (c) does **not** hold, give three examples in which $f(x)$ could be.

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Tip of the Week

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