PILOT

Problem Set 3

Differential Equations

Spring 2025

As we are familiarizing ourselves with the techniques of solving ODEs, let's see how these methods can be applied into various system, of how we can model by first order ODEs. As we wrap up, let's remark on the key components so far in the class:

- Concepts:
  - Existence
  - Uniqueness
  - General Solutions
  - Specific Solutions (IVPs)
- Methods to solve first order ODEs:
  - Separable ODEs
  - Integrating Factor
  - Exact ODEs
  - Autonomous ODEs
- Behavior analyses:
  - Directional Field
  - End Behavior
  - Phase Line
  - Bifurcation Diagram

Tip of the Week

Still looking for off-campus housing for the summer or fall? Check out the Johns Hopkins Off-Campus website which offers resources like apartment listings and roommate matches: https://offcampushousing.jhu.edu/. Make sure to talk with upperclassmen to learn more about options, pricing, and management experiences before you sign a lease.

1. (Stability of Autonomous ODEs). Draw the phase line and determine the stability of each equilibrium for the following autonomous differential equations:

   (a) $$y' = y^4 - 3y^3 + 2y^2.$$

   (b) $$y' = y^{2025} - 1.$$

   (c) $$y' = y^2 + 2x + C, \text{ where } C \in \mathbb{R} \text{ is a constant.}$$

   For part (c), determine the bifurcation values for the parameter C and sketch a bifurcation diagram.

2. (Integrating Factor & Exactness). Let a differential equation be defined as follows:

$$\frac{dy}{dx} = e^{2x} + y - 1.$$

   (a) What is the integrating factor ($\mu(x)$) for the equation? Solve for the general solution.

   (b) Is the equation *exact*? If not, make it exact, then find the general solution.

   (c) Do solutions from part (a) and (b) agree?

3. (Decay and Dating). Carbon-14, a radioactive isotope of carbon, is an effective tool in dating the age of organic compounds, as it decays with a relatively long period. Let $Q(t)$ denote the amount of carbon-14 at time t, we suppose that the decay of $Q(t)$ satisfies the following differential equation:

$$\frac{dQ}{dt} = -\lambda Q \text{ where } \lambda \text{ is the rate of decay constant.}$$

   (a) Let the half-life of carbon-14 be τ, find the rate of decay, λ.

   (b) Suppose that a piece of remain is discovered to have 10% of the original amount of carbon-14, find the age of the remain in terms of τ.

4. (End Behavior). Given following IVP:

$$\begin{cases} y' + \dfrac{1}{2}y = \sin t, \\ y(0) = 1. \end{cases}$$

(a) Find the specific solution of the equation.

(b) Describe the end behavior of the solution.