Final Review Problem Set: Solutions

Differential Equations

Spring 2025

1. Let systems of differential equations be defined as follows, find the general solutions to the equations:

(a) $$\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x} = (x_1, x_2).$$

(b) $$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}, \qquad \mathbf{x} = (x_1, x_2).$$

(c) $$\mathbf{x}' = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 1 & 3 \\ 0 & 4 & 1 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x} = (x_1, x_2, x_3).$$

Solution:

(a) Here, we notice that the liner system is diagonal, so we can simply solve for each entry as $x_1 = e^{3t}$ and $x_2 = e^{2t}$. Hence, the solution is:

$$\mathbf{x} = \boxed{C_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}.$$

(b) We first solve for the eigenvalues and eigenvectors:

$$0 = \det \begin{pmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{pmatrix} = (3-\lambda)(-1-\lambda) + 8 = \lambda^2 - 2\lambda + 5.$$

Hence, the eigenvalues are $\lambda = 1 \pm 2\mathrm{i}$. Since they are complex conjugates, we pick $\lambda_1 = 1 - 2i$, so $\boldsymbol{\xi}^{(1)}$ satisfies $\begin{pmatrix} 2+2\mathrm{i} & -2 \\ 4 & -2+2\mathrm{i} \end{pmatrix} . \xi^{(1)} = \mathbf{0}$, so we have $(2+2i)\zeta_1^{(1)} = 2\zeta_2^{(1)}$, so we the eigenvector is $\boldsymbol{\xi}^{(1)} = (1, 1+\mathrm{i})$. We can get our solution:

$$\mathbf{x} = e^{(1-2i)t} \begin{pmatrix} 1+\mathrm{i} \\ 1 \end{pmatrix} = e^t \big(\cos(2t) - \mathrm{i}\sin(2t)\big) \begin{pmatrix} 1 \\ 1+\mathrm{i} \end{pmatrix}$$

$$= e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + \mathrm{i}e^t \begin{pmatrix} -\sin 2t \\ \cos 2t - \sin 2t \end{pmatrix}.$$

Hence, the solution is:

$$\mathbf{x} = \boxed{C_1 e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + C_2 e^t \begin{pmatrix} -\sin 2t \\ \cos 2t - \sin 2t \end{pmatrix}}.$$

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(c) Again, we first find the eigenvalues of the equation, *i.e.*:

$$\det\begin{pmatrix}1-\lambda & 0 & 4\\ 1 & 1-\lambda & 3\\ 0 & 4 & 1-\lambda\end{pmatrix} = 0,$$

which is $(1-\lambda)^3 + 16 - 12(1-\lambda) = -\lambda^3 + 3\lambda^2 + 9\lambda + 5 = -(\lambda+1)^2(\lambda-5) = 0$. Hence, the eigenvalues are $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 5$. Now, we look for eigenvectors.

- For $\lambda_1 = -1$, we have $\begin{pmatrix}2 & 0 & 4\\ 1 & 2 & 3\\ 0 & 4 & 2\end{pmatrix}\boldsymbol{\xi}_1 = \mathbf{0}$, which is $x\begin{pmatrix}-4\\ -1\\ 2\end{pmatrix}$.
- For $\lambda_2 = -1$, we have $\begin{pmatrix}2 & 0 & 4\\ 1 & 2 & 3\\ 0 & 4 & 2\end{pmatrix}\boldsymbol{\eta} = \begin{pmatrix}-4\\ -1\\ 2\end{pmatrix}$, which is $\boldsymbol{\eta} = \begin{pmatrix}4x\\ x+1\\ -2x-1\end{pmatrix} = \begin{pmatrix}0\\ 1\\ -1\end{pmatrix}$.
- For $\lambda_3 = 5$, we have $\begin{pmatrix}-4 & 0 & 4\\ 1 & -4 & 3\\ 0 & 4 & -4\end{pmatrix}\boldsymbol{\xi}_3 = \mathbf{0}$, which is $x\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$.

Hence, the solution is:

$$\mathbf{x} = \boxed{C_1e^{-t}\begin{pmatrix}-4\\ -1\\ 2\end{pmatrix} + C_2\left(te^{-t}\begin{pmatrix}-4\\ -1\\ 2\end{pmatrix} + e^{-t}\begin{pmatrix}0\\ 1\\ -1\end{pmatrix}\right) + C_3e^{5t}\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}}.$$

2. Solve the following initial value problem:

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Solution:
Here, we first find the eigenvalues for the matrix, that is:

$$\det \begin{pmatrix} 1-\lambda & -4 \\ 4 & -7-\lambda \end{pmatrix} = 0.$$

Therefore, the polynomial is $(1-\lambda)(-7-\lambda)+16 = (\lambda+3)^2 = 0$, hence the eigenvalues is $\lambda_1 = \lambda_2 = -3$. Then, we look for the eigenvectors.

- For $\lambda_1 = -3$, we have $\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0}$, which is $\boldsymbol{\xi}_1 = x \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
- For $\lambda_2 = -3$, we have $\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \boldsymbol{\eta} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, which is $\boldsymbol{\eta} = \begin{pmatrix} x \\ x-1/4 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/4 \end{pmatrix}$.

Hence, the general solution is:

$$\mathbf{x} = C_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \left(te^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-3t} \begin{pmatrix} 0 \\ -1/4 \end{pmatrix} \right).$$

By the initial condition, we have $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, so:

$$\mathbf{x}(0) = \begin{pmatrix} C_1 + 0 \\ C_1 - C_2/4 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Therefore, $C_1 = 3$ and $C_2 = 4$, so the particular solution is:

$$\mathbf{x}(t) = \boxed{\begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-3t} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} te^{-3t}}.$$

3. For the following non-linear systems, find all equilibrium(s) and classify their stability locally.

(a)
$$\begin{cases} \dfrac{dx}{dt} = x - y^2, \\ \dfrac{dy}{dt} = x + x^2 - 2y. \end{cases}$$

(b)
$$\begin{cases} \dfrac{dx}{dt} = 2x + 3y^2, \\ \dfrac{dy}{dt} = x + 4y^2. \end{cases}$$

Solution:

(a) For the first case, we notice that the equilibrium points are if:

$$\begin{cases} x - y^2 = 0, \\ x + x^2 - 2y = 0. \end{cases}$$

Note that this will be two parabolas, and there are at most two intersections, and we observe the intersections $(0,0)$ and $(1,1)$. Also to note, the Jacobian matrix is:

$$J = \begin{pmatrix} 1 & -2y \\ 1+2x & -2 \end{pmatrix}.$$

- For the $(0,0)$ case, we denoting $\mathbf{x} = (x,y)$, we verify the linear approximation as:

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} \mathbf{x},$$

and we note that the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -2$, and by:

$$\lambda_2 < 0 < \lambda_1,$$

we know that we have a unstable saddle point at $(0,0)$.

- For the $(1,1)$ case, we have the linear approximation as:

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x-1 \\ y-1 \end{pmatrix},$$

and we note the eigenvalues are $\lambda = \frac{-1 \pm i\sqrt{15}}{2}$, which is complex with a negative real part, so we have a $\boxed{\text{asymptotically stable spiral point}}$.

(b) Here, we note that the equilibrium(s) is achieved if and only if $x' = y' = 0$, that is:

$$\begin{cases} 2x + 3y^2 = 0, \\ x + 4y^2 = 0. \end{cases}$$

In particular, we consider $z = y^2$, so we have a system of linear equations, which simplifies to $x = y = 0$, hence the only equilibrium is at $(x,y) = (0,0)$.

Then, we consider the system locally, denoting $\mathbf{x} = (x,y)$, that is:

$$\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{x},$$

where the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 0$. Note that one eigenvalue is zero and the other is positive, then the critical point is $\boxed{\text{unstable node or spiral point}}$.

4. Let a system of equations for $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ be:

$$\mathbf{x}' = \begin{pmatrix} F(\mathbf{x}) \\ F(\mathbf{x}) \end{pmatrix}$$

Suppose that $F(x_1, x_2) = \sin x_1 + \csc(3x_2)$.

(a) Find the set of all equilibrium(s) for $\mathbf{x}$.

(b) Find the set in which the equilibrium(s) is locally linear.

Now, $F : \mathbb{R}^2 \to \mathbb{R}$ is not necessarily well-behaved.

(c) Construct a function F such that $\mathbf{x}$ has a equilibrium that is <u>not</u> locally linear.
Hint: Consider the condition in which a non-linear system is locally linear.

Solution:

(a) Here, we note that the equilibrium is when $F(\mathbf{x}) = 0$, *i.e.*, $\sin x_1 + \csc(3x_2) = 0$. Here, we note that the image of $\sin x_1$ is $[-1, 1]$ and the image of $\sec(3x_2)$ is $(-\infty, -1] \sqcup [1, \infty)$, this implies that $\sin x_1 + \sec(3x_2)$ is zero only if $\sin x_1 = \pm 1$ and $\sec(3x_2) = \mp 1$, correspondingly.
First, we consider the set in which x_1 is $+1$, that is:

$$\left\{ \frac{(4k+1)\pi}{2} : k \in \mathbb{Z} \right\}.$$

Correspondingly, we consider the set in which x_2 is -1, that is:

$$\left\{ \frac{(4k+3)\pi}{6} : k \in \mathbb{Z} \right\}.$$

Then, we consider the set in which x_1 is -1, that is:

$$\left\{ \frac{(4k+3)\pi}{2} : k \in \mathbb{Z} \right\}.$$

Likewise, we consider the set in which x_2 is $+1$, that is:

$$\left\{ \frac{(4k+1)\pi}{6} : k \in \mathbb{Z} \right\}.$$

Therefore, set theoretically, we have the set of all equilibriums as:

$$\boxed{\left\{ \frac{(4k+1)\pi}{2} : k \in \mathbb{Z} \right\} \times \left\{ \frac{(4k+3)\pi}{6} : k \in \mathbb{Z} \right\} \cup \left\{ \frac{(4k+3)\pi}{2} : k \in \mathbb{Z} \right\} \times \left\{ \frac{(4k+1)\pi}{6} : k \in \mathbb{Z} \right\}}.$$

(b) Note that $\sin x_1$ is (twice) differentiable over the entire domain $\mathbb{R}$ and $\csc(3x_2)$ is (twice) differentiable on all neighborhoods when $\csc(3x_2)$ is ∓ 1, hence the partial derivatives of $F(\mathbf{x})$ with respect to x_1 or x_2 are (twice) differentiable on the neighborhood on all equilibriums, hence the set in which the equilibrium(s) is locally linearly is the same from part (a), namely:

$$\boxed{\left\{ \frac{(4k+1)\pi}{2} : k \in \mathbb{Z} \right\} \times \left\{ \frac{(4k+3)\pi}{6} : k \in \mathbb{Z} \right\} \cup \left\{ \frac{(4k+3)\pi}{2} : k \in \mathbb{Z} \right\} \times \left\{ \frac{(4k+1)\pi}{6} : k \in \mathbb{Z} \right\}}.$$

(c) Clearly, we must enforce that $F(\mathbf{x})$ is $\boxed{\text{not twice differentiable}}$ with some partial derivatives near the equilibrium point(s). One trivial example could be using the absolute value, such as $F(\mathbf{x}) = |x_1| + |x_2|$, where $(0, 0)$ is a equilibrium but it is <u>not differentiable</u>.
For capable readers, we invite them to look for more functions, such as the Weierstrass Function, a continuous function that is *nowhere* differentiable:

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \cos(3^k x).$$

5. Let the following systems of (x, y) be functions of variable t:

(a)
$$\begin{cases} x' = (1+x)\sin y, \\ y' = 1 - x - \cos y. \end{cases}$$

(b)
$$\begin{cases} x' = x - y, \\ y' = x - 2y + x^2. \end{cases}$$

Identify the corresponding linear system, then evaluate the stability for the equilibrium at $(0, 0)$ by showing it is locally linear.

Solution:

(a) We evaluate x and y both at 0 for the differential equation, and $x' = y' = 0$, so $(0, 0)$ is a equilibrium. Then, we can find the Jacobian Matrix:

$$\mathrm{J} = \begin{pmatrix} \sin y & (1+x)\cos y \\ -1 & \sin y \end{pmatrix},$$

and this implies that the linear system is:

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}.$$

As we evaluate J at $(0, 0)$ and take its determinant, we have:

$$\det\left(\mathrm{J}\big|_{(0,0)}\right) = \det \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 1 \neq 0.$$

Hence, the $(0, 0)$ is locally linear.

Here, we have the eigenvalues as $\lambda^2 + 1 = 0$, so they are purely imaginary, so we have an $\boxed{\text{indeterminate spiral or center point}}$.

(b) We evaluate x and y both at 0 for the differential equation, and $x' = y' = 0$, so $(0, 0)$ is a equilibrium. Then, we consider the Jacobian matrix as:

$$\mathrm{J} = \begin{pmatrix} 1 & -1 \\ 2x+1 & -2 \end{pmatrix}.$$

Now, we evaluate the matrix at $(0, 0)$ and take its determinant:

$$\det(\mathrm{J}\big|_{(0,0)}) = \det \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix} = -1 \neq 0.$$

Hence, the system is locally linear, and the linear system locally at $(0, 0)$ should be:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

We find its eigenvalue as:

$$0 = \det \begin{pmatrix} 1-\lambda & -1 \\ 1 & -2-\lambda \end{pmatrix} = (1-\lambda)(-2-\lambda) + 1 = \lambda^2 + \lambda - 1.$$

By using the quadratic formula, we have the eigenvalues as $\lambda = \frac{-1 \pm \sqrt{5}}{2}$.

Thus, we have $\lambda_1 < 0 < \lambda_2$, so we have a $\boxed{\text{unstable saddle point}}$.

6. Determine the periodic solution, if there are any, of the following system:

$$\begin{cases} x' = y + \dfrac{x}{\sqrt{x^2+y^2}}(x^2+y^2-2), \\ y' = -x + \dfrac{y}{\sqrt{x^2+y^2}}(x^2+y^2-2). \end{cases}$$

Solution:

Here, we recall the formula converting between polar coordinates and Cartesian coordinates:

$$\begin{cases} x = r\cos\theta, & y = r\sin\theta, \\ rr' = xx' + yy', & r^2\theta' = xy' - yx'. \end{cases}$$

Now, we are able to convert the system as:

$$\begin{cases} rr' = x\left[y + \dfrac{x}{\sqrt{x^2+y^2}}(x^2+y^2-2)\right] + y\left[-x + \dfrac{y}{\sqrt{x^2+y^2}}(x^2+y^2-2)\right], \\ r^2\theta' = x\left[-x + \dfrac{y}{\sqrt{x^2+y^2}}(x^2+y^2-2)\right] - y\left[y + \dfrac{x}{\sqrt{x^2+y^2}}(x^2+y^2-2)\right]. \end{cases}$$

Here, by simple deductions, we trivially have:

$$rr' = \frac{x^2+y^2}{\sqrt{x^2+y^2}}(x^2+y^2-2) = \frac{r^2}{r}(r^2-2) \rightsquigarrow r' = r^2 - 2.$$
$$r^2\theta' = -x^2 - y^2 = -r^2 \rightsquigarrow \theta' = -1.$$

Thereby, we consider the radius as:

$$r' = r^2 - 2 = (r-\sqrt{2})(r+\sqrt{2}).$$

Hence, we note that the critical point is $\boxed{r = \sqrt{2}}$ (since r must be positive). Note that $r' < 0$ for $0 < r < \sqrt{2}$ and $r' > 0$ for $r > \sqrt{2}$. Hence, this is an unstable limit cycle.