PILOT Midterm 2 Review

Differential Equations

Johns Hopkins University

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As you prepare for the midterm, please consider the following resources:

- PILOT webpage for ODEs: `https://jhu-ode-pilot.github.io/SP25/`
    - Problem sets 5 to 9 will be associated with this midterm. (Except for the first three question on PSet 5 and last question in PSet 9.)
    - Find the review problem set for midterm 2.
    - Extra material: Spring Break Extra Practice Set (**Harder**).
- Review the *homework sets* provided by the instructor.
- Join the PILOT Midterm 2 Review Session. (You are here.)

Plan for today:

1. Go over all contents that we have covered for this semester so far.
2. In the end, we will open the poll to you. Please indicate which problems from the PSets or Review Set that you want us to go over.

Contents:

1 Second Order ODEs
- Linear Homogeneous Cases
- Linear Independence
- Existence and Uniqueness Theorem
- Superposition Theorem
- Abel's Formula
- Reduction of Order
- Non-homogeneous Cases

2 Higher Order ODEs
- Existence and Uniqueness Theorem
- Homogeneous Cases
- Linear Independence
- Abel's Formula
- Non-Homogeneous Cases

3 Review Problems

Part 1: Contents Review

We will get through all contents over this semester.

- Feel free to download the slide deck from the webpage and annotate on it.
- If you have any questions, ask by the end of each chapter.

Consider the linear homogeneous ODE:

$$y'' + py' + qy = 0.$$

Its characteristic equation is:

$$r^2 + pr + q = 0.$$

With solutions r_1 and r_2, the general solution is:

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

If the solutions r_1 and r_2 are complex, by Euler's Formula ($e^{it} = \cos t + i \sin t$), it can be written as $r_1 = \lambda + i\beta$ and $r_2 = \lambda - i\beta$, then the solution is:

$$y(t) = c_1 e^{\lambda t} \cos(\beta t) + c_2 e^{\lambda t} \sin(\beta t).$$

If the solutions r_1 and r_2 are repeated, the solution is:

$$y(t) = c_1 e^{rt} + c_2 t e^{rt}.$$

To form a fundamental set of solutions, the solutions need to be linearly independent, in which the Wronskian (W) must be non-zero, meaning that:

$$W[y_1, y_2] = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}.$$

Consider IVP in form:

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t), \\ y(t_0) = y_1, y'(t_0) = y_2. \end{cases}$$

The interval I containing t_0 has $p(t)$, $q(t)$, and $g(t)$ continuous on it. Then, there is a unique solution $y(t)$ and twice differentiable on the interval I.

If $y_1(t)$ and $y_2(t)$ are solutions to $l[y] = 0$, then the solution $c_1 y_1(t) + c_2 y_2(t)$ are also solutions for all constants $c_1, c_2 \in \mathbb{R}$.

Consider the equation $y'' + py' + qy = 0$, the Wronskian for the solutions are:

$$W[y_1, y_2] = C \exp\left(-\int p\,dt\right),$$

where C is independent of t but depends on y_1 and y_2.

For non-linear second order homogeneous ODEs, when one solution $y_1(t)$ is given, the other solution is in form:

$$y_2(t) = u(t) \times y_1(t).$$

Let the differential equation be:

$$Ay''(t) + By'(t) + Cy(t) = g(t),$$

where $g(t)$ is a smooth function. Let $y_1(t)$ and $y_2(t)$ be the two homogeneous solutions, then the non-homogeneous cases can be solved by the following approaches:

- Undetermined Coefficient: A guess of particular solution will be made based on the terms appearing in the non-homogeneous part, or $g(t)$. Some brief strategies are:

Non-homogeneous Comp. in $g(t)$		**Guess**
Polynomials:	$\sum_{i=0}^{d} a_i t^i$	$\sum_{i=0}^{d} C_i t^i$
Trig.:	$\sin(at)$ and $\cos(at)$	$C_1 \sin(ax) + C_2 \sin(ax)$
Exp.:	e^{at}	Ce^{at}

Note that the guess are additive and multiplicative. Moreover, if the non-homogeneous part is already appearing in the homogeneous solutions, an extra t needs to be multiplied on the non-homogeneous case.

- Variation of Parameters: The particular solution is:
$$y_p = y_1(t) \int \frac{-y_2(t) \times g(t)}{W} dt + y_2(t) \int \frac{y_1(t) \times g(t)}{W} dt.$$

For higher order IVP in form:

$$\begin{cases} y^{(n)} + P_{n-1}(t)y^{(n-1)} + \cdots + P_1(t)y' + P_0(t)y = g(t), \\ y(t_0) = y_0,\ y'(t_0) = y_1,\ \cdots\ , y^{(n-1)}(t_0) = y_{n-1}. \end{cases}$$

If $P_0(t), P_1(t), \cdots, P_{n-1}(t)$, and $g(t)$ are continuous on an interval I containing t_0. Then there exists a unique solution for $y(t)$ on I.

The higher order homogeneous ODEs are in form:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

By computing the characteristic equation:

$$r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0.$$

With solutions $r_1, r_2, \cdots, r_n$, the general solution is:

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}.$$

Note that the complex solutions can still be converted to sines and cosines, while repeated roots multiply a t on the repeated solutions.

To obtain the fundamental set of solutions, the Wronskian (W) must be non-zero, where Wronskian is:

$$W[y_1, y_2, \cdots, y_n] = \det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{pmatrix}.$$

- Alternation to the Wronskian: By definition of linear independence, $f_1, f_2, \cdots, f_n$ are independent on I is equivalent to the expression where $k_1 f_1 + k_2 f_2 + \cdots + k_n f_n = 0$ if and only if $k_i = 0$.

For higher order ODEs in the form of:

$$\begin{cases} y^{(n)} + P_{n-1}(t)y^{(n-1)} + \cdots + P_1(t)y' + P_0(t)y = g(t), \\ y(t_0) = y_0,\ y'(t_0) = y_1,\ \cdots\ , y^{(n-1)}(t_0) = y_{n-1}. \end{cases}$$

Its Wronskian is:

$$W[y_1, y_2, \cdots, y_n] = Ce^{\int P_{n-1}(t)dt},$$

where C is independent of t but depend on $y_1, y_2, \cdots, y_n$.

Let the differential equation be:

$$L[y^{(n)}(t), y^{(n-1)}(t), \cdots, y(t)] = g(t),$$

where $g(t)$ is a smooth function. Let $y_1(t)$, $y_2(t)$, $\cdots$, $y_n(t)$ be all homogeneous solutions, then the non-homogeneous cases can be solved by the following approaches:

- Undetermined Coefficient: Same as in degree 2, a guess of particular solution will be made based on the terms appearing in the non-homogeneous part, or $g(t)$. Some brief strategies are:

Non-homogeneous Comp. in $g(t)$		**Guess**
Polynomials:	$\sum_{i=0}^{d} a_i t^i$	$\sum_{i=0}^{d} C_i t^i$
Trig.:	$\sin(at)$ and $\cos(at)$	$C_1 \sin(ax) + C_2 \sin(ax)$
Exp.:	e^{at}	Ce^{at}

Note that the guess are additive and multiplicative. Moreover, if the non-homogeneous part is already appearing in the homogeneous solutions, an extra t needs to be multiplied on the non-homogeneous case.

- Variation of Parameters: The particular solution is:
  $y_p = y_1(t)\int \frac{W_1 g}{W}dt + y_2(t)\int \frac{W_2 g}{W}dt + \cdots + y_n(t)\int \frac{W_n g}{W}dt,$
  where W_i is defined to be the Wronskian with the i-th column alternated into $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$.

Part 2:
Open Poll

We will work out some sample questions.

- If you have a problem that you are interested with, tell us now.
- Otherwise, we will work through the practice problem set sequentially.
- We are also open to conceptual questions with the course.

1. Solve the following second order differential equations for $y = y(x)$:

$$\begin{array}{ll} \text{(a)} & y'' + y' - 132y = 0. \\ \text{(b)} & y'' - 4y' = -4y. \\ \text{(c)} & y'' - 2y' + 3y = 0. \end{array}$$

2. Given a differential equation for $y = y(t)$ being:

$$t^3 y'' + ty' - y = 0.$$

   1. Verify that $y_1(t) = t$ is a solution to the differential equation.
   2. Find the full set of solutions using reduction of order.
   3. Show that the set of solutions from part (b) is linearly independent.

3 Given the following second order initial value problem:

$$\begin{cases} \dfrac{d^2y}{dx^2} + \cos(1-x)y = x^2 - 2x + 1, \\ y(1) = 1, \qquad \dfrac{dy}{dx}(1) = 0. \end{cases}$$

Prove that the solution $y(x)$ is symmetric about $x = 1$, *i.e.*, satisfying that $y(x) = y(2-x)$.
Hint: Consider the interval in which the solution is unique.

4 Solve the general solution for $y = y(t)$ to the following second order non-homogeneous ODEs.

(a) $$y'' + 2y' + y = e^{-t}.$$

(b) $$y'' + y = \tan t.$$

5 Solve for the general solution to the following higher order ODE.

(a) $$4\frac{d^4y}{dx^4} - 24\frac{d^3y}{dx^3} + 45\frac{d^2y}{dx^2} - 29\frac{dy}{dx} + 6y = 0.$$

(b) $$\frac{d^4y}{dx^4} + y = 0.$$

Hint: Consider the 8-th root of unity, *i.e.*, ζ_8, and verify which roots satisfies the polynomial.

6. Let a third order differential equation be as follows:

$$\ell\left[y(t)\right] = y^{(3)}(t) + 3y''(t) + 3y'(t) + y(t).$$

Let $\ell\left[y(t)\right] = 0$ be trivial initially.

1. Find the set of all linearly independent solutions.

Then, assume that $\ell\left[y(t)\right]$ is non-trivial.

2. Find the particular solution to $\ell\left[y(t)\right] = \sin t$.
3. Find the particular solution to $\ell\left[y(t)\right] = e^{-t}$.
4. Suppose that $\ell\left[y_1(t)\right] = f(t)$ and $\ell\left[y_2(t)\right] = g(t)$ where $f(t)$ and $g(t)$ are "good" functions. Find an expression to $y_3(t)$ such that $\ell\left[y_3(t)\right] = f(t) + g(t)$.

Good luck on your second midterm exam.