Midterm 2 Review Problem Set: Solutions

Differential Equations

Spring 2025

1. Solve the following second order differential equations for $y = y(x)$:

   (a) $$y'' + y' - 132y = 0.$$

   (b) $$y'' - 4y' = -4y.$$

   (c) $$y'' - 2y' + 3y = 0.$$

Solution:

(a) We find the characteristic polynomial as $r^2 + r - 132 = 0$, which can be trivially factorized into:

$$(r-11)(r+12) = 0,$$

so with roots $r_1 = 11$ and $r_2 = -12$, we have the general solution as:

$$y(x) = \boxed{C_1 e^{11x} + C_2 e^{-12x}}.$$

(b) We turn the equation to the standard form $y'' - 4y' + 4 = 0$, and find the characteristic polynomial as $r^2 - 4r + 4 = 0$, which can be factorized into:

$$(r-2)^2 = 0,$$

so with roots $r_1 = r_2 = 2$ (repeated roots), we have the general solution as:

$$y(x) = \boxed{C_1 e^{2x} + C_2 x e^{2x}}.$$

(c) We find the characteristic polynomial as $r^2 - 2r + 3 = 0$, which the quadratic formula gives:

$$r = \frac{2 \pm \sqrt{2^2 - 4 \times 3}}{2} = 1 \pm \mathrm{i}\sqrt{2}$$

so with roots $r_1 = 1 + \mathrm{i}\sqrt{2}$ and $r_2 = 1 - \mathrm{i}\sqrt{2}$, we would have the solution:

$$y(x) = C_1 e^{(1+\mathrm{i}\sqrt{2})x} + C_2 e^{(1-\mathrm{i}\sqrt{2})x}.$$

To obtain real solution, we apply Euler's identity:

$$y_1(x) = e^x\big(\cos(\sqrt{2}x) - \mathrm{i}\sin(\sqrt{2}x)\big) \text{ and } y_2(x) = e^x\big(\cos(-\sqrt{2}x) - \mathrm{i}\sin(-\sqrt{2}x)\big).$$

By the *principle of superposition*, we can linearly combine the solutions:

$$\widetilde{y_1}(x) = \frac{1}{2}(y_1 + y_2) = e^x \cos(\sqrt{2}x) \text{ and } \widetilde{y_2}(x) = \frac{1}{2}(y_2 - y_1) = e^x \sin(\sqrt{2}x).$$

One can verify that $\widetilde{y_1}$ and $\widetilde{y_2}$ are linearly independent by taking Wronskian,*i.e.*:

$$\mathrm{W}[\widetilde{y_1}, \widetilde{y_2}] = \det\begin{pmatrix} e^x\cos(\sqrt{2}x) & e^x\sin(\sqrt{2}x) \\ e^x\cos(\sqrt{2}x) - \sqrt{2}e^x\sin(\sqrt{2}x) & e^x\sin(\sqrt{2}x) + \sqrt{2}e^x\cos(\sqrt{2}x) \end{pmatrix} = \sqrt{2}e^{2x} \neq 0.$$

Now, they are linearly independent, so we have the general solution as:

$$y(x) = \boxed{C_1 e^x \cos(\sqrt{2}x) + C_2 e^x \sin(\sqrt{2}x)}.$$

2. Given a differential equation for $y = y(t)$ being:
$$t^3 y'' + ty' - y = 0.$$
   (a) Verify that $y_1(t) = t$ is a solution to the differential equation.
   (b) Find the full set of solutions using reduction of order.
   (c) Show that the set of solutions from part (b) is linearly independent.

Solution:

(a) *Proof.* We note that the left hand side is:
$$t^3 y_1'' + ty_1' - y_1 = t^3 \cdot 0 + t \cdot 1 - t = t - t = 0.$$
Hence $y_1(t) = t$ is a solution to the differential equation. □

(b) By reduction of order, we assume that the second solution is $y_2(t) = tu(t)$, then we plug $y_2(t)$ into the equation to get:
$$2t^3 u'(t) + t^4 u''(t) + tu(t) + t^2 u'(t) = t^4 u''(t) + (2t^3 + t^2) u'(t) = 0.$$
Here, we let $\omega(t) = u'(t)$ to get a first order differential equation:
$$t^2 \omega'(t) = (-2t - 1)\omega(t).$$
Clearly, this is separable, and we get that:
$$\frac{\omega'(t)}{\omega(t)} = -\frac{2t+1}{t^2} = -\frac{2}{t} - \frac{1}{t^2},$$
which by integration, we have obtained that:
$$\log\left(\omega(t)\right) = -2\log t + \frac{1}{t} + C.$$
By taking exponentials on both sides, we have:
$$\omega(t) = \exp\left(-2\log t + \frac{1}{t} + C\right) = \widetilde{C} e^{1/t} \cdot \frac{1}{t^2}.$$
Recall that we want $u(t)$ instead of $\omega(t)$, so we have:
$$u(t) = \int \omega(t) dt = \widetilde{C} \int e^{1/t} \cdot \frac{1}{t^2} dt = -\widetilde{C} e^{1/t} + D.$$
By multiplying t, we obtain that:
$$y_2 = -\widetilde{C} t e^{1/t} + Dt,$$
where Dt is repetitive in y_1, so we get:
$$y(t) = \boxed{C_1 t + C_2 t e^{1/t}}.$$

(c) *Proof.* We calculate Wronskian as:
$$W[t, te^{1/t}] = \det \begin{pmatrix} t & te^{1/t} \\ 1 & e^{1/t} - \frac{e^{1/t}}{t} \end{pmatrix} = -e^{1/t} \neq 0,$$
hence the set of solutions is linearly independent. □

3. Given the following second order initial value problem:

$$\begin{cases} \dfrac{d^2y}{dx^2} + \cos(1-x)y = x^2 - 2x + 1, \\ y(1) = 1, \qquad \dfrac{dy}{dx}(1) = 0. \end{cases}$$

Prove that the solution $y(x)$ is symmetric about $x = 1$, *i.e.*, satisfying that $y(x) = y(2-x)$.
Hint: Consider the interval in which the solution is unique.

Solution:
Note that I deliberately messed up with all the messy functions. Not only haven't I found a solution to the system, Wolfram cannot have an elementary solution as well. Hence, we need to think, alternatively, on some theorems.

Proof. Here, we suppose that $y(x)$ is a solution, and we want to show that $y(2-x)$ is also a solution. First we note that we can think of taking the derivatives of $y(2-x)$, by the chain rule:

$$\frac{d}{dx}[y(2-x)] = -y'(2-x),$$
$$\frac{d^2}{dx^2}[y(2-x)] = y''(2-x).$$

Now, if we plug in $y(2-x)$ into the system of equations, we have:

- First, for the differential equation, we have:

$$\begin{aligned} \frac{d^2}{dx^2}[y(x-2)] + \cos(1-x)y(x-2) &= y''(2-x) + \cos(x-1)y(2-x) \\ &= y''(2-x) + \cos\big(1-(2-x)\big)y(2-x) \\ &= y''(x) + \cos(1-x)y(x) \\ &= x^2 - 2x + 1 = (x-1)^2 = (1-x)^2 \\ &= \big((2-x)-1\big)^2 = (2-x)^2 - 2(2-x) + 1. \end{aligned}$$

- For the initial conditions, we trivially have that:

$$y(1) = y(2-1) \text{ and } y'(1) = y'(2-1).$$

Hence, we have shown that $y(2-x)$ is a solution if $y(x)$ is a solution.
Again, we observe the original initial value problem that:

$$\cos(1-x) \text{ and } x^2 - 2x + 1 \text{ are continuous on } \mathbb{R}.$$

Therefore, by the *existence and uniqueness theorem for second order linear case*, there could be only one solution, which forces that:

$$y(x) = y(2-x),$$

so the solution is symmetric about $x = 1$, as desired. □

4. Solve the general solution for $y = y(t)$ to the following second order non-homogeneous ODEs.

(a)
$$y'' + 2y' + y = e^{-t}.$$

(b)
$$y'' + y = \tan t.$$

Solution:

(a) First, we look for homogeneous solution, *i.e.*, $y'' + 2y' + y = 0$, whose characteristic equation is:
$$r^2 + 2r + 1 = (r+1)^2 = 0,$$
with root(s) being -1 with multiplicity of 2, so the general solution to homogeneous case is:
$$y_g(t) = C_1 e^{-t} + C_2 t e^{-t}.$$
Notice that the non-homogeneous part is e^{-t}, but we have e^{-t} and te^{-t} as general solutions already, so we have our guess of particular solution as:
$$y_p(t) = At^2 e^{-t}.$$
By taking the derivatives, we have:
$$y_p'(t) = A(2te^{-t} - t^2e^{-t}) \qquad \text{and} \qquad y_p''(t) = A(2e^{-t} - 4te^{-t} + t^2 e^t).$$
We simply plug in the particular solution, so we have:
$$\begin{aligned} A(2e^{-t} - 4te^{-t} + t^2e^t) + 2A(2te^{-t} - t^2e^{-t}) + At^2e^{-t} &= e^{-t} \\ 2Ae^{-t} &= e^{-t} \\ A &= \frac{1}{2}. \end{aligned}$$
Hence, our solution to the non-homogeneous case is:
$$y(t) = \boxed{C_1e^{-t} + C_2te^{-t} + \frac{1}{2}t^2e^{-t}}.$$

(b) Here, we still look for homogeneous solutions, *i.e.*, $y'' + y = 0$, whose characteristic equation is:
$$r^2 + 1 = 0,$$
with roots $\pm \mathrm{i}$. Since we are dealing with real valued functions, we have the general solution as:
$$y_g = C_1 \sin t + C_2 \cos t.$$
Note that $\tan t$ does not work with undetermined coefficients, we must use the variation of parameters, the Wronskian of our solution is:
$$\mathrm{W}[\sin t, \cos t] = \det \begin{pmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{pmatrix} = -\sin^2 t - \cos^2 t = -1.$$
Now, we may use the formula, namely getting the particular solution as:
$$\begin{aligned} y_p &= \sin t \int \frac{-\cos t \cdot \tan t}{-1} dt + \cos t \int \frac{\sin t \cdot \tan t}{-1} dt \\ &= \sin t \int \sin t dt - \cos t \int \frac{\sin^2 t}{\cos t} dt \\ &= \sin t(-\cos t + C) - \cos t \int \frac{1 - \cos^2 t}{\cos t} dt \end{aligned}$$

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\begin{aligned}
&= -\sin t \cos t - \cos t \left(\int \sec t dt - \int \cos t dt \right) \\
&= -\sin t \cos t - \cos t \left(\log|\sec t + \tan t| - \sin t + C \right) \\
&= -\sin t \cos t + \sin t \cos t - \cos t \log|\sec t + \tan t| \\
&= -\cos t \log|\sec t + \tan t|.
\end{aligned}
$$

Hence, our solution to the non-homogeneous case is:

$$y(t) = \boxed{C_1 \sin t + C_2 \cos t - \cos t \log|\sec t + \tan t|}.$$

5. Solve for the general solution to the following higher order ODE.

(a) $$4\frac{d^4y}{dx^4} - 24\frac{d^3y}{dx^3} + 45\frac{d^2y}{dx^2} - 29\frac{dy}{dx} + 6y = 0.$$

(b) $$\frac{d^4y}{dx^4} + y = 0.$$

Hint: Consider the 8-th root of unity, *i.e.*, ζ_8, and verify which roots satisfies the polynomial.

Solution:

(a) Note that we obtain the characteristic equation as:
$$4r^4 - 24r^3 + 45r^2 - 29r + 6 = 0.$$
To obtain our roots, we use the **Rational Root Theorem**, so if the characteristic equation has any rational root, it must have been one (or more) of the following:
$$\pm 1, \pm 2, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}.$$
From plugging in the values, we notice that 2 and 3 are roots of the characteristic equation, by division, we have:
$$\frac{4r^4 - 24r^3 + 45r^2 - 29r + 6}{(r-2)(r-3)} = 4r^2 - 4r + 1 = (2r-1)^2.$$
Now, we know that the roots are 2, 3, and $1/2$ with multiplicity 2, thus the solution to the differential equation is:
$$y(x) = \boxed{C_1e^{2x} + C_2e^{3x} + C_3e^{x/2} + C_4xe^{x/2}}.$$
Again, we invite readers to verify the ***Rational Root Theorem****.*

(b) For this general solution, we trivially obtain that the characteristic polynomial is:
$$r^4 + 1 = 0.$$
Recall that the root of unity address for the case when $r^n = 1$, so we consider the 8th root of unity, in which $(\zeta_8)^8 = 1$. Now, recall **Euler's Identity** and **deMoivre's formula**, we note that only the odd powers of the 8th root of unity satisfies that $r^4 = -1$, namely, are:
$$\zeta_8 = \cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \mathrm{i}\frac{\sqrt{2}}{2},$$
$$\zeta_8^3 = \cos\left(\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} + \mathrm{i}\frac{\sqrt{2}}{2},$$
$$\zeta_8^5 = \cos\left(\frac{5\pi}{4}\right) + \mathrm{i}\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} - \mathrm{i}\frac{\sqrt{2}}{2},$$
$$\zeta_8^7 = \cos\left(\frac{7\pi}{4}\right) + \mathrm{i}\sin\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2} - \mathrm{i}\frac{\sqrt{2}}{2}.$$
Also, we note that ζ_8 and ζ_8^7 are complex conjugates, whereas ζ_8^3 and ζ_8^5 are complex conjugates, so we can linearly combine them to obtain the set of linearly independent solutions, *i.e.*:
$$y(x) = \boxed{\begin{aligned} & e^{(\sqrt{2}/2)x}\left[C_1\cos\left(\frac{\sqrt{2}}{2}x\right) + C_2\sin\left(\frac{\sqrt{2}}{2}x\right)\right] \\ + \; & e^{-(\sqrt{2}/2)x}\left[C_3\cos\left(\frac{\sqrt{2}}{2}x\right) + C_4\sin\left(\frac{\sqrt{2}}{2}x\right)\right]\end{aligned}}.$$

6. Let a third order differential equation be as follows:
$$\ell\big[y(t)\big] = y^{(3)}(t) + 3y''(t) + 3y'(t) + y(t).$$
Let $\ell\big[y(t)\big] = 0$ be trivial initially.
   (a) Find the set of all linearly independent solutions.

   Then, assume that $\ell\big[y(t)\big]$ is non-trivial.
   (b) Find the particular solution to $\ell\big[y(t)\big] = \sin t$.
   (c) Find the particular solution to $\ell\big[y(t)\big] = e^{-t}$.
   (d) Suppose that $\ell\big[y_1(t)\big] = f(t)$ and $\ell\big[y_2(t)\big] = g(t)$ where $f(t)$ and $g(t)$ are "good" functions. Find an expression to $y_3(t)$ such that $\ell\big[y_3(t)\big] = f(t) + g(t)$.

Solution:

(a) Note that the characteristic polynomial can be factorized as perfect cubes:
$$r^3 + 3r^2 + 3r + 1 = (r+1)^3 = 0,$$
its roots are $r = -1$ with multiplicity 3, so the general solution is:
$$y(t) = \boxed{C_1 e^{-t} + C_2 t e^{-t} + C_3 t^2 e^{-t}}.$$
Here, the readers are invited to check, by ***Wronskian****, that set of solutions are linearly independent.*

(b) First, we want to make our guess of particular solution as:
$$y_p(t) = A\sin t + B\cos t,$$
and by taking the derivatives, we have:
$$y_p'(t) = A\cos t - B\sin t, \qquad y_p''(t) = -A\sin t - B\cos t, \text{ and } \qquad y_p'''(t) = -A\cos t + B\sin t.$$
Then, we want to plug in the results into the equation, so:
$$\begin{aligned} \ell\big[y_p(t)\big] &= (-A\cos t + B\sin t) + 3(-A\sin t - B\cos t) + 3(A\cos t - B\sin t) + A\sin t + B\cos t \\ &= (B - 3A - 3B + A)\sin t + (-A - 3B + 3A + B)\cos t \\ &= (-2A - 2B)\sin t + (2A - 2B)\cos t. \end{aligned}$$
Therefore, we can obtain the system that:
$$\begin{cases} -2A - 2B = 1, \\ 2A - 2B = 0, \end{cases}$$
which reduces to $A = -1/4$ and $B = -1/4$, so the solution is:
$$y(t) = \boxed{C_1 e^{-t} + C_2 t e^{-t} + C_3 t^2 e^{-t} - \frac{1}{4}\sin t - \frac{1}{4}\cos t}.$$

(c) Here, note that e^{-t}, te^{-t}, and $t^2 e^{-t}$ are the solutions to homogeneous case, our guess, then, is:
$$y_p(t) = At^3 e^{-t},$$
and by taking the derivatives, we have:
$$y_p'(t) = 3At^2 e^{-t} - At^3 e^{-t}, \qquad y_p''(t) = 6Ate^{-t} - 6At^2 e^{-t} + At^3 e^{-t}, \qquad \text{and}$$
$$y_p'''(t) = 6Ae^{-t} - 18Ate^{-t} + 9At^2 e^{-t} - At^3 e^{-t}.$$

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When we plug the derivatives back to the solutions, we note that:

$$\begin{aligned}\ell[y_p(t)] &= (6Ae^{-t} - 18Ate^{-t} + 9At^2e^{-t} - At^3e^{-t}) \\ &\quad + 3(6Ate^{-t} - 6At^2e^{-t} + At^3e^{-t}) + 3(3At^2e^{-t} - At^3e^{-t}) + (At^3e^{-t}) \\ &= 6Ae^{-t},\end{aligned}$$

which reduces to $A = 1/6$, so the solution is:

$$y(t) = \boxed{C_1e^{-t} + C_2te^{-t} + C_3t^2e^{-t} + \frac{1}{6}t^3e^{-t}}.$$

(d) *Proof.* Here, one should note that the derivative operator is linear, so we have that:

$$\begin{aligned}\ell[y_1(t) + y_2(t)] &= \frac{d^3}{dt^3}[y_1(t) + y_2(t)] + 3\frac{d^2}{dt^2}[y_1(t) + y_2(t)] + 3\frac{d}{dt}[y_1(t) + y_2(t)] + [y_1(t) + y_2(t)] \\ &= y_1'''(t) + 3y_1''(t) + 3y_1'(t) + y_1(t) + y_2'''(t) + 3y_2''(t) + 3y_2'(t) + y_2(t) \\ &= f(t) + g(t),\end{aligned}$$

as desired. $\square$