



## Problem Set 5

### Differential Equations

Spring 2025

We hope that your preparation of the first midterm has been going on well. As we wrapped up the first order differential equation, we will examine a few cases with the existence and uniqueness problems in detail, and build up to the foundations of more classes of differential equations.

1. (“Dilemma” with Existence & Uniqueness Theorem). Let a first order IVP on  $y := y(t)$  be defined as follows:

$$\begin{cases} y' = \frac{2}{t}y, \\ y(1) = 1. \end{cases}$$

- (a) Find the solution to the above initial value problem.  
(b) Recall the theorem on existence and uniqueness, as follows:

For an IVP in simple form:

$$\begin{cases} \frac{dy}{dt} = a(t)y + b(t), \\ y(t_0) = y_0. \end{cases}$$

For some  $I = (\alpha, \beta) \ni t_0$ , if  $a(t)$  and  $b(t)$  are continuous on the interval  $I$ . Then, there exists a unique solution to the IVP on the interval  $I$ .

Show that the IVP in this problem does not satisfy the condition for the existence and uniqueness theorem for  $\mathbb{R}$ .

- (c) Does the above example violates the existence and uniqueness theorem? Why?
2. (Some Criterion over intervals). Suppose we have an initial value problem over  $y := y(t)$ :

$$\begin{cases} y' = F(t, y), \\ y(t_0) = y_0. \end{cases}$$

We suppose that  $F(t, y)$  and  $\frac{\partial}{\partial y}F(t, y)$  are continuous over a region  $I \times J$ . Determine if Picard’s theorem can guarantee the existence of a uniqueness solution.

- (a)  $I = (0, 1)$ ,  $J = (0, 2)$ ,  $t_0 = 0.5$ , and  $y_0 = 1$ .  
(b)  $I = [0, 1]$ ,  $J = [0, 2]$ ,  $t_0 = 0.5$ , and  $y_0 = 1$ .  
(c)  $I = [0, 1]$ ,  $J = [0, 2]$ ,  $t_0 = 1$ , and  $y_0 = 1$ .

- (d)  $I = \bigcup_{i=1}^{\infty} [1/i, 1]$ ,  $J = [0, 2]$ ,  $t_0 = 0.5$ , and  $y_0 = 2$ .
- (e)  $I = \bigcup_{i=1}^{\infty} [1/i, 1]$ ,  $J = [0, 2]$ ,  $t_0 = \delta$ , and  $y_0 = 2$ , where  $\delta$  is any fixed number on  $(0, 1)$ .
3. (Existence of Largest Interval). For the following IVPs, determine the largest interval in which a solution is guaranteed to exist.
- (a) 
$$\begin{cases} (t-3)y' + (\log t)y = 2t, \\ y(1) = 2. \end{cases}$$
- (b) 
$$\begin{cases} (4-t^2)y' + 2ty = 3t^2, \\ y(1) = -3. \end{cases}$$
- (c) 
$$\begin{cases} y' + (\tan t)y = \sin t, \\ y(\pi) = 0. \end{cases}$$
4. (Preliminary to Second Order ODEs). Let a second order differential equation be defined as follows:
- $$y'' - 2y' + y = 0.$$
- (a) Verify that  $y_1 = e^t$  and  $y_2 = te^t$  are two solutions to the above differential equation.
- (b) Verify that any *linear combination* of  $y_1$  and  $y_2$  is a solution to the above differential equation.

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