Problem Set 2

Differential Equations

Spring 2025

As we have dived into the contents of first order linear differential equations, you should have noticed many techniques, such as method of separation, integrating factors, or exactness. Meanwhile, we are about the see practice on these solving techniques and explore an explanation of such concept.

1. (Separable ODE). Solve the following initial value problems (IVPs) on $y = y(x)$, and specify the domain for your solution:

   (a)
   $$\begin{cases} y' = (x \log x)^{-1}, \\ y(e) = -6. \end{cases}$$

   (b)
   $$\begin{cases} y' = y(y+1), \\ y(0) = 1. \end{cases}$$

   Note: Unless otherwise specified, $\log(x) := \log_e(x)$ is the natural logarithm function, which may be written as $\ln(x)$.

2. (Integrating Factor.) Solve for the general solution to the following ODEs with $y = y(t)$:

   (a) $$2y' + y = 3t.$$

   (b) $$y' + \log(t)y = t^{-t}.$$

3. (Linearity of Solutions.) Let $y = y_1(t)$ be a solution to $y' + p(t)y = 0$, and let $y = y_2(t)$ be a solution to $y' + p(t)y = q(t)$. Show that $y = y_1(t) + y_2(t)$ is then also a solution to $y' + p(t)y = q(t)$.

4. (Differential Forms.) This brief digression to "differential forms" aims for the following goals:

   - Legitimize $\dfrac{\partial y}{\partial x} = \dfrac{f(x)}{g(y)} \iff g(y)dy = f(x)dx$ via the differential operator d.
   - Get the foundation of *exactness* for certain differential equation relationship.

   First, consider variables $x_1, x_2, \cdots, x_n$, we may defined the wedge product ($\wedge$) to connect any two variables satisfying that:
   $$x_i \wedge x_j = -x_j \wedge x_i \text{ for all } 1 \le i, j \le n.$$

   (a) Show that $x_i \wedge x_i = 0$ for $1 \le i \le n$.

Now, given any smooth function f, we defined the differential operator (d) as:

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

(b) Suppose $y(x) = e^x$, find dy.

(c) Now, suppose that $\frac{\partial y}{\partial x} = \frac{f(x)}{g(y)}$, can you express dy in terms of the differential form of x.
Note: Since we have just one variable, we have $dy/dx = \partial y/\partial x$, leading to our first goal.

Furthermore, we can apply the differential operator over differential forms with wedge products already. Suppose:

$$\omega = \sum_{i_1,\cdots,i_k} f_{i_1,\cdots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

we may have the differential of ω as:

$$d\omega = \sum_{i_1,\cdots,i_k} (df_{i_1,\cdots,i_k}) dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

(d) Suppose x, y are the variables, and $\omega = 2xy^2dx + 2x^2ydy$, show that $d\omega = 0$.

This then relates to a concept called *exactness* in differential equations. Consider the equation:

$$\frac{dy}{dx} + \frac{F(x,y)}{G(x,y)} = 0,$$

we can rewrite it as $F(x,y)dx + G(x,y)dy = 0$. Exactness enforces that:

$$\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}.$$

Similarly, exactness is considering finding a solution $f(x,y) = c$ such that $F = \frac{\partial f}{\partial x}$ and $G = \frac{\partial f}{\partial y}$.

(e) Show that $df = F(x,y)dx + G(x,y)dy$ and exactness is equivalently $d(df) = 0$.
Note: This implies that the differential equation in part (d) satisfies *exactness*.

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