PILOT

Midterm 2 Review Problem Set

Differential Equations

Spring 2025

Instructions:

The set of questions serves as PILOT practices to midterm 2 for the Spring 2025 term of `AS.110.302` Differential Equations and Applications at Johns Hopkins University.

- The questions are designed to be done without notes and calculators.
- Solutions to selected questions will be discussed during the PILOT review session.

1. Solve the following second order differential equations for $y = y(x)$:
   (a) $$y'' + y' - 132y = 0.$$
   (b) $$y'' - 4y' = -4y.$$
   (c) $$y'' - 2y' + 3y = 0.$$

2. Given a differential equation for $y = y(t)$ being:
   $$t^3y'' + ty' - y = 0.$$
   (a) Verify that $y_1(t) = t$ is a solution to the differential equation.
   (b) Find the full set of solutions using reduction of order.
   (c) Show that the set of solutions from part (b) is linearly independent.

3. Given the following second order initial value problem:
   $$\begin{cases} \dfrac{d^2y}{dx^2} + \cos(1-x)y = x^2 - 2x + 1, \\ y(1) = 1, \qquad \dfrac{dy}{dx}(1) = 0. \end{cases}$$
   Prove that the solution $y(x)$ is symmetric about $x = 1$, *i.e.*, satisfying that $y(x) = y(2-x)$.
   Hint: Consider the interval in which the solution is unique.

4. Solve the general solution for $y = y(t)$ to the following second order non-homogeneous ODEs.
   (a) $$y'' + 2y' + y = e^{-t}.$$
   (b) $$y'' + y = \tan t.$$

5. Solve for the general solution to the following higher order ODE.

   (a) $$4\frac{d^4y}{dx^4} - 24\frac{d^3y}{dx^3} + 45\frac{d^2y}{dx^2} - 29\frac{dy}{dx} + 6y = 0.$$

   (b) $$\frac{d^4y}{dx^4} + y = 0.$$

   Hint: Consider the 8-th root of unity, *i.e.*, ζ_8, and verify which roots satisfies the polynomial.

6. Let a third order differential equation be as follows:
$$\ell\big[y(t)\big] = y^{(3)}(t) + 3y''(t) + 3y'(t) + y(t).$$
   Let $\ell\big[y(t)\big] = 0$ be trivial initially.

   (a) Find the set of all linearly independent solutions.

   Then, assume that $\ell\big[y(t)\big]$ is non-trivial.

   (b) Find the particular solution to $\ell\big[y(t)\big] = \sin t$.

   (c) Find the particular solution to $\ell\big[y(t)\big] = e^{-t}$.

   (d) Suppose that $\ell\big[y_1(t)\big] = f(t)$ and $\ell\big[y_2(t)\big] = g(t)$ where $f(t)$ and $g(t)$ are "good" functions. Find an expression to $y_3(t)$ such that $\ell\big[y_3(t)\big] = f(t) + g(t)$.