PILOT Midterm 1 Review

Differential Equations

Johns Hopkins University

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As you prepare for the midterm, please consider the following resources:

- PILOT webpage for ODEs: `https://jhu-ode-pilot.github.io/SP25/`
  - The first 5 problem sets will be associated with this midterm. (Except for the last question on PSet 5.)
  - Find the review problem set for midterm 1.
- Review the *homework sets* provided by the instructor.
- Join the PILOT Midterm 1 Review Session. (You are here.)

Plan for today:

1. Go over all contents that we have covered for this semester so far.
2. In the end, we will open the poll to you. Please indicate which problems from the PSets or Review Set that you want us to go over.

Contents:

Part 1: Contents Review

We will get through all contents over this semester.

- Feel free to download the slide deck from the webpage and annotate on it.
- If you have any questions, ask by the end of each chapter.

Differential equations can be classified by their properties:

- Ordinary Differential Equations (ODEs) involves ordinary derivatives ($\frac{dy}{dt}$), while Partial Differential Equations (PDEs) involves partial derivatives ($\frac{\partial y}{\partial t}$).
- Single equation involves one unknown and one equation, while System of equations involves multiple unknowns and multiple equations.
- The order of the differential equation is the order of the highest derivatives term.
- Linear differential equations has only linear dependent on the function, while non-linear differential equations has non-linear dependent on the function.

ODEs can be used for modeling. During modeling, it follows the following steps:

1. Construction of the Models,
2. Analysis of the Models,
3. Comparison of the Models with Reality.

The physics model for half life indicates the relationship between half life (τ) of a substance of amount $N(t)$ with initial amount N_0 at a time t is:

$$N(t) = N_0 \left(\frac{1}{2}\right)^{\frac{t}{\tau}},$$

where the rate of decay (λ) and half life (τ) are related by:

$$\tau \times \lambda = \log 2.$$

For ODEs in form $\frac{dy}{dt} + a(t)y = b(t)$, the integrating factor is:

$$\mu(t) = \exp\left(\int a(t)dt\right).$$

For ODEs in form $M(t) + N(y)\frac{dy}{dt} = 0$, it can be separated by:

$$M(t)dt + N(y)dy = 0.$$

The existence and uniqueness for Initial Value Problem (IVP) depend on cases:

- For an IVP in simple form:

$$\begin{cases} \frac{dy}{dt} = a(t)y + b(t), \\ y(t_0) = y_0. \end{cases}$$

  If $a(t)$ and $b(t)$ are continuous on an interval (α, β) and $t_0 \in (\alpha, \beta)$. Then, there exists a uniqueness solution y for (α, β) to the IVP.

Picard's Theorem

- For an IVP in general form:

$$\begin{cases} \dfrac{dy}{dt} = f(t,y), \\ y(t_0) = y_0. \end{cases}$$

For $t_0 \in I = (a,b)$, $y_0 \in J = (c,d)$, if $f(t,y)$ and $\frac{\partial f}{\partial y}(t,y)$ are continuous on interval $I \times J$. Then, there exists a unique solution on a smaller interval $I' \times J' \subset I \times J$, in which $(t_0, y_0) \in I' \times J'$.

Autonomous ODEs are in form of:

$$\frac{dy}{dt} = f(y).$$

The stability (stable/semi-stable/unstable) of equilibrium can be determined by phase lines, *i.e.*, the zeros of the function $f(t)$.

Rational Root Test

Let the polynomial with integer coefficients be defined as:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0,$$

then any rational root $r = p/q$ such that $p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$ satisfies that $p | a_0$ and $q | a_n$.

The logistic population growth model with population (y), growing rate (r), and carrying capacity (k) is given by:

$$\begin{cases} \dfrac{dy}{dt} = r\left(1 - \dfrac{y}{k}\right)y, \\ y(0) = y_0. \end{cases}$$

The solution for Logistic Population Growth is:

$$y(t) = \frac{ky_0}{(k - y_0)e^{-rt} + y_0}.$$

The condition for a function in form $M(x,y) + N(x,y)\dfrac{dy}{dx} = 0$ to be exact is:

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

For solving Exact ODEs, either finding $\int M(x,y)dx + h(y)$ or $\int N(x,y)dy + h(x)$ and taking partials again to fit gives the solution $\Psi(x,y) = C$.
For not exact cases, the integrating factor is:

$$\mu(t) = \exp\left(\int \frac{M_y - N_x}{N} dx\right) \text{ or } \mu(t) = \exp\left(\int \frac{N_x - M_y}{M} dy\right).$$

When a differential equation contains some parameter c, its equilibriums would exhibit different behavior, the bifurcation value is the critical value such that the equilibriums have different stability.

A bifurcation diagram is the vertical concatenation of phase portraits (C-y plot), in which the equilibriums will be marked for respective values of C.

Part 2:
Open Poll

We will work out some sample questions.

- If you have a problem that you are interested with, tell us now.
- Otherwise, we will work through the practice problem set sequentially.
- We are also open to conceptual questions with the course.

1. Find the general solution for $y = y(t)$:
$$y' + 3y = t + e^{-2t},$$
then, describe the behavior of the solution as $t \to \infty$.
2. Given an initial value problem:
$$\begin{cases} \dfrac{dy}{dt} - \dfrac{3}{2}y = 3t + 2e^t, \\ y(0) = y_0. \end{cases}$$
    1. Find the integrating factor $\mu(t)$.
    2. Solve for the particular solution for the initial value problem.
    3. Discuss the behavior of the solution as $t \to \infty$ for different cases of y_0.

3. An autonomous differential equation is given as follows:
$$\frac{dy}{dt} = 4y^3 - 12y^2 + 9y - 2 \quad \text{where } t \geq 0 \text{ and } y \geq 0.$$
Draw a phase portrait and sketch a few solutions with different initial conditions.

4. Determine if the following differential equation is exact. If not, find the integrating factor to make it exact. Then, solve for its general solution:
$$y'(x) = e^{2x} + y(x) - 1.$$

5. For the first-order autonomous ODE:
$$\frac{dy}{dt} = \sin y + C,$$
where $C \in \mathbb{R}$ is a parameter. Determine any and all bifurcation values for the parameter C and sketch a bifurcation diagram.

6. Let an initial value problem be defined as follows:
$$\begin{cases} \left(x^2 \sin(x) + x^3\right) \dfrac{dy}{dx} - \left(\cos(x) + x^2\right) y = 0, \\ y(0) = 1. \end{cases}$$
Show that the solution to the above initial value problem is symmetric about $x = 0$.

Good luck on your first midterm exam.