

PILOT Midterm 2 Review

Differential Equations

Johns Hopkins University

Spring 2025

As you prepare for the midterm, please consider the following resources:

- PILOT webpage for ODEs:
<https://jhu-ode-pilot.github.io/SP25/>
 - Problem sets 5 to 10 will be associated with this midterm.
(Except for the first three question on PSet 5 and last question in PSet 10.)
 - Find the review problem set for midterm 2.
 - Extra material: Spring Break Extra Practice Set (**Harder**).
- Review the *homework sets* provided by the instructor.
- Join the PILOT Midterm 2 Review Session. (You are here.)

Plan for today:

- 1 Go over all contents that we have covered for this semester so far.
- 2 In the end, we will open the poll to you. Please indicate which problems from the PSets or Review Set that you want us to go over.

Contents:

1 Second Order ODEs

- Linear Homogeneous Cases
- Linear Independence
- Existence and Uniqueness Theorem
- Superposition Theorem
- Abel's Formula
- Reduction of Order
- Non-homogeneous Cases

2 Higher Order ODEs

- Existence and Uniqueness Theorem
- Homogeneous Cases
- Linear Independence
- Abel's Formula
- Non-Homogeneous Cases

3 System of First Order Linear ODEs

- Solving for Eigenvalues and Eigenvectors

- Linear Independence
- Abel's Formula

4 Review Problems

Part 1:

Contents Review

We will get through all contents over this semester.

- Feel free to download the slide deck from the webpage and annotate on it.
- If you have any questions, ask by the end of each chapter.

Consider the linear homogeneous ODE:

$$y'' + py' + qy = 0.$$

Its characteristic equation is:

$$r^2 + pr + q = 0.$$

With solutions r_1 and r_2 , the general solution is:

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

If the solutions r_1 and r_2 are complex, by Euler's Formula ($e^{it} = \cos t + i \sin t$), it can be written as $r_1 = \lambda + i\beta$ and $r_2 = \lambda - i\beta$, then the solution is:

$$y(t) = c_1 e^{\lambda t} \cos(\beta t) + c_2 e^{\lambda t} \sin(\beta t).$$

If the solutions r_1 and r_2 are repeated, the solution is:

$$y(t) = c_1 e^{rt} + c_2 t e^{rt}.$$

To form a fundamental set of solutions, the solutions need to be linearly independent, in which the Wronskian (W) must be non-zero, meaning that:

$$W[y_1, y_2] = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}.$$

Consider IVP in form:

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t), \\ y(t_0) = y_1, y'(t_0) = y_2. \end{cases}$$

The interval I containing t_0 has $p(t)$, $q(t)$, and $g(t)$ continuous on it. Then, there is a unique solution $y(t)$ and twice differentiable on the interval I .

If $y_1(t)$ and $y_2(t)$ are solutions to $l[y] = 0$, then the solution $c_1y_1(t) + c_2y_2(t)$ are also solutions for all constants $c_1, c_2 \in \mathbb{R}$.

Consider the equation $y'' + py' + qy = 0$, the Wronskian for the solutions are:

$$W[y_1, y_2] = C \exp \left(- \int p dt \right),$$

where C is independent of t but depends on y_1 and y_2 .

For non-linear second order homogeneous ODEs, when one solution $y_1(t)$ is given, the other solution is in form:

$$y_2(t) = u(t) \times y_1(t).$$

Let the differential equation be:

$$Ay''(t) + By'(t) + Cy(t) = g(t),$$

where $g(t)$ is a smooth function. Let $y_1(t)$ and $y_2(t)$ be the two homogeneous solutions, then the non-homogeneous cases can be solved by the following approaches:

- Undetermined Coefficient: A guess of particular solution will be made based on the terms appearing in the non-homogeneous part, or $g(t)$. Some brief strategies are:

Non-homogeneous Comp. in $g(t)$		Guess
Polynomials:	$\sum_{i=0}^d a_i t^i$	$\sum_{i=0}^d C_i t^i$
Trig.:	$\sin(at)$ and $\cos(at)$	$C_1 \sin(ax) + C_2 \sin(ax)$
Exp.:	e^{at}	Ce^{at}

Note that the guess are additive and multiplicative.

Moreover, if the non-homogeneous part is already appearing in the homogeneous solutions, an extra t needs to be multiplied on the non-homogeneous case.

- Variation of Parameters: The particular solution is:

$$y_p = y_1(t) \int \frac{-y_2(t) \times g(t)}{W} dt + y_2(t) \int \frac{y_1(t) \times g(t)}{W} dt.$$

For higher order IVP in form:

$$\begin{cases} y^{(n)} + P_{n-1}(t)y^{(n-1)} + \cdots + P_1(t)y' + P_0(t)y = g(t), \\ y(t_0) = y_0, y'(t_0) = y_1, \cdots, y^{(n-1)}(t_0) = y_{n-1}. \end{cases}$$

If $P_0(t), P_1(t), \cdots, P_{n-1}(t)$, and $g(t)$ are continuous on an interval I containing t_0 . Then there exists a unique solution for $y(t)$ on I .

The higher order homogeneous ODEs are in form:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0.$$

By computing the characteristic equation:

$$r^n + a_{n-1}r^{n-1} + \cdots + a_1r + a_0 = 0.$$

With solutions r_1, r_2, \dots, r_n , the general solution is:

$$y(t) = c_1e^{r_1t} + c_2e^{r_2t} + \cdots + c_ne^{r_nt}.$$

Note that the complex solutions can still be converted to sines and cosines, while repeated roots multiply a t on the repeated solutions.

To obtain the fundamental set of solutions, the Wronskian (W) must be non-zero, where Wronskian is:

$$W[y_1, y_2, \dots, y_n] = \det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{pmatrix}.$$

- Alternation to the Wronskian: By definition of linear independence, f_1, f_2, \dots, f_n are independent on I is equivalent to the expression where $k_1 f_1 + k_2 f_2 + \cdots + k_n f_n = 0$ if and only if $k_i = 0$.

For higher order ODEs in the form of:

$$\begin{cases} y^{(n)} + P_{n-1}(t)y^{(n-1)} + \cdots + P_1(t)y' + P_0(t)y = g(t), \\ y(t_0) = y_0, y'(t_0) = y_1, \cdots, y^{(n-1)}(t_0) = y_{n-1}. \end{cases}$$

Its Wronskian is:

$$W[y_1, y_2, \cdots, y_n] = Ce^{\int P_{n-1}(t)dt},$$

where C is independent of t but depend on y_1, y_2, \cdots, y_n .

Let the differential equation be:

$$L[y^{(n)}(t), y^{(n-1)}(t), \dots, y(t)] = g(t),$$

where $g(t)$ is a smooth function. Let $y_1(t), y_2(t), \dots, y_n(t)$ be all homogeneous solutions, then the non-homogeneous cases can be solved by the following approaches:

- Undetermined Coefficient: Same as in degree 2, a guess of particular solution will be made based on the terms appearing in the non-homogeneous part, or $g(t)$. Some brief strategies are:

Non-homogeneous Comp. in $g(t)$	Guess
Polynomials: $\sum_{i=0}^d a_i t^i$	$\sum_{i=0}^d C_i t^i$
Trig.: $\sin(at)$ and $\cos(at)$	$C_1 \sin(ax) + C_2 \sin(ax)$
Exp.: e^{at}	Ce^{at}

Note that the guess are additive and multiplicative.

Moreover, if the non-homogeneous part is already appearing in the homogeneous solutions, an extra t needs to be multiplied on the non-homogeneous case.

- Variation of Parameters: The particular solution is:

$$y_p = y_1(t) \int \frac{W_1 g}{W} dt + y_2(t) \int \frac{W_2 g}{W} dt + \cdots + y_n(t) \int \frac{W_n g}{W} dt,$$

where W_i is defined to be the Wronskian with the i -th

column alternated into $\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$.

For a given first order linear ODE in form:

$$\mathbf{x}' = A\mathbf{x},$$

the eigenvalues can be found as the solutions to the characteristic equation:

$$\det(A - Ir) = 0,$$

and the eigenvectors can be then found by solving the linear system that:

$$(A - Ir) \cdot \boldsymbol{\xi} = \mathbf{0}.$$

The solution to the ODE is:

$$\mathbf{x} = c_1 \boldsymbol{\xi}^{(1)} e^{r_1 t} + c_2 \boldsymbol{\xi}^{(2)} e^{r_2 t} + \cdots + c_n \boldsymbol{\xi}^{(n)} e^{r_n t}.$$

Let the solutions form the fundamental matrix $\Psi(t)$, thus the Wronskian is:

$$\det(\Psi(t)).$$

The system is linearly independent if the Wronskian is non-zero.

For the linear system in form:

$$\mathbf{x}' = A\mathbf{x},$$

the Wronskian can be found by the trace of A , which is the sum of the diagonals, that is:

$$W = Ce^{\int \text{trace } A dt} = Ce^{\int (A_{1,1} + A_{2,2} + \cdots + A_{n,n}) dt}.$$

Part 2: Open Poll

We will work out some sample questions.

- If you have a problem that you are interested with, tell us now.
- Otherwise, we will work through the practice problem set sequentially.
- We are also open to conceptual questions with the course.

- 1 Solve the following second order differential equations for $y = y(x)$:

(a) $y'' + y' - 132y = 0.$

(b) $y'' - 4y' = -4y.$

(c) $y'' - 2y' + 3y = 0.$

- 2 Given a differential equation for $y = y(t)$ being:

$$t^3 y'' + t y' - y = 0.$$

- 1 Verify that $y_1(t) = t$ is a solution to the differential equation.
- 2 Find the full set of solutions using reduction of order.
- 3 Show that the set of solutions from part (b) is linearly independent.

- 3 Given the following second order initial value problem:

$$\begin{cases} \frac{d^2y}{dx^2} + \cos(1-x)y = x^2 - 2x + 1, \\ y(1) = 1, \quad \frac{dy}{dx}(1) = 0. \end{cases}$$

Prove that the solution $y(x)$ is symmetric about $x = 1$, *i.e.*, satisfying that $y(x) = y(2 - x)$.

Hint: Consider the interval in which the solution is unique.

- 4 Solve the general solution for $y = y(t)$ to the following second order non-homogeneous ODEs.

(a) $y'' + 2y' + y = e^{-t}.$

(b) $y'' + y = \tan t.$

- 5 Solve for the general solution to the following higher order ODE.

(a) $4\frac{d^4y}{dx^4} - 24\frac{d^3y}{dx^3} + 45\frac{d^2y}{dx^2} - 29\frac{dy}{dx} + 6y = 0.$

(b) $\frac{d^4y}{dx^4} + y = 0.$

Hint: Consider the 8-th root of unity, i.e., ζ_8 , and verify which roots satisfies the polynomial.

- 6 Let a system of differential equations of $x_i(t)$ be as follows:

$$\begin{cases} x_1' = 3x_1 + 2x_2, & x_1(1) = 0, \\ x_2' = x_1 + 4x_2, & x_2(1) = 2. \end{cases}$$

- 1 Solve for the solution to the initial value problem.
- 2 Identify and describe the stability at equilibrium(s).

Good luck on your second midterm exam.