Problem Set 6

Differential Equations

Spring 2025

Welcome to the realm of second order differential equations. As we just got into this part, we will explore the basic cases as well as some foundations of the second order differential equations. For this problem set, we will briefly investigate two core ideas, the linear independence and Euler's theorem.

Tip of the Week

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1. (Second Order Differential Equations). Find the general solution to the following second order differential equations on $y := y(t)$:

   (a) $$y'' - 3y' + 2y = 0.$$

   (b) $$y'' + 12y' - 3y = 0.$$

2. (Second Order IVP). Let an initial value problem for $y = y(t)$ be defined as follows:

$$\begin{cases} 4y'' - y = 0, \\ y(0) = 2,\ y'(0) = \beta, \end{cases}$$

where β is a real constant.

(a) Find the specific solution to the initial value problem. Express your solution with constant β.

(b) Find the value of β such that the solution *converges* to 0 as t tends to infinity.

3. ($L^2([0, 2\pi])$ Space.) Recall that we have defined linear independence of functions, we define *orthogonality* of two real-valued, "square-integrable" functions over $[0, 2\pi]$, f and g, as:

$$\int_0^{2\pi} f(x)g(x)dx = 0.$$

(a) Show that the set $\{\sin x, \cos x\}$ is linearly independent and orthogonal.

(b) Show that if $\{f(x), g(x)\}$ is orthogonal, then $C_1 f(x)$ and $C_2 g(x)$ is orthogonal.

(c)* Note that $\{x, x^2\}$ are linearly independent, construct a basis that is orthogonal.

4. (Preview on Euler's Theorem). In our study of differential equations, our main focus is on *real-valued functions*. But we are about to see complex numbers in our story. **Euler's theorem** states that for any $z \in \mathbb{C}$, we have:

$$\exp(\mathrm{i}z) = \cos(z) + \mathrm{i}\sin(z).$$

   (a) To review on complex numbers, compute/simplify the following expressions:

$$\text{(i)}(2+5\mathrm{i}) \times (1+2\mathrm{i}), \qquad \text{(ii)}\frac{2-3\mathrm{i}}{1+\mathrm{i}}, \qquad \text{(iii)}\overline{2+5\mathrm{i}}, \qquad \text{(iv)}(20+25\mathrm{i}) \times (\overline{20+25\mathrm{i}}).$$

   (b) Write the following complex exponentials in terms of a sum of the real and imaginary parts:

$$\text{(i)}\exp(\mathrm{i}), \qquad \text{(ii)}\exp\left(\frac{\pi\mathrm{i}}{3}\right), \qquad \text{(iii)}\exp(2+2\mathrm{i}).$$

   (c) Express $\sin(z)$ and $\cos(z)$ in terms of exponential functions, where $z \in \mathbb{C}$ is a complex number.

   (d)* Given a function $\varphi \colon \mathbb{R} \to \mathbb{C}$ defined as $\varphi(x) = \exp(\mathrm{i}x)$. We can decompose $\varphi = i_f \circ \tilde{\varphi} \circ \pi_\sim$, where $\pi_\sim$ is surjective, i_φ is injective, and $\tilde{\varphi}$ is bijective, which can be expressed as follows:

$$\begin{tikzcd} \mathbb{R} \arrow[r, two heads, "\pi_\sim"'] \arrow[rrr, bend left, "\varphi"] & X \arrow[r, hook, two heads, "\sim", "\tilde{\varphi}"'] & Y \arrow[r, hook, "i_\varphi"'] & \mathbb{C}, \end{tikzcd}$$

   Find X and Y in the above commutative diagram.
   Hint: Consider $\pi_\sim$ as a projection to an equivalent class, $\tilde{\varphi}$ as a modification of φ, and i_φ as a map from the image to the co-domain.