Final Review Problem Set

Differential Equations

Spring 2025

Instructions:

The set of questions serves as PILOT practices to final exam for the Spring 2025 term of `AS.110.302` Differential Equations and Applications at Johns Hopkins University.

- The questions are designed to be done without notes and calculators.
- Solutions to selected questions will be discussed during the PILOT review session.
- Note that the final is cumulative, while this practice set only contains materials after the second midterm. Please refer to Midterm 1 and 2 Practices for materials covered in the first two midterms.

1. Let systems of differential equations be defined as follows, find the general solutions to the equations:

   (a) $$\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x} = (x_1, x_2).$$

   (b) $$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}, \qquad \mathbf{x} = (x_1, x_2).$$

   (c) $$\mathbf{x}' = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 1 & 3 \\ 0 & 4 & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x} = (x_1, x_2, x_3).$$

2. Solve the following initial value problem:
$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

3. For the following non-linear systems, find all equilibrium(s) and classify their stability locally if they

are locally linear.

(a)
$$\begin{cases} \dfrac{dx}{dt} = x - y^2, \\ \dfrac{dy}{dt} = x + x^2 - 2y. \end{cases}$$

(b)
$$\begin{cases} \dfrac{dx}{dt} = 2x + 3y^2, \\ \dfrac{dy}{dt} = x + 4y^2. \end{cases}$$

4. Let a system of equations for $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ be:
$$\mathbf{x}' = \begin{pmatrix} F(\mathbf{x}) \\ F(\mathbf{x}) \end{pmatrix}$$
Suppose that $F(x_1, x_2) = \sin x_1 + \csc(3x_2)$.
   (a) Find the set of all equilibrium(s) for $\mathbf{x}$.
   (b) Find the set in which the equilibrium(s) is locally linear.

   Now, $F : \mathbb{R}^2 \to \mathbb{R}$ is not necessarily well-behaved.
   (c) Construct a function F such that $\mathbf{x}$ has a equilibrium that is not locally linear.
   Hint: Consider the condition in which a non-linear system is locally linear.

5. Let the following systems of (x, y) be functions of variable t:

(a)
$$\begin{cases} x' = (1 + x)\sin y, \\ y' = 1 - x - \cos y. \end{cases}$$

(b)
$$\begin{cases} x' = x - y, \\ y' = x - 2y + x^2. \end{cases}$$

Identify the corresponding linear system, then evaluate the stability for the equilibrium at $(0, 0)$ by showing it is locally linear.

6. Determine the periodic solution, if there are any, of the following system:
$$\begin{cases} x' = y + \dfrac{x}{\sqrt{x^2 + y^2}}(x^2 + y^2 - 2), \\ y' = -x + \dfrac{y}{\sqrt{x^2 + y^2}}(x^2 + y^2 - 2). \end{cases}$$