PILOT

Problem Set 5

Differential Equations

Spring 2025

We hope that your preparation of the first midterm has been going on well. As we wrapped up the first order differential equation, we will examine a few cases with the existence and uniqueness problems in detail, and build up to the foundations of more classes of differential equations.

Tip of the Week

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1. ("Dilemma" with Existence & Uniqueness Theorem). Let a first order IVP on $y := y(t)$ be defined as follows:

$$\begin{cases} y' = \dfrac{2}{t}y, \\ y(1) = 1. \end{cases}$$

   (a) Find the solution to the above initial value problem.

   (b) Recall the theorem on existence and uniqueness, as follows:

   > For an IVP in simple form:
   >
   > $$\begin{cases} \dfrac{dy}{dt} = a(t)y + b(t), \\ y(t_0) = y_0. \end{cases}$$
   >
   > For some $I = (\alpha, \beta) \ni t_0$, if $a(t)$ and $b(t)$ are continuous on the interval I. Then, there exists a unique solution to the IVP on the interval I.

   Show that the IVP in this problem does not satisfy the condition for the existence and uniqueness theorem for $\mathbb{R}$.

   (c) Does the above example violates the existence and uniqueness theorem? Why?

2. (Some Criterion over intervals). Suppose we have an initial value problem over $y := y(t)$:

$$\begin{cases} y' = F(t,y), \\ y(t_0) = y_0. \end{cases}$$

We suppose that $F(t,y)$ and $\frac{\partial}{\partial y}F(t,y)$ are continuous over a region $I \times J$. Determine if Picard's theorem can guarantee the existence of a uniqueness solution.

(a) $I = (0,1)$, $J = (0,2)$, $t_0 = 0.5$, and $y_0 = 1$.

(b) $I = [0,1]$, $J = [0,2]$, $t_0 = 0.5$, and $y_0 = 1$.

(c) $I = [0,1]$, $J = [0,2]$, $t_0 = 1$, and $y_0 = 1$.

(d) $I = \bigcup_{i=1}^{\infty}[1/i, 1]$, $J = [0,2]$, $t_0 = 0.5$, and $y_0 = 2$.

(e) $I = \bigcup_{i=1}^{\infty}[1/i, 1]$, $J = [0,2]$, $t_0 = \delta$, and $y_0 = 2$, where δ is any fixed number on $(0,1)$.

3. (Existence of Largest Interval). For the following IVPs, determine the largest interval in which a solution is guaranteed to exist.

   (a)
   $$\begin{cases} (t-3)y' + (\log t)y = 2t, \\ y(1) = 2. \end{cases}$$

   (b)
   $$\begin{cases} (4-t^2)y' + 2ty = 3t^2, \\ y(1) = -3. \end{cases}$$

   (c)
   $$\begin{cases} y' + (\tan t)y = \sin t, \\ y(\pi) = 0. \end{cases}$$

4. (Preliminary to Second Order ODEs). Let a second order differential equation be defined as follows:

$$y'' - 2y' + y = 0.$$

   (a) Verify that $y_1 = e^t$ and $y_2 = te^t$ are two solutions to the above differential equation.

   (b) Verify that any *linear combination* of y_1 and y_2 is a solution to the above differential equation.