Problem Set 11

Differential Equations

Spring 2025

We hope that your second midterm was smooth and successful. For this week, we will explore deeper into the linear systems as well as special cases in linear system, *i.e.*, complex or zero eigenvalues. Alias, lets explore further on what eigenvalues represent and the geometries behind it.

1. (Linear System versus Second Order). Let an initial value problem for linear system on $x_1 := x_1(t)$ and $x_2 := x_2(t)$ be defined as follows:
$$\begin{cases} x_1' = 3x_1 - 2x_2, & x_1(0) = 3, \\ x_2' = 2x_1 - 2x_2, & x_2(0) = \dfrac{1}{2}. \end{cases}$$
   (a) Solve for the *general solution* for the linear system by considering $\mathbf{x} = (x_1, x_2)$.

   (b) Transform the *general system* into a single equation of second order. Then solve the second-order equation. Eventually, convert your solution of one variable back to the *general solution* to $x_1(t)$ and $x_2(t)$.

   (c) Find the particular solution using the initial conditions, then graph the parameterized curve on a x_1x_2-plane with $t \geq 0$.

2. (A "Big" Matrix). Let $\mathbf{x} = (x_1, x_2)$ satisfy the following differential equation.
$$\mathbf{x}' = \begin{pmatrix} \dfrac{1}{42} & \dfrac{1}{21} \\ \dfrac{1}{14} & \dfrac{1}{21} \end{pmatrix} \cdot \mathbf{x}.$$
   Hint: Think about the *geometric* interpretation of eigenvalues and eigenvectors and try to *simplify* the matrix. (*Otherwise, the computation is hard.*)

3. (3 Dimensional Complex Eigenvalues). Let $\mathbf{x} = (x_1, x_2, x_3)$ be in dimension 3, we define a linear system as follows:
$$\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} \cdot \mathbf{x}.$$
   Find the general solution to the above system in terms of real valued functions.

4. (Zero Eigenvalue). Let a system of $\mathbf{x} = (x_1, x_2)$ be defined as:

$$\mathbf{x}' = \begin{pmatrix} -3 & -6 \\ 1 & 2 \end{pmatrix} \cdot \mathbf{x}.$$

(a) Find the eigenvalues and eigenvectors for $\begin{pmatrix} -3 & -6 \\ 1 & 2 \end{pmatrix}$.

(b) Give a full set of solutions to the differential equation. Plot some trajectory on the x_1x_2-plane.

(c)* Let A be an arbitrary square matrix. Show that A is non-invertible if and only if A has zero as an eigenvalue.

Note: Please avoid using the definition that the determinant is the product of all eigenvalues. Moreover, consider the geometric implication of eigenvalue to account for invertibility.

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Tip of the Week

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