

PILOT Midterm 1 Review

Differential Equations

Johns Hopkins University

Spring 2026

As you prepare for midterm 1, please consider the following resources:

- PILOT webpage for ODEs:
<https://jhu-ode-pilot.github.io/SP26/>
 - Find the review problem sets for quiz 1.
 - Review the weekly problem sets 1 – 5.
- Review the *homework/webwork* provided by the instructor.
- Join the PILOT Review Session. (You are here.)

Plan for today:

- 1 Go over all contents that we have covered for this semester so far.
- 2 In the end, we will open the poll to you. Please indicate which problems from the Review Set that you want us to go over.

Part 1: Contents Review

We will get through all contents over this semester.

- Feel free to download the slide deck from the webpage and annotate on it.
- If you have any questions, ask by the end of each chapter.

1 Preliminaries

2 First Order ODEs

3 Second Order ODEs

Preliminaries

- Classifications of Differential Equations
 - ODEs vs PDEs
- Modeling Using ODEs
 - Half Life Problem

When having various differential equations, we can classify them by their properties.

ODEs vs PDEs

Ordinary Differential Equations (ODEs) involves ordinary derivatives ($\frac{dy}{dt}$), while Partial Differential Equations (PDEs) involves partial derivatives ($\frac{\partial y}{\partial t}$).

This course focuses on ODEs, and it can also be classified in various different ways:

- **Single equation** involves one unknown and one equation, while **system of equations** involves multiple unknowns and multiple equations.
- The **order** of the differential equation is the order of the highest derivatives term.
- **Linear** differential equations has only linear dependent on the function, while **non-linear** differential equations has non-linear dependent on the function.

ODEs can be used for modeling. During modeling, it often follows the following steps:

- 1 Construction of the Models,
- 2 Analysis of the Models,
- 3 Comparison of the Models with Reality.

An example of modeling is the **half-life problem**.

Half Life Problem

The physics model for half life indicates the relationship between half life (τ) of a substance of amount $N(t)$ with initial amount N_0 at a time t is:

$$N(t) = N_0 \left(\frac{1}{2}\right)^{\frac{t}{\tau}},$$

where the rate of decay (λ) and half life (τ) are related by:

$$\tau \times \lambda = \log 2.$$

First Order ODEs

- Methods of Solving ODEs
 - Separable ODEs
 - Integrating Factor
- Existence and Uniqueness Theorems
- Autonomous ODEs
 - Rational Root Test
- Logistic Population Growth
 - Partial Fractions
- Exactness Problem
 - Integrating Factor for Non-Exact Case
- Bifurcation
 - Bifurcation Diagram

Here, we will introduce various ways of solving ODEs:

Separable ODEs

For ODEs in form $M(t) + N(y) \frac{dy}{dt} = 0$, it can be separated by:

$$M(t)dt + N(y)dy = 0.$$

When the ODE is not separable, we may consider using the **integrating factor**.

Integrating Factor

For ODEs in form $\frac{dy}{dt} + a(t)y = b(t)$, the integrating factor is:

$$\mu(t) = \exp\left(\int a(t)dt\right).$$

The existence and uniqueness for Initial Value Problem (IVP) tells us information on if we can obtain a unique solution over some interval:

- For an IVP in simple form:

$$\begin{cases} \frac{dy}{dt} = a(t)y + b(t), \\ y(t_0) = y_0. \end{cases}$$

If $a(t)$ and $b(t)$ are continuous on an interval (α, β) and $t_0 \in (\alpha, \beta)$. Then, there exists a uniqueness solution y for (α, β) to the IVP.

Picard's Theorem

- For an IVP in general form:

$$\begin{cases} \frac{dy}{dt} = f(t, y), \\ y(t_0) = y_0. \end{cases}$$

For $t_0 \in I = (a, b)$, $y_0 \in J = (c, d)$, if $f(t, y)$ and $\frac{\partial f}{\partial y}(t, y)$ are continuous on interval $I \times J$. Then, there exists a unique solution on a smaller interval $I' \times J' \subset I \times J$, in which $(t_0, y_0) \in I' \times J'$.

Only Contrapositive is Guaranteed to be True

For both theorems, you can conclude that if *there does not exist a solution or the solution is not unique*, then *the conditions must not be satisfied*. You **cannot** conclude that if *the conditions are not satisfied*, then *there is no unique solution*.

Autonomous ODEs are in form of:

$$\frac{dy}{dt} = f(y).$$

The stability (stable/semi-stable/unstable) of equilibrium can be determined by phase lines, *i.e.*, the zeros of the function $f(t)$.

Rational Root Test

Let the polynomial with integer coefficients be defined as:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0,$$

then any rational root $r = p/q$ such that $p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$ satisfies that $p|a_0$ and $q|a_n$.

The logistic population growth model with population (y), growing rate (r), and carrying capacity (k) is given by:

$$\begin{cases} \frac{dy}{dt} = r \left(1 - \frac{y}{k}\right) y, \\ y(0) = y_0, \end{cases}$$

whose general solution is $y(t) = \frac{ky_0}{(k - y_0)e^{-rt} + y_0}$.

Partial Fractions

For a fraction in the form $\frac{C}{(x - a_1)^{n_1}(x - a_2)^{n_2} \cdots (x - a_m)^{n_m}}$, it can be decomposed in terms of:

$$\frac{C_{1,1}}{x - a_1} + \frac{C_{1,2}}{(x - a_1)^2} + \cdots + \frac{C_{1,n_1}}{(x - a_1)^{n_1}} + \cdots + \frac{C_{m,1}}{x - a_m} + \cdots + \frac{C_{m,n_m}}{(x - a_m)^{n_m}}.$$

The condition for a function in form $M(x, y) + N(x, y)\frac{dy}{dx} = 0$ to be exact is:

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

For solving Exact ODEs, either finding $\int M(x, y)dx + h(y)$ or $\int N(x, y)dy + h(x)$ and taking partials again to fit gives the solution $\Psi(x, y) = C$.

Integrating Factor for Non-Exact Case

$$\mu(x) = \exp \left(\int \frac{M_y - N_x}{N} dx \right) \quad \text{or} \quad \mu(y) = \exp \left(\int \frac{N_x - M_y}{M} dy \right).$$

When a differential equation contains some unknown, fixed parameter C , its equilibria would exhibit different behavior, the bifurcation value is the critical value such that the equilibria have different stability.

Bifurcation Diagram

A bifurcation diagram is the vertical concatenation of phase portraits (C - y plot), in which the equilibria will be marked for respective values of C .

Second Order ODEs

- Linear Homogeneous Cases
 - Complex Characteristic Roots
 - Repeated Characteristic Roots
- Linear Independence
 - Definition of Linearly Independence
 - Superposition Principle
- Reduction of Order
 - Product Rule and Chain Rule

Consider the linear homogeneous ODE:

$$y'' + py' + qy = 0.$$

Its characteristic equation is $r^2 + pr + q = 0$, with real, distinct solutions r_1 and r_2 , the general solution is:

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Complex Characteristic Roots

If the solutions are complex, by Euler's Formula ($e^{it} = \cos t + i \sin t$), it can be written as $r_1 = \lambda + i\beta$ and $r_2 = \lambda - i\beta$, then the solution is:

$$y(t) = c_1 e^{\lambda t} \cos(\beta t) + c_2 e^{\lambda t} \sin(\beta t).$$

Repeated Characteristic Roots

If the solutions are repeated, the solution is:

$$y(t) = c_1 e^{rt} + c_2 t e^{rt}.$$

To form a fundamental set of solutions, the solutions need to be linearly independent, in which the Wronskian (W) must be non-zero, meaning that:

$$W[y_1, y_2] = \det \begin{pmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{pmatrix}.$$

Definition of Linearly Independence

By definition, a set of polynomials $\{f_1, f_2, \dots, f_n, \dots\}$ is linearly independent when for $\lambda_1, \lambda_2, \dots, \lambda_n, \dots \in \mathbb{F}$ (typically \mathbb{C}):

$$\lambda_1 f_1 + \lambda_2 f_2 + \dots + \lambda_n f_n + \dots = 0 \iff \lambda_1 = \lambda_2 = \dots = \lambda_n = \dots = 0.$$

Superposition Principle

If $y_1(t)$ and $y_2(t)$ are solutions to $l[y] = 0$, then the solution $c_1 y_1(t) + c_2 y_2(t)$ are also solutions for all constants $c_1, c_2 \in \mathbb{R}$.

For non-linear second order homogeneous ODEs, when one solution $y_1(t)$ is given, the other solution is in form:

$$y_2(t) = u(t) \cdot y_1(t).$$

Product Rule and Chain Rule

- **Product Rule:** $\frac{d}{dx}[f(x) \cdot g(x)] = \frac{df}{dx}(x)g(x) + f(x)\frac{dg}{dx}(x).$
- **Chain Rule:** $\frac{d}{dx}[f(g(x))] = \frac{df}{dx}(g(x)) \cdot \frac{dg}{dx}(x).$

Procedure of Reduction of Order

As long as $y_1(t)$ is a solution, you will be able to reduce the differential equation with respect to y_2 into a differential equation involving only $u''(t)$ and $u'(t)$ terms to solve for $\omega(t) = u'(t)$.

Part 2: Open Poll

We will work out some sample questions.

- If you have a problem that you are interested with, tell us now.
- Otherwise, we will work through selected problems from the practice problem set.
- We are also open to conceptual questions with the course.

Good luck on your first midterm.