



## Midterm 1 Practices

Differential Equations

Summer 2024

### Instructions:

The set of questions serves as PILOT practices to midterm 1 for the Summer 2024 term of AS.110.302 Differential Equations and Applications at Johns Hopkins University.

- Questions marked with asterisk(s) are more challenging, but still accessible, compared to the others.
- The questions are designed to be done without notes and calculators.
- Solutions to the questions will be discussed during the PILOT review session.

1. Find the general solution for  $y = y(t)$ :

$$y' + 3y = t + e^{-2t},$$

then, describe the behavior of the solution as  $t \rightarrow \infty$ .

2. Given an initial value problem:

$$\begin{cases} \frac{dy}{dt} - \frac{3}{2}y = 3t + 2e^t, \\ y(0) = y_0. \end{cases}$$

- (a) Find the integrating factor  $\mu(t)$ .
- (b) Solve for the particular solution for the initial value problem.
- (c) Discuss the behavior of the solution as  $t \rightarrow \infty$  for different cases of  $y_0$ .
3. Suppose  $f(x)$  is non-zero, let an initial value problem be:

$$\begin{cases} \frac{1-y}{x} \cdot \frac{dy}{dx} = \frac{f(x)}{1+y}, \\ y(0) = 0. \end{cases}$$

- (a) Show that the differential equation is **not** linear.

For the next two questions, suppose  $f(x) = \tan x$ .

- (b) State, without justification, the open interval(s) in which  $f(x)$  is continuous.
- (c)\* Show that there exists some  $\delta > 0$  such that there exists a unique solution  $y(x)$  for  $x \in (-\delta, \delta)$ .

Now, suppose that  $f(x)$  is some function, **not** necessarily continuous.

- (d)\*\* Suppose that the condition in (c) does **not** hold, give three examples in which  $f(x)$  could be.

4. An autonomous differential equation is given as follows:

$$\frac{dy}{dt} = 4y^3 - 12y^2 + 9y - 2 \quad \text{where } t \geq 0 \text{ and } y \geq 0.$$

Draw a phase portrait and sketch a few solutions with different initial conditions.

- 5.\* Determine if the following differential equation is exact. If not, find the integrating factor to make it exact. Then, solve for its general solution:

$$y'(x) = e^{2x} + y(x) - 1.$$

6. Let a differential equation be defined as:

$$\frac{dy}{dt} = t - y \text{ and } y(0) = 0.$$

Use Euler's Method with step size  $h = 1$  to approximate  $y(5)$ .

7. Solve the following second order differential equations for  $y = y(x)$ :

(a)  $y'' + y' - 132 = 0.$

(b)  $y'' - 4y' = -4.$

(c)  $y'' - 2y + 3 = 0.$

- 8.\*\* The following system of partial differential equations portrays the propagation of waves on a segment of the 1-dimensional string of length  $L$ , the displacement of string at  $x \in [0, L]$  at time  $t \in [0, \infty)$  is described as the function  $u = u(x, t)$ :

$$\begin{cases} \text{Differential Equation:} & \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, & \text{where } x \in (0, L) \text{ and } t \in [0, \infty); \\ \text{Initial Condition:} & u(x, 0) = g(x), & \text{where } x \in [0, L]; \\ \text{Boundary Condition:} & u(0, t) = u(L, t) = 0, & \text{where } t \in [0, \infty); \end{cases}$$

where  $c$  is a constant and  $g(x)$  has "good" behavior. Apply the method of separation, *i.e.*:

$$u(x, t) = v(x) \cdot w(t),$$

and attempt to obtain a general solution that is non-trivial.

*Hint:* Use the fact that  $\{\sin(nx), \cos(nx)\}_{n \in \mathbb{Z}^+}$  forms an orthonormal basis.

9. Given a differential equation for  $y = y(t)$  being:

$$t^3 y'' + t y' - y = 0.$$

(a) Verify that  $y_1(t) = t$  is a solution to the differential equation.

(b)\* Find the full set of solutions using reduction of order.

(c) Show that the set of solutions from part (b) is linearly independent.

- 10.\*\* Given the following second order initial value problem:

$$\begin{cases} \frac{d^2 y}{dt^2} + \cos(1-x)y = x^2 - 2x + 1, \\ y(1) = 1, \\ \frac{dy}{dt}(1) = 0. \end{cases}$$

Prove that the solution  $y(t)$  is symmetric about  $t = 1$ , *i.e.*, satisfying that  $y(t) = y(2 - t)$ .

*Hint:* Consider the interval in which the solution is unique.