Midterm 2 Practices

Differential Equations

Summer 2024

Instructions:

The set of questions serves as PILOT practices to midterm 2 for the Summer 2024 term of `AS.110.302` Differential Equations and Applications at Johns Hopkins University.

- Questions marked with asterisk(s) are more challenging, but still accessible, compared to the others.
- The questions are designed to be done without notes and calculators.
- Solutions to the questions will be discussed during the PILOT review session.

1. Solve the general solution for $y = y(t)$ to the following second order non-homogeneous ODEs.

   (a) $$y'' + 2y' + y = e^t.$$

   (b) $$y'' + y = \tan t.$$

2. Solve for the general solution to the following higher order ODE.

   (a) $$4\frac{d^4y}{dx^4} - 24\frac{d^3y}{dx^3} + 45\frac{d^2y}{dx^2} - 29\frac{dy}{dx} + 6y = 0.$$

   (b)** $$\frac{d^4y}{dx^4} + y = 0.$$

   Hint: Consider the 8-th root of unity, *i.e.*, ζ_8, and verify which roots satisfies the polynomial.

3. Let a third order differential equation be as follows:

   $$\ell\big[y(t)\big] = y^{(3)}(t) + 3y''(t) + 3y'(t) + y(t).$$

   Let $\ell\big[y(t)\big] = 0$ be trivial initially.

   (a) Find the set of all linearly independent solutions.

   Then, assume that $\ell\big[y(t)\big]$ is non-trivial.

   (b) Find the particular solution to $\ell\big[y(t)\big] = \sin t$.

   (c) Find the particular solution to $\ell\big[y(t)\big] = e^t$.

   (d)* Suppose that $\ell\big[y_1(t)\big] = f(t)$ and $\ell\big[y_2(t)\big] = g(t)$ where $f(t)$ and $g(t)$ are "good" functions. Find an expression to $y_3(t)$ such that $\ell\big[y_3(t)\big] = f(t) + g(t)$.

4. Show the following Laplace transformation by definition.

   (a) $$\mathcal{L}\big\{\sin(at)\big\} = \frac{a}{a^2 + s^2}.$$

   (b)* $$\mathcal{L}\big\{(f * g)(t)\big\} = \mathcal{L}\big\{f(t)\big\} + \mathcal{L}\big\{g(t)\big\}.$$

5. Given the following the results after Laplace transformation $F(s) = \mathcal{L}\{f(t)\}$, find each $f(t)$ prior to the Laplace transformation.

   (a)
$$F(s) = \frac{2s^2+4}{s^3+4s}.$$

   (b)*
$$F(s) = \frac{s^2}{s^2+9} - 1.$$

6. Find the solution of $y = y(t)$ to the following IVP using Laplace transformation:
$$\begin{cases} y'' - 2y' + 2y = e^{-t}, \\ y(0) = 0, \qquad y'(0) = 1. \end{cases}$$

7.** Dirac delta function $\delta(t)$ is heuristically defined as:
$$\delta(t) = \begin{cases} +\infty, & \text{if } t = 0 \\ 0, & \text{if } t \neq 0 \end{cases} \qquad \text{and} \qquad \int_{\mathbb{R}} \delta(t)dt = 1.$$
In *real analysis*, $\delta(t)$ is often called an "approximation to identity", meaning that it "preserves" the original equation after convolution. By the definition of convolution for f and g, here, as:
$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau,$$
prove that $(f * \delta)(t) = f(t)$ for $t > 0$.
Hint: Use the convolution theorem and the Laplace transformation of step functions.

8. Let a system of differential equations be defined as follows, find the general solutions to those equations.

   (a)
$$\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^2.$$

   (b)*
$$\mathbf{x}' = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 1 & 3 \\ 0 & 4 & 1 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^3.$$

9. Solve the following initial value problem for the system of equations:
$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

10.** (*Putnam 2023.*) Determine the smallest positive real number r such that there exists differentiable functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ satisfying:

- $f(0) > 0$,
- $g(0) = 0$,
- $|f'(x)| \leq |g(x)|$ for all x,
- $|g'(x)| \leq |f(x)|$ for all x, and
- $f(r) = 0$.

You may give an answer without a rigorous proof, as the proof is out of scope of the course.
Hint: Assume that the function "moves" the fastest when the cap of the derivatives are "moving" the fastest, then think of constructing a dynamical system relating f and g.