



## Midterm 1 Practices: Solutions

### Differential Equations

Summer 2024

1. Find the general solution for  $y = y(t)$ :

$$y' + 3y = t + e^{-2t},$$

then, describe the behavior of the solution as  $t \rightarrow \infty$ .

**Solution:**

Here, one could note that this differential equation is not separable but in the form of integrating factor problem, then we find the integrating factor as:

$$\mu(t) = \exp\left(\int_0^t 3ds\right) = \exp(3t).$$

By multiplying both sides with  $\exp(3t)$ , we obtain the equation:

$$y'e^{3t} + 3ye^{3t} = te^{3t} + e^{-2t}e^{3t}.$$

Clearly, we observe that the left hand side is the derivative after product rule for  $ye^{3t}$  and the right hand side can be simplified as:

$$\frac{d}{dt}[ye^{3t}] = te^{3t} + e^t.$$

Therefore, we have turned this into an integration problem, so we do the respective integrations, giving us that:

$$\begin{aligned} ye^{3t} &= \int te^{3t} dt + \int e^t dt \\ &= \frac{te^{3t}}{3} - \int \frac{1}{3}e^{3t} dt + e^t + C \\ &= \frac{te^{3t}}{3} - \frac{e^{3t}}{9} + e^t + C. \end{aligned}$$

Eventually, we divide both sides by  $e^{3t}$  to obtain that:

$$y(t) = \boxed{\frac{t}{3} - \frac{1}{9} + e^{-2t} + Ce^{-3t}}.$$

2. Given an initial value problem:

$$\begin{cases} \frac{dy}{dt} - \frac{3}{2}y = 3t + 2e^t, \\ y(0) = y_0. \end{cases}$$

- Find the integrating factor  $\mu(t)$ .
- Solve for the particular solution for the initial value problem.
- Discuss the behavior of the solution as  $t \rightarrow \infty$  for different cases of  $y_0$ .

**Solution:**

(a) As instructed, we look for the integrating factor as:

$$\mu(t) = \exp\left(\int_0^t -\frac{3}{2}ds\right) = \exp\left(-\frac{3}{2}t\right).$$

(b) With the integrating factor, we multiply both sides by  $\mu(t)$  to obtain that:

$$y'e^{-3t/2} - \frac{3}{2}ye^{-3t/2} = 3te^{-3t/2} + 2e^te^{-3t/2}.$$

Clearly, we observe that the left hand side is the derivative after product rule for  $ye^{-3t/2}$  and the right hand side can be simplified as:

$$\frac{d}{dt} [ye^{-3t/2}] = 3te^{-3t/2} + 2e^{-t/2}.$$

Therefore, we have turned this into an integration problem, so we do the respective integrations, giving us that:

$$\begin{aligned} ye^{-3t/2} &= \int 3te^{-3t/2}dt + \int 2e^{-t/2}dt \\ &= -2te^{-3t/2} + 2 \int e^{-3t/2}dt - 4e^{-t/2} + C \\ &= -2te^{-3t/2} - \frac{4}{3}e^{-3t/2} - 4e^{-t/2} + C. \end{aligned}$$

Then, we divide both sides by  $e^{-3t/2}$  to get the general solution:

$$y(t) = -2t - \frac{4}{3} - 4e^t + Ce^{3t/2}.$$

Given the initial condition, we have that:

$$y_0 = 0 - \frac{4}{3} - 4 + C,$$

which implies  $C = 16/3 + y_0$ , leading to the particular solution that:

$$y(t) = -2t - \frac{4}{3} - 4e^t + \left(\frac{16}{3} + y_0\right)e^{3t/2}.$$

(c) We observe that:

$$\lim_{t \rightarrow \infty} y(t) = \lim_{t \rightarrow \infty} \left[-2t - \frac{4}{3} - 4e^t + \left(\frac{16}{3} + y_0\right)e^{3t/2}\right].$$

Note that the important terms are  $e^t$  and  $e^{3t/2}$ , we need to care the critical value  $-16/3$ :

- when  $y_0 > -16/3$ ,  $y(t) \rightarrow \infty$  when  $t \rightarrow \infty$ ,
- when  $y_0 \leq -16/3$ ,  $y(t) \rightarrow -\infty$  when  $t \rightarrow \infty$ .

3. Suppose  $f(x)$  is non-zero, let an initial value problem be:

$$\begin{cases} \frac{1-y}{x} \cdot \frac{dy}{dx} = \frac{f(x)}{1+y}, \\ y(0) = 0. \end{cases}$$

(a) Show that the differential equation is **not** linear.

For the next two questions, suppose  $f(x) = \tan x$ .

(b) State, without justification, the open interval(s) in which  $f(x)$  is continuous.

(c)\* Show that there exists some  $\delta > 0$  such that there exists a unique solution  $y(x)$  for  $x \in (-\delta, \delta)$ .

Now, suppose that  $f(x)$  is some function, **not** necessarily continuous.

(d)\*\* Suppose that the condition in (c) does **not** hold, give three examples in which  $f(x)$  could be.

**Solution:**

(a) *Proof.* We can write the equation as:

$$F(x, y, y') := y' - \frac{xf(x)}{(y+1)(y-1)} = 0,$$

Note that:

$$F(x, (y+1), (y+1)') = y' - \frac{xf(x)}{(y+2)y} \neq 1,$$

so the function is non-linear. □

(b) Here, we should consider that:

$$f(x) = \tan x = \frac{\sin x}{\cos x},$$

so the discontinuities are at when  $\cos x = 0$ , that is:

$$x \in \left\{ \frac{(2k+1)\pi}{2} : k \in \mathbb{Z} \right\}.$$

Hence, we have the intervals in which  $f(x)$  being continuous as:

$$\left\{ \left( \frac{(2k-1)\pi}{2}, \frac{(2k+1)\pi}{2} \right) : k \in \mathbb{Z} \right\}.$$

(c) *Proof.* Here, we want to write our equation in the standard form and obtain that:

$$\begin{aligned} y' &:= f(t, y) = \frac{x \tan x}{(y+1)(y-1)}, \\ \frac{\partial f(t, y)}{\partial y} &= \frac{x \tan x \cdot 2y}{(y^2-1)^2}. \end{aligned}$$

Clear, we note the discontinuities of  $y$  at  $y = \pm 1$ , and  $x$  demonstrated as above, thus we can form a rectangle  $Q = (-\pi/2, \pi/2) \times (-1, 1)$  in which the initial condition  $(0, 0) \in Q$  and  $f(t, y)$  with  $\partial_y f(t, y)$  are continuous on the interval. By the *existence and uniqueness theorem for non-linear case*, we know that there exists some  $\delta$  such that there is a unique solution for  $-\delta < x < \delta$ . □

(d) If the condition in (c) does not hold, by contraposition, this implies that continuity must fail, i.e.,  $xf(x)$  must be discontinuous at  $x = 0$ . Hence, some examples could be:

$$f(x) = \frac{1}{x^2}, \text{ or } \log x, \text{ or } \csc x, \text{ or } \chi_{\{0\}}(x) \text{ etc.}$$

4. An autonomous differential equation is given as follows:

$$\frac{dy}{dt} = 4t^3 - 12t^2 + 9t - 2 \quad \text{where } t \geq 0 \text{ and } y \geq 0.$$

Draw a phase portrait and sketch a few solutions with different initial conditions.

**Solution:**

Recall from Pre-Calculus (or Algebra) the following *Rational root test*:

**Theorem 4.1: Rational Root Test.** Let the polynomial:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0$$

have integer coefficients  $a_i \in \mathbb{Z}$  and  $a_0, a_n \neq 0$ , then any rational root  $r = p/q$  such that  $p, q \in \mathbb{Z}$  and  $\gcd(p, q) = 1$  satisfies that  $p|a_0$  and  $q|a_n$ .  $\lrcorner$

From the theorem, we can note that if the equation has a rational root, it must be one of:

$$r = \pm 1, \pm 2, \pm \frac{1}{2}, \pm \frac{1}{4}.$$

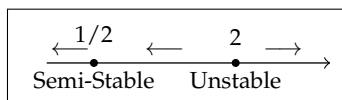
By plugging in, one should notice that  $y = 2$  is a root (one might also notice  $1/2$  is a root as well, but we will get the step slowly), so we can apply the long division (dividing  $y - 2$ ) to obtain that:

$$\frac{4y^3 - 12y^2 + 9y - 2}{y - 2} = 4y^2 - 4y + 1.$$

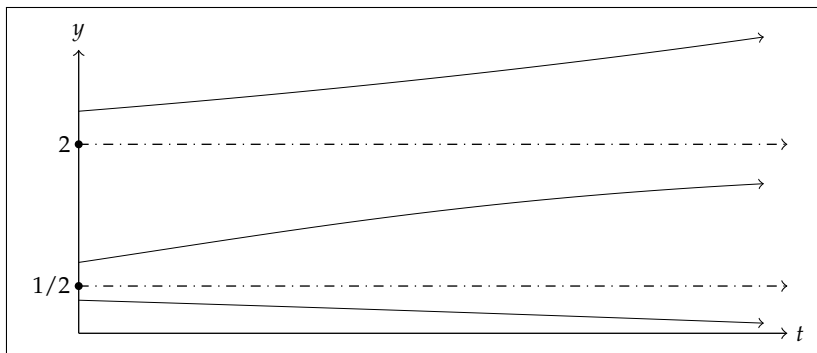
Clear, we can notice that the right hand side is a perfect square (else, you could use the quadratic formula) that:

$$4y^2 - 4y + 1 = (2y - 1)^2.$$

Thus, we now know that the roots are 2 and  $1/2$  (multiplicity 2). Hence, the phase portrait is:



Correspondingly, we can sketch a few solutions (not necessarily in scale):



Note that for the **Theorem 4.1**, it can also be generalized into the following manner (in ring theory):

**Theorem 4.2: Rational Root Theorem.** Let  $R$  be UFD, and polynomial:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in R[x],$$

and let  $r = p/q \in K(R)$  be a root of  $f$  with  $p, q \in R$  and  $\gcd(p, q) = 1$ , then  $p|a_0$  and  $q|a_n$ .  $\lrcorner$

The proof of **Theorem 4.1** and/or **Theorem 4.2** are left as exercises to diligent readers. *Moreover, capable readers should attempt to prove that a polynomial of degree 3 with integer coefficients must have at least one rational root.*

- 5.\* Determine if the following differential equation is exact. If not, find the integrating factor to make it exact. Then, solve for its general solution:

$$y'(x) = e^{2x} + y(x) - 1.$$

**Solution:**

First, we write the equation in the general form:

$$\frac{dy}{dx} + (1 - e^{2x} - y) = 0.$$

Now, we take the partial derivatives to obtain that:

$$\frac{\partial}{\partial y}[1 - e^{2x} - y] = -1,$$

$$\frac{\partial}{\partial x}[1] = 0.$$

Notice that the mixed partials are not the same, the equation is not exact.

Here, we choose the integrating factor as:

$$\begin{aligned}\mu(x) &= \exp\left(\int_0^x \frac{\frac{\partial}{\partial y}[1 - e^{2s} - y] - \frac{\partial}{\partial s}[1]}{1} ds\right) \\ &= \exp\left(\int_0^x -ds\right) = \exp(-x).\end{aligned}$$

Therefore, our equation becomes:

$$(e^{-x}) \frac{dy}{dx} + (e^{-x} - e^x - ye^{-x}) = 0.$$

After multiplying the integrating factor, it would be exact. *We leave the repetitive check as an exercise to the readers.*

Now, we can integrate to find the solution with a  $h(y)$  as function:

$$\varphi(x, y) = \int (e^{-x} - e^x - ye^{-x}) dx = -e^{-x} - e^x + ye^{-x} + h(y).$$

By taking the partial derivative with respect to  $y$ , we have:

$$\partial_y \varphi(x, y) = e^{-x} + h'(y),$$

which leads to the conclusion that  $h'(y) = 0$  so  $h(y) = C$ .

Then, we can conclude that the solution is now:

$$\varphi(x, y) = -e^{-x} - e^x + ye^{-x} + C = 0,$$

which is equivalently:

$$y(x) = \boxed{\tilde{C}e^x + 1 + e^{2x}}.$$

6. Let a differential equation be defined as:

$$\frac{dy}{dt} = t - y \text{ and } y(0) = 0.$$

Use Euler's Method with step size  $h = 1$  to approximate  $y(5)$ .

**Solution:**

With  $y(0) = 0$ , we have  $y'(0) = 0 - 0 = 0$ . We do the following steps:

- We approximate  $y(1) \approx y(0) + 1 \cdot y'(0) = 0$ , then we have  $y'(1) \approx 1 - 0 = 1$ .
- We approximate  $y(2) \approx y(1) + 1 \cdot y'(1) \approx 1$ , then we have  $y'(2) \approx 2 - 1 = 1$ .
- We approximate  $y(3) \approx y(2) + 1 \cdot y'(2) \approx 2$ , then we have  $y'(3) \approx 3 - 2 = 1$ .
- We approximate  $y(4) \approx y(3) + 1 \cdot y'(3) \approx 3$ , then we have  $y'(4) \approx 4 - 3 = 1$ .
- We approximate  $y(5) \approx y(4) + 1 \cdot y'(4) \approx 4$ .

Then, we have approximated that:

$$y(5) \approx \boxed{4}.$$

7. Solve the following second order differential equations for  $y = y(x)$ :

(a)  $y'' + y' - 132 = 0.$

(b)  $y'' - 4y' = -4.$

(c)  $y'' - 2y + 3 = 0.$

**Solution:**

(a) We find the characteristic polynomial as  $r^2 + r - 132 = 0$ , which can be trivially factorized into:

$$(r - 11)(r + 12) = 0,$$

so with roots  $r_1 = 11$  and  $r_2 = -12$ , we have the general solution as:

$$y(x) = \boxed{C_1 e^{11x} + C_2 e^{-12x}}.$$

(b) We turn the equation to the standard form:

$$y'' - 4y' + 4 = 0.$$

We find the characteristic polynomial as  $r^2 - 4r + 4 = 0$ , which can be immediately factorized into:

$$(r - 2)^2 = 0,$$

so with roots  $r_1 = r_2 = 2$  (repeated roots), we have the general solution as:

$$y(x) = \boxed{C_1 e^{2x} + C_2 x e^{2x}}.$$

(c) We find the characteristic polynomial as  $r^2 - 2r + 3 = 0$ , which the quadratic formula gives:

$$r = \frac{2 \pm \sqrt{2^2 - 4 \times 3}}{2} = 1 \pm i\sqrt{2}$$

so with roots  $r_1 = 1 + i\sqrt{2}$  and  $r_2 = 1 - i\sqrt{2}$ , we would have the solution:

$$y(x) = C_1 e^{(1+i\sqrt{2})x} + C_2 e^{(1-i\sqrt{2})x}.$$

To obtain real solution, we apply Euler's identity:

$$y_1(x) = e^x (\cos(\sqrt{2}x) - i \sin(\sqrt{2}x)) \text{ and } y_2(x) = e^x (\cos(-\sqrt{2}x) - i \sin(-\sqrt{2}x)).$$

By the *principle of superposition*, we can linearly combine the solutions to be different solutions, so we have:

$$\tilde{y}_1(x) = \frac{1}{2}(y_1 + y_2) = e^x \cos(\sqrt{2}x),$$

$$\tilde{y}_2(x) = \frac{1}{2}(y_2 - y_1) = e^x \sin(\sqrt{2}x).$$

One can verify that  $\tilde{y}_1$  and  $\tilde{y}_2$  are linearly independent by taking Wronskian, *i.e.*:

$$\begin{aligned} W[\tilde{y}_1, \tilde{y}_2] &= \det \begin{pmatrix} e^x \cos(\sqrt{2}x) & e^x \sin(\sqrt{2}x) \\ e^x \cos(\sqrt{2}x) - \sqrt{2}e^x \sin(\sqrt{2}x) & e^x \sin(\sqrt{2}x) + \sqrt{2}e^x \cos(\sqrt{2}x) \end{pmatrix} \\ &= \sqrt{2}e^{2x} \cos^2(\sqrt{2}x) + \sqrt{2}e^{2x} \sin^2(\sqrt{2}x) = \sqrt{2}e^{2x} \neq 0. \end{aligned}$$

Now, they are linearly independent, so we have the general solution as:

$$y(x) = \boxed{C_1 e^x \cos(\sqrt{2}x) + C_2 e^x \sin(\sqrt{2}x)}.$$

- 8.\*\* The following system of partial differential equations portraits the propagation of waves on a segment of the 1-dimensional string of length  $L$ , the displacement of string at  $x \in [0, L]$  at time  $t \in [0, \infty)$  is described as the function  $u = u(x, t)$ :

$$\left\{ \begin{array}{ll} \text{Differential Equation:} & \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad \text{where } x \in (0, L) \text{ and } t \in [0, \infty); \\ \text{Initial Conditions:} & u(x, 0) = \sin\left(\frac{2\pi x}{L}\right), \\ & \frac{\partial u}{\partial t}(x, 0) = \sin\left(\frac{5\pi x}{L}\right), \quad \text{where } x \in [0, L]; \\ \text{Boundary Conditions:} & u(0, t) = u(L, t) = 0, \quad \text{where } t \in [0, \infty); \end{array} \right.$$

where  $c$  is a constant and  $g(x)$  has “good” behavior. Apply the method of separation, i.e.,  $u(x, t) = v(x) \cdot w(t)$ , and attempt to obtain a general solution that is non-trivial.

Hint: Use the fact that  $\{\sin(n\pi x/L), \cos(n\pi x/L)\}_{n \in \mathbb{Z}^+}$  forms an orthonormal basis.

**Solution:**

With the method of separation, we insert the separations back to the system of equation to obtain:

$$v(x)w''(t) = c^2 v''(x)w(t).$$

Now, we apply the separation and set the common ratio to be  $\lambda$ :

$$\frac{1}{c^2} \cdot \frac{v''(x)}{v(x)} = \frac{w''(t)}{w(t)} = \lambda.$$

Reformatting the boundary condition gives use the following initial value problem:

$$\left\{ \begin{array}{l} v''(x) - \lambda v(x) = 0, \\ v(0) = v(L) = 0. \end{array} \right.$$

As a second order linear ordinary differential equation, we discuss all following cases:

- If  $\lambda = 0$ , then  $v(x) = a + Bx$  and by the initial condition,  $A = B = 0$ , which gives the trivial solution, i.e.,  $v(x) = 0$ ;
- If  $\lambda = \mu^2 > 0$ , then we have  $v(x) = Ae^{-\mu x} + Be^{\mu x}$  and again giving that  $A = B = 0$ , or the trivial solution;
- Eventually, if  $\lambda = -\mu^2 < 0$ , then we have the solution as:

$$v(x) = A \sin(\mu x) + B \cos(\mu x),$$

and the initial conditions gives us that:

$$\left\{ \begin{array}{l} v(0) = B = 0, \\ v(L) = A \sin(\mu L) + B \cos(\mu L) = 0, \end{array} \right.$$

where  $A$  is arbitrary,  $B = 0$ , and  $\mu L = m\pi$  positive integer  $m$ .

Overall, the only non-trivial solution would be:

$$v_m(x) = A \sin(\mu_m x), \text{ where } \mu_m = \frac{m\pi}{L}.$$

Eventually, by inserting back  $\lambda = -\mu_m^2$ , we have  $\lambda = -m^2\pi^2/L^2$ , giving the solution to  $w_m(t)$ , another second order linear ordinary differential equation, as:

$$w_m(t) = C \cos(\mu_m ct) + D \sin(\mu_m ct), \text{ with } C, D \in \mathbb{R}.$$

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By the *principle of superposition*, we can have our solution in the form:

$$u(x, t) = \sum_{m=1}^{\infty} [a_m \cos(\mu_m ct) + b_m \sin(\mu_m ct)] \sin(\mu_m x),$$

where our coefficients  $a_m$  and  $b_m$  have to be chosen to satisfy the initial conditions for  $x \in [0, L]$ :

$$u(x, 0) = \sum_{m=1}^{\infty} a_m \sin(\mu_m x) = \sin\left(\frac{2\pi x}{L}\right),$$

$$\frac{\partial u}{\partial t}(x, 0) = \sum_{m=1}^{\infty} c\mu_m b_m \sin(\mu_m x) = \sin\left(\frac{5\pi x}{L}\right).$$

Since we are hinted that  $\{\sin(n\pi x/L), \cos(n\pi x/L)\}_{n \in \mathbb{Z}^+}$  forms an orthonormal basis, we now know that except for the following:

$$a_2 = 1 \text{ and } b_5 = 1,$$

all the other coefficients are zero, so we have:

$$u(x, t) = \cos\left(\frac{2\pi ct}{L}\right) \sin\left(\frac{2\pi x}{L}\right) + \sin\left(\frac{5\pi ct}{L}\right) \sin\left(\frac{5\pi x}{L}\right).$$

9. Given a differential equation for  $y = y(t)$  being:

$$t^3 y'' + t y' - y = 0.$$

- (a) Verify that  $y_1(t) = t$  is a solution to the differential equation.  
 (b)\* Find the full set of solutions using reduction of order.  
 (c) Show that the set of solutions from part (b) is linearly independent.

**Solution:**

(a) *Proof.* We note that the left hand side is:

$$t^3 y_1'' + t y_1' - y_1 = t^3 \cdot 0 + t \cdot 1 - t = t - t = 0.$$

Hence  $y_1(t) = t$  is a solution to the differential equation.  $\square$

(b) By reduction of order, we assume that the second solution is  $y_2(t) = tu(t)$ , then we plug  $y_2(t)$  into the equation to get:

$$2t^3 u'(t) + t^4 u''(t) + tu(t) + t^2 u'(t) = t^4 u''(t) + (2t^3 + t^2)u'(t) = 0.$$

Here, we let  $\omega(t) = u'(t)$  to get a first order differential equation:

$$t^2 \omega'(t) = (-2t - 1)\omega(t).$$

Clearly, this is separable, and we get that:

$$\frac{\omega'(t)}{\omega(t)} = -\frac{2t+1}{t^2} = -\frac{2}{t} - \frac{1}{t^2},$$

which by integration, we have obtained that:

$$\log(\omega(t)) = -2\log t + \frac{1}{t} + C.$$

By taking exponentials on both sides, we have:

$$\omega(t) = \exp\left(-2\log t + \frac{1}{t} + C\right) = \tilde{C}e^{1/t} \cdot \frac{1}{t^2}.$$

Recall that we want  $u(t)$  instead of  $\omega(t)$ , so we have:

$$u(t) = \int \omega(t) dt = \tilde{C} \int e^{1/t} \cdot \frac{1}{t^2} dt = -\tilde{C}e^{1/t} + D.$$

By multiplying  $t$ , we obtain that:

$$y_2 = -\tilde{C}te^{1/t} + Dt,$$

where  $Dt$  is repetitive in  $y_1$ , so we get:

$$y(t) = \boxed{C_1 t + C_2 t e^{1/t}}.$$

(c) *Proof.* We calculate Wronskian as:

$$W[t, te^{1/t}] = \det \begin{pmatrix} t & te^{1/t} \\ 1 & e^{1/t} - \frac{e^{1/t}}{t} \end{pmatrix} = -e^{1/t} \neq 0,$$

hence the set of solutions is linearly independent.  $\square$

10.\*\* Given the following second order initial value problem:

$$\begin{cases} \frac{d^2 y}{dx^2} + \cos(1-x)y = x^2 - 2x + 1, \\ y(1) = 1, \\ \frac{dy}{dx}(1) = 0. \end{cases}$$

Prove that the solution  $y(x)$  is symmetric about  $x = 1$ , i.e., satisfying that  $y(x) = y(2-x)$ .

*Hint:* Consider the interval in which the solution is unique.

**Solution:**

Note that I deliberately messed up with all the messy functions. Not only haven't I found a solution to the system, Wolfram cannot have an elementary solution as well. Hence, we need to think, alternatively, on some theorems.

*Proof.* Here, we suppose that  $y(x)$  is a solution, and we want to show that  $y(2-x)$  is also a solution. First we note that we can think of taking the derivatives of  $y(2-x)$ , by the chain rule:

$$\begin{aligned} \frac{d}{dx}[y(2-x)] &= -y'(2-x), \\ \frac{d^2}{dx^2}[y(2-x)] &= y''(2-x). \end{aligned}$$

Now, if we plug in  $y(2-x)$  into the system of equations, we have:

- First, for the differential equation, we have:

$$\begin{aligned} \frac{d^2}{dx^2}[y(2-x)] + \cos(1-x)y(2-x) &= y''(2-x) + \cos(1-x)y(2-x) \\ &= y''(2-x) + \cos(1-(2-x))y(2-x) \\ &= y''(x) + \cos(1-x)y(x) \\ &= x^2 - 2x + 1 = (x-1)^2 = (1-x)^2 \\ &= ((2-x)-1)^2 = (2-x)^2 - 2(2-x) + 1. \end{aligned}$$

- For the initial conditions, we trivially have that:

$$y(1) = y(2-1) \text{ and } y'(1) = y'(2-1).$$

Hence, we have shown that  $y(2-x)$  is a solution if  $y(x)$  is a solution.

Again, we observe the original initial value problem that:

$$\cos(1-x) \text{ and } x^2 - 2x + 1 \text{ are continuous on } \mathbb{R}.$$

Therefore, by the *existence and uniqueness theorem for second order linear case*, there could be only one solution, which forces that:

$$y(x) = y(2-x),$$

so the solution is symmetric about  $x = 1$ , as desired. □