



Final Practices: Solutions

Differential Equations

Summer 2024

1. Solve the following initial value problem, represent your solution as a fundamental matrix:

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Solution:

Here, we first find the eigenvalues for the matrix, that is:

$$\det \begin{pmatrix} 1 - \lambda & -4 \\ 4 & -7 - \lambda \end{pmatrix} = 0.$$

Therefore, the polynomial is $(1 - \lambda)(-7 - \lambda) + 16 = (\lambda + 3)^2 = 0$, hence the eigenvalues is $\lambda_1 = \lambda_2 = -3$. Then, we look for the eigenvectors.

- For $\lambda_1 = -3$, we have $\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \xi_1 = \mathbf{0}$, which is $\xi_1 = x \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
- For $\lambda_2 = -3$, we have $\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \eta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, which is $\eta = \begin{pmatrix} x \\ x - 1/4 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/4 \end{pmatrix}$.

Hence, the general solution is:

$$\mathbf{x} = C_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \left(t e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-3t} \begin{pmatrix} 0 \\ -1/4 \end{pmatrix} \right).$$

By the initial condition, we have $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, so:

$$\mathbf{x}(0) = \begin{pmatrix} C_1 + 0 \\ C_1 - C_2/4 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Therefore, $C_1 = 3$ and $C_2 = 4$, so the particular solution is:

$$\mathbf{x}(t) = \boxed{\begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-3t} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} t e^{-3t}}.$$

2.* Let a system of differential equations be defined as follows, find its general solutions:

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 1 & 3 \\ 0 & 4 & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^3.$$

Solution:

Again, we first find the eigenvalues of the equation, *i.e.*:

$$\det \begin{pmatrix} 1-\lambda & 0 & 4 \\ 1 & 1-\lambda & 3 \\ 0 & 4 & 1-\lambda \end{pmatrix} = 0,$$

which is $(1-\lambda)^3 + 16 - 12(1-\lambda) = -\lambda^3 + 3\lambda^2 + 9\lambda + 5 = -(\lambda+1)^2(\lambda-5) = 0$.

Hence, the eigenvalues are $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 5$. Now, we look for eigenvectors.

- For $\lambda_1 = -1$, we have $\begin{pmatrix} 2 & 0 & 4 \\ 1 & 2 & 3 \\ 0 & 4 & 2 \end{pmatrix} \xi_1 = \mathbf{0}$, which is $x \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix}$.
- For $\lambda_2 = -1$, we have $\begin{pmatrix} 2 & 0 & 4 \\ 1 & 2 & 3 \\ 0 & 4 & 2 \end{pmatrix} \eta = \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix}$, which is $\eta = \begin{pmatrix} 4x \\ x+1 \\ -2x-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.
- For $\lambda_3 = 5$, we have $\begin{pmatrix} -4 & 0 & 4 \\ 1 & -4 & 3 \\ 0 & 4 & -4 \end{pmatrix} \xi_3 = \mathbf{0}$, which is $x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Hence, the solution is:

$$\mathbf{x} = \left[C_1 e^{-t} \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix} + C_2 \left(t e^{-t} \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix} + e^{-t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right) + C_3 e^{5t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right].$$

- 3.** Let $\text{Id} \in \mathcal{L}(\mathbb{R}^n)$ be the identity map in an n -dimensional Euclidean space, show that the following equality holds for matrix exponential:

$$\exp(\text{Id}) = e \cdot \text{Id}.$$

Hint: Consider the matrix exponential and the Taylor expansion of $\exp(x)$.

Solution:

Proof. Here, we first note that, by definition:

$$\text{Id}^k = \text{Id} \text{ for all } k \in \mathbb{N},$$

thus, we want to expand the matrix exponential as follows:

$$\begin{aligned}\exp(\text{Id}) &= \sum_{k=0}^{\infty} \frac{1}{k!} \text{Id}^k \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \text{Id} \\ &= \left(\sum_{k=0}^{\infty} \frac{1}{k!} \right) \text{Id}.\end{aligned}$$

Recall that the Taylor expansion of e^x at 0 is:

$$e^x \sim \sum_{k=0}^{\infty} \frac{1}{k!} e^0 (x - 0)^k = \sum_{k=0}^{\infty} \frac{1}{k!} x^k.$$

Evaluating the above equation at 1 gives that:

$$\sum_{k=0}^{\infty} \frac{1}{k!} = e^1 = e,$$

and hence, we have the matrix exponential as:

$$\exp(\text{Id}) = e \cdot \text{Id},$$

as desired. □

4. Let M be a square matrix, M is defined to be *nilpotent* if $M^k = 0$ for some positive integer k .

(a) Show that $N = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is nilpotent, then write down the result of $\exp(N)$.

Now, suppose that $N \in \mathcal{L}(\mathbb{R}^n)$ is a square matrix and is *nilpotent*.

(b)* If all the entries in N are rational, show that $\exp(N)$ has rational entries.

(c)** Suppose that $\text{Id}_n \in \mathcal{L}(\mathbb{R}^n)$ is the identity matrix, prove that $\text{Id}_n + N$ is invertible.

Hint: Use the differences of squares for matrices.

Solution:

(a) *proof of N is nilpotent.* Here, we want to do the matrix multiplication:

$$N^2 = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$N^3 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now, we have shown that $N^3 = 0$, or the zero matrix, hence N is nilpotent. □

Then, we want to calculate the matrix exponential, that is:

$$\exp(N) = \sum_{k=0}^{\infty} \frac{1}{k!} N^k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}.$$

(b) *Proof.* By the definition that N is nilpotent, we know that $N^m = 0$ for some finite positive integer m , hence, we can make the (countable) infinite sum into a finite sum:

$$\exp(N) = \sum_{k=0}^{\infty} \frac{1}{k!} N^k = \sum_{k=0}^m \frac{1}{k!} N^k,$$

thus all the entries are sum and non-zero divisions of rational numbers, while rational numbers are closed under addition and non-zero divisions, hence, all entries of $\exp(N)$ is rational. □

(c) *Proof.* Here, we recall the differences of squares still works when commutativity for multiplications fails, hence the we can still use it for matrix multiplication, namely, for all $m \in \mathbb{Z}^+$:

$$(\text{Id}_n + N) \cdot (\text{Id}_n - N) \cdot (\text{Id}_n + N^2) \cdots (\text{Id}_n + N^{2^m}) = \text{Id}_n - N^{2^{m+1}}$$

Since N is *nilpotent*, this implies that we have some k such that $N^\ell = 0$ for all $\ell \geq k$. Meanwhile, note that $2^\ell \geq \ell$ for all positive integer ℓ . (This can be proven by induction.) Therefore, we select $m + 1 \geq k$ so that $N^{2^{m+1}} = 0$, and we have:

$$(\text{Id}_n + N) \cdot [(\text{Id}_n - N) \cdot (\text{Id}_n + N^2) \cdots (\text{Id}_n + N^{2^m})] = \text{Id}_n,$$

thus $\text{Id}_n + N$ is invertible. □

Note that the elements of all n -by- n matrices can be considered as a *ring*, while *nilpotent* can be defined more generally for *rings*. We invite capable readers to investigate more properties of *nilpotent* elements of *rings* in the discipline of *Modern Algebra*.

5. Suppose a matrix $M \in \mathcal{L}(\mathbb{R}^2)$ is a *rotational matrix* by an angle θ (counter-clockwise), then:

$$M = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

(a)* Show that $M^T = M^{-1}$.

(b)** Let $\theta = 2\pi/k$ be fixed, where k is an integer. Find the least positive integer n such that $M^n = \text{Id}_2$. Here, n is called the *order* of M .

Hint: Consider the rotational matrix geometrically, rather than arithmetically.

(c)** Let $\theta = \pi/2$, calculate the matrix exponential $\exp(M)$.

Hint: Consider the *order* of M and the Taylor series of e^x , e^{-x} , $\sin x$ and $\cos x$.

Solution:

(a) *Proof.* Here, we recall the method of inverting a matrix:

$$M^{-1} = \frac{1}{\det M} \begin{pmatrix} \cos \theta & -(-\sin \theta) \\ -\sin \theta & \cos \theta \end{pmatrix} = \frac{1}{\cos^2 \theta + \sin^2 \theta} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} = M^T. \quad \square$$

(b) Look, we want to analyze this geometrically, if $\theta = 2\pi/k$, then that implies that M is a counter-clockwise rotation of $2\pi/k$, and since a full revolution is 2π , this implies a rotation of k times will make restore to the original vector, i.e., $M^k = \text{Id}_2$. Moreover, for any positive integer less than k , we cannot rotate back to 2π , which implies that the order of M is $\boxed{2}$.

(c) Here, we construct the matrix exponential, note that the order of M is 4, we have:

$$\exp(M) = \sum_{k=0}^{\infty} \frac{1}{k!} M^k.$$

Here, we want to consider each entry respectively, since each entry is finite and since M has order 4, the absolute value of the sum of the entries must be finite, so each entry converges *absolutely*, hence we are free to change the order of the sum, so we have:

$$\exp(M) = \sum_{k=0}^{\infty} \frac{1}{(4k+1)!} M + \sum_{k=0}^{\infty} \frac{1}{(4k+2)!} M^2 + \sum_{k=0}^{\infty} \frac{1}{(4k+3)!} M^3 + \sum_{k=0}^{\infty} \frac{1}{(4k)!} \text{Id}.$$

For the 4 sums of factorials, we note that the Taylor series of e^x , e^{-x} , $\sin x$ and $\cos x$ at 0 evaluated at $x = 1$ are, respectively:

$$\begin{aligned} e^1 &= +\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots \\ e^{-1} &= +\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \dots \\ \sin 1 &= \quad +\frac{1}{1!} \quad \quad -\frac{1}{3!} \quad \quad +\frac{1}{5!} - \dots \\ \cos 1 &= +\frac{1}{0!} \quad \quad -\frac{1}{2!} \quad \quad +\frac{1}{4!} \quad \quad - \dots \end{aligned}$$

Since the first series converges, we know that the later three series converges *absolutely*, so we are free to move around terms.

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From the expressions, by columns, we can observe that:

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{1}{(4k+1)!} &= \frac{e^1 - e^{-1}}{4} + \frac{\sin 1}{2}, & \sum_{k=0}^{\infty} \frac{1}{(4k+2)!} &= \frac{e^1 + e^{-1}}{4} - \frac{\cos 1}{2}, \\ \sum_{k=0}^{\infty} \frac{1}{(4k+3)!} &= \frac{e^1 - e^{-1}}{4} - \frac{\sin 1}{2}, & \sum_{k=0}^{\infty} \frac{1}{(4k+4)!} &= \frac{e^1 + e^{-1}}{4} + \frac{\sin 1}{2}.\end{aligned}$$

Now, we shall also evaluate the matrices generated by M , that is:

$$\begin{aligned}M &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & M^2 &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \\ M^3 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & M^4 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.\end{aligned}$$

Therefore, considering the four entries $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have:

$$\begin{aligned}a &= -\frac{e+1/e}{4} + \frac{\cos 1}{2} + \frac{e+1/e}{4} + \frac{\sin 1}{2} = \frac{\cos 1 + \sin 1}{2}, \\ b &= -\frac{e-1/e}{4} - \frac{\sin 1}{2} + \frac{e-1/e}{4} - \frac{\sin 1}{2} = -2 \sin 1, \\ c &= \frac{e-1/e}{4} + \frac{\sin 1}{2} - \frac{e-1/e}{4} + \frac{\sin 1}{2} = 2 \sin 1, \\ d &= -\frac{e+1/e}{4} + \frac{\cos 1}{2} + \frac{e+1/e}{4} + \frac{\sin 1}{2} = \frac{\cos 1 + \sin 1}{2}.\end{aligned}$$

Therefore, the matrix exponential is:

$$\exp(M) = \begin{pmatrix} \frac{\cos 1 + \sin 1}{2} & -2 \sin 1 \\ 2 \sin 1 & \frac{\cos 1 + \sin 1}{2} \end{pmatrix}.$$

In particular, mathematicians has considered the *rotation* and *flipping* of regular polygons as the *dihedral groups*, where symmetries and combinatorics play an important role. Please think of ways you may “manipulate” a polygon such that the polygon looks the same.

6. Let a non-linear system be:

$$\frac{dx}{dt} = x - y^2 \text{ and } \frac{dy}{dt} = x + x^2 - 2y.$$

Verify that $(0,0)$ is a critical point and classify its type and stability.

Solution:

proof that $(0,0)$ is critical point. The verification of $(0,0)$ being a critical point is trivial. We check that dx/dt and dy/dt evaluated at $(0,0)$ are:

$$\left. \frac{dx}{dt} \right|_{(0,0)} = 0 \text{ and } \left. \frac{dy}{dt} \right|_{(0,0)} = 0,$$

and hence $(0,0)$ is a critical point. □

In particular, denoting $\mathbf{x} = (x, y)$, we verify the linear proportion as:

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} \mathbf{x},$$

and we note that the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -2$, and by:

$$\lambda_2 < 0 < \lambda_1,$$

we know that we have a unstable saddle point at $(0,0)$.

7. Let a system of non-linear differential equations be defined as follows:

$$\begin{cases} x' = 2x + 3y^2, \\ y' = x + 4y^2. \end{cases}$$

Find all equilibrium(s) and classify their stability locally.

Solution:

Here, we note that the equilibrium(s) is achieved if and only if $x' = y' = 0$, that is:

$$\begin{cases} 2x + 3y^2 = 0, \\ x + 4y^2 = 0. \end{cases}$$

In particular, we consider $z = y^2$, so we have a system of linear equations, that is:

$$\begin{cases} 2x + 3z = 0, \\ x + 4z = 0. \end{cases}$$

Meanwhile, the above system simplifies to $x = y = 0$, hence the only equilibrium is at $(x, y) = (0, 0)$.

Then, we consider the system locally, denoting $\mathbf{x} = (x, y)$, that is:

$$\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{x},$$

where the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 0$. Note that one eigenvalue is zero and the other is positive, then the critical point is unstable.

8. Let a system of equations for $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ be:

$$\mathbf{x}' = \begin{pmatrix} F(\mathbf{x}) \\ F(\mathbf{x}) \end{pmatrix}$$

Suppose that $F(x_1, x_2) = \sin x_1 + \csc(3x_2)$.

- (a) Find the set of all equilibrium(s) for \mathbf{x} .
- (b) Find the set in which the equilibrium(s) is locally linear.

Now, $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ is not necessarily well-behaved.

- (c)** Construct a function F such that \mathbf{x} has a equilibrium that is not locally linear.

Hint: Consider the condition in which a non-linear system is locally linear.

Solution:

- (a) Here, we note that the equilibrium is when $F(\mathbf{x}) = 0$, i.e., $\sin x_1 + \csc(3x_2) = 0$. Here, we note that the image of $\sin x_1$ is $[-1, 1]$ and the image of $\sec(3x_2)$ is $(-\infty, -1] \cup [1, \infty)$, this implies that $\sin x_1 + \sec(3x_2)$ is zero only if $\sin x_1 = \pm 1$ and $\sec(3x_2) = \mp 1$, correspondingly.

First, we consider the set in which x_1 is $+1$, that is:

$$\left\{ \frac{(4k+1)\pi}{2} : k \in \mathbb{Z} \right\}.$$

Correspondingly, we consider the set in which x_2 is -1 , that is:

$$\left\{ \frac{(4k+3)\pi}{6} : k \in \mathbb{Z} \right\}.$$

Then, we consider the set in which x_1 is -1 , that is:

$$\left\{ \frac{(4k+3)\pi}{2} : k \in \mathbb{Z} \right\}.$$

Likewise, we consider the set in which x_2 is $+1$, that is:

$$\left\{ \frac{(4k+1)\pi}{6} : k \in \mathbb{Z} \right\}.$$

Therefore, set theoretically, we have the set of all equilibriums as:

$$\left\{ \frac{(4k+1)\pi}{2} : k \in \mathbb{Z} \right\} \times \left\{ \frac{(4k+3)\pi}{6} : k \in \mathbb{Z} \right\} \cup \left\{ \frac{(4k+3)\pi}{2} : k \in \mathbb{Z} \right\} \times \left\{ \frac{(4k+1)\pi}{6} : k \in \mathbb{Z} \right\}.$$

- (b) Note that $\sin x_1$ is (twice) differentiable over the entire domain \mathbb{R} and $\csc(3x_2)$ is (twice) differentiable on all neighborhoods when $\csc(3x_2)$ is ∓ 1 , hence the partial derivatives of $F(\mathbf{x})$ with respect to x_1 or x_2 are (twice) differentiable on the neighborhood on all equilibriums, hence the set in which the equilibrium(s) is locally linearly is the same from part (a), namely:

$$\left\{ \frac{(4k+1)\pi}{2} : k \in \mathbb{Z} \right\} \times \left\{ \frac{(4k+3)\pi}{6} : k \in \mathbb{Z} \right\} \cup \left\{ \frac{(4k+3)\pi}{2} : k \in \mathbb{Z} \right\} \times \left\{ \frac{(4k+1)\pi}{6} : k \in \mathbb{Z} \right\}.$$

- (c) Clearly, we must enforce that $F(\mathbf{x})$ is not twice differentiable with some partial derivatives near the equilibrium point(s). One trivial example could be using the absolute value, such as $F(\mathbf{x}) = |x_1| + |x_2|$, where $(0,0)$ is a equilibrium but it is not differentiable.

For capable readers, we invite them to look for more functions, such as the Weierstrass Function, a continuous function that is *nowhere* differentiable:

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \cos(3^k x).$$

9. Let a system of (x, y) be functions of variable t , and they have the following relationship:

$$x' = (1 + x) \sin y \text{ and } y' = 1 - x - \cos y.$$

- (a) Identify the corresponding linear system.
 (b) Evaluate the stability for the equilibrium at $(0, 0)$ by showing it is locally linear.

Solution:

(a) Here, since we can write:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} (1+x) \sin y \\ 1 - \cos y \end{pmatrix},$$

this implies that the linear system is:

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}.$$

(b) $(0, 0)$ is *locally linear*. We find the Jacobian Matrix, that is:

$$\mathbf{J} = \begin{pmatrix} \sin y & (1+x) \cos y \\ -1 & \sin y \end{pmatrix}.$$

As we evaluate \mathbf{J} at $(0, 0)$ and take its determinant, we have:

$$\det(\mathbf{J}|_{(0,0)}) = \det \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 1 \neq 0.$$

Hence, the $(0, 0)$ is locally linear. □

Note that we have found the linear system in part (a), whose eigenvalues are $\lambda_1 = \lambda_2 = 0$. Since $x' = 0$, it indicates that x is a constant, whereas for $y' = -x$ indicates that it will be a unstable almost everywhere for all neighborhoods of $(0, 0)$.

In particular, readers could illustrate the “slope field” for the linear system in (a), and they should notice that except for $x = 0$ being entirely stable, all other trajectory would shift vertically at a constant rate. However, the line $x = 0$ will always be insignificant enough (having *Lebesgue measure* 0), hence we claim that it is unstable almost everywhere. For interested readers, please explore *Lebesgue measure* as a way to determine how large a subset is in Euclidean space.

10.** Let a locally linear system be defined as:

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix} \mathbf{x} + \mathbf{f}(\mathbf{x}),$$

where $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a vector-valued function. Find the necessary condition(s) in which the equilibrium(s) have a stable *center* in linear system. Then, state the stability and type (if possible).

Hint: Consider the solution for the linear case or matrix exponential.

Solution:

Without loss of generality, we assume that the system of \mathbf{x} has equilibrium(s), else the statement is vacuously true. Now, we start to evaluate the additional conditions with such assumption:

- (i) Note that the system needs to be locally linearly, *i.e.*, we must have $\mathbf{f}(\mathbf{x})$ being twice differentiable with respect to partial derivatives.
- (ii) Moreover, we need to worry about the linear system to have a *stable center*, that is:

$$\mathbf{x}' = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix} \mathbf{x}.$$

Note that the eigenvalues would be the solutions to $(\lambda - r)^2 + \mu^2 = 0$, that is $r = \lambda \pm i\mu$, which is a pair of complex conjugates. Here, in to be stable, we want $\lambda \leq 0$, and for center, this forces $\lambda = 0$.

Note that even the linear system is a stable center, the stability of the non-linear system is *indeterminate*, and the type is *center or spiral point*.