



Midterm 2 Practices

Differential Equations

Summer 2024

Instructions:

The set of questions serves as PILOT practices to midterm 2 for the Summer 2024 term of AS.110.302 Differential Equations and Applications at Johns Hopkins University.

- Questions marked with asterisk(s) are more challenging, but still accessible, compared to the others.
- The questions are designed to be done without notes and calculators.
- Solutions to the questions will be discussed during the PILOT review session.

1. Solve the general solution for $y = y(t)$ to the following second order non-homogeneous ODEs.

(a)
$$y'' + 2y' + y = e^t.$$

(b)
$$y'' + y = \tan t.$$

2. Solve for the general solution to the following higher order ODE.

(a)
$$4 \frac{d^4 y}{dx^4} - 24 \frac{d^3 y}{dx^3} + 45 \frac{d^2 y}{dx^2} - 29 \frac{dy}{dx} + 6y = 0.$$

(b)**
$$\frac{d^4 y}{dx^4} + y = 0.$$

Hint: Consider the 8-th root of unity, i.e., ζ_8 , and verify which roots satisfies the polynomial.

3. Let a third order differential equation be as follows:

$$\ell[y(t)] = y^{(3)}(t) + 3y''(t) + 3y'(t) + y(t).$$

Let $\ell[y(t)] = 0$ be trivial initially.

(a) Find the set of all linearly independent solutions.

Then, assume that $\ell[y(t)]$ is non-trivial.

(b) Find the particular solution to $\ell[y(t)] = \sin t$.

(c) Find the particular solution to $\ell[y(t)] = e^t$.

(d)* Suppose that $\ell[y_1(t)] = f(t)$ and $\ell[y_2(t)] = g(t)$ where $f(t)$ and $g(t)$ are "good" functions.

Find an expression to $y_3(t)$ such that $\ell[y_3(t)] = f(t) + g(t)$.

4. Show the following Laplace transformation by definition.

(a)
$$\mathcal{L}\{\sin(at)\} = \frac{a}{a^2 + s^2}.$$

(b)*
$$\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}.$$

5. Given the following the results after Laplace transformation $F(s) = \mathcal{L}\{f(t)\}$, find each $f(t)$ prior to the Laplace transformation.

(a)
$$F(s) = \frac{2s^2 + 4}{s^3 + 4s}.$$

(b)*
$$F(s) = \frac{s^2}{s^2 + 9} - 1.$$

6. Find the solution of $y = y(t)$ to the following IVP using Laplace transformation:

$$\begin{cases} y'' - 2y' + 2y = e^{-t}, \\ y(0) = 0, \quad y'(0) = 1. \end{cases}$$

- 7.** Dirac delta function $\delta(t)$ is heuristically defined as:

$$\delta(t) = \begin{cases} +\infty, & \text{if } t = 0 \\ 0, & \text{if } t \neq 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t) dt = 1.$$

In *real analysis*, $\delta(t)$ is often called an “approximation to identity”, meaning that it “preserves” the original equation after convolution. By the definition of convolution for f and g , here, as:

$$(f * g)(t) = \int_0^t f(t - \tau)g(\tau) d\tau,$$

prove that $(f * \delta)(t) = f(t)$ for $t \geq 0$.

Hint: Use the convolution theorem and the Laplace transformation of step functions.

8. Let a system of differential equations be defined as follows, find the general solutions to those equations.

(a)
$$\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^2.$$

(b)*
$$\mathbf{x}' = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 1 & 3 \\ 0 & 4 & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^3.$$

9. Solve the following initial value problem for the system of equations:

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

- 10.** (Putnam 2023.) Determine the smallest positive real number r such that there exists differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ satisfying:

- $f(0) > 0$,
- $g(0) = 0$,
- $|f'(x)| \leq |g(x)|$ for all x ,
- $|g'(x)| \leq |f(x)|$ for all x , and
- $f(r) = 0$.

You may give an answer without a rigorous proof, as the proof is out of scope of the course.

Hint: Assume that the function “moves” the fastest when the cap of the derivatives are “moving” the fastest, then think of constructing a dynamical system relating f and g .