Midterm 1 Practices

Differential Equations

Summer 2024

Instructions:

The set of questions serves as PILOT practices to midterm 1 for the Summer 2024 term of `AS.110.302` Differential Equations and Applications at Johns Hopkins University.

- Questions marked with asterisk(s) are more challenging, but still accessible, compared to the others.
- The questions are designed to be done without notes and calculators.
- Solutions to the questions will be discussed during the PILOT review session.

1. Find the general solution for $y = y(t)$:
$$y' + 3y = t + e^{-2t},$$
then, describe the behavior of the solution as $t \to \infty$.

2. Given an initial value problem:
$$\begin{cases} \dfrac{dy}{dt} - \dfrac{3}{2}y = 3t + 2e^{t}, \\ y(0) = y_0. \end{cases}$$
   (a) Find the integrating factor $\mu(t)$.

   (b) Solve for the particular solution for the initial value problem.

   (c) Discuss the behavior of the solution as $t \to \infty$ for different cases of y_0.

3. Suppose $f(x)$ is non-zero, let an initial value problem be:
$$\begin{cases} \dfrac{1-y}{x} \cdot \dfrac{dy}{dx} = \dfrac{f(x)}{1+y}, \\ y(0) = 0. \end{cases}$$
   (a) Show that the differential equation is **not** linear.

   For the next two questions, suppose $f(x) = \tan x$.

   (b) State, <u>without</u> justification, the open interval(s) in which $f(x)$ is continuous.

   (c)* Show that there exists some $\delta > 0$ such that there exists a unique solution $y(x)$ for $x \in (-\delta, \delta)$.

   Now, suppose that $f(x)$ is some function, **not** necessarily continuous.

   (d)** Suppose that the condition in (c) does **not** hold, give three examples in which $f(x)$ could be.

4. An autonomous differential equation is given as follows:
$$\frac{dy}{dt} = 4t^3 - 12t^2 + 9t - 2 \text{ where } t \geq 0 \text{ and } y \geq 0.$$
Draw a phase portrait and sketch a few solutions with different initial conditions.

5.* Determine if the following differential equation is exact. If not, find the integrating factor to make it exact. Then, solve for its general solution:
$$y'(x) = e^{2x} + y(x) - 1.$$

6. Let a differential equation be defined as:
$$\frac{dy}{dt} = t - y \text{ and } y(0) = 0.$$
Use Euler's Method with step size $h = 1$ to approximate $y(5)$.

7. Solve the following second order differential equations for $y = y(x)$:
   (a) $$y'' + y' - 132 = 0.$$
   (b) $$y'' - 4y' = -4.$$
   (c) $$y'' - 2y + 3 = 0.$$

8.** The following system of partial differential equations portraits the propagation of waves on a segment of the 1-dimensional string of length L, the displacement of string at $x \in [0, L]$ at time $t \in [0, \infty)$ is described as the function $u = u(x, t)$:
$$\begin{cases} \text{Differential Equation:} & \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0, & \text{where } x \in (0, L) \text{ and } t \in [0, \infty); \\ \text{Initial Condition:} & u(x, 0) = g(x), & \text{where } x \in [0, L]; \\ \text{Boundary Condition:} & u(0, t) = u(L, t) = 0, & \text{where } t \in [0, \infty); \end{cases}$$
where c is a constant and $g(x)$ has "good" behavior. Apply the method of separation, *i.e.*:
$$u(x, t) = v(x) \cdot w(t),$$
and attempt to obtain a general solution that is non-trivial.
Hint: Use the fact that $\{\sin(nx), \cos(nx)\}_{n \in \mathbb{Z}^+}$ forms an orthonormal basis.

9. Given a differential equation for $y = y(t)$ being:
$$t^3 y'' + t y' - y = 0.$$
   (a) Verify that $y_1(t) = t$ is a solution to the differential equation.
   (b)* Find the full set of solutions using reduction of order.
   (c) Show that the set of solutions from part (b) is linearly independent.

10.** Given the following second order initial value problem:
$$\begin{cases} \dfrac{d^2 y}{dt^2} + \cos(1 - x) y = x^2 - 2x + 1, \\ y(1) = 1, \\ \dfrac{dy}{dt}(1) = 0. \end{cases}$$
Prove that the solution $y(t)$ is symmetric about $t = 1$, *i.e.*, satisfying that $y(t) = y(2 - t)$.
Hint: Consider the interval in which the solution is unique.