



Midterm 1 Practices (Printable Version)

Differential Equations

Summer 2024

Instructions:

The set of questions serves as PILOT practices to midterm 1 for the Summer 2024 term of AS.110.302 Differential Equations and Applications at Johns Hopkins University.

- Questions marked with asterisk(s) are more challenging, but still accessible, compared to the others.
- The questions are designed to be done without notes and calculators.
- Solutions to the questions will be discussed during the PILOT review session.

1. Find the general solution for $y = y(t)$:

$$y' + 3y = t + e^{-2t},$$

then, describe the behavior of the solution as $t \rightarrow \infty$.

2. Given an initial value problem:

$$\begin{cases} \frac{dy}{dt} - \frac{3}{2}y = 3t + 2e^t, \\ y(0) = y_0. \end{cases}$$

- (a) Find the integrating factor $\mu(t)$.
- (b) Solve for the particular solution for the initial value problem.
- (c) Discuss the behavior of the solution as $t \rightarrow \infty$ for different cases of y_0 .

3. Suppose $f(x)$ is non-zero, let an initial value problem be:

$$\begin{cases} \frac{1-y}{x} \cdot \frac{dy}{dx} = \frac{f(x)}{1+y}, \\ y(0) = 0. \end{cases}$$

(a) Show that the differential equation is **not** linear.

For the next two questions, suppose $f(x) = \tan x$.

(b) State, without justification, the open interval(s) in which $f(x)$ is continuous.

(c)* Show that there exists some $\delta > 0$ such that there exists a unique solution $y(x)$ for $x \in (-\delta, \delta)$.

Now, suppose that $f(x)$ is some function, **not** necessarily continuous.

(d)** Suppose that the condition in (c) does **not** hold, give three examples in which $f(x)$ could be.

4. An autonomous differential equation is given as follows:

$$\frac{dy}{dt} = 4y^3 - 12y^2 + 9y - 2 \quad \text{where } t \geq 0 \text{ and } y \geq 0.$$

Draw a phase portrait and sketch a few solutions with different initial conditions.

- 5.* Determine if the following differential equation is exact. If not, find the integrating factor to make it exact. Then, solve for its general solution:

$$y'(x) = e^{2x} + y(x) - 1.$$

6. Let a differential equation be defined as:

$$\frac{dy}{dt} = t - y \text{ and } y(0) = 0.$$

Use Euler's Method with step size $h = 1$ to approximate $y(5)$.

7. Solve the following second order differential equations for $y = y(x)$:

(a) $y'' + y' - 132y = 0.$

(b) $y'' - 4y' = -4y.$

(c) $y'' - 2y' + 3y = 0.$

- 8.** The following system of partial differential equations portraits the propagation of waves on a segment of the 1-dimensional string of length L , the displacement of string at $x \in [0, L]$ at time $t \in [0, \infty)$ is described as the function $u = u(x, t)$:

$$\left\{ \begin{array}{ll} \text{Differential Equation:} & \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad \text{where } x \in (0, L) \text{ and } t \in [0, \infty); \\ \text{Initial Conditions:} & u(x, 0) = \sin\left(\frac{2\pi x}{L}\right), \\ & \frac{\partial u}{\partial t}(x, 0) = \sin\left(\frac{5\pi x}{L}\right), \quad \text{where } x \in [0, L]; \\ \text{Boundary Conditions:} & u(0, t) = u(L, t) = 0, \quad \text{where } t \in [0, \infty); \end{array} \right.$$

where c is a constant and $g(x)$ has “good” behavior. Apply the method of separation, *i.e.*, $u(x, t) = v(x) \cdot w(t)$, and attempt to obtain a general solution that is non-trivial.

Hint: Use the fact that $\{\sin(n\pi x/L), \cos(n\pi x/L)\}_{n \in \mathbb{Z}^+}$ forms an orthonormal basis.

9. Given a differential equation for $y = y(t)$ being:

$$t^3 y'' + t y' - y = 0.$$

- (a) Verify that $y_1(t) = t$ is a solution to the differential equation.
- (b)* Find the full set of solutions using reduction of order.
- (c) Show that the set of solutions from part (b) is linearly independent.

10.** Given the following second order initial value problem:

$$\begin{cases} \frac{d^2y}{dt^2} + \cos(1-x)y = x^2 - 2x + 1, \\ y(1) = 1, \\ \frac{dy}{dt}(1) = 0. \end{cases}$$

Prove that the solution $y(x)$ is symmetric about $x = 1$, i.e., satisfying that $y(x) = y(2-x)$.

Hint: Consider the interval in which the solution is unique.