Midterm 2 Practices: Solutions

Differential Equations

Summer 2024

1. Solve the general solution for $y = y(t)$ to the following second order non-homogeneous ODEs.

   (a) $$y'' + 2y' + y = e^{-t}.$$

   (b) $$y'' + y = \tan t.$$

Solution:

(a) First, we look for homogeneous solution, *i.e.*, $y'' + 2y' + y = 0$, whose characteristic equation is:

$$r^2 + 2r + 1 = (r+1)^2 = 0,$$

with root(s) being -1 with multiplicity of 2, so the general solution to homogeneous case is:

$$y_g(t) = C_1 e^{-t} + C_2 t e^{-t}.$$

Notice that the non-homogeneous part is e^{-t}, but we have e^{-t} and te^{-t} as general solutions already, so we have our guess of particular solution as:

$$y_p(t) = At^2 e^{-t}.$$

By taking the derivatives, we have:

$$y_p'(t) = A(2te^{-t} - t^2 e^{-t}) \qquad \text{and} \qquad y_p''(t) = A(2e^{-t} - 4te^{-t} + t^2 e^{t}).$$

We simply plug in the particular solution, so we have:

$$\begin{aligned} A(2e^{-t} - 4te^{-t} + t^2e^{t}) + 2A(2te^{-t} - t^2e^{-t}) + At^2e^{-t} &= e^{-t} \\ 2Ae^{-t} &= e^{-t} \\ A &= \frac{1}{2}. \end{aligned}$$

Hence, our solution to the non-homogeneous case is:

$$y(t) = \boxed{C_1 e^{-t} + C_2 t e^{-t} + \frac{1}{2} t^2 e^{-t}}.$$

(b) Here, we still look for homogeneous solutions, *i.e.*, $y'' + y = 0$, whose characteristic equation is:

$$r^2 + 1 = 0,$$

with roots $\pm \mathrm{i}$. Since we are dealing with real valued functions, we have the general solution as:

$$y_g = C_1 \sin t + C_2 \cos t.$$

Note that $\tan t$ does not work with undetermined coefficients, we must use the variation of parameters, the Wronskian of our solution is:

$$\mathrm{W}[\sin t, \cos t] = \det \begin{pmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{pmatrix} = -\sin^2 t - \cos^2 t = -1.$$

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Now, we may use the formula, namely getting the particular solution as:

$$\begin{aligned}
y_p &= \sin t \int \frac{-\cos t \cdot \tan t}{-1} dt + \cos t \int \frac{\sin t \cdot \tan t}{-1} dt \\
&= \sin t \int \sin t dt - \cos t \int \frac{\sin^2 t}{\cos t} dt \\
&= \sin t(-\cos t + C) - \cos t \int \frac{1 - \cos^2 t}{\cos t} dt \\
&= -\sin t \cos t + \cancel{C \sin t} - \cos t \left(\int \sec t dt - \int \cos t dt \right) \\
&= -\sin t \cos t - \cos t \left(\log|\sec t + \tan t| - \sin t + C \right) \\
&= -\sin t \cos t + \sin t \cos t - \cancel{C \cos t} - \cos t \log|\sec t + \tan t| \\
&= -\cos t \log|\sec t + \tan t|.
\end{aligned}$$

Hence, our solution to the non-homogeneous case is:

$$y(t) = \boxed{C_1 \sin t + C_2 \cos t - \cos t \log|\sec t + \tan t|}.$$

2. Solve for the general solution to the following higher order ODE.

(a)
$$4\frac{d^4y}{dx^4} - 24\frac{d^3y}{dx^3} + 45\frac{d^2y}{dx^2} - 29\frac{dy}{dx} + 6y = 0.$$

(b)**
$$\frac{d^4y}{dx^4} + y = 0.$$

Hint: Consider the 8-th root of unity, *i.e.*, ζ_8, and verify which roots satisfies the polynomial.

Solution:

(a) Note that we obtain the characteristic equation as:
$$4r^4 - 24r^3 + 45r^2 - 29r + 6 = 0.$$
To obtain our roots, we use the **Rational Root Theorem**, so if the characteristic equation has any rational root, it must have been one (or more) of the following:
$$\pm 1, \pm 2, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}.$$
From plugging in the values, we notice that 2 and 3 are roots of the characteristic equation, by division, we have:
$$\frac{4r^4 - 24r^3 + 45r^2 - 29r + 6}{(r-2)(r-3)} = 4r^2 - 4r + 1 = (2r-1)^2.$$
Now, we know that the roots are 2, 3, and $1/2$ with multiplicity 2, thus the solution to the differential equation is:
$$y(x) = \boxed{C_1e^{2x} + C_2e^{3x} + C_3e^{x/2} + C_4xe^{x/2}}.$$
Again, we invite readers to verify the ***Rational Root Theorem****, in which more details could be found on Question 4 from* ***Midterm 1 Practices: Solutions****.*

(b) For this general solution, we trivially obtain that the characteristic polynomial is:
$$r^4 + 1 = 0.$$
Recall that the root of unity address for the case when $r^n = 1$, so we consider the 8th root of unity, in which $(\zeta_8)^8 = 1$. Now, recall **Euler's Identity** and **deMoivre's formula**, we note that only the odd powers of the 8th root of unity satisfies that $r^4 = -1$, namely, are:
$$\zeta_8 = \cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \mathrm{i}\frac{\sqrt{2}}{2},$$
$$\zeta_8^3 = \cos\left(\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} + \mathrm{i}\frac{\sqrt{2}}{2},$$
$$\zeta_8^5 = \cos\left(\frac{5\pi}{4}\right) + \mathrm{i}\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} - \mathrm{i}\frac{\sqrt{2}}{2},$$
$$\zeta_8^7 = \cos\left(\frac{7\pi}{4}\right) + \mathrm{i}\sin\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2} - \mathrm{i}\frac{\sqrt{2}}{2}.$$
Also, we note that ζ_8 and ζ_8^7 are complex conjugates, whereas ζ_8^3 and ζ_8^5 are complex conjugates, so we can linearly combine them to obtain the set of linearly independent solutions, *i.e.*:
$$y(x) = \boxed{\begin{aligned} & e^{(\sqrt{2}/2)x}\left[C_1\cos\left(\frac{\sqrt{2}}{2}x\right) + C_2\sin\left(\frac{\sqrt{2}}{2}x\right)\right] \\ + \; & e^{-(\sqrt{2}/2)x}\left[C_3\cos\left(\frac{\sqrt{2}}{2}x\right) + C_4\sin\left(\frac{\sqrt{2}}{2}x\right)\right]\end{aligned}}.$$

3. Let a third order differential equation be as follows:
$$\ell\big[y(t)\big] = y^{(3)}(t) + 3y''(t) + 3y'(t) + y(t).$$
Let $\ell\big[y(t)\big] = 0$ be trivial initially.

(a) Find the set of all linearly independent solutions.

Then, assume that $\ell\big[y(t)\big]$ is non-trivial.

(b) Find the particular solution to $\ell\big[y(t)\big] = \sin t$.

(c) Find the particular solution to $\ell\big[y(t)\big] = e^{-t}$.

(d)* Suppose that $\ell\big[y_1(t)\big] = f(t)$ and $\ell\big[y_2(t)\big] = g(t)$ where $f(t)$ and $g(t)$ are "good" functions. Find an expression to $y_3(t)$ such that $\ell\big[y_3(t)\big] = f(t) + g(t)$.

Solution:

(a) Note that the characteristic polynomial can be factorized as perfect cubes:
$$r^3 + 3r^2 + 3r + 1 = (r+1)^3 = 0,$$
its roots are $r = -1$ with multiplicity 3, so the general solution is:
$$y(t) = \boxed{C_1e^{-t} + C_2te^{-t} + C_3t^2e^{-t}}.$$
Here, the readers are invited to check, by ***Wronskian****, that set of solutions are linearly independent.*

(b) First, we want to make our guess of particular solution as:
$$y_p(t) = A\sin t + B\cos t,$$
and by taking the derivatives, we have:
$$y_p'(t) = A\cos t - B\sin t, \qquad y_p''(t) = -A\sin t - B\cos t, \text{ and } \qquad y_p'''(t) = -A\cos t + B\sin t.$$
Then, we want to plug in the results into the equation, so:
$$\begin{aligned}\ell\big[y_p(t)\big] &= (-A\cos t + B\sin t) + 3(-A\sin t - B\cos t) + 3(A\cos t - B\sin t) + A\sin t + B\cos t\\ &= (B - 3A - 3B + A)\sin t + (-A - 3B + 3A + B)\cos t\\ &= (-2A - 2B)\sin t + (2A - 2B)\cos t.\end{aligned}$$
Therefore, we can obtain the system that:
$$\begin{cases}-2A - 2B = 1,\\ 2A - 2B = 0,\end{cases}$$
which reduces to $A = -1/4$ and $B = -1/4$, so the solution is:
$$y(t) = \boxed{C_1e^{-t} + C_2te^{-t} + C_3t^2e^{-t} - \frac{1}{4}\sin t - \frac{1}{4}\cos t}.$$

(c) Here, note that e^{-t}, te^{-t}, and t^2e^{-t} are the solutions to homogeneous case, our guess, then, is:
$$y_p(t) = At^3e^{-t},$$
and by taking the derivatives, we have:
$$\begin{aligned}&y_p'(t) = 3At^2e^{-t} - At^3e^{-t}, \qquad y_p''(t) = 6Ate^{-t} - 6At^2e^{-t} + At^3e^{-t}, \qquad \text{and}\\ &y_p'''(t) = 6Ae^{-t} - 18Ate^{-t} + 9At^2e^{-t} - At^3e^{-t}.\end{aligned}$$

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When we plug the derivatives back to the solutions, we note that:

$$\begin{aligned}\ell[y_p(t)] &= (6Ae^{-t} - 18Ate^{-t} + 9At^2e^{-t} - At^3e^{-t}) \\ &\quad + 3(6Ate^{-t} - 6At^2e^{-t} + At^3e^{-t}) + 3(3At^2e^{-t} - At^3e^{-t}) + (At^3e^{-t}) \\ &= 6Ae^{-t},\end{aligned}$$

which reduces to $A = 1/6$, so the solution is:

$$y(t) = \boxed{C_1e^{-t} + C_2te^{-t} + C_3t^2e^{-t} + \frac{1}{6}t^3e^{-t}}.$$

(d) *Proof.* Here, one should note that the derivative operator is linear, so we have that:

$$\begin{aligned}\ell[y_1(t) + y_2(t)] &= \frac{d^3}{dt^3}[y_1(t) + y_2(t)] + 3\frac{d^2}{dt^2}[y_1(t) + y_2(t)] + 3\frac{d}{dt}[y_1(t) + y_2(t)] + [y_1(t) + y_2(t)] \\ &= y_1'''(t) + 3y_1''(t) + 3y_1'(t) + y_1(t) + y_2'''(t) + 3y_2''(t) + 3y_2'(t) + y_2(t) \\ &= f(t) + g(t),\end{aligned}$$

as desired. □

4. Show the following Laplace transformation by definition.

(a)
$$\mathcal{L}\{\sin(at)\} = \frac{a}{a^2+s^2}.$$

(b)*
$$\mathcal{L}\{(f*g)(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}.$$

Solution:

(a) *Proof.* Here, we do the Laplace transformation via definition:

$$\begin{aligned}
\mathcal{L}\{\sin(at)\} &= \int_0^\infty e^{-st}\sin(at)dt \\
&= -\frac{1}{s}e^{-st}\sin(at)\Big|_{t=0}^{t=\infty} + \frac{a}{s}\int_0^\infty e^{-st}\cos(at)dt \\
&= \frac{a}{s}\left[-\frac{1}{s}e^{-st}\cos(at)\Big|_{t=0}^{t=\infty} - \frac{a}{s}\int_0^\infty e^{-st}\sin(at)dt\right] \\
&= \frac{a}{s^2} - \frac{a^2}{s^2}\mathcal{L}\{\sin(at)\}.
\end{aligned}$$

Thus, we have the Laplace transformation as:

$$\mathcal{L}\{\sin(at)\} = \frac{a/s^2}{a^2/s^2+1} = \frac{a}{s^2+a^2},$$

as desired. □

(b) *Proof.* Recall that the convolution notation in this course is that:

$$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau,$$

so we can apply the Laplace transformation function. For simplicity, we assume that f and g behaves well enough, *i.e.*, they satisfies the conditions for Fubinii's Theorem, thus:

$$\begin{aligned}
\mathcal{L}\{(f*g)(t)\} &= \int_0^\infty e^{-st}(f*g)(t)dt \\
&= \int_0^\infty e^{-st}\int_0^\infty f(\tau)g(t-\tau)d\tau dt \\
&= \int_0^\infty f(\tau)\int_0^\infty e^{-st}g(t-\tau)dtd\tau \\
&= \int_0^\infty f(\tau)e^{-s\tau}\int_0^\infty e^{-s(t-\tau)}g(t-\tau)dtd\tau \\
&= \int_0^\infty f(\tau)e^{-s\tau}d\tau\cdot\int_0^\infty e^{-s(t-\tau)}g(t-\tau)d(t-\tau) \\
&= \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\},
\end{aligned}$$

as desired. □

5. Given the following the results after Laplace transformation $F(s) = \mathcal{L}\{f(t)\}$, find each $f(t)$ prior to the Laplace transformation.

(a) $$F(s) = \frac{2s^2+4}{s^3+4s}.$$

(b)* $$F(s) = \frac{s^2}{s^2+9} - 1.$$

Solution:

(a) For this equation, one should notice that we can factor our denominator here and use partial factions, as:
$$F(s) = \frac{2s^2+4}{s(s^2+4)} = \frac{1}{s} + \frac{s}{s^2+4}.$$
By finding the inverse, we have:
$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \boxed{1+\cos(2t)}.$$

(b) Here, notice that we can combine the -1 into the function, as:
$$F(s) = \frac{s^2}{s^2+9} - \frac{s^2+9}{s^2+9} = \frac{-9}{s^2+9}.$$
Note that the Laplace transformation is linear, so does its inverse, so we have:
$$f(t) = \mathcal{L}^{-1}\left\{\frac{-9}{s^2+9}\right\} = -3\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = \boxed{-3\sin(3t)}.$$

6. Find the solution of $y = y(t)$ to the following IVP using Laplace transformation:

$$\begin{cases} y'' - 2y' + 2y = e^{-t}, \\ y(0) = 0, \qquad y'(0) = 1. \end{cases}$$

Solution:

Here, we want to use the Laplace transformation for derivatives, for simplicity, we denote $Y = \mathcal{L}\{y\}$:

$$\begin{aligned} \mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} &= \mathcal{L}\{e^{-t}\} \\ s^2Y - sy(0) - y'(0) - 2sY + 2y(0) + 2Y &= \frac{1}{s+1} \\ (s^2 - 2s + 1)Y &= \frac{1}{s+1} + 1 = \frac{s+2}{s+1} \\ Y &= \frac{s+2}{(s+1)(s^2-2s+2)} \\ &= \frac{1}{5}\frac{1}{s+1} + \frac{1}{5}\frac{-s+8}{(s-1)^2+1} \\ &= \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{s-1}{(s-1)^2+1} + \frac{7}{5}\frac{1}{(s-1)^2+1}. \end{aligned}$$

By taking the inverse of Laplace, we have:

$$y(t) = \boxed{\frac{1}{5}r^{-t} - \frac{1}{5}e^t\cos t + \frac{7}{5}e^t \sin t}.$$

7.** Dirac delta function $\delta(t)$ is heuristically defined as:

$$\delta(t) = \begin{cases} +\infty, & \text{if } t = 0 \\ 0, & \text{if } t \neq 0 \end{cases} \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(t)dt = 1.$$

In *real analysis*, $\delta(t)$ is often called an "approximation to identity", meaning that it "preserves" the original equation after convolution. By the definition of convolution for f and g, here, as:

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau,$$

prove that $(f * \delta)(t) = f(t)$ for $t \geq 0$.

Hint: Use the convolution theorem and the Laplace transformation of step functions.

Solution:

If you have consulted with a few other texts, you might observe that the convolution formula here is different from the convolution formula in analysis textbooks, *i.e.*:

$$(f * g)(t) = \int_0^{\infty} f(\tau)g(t-\tau)d\tau.$$

In fact, the above definition allows better property, known as "approximation to identity" for the entire domain. However, we may show the version that you see in the ODEs course with a weaker conclusion.

Proof. Here, we apply the Laplace transformation on $f * \delta$, which is:

$$\begin{aligned} \mathcal{L}\{(f * \delta)(t)\} &= \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{\delta(t)\} \\ &= \mathcal{L}\{f(t)\} \cdot e^{-0s} \\ &= \mathcal{L}\{u(t) \cdot f(t)\}. \end{aligned}$$

Hence, we have demonstrated that:

$$(f * \delta)(t) = \begin{cases} 0, & \text{when } t < 0 \\ f(t), & \text{when } t \geq 0 \end{cases}$$

which implies that $(f * \delta)(t) = f(t)$ for $t \geq 0$. □

If you are interested in the concept of Dirac delta function, you can look up the conditions to be a "good kernel", which proceeds further to a narrower class of kernels as approximations to the identity. Moreover, we suggest consulting some constructions of the kernels, such as the *shrinking function* or the series of *Gaussian bell curve*.

8. Let a system of differential equations be defined as follows, find the general solutions to the equation.

$$\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x} \in \mathbb{R}^2.$$

Solution: The question should be trivial, we first find the eigenvalues for the equation, *i.e.*:

$$\det \begin{pmatrix} 3-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix} = 0,$$

which is $(3-\lambda)(2-\lambda) = 0$, that is $\lambda_1 = 3$ and $\lambda_2 = 2$. Then, we look for the eigenvectors.

- For $\lambda_1 = 3$, we have $\begin{pmatrix} 3-3 & 0 \\ 0 & 2-3 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0}$, which is $\boldsymbol{\xi}_1 = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
- For $\lambda_2 = 2$, we have $\begin{pmatrix} 3-2 & 0 \\ 0 & 2-2 \end{pmatrix} \boldsymbol{\xi}_2 = \mathbf{0}$, which is $\boldsymbol{\xi}_2 = x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Hence, the solution is:

$$\mathbf{x} = \boxed{C_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}.$$

9. Let a system of differential equations of $x_i(t)$ be as follows:
$$\begin{cases} x_1' = 3x_1 + 2x_2, & x_1(1) = 0, \\ x_2' = x_1 + 4x_2, & x_2(1) = 2. \end{cases}$$

(a) Solve for the solution to the initial value problem.

(b) Identify and describe the stability at equilibrium(s).

Solution:

(a) Here, we denote $\mathbf{x} = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^\intercal$, so our system becomes:
$$\mathbf{x}' = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(1) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$
Here, the eigenvalues are solutions to:
$$\det \begin{pmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{pmatrix} = 0,$$
which simplifies to $\lambda^2 - 7\lambda + 10 = 0$, and further gives $\lambda_1 = 2$, $\lambda_2 = 5$. Then, we look for eigenvectors of the matrix:

- For $\lambda_1 = 2$, we have $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \boldsymbol{\xi} = \mathbf{0}$, which gives that $\boldsymbol{\xi}_1 = x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.
- For $\lambda_2 = 5$, we have $\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \boldsymbol{\xi}_2 = \mathbf{0}$, which gives that $\boldsymbol{\xi}_2 = x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Now, the general solution must be:
$$\mathbf{x} = C_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^2 t + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^5 t,$$
and by plugging in the initial condition, we have:
$$\begin{cases} -2C_1e^2 + C_2e^5 = 0, \\ C_1e^2 + C_2e^5 = 2. \end{cases}$$
In which the solution is $C_1 = \frac{2}{3e^2}$ and $C_2 = \frac{4}{3e^5}$, so the solution is:
$$\boxed{\begin{cases} x_1 = -\frac{4}{3}e^{2t-2} + \frac{4}{3}e^{5t-5}, \\ x_2 = \frac{2}{3}e^{2t-2} + \frac{4}{3}e^{5t-5}. \end{cases}}$$

(b) Now, we consider the equilibrium at $\mathbf{x} = \begin{pmatrix} 0 & 0 \end{pmatrix}^\intercal$, in which we note that both eigenvalues are positive, meaning that this is an $\boxed{\text{unstable node}}$.

10.** (*Putnam 2023.*) Determine the smallest positive real number r such that there exists differentiable functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ satisfying:

- $f(0) > 0$,
- $g(0) = 0$,
- $|f'(x)| \leq |g(x)|$ for all x,
- $|g'(x)| \leq |f(x)|$ for all x, and
- $f(r) = 0$.

You may give an answer without a rigorous proof, as the proof is out of scope of the course.
Hint: Assume that the function "moves" the fastest when the cap of the derivatives are "moving" the fastest, then think of constructing a dynamical system relating f and g.

Solution:
Here, we first provide a "simplified" case, *i.e.*, we are constructing a dynamical system in which we pick equality for the inequality, that is:

$$\begin{cases} |f'(x)| = |g(x)|, \text{ and} \\ |g'(x)| = |f(x)|. \end{cases}$$

Without loss of generality, we may assume that f and g are non-negative before r, so the system becomes:

$$\begin{cases} f' = -g \\ g' = f \end{cases},$$

or equivalently, $\mathbf{y} = \begin{pmatrix} f \\ g \end{pmatrix}$ that $\mathbf{y} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{y}$. Clearly, we observe the eigenvalues are $\pm\mathrm{i}$ as the polynomial is $\lambda^2 + 1 = 0$. Moreover, the eigenvectors for $\lambda_1 = \mathrm{i}$ is when $\begin{pmatrix} -\mathrm{i} & -1 \\ 1 & -\mathrm{i} \end{pmatrix} \boldsymbol{\xi} = \mathbf{0}$, in which we have $\boldsymbol{\xi} = y \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix}$, and that solution is:

$$\mathbf{y} = \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix} e^{ix} = \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix} (\cos x + \mathrm{i} \sin x) = \begin{pmatrix} -\sin x \\ \cos x \end{pmatrix} + \mathrm{i} \begin{pmatrix} \cos x \\ \sin x, \end{pmatrix}$$

and by conjugation, the solution should be:

$$\begin{pmatrix} f \\ g \end{pmatrix} = C_1 \begin{pmatrix} -\sin x \\ \cos x \end{pmatrix} + C_2 \begin{pmatrix} \cos x \\ \sin x \end{pmatrix}.$$

Note that with the given initial condition that $g(0) = 0$, this enforces $C_1 = 0$, thus $f(x) = C\cos x$ and $g(x) = C \sin x$, and we know that $f(r)$ is zero first at $r = \boxed{\pi/2}$.

The above version has some reasoning, but is not a rigorous proof at all, since this does not consider if r could be smaller than $\pi/2$. For students with interests, we provide the complete proof from the Putnam competition from Victor Lie, as follows.

Proof. Without loss of generality, we assume $f(x) > 0$ for all $x \in [0, r)$ as it is the first positive zero. By the fundamental theorem of calculus, we have:

$$|f'(x)| \leq |g(x)| \leq \left| \int_0^x g(s) ds \right| \leq \int_0^x |g(s)| ds \leq \int_0^t |f(s)| ds.$$

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Now, as we denote $F(x) = \int_0^x f(s)ds$, we have:

$$f'(x) + F(x) \geq 0 \text{ for } x \in [0, r].$$

For the sake of contradiction, we suppose $r < \pi/2$, then we have:

$$f'(x)\cos x + F(x)\cos x \geq 0 \text{ for } x \in [0, r].$$

Notice that the left hand side is the derivative of $f(x)\cos x + F(x)\sin x$, so an integration on $[y, r]$ gives:

$$F(r)\sin r \geq f(y)\cos y + F(y)\sin(y).$$

With some rearranging, we can have:

$$F(r)\sin r\sec^2 y \geq f(y)\sec y + F(y)\sin y\sec^2 y$$

Again, we integrate both sides with respect to y on $[0, r]$, which gives:

$$F(r)\sin^2 r \geq F(r),$$

and this is impossible, so we have a contradiction.

Hence we must have $r \geq \pi/2$, and since we have noted the solution $f(x) = C\cos x$ and $g(x) = C\sin x$, we have proven that $r = \pi/2$ is the smallest case. $\square$