



Midterm 2 Practices: Solutions

Differential Equations

Summer 2024

1. Solve the general solution for $y = y(t)$ to the following second order non-homogeneous ODEs.

(a) $y'' + 2y' + y = e^{-t}.$

(b) $y'' + y = \tan t.$

Solution:

(a) First, we look for homogeneous solution, i.e., $y'' + 2y' + y = 0$, whose characteristic equation is:

$$r^2 + 2r + 1 = (r + 1)^2 = 0,$$

with root(s) being -1 with multiplicity of 2, so the general solution to homogeneous case is:

$$y_g(t) = C_1 e^{-t} + C_2 t e^{-t}.$$

Notice that the non-homogeneous part is e^{-t} , but we have e^{-t} and $t e^{-t}$ as general solutions already, so we have our guess of particular solution as:

$$y_p(t) = A t^2 e^{-t}.$$

By taking the derivatives, we have:

$$y'_p(t) = A(2t e^{-t} - t^2 e^{-t}) \quad \text{and} \quad y''_p(t) = A(2e^{-t} - 4t e^{-t} + t^2 e^{-t}).$$

We simply plug in the particular solution, so we have:

$$A(2e^{-t} - 4t e^{-t} + t^2 e^{-t}) + 2A(2t e^{-t} - t^2 e^{-t}) + A t^2 e^{-t} = e^{-t}$$

$$2A e^{-t} = e^{-t}$$

$$A = \frac{1}{2}.$$

Hence, our solution to the non-homogeneous case is:

$$y(t) = \boxed{C_1 e^{-t} + C_2 t e^{-t} + \frac{1}{2} t^2 e^{-t}}.$$

(b) Here, we still look for homogeneous solutions, i.e., $y'' + y = 0$, whose characteristic equation is:

$$r^2 + 1 = 0,$$

with roots $\pm i$. Since we are dealing with real valued functions, we have the general solution as:

$$y_g = C_1 \sin t + C_2 \cos t.$$

Note that $\tan t$ does not work with undetermined coefficients, we must use the variation of parameters, the Wronskian of our solution is:

$$W[\sin t, \cos t] = \det \begin{pmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{pmatrix} = -\sin^2 t - \cos^2 t = -1.$$

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Now, we may use the formula, namely getting the particular solution as:

$$\begin{aligned}
 y_p &= \sin t \int \frac{-\cos t \cdot \tan t}{-1} dt + \cos t \int \frac{\sin t \cdot \tan t}{-1} dt \\
 &= \sin t \int \sin t dt - \cos t \int \frac{\sin^2 t}{\cos t} dt \\
 &= \sin t (-\cos t + C) - \cos t \int \frac{1 - \cos^2 t}{\cos t} dt \\
 &= -\sin t \cos t + \cancel{C \sin t} - \cos t \left(\int \sec t dt - \int \cos t dt \right) \\
 &= -\sin t \cos t - \cos t (\log |\sec t + \tan t| - \sin t + C) \\
 &= -\sin t \cos t + \sin t \cos t - \cancel{C \cos t} - \cos t \log |\sec t + \tan t| \\
 &= -\cos t \log |\sec t + \tan t|.
 \end{aligned}$$

Hence, our solution to the non-homogeneous case is:

$$y(t) = \boxed{C_1 \sin t + C_2 \cos t - \cos t \log |\sec t + \tan t|}.$$

2. Solve for the general solution to the following higher order ODE.

(a) $4\frac{d^4y}{dx^4} - 24\frac{d^3y}{dx^3} + 45\frac{d^2y}{dx^2} - 29\frac{dy}{dx} + 6y = 0.$

(b)** $\frac{d^4y}{dx^4} + y = 0.$

Hint: Consider the 8-th root of unity, i.e., ζ_8 , and verify which roots satisfies the polynomial.

Solution:

(a) Note that we obtain the characteristic equation as:

$$4r^4 - 24r^3 + 45r^2 - 29r + 6 = 0.$$

To obtain our roots, we use the **Rational Root Theorem**, so if the characteristic equation has any rational root, it must have been one (or more) of the following:

$$\pm 1, \pm 2, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2}.$$

From plugging in the values, we notice that 2 and 3 are roots of the characteristic equation, by division, we have:

$$\frac{4r^4 - 24r^3 + 45r^2 - 29r + 6}{(r-2)(r-3)} = 4r^2 - 4r + 1 = (2r-1)^2.$$

Now, we know that the roots are 2, 3, and 1/2 with multiplicity 2, thus the solution to the differential equation is:

$$y(x) = \boxed{C_1 e^{2x} + C_2 e^{3x} + C_3 e^{x/2} + C_4 x e^{x/2}}.$$

Again, we invite readers to verify the **Rational Root Theorem**, in which more details could be found on Question 4 from **Midterm 1 Practices: Solutions**.

(b) For this general solution, we trivially obtain that the characteristic polynomial is:

$$r^4 + 1 = 0.$$

Recall that the root of unity address for the case when $r^n = 1$, so we consider the 8th root of unity, in which $(\zeta_8)^8 = 1$. Now, recall **Euler's Identity** and **deMoivre's formula**, we note that only the odd powers of the 8th root of unity satisfies that $r^4 = -1$, namely, are:

$$\zeta_8 = \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2},$$

$$\zeta_8^3 = \cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2},$$

$$\zeta_8^5 = \cos\left(\frac{5\pi}{4}\right) + i \sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2},$$

$$\zeta_8^7 = \cos\left(\frac{7\pi}{4}\right) + i \sin\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2}.$$

Also, we note that ζ_8 and ζ_8^7 are complex conjugates, whereas ζ_8^3 and ζ_8^5 are complex conjugates, so we can linearly combine them to obtain the set of linearly independent solutions, i.e.:

$$y(x) = \begin{bmatrix} e^{(\sqrt{2}/2)x} \left[C_1 \cos\left(\frac{\sqrt{2}}{2}x\right) + C_2 \sin\left(\frac{\sqrt{2}}{2}x\right) \right] \\ + e^{-(\sqrt{2}/2)x} \left[C_3 \cos\left(\frac{\sqrt{2}}{2}x\right) + C_4 \sin\left(\frac{\sqrt{2}}{2}x\right) \right] \end{bmatrix}.$$

3. Let a third order differential equation be as follows:

$$\ell[y(t)] = y^{(3)}(t) + 3y''(t) + 3y'(t) + y(t).$$

Let $\ell[y(t)] = 0$ be trivial initially.

(a) Find the set of all linearly independent solutions.

Then, assume that $\ell[y(t)]$ is non-trivial.

(b) Find the particular solution to $\ell[y(t)] = \sin t$.

(c) Find the particular solution to $\ell[y(t)] = e^{-t}$.

(d)* Suppose that $\ell[y_1(t)] = f(t)$ and $\ell[y_2(t)] = g(t)$ where $f(t)$ and $g(t)$ are "good" functions. Find an expression to $y_3(t)$ such that $\ell[y_3(t)] = f(t) + g(t)$.

Solution:

(a) Note that the characteristic polynomial can be factorized as perfect cubes:

$$r^3 + 3r^2 + 3r + 1 = (r + 1)^3 = 0,$$

its roots are $r = -1$ with multiplicity 3, so the general solution is:

$$y(t) = \boxed{C_1 e^{-t} + C_2 t e^{-t} + C_3 t^2 e^{-t}}.$$

Here, the readers are invited to check, by **Wronskian**, that set of solutions are linearly independent.

(b) First, we want to make our guess of particular solution as:

$$y_p(t) = A \sin t + B \cos t,$$

and by taking the derivatives, we have:

$$y_p'(t) = A \cos t - B \sin t, \quad y_p''(t) = -A \sin t - B \cos t, \quad \text{and} \quad y_p'''(t) = -A \cos t + B \sin t.$$

Then, we want to plug in the results into the equation, so:

$$\begin{aligned} \ell[y_p(t)] &= (-A \cos t + B \sin t) + 3(-A \sin t - B \cos t) + 3(A \cos t - B \sin t) + A \sin t + B \cos t \\ &= (B - 3A - 3B + A) \sin t + (-A - 3B + 3A + B) \cos t \\ &= (-2A - 2B) \sin t + (2A - 2B) \cos t. \end{aligned}$$

Therefore, we can obtain the system that:

$$\begin{cases} -2A - 2B = 1, \\ 2A - 2B = 0, \end{cases}$$

which reduces to $A = -1/4$ and $B = -1/4$, so the solution is:

$$y(t) = \boxed{C_1 e^{-t} + C_2 t e^{-t} + C_3 t^2 e^{-t} - \frac{1}{4} \sin t - \frac{1}{4} \cos t}.$$

(c) Here, note that e^{-t} , $t e^{-t}$, and $t^2 e^{-t}$ are the solutions to homogeneous case, our guess, then, is:

$$y_p(t) = A t^3 e^{-t},$$

and by taking the derivatives, we have:

$$\begin{aligned} y_p'(t) &= 3A t^2 e^{-t} - A t^3 e^{-t}, & y_p''(t) &= 6A t e^{-t} - 6A t^2 e^{-t} + A t^3 e^{-t}, & \text{and} \\ y_p'''(t) &= 6A e^{-t} - 18A t e^{-t} + 9A t^2 e^{-t} - A t^3 e^{-t}. \end{aligned}$$

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When we plug the derivatives back to the solutions, we note that:

$$\begin{aligned}\ell[y_p(t)] &= (6Ae^{-t} - 18Ate^{-t} + 9At^2e^{-t} - At^3e^{-t}) \\ &\quad + 3(6Ate^{-t} - 6At^2e^{-t} + At^3e^{-t}) + 3(3At^2e^{-t} - At^3e^{-t}) + (At^3e^{-t}) \\ &= 6Ae^{-t},\end{aligned}$$

which reduces to $A = 1/6$, so the solution is:

$$y(t) = \boxed{C_1e^{-t} + C_2te^{-t} + C_3t^2e^{-t} + \frac{1}{6}t^3e^{-t}}.$$

(d) *Proof.* Here, one should note that the derivative operator is linear, so we have that:

$$\begin{aligned}\ell[y_1(t) + y_2(t)] &= \frac{d^3}{dt^3}[y_1(t) + y_2(t)] + 3\frac{d^2}{dt^2}[y_1(t) + y_2(t)] + 3\frac{d}{dt}[y_1(t) + y_2(t)] + [y_1(t) + y_2(t)] \\ &= y_1'''(t) + 3y_1''(t) + 3y_1'(t) + y_1(t) + y_2'''(t) + 3y_2''(t) + 3y_2'(t) + y_2(t) \\ &= f(t) + g(t),\end{aligned}$$

as desired. □

4. Show the following Laplace transformation by definition.

(a)
$$\mathcal{L}\{\sin(at)\} = \frac{a}{a^2 + s^2}.$$

(b)*
$$\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}.$$

Solution:

(a) *Proof.* Here, we do the Laplace transformation via definition:

$$\begin{aligned} \mathcal{L}\{\sin(at)\} &= \int_0^\infty e^{-st} \sin(at) dt \\ &= -\frac{1}{s} e^{-st} \sin(at) \Big|_{t=0}^{t=\infty} + \frac{a}{s} \int_0^\infty e^{-st} \cos(at) dt \\ &= \frac{a}{s} \left[-\frac{1}{s} e^{-st} \cos(at) \Big|_{t=0}^{t=\infty} - \frac{a}{s} \int_0^\infty e^{-st} \sin(at) dt \right] \\ &= \frac{a}{s^2} - \frac{a^2}{s^2} \mathcal{L}\{\sin(at)\}. \end{aligned}$$

Thus, we have the Laplace transformation as:

$$\mathcal{L}\{\sin(at)\} = \frac{a/s^2}{a^2/s^2 + 1} = \frac{s}{s^2 + a^2},$$

as desired. □

(b) *Proof.* Recall that the convolution notation in this course is that:

$$(f * g)(t) = \int_0^t f(t - \tau)g(\tau) d\tau,$$

so we can apply the Laplace transformation function. For simplicity, we assume that f and g behaves well enough, *i.e.*, they satisfies the conditions for Fubini's Theorem, thus:

$$\begin{aligned} \mathcal{L}\{(f * g)(t)\} &= \int_0^\infty e^{-st} (f * g)(t) dt \\ &= \int_0^\infty e^{-st} \int_0^\infty f(\tau)g(t - \tau) d\tau dt \\ &= \int_0^\infty f(\tau) \int_0^\infty e^{-st} g(t - \tau) dt d\tau \\ &= \int_0^\infty f(\tau) e^{-s\tau} \int_0^\infty e^{-s(t-\tau)} g(t - \tau) dt d\tau \\ &= \int_0^\infty f(\tau) e^{-s\tau} d\tau \cdot \int_0^\infty e^{-s(t-\tau)} g(t - \tau) d(t - \tau) \\ &= \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}, \end{aligned}$$

as desired. □

5. Given the following the results after Laplace transformation $F(s) = \mathcal{L}\{f(t)\}$, find each $f(t)$ prior to the Laplace transformation.

(a)
$$F(s) = \frac{2s^2 + 4}{s^3 + 4s}.$$

(b)*
$$F(s) = \frac{s^2}{s^2 + 9} - 1.$$

Solution:

- (a) For this equation, one should notice that we can factor our denominator here and use partial fractions, as:

$$F(s) = \frac{2s^2 + 4}{s(s^2 + 4)} = \frac{1}{s} + \frac{s}{s^2 + 4}.$$

By finding the inverse, we have:

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2 + 4}\right\} = \boxed{1 + \cos(2t)}.$$

- (b) Here, notice that we can combine the -1 into the function, as:

$$F(s) = \frac{s^2}{s^2 + 9} - \frac{s^2 + 9}{s^2 + 9} = \frac{-9}{s^2 + 9}.$$

Note that the Laplace transformation is linear, so does its inverse, so we have:

$$f(t) = \mathcal{L}^{-1}\left\{\frac{-9}{s^2 + 9}\right\} = -3\mathcal{L}^{-1}\left\{\frac{3}{s^2 + 9}\right\} = \boxed{-3\sin(3t)}.$$

6. Find the solution of $y = y(t)$ to the following IVP using Laplace transformation:

$$\begin{cases} y'' - 2y' + 2y = e^{-t}, \\ y(0) = 0, \quad y'(0) = 1. \end{cases}$$

Solution:

Here, we want to use the Laplace transformation for derivatives, for simplicity, we denote $Y = \mathcal{L}\{y\}$:

$$\begin{aligned} \mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} &= \mathcal{L}\{e^{-t}\} \\ s^2Y - sy(0) - y'(0) - 2sY + 2y(0) + 2Y &= \frac{1}{s+1} \\ (s^2 - 2s + 1)Y &= \frac{1}{s+1} + 1 = \frac{s+2}{s+1} \\ Y &= \frac{s+2}{(s+1)(s^2-2s+2)} \\ &= \frac{1}{5} \frac{1}{s+1} + \frac{1}{5} \frac{-s+8}{(s-1)^2+1} \\ &= \frac{1}{5} \frac{1}{s+1} - \frac{1}{5} \frac{s-1}{(s-1)^2+1} + \frac{7}{5} \frac{1}{(s-1)^2+1}. \end{aligned}$$

By taking the inverse of Laplace, we have:

$$y(t) = \boxed{\frac{1}{5}e^{-t} - \frac{1}{5}e^t \cos t + \frac{7}{5}e^t \sin t}.$$

7.** Dirac delta function $\delta(t)$ is heuristically defined as:

$$\delta(t) = \begin{cases} +\infty, & \text{if } t = 0 \\ 0, & \text{if } t \neq 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t) dt = 1.$$

In *real analysis*, $\delta(t)$ is often called an “approximation to identity”, meaning that it “preserves” the original equation after convolution. By the definition of convolution for f and g , here, as:

$$(f * g)(t) = \int_0^t f(t - \tau)g(\tau)d\tau,$$

prove that $(f * \delta)(t) = f(t)$ for $t \geq 0$.

Hint: Use the convolution theorem and the Laplace transformation of step functions.

Solution:

If you have consulted with a few other texts, you might observe that the convolution formula here is different from the convolution formula in analysis textbooks, *i.e.*:

$$(f * g)(t) = \int_0^{\infty} f(\tau)g(t - \tau)d\tau.$$

In fact, the above definition allows better property, known as “approximation to identity” for the entire domain. However, we may show the version that you see in the ODEs course with a weaker conclusion.

Proof. Here, we apply the Laplace transformation on $f * \delta$, which is:

$$\begin{aligned} \mathcal{L}\{(f * \delta)(t)\} &= \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{\delta(t)\} \\ &= \mathcal{L}\{f(t)\} \cdot e^{-0s} \\ &= \mathcal{L}\{f(t)\}. \end{aligned}$$

Hence, we have demonstrated that:

$$(f * \delta)(t) = \begin{cases} 0, & \text{when } t < 0 \\ f(t), & \text{when } t \geq 0 \end{cases}$$

which implies that $(f * \delta)(t) = f(t)$ for $t \geq 0$. □

If you are interested in the concept of Dirac delta function, you can look up the conditions to be a “good kernel”, which proceeds further to a narrower class of kernels as approximations to the identity. Moreover, we suggest consulting some constructions of the kernels, such as the *shrinking function* or the series of *Gaussian bell curve*.

8. Let a system of differential equations be defined as follows, find the general solutions to those equations.

(a)
$$\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^2.$$

(b)*
$$\mathbf{x}' = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 1 & 3 \\ 0 & 4 & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^3.$$

Solution:

(a) The first question show be trivial, we first find the eigenvalues for the equation, *i.e.*:

$$\det \begin{pmatrix} 3 - \lambda & 0 \\ 0 & 2 - \lambda \end{pmatrix} = 0,$$

which is $(3 - \lambda)(2 - \lambda) = 0$, that is $\lambda_1 = 3$ and $\lambda_2 = 2$. Then, we look for the eigenvectors.

- For $\lambda_1 = 3$, we have $\begin{pmatrix} 3 - 3 & 0 \\ 0 & 2 - 3 \end{pmatrix} \xi_1 = \mathbf{0}$, which is $\xi_1 = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
- For $\lambda_2 = 2$, we have $\begin{pmatrix} 3 - 2 & 0 \\ 0 & 2 - 2 \end{pmatrix} \xi_2 = \mathbf{0}$, which is $\xi_2 = x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Hence, the solution is:

$$\mathbf{x} = C_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(b) Again, we first find the eigenvalues of the equation, *i.e.*:

$$\det \begin{pmatrix} 1 - \lambda & 0 & 4 \\ 1 & 1 - \lambda & 3 \\ 0 & 4 & 1 - \lambda \end{pmatrix} = 0,$$

which is $(1 - \lambda)^3 + 16 - 12(1 - \lambda) = -\lambda^3 + 3\lambda^2 + 9\lambda + 5 = -(\lambda + 1)^2(\lambda - 5) = 0$.

Hence, the eigenvalues are $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 5$. Now, we look for eigenvectors.

- For $\lambda_1 = -1$, we have $\begin{pmatrix} 2 & 0 & 4 \\ 1 & 2 & 3 \\ 0 & 4 & 2 \end{pmatrix} \xi_1 = \mathbf{0}$, which is $x \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix}$.
- For $\lambda_2 = -1$, we have $\begin{pmatrix} 2 & 0 & 4 \\ 1 & 2 & 3 \\ 0 & 4 & 2 \end{pmatrix} \eta = \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix}$, which is $\eta = \begin{pmatrix} 4x \\ x + 1 \\ -2x - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.
- For $\lambda_3 = 5$, we have $\begin{pmatrix} -4 & 0 & 4 \\ 1 & -4 & 3 \\ 0 & 4 & -4 \end{pmatrix} \xi_3 = \mathbf{0}$, which is $x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Hence, the solution is:

$$\mathbf{x} = C_1 e^{-t} \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix} + C_2 \left(t e^{-t} \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix} + e^{-t} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right) + C_3 e^{5t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

9. Solve the following initial value problem for the system of equations:

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Solution:

Here, we first find the eigenvalues for the matrix, that is:

$$\det \begin{pmatrix} 1 - \lambda & -4 \\ 4 & -7 - \lambda \end{pmatrix} = 0.$$

Therefore, the polynomial is $(1 - \lambda)(-7 - \lambda) + 16 = (\lambda + 3)^2 = 0$, hence the eigenvalues is $\lambda_1 = \lambda_2 = -3$. Then, we look for the eigenvectors.

- For $\lambda_1 = -3$, we have $\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \xi_1 = \mathbf{0}$, which is $\xi_1 = x \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
- For $\lambda_2 = -3$, we have $\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \eta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, which is $\eta = \begin{pmatrix} x \\ x - 1/4 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/4 \end{pmatrix}$.

Hence, the general solution is:

$$\mathbf{x} = C_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \left(t e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-3t} \begin{pmatrix} 0 \\ -1/4 \end{pmatrix} \right).$$

By the initial condition, we have $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, so:

$$\mathbf{x}(0) = \begin{pmatrix} C_1 + 0 \\ C_1 - C_2/4 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Therefore, $C_1 = 3$ and $C_2 = 4$, so the particular solution is:

$$\mathbf{x}(t) = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-3t} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} t e^{-3t}.$$

10.** (Putnam 2023.) Determine the smallest positive real number r such that there exists differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ satisfying:

- $f(0) > 0$,
- $g(0) = 0$,
- $|f'(x)| \leq |g(x)|$ for all x ,
- $|g'(x)| \leq |f(x)|$ for all x , and
- $f(r) = 0$.

You may give an answer without a rigorous proof, as the proof is out of scope of the course.

Hint: Assume that the function “moves” the fastest when the cap of the derivatives are “moving” the fastest, then think of constructing a dynamical system relating f and g .

Solution:

Here, we first provide a “simplified” case, *i.e.*, we are constructing a dynamical system in which we pick equality for the inequality, that is:

$$\begin{cases} |f'(x)| = |g(x)|, \text{ and} \\ |g'(x)| = |f(x)|. \end{cases}$$

Without loss of generality, we may assume that f and g are non-negative before r , so the system becomes:

$$\begin{cases} f' = -g \\ g' = f \end{cases},$$

or equivalently, $\mathbf{y} = \begin{pmatrix} f \\ g \end{pmatrix}$ that $\mathbf{y}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{y}$. Clearly, we observe the eigenvalues are $\pm i$ as the polynomial is $\lambda^2 + 1 = 0$. Moreover, the eigenvectors for $\lambda_1 = i$ is when $\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \boldsymbol{\xi} = \mathbf{0}$, in which

we have $\boldsymbol{\xi} = y \begin{pmatrix} i \\ 1 \end{pmatrix}$, and that solution is:

$$\mathbf{y} = \begin{pmatrix} i \\ 1 \end{pmatrix} e^{ix} = \begin{pmatrix} i \\ 1 \end{pmatrix} (\cos x + i \sin x) = \begin{pmatrix} -\sin x \\ \cos x \end{pmatrix} + i \begin{pmatrix} \cos x \\ \sin x \end{pmatrix}$$

and by conjugation, the solution should be:

$$\begin{pmatrix} f \\ g \end{pmatrix} = C_1 \begin{pmatrix} -\sin x \\ \cos x \end{pmatrix} + C_2 \begin{pmatrix} \cos x \\ \sin x \end{pmatrix}.$$

Note that with the given initial condition that $g(0) = 0$, this enforces $C_1 = 0$, thus $f(x) = C \cos x$ and $g(x) = C \sin x$, and we know that $f(r)$ is zero first at $r = \boxed{\pi/2}$.

The above version has some reasoning, but is not a rigorous proof at all, since this does not consider if r could be smaller than $\pi/2$. For students with interests, we provide the complete proof from the Putnam competition from Victor Lie, as follows.

Proof. Without loss of generality, we assume $f(x) > 0$ for all $x \in [0, r)$ as it is the first positive zero. By the fundamental theorem of calculus, we have:

$$|f'(x)| \leq |g(x)| \leq \left| \int_0^x g(s) ds \right| \leq \int_0^x |g(s)| ds \leq \int_0^t |f(s)| ds.$$

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Now, as we denote $F(x) = \int_0^x f(s)ds$, we have:

$$f'(x) + F(x) \geq 0 \text{ for } x \in [0, r].$$

For the sake of contradiction, we suppose $r < \pi/2$, then we have:

$$f'(x) \cos x + F(x) \cos x \geq 0 \text{ for } x \in [0, r].$$

Notice that the left hand side is the derivative of $f(x) \cos x + F(x) \sin x$, so an integration on $[y, r]$ gives:

$$F(r) \sin r \geq f(y) \cos y + F(y) \sin(y).$$

With some rearranging, we can have:

$$F(r) \sin r \sec^2 y \geq f(y) \sec y + F(y) \sin y \sec^2 y$$

Again, we integrate both sides with respect to y on $[0, r]$, which gives:

$$F(r) \sin^2 r \geq F(r),$$

and this is impossible, so we have a contradiction.

Hence we must have $r \geq \pi/2$, and since we have noted the solution $f(x) = C \cos x$ and $g(x) = C \sin x$, we have proven that $r = \pi/2$ is the smallest case. \square