Exam 1 Review Problem Set 1

Differential Equations

Summer 2025

Instructions:

The set of questions serves as PILOT practices to Exam 1 for the Summer 2025 term of `AS.110.302` Differential Equations and Applications at Johns Hopkins University.

- Questions marked with an asterisk are more challenging.
- The questions are designed to be done without notes and calculators.
- Solutions to selected questions will be discussed during the PILOT review session.

1. Solve the following initial value problem (IVP) on $y = y(x)$, and specify the domain for your solution:

$$\begin{cases} y' = (x \log x)^{-1}, \\ y(e) = -6. \end{cases}$$

2. Suppose $f(x)$ is non-zero, let an initial value problem be:

$$\begin{cases} \dfrac{1-y}{x} \cdot \dfrac{dy}{dx} = \dfrac{f(x)}{1+y}, \\ y(0) = 0. \end{cases}$$

   (a) Show that the differential equation is **not** linear.

   For the next two questions, suppose $f(x) = \tan x$.

   (b) State, <u>without</u> justification, the open interval(s) in which $f(x)$ is continuous.

   (c)* Show that there exists some $\delta > 0$ such that there exists a unique solution $y(x)$ for $x \in (-\delta, \delta)$.

   Now, suppose that $f(x)$ is some function, **not** necessarily continuous.

   (d)** Suppose that the condition in (c) does **not** hold, give three examples in which $f(x)$ could be.

3. Draw the phase line and determine the stability of each equilibrium for the following autonomous differential equations:

   (a) $$y' = y^4 - 3y^3 + 2y^2.$$

   (b) $$y' = y^{2025} - 1.$$

4. Let a differential equation be defined as follows:

$$\frac{dy}{dx} = e^{2x} + y - 1.$$

   (a) What is the integrating factor ($\mu(x)$) for the equation? Solve for the general solution.

   (b) Is the equation *exact*? If not, make it exact, then find the general solution.

   (c) Do solutions from part (a) and (b) agree?

5.* This brief digression to "differential forms" aims for the following goals:

- Legitimize $\frac{\partial y}{\partial x} = \frac{f(x)}{g(y)} \iff g(y)dy = f(x)dx$ via the differential operator d.
- Get the foundation of *exactness* for certain differential equation relationship.

First, consider variables $x_1, x_2, \cdots, x_n$, we may defined the wedge product ($\wedge$) to connect any two variables satisfying that:

$$x_i \wedge x_j = -x_j \wedge x_i \text{ for all } 1 \leq i, j \leq n.$$

   (a) Show that $x_i \wedge x_i = 0$ for $1 \leq i \leq n$.

Now, given any smooth function f, we defined the differential operator (d) as:

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

   (b) Suppose $y(x) = e^x$, find dy.

   (c) Now, suppose that $\frac{\partial y}{\partial x} = \frac{f(x)}{g(y)}$, can you express dy in terms of the differential form of x.
   Note: Since we have just one variable, we have $dy/dx = \partial y/\partial x$, leading to our first goal.

Furthermore, we can apply the differential operator over differential forms with wedge products already. Suppose:

$$\omega = \sum_{i_1, \cdots, i_k} f_{i_1, \cdots, i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

we may have the differential of ω as:

$$d\omega = \sum_{i_1, \cdots, i_k} (df_{i_1, \cdots, i_k}) dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

   (d) Suppose x, y are the variables, and $\omega = 2xy^2dx + 2x^2ydy$, show that $d\omega = 0$.

This then relates to a concept called *exactness* in differential equations. Consider the equation:

$$\frac{dy}{dx} + \frac{F(x,y)}{G(x,y)} = 0,$$

we can rewrite it as $F(x,y)dx + G(x,y)dy = 0$. Exactness enforces that:

$$\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}.$$

Similarly, exactness is considering finding a solution $f(x,y) = c$ such that $F = \frac{\partial f}{\partial x}$ and $G = \frac{\partial f}{\partial y}$.

(e) Show that $df = F(x, y)dx + G(x, y)dy$ and exactness is equivalently $d(df) = 0$.
Note: This implies that the differential equation in part (d) satisfies *exactness*.

6.* Determine if the following differential equation is exact. If not, find the integrating factor to make it exact. Then, solve for its general solution:,

$$y'(x) = e^{2x} + y(x) - 1.$$

7. For the first-order autonomous ODE:

$$\frac{dy}{dt} = \sin y + C,$$

where $C \in \mathbb{R}$ is a parameter. Determine any and all bifurcation values for the parameter C and sketch a bifurcation diagram.

8. Let a first order IVP on $y := y(t)$ be defined as follows:

$$\begin{cases} y' = \dfrac{2}{t}y, \\ y(1) = 1. \end{cases}$$

(a) Find the solution to the above initial value problem.

(b) Recall the theorem on existence and uniqueness, as follows:

For an IVP in simple form: $$\begin{cases} \dfrac{dy}{dt} = a(t)y + b(t), \\ y(t_0) = y_0. \end{cases}$$ For some $I = (\alpha, \beta) \ni t_0$, if $a(t)$ and $b(t)$ are continuous on the interval I. Then, there exists a unique solution to the IVP on the interval I.

Show that the IVP in this problem does not satisfy the condition for the existence and uniqueness theorem for $\mathbb{R}$.

(c) Does the above example violates the existence and uniqueness theorem? Why?

9. Solve the following second order differential equations for $y = y(x)$:

(a) $$y'' + y' - 132y = 0.$$

(b) $$y'' - 4y' = -4y.$$

(c) $$y'' - 2y' + 3y = 0.$$

10.* Given the following second order initial value problem:

$$\begin{cases} \dfrac{d^2y}{dx^2} + \cos(1-x)y = x^2 - 2x + 1, \\ y(1) = 1, \\ \dfrac{dy}{dx}(1) = 0. \end{cases}$$

Prove that the solution $y(x)$ is symmetric about $x = 1$, *i.e.*, satisfying that $y(x) = y(2-x)$.
Hint: Consider the interval in which the solution is unique.