Exam 3 Review Problem Set 5: Solutions

Differential Equations

Summer 2025

1. Let an initial value problem for linear system on $x_1 := x_1(t)$ and $x_2 := x_2(t)$ be defined as follows:

$$\begin{cases} x_1' = 3x_1 - 2x_2, & x_1(0) = 3, \\ x_2' = 2x_1 - 2x_2, & x_2(0) = \dfrac{1}{2}. \end{cases}$$

   (a) Solve for the *general solution* for the linear system by considering $\mathbf{x} = (x_1, x_2)$.

   (b) Transform the *general system* into a single equation of second order. Then solve the second-order equation. Eventually, convert your solution of one variable back to the *general solution* to $x_1(t)$ and $x_2(t)$.

   (c) Find the particular solution using the initial conditions, then graph the parameterized curve on a x_1x_2-plane with $t \geq 0$.

Solution:

(a) Here, when we consider the general linear system, we have the matrix as:

$$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} .\mathbf{x}.$$

Hence, we look for the eigenvalues for the matrix, that is when:

$$0 = \det \begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix} = (3-\lambda)(-2-\lambda) - (-2) \times 2 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1).$$

Hence the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$, we solve for the eigenvectors:

- For $\lambda_1 = 2$, we want $\boldsymbol{\xi}^{(1)}$ satisfy that $\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} .\boldsymbol{\xi}^{(1)} = \mathbf{0}$, that is $\xi_1^{(1)} = 2\xi_2^{(1)}$, so the eigenvector is $(2, 1)$.
- For $\lambda_2 = -1$, we want $\boldsymbol{\xi}^{(2)}$ satisfy that $\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} .\boldsymbol{\xi}^{(2)} = \mathbf{0}$, that is $2\xi_1^{(2)} = \xi_2^{(2)}$, so the eigenvector is $(1, 2)$.

Therefore, the general solution for the system of differential equation is:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{x} = C_1 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \boxed{\begin{pmatrix} 2C_1 e^{2t} + C_2 e^{-t} \\ C_1 e^{2t} + 2C_2 e^{-t}. \end{pmatrix}}.$$

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(b) By manipulating the first differential equation, we are able to obtain that:

$$x_2 = \frac{3}{2}x_1 - \frac{1}{2}x_1',$$

whose derivative is correspondingly:

$$x_2' = \frac{3}{2}x_1' - \frac{1}{2}x_1''.$$

Then, we substitute into the second differential equation to obtain that:

$$\frac{3}{2}x_1' - \frac{1}{2}x_1'' = 2x_1 - 2\left(\frac{3}{2}x_1 - \frac{1}{2}x_1'\right),$$

$$x_1'' - x_1' - 2x_1 = 0.$$

By solving the second order linear differential equation on $x_1 = x_1(t)$, its characteristic equation is $r^2 - r - 2 = 0$, which factors to $(r-2)(r+1)$, so the solution is:

$$x_1 = \boxed{D_1 e^{2t} + D_2 e^{-t}}.$$

Now, we take the derivative of x_1 to obtain that:

$$x_1' = 2D_1 e^{2t} - D_2 e^{-t},$$

so we can plug it into equation for x_2, that is:

$$x_2 = \frac{3}{2}(D_1 e^{2t} + D_2 e^{-t}) - \frac{1}{2}(2D_1 e^{2t} - D_2 e^{-t}) = \boxed{\frac{1}{2}D_1 e^{2t} + 2D_2 e^{-t}}.$$

Note: We hope that the readers have already realized that this is equivalent to the solution in part (a), simply having $D_1 = 2C_1$ and $D_2 = C_2$.

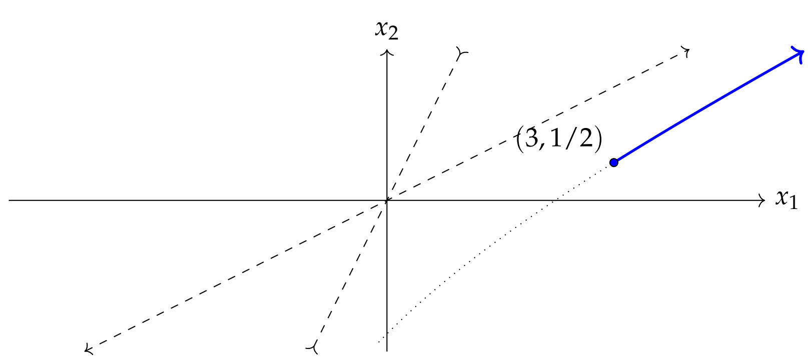
(c) Since the results are equal up to multiplying constants, we use the results from part (a). By plugging in the solutions, we have:

$$\begin{cases} 3 = x_1(0) = 2C_1 + C_2, \\ \frac{1}{2} = x_2(0) = C_1 + 2C_2. \end{cases}$$

This induces the solution that $3C_1 + 3C_2 = 7/2$, so $C_1 + C_2 = 7/6$, and thus $C_1 = 11/6$ and $C_2 = -2/3$, so the particular solution is:

$$x_1 = \boxed{\frac{11}{3}e^{2t} - \frac{2}{3}e^{-t}} \qquad x_2 = \boxed{\frac{11}{6}e^{2t} - \frac{4}{3}e^{-t}}.$$

The graph on the x_1x_2-plane can be visualized as follows:



2. Solve the following initial value problem, represent your solution as a fundamental matrix:

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Solution:
Here, we first find the eigenvalues for the matrix, that is:

$$\det \begin{pmatrix} 1-\lambda & -4 \\ 4 & -7-\lambda \end{pmatrix} = 0.$$

Therefore, the polynomial is $(1-\lambda)(-7-\lambda)+16 = (\lambda+3)^2 = 0$, hence the eigenvalues is $\lambda_1 = \lambda_2 = -3$. Then, we look for the eigenvectors.

- For $\lambda_1 = -3$, we have $\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \boldsymbol{\xi}_1 = \mathbf{0}$, which is $\boldsymbol{\xi}_1 = x \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
- For $\lambda_2 = -3$, we have $\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \boldsymbol{\eta} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, which is $\boldsymbol{\eta} = \begin{pmatrix} x \\ x-1/4 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/4 \end{pmatrix}$.

Hence, the general solution is:

$$\mathbf{x} = C_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \left(te^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-3t} \begin{pmatrix} 0 \\ -1/4 \end{pmatrix} \right).$$

By the initial condition, we have $\mathbf{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, so:

$$\mathbf{x}(0) = \begin{pmatrix} C_1 + 0 \\ C_1 - C_2/4 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Therefore, $C_1 = 3$ and $C_2 = 4$, so the particular solution is:

$$\mathbf{x}(t) = \boxed{\begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-3t} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} te^{-3t}}.$$

3. Let a system of $\mathbf{x} = (x_1, x_2)$ be defined as:

$$\mathbf{x}' = \begin{pmatrix} -3 & -6 \\ 1 & 2 \end{pmatrix} \cdot \mathbf{x}.$$

(a) Find the eigenvalues and eigenvectors for $\begin{pmatrix} -3 & -6 \\ 1 & 2 \end{pmatrix}$.

(b) Give a full set of solutions to the differential equation. Plot some trajectory on the x_1x_2-plane.

(c)* Let A be an arbitrary square matrix. Show that A is non-invertible if and only if A has zero as an eigenvalue.

Note: Please avoid using the definition that the determinant is the product of all eigenvalues. Moreover, consider the geometric implication of eigenvalue to account for invertibility.

Solution:

(a) Similarly, we find eigenvalue by:

$$0 = \det \begin{pmatrix} -3-\lambda & -6 \\ 1 & 2-\lambda \end{pmatrix} = (-3-\lambda)(2-\lambda) + 6 = \lambda^2 + \lambda = \lambda(\lambda+1),$$

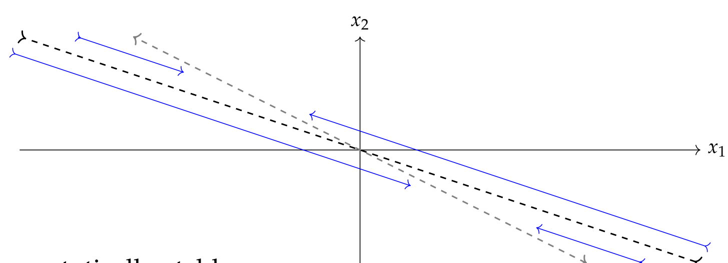
so the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -1$. Now, we look for the eigenvectors, as follows:

- For $\lambda_1 = 0$, we want $\begin{pmatrix} -3 & -6 \\ 1 & 2 \end{pmatrix} .\boldsymbol{\xi}^{(1)} = \mathbf{0}$, so $\xi_1^{(1)} = -2\xi_2^{(1)}$, so the eigenvector is $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.
- For $\lambda_2 = -1$, we want $\begin{pmatrix} -2 & -6 \\ 1 & 3 \end{pmatrix} .\boldsymbol{\xi}^{(2)} = \mathbf{0}$, so $\xi_1^{(2)} = -3\xi_2^{(2)}$, so the eigenvector is $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

(b) With the eigenvectors and eigenvalues, we trivially obtain the full set of solutions as:

$$\boxed{\left\{ C_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix}, C_2 e^{-t} \begin{pmatrix} 3 \\ -1 \end{pmatrix} \right\}}.$$

Graphically, the plot of some trajectories can be visualized as:



The system is asymptotically stable.

(c) Here, we advised readers not to consider the definition of determinant being the product of all eigenvalues, since this equivalence is trivial by that definition. Alternatively, we want readers to think of the definition of determinants in the scope of this class, *i.e.*, the root of $\det(A - \lambda\,\mathrm{Id})$.

Proof. Here, we have the following equivalences:

$$A \text{ is non-invertible} \iff \det(A) = 0 \iff \det(A - 0\,\mathrm{Id}) = 0 \iff 0 \text{ is an eigenvalue of } A.$$

As desired. □

Geometrically, think of eigenvalue monitoring the action in terms of a scalar multiplication, so having a zero eigenvalue collapse a dimension, making the matrix unable to be bijective. Conversely, if the map is not injective, we can easily form the kernel as a subspace whose eigenvalue is zero.

4. Let M be a square matrix, M is defined to be *nilpotent* if $M^k = 0$ for some positive integer k.

(a) Show that $N = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is nilpotent, then write down the result of $\exp(N)$.

Now, suppose that $N \in \mathcal{L}(\mathbb{R}^n)$ is a square matrix and is *nilpotent*.

(b) If all the entries in N are rational, show that $\exp(N)$ has rational entries.

(c)* Suppose that $\mathrm{Id}_n \in \mathcal{L}(\mathbb{R}^n)$ is the identity matrix, prove that $\mathrm{Id}_n + N$ is invertible.
Hint: Use the differences of squares for matrices.

Solution:

(a) *proof of N is nilpotent.* Here, we want to do the matrix multiplication:

$$N^2 = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$N^3 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now, we have shown that $N^3 = 0$, or the zero matrix, hence N is nilpotent. □

Then, we want to calculate the matrix exponential, that is:

$$\exp(N) = \sum_{k=0}^{\infty} \frac{1}{k!} N^k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}.$$

(b) *Proof.* By the definition that N is nilpotent, we know that $N^m = 0$ for some finite positive integer m, hence, we can make the (countable) infinite sum into a finite sum:

$$\exp(N) = \sum_{k=0}^{\infty} \frac{1}{k!} N^k = \sum_{k=0}^{m} \frac{1}{k!} N^k,$$

thus all the entries are sum and non-zero divisions of rational numbers, while rational numbers are closed under addition and non-zero divisions, hence, all entries of $\exp(N)$ is rational. □

(c) *Proof.* Here, we recall the differences of squares still works when commutativity for multiplications fails, hence the we can still use it for matrix multiplication, namely, for all $m \in \mathbb{Z}^+$:

$$(\mathrm{Id}_n + N) \cdot (\mathrm{Id}_n - N) \cdot (\mathrm{Id}_n + N^2) \cdots (\mathrm{Id}_n + N^{2^m}) = \mathrm{Id}_n - N^{2^{m+1}}$$

Since N is *nilpotent*, this implies that we have some k such that $N^\ell = 0$ for all $\ell \geq k$. Meanwhile, note that $2^\ell \geq \ell$ for all positive integer ℓ. (This can be proven by induction.) Therefore, we select $m + 1 \geq k$ so that $N^{2m+1} = 0$, and we have:

$$(\mathrm{Id}_n + N) \cdot \left[(\mathrm{Id}_n - N) \cdot (\mathrm{Id}_n + N^2) \cdots (\mathrm{Id}_n + N^{2^m}) \right] = \mathrm{Id}_n,$$

thus $\mathrm{Id}_n + N$ is invertible. □

Note that the elements of all n-by-n matrices can be considered as a *ring*, while *nilpotent* can be defined more generally for *rings*. We invite capable readers to investigate more properties of *nilpotent* elements of *rings* in the discipline of *Abstract Algebra*.

5. Suppose a matrix $M \in \mathcal{L}(\mathbb{R}^2)$ is a *rotational matrix* by an angle θ (counter-clockwise), then:

$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

(a) Show that $M^\intercal = M^{-1}$.

(b)* Let $\theta = 2\pi/k$ be fixed, where k is an integer. Find the least positive integer n such that $M^n = \mathrm{Id}_2$. Here, n is called the *order* of M.

Hint: Consider the rotational matrix geometrically, rather than arithmetically.

(c)* Let $\theta = \pi/2$, calculate the matrix exponential $\exp(M)$.

Hint: Consider the *order* of M and the Taylor series of e^x, e^{-x}, $\sin x$ and $\cos x$.

Solution:

(a) *Proof.* Here, we recall the method of inverting a matrix:

$$M^{-1} = \frac{1}{\det M}\begin{pmatrix} \cos\theta & -(-\sin\theta) \\ -\sin\theta & \cos\theta \end{pmatrix} = \frac{1}{\cos^2\theta + \sin^2\theta}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = M^\intercal.$$

□

(b) Look, we want to analyze this geometrically, if $\theta = 2\pi/k$, then that implies that M is a counter-clockwise rotation of $2\pi/k$, and since a full revolution is 2π, this implies a rotation of k times will make restore to the original vector, *i.e.*, $M^k = \mathrm{Id}_2$. Moreover, for any positive integer less than k, we cannot rotate back to 2π, which implies that the order of M is $\boxed{k}$.

(c) Here, we construct the matrix exponential, note that the order of M is 4, we have:

$$\exp(M) = \sum_{k=0}^{\infty} \frac{1}{k!} M^k.$$

Here, we want to consider each entry respectively, since each entry is finite and since M has order 4, the absolute value of the sum of the entries must be finite, so each entry converges *absolutely*, hence we are free to change the order of the sum, so we have:

$$\exp(M) = \sum_{k=0}^{\infty} \frac{1}{(4k+1)!} M + \sum_{k=0}^{\infty} \frac{1}{(4k+2)!} M^2 + \sum_{k=0}^{\infty} \frac{1}{(4k+3)!} M^3 + \sum_{k=0}^{\infty} \frac{1}{(4k)!}\,\mathrm{Id}\,.$$

For the 4 sums of factorials, we note that the Taylor series of e^x, e^{-x}, $\sin x$ and $\cos x$ at 0 evaluated at $x = 1$ are, respectively:

$$\begin{array}{lcccccccc}
e^1 & = & +\frac{1}{0!} & +\frac{1}{1!} & +\frac{1}{2!} & +\frac{1}{3!} & +\frac{1}{4!} & +\frac{1}{5!} & +\cdots \\
e^{-1} & = & +\frac{1}{0!} & -\frac{1}{1!} & +\frac{1}{2!} & -\frac{1}{3!} & +\frac{1}{4!} & -\frac{1}{5!} & +\cdots \\
\sin 1 & = & & +\frac{1}{1!} & & -\frac{1}{3!} & & +\frac{1}{5!} & -\cdots \\
\cos 1 & = & +\frac{1}{0!} & & -\frac{1}{2!} & & +\frac{1}{4!} & & -\cdots
\end{array}$$

Since the first series converges, we know that the later three series converges *absolutely*, so we are free to move around terms.

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From the expressions, by columns, we can observe that:

$$\sum_{k=0}^{\infty}\frac{1}{(4k+1)!}=\frac{e^1-e^{-1}}{4}+\frac{\sin 1}{2}, \qquad \sum_{k=0}^{\infty}\frac{1}{(4k+2)!}=\frac{e^1+e^{-1}}{4}-\frac{\cos 1}{2},$$

$$\sum_{k=0}^{\infty}\frac{1}{(4k+3)!}=\frac{e^1-e^{-1}}{4}-\frac{\sin 1}{2}, \qquad \sum_{k=0}^{\infty}\frac{1}{(4k)!}=\frac{e^1+e^{-1}}{4}+\frac{\sin 1}{2}.$$

Now, we shall also evaluate the matrices generated by M, that is:

$$M=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \qquad M^2=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix},$$

$$M^3=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \qquad M^4=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Therefore, considering the four entries $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, we have:

$$\begin{aligned}
a &= -\frac{e+1/e}{4}+\frac{\cos 1}{2}+\frac{e+1/e}{4}+\frac{\sin 1}{2}=\frac{\cos 1+\sin 1}{2},\\
b &= -\frac{e-1/e}{4}-\frac{\sin 1}{2}+\frac{e-1/e}{4}-\frac{\sin 1}{2}=-2\sin 1,\\
c &= \frac{e-1/e}{4}+\frac{\sin 1}{2}-\frac{e-1/e}{4}+\frac{\sin 1}{2}=2\sin 1,\\
d &= -\frac{e+1/e}{4}+\frac{\cos 1}{2}+\frac{e+1/e}{4}+\frac{\sin 1}{2}=\frac{\cos 1+\sin 1}{2}.
\end{aligned}$$

Therefore, the matrix exponential is:

$$\exp(M)=\boxed{\begin{pmatrix}\dfrac{\cos 1+\sin 1}{2} & -2\sin 1\\ 2\sin 1 & \dfrac{\cos 1+\sin 1}{2}\end{pmatrix}}.$$

In particular, mathematicians has considered the *rotation* and *flipping* of regular polygons as the *dihedral groups*, where symmetries and combinatorics play an important role. Please think of ways you may “manipulate” a polygon such that the polygon looks the same.

6. Let a system of non-linear differential equations be defined as follows:

$$\begin{cases} x' = x - y^2, \\ y' = x + x^2 - 2y. \end{cases}$$

Find all equilibrium(s) and classify their stability locally.

Solution: For the first case, we notice that the equilibrium points are if:

$$\begin{cases} x - y^2 = 0, \\ x + x^2 - 2y = 0. \end{cases}$$

Note that this will be two parabolas, and there are at most two intersections, and we observe the intersections $(0, 0)$ and $(1, 1)$. Also to note, the Jacobian matrix is:

$$J = \begin{pmatrix} 1 & -2y \\ 1 + 2x & -2 \end{pmatrix}.$$

- For the $(0, 0)$ case, we denoting $\mathbf{x} = (x, y)$, we verify the linear approximation as:

$$\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix} \mathbf{x},$$

  and we note that the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -2$, and by:

$$\lambda_2 < 0 < \lambda_1,$$

  we know that we have a unstable saddle point at $(0, 0)$.

- For the $(1, 1)$ case, we have the linear approximation as:

$$\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x - 1 \\ y - 1 \end{pmatrix},$$

  and we note the eigenvalues are $\lambda = \frac{-1 \pm i\sqrt{15}}{2}$, which is complex with a negative real part, so we have a $\boxed{\text{asymptotically stable spiral point}}$.

7. Let a system of (x, y) be functions of variable t, and they have the following relationship:

$$x' = (1+x)\sin y \text{ and } y' = 1 - x - \cos y.$$

(a) Identify the corresponding linear system.

(b) Evaluate the stability for the equilibrium at $(0,0)$ by showing it is locally linear.

Solution:

(a) Here, since we can write:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} (1+x)\sin y \\ 1 - \cos y \end{pmatrix},$$

this implies that the linear system is:

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}}.$$

(b) $(0,0)$ *is locally linear.* We find the Jacobian Matrix, that is:

$$\mathbf{J} = \begin{pmatrix} \sin y & (1+x)\cos y \\ -1 & \sin y \end{pmatrix}.$$

As we evaluate $\mathbf{J}$ at $(0,0)$ and take its determinant, we have:

$$\det\left(\mathbf{J}\big|_{(0,0)}\right) = \det\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 1 \neq 0.$$

Hence, the $(0,0)$ is locally linear. □

Note that we have found the linear system in part (a), whose eigenvalues are $\lambda_1 = \lambda_2 = 0$. Since $x' = 0$, it indicates that x is a constant, whereas for $y' = -x$ indicates that it will be a $\boxed{\text{unstable almost everywhere}}$ for all neighborhoods of $(0,0)$.

In particular, readers could illustrate the "slope field" for the linear system in (a), and they should notice that except for $x = 0$ being entirely stable, all other trajectory would shift vertically at a constant rate. However, the line $x = 0$ will always be insignificant enough (having *Lebesgue measure* 0), hence we claim that it is unstable almost everywhere. For interested readers, please explore *Lebesgue measure* as a way to determine how large a subset is in Euclidean space.

8. Suppose D and R are two parties on a non-existing country on the center of Mars. For the simplicity of this problem, they, *unfortunately*, have no elections. Therefore, we can model the amount of the supporter for each party (in millions), denoted x_D and x_R with the following relationship:
$$\begin{cases} \dfrac{dx_D}{dt} = x_D(1 - x_D - x_R), \\ \dfrac{dx_R}{dt} = x_R(3 - 2x_D - 4x_R). \end{cases}$$
Find all possible endings (say arbitrarily long after, that is $t \to \infty$) of the number of supporters (in millions) for the two parties.

Solution:
Alright, we do assume that these parts are just having random letters, so we do not dive into actual politics. This is a typical "Competing Species" models, in which the two parties are competing for the limited resources.
First find the critical points, that is:
$$\begin{cases} x_D(1 - x_D - x_R) = 0, \\ x_R(3 - 2x_D - 4x_R) = 0. \end{cases}$$
We know that we could have many cases:

- When $x_D = x_R = 0$, we have both equations being 0.
- When $x_D = 0$, we can also have $x_R = 3/4$.
- When $x_R = 0$, we can also have $x_D = 1$.
- Or, we can have $1 - x_D - x_R = 0$ and $3 - 2x_D - 4x_R = 0$, and in this case, we have $x_D = x_R = 1/2$.

Here, we note that the Jacobian matrix is:
$$J[x_D', x_R'] = \begin{pmatrix} 1 - 2x_D & -x_D \\ -2x_R & 3 - 8x_R \end{pmatrix}.$$
It can be easily verified that the system are locally linear at all of the critical points, and we leave this as an exercise to the readers.
Here, as we inspect these points, we could conclude that:

- Case 1: $x_D = x_R = 0$. This is when two party has no supporting population, it they remain uncared.
- Case 2: $x_D = 0$ and $x_R = 3/4$. This is when the R party got the support initially, and they keep absolute advantage over the D party.
- Case 3: $x_R = 0$ and $x_D = 1$. This is when the D party got the support initially, and they keep absolute advantage over the R party.
- Case 4: $x_D = x_R = 1/2$. This is when two party all got some initial support. While after some political campaigns (maybe also fightings), they got to a balanced equilibrium.

If the readers are still confused, we recommend them taking a look at the *directional field*. A website of the directional field can be found here.

9. Suppose the tariff system in Mars (between all countries there) is based on the same formula, which is as follows:
$$\Delta\tau_i = \frac{x_i - m_i}{\varepsilon \times \varphi \times m_i}.$$
Here, $\Delta\tau_i$ means the change in tariff, x_i means the total import sale into your country from country i, m_i means the total export sale from your country to the country i, and $\varepsilon \times \varphi$ is 2.
Furthermore, a numerical estimation method in ODEs is called *Euler's Method*, and we will use the reverse of that to obtain an ODE model that:
$$\frac{d\tau(t)}{dt} \approxeq \frac{x(t) - m(t)}{2m(t)}.$$
Now, suppose there is another county, and you want to analyze the trends of tariffs with that country. With ϑ denoting their country's tariff on your country's import, we can create a system.
$$\begin{cases} \dfrac{d\tau(t)}{dt} = \dfrac{x(t) - m(t)}{2m(t)}, \\ \dfrac{d\vartheta(t)}{dt} = \dfrac{m(t) - x(t)}{2x(t)}. \end{cases}$$
For the simplicity of economics, we can model the import sale and export sale as:
$$x(t) = a - b\tau(t) \qquad \text{and} \qquad m(t) = c - d\vartheta(t),$$
where a, b, c, d are positive real constants.
   (a) Write down the system of differential equations to model the tariffs as a vector $\mathbf{x}(t) = \big(\tau(t), \vartheta(t)\big)$.
   (b) Find the set of all equilibrium points on this nonlinear system.
   (c)* Interpret some issues with the assumptions of this model.

Solution:

(a) We can just very simply replace the factors into:
$$\boxed{\begin{pmatrix} \tau \\ \vartheta \end{pmatrix}' = \begin{pmatrix} \dfrac{(a-c) + (d\vartheta - b\tau)}{2(c - d\vartheta)} \\ \dfrac{(c-a) + (b\tau - d\vartheta)}{2(a - b\tau)} \end{pmatrix}}.$$

(b) The find the set of all equilibrium points, we let both derivatives to be zero, and the readers should observe that it is simply:
$$a + d\vartheta = c + b\tau.$$
In set notation, it can be written as:
$$\boxed{\{(\tau, \vartheta) : a + d\vartheta = c + b\tau\}}.$$
If you are familiar with high school algebra, you should notice that this a line with negative slope on the $\tau\vartheta$-plane.

(c) For this model, it does exhibit some unreasonable assumptions:
   - Note that $x(t)$ and $m(t)$ are a rough approximation of the "indifference curve," even if the tariff is very high, the normal economic model would not have negative production, and it should not be linear over the whole scope.
   - Still, we are approximating τ and ϑ as continuous time variables, the rate of change for the tariffs would be discrete.

10.* Let a locally linearly system be defined as:

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix} \mathbf{x} + \mathbf{f}(\mathbf{x}),$$

where $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is a vector-valued function. Find the necessary condition(s) in which the equilibrium(s) have a stable *center* in linear system. Then, state the stability and type (if possible).
Hint: Consider the solution for the linear case or matrix exponential.

Solution:
Without loss of generality, we assume that the system of $\mathbf{x}$ has equilibrium(s), else the statement is vacuously true. Now, we start to evaluate the additional conditions with such assumption:

(i) Note that the system needs to be locally linearly, *i.e.*, we must have $\boxed{\mathbf{f}(\mathbf{x}) \text{ being twice differentiable with respect to partial derivatives}}$.

(ii) Moreover, we need to worry about the linear system to have a *stable center*, that is:

$$\mathbf{x}' = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix} \mathbf{x}.$$

Note that the eigenvalues would be the solutions to $(\lambda - r)^2 + \mu^2 = 0$, that is $r = \lambda \pm i\mu$, which is a pair of complex conjugates. Here, in to be stable, we want $\lambda \leq 0$, and for center, this forces $\boxed{\lambda = 0}$.

Note that even the linear system is a stable center, the stability of the non-linear system is $\boxed{\text{indeterminate}}$, and the type is $\boxed{\text{center or spiral point}}$.