PILOT Exam 2 Review

Differential Equations

Johns Hopkins University

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As you prepare for exam 2, please consider the following resources:

- PILOT webpage for ODEs: `https://jhu-ode-pilot.github.io/SU25/`
  - Find the review problem sets for exam 2.
  - Consult the archives page for PILOT sets from the semester.
- Review the *homework/quiz sets* provided by the instructor.
- Join the PILOT Exam 2 Review Session. (You are here.)

Plan for today:

1. Go over all contents that we have covered for this semester so far.
2. In the end, we will open the poll to you. Please indicate which problems from the Review Set that you want us to go over.

Part 1:
Contents Review

We will get through all contents over this semester.

- Feel free to download the slide deck from the webpage and annotate on it.
- If you have any questions, ask by the end of each chapter.

Second Order ODEs (Continued)

- Non-homogeneous Cases
  - Variation of Parameters
  - Undetermined Coefficients

Let the differential equation be:

$$Ay''(t) + By'(t) + Cy(t) = g(t),$$

where $g(t)$ is a smooth function. Let $y_1(t)$ and $y_2(t)$ be the two homogeneous solutions, then the non-homogeneous cases can be solved by the following approaches:

Variation of Parameters

The particular solution of the differential equation can be written as the integrals of respective parts.

$$y_p = y_1(t) \int \frac{-y_2(t) \cdot g(t)}{W} dt + y_2(t) \int \frac{y_1(t) \cdot g(t)}{W} dt.$$

Another approach is less calculation intensive, but requires the function $g(t)$ to be constrained in certain forms.

Undetermined Coefficients

A guess of particular solution will be made based on the terms appearing in the non-homogeneous part, or $g(t)$. Some brief strategies are:

Non-homogeneous Comp. in $g(t)$		**Guess**
Polynomials:	$\sum_{i=0}^{d} a_i t^i$	$\sum_{i=0}^{d} C_i t^i$
Trig.:	$\sin(at)$ and $\cos(at)$	$C_1 \sin(ax) + C_2 \sin(ax)$
Exp.:	e^{at}	Ce^{at}

Note that the guess are additive and multiplicative. Moreover, if the non-homogeneous part already appears in the homogeneous solutions, an extra t needs to be multiplied on the non-homogeneous case.

Higher Order ODEs

- Existence and Uniqueness Theorem
- Homogeneous Cases
  - Complex Characteristic Roots
  - Repeated Characteristic Roots
- Linear Independence
  - Definition of Linearly Independence
- Abel's Formula
- Non-Homogeneous Cases
  - Variation of Parameters
  - Undetermined Coefficients

For higher order IVP in form:

$$\begin{cases} y^{(n)} + P_{n-1}(t)y^{(n-1)} + \cdots + P_1(t)y' + P_0(t)y = g(t), \\ y(t_0) = y_0,\ y'(t_0) = y_1,\ \cdots\ , y^{(n-1)}(t_0) = y_{n-1}. \end{cases}$$

If $P_0(t), P_1(t), \cdots, P_{n-1}(t)$, and $g(t)$ are continuous on an interval I containing t_0. Then there exists a unique solution for $y(t)$ on I.

Only Contrapositive is Guaranteed to be True

Again, for this theorem, you can conclude that if *there does not exist a solution or the solution is not unique*, then *the conditions must not be satisfied*. You **cannot** conclude that if *the conditions are not satisfied*, then *there is no unique solution*.

The higher order homogeneous ODEs are in form:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

By computing the characteristic equation $r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0$, with solutions $r_1, r_2, \cdots, r_n$, the general solution is $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}$.

Complex Characteristic Roots

If the solutions are complex, by Euler's Formula ($e^{it} = \cos t + i \sin t$), it can be written as $r_1 = \lambda + i\beta$ and $r_2 = \lambda - i\beta$, then the solution is:

$$y(t) = c_1 e^{\lambda t}\cos(\beta t) + c_2 e^{\lambda t}\sin(\beta t) + \text{ rest of the solutions.}$$

Repeated Characteristic Roots

If the solutions are repeated with multiplicity m, the solution is:

$$y(t) = c_1 e^{rt} + c_2 t e^{rt} + \cdots c_m t^{m-1} e^{rt} + \text{ rest of the solutions.}$$

To obtain the fundamental set of solutions, the Wronskian (W) must be non-zero, where Wronskian is:

$$W[y_1, y_2, \cdots, y_n] = \det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{pmatrix}.$$

Definition of Linearly Independence

By definition, a set of polynomials $\{f_1, f_2, \cdots, f_n, \cdots\}$ is linearly independent when for $\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots \in \mathbb{F}$ (typically $\mathbb{C}$):

$$\lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_n f_n + \cdots = 0 \iff \lambda_1 = \lambda_2 = \cdots = \lambda_n = \cdots = 0.$$

For higher order ODEs in the form of:

$$\begin{cases} y^{(n)} + P_{n-1}(t)y^{(n-1)} + \cdots + P_1(t)y' + P_0(t)y = g(t), \\ y(t_0) = y_0,\ y'(t_0) = y_1,\ \cdots\ , y^{(n-1)}(t_0) = y_{n-1}. \end{cases}$$

Its Wronskian is:

$$W[y_1, y_2, \cdots, y_n] = Ce^{\int -P_{n-1}(t)dt},$$

where C is independent of t but depend on $y_1, y_2, \cdots, y_n$.

Let the differential equation be:

$$L[y^{(n)}(t), y^{(n-1)}(t), \cdots, y(t)] = g(t),$$

where $g(t)$ is a smooth function. Let $y_1(t)$, $y_2(t)$, $\cdots$, $y_n(t)$ be all homogeneous solutions, then the non-homogeneous cases can be solved by the following approaches:

Variation of Parameters

The particular solution is:

$$y_p = y_1(t)\int \frac{W_1 g}{W}dt + y_2(t)\int \frac{W_2 g}{W}dt + \cdots + y_n(t)\int \frac{W_n g}{W}dt,$$

where W_i is defined to be the Wronskian with the i-th column alternated into $\begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}^\intercal$.

Undetermined Coefficients

Same as in degree 2, a guess of particular solution will be made based on the terms appearing in the non-homogeneous part, or $g(t)$. Some brief strategies are:

Non-homogeneous Comp. in $g(t)$		**Guess**
Polynomials:	$\sum_{i=0}^{d} a_i t^i$	$\sum_{i=0}^{d} C_i t^i$
Trig.:	$\sin(at)$ and $\cos(at)$	$C_1 \sin(ax) + C_2 \cos(ax)$
Exp.:	e^{at}	Ce^{at}

Again, the guess are additive and multiplicative. Moreover, if the non-homogeneous part already appears in the homogeneous solutions, an extra t needs to be multiplied on the non-homogeneous case.

Laplace Transformation

- Laplace Transformation
  - Properties of Laplace Transformation
- Elementary Laplace Transformations
- Step Functions:
  - Second Shifting Theorem
- Impulse Functions
  - Laplace Transformation of Impulse Function
- Convolution

The Laplace Transformation of a function f is defined as:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t) dt.$$

Properties of Laplace Transformation

1. Laplace Transformation is a linear operator: $\mathcal{L}\{f + \lambda g\} = \mathcal{L}\{f\} + \lambda \mathcal{L}\{g\}$.
2. Laplace Transformation for derivatives:

$$\begin{aligned} \mathcal{L}\{f'(t)\} &= s\mathcal{L}\{f(t)\} - f(0), \\ \mathcal{L}\{f''(t)\} &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0), \\ &\vdots \\ \mathcal{L}\{f^{(n)}(t)\} &= s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0). \end{aligned}$$

3. First Shifting Theorem: $\mathcal{L}\{e^{ct} f(t)\} = F(s - c)$.

Here are the Laplace Transformation of some elementary functions, which can also be calculated by definition:

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}, s > 0$
e^{at}	$\frac{1}{s-a}, s > a$
$t^n, n \in \mathbb{Z}_{>0}$	$\frac{n!}{s^{n+1}}, s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}, s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}, s > 0$
$\sinh(at)$	$\frac{a}{s^2-a^2}, s > 0$
$\cosh(at)$	$\frac{s}{s^2-a^2}, s > 0$
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$

Laplace Transformations can be used for solving IVP, with the derivative rules and inverse operation.

The step functions are defined by:

$$u_c(t) = u(t-c) = \begin{cases} 0, & t < c, \\ 1, & t \geq c. \end{cases}$$

And the Laplace Transformations of the step function is:

$$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}.$$

Second Shifting Theorem

The step function forms the Second Shifting Theorem:

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s).$$

The idealized unit impulse function $\delta(t)$, or *Dirac delta function*, satisfies the properties that:

$$\delta(t) = 0 \text{ for } t \neq 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(t)dt = 1.$$

There is no ordinary function satisfying the Dirac delta function, it is a generalized function (or *distribution*).
A unit impulse at an arbitrary point $t = t_0$, denoted by $\delta(t - t_0)$, follows that:

$$\delta(t) = 0 \text{ for } t \neq t_0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(t - t_0)dt = 1.$$

Laplace Transformation of Impulse Function

The Laplace Transformation of the impulse function is:

$$\mathcal{L}\{\delta(t - c)\} = e^{-cs}.$$

The convolution of f and g, denoted $(f * g)$, is defined as:

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau.$$

Properties of Convolution with Laplace Transformation

1. Commutativity: $f * g = g * f$;
2. Distributivity: $f * (g + h) = f * g + f * h$;
3. Associativity: $(f * g) * h = f * (g * h)$;
4. Zero Property: $f * 0 = 0 * f = 0$, where 0 is a function that maps any input to 0.
5. Approximation to Identity: $f * \delta = \delta * f = f$.

The Laplace Transformation of the convolution of f and g is:

$$\mathcal{L}\{(f * g)(t)\} = F(s)G(s).$$

System of First Order Linear ODEs

- Solving for Eigenvalues and Eigenvectors
- Linear Independence
  - Abel's Formula

For a given first order linear ODE in form:

$$\mathbf{x}' = A\mathbf{x},$$

the eigenvalues can be found as the solutions to the characteristic equation:

$$\det(A - Ir) = 0,$$

and the eigenvectors can be then found by solving the linear system that:

$$(A - Ir) \cdot \boldsymbol{\xi} = \mathbf{0}.$$

Suppose that the eigenvalues are distinct and the eigenvectors are linearly independent, the solution to the ODE is:

$$\mathbf{x} = c_1\boldsymbol{\xi}^{(1)}e^{r_1 t} + c_2\boldsymbol{\xi}^{(2)}e^{r_2 t} + \cdots + c_n\boldsymbol{\xi}^{(n)}e^{r_n t}.$$

Let the solutions form the fundamental matrix $\Psi(t)$, thus the Wronskian is:

$$\det\left(\Psi(t)\right).$$

The system is linearly independent if the Wronskian is non-zero.

Abel's Formula

For the linear system in form:

$$\mathbf{x}' = A\mathbf{x},$$

the Wronskian can be found by the trace of A, which is the sum of the diagonals, that is:

$$W = Ce^{\int \text{trace}\, A dt} = Ce^{\int (A_{1,1}+A_{2,2}+\cdots+A_{n,n})dt}.$$

Part 2:
Open Poll

We will work out some sample questions.

- If you have a problem that you are interested with, tell us now.
- Otherwise, we will work through selected problems from the practice problem set.
- We are also open to conceptual questions with the course.

Good luck on your second exam.