

## Example proofs

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### Fb Examples:

① Let  $f = \langle \text{Fun } g \rightarrow \text{Fun } x \rightarrow x = (g \ 1) \rangle$  In

$f(\text{Fun } y \rightarrow y + 1) \leftarrow \text{Call ths EXP}$

#### Lemma 1:

$$\begin{array}{c} \text{Fun } x \rightarrow x = (\text{Fun } y \rightarrow y + 1) \ 1 \Rightarrow \star \\ \parallel \\ \text{Fun } x \rightarrow x = (g \ 1) \ \square \ \langle \text{Fun } y \rightarrow y + 1 \rangle / g \end{array}$$

#### Lemma 2:

$$\begin{array}{c} \text{Fun } g \rightarrow \text{Fun } x \rightarrow x = (g \ 1) \Rightarrow \star \quad \langle \text{Fun } y \rightarrow y + 1 \rangle \Rightarrow \star \quad \text{Lemma 1} \\ \text{Fun } g \rightarrow \text{Fun } x \rightarrow x = (g \ 1) \ \square \ \langle \text{Fun } y \rightarrow y + 1 \rangle / g \Rightarrow \text{Fun } x \rightarrow x = (\text{Fun } y \rightarrow y + 1) \ 1 \\ f(\text{Fun } y \rightarrow y + 1) \ \square \ \langle \text{Fun } g \rightarrow \text{Fun } x \rightarrow x = (g \ 1) \rangle / f \end{array}$$

$$\begin{array}{c} \text{Fun } g \rightarrow \text{Fun } x \rightarrow x = (g \ 1) \Rightarrow \star \\ \text{EXP} \Rightarrow \text{Fun } x \rightarrow x = (\text{Fun } y \rightarrow y + 1) \ 1 \quad \text{Lemma 2} \end{array}$$

②  $\langle \text{Fun } x \rightarrow \text{Fun } x \rightarrow x + x \rangle \ 4 \ 5$

#### Lemma 1:

$$\begin{array}{c} \text{Fun } x \rightarrow x + x \Rightarrow \text{Fun } x \rightarrow x + x \\ \parallel \\ (\text{Fun } x \rightarrow x + x) \ \square \ 4 \ 5 \end{array}$$

$$\begin{array}{c} \text{Fun } x \rightarrow \text{Fun } x \rightarrow x + x \Rightarrow \star \quad 4 \Rightarrow 4 \quad \text{Lemma 1} \\ \text{Fun } x \rightarrow \text{Fun } x \rightarrow x + x \ 4 \Rightarrow \text{Fun } x \rightarrow x + x \ 5 \Rightarrow 5 \\ (\text{Fun } x \rightarrow \text{Fun } x \rightarrow x + x) \ 4 \ 5 \Rightarrow 10 \end{array}$$

### FbR Examples:

① Let  $a = fa = \text{Fun } x \rightarrow \text{Not } x \}$  In (a.a True)

#### Lemma 1:

$$\begin{array}{c} \text{True} \Rightarrow \text{True} \\ \text{Not True} \Rightarrow \text{False} \\ \parallel \\ \text{Not } x \ \square \ \langle \text{True} \ 1/x \rangle \end{array}$$

#### Lemma 2:

$$\begin{array}{c} \text{Fun } x \rightarrow \text{Not } x \Rightarrow \star \\ fa = \text{Fun } x \rightarrow \text{Not } x \ 3 \Rightarrow \star \\ fa = \text{Fun } x \rightarrow \text{Not } x \ 3. a \Rightarrow \text{Fun } x \rightarrow \text{Not } x \ \text{True} \Rightarrow \text{True} \quad \text{Lemma 1} \\ fa = \text{Fun } x \rightarrow \text{Not } x \ 3. a \ \text{True} \Rightarrow \text{False} \\ \parallel \\ (a.a \ \text{True}) \ \square \ fa = \text{Fun } x \rightarrow \text{Not } x \ 3 / a \end{array}$$

$$\begin{array}{c} \text{Fun } x \rightarrow \text{Not } x \Rightarrow \star \\ \{fa = \text{Fun } x \rightarrow \text{Not } x\} \Rightarrow \star \quad \text{Lemma 2} \\ \text{Let } a = fa = \text{Fun } x \rightarrow \text{Not } x \} \text{ In (a.a True)} \Rightarrow \text{False} \end{array}$$

②  $\langle \text{Fun } x \rightarrow \text{Fun } y \rightarrow fa = x.c + y \rangle \ \{c = 1\} \ 5$

#### Lemma 1:

$$\begin{array}{c} \text{Fun } y \rightarrow fa = \{c = 1\}. c + y \ 3 \Rightarrow \star \\ \parallel \\ \text{Fun } y \rightarrow \{fa = x.c + y\} \ \square \ \{fc = 1\} / x \end{array}$$

#### Lemma 2:

$$\begin{array}{c} \text{I} \Rightarrow \text{I} \\ \{fc = 1\} \Rightarrow \{fc = 1\} \quad \text{Lemma 1} \\ (\text{Fun } x \rightarrow \text{Fun } y \rightarrow fa = x.c + y \ 3) \Rightarrow \star \quad \{fc = 1\} \Rightarrow \star \\ (\text{Fun } x \rightarrow \text{Fun } y \rightarrow fa = x.c + y \ 3) \ \square \ fc = 1 \Rightarrow \text{Fun } y \rightarrow fa = \{c = 1\}. c + y \ 3 \end{array}$$

#### Lemma 3:

$$\begin{array}{c} \text{I} \Rightarrow \text{I} \\ \{fc = 1\} \Rightarrow \{fc = 1\} \quad \text{Lemma 1} \\ \{fc = 1\}. c \Rightarrow 1 \quad 5 \Rightarrow 5 \\ \{fc = 1\}. c + 5 \Rightarrow 6 \\ \{fc = 1\}. c + 5 \Rightarrow \{fa = 6\} \\ \parallel \\ fa = \{fc = 1\}. c + y \ 3 \ \square \ 5 / y \end{array}$$

$$\begin{array}{c} \text{Lemma 2} \quad 5 \Rightarrow 5 \quad \text{Lemma 3} \\ (\text{Fun } x \rightarrow \text{Fun } y \rightarrow fa = x.c + y \ 3) \ \{c = 1\} \ 5 \Rightarrow fa = 6 \ 3 \end{array}$$

### FbV Examples:

①  $\langle \text{Fun } x \rightarrow \text{Match } x \text{ With } \langle \text{Ps}(g) \rightarrow g + 5 \mid \langle \text{Nin}(h) \rightarrow h + 3 \rangle \rangle \text{ Ps}(4) \rangle$

#### Lemma 1:

$$\begin{array}{c} \text{4} \Rightarrow 4 \quad 5 \Rightarrow 5 \\ \text{Ps}(4) \Rightarrow \text{Ps}(4) \quad (g+5)/4 \Rightarrow 4+5 \Rightarrow 9 \\ \text{Match Ps}(4) \text{ With } \langle \text{Ps}(g) \rightarrow g + 5 \mid \langle \text{Nin}(h) \rightarrow h + 3 \rangle \rangle \text{ Ps}(4) \Rightarrow 9 \\ (\text{Match } x \text{ With } \langle \text{Ps}(g) \rightarrow g + 5 \mid \langle \text{Nin}(h) \rightarrow h + 3 \rangle \rangle \text{ Ps}(4)) / x \end{array}$$

$$\langle \text{Fun } x \rightarrow \text{Match } x \text{ With } \langle \text{Ps}(g) \rightarrow g + 5 \mid \langle \text{Nin}(h) \rightarrow h + 3 \rangle \rangle \text{ Ps}(4) \rangle \Rightarrow 9$$

② Let  $v = \text{Id}(\text{Fun } x \rightarrow x)$  In  $\text{Match } v \text{ With } \langle \text{Id}(f) \rightarrow f \circ \rangle$

#### Lemma 1:

$$\begin{array}{c} \text{Fun } x \rightarrow x \Rightarrow \star \\ \text{Id}(\text{Fun } x \rightarrow x) \Rightarrow \star \quad \text{Lemma 1} \\ \text{Match } \text{Id}(\text{Fun } x \rightarrow x) \text{ With } \langle \text{Id}(f) \rightarrow f \circ \rangle \Rightarrow \langle \text{Fun } x \rightarrow x \rangle / f \circ \end{array}$$

#### Lemma 2:

$$\begin{array}{c} \text{I} \Rightarrow \text{I} \\ \{f \circ = \text{Fun } x \rightarrow x\} \Rightarrow \star \\ (\text{Match } v \text{ With } \langle \text{Id}(f) \rightarrow f \circ \rangle) / v \end{array}$$

#### Lemma 3:

$$\begin{array}{c} \text{I} \Rightarrow \text{I} \\ \{f \circ = \text{Fun } x \rightarrow x\} \Rightarrow \star \\ (\text{Match } v \text{ With } \langle \text{Id}(f) \rightarrow f \circ \rangle) \Rightarrow \star \end{array}$$

③ Let  $x = \text{Fun } y : \text{Int} \rightarrow y = 0$  In  $x \ 1 + 2$

#### Lemma 1:

$$\begin{array}{c} \text{y: Int}, \text{y: Int} \vdash y: \text{Int} \quad \text{x: Int}, \text{y: Int} \vdash 0: \text{Int} \\ \text{x: Int}, \text{y: Int} \vdash y = 0 : \text{Bool} \\ \text{x: Int} \vdash \text{Fun } y : \text{Int} \rightarrow y = 0 : \text{Bool} \quad \text{x: Int} \vdash x : \text{Int} \quad \text{x: Int} \vdash x + x : \text{Int} \\ \text{x: Int} \vdash \text{If } (\text{Fun } y : \text{Int} \rightarrow y = 0) \text{ Then } x \text{ Else } x + x : \text{Int} \\ \vdash \text{Fun } x : \text{Int} \rightarrow \text{If } (\text{Fun } y : \text{Int} \rightarrow y = 0) \text{ Then } x \text{ Else } x + x : \text{Int} \rightarrow \text{Int} \end{array}$$

④ Let  $x = \text{Fun } y : \text{Int} \rightarrow y = 1$  In  $x \ 1 + 2$

#### Lemma 1:

$$\begin{array}{c} \text{y: Int}, \text{y: Int} \vdash y: \text{Int} \quad \text{x: Int}, \text{y: Int} \vdash 1: \text{Int} \quad \text{x: Int}, \text{y: Int} \vdash 2: \text{Int} \\ \text{y: Int}, \text{y: Int} \vdash y = 1 : \text{Bool} \\ \text{x: Int} \vdash \text{Fun } y : \text{Int} \rightarrow y = 1 : \text{Bool} \quad \text{x: Int} \vdash x : \text{Int} \quad \text{x: Int} \vdash x + x : \text{Int} \\ \text{x: Int} \vdash \text{If } (\text{Fun } y : \text{Int} \rightarrow y = 1) \text{ Then } x \text{ Else } x + x : \text{Int} \\ \vdash \text{Fun } x : \text{Int} \rightarrow \text{If } (\text{Fun } y : \text{Int} \rightarrow y = 1) \text{ Then } x \text{ Else } x + x : \text{Int} \rightarrow \text{Int} \end{array}$$

### STFbR Examples:

①  $\langle \text{Fun } r : fa : \{fb : \text{Bool}\} \rangle \rightarrow ft = r.a.b \} \cap \{fa = fb = \text{True}\}; c = 1 \}$

#### Lemma 1:

$$\begin{array}{c} r : fa : \{fb : \text{Bool}\} \vdash r : fa : \{fb : \text{Bool}\} \\ r : fa : \{fb : \text{Bool}\} \vdash r.a : fb : \text{Bool} \\ r : fa : fb : \text{Bool} \} \vdash r.a : fb : \text{Bool} \\ r : fa : fb : \text{Bool} \} \vdash ft = r.a.b : \text{Int} : \text{Bool} \\ \vdash \langle \text{Fun } r : fa : \{fb : \text{Bool}\} \rangle \rightarrow ft = r.a.b \} \cap \{fa = fb = \text{True}\}; c = 1 \} : \{ft : \text{Bool}\} \end{array}$$

#### Lemma 2:

$$\begin{array}{c} \text{I} \Rightarrow \text{I} \\ \{fa = fb : \text{Bool}\} \vdash fa = fb : \text{Bool} \\ fa = fb : \text{Bool} \} \vdash fa = fb : \text{Bool} \\ fa = fb : \text{Bool} \} \vdash fa = fb : \text{Bool} \} \vdash ft = r.a.b : \text{Int} : \text{Bool} \\ \vdash \langle \text{Fun } r : fa : \{fb : \text{Bool}\} \rangle \rightarrow ft = r.a.b \} \cap \{fa = fb = \text{True}\}; c = 1 \} : \{ft : \text{Bool}\} \end{array}$$

#### Lemma 3:

$$\begin{array}{c} \text{I} \Rightarrow \text{I} \\ \{fa = fb : \text{Bool}\} \vdash fa = fb : \text{Bool} \\ fa = fb : \text{Bool} \} \vdash fa = fb : \text{Bool} \\ fa = fb : \text{Bool} \} \vdash fa = fb : \text{Bool} \} \vdash ft = r.a.b : \text{Int} : \text{Bool} \\ \vdash \langle \text{Fun } r : fa : \{fb : \text{Bool}\} \rangle \rightarrow ft = r.a.b \} \cap \{fa = fb = \text{True}\}; c = 1 \} : \{ft : \text{Bool}\} \end{array}$$

### STFb Examples:

①  $\text{Fun } f \rightarrow \text{Fun } g \rightarrow \text{If } f \circ \text{ Then } g \text{ f Else } 0$  (Ref 42)

#### Lemma 1:

$$\begin{array}{c} \text{4} \Rightarrow 4 \quad 5 \Rightarrow 5 \\ \text{Ps}(4) \Rightarrow \text{Ps}(4) \quad (g+5)/4 \Rightarrow 4+5 \Rightarrow 9 \\ \text{Match Ps}(4) \text{ With } \langle \text{Ps}(g) \rightarrow g + 5 \mid \langle \text{Nin}(h) \rightarrow h + 3 \rangle \rangle \text{ Ps}(4) \Rightarrow 9 \\ (\text{Match } x \text{ With } \langle \text{Ps}(g) \rightarrow g + 5 \mid \langle \text{Nin}(h) \rightarrow h + 3 \rangle \rangle \text{ Ps}(4)) / x \end{array}$$

#### Lemma 2:

$$\begin{array}{c} \text{I} \Rightarrow \text{I} \\ \{fa = fb : \text{Bool}\} \vdash fa = fb : \text{Bool} \\ fa = fb : \text{Bool} \} \vdash fa = fb : \text{Bool} \\ fa = fb : \text{Bool} \} \vdash fa = fb : \text{Bool} \} \vdash ft = r.a.b : \text{Int} : \text{Bool} \\ \vdash \langle \text{Fun } r : fa : \{fb : \text{Bool}\} \rangle \rightarrow ft = r.a.b \} \cap \{fa = fb = \text{True}\}; c = 1 \} : \{ft : \text{Bool}\} \end{array}$$

#### Lemma 3:

$$\begin{array}{c} \text{I} \Rightarrow \text{I} \\ \{fa = fb : \text{Bool}\} \vdash fa = fb : \text{Bool} \\ fa = fb : \text{Bool} \} \vdash fa = fb : \text{Bool} \\ fa = fb : \text{Bool} \} \vdash fa = fb : \text{Bool} \} \vdash ft = r.a.b : \text{Int} : \text{Bool} \\ \vdash \langle \text{Fun } r : fa : \{fb : \text{Bool}\} \rangle \rightarrow ft = r.a.b \} \cap \{fa = fb = \text{True}\}; c = 1 \} : \{ft : \text{Bool}\} \end{array}$$

### AFb Examples:

①  $\text{Fun } f \rightarrow \text{Fun } g \rightarrow \text{If } f \circ \text{ Then } g \text{ f Else } 0$  (Ref 42)

#### Lemma 1:

$$\begin{array}{c} \text{4} \Rightarrow 4 \quad 5 \Rightarrow 5 \\ \text{Ps}(4) \Rightarrow \text{Ps}(4) \quad (g+5)/4 \Rightarrow 4+5 \Rightarrow 9 \\ \text{Match Ps}(4) \text{ With } \langle \text{Ps}(g) \rightarrow g + 5 \mid \langle \text{Nin}(h) \rightarrow h + 3 \rangle \rangle \text{ Ps}(4) \Rightarrow 9 \\ (\text{Match } x \text{ With } \langle \text{Ps}(g) \rightarrow g + 5 \mid \langle \text{Nin}(h) \rightarrow h + 3 \rangle \rangle \text{ Ps}(4)) / x \end{array}$$

#### Lemma 2:

$$\begin{array}{c} \text{I} \Rightarrow \text{I} \\ \{fa = fb : \text{Bool}\} \vdash fa = fb : \text{Bool} \\ fa = fb : \text{Bool} \} \vdash fa = fb : \text{Bool} \\ fa = fb : \text{Bool} \} \vdash fa = fb : \text{Bool} \} \vdash ft = r.a.b : \text{Int} : \text{Bool} \\ \vdash \langle \text{Fun } r : fa : \{fb : \text{Bool}\} \rangle \rightarrow ft = r.a.b \} \cap \{fa = fb = \text{True}\}; c = 1 \} : \{ft : \text{Bool}\} \end{array}$$

#### Lemma 3:

$$\begin{array}{c} \text{I} \Rightarrow \text{I} \\ \{fa = fb : \text{Bool}\} \vdash fa = fb : \text{Bool} \\ fa = fb : \text{Bool} \} \vdash fa = fb : \text{Bool} \\ fa = fb : \text{Bool} \} \vdash fa = fb : \text{Bool} \} \vdash ft = r.a.b : \text{Int} : \text{Bool} \\ \vdash \langle \text{Fun } r : fa : \{fb : \text{Bool}\} \rangle \rightarrow ft = r.a.b \} \cap \{fa = fb = \text{True}\}; c = 1 \} : \{ft : \text{Bool}\} \end{array}$$

### AFbV Examples:

①  $\text{Fun } f \rightarrow \text{Fun } g \rightarrow \text{If } f \circ \text{ Then } g \text{ f Else } 0$  (Ref 42)

#### Lemma 1:

$$\begin{array}{c} \text{4} \Rightarrow 4 \quad 5 \Rightarrow 5 \\ \text{Ps}(4) \Rightarrow \text{Ps}(4) \quad (g+5)/4 \Rightarrow 4+5 \Rightarrow 9 \\ \text{Match Ps}(4) \text{ With } \langle \text{Ps}(g) \rightarrow g + 5 \mid \langle \text{Nin}(h) \rightarrow h + 3 \rangle \rangle \text{ Ps}(4) \Rightarrow 9 \\ (\text{Match } x \text{ With } \langle \text{Ps}(g) \rightarrow g + 5 \mid \langle \text{Nin}(h) \rightarrow h + 3 \rangle \rangle \text{ Ps}(4)) / x \end{array}$$