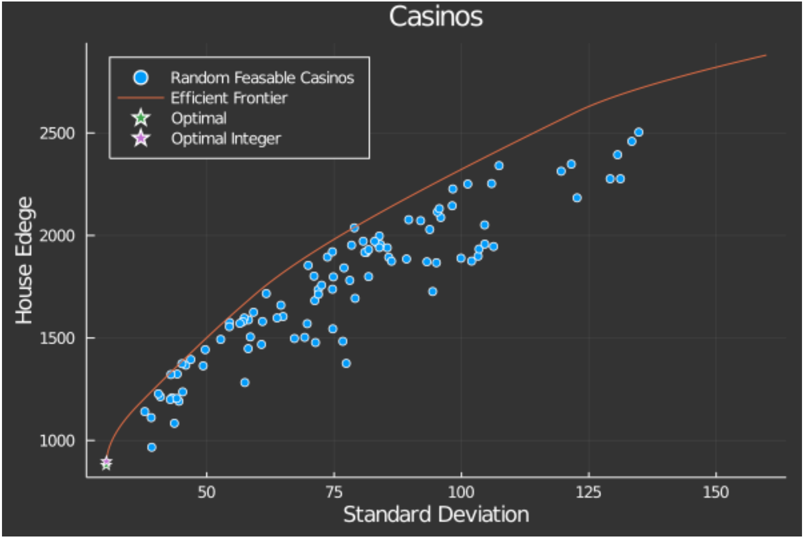
Convex Optimization and Efficient Frontiers

James Haas

Math 662

October 31, 2020

Overview



Casinos
Random Feasable Casinos
Efficient Frontier
Optimal
Optimal Integer
House Edege
2500
2000
1500
1000
50
75
100
125
150
Standard Deviation

Definitions

Definition

A set C is convex if for any $x_1, x_2 \in C$ and any $\theta \in [0, 1]$ we have

$$\theta x_1 + (1 - \theta) x_2 \in C$$

Definition

$\theta_1 x_1 + \theta_2 x_2 + \cdots \theta_n x_n$ is a Convex Combination if $\sum \theta_i = 1$

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Proposition

Every Convex set Contains All of its Convex Combinations

Convex Combination Proof

Proposition

Every Convex set Contains All of its Convex Combinations

Base Case

Let C be a Convex set then by definition our base case is true

Assume

For any $x_1, \ldots, x_n \in C$ and $\sum_{i=1}^{n} \theta_i = 1$ we have $\sum_{i=1}^{n} \theta_i x_i \in C$

- Choose any $x_1, \ldots, x_n, x_{n+1} \in C$ and $\theta_1, \ldots, \theta_n, \theta_{n+1}$ such that $\sum \theta_i = 1$
- $\sum_{i=1}^{n+1} \theta_i x_i = \theta_1 x_1 + \sum_{i=2}^{n} \theta_i x_i$
- Note that $\sum_{i=2}^{n} \theta_i = 1 - \theta_1 \longrightarrow \sum_{i=2}^{n} \frac{\theta_i}{1-\theta_1} = 1$
- $\sum_{i=1}^{n+1} \theta_i x_i = \theta_1 x_1 + (1-\theta_1) \sum_{i=2}^{n} \frac{\theta_i}{1-\theta_1} x_i$
- Since $x_1 \in C$ and convex combinations of n elements is also an element of C we see that $\sum_{i=1}^{n+1} \theta_i x_i \in C$

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Basic Convex Sets

- The whole Euclidean space $\mathbf{R}^n$.
- The Cartesian product $A \times B$ of any two convex sets $A \subset \mathbf{R}^n$ and $B \subset R^m$.
- The solution set of a system of linear equations or inequalities.

- All hyperplanes $\{x \in \mathbf{R}^n \big| a'x = b\}$
- All half-spaces $\{x \in \mathbf{R}^n \big| a'x \leq b\}$
- The solid ball $B_r(x_0) := \{x \in \mathbf{R}^n | \|x - x_0\| \leq r\}$ with center x_0 and radius r

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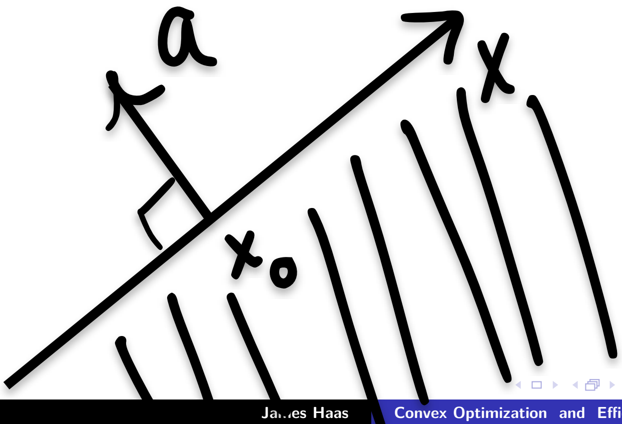
Showing A Set Is Convex

Let $C = \{x \in \mathbf{R}^n \,|\, a'x \leq b\}$ choose any $x_1, x_2 \in C$ and $\theta \in [0, 1]$

$$a'(\theta x_1 + (1-\theta)x_2) = \theta a'x_1 + (1-\theta)a'x_1 \leq$$

$$\leq \theta b + (1-\theta)b = b(\theta + 1 - \theta) = b$$

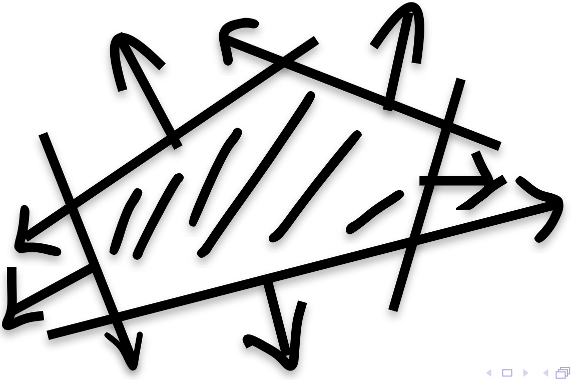


Polyhedron

Def

A polyhedron is the solution set of a finite number of linear equalities and inequalities

$$\mathcal{P} = \{x \big| a_j' x \leq b_j, j = 1 : m, c_j' x = d_j, j = 1 : p\}$$



The Intersection of Convex Sets

Let $C = \cap_{i=1}^{n} C_i$ be the intersection of n convex sets.

- Choose any $x_1, x_2 \in C$
- Hence $x_1, x_2 \in C_k$ and $x_1, x_2 \in C_j$ any two arbitrary sets who's intersection make up C
- fix any $\theta \in [0,1]$ since both C_k and C_j are convex $\theta x_1 + (1-\theta)x_2 \in C_k, C_j$
- Hence since both C_k and C_j are arbitrary $\theta x_1 + (1-\theta)x_2 \ in \cap_{i=1}^{n} C_i = C$ therefore C is convex

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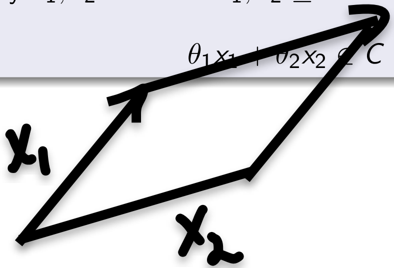
Cone

A set C is a cone if for any $x \in$ and $\theta \geq 0$ we have $\theta x \in C$

Convex Cone

A set C is a convex cone if it is convex and is a cone which means that for any $x_1, x_2 \in C$ and $\theta_1, \theta_2 \geq 0$ we have,

$$\theta_1 x_1 + \theta_2 x_2 \in C$$



Positive Semidefinite Cone

Positive Semidefinite Matrices

$$S_+^n := \{A \big| x'Ax \geq 0 \forall x \in C\mathbf{R}^n/0\}$$

- Is this set Convex?
- Yes. Choose any $A, B \in S_+^n$ and $theta \in [0,1]$

$$x'(\theta A + (1-\theta)B)x = \theta x'Ax + (1-\theta)x'Bx \geq 0$$

- Is this set a Cone?
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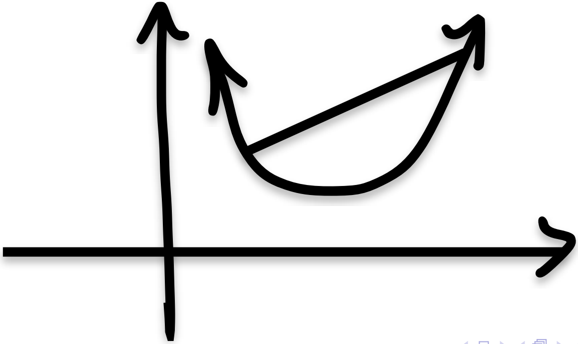
Def

A function $f : X \to \mathbf{R}$ is convex if the domain X is a convex set and if for all $x, y \in X$, and $\theta \in [0,1]$ we have

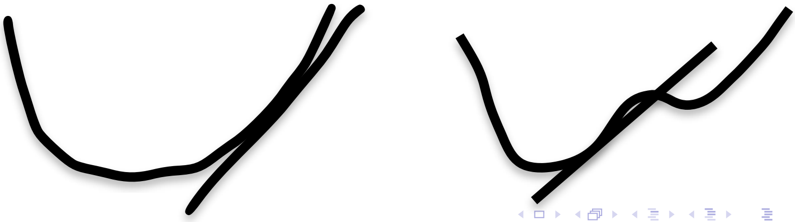
$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$



Suppose f is differentiable. Then f is convex if and only if the domain of f is convex and

$$f(x) \geq f(y) + \nabla f(y)^T (x - y)$$

For any x, y in the domain of f



$\Longrightarrow$

- Let $f : X \to \mathbf{R}$ be a differentiable and convex
- then by definition $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$ for any $x, y \in X$ a convex domain and $\theta \in [0,1]$
- Rearrange terms in both sides of the inequality

$$f(\theta x + (1-\theta)y) = f(y + \theta(x-y))$$

$$\theta f(x) + (1-\theta)f(y) = \theta f(y) + \theta(f(x) - f(y))$$

- Getting closer. $\theta f(y) + \theta(f(x) - f(y)) \geq f(y + \theta(x-y))$
- $$f(x) \geq f(y) + \frac{f(y + \theta(x-y)) - f(y)}{\theta}$$
- Finally take $\lim_{\theta \to 0}$ of both sides and we have

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- $$f(x) \geq f(y) + \frac{f(y + \theta(x-y)) - f(y)}{\theta}$$
- Finally take $\lim_{\theta \to 0}$ of both sides and we have

$$f(x) \geq f(y) + \nabla f(y)^T(x-y)$$

- The domain of f is convex and $f(x) \geq f(y) + \nabla f(y)^T(x-y)$
- Assume that this holds for all $x, y \in X$ the convex domain of f. Now choose any $x \neq y$ and $\theta \in [0,1]$.
- Let $z = \theta x + (1-\theta)y$

$$f(x) \geq f(z) + \nabla f(z)^T(x-z) \qquad f(y) \geq f(z) + \nabla f(z)^T(y-z)$$

- Multiply the first inequality by θ and the second by $(1-\theta)$

$$\theta f(x) \geq \theta f(z) + \theta \nabla f(z)^T(x-z)$$

$$(1-\theta)f(y) \geq (1-\theta)f(z) + (1-\theta)\nabla f(z)^T(y-z)$$

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- Now Add both the inequalities

$$\theta f(x)+(1-\theta)f(y) \geq f(z)+\nabla f(z)^T((1-\theta)(y-z)+\theta(x-z))$$

- Now something cool happens note $z=\theta x+(1-\theta)y$

$$(1-\theta)(y-z)+\theta(x-z)=\theta x+(1-\theta)y-\theta z-(1-\theta)z=$$

$$=\theta x+(1-\theta)y-z=z-z=0$$

- Therefore $\theta f(x)+(1-\theta)f(y) \geq f(z)=f(\theta x+(1-\theta)y)$

- Now Add both the inequalities

$$\theta f(x) + (1-\theta) f(y) \geq f(z) + \nabla f(z)^T ((1-\theta)(y-z) + \theta(x-z))$$

- Now something cool happens note $z = \theta x + (1-\theta)y$

$$(1-\theta)(y-z) + \theta(x-z) = \theta x + (1-\theta)y - \theta z - (1-\theta)z =$$

$$= \theta x + (1-\theta)y - z = z - z = 0$$

- Therefore $\theta f(x) + (1-\theta) f(y) \geq f(z) = f(\theta x + (1-\theta)y)$

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Second-order Condition

If f is twice differentiable at each point in its domain then f is convex if and only if the domain of f is convex and its Hessian is positive semidefinte

$$\nabla^2 f(x) \succeq 0$$

Using the Second-order Condition

Positive semidefinite Quadratic functions

$$f(x) = \frac{1}{2}x^T P x + q^T x + r$$

- $P \in S^n_+$, $q \in \mathbf{R}^n$ and $r \in \mathbf{R}$
- $\nabla f(x) = Px + q$
- $\nabla f(x)^2 = P \succeq 0$

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$$\begin{gathered}\text{minimize } f_0(x)\\ \text{subject to}\\ f_i(x) \leq 0, \quad i = 1, \dots, m\\ h_i(x) = 0, \quad i = 1, \dots, p\end{gathered}$$

The optimal value p^* of the problem is defined as

$$p^* = \inf\{f_0(x)\big|f_i(x) \leq 0,\ i = 1, \dots, m,\ h_i(x) = 0,\ i = 1, \dots, p\}$$

Note that is a problem is infeasible $p^* = \infty$

- The objective function must be convex,
- The inequality constraint functions must be convex,
- The equality constraint functions $h_i(x) = a_i^T x - b_i$ must be affine.

$$\begin{gathered}\text{minimize } f_0(x)\\ \text{subject to}\\ f_i(x) \leq 0, \quad i = 1, \ldots, m\\ h_i(x) = 0, \quad i = 1, \ldots, p\end{gathered}$$

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Casino Problem

I am opening a casino and I have 3 types of games that I can have which are slot machines, craps, and blackjack tables. I can not exceed 256 seats and I can only have 35 dealers on the floor at a time. My investors tell me I need to have at least 5 blackjack tables, and at least 3 craps tables. My goal is to use convex optimization to trace the efficient frontier of possible casinos based on the expected edge of the house and standard deviation.

Game	Seats	Dealers
Slot Machine A	1	0
Slot Machine B	1	0
Slot Machine C	1	0
Black Jack	6	1
Craps	9	3

Boyd, Stephen P. *Convex Optimization*. Cambridge University Press, 2004