THE INTERACTION ALGORITHM AND PRACTICAL FOURIER ANALYSIS

By I. J. GOOD*

[Received August, 1957]

SUMMARY

THE interactions in a complete factorial experiment are linear functions of the observations. They can be calculated by means of a stream-lined algorithm, of which the first example was the adding-and-subtracting algorithm of Yates (1937). The relationship between the two methods of calculating the interactions is given a precise and succinct form, which surprisingly does not seem to have been given in the existing literature. The algorithm is logically simplest for the t^n experiment. The inverse algorithm is specified and also the method of obtaining the divisors for the analysis of variance. Finally some analogous short cuts in practical Fourier analysis are described.

1. INTRODUCTION

Yates (1937, pp. 15 and 29) introduced a simple adding-and subtracting algorithm for calculating the interactions of a 2^n factorial experiment, starting with the original observations. (We shall throughout use the word "interaction" to include the mean and main effects, which are the interactions of order -1 and 0.) The algorithm was extended by Box et al. (1954, pp. 363–366) to the 3^n factorial experiment. These algorithms imply certain linear formulae for the interactions which can be written down at once from Theorem 1 below. The formulae look more obvious than they are, and this may explain why the theorem does not seem to be already in the literature. We shall also specify the inverse algorithm that can be used for recovering the observations from a knowledge of the interactions. Applications for the inverse algorithm are that it gives a completely convincing check on the calculation of the interactions, and that by equating non-significant high-order interactions to zero and then applying the inverse algorithm we can smooth the original observations. By smoothing the observations we mean adjusting them with the intention of making the adjusted values closer than the original ones were to the expectations on the true, but unknown, hypothesis.

2. DESCRIPTION OF THE DIRECT ALGORITHMS

In a t^n factorial experiment we have t^n observations (real numbers) $x_{\mathbf{r}}$

$$(\mathbf{r} = (r_1, r_2, \ldots, r_n);\ r_i = 0, 1, \ldots, t-1;\ i = 1, 2, \ldots, n).$$

These are written down in a column of t^n entries, in the dictionary sequence for the suffixes $\mathbf{r}$. A simple rule is then used for obtaining a second column of numbers, of the same length as the first column. This rule is repeated to give a third column and so on, the rule being used n times in all, and the last column gives the interactions. The rule for

* Address: 25 Scott House, Princess Elizabeth Way, Cheltenham, Glos.

Downloaded from https://academic.oup.com/jrsssb/article/20/2/361/7027226 by guest on 18 May 2025

getting from one column to the next may be expressed as a matrix multiplication of a column vector of t^n components. The matrix is t^n by t^n and consists very largely of zeros. If this matrix is denoted by A then the interactions form a vector $\mathbf{y}$ defined by

$$\mathbf{y} = A^n\mathbf{x},$$

or in more detail by

$$y_{\mathbf{r}} = \sum_{\mathbf{s}} (A^n)_{\mathbf{r},\mathbf{s}}\, x_{\mathbf{s}},$$

where $\mathbf{s} = (s_1, s_2, \ldots, s_n)$, $s_i = 0, 1, \ldots, t-1$, and the components of $\mathbf{x}$ and $\mathbf{y}$ and the rows and columns of A are arranged in the dictionary sequence of the suffixes $\mathbf{r}$ and $\mathbf{s}$. For example, if $n = 3$ in the Yates algorithm, where $t = 2$, we have

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

It is because in this case all the elements are 1, -1 or 0 that we get an adding-and-subtracting algorithm. For the 3^n experiment some doublings are also required. (The interactions are often defined as t^{-n} times the values given by these algorithms.)

In order to write down explicit expressions for the interactions all that is required is the calculation of A^n, and the inverse process can be obtained by replacing A by A^{-1}. When doing an analysis of variance the square of each interaction is to be divided by the sum of the squares of the elements in the corresponding row of A^n (Yates, 1937, p. 91).

If this were all that there were to say this paper would not have been written. But there is more to say because of the special form of A. In fact A^n is equal to the n^{th} "direct power" of a t by t matrix, an expression that depends on the following definition.

3. Direct Products of Matrices

The direct products

$$M^{(1)} \times M^{(2)} \times \ldots \times M^{(n)}$$

of n matrices, each t by t, is the t^n by t^n matrix A whose elements are defined by

$$A_{\mathbf{r},\mathbf{s}} = \prod_{i=1}^{n} M^{(i)}{}_{r_i, s_i};$$

$\mathbf{r}$ and $\mathbf{s}$ and the sequence of rows and columns of A are as previously defined. (As a matter of fact the matrices $M^{(i)}$ do not need to be all of the same order.) This is the

Downloaded from https://academic.oup.com/jrsssb/article/20/2/361/7027226 by guest on 18 May 2025

definition given, for example, by Madelung (1943, p. 85), who also points out that

$$(M^{(1)} \times M^{(2)} \times \ \ldots\ )(L^{(1)} \times L^{(2)} \times \ \ldots\ ) = (M^{(1)} L^{(1)}) \times (M^{(2)} L^{(2)}) \times \ \ldots\ .$$

In particular

$$(M \times M \times \ \ldots\ )(M^{-1} \times M^{-1} \times \ \ldots\ ) = I_t \times I_t \times \ \ldots\ = I_{t^n},$$

where I_u is the u by u identity matrix. Thus

$$(M \times M \times \ \ldots\ )^{-1} = M^{-1} \times M^{-1} \times \ \ldots$$

or, more briefly,

$$(M^{[n]})^{-1} = (M^{-1})^{[n]},$$

where the symbol $[n]$ indicates the n^{th} direct power, that is the operation of taking a direct product of n identical matrices.

We can now state our main result.

4. Theorem 1

Let M be any t by t matrix. Then

$$M^{[n]} = A^n,$$

where

$$A = \{M_{r_1, s_n} \delta_{r_2}^{s_1} \delta_{r_3}^{s_2} \ \ldots\ \delta_{r_n}^{s_{n-1}}\}$$

and δ_a^b is the Kronecker delta, $\delta_a^b = 0$ if $a \neq b$, $\delta_a^a = 1$.

Interpretation of Theorem 1.—The matrix A has at most t^{n+1} non-zero elements, so that it can be applied n times to a column vector of length t^n with at most nt^{n+1} ordinary multiplications of pairs of numbers. But if A^n or $M^{[n]}$ were formed first, and then multiplied by the vector, then t^{2n} ordinary multiplications may be required. For example, in order to compute the mod 2 three-dimensional discrete Fourier transform

$$y_{\mathbf{r}} = \sum_{s_1, s_2, s_3}^{0,1} x_{\mathbf{s}}(-1)^{r_1 s_1 + r_2 s_2 + r_3 s_3}$$

we could multiply the vector $\mathbf{x}$ by the matrix (Yates, 1937, p. 11 or Box et al., 1954, Table *M*)

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

Downloaded from https://academic.oup.com/jrsssb/article/20/2/361/7027226 by guest on 18 May 2025

but it is easier to multiply three times by the 8 by 8 matrix mentioned in Section 2. In this example, M is the two-by-two matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

The fact that the interactions in a 2^n factorial experiment could be regarded as an n-dimensional mod 2 discrete Fourier transform was pointed out by Good (1953). Theorem 1 is useful for t^n factorial experiments since the interactions seem always to be expressible in the form $M^{[n]}\mathbf{x}$.

Proof of Theorem 1.—We shall prove by mathematical induction that, if $1 \leqslant \nu \leqslant n$,

$$(A^{\nu})_{\mathbf{r},\,\mathbf{s}} = M_{r_1,\,s_{n+1-\nu}} M_{r_2,\,s_{n+2-\nu}} \ \ldots \ M_{r_\nu,\,s_n} \times \delta_{r_{\nu-1}}{}^{s_1}\, \delta_{r_{\nu+2}}{}^{s_2} \ \ldots \ \delta_{r_n}{}^{s_{n-\nu}},$$

which is clearly true when $\nu = 1$, and our theorem will follow on putting $\nu = n$. By the inductive hypothesis

$$\begin{aligned}
(A^{\nu+1})_{\mathbf{r},\,\mathbf{s}} &= (A^{\nu} \,.\, A)_{\mathbf{r},\,\mathbf{s}} \\
&= \sum_{\mathbf{u}} \{M_{r_1,\,u_{n+1-\nu}} M_{r_2,\,u_{n+2-\nu}} \ \ldots \ M_{r_\nu,\,u_n}\, \delta_{r_{\nu+1}}{}^{u_1}\, \delta_{r_{\nu+2}}{}^{u_2} \ \ldots \ \delta_{r_n}{}^{u_{n-\nu}} \\
&\qquad\qquad \times M_{u_1,\,s_n}\, \delta_{u_2}{}^{s_1}\, \delta_{u_3}{}^{s_2} \ \ldots \ \delta_{u_n}{}^{s_{n-1}}\} \\
&= \Big(\sum_{u_1} M_{u_1,\,s_n}\, \delta_{r_{\nu+1}}{}^{u_1}\Big) \\
&\qquad \Big(\sum_{u_2,\,u_3,\,\ldots\,u_{n-\nu}} \delta_{r_{\nu+2}}{}^{u_2}\, \delta_{r_{\nu+3}}{}^{u_3} \ \ldots \ \delta_{r_n}{}^{u_{n-\nu}}\, \delta_{u_2}{}^{s_1}\, \delta_{u_3}{}^{s_2} \ \ldots \ \delta_{u_{n-\nu}}{}^{s_{n-\nu-1}}\Big) \\
&\qquad\qquad \times \Big(\sum_{u_{n-\nu+1},\,\ldots,\,u_n} M_{r_1,\,u_{n+1-\nu}} \ \ldots \ M_{r_\nu,\,u_n}\, \delta_{u_{n-\nu+1}}{}^{s_{n-\nu}} \ \ldots \ \delta_{u_n}{}^{s_{n-1}}\Big) \\
&= M_{r_{\nu+1},\,s_n} M_{r_1,\,s_{n-\nu}} M_{r_2,\,s_{n-\nu+1}} \ \ldots \ M_{r_\nu,\,s_{n-1}}\, \delta_{r_{\nu+2}}{}^{s_1} \ \ldots \ \delta_{r_n}{}^{s_{n-\nu-1}},
\end{aligned}$$

and the induction is complete.

It seems worthwhile to notice in passing the analogy with multidimentional integral transforms, a special case of which is the ordinary multidimensional Fourier transform usually known to statisticians as the characteristic function for a multivariate distribution. The transformation from $f(x_1, x_2, \ \ldots \ , x_n)$ to

$$\int\int \ \ldots \ \int f(x_1, x_2, \ \ldots \ , x_n)\, K(x_1, t_1)\, K(x_2, t_2) \ \ldots \ K(x_n, t_n)\, dt_1 dt_2 \ \ldots \ dt_n$$

can be obtained by n applications of the transformation that takes f into

$$\int\int \ \ldots \ \int f(x_1, \ \ldots \ , x_n)\, K(x_1, t_n)\, \delta(x_2 - t_1)\, \delta(x_3 - t_2) \ \ldots \ \delta(x_n - t_{n-1})\, dx_1 \ \ldots \ dx_n,$$

where δ is now the Dirac delta function. In other words we have extracted an n^{th} root of the original transformation.

Downloaded from https://academic.oup.com/jrsssb/article/20/2/361/7027226 by guest on 18 May 2025

5. Theorem 2, the Inverse Algorithm

If $\mathbf{y} = A^n\mathbf{x}$, then $\mathbf{x} = B^n\mathbf{y}$, where B is obtained from M^{-1} in the same way as A is obtained from M. The proof is immediate from the last equation in Section 3, combined with Theorem 1.

6. Examples

For the 2^n factorial experiment,

$$M^{-1} = \tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

so that Yates's adding-and-subtracting algorithm, if applied to the interactions, will give 2^n times the original observations. (See Kempthorne, 1952, p. 259). This result is not surprising in view of the near identity of the discrete Fourier transform and its inverse. Or we may say that it is because M is symmetric and is a multiple of an orthogonal matrix.

For the 3^n factorial experiment (Box et al., 1954, pp. 363–366), we have

$$M = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}, \quad M^{-1} = \tfrac{1}{6}\begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -2 \\ 2 & 3 & 1 \end{bmatrix}.$$

The following table is an artificial numerical example for a 3^2 experiment. The first column gives the levels of the treatments, the second one the observations, the third and fourth the results of applying the direct algorithm and the last two the results of applying the inverse algorithm. The last column is seen to consist of the original observations multiplied by 36.

00	3	14	42	36	108
01	7	20	14	−6	252
02	4	8	−6	18	144
10	1	1	−6	96	36
11	8	10	2	−36	288
12	11	3	12	−48	396
20	2	−7	−18	120	72
21	1	−4	−16	6	36
22	5	5	6	−78	180

7. The Divisors for the Analysis of Variance

For the analysis of variance the divisor corresponding to a given interaction is the sum of the squares of the elements of the appropriate row of the matrix $M^{[n]}$. This is equal to the corresponding diagonal element in $M^{[n]}(M^{[n]})'$. But $M^{[n]}(M^{[n]})' = (MM')^{[n]}$. Usually the rows of M are orthogonal to each other, and MM' is a diagonal matrix, but even if it were not the diagonal elements of its n^{th} direct power could be at once written down. In fact if the diagonal elements of MM' are $\mu_0, \mu_1, \ldots, \mu_{t-1}$, then the divisor for interaction y_r is

$$\prod_{i=2}^{n} \mu_{r_i} = \mu_0^{\alpha_0}\mu_1^{\alpha_1} \ldots \mu_{t-1}^{\alpha_{t-1}},$$

Downloaded from https://academic.oup.com/jrsssb/article/20/2/361/7027226 by guest on 18 May 2025

where α_j ($j = 0, 1, \ldots, t-1$) is the number of components of $\mathbf{r}$ that are equal to j, i.e. the number of "terms of degree" j in the interaction. For example, for the 3^n experiment the divisor is $3^{\alpha_0} 2^{\alpha_1} 6^{\alpha_2} = 2^{\alpha_1+\alpha_2} 3^{\alpha_0+\alpha_2} = 2^m 3^{n-\alpha_1}$, where m is the order of the interaction, i.e. the number of non-zero components of $\mathbf{r}$. This formula is given on page 365 of Box et al. (1954).

For the 4^n experiment, we can obtain M from say, Cochran and Cox (1950, p. 134)

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & -1 & 1 & 3 \\ 1 & -1 & -1 & 1 \\ -1 & 3 & -3 & 1 \end{bmatrix}, \quad MM' = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 20 \end{bmatrix},$$

and we see that the divisors for the analysis of variance are $4^n 5^{\alpha_1+\alpha_3}$. We may conveniently deduce the value of M^{-1}, since

$$M^{-1} = M' \cdot \frac{1}{20} \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 5 & -3 & 5 & -1 \\ 5 & -1 & -5 & 3 \\ 5 & 1 & -5 & -3 \\ 5 & 3 & 5 & 1 \end{bmatrix}.$$

For the t^n experiment ($t \leqslant 75$) we can read off M from the tables of orthogonal polynomials given by Fisher and Yates (1953, Table XXIII), as was mentioned by Cochran and Cox. For example, for the 5^n experiment,

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 2 & -1 & -2 & -1 & 2 \\ -1 & 2 & 0 & -2 & 1 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix}, \quad MM' = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 14 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 70 \end{bmatrix},$$

$$M^{-1} = \frac{1}{70} \begin{bmatrix} 14 & -14 & 10 & -7 & 1 \\ 14 & -7 & -5 & 14 & -4 \\ 14 & 0 & -10 & 0 & 6 \\ 14 & 7 & -5 & -14 & -4 \\ 14 & 14 & 10 & 7 & 1 \end{bmatrix},$$

Downloaded from https://academic.oup.com/jrsssb/article/20/2/361/7027226 by guest on 18 May 2025

and the divisors for the analysis of variance are

$$2^m\, 5^{n-\alpha_2}\, 7^{\alpha_2+\alpha_4}.$$

8. Other Factorial Experiments

For the general $t_1\ t_2\ \ldots\ t_n$ factorial experiment, where the t_i's are not all equal, the interactions can still be expressed in the form

$$\mathbf{y} = (M^{(1)} \times M^{(2)} \times \ \ldots\ \times M^{(n)})\,\mathbf{x},$$

where $M^{(i)}$ is a t_i by t_i matrix. Now

$$M^{(1)} \times M^{(2)} \times \ \ldots\ \times M^{(n)} = C^{(1)}\,C^{(2)}\ \ldots\ C^{(n)},$$

where

$$\begin{aligned} C^{(1)} &= M^{(1)} \times I_{t_2} \times \ \ldots\ \times I_{t_n} \\ C^{(2)} &= I_{t_1} \times M^{(2)} \times \ \ldots\ \times I_{t_n} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ C^{(n)} &= I_{t_1} \times I_{t_2} \times \ \ldots\ \times M^{(n)}. \end{aligned}$$

Thus

$$\mathbf{y} = C^{(1)}\,C^{(2)}\ \ldots\ C^{(n)}\,\mathbf{x}.$$

Now the number of non-zero elements in $C^{(i)}$ is at most $t_i\,t_1\,t_2\,t_3\ \ldots\ t_n$, so the total number of ordinary multiplications, if $\mathbf{y}$ is obtained from $\mathbf{x}$ by the successive application of the $C^{(i)}$'s, is at most $(\Sigma\, t_i)(\Pi\, t_i)$ instead of possibly as many as $(\Pi\, t_i)^2$ if the product of the matrices is used as the multiplier. Thus the calculation of the interactions for the general complete factorial experiment can be "streamlined" but with rather more trouble than for the t^n experiment because the $C^{(i)}$'s are not all the same matrix. We can also see as in Section 3 that

$$(M^{(1)} \times M^{(2)} \times \ \ldots\ \times M^{(n)})^{-1} = (M^{(1)})^{-1} \times (M^{(2)})^{-2} \times \ \ldots\ \times (M^{(n)})^{-1}$$

so that we have the inversion formula

$$\begin{aligned} \mathbf{x} &= \{(M^{(1)})^{-1} \times (M^{(2)})^{-1} \times \ \ldots\ \times (M^{(n)})^{-1}\}\,\mathbf{y} \\ &= D^{(1)}\,D^{(2)}\ \ldots\ D^{(n)}\,\mathbf{y}, \end{aligned}$$

where

$$D^{(1)} = (M^{(1)})^{-1} \times I_{t_2} \times \ \ldots\ \times I_{t_n}, \text{ etc.}$$

As in Section 7, the divisor, in the analysis of variance, for interaction y_r is $\prod_{i=1}^{n} \mu_{r_i}^{(i)}$, where $\mu_0^{(i)}, \mu_1^{(i)}, \ \ldots\ , \mu_{t-1}^{(i)}$ are the diagonal elements of $M^{(i)}\,M^{(i)\prime}$.

9. Connections with Practical Fourier Analysis

We shall now show that the calculations in practical Fourier analysis can be streamlined by means of analogous algorithms. The problem of practical Fourier analysis (Whittaker and Robinson, 1944, Chapter X), also known as trigonometrical interpolation or harmonic analysis, is that of calculating a_s^*'s such that the expressions

$$\frac{1}{t} \sum_{s=0}^{t-1} a_s^*\, \omega^{-rs} \qquad (\omega = e^{2\pi i/t})$$

Downloaded from https://academic.oup.com/jrsssb/article/20/2/361/7027226 by guest on 18 May 2025

take assigned values a_r ($r = 0, 1, \ldots, t-1$). The problem is algebraically solved by the equations

$$a_s^* = \sum_{r=0}^{t-1} a_r \omega^{rs}$$

and we call a_s^*, regarded as a function of s, the mod t discrete Fourier transform of a_r regarded as a function of r.

The corresponding problem in n dimensions is that of calculating a_s^*'s such that the expressions

$$\frac{1}{\tau} \sum_{\mathbf{s}} a_{\mathbf{s}}^* \, \omega_1^{-r_1 s_1} \omega_2^{-r_2 s_2} \ldots \omega_n^{-r_n s_n} \qquad (\omega_\nu = e^{2\pi i/t_\nu}, \ \tau = \Pi\, t_\nu)$$

take assigned values

$$a_{\mathbf{r}}[\mathbf{s} = (s_1, s_2, \ldots, s_n), \ \mathbf{r} = (r_1, r_2, \ldots, r_n),$$
$$r_\nu, \ s_\nu = 0, 1, 2, \ldots, t_\nu - 1, \ \nu = 1, 2, \ldots, n].$$

The problem is algebraically solved by taking

$$a_{\mathbf{s}} = \sum_{\mathbf{r}} a_{\mathbf{r}} \, \omega_1^{r_1 s_1} \omega_2^{r_2 s_2} \ldots \omega_n^{r_n s_n},$$

the n-dimensional discrete Fourier transform moduli $t_1, t_2, \ldots, t_n$.

Chapter X of Whittaker and Robinson (1944) is mainly concerned with the numerical calculation of the one-dimensional discrete Fourier transform. We shall first show how the calculation of a multidimensional discrete Fourier transform can be streamlined. Then we recall how a one-dimensional mod t discrete Fourier transform, where t is composite, can be expressed as a multidimensional transform, so that its calculation can thereby be streamlined. This last result provides a reason for using composite moduli in practical Fourier analysis. We shall also see that it is advisable for the moduli to be composite for multidimensional transforms.

The discussion following is very far from complete. The intention is to give a unified treatment of one particular method of saving arithmetic that is closely related to earlier sections. Clearly if the a_r's are real then a^*_{-s} is the complex conjugate of a^*_s, a remark that can be used to save about half the arithmetic when $t > 2$. Furthermore there are simple relations between the sines and cosines of angles differing by or adding to multiples of $\frac{1}{2}\pi$. If t is a multiple of 2, 3 or 4, then some of the angles have simple sines or cosines. It would take us too far afield to try to list all such savings and to evaluate them. With this caution it is hoped that the remainder of this paper may be of some interest and practical value.

10. Multidimensional Transforms with All Moduli Equal

We first take $t_1 = t_2 = \ldots = t_n = t$. (The case $t = 2$ corresponds to the Yates algorithm.) Write $\mathbf{a}$ and $\mathbf{a}^*$ for the vectors whose components are a_r and a_s^*. The t^n discrete Fourier transform may then be written

$$\mathbf{a}^* = (\Omega \times \Omega \times \ldots \times \Omega)\, \mathbf{a} = \Omega^{[n]}\, \mathbf{a},$$

where $\Omega = \{\omega^{rs}\}$ ($\omega = e^{2\pi i/t}$, $r, s = 0, 1, 2, \ldots, t-1$). Therefore, by Theorem 1,

Downloaded from https://academic.oup.com/jrsssb/article/20/2/361/7027226 by guest on 18 May 2025

$\mathbf{a}^*$ can be obtained from $\mathbf{a}$ by multiplying n times by a matrix, thus:

$$\mathbf{a}^* = \Psi^n \mathbf{a}$$

where

$$\Psi = \{\omega^{r_1 s_n} \delta_{r_2 s_1} \delta_{r_3 s_2} \ \ldots \ \delta_{r_n s_{n-1}}\}$$

$$(r_\nu, s_\nu = 0, 1, \ \ldots \ , t-1; \ \nu = 1, 2, \ \ldots \ , n).$$

For example, if $t = 3$, $n = 2$, we have

$$\Psi = \begin{array}{c|cccccccccc} & 00 & 01 & 02 & 10 & 11 & 12 & 20 & 21 & 22 \\ \hline 00 & 1 & \omega & \omega^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 01 & 0 & 0 & 0 & 1 & \omega & \omega^2 & 0 & 0 & 0 \\ 02 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \omega & \omega^2 \\ 10 & 1 & \omega & \omega^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 11 & 0 & 0 & 0 & 1 & \omega & \omega^2 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \omega & \omega^2 \\ 20 & 1 & \omega & \omega^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 21 & 0 & 0 & 0 & 1 & \omega & \omega^2 & 0 & 0 & 0 \\ 22 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \omega & \omega^2 \end{array}$$

where $\omega = \frac{1}{2}(-1 + i\sqrt{3})$, and where the matrix has been bordered with the names of the rows and columns in dictionary sequence.

11. General Multidimensional Transform

The general discrete Fourier transform, moduli $t_1, t_2, \ \ldots \ , t_n$, can be written

$$\mathbf{a}^* = (\Omega^{(1)} \times \Omega^{(2)} \times \ \ldots \ \times \Omega^{(n)})\, \mathbf{a}$$

where

$$\Omega^{(\nu)} = \{\omega_\nu^{rs}\} \qquad (r, s = 0, 1, \ \ldots \ , t_\nu; \ \nu = 1, 2, \ \ldots \ , n; \ \omega_\nu = e^{2\pi i/t_\nu}).$$

Therefore, as in Section 8,

$$\mathbf{a}^* = \Phi^{(1)} \Phi^{(2)} \ \ldots \ \Phi^{(n)} \mathbf{a}$$

where

$$\Phi^{(1)} = \Omega^{(1)} \times I_{t_2} \times \ \ldots \ \times I_{t_n}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\Phi^{(n)} = I_{t_1} \times I_{t_2} \times \ \ldots \ \times \Omega^{(n)}.$$

Thus the components of $\mathbf{a}^*$ can be obtained by means of $(\Sigma\, t_\nu)(\Pi\, t_\nu)$ ordinary multiplica-

Downloaded from https://academic.oup.com/jrsssb/article/20/2/361/7027226 by guest on 18 May 2025

tions of complex numbers, instead of $(\Pi\, t_\nu)^2$, and the calculation is logically simpler when $t_1 = t_2 = \ \ldots \ = t_n$ since each of the n matrix multiplications is then the same one.

12. The One-dimensional Transform

It was pointed out by Good (1951) that when $t_1, t_2, \ \ldots \ , t_n$ are mutually prime the n-dimensional discrete Fourier transform can be expressed as a one-dimensional discrete Fourier transform mod τ, where $\tau = t_1\, t_2 \ \ldots \ t_n$. The same method can be used for the converse process, the purpose of which is to be in a position to apply the above streamlined algorithm and thereby to reduce the number of ordinary multiplications required from τ^2 to $(t_1 + t_2 + \ \ldots \ + t_n)\,\tau$. For example, with

$$\tau = 5 \,.\, 7 \,.\, 8 \,.\, 9 = 2520$$

the numerical work would be cut by a factor of over 80.

For convenience we recall here the method of identifying one-dimensional and multi-dimensional transforms.

We set up a one-one correspondence between r (not in Clarendon type) and $\mathbf{r}$ (in Clarendon type) by means of the equations

$$\mathbf{r} = (r_1, r_2, \ \ldots \ , r_n) \qquad (r_\nu = 0, 1, \ \ldots \ , t_\nu - 1),$$

$$r \equiv \frac{\tau}{t_1} r_1 + \frac{\tau}{t_2} r_2 + \ \ldots \ + \frac{\tau}{t_n} r_n \qquad (\text{mod } \tau,\ 0 \leqslant r \leqslant \tau - 1)$$

and we set up a one-one correspondence between s and $\mathbf{s}$ by means of the equations

$$\mathbf{s} = (s_1, s_2, \ \ldots \ , s_n) \qquad (s_\nu = 0, 1, \ \ldots \ , t_\nu - 1).$$

$$s \equiv \frac{\tau s_1}{t_1}\left(\frac{\tau}{t_1}\right)^{-1}_{t_1} + \ \ldots \ + \frac{\tau s_n}{t_n}\left(\frac{\tau}{t_n}\right)^{-1}_{t_n} \qquad (\text{mod } \tau,\ 0 \leqslant s \leqslant \tau - 1),$$

where the suffixes indicate that the reciprocals are mod t_ν. (The last congruence above is the solution of the Chinese remainder problem, $s \equiv s_\nu \ (\text{mod } t_\nu)$, $\nu = 1, 2, \ \ldots \ , n$). Denote by $\mathbf{a}$ and $\mathbf{a}^*$ the vectors whose components are $a_{\mathbf{r}}$ and $a_{\mathbf{s}}^*$, where the sequence of the components is the dictionary sequence of the suffices. Denote by $\hat{\mathbf{a}}$ and $\hat{\mathbf{a}}^*$ the same vectors but with components $\hat{a}_r$ and $\hat{a}_s^*$, arranged in the sequence $r = 0, 1, 2, \ \ldots \ , \tau - 1$, $s = 0, 1, 2, \ \ldots \ , \tau - 1$, and where by definition $\hat{a}_r = a_{\mathbf{r}}$ and $\hat{a}_s^* = a_{\mathbf{s}}^*$. (In other words hats will be worn for the one-dimensional interpretation.) Then the equations

$$\hat{a}_s^* = \sum_r \omega^{rs}\, \hat{a}_r$$

and

$$a_{\mathbf{s}}^* = \sum_{\mathbf{r}} \omega_1^{r_1 s_1}\, \omega_2^{r_2 s_2} \ \ldots \ \omega_n^{r_n s_n}\, a_{\mathbf{r}}$$

are equivalent, since $rs \equiv \sum_\nu r_\nu s_\nu \tau / t_\nu \ (\text{mod } \tau)$ and $\omega_\nu = \omega^{\tau/t_\nu}$ $(\nu = 1, 2, \ \ldots \ , n)$. Therefore, for certain permutation matrices P and Q, that are specified by the above remarks,

Downloaded from https://academic.oup.com/jrsssb/article/20/2/361/7027226 by guest on 18 May 2025

we have

$$\begin{aligned}\hat{a}^* &= \Omega\,\hat{\mathbf{a}}\\ &= P(\Omega^{(1)} \times \Omega^{(2)} \times \ \ldots \ \times \Omega^{(n)})\, Q^{-1}\,\hat{\mathbf{a}}\\ &= P\,.\,\Phi^{(1)}\,.\,\Phi^{(2)} \ \ldots \ \Phi^{(n)}\,.\,Q^{-1}\,\hat{\mathbf{a}}.\end{aligned}$$

Thus the one-dimensional transform can be calculated by means of $(\Sigma\, t_\nu)\,\tau$ ordinary complex multiplications.

The following tables give an example, with $\tau = 24$, of the one-one relationship between $\mathbf{r}$ and r and between $\mathbf{s}$ and s. We have $t_1 = 3$, $t_2 = 8$, and the row and column co-ordinates are the two components of the vector $\mathbf{r}$ (or $\mathbf{s}$).

	0	1	2	3	4	5	6	7
0	0	3	6	9	12	15	18	21
1	8	11	14	17	20	23	2	5
2	16	19	22	1	4	7	10	13

	0	1	2	3	4	5	6	7
0	0	9	18	3	12	21	6	15
1	16	1	10	19	4	13	22	7
2	8	17	2	11	20	5	14	23

13. The Multidimensional Transform with Composite Moduli

We now reconsider the general multidimensional discrete Fourier transform with moduli $t_1, t_2, \ \ldots \ , t_n$ and we suppose that some at least of these moduli are composite. The previous algorithm for the general multidimensional transform in effect treats each of the n dimensions separately. But we have just seen that the calculation of a one-dimensional transform can be further speeded up when its modulus has at least two distinct prime factors. We now give algebraic expression to a method of taking advantage of both algorithms.

Let the symbols $\mathbf{a}$, $\mathbf{a}^*$, $\Omega^{(\nu)}$ $(\nu = 1, 2, \ \ldots \ , n)$ have the previous meanings, and let t_ν be resolved into m_ν mutually prime factors $t_\nu = t_{\nu,1}\, t_{\nu,2} \ \ldots \ t_{\nu,m_\nu}$. Let

$$\omega_{\nu,\mu} = \exp\,(2\pi i/t_{\nu,\mu}) \qquad (\mu = 1, 2, \ \ldots \ , m_\nu;\ \nu = 1, 2, \ \ldots \ , n)$$

$$\Omega^{(\nu,\mu)} = \{\omega_{\nu,\mu}{}^{rs}\} \qquad \text{(where } r, s = 0, 1, \ \ldots \ , t_{\nu,\mu} \text{ just for the moment)}$$

$$\begin{aligned}\Phi^{(\nu,1)} &= \Omega^{(\nu,1)} \times I_{t_{\nu,2}} \times \qquad \ldots \times I_{t_{\nu,m_\nu}}\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \Phi^{(\nu,m_\nu)} &= I_{t_{\nu,1}} \times I_{t_{\nu,2}} \times \ \ldots \ \times \Omega^{(\nu,m_\nu)}.\end{aligned}$$

We already know how to find permutation matrices $P^{(\nu)}$ and $Q^{(\nu)}$ such that

$$\Omega^{(\nu)} = P^{(\nu)}\,\Phi^{(\nu,1)}\,\Phi^{(\nu,2)} \ \ldots \ \Phi^{(\nu,m_\nu)}\,(Q^{(\nu)})^{-1}.$$

Hence the equation

$$\mathbf{a}^* = (\Omega^{(1)} \times \Omega^{(2)} \times \ \ldots \ \times \Omega^{(n)})\,\mathbf{a}$$

can be written in a form that requires at most

$$\prod_{\nu=2}^{n} (t_\nu(t_{\nu,1} + t_{\nu,2} + \ \ldots \ + t_{\nu,m_\nu}))$$

ordinary multiplications of complex numbers.

I am indebted to Mr. M. M. Crum for some useful suggestions and for some stimulating discussion.

Downloaded from https://academic.oup.com/jrsssb/article/20/2/361/7027226 by guest on 18 May 2025

REFERENCES

BOX, G. E. P., CONNOR, L. R., COUSINS, W. R., DAVIES, O. L., HIMSWORTH, F. R. & SILLITO, G. P. (1954), *The Design and Analysis of Industrial Experiments*. Edinburgh: Oliver & Boyd.
COCHRAN, W. G. & COX, G. M. (1950), *Experimental Designs*. New York: Wiley.
FISHER, R. A. & YATES, F. (1953), *Statistical Tables*, 4th ed. Edinburgh: Oliver & Boyd.
GOOD, I. J. (1951), "Random motion on a finite Abelian group", *Proc. Camb. Phil. Soc.*, **47**, 756–62.
—— —— (1953), Review of M. H. Quenouille's *Design and Analysis of Experiment*, *Annals of Eugenics*, **18**, 263–266.
KEMPTHORNE, O. (1952), *The Design and Analysis of Experiments*. New York: Wiley.
MADELUNG, E. (1943), *Die Mathematischen Hilfsmittel des Physikers*. New York: Dover.
WHITTAKER, E. T. & ROBINSON, G. (1944), *The Calculus of Observations*. 4th ed. London: Blackie.
YATES, F. (1937), *The Design and Analysis of Factorial Experiment*. Harpenden: Imperial Bureau of Soil Science.

Downloaded from https://academic.oup.com/jrsssb/article/20/2/361/7027226 by guest on 18 May 2025