SOME IMPROVEMENTS IN PRACTICAL FOURIER ANALYSIS AND THEIR APPLICATION TO X-RAY SCATTERING FROM LIQUIDS.

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INTRODUCTION.

One of the more recent applications of Fourier analysis occurs in the quantitative investigation of liquids by x-rays following the theory of Zernike and Prins.[1] Although the following method was developed for this application, it is equally applicable to any problem requiring a Fourier analysis. If a modern mechanical analyzer (e.g. Henrici [2] or Michelson [3]) is available, the evaluation of a Fourier integral presents no difficulty. It is our purpose to show that, for occasional analyses at least, one need not depend upon such costly instruments, even when the required number of coefficients is very large. Like all other arithmetical methods we make use of the symmetry of the trigonometric functions in the four quadrants of a circle. The great reduction in the number of operations, which this allows, has been pointed out by Runge.[4] Since, however, the labor varies approximately as the square of the number of ordinates, the available standard forms become impractical for a large number of coefficients. We shall show that, by a certain transformation process, it is possible to double the number of ordinates with only slightly more than double the labor.

In the technique of numerical analysis the following improvements suggested by Lanczos were used: (1) a simple matrix scheme for any even number of ordinates can be used in place of available standard forms; (2) a transposition of odd ordinates into even ordinates reduces an analysis for $2n$ co-

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[1] Zernike and Prins, *Zeits. f. Phys.*, **41**, 184 (1927); *Zeits. f. Phys.*, **56**, 617 (1929).

[2] Henrici, *Phil. Mag.*, **38**, 110 (1894).

[3] Michelson and Stratton, *Phil. Mag.*, **45**, 85 (1898).

[4] Runge, *Zeit. für Math. und Physik*, **48**, 443 (1903); *Zeit. für Math. und Physik*, **53**, 117 (1905).

efficients to two analyses for n coefficients; (3) by using intermediate ordinates it is possible to estimate, before calculating any coefficients, the probable accuracy of the analysis; (4) any intermediate value of the Fourier integral can be determined from the calculated coefficients by interpolation. The first two improvements reduce the time spent in calculation and the probability of making errors, the third tests the accuracy of the analysis, and the fourth improvement allows the transform curve to be constructed with arbitrary exactness. Adopting these improvements the approximate times for Fourier analyses are: 10 minutes for 8 coefficients, 25 minutes for 16 coefficients, 60 minutes for 32 coefficients, and 140 minutes for 64 coefficients. Interpolation for an intermediate point requires 5 to 10 minutes.

IMPROVEMENTS IN PRACTICAL FOURIER ANALYSIS.

1. Introduction.

Let $y = f(x)$ be a finite, single-valued and continuous function in the range $-\pi$ to π. This function may be expanded into the infinite Fourier series

$$f(x) = \tfrac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots, \qquad (1)$$

where

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos kx \, dx, \qquad (2)$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin kx \, dx. \qquad (3)$$

The arithmetical method of evaluating the coefficients makes use of a finite number of ordinates y_k instead of the entire curve. The number of ordinates required depends solely on the convergence of the Fourier series and cannot be decided by merely looking at the graph of $f(x)$. Mathematically an infinite number of ordinates is required for the evaluation of the coefficients a_k and b_k. This, however, is only the case if $f(x)$ shall be reproduced with absolute accuracy. If $f(x)$ is to be reproduced only within the experimental errors, the situation is greatly simplified. The number of ordinates required for the evaluation of the coefficients need not exceed the number of terms which are practically present in the Fourier analysis of $f(x)$.[5] This means that within the limit of the expected accuracy $f(x)$ can be replaced by a finite sum of the form (1). If this sum contains, let us say, 21 cosine terms and 20 sine terms, then 41 readings will be sufficient for evaluation of the a_k and b_k. Since, however, we do not know in advance how many terms of the series have to be taken into consideration, we might better employ a larger number, say 50 readings. We could then calculate the coefficients beyond the twentieth overtone and find by calculation that the amplitudes of the higher overtones are negligible. An absolute guarantee can never be given, however, since it is possible that some overtones of the series drop out although some higher overtones are still present. Hence it is desirable to corroborate by some additional tests the evidence obtained by the vanishing of the last evaluated coefficients. In a later section two independent tests will be described which allow us to decide whether or not the number of readings was chosen sufficiently high. Even these tests, however, have only plausibility value and cannot be construed as an absolute guarantee for the lack of higher overtones. It is advisable to start the analysis with these two tests before beginning the numerical work which leads to the complete set of Fourier coefficients.

If after completing an analysis the number of coefficients is found to be too small, one can make use of the work already done by doubling the number of ordinates. The operations with the old ordinates remain unchanged and we have to add only the operations with the new ordinates. The additional labor is only slightly greater than that required for the original analysis and we now have twice the number of harmonic components as well as more accurate values for the previous components.

2. The Multiplication Matrix.

The principle of trigonometric interpolation is to find a finite series

$$\tfrac{1}{2}a_0 + a_1 \cos x + \cdots + a_{2n-1} \cos (2n - 1)x + \tfrac{1}{2}a_{2n} \cos 2nx + b_1 \sin x + \cdots + b_{2n-1} \sin (2n - 1)x, \qquad (4)$$

[5] Lanczos, *Journal of Math. and Phys.*, **17**, 135 (1938).

which shall fit the given function $f(x)$ in the $4n$ equidistant points

$$x_k = k\frac{x}{2n} \quad (k = 0, \pm 1, \pm 2, \cdots \pm 2n). \tag{5}$$

Owing to the periodicity condition $y_{2n} = y_{-2n}$. The number of readings has been chosen as a multiple of 4 in order to get full benefit of the symmetry properties of the trigonometric functions in the 4 quadrants of the circle. The origin is put in the middle of the range. The details of the numerical work are discussed in Runge's book.[6]

The analysis is first divided into an even function (cosine analysis) and an odd function (sine analysis) by taking sums and differences of the original ordinates. Let the $4n$ equidistant ordinates be

$$y_{-(2n-1)} \quad y_{-(2n-2)} \quad \cdots \quad y_{-1} \quad y_0 \quad y_1 \quad \cdots \quad y_{2n-1} \quad y_{2n}. \tag{6}$$

They are arranged in reverse order in the following manner and the sums and differences obtained as shown

$$\begin{array}{lcccccc}
 & y_{2n} & y_{2n-1} & y_{2n-2} & \cdots & y_1 & y_0 \\
 & & y_{-(2n-1)} & y_{-(2n-2)} & \cdots & y_{-1} & \\
\hline
\text{Sums} & u_{2n} & u_{2n-1} & u_{2n-2} & \cdots & u_1 & u_0 \\
\text{Differences} & & v_{2n-1} & v_{2n-2} & \cdots & v_1 &
\end{array} \tag{7}$$

For example let $n = 4$ and the 16 equidistant ordinates be denoted as

$$y_{-7} \quad y_{-6} \quad \cdots \quad y_{-1} \quad y_0 \quad y_1 \quad \cdots \quad y_7 \quad y_8. \tag{8}$$

The sums and differences are obtained by writing the ordinates in reverse order and in a double row, the first going from left to right and the second from right to left

$$\begin{array}{lcccccc}
 & y_8 & y_7 & y_6 & \cdots & y_1 & y_0 \\
 & & y_{-7} & y_{-6} & \cdots & y_{-1} & \\
\hline
\text{Sums} & u_8 & u_7 & u_6 & \cdots & u_1 & u_0 \\
\text{Differences} & & v_7 & v_6 & \cdots & v_1 &
\end{array} \tag{9}$$

[6] Runge and Koenig, "Vorlesungen über Numerisches Rechnen" (Springer, Berlin, 1924), page 211.

Then u_k enters only in the cosine and v_k only in the sine analysis. The coefficients of the complete harmonic analysis are given by the relations

$$a_k = \frac{2}{2n} A_k \tag{10}$$

and

$$b_k = \frac{2}{2n} B_k, \tag{11}$$

where

$$A_k = \sum_{\alpha=0}^{2n} u_\alpha \cos \frac{\pi}{2n} k\alpha \quad (k = 0, 1, 2, \cdots 2n) \tag{12}$$

and

$$B_k = \sum_{\alpha=1}^{2n-1} v_\alpha \sin \frac{\pi}{2n} k\alpha \quad (k = 1, 2, \cdots (2n - 1)). \tag{13}$$

The problem is to evaluate these sums A_k and B_k with a minimum amount of computation. To do this it is advantageous to pair the ordinates once more.

This is possible on the basis that:

$$\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} \frac{\pi}{2n} k(2n - \alpha) = -\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} \frac{\pi}{2n} k\alpha \text{ for } k \text{ odd,}$$

$$\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\} \frac{\pi}{2n} k(2n - \alpha) = \left.\begin{matrix}\cos\\ -\sin\end{matrix}\right\} \frac{\pi}{2n} k\alpha \text{ for } k \text{ even.}$$

$$\begin{array}{llllll}
 & u_0 & u_1 & u_2 & \cdots & u_{n-1} \quad u_n \\
 & u_{2n} & u_{2n-1} & u_{2n-2} & \cdots & u_{n+1} \swarrow \\
\hline
\text{Sums} & u_0'' & u_1'' & u_2'' & \cdots & u_{n-1}'' \quad u_n'' \\
\text{Differences} & u_1' & u_1' & u_2' & \cdots & u_{n-1}'
\end{array} \tag{14}$$

$$\begin{array}{lllll}
 & v_1 & v_2 & \cdots & v_{n-1} \quad v_n \\
 & v_{2n-1} & v_{2n-2} & \cdots & v_{n+1} \swarrow \\
\hline
\text{Sums} & v_1' & v_2' & \cdots & v_{n-1}' \quad v_n' \\
\text{Differences} & v_1'' & v_2'' & \cdots & v_{n-1}''
\end{array} \tag{15}$$

Referring to our example in which $n = 4$ (16 ordinates) we have

$$\begin{array}{lccccc} & u_0 & u_1 & u_2 & u_3 & u_4 \\ & u_8 & u_7 & u_6 & u_5 \swarrow & \\ \hline \text{Sums} & u_0'' & u_1'' & u_2'' & u_3'' & u_4'' \\ \text{Differences} & u_0' & u_1' & u_2' & u_3' & \end{array} \tag{16}$$

$$\begin{array}{lcccc} & v_1 & v_2 & v_3 & v_4 \\ & v_7 & v_6 & v_5 \swarrow & \\ \hline \text{Sums} & v_1' & v_2' & v_3' & v_4' \\ \text{Differences} & v_1'' & v_2'' & v_3'' & \end{array} \tag{17}$$

The formulæ for the evaluation of the Fourier coefficients now become

$$A_k = \sum_{\alpha=1}^{n} \left.\begin{matrix} u_\alpha' \\ u_\alpha'' \end{matrix}\right\} \cos \frac{\pi}{2n} k\alpha \begin{matrix} (k = 1, 3, \cdots 2n - 1) \\ (k = 0, 2, \cdots 2n), \end{matrix} \tag{18}$$

$$B_k = \sum_{\alpha=1}^{n} \left.\begin{matrix} v_\alpha' \\ v_\alpha'' \end{matrix}\right\} \sin \frac{\pi}{2n} k\alpha \begin{matrix} (k = 1, 3, \cdots 2n - 1) \\ (k = 2, 4, \cdots 2n - 2). \end{matrix} \tag{19}$$

For the *odd* coefficients the *primed* and for the *even* coefficients the *double primed* quantities are employed. Hence by separating the even and odd coefficients we have reduced the range of the sum index α from $2n$ to n.

For our example the 8 Fourier coefficients for the cosine analysis are given by $a_k = \frac{2}{8} A_k$ where

$$A_k = \sum_{\alpha=0}^{4} \left.\begin{matrix} u_\alpha' \\ u_\alpha'' \end{matrix}\right\} \cos \frac{\pi}{8} k\alpha \begin{matrix} (k = 1, 3, 5, 7) \\ (k = 0, 2, 4, 6, 8), \end{matrix} \tag{20}$$

$$B_k = \sum_{\alpha=1}^{4} \left.\begin{matrix} v_\alpha' \\ v_\alpha'' \end{matrix}\right\} \sin \frac{\pi}{8} k\alpha \begin{matrix} (k = 1, 3, 5, 7) \\ (k = 2, 4, 6). \end{matrix} \tag{21}$$

However, the range of the fixed index k still extends until $2n$. If we separate the even and odd values of the sum index α, then a reduction to n is also possible here. Thus we define the following set of quantities

$$A_k' = \sum_\alpha \left.\begin{matrix} u_\alpha' \\ u_\alpha'' \end{matrix}\right\} \cos\frac{\pi}{2n} k\alpha \quad (\alpha = 1, 3, 5, \cdots n - 1), \tag{22}$$

$$A_k'' = \sum_\alpha \left.\begin{matrix} u_\alpha' \\ u_\alpha'' \end{matrix}\right\} \cos\frac{\pi}{2n} k\alpha \quad (\alpha = 0, 2, 4, \cdots n), \tag{23}$$

$$B_k' = \sum_\alpha \left.\begin{matrix} v_\alpha' \\ v_\alpha'' \end{matrix}\right\} \sin\frac{\pi}{2n} k\alpha \quad (\alpha = 1, 3, 5, \cdots n - 1), \tag{24}$$

$$B_k'' = \sum_\alpha \left.\begin{matrix} v_\alpha' \\ v_\alpha'' \end{matrix}\right\} \sin\frac{\pi}{2n} k\alpha \quad (\alpha = 2, 4, 6, \cdots n). \tag{25}$$

In these formulæ the index k runs only from 0 to n for the A_k and from 1 to n for the B_k quantities. The calculation of these sums gives all the coefficients directly by the formulæ

$$A_k = A_k'' + A_k', \qquad A_{2n-k} = A_k'' - A_k' \qquad (k = 0, 1, 2, \cdots n), \tag{26}$$

$$B_k = B_k' + B_k'', \qquad B_{2n-k} = B_k' - B_k'' \qquad (k = 1, 2, \cdots n). \tag{27}$$

In our example the 16 coefficients are determined by the following 16 simple sums

$$A_k' = u_1' \cos k\frac{\pi}{8} + u_3' \cos 3k\frac{\pi}{8} \quad (k = 1, 3)$$

$$A_k' = u_1'' \cos k\frac{\pi}{8} + u_3'' \cos 3k\frac{\pi}{8} \quad (k = 0, 2), \tag{28}$$

$$A_k'' = u_0' + u_2' \cos 2k\frac{\pi}{8} + u_4' \cos 4k\frac{\pi}{8} \quad (k = 1, 3)$$

$$A_k'' = u_0'' + u_2'' \cos 2k\frac{\pi}{8} + u_4'' \cos 4k\frac{\pi}{8} \quad (k = 0, 2, 4), \tag{29}$$

$$B_k' = v_1' \sin k\frac{\pi}{8} + v_3' \sin 3k\frac{\pi}{8} \quad (k = 1, 3)$$

$$B_k' = v_1'' \sin k\frac{\pi}{8} + v_3'' \sin 3k\frac{\pi}{8} \quad (k = 2, 4), \tag{30}$$

$$B_k'' = v_2' \sin 2k\frac{\pi}{8} + v_4' \sin 4k\frac{\pi}{8} \quad (k = 1, 3)$$

$$B_k'' = v_2'' \sin 2k\frac{\pi}{8} \qquad (k = 2). \tag{31}$$

The 9 coefficients of the cosine analysis are

$$A_k = A_k'' + A_k', \qquad A_{8-k} = A_k'' - A_k' \qquad (k = 0, 1, 2, 3, 4). \tag{32}$$

The 7 coefficients of the sine analysis are

$$B_k = B_k' + B_k'', \qquad B_{8-k} = B_k' - B_k'' \qquad (k = 1, 2, 3, 4). \tag{33}$$

From this scheme we see that the problem of calculation reduces to a small number of multiplications ($u_\alpha \cos \alpha k(\pi/8)$) and additions.

The practical evaluation of the Fourier coefficients is greatly facilitated by the application of a matrix scheme. All the cosine coefficients can be evaluated by one matrix, and all the sine coefficients by another matrix. Both matrices are constructed along similar lines. We describe first the matrix with the help of which the cosine coefficients can be evaluated.

Cosine Matrix.

We use the integers 0, 1, 2, 3, $\cdots$ n as a symbolic notation. An arbitrary integer k shall stand for $\cos k(\pi/2n)$. We first write down the following guiding line:

$$0, 1, 2, \cdots n - 1, n, -(n - 1), \cdots -1, -0, -1, \cdots -(n - 1), n, n - 1, \cdots 1. \tag{34}$$

This line represents a complete cycle of $\cos k(\pi/2n)$ values, k running from 0 to $4n - 1$. We now construct a matrix of $n + 1$ rows and $n + 1$ columns. We start with the row "zero," by putting down zero $n + 1$ times. Then the row "one" follows, by writing down the numbers of the guiding line: 0, 1, $\cdots$, until n. Then the row "two" follows by selecting every second number of the guiding line. Then the row "three" follows by selecting every third number of the guiding line, starting always with zero. Thus we continue, until the row "n" is reached. Whenever we reach the end of the line, we jump back to the beginning, and continue.

Example: $n = 4$.
Guiding line:

$$0, 1, 2, 3, 4, -3, -2, -1, -0, -1, -2, -3, 4, 3, 2, 1. \quad (35)$$

Cosine Matrix ($n = 4$).

$$\begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & -2 & -0 \\ 0 & 3 & -2 & -1 & 4 \\ 0 & 4 & -0 & 4 & 0 \end{matrix} \qquad (36)$$

The next step is the separation of even and odd coefficients. We break the constructed matrix into 4 smaller matrices, which are characterized as follows:

(a)	Even rows / Even columns	(b)	Even rows / Odd columns
(c)	Odd rows / Even columns	(d)	Odd rows / Odd columns.

The "zero" is considered as one of the even numbers.

Example: $n = 4$.
For the example above we get the following 4 smaller matrices:

$$(a) \begin{matrix} 0 & 0 & 0 \\ 0 & 4 & -0 \\ 0 & -0 & 0 \end{matrix} \qquad (b) \begin{matrix} 0 & 0 \\ 2 & -2 \\ 4 & 4 \end{matrix}$$

$$(c) \begin{matrix} 0 & 2 & 4 \\ 0 & -2 & 4 \end{matrix} \qquad (d) \begin{matrix} 1 & 3 \\ 3 & -1 \end{matrix} \qquad (37)$$

Since the actual value of the symbol "4" is zero, the third row of (b) and the third column of (c) can be omitted.

Each one of these matrices has to be used in connection with a proper set of ordinates:

(a) uses the even u'' (i.e. u_0'', u_2'', u_4''),
(b) uses the odd u'' (i.e. u_1', u_3'),
(c) uses the even u' (i.e. u_0', u_2'),
(d) uses the odd u' (i.e. u_1', u_3').

For the actual multiplications it is convenient to set the proper u-values at the top of the matrices, and to have at hand a table which gives the numerical values of the integer symbols. In the present example this table would be:

$$
\begin{aligned}
0 &= \cos 0\frac{\pi}{8} = 1, & 1 &= \cos\frac{\pi}{8} = 0.92388,\\
2 &= \cos 2\frac{\pi}{8} = 0.70711, & 3 &= \cos 3\frac{\pi}{8} = 0.38268,\\
4 &= \cos 4\frac{\pi}{8} = 0.
\end{aligned}
\tag{38}
$$

The horizontal sums resulting from the four multiplication schemes are denoted as follows:

from (a): A'' with even subscripts (i.e. A_0'', A_2'', A_4''),
from (b): A'' with odd subscripts (i.e. A_1'', A_3''),
from (c): A' with even subscripts (i.e. A_0', A_2'),
from (d): A' with odd subscripts (i.e. A_1', A_3').

The final Fourier coefficients are then

$$a_k = \tfrac{1}{4}A_k \qquad \text{and} \qquad a_{8-k} = \tfrac{1}{4}A_{8-k},$$

where

$$
\begin{aligned}
A_k &= A_k' + A_k'' \quad (k = 0, 1, 2, 3, 4),\\
A_{8n-k} &= A_k'' - A_k' \quad (k = 0, 1, 2, 3).
\end{aligned}
\tag{39}
$$

The complete analysis requires only six multiplications.

Sine Matrix.

We again use the integers $1, 2, \cdots n$ as a symbolic notation. An arbitrary integer k shall stand for $\sin k(\pi/2n)$. The guiding line is written as follows:

$$
\begin{aligned}
&1, 2, \cdots n-1, n, n-1, \cdots 2, 1, 0, -1, -2,\\
&\qquad \cdots -(n-1), -n, -(n-1), \cdots -1, 0.
\end{aligned}
\tag{40}
$$

This line represents a complete cycle of $\sin k(\pi/2n)$ values, k running from 1 to $4n$.

We construct a matrix of n rows and n columns. The "zero" row and column is now missing. Otherwise the construction proceeds exactly as before in the cosine case.

Example: $n = 4$.

Guiding line:

$$1, 2, 3, 4, 3, 2, 1, 0, -1, -2, -3, -4, -3, -2, -1, 0. \quad (41)$$

Sine Matrix ($n = 4$).

$$\begin{array}{rrrr} 1 & 2 & 3 & 4 \\ 2 & 4 & 2 & 0 \\ 3 & 2 & -1 & -4 \\ 4 & 0 & -4 & 0 \end{array} \qquad (42)$$

We separate the even and odd rows and columns as before:

(a) Even rows / Even columns (b) Even rows / Odd columns

(c) Odd rows / Even columns (d) Odd rows / Odd columns.

Example: $n = 4$.

$$(a)\ \begin{array}{rr} 4 & 0 \\ 0 & 0 \end{array} \qquad (b)\ \begin{array}{rr} 2 & 2 \\ 4 & -4 \end{array}$$

$$(c)\ \begin{array}{rr} 2 & 4 \\ 2 & -4 \end{array} \qquad (d)\ \begin{array}{rr} 1 & 3 \\ 3 & -1 \end{array} \qquad (43)$$

(a) uses the even v'' (i.e. v_2'', v_4''),
(b) uses the odd v'' (i.e. v_1'', v_3''),
(c) uses the even v' (i.e. v_2', v_4'),
(d) uses the odd v' (i.e. v_1', v_3').

We put the proper v-values at the top of the matrices. Moreover, we have to have a table at hand which gives the values of the integer symbols. In the present example this table would look as follows:

$$0 = \sin 0 \frac{\pi}{8} = 0, \qquad 1 = \sin \frac{\pi}{8} = 0.38268,$$

$$2 = \sin 2 \frac{\pi}{8} = 0.70711, \qquad 3 = \sin 3 \frac{\pi}{8} = 0.92388,$$

$$4 = \sin 4 \frac{\pi}{8} = 1. \tag{44}$$

The results of the 4 multiplication schemes are denoted as follows:

from (a): B'' with even subscripts (i.e. B_2'', B_4''),
from (b): B'' with odd subscripts (i.e. B_1'', B_3''),
from (c): B' with even subscripts (i.e. B_2', B_4'),
from (d): B' with odd subscripts (i.e. B_1', B_3').

The final Fourier coefficients are then

$$b_k = \tfrac{1}{4}B_k \quad \text{and} \quad b_{8-k} = \tfrac{1}{4}B_{8-k},$$

where

$$B_k = B_k' + B_k'' \quad (k = 1, 2, 3, 4)$$

and

$$B_{8n-k} = -B_k'' + B_k' \quad (k = 1, 2, 3). \tag{45}$$

Again the complete analysis requires only six multiplications.

3. The Transformation of Ordinates.

Any arithmetical scheme for harmonic analysis, including the matrix scheme, becomes increasingly complicated as the number of required coefficients increases. The number of operations involved in an analysis for $2n$ coefficients is approximately four times the number of operations for n coefficients. We shall now describe a method which eliminates the necessity of complicated schemes by reducing either a cosine or a sine analysis for $4n$ coefficients to two analyses for $2n$ coefficients. If desired, this reduction process can be applied twice or three times.

The method is based upon the following observation. Let us select every second column of the multiplication matrix;

that is, let us assign only even values to the sum index α.

$$\alpha = 0, 2, 4, 6, \cdots 4n. \tag{46}$$

If we now put

$$\alpha = 2\alpha' \quad (\alpha' = 0, 1, 2, \cdots 2n), \tag{47}$$

we obtain

$$\frac{\pi}{4n}\alpha k = \frac{\pi}{2n}\alpha' k, \tag{48}$$

which shows that the contribution of the even ordinates to a Fourier analysis of $4n$ coefficients is exactly the same as the contribution of all the ordinates to a Fourier analysis of $2n$ coefficients. In regard to the contribution of even ordinates the matrix scheme for one-half the number of ordinates can replace the matrix scheme for the total number of ordinates.

We now wish to show that the contribution of the odd ordinates is also reducible to the same matrix scheme for one-half the number of ordinates. We will then be able to make an analysis for $4n$ coefficients by using twice the matrix scheme for $2n$ coefficients; once to get the contribution of the even ordinates, and once to get the contribution of the odd ordinates. This is possible by means of a transformation process introducing a phase difference, which transposes the odd ordinates, as far as their contribution to the Fourier coefficients is concerned, to the place of the even ordinates.

The contribution of the odd ordinates is, from equations (12) and (13)

$$A_k' = \sum_{\beta=1}^{4n-1} u_\beta \cos \frac{\pi}{4n} k\beta \quad (\beta = 1, 3, 5, \cdots 4n - 1), \tag{49}$$

$$B_k' = \sum_{\beta=1}^{4n-1} v_\beta \sin \frac{\pi}{4n} k\beta \quad (\beta = 1, 3, 5, \cdots 4n - 1). \tag{50}$$

Let $\beta = 2\alpha + 1$ where

$$\alpha = 0, 1, 2, \cdots 2n - 1. \tag{51}$$

Then

$$A_k' = \sum_{\alpha=0}^{2n-1} u_{2\alpha+1} \cos \frac{\pi}{2n} k(\alpha + \tfrac{1}{2}), \tag{52}$$

$$B_k' = \sum_{\alpha=1}^{2n-1} v_{2\alpha+1} \sin \frac{\pi}{2n} k(\alpha + \tfrac{1}{2}). \tag{53}$$

We now define a new set of u_k and v_k ordinates called the transformed ordinates and denote these by $\bar{u}_k$ and $\bar{v}_k$. They are defined by the equations

$$
\begin{array}{llll}
u_1 = 2\bar{u}_0 + \bar{u}_1 & & v_1 = 2\bar{v}_0 - \bar{v}_1 \\
u_3 = \bar{u}_1 + \bar{u}_2 & & v_3 = \bar{v}_1 - \bar{v}_2 \\
u_5 = \bar{u}_2 + \bar{u}_3 & & v_5 = \bar{v}_2 - \bar{v}_3 \\
\cdot \quad \cdot \quad \cdot & & \cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot & & \cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot & & \cdot \quad \cdot \quad \cdot \\
u_{2k+1} = \bar{u}_k + \bar{u}_{k+1} & & v_{2k+1} = \bar{v}_k - \bar{v}_{k+1} \\
\cdot \quad \cdot \quad \cdot & & \cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot & & \cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot & & \cdot \quad \cdot \quad \cdot \\
u_{4n-3} = \bar{u}_{2n-2} + \bar{u}_{2n-1} & & v_{4n-3} = \bar{v}_{2n-2} - \bar{v}_{2n-1} \\
u_{4n-1} = \bar{u}_{2n-1} + \bar{u}_{2n} & & v_{4n-1} = \bar{v}_{2n-1} - \bar{v}_{2n} \\
\bar{u}_{2n} = 0 & & \bar{v}_{2n} = 0
\end{array}
\tag{54}
$$

The evaluation of the transformed ordinates is a quick and simple process. As seen from (54) $u_{4n-1} = \bar{u}_{2n-1}$. On starting with the last ordinate $u_{2n} = 0$ we proceed successively from right to left.

Consider the case $2n = 8$ (i.e. 16 ordinates). The transforming process for the odd ordinates can be carried out in a scheme like the following:

$$
\begin{array}{lccccc}
 & u_1 & u_3 & u_5 & u_7 & \\
 & \bar{u}_1 \nwarrow & \bar{u}_2 \nwarrow & \bar{u}_3 \nwarrow & 0 \nwarrow & \\
\hline
\text{Differences} & 2\bar{u}_0 & \bar{u}_1 & \bar{u}_2 & \bar{u}_3 & \bar{u}_4 = 0 \\
 & & & & & \\
 & v_1 & v_3 & v_5 & v_7 & \\
 & \bar{v}_1 \nwarrow & \bar{v}_2 \nwarrow & \bar{v}_3 \nwarrow & 0 \nwarrow & \\
\hline
\text{Sums} & 2\bar{v}_0 & \bar{v}_1 & \bar{v}_2 & \bar{v}_3 & \bar{v}_4 = 0
\end{array}
\tag{55}
$$

To check the calculation we have

$$
\begin{aligned}
2\bar{u}_0 &= u_1 - u_3 + u_5 - u_7 \\
2\bar{v}_0 &= v_1 + v_3 + v_5 + v_7.
\end{aligned}
\tag{56}
$$

Owing to the definition of the transformed ordinates $\bar{u}_k$ and $\bar{v}_k$ in equations (54) the contribution of the odd ordinates becomes, by substitution in (52) and (53)

$$A_k' = 2\bar{u}_0 \cos\frac{\pi}{4n}k + \bar{u}_1\left(\cos\frac{\pi}{4n}k + \cos\frac{\pi}{4n}3k\right)$$

$$+ \bar{u}_2\left(\cos\frac{\pi}{4n}3k + \cos\frac{\pi}{4n}5k\right) + \cdots \quad (57)$$

$$= 2\cos\frac{\pi}{4n}k\sum_{\alpha=0}^{2n}\bar{u}_\alpha \cos\frac{\pi}{2n}k\alpha,$$

$$B_k' = 2\bar{v}_0 \sin\frac{\pi}{4n}k + \bar{v}_1\left(\sin\frac{\pi}{4n}3k - \sin\frac{\pi}{4n}k\right)$$

$$+ \bar{v}_2\left(\sin\frac{\pi}{4n}5k - \sin\frac{\pi}{4n}3k\right) + \cdots \quad (58)$$

$$= 2\sin\frac{\pi}{4n}k\sum_{\alpha=0}^{2n}\bar{v}_\alpha \cos\frac{\pi}{2n}k\alpha.$$

Thus we see that, apart from the weight factors $2\cos(\pi/4n)k$ and $2\sin(\pi/4n)k$, the calculation with the transformed ordinates is identical to the cosine analysis of half the number of ordinates. The final coefficients of the analysis are then given by equations (26) and (27).

For example, let us consider again the case of 32 ordinates. These y_k are paired and reduced, as outlined previously, to u_k requiring a cosine analysis for 16 coefficients and to v_k requiring a sine analysis for 16 coefficients.

The even u_k are analyzed directly using the cosine scheme for 8 coefficients. The odd u_k are transformed, analyzed by the cosine scheme for 8 coefficients, and multiplied by the weight factor $2\cos(\pi/16)k$. If we call the result of the analysis of the even u_k, A_k'', and of the odd u_k, A_k'; then the final cosine coefficients are

$$A_k = A_k'' + A_k' \quad \text{and} \quad A_{16-k} = A_k'' - A_k' \quad (k = 0, 1, 2, \cdots 8). \quad (59)$$

Similarly the even v_k are analyzed directly using the sine scheme for 8 coefficients. The odd v_k are transformed, analyzed by the cosine scheme for 8 coefficients, and multiplied by the weight factor $2\sin(\pi/16)k$. If we call the result of the analysis of the even v_k, B_k'', and of the odd v_k, B_k'; then the final sine coefficients are

$$B_k = B_k' + B_k'', \qquad B_{16-k} = B_k' - B_k'' \qquad (k = 0, 1, 2, \cdots 8). \tag{60}$$

For 64 ordinates we might apply the reduction process twice. The cosine analysis for 32 coefficients is then accomplished by making 4 cosine analyses for 8 coefficients, and the sine analysis for 32 coefficients by making 1 sine and 3 cosine analyses for 8 coefficients.

The great advantage of the transformation process is its ability to reduce large matrices (or complicated standard forms) to small matrices (or simple standard forms). For analyses requiring a large number of coefficients this reduction more than compensates for the small amount of labor involved in the transformation process and in the multiplication by weight factors. In addition, the transformation process makes it much easier to check the analyses and to locate errors in computation. Finally, if we have made an analysis for $2n$ coefficients and find the convergence unsatisfactory we can double the number of coefficients by making one additional analysis for $2n$ coefficients. In this way we obtain the full benefit of our original calculations.

(*To be continued in May issue.*)