Tensors

Johannes Hahn Andrea Hanke

May 16, 2020

1 Intro

We fix a euclidean vector space V, mostly with $V = \mathbb{R}^3$ in mind. We are interested in a classification of reasonable subspaces and linear maps between spaces of tensors, i.e. subspaces of $V \otimes V \otimes \ldots \otimes V$. We understand "reasonable" to mean that the object in consideration should be invariant under change of coordinates, i.e. invariant under the action of the group $O(V)$ of isometries.

2 Results from representation theory

2.1 Definition:
Let G be a group. A representation of G is a vector space W together with a family of linear maps $\phi_g : W \to W$ such that

$$\forall g, h \in G : \phi_g \circ \phi_h = \phi_{gh}$$

holds. With these maps we can view the elements of G as linear operators acting on V and if we want to simplify notation we will simply write this action as multiplication, i.e. we will write gw instead of $\phi_g(w)$ for group elements $g \in G$ and vectors $w \in W$.

2.2 Example:
The vector space V is a representation for the group $G = O(V)$.

2.3 Example (Trivial representations)**:**
Any vector space can be upgraded to a representation for any group, the so called trivial representation, by defining ϕ_g to be the identity map id_V for all $g \in G$.

2.4 Theorem (New representations from old)**:**
If (W, ϕ_g), (W', ϕ'_g) are representations of G, then the following vector spaces are also representations w.r.t. the given actions of G:

a.) any subspace $U \leq W$ with $\forall g \in G : \phi_g(U) \subseteq U$ (a so called invariant subspace) with $gu := \phi_g(u)$,

b.) any quotient W/U by an invariant subspace with $g(w+U) := \phi_g(w) + U$

c.) the direct product $W_1 \times W_2$ with $g(w_1, w_2) := (\phi_g(w_1), \phi'_g(w_2))$, and

d.) the tensor product $W_1 \otimes W_2$ with $g(w_1 \otimes w_2) := \phi_g(w_1) \otimes \phi_g(w_2)$.

2.5 Example (Trivial sub-representations)**:**
Every representation has two obvious subspaces that are invariant, namely the whole space and the subspace $\{0\}$.

2.6 Definition (Irreducible representations)**:**
Let $W \neq \{0\}$ be a representation of G. If the two obvious invariant subspaces W are the only invariant subspaces, then W is said to be an irreducible representation.

2.7 Example (One-dimensional representations)**:**
Every one-dimensional representation is irreducible for dimension reasons.

2.8 Example:
The euclidean space V is irreducible as a representation for $O(V)$. To see this, assume that $U \leq V$ is any non-zero, invariant subspace. Let $u \in U \setminus \{0\}$.

2.9 Theorem (Schur's lemma)**:**