G-graded algebras and Clifford theory

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1 G-graded algebras

1.1 Definition (Graded algebras)**:**
Let G be a group. A *G-graded k-algebra* is a k-algebra A endowed with a decomposition $A = \bigoplus_{g\in G} A_g$ into k-submodules such that $1 \in A_1$ and $\forall g,h : A_g A_h \subseteq A_{gh}$.

A is a *crossed* G-graded algebra if each A_g contains a unit.

1.2: The homogeneous components of a crossed G-graded algebra all have the same dimension because $A_g = u_g A_1 = A_1 u_g$ if $u_g \in A_g \cap A^\times$. In particular: The units u_g are unique up to multiplication with a unit from $A_1^\times$ and every crossed graded algebra is fully graded. The converse need not hold.

1.3 Example (Group algebras and twisted group algebras)**:**
$k[G]$ is a crossed G-graded algebra. More generally $k_\alpha[G]$ is a crossed G-graded algebra for all 2-cocycles $\alpha \in Z^2(G, k^\times)$.

1.4 Example (Quotient groups)**:**
If A is G-graded and $\phi : G \to \Gamma$ is a group homomorphism, then A is also Γ-graded via

$$A_\gamma := \bigoplus_{g\in\phi^{-1}(\gamma)} A_g$$

If A is crossed as a G-graded algebra and ϕ surjective, then it is also crossed as a Γ-graded algebra.

1.5 Example (Subgroups)**:**
If A is a G-graded algebra and $X \subseteq G$ a subset, then define

$$A_X := \bigoplus_{g\in X} A_g$$

Note that $A_X A_Y \subseteq A_{XY}$ so that if $H \leq G$ is a subgroup, then is a H-graded subalgebra of A. If A is fully graded/crossed, then A_H is fully graded/crossed too.
Furthermore: If $X = gH$ is a coset, then A_X is a $A_{{}^gH}$-A_H-bimodule.

1.6 Example (Tensorproducts of graded algebras):

- Let A and B be two (fully graded / crossed) G-graded k-algebras. Then $A \otimes_k B$ is a (fully graded / crossed) $G \times G$-graded algebra via $(A \otimes B)_{g,h} = A_g \otimes_k B_h$.
- The diagonal subalgebra of the tensor product
$$A \odot B := \sum_{g \in G} A_g \otimes B_g \subseteq A \otimes B$$
is a (fully graded / crossed) G-graded algebra.
- $\odot$ is associative and $k[G]$ is the neutral element.
- Note that if $A = k_\alpha[G]$, $B = k_\beta[G]$ for $\alpha, \beta \in Z^2(G, k^\times)$, then $A \odot B \cong k_{\alpha\beta}[G]$.
- This shows that $H^2(G, k^\times)$ acts on the category of G-graded algebras via $A \mapsto k_\alpha[G] \odot A$.
- Moreover: If $V \in A-\mathsf{Mod}, W \in B-\mathsf{Mod}$, then $V \otimes_k W$ is naturally a $A \odot B$-module.

1.7 Proposition (Conjugation action):
Let A be a crossed G-graded algebra. Then
Conjugation with $u_g \in A_g \cap A^\times$ defines a homomorphism $G \to Out(A_1)$ or more generally to $\mathrm{gradedAut}(A)/\operatorname{Inn}(A_1)$.
Therefore G acts by conjugation on everything that is invariant under multiplication with A_1 in an appropriate sense, i.e. the center of A_1, the set of two sided ideals of A_1, $\operatorname{Hom}_{A_1}(X, Y)$ for modules etc.

Proof. $u_g u_h$ is a unit in $A_g A_h \subseteq A_{gh}$ and therefore $u_g u_h = u_{gh} a$ for some $a \in A_1^\times$. Therefore $\kappa_{u_{gh}} \equiv \kappa_{u_g} \circ \kappa_{u_h} \mod Inn(A_1)$ where κ_u is conjugation with u. □

2 Clifford theory

Convention:
Fix a finite group G and a crossed G-graded k-algebra A for k some commutative ring. Assume that k and A are nice enough such that the Krull-Schmidt theorem holds whenever we want it to hold. That is satisfied if k is a complete DVR and A_1 is finitely generated as a k-module for example.
In particular, the block decompositions actually exist.

2.1: Remember that G acts by conjugation on $C_A(A_1)$. In particular it acts on $Z(A_1)$ and also on $Bl(A_1)$.

2.2 Definition:
Let $b \in Bl(A_1)$ be a block. Define the inertial group $I_G(b)$ as the stabiliser of b w.r.t. the conjugation action.

2.3: In the example of $A = k[\Gamma]$ with the Γ/N-grading, A_1 is just $k[N]$. The inertial group $I_\Gamma(b)$ is usually defined as the stabiliser of the conjugation action of Γ on $k[N]$. Since N acts trivially on $C_{k[\Gamma]}(k[N])$, this factors through $G = \Gamma/N$ and gives exactly the same conjugation action on $C_{k[\Gamma]}(k[N])$. Therefore $N \leq I_\Gamma(b)$ and we can view $I_\Gamma(b)$ as a subgroup of G instead. Then we get $I_G(b) = I_\Gamma(b)/N$.

Moreover $C_\Gamma(N) \leq I_\Gamma(b)$ in that example.

2.1 The Fong-Reynolds theorem

2.4 Definition (Covering of blocks)**:**
We say a block $B \in Bl(A)$ covers $b \in Bl(A_1)$ iff $e_b e_B \neq 0$.

2.5 Lemma (Covering in terms of bimodules)**:**
TFAE:

a.) $e_b e_B \neq 0$

b.) $b \cdot B \neq 0$

c.) $B \cdot b \neq 0$

d.) $b \mid \operatorname{Res}^{A \otimes A^{op}}_{A_1 \otimes A_1^{op}}(B)$

Proof. The first three are obviously equivalent because $b = e_b A_1 = A_1 e_b$ and $B = Ae_b = e_B A$.
We also have

$$b \mid A_1 \mid \operatorname{Res}^{A \otimes A^{op}}_{A_1 \otimes A_1^{op}}(A) = \bigoplus_{B \in Bl(A)} \operatorname{Res}^{A \otimes A^{op}}_{A_1 \otimes A_1^{op}}(B)$$

so that $b \mid \operatorname{Res}^{A \otimes A^{op}}_{A_1 \otimes A_1^{op}}(\tilde{B})$ for some block $\tilde{B}$ of A by the Krull-Schmidt theorem.
Since e_B annihilates all blocks of A other than B, $e_B b \neq 0 \iff b \mid \operatorname{Res}^{A \otimes A^{op}}_{A_1 \otimes A_1^{op}}(B)$. $\square$

2.6 Lemma (Covering in terms of modules)**:**
TFAE:

a.) B covers b.

b.) There exists an indecomposable $0 \neq M \in A-\mathsf{mod}$ with $BM \neq 0 \neq b \operatorname{Res}^G_1(M)$.

c.) There exists an indecomposable $0 \neq N \in A_1-\mathsf{mod}$ with $B \operatorname{Ind}^G_1(N) \neq 0 \neq bN$.

Proof. a. $\Longrightarrow$ b.+c. follows by setting $M' := B$ and $N' := b$. By the previous lemma $bM' \neq 0$ and $BN' \neq 0$, therefore any indecomposable summand $M \mid M'$ and $N \mid N'$ does it.

b. $\implies$ a. If $BM \neq 0 \neq b\,\mathrm{Res}_1^G(M)$, then $M = e_B M$ so that $0 \neq e_b M = e_b e_B M$ so that $e_b e_B \neq 0$.

c. $\implies$ a. If $bN \neq 0 \neq B\,\mathrm{Ind}_1^G(N)$, then $N = e_b N$. As a k-module $\mathrm{Ind}_1^G(N) = \bigoplus_{g\in G} u_g \otimes N$. Therefore there must be a $g \in G$ such that $0 \neq e_B(u_g \otimes N) = e_B(u_g e_b \otimes N)$. Thus $e_B e_b \neq 0$. $\square$

2.7 Lemma:
In that case, every block of A covers exactly one G-conjugation class of blocks of A_1 and every block of A_1 is covered by at least one block of A.

Proof. Step 1: Every $B \in Bl(A)$ covers at least one $b \in Bl(A_1)$, because

$$0 \neq e_B = e_B \cdot 1 = \sum_{b \in Bl(A_1)} e_B e_b$$

Every $b \in Bl(A_1)$ is covered by at least one $B \in Bl(A)$, because

$$0 \neq e_b = 1 \cdot e_b = \sum_{B \in Bl(A)} e_B e_b$$

Step 2: If B covers b, then it covers the whole conjugation class: $e_B e_b \neq 0 \implies e_B {}^g e_b = {}^g e_B {}^g e_b = {}^g(e_B e_b) \neq 0$ so that B also covers ${}^g b$.
Step 3: If B covers b, then it does not cover more than the conjugation class.
Let $C \subseteq Bl(A_1)$ be a G-conjugation class. Then

$$e_C := \sum_{b \in C} e_b$$

is a idempotent of $Z(A_1)$ that is G-invariant, i.e. it commutes with A_1 and all u_g. Therefore $e_C \in Z(A)$. It is therefore a sum of block idempotents of A.
If B covers b, then $e_B e_b \neq 0$ and thus $e_B e_C = \sum_{b \in C} e_B e_b \neq 0$ because the $e_B e_b$ all pairwise orthogonal idempotents. Therefore $e_B \leq e_C$. Conversely, if $e_B \leq e_C$, then $e_B e_C \neq 0$ so that $e_B e_b \neq 0$ for some (all by Step 2) $b \in C$.
Thus, B cannot cover blocks from two different G-conjugacy classes because e_C and $e_{C'}$ are orthogonal if $C \neq C'$. $\square$

2.8 Example (Principal blocks)**:**
Let $N \trianglelefteq \Gamma$ be a normal subgroup. Consider $A = k[\Gamma]$ with the Γ/N-grading as above. Then the principal block $b_0 \in Bl(k[N])$ is G-invariant and covered by the principal block $B_0 \in Bl(k[\Gamma])$.

Proof. Let $\nu : k[\Gamma] \to k$ be the augmentation map. b_0 is the unique block b of $k[N]$ with $\nu(e_b) = 1$. Since ν is Γ-equivariant, $\nu({}^\gamma e_{b_0}) = 1$ so that ${}^\gamma e_{b_0} = e_{b_0}$ for all $g \in \Gamma$.
Because $\nu(e_{B_0}) = 1$ as well, $0 \neq 1 = \nu(e_{B_0})\nu(e_{b_0}) = \nu(e_{B_0} e_{b_0})$ so that $e_{B_0} e_{b_0} \neq 0$. $\square$

2.9 Theorem (Clifford-Fong-Reynolds correspondence)**:**
Let $b \in Bl(A_1)$ be a block and $T := I_G(b)$ its inertial group.
There is a bijection

$$\{\, \beta \in Bl(A_T) \mid \beta \text{ covers } b \,\} \to \{\, B \in Bl(A) \mid B \text{ covers } b \,\}$$

the Fong-Reynolds correspondence or Clifford correspondence for blocks, given by

- the trace map tr_T^G on block idempotents. (Remember that $C_A(A_1)$ is a G-algebra and contains $Z(A_1)$ and $Z(A)$)
- induction $\mathrm{Ind}_{A_T \otimes A_T^{op}}^{A \otimes A^{op}}$ on bimodules.
- $\beta \mapsto \beta^{G/T \times G/T}$ on k-algebras.
- $\beta \mapsto A\beta A$ on subsets of A.

The Clifford correspondent $B \in Bl(A)$ of $\beta \in Bl(A_T)$ is Morita equivalent to β, via

$$\mathrm{Ind}_{A_T}^{A} : \beta\mathsf{-Mod} \leftrightarrows B\mathsf{-Mod} : e_\beta \,\mathrm{Res}_{A_T}^{A}$$

and $Z(\beta) \xrightarrow[\cong]{\mathrm{tr}_T^G} Z(B)$ is the isomorphism (of k-algebras) induced by this equivalence.

Proof. The block ${}^g\beta \in Bl({}^gA_T) = Bl(A_{{}^gT})$ covers ${}^gb \in Bl(A_1)$ and only that, because β covers b and only b, because b is invariant under conjugation by T.

Step 0: These idempotents are pairwise orthogonal as elements of A, because if $g, g' \in G$ are arbitrary and $\beta, \beta' \in Bl(A_T)$ both cover b, then

$$\begin{aligned}
({}^ge_\beta)({}^{g'}e_{\beta'}) \cdot 1 &= {}^ge_\beta \sum_{b_1 \in Bl(A_1)} \underbrace{{}^{g'}e_{\beta'} \cdot e_{b_1}}_{\neq 0 \iff b_1 = {}^{g'}b} \\
&= {}^ge_\beta \cdot {}^{g'}e_{\beta'} \cdot {}^{g'}e_b \\
&= \underbrace{{}^ge_\beta \cdot {}^{g'}e_b}_{\neq 0 \iff {}^gb = {}^{g'}b} \cdot {}^{g'}e_{\beta'} \\
&= \delta_{gT, g'T} \cdot ({}^ge_\beta \cdot {}^ge_b) \cdot {}^ge_{\beta'} \\
&= \delta_{gT, g'T} \cdot {}^g(e_\beta e_b e_{\beta'}) \\
&= \delta_{gT, g'T} \cdot {}^g(e_b e_\beta e_{\beta'}) \\
&= \delta_{gT, g'T} \cdot \delta_{\beta, \beta'} {}^g(e_b e_\beta)
\end{aligned}$$

Therefore $e_B := \mathrm{tr}_T^G(e_\beta) = \sum_{gT} {}^ge_\beta$ is also an idempotent. By construction it is also G-invariant and because $e_\beta \in Z(A_T)$, it commutes with A_1. Hence it is central in A. We will prove that it is an indecomposable central idempotent. We let $B := Ae_B = e_BA$ be the ideal generated by e_B.

Step 1: First we claim that

$$B = Ae_\beta A \tag{$*$}$$

This follows from $e_\beta = e_\beta e_B \in Ae_B$ on one hand and ${}^g e_\beta = u_g e_\beta u_g^{-1} \in Ae_\beta A$ on the other.

Step 2: Furthermore $e_B = \sum_{gT} {}^g e_\beta$ is a decomposition of the identity of B into orthogonal idempotents. Therefore $B = \bigoplus_{gT,hT} ({}^g e_\beta)B({}^h e_\beta)$ as k-modules. Note that

$$({}^g e_\beta)B({}^h e_\beta) = u_g \beta u_h^{-1} \tag{$**$}$$

because $u_g\beta u_h^{-1} = (u_g e_\beta u_g^{-1})(u_g \beta u_h^{-1})(u_h e_\beta u_h^{-1}) \subseteq ({}^g e_\beta)(A\beta A)({}^h e_\beta) \overset{(*)}{=} ({}^g e_\beta)B({}^h e_\beta)$ and conversely $B = A\beta A = \sum_{gT,hT}(u_g A_T)\beta(A_T u_h^{-1}) = \sum_{gT,hT} u_g \beta u_h^{-1}$.

Step 3: The decompositions $B = \bigoplus u_g \beta u_h^{-1}$ and $A = \bigoplus_{gT} u_g A_T = \bigoplus_{hT} A_T u_h^{-1}$ together prove $B \cong \operatorname{Ind}_{A_1 \otimes A_1 op}^{A \otimes A^{op}}(\beta) = A \otimes_{A_T} \beta \otimes_{A_T} A$ as bimodules. This also proves that $\beta^{G/T \times G/T} \to B, (a_{gT,hT}) \mapsto \sum_{gT,hT} u_g a_{g,h} u_h^{-1}$ is a isomorphism of algebras.

This proves that B is a matrix algebra over β. Its two-sided ideals are therefore in bijection with the two-sided ideals of β and no proper decomposition exists. In particular, B is indecomposable, i.e. a block.

Now that we have established well-definedness of the map, we still need to show that it is bijective. Injectivity follows from the orthogonality proved in step 0.

If $B \in Bl(A)$ covers b, then $0 \neq e_B e_b = \sum_{\beta \in Bl(A_T)} e_B e_\beta e_b$ so that there must be a $\beta \in Bl(A_T)$ with $e_\beta e_b \neq 0$ and $e_B e_\beta \neq 0$. But then $e_B \operatorname{tr}_T^G(e_\beta) \neq 0$ because the summands in $\operatorname{tr}_T^G(e_\beta) = \sum_{gT} {}^g e_\beta$ are pairwise orthogonal idempotents. Since $\operatorname{tr}_T^G(e_\beta)$ is itself a block idempotent, it must be equal to e_B. This proves surjectivity.

Finally we have to prove the statement about the explicit shape of Morita equivalences. We have already seen that $B \overset{(*)}{=} Ae_\beta A = Be_\beta B$ and $e_\beta B e_\beta \overset{(**)}{=} u_1 \beta u_1^{-1} = \beta$. Therefore a Morita equivalence $\beta \leftrightarrow B$ is given by tensoring with the corresponding bimodules:

$$Be_\beta \otimes_\beta - : \beta\text{−}\mathsf{Mod} \leftrightarrows B\text{−}\mathsf{Mod} : e_\beta B \otimes_B -$$

For all $N \in B\text{−}\mathsf{Mod}$:

$$\begin{aligned} e_\beta \operatorname{Res}_{A_T}^A(N) &= e_\beta A \otimes_A N \\ &= e_\beta \bigoplus_{\tilde{B} \in Bl(A)} \tilde{B} \otimes_{\tilde{B}} e_{\tilde{B}} N \\ &= e_\beta B \otimes_B N \end{aligned}$$

Now we observe that B is the unique block of A with $e_{\tilde{B}} e_\beta \neq 0$ because $e_{\tilde{B}} e_\beta \neq 0 \implies$

$\forall g : e_{\tilde{B}}{}^g e_\beta \neq 0 \implies e_{\tilde{B}} \operatorname{tr}_T^G(e_\beta) \neq 0 \implies \tilde{B} = B$. Thus for all $M \in \beta-\mathsf{Mod}$:

$$\begin{aligned}
\operatorname{Ind}_{A_T}^A(M) &= A \otimes_{A_T} M \\
&= \bigoplus_{\tilde{B} \in Bl(A)} \tilde{B} \otimes_{A_T} M && \text{because } A = \bigoplus \tilde{B} \text{ as } A\text{-}A_T\text{-bimodules} \\
&= \bigoplus_{\substack{\tilde{B} \in Bl(A) \\ \tilde{\beta} \in Bl(A_T)}} \tilde{B} e_{\tilde{\beta}} \otimes_{\tilde{\beta}} e_{\tilde{\beta}} M && \text{because } A_T = \bigoplus \tilde{\beta} \text{ as } A_T\text{-}A_T\text{-bimodules} \\
&= \bigoplus_{\tilde{B}} \tilde{B} e_\beta \otimes_\beta M && \text{because } M \in \beta-\mathsf{Mod} \\
&= B e_\beta \otimes_\beta M
\end{aligned}$$

For the statement about the trace map just observe that $({}^g z) \cdot ({}^{g'} z') = ({}^g z {}^g e_\beta)({}^{g'} e_\beta {}^{g'} z') = \delta_{gT, g'T} {}^g(zz')$ by step 0. This immediately proves multiplicativity of the trace map and we have $\operatorname{tr}_T^G(e_\beta) = e_B$ by construction of B so that it is indeed a homomorphism of k-algebras.
Therefore multiplication by $\operatorname{tr}_T^G(z) \in Z(B)$ acts as $\operatorname{Ind}_T^G(z)$ on $\operatorname{Ind}_T^G(X)$:

$$\begin{aligned}
\operatorname{tr}_T^G(z) \cdot u_h \otimes x &= \sum_g u_g z u_g^{-1} u_h \otimes x \\
&= \sum_g u_h (u_h^{-1} u_g) z (u_h^{-1} u_g)^{-1} \otimes x \\
&= \sum_g u_h u_{h^{-1}g}^{-1} z u_{h^{-1}g}^{-1} \otimes x && \text{because } z \in C_{A_T}(A_1) \wedge u_x u_y \equiv u_{xy} \mod A_1 \\
&= \sum_g u_h u_g z u_g^{-1} \otimes x \\
&= \sum_g u_h \underbrace{({}^g z)^1 e_\beta}_{=\delta_{g,1} z} \otimes x \\
&= u_h z e_\beta x \\
&= u_h \otimes zx \qquad \square
\end{aligned}$$

2.10: Choosing different units $v_g \in A^\times \cap A_{gT}$ instead of u_g will result in a different isomorphism $B \leftrightarrow \beta^{G/T \times G/T}$. The isomorphisms will differ by conjugation with a diagonal matrix with entries from $A_T^\times$.

2.11 Lemma and definition:
Let $b \in Bl(A_1)$ and $H \leq G$ be arbitrary. Then we define the full subcategory

$$\mathsf{mod}(H|b) := \bigoplus_{\substack{\beta \in Bl(A_H) \\ \beta \text{ covers } b}} \beta-\mathsf{mod} = \left\{ M \in A_H-\mathsf{mod} \;\middle|\; Res_1^H(M) \in \bigoplus_{g \in H/I_H(b)} {}^g b-\mathsf{mod} \right\}$$

Furthermore set $T := I_G(b)$. Then with this notation $\mathrm{Ind}_{A_T}^A$ is an equivalence

$$\mathsf{mod}(T|b) \to \mathsf{mod}(G|b)$$

Proof. This follows then directly from 2.6 and the Fong-Reynolds theorem. □

2.2 Clifford's theorem for simple modules

2.12 Definition (Conjugated modules and inertial subgroups of modules)**:**
G acts on $A_1\mathrm{-Mod}$ (more precisely it acts on the isomorphism classes) as before by letting ${}^gV := A_g \otimes_{A_1} V$. Note that $A_g \otimes_{A_1} A_h \cong A_{gh}$ via multiplication inside A so that this really is an G-action.
The *inertial group* of $V \in A_1\mathrm{-Mod}$ is defined as

$$I_G(V) := \{\, g \in G \mid {}^gV \cong V \,\}$$

2.13: Obviously, V is f.g. / projective / simple / indecomposable / ... iff ${}^gV$ is the same because the conjugation action is given by self-equivalences of $A_1\mathrm{-Mod}$.

2.14: If V is a A_1-submodule of some $\mathrm{Res}_1^G(W)$, then the multiplication $A_gV = u_gV$ is well-defined and also an A_1-submodule of $\mathrm{Res}_1^G(W)$. It is isomorphic to ${}^gV$ as one easily verifies.

2.15 Example:
Let $A = k[\Gamma]$ with the Γ/N-grading of some normal subgroup $N \trianglelefteq \Gamma$.
Also note that $C_\Gamma(N)N \leq \mathit{Stab}(\Gamma, V)$. For $g \in C_\Gamma(N)$ the isomorphism is given simply by $V \to A_{gN} \otimes_{A_N} V, v \mapsto g \otimes v$.

2.16 Lemma (Clifford's first theorem; Semisimple A- and A_1-modules)**:**
Let k be a field and A finite-dimensional.
If V is a simple A-module, then $\mathrm{Res}_1^G(V)$ is a semisimple A_1-module and its simple constituents are a single G-orbit.

Proof. Let $0 \neq U \leq \mathrm{Res}_1^G(V)$ be any simple A_1-submodule. Then $A_gU \subseteq V$ is a A_1-submodule which is isomorphic to ${}^gU$ and therefore itself simple. Now $\sum_{g\in G} A_gU$ is a non-zero A-submodule of V. By simplicity $V = \sum_{g\in G} A_gU$ so that $\mathrm{Res}_1^G(V)$ is semisimple. □

2.17 Definition:
For $H \leq G$ and $U \in \mathrm{Irr}(A_1)$ define

$$\mathrm{Irr}(A_H|U) := \left\{\, V \in \mathrm{Irr}(A_H) \;\middle|\; U \leq \mathrm{Res}_1^H(V) \,\right\}$$

2.18 Theorem (Clifford's theorem for irreducible modules)**:**
Let k be a field and A finite-dimensional. If $U \in A_1-\mathsf{mod}$ is irreducible and $T := I_G(U)$ the stabiliser of its isomorphism class, then

$$\mathrm{Ind}_T^G : \mathrm{Irr}(T|U) \to \mathrm{Irr}(G|U)$$

is a well-defined bijection.

Proof. Consider $J_1 := J(A_1)$ and $J := J_1 A = \bigoplus_{g \in G} J_1 u_g$. Because conjugation with u_g is an automorphism of A_1, we find $u_g J_1 u_g^{-1} = J_1$ so that indeed $u_g J_1 = J_1 u_g$ and thus $J_1 A = A J_1$ is a two-sided, graded ideal of A. Then $\overline{A} := A/J$ is also a G-graded algebra, but $\overline{A}_1 = A_1/J_1$ is now semisimple.

The blocks of $\overline{A}_1$ are canonically isomorphic to $\mathrm{Irr}(\overline{A}_1) = \mathrm{Irr}(A_1)$ as G-sets. And now we really have $T = I_G(b_U)$ where $b_U \in Bl(\overline{A}_1)$ is the block containing U.

By the Fong-Reynolds theorem and its corollary, Ind_T^G is an equivalence

$$\mathsf{mod}(\overline{A}_T|b_U) \to \mathsf{mod}(\overline{A}_G|b_U)$$

As an equivalence it maps the simple objects in the left category bijectively to the simple objects in the right category. Note that all simple A_H-modules are naturally $\overline{A}_H$-modules, because by Clifford's first theorem, their restrictions to A_1 are all semisimple. Therefore the simple objects in $\mathsf{mod}(\overline{A}_H|b_U)$ are exactly $\mathrm{Irr}(A_H|b)$. □

2.3 Clifford's theorem for indecomposable modules

2.19 Theorem (Fong's first reduction; Clifford correspondence for indecomposable, relative projective modules)**:**
Let $U \in A_1-\mathsf{Mod}$ be indecomposable and $T := Stab_G(U)$ its stabiliser.

a.) Ind_T^G induces an equivalence of additive categories

$$\mathrm{add}(\mathrm{Ind}_1^T(U)) \to \mathrm{add}(\mathrm{Ind}_1^G(U))$$

Furthermore: If the Krull-Schmidt theorem is satisfied, then

b.) If

$$\mathrm{Ind}_1^T(U) = V_1 \oplus \cdots \oplus V_m$$

is the decomposition into indecomposables, then all $\mathrm{Ind}_T^G(V_i)$ are indecomposable and $\mathrm{Ind}_T^G(V_i) \cong \mathrm{Ind}_T^G(V_j) \iff V_i \cong V_j$.

Proof. a. Let $b \in Bl(A_1)$ be the block of U. Then $\mathrm{Ind}_1^H(U) \in \mathsf{mod}(A_H|b)$ by 2.6 for all $H \leq G$.
Obviously $\mathsf{mod}(A_H|b) = \bigoplus_{\beta \text{ covers } b} \beta-\mathsf{mod}$ is closed under taking direct summands and direct sums. Therefore $\mathrm{add}(\mathrm{Ind}_1^H(U)) \subseteq \mathsf{mod}(A_H|b)$.
The result follows from the Fong-Reynolds theorem and its corollary.
b. and c. obviously from a. □

2.4 Stable Clifford theory

2.20 Theorem (Fong's second reduction)**:**
Let $k \in \{\mathcal{O}, \mathbb{F}\}$ and assume that $\mathbb{F}$ is a splitting field for A.
Let $b \in Bl(A_1)$ be a G-stable block "of defect zero", i.e. isomorphic to a matrix algebra $b \cong \mathrm{End}_k(U)$ for some f.g. projective k-module U.

a.) Then there exists a 2-cocycle $\xi \in Z^2(G, k^\times)$ such that $\mathsf{mod}(A|b)$ is equivalent to $k_\xi[G]$.

b.) More precisely: $\bigoplus_{B \text{ covers } b} B$ is isomorphic to $b \otimes_k k_\xi[G]$ and

$$U \otimes_k - : k_\xi[G]\text{−}\mathsf{Mod} \to \mathsf{mod}(A|b)$$

is therefore an equivalence.

c.) In particular: Every block $B \in Bl(A)$ that covers b is Morita equivalent to a block of $k_\xi[G]$.

Proof. Step 0: e_b is G-stable and therefore central in A and $e_b A e_b = bA = \bigoplus_{B \text{ covers } b} B$ is therefore a two-sided graded ideal of A. Therefore we can wlog assume that $A_1 = b$ and all blocks of A cover b.

Step 1: The conjugation outer action $G \to \mathrm{Out}(A_1)$ is trivial, because as a matrix algebra A_1 satisfies $\mathrm{Aut}(A_1) = \mathrm{Inn}(A_1)$ by the Skolem Noether theorem. Therefore proposition **??** shows that

$$A = A_1 \otimes_{Z(A_1)} k_\xi[G] = b \otimes_k k_\xi[G]$$

for some $\xi \in Z^2(G, k^\times)$. □

2.21 Theorem (Clifford)**:**
Let $k = \mathbb{F}$ be an algebraically closed field and A finite-dimensional.
Furthermore let $U \in \mathrm{Irr}(A_1)$ be simple and G-stable.

a.) There exists a 2-cocycle $\xi \in Z^2(G, \mathbb{F}^\times)$ such that U extends to an $\mathbb{F}_{\xi^{-1}}[G] \odot A$-module $\widehat{U}$.

b.) There is a bijection

$$\begin{cases} \mathrm{Irr}(A|U) & \leftrightarrow & \mathrm{Irr}(\mathbb{F}_\xi[G]) \\ W \otimes_{\mathbb{F}} \widehat{U} & \leftarrow & W \end{cases}$$

In particular $e(V) = \dim(W)$ for all $V \in \mathrm{Irr}(A|U)$.

Proof. Again the theorem follows from the block-version applied to the G-graded algebra $\overline{A} = A/A \cdot J(A_1)$ instead of A which has the same irreducible modules and semisimple degree-1-piece with blocks corresponding to simple A_1-modules. □