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# Abstract

Let $k$ be an algebraically closed field of characteristic different from 2. In §§1–4 we study the geometry of a Kummer surface in $\mathbb{P}^3$ (i.e. a quartic surface with 16 nodes as its only singularities) and of its minimal desingularization. The 16 nodes of a Kummer surface give rise to a configuration of 16 points and 16 planes in $\mathbb{P}^3$ such that each plane contains exactly six points and each point belongs to exactly six planes (a **(16,6) configuration** for short). A Kummer surface is uniquely determined by the set of its nodes. The philosophy of the first part of this work is to understand (and classify) Kummer surfaces by studying (16,6) configurations. §1 is devoted to classifying (16,6) configurations and studying their manifold symmetries and the underlying questions about finite subgroups of $PGL_4(k)$.

In §2 we use this information to give a complete classification of Kummer surfaces together with explicit equations and the explicit description of their singularities. The moduli space of Kummer surfaces is isomorphic to $\mathcal{M}_2$ (the moduli space of curves of genus 2) but, unlike the case of curves, there exists a fine moduli space for Kummer surfaces. In other words, there exists a universal family over $\mathcal{M}_2$, which we write down explicitly, whose fibers are Kummer surfaces, such that each isomorphism class of a Kummer surface appears exactly once among the fibers. §§3–4 explore the beautiful connections with the theory of K3 surfaces and abelian varieties. We determine the possible Picard numbers for Kummer surfaces in $\mathbb{P}^3$ and for their minimal desingularizations.



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**Keywords.** Kummer surfaces, (16,6) configurations, K3 surfaces, abelian surfaces.

## §0. Introduction.

Let $k$ be an algebraically closed field of characteristic different from 2. A Kummer surface in $\mathbb{P}^3$ is a quartic surface with 16 nodes. A quartic surface in $\mathbb{P}^3$ with only nodes as singularities can have at most 16 nodes. In this sense Kummer surfaces are extremal, so we may expect certain special geometric characteristics. These 16 nodes together with the 16 planes which correspond to the 16 nodes of the dual Kummer surface (called special planes or tropes by the classical authors) form a (16,6) configuration in $\mathbb{P}^3$ which determines the Kummer surface uniquely (Lemma 2.16). This configuration has many symmetries and is a beautiful combinatorial object to study. The purpose of this work is to understand in detail the geometry of a Kummer surface in $\mathbb{P}^3$.

Kummer surfaces appear in many different contexts: they are related to abelian surfaces and to the quadric line complex. Different authors define a Kummer surface in different ways; for example, sometimes it is defined to be the quotient of an abelian surface by the involution. The minimal desingularization of a Kummer surface, which is always a K3 surface, is often called a non-singular Kummer surface. The fact that the quotient of the Jacobian of a non-singular curve of genus 2 by the involution is a Kummer surface in our sense is well-known and easy to prove using theta divisors. The converse, namely, the statement that any Kummer surface can be obtained from some principally polarized abelian variety by taking a quotient by the involution, is also known. We give a new point of view of this in §4. Our approach is based on an *a priori* classification of (16,6) configurations which we call **non-degenerate** (this means that every two special planes have exactly two points of the configuration in common). We prove that all the non-degenerate (16,6) configurations of points and planes in $\mathbb{P}^3$ are combinatorially the same. In other words, for a (16,6) configuration of points and planes in $\mathbb{P}^3$ imposing the non-degeneracy hypothesis determines the incidence matrix uniquely. Thus, in contrast to the traditional approach to Kummer surfaces, we do not need the theory of theta divisors to conclude that the (16,6) configuration associated with a Kummer surface has a certain specified incidence matrix (which we call a (16,6) configuration of type (*)). The classical authors [12] assumed this fact, and the classification of non-degenerate (16,6) configurations in $\mathbb{P}^3$ without proofs.

§1 is devoted to a systematic study of (16,6) configurations. We show that a (16,6) configuration associated with a Kummer surface is always non-degenerate. We define (16,6) configurations of type (*) (this definition is purely combinatorial: it amounts to specifying a particular incidence matrix) and prove that any non-degenerate (16,6) configuration of points and planes in $\mathbb{P}^3$ is of type (*). To do that, we first classify *abstract* non-degenerate (16,6) configurations (which are, by definition, $16\times16$ matrices $(a_{ij})$ whose entries are zeroes and ones, with exactly 6 ones in each row and each column, such that for every two rows $i$ and $j$ there are exactly two columns $l$ and $l'$ such that $a_{il} = a_{jl'} = 1$). We prove that there are

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exactly three abstract non-degenerate (16,6) configurations: the one of type (*) and two "exotic" ones. We then show that the exotic ones do not occur as configurations of points and planes in $\mathbb{P}^3$.

In our work we are not concerned with the quadric line complex, but we hope that this point of view may be useful for stating the precise conditions the "general" quadric hypersurface in $\mathbb{P}^5$ needs to satisfy in order that the corresponding line complex give rise to a Kummer surface. We describe a parameter space $V$ for the set of all the non-degenerate (16,6) configurations viewed as subschemes of $\mathbb{P}^3$. The moduli of the non-degenerate (16,6) configurations modulo automorphisms of $\mathbb{P}^3$ is the quotient of an open subset $U$ of $\mathbb{P}^3$ (which we describe explicitly) by the normalizer $N$ of a certain finite subgroup of $PGL_4(k)$: the group $F_0 \cong \mathcal{C}_2^4$ (Theorem 1.68). The group $N$ is isomorphic to a non-split extension of the symmetric group $S_6$ by $\mathcal{C}_2^4$ (Theorem 1.64). For any non-degenerate (16,6) configuration, after a suitable choice of coordinates, $F_0$ acts as a group of automorphisms of the configuration. For any sufficiently general point $(a, b, c, d) \in \mathbb{P}^3$ (namely, for any point belonging to the set $U$) the 16 points of its orbit under $F_0$ form a (16,6) configuration. There are, for a general $(a, b, c, d) \in U$, 720 distinct (16,6) configurations of this form which are moved into each other by elements of $N$ ($N$ preserves the set $U$). It is easy to show that a Kummer surface is determined uniquely by its (16,6) configuration. Thus the moduli space of Kummer surfaces is given by $U/N$. Since there exists a bijection between Kummer surfaces and non-singular curves of genus 2, $U/N \cong \mathcal{M}_2$ gives a new description of $\mathcal{M}_2$ as a quotient of an open subset of $\mathbb{P}^3$ by a finite group. In §2 we give a complete classification of Kummer surfaces together with explicit equations and the explicit description of their singularities. These equations appear already in [12] but it is not stated or proved there that they describe a universal family of Kummer surfaces over $U/N$ (that is, that every Kummer surface appears exactly once in this family). The same reasoning also yields the Hilbert scheme of Kummer surfaces as subschemes of $\mathbb{P}^3$, given by $(U \times PGL_4(k))/(N \times \mathcal{C}_2^4) \cong V/((\mathcal{C}_2 \ltimes (S_4 \times S_4)) \ltimes A_5)$, up to a birational map. To construct the true Hilbert scheme, one needs a more precise knowledge of automorphism groups of non-generic (16,6) configurations.

In §§3–4 we explore the beautiful connections with the theory of K3 surfaces and abelian surfaces. The rank $\rho(S)$ of the Néron-Severi group of a Kummer surface can be 1, 2, 3 or 4 if $char\ k = 0$. If $char\ k = p > 0$, we can also have, in addition to the above possibilities, $\rho(S) = 6$.

This work is part of the author's Ph. D. thesis.

I would like to thank Fabrizio Catanese, Frans Oort, Fernando Serrano, Mark Spivakovsky and Angelo Vistoli for discussions, suggestions and ideas which were a great help to me in this work. I am very grateful to my thesis advisor Robin Hartshorne whose close attention and numerous comments at all stages of this project were very important in shaping and improving this paper.

## §1. THE CLASSIFICATION OF (16,6) CONFIGURATIONS.

Let $k$ be an algebraically closed field of characteristic $\neq 2$. This section is devoted to the study of (16,6) configurations. We define (16,6) configurations of type (*) (this definition is purely combinatorial: it amounts to specifying a particular incidence matrix) and the notion of a non-degenerate (16,6) configuration. The motivation for studying such configurations is the fact (which will be proved below) that the singular locus of a Kummer surface always forms such a configuration and, conversely, any non-degenerate (16,6) configuration of points and planes in $\mathbb{P}^3$ is of type (*) and is the singular locus of a uniquely determined Kummer surface. Thus classifying Kummer surfaces is equivalent to classifying non-degenerate (16,6) configurations. The main point of this section is to prove that any non-degenerate (16,6) configuration of points and planes in $\mathbb{P}^3$ is of type (*) and, up to a linear automorphisms of $\mathbb{P}^3$, looks like the orbit of a sufficiently general point $(a, b, c, d) \in \mathbb{P}^3$ (where we give a precise definition (1.2.1) of "sufficiently general") under a certain finite subgroup $F_0$ of $PGL_4(k)$ of order 16. Hudson [12, p. 7] makes this claim without proof and without mentioning the non-degeneracy hypothesis, without which the statement is false, (see Example 1.11). We introduce the notion of an *abstract* (16,6) configuration, in order to separate the combinatorial aspects of the problem from the geometric ones. This section contains the following subsections:

I. The classification of abstract non-degenerate (16,6) configurations.

II. The (8,4) configuration.

III. A non-degenerate (16,6) configuration of points and planes in $\mathbb{P}^3$ is of type (*).

IV. A nondegenerate (16,6) configuration in $\mathbb{P}^3$ is of the form $(a, b, c, d)$ of (1.4.1).

V. Moduli of non-degenerate (16,6) configurations.

**Definition 1.1.** *A* **(16,6) configuration** *is a set of 16 planes and 16 points in $\mathbb{P}^3_k$ such that every plane contains exactly 6 of the 16 points and every point lies in exactly 6 of the 16 planes.*

When talking about a given (16,6) configuration, we shall sometimes refer to the 16 planes as **special planes** (this is the classical terminology).

The purpose of this section is to classify the (16,6) configurations satisfying an additional assumption of non-degeneracy:

**Definition 1.2.** *A (16,6) configuration is* **non-degenerate** *if every two special planes share exactly two points of the configuration and every pair of points is contained in exactly two special planes (note that the second part of the definition follows automatically from the first by a counting argument).*

We classify these configurations both as subsets of $\mathbb{P}^3$ and as subsets of $\mathbb{P}^3$ modulo projective linear transformations. The results of this section will allow us to completely classify Kummer surfaces, both as subvarieties of $\mathbb{P}^3$ and as abstract surfaces. Since there is a bijection between the set of Kummer surfaces and the set of non-singular curves of genus 2 (Propositions 4.22 and 4.23), we also get a new way of expressing $\mathcal{M}_2$ as a quotient of an open subset of $\mathbb{P}^3$ by a finite group.

In $\mathbb{P}^3$ with homogeneous coordinates $(x:y:z:t)$, a plane $H$ is represented by:

$$H : ax + by + cz + dt = 0,$$

where $a, b, c, d \in k$ and not all of $a$, $b$, $c$, $d$ are simultaneously 0. We impose the following conditions on $(a, b, c, d)$.

$$\begin{aligned} ad &\neq \pm bc \\ ac &\neq \pm bd \\ ab &\neq \pm cd \\ a^2 + d^2 &\neq b^2 + c^2 \\ a^2 + c^2 &\neq b^2 + d^2 \\ a^2 + b^2 &\neq c^2 + d^2 \\ a^2 + b^2 + c^2 + d^2 &\neq 0. \end{aligned} \tag{1.2.1}$$

**Lemma 1.3.** *The plane $H$ above contains the following 6 points: $(d, -c, b, -a)$, $(d, c, -b, -a)$, $(c, d, -a, -b)$, $(-c, d, a, -b)$, $(-b, a, d, -c)$, $(b, -a, d, -c)$. If $a$, $b$, $c$, $d$ satisfy the conditions (1.2.1), the above 6 points are all distinct.*

*Proof.* Straightforward verification.

Let $A$ denote the linear transformation in $\mathbb{P}^3$ which interchanges $x$ with $t$ and $z$ with $y$; $B$ the one which interchanges $y$ with $t$ and $z$ with $x$; $C$ the one which interchanges $x$ with $y$ and $z$ with $t$.

Let $A'$ denote the linear transformation in $\mathbb{P}^3$ which changes the sign of $z$ and $y$ (and leaves $x$ and $t$ unchanged). Let $B'$ be the one which changes the sign of $z$ and $x$; $C'$ the one which changes the sign of $x$ and $y$.

Let $\mathcal{C}_2$ denote the finite group with two elements.

**1.4 Note.** $C = AB$, $C' = A'B'$. Furthermore, all of the $A$, $B$, $A'$, $B'$ are of order 2 and pairwise commute with each other. Therefore, they generate a group isomorphic to $\mathcal{C}_2^4$. We denote this group by $F_0$. The reason for calling it $F_0$ will become apparent in Remark 1.13, where $F_0$ will appear as an index 2 subgroup in a group of automorphisms of a K3 surface. This way of thinking about $F_0$ will also give a conceptual description of the incidence relations between points and planes in a (16,6) configuration.

We have a 1–1 correspondence between the elements of $F_0$ and the orbit of $(a, b, c, d)$ under $F_0$: we associate each element of $F_0$ to the image of $(a, b, c, d)$ under this element (see (1.4.1) below).

$$\begin{aligned} (a, b, c, d) &\longleftrightarrow (1) \\ (d, -c, b, -a) &\longleftrightarrow (AB') \\ (d, c, -b, -a) &\longleftrightarrow (AC') \\ (c, d, -a, -b) &\longleftrightarrow (BC') \\ (-c, d, a, -b) &\longleftrightarrow (BA') \end{aligned}$$

$$
\begin{aligned}
(-b,a,d,-c) &\longleftrightarrow (CA')\\
(b,-a,d,-c) &\longleftrightarrow (CB')\\
(d,c,b,a) &\longleftrightarrow (A)\\
(c,d,a,b) &\longleftrightarrow (B)\\
(b,a,d,c) &\longleftrightarrow (C)\\
(a,-b,-c,d) &\longleftrightarrow (A')\\
(-a,b,-c,d) &\longleftrightarrow (B')\\
(-a,-b,c,d) &\longleftrightarrow (C')\\
(d,-c,-b,a) &\longleftrightarrow (AA')\\
(-c,d,-a,b) &\longleftrightarrow (BB')\\
(-b,-a,d,c) &\longleftrightarrow (CC')
\end{aligned}
\tag{1.4.1}
$$

**1.5 Note.** Write $F_0$ as a direct product of two order 4 subgroups:

$$F_0 = \langle 1, AB', BC', CA' \rangle \times \langle 1, AC', BA', CB' \rangle.$$

We can arrange the elements of $F_0$ in the following diagram where each entry is the product of the topmost entry in its column with the leftmost entry in its row.

$$
\begin{array}{cccc}
1 & AB' & BC' & CA' \\
AC' & A' & C & BB' \\
BA' & CC' & B' & A \\
CB' & B & AA' & C'
\end{array}
\tag{1.5.1}
$$

**Proposition 1.6.** *The orbit under $F_0$ of the point $(a,b,c,d)$, satisfying the conditions (1.2.1), consists of exactly 16 distinct points and the orbit of the plane $H = \{ax + by + cz + dt = 0\}$ of 16 distinct planes. These two orbits together form a $(16,6)$ configuration.*

*Proof.* The distinctness is straightforward. Lemma 1.7, which we prove below immediately following the Proposition, describes explicitly the 6 planes containing each point and the 6 points contained in each plane (note the symmetry of the incidence diagram due to the group action of $F_0$ on the entire situation).

Conditions (1.2.1) are precisely the conditions needed to guarantee that no plane contains more than 6 points and no point is contained in more than 6 planes. To prove this statement, we check it directly for one plane. Then we conclude it for every plane by the transitivity of the action of $F_0$ on the set of planes. The statement for points is dual, hence follows formally from the same reasoning. □

**Lemma 1.7. Incidence Diagram.** *In the configuration of 16 planes and 16 points of Proposition 1.6 the following holds: there exists a 1-1 correspondence from both the set of planes and the set of points to the sixteen elements of the 4×4 matrix shown on the diagram (1.5.1), such that the following incidence relations are satisfied. Six points lying on any given plane are given by the row and column* *containing that plane, with the element corresponding to the given plane removed. The same rule describes the six planes containing any given point. For example, the plane $\star$ below contains the 6 points $\circ$.*

$$(*) \qquad \begin{array}{cccc} \cdot & \circ & \cdot & \cdot \\ \circ & \star & \circ & \circ \\ \cdot & \circ & \cdot & \cdot \\ \cdot & \circ & \cdot & \cdot \end{array}$$

*Proof.* We identify both the set of points and the set of planes in our configuration with entries in the $4 \times 4$ diagram (1.5.1) via (1.4.1). Elements of $F_0$ act on the $4 \times 4$ diagram (1.5.1) by multiplication of each entry. Multiplying the matrix (1.5.1) by an element of $F_0$ is equivalent to some permutation of rows and of columns. Since every entry of (1.5.1) can be moved to any other by these operations, it is sufficient to verify the Lemma for the six planes containing one particular point, say, $(a, b, c, d)$ (the statement for the six points contained in a given plane is dual, hence follows formally by the same reasoning). Now the Lemma is immediate from Notes 1.4 and 1.5.

**1.8. Examples.** (a) The plane $(A) = (d, c, b, a)$ contains the points $(C')$, $(B')$, $(BB')$, $(BA')$, $(CA')$, $(CC')$.

(b) The plane $(1) = (a, b, c, d)$ contains the points $(AB')$, $(BC')$, $(CA')$, $(AC')$, $(BA')$, $(CB')$.

**Definition 1.9.** *A (16,6) configuration satisfying the incidence relations described in Lemma 1.7 will be called* **a (16,6) configuration of type (*)**.

**Corollary 1.10.** *A* $(16, 6)$ *configuration of type (*) is non-degenerate.*

*Proof.*

$$\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \star & \cdot & \circ \\ \cdot & \circ & \cdot & \star \\ \cdot & \cdot & \cdot & \cdot \end{array} \qquad\qquad \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \star & \circ & \star & \circ \\ \cdot & \cdot & \cdot & \cdot \end{array}$$

Denote the planes by $p_{jl}$, $1 \leq j, l \leq 4$ and the points by $t_{jl}$, $1 \leq j, l \leq 4$, where $p_{jl}$ is the plane in the $j$-th row and the $l$-th column of the diagram; similarly for points. Take two distinct planes, $p_{jl}$ and $p_{j'l'}$. If $j \neq j'$ *and* $l \neq l'$, then the planes $p_{jl}$ and $p_{j'l'}$ have the two points $t_{jl'}$ and $t_{j'l}$ in common (and no others). If $j = j'$ then $l \neq l'$ and the planes $p_{jl}$ and $p_{jl'}$ have in common the two points in the $j$-th row whose indices differ from $l$ and $l'$ (and no other points in common). Similarly for the case $l = l'$. The statement for points is dual.

Subsections I–III of this section are devoted to proving the converse to Corollary 1.10: any non-degenerate (16,6) configuration in $\mathbb{P}^3$ is of type (*). On the other hand, we do not know much about *degenerate* (16,6) configurations (as we prove below, only the non-degenerate ones occur as singular loci of Kummer surfaces). We give an example of a (16,6) configuration, whose 16 points are divided into four subsets of four collinear points (in particular, it is degenerate).

**Example 1.11.** Let $Q \in \mathbb{P}^3_k$ be a non-singular quadric surface. Take 8 lines $L_j$, $1 \le j \le 8$ of one ruling and 4 lines $L'_m$, $1 \le m \le 4$ of the other ruling. Then the 16 points

$$\{L_j \cap L'_m | 1 \le j \le 4, 1 \le m \le 2\} \cup \{L_j \cap L'_m | 5 \le j \le 8, 3 \le m \le 4\}$$

and the 16 tangent planes to $Q$ at the points

$$\{L_j \cap L'_m | 1 \le j \le 4, 3 \le m \le 4\} \cup \{L_j \cap L'_m | 5 \le j \le 8, 1 \le m \le 2\}$$

form a degenerate (16,6) configuration.

**Definition 1.12.** *An* **abstract** *(16,6) configuration is a* $16 \times 16$ *matrix* $(a_{ij})_{1 \le j \le 16}^{1 \le i \le 16}$ *whose entries are zeroes and ones, with exactly six ones in each row and in each column. The rows of the matrix will be called* **points** *of the configuration and the columns* **planes** *of the configuration. We shall say that the* $i$*-th point belongs to the* $j$*-th plane if and only if* $a_{ij} = 1$.

The concept of a *nondegenerate* (16,6) configuration has an obvious meaning for abstract configurations. Alternatively, one may think of an abstract (16,6) configuration as a graph with 32 vertices, which are divided into two sets of 16: the vertices in one set are called points, the vertices in the other planes. There are exactly 6 arcs coming out of each vertex and an arc always connects a point and a plane (i.e. no arc connects a point with a point or a plane with a plane). A point is said to belong to a plane if and only if the corresponding vertices of the graph are connected by an arc. It is obvious that the languages of $16 \times 16$ matrices and graphs are equivalent to each other and we shall use them interchangeably. For example, the configuration of type (*) can be thought of as a 5-dimensional cube, all of whose vertices have coordinates zeroes and ones. The vertices whose sum of coordinates is even are points, while those whose sum of coordinates is odd are planes. The arcs of the graph are the edges and the main diagonals of the cube (we shall prove below the equivalence of this description with the one in Lemma 1.7; see also the next Remark).

**Remark 1.13.** We may think of Lemma 1.7 as describing a particular nondegenerate abstract (16,6) configuration, the one we have called of type (*). This description in terms of a $4 \times 4$ diagram is also due to Hudson [12]. At first sight, it seems that we have several accidental, ad hoc descriptions, such as the $4 \times 4$ diagram (*) or the 5-dimensional cube of the previous remark. In fact, there is a more systematic way of thinking about it [6]. Namely, there is a way of associating an abstract (16,6) configuration to any pair $(F, M)$, where $F$ is a group of order 16 and $M$ is a set consisting of 6 distinct elements of $F$. Not every abstract (16,6) configuration appears in this way (not even every non-degenerate one), and the same configuration may arise from different pairs $(F, M)$. For example, the diagram (*) corresponds to $F = \mathcal{C}_4^2$ (where $\mathcal{C}_n$ denotes the cyclic group with $n$ elements) and the set $M$ is

$$\{(0,1),(0,2),(0,3),(1,0),(2,0),(3,0)\}.$$

The description in terms of the 5-dimensional cube corresponds to $F = C_2^4$ and

$$M = \{(0,0,0,0),(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1),(1,1,1,1)\}.$$

This last description is the one with the most geometric meaning. The group $C_2^4$ may be identified, as we already saw, with the group $F_0 \in PGL_4(k)$, which acts by automorphism on the (16,6) configuration and also on the Kummer surface. We can also identify $C_2^4$ with the group of 2-torsion points on an abelian surface $A$ and the set $M$ with the set of Weierstrass points on a curve of genus 2 embedded in $A$ as a $\Theta$-divisor passing through the origin. Even the 5-dimensional cube itself, which can be identified with $C_2^5$, appears as a group of automorphisms of the desingularization of the Kummer surface, which is embedded in $\mathbb{P}^5$ (the action of $C_2^5$ on $\mathbb{P}^5$ is the obvious one: changing the signs of the coordinates).

We divide the proof that every non-degenerate (16,6) configuration in $\mathbb{P}^3$ has the form (1.4.1) (up to an automorphism of $\mathbb{P}^3$) into three parts. First, we classify the abstract non-degenerate (16,6) configurations (there turn out to be exactly three of them). Then we show that only one of them (the one of type (*)) can be realized by an actual configuration of points and planes in $\mathbb{P}^3$. Finally, we prove that any (16,6) configuration of type (*) in $\mathbb{P}^3$ is of the form (1.4.1).

## I. The classification of abstract non-degenerate (16,6) configurations.

In this subsection we prove that there are exactly three non-isomorphic non-degenerate abstract (16,6) configurations.

*Acknowledgement.* I thank Fabrizio Catanese for giving me the idea for this classification.

Consider a non-degenerate abstract (16,6) configuration.

**Definition 1.14.** *A* **line** *is a pair of points.*

**Remark 1.15.** Since our configuration is non-degenerate, the lines are in one-to-one correspondence with pairs of planes. In other words, two planes intersect in a unique line.

**Lemma 1.16.** *Fix a plane $H$ of our configuration and a point $P \notin H$. Let $P_1, \dots, P_6$ be the points contained in $H$. The 6 planes passing through $P$ cut out 6 lines on $H$. These six lines and the points $P_1, \dots P_6$ form either a hexagon or a pair of triangles:*

*Proof.* Since our configuration is non-degenerate, each $P_i$ is contained in exactly two of the six lines. It is immediate to verify that the only abstract (6,2) configurations of six lines and six points are a hexagon and a pair of triangles. □

**Definition 1.17.** *A* **6-side** *is a configuration of six points and six lines in a plane which is either a hexagon or a pair of triangles.*

**Definition 1.18.** *If $Q \in H$ is a point, an* **angle** *with vertex $Q$ is a pair of lines in $H$ both containing $Q$.*

By Lemma 1.16, to each $P \notin H$ we have naturally associated a 6-side in $H$.

**Lemma 1.19.** *Consider the ten 6-sides in $H$ associated to the ten points of the configuration lying outside of $H$.*

*(a) Fix a point $P_i \in H$. To each of the ten 6-sides associate the angle with vertex $P_i$ given by the two lines of the 6-side passing through $P_i$. This defines a bijection between the ten 6-sides and the ten angles in $H$ with vertex $P_i$.*

*(b) Two distinct 6-sides have exactly two disjoint lines in common. This defines a bijection between the set of pairs of 6-sides and the set of pairs of disjoint lines in $H$.*

*Proof.* (a) There are five lines in $H$ passing through $P_i$, hence there are $\binom{5}{2} = 10$ angles with vertex $P_i$. The bijection of (a) is given as follows. The ten 6-sides correspond to the ten points $P \notin H$. Each point $P \notin H$ determines a line $(P, P_i)$. The line $(P, P_i)$ determines uniquely the pair of planes containing that line. Intersecting each of these planes with $H$, we get a uniquely defined angle with vertex in $P_i$. It is obvious that the correspondence we described is a bijection at each step.

(b) We have $\binom{10}{2} = 45$ pairs of 6-sides. The total number of lines in $H$ is $\binom{6}{2} = 15$. Once a line is fixed, the number of lines disjoint from it is $\binom{4}{2} = 6$. The number of pairs of disjoint lines is therefore $\frac{15 \cdot 6}{2} = 45$. The bijection of (b) is described as follows. A pair of 6-sides determines a pair of points away from $H$, i.e. a line disjoint from $H$. This line determines uniquely the pair of planes containing it. Intersecting these two planes with $H$ we get a uniquely determined pair of disjoint lines. These are exactly the lines shared by the two 6-sides. The above correspondence is a bijection because each step can be reversed. □

**Theorem 1.20.** *There are exactly three non-isomorphic non-degenerate abstract (16,6) configurations.*

*Proof.* Consider a non-degenerate abstract (16,6) configuration. Fix a plane $H$ and consider the ten 6-sides of Lemma 1.19. Clearly, this collection of 6-sides completely determines our abstract (16,6) configuration. Conversely, given a collection of ten 6-sides on a plane $H$ with six points, satisfying (a) and (b) of Lemma 1.19, we can reconstruct the (16,6) configuration from which this collection comes. Indeed, the collection of ten 6-sides determines a configuration of sixteen points and sixteen planes with certain incidence relations: we take a point for each of the ten 6-sides, and the six points of $H$. In addition to $H$ itself, we take a plane for each of the fifteen lines in $H$. We say that a point of $H$ belongs to a plane different from $H$, if it belongs to the corresponding line in $H$. We say that a point away from $H$ belongs to a plane different from $H$ if the 6-side corresponding to the point contains the line corresponding to the plane. It remains to check that this is indeed a non-degenerate (16,6) configuration. Take a line $L$ in $H$ and a point $P_i \in L$. There are exactly four angles with vertex $P_i$ which contain $L$. Hence by (a) there are exactly four 6-sides containing $L$. Hence, in addition to the two points of $L$, the plane passing through $L$ contains four other points, so each plane contains six points. Each point $P$ away from $H$ belongs to six planes by definition: namely, the six planes given by the six sides of the 6-side corresponding to $P$. Each point $P_i$ on $H$ belongs to $H$ and to the five planes corresponding to the five lines through $P_i$. Thus we have a (16,6) configuration. Non-degeneracy is immediate from (b).

Thus, to classify the non-degenerate (16,6) configurations we need to classify the collections of ten 6-sides on $H$ satisfying (a) and (b) of Lemma 1.19. We have not yet proved that for a given (16,6) configuration this collection of ten 6-sides is independent of the choice of the plane $H$, but this is, in fact, true and we shall prove it in the end to complete our classification.

Let us denote the six points of $H$ by $1, \ldots, 6$ to simplify the notation. The number of different triangle pairs one can form with six points is $\frac{1}{2}\binom{6}{3} = 10$. The collection of all the ten possible triangle pairs satisfies (a) and (b) of Lemma 1.19 and so gives rise to a non-degenerate (16,6) configuration. This is the (16,6) configuration of type (*), as one can check directly from the definitions. Suppose we are not in this situation, that is, we are given a collection of ten 6-sides, satisfying (a) and (b) and at least one of these 6-sides is a hexagon. Let (123456) denote the hexagon with sides $(12), (23), \ldots, (61)$. Assume (without loss of generality) that (123456) belongs to our collection. Denote it by $E_0$. Let $\widehat{ijk}$ denote the angle $((ij), (jk))$ with vertex $j$. By (a), there is a unique 6-side in our collection containing $\widehat{214}$, call it $E_1$. Let $E_2$ denote the unique 6-side containing $\widehat{215}$.

If $E_1$ is a triangle pair, it must be (124)(356). Suppose $E_1$ is a hexagon. We claim that $E_1 = (126354)$. Indeed, suppose not. $E_1$ must contain exactly one other line (apart from (14)) passing through 4. Suppose $(45) \in E_1$. Then 5 must be connected to either 3 or 6. It cannot be 6 for then $E_1$ would intersect $E_0$ in three lines, contradicting (b). Then $(35) \in E_1$ and $E_1 = (126354)$, as desired. Hence we may assume $(45) \notin E_1$. Now if $(34) \in E_1$ then we must have $(56) \in E_1$, contradicting (b). Finally, if $(46) \in E_1$ then $E_1$ intersects $E_0$ only in one line, again contradicting (b). This completes the proof that $E_1$ is either (124)(356) or (126354). An analogous argument shows that $E_2$ is either (125)(346) or (126435).

Let $E_3$ denote the 6-side associated to the angle $\widehat{213}$. It must be a hexagon, otherwise it would intersect $E_0$ in four lines which is impossible. Applying conditon (b) to the intersection $E_3 \cap E_0$ as above, one gets the following three possibilities for $E_3$: (125463), (126453) and (124653).

$E_1$ and $E_2$ cannot both be hexagons for then they would intersect in three lines. We are left with the following cases.

*Case 1.* $E_1 = (124)(356)$, $E_2 = (125)(346)$. Then, since by (a) $E_3$ cannot have an angle in common with $E_1$ or $E_2$, we must have $E_3 = (126453)$.

*Case 2.* $E_1 = (124)(356)$, $E_2 = (126435)$. Again by (a) we get $E_3 = (125463)$.

*Case 3.* $E_1 = (126354)$, $E_2 = (125)(346)$. By (a), $E_3 = (124653)$.

Cases 2 and 3 can be carried one into another by the permutation of the points which interchanges 1 with 2, 3 with 6 and 4 with 5. Therefore we may forget Case 3. We start with Case 2.

We apply (a) and (b) in the same way as before. The remaining 6-sides are now uniquely determined. The 6-side containing $\widehat{354}$ is (135426). The one containing $\widehat{154}$ is (154)(236). We continue listing the 6-sides, each one next to an angle it contains to indicate how we got it. The verification that each possibility is, indeed, unique is straightforward and we leave it to the reader.

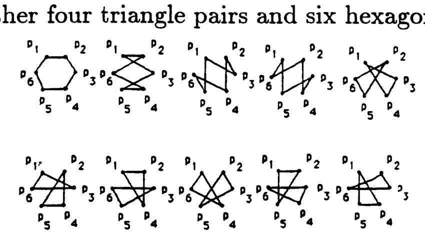
$$\widehat{314} \to (134)(256)$$
$$\widehat{416} \to (146)(235)$$
$$\widehat{324} \to (132465)$$
$$\widehat{436} \to (152436).$$

Now, consider Case 1. We have a unique possibility for the 6-sides containing $\widehat{154}$ (it is (145)(236)) and $\widehat{254}$ (it is (136)(245)). For the 6-side containing $\widehat{324}$ we have two possibilities: (132465) and (153246). The second possibility can be transformed into Case 2 by the cyclic permutation of the points $1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 1$, so we may ignore it. From here on, again, our collection of 6-sides is uniquely determined:

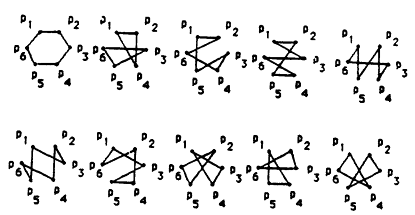
$$\widehat{325} \to (146)(325)$$
$$\widehat{134} \to (134)(256)$$
$$\widehat{435} \to (153426).$$

To sum up, we have shown that there are exactly three non-equivalent collections of ten 6-sides satisfying (a) and (b) of Lemma 1.19. One such collection consists of all the ten possible triangle pairs. The other two, one having six triangle pairs and four hexagons, the other four triangle pairs and six hexagons, are given by

(A)



(B)



This proves that, apart from the configuration of type (*) there are at most two and at least one "exotic" non-degenerate abstract (16,6) configurations. We now prove that there are exactly two exotic ones. That is, if the collection of ten 6-sides associated with a plane $H$ of our configuration has form (A) then it has form (A) for any other plane of the configuration, and similarly for (B).

**Definition 1.21.** *A* **Rosenhain tetrahedron** *in an abstract (16,6) configuration is a set of four points and four planes such that each plane contains exactly three points and each point belongs to exactly three planes (in other words, it is a (4,3) subconfiguration). The four points are called* **vertices** *of the tetrahedron. An* **edge** *is a pair of vertices, and a* **face** *a triple of vertices.*

To complete the proof of Theorem 1.20, it is sufficient to prove the following Lemma.

**Lemma 1.22.** *Consider a collection of ten 6-sides associated to a plane $H$ of an abstract non-degenerate (16,6) configuration. Let $n$ be the number of triangle pairs among these 6-sides (so that $n$ can be 4, 6 or 10). Then the (16,6) configuration has exactly $8n$ distinct Rosenhain tetrahedra.*

Lemma 1.22 implies that the number of triangle pairs in a collection of 6-sides associated with a plane $H$ of the configuration is an invariant of the configuration and does not depend on the choice of $H$.

*Proof.* We start with a preliminary Lemma.

**Lemma 1.23.** *A Rosenhain tetrahedron can intersect a plane of the configuration either in a face or in a point (in other words, the intersection can be neither an edge nor empty).*

*Proof.* Let $P_1, P_2, P_3, P_4$ be the vertices of the tetrahedron, $H_1, H_2, H_3, H_4$ the planes of the tetrahedron, and $H$ a plane of the configuration. We number the points and the planes so that $P_i \notin H_i$.

First, suppose the intersection is empty, i.e. $P_i \notin H$ for $1 \leq i \leq 4$. Each of the $H_i$ intersect $H$ in two points, and since $H_i \cap H_j \subset \{P_1, P_2, P_3, P_4\}$ for $1 \leq i, j \leq 4$, the four sets $H \cap H_i$ must be disjoint. Then $H$ contains eight points which is a contradiction.

Next, suppose the intersection is the edge $(P_1, P_2)$. Then the line $(P_1, P_2)$ is contained in the planes $H_3$, $H_4$ and $H$, hence $H$ must coincide either with $H_3$ or with $H_4$. But then our tetrahedron intersects $H$ in a face, which is a contradiction. This proves Lemma 1.23.

To count the number of Rosenhain tetrahedra in our configuration we fix a plane $H$. We have to count the tetrahedra which intersect $H$ in a face and those which intersect $H$ in a point. Each triangle pair appearing in our collection of 6-sides gives rise to exactly two tetrahedra with a face in $H$ and all such tetrahedra arise in this way. Thus the number of tetrahedra which intersect $H$ in a face is $2n$.

It is a little harder to count the tetrahedra with one vertex in $H$. Let $P_1$,$P_2$, $P_3$,$P_4$, $H_1, H_2, H_3, H_4$ be such a tetrahedron (we use the same notation as in Lemma 1.23).

Say, $P_4 \in H$. Consider the three 6-sides in $H$ associated with the points $P_1$, $P_2$ and $P_3$. Each of them contains an angle with vertex $P_4$ and the line $L := H \cap H_4$, disjoint from this angle. Apart from $P_4$, there are only three points in $H$ which are not in $L$; call them $Q_1$, $Q_2$ and $Q_3$. Thus, after renumbering, we have that the 6-side of $P_1$ contains $L$ and $\widehat{Q_2P_4Q_3}$, that of $P_2$ contains $L$ and $\widehat{Q_1P_4Q_3}$, and that of $P_3$ contains $L$ and $\widehat{Q_1P_4Q_2}$. Since there are exactly four 6-sides in our collection containing a given line (by (a) of Lemma 1.19), there is exactly one other 6-side containing the line $L$; call it $E$. Moreover, for every $i$, $1 \le i \le 3$, the pair of lines $L, (P_4Q_i)$ already occurs in two of the three 6-sides of $P_1$, $P_2$ and $P_3$. Hence, by (b) of Lemma 1.19, $E$ cannot contain any of the lines $P_4Q_i$, $1 \le i \le 3$. Since $E$ must contain some angle with vertex $P_4$, it must contain the triangle $\{P_4\} \cup L$. In particular, $E$ is a triangle pair. Thus we have associated to a tetrahedron with one vertex in $H$ a triangle pair $E$ together with a preferred point $P_4$. One can easily check that this association is a bijection, since all the steps in the above argument can be reversed. In other words, given a triangle pair $E$ and a point $P_4$ in $H$, one can reconstruct a uniquely defined tetrahedron with one vertex $P_4$ in $H$ as above. Hence the number of tetrahedra with one vertex in $H$ is the number of triangle pairs times the number of points in $H$, which is $6n$, as desired. This proves Lemma 1.22 and Theorem 1.20. □

## II. The (8,4) configuration.

Before we prove that the "exotic" abstract configurations (A) and (B) cannot be realized as configurations of planes and points in $\mathbb{P}^3$, we need a more detailed study of the projective geometry which enters into the construction of the (16,6) configuration of type (*).

In this subsection, we study (8,4) configurations, particularly those we call the standard ones (Definition 1.29). Standard (8,4) configurations appear in many different ways as subconfigurations of non-degenerate (16,6) configurations. They may be thought of as basic building blocks of (16,6) configurations, particularly of the ones of type (*). The importance of studying (8,4) configurations, apart from their intrinsic beauty, is that they provide an explanation for the following, apparently surprising, dimension count. The study of (8,4) configurations and their moduli explains both why the type (*) configurations have 3-dimensional moduli and why the other two types of non-degenerate (16,6) configurations do not exist at all.

One may try to compute the dimension of the moduli space of (16,6) configurations by the following naïve dimension count. Each of the 16 points of a (16,6) configuration moves in a 3-dimensional space $\mathbb{P}^3_k$. If we take into account the action of the 15-dimensional group $PGL_4(k)$, we see that the space of arbitrary 16-tuples of points modulo automorphisms of $\mathbb{P}^3$ has dimension $3 \cdot 16 - 15 = 33$. On the other hand, to require six points to lie on a plane imposes three conditions on our set of 16 points, so we have $3 \cdot 16 = 48$ conditions. Thus, apparently, we have many more conditions than variables. Since we know that (16,6) configurations exist (Proposition 1.6), these 48 conditions cannot be independent. The explanation of which conditions are superfluous is provided by the study of (8,4) configurations.

In order to define standard (8,4) configurations we need a preliminary theorem (Theorem 1.25 below). This theorem, which will show that some of the 48 coplanarity conditions imposed by the special planes of the (16,6) configuration follow automatically from the others, may be thought of as a 3-dimensional analogue of the Desargues theorem. We recall the classical, 2-dimensional version first.

**Theorem 1.24 (Desargues).** [10, p.13] *Let two triangles $ABC$ and $A'B'C'$ be such that the lines joining corresponding vertices, namely $AA'$, $BB'$ and $CC'$, pass through a point $O$. Then the three pairs of corresponding sides intersect in three points*

$$P = AB.A'B'$$
$$R = BC.B'C'$$
$$Q = AC.A'C'$$

*which lie on a straight line.*

**Theorem 1.25.** *Let $P_1$, $P_2$, $P_3$ and $P_4$ be four points in $\mathbb{P}^3$, no three of which are collinear, all contained in a plane $H$. For each of the six lines $P_iP_j$, $1 \leq i < j \leq 4$, take a plane $H_{ij}$ in $\mathbb{P}^3$, containing $P_iP_j$ and different from $H$. For each of the four triangles $P_iP_jP_k$, $1 \leq i < j < k \leq 4$, consider the point $P_{ijk} := H_{ij} \cap H_{ik} \cap H_{jk}$. Then the points $P_{123}$, $P_{124}$, $P_{134}$ and $P_{234}$ are coplanar.*

*Proof.* First of all, we reformulate the classical Desargues theorem in order to put it in the form in which we shall use it.

**Lemma 1.26.** *Let $L$ be a line in $\mathbb{P}^2_k$ and $P_1$, $P_2$, $P_3$, $P_4$, $P_5$ an (ordered) set of points in $L$. Let $L_1$ be a line passing through $P_1$, different from $L$. Let $Q_1, Q_2 \in L_1$ be two distinct points, different from $P_1$. Let $L_2$, $L_3$ be the lines joining $Q_1$ with $P_2$ and $P_3$, respectively. Let $L_4$, $L_5$ be the lines joining $Q_2$ with $P_4$ and $P_5$, respectively. Put*

$$R_1 : = L_2 \cap L_4$$
$$R_2 : = L_3 \cap L_5$$

*Let $L_6$ be the line joining $R_1$ with $R_2$ and put*

$$P_6 := L \cap L_6.$$

*Then the point $P_6$ depends only on the points $P_1$, $P_2$, $P_3$, $P_4$, and $P_5$, but not on the line $L_1$ nor on the choice of the points $Q_1$ and $Q_2$. Moreover, the permutation of the $P_i$'s, $1 \leq i \leq 5$, which interchanges $P_3$ with $P_4$ and leaves $P_1$, $P_2$ and $P_5$ unchanged, preserves $P_6$.*

In other words, we may think of Desargues theorem as giving a well defined way (via Lemma 1.26) of associating the point $P_6$ to the ordered 5-tuple of points $P_1$, $P_2$, $P_3$, $P_4$, $P_5$.

**Warning.** The last statement of Lemma 1.26 does not hold for the permutations $P_2 \leftrightarrow P_4$ or $P_3 \leftrightarrow P_5$.

*Proof.* Since the line $L_1$ can be moved to any other line different from $L$ by a linear automorphism of the plane, which fixes $L$ pointwise, we may fix $L_1$ once and for all without affecting $P_6$. It is sufficient to prove that $P_6$ is independent of the choice of $Q_1$: the independence of $Q_2$ will follow by symmetry. Now, there exists a linear automorphism of the plane which fixes $L$ pointwise, maps $L_1$ to itself, fixes $Q_2$ and moves $Q_1$ into any other point on $L_1$ distinct from $P_1$ and $Q_2$ (namely, we may take a homotethy centered at $Q_2$ in the coordinates in which $L$ is the line at infinity). This proves that $P_6$ does not depend on the choice of $Q_1$, as desired.

To prove the second statement of the Lemma, we choose a coordinate system in the plane such that $P_1$ is the origin of the affine coordinate chart with coordinates $x, y$ and the lines $L$ and $L_1$ are given by $x = 0$ and $y = 0$, respectively. Furthermore, we may assume that $P_1 = (0,1)$, $P_5 = (0,\infty)$, $Q_1 = (\infty, 0)$, $Q_2 = (1,0)$ (the reference to points at infinity should be self-explanatory. One could write it more precisely in terms of projective coordinates but we prefer the classical notation because it's simpler). Then the points $P_3$ and $P_4$ are

$$P_3 = (0, a)$$
$$P_4 = (0, b)$$

for some $a, b \in k \setminus \{0,1\}$, such that $a \neq b$. One computes easily that in this coordinate system $R_1 = (\frac{a-1}{a}, 1)$, $R_2 = (1, b)$ and $P_6 = (0, (a-1)(b-1))$. Since this expression is symmetric in $a$ and $b$, interchanging $P_3$ with $P_4$ does not affect $P_6$, as desired. $\square$

**Remark 1.27.** The first part of Lemma 1.26 (i.e. the statement that $P_6$ depends only on the odered set $P_1, \ldots, P_5$) is equivalent to Desargues theorem. Indeed, we can reformulate the statement that $P_6$ does not depend on $Q_1$ as follows. Keep the above notation. Fix $Q_2 \in L_1$ and consider another point $Q_1' \in L_1$, different from $P_1$ and $Q_2$. Let $L_2'$ and $L_3'$ be the lines joining $Q_1'$ to $P_2$ and $P_3$, respectively. Put

$$R_1' : = L_2 \cap L_4$$
$$R_2' : = L_3 \cap L_5$$

Let $L_6'$ be the line joining $R_1'$ with $R_2'$. Lemma 1.26 says that $L_6' \cap L = P_6$. This is the same as saying that $L_6 \cap L_6' \in L$, or, in other words, that the points $P_2$, $P_3$ and $L_6 \cap L_6'$ are collinear. But this is nothing but Desargues theorem, applied to the triangles $Q_1 R_1 R_2$ and $Q_1' R_1' R_2'$, where the role of the point $O$ is played by $Q_2$.

*Proof of Theorem 1.25, continued.* As a matter of notation, let us write

$$P_1' : = P_{234}$$
$$P_2' : = P_{134}$$
$$P_3' : = P_{124}$$
$$P_4' : = P_{123}.$$

It is sufficient to prove that the lines $P_1'P_3'$ and $P_2'P_4'$ have a non-empty intersection. Since both of these lines meet the line $H_{13} \cap H_{24}$, and the three lines are not contained in a plane, our problem is equivalent to proving that

$$P_1'P_3' \cap H_{13} \cap H_{24} = P_2'P_4' \cap H_{13} \cap H_{24}. \tag{1.27.1}$$

Let $L := H_{13} \cap H_{24}$. Since both of the lines $P_1P_2' \subset H_{13}$ and $P_4P_3' \subset H_{24}$ are contained in $H_{14}$, they have a non-empty intersection. Since they both meet $L$ and $L \not\subset H_{14}$, we must have

$$P_1P_2' \cap L = P_4P_3' \cap L.$$

By an analogous reasoning,

$$\begin{aligned}
P_1P_4' \cap L &= P_2P_3' \cap L \\
P_3P_2' \cap L &= P_4P_1' \cap L \\
P_3P_4' \cap L &= P_2P_1' \cap L,
\end{aligned}$$

and each of the above intersections consists of exactly one point. We put

$$\begin{aligned}
\tilde{P}_1 &: = P_1P_3 \cap P_2P_4 \\
\tilde{P}_2 &: = P_1P_2' \cap L = P_4P_3' \cap L \\
\tilde{P}_3 &: = P_1P_4' \cap L = P_2P_3' \cap L \\
\tilde{P}_4 &: = P_3P_2' \cap L = P_4P_1' \cap L \\
\tilde{P}_5 &: = P_3P_4' \cap L = P_2P_1' \cap L \\
\tilde{P}_6 &: = P_2'P_4' \cap L.
\end{aligned}$$

Now, since $P_2' = P_1P_2' \cap P_3P_2'$, $P_4' = P_1P_4' \cap P_3P_4'$, we have that $\tilde{P}_6$ is the point of $L$ associated to the quintuple $\tilde{P}_1, \ldots, \tilde{P}_5$ by Lemma 1.26 (applied to the plane $H_{13}$, the role of $Q_1$, $Q_2$ being played by $P_1$ and $P_3$). On the other hand, applying Lemma 1.26 to the plane $H_{24}$ and the permuted quintuple $\tilde{P}_1, \tilde{P}_2, \tilde{P}_4, \tilde{P}_3, \tilde{P}_5$, we get that $\tilde{P}_6 = P_1'P_3' \cap L$, which proves (1.27.1). □

**Note 1.28.** Consider the set of eight points $P_i$, $P_i'$, $1 \le i \le 4$ and the eight planes $H$, $H_{ij}$, $1 \le i < j \le 4$, and the plane $H'$ containing $P_1'$, $P_2'$, $P_3'$ and $P_4'$ as in Theorem 1.25. In this set of points and planes each plane contains four points and each point is contained in four planes. This motivates the following definition.

**Definition 1.29.** *An* **(8,4) configuration** *is a configuration of eight points and eight planes, such that every plane contains exactly four points and every point lies on exactly four planes.*

The (8,4) configuration of Theorem 1.25 has the following additional property. Both the set of points and the set of planes can be divided into four pairs each, such that every plane has no points in common with the plane it is paired to and intersects any other plane in exactly two points, and the dual statement holds for points (in the above notation $P_i$ is paired with $P_i'$, $H_{ij}$ with $H_{\{1,2,3,4\}\setminus\{i,j\}}$ and $H$ with $H'$. It is easy to see that there is a unique such abstract (8,4) configuration and that every such (8,4) configuration of points and planes in $\mathbb{P}^3$ is of the form described in Theorem 1.25. This abstract (8,4) configuration is described by the following incidence diagram.

**1.30. Incidence diagram of an (8,4) configuration.** Let us denote the eight points by 1, 2, 3, 4, 1′, 2′, 3′, 4′, and the eight planes by **1**, **2**, **3**, **4**, **1′**, **2′**, **3′**, **4′**. The incidence relations are given by the following diagram:

$$\text{(1.30.1)} \qquad \begin{array}{cc} 1 & 1' \\ 2 & 2' \\ 3 & 3' \\ 4' & 4 \end{array} \qquad\qquad \begin{array}{cc} \mathbf{1} & \mathbf{1'} \\ \mathbf{2} & \mathbf{2'} \\ \mathbf{3} & \mathbf{3'} \\ \mathbf{4'} & \mathbf{4} \end{array}$$

where the rules of incidence are the same as in diagram (*): a plane in the $i$-th row and $j$-th column contains all the points in the $i$-th row and all the points in the $j$-th column, except for the point in the $i$-th row and $j$-th column.

**Definition 1.31.** *An (8,4) configuration with the incidence relations of 1.30 is called the* **standard** *(8,4) configuration.*

**Remark 1.32.** If we identify any two (8,4) configurations which can be mapped to each other by an element of $PGL_4(k)$, the standard (8,4) configurations form a 2-dimensional family. Indeed, in the notation of Theorem 1.25, the four points $P_i$, $1 \le i \le 4$ can be fixed arbitrarily. So can the point $P_1'$ (this is equivalent to fixing the three planes $H_{23}$, $H_{24}$ and $H_{34}$). Now, the three planes $H_{12}$, $H_{13}$ and $H_{14}$ give us three parameters, but we have the action on the space of configurations of the 1-parameter subgroup of $PGL_4(k)$ which fixes $P_1'$ and the plane $H$ pointwise; so the final answer is $3 - 1 = 2$.

**Remark 1.33.** There are many standard (8,4) configurations contained in a (16,6) configuration of type (*). To see this, fix a (16,6) configuration of type (*). Identify both the set of planes and the set of points with elements of $C_4^2$, as in (*) and (1.5.1). Then all of the following (and all of their images under permutations of rows or columns of the diagram) are standard (8,4) subconfigurations of our (16,6) configuration. In each diagram, a $\circ$ denotes the fact that the corresponding point, but not the corresponding plane, belongs to the (8,4) configuration in question. A $\star$ means that the corresponding plane, but not the corresponding point, belongs to the (8,4) configuration. Finally, a $\heartsuit$ means that both the corresponding plane and the point belong to the (8,4) configuration.

$$\text{(1.33.1)} \qquad \begin{array}{cccc} \heartsuit & \heartsuit & \cdot & \cdot \\ \heartsuit & \heartsuit & \cdot & \cdot \\ \heartsuit & \heartsuit & \cdot & \cdot \\ \heartsuit & \heartsuit & \cdot & \cdot \end{array} \qquad\qquad \begin{array}{cccc} \star & \star & \circ & \circ \\ \star & \star & \circ & \circ \\ \circ & \circ & \star & \star \\ \circ & \circ & \star & \star \end{array}$$

(one checks immediately from definitions that in each case we have a standard (8,4) subconfiguration).

Next, we use Theorem 1.25 and Remark 1.32 to show that the (16,6) configurations of type (*) form a 3-dimensional family. Below (Theorem 1.68) we shall give another proof of this fact. Of course, we expect this to be the case since the point $(a, b, c, d)$ of (1.4.1) moves in a 3-dimensional space: namely, the open subset of $\mathbb{P}^3$ defined by (1.2.1). However, we have not yet proved that any non-degenerate (16,6) configuration is of the form (1.4.1), nor that only finitely many points $(a, b, c, d)$ give rise to isomorphic configurations (all of this will be proved below).

Consider a (16,6) configuration of type (*) and look at the (8,4) subconfiguration given by the first diagram in (1.33.1). We denote the points and planes of the configuration as follows. The first diagram shows points, the second planes. The notation for each point or plane is shown in the position on the diagram corresponding to that point or plane.

$$\begin{matrix} 1 & 1' & \cdot & \cdot \\ 2 & 2' & \cdot & \cdot \\ 3 & 3' & P & \cdot \\ 4' & 4 & R & \cdot \end{matrix} \qquad\qquad \begin{matrix} \mathbf{1} & \mathbf{1'} & \cdot & \cdot \\ \mathbf{2} & \mathbf{2'} & \cdot & \cdot \\ \mathbf{3} & \mathbf{3'} & \cdot & \cdot \\ \mathbf{4'} & \mathbf{4} & \cdot & \cdot \end{matrix} \tag{1.33.2}$$

We ask the following question. Given the (8,4) configuration 1, 2, 3, 4, $1'$, $2'$, $3'$, $4'$ as above, what (if any) are the (16,6) configurations of type (*) containing the given (8,4) configuration in the position specified by (1.33.2)? We want to show that such (16,6) configurations always exist and form a 1-dimensional family. In other words, after we specify one more parameter, in addition to an already specified (8,4) configuration 1, 2, 3, 4, $1'$, $2'$, $3'$, $4'$, we will be able to reconstruct a unique (16,6) configuration from that data.

Consider the line $PR$. It meets the already specified lines $11'$, $22'$, $\mathbf{3}\cap\mathbf{3'}$ and $\mathbf{4}\cap\mathbf{4'}$. One might think that this would determine the line $PR$ up to two possibilities, but this is not so because the four lines above are not in general position. Let $G := G(4,2)$ be the Grassmanian of lines in $\mathbb{P}^3$. For a line $L \subset \mathbb{P}^3$ let $\sigma_1(L)$ denote the Schubert cell in $G$ consisting of all the lines which meet $L$, and $\sigma_1$ the divisor class of $\sigma_1(L)$ for any $L$ (of course, it is independent of $L$). Recall that $\sigma_1$ is a very ample divisor. It defines the embedding of $G$ as a quadric hypersurface in $\mathbb{P}^5$ which is nothing but the Plücker embedding. It is well known by Schubert calculus [8, Chapter 1, §5, p. 206] that

$$\sigma_1^4 = 2, \tag{1.33.3}$$

so for a generically chosen set of four lines there are exactly two lines in $\mathbb{P}^3$ which meet all four.

**Definiton 1.34.** *Four lines $L_i \in \mathbb{P}^3$, $1 \leq i \leq 4$, are said to be* **independent** *if* $\dim(\cap_{i=1}^4 \sigma_1(L_i)) = 0$ *in $G$.*

**Lemma 1.35.** *(a) Let $L_i$, $1 \leq i \leq 3$ be three pairwise disjoint lines in $\mathbb{P}^3$. Then $\cap_{i=1}^3 \sigma_1(L_i)$ is a non-singular plane conic curve contained in $G$.*

*(b) Let $L_i$, $1 \leq i \leq 4$ be four pairwise disjoint lines in $\mathbb{P}^3$, which are not independent. Then $\cap_{i=1}^4 \sigma_1(L_i)$ is a non-singular plane conic in $G$. It coincides with the intersection of any three of $\sigma_1(L_i)$. In other words, if a line in $\mathbb{P}^3$ meets any three of the $L_i$'s, it automatically meets the fourth.*

*Proof.* (a) *Claim.* Any set of three pairwise disjoint lines can be moved into any other such set by an element of $PGL_4(k)$.

*Proof of Claim* The first two lines $L_1$ and $L_2$ can be moved to the coordinate lines $x = y = 0$ and $z = t = 0$. Now, the third line is defined by two linear equations

$$\begin{aligned} a_1x + a_2y + a_3z + a_4w &= 0 \\ b_1x + b_2y + b_3z + b_4w &= 0. \end{aligned} \tag{1.35.1}$$

The fact that the third line does not meet the first two means that the two determinants

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix}$$

are non-zero. But then there exists a matrix in $PGL_4(k)$ of the form

$$\begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

where $*$ denotes possible non-zero entries, which fixes $L_1$ and $L_2$ and which transforms the equations 1.35.1 into the form

$$\begin{aligned} x + z &= 0 \\ y + w &= 0. \end{aligned}$$

This proves the Claim.

To prove (a) it remains to check the statement for some particular set of three pairwise disjoint lines. Let $x_1, \ldots, x_4$ be a coordinate system on $\mathbb{C}^4$ and let $w_{ij} := x_i \wedge x_j$ be the Plücker coordinates on $\mathbb{P}^5$. We order the $w_{ij}$ as follows: ($w_{12}$, $w_{13}$, $w_{14}$, $w_{34}$, $w_{24}$, $w_{23}$). Consider the three pairwise disjoint lines with coordinates (1,0,0,0,0,0), (0,0,0,1,0,0) and (1,1,0,1,1,0,0). The corresponding Schubert cycles are given by equations $w_{34} = 0$, $w_{12} = 0$ and $w_{12} - w_{13} + w_{34} - w_{24} = 0$. The equation of $G$ in these coordinates is $w_{12}w_{34} - w_{13}w_{24} + w_{14}w_{23} = 0$, so that the three Schubert cycles cut out a non-singular conic on $G$, as desired. This proves (a).

(b) Since by (a) the intersection of the $\sigma_1(L_i)$ for any three of the lines $L_i$ is an irreducible curve, and since $\cap_{i=1}^4 \sigma_1(L_i)$ is at least 1-dimensional, $\cap_{i=1}^4 \sigma_1(L_i)$ must equal the intersection of any three of the $\sigma_1(L_i)$. This proves (b). □

**Remark 1.36.** Let $L_i$, $1 \le i \le 4$ be four *pairwise disjoint* lines in $\mathbb{P}^3$. Then the $L_i$ are independent if and only if the corresponding four points in $G$ span a 3-plane. To see this, view $\mathbb{P}^5$ as $\mathbb{P}(\bigwedge^2 \mathbb{C}^4)$. Choosing an identification $\mathbb{C}^6 \cong \bigwedge^2 \mathbb{C}^4$ induces a skew-symmetric bilinear form on $\mathbb{C}^6$, given by the exterior product of two 2-forms. This form induces a map $\mathbb{C}^6 \to (\mathbb{C}^6)^*$ which sends every vector to the 5-dimensional subspace orthogonal to it. This gives a map $D : \mathbb{P}^5 \to (\mathbb{P}^5)^*$ which maps $G$ to itself. For $L \in G$, $D(L)$ is the plane corresponding to $\sigma_1(L)$ in $(\mathbb{P}^5)^*$. By Lemma 1.35, the intersection of the four hyperplanes corresponding to $\sigma_1(L_i)$ in $\mathbb{P}^5$ is either a plane or a line not contained in $G$ (i.e. it cannot be a line contained in $G$). Hence saying that the lines $L_i$, $1 \le i \le 4$ are independent means that the hyperplanes $\sigma_1(L_i)$ intersect in a line in $\mathbb{P}^5$. By duality (using the map $D$), this is equivalent to saying that the four points $L_i \in G$ span a 3-plane in $\mathbb{P}^5$.

**Lemma 1.37.** *Consider a standard (8,4) configuration* $1, 2, 3, 4, 1', 2', 3', 4'$ *as in (1.30.1). Then the lines* $11'$, $22'$, $\mathbf{3} \cap \mathbf{3'}$ *and* $\mathbf{4} \cap \mathbf{4'}$ *are not independent. Furthermore, either these four lines are pairwise disjoint or* $11'$ *meets* $22'$, $\mathbf{3} \cap \mathbf{3'}$ *meets* $\mathbf{4} \cap \mathbf{4'}$ *and all the other pairs of lines are disjoint. In the second case,* $11' \cap 22'$ *lies in the plane spanned by* $\mathbf{3} \cap \mathbf{3'}$ *and* $\mathbf{4} \cap \mathbf{4'}$, *and* $(\mathbf{3} \cap \mathbf{3'}) \cap (\mathbf{4} \cap \mathbf{4'})$ *lies in the plane spanned by* $11'$ *and* $22'$. *In particular, any line which meets three of the above four lines must automatically meet the fourth.*

*Proof.* First of all, for any $i, j$, $1 \le i, j \le 4$, the lines $ii'$ and $\mathbf{i} \cap \mathbf{i'}$ are disjoint. Indeed, because of the automorphisms of the situation, it is sufficient to prove that $11' \cap \mathbf{33'} = \emptyset$ and $11' \cap \mathbf{11'} = \emptyset$. Suppose $11' \cap \mathbf{3} \cap \mathbf{3'} = P \in \mathbb{P}^3$. Then, $P \ne 1$, since $1 \notin \mathbf{3'}$. Hence $11'$ contains two distinct points, 1 and $P$, which lie in $\mathbf{3}$. Hence $11' \subset \mathbf{2}$, so $1' \in \mathbf{3}$, which is a contradiction.

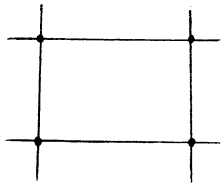
Next, suppose that $11' \cap \mathbf{11'} = P \in \mathbb{P}^3$. Then $P \ne 1$ since $1 \notin \mathbf{1}$. Then $1', P \in \mathbf{1} \implies 11' \in \mathbf{1} \implies 1 \in \mathbf{1}$, a contradiction.

Next, we prove that $11'$ meets $22'$ if and only if $\mathbf{3} \cap \mathbf{3'}$ meets $\mathbf{4} \cap \mathbf{4'}$. It is sufficient to prove "only if" ("if" will follow by duality). Suppose, then, that $11'$ meets $22'$. Then $1, 1', 2, 2'$ are coplanar, so $12$ meets $1'2'$. But $12 = \mathbf{3} \cap \mathbf{4'}$ and $1'2' = \mathbf{3'} \cap \mathbf{4}$. Hence $\mathbf{3} \cap \mathbf{4'}$ meets $\mathbf{3'} \cap \mathbf{4}$. Thus, the four planes $\mathbf{3}, \mathbf{3'}, \mathbf{4}, \mathbf{4'}$ pass through the same point, which implies that $\mathbf{3} \cap \mathbf{3'}$ meets $\mathbf{4} \cap \mathbf{4'}$, as desired.

Suppose that $11'$ meets $22'$ (consequently, $\mathbf{3} \cap \mathbf{3'}$ meets $\mathbf{4} \cap \mathbf{4'}$). Then, $\mathbf{3}, \mathbf{3'}, \mathbf{4}, \mathbf{4'}$ pass through the same point. Both the lines $12 = \mathbf{3} \cap \mathbf{4'}$ and $1'2' = \mathbf{3'} \cap \mathbf{4}$ meet $11'$ and $22'$, but do not pass through $11' \cap 22'$. Hence, both must lie in the plane spanned by $11'$ and $22'$, which implies that $12 \cap 1'2' = (\mathbf{3} \cap \mathbf{4'}) \cap (\mathbf{3'} \cap \mathbf{4})$ lies in the plane spanned by $11'$ and $22'$. The fact that $11' \cap 22'$ belongs to the plane spanned by $\mathbf{3} \cap \mathbf{3'}$ and $\mathbf{4} \cap \mathbf{4'}$ follows by duality.

Thus, to prove Lemma 1.37 it remains to consider the case when all the lines $11'$, $22'$, $\mathbf{3} \cap \mathbf{3'}$ and $\mathbf{4} \cap \mathbf{4'}$ are pairwise disjoint. By Lemma 1.35 it is sufficient to prove that $11'$, $22'$, $\mathbf{3} \cap \mathbf{3'}$ and $\mathbf{4} \cap \mathbf{4'}$ are not independent. First, we note that the four lines $12'$, $21'$, $34$ and $3'4'$ are pairwise disjoint but not independent (since each of the four meets each of the four lines $12$, $34'$, $1'2'$, $3'4$). To prove the pairwise disjointedness, note that $12 \cap 1'2' = \emptyset$, for otherwise the four points $1, 2, 1', 2'$ would be coplanar and $11'$ would meet $22'$. To prove $12' \cap 34 = \emptyset$, note that otherwise $1, 2', 3, 4$ would be coplanar, so we would have $2' \in \mathbf{4}'$, a contradiction.

Consider $Q := \sigma_1(12) \cap \sigma_1(1'2') \subset G$. Since $12 \cap 1'2' = \emptyset$, $Q$ is a nonsingular quadric surface contained in $\mathbb{P}^3$, given by the intersection of the two hyperplanes in $\mathbb{P}^5$ corresponding to $\sigma_1(12)$ and $\sigma_1(1'2')$. $11'$, $22'$, $\mathbf{3} \cap \mathbf{3}'$, $\mathbf{4} \cap \mathbf{4}'$, $12'$, $21'$, $34$ and $3'4'$ give eight points in $Q$. Consider the line $12'$; we have $12' \cap 11' = \{1\}$. The line $l$ in $G$ joining the points $12'$ and $11'$ corresponds to the pencil of lines in $\mathbb{P}^3$ passing through 1 and contained in the plane spanned by the points 1, $1'$ and $2'$. Since $Q$ is the intersection of two hyperplane sections of $G$, we must have $l \subset Q$. Consider the points $11'$, $22'$, $12'$, $1'2 \in Q$ and the four lines in $Q$ joining $11'$ with $12'$, $12'$ with $22'$, $22'$ with $1'2$ and $1'2$ with $11'$. We get the following configuration of points and lines in $Q$.



Since the only lines in $Q$ are the lines of the two rulings, $\overline{(11')(12')}$ and $\overline{(22')(1'2)}$ belong to one ruling and $\overline{(12')(22')}$ and $\overline{(1'2)(11')}$ to the other. Similarly, there are four lines: those joining $\mathbf{3} \cap \mathbf{3}'$ with 34, $\mathbf{4} \cap \mathbf{4}'$ with $3'4'$ in one ruling, and $\overline{(\mathbf{3} \cap \mathbf{3}')(3'4')}$ and $\overline{(\mathbf{4} \cap \mathbf{4}')(34)}$ in the other. We know that $12'$, $1'2$, $34$, $3'4'$ are not independent, hence they must lie on a nonsingular plane conic in $Q$ by Remark 1.36. We wish to prove the same for $11'$, $22'$, $\mathbf{3} \cap \mathbf{3}'$ and $\mathbf{4} \cap \mathbf{4}'$. Thus, it remains to prove the following Lemma about quadric surfaces in $\mathbb{P}^3$.

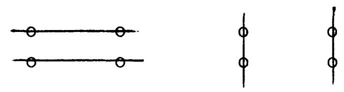
**Lemma 1.38.** *Let $E_1$, $E_2$, $E_3$, $E_4$ be four distinct lines in one ruling of a quadric surface $Q \subset \mathbb{P}^3$, and $F_1$, $F_2$, $F_3$, $F_4$ four distinct lines of another ruling. Then, $P_1 = E_1 \cap F_1$, $P_2 = E_2 \cap F_2$, $P_3 = E_3 \cap F_3$, $P_4 = E_4 \cap F_4$ are coplanar if and only if $P_1' = E_1 \cap F_2$, $P_2' = E_2 \cap F_1$, $P_3' = E_3 \cap F_4$, $P_4' = E_4 \cap F_3$ are coplanar.*

*Proof.* Consider the linear system consisting of all the curves of type (2,2) on $Q$. It is sufficient to prove that any curve of type (2,2) on $Q$ passing trough seven of the points $\{P_i, P_i'\}$, $1 \le i \le 4$, automatically passes through the eighth. Indeed, if that is so, consider the plane containing all the $P_i$, $1 \le i \le 4$, and another containing $P_1'$, $P_2'$, $P_3'$. The union of these two planes is a quadric surface cutting $Q$ in a curve of type (2,2) containing $P_1$, $P_2$, $P_3$, $P_4$, $P_1'$, $P_2'$, $P_3'$. If we can prove that it automatically contains $P_4'$, then the $P_i'$ are coplanar.

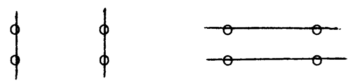
To prove this claim about curves of type (2,2), we order the set $\{P_i, P_i'\}$, $1 \le i \le 4$, in some way:

$$\{P_i, P_i'\}_{1 \le i \le 4} = \{R_1, R_2, \dots, R_8\}.$$

It is easy to check directly that each of the $R_1, \dots, R_7$ imposes a linearly independent base condition on our linear system by producing a curve of type (2,2) passing through $R_1, \dots, R_i$, but not $R_{i+1}$, for $i \le 6$. The quadrics in $\mathbb{P}^3$ form a 10-dimensional family; hence, the curves of type (2,2) on $Q$ form a 9-dimensional family. Thus, the dimension of the linear system (2,2) with base points $R_1, \ldots, R_i$, $i \leq 7$, is $9-i$. For $i = 7$ the dimension is 2; hence, if we add another base point $R_8$, the dimension cannot become more than 2. On the other hand, the dimension is at least 2, since there are two curves of type (2,2) passing through the eight points, given by the following two sets of four lines:



and



so there are at least two linearly independent quadrics passing through all the eight points. Hence

$$h^0(2E_1 + 2F_1 - \sum_{i=1}^{7} R_i) = h^0(2E_1 + 2F_1 - \sum_{i=1}^{8} R_i) = 2,$$

so any (2,2) curve passing through $R_1, \ldots, R_7$ passes through $R_8$, as desired. This completes the proof of Lemmas 1.38 and 1.37. □

Now, consider a standard (8,4) configuration $1, 2, 3, 4, 1', 2', 3', 4'$ as above and let $L$ be any line in $\mathbb{P}^3$ which

(a) meets all four of $11'$, $22'$, $\mathbf{3} \cap \mathbf{3'}$ and $\mathbf{4} \cap \mathbf{4'}$
(b) may or may not meet the two lines $34' = \mathbf{1} \cap \mathbf{2}$ and $3'4 = \mathbf{1'} \cap \mathbf{2'}$
(c) does not meet any other lines $ij$, $ij'$, $i'j'$, $\mathbf{i} \cap \mathbf{j}$, $\mathbf{i} \cap \mathbf{j'}$, $\mathbf{i'} \cap \mathbf{j'}$.

First of all, we must prove that such a line $L$ exists. For this it is sufficient to prove that each of the 24 lines mentioned in (c) is independent from any three of the four lines mentioned in (a). For this we need another elementary lemma about Schubert cycles.

**Lemma 1.39.** *Let $L_i$, $1 \leq i \leq 3$ be three pairwise disjoint lines in $\mathbb{P}^3$. Let $L'$ be a line, distinct from the $L_i$'s, which meets either one or two of the $L_i$'s. Then the four lines $L'$, $L_1$, $L_2$, $L_3$ are independent.*

*Proof.* First, consider the case $L' \cap L_1 \neq \emptyset$, $L' \cap L_i = \emptyset$, $i = 2, 3$. Let $L$ be a line in $\mathbb{P}^3$ which meets all four of $L'$, $L_i$, $1 \leq 3$. Then $L$ is either contained in the plane $H$ spanned by $L'$ and $L_1$ or passes through $L' \cap L_1$. In the first case, note that $L_2, L_3 \not\subset H$ since the $L_i$'s are pairwise disjoint. Now, $L_2 \cap H \neq L_3 \cap H$ since $L_2 \cap L_3 = \emptyset$, so $L$ must pass through two distinct points $L_2 \cap H$ and $L_3 \cap H$. Hence $L$ is uniquely determined.

If $L' \cap L_1 \in L$, consider the linear projection $\pi : \mathbb{P}^3 \setminus \{L' \cap L_1\} \to \mathbb{P}^2$ from $L' \cap L_1$. Again by disjointness of the $L_i$'s, $\pi(L_2)$ and $\pi(L_3)$ are two distinct lines in $\mathbb{P}^2$, hence intersect in a unique point. In order to meet both $L_2$ and $L_3$, $L$ must be the preimage of that point in $\mathbb{P}^3$ under $\pi$, hence $L$ is uniquely determined.

Next, consider the case $L' \cap L_1, L' \cap L_2 \neq \emptyset$. Since $L \neq L'$, $L$ must either pass through $L' \cap L_1$ and be contained in the plane $H$ generated by $L'$ and $L_2$ or viceversa. Say, the former holds. But $L$ must also pass through the point $L_3 \cap H$, which is well defined and distinct from $L' \cap L_1$ by disjointness of the $L_i$'s. This shows, again, that $L$ is determined up to two possibilities.

In both cases, we have proved that $\sigma_1(L') \cap (\cap_{i=1}^{3} \sigma_1(L_i))$ is a finite set, which means, by definition, that the four lines in question are independent. □

**Lemma 1.40.** *Fix a standard (8,4) configuration 1,2,3,4,1′,2′,3′,4′ in $\mathbb{P}^3$. The set of lines $L$ in $\mathbb{P}^3$ satisfying (a)–(c) above are parametrized by a plane conic, with a finite set of points removed, in the Grassmanian $G(4,2)$.*

*Proof.* By Lemmas 1.35 and 1.37, we only have to show that every one of the 24 lines mentioned in (c) is independent from any three of the four lines of (a). Up to relabeling, a line mentioned in (c) must have one of the following forms:

(1) $13 = \mathbf{2} \cap \mathbf{4'}$ (there are eight such lines)
(2) $1'3 = \mathbf{1} \cap \mathbf{3'}$ (eight lines)
(3) $1'2 = \mathbf{1} \cap \mathbf{2'}$ (two lines)
(4) $3'4' = \mathbf{3} \cap \mathbf{4}$ (two lines)
(5) $33'$ (two lines)
(6) $\mathbf{1} \cap \mathbf{1'}$ (two lines).

Let $L'$ be one of the lines (1)–(6) above. By Lemma 1.35(b), it is sufficient to prove independence of $L'$ from any three lines from among the four lines mentioned in (a). In the cases (1), (3) and (4) $L'$ meets exactly two of the lines in (a); in case (2) exactly one of the lines of (a). Hence in those cases we get independence by Lemma 1.40. Consider case (5). By definition of the standard (8,4) configuration, $L'$ is disjoint from $\mathbf{1} \cap \mathbf{1'}$ (otherwise, $L' = 33'$ would have to be contained in either $\mathbf{1}$ or $\mathbf{1'}$, which is a contradiction). Similarly, $L' \cap \mathbf{2} \cap \mathbf{2'} = \emptyset$. If $L'$ meets either one or two of $11'$ and $22'$, the independence follows from Lemma 1.40. Hence we may assume that $L' \cap 11' = L' \cap 22' = \emptyset$. Then to prove that $L'$ is independent from some three of the four lines of (a), it is sufficient (by Lemma 1.35(b)) to find a line in $\mathbb{P}^3$ which meets all the lines of (a) but not $L'$. We take the line 12.

The reasoning in case (6) is analogous. The line $\mathbf{3} \cap \mathbf{3'}$ meets all the four lines in (a) but not $L'$. □

Take a line $L$ satisfying (a)–(c). Let $P := L \cap \mathbf{3} \cap \mathbf{3'}$, $R := L \cap \mathbf{4} \cap \mathbf{4'}$.

**Proposition 1.41.** *Let us consider a standard (8,4) configuration as in (1.30.1). Let $L$ be a line satisfying (a)–(c). Let $P$, $R$ be as above. Then there exists a unique (16,6) configuration of type (*) containing the points 1, 2, 3, 4, 1′, 2′, 3′, 4′, $P$, $R$ and the planes* **1, 2, 3, 4, 1′, 2′, 3′, 4′** *in the positions indicated by the diagram (1.33.2).*

*Proof.* We construct the (16,6) configuration in stages. The uniqueness will be obvious by construction and we only have to check at each stage that all the required incidence relations are satisfied and that there are no unexpected incidence relations (i.e. that each point belongs to those and only to those among the special planes which are prescribed by the diagram (*)).

Recall the notation of diagram (1.33.1). By assumption, we start out with

$$\begin{array}{cccc} \heartsuit & \heartsuit & \cdot & \cdot \\ \heartsuit & \heartsuit & \cdot & \cdot \\ \heartsuit & \heartsuit & \circ & \cdot \\ \heartsuit & \heartsuit & \circ & \cdot \end{array}$$

already constructed. Next, we add the two planes spanned by $(PR, 11')$ and $(PR, 22')$ (call them $\mathbf{H}$ and $\mathbf{I}$, respectively):

$$\begin{array}{cccc} \heartsuit & \heartsuit & \mathbf{H} & \cdot \\ \heartsuit & \heartsuit & \mathbf{I} & \cdot \\ \heartsuit & \heartsuit & \circ & \cdot \\ \heartsuit & \heartsuit & \circ & \cdot \end{array} \tag{1.41.1}$$

We have to check that the planes $\mathbf{H}$ and $\mathbf{I}$ do not contain any "extra" points. Because of the automorphisms of the situation it is sufficient to check that $\mathbf{H}$ does not contain the following two points:

$$\begin{array}{cccc} \cdot & \cdot & \mathbf{H} & \cdot \\ 2 & \cdot & \cdot & \cdot \\ 3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}$$

If $2 \in \mathbf{H}$ then the line $12 \subset \mathbf{H}$, hence $12 \cap PR \neq \emptyset$, which contradicts the choice of the line $PR$. If $3 \in \mathbf{H}$, then $13 \subset \mathbf{H} \implies 13 \cap PR \neq \emptyset$, a contradiction.

The two points $H$ and $I$ in the upper half of the third column

$$\begin{array}{cccc} \heartsuit & \heartsuit & H & \cdot \\ \heartsuit & \heartsuit & I & \cdot \\ \heartsuit & \heartsuit & \circ & \cdot \\ \heartsuit & \heartsuit & \circ & \cdot \end{array}$$

can be defined as intersections of triples of planes:

$$\begin{array}{cccc} \star & \star & H & \cdot \\ \cdot & \cdot & \star & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} \quad \text{and} \quad \begin{array}{cccc} \cdot & \cdot & \star & \cdot \\ \star & \star & I & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array} \tag{1.41.2}$$

where the triples of planes marked by $\star$ in (1.41.2) do not have a line in common (if $\mathbf{1} \cap \mathbf{1'} \cap \mathbf{I}$ were a line, this line would contain $2'$, a contradiction since $2' \notin \mathbf{1}$). To prove that $H$ and $I$ belong only to those special planes which are prescribed by the (16,6) configuration, it is enough to check that $H \notin \mathbf{2}, \mathbf{3}$. Since all such arguments are very similar to each other, we go over them briefly. $\mathbf{2}$, $\mathbf{1'}$ and $\mathbf{I}$ do not have a line in common (otherwise we would have $12' \cap PR \neq \emptyset$), hence $\mathbf{2} \cap \mathbf{1'} \cap \mathbf{I} = 2'$. If $H \in \mathbf{2}$ then $H = \mathbf{2} \cap \mathbf{1'} \cap \mathbf{I} = 2'$ which is a contradiction since $2' \notin \mathbf{1}$ but $H \in \mathbf{1}$.

Similarly, $\mathbf{1}$, $\mathbf{3}$ and $\mathbf{I}$ do not have a line in common, hence $\mathbf{1} \cap \mathbf{3} \cap \mathbf{I} = 2$. If $H \in \mathbf{3}$ then $H = 2$ which is impossible since $2 \notin \mathbf{1}'$.

To define the planes $\mathbf{P}$, $\mathbf{R}$ in the lower half of the third column,

$$\begin{array}{cccc} \heartsuit & \heartsuit & \heartsuit & \cdot \\ \heartsuit & \heartsuit & \heartsuit & \cdot \\ \heartsuit & \heartsuit & \mathbf{P} & \cdot \\ \heartsuit & \heartsuit & \mathbf{R} & \cdot \end{array}$$

we must show that the five points $4'$, $4$, $H$, $I$, $P$ are coplanar. (Here and below, we tacitly assume all the facts which follow from the known facts by permutations of rows or permutations of columns of the diagram, which amounts to relabeling points or planes). For that it is enough to see that $4'$, $4$, $I$, $P$ are coplanar (by permuting the first two rows). Now, the line $IP$ meets the lines $11'$, $\mathbf{2} \cap \mathbf{2}'$, $\mathbf{3} \cap \mathbf{3}'$, hence it must meet $44'$ by Lemma 1.35; thus, the points $4'$, $4$, $I$, $P$ are coplanar, as desired.

To check that there are no extra incidence relations satisfied by $\mathbf{P}$ and $\mathbf{R}$, it is sufficient, because of the automorphisms, to prove that $1 \notin \mathbf{P}$ and $4 \notin \mathbf{P}$. Suppose that $1 \in \mathbf{P}$. The three points 1, 3 and $R$ are not collinear by the assumption (c) on the line $PR$. Hence these three points span the plane $\mathbf{4}'$, so $\mathbf{4}' = \mathbf{P}$, which is a contradiction since $3' \notin \mathbf{4}'$.

Similarly, suppose $4 \in \mathbf{P}$. Then, since 3, 4 and $P$ are not collinear by (c), they span $\mathbf{4}'$, so $\mathbf{4}' = \mathbf{P}$, a contradiction.

We have now constructed:

$$\begin{array}{cccc} \heartsuit & \heartsuit & \heartsuit & \cdot \\ \heartsuit & \heartsuit & \heartsuit & \cdot \\ \heartsuit & \heartsuit & \heartsuit & \cdot \\ \heartsuit & \heartsuit & \heartsuit & \cdot \end{array} \tag{1.41.3}$$

Consider the three planes in the top row of (1.41.3). They do not intersect in a line, since such a line would have to contain 1 and $1'$, which is impossible because $1 \notin \mathbf{1} \implies 11' \not\subset \mathbf{1}$. Of course, the same reasoning holds for the second, third and fourth row of (1.41.3). Define the points $J$, $K$, $Q$ and $S$ to be the intersection points of the three planes in the first, second, third and fourth row of (1.41.3), respectively:

$$\begin{array}{cccc} \heartsuit & \heartsuit & \heartsuit & J \\ \heartsuit & \heartsuit & \heartsuit & K \\ \heartsuit & \heartsuit & \heartsuit & Q \\ \heartsuit & \heartsuit & \heartsuit & S \end{array}$$

Similarly, we can define the planes $\mathbf{J}$, $\mathbf{K}$, $\mathbf{Q}$, and $\mathbf{S}$ as the planes spanned by the triples of points $11'H$, $22'I$, $33'P$, $4'4R$, respectively. The proof that the three points in each case are not collinear is dual to the above proof that $\mathbf{1}$, $\mathbf{1}'$ and $\mathbf{H}$ do not have a line in common.

To check that the points $1$, $1'$, $H$, $K$, $Q$ and $S$ are coplanar (the dual statement about 6 planes follows by duality), it is enough to check that $1$, $1'$, $H$ and $K$ are coplanar. This is the same as proving that the lines $11'$ and $HK$ intersect. Now, $HK$ meets the three lines $\mathbf{1} \cap \mathbf{1'}$, $22'$ and $\mathbf{2} \cap \mathbf{2'}$, hence it must meet $11'$ by Lemma 1.37. (Note that the four line set $\mathbf{1} \cap \mathbf{1'}$, $\mathbf{2} \cap \mathbf{2'}$, $33'$ and $4'4$ can be transformed by relabeling into $11'$, $22'$, $\mathbf{1} \cap \mathbf{1'}$, and $\mathbf{2} \cap \mathbf{2'}$.)

To check that $J$, $K$, $Q$ and $S$ do not belong to any extra planes it is sufficient, because of automorphisms of the situation, to check that $K \notin \mathbf{1}, \mathbf{H}, \mathbf{K}$ (provided our proof only uses the incidence relations or lack thereof verified up to now, and not the original definition of $P$ and $R$, in which the rows of the diagram do not play symmetric roles):

$$\begin{array}{cccc} \mathbf{1} & \cdot & \mathbf{H} & \cdot \\ \cdot & \cdot & \cdot & K \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}$$

Indeed, the planes $\mathbf{1}$, $\mathbf{2'}$ and $\mathbf{I}$ do not have a line in common, otherwise such a line would have to contain both 2 and $1'$, hence $1' \in \mathbf{I}$, which would be a contradiction. Thus $\mathbf{1} \cap \mathbf{2'} \cap \mathbf{I} = 2$. Now, suppose $K \in \mathbf{1}$. Then $K = \mathbf{1} \cap \mathbf{I} \cap \mathbf{2} \cap \mathbf{2'} = 2$ which is a contradiction since $2 \notin \mathbf{2}$.

If $K \in \mathbf{H}$ then $K = \mathbf{H} \cap \mathbf{2} \cap \mathbf{2'} \implies K = J \implies J \in \mathbf{J}$, a contradiction.

Finally, suppose $K \in \mathbf{K}$. Since 2, $2'$ and $I$ are not collinear, one of the three lines spanned by pairs of these points does not contain $K$. Say, $K \notin 22'$. Then $\mathbf{K}$ is the plane spanned by 2, $2'$ and $K$, hence $\mathbf{K} = \mathbf{I}$. This is a contradiction since $I \notin \mathbf{I}$.

The proof that $\mathbf{J}$, $\mathbf{K}$, $\mathbf{Q}$ and $\mathbf{S}$ satisfy no extra incidence relations is dual to the one for $J$, $K$, $Q$ and $S$. Hence the points and planes in (1.41.3) satisfy all the required incidence relations and no extra ones. This completes the definition of the desired (16,6) configuration and the proof of Proposition 1.41. □

### III. A non-degenerate (16,6) configuration of points and planes in $\mathbb{P}^3$ is of type (*).

In this subsection we prove that a non-degenerate (16,6) configuration of points and planes in $\mathbb{P}^3$ is of type (*). We also solve a closely related classification problem: classify all the configurations of 10 conics and 15 points in the plane such that each conic contains exactly six points and each point belongs to exactly four conics, with the added condition that every two conics have exactly two points in common and every two points belong to exactly two conics. The connection of this problem to (16,6) configurations and Kummer surfaces is as follows. We shall prove in the next section that every non-degenerate (16,6) configuration in $\mathbb{P}^3$ is the singular locus of a unique Kummer surface. We shall also review the well known fact that six nodes of a Kummer surface contained in a special plane lie on a conic (one can also easily see this directly from the explicit description (1.4.1) of a non-degenerate (16,6) configuration, once we establish that they are all of the form (1.4.1)). Fixing a point $P$ of a (16,6) configuration and projecting the remaining 15 nodes and the 10 conics not passing through $P$ from $P$ to a plane, we get a configuration of 15 points and 10 conics on a plane described above. *A priori* the possible incidence relations for such a configuration could have three combinatorial types, corresponding to (*), (A) and (B) of Theorem 1.20. In this subsection we show that type (A) does not occur, while the configurations of type (B) form a 2-dimensional family, which we describe explicitly (of course, the configurations corresponding to type (*) form a 3-dimensional family and are in 1-1 correspondence with non-degenerate (16,6) configurations and Kummer surfaces).

**Theorem 1.42.** *A non-degenerate (16,6) configuration of points and planes in $\mathbb{P}^3$ is of type (*).*

*Proof.* We have to prove that the two exotic abstract (16,6) configurations (A) and (B) of Theorem 1.20 cannot be realized as configurations of points and planes in $\mathbb{P}^3$.

(B): Let us use the notation of Theorem 1.20. Number the ten points which do not lie in $H$ in the order in which they appear in the diagram (B), $P_7$ through $P_{16}$. The diagram (B) completely describes all the incidence relations between the $P_i$'s and the sixteen planes involved. In particular, if we ignore the points $P_7, P_8, P_9, P_{10}$, the remaining twelve points and the incidence relations they satisfy are indistinguishable from a part of a (16,6) configuration of type (*). One checks directly from definitions that these twelve points and sixteen planes form the following incidence diagram (in the notation of the preceding subsection):

$$\begin{matrix} \heartsuit & \heartsuit & \star & \star \\ \heartsuit & \heartsuit & \star & \star \\ \heartsuit & \heartsuit & \heartsuit & \heartsuit \\ \heartsuit & \heartsuit & \heartsuit & \heartsuit \end{matrix} \tag{1.42.1}$$

where the positions of the points are

$$\begin{matrix} P_1 & P_{12} & \cdot & \cdot \\ P_4 & P_{15} & \cdot & \cdot \\ P_2 & P_{16} & P_{11} & P_{14} \\ P_{13} & P_5 & P_3 & P_6 \end{matrix}$$

and where the planes (where $H_{ij}$ denotes the plane whose intersection with $H$ in (B) is $\{P_i, P_j\}$:

$$\begin{matrix} H_{24} & H_{15} & H_{13} & H_{16} \\ H_{12} & H_{45} & H_{34} & H_{46} \\ H_{14} & H_{25} & H_{23} & H_{26} \\ H & H_{36} & H_{56} & H_{35} \end{matrix}$$

In particular, we have the data

$$\begin{matrix} \heartsuit & \heartsuit & \cdot & \cdot \\ \heartsuit & \heartsuit & \cdot & \cdot \\ \heartsuit & \heartsuit & \circ & \cdot \\ \heartsuit & \heartsuit & \circ & \cdot \end{matrix} \tag{1.42.2}$$

of Proposition 1.41. It is obvious from the proof of Proposition 1.41 that (1.42.2) determines the data (1.42.1) completely. In particular, by Proposition 1.41 there exists a (16,6) configuration of type (*) whose sixteen planes are the same as in our configuration (B) and which contains points $P_i$, $1 \leq i \leq 6$ and $11 \leq i \leq 16$. In particular, the planes $H_{12}$, $H_{45}$, $H_{34}$ and $H_{23}$ do not pass through the same point (otherwise that point would have to be

$$\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \circ \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}$$

in the configuration of type (*), which would be a contradiction since $\circ \notin H_{23}$). But by (B) the planes $H_{i,i+1}$, $1 \leq i \leq 4$ pass through the point $P_7$, which is a contradiction.

(A): Let the notation $P_i$, $1 \leq i \leq 16$ and $H_{ij}$, $1 \leq i < j \leq 6$ be as above. Using the dictionary between the diagram (A) and the incidence relations, we check that the points $P_2$, $P_3$, $P_5$, $P_6$, $P_8$, $P_{14}$, $P_{15}$, $P_{16}$ and the planes $H$, $H_{14}$, $H_{23}$, $H_{25}$, $H_{26}$, $H_{35}$, $H_{36}$, $H_{56}$ form an (8,4) configuration, which we can represent by the diagrams

$$\begin{array}{cccc} P_2 & P_3 & P_5 & P_{16} \\ P_8 & P_{14} & P_{15} & P_6 \end{array} \quad \text{and} \quad \begin{array}{cccc} H_{35} & H_{25} & H_{23} & H \\ H_{26} & H_{36} & H_{56} & H_{14} \end{array}$$

In particular, by Lemma 1.35 a line meeting any three of the lines $P_2P_8$, $H_{25} \cap H_{36}$, $H_{23} \cap H_{56}$ and $P_{16}P_6$ must meet the fourth. Consider the line $P_1P_7$. By (A), $P_1P_7, P_2P_8 \subset H_{12}$, so $P_1P_7 \cap P_2P_8 \neq \emptyset$. Also by (A), $P_1P_7, P_6P_{16} \subset H_{46} \implies P_1P_7 \cap P_6P_{16} \neq \emptyset$; $P_7 \in H_{23} \cap H_{56} \implies P_1P_7 \cap (H_{23} \cap H_{56}) \neq \emptyset$. Hence by Lemma 1.35, $P_1P_7 \cap (H_{25} \cap H_{36}) \neq \emptyset$. Now, $H_{25} \cap H_{36} = P_{10}P_{11}$ by (A), so the points $P_1$, $P_7$, $P_{10}$, $P_{11}$ are coplanar. By (A), $P_1, P_7, P_{10} \in H_{12}$, hence $P_{11} \in H_{12}$, which contradicts (A). $\square$

We end this subsection by discussing a closely related problem about configurations which we shall not use in the sequel. It is well known (and easy to prove–see Proposition 2.16) that in a (16,6) configuration coming from the set of nodes of a Kummer surface (which, as we shall prove means precisely all the non-degenerate (16,6) configurations) the six points belonging to each special plane lie on a conic. Consider a non-degenerate (16,6) configuration in $\mathbb{P}^3$, such that the six points belonging to each special plane lie on a conic. Fix one of the points $P$ and consider the linear projection from $P$:

$$\mathbb{P}^3 \setminus \{P\} \to \mathbb{P}^2.$$

The remaining fifteen nodes will project to fifteen points in the plane. The ten conics corresponding to the ten planes not passing through $P$ will project to conics. Thus we obtain a configuration of fifteen points and ten conics in the plane such that each conic contains exactly six points, each point is contained in exactly four conics and each two conics have exactly two points of the configuration in common. Thus a question closely related to classification of (16,6) configurations is that of classifying the configurations of points and conics in the plane as above. By Theorem

1.20, there are at most three possible combinatorial types for these configurations. Those corresponding to (16,6) configurations of type (*), as we will show, always come from Kummer surfaces and, conversely, determine a unique Kummer surface. Hence that classification problem is the same as classifying Kummer surfaces, which is done below (in particular, we show that there is a 3-dimensional family of such configurations). We now prove

**Theorem 1.43.** *There does not exist a configuration of fifteen points and ten conics as above corresponding to the diagram (A). On the other hand, there is a 2-dimensional family of such point-conic configurations corresponding to the diagram (B).*

*Proof.* We may view (A) as the dual of an actual geometric diagram of six lines and fifteen points in the plane. By definition, the six points dual to the six lines of any 6-side appearing in (A) lie on a conic. This defines the incidence relations between the conics and the planes. Alternatively (dualizing), we may view (A) itself as an actual diagram of six points and fifteen lines in the plane such that the six lines of any 6-side are tangent to a conic. Recall Pascal's theorem and its dual, Brianchon's theorem:

**Theorem (Pascal).** *Consider a hexagon in a plane. This hexagon can be inscribed in a conic if and only if the three intersection points of the pairs of opposite sides are collinear.*

**Theorem (Brianchon).** *Consider a hexagon in a plane. Then a conic can be inscribed in it if and only if the three lines connecting the three pairs of opposing vertices pass through the same point.*

We apply Brianchon's theorem to the conics number 1, 3 and 4 in (A). We get that the lines $P_1P_4$, $P_2P_5$, $P_3P_6$ pass through the same point; $P_1P_4$, $P_2P_3$, $P_5P_6$ pass through the same point and $P_1P_4$, $P_2P_6$, $P_3P_5$ pass through the same point. In other words, all of the points $P_2P_5 \cap P_3P_6$, $P_2P_3 \cap P_5P_6$ and $P_2P_6 \cap P_3P_5$ lie on the line $P_1P_4$. Hence these three points are collinear, which is absurd. This completes the proof for (A).

For (B), we consider the dual diagram to (B) of six lines and fifteen points in a plane and apply Pascal's theorem. Now, each of the four hexagons in (B) gives us the same collinearity condition. Namely, let us denote by $P_{ij}$, $1 \leq j < j \leq 6$ the point dual to the line $P_iP_j$ in (B) and by $L_i$, $1 \leq i \leq 6$, the line dual to $P_i$. Then if we assume that each of the four sextuples of points corresponding to the four hexagons in (B) lie on a conic, we get from Pascal's theorem that the poinnts $P_{14}$, $P_{25}$ and $P_{36}$ are collinear (of course, the line $L$ which contains them must be distinct from all of the $L_i$, $1 \leq i \leq 6$). Conversely, requiring that $P_{14}$, $P_{25}$ and $P_{36}$ be collinear guarantees (by Pascal's theorem applied in the converse direction) that each of the four sextuples of points dual to the sides of the hexagons in (B) lies on a conic. Thus to classify the configurations of points and conics of type (B) as above, it is enough to solve the following problem. Take three distinct points $P_{14}$, $P_{25}$ and

$P_{36}$ lying on a line $L$. Choose lines $L_i$, $1 \leq i \leq 6$, all distinct from $L$, such that

$$P_{ij} = L_i \cap L_j \tag{1.43.1}$$

for $(ij) = (14)$, $(25)$ and $(36)$. Define the remaining $P_{ij}$ by (1.43.1). Consider the sextuple of points $P_{14}$, $P_{25}$, $P_{13}$, $P_{13}$, $P_{43}$, $P_{26}$, $P_{56}$. We want to classify all the situations where this sextuple *and all of its six images under the stabilizer in* $S_6$ of the unordered triple (14)(25)(36) of unordered pairs, lie on a conic.

Now, the choice of the lines $L$, $L_i$ and points $P_{ij}$ depends on three parameters. Indeed, all the choices of the line $L$ and the three points on it are equivalent to each other. Once $L$ and $P_{14}$, $P_{25}$ and $P_{36}$ are fixed, we are still free to choose $L_1$, $L_2$ (which amounts to choosing $P_{12}$. After these are fixed, we still have a 1-parameter group of linear automorphisms of the plane acting on the situation, so we are free to put $P_{23}$ wherever we like in $L_2 \setminus \{P_{12}, P_{25}\}$. After that there are no more choices and the three remaining lines give the three parameters on which the situation depends. One might think that the six conics described above give six conditions, but it turns out that all the six conditons are equivalent, so the desired configuration of fifteen points and six conics exists and has 2-dimensional moduli. More precisely, normalize the configuration of seven lines and fifteen points above as follows in order to simplify the calculation. Let us choose a coordinate system on the plane and let $(x, y)$ be the affine coordinates on one of the coordinate charts. We let $L$ be the line at infinity and the points $P_{14}$, $P_{25}$ the points of intersection of $L$ with the $x$- and the $y$-axes, respectively. Let $L_1$ and $L_2$ be the $x$- and the $y$-axes. Let $L_4 := \{y = 1\}$ and $L_5 := \{x = 1\}$. In particular, we have $P_{12} = (0,0)$, $P_{15} = (1,0)$, $P_{24} := (0,1)$ and $P_{45} = (1,1)$. Now, take $a, b \in k$, $a, b \neq 0, 1$ and $c \in k$, $c \neq 0$. Let $P_{13} := (a, 0)$, $P_{23} := (0, b)$ and $P_{26} = (0, b + c)$. Now the rest of the configuration is uniquely determined. We have $P_{16} = (a + \frac{ca}{b}, 0)$, $L_3 = \{\frac{x}{a} + \frac{b}{y} = 1\}$, $L_6 = \{\frac{x}{a} + \frac{y}{b} = 1 + \frac{c}{b}\}$, $P_{36}$ is the intersection point of $L_3$ with the line at infinity,

$$\begin{aligned}
P_{34} &= (a(1 - \frac{1}{b}), 1) \\
P_{35} &= (1, b(1 - \frac{1}{a})) \\
P_{46} &= (\frac{a}{b}(b + c - 1), 1) \\
P_{56} &= (1, (b + c - \frac{b}{a}))
\end{aligned}$$

Changing to homogeneous coordiantes now, we have to find the conditions on $a, b$ and $c$ for each of the following six sextuples of points to lie on a conic:

$$(0, b + c, 1), (1, b + c - \frac{b}{a}, 1), (a - \frac{a}{b}, 1, 1), (a, 0, 1), (0, 1, 0), (1, 0, 0)$$
$$(0, b, 1), (1, b - \frac{b}{a}, 1), (\frac{a}{b}(b + c - 1), 1), (a + \frac{ca}{b}, 0, 1), (0, 1, 0), (1, 0, 0)$$

$$
\begin{aligned}
&(0,b+c,1),(0,b,1),(1,1,1),(1,0,1),(1,-\frac{b}{a},0),(1,0,0)\\
&(1,b+c-\frac{b}{a},1),(1,b-\frac{b}{a},1),(0,1,1),(0,0,1),(1,-\frac{b}{a},0),(1,0,0)\\
&(0,1,1),(1,1,1),(a,0,1),(a+\frac{ca}{b},0,1),(1,-\frac{b}{a},0),(0,1,0)\\
&(0,0,1),(1,0,1),(a-\frac{a}{b},1,1),(\frac{a}{b}(b+c-1),1,1),(1,\frac{b}{a},0),(0,1,0).
\end{aligned}
\tag{1.43.2}
$$

Each of the sextuples $\{(X_i,Y_i,Z_i)\}_{1\le i\le 6}$ above lies on a conic if and only if the $6\times 6$ matrix with rows $(X_i^2,Y_i^2,Z_i^2,X_iY_i,X_iZ_i,Y_iZ_i)$, $1\le i\le 6$, is singular. Thus we have to calculate the determinants of six $6\times 6$ matrices. Because of the large number of zeroes and ones (note that each zero in (1.43.2) gives three zeroes in the matrix of quadratic forms), this calculation can easily be done by hand. The answer is that all of the six determinants are equal and are given by the expression

$$a+b-ac-2ab. \tag{1.43.3}$$

Thus we get the desired configuration of points and conics if and only if $a+b-ac-2ab=0$. Hence the 15 point, 10 conic configurations in question are parametrized by the non-singular surface in $(k\setminus\{0,1\})^2\times k^*$ defined by the equation 1.43.3. This completes the proof of Theorem 1.43.

## IV. A non-degenerate (16,6) configuration in $\mathbb{P}^3$ is of the form $(a,b,c,d)$ of (1.4.1).

**Definition 1.44.** *We say that a set of planes is in* **general linear position** *if no four of them pass through the same point.*

**Theorem 1.45.** *Any non-degenerate (16,6) configuration in $\mathbb{P}^3$ is of the form described in Proposition 1.6, up to an automorphism of $\mathbb{P}^3$.*

*Proof.* By Theorem 1.42, any non-degenerate (16,6) configuration in $\mathbb{P}^3$ is of type (*).

Let $U$ denote the Zariski open subset of $\mathbb{P}^{3^*}$ defined by the conditions (1.2.1). As usual, let us denote a plane in $\mathbb{P}^3$ by a 4-tuple of elements of $k$. We may view this 4-tuple as a point in $\mathbb{P}^{3^*}$. Let $V$ denote the open subset of $(\mathbb{P}^{3^*})^6$ consisting of all the ordered 6-tuples of planes in general linear position. Consider the map

$$\phi: U\times PGL_4(k)\to V,$$

defined as follows (here $PGL_4(k)$ is the group of non-singular $4\times 4$ matrices with entries in $k$, modulo multiplication by non-zero elements of $k$). Let $(a,b,c,d)\in U$ and $M\in PGL_4(k)$. Then $M$ acts on planes in $\mathbb{P}^3$ as a projective linear transformation. We define $\phi((a,b,c,d),M)$ to be the image under $M$ of the *ordered* 6-tuple of planes 1, $AB'$, $BC'$, $AC'$, $CC'$, $AA'$ (in the notation of Notes 1.4 and 1.5). Let $\mathcal{S}$ denote the set of ordered pairs consisting of a (16,6) configuration $\Omega$ of type (*), together with an incidence-preserving bijection $b_\star$ from the set of planes of $\Omega$ to the entries of the diagram (*). We define a map

$$\theta : \mathcal{S} \to V$$

as follows. Consider an element $(\Omega, b_\star) \in \mathcal{S}$. Consider the 6 planes marked by $\star$ in the following diagram.

$$\begin{matrix} \star & \star & \star & \cdot \\ \star & \cdot & \cdot & \cdot \\ \cdot & \star & \cdot & \cdot \\ \cdot & \cdot & \star & \cdot \end{matrix} \tag{1.45.1}$$

We define $\theta(\Omega, b_\star)$ to be the ordered sextuple of planes of $\Omega$ corresponding to the sextuple of stars in (1.45.1) under $b_\star$.

**Lemma 1.46.** *The six planes of (1.45.1) are in general linear position.*

*Proof.* By (1.45.1), among any four of our six planes, there are three which pass through the same point of the (16,6) configuration. Hence, if four of our planes passed through the same point in $\mathbb{P}^3$, this point would belong to the (16,6) configuration. But no point of the (16,6) configuration belongs to four of our 6 planes by (1.5.1), which proves that they are in general linear position. □

**Note on proofs by inspection of the diagram.** The proof of Lemma 1.46 is an example of what we consider an acceptable proof by inspection of the diagram. There are two statements in the proof, each of which involves checking $\binom{6}{4} = 15$ cases, and each case takes about one second to check. Writing down a proof of these statements would be tedious and pointless. On the other hand, if the inspection involves something like $\binom{16}{2} = 120$ cases, then we feel that it is necessary to give an analytical proof.

Let $\eta : U \times PGL_4(k) \to \mathcal{S}$ be the map defined as follows. Let $(a, b, c, d) \in U$ and $M \in PGL_4(k)$. $M$ acts on planes in $\mathbb{P}^3$ as a projective linear transformation. We define $\eta((a, b, c, d), M)$ to be the image under $M$ of the orbit of the plane $H = \{ax + by + cz + dt = 0\}$ under $F_0$. The bijection from the set of planes to elements of the diagram (*) is defined by (1.4.1) and (1.5.1). $\eta$ is well defined by Proposition 1.6 and Lemma 1.7. Clearly, $\phi = \theta \circ \eta$.

**Lemma 1.47.** *No three points of a (16,6) configuration of type (*) are collinear. No three special planes intersect in a line.*

*Proof.* Suppose we have three collinear points $P_1$, $P_2$, $P_3$ in a (16,6) configuration of type (*). By Corollary 1.11, $P_1$ and $P_2$ belong to two planes of the (16,6) configuration. But then $P_3$ also belongs to the same two planes, which contradicts Corollary 1.11.

The proof of the second assertion is dual and we omit it. Lemma 1.47 is proved. □

**Lemma 1.48.** *The map $\theta$ is injective.*

*Proof.* We have to prove that the 6 planes of (1.45.1) together with their position in the incidence diagram completely determine our (16,6) configuration of type (*). Indeed, the following ten points of the configuration are uniquely determined, since each of them is an intersection point of three of our planes (and since by the previous lemma no three of our planes have a line in common).

$$\begin{array}{cccc} \circ & \circ & \circ & \circ \\ \cdot & \circ & \circ & \cdot \\ \circ & \cdot & \circ & \cdot \\ \circ & \circ & \cdot & \cdot \end{array} \tag{1.48.1}$$

Each of the remaining 10 special planes contains three or more of the points marked by $\circ$ above. Since no three of our ten points are collinear, all the special planes and hence the entire configuration (together with the bijection $b_\star$) is uniquely determined. This completes the proof. □

We continue with the proof of Theorem 1.45.

**Lemma 1.49.** *$\phi$ is a 16-to-1 map onto its image. If*

$$\phi((a,b,c,d),M) = \phi((a',b',c',d'),M') \tag{1.49.1}$$

*then ${M'}^{-1}M \in F_0$, so that $(a,b,c,d)$ and $(a',b',c',d')$ define the same (16,6) configuration and ${M'}^{-1}M$ is its automorphism.*

*Proof.* Since the six planes in general position, placed in the incidence diagram as in (1.45.1), determine the (16,6) configuration uniquely, ${M'}^{-1}M$ must send the (16,6) configuration given by $(a,b,c,d)$ (cf. Proposition 1.6) into that given by $(a',b',c',d')$. Clearly, if $\sigma \in F_0$,

$$\phi((a,b,c,d),M) = \phi(\sigma(a,b,c,d),M\sigma^{-1}).$$

Thus, to prove that $\phi$ is 16-to-1, it remains to show the converse: (1.49.1) implies that ${M'}^{-1}M \in F_0$.

Suppose (1.49.1) holds. Let us denote $T := {M'}^{-1}M$. $T$ maps the (16,6) configuration $(a,b,c,d)$ to the (16,6) configuration $(a',b',c',d')$. Moreover, $T$ maps the following six planes into the corresponding planes of the target configuration:

$$\begin{aligned} T(a,b,c,d) &= (a',b',c',d') \\ T(d,-c,b,-a) &= (d',-c',b',-a') \\ T(c,d,-a,-b) &= (c',d',-a',-b') \\ T(d,c,-b,-a) &= (d',c',-b',-a') \\ T(-b,-a,d,c) &= (-b',-a',d',c') \\ T(d,-c,-b,a) &= (d',-c',-b',a'). \end{aligned} \tag{1.49.2}$$

Let us denote the above six planes by $v_j$, $1 \le j \le 6$:

$$\begin{aligned} v_1 &= (a, b, c, d) \\ v_2 &= (d, -c, b, -a) \\ v_3 &= (c, d, -a, -b) \\ v_4 &= (d, c, -b, -a) \\ v_5 &= (-b, -a, d, c) \\ v_6 &= (d, -c, -b, a). \end{aligned}$$

Let $\sigma_j$, $2 \le j \le 6$ denote the five elements of the group $F_0$, defined by

$$v_j = \sigma_j v_1, \qquad 2 \le j \le 6.$$

(namely, $\sigma_2 = AB'$, $\sigma_3 = BC'$, etc.)

By (1.49.2), $T\sigma_j(v_1) = \sigma_j T(v_1)$ for $2 \le j \le 6$. Since the six planes $Tv_j$, $1 \le j \le 6$ determine the (16,6) configuration $(a', b', c', d')$ uniquely (together with the identification of planes and points with the entries of (1.45.1)), we have

$$T\sigma v_1 = \sigma T v_1 \qquad \text{for any } \sigma \in F_0.$$

Any special plane $v$ can be written $v = \delta v_1$, for some $\delta \in F_0$. Hence, for any plane $v = \delta v_1$ belonging to the (16,6) configuration $(a, b, c, d)$ and for any $\sigma \in F_0$,

$$T\sigma v = T\sigma\delta v_1 = \sigma\delta T v_1 = \sigma T \delta v_1 = \sigma T v. \tag{1.49.3}$$

Since a (16,6) configuration contains five planes in general position, (1.49.3) implies that $T\sigma = \sigma T$ for any $\sigma \in F_0$. In other words, $T$ belongs to the centralizer of $F_0$ in $PGL_4(k)$. To complete the proof of Lemma 1.49, it remains to prove

**Lemma 1.50.** *$F_0$ is its own centralizer in $PGL_4(k)$.*

*Proof.* Pick and fix once and for all an element $i$ of $k$ such that $i^2 + 1 = 0$. We have the 4–1 covering

$$\pi : SL_4(k) \to PGL_4(k),$$

whose kernel is $\{1, -1, i, -i\}$ (here we identify constant multiples of the identity matrix with elements of $k$). Consider the matrices

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad e_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$e_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and } e_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{1.50.1}$$

in $SL_4(k)$. The $e_j$, $1 \leq j \leq 4$, are a particular choice of lifting of $C'$, $A'$, $C$ and $A$ to $SL_4(k)$. An element of $PGL_4(k)$ lies in the centralizer of $F_0$ if and only if for some (hence any) lifting $T$ of it to $SL_4(k)$, and for any $j$, $1 \leq j \leq 4$, one of the following holds:

$$e_jT = \pm Te_j \tag{1.50.2}$$

$$e_jT = \pm iTe_j. \tag{1.50.3}$$

We want to show that if an element $T \in SL_4(k)$ satisfies the above commutation relations then $T$ is, up to a constant multiple, a product of some of the $e_j$. First, note that for any $j$, $1 \leq j \leq 4$, there are no matrices $T$ in $SL_4(k)$ such that $e_jT = \pm iTe_j$. In other words, we always have (1.50.2) and never (1.50.3). A matrix in $SL_4(k)$ either commutes or anticommutes with both $e_1$ and $e_2$ if and only if it has one of the following forms:

$$\begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \begin{pmatrix} 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \\ * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & * & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \\ 0 & * & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix},$$

where $*$ denotes a non-zero entry. Multiplying by a product of some of the $e_j$, we may assume that $T$ is diagonal, i.e.

$$T = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix},$$

where $\alpha\beta\gamma\delta = 1$. Then the commutation relation $e_3T = \pm Te_3$ or says that one of the following holds:

$$\begin{aligned} &\text{either } \alpha = \beta \quad \text{and } \gamma = \delta \\ &\quad\text{or } \alpha = -\beta \text{ and } \gamma = -\delta \end{aligned} \tag{1.50.4}$$

From the commutation relation $e_4T = \pm Te_4$ we get

$$\begin{aligned} &\text{either } \alpha = \delta \quad \text{and } \gamma = \beta \\ &\quad\text{or } \alpha = -\delta \text{ and } \gamma = -\beta \end{aligned} \tag{1.50.5}$$

Together, (1.50.4) and (1.50.5) imply that up to multiplication by a constant, $T$ is a diagonal matrix with eigenvalues 1,1,-1,-1. Hence $T$ has the form $e_1^{\epsilon_1}e_2^{\epsilon_2}$, where $\epsilon_j \in \{0,1\}$. Lemma 1.50 (hence also Lemma 1.49) is proved. $\square$

**Remark 1.51.** In order to prove Theorem 1.45 we only need that $\phi$ is finite-to-one, but this more precise version of Lemma 1.49 will be useful later for constructing the moduli space of (16,6) configurations.

We now return to the proof of Theorem 1.45.

$\phi$ is a finite-to-one morphism between two irreducible quasi-projective varieties of the same dimension, so $Im(\phi)$ is Zariski-dense in $V$. (Below we shall prove that $Im(\phi) = V$, so $\phi$ is, in fact, a 16-1 covering of $V$). Note that $\mathcal{S}$ is irreducible, since $\theta$ is injective. Theorem 1.45 is the statement that $\eta$ is surjective. We consider an arbitrary (16,6) configuration of type (*) (denote it by $\Omega$). To prove that $\eta$ is surjective it is enough to prove that $\theta(\Omega) \in Im(\phi)$, since $\theta$ is injective by Lemma 1.48. Given $\Omega$, look at the six planes of (1.45.1). Let us denote these 6 planes by $\{w_i\}_{1\leq i\leq 6}$, $w_i \in \mathbb{P}^{3^*}$. By Lemma 1.46 the $w_i$, $1 \leq i \leq 6$, are in general linear position, so they determine a point in $V$. Let us denote $\theta(\Omega)$ by $\omega$. In any case, $\omega$ belongs to the closure of $Im(\phi)$. Let $w_i$, $1 \leq i \leq 16$ denote the special planes of $\Omega$. Thus, we may think of $\Omega$ as a point in $(\mathbb{P}^{3^*})^{16}$. For any $\Lambda \in Im(\phi)$, there exists $T \in PGL_4(k)$ and $(a,b,c,d) \in U$ such that $\Lambda = \phi((a,b,c,d),T)$. In particular, $T$ moves $\Lambda$ to the six planes of the (16,6) configuration $(a,b,c,d)$ given by (1.4.1). Then $TF_0T^{-1}$ acts as a group of automorphisms of the (16,6) configuration which is the image of $(a,b,c,d)$ under $T$. This group is defined canonically in the sense that it does not depend on the choice of lifting $((a,b,c,d),T)$ (by Lemma 1.49, a different choice of lifting would modify $T$ by an element of $F_0$). As an abstract group, $TF_0T^{-1}$ is just $\mathbb{F}_2^4$. Let us take $\sigma_0 \in F_0$. Let $\sigma = T\sigma_0T^{-1}$. The action of $\sigma$ on $\Lambda$ is completely determined by any set of 5 special planes of $\Lambda$ in general position together with their image under $\sigma$. In other words, the image of $\sigma$ in $PGL_4(k)$ is an algebraic function of $\Lambda$, as long as the five planes remain in general position. Since $\Omega$ is also a (16,6) configuration of type (*), it contains 5 special planes in general position, which are limits of corresponding sets of 5 planes of configurations in $Im(\phi)$. This shows that the action of $\mathbb{F}_2^4$ on the (16,6) configurations $\Lambda \in Im(\phi)$ extends to $\Omega$ by continuity. The resulting action of $\mathbb{F}_2^4$ on the (16,6) configuration $\Omega$ gives an injection

$$\psi : \mathbb{F}_2^4 \hookrightarrow PGL_4(k).$$

If we can show that the subgroup $Im(\psi) \subset PGL_4(k)$ is conjugate to $F_0$, the proof of Theorem 1.45 will be complete: since $\Omega$ is the orbit of one plane under $Im(\psi)$, moving $Im(\psi)$ to $F_0$ by conjugation is equivalent to moving $\Omega$ to a (16,6) configuration of type (1.4.1). It remains to prove

**Lemma 1.52.** *Any subgroup of $PGL_4(k)$, isomorphic to $\mathbb{F}_2^4$, is conjugate to $F_0$.*

*Proof.* Let $F'$ be such a subgroup and let $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ be a set of generators of $F'$. We have the 4-1 covering

$$\pi : SL_4(k) \to PGL_4(k)$$

whose kernel is generated by

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}.$$

Lift the $\epsilon_j$'s in an arbitrary way to elements $e_j \in SL_4(k)$, $1 \le j \le 4$. Then $e_j^2 = \pm 1$ or $\pm i$ (where $i$ is an element of $k$ such that $i^2 = -1$, which we fix once and for all; here and below we identify an element $a \in k$ with $a$ times the identity matrix in $SL_4(k)$). Multiplying by $i$, if necessary, we may assume that $e_j^2 = 1$ or $i$. The first step is to rule out the case $e_j^2 = i$.

**Sublemma 1.53.** $e_j^2 \neq i$, $1 \le j \le 4$.

*Proof.* Suppose the contrary. Say, $e_1^2 = i$. Then $e_1$ is a diagonalizable matrix whose eigenvalues are $\pm\sqrt{i}$ (this notation should cause no confusion; again, we choose and fix once and for all an element $\sqrt{i}$ of $k$ whose square is $i$). Since $\det e_1 = 1$, after conjugating by an element of $SL_4(k)$ and multiplying by $-1$, if necessary, we may assume that

$$e_1 = \begin{pmatrix} \sqrt{i} & 0 & 0 & 0 \\ 0 & \sqrt{i} & 0 & 0 \\ 0 & 0 & \sqrt{i} & 0 \\ 0 & 0 & 0 & -\sqrt{i} \end{pmatrix}.$$

By construction of the $e_j$'s, we have for any $j, l$ such that $1 \le j, l \le 4$

$$e_j e_l = \pm e_l e_j \quad \text{or} \tag{1.53.1}$$

$$e_j e_l = \pm i e_l e_j. \tag{1.53.2}$$

Let $j = 1$ in the above commutation relations. Then $\det e_l = 1$ implies that $e_1 e_l = e_l e_1$ ((1.53.2) is impossible and the only matrices anticommuting with $e_1$ have determinant 0). Hence for $2 \le l \le 4$, $e_l$ has the form

$$\begin{pmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}, \tag{1.53.3}$$

where $*$ denotes a possible non-zero entry.

We may now assume that $e_l^2 = 1$ for $2 \le l \le 4$ (if not, multiply $e_l$ by $e_1$ and by an appropriate element of $Ker\ \pi$). We may also assume that the lower righthand entry in (1.53.3) is 1 and the upper lefthand 3×3 matrix belongs to $SL_3(k)$. We may use this $3 \times 3$ matrix to identify the $e_l$, $2 \le l \le 4$ with elements of $SL_3(k)$. It remains to show that there do not exist three elements $e_l$, $2 \le l \le 4$ of $SL_3(k)$ whose squares are equal to 1, which satisfy the commutation relations (1.53.1)–(1.53.3), and which generate a group of order 8 or more.

Each of the $e_l$ is a diagonalizable matrix with eigenvalues 1,-1,-1. Conjugating by an element of $SL_3(k)$, we may assume that

$$e_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then the commutation relations (1.53.1)–(1.53.3) together with $\det e_l = 1$ imply that both $e_3$ and $e_4$ commute with $e_2$. Then $e_3$ and $e_4$ must have the form

$$\begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}. \tag{1.53.4}$$

Conjugating by an element of $SL_3(k)$ of the form (1.53.4), we may assume that

$$e_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore, by (1.53.1)–(1.53.2), $e_4$ must commute with both $e_2$ and $e_3$, hence $e_4$ must be a diagonal matrix with eigenvalues 1,-1,-1. But then $e_4$ belongs to the subgroup of $SL_4(k)$ generated by $e_2$ and $e_3$, which is a contradiction. The Sublemma is proved. □

From now on we assume that $e_j^2 = 1$, $1 \leq j \leq 4$. Then each of the $e_j$ is a diagonalizable matrix with eigenvalues 1,1,-1,-1. Conjugating by an element of $SL_4(k)$, we may assume that

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Now (1.53.1)–(1.53.2) imply that for $2 \leq j \leq 4$, $e_1 e_j = \pm e_j e_1$. Next, we may assume that at least two of the $e_j$, $2 \leq j \leq 4$, actually commute with $e_1$. Indeed, suppose one of the $e_j$ (say, $e_4$) anticommutes with $e_1$. Multiplying any other $e_j$ which anticommutes with $e_1$ by $e_4$ or $ie_4$, if necessary, we may assume that $e_2$ and $e_3$ commute with $e_1$. Hence, $e_2$ and $e_3$ have the form

$$\begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}. \tag{1.53.5}$$

Conjugating the $e_j$ by an element of $SL(4, k)$ of the form (1.53.5), we may assume that

$$e_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.53.6}$$

Now, any matrix which commutes with both $e_1$ and $e_2$ is diagonal. If, in addition, such a matrix has eigenvalues 1,1,-1,-1, then it belongs to the subgroup of $SL_4(k)$ generated by $e_1$ and $e_2$. Hence neither $e_3$ nor $e_4$ may commute with both $e_1$ and $e_2$. Then $e_3$ anticommutes with $e_2$. If $e_2$ also anticommutes with $e_4$, replace $e_4$ by $e_3e_4$ or $ie_3e_4$ (the $i$ is needed to make the square come out to 1). Hence, we may assume that $e_2$ commutes with $e_4$ and so $e_1$ anticommutes with $e_4$. Finally, if $e_3$ anticommutes with $e_4$, replace $e_4$ by $e_2e_4$, so we may assume, in addition, that $e_3$ commutes with $e_4$.

Since $e_3$ commutes with $e_1$ and anticommutes with $e_2$, $e_3$ must have the form

$$\begin{pmatrix} 0 & * & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \end{pmatrix}.$$

Conjugating by a diagonal matrix in $SL_4(k)$, we may assume that

$$e_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Now the commutation relations together with $e_4^2 = 1$ imply that $e_4$ must have the form

$$\begin{pmatrix} 0 & 0 & 0 & \alpha \\ 0 & 0 & \beta & 0 \\ 0 & \beta^{-1} & 0 & 0 \\ \alpha^{-1} & 0 & 0 & 0 \end{pmatrix},$$

where $\alpha, \beta \in k$. Finally, conjugating by the matrix

$$\begin{pmatrix} \sqrt{\alpha} & 0 & 0 & 0 \\ 0 & \sqrt{\beta} & 0 & 0 \\ 0 & 0 & (\sqrt{\beta})^{-1} & 0 \\ 0 & 0 & 0 & (\sqrt{\alpha})^{-1} \end{pmatrix},$$

we can achieve the situation when

$$e_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

This completes the proof of Lemma 1.52 and hence also of Theorem 1.45. □

## V. Moduli of non-degenerate (16,6) configurations.

In this subsection we prove that the map $\phi : U \times PGL_4(k) \to V$ of Theorem 1.45 is actually surjective (and therefore that $\theta : \mathcal{S} \hookrightarrow V$ is an isomorphism). This is needed to give a complete classification of non-degenerate (16,6) configurations.

**Proposition 1.54.** *The map $\phi : U \times PGL_4(k) \to V$ of Theorem 1.45 is surjective.*

*Proof.* To say that $\phi$ is surjective is equivalent to the following statement. Given any set of six planes in $\mathbb{P}^3$ in general position together with a bijection to the set of stars in (1.45.1), there exists a (16,6) configuration of type (*) which contains these 6 planes in the place in the incidence diagram prescribed by (1.45.1). Let us denote the six planes by $w_i$, $1 \le i \le 6$, and let $\omega$ denote the point of $V$ given by the $\{w_i\}_{1\le i\le 6}$. In any case, $Im(\phi)$ is dense in $V$, so $\omega$ lies in the closure of the set of points of $V$ corresponding to (16,6) configurations of type (*). The idea is that we may construct the remaining 10 planes from the $w_i$'s by the same rule as before (cf. (1.48.1)) and then prove that they form a (16,6) configuration by continuity, using the uniqueness of the (16,6) configuration with the given 6 planes.

First of all, since our planes are in general position, the 10 points given by the diagram (1.48.1) are well-defined. (The definition of these 10 points has nothing to do with the (16,6) configuration: the diagram (1.48.1) is merely a shorthand for writing "consider the points $w_1 \cap w_2 \cap w_3$, $w_2 \cap w_3 \cap w_4$, etc." Of course, we do not consider all the possible intersection points of triples of the $w_i$'s—just those prescribed by the diagram (1.48.1).)

**Lemma 1.55.** *Among the 10 points given by circles in the diagram (1.48.1), no three are collinear.*

*Proof.* Let the three points in question be given by

$$P_1 := w_{i_1} \cap w_{i_2} \cap w_{i_3} \tag{1.55.1}$$

$$P_2 := w_{j_1} \cap w_{j_2} \cap w_{j_3} \tag{1.55.2}$$

$$P_3 := w_{k_1} \cap w_{k_2} \cap w_{k_3} \tag{1.55.3}$$

We give a proof by contradiction. Suppose that the $P_i$ are collinear. Let us introduce the following notation for the sets of indices:

$$\begin{aligned} M_1 &:= \{i_1, i_2, i_3\} \\ M_2 &:= \{j_1, j_2, j_3\} \\ M_3 &:= \{k_1, k_2, k_3\}. \end{aligned}$$

First of all, we observe that no two of the sets $M_l$ are disjoint. This follows from diagrams (1.45.1) and (1.48.1): for any triple of the six planes whose intersection appears in the diagram (1.48.1) the intersection of the complementary triple does not appear in (1.48.1). Consider two cases.

*Case 1.* $\#(M_1 \cap M_2) = 2$. (Here and below, $\#$ denotes the cardinality of a finite set.)

Without loss of generality we may take $i_1 = j_1$ and $i_2 = j_2$. Since the $P_i$ are collinear and $P_1, P_2 \in w_{i_1} \cap w_{i_2}$, we have $P_3 \in w_{i_1} \cap w_{i_2}$. Hence, three of our 10 points belong to the same pair $w_{i_1} \cap w_{i_2}$ of our six planes, which is a contradiction to diagrams (1.45.1) and (1.48.1) (note that since the $w_i$'s are in general position, the representations (1.55.1)–(1.55.3) of the $P_l$'s as intersections of three of the $w_i$'s are unique).

*Case 2.*

$$\#(M_l \cap M_p) = 1 \qquad \text{for} \quad 1 \leq l < p \leq 3. \tag{1.55.4}$$

Since we only have 6 planes altogether, (1.55.4) implies that $\cap_{l=1}^{3} M_l = \emptyset$. Hence we may assume, without loss of generality, that $i_1 = j_1$, $j_2 = k_2$ and $k_3 = i_3$. Then $P_1, P_2 \in w_{i_1}$, hence $P_3 \in w_{i_1}$ by collinearity. But then $w_{i_1}$, and $w_{k_l}$, $1 \leq l \leq 3$, all pass through $P_3$, which contradicts the fact that the $w_i$ are in general position. $\square$

Now we can finish the proof of Proposition 1.54. The 10 points of diagram (1.48.1) depend algebraically on the 6 planes $\{w_j\}_{1 \leq j \leq 6}$. Since no three of the 10 points are collinear, they determine the 10 planes, which are denoted by points in (1.45.1). Let us denote these 10 planes by $w_j$, $7 \leq j \leq 16$. The $\{w_j\}_{7 \leq j \leq 16}$, again, depend algebraically on the $\{w_j\}_{1 \leq j \leq 6}$. The next step is to check that the planes $\{w_j\}_{1 \leq j \leq 16}$ are all distinct. Let us denote the 10 points of diagram (1.48.1) by $v_j$, $1 \leq j \leq 10$, as follows:

$$\begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \\ \cdot & v_5 & v_6 & \cdot \\ v_7 & \cdot & v_8 & \cdot \\ v_9 & v_{10} & \cdot & \cdot \end{array} \tag{1.55.5}$$

**Definition 1.56.** *Let $P = P_{jl}$, $1 \leq j, l \leq 4$ denote the entry in the $j$-th row and the $l$-th column of the $4 \times 4$ diagram (*).* **The incidence set** *of $P$ is defined to be*

$$Inc(P) := \{P_{j'l'} |\ either\ j = j'\ or\ l = l',\ but\ (j,l) \neq (j',l')\}.$$

*We say that a subset of the points $\{v_i\}_{1 \leq i \leq 10}$ is* **strongly coplanar** *if it is contained in $Inc(P_{jl})$ for some $j, l$, $1 \leq j, l \leq 4$.*

**Lemma 1.57.** *Consider four points $v_{j_1}, v_{j_2}, v_{j_3}, v_{j_4}$, $1 \leq j_1 < j_2 < j_3 < j_4 \leq 10$. Assume that three of these four points are strongly coplanar, but all four together are not strongly coplanar. Then these four points are not contained in a plane in $\mathbb{P}^3$.*

**Corollary 1.58.** *All the planes $w_j$, $1 \leq j \leq 16$ are distinct.*

We first prove the Corollary assuming the Lemma.

*Proof.* Suppose that for some $j, l$, $1 \leq j < l \leq 16$, we have $w_j = w_l$. Let $P, P'$ denote the entries in the $4 \times 4$ diagram (*) corresponding to $w_j$, $w_l$ respectively. We have $\#\{v_p | 1 \leq p \leq 10, v_p \in Inc(P)\} \geq 3$ and similarly for $P'$. On the other hand, $\#\{p \mid 1 \leq p \leq 10, v_p \in Inc(P) \cap Inc(P')\} \leq 2$. Hence, one of the two planes (say, $w_j$) must contain a point $v_p$, $1 \leq p \leq 10$, such that $v_p \notin Inc(P)$. Take $p_1$, $p_2$ and $p_3$, $1 \leq p_1, p_2, p_3 \leq 10$, such that $v_{p_t} \in Inc(P)$ for $1 \leq t \leq 3$. By construction, $v_{p_t} \in w_j$, $1 \leq t \leq 3$, and $\{v_{p_1}, v_{p_2}, v_{p_3}\} \not\subset Inc(Q)$ for any entry of the diagram $Q$ different from $P$. Then the four points $v_p$, $v_{p_1}$, $v_{p_2}$ and $v_{p_3}$ satisfy the assumptions of Lemma 1.57. But then by Lemma 1.57 they cannot all be contained in the plane $w_j$, which is a contradiction. $\square$

**Remark 1.59.** For a subset $M$ of the entries of the $4 \times 4$ diagram, let the **span** of $M$ be the set of all rows and all columns of the matrix which contain at least one element of $M$. It is easy to see from definitions that a four element subset $M$ of $\{v_j\}_{1 \le j \le 10}$ does not contain three strongly coplanar points if and only if the span of $M$ is either the set of two rows and two columns of (*) or the entire matrix (*). For example, no three of the points $v_4$, $v_6$, $v_7$, $v_{10}$ are strongly coplanar, and hence all four are allowed to be coplanar by Lemma 1.57. In a (16,6) configuration of (1.4.1), these four points will be coplanar if and only if $(a^2 - d^2)(b^2 - c^2) = 0$. Note that $(a^2 - d^2)(b^2 - c^2) = 0$ is not ruled out by the conditions (1.2.1). In general, in a (16,6) configuration (1.4.1), the conditions that four points are coplanar if and only if they are strongly coplanar, are

$$\begin{aligned} &(1') \ abcd \neq 0 \qquad \text{and} \\ &(2') \ \text{all of the } \pm a^2, \pm b^2, \pm c^2, \pm d^2 \quad \text{are distinct.} \end{aligned} \tag{1.59.1}$$

The Kummer surfaces $S$ whose singular loci do not satisfy (1.59.1) have the rank $\rho(S)$ of Néron-Severi group equal to 2 (for a generic Kummer surface, $\rho(S) = 1$) [7]. By identifying the moduli space of Kummer surfaces with $\mathcal{M}_2$, the locus of Kummer surfaces which do not satisfy (1.59.1) corresponds to the locus of bielliptic curves in $\mathcal{M}_2$ [7].

*Proof of Lemma 1.57.* Consider four points $\{v_{j_l}\}_{1 \le l \le 4}$ and assume that the first three of them are strongly coplanar.

*Case 1.* There exists $p$, $1 \le p \le 6$, such that $v_{j_l} \in Inc(w_p)$ for $1 \le l \le 3$. Then $v_{j_4} = w_{k_1} \cap w_{k_2} \cap w_{k_3}$, where $1 \le k_1, k_2, k_3 \le 6$ and $p \notin \{k_1, k_2, k_3\}$. Since the planes $w_j$, $1 \le j \le 6$ are in general position, $v_{j_4} \notin w_p$, as desired.

*Case 2.* No three of the $v_{j_l}$, $1 \le l \le 4$, are contained in a plane $w_p$ with $1 \le p \le 6$. Associated with each $v_{j_l}$ we have a three element set $M_l = \{k_{1l}, k_{2l}, k_{3l}\} \subset \{1, \dots, 6\}$, defined by

$$v_{j_l} = w_{k_{1l}} \cap w_{k_{2l}} \cap w_{k_{3l}}.$$

We are assuming that for any $j$, $1 \le j \le 4$,

$$\bigcap_{\substack{1 \le l \le 4 \\ l \neq j}} M_l = \emptyset. \tag{1.59.2}$$

By (1.63.2), $\cup_{j=1}^4 M_j = \{1, \dots, 6\}$. Then (1.59.2) implies that, up to renumbering, we have only two possibilities:

(a)

$$\#(M_j \cap M_l) = 1 \qquad \text{for any } j, l, \ 1 \le j < l \le 4. \tag{1.59.3}$$

(b)

$$\begin{aligned} \#(M_1 \cap M_2) &= \#(M_3 \cap M_4) = 2 \\ \#(M_1 \cap M_3) &= \#(M_2 \cap M_4) = 1 \\ \#(M_1 \cap M_4) &= \#(M_2 \cap M_3) = 0. \end{aligned} \tag{1.59.4}$$

*Case 2(a).* Without loss of generality, assume that $k_{11} = k_{12}$, $k_{22} = k_{23}$ and $k_{33} = k_{31}$. First of all, we claim that $j_l \neq 4$ for $1 \leq l \leq 4$ (cf. (1.55.5)). Indeed, suppose $j_1 = 4$. Then by (1.59.3) we have $\{1,2,3\} \cap \{j_l | 1 \leq l \leq 4\} = \emptyset$. Also by (1.59.3), every $p$, $1 \leq p \leq 6$, is contained in exactly two of the sets $M_l$, $1 \leq l \leq 4$. In particular, letting $p$ range from 1 to 3, we get that exactly one point $v_{j_l}$, $2 \leq l \leq 4$, is contained in each of the first three columns of (1.55.5) (and all of the $v_{j_l}$, $2 \leq l \leq 4$ are in the last three rows of (1.55.5). If the span of $\{v_{j_l}\}_{2 \leq l \leq 4}$ contains the last three rows of (1.55.5), then the span of $\{v_{j_l}\}_{1 \leq l \leq 4}$ is the whole matrix (1.55.5) and so no three of $\{v_{j_l}\}_{1 \leq l \leq 4}$ are strongly coplanar, which is a contradiction. Suppose the $p$-th row of (1.55.5) is not in the span of $\{v_{j_l}\}_{2 \leq l \leq 4}$ for some $p$, $2 \leq p \leq 4$. But then $p+2$ belongs to only one of the sets $M_l$, $1 \leq l \leq 4$, which contradicts (1.59.3).

Thus $j_l \neq 4$ for any $l$, $1 \leq l \leq 4$. Next we show that the first three columns of (1.55.5) are in the span of $\{v_{j_l}\}_{1 \leq l \leq 4}$. Indeed, suppose the $p$-th column of (1.55.5) is not in the span of $\{v_{j_l}\}_{1 \leq l \leq 4}$ for some $p$, $1 \leq p \leq 3$. We have exactly two planes $w_{q_1}$ and $w_{q_2}$ with $1 \leq q_1, q_2 \leq 6$ in the $p$-th column. Say $w_{q_1}$ and $w_{q_2}$ lie in the rows numbered $q_1'$ and $q_2'$, $1 \leq q_1', q_2' \leq 4$. Denote $\{1,2,3\} \setminus \{p\} := \{p_1, p_2\}$. Since each of $q_1$, $q_2$ must belong to exactly two of the sets $M_l$ by (1.59.3) and since there are no points $v_{j_l}$, $1 \leq l \leq 4$, in the $p$-th and the 4-th columns, the span of $\{v_{j_l}\}_{1 \leq l \leq 4}$ must consist of the columns $p_1$ and $p_2$ and rows $q_1'$ and $q_2'$. Hence no three of $\{v_{j_l}\}_{1 \leq l \leq 4}$ are strongly coplanar which is a contradiction. This proves that the span of $\{v_{j_l}\}_{1 \leq l \leq 4}$ contains the first three columns of (1.55.1) (but not the fourth column).

Write $\{1,2,3\} = \{p, p_1, p_2\}$, where two of the points $\{v_{j_l}\}_{1 \leq l \leq 4}$ lie in the $p$-th column and exactly one point $v_{j_l}$ lies in each of the columns $p_1$ and $p_2$. Permuting the first three rows and the last three columns in (1.55.1) we may assume that $p = 1$. Then by (1.59.3), $5, 6 \notin \{j_1, j_2, j_3, j_4\}$. Then for some $q \in \{1,3,4\}$, the $q$-th row of (1.55.1) contains at least two of the points $\{v_{j_l}\}_{1 \leq l \leq 4}$, say in columns $p_1'$ and $p_2'$. Let $p_3'$ denote the unique element in $\{1,2,3\} \setminus \{p_1', p_2'\}$. Then there is a plane $w_t$ in the $p_3'$-th column and the $q$-th row of (1.55.1), $1 \leq t \leq 6$. By (1.59.3) $t$ belongs to exactly two of the sets $M_l$, $1 \leq l \leq 4$. Since we already have two of the points $\{v_{j_l}\}_{1 \leq l \leq 4}$ in the $q$-th row, we may have none in the $p_3'$-th column, except, possibly the point in the $p_3'$th row and the $q$-th column. But then we have three of the points $\{v_{j_l}\}_{1 \leq l \leq 4}$ in the $q$-th row. By (1.55.1) we get that $q = 1$ and the three points are $v_1$, $v_2$, $v_3$. But then the span of $\{v_{j_l}\}_{1 \leq l \leq 4}$ cannot contain both the 3rd and the 4th rows of (1.55.1), hence either 5 or 6 belongs only to one of the sets $M_l$, $1 \leq l \leq 4$, which is a contradiction. Hence, Case 2(a) cannot occur.

*Case 2(b)* This case is also impossible. Indeed, one checks easily that it is impossible to split the set $\{1,2,3,4,5,6\}$ into two three element sets $M_1$ and $M_4$ in such a way that $\cap_{p \in M_l} Inc(w_p) \neq \emptyset$ for $l = 1, 4$. This completes the proof of Lemma 1.57. □

Now, for any point $\mu \in U \times PGL_4(k)$ we have a canonically defined action of the group $\mathbb{F}_2^4$ on the (16,6) configuration corresponding to $\mu$. Fix a coordinate system on $\mathbb{P}^3$. Viewing linear transformations of $\mathbb{P}^3$ as elements of $\mathbb{P}^{15}$, we therefore get a map

$$\lambda : PGL_4(k) \times \mathbb{F}_2^4 \to \mathbb{P}^{15}.$$

Explicitly, $\lambda$ is given by

$$\lambda(M, \sigma) = M\sigma M^{-1},$$

where we view the righthand side as a point in $\mathbb{P}^{15}$ via our fixed coordinate system. Let $D$ be any curve in $V$ passing through $\omega$, such that the generic point of $D$ is contained in $Im(\phi)$. If we could prove that the action of $\mathbb{F}_2^4$ as the automorphism group of a generic (16,6) configuration in $D$ extends by continuity to automorphisms of the set $\{w_j\}_{1 \leq j \leq 16}$, the proof of Proposition 1.54 would be complete by Lemma 1.52; the incidence relations (*) would follow by continuity and the non-degeneracy conditions (1.2.1) by Lemma 1.47.

We introduce the following notation. For a point $\omega' \in D \cap Im(\phi)$ and an element $\sigma \in F_0$, we write $\phi^{-1}(\omega')(\sigma)$ to denote $M\sigma M^{-1}$, where $M \in PGL_4(k)$ is defined by

$$\phi((a, b, c, d), M) = \omega'. \tag{1.59.5}$$

Note that while (1.59.5) does not determine $M$ uniquely ($\phi$ is a 16 to 1 map), the expression $M\sigma M^{-1}$ is unambiguous, since all the possible choices for $M$ differ from each other by an element of $F_0$. Fix $\sigma$. Then

$$\omega' \to \phi^{-1}(\omega')(\sigma)$$

defines a map from $D \setminus \omega$ to $\mathbb{P}^{15}$. By the properness of $\mathbb{P}^{15}$, the map $\phi^{-1}(\cdot)(\sigma)$ extends to all of $D$. It remains to prove that $\phi^{-1}(\omega)(\sigma)$, viewed as a $4 \times 4$ matrix modulo multiplication by constant, is non-singular for every $\sigma \in F_0$. Again, the group relations such as $\sigma^2 = 1$ and the commutativity relations will follow from those at the generic point of $D$ by continuity.

We will now deduce non-singularity of $\phi^{-1}(\omega)(\sigma)$ from Lemma 1.57. Let $\omega'$ denote the generic point of $D$. Then the linear transformation $\phi^{-1}(\omega')(\sigma)$ maps the six planes given by $\omega'$ to another set of six planes of the (16,6) configuration corresponding to $\omega'$ (recall that $\omega' \in Im(\phi)$ by assumption). Similarly, the 10 points of the diagram (1.48.1) are moved to another set of ten points of the same (16,6) configuration. It is easy to see from Note 1.5 that the group of transformations of the diagram (1.48.1) induced by $F_0$ is the group generated by permutations of pairs of columns and permutations of pairs of rows. Clearly the new ten points must have at least 4 points in common with the old ten points.

**Lemma 1.60.** *The image of the ten points of (1.48.1) under $\phi^{-1}(\omega')(\sigma)$ intersected with the original ten points contains four points which are not strongly coplanar but three of which are strongly coplanar.*

*Proof.* Let $K$ denote the set $\{$old 10 points$\} \cap \{$new 10 points$\}$. If $\phi^{-1}(\omega')(\sigma)$ fixes the first row of (1.48.1) then the four points in the first row satisfy the conclusion of the Lemma and there is nothing to prove. Hence we may assume that $\phi^{-1}(\omega')(\sigma)$ induces a non-trivial permutation of the rows of (1.5.1). Then, $K$ contains at least two points in the first row of (1.5.1) and at least two points in the image of the first row under $\phi^{-1}(\omega')(\sigma)$. If $\phi^{-1}(\omega')(\sigma)$ induces a non-trivial permutation of the columns then these four points form a parallelogram but not a rectangle, hence they again satisfy the conclusion of the Lemma. Finally, if $\phi^{-1}(\omega')(\sigma)$ permutes rows but not columns, $K$ contains 4 points forming a rectangle in the first row and its image plus at least one more point, hence the Lemma is true in this case, too. $\square$

Consider $Im(\phi^{-1}(\omega')(\sigma))$ in $\mathbb{P}^3$. Since $Im(\phi^{-1}(\omega')(\sigma))$ in $\mathbb{P}^3$ contains the above four points, $Im(\phi^{-1}(\omega)(\sigma))$ contains the corresponding four points in the diagram (1.48.1) by continuity. By Lemma 1.57 these four points do not lie in a plane, which proves that $\phi^{-1}(\omega)(\sigma)$ is non-degenerate. This completes the proof of Proposition 1.54. $\square$

To complete the classification of the (16,6) configurations of type (*) it remains to see which choices of $(a,b,c,d) \in U$ lead to projectively isomorphic (16,6) configurations. Clearly, any permutation of the coordinates and changing the sign of any coordinate $(a,b,c,d)$ will give isomorphic configuration. In other words, let $W \cong S_4 \ltimes \mathbb{F}_2^3$ denote the subgroup of $PGL_4(k)$ generated by all the changes of signs of the coordinates and all the permutations of the coordinates. Then any point in the orbit of $(a,b,c,d)$ under $W$ gives an isomorphic (16,6) configuration. Note that $W \subset N$, where $N = N(F_0)$ denotes the normalizer of $F_0$ in $PGL_4(k)$.

**Remark 1.61.** Let $N := N(F_0)$ be as above. Then $N$ preserves $U$ (since the set $U$ can be defined purely in terms of $F_0$). Two points $(a,b,c,d), (a',b',c',d') \in U$ give rise to projectively isomorphic (16,6) configurations if they belong to the same orbit of $N$. Indeed, let $M \in N$ be a transformation such that $M(a,b,c,d) = (a',b',c',d')$. Then, since $M \in N$, the orbit of $(a,b,c,d)$ under $F_0$ maps to the orbit of $(a',b',c',d')$ under $F_0$. Hence $M$ gives an isomorphism of the two (16,6) configurations. To complete our classification of (16,6) configurations up to automorphisms of $\mathbb{P}^3$, it remains to prove the converse: if two configurations of the form $(a,b,c,d)$ as in (1.4.1) are projectively isomorphic, they are mapped into each other by an element of $N$.

To prove this, we first determine $N$. We have a natural action of $N$ on $F_0$ by conjugation. View $F_0$ as a 4-dimensional vector space over $\mathbb{F}_2$ on which $N$ acts by linear automorphisms. By Lemma 1.50, we have an injection

$$\frac{N}{F_0} \hookrightarrow GL_4(\mathbb{F}_2) \tag{1.61.1}$$

(note that $GL_4(\mathbb{F}_2) = PGL_4(\mathbb{F}_2) = SL_4(\mathbb{F}_2)$). The point of the next theorem is to show that $\frac{N}{F_0}$ can be identified with $Sp_4(\mathbb{F}_2)$ for a suitable symplectic structure on $F_0 \cong \mathbb{F}_2^4$.

Next we describe the symplectic structure on $F_0$ which is preserved by $N$. Namely, consider any four element subgroup $H$ of $F_0$. We may think of such a subgroup as a 2-plane in our 4-dimensional vector space $\mathbb{F}_2^4$.

**Definition 1.62.** *$H$ is an* **isotropic plane** *if the elements of $H$, lifted to $SL_4(k)$, pairwise commute with each other (rather than anticommute).*

Note that this property is well-defined (i.e. independent of the lifting to $SL_4(k)$) and is preserved by conjugation.

**Lemma 1.63.** *The above definition of isotropic planes gives a symplectic structure on $F_0 \equiv \mathbb{F}_2^4$. Namely, if we let $e_1$, $e_2$, $e_3$, $e_4$ of (1.50.1) be the four coordinate vectors on $\mathbb{F}_2^4$, our symplectic form is given by $f(x,y) = x_1y_4 + x_2y_3 + x_4y_1 + x_3y_2$.*

(We alternate between the additive and multiplicative notation for the group law on $F_0$ and hope to make this as little confusing as possible. When we write the group law additively, we are thinking of $F_0$ as $\mathbb{F}_2^4$, and when we write it multiplicatively, we think of $F_0$ as a subgroup of $PGL_4(k)$.)

*Proof of Lemma 1.63.* First of all, if there exists a symplectic form $f$ whose isotropic planes are as defined above, we must have

$$\begin{aligned} &f(e_1,e_2) = f(e_1,e_3) = f(e_2,e_4) = f(e_3,e_4) = 0\\ &f(e_1,e_4) = f(e_2,e_3) = 1. \end{aligned}$$

Thus if such a symplectic form $f$ exists, it must be given by the formula of the Lemma. Next, take any two elements of $F_0$:

$$x = \prod_{j=1}^{4} e_j^{x_j} \quad \text{and}$$
$$y = \prod_{j=1}^{4} e_j^{y_j},$$

where we may think of the $x_j$ and $y_j$ as elements of $\mathbb{F}_2$. Note a slight abuse of notation: we are not very careful about distinguishing between the elements $e_j \in F_0$ and their representatives in $SL_4(k)$. Since the commutativity properties we are investigating are independent of the choice of liftings to $SL_4(k)$, this should cause no confusion. We let the symplectic form $f$ be defined as in the Lemma. Let $\tilde{f} : \mathbb{F}_2^4 \times \mathbb{F}_2^4 \to \mathbb{F}_2$ be defined by

$$\tilde{f}(x,y) = \begin{cases} 0 & \text{if } x \text{ and } y \text{ commute in } SL_4(k)\\ 1 & \text{otherwise.} \end{cases}$$

We wish to prove that $\tilde{f} \equiv f$. It is easy to see that $\tilde{f}$ is bilinear in $x$ and $y$. We have

$$f(e_j, e_l) = \tilde{f}(e_j, e_l) \tag{1.63.1}$$

for $1 \le j, l \le 4$. Since both sides of (1.63.1) are bilinear in $x$ and $y$ and they agree on coordinate vectors, we get $f = \tilde{f}$, as desired. Lemma 1.63 is proved. □

**Theorem 1.64.**

(1) *$N$ is the subgroup of $PGL_4(k)$ generated by the matrices*

$$\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & i & 0\\ 0 & 0 & 0 & i \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 1 & -i & i\\ -1 & -1 & i & i\\ i & -i & 1 & -1\\ i & i & -1 & -1 \end{pmatrix}.$$

(2) $\#N = 2^8 \cdot 3^2 \cdot 5 = 11520$. *$N$ is a non-split extension of $S_6$ by $\mathbb{F}_2^4$. That is, we have a non-split exact sequence*

$$1 \to F_0 \to N \to S_6 \to 1, \tag{1.64.1}$$

*In particular, $N$ is* **not** *a semi-direct product of $\mathbb{F}_2^4$ with $S_6$.*

(3) *The action of $S_6$ on $F_0$ by conjugation induced by (1.64.1) is given by identifying $S_6 \cong Sp_4(2)$, acting on the 4-dimensional vector space $\mathbb{F}_2^4$.*

**Remark 1.65.** It is well-known that $Sp_4(\mathbb{F}_2) \cong S_6$ [4, p. [4], lines 5, 22]. To visualize this identification, consider the set $K = \{1, 2, 3, 4, 5, 6\}$ of 6 elements. The 15 pairs $(a, b)$, $a, b \in K$ can be identified with the 15 points of the 3-dimensional projective space $\mathbb{P}^3_{\mathbb{F}_2}$. This projective space has 15 planes and 35 lines. Each line contains exactly 3 points. To give a non-degenerate symplectic form on $\mathbb{F}_2^4$ is equivalent to fixing an isomorphism $\mathbb{P}^3_{\mathbb{F}_2} \to (\mathbb{P}^3_{\mathbb{F}_2})^*$ such that the image of each point in $\mathbb{P}^3_{\mathbb{F}_2}$ is a plane which contains this point. This is also equivalent to specifying all the isotropic planes in $\mathbb{F}_2^4$ or, equivalently, their projectifications in $\mathbb{P}^3_{\mathbb{F}_2}$, which we call isotropic lines. To identify the set of 15 pairs $(a, b)$, $a, b \in K$, with the 15 points of the projective space $\mathbb{P}^3_{\mathbb{F}_2}$ equipped with a symplectic structure in the above sense, we need to specify the sets of isotropic and non-isotropic lines in $\mathbb{P}^3_{\mathbb{F}_2}$. Specifying a line is equivalent to specifying its three points. The isotropic lines are defined to be the **synthemes**—the lines of the form $\{(a, b), (c, d), (e, f)\}$, with all the $a, b, c, d, e, f$ distinct. The non-isotropic lines are of the form $\{(a, b), (b, c), (a, c)\}$. There are 15 isotropic lines and 20 non-isotropic ones.

*Proof of Theorem 1.64.* $\frac{N}{F_0}$ acts by conjugation on $F_0 \cong \mathbb{F}_2^4$. If two elements in $F_0$ have the property that their pullbacks to $SL_4(k)$ commute then they still commute after conjugation by an element of $N$. Hence, conjugation by elements of $N$ preserves the symplectic structure on $\mathbb{F}_2^4$. Therefore the injection (1.61.1) induces an injection

$$\psi : \frac{N}{F_0} \hookrightarrow Sp_4(\mathbb{F}_2). \tag{1.65.1}$$

(3) of Theorem 1.64 asserts that $\psi$ is an isomorphism.

To show that $\psi$ is an isomorphism, it is sufficient to show that the groups on the left and the righthandside have the same order, that is, $\#N = 2^8 \cdot 3^2 \cdot 5$. For that it is sufficient to show two things:

(a) The stabilizer $I$ in $N$ of an isotropic line in $\mathbb{P}^3_{\mathbb{F}_2}$ has order at least $2^8 \cdot 3$ (in fact, Theorem 1.64 will imply that $\frac{I}{F_0} \cong S_4 \times \mathbb{F}_2$ [4, p. [4], line 44]).

(b) $N$ acts transitively on the set of isotropic lines.

Since there are 15 isotropic lines, (a) and (b) together will show that $2^4 \cdot 3 \cdot 15 \leq \#\frac{N}{F_0}$. Since we already know that $\#\frac{N}{F_0} | \#S_6$, we will obtain that $\#\frac{N}{F_0} = \#S_6$ and $\psi$ is an isomorphism.

To prove (a), it is sufficient to prove that $\#\frac{I}{F_0} \geq 2^4 \cdot 3$. Now, we consider a specific isotropic line $L$: the one corresponding to the subgroup of $F_0$ generated by

$e_1$ and $e_2$ of (1.50.1). Then $I$ contains the subgroup $W \cong S_4 \ltimes \mathbb{F}_2^4$, which is the subgroup generated by all the permutations of coordinates and the matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The injection $F_0 \hookrightarrow W$ breaks up into $\mathbb{F}_2^2 \hookrightarrow S_4$ and $\mathbb{F}_2^2 \hookrightarrow \mathbb{F}_2^3$. If we identify the subgroup of $S_4$ generated by $e_3$ and $e_4$ with $\mathbb{F}_2^2$, we have

$$\frac{S_4}{\mathbb{F}_2^2} \cong S_3.$$

Hence $\frac{W}{F_0} \cong S_3 \times \mathbb{F}_2$ and $\#\frac{W}{F_0} = 12$. To prove (a), it remains to exhibit an element $\sigma$ of $I$ which has order at least 4 modulo $W$ (by this we mean that $\sigma^n \notin W$ for $n < 4$). We may take

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & i & 0 \end{pmatrix}. \tag{1.65.2}$$

One can easily check directly that $\sigma \in I$ and that the order of $\sigma$ modulo $W$ is 4 (we have $\sigma^4 = 1$). This proves (a).

**Remark 1.66.** If the reader is curious as to how one finds such a matrix, we observe that every element of $F_0$ pointwise fixes two lines in $\mathbb{P}_k^3$ and is uniquely determined by these two lines. We thus have 16 pairs of lines in $\mathbb{P}_k^3$, which we write down explicitly (we specify each line by giving a pair of points belonging to this line):

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} : \quad ((1,0,0,1),(0,1,1,0)) \text{ and } ((1,0,0,-1),(0,1,-1,0))$$

$$B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} : \quad ((1,0,1,0),(0,1,0,1)) \text{ and } ((1,0,-1,0),(0,1,0,-1))$$

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} : \quad ((1,1,0,0),(0,0,1,1)) \text{ and } ((1,-1,0,0),(0,0,1,-1))$$

$$A' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : \quad ((1,0,0,0),(0,0,0,1)) \text{ and } ((0,1,0,0),(0,0,1,0))$$

$$B' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} : \quad ((1,0,0,0),(0,0,1,0)) \text{ and } ((0,1,0,0),(0,0,0,1))$$

$$C' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} : \quad ((1,0,0,0),(0,1,0,0)) \text{ and } ((0,0,0,1),(0,0,1,0))$$

$$AA' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} : \quad ((1,0,0,1),(0,1,-1,0)) \text{ and } ((1,0,0,-1),(0,1,1,0))$$

$$BB' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} : \quad ((1,0,1,0),(0,1,0,-1)) \text{ and } ((1,0,-1,0),(0,1,0,1))$$

$$CC' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} : \quad ((1,1,0,0),(0,0,1,-1)) \text{ and } ((0,0,1,1),(1,-1,0,0))$$

$$AB' = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} : \quad ((1,0,0,i),(0,1,-i,0)) \text{ and } ((1,0,0,-i),(0,1,i,0))$$

$$BC' = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} : \quad ((1,0,i,0),(0,1,0,i)) \text{ and } ((1,0,-i,0),(0,1,0,-i))$$

$$CA' = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} : \quad ((1,i,0,0),(0,0,1,-i)) \text{ and } ((1,-i,0,0),(0,0,1,i))$$

$$AC' = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} : \quad ((1,0,0,i),(0,1,i,0)) \text{ and } ((1,0,0,-i),(0,1,-i,0))$$

$$BA' = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} : \quad ((1,0,i,0),(0,1,0,-i)) \text{ and } ((1,0,-i,0),(0,1,0,i))$$

$$CB' = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} : \quad ((1,i,0,0),(0,0,1,i)) \text{ and } ((1,-i,0,0),(0,0,1,-i)).$$

An element $\alpha \in PGL_4(k)$ lies in $N$ if and only if it maps this configuration of 16 pairs of lines into itself. This approach allows to write down explicit elements of $N$. Since we already conjecture that $\frac{N}{F_0} \cong S_6$ with the identifications of Remark 1.65, we may look for specific types of permutations: transpositions, 4-cycles, etc. In this case we first wrote down the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \in I,$$

which is a triple transposition modulo $F_0$ (i.e. something of the form (12)(34)(56)), then modified it by an element of $W \cong S_4 \ltimes \mathbb{F}_2^3 \cong (S_3 \times \mathbb{F}_2) \ltimes F_0$ to make it into a 4-cycle.

We now claim that (b) is a consequence of the proof of Lemma 1.52. Indeed, let $L'$ be any other isotropic line in $\mathbb{P}^3_{\mathbb{F}_2}$. Let $e'_1$ and $e'_2$ be the generators of the 4 element subgroup corresponding to $L'$ and complete $(e'_1, e'_2)$ in an arbitrary way to a set of generators $(e'_1, e'_2, e'_3, e'_4)$ of $F_0$. We can now apply the proof of Lemma 1.52 to show that there exists $\alpha \in PGL_4(k)$ such that

$$\begin{aligned} &\alpha^{-1} e'_1 \alpha = e_1, \\ &\alpha^{-1} e'_2 \alpha = e_2 \qquad \text{and} \\ &\alpha^{-1} e'_j \alpha \in F_0 \qquad \text{for } j = 3, 4. \end{aligned} \tag{1.66.1}$$

Then $\alpha \in N$ and $\alpha$ moves $L$ to $L'$. In other words, in the proof of Lemma 1.52 we started out with four arbitrary matrices whose squares were constant multiples of the identity and which pairwise commuted in $PGL_4(k)$. We showed that they can be moved by conjugation to $e_1, e_2, \tilde{e}_3, \tilde{e}_4$, where $\tilde{e}_j \in F_0$, provided the first two of them commute in $SL_4(k)$. In particular, we may take these four matrices to be $e'_j$, $1 \le j \le 4$ to get (1.66.1). This completes the proof of (b) and hence of (3) of Theorem 1.64.

Since $\psi$ of (1.65.1) is an isomorphism, we get the exact sequence (1.64.1). To prove (2), it remains to prove that the sequence (1.64.1) does not split, i.e. there is no section $S_6 \to N$. Let $L$ be the isotropic line corresponding to $(e_1, e_2)$, as above, and $I$ the stabilizer of $L$. By (3) we now know that $\frac{I}{F_0} \cong S_4 \times \mathbb{F}_2$ [4, p. [4], line 44]. That is, (1.64.1) restricts to the exact sequence

$$1 \to F_0 \to I \to S_4 \times \mathbb{F}_2 \to 1. \tag{1.66.2}$$

To show that (1.64.1) does not split, it is enough to show that (1.66.2) does not split. On the other hand, we have $F_0 \subset W \subset I$ and the induced exact sequence

$$1 \to F_0 \to W \xrightarrow{\pi} S_3 \times \mathbb{F}_2 \to 1. \tag{1.66.3}$$

does split (for instance, we may send the generator of $1 \times \mathbb{F}_2$ to

$$\sigma_1 := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the elements of $S_3 \times 1$ to permutations on the last three variables). Here and below the identity element in any group will be denoted by 1, even when the group is $\mathbb{F}_2$. Let $t_1 := \pi(\sigma_1)$. Suppose there was a section $s : S_4 \times \mathbb{F}_2 \to I$. Then $s(t_1)$ would be an element of $W$ which commutes (in $PGL_4(k)$) with both $e_1$ and $e_2$. The only elements of $W$ with this property are $S_4 \times 1$ conjugates of $\sigma_1$. Conjugating the whole situation by an element of $S_4 \times 1$, if necessary, we may assume that $s(t_1) = \sigma_1$. Then $\sigma_1$ must commute (in $PGL_4(k)$) with all of $s(S_4 \times \mathbb{F}_2)$. Hence all the elements of $s(S_4 \times \mathbb{F}_2)$ are of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}. \tag{1.66.4}$$

Now, by the proof of (3) we have $\#I = 2^8 \cdot 3$ and $[N : I] = 15$. Hence the element $\sigma$ of (1.65.2) generates $I$ over $W$. This shows that $I$ is the group consisting of all the matrices $\beta$ of the following form. $\beta$ has exactly one non-zero entry in each row and in each column. The non-zero entries belong to the set $\{1, -1, i, -i\}$ and an even number of the non-zero entries belongs to the set $\{1, -1\}$. $W$ is the subgroup of all those matrices all of whose entries belong to $\{1, -1\}$. We have an obvious lifting of $\pi$ of (1.66.3) to $W$, compatible with $s$: just let $S_3 \times 1$ act by permutation of the last three variables in $\mathbb{P}^3_k$. $s$ can only differ from this standard lifting by an element of $F_0$, hence an element of $s(S_3 \times \mathbb{F}_2)$ cannot have any entries equal to $i$ or $-i$. Since all the elements of $s(S_3 \times \mathbb{F}_2)$ must have form (1.66.4), the subgroup $s(S_3 \times \mathbb{F}_2)$ is the same as the standard one modulo the subgroup $(e_1, e_2, \sigma_1)$. On the other hand, $s(S_4 \times \mathbb{F}_2)$ cannot be contained in $W$, so there exists an element $\alpha$ of $s(S_4 \times \mathbb{F}_2)$ two of whose entries are plus or minus $i$. Multiplying $\alpha$ by an element of $s(S_3 \times \mathbb{F}_2)$, we may assume that $\alpha$ is a diagonal matrix, two of whose entries are plus or minus 1, and two others plus or minus $i$. But then $\alpha^2 \in F_0$, and $\alpha^2 \neq 1$, which is a contradiction. This proves (2) of Theorem 1.64.

Finally, to give a minimal set of generators for $N$ we use the exact sequence (1.64.1) and the fact that it does not split. It is well known (and easy to prove) that $S_6$ is generated by any pair of the form (simple transposition, 6-cycle), provided the simple transposition does not commute with the cube of the 6-cycle. We check that the images of the two matrices of (1) in $S_6$ are a simple transposition and a 6-cycle, using the identification of Remark 1.65. The first matrix has order 2 and fixes a plane in $\mathbb{P}^3_{\mathbb{F}_2}$ (namely the 8 element subgroup $(e_1, e_2, e_3)$) pointwise. This means it is a simple transposition (the simple transposition (12) fixes the point (12) and all the points of the form $(j, l)$, where $3 \leq j, l \leq 6$. The other elements of $S_6$ of order 2 are of the type (12)(34) and (12)(34)(56), and none of them fixes 7 points in $\mathbb{P}^3_{\mathbb{F}_2}$). For the second matrix, one checks directly that the orbit of $e_1$ under it in $\mathbb{P}^3_{\mathbb{F}_2}$ has 6 elements and that it does not fix any points in $\mathbb{P}^3_{\mathbb{F}_2}$. The only other elements of $S_6$ which have order 6 are of the form (12)(345) and they always fix a point ((12) in this example) in $\mathbb{P}^3_{\mathbb{F}_2}$. Finally, to say that the transposition does not commute with the cube of the 6-cycle is the same as saying that the orbit of the transposition under conjugation by the powers of the 6-cycle has six elements (rather than three). Let (12) denote $e_1$ viewed as a point in $\mathbb{P}^3_{\mathbb{F}_2}$, under the identification of Remark 1.69. Then

$$\alpha_1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$$

pointwise fixes all the isotropic lines passing through $e_1$. Hence $\alpha_1$ is the transposition (12). Since the orbit of $e_1$ under our 6-cycle consists of 6 elements and since the action of $S_6$ on the set of transpositions by conjugation is compatible with its action on $\mathbb{P}^3_{\mathbb{F}_2}$, our proof is complete. □

**Remark 1.67.** Although the sequence (1.64.1) does not split, $N$ contains a subgroup $\tilde{N}$ of index 2, which contains $F_0$ and is isomorphic to $A_6 \ltimes F_0$. There is a beautiful conceptual explanation of this fact, which we got from a paper of Mukai [15, p.190]. Let $V$ denote the 4-dimensional vector space over $k$ with a basis $x, y, z, t$. Consider the basis for $\bigwedge^2 V$ given by

$$\begin{aligned} w_1 &= x \wedge y + z \wedge t & w_4 &= -i(x \wedge y - z \wedge t) \\ w_2 &= x \wedge z + t \wedge y & w_5 &= -i(x \wedge z - t \wedge y) \\ w_3 &= x \wedge t + y \wedge z & w_6 &= -i(x \wedge t - y \wedge z) \end{aligned}$$

We have a natural homomorphism $\lambda_1 : PGL_4(k) \hookrightarrow PGL_6(k)$, since a projective linear transformation of $V$ induces a transformation of $\bigwedge^2 V$. The image of $F_0$ in $PGL_6(k)$ consists of the identity and the fifteen involutions which change sign of two of the $w_i$'s. The group $\lambda_1(N)$ acts by permuting the $w_i$'s and changing the sign of some of the $w_i$'s. Let $\tilde{N}$ denote the commutator subgroup of $N$. Since $Sp_4(\mathbb{F}_2)$ acts transitively on $\mathbb{F}_2^4$, we have $F_0 \subset \tilde{N}$ and $\frac{\tilde{N}}{F_0} = A_6$. In particular, $[N : \tilde{N}] = 2$. Moreover, $\lambda_1(\tilde{N})$ is contained in the subgroup

$$A_6 \ltimes \mathbb{F}_2^4 \subset PGL_6(k), \tag{1.67.1}$$

generated by all the even permutations of coordinates and by the changes of signs of pairs of coordinates. Since $\tilde{N}$ and $A_6 \ltimes \mathbb{F}_2^4$ have the same order, they must be isomorphic. From this information one can calculate explicit generators of $A_6 \subset PGL_4(k)$, but we shall not do it here.

**Theorem 1.68.** *Two points $(a,b,c,d),(a',b',c',d') \in U$ give rise to isomorphic (16,6) configurations if and only if they belong to the same orbit of $N$.*

*Proof.* "If" is trivial. We prove "only if". Let $T$ denote the isomorphism carrying one (16,6) configuration into the other. Then the group $TF_0T^{-1}$ acts on the (16,6) configuration $(a',b',c',d')$ as a group of automorphisms, transitively on the set of points and on the set of planes.

*Claim.* A linear automorphism of a (16,6) configuration $(a,b,c,d)$ normalizes $F_0$.

*Proof of Claim.* Let $\delta_j$, $1 \leq j \leq 16$ denote the elements of $F_0$ and let $\alpha$ denote an automorphism of the (16,6) configuration $(a,b,c,d)$. We identify elements of $F_0$ with points of the (16,6) configuration, as in (1.4.1). Namely, we identify $\delta_i$ with the point $\delta_i(a,b,c,d)$. Multiplying $\alpha$ by an element of $F_0$, if necessary, we may assume that $\alpha$ fixes the point $(a,b,c,d) \in \mathbb{P}^3_k$. Consider the group $F_0' := \alpha^{-1}F_0\alpha \cong \mathbb{F}_2^4$. Then the orbit of $(a,b,c,d)$ under $F_0'$ is the same as under $F_0$. Let us denote the point $(a,b,c,d)$ by $v_1$. Let $v_j := \delta_j v_1$, $1 \leq j \leq 16$. Let $\delta_j' := \alpha^{-1}\delta_j\alpha$. It is sufficient to show that for each $j$, $1 \leq j \leq 16$, $\delta_j' = \delta_l$, where $l$ is the index defined by $\delta_j' v_1 = v_l$. For $1 \leq j \leq 16$, let $l(j)$ be the index defined by $\delta_j' v_1 = \delta_{l(j)} v_1$. If we show that $\delta_j' = \delta_{l(j)}$, the Claim will be proved; to prove it we need some sublemmas.

**Sublemma 1.69.** *The bijection $\delta_j \to \delta_{l(j)}$ is an automorphism of $F_0$.*

*Proof of Sublemma.* Since $\alpha$ is an automorphism of the (16,6) configuration, it must preserve the incidence relations of Lemma 1.7. Now we can reinterpret these incidence relations viewing $F_0$ as a 4-dimensional vector space over $\mathbb{F}_2$. Namely, we have a preferred pair of non-isotropic planes (the planes $(1, AB', BC', CA')$ and $(1, AC', BA', CB')$, in the notation of (1.5.1). The identification (1.4.1) induces a group structure on the points of the (16,6) configuration. The Sublemma is equivalent to saying that the action of $\alpha$ is equivariant with respect to this group structure.

**Sublemma 1.70.** *Let $\alpha$ be a permutation of entries of the $4 \times 4$ diagram (1.5.1) such that:*

(1) *$\alpha$ fixes 1*
(2) *for any entry $\delta$ in the diagram $\alpha$ sends the six element set $Inc(\delta)$ (cf. Definition 1.56) to itself.*

*Then $\alpha$ is a product of a permutation of columns, a permutation of rows and, possibly, the transposition of the $4 \times 4$ square.*

*Proof of Sublemma 1.70.* Let $H$ denote the subgroup of permutations of entries of our $4 \times 4$ square, generated by the permutations of rows fixing 1, permutations of columns fixing 1, and the transposition of the $4 \times 4$ matrix. We are free to multiply $\alpha$ by an element of $H$. Hence, we may assume that

$$\#(\{AB', BC', CA'\} \cap \alpha(\{AB', BC', CA'\})) \geq 2 \quad \text{and}$$
$$\#(\{AC', BA', CB'\} \cap \alpha(\{AC', BA', CB'\})) \geq 2.$$

First, we prove that $\alpha$ moves each of the sets $\{AB', BC', CA'\}$ and $\{AB', BC', CA'\}$ into itself. Indeed, suppose the contrary. Then, modifying $\alpha$ by an element of $H$, we may assume furthermore that $\alpha$ fixes $BA'$, $CB'$, $BC'$, $CA'$ and interchanges $AC'$ with $AB'$. But this is impossible, since $\alpha$ must map the set $\{AC', BA', B, AA', C'\}$ into itself. We get a contradiction, so $\alpha$ must map each of the sets $\{AB', BC', CA'\}$ and $\{AB', BC', CA'\}$ into itself. Modifying $\alpha$ by an element of $H$, we may assume that $\alpha$ fixes the elements $AB'$, $BC'$, $CA'$, $AC'$, $BA'$ and $CB'$. But then we may apply the preceding argument to each of these 6 elements instead of 1, to show that $\alpha$ must map every row to itself and every column to itself. Hence $\alpha$ is the identity and Sublemma 1.70 is proved. $\square$

We continue with the proof of Sublemma 1.69. Let $L$ and $L'$ denote the non-isotropic planes $(1, AB', BC', CA')$ and $(1, AC', BA', CB')$. By Sublemma 1.70, $\alpha$ must either act by permutations on both the set of 4 translates of $L$ and the set of four translates of $L'$ in $\mathbb{P}^3$ or interchange the sets of translates of $L$ and $L'$ (always mapping the pair $(L, L')$ to itself). Now, a map from $\mathbb{F}_2^2$ to $\mathbb{F}_2^2$ which fixes the identity element and is an isomorphism of sets, is an isomorphism of groups. Hence $\alpha$ is a group homomorphism restricted to each of the order 4 subgroups $L$ and $L'$. Since $\alpha$ preserves (or, possibly, interchanges) the set of rows and the set of columns of the $4 \times 4$ square, we have for any $\delta \in L$, $\delta' \in L'$,

$$\alpha(\delta\delta') = \alpha(\delta)\alpha(\delta').$$

Hence, $\alpha$ is a group homomorphism and Sublemma 1.69 is proved. $\square$

To show that $\delta'_j = \delta_{l(j)}$ for $1 \leq j \leq 16$, it is sufficient to show that the two transformations agree when restricted to our (16,6) configuration. By definition, they agree on $v_1$. It remains to show that they agree on $\delta_n v_1$ for $2 \leq n \leq 16$. We have

$$\delta'_j(\delta_n v_1) = \alpha^{-1}\delta_j \alpha \delta_n v_1.$$

Let $m$ be the unique index from 1 to 16 such that $n = l(m)$. Then

$$\alpha^{-1}\delta_j\alpha\delta_n v_1 = \alpha^{-1}\delta_j\alpha\alpha^{-1}\delta_m\alpha v_1 = \alpha^{-1}\delta_j\delta_m v_1 = \delta_{l(j)}\delta_n v_1,$$

where the last equality is by Sublemma 1.69. This completes the proof of Claim.

We continue the proof of Theorem 1.68. We have shown that $TF_0T^{-1} \subset N$. $N$ contains many different subgroups isomorphic to $\mathbb{F}_2^4$, but we want to show that if the orbit under such a subgroup of the point $(a, b, c, d)$ is the same as under $F_0$, then this subgroup must equal $F_0$.

Let $F'_0 := TF_0T^{-1}$. $F'_0$ is contained in a 2-Sylow subgroup of $N$. Recall the subgroup $I$ of (1.66.2). Since $[N : I] = 15$, we may, after conjugating the whole situation by an element of $N$, assume that $F'_0 \subset I$. By the proof of Sublemma 1.69, conjugation by an element of $F'_0$ must preserve a specific pair of non-isotropic planes. Namely, it must preserve the pair of sets $\{AB', BC', CA'\}$ and $\{AC', BA', CB'\}$

(possibly interchanging the two sets with each other). Each element of $F_0$ pointwise fixes two lines in $\mathbb{P}^3_k$ and is completely determined by these two lines.

By the proof of Theorem 1.64, $I$ consists of matrices with exactly one non-zero entry in each row and in each column, and the non-zero entries belong to the set $\{1,-1,i,-i\}$, with either none or two of them belonging to $\{i,-i\}$. By the table of Remark 1.66, if such a matrix has two entries belonging to $\{i,-i\}$, it cannot preserve the set $\{AB',BC',CA',AC',BA'\}$. Hence, all the elements of $F_0'$ have all their non-zero entries in $\{1,-1\}$. In other words, $F_0' \subset W \cong S_4 \ltimes \mathbb{F}_2^3 \cong (S_3 \times \mathbb{F}_2) \ltimes F_0$. It remains to show that $F_0$ is the unique subgroup of $W$ isomorphic to $\mathbb{F}_2^4$. This will imply that $F_0' = F_0$, hence $T \in N$, as desired.

Suppose $F_0' \neq F_0$. Then by Lemma 1.48, $F_0'$ is conjugate to $F_0$ in $PGL_4(k)$. In particular, the fixed point set of each element of $F_0'$ consists of two disjoint lines in $\mathbb{P}^3_k$. Now, it is easy to see that conjugating by an element of $F_0$, any element of $W$ of order 2 can be transformed into one of the following:

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$
$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Only one of the above five elements (the fourth one) has two disjoint lines as its fixed point set. Call this element $\delta$. The subgroup of $F_0$ fixed by the conjugation by $\delta$ is the subgroup generated by $C'$ and $CA'$. Since $F_0'$ is commutative, $F_0' \cap F_0 \subset (C', CA')$.

Conjugating everything by an element of $F_0$ we may assume $\delta \in F_0'$. Moreover, every other element of $F_0'$, which is not in $F_0$, is conjugate to $\delta$ by an element of $F_0$. But since $\delta$ commutes with an order four subgroup of $F_0$, its orbit under conjugation by $F_0$ contains at most four elements. But then $\#F_0' \leq \#(C', CA') + 4 = 8$, which is a contradiction. This completes the proof of Theorem 1.68. $\square$

Thus we have a bijection (given explicitly by (1.4.1)) between all the (16,6) configurations modulo an automorphism of $\mathbb{P}^3_k$ and the quotient of $U$ by the finite group $N$.

**1.71 Note.** See (4.15) for an intepretation of (16,6) configurations of type (*) in terms of 2-torsion points on an abelian surface and Weierstrass points on a curve of genus 2.

**Remark 1.72.** Theorem 1.68 also yields a description of the Hilbert scheme of (16,6) configurations viewed as subschemes of $\mathbb{P}^3_k$ up to birational equivalence. Namely, let $\tilde{S}$ denote the Hilbert scheme in question. By the Claim in the proof of Theorem 1.72, a linear automorphism of a (16,6) configuration of the form $(a,b,c,d)$

of (1.4.1) belongs to $N$. From this one deduces easily that the group of automorphisms of a *generic* such configuration is $F_0$. Therefore by Theorem 1.68 $\tilde{\mathcal{S}}$ is birationally equivalent to (and dominated by) $(U \times PGL_4(k))/(N \times \mathcal{C}_2^4)$. Here $\mathcal{C}_2^4$ acts trivially on $PGL_4(k)$ and acts as the group $F_0$ on $U$. $N$ acts on $U$ as a subgroup of $PGL_4(k)$ and on $PGL_4(k)$ by $T(M) := MT^{-1}$ for $M \in PGL_4(k)$, $T \in N$.

There is, however, another description of $\mathcal{C}$ in terms of the variety $V$ of Theorem 1.45. Consider the natural maps

$$\begin{tikzcd} U \times PGL_4(k) \arrow[r, "\eta"] & \mathcal{S} \arrow[r, "\theta"] \arrow[d, "\pi"'] & V \\ & \tilde{\mathcal{S}} & \end{tikzcd} \tag{1.72.1}$$

where $\phi$, $\eta$ and $\theta$ are defined in the proof of Theorem 1.45 and $\pi$ is the map which forgets the bijection to the entries of the diagram (*). By the above, $\theta$ is an isomorphism, $\phi$ and $\eta$ are 16-1 surjective étale covers and $\tilde{\eta}$ is the quotient by the action of $N \times \mathcal{C}_2^4$ described above. We identify $\mathcal{S}$ with $V$ via $\theta$ and wish to express the map $\pi$ as the quotient by a group action. In other words, we are asking the question: in a given (16,6) configuration, how many ways are there of choosing an ordered sextuple of planes as in (1.45.1)? We claim that $\pi$ is the quotient of $V$ by the group (where for each semidirect product, the action of the first factor on the second is described below)

$$(\mathcal{C}_2 \ltimes (S_4 \times S_4)) \ltimes A_5, \tag{1.72.2}$$

where the stabilizer of the generic point of $V$ is isomorphic to $S_3$. In particular, $\pi$ is a 11520-1 map, which agrees with $\eta$ being 16-1 and $\tilde{\eta}$ 184320-1 maps, respectively. We very briefly sketch the proof, giving barely enough details for the reader to be able to reconstruct the arguments.

Indeed, fix a non-degenerate (16,6) configuration together with a bijection from the set of planes with to entries in the diagram (*). First of all, it is easy to check that as *unordered* sextuples of planes all the sextuples combinatorially isomorphic to the on in (1.45.1) are obtained from (1.45.1) by a composition of a permutation of rows, a permutation of columns and, possibly, the transposition of the matrix (this is why $\mathcal{C}_2 \ltimes (S_4 \times S_4)$ appears in (1.72.2); here $\mathcal{C}_2$ acts on $(S_4 \times S_4)$ by interchanging the factors). To see that, observe that, up to an element of $\mathcal{C}_2 \ltimes (S_4 \times S_4)$, our sextuple of planes must contain three planes which are not all in the same row or column but which pass through the same point of the configuration:

$$\begin{matrix} \cdot & \star & \star & \cdot \\ \star & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{matrix}$$

Then one simply checks all the possibilities for the remaining three planes, using automorphisms of the situation and the fact that every point of the configuration is contained in either one or three of our planes.

It remains to decide which permutations of the six planes of (1.45.1) are induced by automorphisms of the abstract (16,6) configuration. The reason we do not get the full symmetric group $S_6$ is that not all triples of planes in (1.45.1) are equivalent to each other in our configuration. Some triples of planes intersect in a point of the configuration while others do not. Let us think of our planes as a six element set and recall the identification $S_6 \cong SP_4(\mathbb{F}_2)$ of Remark 1.65. Under this identification we may think of a triple of elements $(i, j, k)$ as specifying a uniquely defined non-isotropic line in $\mathbb{P}^3_{\mathbb{F}_2}$ (namely, the line $(i, j), (i, k), (j, k)$). Among all the possible triples of planes of (1.45.1) there are exactly ten which intersect in a point of the configuration (the ten corresponding points of the configuration are shown in (1.48.1)). In terms of $\mathbb{P}^3_{\mathbb{F}_2}$ with a symplectic structure we have ten preferred non-isotropic lines such that each point of $\mathbb{P}^3_{\mathbb{F}_2}$ belongs to exactly two lines. The subgroup of $S_6$ we want to find is precisely the subgroup which maps this set of ten lines to itself.

For each point $x \in \mathbb{P}^3_{\mathbb{F}_2}$ consider the plane $H_x$ generated by the two non-isotropic lines from our set passing through $x$. Let $L_x$ be the unique isotropic line contained in $L_x$ and passing through $x$. One checks directly that as $x$ runs over the points of $\mathbb{P}^3_{\mathbb{F}_2}$, each of the lines $L_x$ arises from exactly three different points $x$. Hence the set $M := \{L_x \mid x \in \mathbb{P}^3_{\mathbb{F}_2}\}$ has cardinality 5. One checks directly from definitions that the five isotropic lines of $M$ form a **total** [4, p. [4]]: a set of five (pairwise disjoint) isotropic lines whose union equals $\mathbb{P}^3_{\mathbb{F}_2}$. It is easy to prove directly that any permutation of the planes of (1.45.1) which preserves the incidence relations imposed by (1.45.1) must be even. Hence the desired subgroup of $S_6$ is contained in the stabilizer of a total in $A_6$, which is $A_5$ [4, p. [4]]. On the other hand, there is no difficulty in explicitly producing enough elements of the subgroup in question to show that it equals $A_5$. This is not the obvious $A_5 \subset S_6$ consisting of even permutations stabilizing one of the six elements of our set, but its image under the outer automorphism of $S_6$ which interchanges pairs $(ij)$ with synthemes $(ab)(cd)(ef)$ [4, p. [4]]. $\mathcal{C}_2 \ltimes (S_4 \times S_4)$ acts on $A_5$ as follows. Let $g \in \mathcal{C}_2 \ltimes (S_4 \times S_4)$, $h \in A_5$. We apply $g$ to the diagram (1.45.1), then apply $h$ to the images of the ordered set of six planes, then apply $g^{-1}$ to the diagram (1.45.1). Call the resulting transformation $ghg^{-1}$. This defines the semidirect product (1.72.2) and completes our construction of the finite group in question.

To see that the stabilizer of every point of $\mathcal{S}$ under the action of $(\mathcal{C}_2 \ltimes (S_4 \times S_4)) \ltimes A_5$ is $S_3$, consider an element $(\alpha, \beta, \gamma, \delta)$ of the stabilizer (here $\alpha \in \mathcal{C}_2$, $\beta, \gamma \in S_4$ and $\delta \in A_5$). Then $\alpha$ must fix the first row of (1.45.1) and $\beta$ the last column. Moreover, the unordered set of six planes must remain unchanged by $(\alpha, \beta, \gamma, \delta)$, so after a suitable renumbering of rows and columns in (1.45.1) we must have

$$\alpha = \beta^{-1} \tag{1.72.3}$$

and $\gamma$ is the identity element of $\mathcal{C}_2$. In (1.72.3) we view both $\alpha$ and $\beta$ as elements of $S_4$ under a suitable bijection between the set of rows and the set of columns. We may identify the set of possible $\alpha$'s with $S_3$. Once $\alpha \in S_3$ is chosen, $\beta$ is uniquely determined by (1.72.3) and $\delta$ is the element of $A_5$ inverse to the permutation of the planes induced by $\alpha \circ \beta$ (where we view $\alpha$ and $\beta$ as elements of the group (1.72.2). Hence the stabilizer is isomorphic to $S_3$. The map $S_3 \hookrightarrow (\mathcal{C}_2 \ltimes (S_4 \times S_4)) \ltimes A_5$ is the trivial homomorphism on the first component, and the obvious embedding $S_3 \hookrightarrow S_4$ on the second and third component. The map $S_3 \hookrightarrow A_5$ is given by splitting the six element set of (1.45.1) into two triples and embedding $S_3$ diagonally into the stabilizer $S_3 \times S_3$ of the pair of triples.

This completes our sketch of the proof that the Hilbert scheme of Kummer surfaces in $\mathbb{P}^3_k$ birationally dominates $V/((\mathcal{C}_2 \ltimes (S_4 \times S_4)) \ltimes A_5)$. To give a more precise description of the Hilbert scheme (rather than just its birational equivalence class) one must know which subgroups of $N$ occur as automorphism groups of non-generic (16,6) configurations and what are the loci of those configurations in $V$. We leave this problem as a task for the future. Another way of clarifying the meaning of diagram (1.72.1) is that it expresses $N \equiv \frac{N \times \mathcal{C}_2^4}{\mathcal{C}_2^4}$ as a homogeneous space for $(\mathcal{C}_2 \ltimes (S_4 \times S_4)) \ltimes A_5$, such that the stabilizer of every element is $S_3$. In principle, one could unravel all the definitions and write down explicit formulas for the action of $(\mathcal{C}_2 \ltimes (S_4 \times S_4)) \ltimes A_5$ on $\mathcal{S}$, but in practice this task is quite tedious.

## §2. THE CLASSIFICATION OF KUMMER SURFACES IN $\mathbb{P}^3$.

Let $k$ be an algebraically closed field of characteristic different from 2.

In this Chapter we study Kummer surfaces in $\mathbb{P}^3$ and their equations. First, we define Kummer surfaces and recall the classical notion of enveloping cone at a node. We prove that the enveloping cone at each node of a Kummer surface is a union of six planes passing through the node, no three of which have a line in common. The planes in $\mathbb{P}^3$ which belong to the enveloping cone at some node of the Kummer surface are called **special planes** for that surface. We show that there are exactly sixteen special planes and that together with the set of nodes they form a non-degenerate (16,6) configuration which uniquely determines the Kummer surface. Conversely, we use the classification of non-degenerate (16,6) configurations from the previous chapter to give explicit equations for a Kummer surface which gives rise to this configuration. These equations describe the universal family of Kummer surfaces over $\mathcal{M}_2$, thus giving a complete classification of Kummer surfaces. These equations were discovered by Hudson [12], although she does not prove the fact that all the Kummer surfaces can be defined by equations of that form.

**Definition 2.1.** *Let $S$ be a surface and $P$ a singular point of $S$. We say that $P$ is a* **node** *of $S$ if the tangent cone to $S$ at $P$ is a non-degenerate quadratic cone. In other words, the formal completion $\hat{O}_{S,P}$ of the local ring of $S$ at $P$ is isomorphic to $\frac{k[[x,y,z]]}{(x^2+y^2+z^2)}$.*

**Definition 2.2.** *A* **Kummer surface** *in $\mathbb{P}^3$ is a reduced, irreducible surface of degree 4 having 16 nodes and no other singularities.*

Let $S$ be a Kummer surface in $\mathbb{P}^3$. Assume that $P = (0,0,0,1)$ is a node of $S$. Then if $(x : y : z : t)$ are the coordinates in $\mathbb{P}^3$, the defining equation $f$ of $S$ has the form

$$f(x,y,z,t) = t^2 f_2(x,y,z) + 2t f_3(x,y,z) + f_4(x,y,z) = 0 \tag{2.2.1}$$

where $\deg f_i = i$. Since $P$ is a node of $S$, $f_2$ is a non-degenerate quadratic form in $x,y,z$. The tangent cone to $S$ at $P$ is defined by the equation $f_2 = 0$.

**Definition 2.3.** *Let $S$ be a Kummer surface and $P$ a node of $S$. Let $l$ be a line in $\mathbb{P}^3$ passing through $P$. If $l \not\subset S$, $\deg S.l = 4$. Since $l$ passes through $P$ which is a double point, $l$ will intersect $S$ in two more points $P', P''$, one or both of which may be equal to $P$. We say that $l$ belongs to the* **enveloping cone** *$V$ of $S$ at $P$ if either $l \subset S$ or, in the case that $l \not\subset S$, $P' = P''$ (in other words, $P'$ is a node, or $l$ is tangent to $S$ at $P'$, or $l$ has a fourfold contact with $S$ at $P'$).*

*Consider the linear projection $p : \mathbb{P}^3 \setminus \{P\} \to \mathbb{P}^2$. The image of the enveloping cone under $p$ will be called the* **projectivized enveloping cone**.

**Remark 2.4.** We shall see below that the case $l \subset S$ is impossible (4.30).

**Lemma 2.5.** *Fix a node $P$ of $S$. A generic hyperplane section of $S$ passing through $P$ is a plane quartic curve with exactly one node.*

*Proof.* Let us consider the section $C$ of $S$ by a generic plane $H$ passing through the node $P$. To prove that the $C$ is nonsingular outside $P$ is to prove that $H$ does not pass through another node and that it is not tangent to the Kummer surface at any nonsingular point. Since the number of nodes of $S$ is finite, we can ensure that the plane does not pass through any other node. Let us suppose that all the planes through the node are tangent to $S$ at some nonsingular point. Consider the Gauss map $d : S - \{\text{nodes}\} \to \mathbb{P}^{3*}$. Our assumption implies that $S^* \simeq d(S - \{\text{nodes}\})$ contains all the planes passing through the node; thus, $S^*$ contains a plane $W$ in $\mathbb{P}^{3*}$. Since $S$ is irreducible, so is $S^*$. Hence $S^* \simeq W$. Therefore, $S$ is a strange surface (i.e. a surface such that all the planes tangent to the surface at a nonsingular point pass through the same point of $\mathbb{P}^3$). Hence when we project any point $Q$ of $S$ from the node $P$, the tangent plane at $Q$ contains $P$. Therefore the projection is ramified at $Q$. Thus the projection from $P$ is ramified at every point of $S$, so it must be inseparable. But the projection has degree 2, and to be inseparable it must have degree $np^r$, where $p = char\, k$, $r > 0$. This is a contradiction since, by assumption, $p \neq 2$. This shows that $S \cap H$ is nonsingular outside $P$.

Next, we show that $C$ has a node at $P$. We may choose $H$ so that it intersects the tangent cone to $S$ at $P$ in two distinct lines. Choose the coordinates $(x : y : z : t)$ on $\mathbb{P}^3$ so that $H$ is defined by $x = 0$ and $f_2 \equiv yz \mod x$. Then by (2.2.1) $H \cap S$ has a node at $P$. Since $H \cap S$ is a quartic curve and its only singularity is a node at $P$, it must be irreducible. Indeed, any intersection point of irreducible components of $H \cap S$ must be a singularity of $H \cap S$. Hence $H \cap S$ can have at most two irreducible components whose only intersection point is $P$ and the intersection number there is 1. Hence the product of the degrees of the irreducible components is 1 and the sum of the degrees is 4, which is absurd. This completes the proof of Lemma 2.5. $\square$

**Theorem 2.6.** *The enveloping cone to the Kummer surface $S$ given by the equation (2.2.1) at the node $P = (0,0,0,1)$ is defined by the equation $f_3^2 - f_2 \cdot f_4 = 0$. The enveloping cone is a union of 6 planes. Each of the 6 planes contains 6 nodes of $S$ (including $P$). The section of $S$ by each of these planes is a conic with a multiplicity 2 structure.*

*Proof.* Let us determine the equation of the enveloping cone to the Kummer surface $S$ at $P = (0,0,0,1)$. $S$ is given by the equation (2.2.1). Take a line $l$ passing through $P$. Making a linear homogeneous change of coordinates in $(x, y, z)$, if necessary, we may assume that $l$ is not contained in the plane $z = 0$, and hence that $l$ is defined by the equations

$$\begin{aligned} x &= \alpha z \\ y &= \beta z \end{aligned} \tag{2.6.1}$$

where $\alpha, \beta \in k$. Substituting (2.6.1) in (2.2.1), and factoring out $z^2$ we obtain a quadratic form $q$ in $t$ and $z$. $l$ belongs to the enveloping cone if and only $q$ is a square of a linear form in $t$ and $z$ or $f$ is identically zero on $l$. This is true if and only if the discriminant of $q$ vanishes identically on $l$. But the discriminant of $q$ equals $\frac{f_3^2 - f_2 f_4}{z^4}$. Thus, the equation of the enveloping cone is

$$f_3^2 - f_2 f_4 = 0. \tag{2.6.2}$$

In particular, the enveloping cone is a variety of degree 6. There are at most finitely many lines in $S$ passing through $P$ (in fact, there are no lines in $S$ at all, but we have not proved that yet). Indeed, otherwise the intersection of $S$ with its tangent cone at $P$ would have to contain an infinite set of lines, hence $S$ would coincide with its tangent cone. But then $S$ would be a quadric surface, which is absurd.

**Lemma 2.7.** *The enveloping cone $V$ to the Kummer surface $S$ at the node $P = (0,0,0,1)$ is reduced.*

*Proof.* $\deg V = 6$, $\deg(V \cap S) = 24$. $V \cap S$ is a scheme of multiplicity greater or equal than 2 by definition of $V$. The projectivized enveloping cone is a plane curve of degree 6 by (2.6.2). To prove it is reduced, it is sufficient to prove that its intersection with some line $L$ in $\mathbb{P}^2$ consists of six distinct points. Let us take a general hyperplane section of $S$ through $P$ containing $L$ (denote it by $H$). $H \cap S$ is a quartic curve $C$ with one node at $P$. Consider its normalization $\tilde{C}$. Consider the map $\tilde{C} \to L$ induced by the linear projection from $P$ to $L$ in the plane $H$. Then $\pi$ is a 2-1 covering of $L$ by $\tilde{C}$. Since $char\, k \neq 2$, $\pi$ is separable. Since $g(\tilde{C}) = 2$, we obtain by Hurwitz's Theorem that $\pi$ has six distinct branch points, as desired. □

**Lemma 2.8.** *Let $S$ be an irreducible quartic surface in $\mathbb{P}^3$ having only nodes as singularities.*

(1) *If three nodes of $S$ are collinear, then the line joining them is contained in $S$.*
(2) *It is not possible for four nodes of $S$ to be collinear.*

*Proof.*

(1) Let $L$ be a line containing three nodes of $S$. If $L \not\subset S$, then $L$ has intersection multiplicity $\geq 6$ with $S$, which is impossible.
(2) Suppose that $P_1$, $P_2$, $P_3$, $P_4$ are four nodes of $S$ lying on a line $L$. By (1), the line $L$ must be contained in $S$. Since $S$ has only nodes as singularities, a general point of $L$ is smooth on $S$, so a general plane $H$ containing $L$ will intersect $S$ in a curve $C$ consisting of the reduced line $L$ plus another curve $C'$ of degree 3 not containing $L$ as a component. This $C'$ can intersect $L$ in at most 3 points. On the other hand, $C = S \cap H$ must be singular at the four points $P_1$, $P_2$, $P_3$, $P_4$ on $L$, which is a contradiction.

**Proposition 2.9.** *Let $S$ be an irreducible surface of degree 4 in $\mathbb{P}^3$ having only nodes as singularities. Let $P_1$ be one of the nodes and consider the projection $\phi$ from $S - P_1$ to a plane $H$ not containing $P_1$. Let $\tilde{S}$ be the blowing-up of $S$ at $P_1$. Then $\phi$ extends to a morphism $\tilde{\phi} : \tilde{S} \to H$, where $\tilde{\phi}$ is a generically finite morphism of degree 2. We are interested in the branch locus $B$ of $\tilde{\phi}$ in $H$, which is defined as $B = \{Q \in H \mid \tilde{\phi}^{-1}(Q)$ is not two distinct points$\}$.*

(1) *Let $P_2 \in S$ be another node, and suppose that the line $L = P_1P_2$ is not contained in $S$. Then $\phi(P_2)$ is a node of $B$, and $L$ meets $S$ only at $P_1$ and $P_2$.*

(2) *Again let* $P_2 \in S$ *be another node of* $S$, *and suppose that* $L = P_1P_2$ *is contained in* $S$, *but that* $L$ *contains no further nodes of* $S$. *Then,* $\phi(P_2)$ *is an ordinary cusp of* $B$.

(3) *Suppose that* $P_2$ *and* $P_3$ *are nodes of* $S$, *and that* $P_1$, $P_2$, $P_3$ *are collinear. Thus, by Lemma 2.8, the line* $L = P_1P_2P_3$ *is contained in* $S$ *and contains no further node of* $S$. *In this case, the singularity of* $B$ *at* $\phi(P_2)$ *is either a double point analytically isomorphic to* $y^2 = x^r$, *for some* $r \geq 4$, *or a multiplicity two structure on a nonsingular curve.*

(4) *Suppose that a line* $L$ *passing through* $P_1$ *is contained in* $S$, *but contains no nodes other than* $P_1$. *Let* $Q \in L$. *Then* $\phi(Q)$ *is a singular point of* $B$ *of multiplicity at least 2.*

*Proof.* The linear projection from $\mathbb{P}^3 \setminus \{P_1\}$ to $H$ extends to the blowing up of $\mathbb{P}^3$ at $P$. Hence $\phi$ extends to a map $\tilde{\phi} : \tilde{S} \to H$. Let $l \subset \mathbb{P}^3$ be a line passing through $P_1$, not contained in the tangent cone to $S$ at $P_1$. Then $(l.S)_{P_1} = 2$ and $l.S = 4$, so the total intersection number of $l$ with $S$ away from $P_2$ is 2. In other words, $\tilde{\phi}$ is a morphism of degree 2. Since *char* $k \neq 2$, $\tilde{\phi}$ is separable, hence generically 2-1.

Taking the point $P_1$ to be $(0,0,0,1)$, we may write the equation of $S$ as in (2.2.1). We take the plane $H$ to be the plane $t = 0$. Then $(x : y : z)$ form a system of homogeneous coordinates on $H$. Take a point $Q = (x,y,z) \in H$. We may choose coordinates in such a way that $z \neq 0$, so we may write $Q = (x,y,1)$. The inverse image of $Q$ in $S$ is given by those ratios $(t : x : y : 1)$ obtained by solving the equation (2.2.1) for $t$. Hence $Q \in B$ if and only if (2.2.1), viewed as a quadratic equation in $t$, has a double root. This occurs when the discriminant $\Delta = {f_3}^2 - f_2f_4 = 0$. So $B$ is given by $\Delta(x,y,1) = 0$ locally near the point $Q$. Since the description of $B$ as the locus defined by the equation $\Delta = {f_3}^2 - f_2f_4$ is independent of the coordinate system, it is valid even for $z = 0$. We shall now compute $\Delta$ in each of the three cases above.

(1) Let $P_2 \in S$ be another node. We may take $P_2 = (0,0,1,0)$ so that the line $L = P_1P_2$ is the line $x = y = 0$. We assume $L \not\subset S$. Since $P_2$ is a node, looking at (2.2.1) as a polynomial in $z$, there are no $z^3$ or $z^4$ terms. Hence we may write

$$f_2(x,y,z) = a_2(x,y) + a_1(x,y)z + a_0z^2$$
$$f_3(x,y,z) = b_3(x,y) + b_2(x,y)z + b_1(x,y)z^2$$
$$f_4(x,y,z) = c_4(x,y) + c_3(x,y)z + c_2(x,y)z^2,$$

where, as usual, the subscripts denote degree of the form.

Since $P_2$ is a node, the coefficient of $z^2$ in (2.2.1), namely,

$$g_2(x,y,t) = a_0t^2 + 2b_1(x,y)t + c_2(x,y),$$

must be a non-degenerate quadratic form in $x$, $y$, $t$. Furthermore, since the line $L \not\subset S$, $L$ cannot be in the tangent cone to $S$ at $P_2$ (otherwise it would have intersection multiplicity $\geq 3$ with $S$ at $P_2$, in which case $L.S \geq 5$). In other words, $g_2(0,0,1) \neq 0$, hence $a_0 \neq 0$.

Now we can compute $\Delta(x,y,1) = f_3(x,y,1)^2 - f_2(x,y,1)f_4(x,y,1)$. To study the singularity at $(0,0,1)$ in $H$, we look for the quadratic form in $x$, $y$, which is

$$\Delta(x,y,1)_2 = b_1(x,y)^2 - a_0 c_2(x,y).$$

This $\Delta(x,y,1)_2$ is exactly the discriminant of $g_2(x,y,t)$ regarded as a polynomial in $t$ over the field $k(x,y)$. Geometrically, $\Delta(x,y,1)_2$ represents the branch points in the $(x,y)$-line of the projection of the conic $g_2 = 0$ of the $(x,y,t)$-plane from the point $(0,0,1)$. Since $g_2 = 0$ is a nondegenerate conic, and the point $(0,0,1)$ is not in that conic, there are exactly two branch points. Thus, $\Delta(x,y,1)_2$ is a product of two distinct linear forms in $(x,y)$, and we conclude that our branch curve $B$ has a node at the point $(0,0,1) = \phi(P_2)$.

(2) Now suppose $P_2$ is a second node of $S$, the line $L = P_1P_2$ is contained in $S$, but that $L$ contains no further node of $S$. Then, we shall show that $\phi(P_2)$ is an ordinary cusp of $B$.

As in (1) we may assume that $P_2 = (0,0,1,0)$. Using the same notation as above, in this case $L$ belongs to the tangent cone at $P_2$, so $g_2(0,0,1) = 0$, which implies that $a_0 = 0$. Then, since $g_2$ is nondegenerate, $b_1(x,y) \neq 0$. Then the singularity of $\Delta(x,y,1)$ at $(0,0,1)$ begins with $\Delta(x,y,1)_2 = b_1(x,y)^2$, the square of a nonzero linear form. To show that $\phi(P_2)$ is an ordinary cusp of $B$ it will be sufficient to show that $b_1(x,y)$ does not divide the cubic term of $\Delta(x,y,1)$ which is

$$\Delta(x,y,1)_3 = 2b_1(x,y)b_2(x,y) - a_1(x,y)c_2(x,y).$$

Since $b_1$ divides the first term, we must show that $b_1(x,y)$ does not divide either $a_1(x,y)$ or $c_2(x,y)$.

As for $c_2(x,y)$, let us consider

$$g_2(x,y,t) = 2b_1(x,y)t + c_2(x,y).$$

If $b_1$ divides $c_2$, then $g_2$ has a linear factor, which is impossible since $g_2$ is a nondegenerate quadratic form.

As for $a_1(x,y)$, we must use the hypothesis that no further singularity of $S$ lies on $L$. We compute the partial derivatives of $F(x,y,z,t)$ and substitute $x = y = 0$, to find

$$\begin{aligned}
F_t(0,0,z,t) &= 2a_0 z^2 t \\
F_x(0,0,z,t) &= \left(\frac{\partial}{\partial x}(a_1(x,y))\right) zt^2 + 2\left(\frac{\partial}{\partial x}(b_1(x,y))\right) z^2 t \\
F_y(0,0,z,t) &= \left(\frac{\partial}{\partial y}(a_1(x,y))\right) zt^2 + 2\left(\frac{\partial}{\partial y}(b_1(x,y))\right) z^2 t \\
F_z(0,0,z,t) &= 2a_0 zt^2.
\end{aligned}$$

The first and last are zero because $a_0 = 0$. If $b_1(x, y)$ divides $a_1(x, y)$, we can write $a_1(x, y) = \lambda b_1(x, y)$, $\lambda \neq 0$. Then, taking $(z, t) = (-\frac{1}{2}\lambda, 1)$, we obtain a point $P_3 \in L$, $P_3 \neq P_1, P_2$, at which all four partials vanish, hence another singular point of $S$ on $L$.

(3) Finally, suppose $P_2$ and $P_3$ are nodes of $S$ lying on the line $L$ contained in $S$. Using the same notation as in part (2), we may assume that $P_2 = (0, 0, 1, 0)$ and $P_3 = (0, 0, -\frac{1}{2}, 1)$ for example. Since the four partials vanish at $P_3$, we conclude from the above that

$$\frac{\partial}{\partial x} a_1(x, y) = \frac{\partial}{\partial x} b_1(x, y)$$
$$\frac{\partial}{\partial y} a_1(x, y) = \frac{\partial}{\partial y} b_1(x, y)$$

from which it follows that the linear forms $a_1(x, y)$ and $b_1(x, y)$ are equal.

Then $b_1(x, y)$ divides $\Delta(x, y, 1)_3$, so we have a double point of $B$ which is neither a node nor an ordinary cusp. If reduced, it is a double point analytically isomorphic to $y^2 = x^r$, for some $r \geq 4$ ([11, I, Exercise 5.14]). If nonreduced, it is a multiplicity two structure on a nonsingular curve passing through $\phi(P_2)$.

(4) Using the same notation as before, let

$$F(x, y, z, t) = t^2 f_2 + 2t f_3 + f_4,$$

and write

$$\begin{aligned} f_2 &= a_2(x, y) + a_1(x, y)z + a_0 z^2 \\ f_3 &= b_3(x, y) + b_2(x, y)z + b_1(x, y)z^2 + b_0 z^3 \\ f_4 &= c_4(x, y) + c_3(x, y)z + c_2(x, y)z^2 + c_1(x, y)z^3 + c_0 z^4 \end{aligned}$$

Since $L \subset S$, we must have $F(0, 0, z, t)$ identically 0.

$$F(0, 0, z, t) = a_0 z^2 t + 2b_0 z^3 t + c_0 z^4.$$

Hence $a_0 = b_0 = c_0 = 0$. Now we compute $\Delta(x, y, 1) = {f_3}^2 - f_2 f_4$. $\Delta(x, y, 1)$ has no constant term, and no linear term in $x$, $y$. It begins with quadratic terms,

$$\Delta(x, y, 1)_2 = b_1(x, y)^2 - a_1(x, y)c_1(x, y).$$

Thus, $B$ has a singularity of multiplicity $\geq 2$ at $\phi(Q)$.

We continue the proof of Theorem 2.6. We now prove that if $S$ is a Kummer surface in $\mathbb{P}^3$, its enveloping cone consists of six planes.

Let $P_1$ be one of the nodes. Projecting from $P_1$, we must show that the branch locus consists of six distinct lines.

By Lemma 2.7, $B$ is a reduced curve of degree 6. By Proposition 2.9, the other 15 nodes of $S$ project to singular points of $B$. If the line $P_1 P_i$ contains no other node, the corresponding point of $B$ is a node or ordinary cusp. If two nodes besides $P_1$ lie on a line, the corresponding point of $B$ is a double point analytically isomorphic to $y^2 = x^r$, for some $r \geq 2$.

Let $\tilde{B}$ be the normalization of $B$. Then the arithmetic genus of $\tilde{B}$ is given by the formula

$$p_a(\tilde{B}) = p_a(B) - \frac{1}{2}\sum_i r_i(r_i - 1),$$

where $p_a(B) = \frac{1}{2}(6-1)(6-2) = 10$ and the sum is taken over all the multiple points of $B$, of multiplicity $r_i$, including infinitely near points [11, V, 3.9.2]. Each node or cusp of $B$ contributes 1 to the sum $\frac{1}{2}\sum_i r_i(r_i-1)$, while a double point of type $y^2 = x^r$, $r \geq 4$, contributes at least 2. Thus, the images of the 15 nodes other than $P_1$ in $B$ contribute at least 15 to the sum $\frac{1}{2}\sum_i r_i(r_i-1)$. Hence, $p_a(\tilde{B}) \leq -5$.

Now, $\tilde{B}$ is the disjoint union of the normalizations $\tilde{B}_i$ of the irreducible components $B_i$ of $B$. The arithmetic genus of $\tilde{B}$ can be computed as

$$p_a(\tilde{B}) = \sum p_a(\tilde{B}_i) + 1 - \#(\text{irred. comp.}).$$

Since $B$ has degree 6, the number of irreducible components is $\leq 6$. On the other hand, each $\tilde{B}_i$ is irreducible and nonsingular so $p_a(\tilde{B}_i) \geq 0$. Thus, $p_a(\tilde{B}) \geq -5$.

We can conclude that $p_a(\tilde{B}) = -5$, and that there are 6 irreducible components, each one rational. Since $B$ has degree 6, $B$ is just a union of 6 lines, so the enveloping cone itself is a union of six planes passing through $P$.

Each of the six lines of the projectivized enveloping cone $C$ contains exactly five singularities of $C$, hence every plane of the enveloping cone at $P$ contains exactly five other nodes of $S$ besides $P$. By definition, each of the planes $H$ of the enveloping cone is tangent to $S$ at every point of $H \cap S$. Hence $H \cap S$ is a plane quartic which is not reduced at any point, so it must be either a double conic or a quadruple line. The latter is impossible since $H \cap S$ contains 6 nodes of $S$ and no three nodes are collinear. For the same reason $H \cap S$ cannot be a union of two double lines (i.e. a *reducible* double conic). Hence $H \cap S$ is an irreducible conic with multiplicity 2 structure, as desired. This completes the proof of Theorem 2.6. $\square$

**Definition 2.10.** *The lines* $\overline{P_1P_i}$, $2 \leq i \leq 16$, *are called special lines.*

**Corollary 2.11.** *If $S$ is a Kummer surface, no three nodes of $S$ are collinear. For any two nodes, the line joining them is not contained in $S$. All special lines contain exactly two nodes. All the special lines are distinct. The Kummer surface $S$ does not contain any line passing through one or two of its nodes.*

*Proof.* Indeed, since all the singularities of $B$ are nodes, cases (2), (3) or (4) of Proposition 2.9 cannot occur. Thus, on the one hand, no three nodes of $S$ are collinear and on the other, a line which contains one or two nodes cannot be contained in $S$.

**Note 2.12.** Below we shall prove that a Kummer surface cannot contain any lines at all (4.30).

**Remark 2.13.** We can rephrase what we have said above in terms of the blowing-up $\tilde{S}$ of $S$ at $P$. By definition of blowing-up, $\tilde{S}$ is naturally embedded in $\mathbb{P}^3 \times \mathbb{P}^2$, so the projection on the second factor gives a map $\psi : \tilde{S} \to \mathbb{P}^2$. Identify $\mathbb{P}^2$ with the set of lines in $\mathbb{P}^3$ passing through $P_1$. Then $\psi$ is a double cover whose branch locus is the plane curve of degree 6 consisting of 6 lines—the projectivized enveloping cone. The preimages of the other 15 nodes of $S$ in $\tilde{S}$ lie above the 15 nodes of the branch locus of $\psi$.

**Definition 2.14.** *A plane which is a component of the enveloping cone of $S$ at some node is called a special plane.*

**Corollary 2.15.** *Any plane containing 5 nodes of $S$ must contain 6, and it must be one of the 6 planes in the enveloping cone of each of its nodes.*

*Proof.* Let $H$ be a plane containing 5 nodes. Let us consider the projectivized enveloping cone from any one of the 5 nodes. Denote this node by $P$. The images of the other 4 nodes under the linear projection from $P$ to $\mathbb{P}^2$ are collinear. We want to prove that this line is one of the 6 lines, $L_i$, $1 \leq i \leq 6$, of the projectivized enveloping cone. Each node belongs to 2 of the $L_i$. Thus, one of the lines $L_i$ must contain two of the nodes. Hence, it contains the 4 nodes; so the line we have started with is one of the $L_i$. Then $H$ is a special plane at the point $P$. □

**Proposition 2.16.** *The union of the sixteen enveloping cones at the sixteen nodes of $S$ consists of sixteen planes. Each plane cuts out a conic on $S$, containing six nodes. Each node lies on exactly six of the sixteen conics. Together the nodes of $S$ and the 16 special planes form a non-degenerate (16,6) configuration.*

*Proof.* Each of the 16 nodes gives rise to 6 special planes; each special plane contains 6 nodes by Theorem 2.6 and it is a special plane for each one of its nodes. Hence the total number of special planes is $\frac{16\cdot 6}{6} = 16$, as desired. Hence the set of nodes and special planes form a (16,6) configuration. By Corollary 2.11, every pair of special planes can have at most two nodes in common. Since there are as many pairs of nodes as there are pairs of special planes, and since each pair of nodes lies on exactly two planes, this implies that any two special planes have exactly two nodes in common. Hence our (16,6) configuration is non-degenerate. □

**Remark 2.17.** If one of the special planes not passing through $P$ is taken to be $t = 0$, then $f_4 = 0$ of (2.2.1) is the equation of the double conic passing through 6 nodes in the special plane.

**Corollary 2.18.** *The nodes and special planes of $S$ form a (16,6) configuration of type (*), and hence, up to an automorphism of $\mathbb{P}^3$, a (16,6) configuration of the form described in Proposition 1.6. We call this the (16,6) configuration* **associated** *to the Kummer surface $S$.*

*Proof.* It follows from Proposition 2.16, and the fact that any nondegenerate (16,6) configuration is of type (*) and hence is of the form described by (1.6) up to an isomorphism of $\mathbb{P}^3$ (Theorem 1.42 and Theorem 1.45).

**Lemma 2.19.** *Two Kummer surfaces $S$ and $S'$ whose singular loci coincide must themselves coincide.*

*Proof.* Since $S$ and $S'$ have all the nodes in common and the set of nodes completely determines the associated (16,6) configuration, the sets of special planes of $S$ and $S'$ coincide. The special conics are, set-theoretically, sections of the Kummer surface by the special planes, hence $S$ and $S'$ must have the 16 special conics in common. If $S \neq S'$ then $\deg(S \cap S') \geq 32$, which is a contradiction. Hence $S$ and $S'$ coincide. $\square$

## The universal equation of a Kummer surface in $\mathbb{P}^3$.

**Theorem 2.20.** *Given a non-degenerate (16,6) configuration, there exists a Kummer surface $S$ whose associated (16,6) configuration is the given one.*

*Proof.* By Theorem 1.45 any (16,6) configuration of type (*) is of the form $(a,b,c,d)$ described in Proposition 1.6, for a suitable system of coordinates in $\mathbb{P}^3$. We shall fix such a suitable coordinate system once and for all. The (16,6) configuration of Proposition 1.6 was constructed as the orbit of a point $(a,b,c,d)$ in an open subset $U \subset \mathbb{P}^3$ (see (1.2.1)) under the action of the finite group $F_0 \subset PGL_4(k)$ (see Note 1.4). Hence $F_0$ maps this (16,6) configuration to itself. Therefore if there exists a Kummer surface $S$ whose associated (16,6) configuration is the given one, by Lemma 2.19 such a surface must be mapped to itself by elements of $F_0$. In other words, the defining equation $f$ of $S$ is a quartic form, invariant up to a scalar multiple under changing the signs of any two coordinates at a time and under interchanging any two pairs of coordinates (e.g. $x \leftrightarrow y$, $z \leftrightarrow t$). We look for a solution of the form

(2.20.1)

$$f = x^4 + y^4 + z^4 + t^4 + 2Dxyzt + A(x^2t^2 + y^2z^2) + B(y^2t^2 + x^2z^2) + C(z^2t^2 + x^2y^2)$$

(this equation was first written down by Hudson [12, p. 81]).

Consider the point $P = (a,b,c,d) \in U$ (cf. (1.2.1)) which gives our (16,6) configuration. Acting on $(a,b,c,d)$ by an element of the normalizer $N$ of $F_0$ we may assume that all the $a$, $b$, $c$, $d$ are different from 0.

The partial derivatives of $f$ with respect to $x$, $y$, $z$, $t$, divided by 2 (denote them by $f_x$, $f_y$, $f_z$, $f_w$), are

$$\begin{aligned} f_x &= 2x^3 + Dyzt + Axt^2 + Bxz^2 + Cxy^2 \\ f_y &= 2y^3 + Dxzt + Ayz^2 + Byt^2 + Cyx^2 \\ f_z &= 2z^3 + Dxyt + Azy^2 + Bzx^2 + Czt^2 \\ f_t &= 2t^3 + Dxyz + Ax^2t + By^2t + Cz^2t. \end{aligned} \tag{2.20.2}$$

$P$ is a singular point of $S$ if and only if all of the $f_x, f_y, f_z, f_w$ vanish on $(a,b,c,d)$. Substituting $a,b,c,d$ for $x,y,z,w$, and setting $f_x = f_y = f_z = f_w = 0$, we may view (2.20.2) as a system of four inhomogeneous linear equations with the unknowns $A$,

$B$, $C$, $D$:

$$\begin{aligned}
ad^2A + ac^2B + ab^2C + bcdD &= -2a^3\\
bc^2A + bd^2B + a^2bC + acdD &= -2b^3\\
b^2cA + a^2cB + cd^2C + abdD &= -2c^3\\
a^2dA + b^2dB + c^2dC + abcD &= -2d^3.
\end{aligned}$$

The trick to solving these equations is to subtract $a$ times the first equation from $b$ times the second equation. We obtain

$$(a^2d^2 - b^2c^2)A + (a^2c^2 - b^2d^2)B = b^4 - a^4.$$

Subtracting $c$ times the third equation from $d$ times the fourth equation, we get

$$(a^2d^2 - b^2c^2)A - (a^2c^2 - b^2d^2)B = c^4 - d^4.$$

Hence

$$\begin{aligned}
A &= \frac{b^4 + c^4 - a^4 - d^4}{a^2d^2 - b^2c^2}\\
B &= \frac{c^4 + a^4 - b^4 - d^4}{b^2d^2 - c^2a^2}.
\end{aligned} \tag{2.20.3}$$

By the symmetry of the situation,

$$C = \frac{a^4 + b^4 - c^4 - d^4}{c^2d^2 - a^2b^2}. \tag{2.20.4}$$

Once we know $A$, $B$ and $C$, we can calculate $D$ directly. The answer is

(2.20.5)

$$D = \frac{abcd(d^2 + a^2 - b^2 - c^2)(d^2 + b^2 - c^2 - a^2)(d^2 + c^2 - a^2 - b^2)(a^2 + b^2 + c^2 + d^2)}{(a^2d^2 - b^2c^2)(b^2d^2 - c^2a^2)(c^2d^2 - a^2b^2)}$$

Thus if the desired Kummer surface $S$ with equation (2.20.1) exists, $A$, $B$, $C$ and $D$ must be given by (2.20.3)–(2.20.5). Let us take (2.20.3)–(2.20.5) as the definition of $A$, $B$, $C$ and $D$ and prove that (2.20.1) is an equation of a Kummer surface whose singular locus is the $F_0$-orbit of $(a, b, c, d)$.

First, we prove a useful identity satisfied by $A$, $B$, $C$ and $D$.

**Lemma 2.21.**

$$4 - A^2 - B^2 - C^2 + ABC + D^2 = 0. \tag{2.21.1}$$

*Proof.* (2.21.1) can be rewritten

$$(A+2)(B+2)(C+2) = (A+B+C+2+D)(A+B+C+2-D). \tag{2.21.2}$$

To prove (2.21.2), write

$$A+2 = \frac{(b^2-c^2-a^2+d^2)(b^2-c^2+a^2-d^2)}{(a^2d^2-b^2c^2)}.$$

Hence by the symmetry of the situation,

$$\begin{aligned}&(A+2)(B+2)(C+2) = \\ &= -\frac{(a^2+b^2-c^2-d^2)^2(a^2+c^2-b^2-d^2)^2(a^2+d^2-c^2-b^2)^2}{(a^2d^2-b^2c^2)(b^2d^2-c^2a^2)(c^2d^2-a^2b^2)}.\end{aligned}$$

In order to transform the right hand side of (2.21.2), write

(2.21.3)
$$\begin{aligned}&A+B+C+2\pm D = A+2\pm D+ \\ &+\frac{(c^4+a^4-b^4-d^4)(c^2d^2-a^2b^2)+(a^4+b^4-c^4-d^4)(b^2d^2-a^2c^2)}{(c^2d^2-a^2b^2)(b^2d^2-a^2c^2)}\end{aligned}$$

The numerator of the last term of (2.21.3) can be rewritten as

$$\begin{aligned}&(a^4-d^4)(b^2d^2-c^2a^2+c^2d^2-a^2b^2)+(b^4-c^4)(b^2d^2-c^2a^2-c^2d^2+a^2b^2) = \\ &= (a^4-d^4)(b^2+c^2)(d^2-a^2)+(b^4-c^4)(b^2-c^2)(a^2+d^2) = \\ &= (b^2+c^2)(a^2+d^2)(a^2+b^2-c^2-d^2)(b^2+d^2-a^2-c^2).\end{aligned}$$

Dividing both sides of (2.21.2) by $(a^2+b^2-c^2-d^2)^2(b^2+d^2-a^2-c^2)^2$ and clearing the denominators, it remains to prove the identity

(2.21.4)
$$\begin{aligned}&-(a^2+d^2-b^2-c^2)^2(a^2d^2-b^2c^2)(b^2d^2-c^2a^2)(c^2d^2-a^2b^2) = \\ &= ((b^2d^2-c^2a^2)(c^2d^2-a^2b^2)+(b^2+c^2)(a^2+d^2)(a^2d^2-b^2c^2)+ \\ &+abcd(a^2+d^2-c^2-b^2)(a^2+d^2+c^2+b^2)) \\ &((b^2d^2-c^2a^2)(c^2d^2-a^2b^2)+(b^2+c^2)(a^2+d^2)(a^2d^2-b^2c^2)- \\ &-abcd(a^2+d^2-c^2-b^2)(a^2+d^2+c^2+b^2)).\end{aligned}$$

Now,

$$\begin{aligned}&(b^2d^2-c^2a^2)(c^2d^2-a^2b^2)+(b^2+c^2)(a^2+d^2)(a^2d^2-b^2c^2) = \\ &(a^2+d^2-b^2-c^2)(a^2b^2c^2+a^2d^2c^2+a^2b^2d^2+d^2b^2c^2).\end{aligned}$$

Hence we may divide both sides of (2.21.4) by $(a^2+d^2-b^2-c^2)^2$. It remains to prove that

(2.21.5)
$$\begin{aligned}&-(a^2d^2-b^2c^2)(b^2d^2-c^2a^2)(c^2d^2-a^2b^2)=\\&=(a^2b^2c^2+a^2d^2c^2+a^2b^2d^2+d^2b^2c^2)^2-a^2b^2c^2d^2(a^2+b^2+c^2+d^2)^2.\end{aligned}$$

It is easy to see that both sides of (2.21.5) are equal to

$$a^4c^4b^4+a^4c^4d^4+a^4d^4b^4+d^4c^4b^4-a^6b^2c^2d^2-b^6a^2c^2d^2-c^6b^2a^2d^2-d^6b^2c^2a^2$$

and the lemma is proved. □

By construction, $S$ has a singularity at $(a,b,c,d)$ and hence also at every other point in the $F_0$-orbit of $(a,b,c,d)$. In particular, $S$ has at least sixteen distinct singular points. We want to prove that all of these singularities are nodes and that there are no other singularities. First of all, note that under our assumptions (all $a,b,c,d\neq 0$) there are no singularities in the plane $t=0$. Indeed, by (2.20.2) if $t=f_t=0$ then $D=0$. But by (2.20.5) $D$ is a product of $abcd$ with a non-zero quantity (cf. (1.2.1)), so $D\neq 0$.

Therefore we may restrict attention to the coordinate chart $Spec\, k[\bar{x},\bar{y},\bar{z}]$, where denotes passing to the inhomogeneous coordinates. In that coordinate chart the singular locus of $S$ is defined by the inhomogeneous equation $\bar{f}$ together with its partial derivatives:

(2.21.6)
$$\begin{aligned}\bar{f}&=\bar{x}^4+\bar{y}^4+\bar{z}^4+1+2D\bar{x}\bar{y}\bar{z}+A(\bar{x}^2+\bar{y}^2\bar{z}^2)+B(\bar{y}^2+\bar{x}^2\bar{z}^2)+C(\bar{z}^2+\bar{x}^2\bar{y}^2)=0\\ \bar{f}_{\bar{x}}&=2\bar{x}^3+D\bar{y}\bar{z}+A\bar{x}+B\bar{x}\bar{z}^2+C\bar{x}\bar{y}^2=0\\ \bar{f}_{\bar{y}}&=2\bar{y}^3+D\bar{x}\bar{z}+A\bar{y}\bar{z}^2+B\bar{y}+C\bar{y}\bar{x}^2=0\\ \bar{f}_{\bar{z}}&=2\bar{z}^3+D\bar{x}\bar{y}+A\bar{z}\bar{y}^2+B\bar{z}\bar{x}^2+C\bar{z}=0\end{aligned}$$

We may replace $\bar{f}$ above by

$$g:=\frac{1}{4}(\bar{f}-\bar{x}\bar{f}_{\bar{x}}-\bar{y}f_{\bar{y}}-\bar{z}f_{\bar{z}})=2+D\bar{x}\bar{y}\bar{z}+A\bar{x}^2+B\bar{y}^2+C\bar{z}^2.$$

**Lemma 2.22.** *Let $f\in(x,y,z)k[x,y,z]_{(x,y,z)}$ and let $f_x,f_y,f_z$ denote the partial derivatives of $f$. The surface defined by $f$ has a node at the origin if and only if*

$$(f,f_x,f_y,f_z)k[x,y,z]_{(x,y,z)}=(x,y,z)k[x,y,z]_{(x,y,z)}. \tag{2.22.1}$$

*Proof.* "Only if" is trivial. We prove "if". Suppose (2.22.1) holds. Then the surface defined by $f$ has a singularity at the origin. Since some of the $f_x,f_y,f_z$ have multiplicity 1 at the origin, $f$ must have multiplicity 2 there. Write

$$f=f_2(x,y,z)+h(x,y,z),$$

where $f_2$ is a quadratic form in $x, y, z$ and $h \in (x, y, z)^3 k[x, y, z]_{(x,y,z)}$. Then all the partial derivatives of $h$ lie in $(x, y, z)^2 k[x, y, z]_{(x,y,z)}$, so (2.22.1) implies that the partial derivatives of $f_2$ generate the $k$-vector space $\frac{(x,y,z)}{(x,y,z)^2}$. But then $f_2$ defines a non-degenerate quadratic cone, so the singularity in question is a node. $\square$

Let $I \subset k[\bar{x}, \bar{y}, \bar{z}]$ be the ideal generated by $g, \bar{f}_{\bar{x}}, \bar{f}_{\bar{y}}$ and $\bar{f}_{\bar{z}}$. Since $S$ has at least 16 singularities in $t \neq 0$, $length\frac{k[\bar{x},\bar{y},\bar{z}]}{I} \geq 16$. $length\frac{k[\bar{x},\bar{y},\bar{z}]}{I}$ is finite if and only if all the singularities of $S$ are isolated and in that case it equals the sum of local contributions from all the singularities. Hence, by Lemma 2.22, $length\frac{k[\bar{x},\bar{y},\bar{z}]}{I} = 16$ if and only if all the 16 singularities are nodes and there are no other singularities. Thus it suffices to prove that

$$length\frac{k[\bar{x},\bar{y},\bar{z}]}{I} = 16. \tag{2.22.2}$$

We prove (2.22.2) by producing a set of generators of $\frac{k[\bar{x},\bar{y},\bar{z}]}{I}$ as a $k$-vector space.

Since all the equations in (2.21.6) have degree 3, the elements $1$, $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{x}^2$, $\bar{y}^2$, $\bar{z}^2$, $\bar{x}\bar{y}$, $\bar{y}\bar{z}$, $\bar{x}\bar{z}$ are $k$-linearly independent in $\frac{k[\bar{x},\bar{y},\bar{z}]}{I}$. They generate the 10-dimensional vector subspace of $\frac{k[\bar{x},\bar{y},\bar{z}]}{I}$ of all the polynomials of degree at most 2. Consider the vector subspace of $I$ of all the polynomials of degree at most 3 modulo those of degree 2 or less. This space is the subspace of the 10-dimensional vector space of the homogeneous cubic forms in $\bar{x}, \bar{y}, \bar{z}$, generated by the degree 3 parts of the equations (2.21.6):

$$\begin{aligned} 2\bar{x}^3 + B\bar{x}\bar{z}^2 + C\bar{x}\bar{y}^2 &= 0 \\ 2\bar{y}^3 + A\bar{y}\bar{z}^2 + C\bar{y}\bar{x}^2 &= 0 \\ 2\bar{z}^3 + A\bar{z}\bar{y}^2 + B\bar{z}\bar{x}^2 &= 0 \\ \bar{x}\bar{y}\bar{z} &= 0 \end{aligned} \tag{2.22.3}$$

It is obvious that these four equations are linearly independent. Hence we get 10-4=6 more generators of $\frac{k[\bar{x},\bar{y},\bar{z}]}{I}$, say, $\bar{x}^3$, $\bar{y}^3$, $\bar{z}^3$, $\bar{x}\bar{z}^2$, $\bar{y}\bar{z}^2$, $\bar{z}\bar{y}^2$. Altogether, we have 16 generators which generate the subspace of $\frac{k[\bar{x},\bar{y},\bar{z}]}{I}$ of all elements represented by polynomials of degree 3 or less. Let us denote all these generators by $\omega_1, \ldots, \omega_{16}$. We want to prove that every polynomial $h \in k[x, y, z]$ is of the form:

$$h = \sum_{i=1}^{16} a_i\omega_i + g \quad \text{with} \quad u \in I.$$

To prove this it is sufficient to prove that for every polynomial $h \in k[\bar{x}, \bar{y}, \bar{z}]$, $\deg h = 4$, there exists $u \in I$ such that:

$$\deg(h - u) \leq 3.$$

For each $h \in k[\bar{x}, \bar{y}, \bar{z}]$, let $\tilde{h}$:=terms of degree $\deg h$. Let

$$\tilde{I} := \{\tilde{h} \mid h \in I\}$$

and

$$\tilde{I}_4 := \{\tilde{h} \in \tilde{I} \mid \deg h = 4\}.$$

$\tilde{I}_4 \subset$homogeneous forms in $\bar{x}$, $\bar{y}$, $\bar{z}$ of degree 4, which is a 15-dimensional $k$-vector space. We want to prove that this inclusion is actually an equality, that is, $\tilde{I}_4$ contains all the monomials in $\bar{x}$, $\bar{y}$ and $\bar{z}$ of degree 4. We know that $\bar{x}^2\bar{y}z$, $\bar{x}\bar{y}^2\bar{z}$, $\bar{x}\bar{y}\bar{z}^2 \in \tilde{I}_4$, since $\bar{x}\bar{y}\bar{z} \in \tilde{I}_3$.

Multiplying the first equation in (2.22.3) by $\bar{z}$, the third equation by $\bar{x}$, and using the fact that $\bar{x}\bar{z}\bar{y}^2 \in \tilde{I}_4$, we get

$$\begin{aligned} &2\bar{x}^3\bar{z} + B\bar{z}^3\bar{x} \in \tilde{I}_4 \\ &2\bar{z}^3\bar{x} + B\bar{x}^3\bar{z} \in \tilde{I}_4. \end{aligned} \tag{2.22.4}$$

Since

(2.22.5)

$$B^2 - 4 = (B-2)(B+2) =$$
$$\frac{(a^2+b^2-c^2-d^2)(a^2+d^2-b^2-c^2)(a^2+c^2-b^2-d^2)(a^2+b^2+c^2+d^2)}{(b^2d^2-a^2c^2)^2} \neq 0,$$

(2.22.4) implies that $\bar{x}^3\bar{z}, \bar{z}^3\bar{x} \in \tilde{I}_4$. By symmetry, $\bar{x}^3\bar{y}, \bar{x}\bar{y}^3$, $\bar{z}\bar{y}^3$, $\bar{y}\bar{z}^3 \in \tilde{I}_4$. To prove that the remaining 6 monomials of degree 4 lie in $\tilde{I}_4$, observe that although $I$ is generated by $\bar{f}_{\bar{x}}$, $\bar{f}_{\bar{x}}$, $\bar{f}_{\bar{z}}$ and $g$, $\tilde{I}$ is not generated by $\tilde{\bar{f}}_{\bar{x}}$, $\tilde{\bar{f}}_{\bar{x}}$, $\tilde{\bar{f}}_{\bar{z}}$ and $\tilde{g}$. Namely, each of the polynomials

$$\begin{aligned} &D\bar{z}\bar{y}\bar{f}_{\bar{x}} - (2\bar{x}^2 + B\bar{z}^2 + C\bar{y}^2)g \\ &D\bar{z}\bar{x}\bar{f}_{\bar{y}} - (2\bar{y}^2 + A\bar{z}^2 + C\bar{x}^2)g \\ &D\bar{x}\bar{y}\bar{f}_{\bar{z}} - (2\bar{z}^2 + B\bar{x}^2 + A\bar{y}^2)g \end{aligned} \tag{2.22.6}$$

lies in $I$ and has degree 4, but its highest order part does not lie in the ideal generated by the cubic forms of (2.22.3). Calculating explicitly the higher order parts of the polynomials of (2.22.6) we obtain

$$\begin{aligned} &D\bar{y}^2\bar{z}^2 - 2B\bar{x}^2\bar{y}^2 - B^2\bar{y}^2\bar{z}^2 - BC\bar{y}^4 - 2C\bar{z}^2\bar{x}^2 - BC\bar{z}^4 - C^2\bar{y}^2\bar{z}^2 \in \tilde{I}_4 \\ &D\bar{x}^2\bar{z}^2 - 2A\bar{x}^2\bar{y}^2 - A^2\bar{x}^2\bar{z}^2 - AC\bar{x}^4 - 2C\bar{z}^2\bar{y}^2 - AC\bar{z}^4 - C^2\bar{x}^2\bar{z}^2 \in \tilde{I}_4 \\ &D\bar{x}^2\bar{y}^2 - 2A\bar{x}^2\bar{z}^2 - A^2\bar{x}^2\bar{y}^2 - AB\bar{x}^4 - 2B\bar{y}^2\bar{z}^2 - BA\bar{y}^4 - B^2\bar{x}^2\bar{y}^2 \in \tilde{I}_4. \end{aligned} \tag{2.22.7}$$

Multiplying the first three polynomials of (2.22.3) by $\bar{x}$, $\bar{y}$ and $\bar{z}$, respectively, we get

$$\begin{aligned} &2\bar{x}^4 + B\bar{x}^2\bar{z}^2 + C\bar{x}^2\bar{y}^2 \in \tilde{I}_4 \\ &2\bar{y}^4 + A\bar{z}^2\bar{y}^2 + C\bar{x}^2\bar{y}^2 \in \tilde{I}_4 \\ &2\bar{z}^4 + A\bar{z}^2\bar{y}^2 + B\bar{x}^2\bar{z}^2 \in \tilde{I}_4 \end{aligned} \tag{2.22.8}$$

To say that all the monomials $\bar{x}^4$, $\bar{y}^4$, $\bar{z}^4$, $\bar{x}^2\bar{y}^2$, $\bar{x}^2\bar{z}^2$, $\bar{y}^2\bar{z}^2$ belong to the vector space generated by the six forms of (2.22.7)–(2.22.8) is equivalent to saying that the matrix

$$\begin{pmatrix} 2 & 0 & 0 & C & B & 0 \\ 0 & 2 & 0 & C & 0 & A \\ 0 & 0 & 2 & 0 & B & A \\ 0 & -BC & -BC & -2B & -2C & D-B^2-C^2 \\ -AC & 0 & -AC & -2A & D-A^2-C^2 & -2C \\ -AB & -AB & 0 & D-A^2-B^2 & -2A & -2B^2 \end{pmatrix}$$

is non-singular. One computes the determinant explicitly, using Gaussian elimination and the identity (2.21.1). The answer is

$$2D^2(C^2-4)(A^2-4)(B^2-4) \neq 0$$

by (2.22.5) and our assumption that $abcd \neq 0$. This completes the proof of Theorem 2.20.

**Remark 2.23.** In fact, our proof of Theorem 2.20 shows that any $F_0$-invariant surface in $\mathbb{P}^3$ with equation of the form (2.20.1) and with at least one singularity in the set $U$, is a Kummer surface. Surfaces of the form (2.20.1) form an irreducible component in the Hilbert scheme of $F_0$-invariant quartics. The locus of Kummer surfaces is a subvariety of codimension 1 in that connected component, defined by the equation (2.21.1).

**Example 2.24.** Assume that *char* $k \neq 2,3$. Then the point $(a,b,c,d) = (1,1,1,0)$ satisfies conditions (1.2.1) and so belongs to the set $U$. Putting $(a,b,c,d) = (1,1,1,0)$ in (2.20) we obtain an explicit equation of a Kummer surface:

$$F(x,y,z,t) = x^4+y^4+z^4+t^4-x^2y^2-x^2z^2-x^2t^2-y^2t^2-y^2z^2-z^2t^2.$$

We can verify directly that this is a Kummer surface. Note that $F$ is invariant under the full symmetric group on $x$, $y$, $z$, $t$, and also under replacing any combination of coordinates by their negatives.

To verify that $F$ defines a Kummer surface, the first step is to locate its singularities. Taking partial derivatives, we find

$$\begin{aligned} F_x &= 4x^3 - 2x(y^2+z^2+t^2) \\ F_y &= 4y^3 - 2y(x^2+z^2+t^2) \\ F_z &= 4z^3 - 2z(x^2+y^2+t^2) \\ F_t &= 4t^3 - 2t(x^2+y^2+z^2) \end{aligned}$$

First, we consider the case $x=0$, so that $F_x = 0$. If $y=0$ also, then from $F_z = F_t = 0$ we find that $z=t=0$, a contradiction. Therefore $y \neq 0$. By symmetry, $z$, $t \neq 0$. Then the equations $F_y = F_z = F_t = 0$ can be easily solved to give $y = \pm 1$, $z = \pm 1$, $t = \pm 1$. Using the symmetries of $F$, we find that there is a singularity at (1,1,1,0) and at all the images of (1,1,1,0) under permutations and changing signs, a total of 16 singular points.

Next we show that there are no other singularities. The above calculations show that we have found all the singularities with at least one coordinate equal to zero. It remains to show that for $x$, $y$, $z$, $t$ all nonzero, there are no singular points. In that case, the equation $F_x = 0$ implies

$$2x^2 - y^2 - z^2 - t^2 = 0.$$

We can view this and its permutations $F_y = 0$, $F_z = 0$, $F_t = 0$ as a system of linear equations in $x^2$, $y^2$, $z^2$, $t^2$, whose matrix is

$$\begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 \\ -1 & -1 & 2 & -1 \\ -1 & -1 & -1 & 2 \end{pmatrix}$$

Its determinant is -27, so there are no nonzero solutions. Thus, the surface has no singularities except the 16 noted above.

Finally, we check that the singular points are nodes. By symmetry, it is enough to check one of them, say (1,1,1,0). Making a change of coordinates

$$\begin{aligned} x' &= x - z \\ y' &= y - z \\ z' &= z \\ t' &= t, \end{aligned}$$

our node becomes the point (0,0,1,0). Expanding with respect to $z'$, the coefficient of $z'^2$ is

$$4x'^2 + 4y'^2 - 4x'y' - 3t'^2$$

which is a nondegenerate form in $x'$, $y'$, $t'$. Thus, $F = 0$ defines a Kummer surface.

## §3. Divisors on a Kummer surface and its minimal desingularization.

Let $\pi : X \to S$ be the blowing-up of the Kummer surface $S$ at the 16 nodes $\{P_i\}_{1 \leq i \leq 16}$. $X$ is nonsingular, since a node is analytically isomorphic to a cone and if we blow up the vertex of the cone we desingularize it.

In this chapter we prove that $X$ is a K3 surface, with Picard number $\rho$ ($\rho$ :=rank NS(X)) satisfying $17 \leq \rho \leq 20$, if $char\, k = 0$, and $17 \leq \rho \leq 22$, if $char\ k > 0$. We study the geometric meaning of Rosenhain tetrahedra (Definition 1.21), whose number we computed in Lemma 1.23. We show how to associate a divisor $D \subset X$ to each Rosenhain tetrahedron. The linear equivalence class of $D$ is independent of the choice of the Rosenhain tetrahedron. The linear system $|D|$ defines an embedding of $X$ in $\mathbb{P}^5$ as a degree 8 surface, though we shall not prove it here. We also study the relations between the groups $W(S)$ (Weil divisors on $S$), $Pic(S)$ (Cartier divisors of $S$), and $Pic(X)$. More results on $W(S)$ will be given in Chapter 4.

**Lemma 3.1.** *Let $S$ be any surface of degree $d$ in $\mathbb{P}^3$. Then $h^1(S, \mathcal{O}_S) = 0$.*

*Proof.* Consider the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-d) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_S \to 0$$

and the corresponding long exact sequence for cohomology. Since $\mathbb{P}^3$ has no intermediate cohomology, $H^1(S, \mathcal{O}_S) = 0$, as desired. □

**Proposition 3.2.** *Let $S$ be a normal surface with a singular point $P$. Let $\pi : \tilde{S} \to S$ be the blowing up at $P$. Assume that $\pi^{-1}(P) \cong \mathbb{P}^1$ and that $\tilde{S}$ is non-singular at every point of $\pi^{-1}(P)$. We have $\pi_*(\mathcal{O}_{\tilde{S}}) \simeq \mathcal{O}_S$ and $R^i\pi_*(\mathcal{O}_{\tilde{S}}) = 0$ for $i > 0$. In particular, $H^i(S, \mathcal{O}_S) = H^i(\tilde{S}, \mathcal{O}_{\tilde{S}})$, for $i \geq 0$.*

*Proof.* Let $E = \pi^{-1}(P)$. $\tilde{S} - E \overset{\pi}{\simeq} S - P$, hence the natural map $\mathcal{O}_S \to \pi_*(\mathcal{O}_{\tilde{S}})$ is an isomorphism except, possibly, at $P$. Moreover, the sheaves $\mathcal{F}^i = R^i\pi_*(\mathcal{O}_{\tilde{S}})$, $i > 0$, have support at $P$. Let us use the theorem on formal functions [11, III, 11.1]. It says that $\hat{\mathcal{F}}^i = \varprojlim H^i(E_n, \mathcal{O}_{E_n})$, where $E_n$ is the closed subscheme of $\tilde{S}$ defined by $\mathcal{I}^n$ ($\mathcal{I}$ := ideal of $E$, and $\hat{}$ is the completion of the stalks at $P$). We have that $\frac{\mathcal{I}}{\mathcal{I}^2} \simeq \mathcal{O}_E(m)$ for some $m > 0$, and, since $\frac{\mathcal{I}^n}{\mathcal{I}^{n+1}} = S^n(\frac{\mathcal{I}}{\mathcal{I}^2})$, we obtain that $\frac{\mathcal{I}^n}{\mathcal{I}^{n+1}} \simeq \mathcal{O}_E(mn)$. Since $E \simeq \mathbb{P}^1$, $H^i(E, \mathcal{O}_E(mn)) = 0$, for all $i > 0$, $n > 0$, $m > 0$. Since we have, for each $n$, natural exact sequences:

$$0 \to \frac{\mathcal{I}^n}{\mathcal{I}^{n+1}} \to \mathcal{O}_{E_{n+1}} \to \mathcal{O}_{E_n} \to 0,$$

where $E_1 = E$, we obtain from the long exact sequence of cohomology, using induction on n, that $H^i(\mathcal{O}_{E_n}) = 0$, for all $i > 0$, $n > 0$. Thus, $\hat{\mathcal{F}}^i = 0$, for all $i > 0$. Since $\mathcal{F}^i$ is a coherent sheaf with support at $P$, $\mathcal{F}^i = \hat{\mathcal{F}}^i$, so $\mathcal{F}^i = 0$. Since $S$ is normal and $\pi$ is birational, $\pi_*\mathcal{O}_{\tilde{S}} \simeq \mathcal{O}_S$. By [11, III, Ex. 8.1] we conclude that $H^i(S, \mathcal{O}_S) \simeq H^i(\tilde{S}, \mathcal{O}_{\tilde{S}})$, for all $i \geq 0$. □

**Definition 3.3.** *A* **K3 surface** *$X$ is a nonsingular surface with $q(X) = h^1(X, \mathcal{O}_X)=0$ and canonical bundle $K_X=0$.*

**Theorem 3.4.** *$X$ is a K3 surface.*

*Proof.* $q(X) = h^1(X, \mathcal{O}_X) = 0$, because $h^1(S, \mathcal{O}_S) = 0$ (by Lemma 3.1), and cohomology remains unchanged by the blowing up of the nodes (by Proposition 3.2). We define the canonical divisor of $S$, $K_S$, as the divisor corresponding to the Grothendieck dualizing sheaf ([11, III.7]). Since $S$ is a hypersurface, it is Gorenstein, so the dualizing sheaf is locally free of rank 1 and $K_S$ is a well defined element of $Pic(S)$. Since $X - \cup E_i \simeq S - \cup P_i$, where $E_i = \pi^{-1}(P_i)$, and since we have $K_{S-\cup P_i} = 0$ because $K_S = 0$, we deduce that $K_X = \sum_{i=1}^{16} \alpha_i E_i$ for some $\alpha_i \in \mathbb{Z}$. By the adjunction formula, $E_i(E_i + K_X) = -2$, which implies that $K_X.E_i = -2\alpha_i = 0$, $1 \leq i \leq 16$. Thus $\alpha_i = 0$, $1 \leq i \leq 16$, so $K_X = 0$. $\square$

**3.5 Notation.** We denote by $b_i(Z)$ (resp. $\chi(Z) = \sum_i (-1)^i b_i(Z)$) the Betti numbers (resp. the Euler-Poincaré characteristic) of an algebraic variety $Z$ computed in étale or classical topology. Recall that one can define $b_i(Z)$ and $\chi(Z)$ of every scheme $Z$ of finite type over $k$ by using the l-adic étale cohomology:

$$b_i(Z,l) = \dim H^i_{\acute{e}t}(Z, \mathbb{Q}_l)$$
$$\chi(Z,l) = \sum_i (-1)^i b_i(Z,l)$$

If $Z$ is smooth and projective of dimension $d$, then $\chi(Z) = c_d(Z)$, where $c_d(Z)$ is the top Chern number of $Z$.

**Remark 3.6.** For a $K3$ surface $X$ over an algebraically closed field $k$ of arbitrary characteristic $b_2(X) = 22$, $b_1(X)=0$, $\chi(X, \mathcal{O}_X)=2$, $c_2(X) = \chi(X)=12\chi(X, \mathcal{O}_X)$ by Noether's formula. Thus $\chi(X)=24$. $h^2(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X(K_X))=1$, since $K_X \sim 0$. Hence $h^1(X, \mathcal{O}_X)=0$, [5, pp. 73–74].

**Definition 3.7.** *Let $F$ be a surface. The factor group of classes of divisors on $F$, modulo algebraic equivalence, is the* **Néron-Severi group of** $F$, *$NS(F)$. The* **Picard number of** $F$, *$\rho(F)$, is the rank of the Néron-Severi group of $F$. Let $F$ be nonsingular. $NS(F)$ is a finitely generated abelian group* [5, 0.7.2, p.69].

**3.8 Note.** Let *char* $k = 0$. Let $F$ be a nonsingular surface. Consider the exponential sequence

$$0 \to \mathbb{Z} \xrightarrow{f_1} \mathcal{O}_F \xrightarrow{f_2} \mathcal{O}_F{}^* \to 0.$$

Taking cohomology, we obtain:

$$\cdots \to Pic\, F \xrightarrow{\delta} H^2(F, \mathbb{Z}) \xrightarrow{f_1{}^*} H^2(F, \mathcal{O}_F) \to \dots$$

Let $S_F = Im(\delta) = Ker(f_1{}^*) \subset H^2(F, \mathbb{Z})$. The group $S_F$, consisting of algebraic cocycles, is the Néron-Severi group of $F$. From the exact sequence of cohomology above, it follows that

$$\rho(F) \leq h^{1,1} = b_2 - 2p_g,$$

where $b_2$ is the second Betti number of $F$ and $p_g = \dim H^2(F, \mathcal{O}_F)$ is the geometric genus of $F$.

For a $K3$ surface $X$, the map $c_1 : Pic(X) \to H^2(X, \mathbb{Z})$ is injective. Hence $Pic(X)$ is mapped isomorphically onto the Picard lattice. In particular, a divisor is never homologous to 0, unless it is linearly equivalent to 0.

**Lemma 3.9.** *Let $F$ be a nonsingular surface. Then for $l \neq char\ k$*

$$\rho(F) \leq b_2(F, l) = c_2(F) + 2q - 2,$$

*where $q = \dim(Pic_F{}^0)$.*

*Proof.* [14, Chapter V, Corollary 3.28]. □

**Proposition 3.10.** *Let $X$ be a $K3$ surface. Then,*

$$Pic(X) \simeq NS(X) \simeq Num(X) \simeq \mathbb{Z}^\rho,$$

*where $\rho(X) \leq 22$, $Num\,X = \frac{Pic(X)}{Pic^n(X)}$, $Pic^n(X)$ = subgroup of $Pic(X)$ of divisor classes numerically equivalent to 0, and $\rho$ is the Picard number of $X$.*

*Proof.* [5, Proposition 1.2.3]. □

**3.11 Remarks.** Note 3.8, Lemma 3.9 and Proposition 3.10 show that the possibilities for $\rho(X)$, where $X$ is a $K3$ surface, are:

(1) If *char* $k$=0,

$$1 \leq \rho(X) \leq h^{1,1}(X) = b_2 - 2p_g = 22 - 2 \cdot 1 = 20.$$

(2) If $char k = p > 0$,

$$1 \leq \rho(X) \leq 22.$$

**Proposition 3.12.** *Let $X$ be the blowing up of a Kummer surface $S$ in $\mathbb{P}^3$. If char $k$=0, then $17 \leq \rho(X) \leq 20$. If $char k = p > 2$, then $17 \leq \rho(X) \leq 22$.*

*Proof.* By Theorem 3.4, $X$ is a K3 surface. Since $X$ is the minimal desingularization of the Kummer surface $S$, we have 16 exceptional curves $E_i$, and the full preimage $\tilde{H}$ of the hyperplane section $H$ on $S$. Since $E_i^2 = -2$, $E_i.E_j = 0$ for $i \neq j$ and $\tilde{H}^2 = 4 \neq 0$, the $17 \times 17$ intersection matrix of $E_i$ and $\tilde{H}$ is clearly non-degenerate. Hence all the $E_i$ and $\tilde{H}$ are $\mathbb{Z}$-linearly independent in $NS(X)$, so $\rho(X) \geq 17$. If *char* $k$=0, by Remark 3.11, (1), $\rho(X) \leq 20$. If $char\ k = p > 2$, by Remark 3.11, (2), $\rho(X) \leq 22$. □

**3.13 Notation.** We denote by $Pic(S)$ the group of Cartier divisors of $S$, which is, by definition, the group generated by locally principal subschemes modulo linear equivalence. Let $W(S)$ denote the group of Weil divisors of $S$ modulo linear equivalence. Let us define the map $\pi^* : Pic(S) \to Pic(X)$. Let $E$ be an effective Cartier divisor on $S$. Then $E$ can be considered as a subscheme of $S$. Let $\mathcal{I}$ be the sheaf of ideals defining $E$. We define $\pi^*(E)$ as the subscheme defined by the inverse image ideal sheaf $\pi^{-1}\mathcal{I}.\mathcal{O}_X$. Let $\tilde{\pi}$ be the map

$$\tilde{\pi} : \{\text{effective curves on S}\} \rightarrow W(X),$$

defined by $\tilde{\pi}(D)$=strict transform of $D$, which is obtained as the closure in $X$ of $\pi^{-1}(D \mid S - \{\text{singular points}\})$. Let $D$ be an effective Cartier divisor in $S$. $\pi^*(D)$ differs from $\tilde{\pi}(D)$ by a linear combination of the exceptional divisors $E_i$. Let $E$ be a Cartier divisor on $X$. Let $\pi(E)$ be the Weil divisor on $S$ which is the closure of the image of $\pi(E|_{X-\cup E_i})$ under the isomorphism $\pi : X - \cup E_i \rightarrow S - \cup P_i$ (note that on a nonsingular variety Weil divisors=Cartier divisors, and the group of Weil divisors on $S$ is naturally bijective to the group of Weil divisors on $S - \cup P_i$).

**Lemma 3.14.** *If $F$ is a Weil divisor on $S$, then $2F$ is a Cartier divisor on $S$.*

*Proof.* Let $F$ be a Weil divisor on $S$. We have to show that $2F$ is locally principal at every point of $S$. At the non-singular points of $F$ this is obvious, since $F$ is a Weil divisor, hence a Cartier divisor. Let $P$ be a node of $S$. We want to show that the natural image of $2F$ in $\mathcal{O}_{S,P}$ is a Cartier divisor. For that it is sufficient to show that the divisor class group $Cl(Spec\,\mathcal{O}_{S,P})$ (which is, by definition, the Weil divisors on $Spec\,\mathcal{O}_{S,P}$ modulo Cartier divisors on $Spec\,\mathcal{O}_{S,P}$) is equal to $\frac{\mathbb{Z}}{2\mathbb{Z}}$. Consider the natural map

$$\phi : \mathcal{O}_{S,P} \rightarrow \hat{\mathcal{O}}_{S,P}$$

from $\mathcal{O}_{S,P}$ to its formal completion. $\phi$ induces a map

$$\phi_* : Cl(Spec\mathcal{O}_{S,P}) \rightarrow Cl(Spec\hat{\mathcal{O}}_{S,P}).$$

given by sending each prime ideal $I \subset \mathcal{O}_{S,P}$ of height 1 to $I\hat{\mathcal{O}}_{S,P}$. Since $\phi$ is faithfully flat, $\phi_*$ is an injection. Since $Cl(\mathcal{O}_{S,P}) \neq 0$ (a non-trivial element is given by any curve passing through $P$ and non-singular at $P$), to prove that $\phi_*$ is an isomorphism it is sufficient to prove that

$$Cl(Spec\hat{\mathcal{O}}_{S,P}) = \frac{\mathbb{Z}}{2\mathbb{Z}}.$$

But $\hat{\mathcal{O}}_{S,P} = \frac{k[[x,y,z]]}{(x^2+y^2+z^2)}$. Since $\frac{k[[x,y,z]]}{(x^2+y^2+z^2)}$ is also the formal completion of the local ring of the vertex of a non-degenerate quadric cone in $\mathbb{A}^3_k$, applying the above reasoning in the converse direction, the problem reduces to proving

$$Cl\left(Spec\frac{k[x,y,z]_{(x,y,z)}}{(x^2+y^2+z^2)}\right) = \frac{\mathbb{Z}}{2\mathbb{Z}}.$$

This is done in ([11, III, Exercises 6.5.2, 6.11.3]).

**Lemma 3.15.** *Let $S$ be any surface and $P$ a point on $S$. Let $\psi : \tilde{S} \to S$ be the blowing up of $P$ and assume that $E := \psi^{-1}(P)$, taken with its reduced structure, is a smooth curve on $\tilde{S}$. Let $C$ be a curve on $S$ passing through $P$, such that $P$ is a non-singular point of $C$. Let $\tilde{C}$ denote the strict transform of $C$ in $\tilde{S}$. Then $\tilde{C}.E = 1$.*

*Proof.* Locally, we can always embed $S$ in a non-singular variety $Z$. Let $\Psi : \tilde{Z} \to Z$ be the blowing up of $Z$ at $P$. Then $\tilde{S}$ is the strict transform of $S$ under $\Psi$ and, set-theoretically, $E = \Psi^{-1}(P) \cap \tilde{S}$. Now, $\Psi^{-1}(P)$ is a projective space of dimension $\dim Z - 1$, containing the smooth curve $E$. $\tilde{C}$ is a curve in $\tilde{Z}$, intersecting $\Psi^{-1}(P)$ *transversely* in exactly one point, belonging to $E$. Hence the intersection number $E.\tilde{C}$ on $\tilde{S}$ is well defined and equal to 1. □

**Lemma 3.16.** *Let $S$ be a Kummer surface as before. Let $H$ be a special plane of $S$ and $C$ the special conic determined by $H$. Let $P_i$, $1 \le i \le 6$ denote the nodes of $S$ belonging to $C$. Then*

$$\pi^*(H \cap S) = 2\tilde{C} + \sum_{i=1}^{6} E_i,$$

*where $\tilde{C} = \tilde{\pi}(C)$, and $E_i$ are the exceptional divisors associated to $P_1, \dots, P_6$.*

*Proof.* Scheme-theoretically, $H \cap S = 2C$ away from the singular points of $S$. Hence

$$\pi^*(H \cap S) = 2\tilde{C} + \sum_{i=1}^{6} \alpha_i E_i,$$

where $\alpha_i \in \mathbb{Z}$. Since $2C$ is a very ample divisor on $S$, it can be moved by linear equivalence to a divisor which does not pass through any of the $P_i$. Hence $\pi^*(2C).E_i = 0$, $1 \le i \le 6$. Since $C$ is a non-singular curve passing through $P_i$, $\tilde{C}$ intersects $E_i$ transversely in exactly one point by Lemma 3.15. Hence

$$0 = \pi^*(2C).E_i = 2\tilde{C}.E_i + \alpha_i E_i^2 = 2 + \alpha_i E_i^2.$$

Since $E_i^2 = -2$, $\alpha_i = 1$, as desired.

**Theorem 3.17.** *Let $E_i \in Pic(X)$ denote the 16 exceptional divisors. With respect to the intersection pairing in $Pic(X)$ the $E_i$ are all orthogonal to each other and to $\pi^*(Pic(S))$, so there is a natural inclusion:*

$$\pi^*(Pic(S)) \oplus \mathbb{Z}^{16} \subset Pic(X).$$

*Moreover, $2Pic(X) \subset \pi^*(Pic(S)) \oplus \mathbb{Z}^{16}$.*

*Proof.* The first assertion follows from the fact that all the $E_i$ are disjoint and any Cartier divisor on $S$ can be moved by linear equivalence to one not passing through any of the $P_i$.

For the second assertion, take a Cartier divisor $D$ on $X$. Consider the Weil divisor $\pi(D)$ on $S$ which we shall call $F$. Then $2D$ and $\pi^*(2F)$ differ by a linear combination of the $E_i$. □

**Corollary 3.18.** $\rho(X) = \rho(S)+16$. *In char* $k = 0, 1 \le \rho(S) \le 4$; *in char* $k = p > 0$, $1 \le \rho(S) \le 6$.

*Proof.* The first assertion is clear from Theorem 3.17. The possibilities for $\rho(S)$ in *char* $k = 0$, and in *char* $k > 0$ follow from the first assertion and 3.12.

**Lemma 3.19.** *Consider a (16,6) configuration of type (*). Let* $P_0$ *be a point of the configuration and* $h_1$, $h_2$, $h_3$ *three planes containing* $P_0$. *Let* $P_{12}$, $P_{13}$, $P_{23}$ *be the other points on the lines* $h_1h_2$, $h_1h_3$, $h_2h_3$, *respectively. Then there is a plane* $h_0$ *of the (16,6) configuration containing* $P_{12}$, $P_{13}$, $P_{23}$.

*Proof.* The question is purely a matter of the incidence correspondence of the (16,6) configuration, as expressed in 1.7. The first step is to show that the incidence relations of 1.7 have enough automorphisms so that we can reduce the proof to checking only a few cases.

**Sublemma 3.20.** *Any permutation of the rows and the columns of 1.7 preserves the incidence correspondence. Furthermore, rows can be exchanged with columns by reflecting around the main diagonal.*

*Proof.* Trivial.

Let us continue the proof of Lemma 3.19:

Using permutations of rows and columns, we see that the group of automorphisms of the abstract (16,6) configuration is transitive on points. So we may take $P_0$ to be the upper left corner. Similarly, the other three points of the column of $P_0$ may be permuted, those of the row of $P_0$ may be permuted, and first column can be exchanged with first row. This shows that we can reduce the proof to two cases, as follows. Let $P_0$ be any point of the configuration, and $h_1$, $h_2$, $h_3$ any three planes containing $P_0$. It will be sufficient to consider the following two cases:

(1)

$$\begin{array}{cccc} P_0 & \cdot & \cdot & \cdot \\ h_1 & \cdot & \cdot & \cdot \\ h_2 & \cdot & \cdot & \cdot \\ h_3 & \cdot & \cdot & \cdot \end{array}$$

(2)

$$\begin{array}{cccc} P_0 & h_3 & \cdot & \cdot \\ h_1 & \cdot & \cdot & \cdot \\ h_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}$$

Now the proof is easy. There exists a plane $h_0$ containing $P_{12}$, $P_{13}$, $P_{23}$, as follows:

(1)

$$\begin{array}{cccc} h_0 & \cdot & \cdot & \cdot \\ P_{23} & \cdot & \cdot & \cdot \\ P_{13} & \cdot & \cdot & \cdot \\ P_{12} & \cdot & \cdot & \cdot \end{array}$$

(2)

$$\begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & P_{13} & \cdot & \cdot \\ \cdot & P_{23} & \cdot & \cdot \\ P_{12} & h_0 & \cdot & \cdot \end{array}$$

□

**Corollary 3.21.** *There exist 80 Rosenhain tetrahedra in a Kummer surface $S$ (see 1.21 for the definition of a Rosenhain tetrahedron and 1.23 for another proof of Corollary 3.21).*

*Proof.* By 2.18 the nodes and special planes of a Kummer surface form a (16,6) configuration of type (*). By Lemma 3.19, in any (16,6) configuration of type (*) we can find four planes $h_0$, $h_1$, $h_2$, $h_3$, and four points $P_0$, $P_{12}$, $P_{13}$, $P_{23}$, satisfying the required incidence relations to form a Rosenhain tetrahedron. There exist 80 of them since we have $\binom{6}{3} = 20$ choices of the lines $h_1h_2$, $h_1h_3$, $h_2h_3$. We have 20 tetrahedra with vertex $P_0$; each of them contains 4 nodes as vertices. Since there are a total of 16 nodes in $S$, we have

$$\frac{16 \cdot 20}{4} = 80$$

Rosenhain tetrahedra.

**Proposition 3.22.** *For any Rosenhain tetrahedron, let $D$ be the divisor on $X$ formed by the sum of the proper transforms of the 4 conics in which the planes meet $S$ and the four exceptional curves, inverse images of the 4 nodes:*

$$D = \tilde{C}_0 + \tilde{C}_1 + \tilde{C}_2 + \tilde{C}_3 + E_0 + E_{12} + E_{13} + E_{23},$$

*in the above notation. Then, the linear equivalence class of $D$ is independent of the Rosenhain tetrahedron chosen.*

*Proof.* We first show that

$$2D \sim 4\pi^*(H \cap S) - \sum_{i=1}^{16} E_i. \tag{3.22.1}$$

To prove this we use Lemma 3.15 which says that

$$\pi^*(H \cap S) = 2\tilde{C} + \sum_{i=1}^{6} E_i,$$

where $E_i$, $1 \leq i \leq 6$ are the exceptional divisors meeting $\tilde{C}$. We need to verify the count of the $E_i$ in 3.22.1, which uses the incidence correspondence. Let $h_0$, $h_1$, $h_2$ and $h_3$ be the four special planes corresponding to the special conics $C_i$, $0 \leq i \leq 3$.

As in the proof of 3.19 we can reduce to proving 3.22.1 in the two cases described by the following diagrams:

(1)

$$\begin{array}{cccc} P_0 & \cdot & \cdot & \cdot \\ h_1 & \cdot & \cdot & \cdot \\ h_2 & \cdot & \cdot & \cdot \\ h_3 & \cdot & \cdot & \cdot \end{array}$$

(2)

$$\begin{array}{cccc} P_0 & h_3 & \cdot & \cdot \\ h_1 & \cdot & \cdot & \cdot \\ h_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{array}$$

We know by 3.19 that, in both cases, there is a plane $h_0$ of the (16,6) configuration containing $P_{12}$, $P_{13}$, $P_{23}$.

Denote the nodes of $S$ as follows:

$$\begin{array}{cccc} P_0 & P_4 & P_5 & P_6 \\ P_{23} & P_{14} & P_{15} & P_{16} \\ P_{13} & P_{24} & P_{25} & P_{26} \\ P_{12} & P_{34} & P_{35} & P_{36} \end{array}$$

in case (1) and

$$\begin{array}{cccc} P_0 & P_4 & P_{35} & P_{36} \\ P_{14} & P_{13} & P_{15} & P_{16} \\ P_{24} & P_{23} & P_{25} & P_{26} \\ P_{12} & P_{34} & P_5 & P_6 \end{array}$$

in case (2). In either case, $h_0$, $h_1$, $h_2$, $h_3$ contain the following nodes:

$$\begin{aligned} h_0 &: P_{12}, P_{13}, P_{23}, P_4, P_5, P_6 \\ h_1 &: P_0, P_{12}, P_{13}, P_{14}, P_{15}, P_{16} \\ h_2 &: P_0, P_{12}, P_{23}, P_{24}, P_{25}, P_{26} \\ h_3 &: P_0, P_{13}, P_{23}, P_{34}, P_{35}, P_{36} \end{aligned} \tag{3.22.2}$$

Let us substitute this in 3.22.1:

$$\begin{aligned} 4\pi^*(H \cap S) \sim & 2(\tilde{C}_0 + \tilde{C}_1) \\ & + 2(\tilde{C}_2 + \tilde{C}_3) \\ & + 3(E_0 + E_{12} + E_{13} + E_{23}) \\ & + \sum_{l \in J} E_l \end{aligned} \tag{3.22.3}$$

where

$$J = \{4, 5, 6, 14, 15, 16, 24, 25, 26, 34, 35, 36\}.$$

Consequently,

$$2D \sim 4\pi^*(H \cap S) - \sum E_i,$$

where the sum is taken over all the sixteen exceptional divisors. This shows that $2D$ is independent of the choice of the Rosenhain tetrahedron. By 3.10, $Pic(X)$ has no torsion, so the same is true for $D$. The Proposition is proved. $\square$

**Corollary 3.23.** *Let $X$ be the minimal desingularization of the Kummer surface $S$ with exceptional curves $E_i = \pi^{-1}(P_i)$, $1 \leq i \leq 16$, as above. Then $\sum_{i=1}^{16} E_i \in 2Pic(X)$.*

*Proof.* By Proposition 3.22,

$$\sum_{i=1}^{16} E_i \sim 4\pi^*(H) - 2D,$$

so $\sum_{i=1}^{16} E_i \sim 2(2\pi^*(H) - D) \in 2Pic(X)$.

**Remark 3.24.** From (3.22.1) one computes that $D^2 = 8$. Hence by [17] dim $|D| = \frac{D^2}{2} + 1 = 5$, so that $|D|$ defines a rational map $X \to \mathbb{P}^5$. In fact this rational map is a morphism giving a closed embedding of $X$ in $\mathbb{P}^5$ as a complete intersection of three quadrics, but we will not prove it here.

## §4. Geometry of a Kummer surface in $\mathbb{P}^3$ and the associated abelian variety.

In this Chapter we study the relations between Kummer surfaces in $\mathbb{P}^3$ and abelian surfaces. We show that every Kummer surface is the quotient of an abelian surface by the involution and, conversely, the quotient of an abelian surface which is the Jacobian of a non-singular curve of genus 2 by the involution is a Kummer surface. We prove that a Kummer surface is self-dual and contains no lines. For a Kummer surface $S$ with Picard number 1 we prove that all the curves on it must have even degree, that $Pic(S)$ is generated by the hyperplane section of $S$, that the 2-torsion subgroup Tor(W(S)) of the group $W(S)$ of Weil divisors is $(\frac{\mathbb{Z}}{2\mathbb{Z}})^{15}$ and that every Weil divisor on $S$ is a linear combination of the special conics of $S$.

**Definition 4.1.** *An* **abelian surface** *$A$ is a complete algebraic surface over $k$ with a group law $m : A \times A \to A$ such that $m$ and the inverse map*

$$A \to A$$
$$x \to -x$$

*induced by $m$ are both morphisms.*

For an element $a \in A$, let $t_a : A \to A$ denote the translation by $a$. Let $A^*$ denote the dual abelian variety $Pic^0(A)$. For an invertible sheaf $\mathcal{L}$ on $A$, define the map $\phi_{\mathcal{L}} : A \to A^*$ by

$$\phi_{\mathcal{L}}(a) := t_a^*\mathcal{L} \otimes \mathcal{L}^{-1},$$

for $a \in A$.

**Definition 4.2.** *A* **polarization** *on an abelian variety $A$ is an isogeny $\lambda : A \to A^*$ such that $\lambda = \phi_{\mathcal{L}}$ for some ample invertible sheaf $\mathcal{L}$ on $A$. The degree of a polarization is its degree as an isogeny. $\lambda$ is called* **principal** *if it has degree 1.*

**Lemma 4.3.** *Let $A$ be an abelian variety. Let $\mathcal{L}$ be an ample invertible sheaf which defines a polarization $\lambda$. Then another invertible sheaf $\mathcal{M}$ defines the same polarization if and only if there exists $a \in A$ with $\mathcal{M} = t_a{}^*\mathcal{L}$.*

*Proof.* First suppose that $\mathcal{M} = t_a{}^*\mathcal{L}$. For any $b$

$$\phi_{\mathcal{L}}(b) = t_b{}^*\mathcal{L} \otimes \mathcal{L}^{-1}$$
$$\phi_{\mathcal{M}}(b) = t_b{}^*\mathcal{M} \otimes \mathcal{M}^{-1} = t_{a+b}{}^*\mathcal{L} \otimes t_a{}^*\mathcal{L}^{-1}.$$

These two sheaves are isomorphic by the Theorem of the square [13, p. 112].

Conversely, suppose that $\phi_{\mathcal{M}} = \phi_{\mathcal{L}}$. Then, for all $a \in A$,

$$t_a{}^*\mathcal{L} \otimes \mathcal{L}^{-1} \simeq t_a{}^*\mathcal{M} \otimes \mathcal{M}^{-1},$$

so,

$$t_a{}^*(\mathcal{L} \otimes \mathcal{M}^{-1}) \simeq \mathcal{L} \otimes \mathcal{M}^{-1}.$$

Then the sheaf $\mathcal{L} \otimes \mathcal{M}^{-1}$ satisfies the conditions of [13, 9.2, p. 117] and so is an element of $Pic^0(A) = A^*$, [13, p. 118]. Now, since $\lambda$ is an isogeny, it is surjective, so there is an $a$ such that $\phi_{\mathcal{M}}(a) = \mathcal{L} \otimes \mathcal{M}^{-1}$, i.e.

$$t_a{}^*\mathcal{M} \otimes \mathcal{M}^{-1} = \mathcal{L} \otimes \mathcal{M}^{-1},$$

hence $\mathcal{L} = t_a{}^*\mathcal{M}$, and $\mathcal{M} = t_{-a}{}^*\mathcal{L}$.

**Corollary 4.4.** *To give a polarization $\lambda$ on $A$ is equivalent to give an ample invertible sheaf $\mathcal{L}$ on $A$, such that $\phi_{\mathcal{L}}$ is an isogeny, and $\mathcal{L}$ is defined up to translation.*

**Lemma 4.5.** *If $\mathcal{L}$ is effective (i.e. $H^0(\mathcal{L}) \neq 0$), then the following statements are equivalent:*

(1) *$\mathcal{L}$ is ample.*
(2) *$Ker\ \phi_{\mathcal{L}}$ has dimension 0.*
(3) *$\phi_{\mathcal{L}}$ is an isogeny.*

*Proof.* [13, 9.1, p. 117].

**Lemma 4.6.** *If $\mathcal{L} = \mathcal{L}(D)$ is an ample invertible sheaf giving a polarization $\lambda$, the following holds.*

(1) *$\chi(\mathcal{L}) = \frac{D^g}{g!}$, where $g = dim\, A$.*
(2) *$deg\, \phi_{\mathcal{L}} = \chi(\mathcal{L})^2$.*
(3) *There is exactly one integer $r$ for which $H^r(A, \mathcal{L}) \neq 0$.*

*Proof.* [13, 13.3].

Now let $\mathcal{L} = \mathcal{L}(D)$, $D$ not necessarily effective, be an ample divisor on an abelian surface $A$ which gives a polarization. Then $D^2$ must be positive, since $D$ is ample, so $\chi(\mathcal{L}) > 0$. Therefore, the integer $r$ of Lemma 4.6, (3), must be 0 or 2. If $r = 2$, then $H^2(A, \mathcal{L}(D)) \neq 0$. By Serre duality, $H^0(A, \mathcal{L}(-D)) \neq 0$. Then $-D$ is effective, which is impossible for an ample divisor (since $nD$ is also effective for $n >> 0$). We conclude that $r = 0$, so $D$ is necessarily effective. We obtain

**Proposition 4.7.** *On an abelian surface $A$, to give a polarization is equivalent to giving an ample, effective divisor $D$, defined up to translation.*

**Note 4.8.** Let $\mathcal{L}$ give a principal polarization. Then $deg\, \phi_{\mathcal{L}} = 1$, so by 4.6 $\chi(\mathcal{L}) = 1$, $D^2 = 2$ and $h^0(\mathcal{L}(D)) = 1$. Thus, $D$ does not move for linear equivalence. Hence, to give a principal polarization $\lambda$ on an abelian surface is equivalent to giving an effective divisor $D$ with $D^2 = 2$, up to translation. $D$ does not move for linear equivalence, so knowing one such $D$, all others are translates of it. In particular, the properties "$D$ reduced, $D$ nonsingular, $D$ irreducible" are independent of the $D$ chosen.

Next we ask: what kind of curve $D$ can give a principal polarization on an abelian surface $A$?

**Lemma 4.9.** *If $C$ is an irreducible curve on an abelian surface $A$, then $C^2 \geq 0$.*

*Proof.* Suppose $0 \in C$. Let $a \notin C$. Then $t_aC$ and $C$ are distinct and numerically equivalent, so
$$C^2 = C.t_aC \geq 0.$$

**Proposition 4.10.** *Let $D$ give a principal polarization on an abelian surface. Then either*

(1) *$D$ is a nonsingular curve of genus 2, or*
(2) *$D$ is a union of two nonsingular elliptic curves meeting in one point.*

*Proof.* First of all, since $D$ is ample, it is connected. Let $D = \sum_i n_iC_i$, with $C_i$ irreducible. Then,
$$2 = D^2 = \sum_i n_iC_iD, \qquad C_i.D > 0.$$

Hence $D = C_1 + C_2$. By Lemma 4.9, $({C_1}^2) = ({C_2}^2) = 0$ and $(C_1.C_2) = 1$. From ${C_1}^2 = {C_2}^2 = 0$ we conclude that $p_a(C_1) = p_a(C_2) = 1$. An irreducible curve with $p_a = 1$ is either nonsingular elliptic or rational with a singularity. But there are no nonconstant maps of a rational curve to an abelian variety, so $C_1$, $C_2$ are both nonsingular elliptic. This is case (2).

Otherwise, $D$ must be irreducible, say $D = nC$. Then $2 = D^2 = n^2C^2$, so $n = 1$ and $D$ is reduced. In this case, $D$ could be nonsingular of genus 2, or an elliptic curve with just one node or one cusp as singularity. However, the latter case cannot occur. Indeed, the normalization $\tilde{D}$ would be an elliptic curve with a morphism $\tilde{D} \to A$. Any morphism of abelian varieties is, up to translation, equal to a homomorphism. Thus, $D =$ image of $\tilde{D}$ will be the quotient of $\tilde{D}$ by a finite subgroup, hence nonsingular.

**Examples 4.11.**

(1) Let $C$ be a nonsingular curve of genus 2. Then its Jacobian $J$ contains $C$ and $C$ gives a principal polarization of $J$. [13, 6.6, p. 186]. The image of $C$ in $J$ is traditionally called a theta divisor on $J$.
(2) Let $E_1$, $E_2$ be elliptic curves, and let $P_1 \in E_1$, $P_2 \in E_2$ be points. Then $A = E_1 \times E_2$ is an abelian surface, and $D = P_1 \times E_2 + E_1 \times P_2$ gives a principal polarization.

**Proposition 4.12.** *If $A$ is a principally polarized abelian surface whose polarization class is represented by an irreducible nonsingular curve $C$, then $A$ is isomorphic (as principally polarized abelian surface) to the Jacobian $J(C)$.*

*Proof.* The map $C \to A$ induces, by the property of the Jacobian as Albanese variety of $C$, a map $\pi : J \to A$ such that $\pi(C) = C$ [13, 6.1, p. 186]. Let us denote $C \subset J$ by $C_1$ and $C \subset A$ by $C_2$. Then $\pi_*(C_1) = C_2$ in the sense of divisors since the map $C_1 \to C_2$ is an isomorphism. On the other hand, $C_2^2 = 2$ since $C_2$ gives a principal polarization of $A$. By the projection formula,
$$C_1.\pi^*C_2 = \pi_*C_1.C_2 = {C_2}^2 = 2.$$

If $\pi$ were not an isomorphism, then $\pi^*C_2 = C_1 \cup C'$, for some other curve $C'$. We know ${C_1}^2 = 2$ on $J$. Since $C_1$ is ample, $C_1.C' > 0$. Then

$$C_1\pi^*C_2 = {C_1}^2 + C_1.C' = 2 + C_1.C' > 2,$$

which is a contradiction. We conclude that $deg\, \pi = 1$, so $J \simeq A$.

**Proposition 4.13.** *If $A$ is a principally polarized abelian surface whose polarization class is represented by the union of two elliptic curves $E_1 \cup E_2$ meeting at a point, then $A \cong E_1 \times E_2$ with the product polarization.*

*Proof.* Translating, we may assume that the intersection point of $E_1$ and $E_2$ is the zero element of $A$. We may also assume that the intersection point $E_1 \cap E_2$ is the origin of each of the elliptic curves. Then the inclusions $E_1 \to A$ and $E_2 \to A$ are group homomorphisms. Thus we obtain a group homomorphism $E_1 \times E_2 \to A$. The same argument as in the previous proposition shows that deg $\pi = 1$, so $\pi$ is an isomorhism. □

**Corollary 4.14.** *Any principally polarized abelian surface is isomorphic, as a principally polarized abelian variety, either to the Jacobian of a nonsingular curve of genus 2, with its standard theta divisor, or to a product of two elliptic curves with the product polarization.*

**Proposition 4.15.** *Let $B$ be a nonsingular curve of genus 2 and $A = J(B)$ its Jacobian. Fix a Weierstrass point $q_1$ on $B$ and let $C_1$ be the theta divisor on $A$ which is the image of the map*

$$\begin{aligned} &B \hookrightarrow A \\ &q \to q - q_1 \in Pic^0(B) = A. \end{aligned}$$

*Then the sixteen elements of $A_2$ and the sixteen translates of $C_1$ by elements of $A_2$ form an abstract (16,6) configuration of type (*) (a (16,6) configuration of type (*) is just an incidence relation between two sets of 16 objects each, so it makes sense to talk about a (16,6) configuration of type (*), where one of the sets is composed of 16 points and the other set has 16 curves).*

*Proof.* $B$ can be realized as the locus of $y^2 = \prod_{i=1}^6 (x - s_i)$, where $q_i = (s_i, 0)$, $1 \le i \le 6$. Consider the points $P_i := (q_i - q_1), P_{ij} := (q_i + q_j - 2q_1) \in Pic^0(B) = A$, $i = 1, \dots, 6$. Since no pair $q_i + q_j$ is linearly equivalent to another pair $q_k + q_l$, the points $P_i$, $P_{ij}$ are all distinct. Since the hyperelliptic series on $B$ contains divisors $2q_i$, the $P_i$ and $P_{ij}$ are precisely the 16 2-torsion points of $A$. The group law on the points $P_i$, $P_{ij}$ is easily written down:

$$P_i + P_j = (q_i - q_1) + (q_j - q_1) = q_i + q_j - 2q_1 = P_{ij}.$$

Since the meromorphic function $f(x,y) = \frac{y}{(x-s_1)^3} = \sqrt{\frac{(x-s_1)\dots(x-s_6)}{(x-s_1)^6}}$ on $B$ has divisor

$$(f) = \sum_{i=1}^{6} q_i - 6q_1 = \sum_{i=2}^{6} q_i - 5q_1, \tag{4.15.1}$$

we see that

$$P_i + P_{jk} \sim (q_i + q_j + q_k - 3q_1) \sim (-q_l - q_m + 2q_1) \sim -P_{lm} = P_{lm},$$

where $i$, $j$, $k$, $l$, $m$ are all distinct. Similarly,

$$P_{ij} + P_{kl} \sim (q_i + q_j + q_k + q_l - 4q_1) \sim (-q_m + q_1) \sim -P_m = P_m,$$

where $i$, $j$, $k$, $l$, $m$ are all distinct.

The standard theta function $C_1 = \{(q - q_1) \mid q \in B\} \subset A$ contains the 6 half-lattice points $\{P_i\}$, $1 \leq i \leq 6$. Its translate $C_i := C_1 + P_i = \{(q + q_i - 2q_1)/q \in B\}$ contains the 6 half-lattice points $P_1$, $P_i$, $P_{ij}$, $j \neq 1, i$. The theta divisor $C_{ij} := C_1 + P_{ij} = \{(q + q_i + q_j - 3q_1) \mid q \in B\}$ contains the 6 points $P_i$, $P_{ij}$, $P_{lm}$, where $l, m \neq 1, i, j$. Conversely, each of the half-lattice points $P_i$, $P_{ij}$ lies on exactly 6 of the divisors $C_i$, $C_{ij}$:

$$P_i \in C_1, C_i, C_{ij}$$

for $j \neq 1, i$;

$$P_{ij} \in C_i, C_j, C_{ij}, C_{kl}$$

for $k, l \neq 1, j$.

Note: This discussion follows closely the treatment in [8, pp. 784-785].

It remains to see that this configuration is of type (*). By (4.14.1), we may identify $A_2$ with $(\frac{\mathbb{Z}}{2\mathbb{Z}})$ in such a way that the points $P_i$, $1 \leq i \leq 6$ are identified, respectively, with

(4.15.2) $\quad (0,0,0,0), (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)$ and $(1,1,1,1)$.

Hence the (16,6) configuration in question is the configuration associated to the group $\mathcal{C}_2^4$ and the six element set (4.4.2) as in Lemma 1.24. By the criterion of Lemma 1.26, our configuration is non-degenerate. Hence the incidence relations must be of one of the three types described in Theorem 1.20. To prove that it is actually of type (*), one checks directly from definitions that the associated collection of 6-sides of Theorem 1.20 consists of ten triangle pairs (by Theorem 1.20, it is sufficient to check that at least seven of the 6-sides are triangle pairs). See also the list of all the ways a non-degenerate (16,6) configuration may arise from a group of order 16 with a preferred set of six elements.

Let $A$ be a principally polarized abelian variety with principal polarization class $\Theta$. We will now start working toward the main result of this section: if $\Theta$ is irreducible, the linear system $|2\Theta|$ defines a morphism $A \xrightarrow{|2\Theta|} \mathbb{P}^3$ whose image is a Kummer surface isomorphic to the quotient $\frac{A}{\langle 1,-1\rangle}$ of $A$ by the involution $a \to -a$. The correspondence $A \to \frac{A}{\langle 1,-1\rangle}$ is a bijection from the set of principally polarized abelian varieties with irreducible principal polarization and Kummer surfaces up to automorphism of $\mathbb{P}^3$ (Propositions 4.22 and 4.23).

**Note 4.16.** Let $A_2$ be the subgroup of elements $a \in A$ such that $2a{=}0$. Then $A_2 \simeq (\frac{\mathbb{Z}}{2\mathbb{Z}})^4$ [16, p. 64]. Let us consider the abstract surface $\frac{A}{\langle\iota\rangle}$, where $\iota$ is the involution of $A$ defined by

$$\iota(a) = -a, \quad \text{for } a \in A.$$

Since the 2-torsion points are exactly the fixed points of $\iota$, $\frac{A}{\langle\iota\rangle}$ is nonsingular except at the images of the 16 2-torsion points. The abelian surface with the $\frac{\mathbb{Z}}{2\mathbb{Z}}$-action of the involution on it, is, locally near a 2-torsion point, analytically isomorphic to a plane with the action of the antiidentity which sends $a$ to $-a$. The ring of invariants of the $\frac{\mathbb{Z}}{2\mathbb{Z}}$-action on $k[[x,y]]$ is $k[[x^2, y^2, xy]]$ which gives us a hypersurface in $\mathbb{A}^3$ with the equation $uv = t^2$ (once we have made the change of coordinates $u = x^2$, $v = y^2$, $t = uv$). Thus, locally, we have a node. Hence $\frac{A}{\langle\iota\rangle}$ has exactly 16 nodes which come from the sixteen 2-torsion points of $A$.

**Proposition 4.17.** *Let $A$ be an abelian surface and $D$ any effective divisor. Then $|2D|$ has no base points. $h^0(A, 2D) = 4$, so $|2D|$ defines a morphism $A \to \mathbb{P}^3$.*

*Proof.* For any point $a \in A$, we must show that there is an effective divisor $E \sim 2D$ such that $a \notin E$. If $a \notin D$, take $E = 2D$. If $a \in D$, choose a $b$ such that $a+b, a-b \notin D$ (the set of such $b$ is a nonempty open subset of $A$). Then $a \notin D_b + D_{-b} \sim 2D$ by the theorem of the square.

$$h^0(A, \mathcal{O}(2D)) = \chi(\mathcal{O}(2D)) = \frac{(2D)^2}{2} = 2D^2 = 4,$$

as desired. $\square$

**Example 4.18.** Let $A \equiv E_1 \times E_2$ and let

$$D = E_1 \times P_2 + P_1 \times E_2$$

be the principal polarization. Then $|2D|$ is just the product of the linear systems $|2P_1|$ on $E_1$ and $|2P_2|$ on $E_2$ which define 2-1 maps $E_i \to \mathbb{P}^1$. Thus $|2D|$ defines a 4-1 map of $A$ to a quadric surface $\mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3$.

**Lemma 4.19.** *Let $\iota$ denote the involution $a \to -a$ on an abelian variety $A$. Consider the induced map $\iota^* : A^* \to A^*$ given by $D \to \iota^* D$. Then $D + \iota^* D = 0$ in $A^*$.*

*Proof.* Let $F$ be any effective ample divisor on $A$. Then $G := F + \iota^* F$ is also ample and invariant under $\iota^*$. Let $\mathcal{L} := \mathcal{L}(G)$. By Lemma 4.5, $\phi_{\mathcal{L}}$ is an isogeny, in particular, surjective. Hence any $D \in A^*$ may be written as

$$D = t_a^* \mathcal{L} \otimes \mathcal{L}^{-1}.$$

Then, using the Theorem of the square,

$$\iota^* D = \iota^*(t_a^* \mathcal{L} \otimes \mathcal{L}^{-1}) = t_{-a}^* \iota^* \mathcal{L} \otimes \iota^* \mathcal{L}^{-1} = t_{-a}^* \mathcal{L} \otimes \mathcal{L}^{-1} = -D.$$

$\square$

**Lemma 4.20.** *Let $A$ be an abelian surface and $\iota$ the involution $\iota(a) = -a$. Let $D \in PicA$. Then $\iota^* D$ is algebraically equivalent to $D$.*

*Proof.* We have the following exact sequence

$$0 \to Pic^0(A) \to Pic\, A \to Hom(A, A^*),$$

since $\frac{Pic(A)}{Pic^0(A)} \simeq NS(A) \subset Hom(A, A^*)$.

$\iota$ induces the identity map on $Hom(A, A^*)$. Hence $D$ and $\iota^* D$ have the same image in $Hom(A, A^*)$. Therefore, $D - \iota^* D \in Pic^0(A)$, so $D$ and $\iota^* D$ belong to the same connected component of $Pic\, A$. Hence there exists a continuous family of curves joining them, i.e. $D \overset{\text{alg.}}{\sim} \iota^* D$. □

Let $A$ be a principally polarized abelian variety with principal polarization class $D$. By Lemmas 4.5 and 4.19, there exists $a \in A$ such that $\iota^* D = t_a^* D$. Then for any $b \in A$ such that $2b = a$, $t_b^* D$ is invariant under $\iota^*$. Hence, after translation by an element of $A$, we may assume that $D$ is invariant under $\iota^*$.

**Lemma 4.21.** *Let $A$ be a principally polarized abelian surface, $D$ a principal polarization class such that $\iota^* D = D$. Consider the morphism*

$$A \overset{|2D|}{\to} \mathbb{P}^3. \tag{4.21.1}$$

*of Proposition 4.17. This morphism factors through the natural map $A \to \frac{A}{\langle 1, -1 \rangle}$.*

*Proof.* We need to show that any effective divisor $F \in |2D|$ is mapped to itself by the involution. Indeed, since $\iota^*(2D) \sim 2D$, we have $\iota^* F \sim F$. By Lemmas 4.5 and 4.20, $\iota^* F$ is a translate of $F$ by some element of $A$. By Lemma 4.6 (2), there are exactly 16 translates of $F$ which are linearly equivalent to $F$. On the other hand, for any $a \in A_2$, $t_a^*(2D) \sim 2(t_a^* D) \sim t_{2a}^* D + D \sim 2D$, so $t_a^* F \sim F$. Thus, $t_a^* F \sim F$ if and only if $a \in A_2$. In particular, $\iota^* F = t_a^* F$ for some $a \in A_2$. For each $a \in A_2$, consider the set

$$M_a := \{F \in |2D| \mid \iota^* F = t_a^* F\}.$$

Since these sixteen sets completely cover $|2D|$, at least one of them must be Zariski dense in $|2D|$, hence coincide with $|2D|$. We want to show it can only be $M_0$. Indeed, suppose $|2D| = M_a$ with $a \in A_2 \setminus \{0\}$. Since $|2D|$ contains the divisors $D_b + D_{-b}$, $b \in A$, we must have

$$D_b + D_{-b} \sim D_{b+a} + D_{-b+a} \tag{4.21.2}$$

for all $b \in A$. But (4.21.2) is impossible unless $b$ is a point of order 4 on $A$. Hence $\iota^* F = F$ for all $F \in |2D|$, as desired. □

**Proposition 4.22.** *Let $S$ be a Kummer surface in $\mathbb{P}^3$. Then there exists a principally polarized abelian surface $A$ whose principal polarization $D$ is non-singular, such that the image of the map (4.21.1) is $\frac{A}{\langle 1,-1\rangle} \cong S$.*

*Proof.* Let $\pi : X \to S$ be the resolution of singularities of $S$ obtained by blowing up the sixteen nodes. Let $E_i$, $1 \leq i \leq 16$ denote the sixteen exceptional divisors of $\pi$. By Corollary 3.23 $\sum_{i=1}^{16} E_i$ is 2-divisible in $Pic(X)$. Hence, there exists a two-sheeted covering $\tilde{f} : \tilde{A} \to X$ whose branch locus is $\sum_{i=1}^{16} E_i$ [1, p.42]. Let $E_i' := \tilde{f}^{-1}(E_i)$. For each $i$, $1 \leq i \leq 16$, we have $2(E_i^2) = (2E_i')^2$, so that the curves $E_i'$ have self-intersection $-1$. Therefore we may contract the $E_i'$'s and obtain a nonsingular surface $A$. Let us prove that it is an abelian surface by looking at its numerical invariants. We have

$$K_X = \mathcal{O}_X$$

by Theorem 3.4. Hence $K_{\tilde{A}}$ equals the branch locus of $\tilde{f}$, which is $\sum_{i=1}^{16} E_i'$ [1, p. 42]. Since under blowing up a point on a non-singular surface, the new canonical divisor is the pullback of the old plus the exceptional curve taken once, $K_A = \mathcal{O}_A$.

$$\chi(A) = \chi(\tilde{A}) - 16 = 2(\chi(X) - 16) - 16 = 0 \quad \text{since} \quad \chi(X) = 24 \quad (3.7, 4.6),$$

$$b_2(A) = b_2(\tilde{A}) - 16 \geq b_2(X) - 16 = 6 \quad (3.7, 4.6).$$

Thus, $A$ is an abelian surface, by the general classification of surfaces [5, Theorem 1.1.2], [2, p. 197]. The involution exchanging the sheets of $\tilde{f}$ induces an involution $\iota$ on $A$. $S$ is the quotient of $A$ by $\iota$. The map $f : A \to S$ is 2-1 everywhere, and is ramified only at the singularities of $S$.

Now consider a special conic $C$ on $S$. The preimage $C'$ of $C$ in $X$ is a smooth rational curve, which meets exactly six of the $E_i$'s, intersecting each $E_i$ exaclty once and transversely. Let $\tilde{C}$ be the preimage of $C$ in $\tilde{A}$ and $D$ the image of $\tilde{C}$ in $A$. Then (by writing the covering map in local coordinates near each ramification point), $\tilde{C}$ is also non-singular and meets exactly six of the $E_i'$'s, each of them once and transversely. Then the map $\tilde{C} \to C'$ is a 2-1 map to a rational curve ramified at six points, where $\tilde{C}$ is smooth. Hence by the Riemann-Hurwitz formula $\tilde{C}$ is a smooth curve of genus 2. Since $\tilde{C}$ intersects the $E_i'$'s transversely, its image $D \subset A$ is also non-singular irreducible curve of genus 2. Since $2C$ is a hyperplane secton of $S$, $2D$ is the linear system which defines the map $A \to \mathbb{P}^3$. Then $D$ is ample by the Nakai-Moishezon criterion. By the adjunction formula,

$$\chi(\mathcal{L}(D)) = \frac{D^2}{2} = 1,$$

so by Lemma 4.6 $D$ is a principal polarization. This proves Proposition 4.22.

Next, we prove the following converse to Proposition 4.22.

**Proposition 4.23.** *Let $A$ be a principally polarized abelian surface with principal polarization $D$. There are two possibilities.*

(1) *$D$ is irreducible. Then the linear system $|2D|$ defines a 2-1 morphism to $\mathbb{P}^3$ whose image is a Kummer surface.*

(2) *$D$ is the union of two elliptic curves intersecting transversely. Then $|2D|$ defines a 4-1 map to $\mathbb{P}^3$ whose image is a non-singular quadric.*

*Proof.* Say, $|2D|$ defines a morphism of degree $d$ to a surface of degree $n$ (clearly, $|2D|$ has no fixed components, so the morphism it defines is always finite to one). Since $(2D)^2 = 8$, we have $dn = 8$. By Lemma 4.21, the morphism factors through $\frac{A}{\langle\iota\rangle}$, so $d \geq 2$. On the other hand, the image in $\mathbb{P}^3$ cannot be a plane, since the linear sytem $|2D|$ is 3-dimensional. We are left with only two possibilities.

(1) $n = 4$, $d = 2$. The morphism $|2D|$ is a composition of the map $A \to \frac{A}{\langle\iota\rangle}$ with some finite, generically 1-1 map to $\mathbb{P}^3$. Then by Proposition 4.10 and Example 4.18 $D$ is irreducible and non-singular. To prove that, in fact, $|2D|$ defines a closed embedding, it is sufficient to prove that $|2D|$ separates points on $\frac{A}{\langle\iota\rangle}$ and separates tangent vectors at any *non-singular* point of $\frac{A}{\langle\iota\rangle}$. Indeed, that would imply that the image in $\mathbb{P}^3$ is a hypersurface with isolated singularities, hence normal. Since the map $\frac{A}{\langle\iota\rangle} \to \mathbb{P}^3$ is finite onto its image and birational, it must be an isomorphism onto its image.

To prove that $|2D|$ separates points, we have to show that for each $a, b \in A$, $a \neq \pm b$ there exists an $F \in |D|$ such that $a \in F$, $\pm b \notin F$. We look for $F$ of the form $F = D_c + D_{-c}$. First we note that $D_a \neq D_b, D_{-b}$. Indeed, suppose $D_a = D_b$. Then $\phi_{\mathcal{L}}(a) = \phi_{\mathcal{L}}(b)$. But deg $\phi_{\mathcal{L}} = 1$, since $D$ is a principal polarization, so $a = b$, contrary to hypothesis. The proof that $Da \neq D_{-b}$ is entirely similar.

Since $D$ is irreducible, $D_a \cap (D_b \cup D_{-b})$ is a non-empty Zariski open set in $D_a$ (here is where we use the hypothesis that we are in case (1)!). Now, take any $c \in D_a \setminus (D_b \cup D_{-b})$. Then, translating by $a$, $c - a \in D$, so $-a \in D_{-c} \implies a \in D_c$. On the other hand, $c \notin D_b$ implies that $c - b \notin D \implies b \notin D_c$; similarly, $b \notin D_{-c}$, as desired. Hence $|2D|$ separates points on $\frac{A}{\langle\iota\rangle}$.

Next we prove that $|2D|$ separates tangent vectors at every *non-singular* point of $\frac{A}{\langle\iota\rangle}$. Let $a \in A \setminus A_2$, so that the image of $a$ in $\frac{A}{\langle\iota\rangle}$ is non-singular. Consider the set of all the elements of $|2D|$ which pass through $a$ with multiplicity 1. We have to show first of all that this set is not empty and that every tangent direction to $A$ at $a$ appears among the tangents to such elements $F \in |2D|$. For that it is enough to show that at least two distinct tangent directions appear. Again, we study elements of $|2D|$ of the form $D_b + D_{-b}$. Since $D$ is non-singular, if $D_b$ passes through $a$ then it has multiplicity 1 there if and only if $a \notin D_b \cap D_{-b}$. In order that $a \in D_b \setminus D_{-b}$, it is necessary and sufficient that

$$b \in D_a \setminus D_{-a}. \tag{4.23.1}$$

Since $a \notin A_2$, there is a 1-parameter family of $b$'s satisfying (4.22.1). We have to prove that not all the $D_b$ have the same tangent direction at $a$. Suppose the contrary. Then all the $D_b$ for $b$ satisfying (4.22.1) are tangent to each other at $a$. In particular, for any $b, b'$ as in (4.22.1), we have $D_b.D_{b'} = (D_b.D_{b'})_a = 2$ and $D_b$ and $D_{b'}$ have no other points of intersection. Translating by any $c$ such that $2c = b - b'$, we get that $D_c \cap D_{-c}$ is a one point set. But since the set $D_c \cap D_{-c}$ is invariant by the involution, it must be an element of $A_2$. Hence $a - c \in A_2$, so $2a - b + b' = 0$, or $b' = b - 2a$. But this is absurd since $b'$ was chosen from a one-parameter family, independently of $b$. Contradiction, so (1) is proved.

(2) $n = 2$, $d = 4$. Let $Q$ denote the image of $A$ in $\mathbb{P}^3$. $Q$ is a reduced and irreducible quadric surface. First, we prove that $Q$ is non-singular. Suppose the contrary, i.e. $Q$ is a quadric cone. The translations in $A$ by elements of $A_2$ map $2D$ to itself, hence induce automorphisms of the linear system $|2D|$ and, consequently, of $Q$. Any automorphism of $Q$ must fix the vertex $V$ of the cone. Let $a \in A$ be any point mapping to $V$. The orbit of $a$ in $A$ under the above action of $A_2$ consists of sixteen *distinct* points and all of them must map to $V$. This contradicts the fact that we have a 4-1 map.

Hence $Q \cong \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$. Consider a hyperplane section of $Q$ in $\mathbb{P}^3$ of the form $\mathbb{P}^1 \times P_2 \cup P_1 \times \mathbb{P}^1$. Let us denote these two lines by $L_1$ and $L_2$, respectively, and let $F_1$ and $F_2$ be the preimages of $L_1$ and $L_2$ on $A$. We have $F_1 + F_2 = 2D$. Since $L_1.L_2 = 1$ and we have a 4-1 map, $F_1.F_2 = 4$. Since

$$(2D)^2 = 8 = F_1^2 + F_2^2 + 2F_1.F_2,$$

we must have

$$F_1^2 = F_2^2 = 0.$$

The key point is to prove that the $F_i$'s must be *subgroups* of $A$ and hence be disjoint unions of elliptic curves. Indeed, we may assume, after translation, that $F_1$ passes through $0 \in A$. Let $E_1$ denote the connected component of $F_1$ passing through 0. Write $F_1 = E_1 + \tilde{E}_1$. Since $F_1^2 = 0$ and no curve has negative self intersection on $A$, we must have

$$E_1^2 = \tilde{E}_1^2 = E_1.E_2 = 0. \tag{4.23.2}$$

We claim that $E_1$ is mapped to itself under translation by any of its elements and hence is a subgroup of $A$. Indeed, take $a \in E_1$. If $t_a^* E_1 \neq E_1$, we would have $E_1^2 = E_1.t_a^* E_1 \geq (E_1.t^* E_1)_a \geq 1$, which would contradict (4.22.2). Hence $E_1$ is a subgroup of $A$. Since $E_1$ is connected and proper, it must itself be a 1-dimensional abelian variety, i.e. an elliptic curve. Moreover, since $\tilde{E}_1.E_1 = 0$, translating $E_1$ by an element of $\tilde{E}_1$ and arguing as above, we see that $\tilde{E}_1$ must contain a translate of $E_1$, i.e. another elliptic curve of the form $(E_1)_a$, $a \in A$. By a similar argument, we translate $F_2$ so it also passes through 0 and prove that

$$F_2 = E_2 + (E_2)_{a'} + R,$$

where $a' \in A$ and $R$ is some effective divisor (which we shall now prove is 0). Since we have a 4-1 map $F_2 \to L_2$ and it must have degree at least 2 on each of the elliptic curves (there are no birational maps from $E_2$ to $L_2$), we must have $R = \emptyset$. Thus,

$$F_1 = E_1 + (E_1)_a$$
$$F_2 = E_2 + (E_2)_{a'}$$

Since $E_1$ and $E_2$ are subgroups, we have a map (necessarily an isogeny)

$$E_1 \times E_2 \to A.$$

$E_1 + E_2$ is an ample divisor with $(E_1 + E_2)^2 = 2$. Denoting by $\frac{a}{2}$ any element of $A$ such that $2\frac{a}{2} = a$, we have by the theorem of the square

$$F_1 \sim 2(E_1)_{\frac{a}{2}}$$
$$F_2 \sim 2(E_2)_{\frac{a'}{2}}.$$

Hence $2D \sim 2(E_1)_{\frac{a}{2}} + 2(E_2)_{\frac{a'}{2}}$. Hence $D$ is uniquely defined up to translation by an element of $A_2$. In other words, $D$ is a translate of $E_1 + E_2$, hence a union of two elliptic curves, as desired. □

**Definition 4.24.** *Let $S \in \mathbb{P}^3$ be a Kummer surface. The* **Gauss map** *$S \setminus \{nodes\} \to \mathbb{P}^{3^*}$ is the map which sends every non-singular point of $S$ to its tangent plane viewed as a point in $\mathbb{P}^{3^*}$. The closure $S^*$ of the image of the Gauss map in $\mathbb{P}^{3^*}$, is called the* **dual Kummer surface** *to $S$.*

**Remark 4.25.** The Gauss map induces a birational map from $S$ to $S^*$. We shall see below that under this birational map nodes of $S$ map to special conics of $S^*$ and viceversa.

**Lemma 4.26.** *Identify $\mathbb{P}^{3^*}$ with $\mathbb{P}(H^0(A, \mathcal{O}_A(2D)))$. Then $S^*$ is the image of the map $\psi : A^* \to \mathbb{P}^{3^*}$ given by sending each divisor $F \in Pic^0(A)$ to the global section of $|2D|$ (determined uniquely up to a scalar) whose set of zeroes is $F + \iota^*F + 2D$. There exists a (non-canonical) isomorphism $g : H^0(A, \mathcal{O}_A(2D))^* \to H^0(A, \mathcal{O}_A(2D))$ which makes the following diagram commutative:*

$$\begin{array}{ccc} A & \xrightarrow{|2D|} & \mathbb{P}(H^0(A, \mathcal{O}_A(2D))^*) \\ {\scriptstyle \phi_{\mathcal{L}(D)}}\downarrow & & \downarrow{\scriptstyle \mathbb{P}(g)} \\ A^* & \xrightarrow{\psi} & \mathbb{P}(H^0(A, \mathcal{O}_A(2D))) \end{array} \tag{4.26.1}$$

Note that $\psi \circ \phi_{\mathcal{L}(D)}$ is *not* the Gauss map. We shall discuss its relation to the Gauss map in Remark 4.28.

*Proof.* Each tangent plane to $S$ at a non-singular point determines a hyperplane section of $S$, hence an element $F \in |2D|$ and hence a well defined element of $\mathbb{P}(H^0(A, \mathcal{O}_A(2D)))$ (given by any global section of $\mathcal{O}_A(2D)$ whose set of zeroes is $F$). For any point $a \in A \setminus A_2$, $\phi_{\mathcal{L}(D)}(a) = D_a - D$ by definition of $\phi_{\mathcal{L}(D)}$. Hence $\psi \circ \phi_{\mathcal{L}(D)}(a)$ is the section of $\mathcal{O}_{2D}$ whose set of zeroes is $D_a + D_{-a}$. Since $a \notin A_2$, there exists $b \in A$, unique up to sign, which is a singular point of $D_a \cup D_{-a}$. This means that $D_a + D_{-a}$ is the hyperplane section of $S$ by the tangent plane at the image of $b$ in $S$. Hence $\psi \circ \phi_{\mathcal{L}(D)}(A \setminus A_2)$ is contained in the image of the Gauss map.

Conversely, given a $b \in A \setminus A_2$, there exists an $a \in A \setminus A_2$, defined uniquely up to translation, such that $D_a \cap D_{-a} = \{b, -b\}$. This proves that every element in the image of the Gauss map belongs to $\psi \circ \phi_{\mathcal{L}(D)}(A \setminus A_2)$. Since $\psi \circ \phi_{\mathcal{L}(D)}$ is well defined on all of $A$ and since $S^*$ is irreducible, it must be the image of $\psi \circ \phi_{\mathcal{L}(D)}$.

To prove the existence of the isomorphism $g$, it is sufficient to prove that the map $\psi \circ \phi_{\mathcal{L}(D)}$ is given by the linear system $2|D|$. To see that, let us study the preimage of a hyperplane under $\psi \circ \phi_{\mathcal{L}(D)}$. Take any $a \in A \setminus A_2$. The set $\{F \in |2D| \mid a \in F\}$ is in 1-1 correspondence with the planes in $\mathbb{P}^3$ passing through the image of $a$ in $S$, and hence is a 2-plane in $\mathbb{P}(H^0(A, \mathcal{O}_A(2D)))$. In other words, we may choose a basis $(f_1, f_2, f_3, f_4)$ for $\mathbb{P}(H^0(A, \mathcal{O}_A(2D)))$ such that $a$ lies in the set of zeroes of $f_2$, $f_3$ and $f_4$, but not of $f_1$. Let $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be coordinates on $\mathbb{P}(H^0(A, \mathcal{O}_A(2D)))^*$ and consider the hyperplane $\alpha_1 = 0$ in $\mathbb{P}^{3^*} = \mathbb{P}(H^0(A, \mathcal{O}_A(2D)))$. The preimage of this hyperplane in $A^*$ under $\psi$ is

$$\{F \in Pic^0(A) \mid \text{the only effective element of } F + \iota^* F + 2D \text{ passes through } a\}.$$

Hence its preimage in $A$ is

$$\{b \in A \mid a \in D_b \cup D_{-b}\} \tag{4.26.2}$$

The set (4.26.2) is just $D_a \cup D_{-a}$, which is linearly equivalent to $2D$. Hence the linear system defining the map $\psi \circ \phi_{\mathcal{L}(D)}$ is a linear subsystem of $|2D|$. If the image of $\psi \circ \phi_{\mathcal{L}(D)}$ (which is $S^*$) were contained in a plane, all the tangent planes of $S$ would have to pass through the same point, so $S$ would be a strange surface. In particular, all the special planes would pass through the same point, which contradicts the (16,6) configuration. Hence the linear subsystem of $|2D|$ in question is 3-dimensional and so must equal $|2D|$, as desired. □

**Corollary 4.27.** *The Kummer surface is self-dual (i.e $S$ is—non-canonically—isomorphic to $S^*$).*

**Remark 4.28.** For any non-singular point $P \in S$, we may consider its preimage $\{a, -a\}$ in $A$ under the map defined by $|2D|$. Let $Q \in S^*$ be the image of $P$ under the Gauss map and let

$$\{b, -b\} := (\psi \circ \phi_{\mathcal{L}(D)})^{-1}(Q).$$

We may ask how to describe explicitly the relationship between $a$ and $b$ in terms of $A$, without reference to the Kummer surface. From the proof of Lemma 4.26, we get that $b$ is nothing but the intersection point (determined up to sign) of $D_a$ with $D_{-a}$. This relationship is reflexive, i.e. we also have

$$\{a, -a\} = D_b \cap D_{-b}.$$

**Remark 4.29.** It is well known (and easy to prove) that

$$(S^*)^* = S, \tag{4.29.1}$$

the canonical isomorphism being given by the composition of the two Gauss maps (this holds for any hypersurface $S$ whatsoever). Take any special conic of $S$. The special plane corresponding to that conic is tangent to $S$ at every point of the conic, so the Gauss map sends the entire conic to one point. That point must be singular, otherwise it would have a unique image in $(S^*)^*$, contradicting (4.29.1). Hence under the birational correspondence given by the Gauss map, nodes of $S$ correspond to the special conics of $S^*$ and viceversa. In terms of the resolution of singularities $X$ of $S$, there are 32 smooth rational curves on $X$, sixteen of which map to nodes and the other sixteen to special conics of $S$. By the above, the Gauss map is obtained by blowing up the nodes of $S$ and then blowing down the strict transforms of the special conics. Hence we may also view $X$ as a resolution of singularities of $S^*$. The isomorphism $S \cong S^*$ of Lemma 4.26 induces an involution of $X$ which interchanges the sixteen lines mapping to nodes of $S$ with those mapping to special conics. Together with the action of $F_0 \cong \mathcal{C}_2^4$ lifted from $S$, we obtain an intersecting action of $\mathcal{C}_2^5$ on the $K3$ surface $X$.

**Theorem 4.30.** *There are no lines on a Kummer surface.*

*Proof.* Let $A$ be a principally polarized abelian surface such that $\frac{A}{\langle 1,-1\rangle} \equiv S$. Let $f$ denote the map $f : A \to S$. Consider a line $L \subset S$. By Corollary 2.11, $L$ does not pass through any nodes of $S$. Hence the map $f^*L \to L$ induced by $f$ is a nowhere ramified 2-1 cover of the smooth rational curve $L$ by a complete curve. Then $f^*L$ must be a disjoint union of two smooth rational curves. This is impossible since there are no rational curves on an abelian surface. □

**Theorem 4.31.** *Let $S$ be a Kummer surface and $f : A \to \frac{A}{\langle\iota\rangle} \cong S$ the associated abelian surface together with its natural map to $S$. Let $X$ be the minimal desingularization of $S$. Then*

$$\rho(X) = \rho(A) + 16.$$

*Proof.* The map $f : A \to S$ induces $f^* : NS(\frac{A}{\langle\iota\rangle}) \to NS(A)$. For any effective divisor $D$ of $A$, the divisor $D + \iota^*D$ is obviously induced from some divisor of $\frac{A}{\langle\iota\rangle}$ since it is invariant under the involution. By Lemma 4.20, $D \overset{\text{alg.}}{\sim} \iota^*D$, so

$$2D \sim D + \iota^*D \in f^*\left(NS\left(\frac{A}{\langle\iota\rangle}\right)\right)$$

and

$$NS(A) \supset f^*\left(NS\left(\frac{A}{\langle\iota\rangle}\right)\right) \supset 2NS(A).$$

Hence
$$\rho(A) = \rho\left(\frac{A}{\langle\iota\rangle}\right) = \rho(S).$$
Thus, by Theorem 3.17
$$\rho(X) = \rho(A) + 16.$$
This proof follows closely [18, Proposition 1]. □

**Theorem 4.32.** *The curves on a Kummer surface $S \subset \mathbb{P}^3$ with $\rho(S) = 1$ have even degree.*

*Proof.* Let $f : A \overset{|2D|}{\rightarrow} S$ be as above. By the proof of Theorem 4.31, $\rho(A) = \rho(S) = 1$. Since $D^2 = 2$, if $D = nD'$ for any $D' \in Pic\,A$, $n \in \mathbb{N}$, then $n = 1$. Since $NS(A)$ has no torsion, $NS(A) = \mathbb{Z}.D$. Let $H$ be a generic hyperplane section of $S$. Since we have a finite number of nodes, we can choose a generic hyperplane section of $S$ not containing any of them. Let $C$ be a curve on $S$. Let $f^{-1}(C)$ be the inverse image as a subscheme. Write $f^{-1}(C) = nD$ for some $n \in \mathbb{N}$. We have
$$\deg\, C = H.C = \frac{1}{2}(2D.f^{-1}(C)) = D.nD = n.D^2 = 2n$$
so deg $C$ is even. □

**4.33 Remark.** Let $A$ be an abelian surface.

(1) $b_2$=6, $b_1$=4, $K_A \sim 0$, $c_2(A) = \chi(A) = 0 = \chi(A, \mathcal{O}_A)$. [5, pp. 73–74].

(2) Let $char\, k = 0$. Then
$$1 \leq \rho(A) \leq h^{1,1}(A) = b_2 - 2p_g = 6 - 2 \cdot 1 = 4.$$

(3) Let $char\, k = p > 0$. Then
$$1 \leq \rho(A) \leq 6, \qquad \rho(A) \neq 5.$$

By the proof of Theorem 4.31, the same bounds apply to $\rho(S)$.

**Note 4.34.** Recall that the (16,6) configuration associated to a Kummer surface $S$ is of type (*). The nodes and special planes of $S$ (and therefore, the special conics $C_i$, $1 \leq i \leq 16$) have a given incidence relation (1.7).

**Proposition 4.35.** *Let $S$ be a Kummer surface with $\rho(S)$=1. Then $Pic(S)$ is generated by $H$, where $H$ is a hyperplane section of $S$.*

*Proof.* Since deg $H = 4$ and since every curve on $S$ has even degree, either $H$ generates $Pic(S)$ or $Pic(S)$ is generated by a curve of degree 2. Suppose that $Pic(S)$ is generated by a curve $C$ of degree 2. Since $\rho(S) = 1$, $\pi^*(Pic(S)) \subset Pic(X)$ by 3.17 and $Pic(X)$ is a free abelian group by 3.10, we have that $Pic(S) \simeq \mathbb{Z}$. Hence, $2C$ must be a hyperplane section of $S$, since it has degree 4 and $Pic(S) \simeq \mathbb{Z}$. Thus, $C - C_1$ is a 2-torsion element; hence $C - C_1 = \sum_{i,j} n_{i,j}(C_i - C_j)$ by Th. 4.44. Therefore, $C = \sum_{i=1}^{16} l_i C_i$, with $\sum_{i=1}^{16} l_i = 1$.

By adding or substracting an even number of $C_i$,

$$C = \sum_{i=1}^{16} \epsilon_i C_i,\ \epsilon_i \in \{0,1\},\ 2 \nmid \sum_{i=1}^{16} \epsilon_i.$$

We are going to show that $C$ cannot be at the same time a Cartier divisor and be the sum of an odd number of special conics.

If $C$ contains configurations of conics of the form (up to a permutation of rows and columns):

$$\text{(4.35.1)} \qquad \begin{matrix} \star & \star & \cdot & \cdot \\ \star & \star & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{matrix} \qquad\qquad \begin{matrix} \star & \cdot & \cdot & \cdot \\ \cdot & \star & \cdot & \cdot \\ \cdot & \cdot & \star & \cdot \\ \cdot & \cdot & \cdot & \star \end{matrix}$$

we can just substract them because they form a Cartier divisor, since they pass an even number of times through the nodes on them.

To finish the proof we need the following Lemma:

**Lemma 4.36.** *Consider the 4x4 matrix $M$ associated to $C$; that is, we write 1 for each $C_i$ such that $\epsilon_i = 1$. Let us subdivide $M$ into four 2x2 blocks, $A = (a_{ij})$, $B = (b_{ij})$, $D = (d_{ij})$ and $E = (e_{ij})$, as follows:*

$$\begin{matrix} A & B \\ D & E \end{matrix}$$

*If $a_{11} + a_{12} + a_{21} + a_{22}$ is odd, so are $b_{11} + b_{12} + b_{21} + b_{22}$ and $d_{11} + d_{12} + d_{21} + d_{22}$.*

*Proof.* Since $C$ is a Cartier divisor,

$$a_{11} + a_{22} + b_{11} + b_{12} + d_{12} + d_{22} \text{ is even, and}$$

$$a_{21} + a_{12} + b_{21} + b_{22} + d_{12} + d_{22} \text{ is even.}$$

Since $a_{11} + a_{12} + a_{21} + a_{22}$ is odd, then

$$b_{11} + b_{12} \not\equiv b_{21} + b_{22} \pmod 2.$$

By symmetry, the same is true for $d_{11} + d_{12} + d_{21} + d_{22}$.

If $C$ were the sum of an odd number of special conics $C_i$, $M$ would contain an odd number of 1; consequently, one of the 2x2 subblocks would contain an odd number of them, say $A$. By the Lemma 4.36, $B$ and $D$ would contain an odd number of 1. Since the role of $A$ and $E$ is symmetric, $E$ would also have an odd number of them; thus, $M$ would contain an even number of 1. Contradiction.

**4.37 Remark.** Theorem 4.30 follows from Theorem 4.32, for a Kummer surface with $\rho(S)$=1.

**4.38 Note.** For the existence of curves of odd degree on $\frac{A}{\langle 1,-1\rangle}$, where $A$ is a deformation of the product of two isogenous elliptic curves and has Picard number 2, see [7].

**Lemma 4.39.** *The principally polarized abelian surfaces which are not Jacobians of a nonsingular curve of genus 2 are deformations of the latter.*

*Proof.* The moduli space of principally polarized abelian surfaces, $M_A$, is irreducible [3]. Furthermore, the moduli space of the abelian surfaces which are Jacobians of a nonsingular curve of genus 2, $M_{AJ}$, has dimension 3, the same dimension as the moduli space of abelian surfaces. Hence, $M_{AJ}$ is dense in $M_A$. □

**Lemma 4.40.** *Let $A$ be an abelian surface which is not of the form $J(B)$, where $B$ is a nonsingular curve of genus 2. $\frac{A}{\langle 1,-1\rangle}$ is a deformation of a Kummer surface of the form $\frac{J(B)}{\langle 1,-1\rangle}$.*

*Proof.* Let us consider $\frac{A}{\langle 1,-1\rangle}$, where $A$ is an abelian surface. If $A \not\simeq J(B)$, $A$ is a deformation of a Jacobian by 4.38. On the other hand, 2-torsion points are limits of 2-torsion points, the involution is canonical. Hence, we can conclude that $\frac{A}{\langle 1,-1\rangle}$ is a deformation of one coming from the Jacobian of a nonsingular curve of genus 2. □

**4.41 Remark.** The moduli space of Kummer surfaces in $\mathbb{P}^3$, $\mathcal{M}_S$, has projective dimension 18, since the moduli space of Jacobian varieties of a nonsingular curve of genus 2, $\mathcal{M}_{AJ}$, has dimension 3, and $\#Aut(\mathbb{P}^3) = 15$. The $(16,6)$ configuration of the form $(a,b,c,d)$, (see Chapter 1), which we shall denote by $(16,6)_{(a,b,c,d)}$, has projective dimension 18, since $(a,b,c,d)$ moves in a 3-dimensional open set of $\mathbb{P}^3$ and $\#Aut(\mathbb{P}^3)$=15. $\mathcal{M}_S$ is irreducible since so it is the moduli space of abelian varieties $\mathcal{M}_A$. We have a map $\mathcal{M}_S \to (16,6)_{(a,b,c,d)}$, such that to the $(16,6)$ configuration of $S$ corresponds a $(16,6)$ configuration of the form $(a,b,c,d)$. The map $(16,6)_{(a,b,c,d)} \to \mathcal{M}_S$ is defined by sending a (16,6) configuration of the form $(a,b,c,d)$ to a Kummer surface $S$ which is defined by the equation in (2.20.1).

**Lemma 4.42.** *Let $S$ be a Kummer surface in $\mathbb{P}^3$ with $\rho(S)$=1. A non-special plane in $\mathbb{P}^3$ cannot contain more than three nodes. There are non-special planes containing exactly three nodes.*

*Proof.* Let $S$ be of $\rho(S) = 1$. From the (16,6) configuration we see that the only set of 4 nodes which do not belong to any special plane is, up to an automorphism of the (16,6) configuration, $P_1$, $P_2$, $P_5$, $P_6$. Let us prove that there is no plane passing through them.

Following the notation of §1, (1.4.1),

$$\begin{aligned} P_1 &\leftrightarrow (-c,d,-a,b) \\ P_2 &\leftrightarrow (d,-c,b,-a) \\ P_5 &\leftrightarrow (-c,d,a,-b) \\ P_6 &\leftrightarrow (d,-c,-b,a). \end{aligned}$$

$P_1$, $P_2$, $P_5$, $P_6$ do not lie on the same plane, since

$$\det\begin{pmatrix} -c & d & -a & b \\ d & -c & b & -a \\ -c & d & a & -b \\ d & -c & -b & a \end{pmatrix} = (c^2-d^2)(b^2-a^2) \neq 0$$

Note that, if $\rho(S)$=1, $c^2 \neq d^2$ and $b^2 \neq a^2$ (see (Remark 1.59)). According to the (16,6) configuration, there are sets of three nodes not contained in any special plane. The unique plane passing through these three nodes contains no other nodes, which proves the last statement. □

**Theorem 4.43.** *Let $S$ be a Kummer surface with Picard number $\rho(S)=1$, $S \simeq \frac{J(B)}{\langle 1,-1\rangle}$, where $B$ is a nonsingular curve of genus 2. Every Weil divisor $D \in W(S)$ is of the form*

$$D \sim \sum_{i=1}^{16} n_i C_i,$$

*where $C_i$ are special conics, $n_i \in \mathbb{Z}$.*

*Proof.* Let $f : J(B) \to \frac{J(B)}{\langle 1,-1\rangle} \simeq S$. Let $D \in W(S)$. We can assume that $D$ is of degree 0 since, by Theorem 4.32, $D$ has always even degree, and we can just substract a linear combination of the special conics $C_i$ to obtain $\deg D = 0$. Let $P_1$ be the zero of $J(B)$ and $\eta : B \to J(B)$, $\eta(B) = \Theta_1$. Let us denote by ${J(B)^*}_2$ the 2-torsion elements of $J(B)^*$, the dual abelian surface. Its elements are of the form $(\Theta_1+\alpha)-\Theta_1$, where $\alpha \in J(B)_2$. We consider $f^*(D)$. $f^*(D) \in Pic^0(J(B))$=$J(B)^*$, and $f^*(D)$ is a 2-torsion point. Thus, $f^*(D) \in {J(B)^*}_2$. Considering the image of $f^*(D)$ in $S$, we obtain that $D$ is a linear combination of special conics. □

**Proposition 4.44.** *Let $S$ be a Kummer surface with Picard number $\rho(S)=1$. $W(S)$ has 2-torsion.* $\mathrm{Tor}(W(S)) \simeq (\frac{\mathbb{Z}}{2\mathbb{Z}})^{15}$, *with generators $C_i - C_1$, $i = 2,\dots,16$, where $C_j$, $1 \leq j \leq 16$ are special conics.*

*Proof.* It is clear that $W(S)$ is 2-torsion since every Weil divisor of $S$ is a linear combination of special conics. $\mathrm{Tor}(W(S))$ is generated by $C_i - C_1$, $2 \leq i \leq 16$, where $C_j$, $1 \leq j \leq 16$, are special conics, by Theorem 4.43 and the fact that

$$2(C-C') = 2C - 2C' = H - H \sim 0,$$

where $C$, $C'$ are special conics and $H$ is a special plane.

## References


1. W. Barth, C. Peters and A. Van de Ven, *Compact Complex Surfaces*, Springer- Verlag, Grundlehren der Mathematischen Wissenschaften, 1984.
2. E. Bombieri and D. Mumford, *Enriques'classification of surfaces in char.p, III.*, Invent. Math. **35** (1976), 197-232.
3. Ch.- L. Chai, *Compactification of Siegel Moduli Schemes*, London Math. Soc. Lect. notes Ser 107, 1985.
4. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, ATLAS *of Finite Groups*, Clarendon Press, Oxford, 1985.
5. F. Cossec and I. Dolgachev, *Enriques Surfaces*, Vol.I, Birkhäuser, 1989.
6. M. R. Gonzalez-Dorrego, *Construction of (16,6) configurations from a group of order 16*, to appear in Journal of Algebra.
7. M. R. Gonzalez-Dorrego, *Curves on a Kummer surface in* $\mathbb{P}^3$, preprint.
8. P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, New York, 1978.
9. M. Hall and J. K. Senior, *The groups of order* $2^n$ *(*$n \leq 6$*)*, The MacMillan Company, New York, 1964.
10. R. Hartshorne, *Foundations of Projective Geometry*, W. A. Benjamin, Inc., New York, NY 10016, 1967.
11. R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, Graduate Texts in Mathematics, 52.
12. R. Hudson, *Kummer's quartic surface*, Cambridge Univ. Press, 1905.
13. J. Milne, *Abelian varieties*, Arithmetic Geometry, ed. Cornell and Silverman.
14. J. S. Milne, *Étale Cohomology*, Princeton, 1980.
15. S. Mukai, *Finite groups of automorphisms of K3 surfaces and the Mathieu group*, Invent. Math. **94** (1988), 183–221.
16. D. Mumford, *Abelian Varieties*, Oxford University Press, Bombay, 1970.
17. B. Saint-Donat, *Projective models of K3 surfaces*, Amer. J. Math. **96, no. 4**, 602–639.
18. T. Shioda, *Algebraic cycles on certain K3 surfaces in characteristic p*, International Conference in Topology and related topics. Manifolds 1973.



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