GLUEING JACOBIANS

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CONTENTS

I. INTRODUCTION

In [FK91] the authors describe a method for glueing two elliptic curves E_1 and E_2 along their torsion subgroups to produce a genus 2 curve that covers both of them. In this article, we extend this method to genus 3: we glue a genus 1 curve X_1 to the Jacobian variety of a genus 2 curve X_2. This produces an abelian 3-fold which, since all abelian 3-folds are principally polarized, is the Jacobian variety of a genus 3 curve X_3. We determine explicit equations for X_3, given the data of blah. We have implemented this method in Magma, and conclude the paper with several examples.

[Other papers to mention? Howe? Howe, Leprovost, Poonen? Broker, Lauter, Stevenhagen, etc.?]

II. BACKGROUND

Encoding divisors as polynomials as in Mumford and Cantor. Describe construction of the Kummer as in Mueller.

II.1. **Definitions and conventions.** Throughout, let k be a field of characteristic $\neq 2$. A *hyperelliptic curve* over k is a curve C of genus $g \geq 2$ with a model of the form $y^2 = f(x)$, where $f \in k[x]$ has distinct roots. Then $\deg(f)$ is either $2g + 1$ or $2g + 2$—we call the model *odd* or *even* according to the parity of $\deg(f)$. Note that an odd model has the single point $\infty = (1 : 0 : 0)$ at infinity while an even model has two: letting c be the leading

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coefficient of f, then the two points at infinity are $\infty = (1 : \sqrt{c} : 0)$ and $\infty' = (1 : -\sqrt{c} : 0)$. Let

$$\widehat{\infty} = \begin{cases} \infty + \infty' & \text{if } \deg(f) \text{ is even;} \\ 2\infty & \text{if } \deg(f) \text{ is odd.} \end{cases} \tag{1}$$

We denote by $\iota : C \to C$ the hyperelliptic involution that maps $(x, y) \mapsto (x, -y)$.

II.2. **Representing divisors.** Let C be a hyperelliptic curve, $\mathrm{Div}_k(C)$ be the group of k-divisors on C [define?], $\mathrm{Div}_k^0(C)$ be the subgroup of divisors of degree 0, and $\mathrm{Jac}(C)$ be its Jacobian variety. Denote by $\equiv$ the equivalence relation of linear equivalence on $\mathrm{Div}_k(C)$. We describe how points of $\mathrm{Jac}(C)$ can be represented as pairs of polynomials, as presented in [Can87] and [Mum07, §1].

Given a point $P = (u, v)$ on C, then $\iota(P) = (u, -v)$ also lies on C. Since $\mathrm{div}(x - u) = P + \iota(P) - \widehat{\infty}$, then $-P' \equiv P - \widehat{\infty}$. Then each divisor $D \in \mathrm{Div}_k^0(C)$ is linearly equivalent to one the form [TODO: fix this to make it work in the even model case, too]

$$\sum_{i=1}^{r} P_i - r\widehat{\infty} \tag{2}$$

[Again, Cantor and Mumford only consider odd models, so have ∞, not $\widehat{\infty}$.] satisfying the following conditions:

(1) $P_i \notin \{\infty, \infty'\}$ for all i; and
(2) $P_j \neq \iota(P_i)$ for all $j \neq i$, i.e., at most one of P_i and $\iota(P_i)$ appears.

A divisor of this form is called *semireduced*.

Given a semireduced divisor $D = \sum_{i=1}^{r} P_i - r\widehat{\infty}$, we produce a pair $(a(x), b(x))$ of polynomials. Writing $P_i = (u_i, v_i)$ for each i, let $a(x) = \prod_{i=1}^{r}(x - u_i)$ and $b(x)$ be the unique polynomial of degree at most $r - 1$ such that $b(u_i) = v_i$ for all i [TODO: add statement about multiplicities here]. In the case where all the P_i are distinct, we can write b explicitly using Lagrange interpolation as

$$b(x) = \sum_{i=1}^{r} v_i \prod_{j \neq i} \frac{x - u_j}{u_i - u_j}.$$

[Put some statement about a bijection between pairs of polynomials and semireduced divisors here?] By construction $b(x)^2 \equiv f(x) \pmod{x - u_i}$ for each i [again, add statement about multiplicities], so $a(x) \mid (b(x)^2 - f(x))$.

The above observation allows us to construct an affine open patch of the Jacobian by giving explicit equations. As in the discussion following [Mum07, Proposition 1.2], let $k[a_1, \ldots, a_g, b_1, \ldots, b_g]$ be the polynomial ring in $2g$ indeterminates, and define polynomials $a(x), b(x) \in k[a_1, \ldots, a_g, b_1, \ldots, b_g][x]$

$$a(x) = x^g + a_1 x^{g-1} + \cdots + a_g$$
$$b(x) = b_1 x^{g-1} + \cdots + b_g.$$

As above, we must have $b(x)^2 - f(x) \equiv 0 \pmod{a(x)}$. To ensure this, we divide $b(x)^2 - f(x)$ by $a(x)$, and then insist that the remainder be 0 by setting all its coefficients $= 0$. This realizes an affine open patch of $\mathrm{Jac}(X_2)$ as a subvariety of $\mathbb{A}^4$. We illustrate this with an example.

Example 1. Consider the genus 2 hyperelliptic curve

$$X_2 : y^2 = f(x)$$

where

$$f(x) = x(x-5)(x+2)(x-31)(x+15)(x-4) = x^6 - 23x^5 - 351x^4 + 3263x^3 - 1570x^2 - 18600x$$

Then

$$a(x) = x^2 + a_1 x + a_0 \qquad \text{and} \qquad b(x) = b_1 x + b_0 \,.$$

By long division with remainder, we find that

$$b(x)^2 - f(x) \equiv (-a_1^5 - 23a_1^4 + 4a_1^3a_2 + 351a_1^3 + 69a_1^2a_2 + 3263a_1^2 - 3a_1a_2^2 - 702a_1a_2 + a_1b_1^2 + 1570a_1 - 23a$$
$$- a_1^4a_2 - 23a_1^3a_2 + 3a_1^2a_2^2 + 351a_1^2a_2 + 46a_1a_2^2 + 3263a_1a_2 - a_2^3 - 351a_2^2 + a_2b_1^2 + 1570a_2 -$$

Thus an affine patch of the $\mathrm{Jac}(X_2)$ is the surface of $\mathbb{A}^4$ defined by

$$0 = -a_1^5 - 23a_1^4 + 4a_1^3a_2 + 351a_1^3 + 69a_1^2a_2 + 3263a_1^2 - 3a_1a_2^2 - 702a_1a_2 + a_1b_1^2 + 1570a_1 - 23a_2^2 - 3263a_2 -$$
$$0 = -a_1^4a_2 - 23a_1^3a_2 + 3a_1^2a_2^2 + 351a_1^2a_2 + 46a_1a_2^2 + 3263a_1a_2 - a_2^3 - 351a_2^2 + a_2b_1^2 + 1570a_2 - b_2^2 \,.$$

II.3. **Embedding the Kummer variety.** Let X be a curve of genus g and let $\mathrm{Sym}^g(X) = X^g/S_g$ be the g^{th} symmetric power of X. Fixing a divisor $D_0 \in \mathrm{Div}(X)$ of degree g, recall that the map

$$\mathrm{Sym}^g(X) \to \mathrm{Jac}(X)$$
$$\{P_1, \ldots, P_g\} \mapsto [P_1] + \cdots + [P_g] - D_0$$

is surjective.

Let

$$f(x) = f_6x^6 + f_5x^5 + \cdots + f_1x + f_0 \in k[x]$$

be a polynomial with no repeated roots (in the algebraic closure k^{al}). Then

$$X_2 : y^2 = f(x)$$

is a genus 2 hyperelliptic curve over k.

We now show how to realize the Kummer surface of X_2 as a quartic surface in $\mathbb{P}^3$, as described in [Mül10] and [CF96]. Suppose $P_1 = (x, y)$ and $P_2 = (u, v)$ are affine points on X_2. Let

$$\kappa_1 = 1$$
$$\kappa_2 = x + u$$
$$\kappa_3 = xu$$
$$\kappa_4 = \frac{F_0(x,u) - 2yv}{(x-u)^2}$$

where

$$F_0(x,u) = 2f_0 + f_1(x+u) + 2f_2xu + f_3(x+u)xu + 2f_4(xu)^2 + f_5(x+u)xu + 2f_6(xu)^3 .$$

The image of κ is a quartic surface given by

$$K_2(\kappa_1,\kappa_2,\kappa_3)\kappa_4^2 + K_1(\kappa_1,\kappa_2,\kappa_3)\kappa_4 + K_0(\kappa_1,\kappa_2,\kappa_3)\,, \tag{3}$$

where

$$\begin{aligned}
K_2(\kappa_1,\kappa_2,\kappa_3) &= \kappa_2^2 - 4\kappa_1\kappa_3 \\
K_1(\kappa_1,\kappa_2,\kappa_3) &= -4\kappa_1^3 f_0 - 2\kappa_1^2\kappa_2 f_1 - 4\kappa_1^2\kappa_3 f_2 - 2\kappa_1\kappa_2\kappa_3 f_3 - 4\kappa_1\kappa_3^2 f_4 - 2\kappa_2\kappa_3^2 f_5 - 4\kappa_3^3 f_6 \\
K_0(\kappa_1,\kappa_2,\kappa_3) &= -4\kappa_1^4 f_0 f_2 + \kappa_1^4 f_1^2 - 4\kappa_1^3\kappa_2 f_0 f_3 - 2\kappa_1^3\kappa_3 f_1 f_3 - 4\kappa_1^2\kappa_2^2 f_0 f_4 + 4\kappa_1^2\kappa_2\kappa_3 f_0 f_5 \\
&\quad - 4\kappa_1^2\kappa_2\kappa_3 f_1 f_4 - 4\kappa_1^2\kappa_3^2 f_0 f_6 + 2\kappa_1^2\kappa_3^2 f_1 f_5 - 4\kappa_1^2\kappa_3^2 f_2 f_4 + \kappa_1^2\kappa_3^2 f_3^2 \\
&\quad - 4\kappa_1\kappa_2^3 f_0 f_5 + 8\kappa_1\kappa_2^2\kappa_3 f_0 f_6 - 4\kappa_1\kappa_2^2\kappa_3 f_1 f_5 + 4\kappa_1\kappa_2\kappa_3^2 f_1 f_6 - 4\kappa_1\kappa_2\kappa_3^2 f_2 f_5 \\
&\quad - 2\kappa_1\kappa_3^3 f_3 f_5 - 4\kappa_2^4 f_0 f_6 - 4\kappa_2^3\kappa_3 f_1 f_6 - 4\kappa_2^2\kappa_3^2 f_2 f_6 - 4\kappa_2\kappa_3^3 f_3 f_6 - 4\kappa_3^4 f_4 f_6 \\
&\quad + \kappa_3^4 f_5^2 .
\end{aligned}$$

Then the Kummer surface K is given by equation (3) and the map $\kappa = [\kappa_1 : \kappa_2 : \kappa_3 : \kappa_4]$ is the desired map $\mathrm{Jac}(X_2) \to K$.

III. Overview of method

Our construction proceeds as follows. We take as input an elliptic curve X_1 and a genus 2 curve X_2 over a number field k [or more genenerally, any field? characteristic $\neq 2$?] given in Weierstrass form

$$X_1 : y^2 + u(x)y = v(x) \qquad X_2 : y^2 + h(x)y = f(x) .$$

Letting J_2 be the Jacobian variety of X_2, then J_2 is an abelian surface with 16 2-torsion points. The Kummer variety K_2 of X_2 is obtained by forming the quotient of J_2 by the negation map $[-1]$. This quotient map $\pi : J_2 \to K_2$ is injective on the 2-torsion points of J_2, whose images are the singular points of K_2. [Nodes, I guess?] Using the explicit embedding given in [Mül10] (which in turn is a generalization of [CF96]), we can realize K_2 as a quartic surface in $\mathbb{P}^3$.

Fix two nodes T_1, T_2 of K_2. Consider the pencil of planes $\mathcal{H} = \{H_\mu : \mu \in \mathbb{P}^1\}$ passing through T_1 and T_2. The intersection of a plane $H_\mu \in \mathcal{H}$ with K_2 is a quartic plane curve C_μ with two nodes. By the usual degree-genus formula for plane curves, C_μ has genus 1 for each $\mu \in \mathbb{P}^1$. We will endow C_μ with the structure of an elliptic curve and compute its j-invariant as a function of μ.

To a point $Q \in C_\mu$ we associate the line ℓ_Q passing through T_1 and Q. The association $Q \mapsto \ell_Q$ defines a degree 2 map $C_\mu \to \mathbb{P}^1$ ramified at 4 points. Computing the cross-ratio of these 4 points yields the λ-invariant of C_μ, allowing us to find a Legendre model $y^2 = x(x-1)(x-\lambda)$ for C_μ. We can then compute the j-invariant of C_μ using the standard formula

$$j = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda-1)^2} .$$

Note that computing the λ-invariant of C_μ not only endows C_μ with the structure of an elliptic curve, but also with level 2 structure: the Legendre model $E_{\mathrm{Leg}} : y^2 = x(x-1)(x-\lambda)$ comes equipped with the basis $\{(0,0),(1,0)\}$ for $E_{\mathrm{Leg}}[2]$, and we may pull back this basis along the isomorphism $C_\mu \xrightarrow{\sim} E_{\mathrm{Leg}}$ to obtain a basis for $C_\mu[2]$.

Lemma 1. *The composite map*

$$\begin{array}{ccccccc} \varphi : \mathbb{P}^1 & \longrightarrow & \mathcal{M}_1 & \longrightarrow & X(2) & \longrightarrow & X(1) \\ \mu & \longmapsto & C_\mu & \longmapsto & \lambda(C_\mu) & \longmapsto & j(\lambda(C_\mu)) \end{array}$$

has degree 12*.*

Proof. By the classical theory of modular functions, the map $X(2) \to X(1)$, $\lambda \mapsto j(\lambda)$ has degree 6, corresponding to the 6 permutations of $0, 1, \infty$ acted on by S_3. As the map $\mu \mapsto C_\mu$ has degree 1, it suffices to show that the map $\mathcal{M}_1 \to X(2)$ has degree 2. [I think this just follows from the fact that we could've chosen to the other node and taken lines through T_2 and Q to obtain a map to $\mathbb{P}^1$. I guess we have to show that this would produce the same λ...] ∎

Thus the composite map in the above lemma is a rational function of degree 12 in μ. Let $j_1 = j(X_1)$. In order to find a value of μ that yields an elliptic curve C_μ isomorphic to our original curve X_1, we solve the equation $\varphi(\mu) = j_1$. The solutions μ to this equation may not lie in the ground field, so it may be necessary to base change our curve to an algebraic extension. [I think in all the examples so far we've only needed quadratic extensions of the base field...] [One more interesting note: I think in all the examples we've done so far, $\varphi(\mu) - j_1$ has an interesting factorization. The numerator is a product of quadratics, and the denominator is a product of linear factors squared. Is this expected?]

IV. Comparison with analytic construction

The first author has also described a complex analytic method for glueing a genus 1 and a genus 2 curve along their 2-torsion using period matrices. In this section we compare these two methods of glueing.

V. Applications

Constructing abelian three-folds with interesting torsion?

VI. Examples

References

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