

# GLUEING JACOBIANS

JEROEN HANSELMAN, SAM SCHIAVONE, AND JEROEN SIJSLING

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## I. INTRODUCTION

In [FK91] the authors describe a method for glueing two elliptic curves  $E_1$  and  $E_2$  along their torsion subgroups to produce a genus 2 curve that covers both of them. In this article, we extend this method to genus 3: we glue a genus 1 curve  $X_1$  to the Jacobian variety of a genus 2 curve  $X_2$ . This produces an abelian 3-fold which, since all abelian 3-folds are principally polarized, is the Jacobian variety of a genus 3 curve  $X_3$ . We determine explicit equations for  $X_3$ , given the data of blah. We have implemented this method in Magma, and conclude the paper with several examples.

[Other papers to mention? Howe? Howe, Leprovost, Poonen? Brooker, Lauter, Steinhagen, etc.?]

## II. BACKGROUND

Encoding divisors as polynomials as in Mumford and Cantor. Describe construction of the Kummer as in Mueller.

## III. OVERVIEW OF METHOD

Our construction proceeds as follows. We take as input an elliptic curve  $X_1$  and a genus 2 curve  $X_2$  over a number field  $K$  [or more genenerally, any field?] given in Weierstrass form

$$X_1 : y^2 + u(x)y = v(x) \quad X_2 : y^2 + h(x)y = f(x).$$

Letting  $J_2$  be the Jacobian variety of  $X_2$ , then  $J_2$  is an abelian surface with 16 2-torsion points. The Kummer variety  $K_2$  of  $X_2$  is obtained by forming the quotient of  $J_2$  by the negation map  $[-1]$ . This quotient map  $\pi : J_2 \rightarrow K_2$  is injective on the 2-torsion points of  $J_2$ , whose images are the singular points of  $K_2$ . [Nodes, I guess?] Using the explicit

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embedding given in [Mül10] (which in turn is a generalization of [CF96]), we can realize  $K_2$  as a quartic surface in  $\mathbb{P}^3$ .

Fix two nodes  $T_1, T_2$  of  $K_2$ . Consider the pencil of planes  $\mathcal{H} = \{H_\mu : \mu \in \mathbb{P}^1\}$  passing through  $T_1$  and  $T_2$ . The intersection of a plane  $H_\mu \in \mathcal{H}$  with  $K_2$  is a quartic plane curve  $C_\mu$  with two nodes. By the usual degree-genus formula for plane curves,  $C_\mu$  has genus 1 for each  $\mu \in \mathbb{P}^1$ . We will endow  $C_\mu$  with the structure of an elliptic curve and compute its  $j$ -invariant as a function of  $\mu$ .

To a point  $Q \in C_\mu$  we associate the line  $\ell_Q$  passing through  $T_1$  and  $Q$ . The association  $Q \mapsto \ell_Q$  defines a degree 2 map  $C_\mu \rightarrow \mathbb{P}^1$  ramified at 4 points. Computing the cross-ratio of these 4 points yields the  $\lambda$ -invariant of  $C_\mu$ , allowing us to find a Legendre model  $y^2 = x(x-1)(x-\lambda)$  for  $C_\mu$ . We can then compute the  $j$ -invariant of  $C_\mu$  using the standard formula

$$j = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}.$$

Note that computing the  $\lambda$ -invariant of  $C_\mu$  not only endows  $C_\mu$  with the structure of an elliptic curve, but also with level 2 structure: the Legendre model  $E_{\text{Leg}} : y^2 = x(x-1)(x-\lambda)$  comes equipped with the basis  $\{(0,0), (1,0)\}$  for  $E_{\text{Leg}}[2]$ , and we may pull back this basis along the isomorphism  $C_\mu \xrightarrow{\sim} E_{\text{Leg}}$  to obtain a basis for  $C_\mu[2]$ .

**Lemma 1.** *The composite map*

$$\begin{aligned} \varphi : \mathbb{P}^1 &\longrightarrow \mathcal{M}_1 \longrightarrow X(2) \longrightarrow X(1) \\ \mu &\longmapsto C_\mu \longmapsto \lambda(C_\mu) \longmapsto j(\lambda(C_\mu)) \end{aligned}$$

*has degree 12.*

*Proof.* By the classical theory of modular functions, the map  $X(2) \rightarrow X(1)$ ,  $\lambda \mapsto j(\lambda)$  has degree 6, corresponding to the 6 permutations of  $0, 1, \infty$  acted on by  $S_3$ . As the map  $\mu \mapsto C_\mu$  has degree 1, it suffices to show that the map  $\mathcal{M}_1 \rightarrow X(2)$  has degree 2. [I think this just follows from the fact that we could've chosen to the other node and taken lines through  $T_2$  and  $Q$  to obtain a map to  $\mathbb{P}^1$ . I guess we have to show that this would produce the same  $\lambda$ ...]  $\square$

Thus the composite map in the above lemma is a rational function of degree 12 in  $\mu$ . Let  $j_1 = j(X_1)$ . In order to find a value of  $\mu$  that yields an elliptic curve  $C_\mu$  isomorphic to our original curve  $X_1$ , we solve the equation  $\varphi(\mu) = j_1$ . The solutions  $\mu$  to this equation may not lie in the ground field, so it may be necessary to base change our curve to an algebraic extension. [I think in all the examples so far we've only needed quadratic extensions of the base field...] [One more interesting note: I think in all the examples we've done so far,  $\varphi(\mu) - j_1$  has an interesting factorization. The numerator is a product of quadratics, and the denominator is a product of linear factors squared. Is this expected?]

#### IV. APPLICATIONS

Constructing abelian three-folds with interesting torsion?

## V. EXAMPLES

### REFERENCES

- [CF96] J. W. S. Cassels and E. V. Flynn. *Prolegomena to a middlebrow arithmetic of curves of genus 2*, volume 230 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1996.
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