GLUEING JACOBIANS

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I. INTRODUCTION

In [FK91] the authors describe a method for glueing two elliptic curves E_1 and E_2 along their torsion subgroups to produce a genus 2 curve that covers both of them. In this article, we extend this method to genus 3: we glue a genus 1 curve X_1 to the Jacobian variety of a genus 2 curve X_2. This produces an abelian 3-fold which, since all abelian 3-folds are principally polarized, is the Jacobian variety of a genus 3 curve X_3. We determine explicit equations for X_3, given the data of blah. We have implemented this method in Magma, and conclude the paper with several examples.

[Other papers to mention? Howe? Howe, Leprovost, Poonen? Broker, Lauter, Stevenhagen, etc.?]

II. BACKGROUND

Encoding divisors as polynomials as in Mumford and Cantor. Describe construction of the Kummer as in Mueller.

II.1. Representing divisors.

II.2. Embedding the Kummer variety.

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III. Overview of method

Our construction proceeds as follows. We take as input an elliptic curve X_1 and a genus 2 curve X_2 over a number field K [or more genenerally, any field?] given in Weierstrass form

$$X_1 : y^2 + u(x)y = v(x) \qquad X_2 : y^2 + h(x)y = f(x)\,.$$

Letting J_2 be the Jacobian variety of X_2, then J_2 is an abelian surface with 16 2-torsion points. The Kummer variety K_2 of X_2 is obtained by forming the quotient of J_2 by the negation map $[-1]$. This quotient map $\pi : J_2 \to K_2$ is injective on the 2-torsion points of J_2, whose images are the singular points of K_2. [Nodes, I guess?] Using the explicit embedding given in [Mül10] (which in turn is a generalization of [CF96]), we can realize K_2 as a quartic surface in $\mathbb{P}^3$.

Fix two nodes T_1, T_2 of K_2. Consider the pencil of planes $\mathcal{H} = \{H_\mu : \mu \in \mathbb{P}^1\}$ passing through T_1 and T_2. The intersection of a plane $H_\mu \in \mathcal{H}$ with K_2 is a quartic plane curve C_μ with two nodes. By the usual degree-genus formula for plane curves, C_μ has genus 1 for each $\mu \in \mathbb{P}^1$. We will endow C_μ with the structure of an elliptic curve and compute its j-invariant as a function of μ.

To a point $Q \in C_\mu$ we associate the line ℓ_Q passing through T_1 and Q. The association $Q \mapsto \ell_Q$ defines a degree 2 map $C_\mu \to \mathbb{P}^1$ ramified at 4 points. Computing the cross-ratio of these 4 points yields the λ-invariant of C_μ, allowing us to find a Legendre model $y^2 = x(x-1)(x-\lambda)$ for C_μ. We can then compute the j-invariant of C_μ using the standard formula

$$j = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda-1)^2}\,.$$

Note that computing the λ-invariant of C_μ not only endows C_μ with the structure of an elliptic curve, but also with level 2 structure: the Legendre model $E_{\text{Leg}} : y^2 = x(x-1)(x-\lambda)$ comes equipped with the basis $\{(0,0),(1,0)\}$ for $E_{\text{Leg}}[2]$, and we may pull back this basis along the isomorphism $C_\mu \xrightarrow{\sim} E_{\text{Leg}}$ to obtain a basis for $C_\mu[2]$.

Lemma 1. *The composite map*

$$\begin{array}{ccccccc} \varphi : \mathbb{P}^1 & \longrightarrow & \mathcal{M}_1 & \longrightarrow & X(2) & \longrightarrow & X(1) \\ \mu & \longmapsto & C_\mu & \longmapsto & \lambda(C_\mu) & \longmapsto & j(\lambda(C_\mu)) \end{array}$$

has degree 12.

Proof. By the classical theory of modular functions, the map $X(2) \to X(1)$, $\lambda \mapsto j(\lambda)$ has degree 6, corresponding to the 6 permutations of $0, 1, \infty$ acted on by S_3. As the map $\mu \mapsto C_\mu$ has degree 1, it suffices to show that the map $\mathcal{M}_1 \to X(2)$ has degree 2. [I think this just follows from the fact that we could've chosen to the other node and taken lines through T_2 and Q to obtain a map to $\mathbb{P}^1$. I guess we have to show that this would produce the same λ...] $\square$

Thus the composite map in the above lemma is a rational function of degree 12 in μ. Let $j_1 = j(X_1)$. In order to find a value of μ that yields an elliptic curve C_μ isomorphic to our original curve X_1, we solve the equation $\varphi(\mu) = j_1$. The solutions μ to this equation may not lie in the ground field, so it may be necessary to base change our curve to an algebraic extension. [I think in all the examples so far we've only needed quadratic extensions of the base field...] [One more interesting note: I think in all the examples we've done so far, $\varphi(\mu) - j_1$ has an interesting factorization. The numerator is a product of quadratics, and the denominator is a product of linear factors squared. Is this expected?]

IV. Applications

Constructing abelian three-folds with interesting torsion?

V. Examples

References

[CF96] J. W. S. Cassels and E. V. Flynn. *Prolegomena to a middlebrow arithmetic of curves of genus* 2, volume 230 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1996.

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[Mül10] Jan Steffen Müller. Explicit Kummer surface formulas for arbitrary characteristic. *LMS J. Comput. Math.*, 13:47–64, 2010.