# Modeling of Web Conveyance Systems for Multivariable Control

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*Abstract*—**A web is a very long but thin, continuous sheet of flexible material including such industrial products as film, belt, strip, foil, and fabric. As a web is being processed, it is conveyed through a system of rollers that supply support, transport, and control. Ideally, the motion of the web is only in a longitudinal direction, i.e., in the machine direction. This is not the case for real web conveyance systems since misalignment of rollers, lateral disturbances, and web physical properties result in lateral vibrations of the web. This lateral motion results in an inferior product quality, waste, and lower productivity. For this reason it is important to have control over the lateral positioning of the web. To achieve a tight control, lateral web dynamics must be well understood.**

**This paper proposes a distributed parameter model of the lateral dynamic behavior of a moving web in an *n*-roller system. A span of web, moving longitudinally and under tension is modeled as a beam with shear flexibility (Timoshenko beam). A quasi-static simplification based upon spectral separation is made to simplify the model. Boundary conditions at the web-roller interface are considered. Actuator dynamics are introduced through the formulation of the boundary conditions. Closed-loop simulations, using a state-space version of the simplified model, are in agreement with experimental results. A preliminary version of this paper appears in The Proceedings of AIAA/ASME/ASCE/AHS Structures, Structural Dynamics and Materials Conference, April 1987.**

## I. INTRODUCTION

THE term "web" refers to a continuous sheet of flexible material that is much longer than wide and much wider than thick. Examples of webs processed for industrial purposes are paper, polyester, and acetate. As a web is being processed it is conveyed over a system of rollers that provide support, transport, and control. To ensure that the processing is accomplished within the required tolerances, the web must follow a predetermined path as it moves through the conveyance system. Normally this path is chosen to be constant velocity motion in the longitudinal direction (i.e., in the machine direction). Deviation from this path will result in an inferior product quality, waste, and lower productivity. Therefore, it is important to have a tight control on the lateral placement of the web edge.

The problem of controlling lateral edge placement is rather unique, since the web is being continuously forced in the lateral direction. The lateral force causes the web to vibrate back-and-forth, a motion termed "web weave." The mechanism for the generation of this weave is not well understood. Its origin may be a combination of web camber, method of web formation, and interaction of web with machine parts. The controllers presently used to eliminate lateral weave are only able to null the weave at the control sensor locations. They are unable to completely null the weave along a length of web. This control limitation is not only due to the type of control scheme used, but also due to the accepted model for lateral dynamics.

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Although web conveyance systems have been widely used in industry since the days of the industrial revolution, the technical literature on the subject of lateral web dynamics is very limited. The most significant work was done by J. J. Shelton [1], [4], and [5] in the 1960's. His work provides a foundation for modeling the lateral dynamics of a web moving between two rollers. Shelton uses both a string and a simple beam to model the web. The simple beam model [7] considers deflections due only to bending and ignores the effects of shear.

Work by Sievers [3] extends Shelton's two-roller model for lateral web dynamics to an *n*-roller system model. The two web models considered by Sievers were also the string and simple beam models. Simulations using both the *n*-roller string model and the *n*-roller simple beam model with a PID controller in the feedback loop were run and compared to experimental data. Both models agreed with experimental results when low gains were used in the feedback loop, but the simulations deviated from experimental data when higher feedback gains were used. Unfortunately, the high gain operating region is the one applicable for controlling lateral vibrations. In order for the dynamic behavior of the web to be predicted during both high and low gain operation, a model of more complexity needs to be considered.

The literature available on the control of lateral web placement is also very sparse. A consequence of this is that most web handling systems use a proportional-derivative-integral controller with gain settings determined by trial and error. The PID controllers used on the machines do not take advantage of any model-based knowledge. An optimal multivariable control scheme will offer a definite upgrade in control performance since the control input is dependent on the states rather than only one output. Linear quadratic regulator control could provide for a robust, stable control design [10] and [11].

A distributed parameter model of lateral web dynamics for an *n*-roller conveyance system is presented in this paper. A web span between two rollers is modeled as a longitudinally tensioned Timoshenko beam moving with a constant longitudinal velocity. The Timoshenko beam model, [8] and [9], considers deflections due to both bending and shear forces. The consideration of shear is the key element that allows the Timoshenko beam model to predict observed dynamical behavior that is not exhibited by the string and simple beam models. Section III-A develops this web model.

For a complete system description, boundary conditions at the web–roller interface must also be considered. The boundary conditions are developed in Section III-B. Certain rollers in the conveyance system act as actuators. These rollers actively control the placement of the web's edge. The dynamics of the actuating rollers automatically appear in the formulation of the boundary conditions. These results are also included in Section III-B.

Section IV develops a state-space version of the model. This form is determined by using the boundary conditions and mode shapes derived in Section III. The state-space model is formulated for the purpose of multivariable control design. This form of the *n*-roller system makes it convenient for LQR or LQG control techniques to be applied to the system [6].

Simulation results are included in Section V. The plots of

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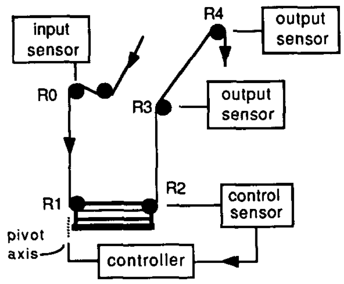


Fig. 1. Web path description.

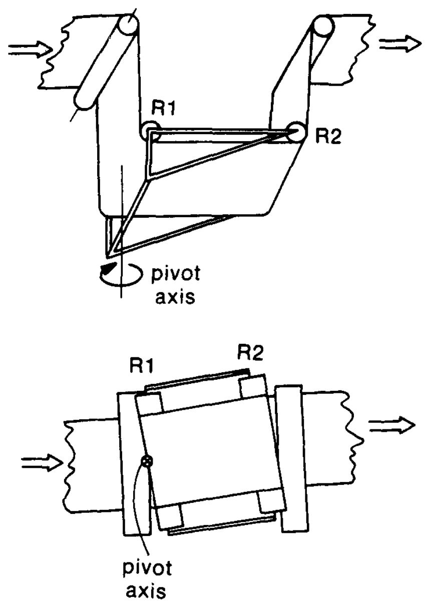


Fig. 2. Two different views of a displacement guide.

simulation data versus experimental data provide a strong argument that the model derived in this paper will predict the dynamic behavior of the web viewed during system operation.

## II. System Description

Fig. 1 is a sketch of an experimental setup developed for purposes of modeling and control. The combination of rollers $R1$ and $R2$ is a displacement guide (see Fig. 2). This guide is the actuator responsible for changing the lateral position of the web. $R1$ and $R2$ are mechanically connected and free to pivot about an axis parallel to the incoming web at $R1$. The rest of the rollers are passive; consequently, they are restricted to rotate about their central axes. A low-frequency lateral disturbance (i.e., a weave) is introduced upstream of the roller $R0$. The amplitude and phase of the weave are monitored by a sensor at $R0$. The weave is also sensed at $R2$ and the signal is sent to a controller that controls the pivot angle of the displacement guide.

The controller type presently used for this type of system is proportional-integral-differential. In experiments conducted on the system in the high-gain region (i.e., when the amplitude at $R2$ has been markedly reduced), it is observed that the weave at $R4$ is always greater than the weave at $R3$. The string and the simple beam models both predict that when weave amplitude is minimized at a roller, the amplitude of the weave at all points downstream of that roller is either equal or less [3]. The observed "weave regeneration phenomenon" is not predicted by either model. An improved model (i.e., one that predicts weave regeneration) is the subject of this paper.

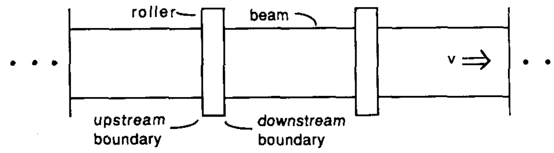


Fig. 3. Web conveyance system modeled as a series of beams joined at boundaries.

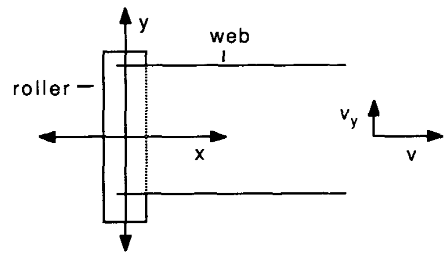


Fig. 4. Coordinate system for one web span.

## III. Modeling of a Web Conveyance System

A web conveyance system is made up of a number of web spans joined at their boundaries by rollers. To model the system, each span of web is treated as a one-dimensional member. The boundaries of each member correspond to where the web leaves a roller and where it contacts the next downstream roller (see Fig. 3).

The beam model proposed to describe lateral web dynamics is the Timoshenko beam model. The Timoshenko beam model considers deflections due to bending and shear forces. One might envision shear deflections of such a wide web to be important, certainly for a web of woven cloth.

### *A. Dynamic Equation of Motion for a Web Segment Modeled as a Timoshenko Beam*

*Nomenclature:*

| | |
|---|---|
| $y$ | lateral deflection of the web's centerline |
| $x$ | distance along length of web |
| $t$ | time |
| $v_y$ | velocity of centerline in $y$ direction |
| $v$ | velocity of centerline in $x$ direction |
| $\theta$ | angle of face rotation |
| $\Psi$ | angle due to shear deformation $= \partial y/\partial x - \theta$ |
| $\alpha$ | angle of web wrap on a roller |
| $T$ | tension |
| $EI$ | bending stiffness |
| $GA$ | shear stiffness |
| $m$ | mass/length |
| $\rho$ | density |
| $J$ | rotary inertia/length |
| $\gamma$ | pivot angle of displacement guide |
| $\partial(\ )/\partial x$ | $= (\ )'$ |
| $\partial(\ )/\partial t$ | $= (\dot{\ })$ |

A convenient method of calculating the equation of motion of a beam moving with a longitudinal velocity and experiencing tension is to use Hamilton's principle [2]. The kinetic energy of the beam is

$$K=\frac{1}{2}\int_0^L m(v^2+v_y^2)\,dx+\frac{1}{2}\int_0^L J\left(\frac{d\theta}{dt}\right)^2 dx$$
$$=\frac{1}{2}\int_0^L m\left(v^2+\left(\frac{\partial y}{\partial t}+v\frac{\partial y}{\partial x}\right)^2\right) dx+\frac{1}{2}\int_0^L J\left(\frac{\partial\theta}{\partial t}+v\frac{\partial\theta}{\partial x}\right)^2 dx. \tag{1}$$

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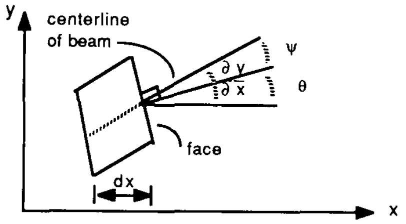


Fig. 5. Effect of shear on beam element.

The potential energy of the beam is

$$V=\frac{1}{2}\int_0^L EI\left(\frac{\partial\theta}{\partial x}\right)^2 dx+\frac{1}{2}\int_0^L AG(\Psi)^2\,dx+\frac{1}{2}\int_0^L T\left(\frac{\partial y}{\partial x}\right)^2 dx. \tag{2}$$

Application of Hamilton's principle to the kinetic and potential energy of the web yields the equation of motion for lateral web dynamics

$$\begin{aligned}0=&(Jv^2-EI)\left(\frac{mv^2}{GA}-\frac{T}{GA}-1\right)\frac{\partial^4 y}{\partial x^4}\\&+(mv^2-T)\frac{\partial^2 y}{\partial x^2}+2mv\frac{\partial^2 y}{\partial x\partial t}+m\frac{\partial^2 y}{\partial t^2}\\&+\left(\frac{5mv^2}{GA}-\frac{EIm}{GA}-\frac{JT}{GA}-\frac{Jmv^2}{GA}-J\right)\frac{\partial^4 y}{\partial x^2\partial t^2}\\&+\frac{Jm}{GA}\frac{\partial^4 y}{\partial t^4}+\frac{4Jmv}{GA}\frac{\partial^4 y}{\partial t^3\partial x}\\&+\left(\frac{4Jmv^3}{GA}-\frac{2EImv}{GA}-\frac{2JvT}{GA}-2Jv\right)\frac{\partial^4 y}{\partial x^3\partial t}.\end{aligned} \tag{3}$$

The above equation is simplified if it can be assumed that a web segment behaves in a quasi-static fashion. The quasi-static assumption is based upon spectral separation; the realization that the natural frequencies of a single span of web are much greater than the disturbance frequencies. A conservative lower bound on the frequency of the first mode of a single span can be calculated by considering a convecting string of length $L$ constrained at both ends ($T \gg mv^2$ for the web processes under consideration)

$$f_0=\frac{1}{2L}\sqrt{\frac{T}{m}}\left(1-\frac{mv^2}{T}\right)\approx\frac{1}{2L}\sqrt{\frac{T}{m}}. \tag{4}$$

For a number of web processes used in industry, it can be shown that the above lower bound frequency is at least 10 times greater than the maximum disturbance observed in the process. Using the quasi-static assumption, the equation of motion for a web is simplified considerably to the ordinary differential equation

$$\frac{d^4y}{dx^4}-K^2\frac{d^2y}{dx^2}=0 \tag{5}$$

where

$$K^2=\frac{T}{EI\left(1+\dfrac{nT}{AG}\right)} \tag{5a}$$

which supports solutions of the form

$$y(x,t)=f_1(t)+f_2(t)x+f_3(t)\sinh(Kx)+f_4(t)\cosh(Kx). \tag{6}$$

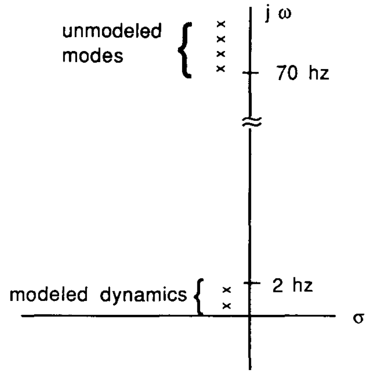


Fig. 6. $S$-plane plot of modeled and unmodeled modes.

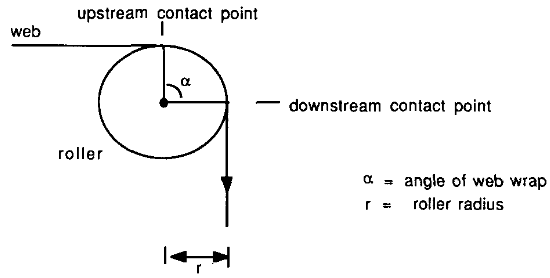


Fig. 7. Upstream and downstream roller contact points.

The approximation introduced by this "spectral separation" is best understood with reference to Fig. 6. At this point in the modeling process, one can already foresee that the dynamics will be separated into two groups. The dynamics of interest, characterized by lateral web motion throughout the entire conveyance system, occurs with characteristic frequencies of a few hertz. A separate set of characteristic modes is being neglected with the quasi-static assumption introduced above. These motions, consisting of lateral web vibrations primarily confined to one span, are stable and much faster (at least 70 Hz) than the ones being modeled. One introduces the simplifying assumption that these vibrations are zero and that each span is instantaneously in static equilibrium, thus (5). This assumption has reduced the distributed parameter model to a lumped parameter model. The assumption is verified in Section IV.

Equation (5) has the form of the equation of static equilibrium of a simple beam in tension. Modeling the web as a Timoshenko beam, as opposed to a simple beam, thus does not change the character of the solution between rollers, but it will change the dynamic behavior of the web allowed at the boundaries. Consequently, the necessity of using a Timoshenko beam model is not obvious at this point but will be made more clear in the next section when the boundary conditions are considered.

### B. Boundary Conditions of a Web Span

The previous section has introduced a model of a web segment in terms of lateral deflections, $y$, from a reference axis. A complete conveyance system has many such segments. Each segment spans the space between two neighboring rollers such that the boundaries occur where the web exits the roller $R_i$ and where the web initially makes contact with the roller $R_{i+1}$. To describe the matching conditions between two neighboring segments at a roller, it is convenient to introduce boundary variables. Let $y_i^u(t)$ and $y_i^d(t)$ be the lateral deflection of the web at the upstream and downstream contact points of the roller $R_i$ (see Fig. 7).

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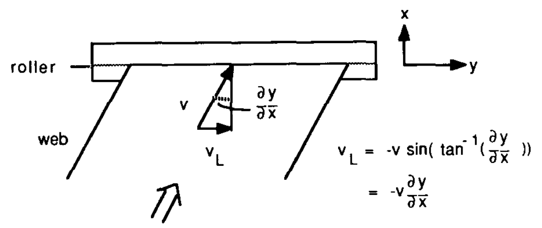


Fig. 8. Velocity boundary condition for web approaching roller.

The lines of contact on the $i$th roller together with the upstream and downstream $x$-axis segments define two distinct planes (the web planes). Angles associated with the upstream web segment (identified with the superscript "$u$") are defined about the normal to the upstream web plane, with an analogous definition for downstream variables. Thus, $\theta_i^u(t)$ and $\theta_i^d(t)$ represent the rotation of the web cross-sectional face at the upstream and downstream contact points of roller-$i$ projected onto the respective web plane.

For a passive roller, two matching conditions for the two web segments at roller-$i$ are necessary. To match the boundary conditions of one web span to its adjoining web span, it is simplest to consider variables that behave continuously from one web span to the next, that is, choose the "matching variables" to be the ones continuous at the upstream *and* downstream web–roller contact points. For the Timoshenko beam model, these variables are lateral displacement and face rotation. The resulting approximate matching conditions for the two web segments at roller $i$ are

$$y_i^u = y_i^d \tag{7}$$

$$\theta_i^u = \theta_i^d. \tag{8}$$

The above equations are valid for roller radius much less than roller spacing. These conditions are reasonable since the length of web wrap on a roller is normally very small compared to the length of the web span. If this condition is violated, one might modify the boundary conditions to account for changes the web undergoes in the region of contact with the roller.

The necessity of using the Timoshenko beam model is inherent in the boundary condition of (8). The analogous boundary condition in the simple beam model is that the slope on the roller be continuous, i.e., $y_i^{u\prime} = y_i^{d\prime}$. The actual slope as it leaves a roller is discontinuous but this is not built into the simple beam model since shear effects are not considered; hence, the Timoshenko beam model does a better job of modeling true dynamical behavior at the boundary.

Two further boundary conditions describe the interaction between the moving web and the roller that it is approaching. The analysis is done assuming that $\partial y_i^u/\partial x$ is very small and that no slip occurs on the rollers. It was observed by Shelton [1] that the lateral velocity of a web is proportional to the local angle between the centerline of the web and the normal to the roller axis (see Fig. 8). Therefore the lateral velocity of the web, upstream of a stationary roller is

$$v_L = \frac{\partial y_i^u}{\partial t} = -v\,\frac{\partial y_i^u}{\partial x}\,. \tag{9}$$

Lateral acceleration is the result of a change in the slope of the web, and can be derived by expanding the expression for $v$ around $L$

$$\begin{aligned}\frac{\Delta v}{\Delta t} = \frac{v_{L-\Delta x} - v_L}{\Delta t} &= v\left(\frac{\Delta x}{\Delta t}\frac{\partial^2 y}{\partial x^2}\bigg|_L + \frac{(\Delta x)^2}{(\Delta t)^2}\frac{\partial^3 y}{\partial x^3}\bigg|_L + \cdots\right)\\ &= v^2\left(\frac{\partial^2 y}{\partial x^2}\bigg|_L + \frac{(\Delta x)^2}{(\Delta t)}\frac{\partial^3 y}{\partial x^3}\bigg|_L + \cdots\right).\end{aligned} \tag{10}$$

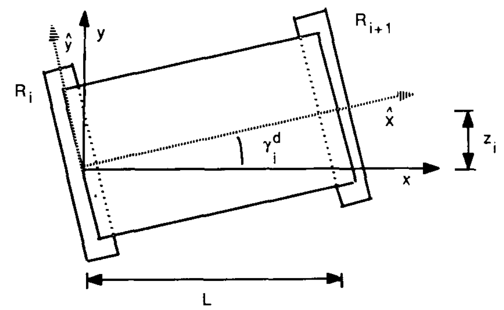


Fig. 9. Coordinate transformation used to introduce control variables.

In the limit as $\Delta t$ goes to 0 (where $v = \Delta x/\Delta t$)

$$a_L = \frac{\partial^2 y}{\partial t^2}\bigg|_L = v^2\,\frac{\partial^2 y}{\partial x^2}\bigg|_L \Rightarrow \frac{\partial^2 y_i^u}{\partial t^2} = v^2\,\frac{\partial^2 y_i^u}{\partial x^2}\,. \tag{11}$$

Since $y(x,t)$ is a smooth differentiable function, the above expansion is valid. Equations (9) and (11) are the lateral velocity and acceleration of the web on the $i$th roller when the roller is the passive type.

Equations (7)–(11) are also true for actuating rollers if the variable $y$ is taken to be the displacement of the web with respect to a coordinate system pivoting with the roller axes, i.e., the displacement of the web with respect to the $\hat{x}$–$\hat{y}$ coordinate system shown in Fig. 9. The $x$–$y$ coordinate is defined by the $R_i$ roller axis and the centerline of a stationary web when no control is applied to the system and all roller axes are parallel. A transformation from the moving $\hat{x}$–$\hat{y}$ coordinate system to the stationary $x$–$y$ coordinate system can be carried out in (7), (8), (10), and (11). The resulting boundary conditions introduce the control variables $z_i(t)$ and $\gamma_i(t)$, where $z_i(t)$ is defined as the lateral deflection of the roller $R_i$, with respect to the stationary coordinate system. The variables $\gamma_i^u$ and $\gamma_i^d$ represent the components of the pivot angle of the $i$th roller projected onto the upstream and downstream web planes, respectively. The transformed boundary conditions are

$$y_i^u = y_i^d \tag{12}$$

$$\theta_i^u = \theta_i^d + \frac{\alpha}{\pi/2}\,(\gamma_i^d - \gamma_i^u) \tag{13}$$

$$\frac{\partial y_i^u}{\partial t} = -v\left(\frac{\partial y_i^u}{\partial x} - \gamma_i^u\right) + \frac{\partial z_i}{\partial t} \tag{14}$$

$$\frac{\partial^2 y_i^u}{\partial t^2} = v^2\,\frac{\partial^2 y_i^u}{\partial x^2} + \frac{\partial^2 z_i}{\partial t^2}\,. \tag{15}$$

These boundary conditions are valid for both passive and active rollers where $y$ is taken to be the displacement of the web with respect to the stationary reference frame.

### C. Web Segment Shape

Any solution of the ordinary differential equation (5) forced only at the boundaries, will be a linear combination of four spatial functions. The solution of the equation can be written in terms of boundary variables in the following form:

$$y(x,\,t) = y_i^d(t)X_1(x) + \theta_i^d(t)X_2(x) + y_{i+1}^u(t)X_3(x) + \theta_{i+1}^u(t)X_4(x). \tag{16}$$

The shape functions of the web modeled as a Timoshenko beam

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are the basis functions $X_1(x)$, $X_2(x)$, $X_3(x)$, and $X_4(x)$. It was convenient to write the solution of (5) in terms of lateral deflections and face rotations at the boundaries. This is a preferred choice since $y_i^d(t)$, $y_{i+1}^u(t)$, $\theta_i^d(t)$, and $\theta_{i+1}^u(t)$ are all continuous at the boundaries. The shape functions can be calculated using the four boundary conditions $y_i^d(t) = y_i(0, t)$, $y_{i+1}^u(t) = y(L, t)$, $\theta_i^d(t) = \theta(0, t)$, and $\theta_{i+1}^u(t) = \theta_{i+1}(L, t)$ and the fact that $y(x, t)$ must be of the form given by (6). Functional expressions of the shape functions are given below and are graphically represented in Figs. 10–13 (the constants $R$ and $a$ are defined below to simplify notation):

$$\begin{aligned} X_1(x) = \{&a \sinh(KL) \sinh(Kx) + a[1-\cosh(KL)] \cosh(Kx) \\ &- a^2 \sinh(KL)Kx + a^2 KL \sinh(KL) \\ &+ a[1-\cosh(KL)]\}/R \end{aligned} \tag{17}$$

$$\begin{aligned} X_2(x) = \{&[1-\cosh(KL) + aKL \sinh(KL)] \sinh(Kx) \\ &+ [\sinh(KL) - aKL \cosh(KL)] \cosh(Kx) \\ &+ a[1-\cosh(KL)]Kx - [\sinh(KL) \\ &- aKL \cosh(KL)]\}/KR \end{aligned} \tag{18}$$

$$\begin{aligned} X_3(x) = \{&-a \sinh(KL) \sinh(Kx) \\ &- a[1-\cosh(KL)] \cosh(Kx) \\ &+ a^2 \sinh(KL)Kx + a[1-\cosh(KL)]\}/R \end{aligned} \tag{19}$$

$$\begin{aligned} X_4(x) = \{&-[1-\cosh(KL)] \sinh(Kx) + a[1-\cosh(KL)]Kx \\ &+ [aKL - \sinh(KL)] \cosh(Kx) \\ &- [aKL - \sinh(KL)]\}/KR \end{aligned} \tag{20}$$

where

$$R = a[2-2\cosh(KL)] + a^2 KL \sinh(KL)$$

$$a = \frac{EI}{GA}\left[1 + \frac{T}{GA}\right]$$

$$K^2 = \frac{T}{EI\left(1 + \frac{T}{GA}\right)}.$$

## IV. State-Space Formulation of Model for Purpose of Multivariable Control Design

This section formulates a state-space model using the closed-loop boundary conditions and the basis functions derived in Section III. A judicious choice of state variables is $y_i^u$, $\dot{y}_i^u$, $z_i$, and $\dot{z}_i$; the displacement and velocity of web at the $i$th roller and the lateral displacement and velocity of the $i$th roller. For state-space representation it is necessary to derive an expression for $\ddot{y}_i^u$ in terms of the states, the control variables, and the inputs. The boundary conditions can be written in matrix form

$$\begin{bmatrix} \theta_i^u \\ y_i^u \end{bmatrix} = \begin{bmatrix} \theta_i^d + \frac{\alpha}{\pi/2}(\gamma_i^d - \gamma_i^u) \\ y_i^d \end{bmatrix} \tag{21}$$

$$\begin{bmatrix} \dot{y}_i^u \\ \ddot{y}_i^u \end{bmatrix} = \begin{bmatrix} -v & 0 \\ 0 & v^2 \end{bmatrix} \begin{bmatrix} y_i^{u\prime} - \gamma_i^u \\ y_i^{u\prime\prime} \end{bmatrix} + \begin{bmatrix} \dot{z}_i \\ \ddot{z}_i \end{bmatrix}. \tag{22}$$

The curvature and the slope of the web centerline at the $i$th roller can be specified in terms of the deflection and face rotation of the web with respect to the local coordinate system, using the basis functions

$$\begin{bmatrix} y_i^{u\prime} \\ y_i^{u\prime\prime} \end{bmatrix} = \begin{bmatrix} X_1'(L_i) & X_2'(L_i) & X_3'(L_i) & X_4'(L_i) \\ X_1''(L_i) & X_2''(L_i) & X_3''(L_i) & X_4''(L_i) \end{bmatrix} \begin{bmatrix} y_{i-1}^d \\ \theta_{i-1}^d \\ y_i^u \\ \theta_i^u \end{bmatrix}. \tag{23}$$

Manipulation of (21)–(23) yields a linear expression for $\ddot{y}_i^u$ in terms of the states, the control variable, and the inputs

$$\ddot{y}_i^u = f(y_i^u, \dot{y}_i^u, y_{i-1}^u, \dot{y}_{i-1}^u, \cdots y_0, \theta_0, z_i, \dot{z}_i, \ddot{z}_i). \tag{24}$$

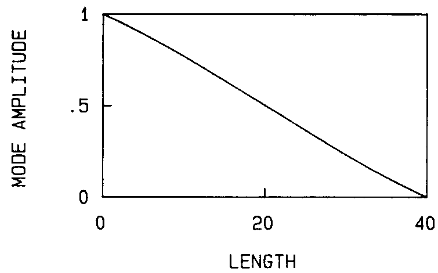

Fig. 10. $X_1(x)$ versus $x$.

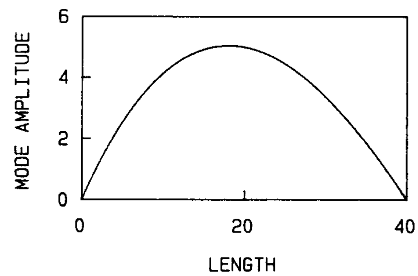

Fig. 11. $X_2(x)$ versus $x$.

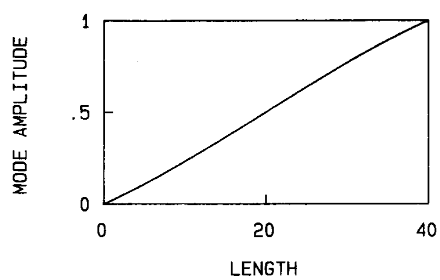

Fig. 12. $X_3(x)$ versus $x$.

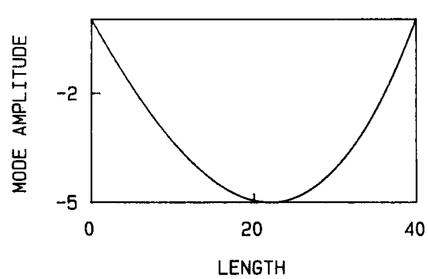

Fig. 13. $X_4(x)$ versus $x$.

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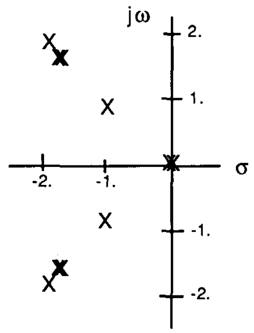


Fig. 14. Typical $s$-plane plot of open-loop eigenvalues of (25).

The inputs $y_0$ and $\theta_0$ are the deflection and face rotation of the web at the upstream location at $R0$. For an $n$-roller system with one displacement guide, a set of $n$ linear equations can be generated.

The system in Fig. 1 is a five-roller conveyance system where the combination of rollers $R1$ and $R2$ is a displacement guide acting as the control actuator. The control variable, $\ddot{z}$, is proportional to the torque applied to the pivot carriage of the displacement guide. The pivot angles $\gamma_i^u$, $\gamma_i^d$ are all equal to zero with the exception of $\gamma_1^d$ and $\gamma_2^u$. Both $\gamma_1^d$ and $\gamma_2^u$ are equal to $z/L_2$. For this particular configuration, the form of the state-space model with external inputs is

$$\begin{bmatrix} \dot{y}_1^u \\ \ddot{y}_1^u \\ \dot{y}_2^u \\ \ddot{y}_2^u \\ \dot{y}_3^u \\ \ddot{y}_3^u \\ \dot{y}_4^u \\ \ddot{y}_4^u \\ \dot{z}_1 \\ \ddot{z}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & 0 & 0 & a_{69} & a_{610} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} & a_{89} & a_{810} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1^u \\ \dot{y}_1^u \\ y_2^u \\ \dot{y}_2^u \\ y_3^u \\ \dot{y}_3^u \\ y_4^u \\ \dot{y}_4^u \\ z_1 \\ \dot{z}_1 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} [\ddot{z}] + \begin{bmatrix} 0 \\ I_{21} \\ 0 \\ I_{41} \\ 0 \\ I_{61} \\ 0 \\ I_{81} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ I_{22} \\ 0 \\ I_{42} \\ 0 \\ I_{62} \\ 0 \\ I_{82} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}. \tag{25}$$

The values of the entries in the above matrices are determined using the $n$-dimensional set of linear equations generated from (24). These values are given explicitly in [3], and are omitted here for brevity. To avoid violation of the assumption of spectral separation, any dynamics in system (25) must be slow compared to the neglected web segment natural frequencies given by (4). An eigenanalysis of (25), using parameters chosen to model the configuration of Fig. 1 was performed. Fig. 14 is a typical example of an $s$-plane plot of these values. The maximum eigenvalue frequency remains at least 10 times less than the frequency given by (4).

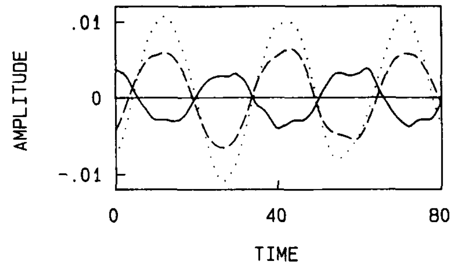


Fig. 15. Plot of data for experiment with velocity = 200 ft/min, weave frequency = 2 cycles/min, tension = 1 lb/in, gain = 60, and input weave amplitude = 0.165 in.

## V. Model Validation

The system in Fig. 1 is a conveyance system built for purposes of control experiments and model validation. A proportional gain controller was used for the experiments

$$z = -Gy_2^u \tag{26}$$

where $y_2^u$ is the displacement of the web at the upstream location on $R2$ and $G$ is the feedback gain.

A sinusoidal input forces the web to weave back-and-forth at a specified frequency. Experiments were run using an acetate web with width 44.5 in. The web span lengths were 29 in, 22 in, 40 in, and 61 in. The velocity of the web was varied between 200 and 800 ft/min, the frequency of the input weave was varied between 1 to 4 cycles/min, and the tension was varied between 0.75 and 1.5 lb/in. The material properties of the web used in the experiments were calculated to be

$$EI = 1.76 \times 10^8 \text{ lb in}^2$$

$$GA = 3.42 \times 10^5 \text{ lb}$$

$$\rho I = 1.18 \text{ lb in}$$

$$m = 7.12 \times 10^{-3} \text{ lb/in.}$$

The string and simple beam models predict that the weave amplitude at $R4$ is always less than the amplitude at $R3$ under all operating conditions (i.e., open loop as well as closed loop). The experimental data did not always exhibit this behavior. In fact, during high gain closed-loop operating conditions, experiments showed that weave amplitude at $R4$ was always greater than weave amplitude at $R3$.

Fig. 15 is a typical example of low-pass filtered experimental data characteristic of high feedback gain operating conditions. Weave regeneration was found in all experiments when the weave was essentially eliminated at the control camera, i.e., when a high feedback gain was used. When the controller attenuated the weave by 25–50 times at $R2$, the amplitude of the weave at $R3$ was always greater than at $R2$, and the amplitude at $R4$ was always greater than at $R3$.

The simulations of the web modeled as a Timoshenko beam were successful at predicting the dynamic behavior of the web observed during experimentation. The boundary conditions and matching conditions derived for the Timoshenko beam model capture necessary dynamical behavior for the web to predict weave regeneration. The simulation plotted in Fig. 16 shows the characteristic behavior of the web, when a high gain is used in the feedback loop.

A comparison between the simulation plotted in Fig. 16 and the experimental data plotted in Fig. 15 shows a close match-up

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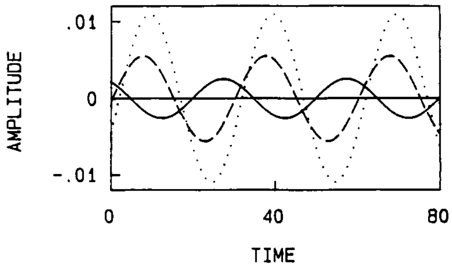


Fig. 16. Simulation of experiment plotted in Fig. 15.

between model and real system. Amplitudes of data and model are in agreement. A small phase shift is present in the outputs at $R3$ and $R4$. The phase shift could be explained by errors dependent on the accuracy of the data used to calculate the input bending slope. To calculate the input, $\theta_0$, for the simulation it was necessary to know the time delay and amplitude difference between the weave at the input sensor at $R2$. The signal-to-noise ratio at the $R2$ sensor was high so the accuracy of the data taken at $R2$ was not very good. This could cause a phase shift and amplitude shift in $\theta_0$, which would explain the phase shift present in the simulations.

Experiments were run using different operating conditions (i.e., different web velocities and input frequencies). The experimental specifications were different, but the data plots and simulation plots showed the same characteristic behavior apparent in the data plots and simulation plots of Figs. 15 and 16. Our results show that there is a very strong correlation between the proposed web model and the real dynamic system.

## VI. Unmodeled Dynamics

Although the Timoshenko beam model applied to the web–roller system is more complex than the string or simple beam model, it is still somewhat idealized. It is unknown which of dozens of unmodeled effects might be important. A list of such potentially important effects might read as follows.

1) Web wringling due to large shear deformation.

2) Web twist, and its effects on bending and shear stiffness and on rotary inertia density, $J$.

3) Lateral/torsional coupling due to deflection of the web normal to the web plane.

4) Slip at the rollers.

5) Anisotropic web elastic behavior.

Insight into which, if any, of these unmodeled effects is important can only be provided by careful experimentation.

## VII. Conclusions

The distributed parameter model of the web presented in this paper is a very general description of the web's behavior. A simplification of the model was possible since the operating conditions of interest allowed for the validity of a quasi-static model. For web processes operating in a region that invalidates the quasi-static assumption, the distributed parameter model is still valid. In future work the distributed model could prove very useful in designing a control scheme for a web conveyance system operating in a region where dynamic modes are excited.

It would be a simple matter to extend the "lumped" modeling ideas of this paper to different path configurations and different types of actuators. An example of a different path configuration might be a system where all roller axes are not parallel in the open-loop state. Boundary conditions for the web between two nonparallel rollers would need to be developed but the methodology for doing this would be similar to that contained in the paper. The dynamics of different actuators for controlling the web's placement might also be useful. Again, similar modeling ideas used to generate the dynamics of the displacement guide could be exploited to develop models for different guides.

The model proposed in this paper considerably extends past work on lateral web dynamics. One of the main contributions of this paper is the formulation of a model that predicts "weave regeneration." Previous models were too simple to capture the necessary dynamical behavior for weave regeneration to be predicted.

Since boundary conditions as well as matching conditions from one web span to the next are considered, the formulation of an $n$-roller system model was possible. This appears to be the first published work concerning a model for lateral dynamics of a complete conveyance system rather than just the dynamics of a web span between two rollers. Consequently, the $n$ roller system model is a second important contribution of the paper.

The model has been manipulated into a form suitable for modern multivariable control design. Control design and experimental implementation are the subject of current work and will be presented in a future paper [6].

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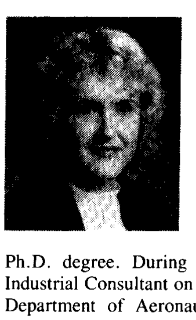


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