$$P(x_0) = 0.2 \rightarrow P(y_0) = 0.6$$
$$\searrow P(y_1) = 0.4$$

$$P(x_1) = 0.8 \longrightarrow P(y_0) = 0.4$$
$$P(y_1) = 0.6.$$

$$P(y_0) = 0.2 \times 0.6 + 0.8 \times 0.4 = 0.12 + 0.32 = 0.44$$
$$P(y_1) = 0.2 \times 0.4 + 0.8 \times 0.6 = 0.08 + 0.48 = 0.56$$

$$P(x_0 | y_0) = \frac{P(x_0 \cap y_0)}{P(y_0)} = \frac{0.12}{0.44} = 0.27.$$

$$P(x_0 | y_1) = \frac{P(x_0 \cap y_1)}{P(y_1)} = \frac{0.08}{0.56} = 0.14$$

$$P(x_1 | y_0) = \frac{P(x_1 \cap y_0)}{P(y_0)} = \frac{0.32}{0.44} = 0.73$$

$$P(x_1 | y_1) = \frac{P(x_1 \cap y_1)}{P(y_1)} = \frac{0.48}{0.56} = 0.86.$$

2 (a) P(lower face is a head)

$$= \frac{2}{5}(1) + \frac{2}{5}\left(\frac{1}{2}\right) + \frac{1}{5}(0) = 0.6$$

(b) P(upper face is head when lower face is head)

$$= \frac{\frac{2}{5}(1)}{0.6} = 0.67$$

(c) P (lower face is a head when the same coin is tose

$$= \frac{2}{4}(1) + \frac{2}{4}\left(\frac{1}{2}\right) = 0.75.$$

(d) $P(= \frac{\frac{2}{4}(1)}{0.75} = \frac{0.5}{0.75} = \frac{2}{3} = 0.67$

Probability of getting black urn in 1st draw
$= \frac{B}{R+B}$.

Now since there is no replacement, one black ball ~~has~~ ~~be~~ is missing.

$\therefore$ Probability of getting black urn in 2nd draw $= \frac{(B-1)}{R+(B-1)}$

Similarly for $(k-1)^{th}$ draw.
probability $= \frac{B-(k-2)}{R+B-(k-2)}$

$= \frac{B-k+2}{R+B-k+2}$

Probability of getting Red urn in k^{th} draw
$= \frac{R}{R+B-(k-1)}$

Total probability $= \left(\frac{B}{R+B}\right)\left(\frac{B-1}{R+B-1}\right)\left(\frac{B-2}{R+B-2}\right)\cdots\left(\frac{B-k+2}{R+B-k+2}\right)\left(\frac{R}{R+B-}\right)$

$$\boxed{\text{Ans} = \frac{B!\,(R+B-k)!\,R}{(B-k+1)!\,(R+B)!}}$$

Let $P(E) = P(\text{second ball is black})$
$P(F) = P(1^{st} \text{ ball is black})$

$\therefore \; P(E \cap F) = P(1^{st} \text{ and } 2^{nd} \text{ ball is black})$

$$= \sum_{r=1}^{\infty} \frac{(n-r)(n-r-1)}{n(n-1)(n-2)}$$

$$P(F) = \cancel{P} \sum_{r=1}^{\infty} \frac{n-r}{n(n-1)} = \frac{1}{2} \quad \left(\text{found in } Q\right).$$

$$P(E \cap F) = \sum_{r=1}^{\infty} \frac{(n-r)(n-r-1)}{n(n-1)(n-2)}$$

$$= \sum_{r=1} \frac{(n-r)^2 - (n-r)}{n(n-1)(n-2)}$$

$$= \frac{\frac{(n-1)(n)(2n-1)}{6} - \frac{(n-1)(n)}{2}}{n(n-1)(n-2)}$$

$$= \frac{\frac{n(n-1)}{6}\left[2n-1-3\right]}{n(n-1)(n-2)}$$

$$= \frac{1}{3}$$

$$\therefore \; P(E/F) = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

$\boxed{\text{Ans} = 2/3}$

~~Prob~~ Probability of choosing an urn $= \frac{1}{n}$

$P(2^{nd}$ ball black$) = P(1^{st}$ ball black and 2^{nd} ball black$)$

$+ P(1^{st}$ ball brown and 2^{nd} ball black$)$

$$= \frac{(n-r)(n-r-1)}{n(n-1)(n-2)} + \frac{(r-1)(n-r)}{n(n-1)(n-2)}$$

$$= \frac{(n-r)\cancel{(n-2)}}{n(n-1)\cancel{(n-2)}}$$

$\therefore$ Total probability $= \sum_{r=1}^{n} \frac{n-r}{n(n-1)}$

$$= \frac{\frac{\cancel{n(n-1)}}{2}}{\cancel{n(n-1)}}$$

$\boxed{\text{Ans} = \frac{1}{2}} = 0.5$

Conditional probability :-

$$P(E/F) = \frac{P(E \cap F)}{P(F)}$$

Without loss of generality, consider the surface area of sphere to be 100 units sq. and one vertex takes 1 sq. unit.

Upper bound on the probability one of vertices is white is obtained as follows:

P(Atleast one vertex is white) = P(One white 7 black)
+ P(2 white 6 black)
+ .... P(8 white 0 black)

$$P(\text{One white 7 black}) = \frac{10}{100} \cdot \left(\frac{90}{99} \cdot \frac{89}{98} \cdot \frac{88}{97} \cdots \frac{84}{93} \right)$$

$$P(\text{2 white 6 black}) = \frac{10}{100} \cdot \frac{9}{99} \left(\frac{90}{98} \cdot \frac{89}{97} \cdot \frac{88}{96} \cdots \frac{85}{93} \right)$$

⋮

Similarly,

$$P(\text{8 white 0 black}) = \frac{10}{100} \cdot \frac{9}{99} \cdot \frac{8}{98} \cdots \frac{3}{93}$$

$$\therefore P(\text{Atleast one white vertex}) = \sum_{i=1}^{8} (i \text{ white } (8-i) \text{ black})$$

$$= \frac{\begin{aligned}[t] &[10(90\cdots84) + (10\cdot9)(90\cdots85) + (10\cdot9\cdot8)(90\cdots86) + (10\cdots7)(90\cdots85) \\ &+ (10\cdots6)(90\cdots88) + (10\cdots5)(90\cdot89) \\ &+ (10\cdots4)(90) + (10\cdots3)]\end{aligned}}{(100\cdots93)}$$

$= 0.05611462\text{7}$, is the upper bound.

P(Atleast one vertex is white) upper bound is less than 1, strictly (Also non-zero)

Hence, P(Cube with all black vertices) = 1 − P(Atleast one vertex white)

is non zero.

This proves, there is atleast one Cube with all black vertices.

(a) To estimate population of fish in Powai Lake :

Let the population be n.

Catch 100 fish from 'n', mark them and let them mix well.

Again catch 100 fish.

Let E be the event that 10 out of the 100 are marked fish.

No. of ways for event E to occur $= {}^nC_{100} \cdot {}^{100}C_{10} \cdot {}^{n-100}C_{90}$

{ ${}^nC_{100}$ to catch 100 out of n first time

${}^{100}C_{10}$ to catch 10 out of marked 100

${}^{n-100}C_{90}$ to catch 90 out of unmarked $n-100$ }

Total no. of ways for the experiment to occur $= {}^nC_{100} \cdot {}^nC_{100}$

{ For catching 100 fish out of n twice i.e, before and after marking }

Thus, $$P(E) = \frac{{}^nC_{100} \cdot {}^{100}C_{10} \cdot {}^{n-100}C_{90}}{{}^nC_{100} \cdot {}^nC_{100}} = \frac{{}^{100}C_{10} \; {}^{n-100}C_{90}}{{}^nC_{100}} \rightarrow (1)$$

(b) Generalisation: Catch and mark 'm' fish. Again catch 'm' fish

Let D be the event that 'p' out of those 'm' are marked.

So, replacing 100 with m & 10 with p in (1), we get

$$P(D) = P_{m,p}(n) = \frac{{}^mC_p \; {}^{n-m}C_{m-p}}{{}^nC_m}$$

The plots for different cases of m and p are given below.

From the plots, we note down the value of n for which the maximum probability (i.e maximum value of $P_{p,m}(n)$) is attained, in each case. They are as follows:

(A) $p = 10;\ m = 100$

$\max(P_{m,p}(n)) = 0.139$ for $n = 1000$

Hence, the estimate of ~~total~~ actual value of n,

$\hat{n}_1 = 1000$

(B) $p = 20;\ m = 100$

$\max(P_{m,p}(n)) = 0.111$ for $n = 499$

Hence, the estimate of actual value of n,

$\hat{n}_2 = 499$

(C) $p = 50;\ m = 100$

$\max(P_{m,p}(n)) = 0.112$ for $n = 200$

Hence, the estimate of actual value of n,

$\hat{n}_3 = 200$

(D) $p = 75;\ m = 100$

$\max(P_{m,p}(n)) = 0.183$ for $n = 133$

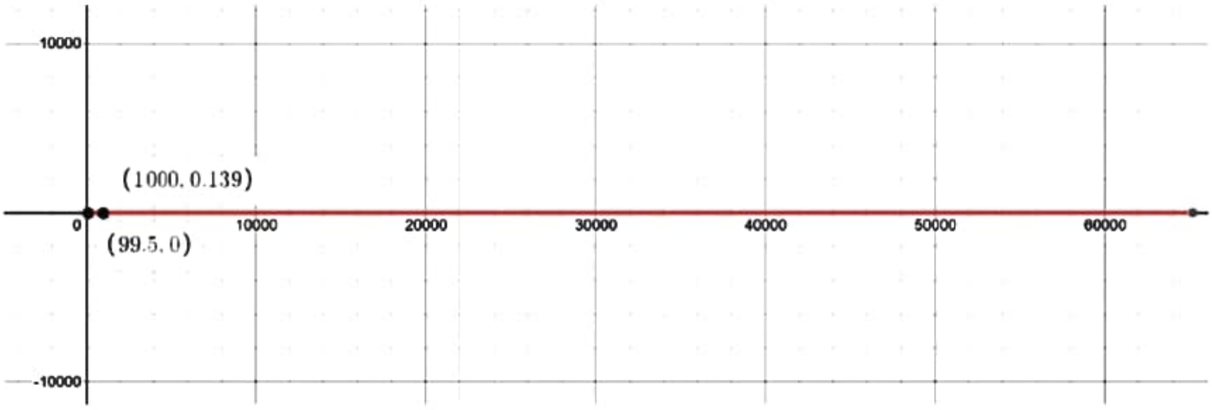
Hence, the estimate of actual value of n,

$\hat{n}_4 = 133$

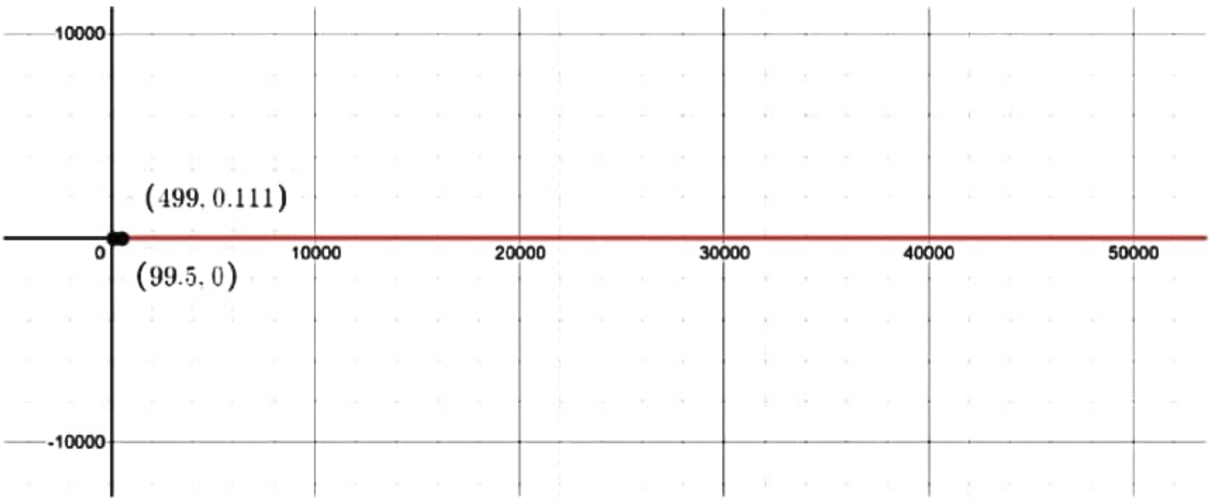
$P_{m,p}(n) = ({}^{m}C_{p})({}^{n-m}C_{m-p}) / {}^{n}C_{m}$

Plots for m=100 and,

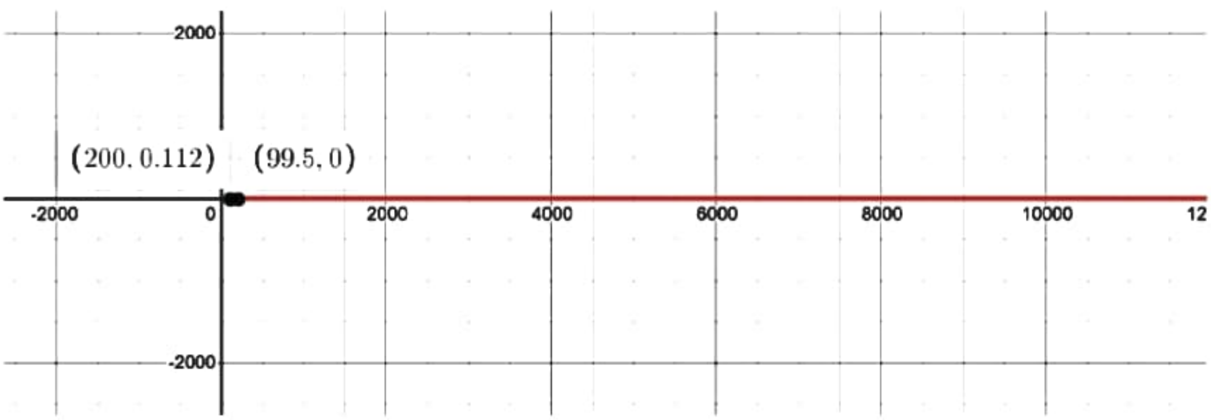
(A) p=10:



(B) p=20:

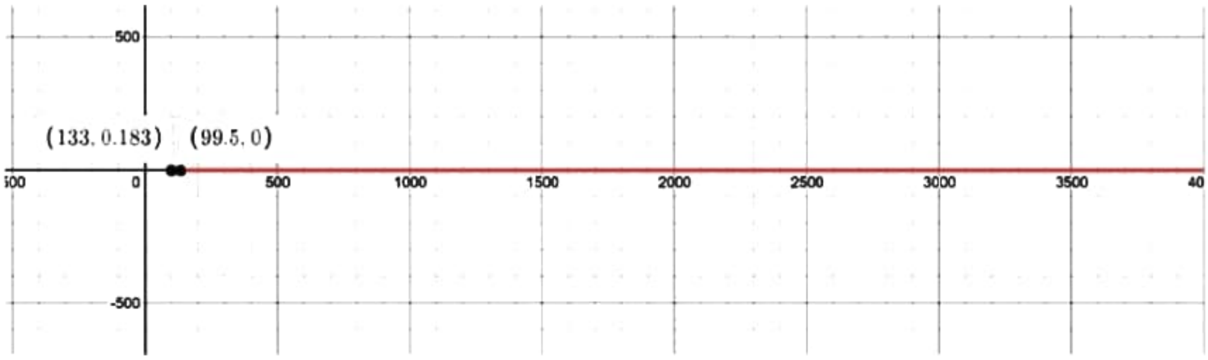


(C) p=50:



(D)

p=75:



Simulation & Execution:

(Python)

```python
import random

N=int(input("Enter population estimate n:"))

def getp(n):
    l1=[]
    l2=[]
    for i in range(100):
        t=random.randint(1,n)
        while t in l1:
            t=random.randint(1,n)
        l1.append(t)
    for i in range(100):
        t=random.randint(1,n)
        while t in l2:
            t=random.randint(1,n)
        l2.append(t)
    counter = 0
    for i in l2:
        if i in l1:
            counter+=1
    return counter

sump=0
for i in range(500):
    sump+=getp(N)
avgp=sump/500
print("for n=",N,"average p=",avgp)
```

```
Python 3.9.6 (tags/v3.9.6:db3ff76, Jun 28
D64)] on win32
Type "help", "copyright", "credits" or "l
>>>
=== RESTART: C:\Users\mahtw\AppData\Local
Enter population estimate n:1000
for n= 1000 average p= 10.008
>>>
=== RESTART: C:\Users\mahtw\AppData\Local
Enter population estimate n:499
for n= 499 average p= 20.212
>>>
=== RESTART: C:\Users\mahtw\AppData\Local
Enter population estimate n:200
for n= 200 average p= 50.342
>>>
=== RESTART: C:\Users\mahtw\AppData\Local
Enter population estimate n:133
for n= 133 average p= 75.144
>>>
```

Upon simulation and sample average calculation, we get

for $\hat{n} = 1000$, $\hat{p}_1 = 10.008$ where $p_1 = 10$

for $\hat{n}_2 = 499$, $\hat{p}_2 = 20.212$ where $p_2 = 20$

for $\hat{n}_3 = 200$, $\hat{p}_3 = 50.342$ where $p_3 = 50$

for $\hat{n}_4 = 133$, $\hat{p}_4 = 75.144$ where $p_4 = 75$

On comparing $\hat{p}_i$ and p_i, it is observed that they are very close, and for practical purposes $\boxed{\hat{p}_i \approx p_i}$. This is because, we have chosen those values as $\hat{n}_i$ which gave max. value for $P_{m,p}(n)$ in their corresponding plots. Therefore, for the chosen value of estimate $\hat{n}_i$, the probability of choosing p_i marked fishes out of m fish is the maximum, compared to all other values for $\hat{n}_i$