Lecture Notes on Measure Theory

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Preface

This book originates from the Measure Theory course for graduate students at Hunan University.

Introduction

Hearistics form Linear Algebra

system of linear equations

$$Ax = b \tag{1}$$

Def. 0.0.1. *A set V along with a number field $\mathbb{F}$ is called a linear space (or vector space) if it is closed under additions and scalar multiplications, i.e.,*

$$\alpha u + \beta v \in V \ , \ \forall u, v \in V \ , \ \alpha, \beta \in \mathbb{F} \tag{2}$$

Theorem 0.0.1. *If V and W are finite dimensional linear spaces, and T is a linear transformation from $V \to W$,*

$$\dim V = \dim \mathit{val}\, T + \dim \ker T$$

Def. 0.0.2. *Compact operator K: bounded set $\to$ compact set.*

Theorem 0.0.2. *Fredholm Alternative*

Assume that V is a normed linear space and $K : V \to V$ is a compact operator. Then:

- *either, $\exists$ nontrivial solution $r \in V$, s.t. $(K - I)r = 0$;*
- *or, $\forall v \in V, \exists! u \in V$, s.t. $(K - I)u = v$.*

Remark 0.0.1. *If K is compact, $\mathit{rank}(\ker(K - I)) < \infty$.*

Proof. If rank $\ker(K - I) = \infty$, we have orthonormal (We consider Hilbert space) $u_1, u_2, \ldots, u_n, \ldots$ s.t. $(K - I)u_n = 0 \implies \{u_1, \ldots, u_n, \ldots\}\exists$ convergent subsequence on a compact set. But $u_1, u_2, \ldots, u_n, \ldots$ orthogonal $\implies$ contradiction. $\square$

Theorem 0.0.3. *If V is Hilbert space, then*

$$\dim(\ker(K - I)) = \dim(\mathit{val}(K - I)^{\perp}) \tag{3}$$

Cardinality

Def. 0.0.3. *For any nonempty set X, power set 2^X is the set family of all X.*

Def. 0.0.4. *For a fintie set A. Card(A) = number of the elements of the A. Two infinite set A , b , we say that Card(A) = Card(B) if $\exists$ bijection $f : A \to B$. And say that*

1. *Card$(A) \leq$ Card(B) if $\exists$ injection $g_1 : A \to B$.*
2. *Card$(A) \geq$ Card(B) if $\exists$ surjection $g_2 : A \to B$.*

Theorem 0.0.4. *Bernstrein theorem*
If Card$(A) \leq$ Card(B) and Card$(A) \geq$ Card(B), then Card(A) = Card(B).

EX. 0.0.1. *Countable set*

1. *Card$(\mathbb{N}) = \aleph_0$.*
2. *Card$(\mathbb{Z}) = \aleph_0$.*
3. *Card$(\mathbb{Q}) = \aleph_0$.*

Def. 0.0.5. *E is a set.*

1. *Interior of E : $E^\circ :=$ the largest open set contained in E.*
2. *Closed of E : $\bar{E} :=$ the smallest closed set containing E.*
3. *Dense in A : $\bar{E} \supset A$.*
4. *Nowhere dense : $\bar{E}^\circ = \phi$.*

Def. 0.0.6.
1^{st} Category : A Countable union of nowhere dense set.
2^{nd} Category : is not of 1^{st} Category.

Def. 0.0.7.

$$C[0,1] := \{\textit{continuous functions on}[0,1]\}. \tag{4}$$

Def. 0.0.8.

$$D[0,1] := \{u \in C[0,1] : \textit{uis diffierentiable at some poinx} \in (0,1)\}. \tag{5}$$

Remark 0.0.2. *$D[0,1]$ is 1^{st} Category.*

Theorem 0.0.5. *Cantor*

For any nonempty set X, $Card(2^X) > Card(X)$.

Proof.

On the one hand, $\forall x \in X$, $\mathrm{Id}(x) = \{x\}$, is obviously an injection $\Longrightarrow$ $\mathrm{Card}(X) \leq \mathrm{Card}(2^X)$.

On the other hand, suppose that $\exists$ a bijection $g : X \to 2^X$. Let $Y := \{x \in X : x \notin g(x)\} \subset X \in 2^X$, then $\exists x_0 \in X$, s.t. $g(x_0) = Y$. Now any attempt to answer the question "Is $x_0 \in Y$?" quickly leads to an absurded uncountable. □

Cor. 0.0.1. *Cardinality continuum*

$$\aleph_0 = Card(\mathbb{N}) < Card(2^{\mathbb{N}}) = Card((0,1)) = Card(\mathbb{R}) = \aleph_1 = \mathfrak{c} \tag{6}$$

Def. 0.0.9. *Denote:* $\mathbb{P}_k = \left\{\sum_{j=0}^{k} a_j x^j : a_j \in \mathbb{Z}\right\}$, *the set of all polynomials with integer coefficients of k-order.*

$\mathbb{R}$*eal number:*

1. *Algebraic number: a root of* $P \in \bigcup_{k=1}^{\infty} \mathbb{P}_k$.
2. *Transcendental number: which is not algebraic number.*

Continuum hypothesis : There are no other cardinalities between $\aleph_0$ and $\aleph_1$.

Review integral

Starting from classical mathematical analysis theory, we discuss some of its limits and orderings, thereby introducing modern analysis theory.

Newton-Leibniz formula : $\int_a^b f(x)\, dx = F(x)|_a^b = F(b) - F(a)$.

Classical theory: Cauchy, Lagranges, Weienstranss.

Riemann integral : partition of domain.

limitations of Riemann integral

· Integrality.

$\exists$ non-integral function, such as Dirichlet function $D(x) = \chi_{\mathbb{Q}}(x)$.

Def. 0.0.10. *Characteristic function (Indicator function in probability theory):*

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}. \tag{7}$$

· Differentiability.

Bounded variation $\implies$ diffierentiable a.e..

Absolute continuity $\implies$ fundamental theory.

· Interchange of limit and integration.

If a sequence of functions $\{u_n\}$ is uniformly convergenl, i.e., $u_n \rightrightarrows u$, as $n \to \infty$ on closed interval $[a, b]$, then

$$\int_a^b u\ dx = \lim_{n\to\infty} u_n\ dx. \tag{8}$$

The condition be too restrictive and not necessary. For example, if $u_n = x^n$ on $[0, 1]$, then

$$\lim_{n\to\infty} u_n = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}.$$

u_n is not uniformly but $\int_0^1 \lim_{n\to\infty} u_n\ dx = 0 = \lim_{n\to\infty} \int_0^1 u_n\ dx = \lim_{n\to\infty} \frac{1}{n+1}$.

Modern theory: Peano, Jordan, Borel.

Lebesgue integral: partition of the range.

Perfect measure

Q: what's the length of an arbitrary set ?

Find a function μ , $\mu : 2^{\mathbb{R}} \to \bar{\mathbb{R}}_+$ [1], satisfying

1. Countably additive. If $\{E_i\}_1^\infty$ is sequence of disjoint sets, then

$$\mu(\bigcup_1^\infty E_i) = \sum_1^\infty \mu(E_i). \tag{9}$$

2. Transformation invariance. If E is congruent to F, i.e., E can be transformed in to F by translations, rotations and reflections, then $\mu(E) = \mu(F)$.

3. If $\mu(Q) = 1$, where Q is the unit cube.

Theorem 0.0.6. *Vital*

Don't exists "perfect measure" in $\mathbb{R}^d$.

[1] extended real line , including $+\infty$

Proof.

Define an equivalence relation by declaring $x \sim y$ iff $x - y \in \mathbb{Q}$(denote $[0,1)/\sim = [x]$).

$$\mu([0,1)) = 1 \tag{10}$$

Let N be a subset of $[0,1]$, containing precisely one member of each equivalence class.

Axiom of choice: If $\{A_\alpha\}$ is a nonempty collection of nonempty sets, then $\prod_\alpha A_\alpha$ is nonempty. ($\prod_\alpha A_\alpha = (a_1, a_2, \ldots)$).

Denote: $R := Q \cap [0,1], \forall r \in R$, define

$$N_r := \{x + r : x \in N \cap [0, 1-r)\} \cup \{x + r - 1 : x \in N \cap [1-r, 1)\}. \tag{11}$$

Claim: $N_r \cap N_s = \phi$, $\forall r.s \in R$, $r \neq s$.

Proof Claim: Assume that $\exists x_0 \in N_r \cap N_s, \forall r, s \in R$.

$$\implies {}^2\begin{cases} y_1 = x_0 - r \in N \\ y_2 = x_0 - s \in N \end{cases} \tag{12}$$

If $y_1 = y_2$, $\implies$ $s = r$.

If $y_1 \neq y_1$, $\implies$ $y_1 + r = x_0 = y_2 + s$ $\implies$ $y_1 - y_2 = s - r \in R \subset \mathbb{Q}$, $\implies$ $y_1, y_2 \in [x]$, contradiction [3].

The claim is proven.

Claim: $\bigcup_{r \in R} N_r = [0,1)$.

Proof Claim: $\forall x \in [0,1), \exists y \in N$ s.t. $x \in [y] \implies x - y \in \mathbb{Q}$.

If $0 < r = x - y \in \mathbb{Q} \implies x = y + r$.

otherwise, $0 < \hat{r} = x - y + 1 \in \mathbb{Q} \implies x \in \mathbb{N}_{\hat{r}}$.

Let $R = \{r_n\}$, $[0,1) = \bigcup_1^\infty N_{r_n}$, R is countable set.

The claim is proven.

$$1 = \mu([0,1)) = \mu\left(\bigcup_1^\infty N_{r_n}\right) = \sum_{n=1}^\infty \mu(N_{r_n}) = \sum_{n=1}^\infty \mu(N) = \begin{cases} 0 & \mu(\mathbb{N}) = 0 \\ \infty & \mu(\mathbb{N}) \neq 0 \end{cases}. \tag{13}$$

The result can be easily extended to higher dimensions by considering $\mathbb{N} \times (0,1)$. $\square$

Weakening conditions.

1' Finitely additive:

[2] or $x_0 - r \in N \& x_0 - s + 1 \in N$, etc.

[3] $y_1, y_2 \in$ same equivalence class but every two are different.

If $E_1, E_2, \ldots, E_n$ is a finite sequence of disjoint sets, then $\mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(E_i)$.

Banach-Tarski paradox:

Let U, V be arbitrary bounded open sets, then $\exists k \in \mathbb{N}_+$, $E_1, E_2, \ldots, E_k \subset U$ and $F_1, F_2, \ldots, F_k \subset V$, s.t.

(a) $E_1, \ldots, E_k$ are mutually disjoint and $\bigcup_{i=1}^{k} E_i = U$

(b) $F_1, \ldots, F_k$ are mutually disjoint and $\bigcup_{i=1}^{k} F_i = V$

(c) E_i is congruent to F_i, $\forall i = 1, 2, \ldots, k$

1" Countable subadditivity:

If $E_1, E_2, \ldots, E_n, \ldots$ are countable sequence, $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$.

Contents

Chapter 1

General Measure Theorem

1.1 Set

Def. 1.1.1. *If E, F are two sets, we define their difference by $E \backslash F := \{x \in E : x \notin F\}$, and their symmetric difference by $E \Delta F := (E \backslash F) \cup (F \backslash E)$.*

When the unverisal set X is clearly understood, then define the complement of E by $E^c := X \backslash E$.

Theorem 1.1.1. *De Morgan's Law*

$$(\bigcup_{\alpha \in \Lambda} A_\alpha)^c = \bigcap_{\alpha \in \Lambda} A_\alpha^c. \tag{1.1}$$

Def. 1.1.2. $\{E_n\}_1^\infty$

limit supprior :

$$\limsup E_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n = \{x : x \in E_n, for\ infinitely\ many\ n\}. \tag{1.2}$$

limit inferior :

$$\liminf E_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} E_n = \{x : x \in E_n, for\ all\ but\ finitely\ many\ n\}. \tag{1.3}$$

Remark 1.1.1.

$$\bigcap_{n=k}^{\infty} E_n \subset \liminf E_n \subset \limsup E_n \subset \bigcup_{n=k}^{\infty} E_n, \forall n. \tag{1.4}$$

Remark 1.1.2.

$$(\limsup E_n)^c = \liminf E_n^c. \tag{1.5}$$

$$(\liminf E_n)^c = \limsup E_n^c. \tag{1.6}$$

Remark 1.1.3.

$$\chi_{\limsup E_n}(x) = \limsup_{n\to\infty} \chi_{E_n}(x). \tag{1.7}$$

$$\chi_{\liminf E_n}(x) = \liminf_{n\to\infty} \chi_{E_n}(x). \tag{1.8}$$

Def. 1.1.3. $\lim E_n$ *exists if* $\limsup E_n = \liminf E_n$.

EX. 1.1.1. *If A,B are some sets.*

$$E_n = \begin{cases} A & n \text{ is odd} \\ B & n \text{ is even} \end{cases}. \tag{1.9}$$

$\limsup E_n = A \cup B$,
$\liminf E_n = A \cap B$.

Remark 1.1.4. *Increasing :* $E_n \subset E_{n+1}, \forall n \in \mathbb{N}_+, \Rightarrow \lim E_n = \bigcup_1^\infty E_n$.

Remark 1.1.5. *Decreasing :* $E_n \supset E_{n+1}, \forall n \in \mathbb{N}_+, \Rightarrow \lim E_n = \bigcap_1^\infty E_n$.

Remark 1.1.6. *Cauchy:* $\{a_n\} \subset \mathbb{R}$ *monotone increasing & bounded,* $\Rightarrow$ $\lim a_n$ *exists and is equal to* $\sup a_n$

Def. 1.1.4. *Let X be a nonempty set. A ring of set on X is nonempty collection $\mathscr{R}$ of subsets of X that is closed under finite union and diffences, i.e.,*

1. *If* $E_1, E_2, ..., E_n \in \mathscr{R}$*, then* $\bigcup_1^n E_i \in \mathscr{R}$*;*
2. *If* $E, F \in \mathscr{R}$*, then* $E \backslash F \in \mathscr{R}$*.*

Def. 1.1.5. *Let X be a nonempty set. An algebra of set on X is nonempty collection $\mathscr{A}$ of subsets of X that is closed under finite union and complements, i.e.,*

1. *If* $E_1, E_2, ..., E_n \in \mathscr{A}$*, then* $\bigcup_1^n E_i \in \mathscr{A}$*;*
2. *If* $E \in \mathscr{R}$*, then* $E^c \in \mathscr{R}$*.*

Def. 1.1.6. *A σ-ring is a ring that is closed under countable union, i.e., if* $E_1, E_2, ... \in \mathscr{R}$*, then* $\bigcup_1^\infty E_i \in \mathscr{R}$*.*

Def. 1.1.7. *A* *σ-algebra* *is an algebra that is closed under countable union, i.e., if* $E_1, E_2, ... \in \mathscr{A}$*, then* $\bigcup_1^\infty E_i \in \mathscr{A}$.

Remark 1.1.7.

1. *A ring is also closed under finite intersection.*
   If $E_1, E_2, ..., E_n \in \mathscr{R}$*, then* $\bigcap_1^n E_i = E\backslash(\bigcup_1^n (E\backslash E_i)) \in \mathscr{R}$*.(Set* $E := \bigcup_1^n E_i \in \mathscr{R}$*.)*

2. *An algebra is also closed under finite intersection.*
   If $E_1, E_2, ..., E_n \in \mathscr{A}$*, then* $\bigcap_1^n E_i = (\bigcup_1^n E_i^c)^c$.

3. *A ring* $\mathscr{R}$ *is a* σ*-ring proivided that it is closed under countable disjiont unions.*

   Proof. "$\Rightarrow$" trivial.
   "$\Leftarrow$" For any sequence $\{E_n\}_1^\infty \subset \mathscr{R}$. $\mathcal{L}$et $F_k = E_k\backslash(\bigcup_1^{k-1} E_i)$, then $F_k \in \mathscr{R}$ and muatually disjiont. We have $\bigcup_1^\infty F_k = \bigcup_1^\infty E_k \in \mathscr{R}$. □

4. *An algebra* $\mathscr{A}$ *is a* σ*-algebra proivided that it is closed under countable disjiont unions.*

5. *Every ring contains the empty set.* $\phi = E\backslash E$.

6. *Every algebra contains the unverisal set.* $X = E \cup E^c$.

7. *Every algebra is ring.*

8. *Every* σ*-algebra is* σ*-ring.*

9. *An algebra can be characterized as ring containing* X.

10. *A* σ*-algebra can be characterized as* σ*-ring containing* X.

Remark 1.1.8. *Ring of sets are really rings.*

EX. 1.1.2. $E \bigoplus F := E\Delta F$ *and* $E \bigodot F := E \bigcap F$ *closed iff* $E \bigcup F$ *and* $E\backslash F$ *closed, where* $0 = \phi$ *and* $1 = X$.

EX. 1.1.3. *For any empty set* X*,* 2^X *and* $\{\phi, X\}$ *are* σ*-algebra.(trivial)*

EX. 1.1.4. *If* X *is uncountable, then*

$$\mathscr{A} := \{E \subset X : E \text{ is countable or } E^c \text{ is countable}\} \tag{1.10}$$

is a σ*-algebra called the* σ*-algebra of countable or co-countable sets.*

Remark 1.1.9.

1. *The intersection of a family of σ-ring is again a σ-ring.*

2. *The intersection of a family of σ-algebra is again a σ-algebra.*

3. *It follows that $\exists!$ smallest σ-ring containing any collection $\mathscr{E}$ of subsets of X.*

4. *It follows that $\exists!$ smallest σ-algebra containing any collection $\mathscr{E}$ of subsets of X.*

Def. 1.1.8. *The smallest (unique) σ-ring containing a give $\mathscr{E} \subset 2^X$ is called the σ-ring generated by $\mathscr{E}$, denote $\mathscr{R}_\sigma(\mathscr{E})$.*

Def. 1.1.9. *The smallest (unique) σ-algebra containing a give $\mathscr{E} \subset 2^X$ is called the σ-algebra generated by $\mathscr{E}$, denote $\mathscr{A}_\sigma(\mathscr{E})$.*

EX. 1.1.5. *$\mathcal{L}$et $X = \mathbb{R}^1$, and $\mathscr{E} := \{[1,3],[2,4]\}$.*

$$\mathscr{R}_\sigma(\mathscr{E}) = \mathscr{E} \cup \{[2,3],[1,4],(3,4],[1,2),[1,2)\cup(3,4],\phi\}.$$

Lemma 1.1.1. *If a collection $\mathscr{E}$ is subset of a σ-ring $\mathscr{R}$, then $\mathscr{R}_\sigma(\mathscr{E}) \subset \mathscr{R}$.*

Proof. Note that $\mathscr{R}$ is σ-ring containing $\mathscr{E}$, then it contains $\mathscr{R}(\mathscr{E})$. □

Lemma 1.1.2. *If a collection $\mathscr{E}$ is subset of a σ-algebra $\mathscr{A}$, then $\mathscr{A}_\sigma(\mathscr{E}) \subset \mathscr{A}$.*

Lemma 1.1.3. *$\mathscr{E} \subset \mathscr{F} \Rightarrow \mathscr{R}_\sigma(\mathscr{E}) \subset \mathscr{R}_\sigma(\mathscr{F})$.*

Proof. trivial. □

Lemma 1.1.4. *$\mathscr{E} \subset \mathscr{F} \Rightarrow \mathscr{A}_\sigma(\mathscr{E}) \subset \mathscr{A}_\sigma(\mathscr{F})$.*

Prop. 1.1.1. *If $\mathscr{E}$ is any class of sets, and $R \in \mathscr{R}_\sigma(\mathscr{E})$, then $\exists$ subcollection $\mathscr{F} \subset \mathscr{E}$, s.t. $R \in \mathscr{R}_\sigma(\mathscr{F})$.*

Proof.
Set
$$\mathscr{G} := \bigcup_{\mathscr{F}\subset\mathscr{E},\mathscr{F} \text{ is countable}} \mathscr{R}_\sigma(\mathscr{F}),$$
we proof that $\mathscr{G} = \mathscr{R}_\sigma(\mathscr{E})$.
"$\subset$" $\forall$ countable $\mathscr{F} \subset \mathscr{E}$, $\Rightarrow \mathscr{R}_\sigma(\mathscr{E}) \subset \mathscr{R}_\sigma(\mathscr{F})$.
"$\supset$"
Claim:

1. $\mathscr{F} \subset \mathscr{G}$;

2. $\mathscr{G}$ is σ-ring.

Proof Claim: 1. $\forall E \in \mathscr{E}$, $\{E\}$ countable, $\{E\} \subset \mathscr{G} \Rightarrow \mathscr{E} \subset \mathscr{G}$.
2. $\forall \{F_n\}_1^\infty \subset \mathscr{G}$, $\exists$ countable $\mathscr{F}_n \subset \mathscr{E}$, s.t. $F_n \in \mathscr{R}_\sigma(\mathscr{F}_n), \forall n \in \mathbb{N}_+$.

$$\bigcup_1^\infty F_n \in \bigcup_1^\infty \mathscr{R}_\sigma(\mathscr{F}_n) \subset \mathscr{R}_\sigma(\bigcup_1^\infty \mathscr{F}_n) \subset \mathscr{G}. \tag{1.11}$$

The claim is proven.
Note that $F_1, F_2 \in \mathscr{R}_\sigma(\mathscr{F}_1 \cup \mathscr{F}_2)$, it follow that $F_1 \backslash F_2 \in \mathscr{R}_\sigma(\mathscr{F}_1 \cup \mathscr{F}_2) \subset \mathscr{G}$, which verified of $\mathscr{G}$ is a σ-ring. □

Def. 1.1.10. *$\forall$ class $\mathscr{E} \subset 2^X$ and a subset $F \subset X$, denote by*

$$\mathscr{E} \cap F := \{E \cap F : E \in \mathscr{E}\}. \tag{1.12}$$

Lemma 1.1.5. *If a σ-ring $\mathscr{R} \subset 2^X$, and $F \subset X$, then $\mathscr{R} \cap F$ is also a σ-ring.*

Proof.
$\forall E_n \cap F \in \mathscr{R} \cap F, n \in \mathbb{N}_+$.
$\cup(E_n \cap F) = (\cup E_n) \cap F \in \mathscr{R} \cap F$.
$(E_1 \cap F) \backslash (E_2 \cap F) = (E_1 \backslash E_2) \cap F \in \mathscr{R} \cap F$. □

Prop. 1.1.2. *If $\mathscr{E}$ is any class of sets on X and $F \subset X$, then $\mathscr{R}_\sigma(\mathscr{E}) \cap F = \mathscr{R}_\sigma(\mathscr{E} \cap F)$.*

Proof.
"$\supset$" Note that $\mathscr{E} \subset \mathscr{R}_\sigma(\mathscr{E}) \Rightarrow \mathscr{E} \cap F \subset \mathscr{R}_\sigma(\mathscr{E}) \cap F$.
From Lemma 1.1.5, we know that is σ-ring,

$$\overset{Lemma 1.1.3}{\Rightarrow} \mathscr{R}_\sigma(\mathscr{E}) \cap F \subset \mathscr{R}_\sigma(\mathscr{E} \cap F). \tag{1.13}$$

"$\subset$" Denote by

$$\mathscr{F} := \{B_n \cup (A_n \backslash F) : B_n \in \mathscr{R}_\sigma(\mathscr{E} \cap F), A_n \in \mathscr{R}_\sigma(\mathscr{E})\}. \tag{1.14}$$

$\forall E \in \mathscr{R}_\sigma(\mathscr{E})$, $E = (E \cap F) \cup (E \backslash F)$ and $E \cap F \subset \mathscr{R}_\sigma(\mathscr{E} \cap F)$,

$$\Rightarrow E \in \mathscr{F}, \tag{1.15}$$

$$\Rightarrow \mathscr{E} \subset \mathscr{F}. \tag{1.16}$$

Now, if 1. $\mathscr{F}$ is σ-ring and 2. $\mathscr{F} \cap F = \mathscr{R}_\sigma(\mathscr{E} \cap F)$, we have $\mathscr{R}_\sigma(\mathscr{E}) \subset \mathscr{F}$, $\mathscr{R}_\sigma(\mathscr{E}) \cap F \subset \mathscr{F} \cap F = \mathscr{R}_\sigma(\mathscr{E} \cap F)$.

2. $\mathscr{F} \cap F = \mathscr{R}_\sigma(\mathscr{E} \cap F)$.

$\forall E \in \mathscr{F} \cap F$, $\forall B_n \in \mathscr{R}_\sigma(\mathscr{E} \cap F)$, $E = (B_n \cap (A_n \backslash F)) \cap F = B_n \cap F \in \mathscr{R}_\sigma(\mathscr{E} \cap F) \cap F$.

$\forall E \in \mathscr{R}_\sigma(\mathscr{E} \cap F) \cap F$, $\exists B_n = E, A_n = \phi$, s.t. $E \in \mathscr{F} \cap F$.

$\Rightarrow \mathscr{F} \cap F = \mathscr{R}_\sigma(\mathscr{E} \cap F) \cap F$.

$$\mathscr{R}_\sigma(\mathscr{E} \cap F) = \mathscr{R}_\sigma(\mathscr{E} \cap F \cap F) \subset \mathscr{R}_\sigma(\mathscr{E} \cap F) \cap F = \mathscr{F} \cap F. \tag{1.17}$$

$\mathscr{R}_\sigma(\mathscr{E} \cap F) \supset \mathscr{R}_\sigma(\mathscr{E} \cap F) \cap F$, $\Rightarrow \mathscr{F} \cap F = \mathscr{R}_\sigma(\mathscr{E} \cap F)$.

1. $\mathscr{F}$ is σ-ring

countable union

$\forall \{F_n\}_1^\infty \subset \mathscr{F}$, $\exists B_n \in \mathscr{R}_\sigma(\mathscr{E} \cap F)$, $A_n \in \mathscr{R}_\sigma(\mathscr{E})$, $F_n = B_n \cup (A_n \backslash F)$.

$$\begin{aligned} \bigcup_1^\infty F_n &= \bigcup_1^\infty B_n \cup (A_n \backslash F) = (\bigcup_1^\infty B_n) \bigcup (\bigcup_1^\infty (A_n \backslash F)) \\ &= (\bigcup_1^\infty B_n) \bigcup ((\bigcup_1^\infty A_n) \backslash F) \in \mathscr{F}. \end{aligned} \tag{1.18}$$

difference

$F_1 \backslash F_2 = (B_1 \cup (A_1 \backslash F)) \backslash (B_2 \cup (A_2 \backslash F)) = (B1 \backslash B_2) \cup ((A_1 \backslash F) \backslash (A_2 \backslash F))$.

$B_1 \backslash B_2 \in \mathscr{R}_\sigma(\mathscr{E} \cap F)$. $(A_1 \backslash F) \cup (A_2 \backslash F) = (A_1 \cap F^c) \cap (A_2 \cap F^c)^c = (A_1 \cap F^c) \cap (A_2^c \cup F) = (A_1 \cap F^c \cap A_2^c) \cup (A_1 \cap F^c \cap F) = A_1 \cap A_2^c \cap F^c = (A_1 \backslash A_2) \backslash F$.

By $A_1 \backslash A_2 \in \mathscr{R}_\sigma(\mathscr{E})$, the difference is closed.

□

Cor. 1.1.1. *If $\mathscr{E}$ is any class of sets on X and $F \subset X$, then $\mathscr{A}_\sigma(\mathscr{E}) \cap F = \mathscr{A}_\sigma(\mathscr{E} \cap F)$.*

Proof. By proposition 1.1.2, trivial. □

Def. 1.1.11. *If X is any metric space, or more generaly any topological space, the σ-algebra generate by the family od all oepn (or closed) sets in X, is called the Borel σ-algebra on X, denote by $\mathscr{B}^X$. It is member are called Borel sets.*

Remark 1.1.10. *$\mathscr{B}^X$ includes all open sets, closed sets, countable unions or intersections of open and closed sets, but is generally not all of 2^X.*

Prop. 1.1.3. *$\mathscr{B}^{\mathbb{R}}$ is generate by each the following:*

1. *the open intervals, $\mathscr{E}_1 := \{(a, b) : a < b\}$;*

2. *the closed intervals,* $\mathscr{E}_2 := \{[a,b] : a < b\}$*;*

3. *the half-open intervals,* $\mathscr{E}_3 := \{(a,b] : a < b\}$*,* $\mathscr{E}_4 := \{[a,b) : a < b\}$*;*

4. *the open rays,* $\mathscr{E}_5 := \{(a,+\infty] : a \in \mathbb{R}\}$*,* $\mathscr{E}_6 := \{(-\infty,a) : a \in \mathbb{R}\}$*;*

5. *the closed rays,* $\mathscr{E}_7 := \{[a,+\infty] : a \in \mathbb{R}\}$*,* $\mathscr{E}_8 := \{(-\infty,a] : a \in \mathbb{R}\}$*.*

Def. 1.1.12. *A subset* $\mathscr{M} \subset 2^X$ *is monotone class if it is closed under countable increasing unions and countable decreasing intersection, i.e.,*

1. *if* $E_n \in \mathscr{M}$ *and* $E_n \subset E_{n+1}$*,* $\forall n \in \mathbb{N}_+$*, then* $\bigcup_1^\infty E_n \in \mathscr{M}$*;*

2. *if* $E_n \in \mathscr{M}$ *and* $E_n \supset E_{n+1}$*,* $\forall n \in \mathbb{N}_+$*, then* $\bigcap_1^\infty E_n \in \mathscr{M}$*;*

Lemma 1.1.6. *σ-ring is a monotone class.*

Proof. trivial. □

Lemma 1.1.7. *A monotone ring is σ-ring.*

Proof. $\forall \{E_n\}_1^\infty \subset \mathscr{R}$, set $F_n = \bigcup_1^n E_j \in \mathscr{R}$. From the definition of monotone class, we have $\bigcup_1^\infty F_n = \bigcup_1^\infty E_n \in \mathscr{R}$. □

Lemma 1.1.8. *A monotone algebra is σ-algebra.*

Proof. By Lemma 1.1.7, trivial. □

Remark 1.1.11. *The intersection of any family of monotone class is also monotone class.*

Remark 1.1.12. $\forall \mathscr{E} \subset 2^X$*,* $\exists!$ *smallest monotone calss containing* $\mathscr{E}$*.*

Def. 1.1.13. *The smallest (unique) monotone class containing any give* $\mathscr{E} \subset 2^X$*, is called the monotone class generated by* $\mathscr{E}$*, denote by* $\mathscr{M}_\sigma(\mathscr{E})$*.*

Theorem 1.1.2. *The monotone class theorem*

1. *If* $\mathscr{E}$ *is a ring of subsets of* X*, then* $\mathscr{M}_\sigma(\mathscr{E}) = \mathscr{R}_\sigma(\mathscr{E})$*.*

2. *If* $\mathscr{E}$ *is an algebra of subsets of* X*, then* $\mathscr{M}_\sigma(\mathscr{E}) = \mathscr{A}_\sigma(\mathscr{E})$*.*

Proof.
"$\subset$" Since $\mathscr{R}_\sigma(\mathscr{E})$ is monotone class, and $\mathscr{E} \subset \mathscr{R}_\sigma(\mathscr{E})$, we have $\mathscr{M}_\sigma(\mathscr{E}) \subset \mathscr{R}_\sigma(\mathscr{E})$.

"$\supset$" It suffice show that $\mathscr{M}_\sigma(\mathscr{E})$ is σ-ring (or σ-algebra).

$\forall E \in \mathscr{M}_\sigma(\mathscr{E})$, define $\mathscr{M}(E) := \{F \in \mathscr{M}_\sigma(\mathscr{E}) : E \backslash F, F \backslash E, E \cup F \in \mathscr{M}_\sigma(\mathscr{E})\}$.

Clearly, $\phi, E \in \mathscr{M}(E)$.
Claim : It is easy to checks that monotone class.
Proof Claim : $\forall E_n \in \mathscr{M}(E), \forall n \in \mathbb{N}+, E_n \subset E_{n+1}$,
1. $(\bigcup_1^\infty E_n) \cup E = \bigcup_1^\infty (E \cup E_n) \in \mathscr{M}_\sigma(\mathscr{E})$;
2. $(\bigcup_1^\infty E_n) \backslash E = (\bigcup_1^\infty E_n \backslash E) \in \mathscr{M}_\sigma(\mathscr{E})$;
3. $E \backslash (\bigcup_1^\infty E_n) = \bigcap_1^\infty (E \backslash E_n) \in \mathscr{M}_\sigma(\mathscr{E})$.
The claim is proven.
Moreover, we can find a symmetric property: $F \in \mathscr{M}(E)$ iff $E \in \mathscr{M}(F)$.
$\forall E \in \mathscr{E}, \Rightarrow F \in \mathscr{M}(E), \forall F \mathscr{E}, \Rightarrow \mathscr{E} \subset \mathscr{M}(E), \Rightarrow \mathscr{M}_\sigma(\mathscr{E}) \subset \mathscr{M}(\mathscr{E})$.
$F \in \mathscr{M}_\sigma(\mathscr{E}) \subset \mathscr{M}(E)$, iff $E \in \mathscr{M}(F)$, $\Rightarrow \mathscr{E} \subset \mathscr{M}(F)$, $\Rightarrow \mathscr{M}_\sigma(\mathscr{E}) \subset \mathscr{M}(\mathscr{F})$.
$\forall E, F \in \mathscr{M}_\sigma(\mathscr{E})$, $\mathscr{M}_\sigma(\mathscr{E}) \subset \mathscr{M}(E) \& \mathscr{M}_\sigma(\mathscr{E}) \subset \mathscr{M}(F)$, $\Rightarrow E \backslash F, F \backslash E, E \cup F \in \mathscr{M}_\sigma(\mathscr{E})$.

□

1.2 Measure and their properties

Def. 1.2.1. *If X is a set equipped with a σ-algebra $\mathscr{A} \subset 2^X$, then $(X, \mathscr{A})$ is called a measurable space .*

Def. 1.2.2. *A positive measure on this space (C.F. Def. 1.2.1) is a set function $\mu : \mathscr{A} \to [0, +\infty]$, satisfying*

1. *$\mu(\phi) = 0$;*
2. *countable additivity, i.e., if $\{E_n\}_1^\infty$ is a sequence of muatually disjiont sets, then $\mu(\bigcup_1^\infty E_n) = \bigcup_1^\infty \mu(E_n)$.*

Def. 1.2.3. *The triplet $(X, \mathscr{A}, \mu)$ is called a measure space .*

EX. 1.2.1. *$\mathcal{L}$et X is an arbitrary set, $\mathscr{A} \subset 2^X$, define $\# : 2^X \to [0, +\infty]$ as*

$$\#(E) = \begin{cases} Card(E) & if\ E\ is\ finite \\ +\infty & other \end{cases}. \tag{1.19}$$

This is called the counting measure.

EX. 1.2.2. *$\mathcal{L}$et X be a nonempty set. Fix $x_0\ inX$, define $\delta_{x_0} : 2^X \to [0, +\infty]$ as*

$$\delta_{x_0}(E) = \begin{cases} 1 & x_0 \in E \\ 0 & x_0 \notin E \end{cases}. \tag{1.20}$$

This is called the point mass or Dirac measure at x_0.

EX. 1.2.3. *Let $\Omega = \{w_i\}_1^\infty$ is a countable set, $\mathscr{A} = 2^\Omega$, and $\{p_i\}_1^\infty \subset [0,1]$ be a sequence of real numbers, s.t. $\sum_1^\infty p_i = 1$, define $P : 2^\Omega \to [0,1]$ as*

$$P(E) = \sum_1^\infty p_j \delta_{w_j}(E). \tag{1.21}$$

This is called discrete probability space $(\Omega, 2^\Omega, P)$.

Def. 1.2.4. *Size of measurs.*

1. *If $\mu(X) < +\infty (\Rightarrow \mu(E) < +\infty, \forall E \in \mathscr{A})$, then μ is said to be finite. In particular, if $\mu(X) = 1$, then $(X, \mathscr{A}, \mu)$ is called a probability measure.*

2. *If $X = \bigcup_1^\infty E_n$, where $E_n \in \mathscr{A}$, and $\mu(E_n) < +\infty, \forall n \in \mathbb{N}_+$, then μ is said to be σ-finite. More generally, E is said to be μ-finite for μ, if $E = \bigcup_1^\infty E_n$, where $E_n \in \mathscr{A}$, and $\mu(E_n) < +\infty$.*

3. *If for each $E \in \mathscr{A}$ with $\mu(E) = +\infty$, $\exists F \in mathscr A$, satisfying $\phi \neq F \subset F$ and $\mu(F) < +\infty$, then μ is said to be semifinite.*

EX. 1.2.4. *Let $(X, \mathscr{A}, \mu)$ be a σ-finite measure space. We have $X = \bigcup_1^\infty E_n$ with $E_n \in \mathscr{A}$,$\mu(E_n) < +\infty$. Define*

$$\nu(E) := \sum_{j=1}^\infty \frac{\mu(E \cap E_j)}{2^j \mu(E_j)}, \tag{1.22}$$

then $(X, \mathscr{A}, \nu)$ is a probability measure space and ν is conditional probability $P(E|E_j)$.

EX. 1.2.5. *Let $f : X \to [0, +\infty]$ be any function . The f determines a measure on 2^X by $\mu_f(E) = \sum_{x \in E} f(x)$.*

1. *μ_f if semifinite iff $f(x) < +\infty$, $\forall x \in X$.*

   Proof.
   "$\Rightarrow$" If not, $\exists x_0$ s.t. $f(x_0) = +\infty \Rightarrow \mu_f(\{x_0\}) = +\infty$.
   "$\Leftarrow$" $\forall E \in 2^X$ with $\mu(E) = +\infty$, $\exists \{x_0\} \subset E$, s.t. $\mu_f(\{x_0\}) = f(x_0) < +\infty$. $\square$

2. *μ_f is σ-finite iff $f(x) < +\infty$,$\forall x \in X \& E = \{x : f(x) > 0\}$ is countable.*

Proof.

"$\Rightarrow$" $\exists \{E_n\}_1^\infty \subset \mathscr{A}$, s.t. $X = \bigcup_1^\infty E_n$, $\mu_f(E_n) < +\infty, \forall n \in \mathbb{N}_+$. $\forall x \in X$, $\exists E_x \in \{E_n\}_1^\infty$, s.t. $x \in E_x$. $\Rightarrow f(x) = \mu_f(\{x\}) \leq \mu_f(E_x) < +\infty$. Denote : $X_n := \{x : f(x) > \frac{1}{n}\}$.

$$E = \bigcup_1^\infty X_n. \tag{1.23}$$

Assume that E is uncountable. $\Rightarrow \exists X_{n_0} \in \{X_n\}_1^\infty$ is uncountable. $X_{n_0} = \bigcup_1^\infty (X_{n_0} \cap E_n) \Rightarrow \exists X_{n_0} \cap E_{n_1}$ is uncountable.

$$\mu_f(X_{n_0} \cap E_{n_1}) \geq \sum_{x \in X_{n_0} \cap E_{n_1}} \frac{1}{n_0} = +\infty. \tag{1.24}$$

Contradictory.

"$\Leftarrow$" Let $E = \{x_n\}_1^\infty$ (E is countable). $X = \bigcup_1^\infty \{x_i\} \cup (X \backslash E)$. $\mu_f(\{x_i\}) = f(x_i) < +\infty, \mu_f(X \backslash E) = 0$.

$\square$

EX. 1.2.6. *Two special space.*

1. *Taking $f \equiv 1$, μ_f is the counting measure.*

2. *Taking $f(x) = \begin{cases} 1 & x = x_0 \\ 0 & x \neq x_0 \end{cases}$, μ_f is the Dirac measure at x_0.*

EX. 1.2.7. *Let X be an infinite set, and $\mathscr{A} = 2^X$. Define*

$$\mu(E) = \begin{cases} 0 & if\ E\ is\ finite \\ \infty & if\ E\ is\ infinite \end{cases}. \tag{1.25}$$

Then μ is finite additive but not a measure.

EX. 1.2.8. *Let X be uncountable and*

$$\mathscr{A} := \{E \subset X : E \text{ is countable or } E^c \text{ is countable}\}. \tag{1.26}$$

Define

$$\mu(E) = \begin{cases} 0 & if\ E\ is\ countable \\ 1 & if\ E^c\ is\ countable \end{cases}. \tag{1.27}$$

Then μ is a measure.

Prop. 1.2.1. *Let $(X, \mathscr{A}, \mu)$ be a measure space.*

1. *Monotone: if $E, F \in \mathscr{A}$ with $E \subset F$, then $\mu(E) \leq \mu(F)$.*

2. *Subtractive: if $E, F \in \mathscr{A}$ with $E \subset F$, then $\mu(F \backslash E) = \mu(F) - \mu(E)$ where $\mu(E) < +\infty$.*

3. *Countable subadditive: if $\{E_n\}_1^\infty \subset \mathscr{A}$ is a sequence of sets, then $\mu(\bigcup_1^\infty E_n) \leq \sum_1^\infty \mu(E_n)$.*

Proof.
$\forall E, F \in \mathscr{A}$ with $E \subset F$, $F = (F \backslash E) \cup E$ and $(F \backslash E) \cap E = \phi$.
$\mu(F) = \mu(F \backslash E) + \mu(E) \Rightarrow \mu(F) \geq \mu(E)$ (μ is positive measure).
$\mu(F) = \mu(F \backslash E) + \mu(E) \Rightarrow \mu(F \backslash E) = \mu(F) - \mu(E)$ if $\mu(E) < \infty$.
Set $F_n = E_n \backslash (\bigcup_1^{n-1} E_j) \subset E_n$, are muatually disjiont $\Rightarrow \mu(F_n) < \mu(E_n)$.

$$\mu(\bigcup_1^\infty E_n) = \mu(\bigcup_1^\infty F_n) = \sum_1^\infty \mu(F_n) \leq \sum_1^\infty \mu(E_n). \tag{1.28}$$

□

Prop. 1.2.2. *Let μ be a nonnegative and additive set function, satisfying $\mu(\phi) = 0$ on a σ-algebra $\mathscr{A}$, consider the following statements, we have 1 $\Leftrightarrow$ 2 $\Rightarrow$ 3 $\Leftrightarrow$ 4.*

1. *Countable additive.*

2. *Continuous frow below. If $\{E_n\}_1^\infty \subset \mathscr{A}$ with $E_n \subset E_{n+1}, \forall n \in \mathbb{N}_+$, then $\mu(\bigcup_1^\infty E_n) = \lim_{n\to\infty} \mu(E_n)$.*

3. *Continuous from above. If $\{E_n\}_1^\infty \subset \mathscr{A}$ with $E_n \supset E_{n+1}, \forall n \in \mathbb{N}_+$, and $\mu(E_1) < \infty$ (or $\exists k, \mu(E_k) < \infty$), then $\mu(\bigcap_1^\infty E_n) = \lim_{n\to\infty} \mu(E_n)$.*

4. *Continuous from above at ϕ. If $\{E_n\}_1^\infty \subset \mathscr{A}$ with $E_n \downarrow \phi$ [1] and $\mu(E_1) < \infty$ (or $\exists k, \mu(E_k) < \infty$), then $\lim_{n\to\infty} \mu(E_n) = 0$.*

In addition, if μ is finite, then all four statements are equivalent.

Note: additive $E \cap F = \phi$, $\mu(E \cup F) = \mu(E) + \mu(F)$. $\Rightarrow$ $E \supset F$, $\mu(E \backslash F \cup F) = \mu(E \backslash F) + \mu(F)$. If $\mu(F) < \infty$, $\mu(E) - \mu(F) = \mu(E \backslash F)$.

[1] $E_n \supset E_{n+1}, \forall n \in \mathbb{N}_+$ and $\lim_{n\to\infty} E_n = \phi$.

Proof.

1 $\Rightarrow$ *2* $\forall \{E_n\}_1^\infty \subset \mathscr{A}$ and $E_n \uparrow$ [2]. Set $E_0 = \phi$.

$\cdot$ If $\mu(E_k) = +\infty$ for some $k \in \mathbb{N}_+$, then the result follows from the monotonously.

$\cdot$ If $\mu(E_k) < +\infty, \forall k \in \mathbb{N}_+$. We set $F_n := E_n \backslash E_{n-1}$ are muatually disjiont.

$$\begin{aligned} \mu(\bigcup_1^\infty E_n) = & \mu(\bigcup_1^\infty F_n) = \sum_1^\infty \mu(F_n) = \sum_1^\infty (\mu(E_n) - \mu(E_{n-1})) \\ = & \lim_{N\to\infty} \sum_1^N (\mu(E_n) - \mu(E_{n-1})) = \lim_{N\to\infty} \mu(E_N) - 0. \end{aligned} \tag{1.29}$$

2 $\Rightarrow$ *1* $\forall \{E_n\}_1^\infty \subset \mathscr{A}$ of muatually disjiont sets. Set $F_n = \bigcup_1^n E_j$ and $E = \bigcup_1^\infty E_n$, then $F_n \uparrow E$.

$$\mu(E) = \lim_{n\to\infty} \mu(F_n) = \lim_{n\to\infty} \sum_1^n \mu(E_j) = \sum_1^\infty \mu(E_n). \tag{1.30}$$

3 $\Rightarrow$ *4* trivial.

4 $\Rightarrow$ *3* $\forall \{E_n\}_1^\infty \subset \mathscr{A}$ with $E_n \downarrow$ and $\mu(E_1) < \infty$. Denote $E := \bigcap_1^\infty E_n$, $\mu(E) < \infty$. Set $F_n = E_n \backslash E$, then $F_n \downarrow \phi$.

$$0 = \lim_{n\to\infty} \mu(F_n) = \lim_{n\to\infty} (\mu(E_n) - \mu(E)) \tag{1.31}$$

2 $\Rightarrow$ *3* $\forall \{E_n\}_1^\infty \subset \mathscr{A}$ with $E_n \downarrow$ and $\mu(E_1) < \infty$. Set $F_n = E_1 \backslash E_n, \forall n \in \mathbb{N}_+$, then $F_n \uparrow$.

$$\bigcup_1^\infty F_n = \bigcup_1^\infty (E_1 \backslash E_n) = E_1 \backslash (\bigcup_1^\infty E_n). \tag{1.32}$$

$$\mu(E_1) - \mu(\bigcap_1^\infty E_n) = \mu(\bigcup_1^\infty F_n) = \lim_{n\to\infty} \mu(F_n) = \lim_{n\to\infty} (\mu(E_1) - \mu(E_n)). \tag{1.33}$$

$\Rightarrow \mu(\bigcap_1^\infty E_n) = \lim_{n\to\infty} \mu(E_n)$.

In addition, we assume that μ is finite.

4 $\Rightarrow$ *1* $\forall$ a sequence $\{E_n\}_1^\infty \subset \mathscr{A}$ of muatually disjoint sets. Set $F_n := \bigcup_1^n E_j$, $E := \bigcup_1^\infty F_n \in \mathscr{A}$, $F_n \uparrow E$, $G_n = E \backslash F_n$, $G_n \downarrow \phi$.

$$\begin{aligned} 0 = & \lim_{n\to\infty} \mu(G_n) = \lim_{n\to\infty} \mu(E \backslash F_n) = \lim_{n\to\infty} (\mu(E) - \mu(F_n)) \\ & = \mu(E) - \lim_{n\to\infty} \sum_1^n \mu(E_j). \end{aligned} \tag{1.34}$$

[2] $E_n \subset E_{n+1}, \forall n \in \mathbb{N}_+$.

$\Rightarrow \mu(E) = \sum_1^\infty \mu(E_j)$. $\square$

Remark 1.2.1. *Some finiteness assumption is necessary as it can happen that* $\mu(E_n) = +\infty, \forall n \in \mathbb{N}_+$, *but* $\mu(\cap E_n) < \infty$. *For example,* $(\mathbb{N}_+, 2^{\mathbb{N}_+}, \#)$, *define* $E_n := \{i : i \geq n\}$. $\#(E_n) = +\infty$, $\cap E_n = \phi$, $\Rightarrow \#(\cap E_n) = 0$.

1.3 Null sets and completion

Def. 1.3.1. *Let* $(X, \mathscr{A}, \mu)$ *be a measure space. A set* $E \in \mathscr{A}$ *is said to be a null set if* $\mu(E) = 0$.

If a statement about points $x \in X$ is true except for some points in a null set. We say it is true almost everywhere (a.e., or almost surely, a.s.). More precisely, we shall speak of a μ-a.e..

Remark 1.3.1. *Any countable union of null set is a null set.*

From monotonicity, if $E \subset F \in \mathscr{A}$ and $\mu(F) = 0$, then $\mu(E) = 0$ proivided that $E \in \mathscr{A}$. But it need not be true generally.

Def. 1.3.2. *A measure whose domain includes all subsets of null set is called complete.*

Theorem 1.3.1. *Suppose* $(X, \mathscr{A}, \mu)$ *is measure space. Let* $\mathcal{N} := \{N \in \mathscr{A} : \mu(N) = 0\}$ *and* $\overline{\mathscr{A}} := \{E \cup F : E \in \mathscr{A}, F \subset N \text{ for some } N \in \mathcal{N}\}$. *The* $\overline{\mathscr{A}}$ *is a* σ*-algebra and* $\exists!$ *extension* $\overline{\mu}$ *of* μ *to a complete measure on* $\mathscr{A}$.

Proof. First, we note $\mathscr{A}$ and $\mathcal{N}$ are closed under countable unions, so is $\mathscr{A}$.

$\forall E \cup F \in \overline{\mathscr{A}}$, where $E \in \mathscr{A}$, $F \subset N \in \mathcal{N}$.

Without loss of generally (WLOG), we assume that $E \cap N = \phi$. (Otherwise, replace F and N by $F \backslash E$ and $N \backslash E$. $F \backslash E \subset N \backslash E$ and $\mu(N \backslash E) = 0$, $E \cup (F \backslash E) = E \cup F$.)

Then, $E \cup F = (E \cup N) \cap (N^c \cup F)$ [3].

$$\Rightarrow (E \cup F)^c = \overbrace{(E \cup N)^c}^{\in \mathscr{A}} \cup \overbrace{(N^c \cup F)}^{N \cap F^c = N \backslash F \subset N \in \mathcal{N}} \tag{1.35}$$

$\overline{\mathscr{A}}$ is also closed under complements. $\Rightarrow \overline{\mathscr{A}}$ is a σ-algebra.

Define

$$\overline{\mu}(E \cup F) = \mu(E). \tag{1.36}$$

[3] $((E \cup N) \cap N^c) \cup ((E \cup N) \cap F) = (E \backslash F) \cup ((E \cup F) \cap F) = E \cup F$

If $E_1 \cup F_1 = E_2 \cup F_2$, where $E_1, E_2 \in \mathscr{A}$, $F_1 \subset N_1 \in \mathcal{N}$, $F_2 \subset N_2 \in \mathcal{N}$, let us consider the following equation and determine whether the equality at the question mark holds.

$$\mu(E_1) = \overline{\mu}(E_1 \cup F_1) \stackrel{?}{=} \overline{\mu}(E_2 \cup F_2) = \mu(E_2). \tag{1.37}$$

$E_1 \subset E_2 \cup F_2 \subset E_2 \cup N_2 \Rightarrow \mu(E_1) \leq \mu(E_2 \cup N_2) = \mu(E_2)$. Likewise, $\mu(E_2) \leq \mu(E_1)$. So that definition of $\overline{\mu}$ make sense.

Claim : $\forall G \subset E \cup F \in \overline{\mathscr{A}}$ with $\overline{\mu}(E \cup F) = 0$ $(\mu(E) = 0, F \subset N \in \mathcal{N})$, $\Rightarrow G \in \overline{\mathscr{A}}$.

Proof Claim: Note that $\mu(E \cup N) = 0$, $G = \phi \cup G, \phi \in \mathscr{A}$ and $G \subset E \cup N \in \mathcal{N}$, $\Rightarrow G \in \overline{\mathscr{A}}$.

The claim is proven.

$\Rightarrow \overline{\mu}$ is complete on $\overline{\mathscr{A}}$.

It is easy to verify that the extension $\overline{\mu}$ is unique. □

Def. 1.3.3. *The measure $\overline{\mu}$ define as above is called the completion of μ and $\overline{\mathscr{A}}$ is the completion of $\mathscr{A}$ with respect to μ.*

EX. 1.3.1. *trivial σ-algebra $\mathscr{A} = \{\phi, X\}$, $\mu \equiv 0$.*
$\overline{\mathscr{A}} = 2^X$, $\overline{\mu} = 0$.

1.4 Outer meausures and extension

Def. 1.4.1. *A nonempty class $\mathscr{E}$ of set is said to be hereditary if whenever $F \subset E$ and $E \in \mathscr{E} \Rightarrow F \in \mathscr{E}$.*

If $\mathscr{E}$ is any class of sets, we denote $\mathscr{H}_\sigma(\mathscr{E})$ the smallest (unique) hereditary σ-ring containing $\mathscr{E}$.

Remark 1.4.1. *If $\mathscr{A}$ is an algebra, then $\mathscr{H}_\sigma(\mathscr{A}) = 2^X$.*

Def. 1.4.2. *An outer measure on a nonempty set X is a set function $\mu^* : 2^X \to [0, +\infty]$ satisfying*

1. *$\mu^*(\phi) = 0$.*
2. *Countable subadditivity. $\forall$ a sequence $\{E_n\}_1^\infty \subset 2^X$, $\mu^*(\bigcup_1^\infty E_n) \leq \sum_1^\infty \mu^*(E_n)$.*
3. *Monotonicity. $\mu^*(E) \leq \mu^*(F)$ if $E \subset F$.*

Def. 1.4.3. *If X be nonempty set. A class $\mathscr{E}$ is said to be a sequential covering class if*

1. *$\phi \in \mathscr{E}$.*
2. *$\forall E \in \mathscr{E}$, $\exists$ countable sequence $\{E_n\}_1^\infty \subset \mathscr{E}$, s.t. $\mathscr{E} \subset \bigcup_1^\infty E_n$.*

EX. 1.4.1. *For $X = \mathbb{R}$, the class $\mathscr{E} := \{(a,b) : a < b\} \cup \phi$ is a sequential covering class.*

Prop. 1.4.1. *Let $\mathscr{E}$ be sequential covering class, a set function $\mu : \mathscr{E} \to [0, +\infty]$ satisfying $\mu(\phi) = 0$. $\forall E \subset X$, define*

$$\mu^*(E) = \inf\{\sum_1^\infty \mu(E_n) : E_n \in \mathscr{E}, E \subset \bigcup_1^\infty E_n\}. \tag{1.38}$$

Then μ^* is outer measure.

Proof.
First, $\forall E \subset X$, $\exists \{E_n\}_1^\infty \subset \mathscr{E}$, since $\mathscr{E}$ is a sequential covering class, $\Rightarrow$ the definition of μ^* make sense.

By take $E_n = \phi$, we find that $0 \le \mu^*(\phi) \le \sum_1^\infty \mu(E_n) = 0$.

$\Rightarrow \mu^*(\phi) = 0$.

$\forall E, F \subset X$ with $E \subset F$, $\forall \{F_n\}_1^\infty \subset \mathscr{E}$ satisfying $E \subset F \subset \bigcup_1^\infty F_n$, $\Rightarrow \{F_n\}_1^\infty$ is also a sequential covering for E.

$\Rightarrow \mu^*(E) \le \mu^*(F)$.

$\forall \{E_n\}_1^\infty \subset 2^X$, $\forall n \in \mathbb{N}_+$, $\varepsilon > 0$, $\exists \{E_n^j\}_{j=1}^\infty$, s.t. $E_n \subset \bigcup_{j=1}^\infty E_n^j$ and $\sum_{j=1}^\infty \mu(E_n^j) \le \mu^*(E_n) + \frac{\varepsilon}{2^n}$.

Note that $\bigcup_{n=1}^\infty E_n \subset \bigcup_{n,j=1}^\infty E_n^j$,

$$\mu^*(\bigcup_{n=1}^\infty E_n) \le \sum_{n,j=1}^\infty \mu(E_n^j) \le \sum_{n=1}^\infty \mu^*(E_n) + \varepsilon. \tag{1.39}$$

The finishes the proof by letting $\varepsilon \to 0$. □

Def. 1.4.4. *Let μ^* be an outer measure. A set $E \subset X$ is called μ^*-measurable if $\forall F \subset X$, $\mu^*(F) = \mu^*(F \cap E) + \mu^*(F \cap E^c)$.*

Def. 1.4.5. *Denote $\mathscr{M}_{\mu^*}^X$ (briefly $\mathscr{M}$) the collection of all μ^*-measurable set on X.*

Remark 1.4.2. *It is sufficient to verify $\mu^*(F) \le \mu^*(F \cap E) + \mu^*(F \cap E^c)$ for $F \subset X$ with $\mu^*(F) < \infty$.*

Theorem 1.4.1. *Caratheodsry's exten theorem*

If $\mu^*$is an outer measure on X, then $\mathscr{M}_{\mu^*}^X$ is a σ-algebra and the restriction of μ^* to μ, denote by $\mu^*|_{\mathscr{M}}$ is a complete measure.

Proof.

Clearly, ϕ,$X \in \mathscr{M}$.

It is easy to check that $\mathscr{M}$ is closed under complements. Since the definition of μ^*-measurability of E is symmetric in E and E^c.

Claim : $\mathscr{M}$ is closed under finite unions.

Proof Claim: $\forall E_1, E_2 \in \mathscr{M}, \forall F \subset X$, note that $E_1 \cup E_2 = (E_1 \cap E_2) \cup (E_1 \cap E_2^c) \cup (E_1^c \cap E_2)$.

$$\begin{aligned} \mu^*(F) = \; & \mu^*(F \cap E_1) + \mu^*(F \cap E_1^c) \\ = \; & \mu^*(F \cap E_1 \cap E_2) + \mu^*(F \cap E_1 \cap E_2^c) \\ & + \mu^*(F \cap E_1^c \cap E_2) + \mu^*(F \cap E_1^c \cap E_2^c) \\ \geq \; & \mu^*(F \cap (E_1 \cap E_2)) + \mu^*(F \cap (E_1 \cap E_2)^c). \end{aligned} \tag{1.40}$$

$\Rightarrow E_1 \cup E_2 \in \mathscr{M}$.

The claim is proven.

Moreover, taking $F = E_1 \cup E_2$, we have $\mu^*(E_1 \cup E_2) = \mu^*(E_1) + \mu^*(E_2)$ proivided $E_1 \cap E_2 = \phi \Rightarrow \mu^*$ is additive.

By induction, we deduce that μ^* is finitely additive & $\mathscr{M}$ is closed under finite unions.

$\forall \{M\}_1^\infty$, WLOG, assume that M_n's are muatually disjiont. (If not, we have $\forall \{F_n\}_1^\infty \subset \mathscr{M}$,$M_1 = F_1$,$M_2 = F_2 \backslash F_1$,. . ..)

Set $E_n := \bigcup_1^n M_j$, $E = \bigcup_1^\infty E_j$. $\forall F \subset X$,

$$\begin{aligned} \mu^*(F) &= \mu^*(F \cap E_n) + \mu^*(F \cap E_n^c) \\ &= \mu^*(F \cap E_n \cap M_n) + \mu^*(F \cap E_n \cap M_n^c) + \mu^*(F \cap E_n^c) \\ &= \mu^*(F \cap M_n) + \mu^*(F \cap E_{n-1}) + \mu^*(F \cap E_n^c) \\ &= \dots \\ &= \sum_{j=1}^n \mu^*(F \cap M_j) + \mu^*(F \cap E_n^c) \\ &\geq \sum_{j=1}^n \mu^*(F \cap M_j) + \mu^*(F \cap E^c). \end{aligned} \tag{1.41}$$

Letting $n \to \infty$, we have $\mu^*(F) \geq \sum_{j=1}^\infty \mu^*(F \cap M_j) + \mu^*(F \cap E^c) \geq \mu^*(\bigcup_{j=1}^\infty (F \cap M_j)) + \mu^*(F \cap E^c) = \mu^*(F \cap E) + \mu^*(F \cap E^c)$.

$\Rightarrow E \in \mathscr{M}$, $\mathscr{M}$ is a σ-algebra.

Set $F = E$. $\Rightarrow \mu^*(E) = \sum_{j=1}^\infty \mu^*(M_j)$, $\Rightarrow \mu^*$ is countable additive on $\mathscr{M}$.

$\forall N \subset X$ with $\mu^*(N) = 0$, $\forall F \subset X$, since $\mu^*(F \cap N) = 0$, $\mu^*(F) \geq \mu^*(F \cap N) + \mu^*(F \cap N^c)$. $\Rightarrow N \in \mathscr{M}$.

$\Rightarrow \mu|_{\mathscr{M}}$ is complete measure. $\square$

Def. 1.4.6. *Let $\mathscr{A} \subset 2^X$ be an algebra. A set function $\mu_0 : \mathscr{A} \to [0, +\infty]$ is called a premeasure if*

1. $\mu_0(\phi) = 0$.
2. *Countably additive.*

Def. 1.4.7. *If μ_0 is a premeasure on algebra $\mathscr{A} \subset 2^X$, then $\forall E \subset X$, $\mu^*(E) = \inf\{\sum_1^\infty \mu_0(E_n) : E_n \in \mathscr{E}, E \subset \bigcup_1^\infty E_n\}$.*

Prop. 1.4.2. *If μ_0 is a premeasure on algebra $\mathscr{A} \subset 2^X$ and the outer measure μ^* is constructed as above, then*

1. $\mu^*|_{\mathscr{A}} = \mu_0$.
2. $\mathscr{A}_\sigma(\mathscr{A}) \subset \mathscr{M}^X_{\mu^*}$.

Proof.

1. $\forall A \in\subset \mathscr{A}$, obviously $\mu^*(A) \leq \mu_0(A)$.

$\forall \{A_n\}_1^\infty \subset \mathscr{A}$ and $A \subset \bigcup_1^\infty A_n$. Set $F_n := A_n \bigcup_{j=1}^{n-1} A_j \in \mathscr{A}$, then $A \subset \bigcup_1^\infty F_n$.

$\sum_1^\infty \mu_0(F_n) = \mu_0(\bigcup_1^\infty F_n) \geq \mu_0(A)$ [4]

$\Rightarrow \mu^*(A) \geq \mu_0(A)$.

2. $\forall A \in \mathscr{A}$, $F \subset X$, $\varepsilon > 0$, $\exists$ a sequence $\{F_n\}_1^\infty \subset \mathscr{A}$ with $F \subset \bigcup_1^\infty F_n$ and $\sum_1^\infty \mu_0(F_n) < \mu^*(F) + \varepsilon$.

$$\mu_0(F_n) \geq \mu_0(F_n \cap A) + \mu_0(F_n \cap A^c). \tag{1.42}$$

$$\sum_1^\infty \mu_0(F_n \cap A) \geq \mu_0(\bigcup_1^\infty F_n \cap A) \geq \mu^*(F \cap A). \tag{1.43}$$

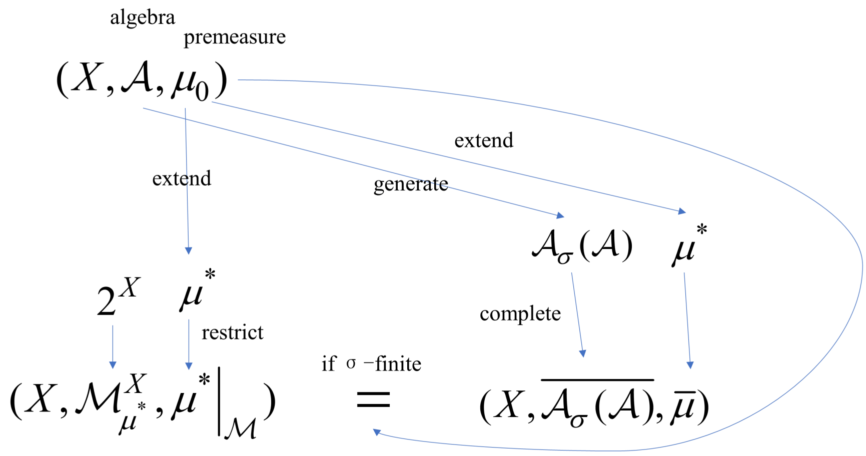
$$\Rightarrow \sum_1^\infty (\mu_0(F_n \cap A) + \mu_0(F_n \cap A^c)) \geq \mu^*(F_n \cap A) + \mu^*(F_n \cap A^c). \tag{1.44}$$

$$\begin{aligned} \mu^*(F) + \varepsilon > & \sum_1^\infty \mu_0(F_n) \geq \sum_1^\infty (\mu_0(F_n \cap A) + \mu_0(F_n \cap A^c)) \\ \geq & \ \mu^*(F_n \cap A) + \mu^*(F_n \cap A^c). \end{aligned} \tag{1.45}$$

Letting $\varepsilon \to 0$, we conclude that $\mu^*(F) \geq \mu^*(F_n \cap A) + \mu^*(F_n \cap A^c)$.

$\Rightarrow A \in mathscrM \Rightarrow \mathscr{A} \subset \mathscr{M} \Rightarrow \mathscr{A}_\sigma(\mathscr{A}) \subset \mathscr{M}$. □

[4] $\mu_0(\bigcup_1^\infty F_n) = \mu_0(\bigcup_1^\infty A_n)$



1.5 Induced measures and approximation

Lemma 1.5.1. *If μ_0 is σ-finite, then so are μ^* on 2^X and $\mu^*|_{\mathscr{M}}$ on $\mathscr{M}$ or $\mathscr{A}_\sigma\mathscr{A}$.*

Proof. $X = \bigcup_1^\infty X_n$ with $\mu_0(X_n) < +\infty$, $X_n \in \mathscr{A}$. Easy for algebra. $\square$

Theorem 1.5.1. *Uniqueness extension theorem*

If μ_0 is σ-finite premeasure on an algebra $\mathscr{A}$, then $\exists!$ measure μ on $\mathscr{A}_\sigma(\mathscr{A})$, s.t. $\mu(E) = \mu_0(E)$, $\forall E \in \mathscr{A}$.

Proof.

The existence is given by Definition 1.4.7 and Proposition 1.4.2. The following proves the uniqueness.

Suppose that μ_1 and μ_2 are both extension of μ_0 on $\mathscr{A}_\sigma(\mathscr{A})$, i.e., $\mu_1(E) = \mu_2(E)$.

Let $\mathscr{E} := \{E \in \mathscr{A}_\sigma(\mathscr{A}) : \mu_1(E) = \mu_2(E)\} \subset \mathscr{A}_\sigma(\mathscr{A})$. [5]

Case I. At least one of μ_1,μ_2 is finite. If a sequence $\{E_n\}_1^\infty \subset \mathscr{E}$ is monotoneous $\Rightarrow \lim_{n\to\infty} E_n$ exists, then from the continuity of μ_1, μ_2 (c.f. Proposition 1.2.2), $\lim_{n\to\infty} \mu_1(E_n) = \lim_{n\to\infty} \mu_2(E_n)$.

$\Rightarrow \mu_1(\lim_{n\to\infty} E_n) = \mu_2(\lim_{n\to\infty} E_n) \Rightarrow \lim_{n\to\infty} E_n \in \mathscr{E} \Rightarrow \mathscr{E}$ is monotone class.

From monotone class theorem 1.1.2, $\mathscr{A}_\sigma(\mathscr{A}) = \mathscr{M}_\sigma(\mathscr{A}) \subset \mathscr{E} \Rightarrow \mathscr{E} = \mathscr{A}_\sigma(\mathscr{A})$.

Case II. In general both μ_1 and μ_2 are not necessarily finite. Since μ_0 is σ-finite, we can find $\{X_n\}_1^\infty \subset \mathscr{A}$, s.t. $\mu_0(X_n) < \infty$, $=\forall n \in \mathbb{N}_+$,$X = \bigcup_1^\infty X_n$.

$\forall X_n \in \{X_n\}_1^\infty \subset \mathscr{A}$, $X_n \cap \mathscr{A}$ is algebra.

[5] $\mathscr{A} \subset \mathscr{E}$

From the pervious proposition (c.f. proposition 1.1.1), $\mathscr{A}_\sigma(X_n \cap \mathscr{A}) = X_n \cap \mathscr{A}_\sigma(\mathscr{A})$.

It follow from the reasoning in Case I, that $\mu_1 \equiv \mu_2$ on $X_n \cap \mathscr{A}_\sigma(\mathscr{A})$.

$\forall A \in \mathscr{A}_\sigma(\mathscr{A})$, $\mu_1(A) = \sum_1^\infty \mu_1(A \cap X_n) = \sum_1^\infty \mu_2(A \cap X_n) = \mu_2(A)$.

□

Lemma 1.5.2. *If $E \subset X$, then*

$$\begin{aligned} \mu^*(E) = &\quad \inf\{\mu^*|_{\mathscr{M}}(F) : E \subset F \in \mathscr{M}\} \\ = &\quad \inf\{\mu^*|_{\mathscr{M}}(F) : E \subset F \in \mathscr{A}_\sigma(\mathscr{A})\}. \end{aligned} \tag{1.46}$$

Proof.

$\forall E \subset X$, recall that

$$\mu^*(E) := \inf\{\sum_1^\infty \mu_0(A_n) : A_n \in \mathscr{A} \& E \subset \bigcup_1^\infty A_n\} \tag{1.47}$$

Without increasing $\sum_1^\infty \mu_0(A_n)$, we can assume that A_n's are muatually disjiont. Note that $\mathscr{A}_\sigma(\mathscr{A}) \subset \mathscr{M}$ and $\mu^*(E) \leq \mu^*(F), E \subset F \in \mathscr{M}$, then

$$\begin{aligned} \mu^*(E) \geq &\quad \inf\{\sum_1^\infty \mu^*|_{\mathscr{M}}(A_n) : A_n \in \mathscr{A}_\sigma(\mathscr{A}), \\ &\qquad\qquad \textit{muatually disjiont } \& E \subset \bigcup_1^\infty A_n = F\} \\ = &\quad \inf\{\mu^*|_{\mathscr{M}}(F) : E \subset F \in \mathscr{A}_\sigma(\mathscr{A})\} \\ \geq &\quad \inf\{\mu^*|_{\mathscr{M}}(F) : E \subset F \in \mathscr{M}\} \\ \geq &\quad \mu^*(E). \end{aligned} \tag{1.48}$$

□

Def. 1.5.1. *Let $E \subset 2^X$ and $F \in \mathscr{A}_\sigma(\mathscr{A})$, we say that F is measurable cover of E if $E \subset F$ and $\forall G \in \mathscr{A}_\sigma(\mathscr{A})$ with $G \subset F \backslash E$, we have $\mu^*|_{\mathscr{M}}(G) = 0$.*

Theorem 1.5.2. *Approximation*

If $E \in 2^X$ is of σ-finite outer measure, then $\exists$ a measurable cover $F \in \mathscr{A}_\sigma(\mathscr{A})$ of E, s.t. $\mu^*|_{\mathscr{M}}(F) = \mu^*(E)$.

Proof.

Case I. $\mu^*(E) < \infty$.

$\forall \varepsilon > 0$, $\exists F \in \mathscr{A}_\sigma(\mathscr{A})$ with $E \subset F_n$, s.t. $\mu^*|_{\mathscr{M}}(F_n) \leq \mu^*(E) + \varepsilon$.

Set $F = \bigcup_1^\infty F_n \in \mathscr{A}_\sigma(\mathscr{A})$, $\Rightarrow$ $\mu^*|_{\mathscr{M}}(F_n) \geq \mu^*|_{\mathscr{M}}(F) \geq \mu^*(E) \Rightarrow$ and $E \subset F$. Letting $\varepsilon \to 0$, we have $\mu^*|_{\mathscr{M}}(F) = \mu^*(E) < \infty$.

Obviously, F is a measurable cover of E.

Case II. $\mu^*(E) = \infty$.

From the fact that E is σ-finite outer measure, $\exists \{E_n\}_1^\infty \subset 2^X$, s.t. $E = \bigcup_1^\infty E_n$ with $\mu^*(E_n) < \infty$. Respect the produce in Case I, $\forall n \in \mathbb{N}_+$, $\exists$ a measurable cover F_n of E_n, s.t. $\mu^*(E_n) = \mu^*|_{\mathscr{M}}(F_n)$.

Set $F = \bigcup_1^\infty F_n$.

$\forall G \subset F \backslash E$ with $G \in \mathscr{A}_\sigma(\mathscr{A})$, let $G_n := G \cap F_n \Rightarrow G = \bigcup_1^\infty G_n$.

$G_n \subset F_n \backslash E \subset F_n \backslash E_n \Rightarrow E_n \cup G_n \subset F_n \Rightarrow En \backslash G_n$.

$\mu^*(E_n) \leq \mu^*|_{\mathscr{M}}(F_n \backslash G_n) = \mu^*|_{\mathscr{M}}(F_n) - \mu^*|_{\mathscr{M}}(G_n)$.

$\Rightarrow \mu^*|_{\mathscr{M}}(G_n) = 0$, $\forall n \in \mathbb{N}_+$. $\square$

Theorem 1.5.3. *If μ_0 is σ-finite premeasure on an algebra $\mathscr{A}$, then*

$$(X, \mathscr{M}_{\mu^*}^X, \mu^*|_{\mathscr{M}}) = (X, \overline{\mathscr{A}_\sigma(\mathscr{A})}, \overline{\mu}). \tag{1.49}$$

Proof.

"$\supset$".

Obviously, $\mathscr{A}_\sigma(\mathscr{A}) \subset \mathscr{M}$ from previous result(c.f. proposition 1.4.2). It is easy to know that $\mu^*|_{\mathscr{M}} = \mu^*|_{\mathscr{A}_\sigma(\mathscr{A})} = \overline{\mu}$.

"$\subset$".

It remains to show that $\mathscr{M} \subset \mathscr{A}_\sigma(\mathscr{A})$. $\forall E \in \mathscr{M}$ with $E = \bigcup_1^\infty E_n$ and $\mu^*|_{\mathscr{M}}(E_n) < \infty$ due to σ-finiteness of $\mu^*|_{\mathscr{M}}$.

WLOG, assume that $E \in \mathscr{M}$ with $\mu^*|_{\mathscr{M}}(E) < \infty$. From the above theorem (c.f. theorem 1.5.2), $\exists$ a measurable cover $F \in \mathscr{A}_\sigma(\mathscr{A})$, s.t. $\mu^*|_{\mathscr{M}}(E) = \mu^*|_{\mathscr{M}}(F) < \infty$.

Consider $F \backslash E \in \mathscr{M}$, $\exists$ a measurable $G \in \mathscr{A}_\sigma(\mathscr{A}) \& F \backslash E \subset G$, s.t. $\mu^*|_{\mathscr{M}}(G) = \mu^*|_{\mathscr{M}}(F \backslash E)$.

Note that $E = \underbrace{(F \backslash G)}_{\in \mathscr{A}_\sigma(\mathscr{A})} \cup \underbrace{(E \cap G)}_{\subset G} \in \mathscr{A}_\sigma(\mathscr{A})$. $\square$

1.6 Measures on the real line

Def. 1.6.1. *A measure on the real line whose domain is the Borel σ-algebra of $\mathbb{R}$, i.e., $\mathscr{B}^{\mathbb{R}}$, is called a Borel measure.*

Def. 1.6.2. *Suppose that μ is a finite Borel measure, define: $F(x) := \mu((-\infty, x]) \, (= P(\{\omega \in \Omega : X(\omega) \leq x\}))$ is called a distribution function of μ.*

Prop. 1.6.1. *Properties os distribution function:*

1. *Monotoneous. $\forall x_1 < x_2$, $F(x_2) - F(x_1) = \mu((x_1, x_2]) \geq 0$.(Increasing)*

2. Right countinuous. $(-\infty, x] = \bigcap_1^\infty (-\infty, x_n]$,$x_n \to x_+$ *as* $n \to \infty$.

Q: Conversely, give $F(x)$ satisfying *1, 2*, is there a Borel measure, s.t. $\mu((a,b]) = F(b) - F(a)$?

Two special case.

1. If $\lim_{x\to\infty} F(x) = 1$, $\lim_{x\to-\infty} F(x) = 0$, then μ is a probability measure.

2. If $F(x) = x$, then μ is the Lebesgue measure.

Def. 1.6.3. *The left open right closed intervals in* $\mathbb{R}$*, i.e., set of the form* $(a,b]$*,* $(a,+\infty)$*,*ϕ*.* $\forall -\infty \le a < b < +\infty$ *are called helf-intervals. Denote* $\mathscr{H}$ *the collection of finite disjiont unions of helf-intervals.*

Remark 1.6.1. $\mathscr{H}$ *is an algebra and* $\mathscr{A}_\sigma(\mathscr{H}) = \mathscr{B}^{\mathbb{R}}$*.*

Prop. 1.6.2. *Let* F *be an increasing and right countinuous function* $\mathbb{R} \to \mathbb{R}$*. Define*

$$\begin{aligned} \mu_0: \quad & \mathscr{H} \quad \to \quad \overline{\mathbb{R}_+}, \\ \mu_0(\bigcup_1^\infty (a_j, b_j]) \quad &= \quad \sum_1^\infty (F(b_j) - F(a_j)), \end{aligned} \tag{1.50}$$

and $\mu_0(\phi) = 0$*, then* μ_0 *is premeasure on* $\mathscr{H}$*.*

Proof.

1 Well-defined.

$\forall I = (a,b] \in \mathscr{H}$ with $I = \bigcup_1^n (a_j, b_j]$, $\Rightarrow a = a_1 < b_1 = a_2 < b_2 = a_3 < b_3 = \ldots < b_n = b$.

$\forall \bigcup_{i=1}^{n_i} I_i = \bigcup_{j=1}^{n_j} J_j \in \mathscr{H}$, where I_i,J_j are helf-intervals (disjiont).

$$\mu_0(\bigcup_{i=1}^{n_i} I_i) = \mu_0(\bigcup_{i=1}^{n_i}\bigcup_{j=1}^{n_j}(I_i \cap J_j)) = \mu_0(\bigcup_{j=1}^{n_j} J_j). \tag{1.51}$$

$\Rightarrow \mu_0$ is well-defined and finitely additive.

2 Countably additive.

WLOG, we consider $J = (a,b]$ or $(a,+\infty)$. $-\infty < a < b < +\infty$.

$$\mu(J) = \mu\left(J \backslash \cup_1^n (a_j, b_j]\right) + \mu\left(\cup_1^n (a_j, b_j]\right) \ge \sum_{j=1}^n \mu((a_j, b_j]) \tag{1.52}$$

$$\Rightarrow \mu_0(J) \ge \sum_1^\infty \mu(a_j, b_j]) \tag{1.53}$$

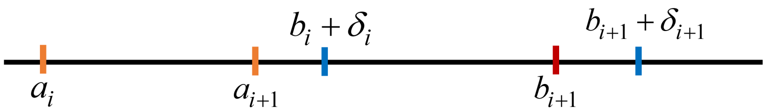
To prove the inverse inequality, we first consider
Case I: a, b are finite, $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t.

$$F(a+\delta) - F(a) < \varepsilon. \tag{1.54}$$

$\forall I = (a_i, b_i]$, $\exists \delta_i > 0$, s.t.

$$F(b_i+\delta_i) - F(b_i) < \frac{\epsilon}{2^i}. \tag{1.55}$$

We find that open intervals $\{(a_i, b_i+\delta_i)\}_1^\infty$ cover the compact set $[a+\delta, b]$. $\exists$ finite subcover, denote by $\{(a_i, b_i+\delta_i)\}_n^1$, satisfying $b_i+\delta_i \in (a_{i+1}, b_{i+1}+\delta_{i+1})$.



$$\begin{aligned}
\mu_0(J) = F(b) - F(a) &\overset{(1.54)}{\leq} F(b) - F(a+\delta) + \varepsilon \leq F(b_n+\delta_n) - F(a+\delta) + \varepsilon \\
&= F(b_n+\delta_n) - F(a_n) + F(a_n) - F(a+\delta) + \varepsilon \\
&\leq F(b_n+\delta_n) - F(a_n) + \sum_1^{n-1}(F(a_{i+1} - F(a_i)) + \varepsilon \\
&\leq F(b_n+\delta_n) - F(a_n) + \sum_1^{n-1}(F(b_i+\delta_i) - F(a_i)) + \varepsilon \\
&\leq \sum_1^{n}(F(b_i+\delta_i) - F(a_i)) + \varepsilon \\
&\overset{(1.55)}{\leq} \sum_1^{n}(F(b_i) - F(a_i)) + 2\varepsilon
\end{aligned} \tag{1.56}$$

Letting $\epsilon \to 0$. □

Theorem 1.6.1. *If $F : \mathbb{R} \to \mathbb{R}$ ia an increasing and right continuous function, then*

1. *$\exists!$ Borel measure μ_F on $\mathbb{R}$, s.t. $\mu_F((a, b]) = F(b) - F(a)$.*
2. *If G is another sunch function, then $\mu_F = \mu_G$ iff $F - G = const.$.*

3. *Conversely, if μ is a Borel measure on $\mathbb{R}$ which is finite on bounded intervals. Define*

$$F(x) = \begin{cases} \mu((0,x]) & x > 0 \\ 0 & x = 0 \\ -\mu((x,0]) & x < 0 \end{cases}, \tag{1.57}$$

then F is increasing and right continuous and $\mu_F = \mu$.

Proof.

From above proposition , we induce a premeasure on $\mathscr{H}$.

Clearly, F, G induce the same premeasure on $\mathscr{H}$ if $F - G = const.$. All these premeasure are σ-finite. From unique extension theorem (c.f. theorem 1.5.1) $\mu_F = \mu_G$ on $\mathscr{B}^{\mathbb{R}}$.

The monotonicity of μ implies the monotonicity of F.

When $x_n \to x$,

$$\mu(\bigcap_1^\infty (0, x_n]) = \mu(\lim_{n\to\infty} (0, x_n]) = \lim_{n\to\infty} \mu((0, x_n]), \tag{1.58}$$

$$\mu(\bigcup_1^\infty (x_n, 0]) = \mu(\lim_{n\to\infty} (x_0, 0]) = \lim_{n\to\infty} \mu((x_n, 0]), \tag{1.59}$$

It is easy to check that the induce measure $\mu_F = \mu$ for each helf-intervals, $\Rightarrow \mu_F = \mu$ on $\mathscr{H}$. Applying unique extension theorem (c.f. theorem 1.5.1) again, we have $\mu_F = \mu$ on $\mathscr{B}^{\mathbb{R}}$.

$\square$

Def. 1.6.4. *We denote the complete induce measure by μ_F, which is called Lebesgue-Stieltjes measure associated to F, $\forall E \in \mathscr{M}^{\mathbb{R}}_{\mu_F}$,*

$$\mu_F(E) = \inf\{\sum_1^\infty \underbrace{\mu_F((a_j, b_j])}_{=F(b_j)-F(a_j)} : E \subset \bigcup_1^\infty (a_j, b_j]\}. \tag{1.60}$$

Lemma 1.6.1. $\forall E \in \mathscr{M}$, $\mu_F(E) = \inf\{\sum_1^\infty \mu_F((a_j, b_j]) : E \subset \bigcup_1^\infty (a_j, b_j)\}$.

Proof.

Denote $\nu(E) = \inf\{\sum_1^\infty \mu_F((a_j, b_j]) : E \subset \bigcup_1^\infty (a_j, b_j]\}$, $\forall E \in \mathscr{M}$.

Suppose that $E \subset \bigcup_1^\infty (a_j, b_j)$. Then $\exists$ a sequence $\{I_j^k = (a_j^k, b_j^k]\}$ of helf-intervals, s.t. $(a_j, b_j) = \bigcup_{k=1}^\infty I_j^k$, satisfying $a_j^1 = a_j$, $b_j^k = a_j^{k+1}$, and $b_j^k \to b_j$ as $k \to \infty$.

$$\Rightarrow E \subset \bigcup_{j,k=1}^\infty I_j^k \Rightarrow \sum_1^\infty \mu_F((a_j, b_j)) = \sum_{j,k=1}^\infty \mu_F(I_j^k) \geq \mu_F(E)$$

$\Rightarrow \nu(E) \geq \mu_F(E)$.

On the other hand, $\forall \varepsilon > 0$, $\exists$ a sequence $\{(a_j, b_j]\}$ of helf-intervals, $E \subset \bigcup_1^\infty (a_j, b_j]$ and $\sum_1^\infty \mu_F((a_j, b_j]) (= F(b_j) - F(a_j)) < \mu_F(E) + \varepsilon$.

Using the fact that F is right continuous, $\forall j \in \mathbb{N}_+$, $\exists \delta_j > 0$, s.t. $F(b_j + \delta_j) - F(b_j) < \frac{\varepsilon}{2^j}$.

$$\begin{aligned} E &\subset \bigcup_1^\infty (a_j, b_j + \delta_j) \\ \Rightarrow \sum_1^\infty \mu_F((a_j, b_j + \delta_j)) - \varepsilon &< \sum_1^\infty \mu_F((a_j, b_j]) < \mu(E) + \varepsilon \end{aligned}$$

$\Rightarrow \nu(E) \leq \mu_F(E)$ by letting $\varepsilon \to 0$. □

Theorem 1.6.2. *If $E \in \mathscr{M}_{\mu_F}^{\mathbb{R}}$, then:*

1. $\mu_F = \inf\{\mu_F(U) : \textit{open } U \supseteq E\}$
2. $\mu_F = \sup\{\mu_F(K) : \textit{compact } K \subseteq E\}$

Proof.

1. $\forall \epsilon > 0$, $\exists (a_j, b_j)$ s.t. $E \subseteq \bigcup (a_j, b_j)$ and $\sum_1^\infty \mu_F(a_j, b_j) \leq \mu(E) + \epsilon$ by the above Lemma.

Set $U = \bigcup_1^\infty (a_j, b_j)$, then U is open and the first "=" is valid.

2. Suppose first that E is bounded. If E is closed, we have done $K = E$. Otherwise, from *1*, $\exists$ open U s.t. $\forall \epsilon > 0$, $\overline{E} \subset U$ and

$$\mu_F(U) < \mu_F(\overline{E} \setminus E) + \varepsilon \tag{1.61}$$

Set $K := E \backslash U = \overline{E} \backslash U$, then K is closed $\Rightarrow$ compact.

Note that $\overline{E} = (\overline{E} \backslash E) \cup E$ [6].

$$\mu(E) > \mu_F(K) = \mu_F(E) - \mu_F(E \cap U) = \mu(E) - (\mu_F(U) - \mu_F(U \backslash E)). \tag{1.62}$$

Tip: $E \cap U = U \backslash (U \backslash E) = U \cap (U \cap E^c)^c = U \cap (U^c \cup E) = U \cap E$.

By Eq. (1.61),

$$\mu(E) > \mu_F(E) - \varepsilon. \tag{1.63}$$

Letting $\varepsilon \to 0$, $\mu_F(K) = \mu(E)$.

In general, E is unbounded. Let $H_n = E \cap (n, n+1]$. Repeating the preceding argument $\forall \epsilon > 0$, $\exists$ compact $K_n \subset H_n$ s.t. $\mu_F(H_n) \leq \mu_F(K_n) + \frac{\epsilon}{2^{n+1}}$.

[6] $\overline{E} \backslash E \subset U$ and $\overline{E} \backslash E \subset U \backslash E$

Set $G_n = \bigcup_{-n}^{n} K_j$, then G_n is compact.

$$\mu_F\left(\bigcup_{-n}^{n} H_j\right) \le \mu_F(G_n) + 2\epsilon \tag{1.64}$$

$$\mu(G_n) \le \lim_{n\to\infty} \mu_F\left(\bigcup_{-n}^{n} H_j\right) = \mu(E) \quad \text{Letting } n \to \infty. \tag{1.65}$$

□

Def. 1.6.5. *Lebesgue measure, denoted by m, is the complete σ-measure associated to the function $F(x) = x$ whose domain is called the class of all Lebesgue measurable sets, denoted by $\mathscr{L}$.*

Prop. 1.6.3. *Every countable set is a Borel set of Lebesgue measure zero.*

Proof. $\{a\} = \bigcap_1^\infty \left(a - \frac{1}{n}, a\right)$ every single point set is Borel set.

$$m(\{a\}) = m\left(\lim_{n\to\infty}\left(a - \frac{1}{n}, a\right)\right) = \lim_{n\to\infty} m\left(a - \frac{1}{n}, a\right) = \lim_{n\to\infty} \frac{1}{n} = 0$$

□

Notation: Let T be a linear transformation on $\mathbb{R}$, given by $T(x) = \alpha x + \beta$, $\alpha \neq 0$, $\beta \in \mathbb{R}$. $\forall E \subseteq \mathbb{R}$, denote $T(E) = \{\alpha x + \beta : x \in E\}$.

Theorem 1.6.3. *Let T be a linear transformation given as above. Then*

$$m^*(T(E)) = |\alpha| m^*(E) \tag{1.66}$$

and $T(E)$ is a Borel set (Lebesgue measure set) iff E is a Borel set (Lebesgue measure set).

Proof.
(1) **Claim:** $T(\mathscr{B}^{\mathbb{R}}) := \{T(E) : E \in \mathscr{B}^{\mathbb{R}}\} = \mathscr{B}^{\mathbb{R}}$.
Proof Claim: It is easy to check that $T(\mathscr{B}^{\mathbb{R}})$ is a σ-algebra.
1. Countable union: $\forall E_n \in T(\mathscr{B}^{\mathbb{R}}), E_n = T(E_n')$, $E_n' \in \mathscr{B}^{\mathbb{R}}$.

$$\bigcup_1^\infty E_n = \bigcup_1^\infty T(E_n') = T\left(\bigcup_1^\infty E_n'\right) \in T(\mathscr{B}^{\mathbb{R}}).$$

2. Complements: $E \in T(\mathscr{B}^{\mathbb{R}}) \Rightarrow \exists B \in \mathscr{B}^{\mathbb{R}}, T(B) = E$.

$$T(\mathscr{B}) = [T(B)]^c = E^c$$

$\forall (a,b) = E \in \mathscr{H}$, assume $\alpha > 0$, $\exists \left(\frac{a-\beta}{\alpha}, \frac{b-\beta}{\alpha}\right) = F \in \mathscr{H}$, such that $T(F) = E$.

$$\Rightarrow \mathscr{H} \subset T(\mathscr{B}^{\mathbb{R}}) \Rightarrow \mathscr{B}^{\mathbb{R}} \subset T(\mathscr{B}^{\mathbb{R}}).$$

Applying the same reasoning to the inverse transformation. We have

$$\mathscr{B}^{\mathbb{R}} \subset T^{-1}(\mathscr{B}^{\mathbb{R}}) \Rightarrow T(\mathscr{B}^{\mathbb{R}}) \subset T(T^{-1}(\mathscr{B}^{\mathbb{R}})) \Rightarrow T(\mathscr{B}^{\mathbb{R}}) \subset \mathscr{B}^{\mathbb{R}}.$$

(2) $\forall E = (a,b) \in \mathscr{H}$, define $m_1(E) = m(T(E)), m_2(E) = |\alpha| m(E)$.($m$ is Lebesgue measure.)

$$m_1(E) = |\alpha b - \alpha a| = |\alpha||b-a| = m_2(E) \quad .$$

From Uniqueness Extension Theorem, $m_1(E) = m_2(E)$ on $\mathscr{B}^{\mathbb{R}}$.

$$\begin{aligned} m^*(T(E)) = &\ \inf\{m(F) : T(E) \subset F \in \mathscr{B}^{\mathbb{R}}\} \\ = &\ \inf\{m(T(T^{-1}(F))) : T(E) \subset F \in \mathscr{B}^{\mathbb{R}}\} \\ = &\ \inf\{|\alpha| m(T^{-1}(F)) : E \subset T^{-1}(F) \in \mathscr{B}^{\mathbb{R}}\} \\ = &\ |\alpha| \inf\{m(G) : E \subset G \in \mathscr{B}^{\mathbb{R}}\} \\ = &\ |\alpha| m^*(E) \end{aligned} \tag{1.67}$$

Thus, if $m^*(N) = 0$, then $m^*(T(N)) = 0$.

Note that every Lebesgue measurable set is a union of a Borel set and a set of outer measure zeros [7].

It is follows that $\mathscr{L}$ is preserved by T.

□

[7] $E = B \cup F$, $F \subset N$, $B \in \mathscr{B}^{\mathbb{R}}$

Chapter 2

Integral Theorem

2.1 Measurable functions

Def. 2.1.1. *Any mapping $f : X \to Y$ between two sets induces a mapping $f^{-1} : Y \to 2^X$, define by $\forall E \in Y$,*

$$f^{-1}(E) := \{x \in X : f(x) \in E\}, \tag{2.1}$$

$f^{-1}(E)$ is called the inverse image under f of the set E.

EX. 2.1.1. *Consider a characteristic function*

$$\chi_E(x) = \begin{cases} 0 & x \notin E \\ 1 & x \in E \end{cases} \tag{2.2}$$

. Obviously, $\chi^{-1}(1) = E$, $\chi^{-1}(0) = E^c$.

Remark 2.1.1. *Induce mappings preserved countable unions, intersections and complements, i.e.,*

$$f^{-1}(\bigcup_1^\infty E_n) = \bigcup_1^\infty f^{-1}(E_n), \tag{2.3}$$

$$f^{-1}(\bigcap_1^\infty E_n) = \bigcap_1^\infty f^{-1}(E_n), \tag{2.4}$$

$$\left[f^{-1}(E)\right]^c = f^{-1}(E^c). \tag{2.5}$$

It is follows that $f^{-1}(\mathscr{Y})$ is a σ-algebra if $\mathscr{Y}$ is a σ-algebra.

Def. 2.1.2. *If* $(X, \mathscr{X})$ *and* $(Y, \mathscr{Y})$ *are measurable space, a mapping* $f : X \to Y$ *is said to be* $(\mathscr{X} - \mathscr{Y})$*-measurable* [1] *if* $f^{-1}(E) \in \mathscr{X}$, $\forall E \in \mathscr{Y}$.

Remark 2.1.2. *If* $f : X \to Y$ *is* $(\mathscr{X} - \mathscr{Y})$*-measurable, and* $g : Y \to Z$ *is* $(\mathscr{Y} - \mathscr{Z})$*-measurable, then the compostion* $g \circ f : X \to Z$ *is* $(\mathscr{X} - \mathscr{Z})$*-measurable.*

Def. 2.1.3. *$\mathcal{L}$et* $(X, \mathscr{A})$ *be a measurable space. A* $\mathbb{R}$*-valued function is called* $\mathscr{A}$*-measurable if* $f^{-1}(B) \in \mathscr{A}$, $\forall B \in \mathscr{B}^{\mathbb{R}}$.

Prop. 2.1.1. *If* $\mathscr{Y}$ *is generated bu* $\mathscr{E}$*, then* f *is* $(\mathscr{X} - \mathscr{Y})$*-measurable iff* $f^{-1}(E) \in \mathscr{E}$, $\forall E \in \mathscr{E}$.

Proof. "$\Rightarrow$" tirival.

"$\Leftarrow$" define: $\mathscr{Y}' := \{E \subset Y : f^{-1}(E) \in \mathscr{X}\}$ is a σ-algebra and $\mathscr{E} \subset \mathscr{Y}'$. $\Rightarrow \mathscr{Y}' \supset \mathscr{Y}$. $\square$

Cor. 2.1.1. *A* $\mathbb{R}$*-valued function on measurable space is* $\mathscr{A}$*-measurable iff* $f^{-1}(E) \in \mathscr{A}$, $\forall E \in \mathscr{E}_i$*, where is define in proposition 1.1.3.*

Cor. 2.1.2. *If* X *and* Y *are metric (or topological) space, then every continuous function is measurable.*

Def. 2.1.4. $\mathbb{R}$*-valued function* $f : X \to \overline{\mathbb{R}}$ *is called* $\mathscr{A}$*-measurable if*

$$f^{-1}(\{\pm\infty\}) \in \mathscr{A} \tag{2.6}$$

and

$$f^{-1}(B) \in \mathscr{A}, \forall B \in \mathscr{B}^{\mathbb{R}}. \tag{2.7}$$

Inparticuler,

1. $f : \mathbb{R} \to \overline{\mathbb{R}}$ *is called Lebesgue measurable function if it is* $(\mathscr{L} - \mathscr{B}^{\mathbb{R}})$*-measurable.*
2. $f : \mathbb{R} \to \overline{\mathbb{R}}$ *is called Borel measurable function if it is* $(\mathscr{B}^{\mathbb{R}} - \mathscr{B}^{\mathbb{R}})$*-measurable.*

Remark 2.1.3. *If* f, g *are Lebesgue measurable, it does not follow that* $g \circ f$ *is Lebesgue measurable. Even if* f *is assumed continuous.*

$$\begin{array}{ccccc} \mathbb{R} & \xrightarrow{f} & \mathbb{R} & \xrightarrow{g} & \mathbb{R} \\ f^{-1}\left(g^{-1}(B)\right) \in \mathscr{B}^{\mathbb{R}} & \not\Leftarrow & g^{-1}(B) \in \mathscr{L} & \Leftarrow & B \in \mathscr{B}^{\mathbb{R}} \end{array}$$

[1] briefly, measurable of $\mathscr{X}$, $\mathscr{Y}$ are understood

Prop. 2.1.2. *If f is a $\overline{\mathbb{R}}$-valued measurable function on a measurable space $(X, \mathscr{A})$ and $g : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is a Borel measurable function , then $g \circ f$ is $\mathscr{A}$-measurable.*

Cor. 2.1.3. *If f is a $\overline{\mathbb{R}}$-valued measurable function on a measurable space $(X, \mathscr{A})$, So are $c \cdot f$, f^n, $|f|^c$, $\forall n \in \mathbb{N}_+$, $c \in \mathbb{R}$.*

Lemma 2.1.1. *If f, g are $\mathbb{R}$-measurable functions on measurable space $(X, \mathscr{A})$, then $\forall c \in \mathbb{R}$, the following sets*

$$A := \{x : f(x) < g(x) + c\}$$

$$B := \{x : f(x) \leq g(x) + c\}$$

$$C := \{x : f(x) = g(x) + c\}$$

are $\mathscr{A}$-measurable.

Proof.
Let $\{r_n\}_1^\infty$ be the set of all rational number on $\mathbb{R}$.
Note that

$$A = \bigcup_{n \in \mathbb{N}_+} \{\overbrace{\{x : f(x) < r_n\}}^{\in \mathscr{A}} \cap \overbrace{\{x : r_n - c < g(x)\}}^{\in \mathscr{A}}\}, \tag{2.8}$$

it follows that $A \in \mathscr{A}$.

Consider $X \backslash B = \{x : f(x) > g(x) + c\}$, We deduce that $X \backslash B \in \mathscr{A} \Rightarrow B \in \mathscr{A}$.

The relation $C = B \backslash A$ implies $C \in \mathscr{A}$. □

Theorem 2.1.1. *If f, g are $\mathbb{R}$-valued measurable functions on a measurable space $(X, \mathscr{A})$. So are $f + g$ and $f \cdot g$.*

Proof.
First, we note that $(f + g)^{-1}(\{\pm\infty\}) = f^{-1}(\{\pm\infty\}) \cup g^{-1}(\{\pm\infty\}) \in \mathscr{A}$.

It remains to consider the case that f, g are finite.

$\forall c \in \mathbb{R}, \{x : f(x) + g(x) \leq c\} = \{x : f(x) \leq -g(x) + c\} \in \mathscr{A}$ from the above lemma (c.f. lemma 2.1.1).

$\Rightarrow f + g$ is $\mathscr{A}$-measurable.

For $f \cdot g$, if $f = g$, it is easy to see that $f \cdot g = f^2$ is $\mathscr{A}$-measurable. Moreover, from previous results, we have $(f+g)^2$, $(f-g)^2$ are $\mathscr{A}$-measurable. If $f \neq g$, thanks to $f \cdot g = \frac{1}{4}[(f + g)^2 - (f - g)^2]$. We know that $f \cdot g$ is also $\mathscr{A}$-measurable. □

Theorem 2.1.2. *If $\{f_n\}_{n=1}^{\infty}$ is a sequence of $\mathbb{R}$-valued measurable functions on a measurable space $(X, \mathscr{A})$. Then the following functions*

$$\begin{aligned} f^*(x) &:= \sup_n f_n(x) \\ f_*(x) &:= \inf_n f_n(x) \\ \overline{f}(x) &:= \limsup_{n\to\infty} f_n(x) \\ \underline{f}(x) &:= \liminf_{n\to\infty} f_n(x) \end{aligned} \tag{2.9}$$

are $\mathscr{A}$-measurable. Moreover, if $\lim f_n(x)$ exists, then it is $\mathscr{A}$-measurable.

Proof.

Obviously, $(f^*)^{-1}(\{-\infty\}) = \bigcap_{n=1}^{\infty}\{f_n^{-1}(\{-\infty\})\} \in \mathscr{A}$.

$(f^*)^{-1}(\{+\infty\}) = \bigcap_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\{x : f_n(x) > m\} \in \mathscr{A}$ [2].

It reduces to show the finite case. $\forall c \in \mathbb{R}$, $\{x : f^*(x) \le c\} = \bigcap_{n=1}^{\infty}\{x : f_n(x) \le c\} \in \mathscr{A} \Rightarrow f^*$ is $\mathscr{A}$-measurable.

The relation

$$f_* = -\sup(-f_n) \tag{2.10}$$

implies f_* is $\mathscr{A}$-measurable.

$$\overline{f} = \inf_k \sup_{n\ge k} f_n \tag{2.11}$$

and

$$\underline{f} = \sup_k \inf_{n\ge k} f_n, \tag{2.12}$$

we have $\overline{f}$, $\underline{f}$ are $\mathscr{A}$-measurable.

Moreover, if $\lim_{n\to\infty} f_n(x)$ exists, then $\lim_{n\to\infty} f_n(x) = \overline{f} = \underline{f}$.

$\Rightarrow \liminf_{n\to\infty} f_n$ is $\mathscr{A}$-measurable. □

Cor. 2.1.4. *If f, g are $\mathbb{R}$-valued measurable functions on a measurable space $(X, \mathscr{A})$, then so are $\max\{f, g\}$, $\min\{f, g\}$.*

Def. 2.1.5. *$\forall\mathbb{R}$-valued function f, we define the positive and negative parts of f by $f^+(x) := \max\{f(x), 0\}$ and $f^-(x) := \max\{-f(x), 0\}$. Then $f = f^+ - f^-$ and $|f| = f^+ + f^-$.*

Prop. 2.1.3. *If $(X, \mathscr{A}, \mu)$ is a complete measure space, f is $\mathscr{A}$-measurable, and $f = g$ μ-a.e., then g is also $\mathscr{A}$-measurable.*

Cor. 2.1.5. *If $(X, \mathscr{A}, \mu)$ is a complete measure space, $\{f_n\}$ is $\mathscr{A}$-measurable, and $\lim_{n\to\infty} f_n(x) = f(x)$, μ-a.e.. Then $f(x)$ is $\mathscr{A}$-measurable.*

[2] $\forall m$, $\exists n$, $f_n(x) > m$

2.2 Simple functions and their integration

Def. 2.2.1. *Let $(X, \mathscr{A}, \mu)$ be a measure space. Define a function $\phi : X \to \mathbb{R}$ is called simple, if $\exists$ disjoint class $\{E_1, E_2, ..., E_n\} \subset \mathscr{A}$ and a finite set $\{c_1, c_2, ..., c_n\} \subset \mathbb{R}$, s.t. $\forall x \in X$,*

$$\phi(x) = \begin{cases} c_j & x \in E_j \\ 0 & x \notin \bigcup_1^n E_j \end{cases}. \tag{2.13}$$

Remark 2.2.1.
1. *The simplest example of a simple function is a characteristic function.*
2. *It is easy to verify that every simple function is $\mathscr{A}$-measurable.*
3. *In fact, for the simple function described as above, we have*

$$\phi(x) = \begin{cases} c_j & x \in E_j \\ 0 & x \notin \bigcup_1^n E_j \end{cases} = \sum_{j=1}^n c_j \chi_{E_j}(x), \tag{2.14}$$

   $\forall x \in E_j$ which is the standard reprsentation of ϕ.
4. *Any linear conbination of simple functions and any finite product of simple function are simple.*

Theorem 2.2.1. *Every $\mathbb{R}$-valued measurable functions f is limit of a sequence $\{\varphi_n\}$ of simple functions. In particular, if f is nonnegative, then each φ_n can be taken nonnegative and the sequence $\{\varphi_n\}$ can be assumed increasing.*

Proof.
First, we assume that $f \geq 0$.
$\forall n \in \mathbb{N}_+$, $x \in X$, set

$$\varphi_n(x) = \begin{cases} \dfrac{k-1}{2^n} & \dfrac{k-1}{2^n} \leq f(x) < \dfrac{k}{2^n},\ k = 1, 2, ..., n \cdot 2^n \\ n & f(x) \geq n \end{cases} \tag{2.15}$$

For $x \in X$,

1. if $f(x) < \infty$, then $\exists \varepsilon > 0$, $\exists n \in \mathbb{N}_+$ sufficiently large, s.t. $0 \leq f(x) - \varphi_n(x) \leq \frac{1}{2^n} < \varepsilon$.

2. if $f(x) = \infty$, then $\varphi_n(x) = n$, $\forall n \in \mathbb{N}_+$, $\Rightarrow \phi_n(x) \to \infty$ as $n \to \infty$.

It is prove the 1^{st} part of theorem.

The 2^{nd} part follows by applying the above result, we proved to the f^+ and f^- of f separately. □

Prop. 2.2.1. *Let $(X, \mathscr{A}, \mu)$ be a measure space and $(X, \bar{\mathscr{A}}, \bar{\mu})$ be its completion. If f is a $\bar{\mathscr{A}}$-measurable function, then $\exists$ a $\mathscr{A}$-measurable function g such that $f = g$ $\bar{\mu}$-a.e.*

Proof.

We first assume that f is a simple function on $(X, \bar{\mathscr{A}}, \bar{\mu})$, $f = \sum_{j=1}^n c_j \chi_{E_j}$ where $E_j \in \bar{\mathscr{A}}$. Then $E_j = A_j \cup B_j$, where $A_j \in \mathscr{A}$, $B_j \in \mathscr{N}_j$ for some μ-null set.

Set $g = \sum_{j=1}^n c_j \chi_{A_j}$. Obviously, g is $\mathscr{A}$-measurable and $f = g$ $\bar{\mu}$-a.e.

For the general case, we can find a sequence $\{\varphi_n\}$ of $\bar{\mathscr{A}}$-measurable simple functions s.t. $\varphi_n(x) \to f(x)$ as $n \to \infty$.

$\exists$ $\mathscr{A}$-measurable simple functions $\{\psi_n\}$ s.t. $\phi_n = \psi_n$ except on some B_n with $B_n \subset \mathcal{N}_n$ and $\mu(\mathcal{N}_n) = 0$. Set $B = \bigcup_1^\infty B_n \Rightarrow \bar{\mu}(B) = 0$. Let $g := \lim_{n\to\infty} \psi_n \chi_{X\setminus B}$, easy to check $\lim_{n\to\infty} \psi_n \chi_{X\setminus B}$ is a $\mathscr{A}$-measurable simple function.

From the previous corollary (c.f. corollary 2.1.5), g is also $\mathscr{A}$-measurable and $f = g$ $\bar{\mu}$-a.e.

□

Def. 2.2.2. *A simple function $\phi = \sum_{j=1}^n c_j \chi_{E_j}$ on a measure space $(X, \mathscr{A}, \mu)$ is said to be integrable if $\mu(E_j) < \infty$ for each index j for which $c_j \neq 0$. The integral of ϕ with respect to μ is defined by*

$$\int \phi \, d\mu = \sum_1^n c_j \mu(E_j). \tag{2.16}$$

Theorem 2.2.2. *If ϕ and ψ are both integrable simple functions, then so is $\alpha\phi + \beta\psi$, $\forall \alpha, \beta \in \mathbb{R}$, and*

$$\int (\alpha\phi + \beta\psi) \, d\mu = \alpha \int \phi \, d\mu + \beta \int \psi \, d\mu. \tag{2.17}$$

Theorem 2.2.3. *If ϕ is an integrable simple function, and $\phi \geq 0$, then $\int \phi \, d\mu \geq 0$.*

Cor. 2.2.1. *Let ϕ and ψ be integrable simple functions.*

1. *If $\phi \geq \psi$, then $\int \phi \, d\mu \geq \int \psi \, d\mu$. In particular, $|\int \phi \, d\mu| \leq \int |\phi| \, d\mu$.*

2. $\int |\phi + \psi| \, d\mu = \int |\phi| \, d\mu + \int |\psi| \, d\mu$.

Def. 2.2.3. *If ϕ is an integrable simple function on a measure space $(X, \mathscr{A}, \mu)$, $\forall E \in \mathscr{A}$, define the integral of ϕ over E as*

$$\int_E \phi \, d\mu = \int \phi \chi_E \, d\mu. \tag{2.18}$$

Cor. 2.2.2. *If ϕ is an integrable simple function on a measure space $(X, \mathscr{A}, \mu)$, and $\exists \alpha, \beta$ such that $\alpha \le \phi \le \beta$ on E, then*

$$\alpha\mu(E) \le \int_E \phi \, d\mu \le \beta\mu(E). \tag{2.19}$$

Prop. 2.2.2. *The set function $v(E) := \int_E \phi \, d\mu$ is a measure on $(X, \mathscr{A})$.*

Proof.

Obviously, $v(\phi) \ge 0$, where ϕ is null ser.

For any $E_n \in \mathscr{A}$ of mutually disjoint sets, set $E = \bigcup_1^\infty E_n$.

$$\begin{aligned} v(E) = & \int_E \phi \, d\mu = \int \phi \chi_E \, d\mu = \int \phi \sum_1^\infty \chi_{E_n} \, d\mu \\ = & \sum_1^\infty \int \phi \chi_{E_n} \, d\mu \\ = & \sum_1^\infty v(E_n). \end{aligned} \tag{2.20}$$

□

EX. 2.2.1. *Let δ_{x_0} be the Dirac measure at x_0.*

$$\int \phi \, d\delta_{x_0} = \sum c_j \delta_{x_0}(E_j) = \begin{cases} 0 & x_0 \notin E_j \\ c_j & x_0 \in E_j \end{cases} = \phi(x_0). \tag{2.21}$$

Generally, $\mu = \sum_1^\infty \delta_{x_j}$, $\int \phi \, d\mu = \sum \phi(x_j)$.

2.3 Integration of measurable functions

Notation: $L^1_+ := \{\text{measurable functions}, f : X \to [0, +\infty]\}$

Def. 2.3.1. *The integral of a function f in L^1_+ is defined by*

$$\int f \, d\mu = \sup \left\{ \int \phi \, d\mu : 0 \le \phi \le f \textit{ and } \phi \textit{ is simple} \right\}. \tag{2.22}$$

Remark 2.3.1. *If f is simple, then the two definitions agree.*

Theorem 2.3.1. *The Monotone Convergence Theorem, MCT*

If $\{f_n\}$ is a sequence in L^1_+, s.t. $f_n \leq f_{n+1}$, $\forall n \in \mathbb{N}$, and $f(x) = \lim_{n\to\infty} f_n(x)$, then

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu. \tag{2.23}$$

Proof.

Note that $\{\int f_n \, d\mu\}_1^\infty$ is increasing from definition. So $\lim_{n\to\infty} \int f_n \, d\mu$ exists.

"$\leq$". For the same reasoning, $\int f_n \, d\mu \leq \int f \, d\mu \implies \lim_{n\to\infty} \int f_n \, d\mu \leq \int f \, d\mu$.

"$\geq$". To prove the reverse inequality, we fix $\alpha \in (0, 1)$.

Let ϕ be a simple function satisfying $0 \leq \phi \leq f$ and let $E_n = \{x \in X : f_n \geq \alpha\phi\}$. Then $\{E_n\}$ is an increasing sequence of measurable sets whose union is X.

Due to $\int f_n \, d\mu = \int_{E_n} f_n \, d\mu \geq \alpha \int_{E_n} \phi \, d\mu = \alpha\nu(E_n)$ [3]. We have

$$\lim_{n\to\infty} \int f_n \, d\mu \geq \alpha \lim_{n\to\infty} \nu(E_n) = \alpha\nu(\lim_{n\to\infty} E_n)^4 = \alpha \int \phi \, d\mu. \tag{2.24}$$

$\implies \lim_{n\to\infty} \int f_n \, d\mu \geq \int \phi \, d\mu$ by letting $\alpha \to 1$.

Taking the supremum of all nonnegative simple functions, we obtain $\lim_{n\to\infty} \int f_n \, d\mu \geq \int f \, d\mu$. □

Theorem 2.3.2. *If $\{f_n\}_1^\infty$ is a sequence in L^1_+ and $f = \sum_1^\infty f_n$, then*

$$\int f \, d\mu = \sum_1^\infty \int f_n \, d\mu. \tag{2.25}$$

Proof.

First, we consider two functions $f_1, f_2 \in L^1_+$.

From the previous Theorem, we can find two increasing sequences $\{\phi_n\}$, $\{\psi_n\}$ of simple functions such that $\phi_n \to f_1, \psi_n \to f_2$.

This follows that $\{\phi_n + \psi_n\}_1^\infty$ is increasing and converges to $f_1 + f_2$.

By MCT,

$$\begin{aligned}\int f_1 \, d\mu + \int f_2 \, d\mu &= \lim_{n\to\infty} \int \phi_n \, d\mu + \lim_{n\to\infty} \int \psi_n \, d\mu \\ &= \lim_{n\to\infty} \int \phi_n + \psi_n \, d\mu \overset{\text{MCT}}{=} \int f_1 + f_2 \, d\mu\end{aligned} \tag{2.26}$$

[3] $v(E)$ is a measure

[4] measure continuity

By induction, we have

$$\int \sum_{i=1}^{N} f_n \, d\mu = \sum_{i=1}^{N} \int f_n \, d\mu. \tag{2.27}$$

Applying MCT again, we conclude that

$$\int \sum_{i=1}^{\infty} f_n \, d\mu = \lim_{N\to\infty} \int \sum_{i=1}^{N} f_n \, d\mu = \lim_{N\to\infty} \sum_{i=1}^{N} \int f_n \, d\mu = \int \sum_{i=1}^{\infty} f_n \, d\mu. \tag{2.28}$$

□

Prop. 2.3.1. *If $f \in L^1_+$, then $\int f \, d\mu = 0$ iff $f = 0$ μ-a.e..*

Proof.

Suppose f is simple. $f = \sum_{j=1}^{N} c_j \chi_{E_j}$. Then $\int f \, d\mu = 0 \Leftrightarrow \forall j = 1, 2, \ldots, N, c_j = 0$ or $\mu(E_j) = 0 \Leftrightarrow f = 0$ μ-a.e..

In general, if $f \in L^1_+$, then there exists an increasing sequence $\{\phi_n\}$ of simple functions in L^1_+ such that $0 \le \phi_n \le f$.

"$\Leftarrow$". If $f = 0$ μ-a.e., then $\phi_n = 0$ μ-a.e. From the above result, $\int \phi_n \, d\mu = 0$. By MCT, $\int f \, d\mu = \lim_{n\to\infty} \int \phi_n \, d\mu = 0$.

"$\Rightarrow$". Consider $E := \{x \in X : f(x) > 0\} = \bigcup_{n=1}^{\infty} \{x \in X : f(x) > \frac{1}{n}\} = \bigcup_{n=1}^{\infty} E_n$. Assume that $\mu(E) \ne 0$, then $\mu(E_n) \ne 0$ for some $n \in \mathbb{N}_+$. However, $\int f \, d\mu \ge \int_{E_n} f \, d\mu \ge \int_{E_n} \frac{1}{n} \, d\mu = \frac{1}{n}\mu(E_n) > 0$, contradiction. □

Cor. 2.3.1. *If $\{f_n\}_1^{\infty}$ is in L^1_+ and $f_n(x)$ increases to $f(x)$ for a.e. $x \in X$, then $\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu$.*

Proof.

Suppose that $f_n(x)$ increases to $f(x)$ in E and $\mu(E \setminus E) = 0$.

Then $f_n\chi_E, f\chi_E \in L^1_+$ and $\lim_{n\to\infty} f_n(x)\chi_E(x) = f(x)\chi_E(x)$.

By MCT, $\int f \, d\mu = \int f\chi_E \, d\mu = \lim_{n\to\infty} \int f_n\chi_E \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu$. □

Remark 2.3.2. *The hypothesis that $\{f_n\}_1^{\infty}$ is increasing or increasing a.e. is essential for MCT.*

For example, if $X = \mathbb{R}, \mu = m$ Lebesgue measure.

$f_{n_1} : \chi_{(n,n+1)}(x) \to 0$ a.s. $n \to \infty$, but $\int_{\mathbb{R}} f_{n_1} \, d\mu = 1$.

$f_{n_2} : n\chi_{(0,\frac{1}{n})}(x) \to 0$ a.s. $n \to \infty$, but $\int_{\mathbb{R}} f_{n_2} \, d\mu = 1$.

Lemma 2.3.1. *Fatou's lemma*

If $\{f_n\}_1^{\infty}$ is a sequence in L^1_+, then

$$\int \liminf_{n\to\infty} f_n \, d\mu \le \liminf_{n\to\infty} \int f_n \, d\mu. \tag{2.29}$$

Proof.

Note that $\inf_{n\geq k} f_n \leq f_j$, $j \geq k$. Then $\int \inf_{n\geq k} f_n\, d\mu \leq \int f_j\, d\mu$, $j \geq k$, which implies $\int \inf_{n\geq k} f_n\, d\mu \leq \inf_{n\geq k} \int f_n\, d\mu$. By MCT,

$$\int \liminf_{n\to\infty} f_n\, d\mu = \lim_{k\to\infty} \int \inf_{n\geq k} f_n\, d\mu \leq \lim_{k\to\infty} \inf_{n\geq k} \int f_n\, d\mu = \liminf_{n\to\infty} \int f_n\, d\mu. \tag{2.30}$$

□

Cor. 2.3.2. *If* $\{f_n\}_1^\infty \subset L_+^1$ *and* $f_n \to f$, *then* $\int f\, d\mu \leq \liminf_{n\to\infty} \int f_n\, d\mu$.

Def. 2.3.2. *Let* f^+ *and* f^- *be the positive and negative parts of a measurable function* f. *If at least one of* $\int f^+\, d\mu$ *and* $\int f^-\, d\mu$ *is finite, define*

$$\int f\, d\mu = \int f^+\, d\mu - \int f^-\, d\mu. \tag{2.31}$$

Moreover, f *is called integrable if* $\int f^+\, d\mu$ *and* $\int f^-\, d\mu$ *are both finite.*

Remark 2.3.3. *Note that* $|f| = f^+ + f^-$. *It follows that* f *is integrable iff* $\int |f|\, d\mu < \infty$. *Easy to check that the set of all integrable functions is a normed linear space (n.l.s.) with the norm*

$$\|f\|_1 := \int |f|\, d\mu, \tag{2.32}$$

which is denoted by $L^1(X, \mathscr{A}, \mu)$ *or briefly* L^1.

Cor. 2.3.3. *If* $f \in L^1$ *and* $\int_E f\, d\mu = 0$ *for any measurable set* E, *then* $f = 0$ *a.e.*

Proof. Set $E = \{x : f(x) > 0\}$. Then $\int_E f\, d\mu = \int_E f^+\, d\mu = 0 \Rightarrow f^+ = 0$ a.e.. The same reasoning shows that $f^- = 0$ μ-a.e. $\Rightarrow f = 0$ μ-a.e. □

Theorem 2.3.3. *The Dominated Convergence Theorem (DCT)*

Let $\{f_n\}_1^\infty \subset L^1$ *such that* $f_n(x) \to f(x)$ *and* $\exists g \in L^1$ *such that* $|f_n| \leq g$ *a.e.,* $\forall n \in \mathbb{N}$. *Then* $f \in L^1$ *and*

$$\lim_{n\to\infty} \int f_n\, d\mu = \int f\, d\mu. \tag{2.33}$$

Proof.

Obviously, $|f_n| \to |f|$ pointwise. From Fatou's Lemma,

$$\int |f|\, d\mu \leq \liminf_{n\to\infty} \int |f_n|\, d\mu \leq \int g\, d\mu \Rightarrow f \in L^1.$$

It follows from $|f_n| \leq g$ a.e. that

$$\left.\begin{array}{r} f_n + g \geq 0 \quad a.e. \\ -f_n + g \geq 0 \quad a.e. \end{array}\right. \in L^1.$$

Applying Fatou's Lemma,

$$\int f\, d\mu + \int g\, d\mu \leq \liminf_{n\to\infty} \int (f_n + g)\, d\mu = \lim_{n\to\infty} \int f_n\, d\mu + \int g\, d\mu. \tag{2.34}$$

$$\int -f\, d\mu + \int g\, d\mu \leq \liminf_{n\to\infty} \int (-f_n + g)\, d\mu = -\lim_{n\to\infty} \int f_n\, d\mu + \int g\, d\mu. \tag{2.35}$$

$$\Rightarrow \int f\, d\mu = \lim_{n\to\infty} \int f_n\, d\mu. \tag{2.36}$$

□

2.4 Modes of covergence

Def. 2.4.1. *$\forall f \in L^1$, we defined the L^1-norm of f by $\|f\|_{L_1} := \int |f|\, d\mu$, (c.f. Remark 2.3.3) which induces distance $\rho(f,g) = \|f-g\|_{L1}$ between two functions in L^1.*

Def. 2.4.2. *A sequence $\{f_n\}_1^\infty$ of functions in L^1 is called Cauchy in L^1 if $\|f_n - f_m\|_{L_1} \to 0$ as m, $n \to \infty$ and $\{f_n\}_1^\infty$ converges to f in L^1 if $\|f_n - f\| \to 0$ as $n \to \infty$.*

Remark 2.4.1. *$\rho(f,g) = 0 \Leftrightarrow f = g$ a.e..*

EX. 2.4.1.

1. *$f_n(x) = \frac{1}{n}\chi_{(0,n)}(x)$, converges uniformly 0.*
2. *$f_n(x) = \chi_{(n,n+1)}(x)$, pointwise 0.*
3. *$f_n(x) = n\chi_{[0,\frac{1}{n}]}(x)$, a.e. 0.*
4. *$f_1(x) = \chi_{[0,1]}(x)$,*
   $f_2(x) = \chi_{[0,\frac{1}{2}]}(x)$, $f_3 = \chi_{[\frac{1}{2},1]}(x)$,
   $f_4(x) = \chi_{[0,\frac{1}{4}]}(x)$, $f_5(x) = \chi_{[\frac{1}{4},\frac{1}{2}]}(x)$, ...
   ...
   $f_n \to 0$ in L^1.

Def. 2.4.3. *A sequence $\{f_n\}_1^\infty$ of measurable functions in measure space $(X, \mathscr{A}, \mu)$ is called Cauchy in measure if* $\forall \epsilon > 0$,

$$\mu\{x : |f_n(x) - f_m(x)| > \epsilon\} \to 0 \quad as \quad n, m \to \infty,$$

and $\{f_n\}_1^\infty$ converges to f in measure if

$$\mu\{x : |f_n(x) - f(x)| > \epsilon\} \to 0 \quad as \quad n \to \infty.$$

Remark 2.4.2. *Example 2.4.1 1, 3, 4, $f_n \to 0$ in measure, but in 2$\{f_n\}$ is not Cauchy in measure.*

Theorem 2.4.1. *Suppose that $\{f_n\}_1^\infty$ is Cauchy in measure. Then $\exists$ a measurable function f such that $f_n \to f$ in measure and $\exists$ a subsequence $\{f_{n_j}\}_{j=1}^\infty$ of $\{f_n\}_1^\infty$ such that $f_{n_j} \to f$ a.e.*

Moreover, if also $f_n \to g$ in measure, then $f = g$ a.e.

Proof.

Using the fact that $\{f_n\}_1^\infty$ is Cauchy in measure, we can find a subsequence $\{f_{n_j}\}(\triangleq g_j)$ of $\{f_n\}$, such that $\forall j$,

$$E_j := \{x : |g_j(x) - g_{j+1}(x)| > \frac{\delta}{2^j}\} \in \mathscr{A}, \tag{2.37}$$

then $\mu(E_j) \le \frac{1}{2^j}$.

Set $F_k = \bigcup_{j=k}^\infty E_j \in \mathscr{A} \Rightarrow \mu(F_k) \le \sum_{j=k}^\infty \frac{1}{2^j} = \frac{1}{2^{k-1}}$. If $x \notin F_k$, then $\forall n > m > k$,

$$\begin{aligned} |g_m(x) - g_n(x)| &= |\sum_{j=m}^{n-1}(g_j(x) - g_{j+1}(x))| \le \sum_{j=m}^{n-1} |g_j(x) - g_{j+1}(x)| \\ &\le \sum_{j=m}^{n-1} \frac{\delta}{2^j} \le \sum_{j=m}^{\infty} \frac{\delta}{2^j} = \frac{\delta}{2^{m-1}}, \end{aligned} \tag{2.38}$$

which implies $\{g_j(x)\}$ is a Cauchy sequence for each $x \notin F_k$. $k \in \mathbb{N}_+$.Let $F = \bigcap_{k=1}^\infty F_k$. Then $\mu(F) = 0$. Define [5]:

$$f(x) = \begin{cases} \lim_{j\to\infty} g_j(x) & x \notin F \\ 0 & x \in F \end{cases}. \tag{2.39}$$

$\Rightarrow f$ measurable.

[5] $x \notin F \implies x \in F^c = \bigcup_1^\infty F_k^c \implies x \in F_{k_1}^c \implies x \notin F_{k_1}$ and g Cauchy $\implies \lim g_j \exists$.

$\forall x \notin F \Rightarrow x \notin F_k, \exists k$. From the above inequality, by taking $n \to \infty$, we have $|g_j(x) - f(x)| \leq \frac{\delta}{2^j} \Rightarrow 0$ as $n \to \infty \Rightarrow x \notin F_k$.

Note that $\forall \varepsilon > 0$,

$$\begin{aligned} &\{x : |f_n(x) - f(x)| \geq \varepsilon\} \subset \\ &\{x : |f_n(x) - f_{n_j}(x)| \geq \frac{\varepsilon}{2}\}(\to 0 \text{ as } n, n_j \to \infty) \\ &\cup \{x : |f_{n_j}(x) - f(x)| \geq \frac{\varepsilon}{2}\}(\to 0 \text{ as } n_j \to \infty). \end{aligned} \tag{2.40}$$

We deduce that $\mu\{x : |f_n(x) - f(x)| \geq \varepsilon\} \to 0$ as $n \to \infty$, $\Rightarrow f_n \to f$ in measure.

Suppose $f_n \to f$, $f_n \to g$ in measure. Likewise, thanks to the $\forall \varepsilon > 0$,

$$\begin{aligned} &\{x : |f(x) - g(x)| \geq \frac{2}{n}\} \subset \\ &\{x : |f(x) - f_k(x)| \geq \frac{1}{n}\} \cup \{x : |g(x) - f_k(x)| \geq \frac{1}{n}\}, \end{aligned} \tag{2.41}$$

$\exists N \in mathbbN_+$, s.t. $\forall k > N$, $\mu(\{x : |f(x) - g(x)| \geq \frac{2}{n}\}) < \frac{\varepsilon}{2^n}$. It follows from $\{x : |f(x) - g(x)| \geq 0\} = \bigcup_1^\infty \{x : |f(x) - g(x)| \geq \frac{2}{n}\}$ that [6] $f = g$ a.e.. □

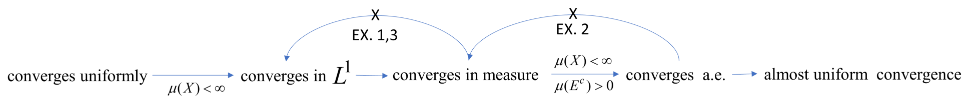
Prop. 2.4.1. *A Cauchy sequence in L^1 is Cauchy in measure. Moreover, if $f_n \to f$ in L^1, then $f_n \to f$ in measure.*

Proof.

Let $E_{n,\varepsilon} := \{x : |f_n(x) - f(x)| \geq \varepsilon\}$, $\forall \varepsilon > 0$.

$$\|f_n - f\|_{L^1} = \int |f_n - f| d\mu \geq \int_{E_{n,\varepsilon}} |f_n - f| d\mu > \varepsilon \mu(E_{n,\varepsilon}), \tag{2.42}$$

$\Rightarrow \mu(E_{n,\varepsilon}) \leq \frac{1}{\varepsilon} \|f_n - f\|_{L^1} \to 0$ as $n \to \infty$. □



Cor. 2.4.1. *If $f_n \to f$ in L^1, then $\exists$ subsequence $\{f_{n_j}\}$ of $\{f_n\}$, $f_{n_j} \to f$ a.e..*

[6] $\mu(\bigcup_1^\infty \{x : |f(x) - g(x)| \geq \frac{2}{n}\}) \leq \sum_{n=1}^\infty \frac{\varepsilon}{2^n} = \varepsilon$

Theorem 2.4.2. *Egoroff's Theorem*

Suppose that $\mu(X) < \infty$ and f_n and f measurable functions $f_n \to f$ a.e., then $\forall \varepsilon > 0$, $\exists$ measurable $E \subset X$ with $\mu(E) \leq \varepsilon$, s.t. $f_n \rightrightarrows f$ uniformly on E^c.

Proof.

WLOG, assume that $f_n \to f$ everywhere on X. $\forall n, k \in \mathbb{N}_+$, let $E_{n,k} := \{x : |f_n(x) - f(x)| > \frac{1}{k}\}$.

Fix $k \in \mathbb{N}$, it is easy to see that $E_{n,k} \downarrow$ as $n \to \infty$ and $\mu\left(\bigcap_{n=1}^{\infty} E_{n,k}\right) = 0$[7].

Thanks to $\mu(X) < \infty$, we have $\forall \varepsilon > 0$, $k \in \mathbb{N}_+$, $\exists n_k \in \mathbb{N}_+$, s.t. $\mu(E_{n_k,k}) < \frac{\varepsilon}{2^k}$. Let $E := \bigcup_{k=1}^{\infty} E_{n_k,k} \Rightarrow \mu(E) \leq \varepsilon$.

$\forall x \notin E$, $x \notin E_{n_k,k}$. $\forall k \in \mathbb{N}_+$, $\Rightarrow \forall \delta > 0$, $\exists N = n_k + 1$, $|f_n(x) - f(x)| \leq \frac{1}{k} < \delta$, $\forall n > N$, which implies $f_n \rightrightarrows f$ uniformly on E^c. $\square$

Littlewood's Three Principles:

1. Every measurable set is "nearly" a union of intervals.

$m(E) = \inf\{m(U) : E \subset U \textit{ and } U \textit{ is open set}\}$.

2. Every measurable function is "nearly" continuous. "Lusin's Theorem" $1 + 3 \implies 2$, Simple $\varphi_n \to f$, $\varphi_n = \sum_{j=1}^{k-1} \chi_{E_j}$. $E_j = \cup(a_j^k, b_j^k)$.

3.Every convergent sequence of measurable function is "nearly" uniformly convergent.

"Egoroff's theorem"

[7] $\bigcap_{n=1}^{\infty} E_{n,k} = \lim_{n\to\infty} E_{n,k}$

Chapter 3

General set Function

3.1 Signed measure and their decompostions

Def. 3.1.1. *$\mathcal{L}$et $(X, \mathscr{A})$ be a measurable space. A signed measure is a set Function*

$$\nu : \mathscr{A} \to [-\infty, +\infty) \text{ or } (-\infty, +\infty] \tag{3.1}$$

s.t.

1. *$\nu(\phi) = 0$.*
2. *Countably additive.*

Remark 3.1.1.

1. *A signed asuumes at most one of $\pm\infty$.*
2. *Every positive measure is a signed measure.*

EX. 3.1.1.

1. *If μ_1,mu_2 are positive measure, and at least one of them is finite, then $\nu := \mu_1 - \mu_2$ is a signed measure.*
2. *If $(X, \mathscr{A}, \mu)$ is a measure space and $f : X \to [-\infty, +\infty]$ is a $\mathscr{A}$-measurable function, s.t. at least one of $\int f^+ \, d\mu$ and $\int f \, d\mu$ is finite, in which case, f is said to be extended integrable. Then the set function $\nu(E) := \int_E f \, d\mu$ is a signed measure.*

Def. 3.1.2. *We call the signed measure $\nu(E) := \int_E f \, d\mu$ the indefinite integral of an extended integrable function f, for which at least one of $\int f^+ \, d\mu$, $\int f^- \, d\mu$ is finite.*

Remark 3.1.2. *We shall see shortly that these are actually the only examples, i.e., every signed measure can be represented as one of these forms.*

Def. 3.1.3. *Let ν be a signed measure on a measurable space $(X, \mathscr{A})$. A set $E \in \mathscr{A}$ is said to be positive if $\forall F \subset E$ and $F \in \mathscr{A}$, $\nu(F) \geq 0$ (negative if $\forall F$, $\nu(F) \leq 0$).*

For a signed measure, a set $E \in \mathscr{A}$ is ν-null if E is both positive and negative for ν.

EX. 3.1.2. *In example 3.1.1 2, $\nu(E) = \int_E f \, d\mu$. E is positive if $f \geq 0$ on E a.e. for μ (or negative if $f \leq 0$ μ-a.e.).*

Lemma 3.1.1. *Any measurable subset of a positive set is positive. A countable union of positive sets is positive.*

Proof.

The first assertion is obvious from the definition of positivity.

For the second one, if $E_1, E_2, \ldots$ are positive, we set

$$F_n = E_n \setminus \bigcup_{k=1}^{n-1} E_k. \tag{3.2}$$

Then $\bigcup_1^n F_k = \bigcup_1^n E_k, \forall n \in \mathbb{N}_+$ and $F = \bigcup_1^\infty F_n = \bigcup_1^\infty E_n \in \mathscr{A}$.

$$\nu(F) = \nu(F \cap \left(\bigcup_1^\infty F_n \right)) = \sum_{n=1}^\infty \nu(F \cap F_n) \geq 0 \implies \bigcup_1^\infty E_n \geq 0.$$

□

Theorem 3.1.1. *Hahn Decomposition Theorem*

If ν is a signed measure on a measurable space $(X, \mathscr{A})$, then there exists a positive set P and a negative set N for ν, such that $P \cup N = X$ and $P \cap N = \emptyset$. If P' and N' is another such pair, then $P \Delta P'$ and $N \Delta N'$ are ν-null.

Proof.

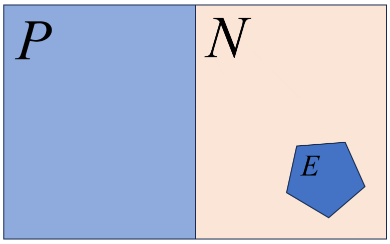
WLOG, assume that ν does not take the value ∞ (otherwise, consider $-\nu$).

Let $M := \sup\{\nu(A) : A \text{ is positive}\}$. There exists a sequence $\{P_j\}_{j=1}^\infty$ of positive sets, such that $\nu(P_j) \to M < \infty$.

Set $P = \bigcup P_j$, then $P \geq 0$ from the above lemma.

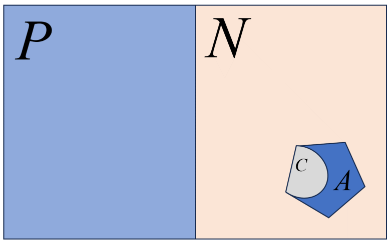
Claim: $X \setminus P$ is negative for ν.

Proof Claim: Assume that $N := X \setminus P$ is not negative.



Step 1: Note that N cannot contain any nonnull positive set. If not, there exists $E \subset N$, and E is positive, then $E \cup P \geq 0$ and $\nu(E \cup P) = \nu(E) + \nu(P) > M + M$, which is impossible.

Step 2: If $A \subset N$ and $A \in \mathscr{A}$, $\nu(A) > 0$, then there exists $B \subset A$, such that $B \in \mathscr{A}$ and $\nu(B) > \nu(A)$.



Indeed, A cannot be positive from Step 1, then there exists $C \subset A$, such that $C \in \mathscr{A}$ and $\nu(C) < 0$.

Set $B := A \cap C$, $A := B \backslash C$ with $B \cap C = \emptyset$. $\nu(B) = \nu(A) - \nu(C)$.

Step 3: We can find a sequence $\{A_j\}$ of measurable subsets of N and a sequence $\{n_j\} \subset \mathbb{N}^+$ as follows:

1. There exists n_1 which is the smallest integer such that there exists $B \subset N$ with $B \in \mathscr{A}$, such that $\nu(B) > \frac{1}{n_1}$ and A_1 is such a set ($A_1 := B$).

2. $\exists n_2$ is the smallest integer, s.t. $\exists B \subset A_1$ with $B \in \mathscr{A}$, s.t. $\nu(B) > \nu(A_1) + \frac{1}{n_2}$ (By step 2, $\nu(B) > \nu(A_1)$, $\exists B \in \mathscr{A}$, and A_2 is such a set ($A_2 := B$).

   ...

j. proceeding inductively, $\exists n_j$ is the smallest integer, s.t. $\exists B \subset A_{j-1}$ with $B \in \mathscr{A}$, s.t. $\nu(B) > \nu(A_{j-1}) + \frac{1}{n_j}$ and A_j is such a set ($A_j := B$).

Step 4: Let $A := \bigcap_{j \in \mathbb{N}_+} A_j \subset N \implies A \in \mathscr{A}$. Continue from the previous step: $+\infty > \nu(A) = \nu\left(\lim_{j\to\infty} A_j\right) \overset{continuous}{=} \lim_{j\to\infty} \nu(A_j) > \lim_{j\to\infty} \sum_{k=1}^{j} \frac{1}{n_k} = \sum_{k=1}^{\infty} \frac{1}{n_k} \implies n_k \to \infty$ as $k \to \infty$.

$\exists n \in \mathbb{N}$, s.t. $\exists B \subset A$ with $B \in \mathscr{A}$, $\nu(B) > \nu(A) + \frac{1}{n}$ [1]. Using $n_k \to \infty$ as $k \to \infty$ for k sufficiently large, we have $n < n_k$, which contradicts the definition of n_k. $\nu(B) > \nu(A_{k-1}) + \frac{1}{n_k}$. n_k is the smallest positive integer satisfying $\nu(B) > \nu(A_{k-1}) + \frac{1}{n_k}$ for k.

Hence, N is negative. Claim proof done.

Finally, suppose that P', N' is another such pair as in the statement of the theorem. Note that $P \backslash P' \subset P$ and $P \backslash P' \subset N' \implies P \backslash P'$ is both positive and negative for ν. $\implies P \setminus P$ is ν-null. Likewise, $P' \backslash P, N \backslash N', N' \setminus N$ are null for ν. $\square$

Def. 3.1.4. *The decomposition $X = P \cup N$ as a union of a positive set and a negative set for ν is called a Hahn decomposition for ν.*

Def. 3.1.5. *We say that two signed measures μ, ν on a measurable space $(X, \mathscr{A})$ are mutually singular, or that ν is singular with respect to μ or vice versa, denoted by $\mu \perp \nu$, if $\exists E, F \in \mathscr{A}$ with $E \cup F = X$, $E \cap F = \phi$ s.t. E is null for ν and F is null for μ.*

Theorem 3.1.2. *The Jordan decomposition Thm:*

If ν is a signed measure, then $\exists$ two positive measures ν^+, ν^- s.t. $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

Proof.

$X = P \cup N$ is a Hahn decomposition for ν.

Define $\nu^+(A) := \nu(A \cap P)$, $\nu^-(A) := -\nu(A \cap N)$,$\forall A \in \mathscr{A}$. Clearly, $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

If also, $\nu = \mu^+ - \mu^-$ satisfying $\mu^+ \perp \mu^-$, where μ^+, μ^- are positive measures. Then $\exists E, F \in \mathscr{A}$ s.t. $X = E \cup F$ and $E \cap F = \phi$, E is null for μ^- and F is null for μ^+. $\implies E, F$ is another Hahn decomposition. $\implies E \triangle P$ and $F \triangle N$ are both null for ν.

$$\begin{aligned} &\nu^+(A) = \nu(A \cap P) = \nu(A \cap E) = \mu^+(A \cap E) \\ &= \mu^+(A \cap E) + \underbrace{\mu^+(A \cap F)}_{=0} = \mu^+(A) \\ &\implies \nu^+(A) = \mu^+(A) \end{aligned} \tag{3.3}$$

Likewise, we have $\nu^- \equiv \mu^-$. $\square$

Def. 3.1.6. *The positive measure ν^+, ν^- as above are called positive and negative variations of ν and $\nu = \nu^+ - \nu^-$ is called the Jordan decomposition for ν.*

[1] $\nu(A) > \nu(A_{k-1})$

Analog: Any bounded variation function is a difference of two non-decreasing functions. Moreover, define

$$|\nu| = \nu^+ + \nu^-$$

the total variation of ν.

3.2 Radon-Nikdogym theorem

Def. 3.2.1. *Suppose that ν is a signed measure and μ is a positive measure on a measurable space $(X, \mathscr{A})$. We say that ν is absolutely continuous w.r.t. μ, denoted by $\nu \ll \mu$, if $\forall E \in \mathscr{A}$ with $\mu(E) = 0$, we have $\nu(E) = 0$.*

Remark 3.2.1.

1. $\nu^+ \ll \mu, \nu^- \ll \mu$.

   $\mu(E) = 0 \implies \mu(E \cap P) = 0 \implies \nu(E \cap P) = 0 = \nu^+(E) \implies \nu^+ \ll \mu$.

2. $\nu \ll \mu \iff \nu^+ \ll \mu$ *and* $\nu^- \ll \mu \iff |\nu| \ll \mu$.

Prop. 3.2.1. *$\mathcal{L}$et ν be a finite signed measure and μ be a positive measure on a measurable space $(X, \mathscr{A})$. Then $\nu \ll \mu$ iff $\forall \varepsilon > 0, \exists \delta > 0$ such that $|\nu(E)| \leq \varepsilon$ whenever $\mu(E) < \delta$.*

Proof.
Since $\nu \ll \mu$ iff $|\nu| \ll \mu$, and $|\nu(E)| \leq |\nu|(E)$, it suffices to assume that $\nu = |\nu|$ is positive.

"$\Leftarrow$". If not true, $\exists E \in \mathscr{A}$ with $\mu(E) = 0$ but $\nu(E) \neq 0$.

Clearly, "$\varepsilon - \delta$ condition" $\Rightarrow \nu \ll \mu$.

"$\Rightarrow$". On the other hand, suppose that $\nu \ll \mu$, but "$\varepsilon - \delta$ condition" does not hold.

Then $\exists \varepsilon_0 > 0, \forall n \in \mathbb{N}$, we can find $E_n \in \mathscr{A}$ such that $\mu(E_n) < \frac{1}{2^n}$ and $\nu(E_n) \geq \varepsilon_0$. Set $F_k = \bigcup_{n \geq k} E_n$, $F = \bigcap_{k=1}^{\infty} F_k \Rightarrow \mu(F) = 0$ [2].

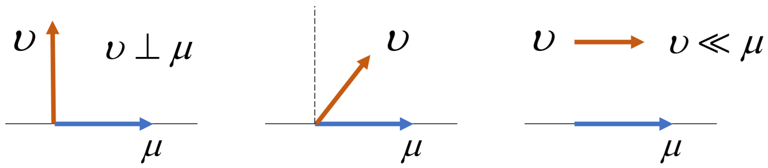
However, $\nu(F_k) \geq \varepsilon_0$, $\forall k \in \mathbb{N}_+$. Hence, $\nu(F) \overset{\text{since } \mu \text{ is finite}}{=} \lim_{k \to \infty} \nu(F_k) \geq \varepsilon_0$. Contradiction. $\square$

Cor. 3.2.1. *If $f \in L^1(\mu)$, then $\forall \varepsilon > 0, \exists \delta > 0$ such that $\int_E f d\mu < \varepsilon$ whenever $\mu(E) < \delta$. ($\nu(E) = \int_E f d\mu$)*

[2] $\mu(F_k) \leq \sum_{n=k}^{\infty} \frac{1}{2^n} = \frac{1}{2^{k-1}}$

Def. 3.2.2. *We use the notation $d\nu = f d\mu$ to express the relationship $\nu(E) = \int_E d\nu = \int_E f\ d\mu$, $\forall E \in \mathscr{A}$.*

Lemma 3.2.1. *Suppose that ν and μ are both finite positive measures on a measurable space $(X, \mathscr{A})$. Then either $\nu \perp \mu$, or $\exists \varepsilon > 0$ and $A \in \mathscr{A}$, s.t. $\mu(A) > 0$ and $\nu - \varepsilon\mu$ is positive on A, i.e. A is positive for $\nu - \varepsilon\mu$.*



Proof.
$\forall n \in \mathbb{N}_+$, note that $\nu - \frac{1}{n}\mu$ is a signed measure. $\exists$ Hahn decomposition $X = P_n \cup N_n$ for $\nu - \frac{1}{n}\mu$. It is easy to check $P_n \uparrow$ and $N_n \downarrow$.

Let $P := \bigcup_1^\infty P_n$ and $N := \bigcap_1^\infty = N_n$. Then N is negative for $\nu - \frac{1}{n}\mu$, $\forall n \in \mathbb{N}_+$.

$\implies \nu(N) \leq \frac{1}{n}\mu(N)$, $\forall n \in \mathbb{N}_+$. $\implies \nu(N) = 0$ ($n \to \infty$, μ bdd.).

If $\mu(P) = 0$, then $\mu \perp \nu$.

Otherwise, if $\mu(P) > 0$[3], then $\exists n_0 \in \mathbb{N}_+$, $\mu(P_{n_0}) > 0$, and $A = P_{n_0}$ is positive for $\nu - \frac{1}{n_0}\mu$ from defintion. □

Theorem 3.2.1. Radon Nikdogym Theorem

Let $(X, \mathscr{A}, \mu)$ be a σ-finite positive measure space. If ν is a signed measure on $(X, \mathscr{A})$ and $\nu \ll \mu$, then there exists a measurable function $f : X \to \mathbb{R}$ such that

$$\nu(E) = \int_E f\, d\mu, \quad \forall E \in \mathscr{A} \quad (\Leftrightarrow d\nu = f d\mu).$$

Moreover, the function is unique in the sense that any such two functions are equal μ-a.e..

Proof.
Case I: Suppose that ν and μ are both finite positive measures. Denote

$$F := \{\text{measurable } f : X \to \mathbb{R} \mid \int_E f\, d\mu \leq \nu(E), \forall E \in \mathscr{A}\}. \tag{3.4}$$

Obviously, F is nonempty since at least $0 \in F$.

Claim: If $f, g \in F$, then $h := \max\{f, g\} \in F$.

[3] $\mu(P) = \lim_{n\to\infty} \mu(P_n)$

Proof Claim: Write

$$A := \{x \in X \mid f(x) > g(x)\}. \tag{3.5}$$

For all $E \in \mathscr{A}$,

$$\int_E h\, d\mu = \int_{E\cap A} f\, d\mu + \int_{E\setminus A} g\, d\mu \le \nu(E\cap A) + \nu(E\setminus A) = \nu(E). \tag{3.6}$$

$\Rightarrow h \in F$.

Let $a := \sup\left\{\int_E f d\mu \mid f \in F\right\}$ $(\le \nu(E) < +\infty)$. We can find a sequence $\{f_n\} \subset F$, s.t. $\int_E f_n d\mu \to a$ $(n \to \infty)$.

Set $g_n(x) := \max\{f_1(x), f_2(x), \dots, f_n(x)\}$ and $f := \sup_n f_n$. Then $g_n \uparrow f$. Which implies $\forall E \in \mathscr{A}, \int_E f_n d\mu \le \int_E g_n d\mu \le \nu(E)$. By MCT (c.f. theorem 2.3.1), $\int_E f d\mu \le \nu(E) \implies f \in F$.

In particular, $\int f_n d\mu \le \int g_n d\mu \le a$.

$$a = \lim_{n\to\infty} \int f_n d\mu \le \int f d\mu \le a \implies \int f d\mu = a. \tag{3.7}$$

The claim is proven.

Claim: $\nu_0(E) := \nu(E) - \int_E f d\mu \equiv 0$ $, \forall E \in \mathscr{A}$.[4]

Proof Claim: According to the above lemma 3.2.1, either $\nu_0 \ll \mu$.

Noting that $\nu_0 \ll \mu$, we have $\nu_0 \equiv 0$. Done.

Otherwise[5], $\exists \epsilon_0 > 0, A \in \mathscr{A}$, s.t. $\mu(A) > 0$ and $\varepsilon_0\mu(A\cap E) \le \nu_0(A\cap E) = \nu(A\cap E) - \int_{A\cap E} f d\mu$ $, \forall E \in \mathscr{A}$.

Let $g = f + \varepsilon_0\chi_A$. $\int_E g d\mu = \varepsilon_0\mu(A\cap E) + \int_E f d\mu \nu(E\setminus A) + \nu(E\cap A) = \nu(E)$.[6]

$\Rightarrow g \in F$.

However, $\int_E g d\mu = \int(f + \varepsilon_0\chi_A)d\mu = a + \varepsilon_0\mu(A) > a$, which is a contradiction.

The claim is proven.

Uniqueness. If also $\nu(E) = \int_E h d\mu$ $, \forall E \in \mathscr{A}$.

$$\Rightarrow \int_E (f-h)d\mu = 0 \ \forall E \in A.$$

$$\Rightarrow f = h\ \mu - a.e. \text{ from the previous Cor.}$$

[4] ν_0 is positive.
[5] $(\exists \varepsilon_0 > 0, A \in \mathscr{A}$, s.t. $\mu(A) > 0$ and $(\nu_0 - \varepsilon_0\mu)(A) \ge 0$ on $A)$.
[6] $\int_E f d\mu = \int_{E\setminus A} f d\mu + \int_{E\cap A} f d\mu \le \nu(E\setminus A) + \nu(E\cap A) - \varepsilon_0\mu(E\cap A)$.

Case II: Suppose that ν, μ are σ-finite positive measures. The X is countable union of disjoint ν-finite (μ-finite) set. $X = \bigcup B_i$: ν-finite, $X = \bigcup C_j$: μ-finite.

By taking intersections, we can obtain a disjoint sequence $\{A_n\}$, s.t. $\nu(A_n), \mu(A_n)$ are both finite, where $A_{i,j} = \bigcap B_i \cap C_j$.

Set $\nu_n(E) = \nu(E \cap A_n)$, $\mu_n(E) = \mu(E \cap A_n)$ $\forall E \in \mathscr{A}$. Obviously, ν_n, μ_n are both finite positive measures on X.

For the result in Case I, $\exists$ measurable function $f_n : X \to \mathbb{R}$ ($f_n : A_n \to \mathbb{R}, f_n : X \backslash A_n \to \{0\}, f_n = 0$, μ-a.e. on A_n^c) s.t. $\nu_n(E) = \int_E f_n d\mu_n$. Let $f = \sum_{n=1}^{\infty} f_n$. Then $\nu(E) = \int_E f d\mu$ as desired. The uniqueness follows easily from Case I.

Case III: In general, if ν is a signed and σ-finite measure. By Jordan Decomposition, we have $\nu = \nu^+ + \nu^-$.

Applying the preceding argument, and subtracting the result.

We complete the proof.

□

Remark 3.2.2. *If μ is not σ-finite, then the R-N is not true. For example, let $X = [0,1]$, $\mathscr{A} = \mathscr{L}$ the collection of all Lebesgue measurable sets. $\nu = m$ Lebesgue measure, and $\mu = \#$ (not σ-finite) the counting measure.*

Suppose there $\exists f$, $m(E) = \int_E f d\#$. Taking $E = \{x\}$. $0 = m(\{x\}) = \int_{\{x\}} f d\# = f(x) \Rightarrow f(x) \equiv 0$ on $[0,1] \Rightarrow m \equiv 0$ on $[0,1]$, which is impossible.

3.3 Derivatives of meausres

Def. 3.3.1. *The unique (in the μ-a.e. sense) function f determined by R-N Theorem is called R-N derivative of ν w.r.t. μ, denoted by $\frac{d\nu}{d\mu} = f$ or $d\nu = f\, d\mu$, which is also called density function in probability Theory.*

Remark 3.3.1.

- *$\frac{d\nu}{d\mu}$ should be considered as a class of functions equal to f μ-a.e.*
- *The formulas suggested by differential notation are generally correct.*

For example,

$$\frac{d(\alpha\nu_1 + \beta\nu_2)}{d\mu} = \alpha\frac{d\nu_1}{d\mu} + \beta\frac{d\nu_2}{d\mu} \tag{3.8}$$

Theorem 3.3.1. *The Chain Rule.*

If λ and μ are σ-finite positive measures satisfying $\lambda \ll \mu$ on a measurable space $(X, \mathscr{A})$, and ν is σ-finite signed measure on $(X, \mathscr{A})$ s.t. $\nu \ll \lambda$. Then

$$\frac{d\nu}{d\mu} = \frac{d\nu}{d\lambda}\,\frac{d\lambda}{d\mu}. \tag{3.9}$$

Proof.

Denote $f := \frac{d\nu}{d\lambda}$ and $g := \frac{d\lambda}{d\mu}$. For simplicity, by considering ν^+, ν^- separately, we assume that ν is positive. It follows easily that $f \geq 0$ $\lambda - a.e.$ $(\nu(E) = \int_E f d\lambda \ , \nu \geq 0 \ , \lambda \geq 0)$.

WLOG, we may assume that $f \ , \ g \geq 0$ everywhere. Let $\{f_n\}$ be an increasing sequence of simple functions converging to f pointwise. By MCT (c.f. theorem 2.3.1),

$$\lim_{n\to\infty} \int_E f_n d\lambda = \int_E f d\lambda \tag{3.10}$$

and

$$\lim_{n\to\infty} \int_E f_n g d\mu = \int_E f g d\mu \text{ (where } f_n = \sum_{j=1}^{k} c_j \chi_{E_j}) \tag{3.11}$$

$\forall F \in \mathscr{A}$, $\int_E \chi_F d\lambda = \lambda(E \cap F) = \int_{E\cap F} g d\mu = \int_E \chi_F g d\mu$, $\Rightarrow \int_E f_n d\lambda = \int_E f_n g d\mu$,

$$\overset{n\to\infty}{\Rightarrow} \int_E f d\lambda = \int_E f g d\mu \ \forall E \in \mathscr{A}. \tag{3.12}$$

$\square$

Cor. 3.3.1. *Change of variable*

If λ, μ are σ-finite positive measures, s.t. $\mu \ll \lambda$ and f is a $\overline{\mathbb{R}}$-valued measurable function, s.t. $\int f d\lambda$ is defined (extended integrable), then

$$\int f d\mu = \int f \frac{d\mu}{d\lambda} d\lambda. \tag{3.13}$$

EX. 3.3.1.

1. Let m be Lebesgue measure, and ν be a σ-finite signed measure on $\mathbb{R}$. If $E_x = [a,x], F(x) = \nu(E_x)$ then $\frac{d\nu}{dm} = F'(x)$. In addition, $\nu \ll m$ and $\nu(E_x) = \int_{E_x} F' \, dm = F(x) - F(a)$.

2. Let $(\mathbb{N}^+, 2^{\mathbb{N}^+}, \#)$ be the counting measure space. If we define $\nu(\{n\}) = \frac{1}{2^n}$, then $\#$ is σ-finite, ν is finite, and $\nu \ll \#$. Hence, $\nu(E) = \int_E f \, d\#$ for some measurable function f, according to R-N Theorem (c.f. theorem 3.2.1).

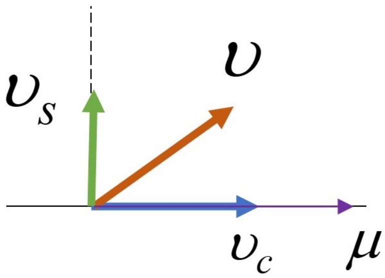
$$\frac{d\nu}{d\#}(n) = f(n) = \frac{1}{2^n}, \ \forall n \in \mathbb{N}^+. \tag{3.14}$$

In general, if $\mu(\{n\}) = a_n \geq 0$, $\forall n \in \mathbb{N}^+$ and $\nu(\{n\}) = b_n$, then $\nu \ll \mu$ iff $\nu(\{n\}) = 0$, $\forall n \in \mathbb{N}^+$ for which $a_n = 0$. Similarly,

$$\frac{d\nu}{d\mu}(n) = f(n) = \frac{b_n}{a_n}, \ a_n \neq 0, \ \forall n \in \mathbb{N}^+. \tag{3.15}$$

Theorem 3.3.2. *The Lebesgue Decomposition Theorem*

Let ν be a σ-finite signed measure and μ be a σ-finite positive measure on a measurable space $(X, \mathscr{A})$. Then $\exists!$ pair of two σ-finite signed measures ν_s and ν_c on $(X, \mathscr{A})$ s.t. $\nu = \nu_s + \nu_c$. Where $\nu_s \perp \mu$ and $\nu_c \ll \mu$.



Proof.

Assume first that ν and μ are finite. Since we may treat $\nu = \nu^+ - \nu^-$ separately. WLOG, also assume that ν is positive.

Note that $\nu \ll \mu + \nu$ $((\mu+\nu)(E) = 0 \Rightarrow \nu(E) = 0)$, $\forall E \in \mathscr{A}$. Since μ and ν are positive measures. Then by R-N theorem (c.f. theorem 3.2.1), $\exists$ a measurable function f, s.t. $\nu(E) = \int_E f \, d(\mu+\nu)$,$\forall E \in \mathscr{A}$.

Using the fact that, ν, μ are positive and $\nu(E) \leq (\mu+\nu)(E)$, $\forall E \in \mathscr{A}$, We deduce that

1. $0 \leq f \leq 1$, $(\mu+\nu)$-a.e..
2. $0 \leq f \leq 1$, ν-a.e..

1: If not, $\nu(F) = \int_F f \, d(\mu+\nu) > \int_F d(\mu+\nu) = (\mu+\nu)(F)$, $\exists F \in \mathscr{A}$. Contradiction.

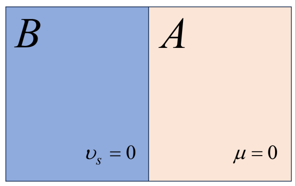
2: $f > 1$ on F, $(\mu+\nu)(F) = 0$, μ, ν positive, $\nu(F) = 0$.

Write

$$A := \{x \in X \,|\, f(x) = 1\} \in \mathscr{A}, \tag{3.16}$$

$$B := \{x \in X \,|\, 0 < f(x) < 1\} = X \backslash A \in \mathscr{A}. \tag{3.17}$$

Then $\nu(A) = \int_A 1 \; d(\mu+\nu) = \mu(A) + \nu(A)$, $\implies \mu(A) = 0$. Set $\nu_s(E) = \nu(E \cap A)$, $\nu_c(E) = \nu(E \cap B)$, $\forall E \in \mathscr{A}$. Obviously, $\nu_s \perp \nu_c$.



It remains to show that $\nu_c \ll \mu$.

$\forall E \in \mathscr{A}$ with $\mu(E) = 0$, $\int_{E\cap B} \, d\nu = \nu(E \cap B) = \int_{E\cap B} f \, (d\mu + d\nu)$. Then $\int_{E\cap B}(1 - f) \, d\nu = 0 \overset{Claim}{\Longrightarrow} \nu(E \cap B) = \nu_c(E) = 0 \implies \nu_c \ll \mu$.

Claim: If $g > 0$(measurable function) ν-a.e. on F , and $\int_F g \, d\nu = 0$, then $\nu(F) = 0$.

Proof claim:

Defined

$$E_n = \left\{ x : g(x) > \frac{1}{n} \right\} \in \mathscr{A}, \; \forall n \in \mathbb{N}. \tag{3.18}$$

Then $\frac{1}{n}\nu(E) < \int_{E_n} g \, d\nu < \int_F g \, d\nu = 0 \Rightarrow \nu(E_n) = 0$, $\forall n \in \mathbb{N}$. Since $E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots$, We deduce $\lim_{n\to\infty} \nu(E_n) = \nu\left(\lim_{n\to\infty} E_n\right) = 0$. $\Rightarrow \nu(F) = 0$. [7]

The claim is proven.

Uniqueness. Suppose that $\nu = \nu_s + \nu_c$ and $\mu_s + \mu_c$ are two Lebesgue Decompositions of ν w.r.t. μ. Note that $\underbrace{\nu_s - \mu_s}_{\perp\mu} = \underbrace{\mu_c - \nu_c}_{\ll\mu} := \lambda$. Then $\lambda = 0$, i.e. $\nu_s = \nu_s$, $\nu_c = \nu_c$. $\square$

Def. 3.3.2. *The decomposition $\nu = \nu_s + \nu_c$, where $\nu_s \perp \mu$ and $\nu_c \ll \mu$ is called Lebesgue decomposition of ν w.r.t. μ in which ν_s is called singular part and ν_c is called the absolutely continuous part of ν w.r.t. μ.*

Remark 3.3.2. *The R-N theorem is applicable for ν_c*

$$\nu = \underbrace{\int f_c \, d\mu}_{=\nu_c} + \nu_s. \tag{3.19}$$

[7] $g > 0$ ν -a.e. on $F \implies \nu(F \triangle \lim_{n\to\infty} E_n) = 0$.

Chapter 4

Condition Expectation

4.1 Condition Expectation

Def. 4.1.1. *Let $(\Omega, \mathscr{F}, P)$ be a probability space. Define conditional probability: the probability of event A given B.*

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \textit{provided } P(B) > 0. \quad (\tilde{P} = \frac{P}{P(B)} \textit{ on } \mathscr{F} \cap B)$$

Def. 4.1.2. *Let X be a random variable (r.v.), i.e., a measurable function on $(\Omega, \mathscr{F})$. Define: Expected Value (mean value) of X,*

$$E(X) = \int_\Omega X\, dP \tag{4.1}$$

Def. 4.1.3. *Given an event $B \in \mathscr{F}$ with $P(B) > 0$. Consider the new probability $\tilde{P} = \frac{P}{P(B)}$ on $\mathscr{F} \cap B$.Expected value of X over B.*

$$E(X|B) = \int_\Omega X\, d\tilde{P} = \frac{1}{P(B)} \int_B X\, dP \tag{4.2}$$

Q: What is a measurable definition of $E(X|Y)$? The expected value of X given another r.v. Y.

Consider first that Y is a simple r.v., i.e., $Y = \sum_{j=1}^n a_j \chi_{A_j}$ where $\{a_j\} \subset \mathbb{R}, \{A_j\} \subset \mathscr{F}$ with $\bigcup A_j = \Omega$ and $A_j \cap A_i = \phi$, $i \neq j$.

$$E(X|Y) = \begin{cases} \frac{1}{P(A_1)} \int_{A_1} X\, dP & \omega \in A_1 \\ \vdots \\ \frac{1}{P(A_n)} \int_{A_n} X\, dP & \omega \in A_n \end{cases} \tag{4.3}$$

Remark 4.1.1.

1. *$E(X|Y)$ is also a (simple) r.v..*

2. *$E(X|Y)$ is $\mathscr{A}_\sigma$-measurable.*

   Def. 4.1.4. *Let $X : \Omega \to \mathbb{R}$ be a r.v. Then $\mathscr{A}_\sigma(X) := \{X^{-1}(B) : B \in \mathscr{B}^{\mathbb{R}}\} \subseteq \mathscr{F}$ is the σ-field generated by X.*

3. *$\int_A X\,dP = \int_A E(X|Y)\,dP, \forall A \in \mathscr{A}_\sigma$.*

   Def. 4.1.5. *Let X, Y be two r.v. on a probability space $(\Omega, \mathscr{F}, P)$. The conditional expectation of X given Y is any $\mathscr{A}_\sigma$-measurable r.v. Z satisfying*

   $$\int_A X\,dP = \int_A Z\,dP\ , \forall A \in \mathscr{A}_\sigma(Y), \tag{4.4}$$

   denote by $E(X|Y)$.

Def. 4.1.6. *Let $(\Omega, \mathscr{F}, P)$ be a probability space. Suppose that $\mathscr{G}$ is a σ-field with $\mathscr{G} \subset \mathscr{F}$. If X is an integrable r.v., then we define $E(X|\mathscr{G})$ to be any $\mathscr{G}$-measurable r.v. Z, s.t. $\int_A X\,dP = \int_A Z\,dP$, $\forall A \in \mathscr{G}$.*

Remark 4.1.2.

1. *$E(X|Y) = E(X|\mathscr{A}_\sigma(Y))$.*

2. *$E(E(X|\mathscr{G})) = E(X)$. ($A = \Omega, \int_\Omega X\,dP = \int_\Omega E(X|\mathscr{G})\,dP$)*

3. *$E(X|\mathscr{W}) = E(X)$. $\mathscr{W} = \{\phi, \Omega\}$, the trivial σ-field.*

4. *$E(X|\mathscr{F}) = E(X)$.*

Theorem 4.1.1. *Let X be an integrable r.v. on a probability space $(\Omega, \mathscr{F}, P)$. Then $\forall\sigma$-field $\mathscr{G} \subset \mathscr{F}$, $\exists!$ $E(X|\mathscr{G})$ (unique in the p-a.s. sense).*

Proof.
$\forall G \in \mathscr{G}$, define:

$$\nu(G) = E(X\chi_G) = \int_G X\,dP \tag{4.5}$$

It is easy to check that ν is finite signed measure on $(\Omega, \mathscr{G})$

Let $\mu = P|_G$ the restrict of P on G, i.e., $\forall G \in \mathscr{G}$,

$$\mu(G) = P(G) \tag{4.6}$$

Then μ is a positive measure on (Ω, G).

It follows easily that $\nu \ll \mu$. Then, by Radon-Nikodym Theorem, $\exists!\mu$-measurable r.v. Y such that $dP = Y d\mu$ which implies $\forall G \in \mathscr{G}$,

$$\int_G X dP = \nu(G) = \int_G d\nu = \int_G Y d\mu. \tag{4.7}$$

Moreover, if $Y \in L^1(\Omega, G, P)$. If $\exists$ another such $\overline{Y}$, then$\int_G (Y - \overline{Y}) dP = 0$, $\forall G \in \mathscr{G} \implies Y = \overline{Y}$ a.s. on G. $\square$

EX. 4.1.1. ***2nd approach: The Least squares method.***

1. Consider a linear subspace $V \subset \mathbb{R}^n$. Given a vector $x \in \mathbb{R}^n$, the least squares problem is to find a vector $z \in V$, such that

$$\|z - x\| = \min_{y \in V} \|y - x\|. \tag{4.8}$$

Def. 4.1.7. *The projection of x onto V is $z = proj_V(x)$.*

$\forall v \in V$, set $i(\tau) = \|z + \tau v - x\|^2$.

$$\begin{aligned} \|z - x\| &= \min_{y \in V} |y - x| \\ \iff & 0 = i'(0) = 2\langle z - x, v\rangle, \forall v \in V \\ \iff & \langle z, v\rangle = \langle x, v\rangle \end{aligned} \tag{4.9}$$

2. Take the linear space $L^2(\Omega, \mathcal{F}, P)$ with the norm

$$\|X\| = \left(\int_\Omega |X|^2 \, dP\right)^{\frac{1}{2}}. \tag{4.10}$$

In addition, $\forall X, Y \in L^2(\Omega, \mathcal{F}, P)$, define inner product

$$\langle X, Y\rangle = \int_\Omega XY \, dP. \tag{4.11}$$

Consider a linear subspace $V = L^2(\Omega, \mathscr{G}, P) \subset L^2(\Omega, \mathscr{F}, P)$ with σ-field $\mathscr{G} \subset \mathscr{F}$. Projection $Z = proj_V(X)$,$X \in L^2(\Omega, \mathscr{F}, P)$.

Set $i(\tau) = \|Z + \tau Y - X\|^2$, $i'(0) = 0$,

$$\langle Z, Y\rangle = \langle X, Y\rangle \ , \forall Y \in V. \tag{4.12}$$

Taking $Y = \chi_A, \forall A \in \mathscr{G}$,

$$\int_A X \, dP = \overbrace{\int_A Z \, dP}^{=E(X|\mathscr{G})} \ , \forall A \in \mathscr{G}. \tag{4.13}$$

(4.12) $\overset{(4.14)}{\Longleftrightarrow}$ (4.13).
In fact, $Y_n = \sum_1^n a_j \chi_{A_j}$, $A_j \in \mathscr{G}$,$Y_n \to Y$.

$$\int_A YX\,dP = \lim_{n\to\infty} \sum_1^n a_j \int_A \chi_{A_j} X\,dP \tag{4.14}$$

Prop. 4.1.1.

1. $\forall a, b \in \mathbb{R}, X, Y$ *are r.v.s ,* $\mathscr{G}$ *is* σ*-field.*

$$E(aX + b|\mathscr{G}) = aE(X|\mathscr{G}) + bE(Y|\mathscr{G}) \quad a.s. \tag{4.15}$$

2. *If* X *is r.v. on* $(\Omega, \mathcal{F}, P)$, $\mathscr{W} = \{\phi, \Omega\}$, *then*

$$E(X|\mathscr{F}) = E(X) \quad a.s., \tag{4.16}$$

$$E(X|\mathscr{W}) = E(X) \quad a.s.. \tag{4.17}$$

3. *If* Y *is* $\mathscr{G}$*-measurable and* X, Y *are integrable, Then*

$$E(XY|\mathscr{G}) = YE(X|\mathscr{G}) \quad a.s.. \tag{4.18}$$

4. $\forall \mathscr{E} \subset \mathscr{G} \subset \mathscr{F}$,

$$E(X|\mathscr{E}) = E(E(X|\mathscr{G})|\mathscr{E}) = E(E(X|\mathscr{E})|\mathscr{G}) \quad a.s.. \tag{4.19}$$

Chapter 5

Product measures

5.1 Product measures

Def. 5.1.1. *Given two measurable spaces* $(\Omega_1, \mathscr{F}_1)$, $(\Omega_2, \mathscr{F}_2)$.

1. The Cartesian product $\Omega_1 \times \Omega_2 := \{(x_1, x_2) | x_1 \in \Omega_1, x_2 \in \Omega_2\}$.

2. A rectangle is a set of the form $A_1 \times A_2$, *where* $A_1 \in \mathscr{F}_1$ *and* $A_2 \in \mathscr{F}_2$.

3. The product σ-algebra $\mathscr{F} := \mathscr{F}_1 \times \mathscr{F}_2$ *is defined by the smallest σ-algebra containing all rectangles, i.e.,*

$$\mathscr{F} := \mathscr{A}_\sigma \left(\{A_1 \times A_2 | A_1 \in \mathscr{F}_1 \text{ and } A_2 \in \mathscr{F}_2\} \right). \tag{5.1}$$

Remark 5.1.1. *The family of all finite unions of rectangles is an algebra.*

Def. 5.1.2. *Suppose that* $(\Omega_1, \mathscr{F}_1, \mu_1)$ *and* $(\Omega_2, \mathscr{F}_2, \mu_2)$ *are measure spaces. Define the premeasure* $\mu(\bigcup_{k=1}^n A_1^k \times A_2^k)$ *as*

$$\mu \left(\bigcup_{k=1}^n A_1^k \times A_2^k \right) := \sum_{k=1}^n \mu_1(A_1^k) \mu_2(A_2^k), \tag{5.2}$$

where $A_1^k \in \mathscr{F}_1$ *and* $A_2^k \in \mathscr{F}_2$.

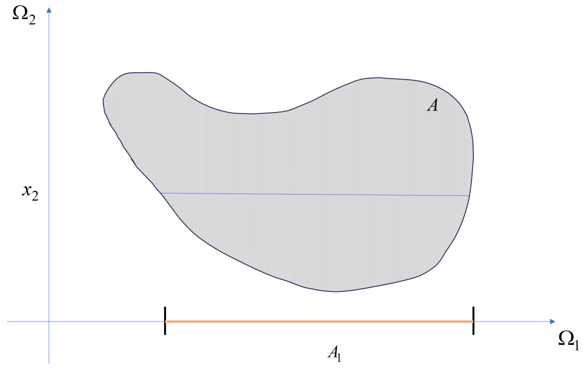
Define

$$\mathscr{A} := \left\{ \bigcup_{k=1}^n A_1^k \times A_2^k \mid A_1^k \in \mathscr{F}_1, A_2^k \in \mathscr{F}_2 \right\}. \tag{5.3}$$

If μ_1, μ_2 *are σ-finite, then* $\exists!$ *induced measure* μ *on the measurable space* $(\Omega_1 \times \Omega_2, \mathscr{F}_1 \times \mathscr{F}_2)$ *which is called the product measure.*

Def. 5.1.3. $\forall A \in \Omega_1 \times \Omega_2$, *sections are defined by*

$$A_1(x_2) := \{x_1 \in \Omega_1 | (x_1, x_2) \in A\}, \quad A_2(x_1) := \{x_2 \in \Omega_2 | (x_1, x_2) \in A\}.$$



Prop. 5.1.1. *If $A \subset \Omega_1 \times \Omega_2$, then $A_1(x_2) \in \mathscr{F}_1$,$\forall x_2 \in \Omega_2$ and $A_2(x_1) \in \mathscr{F}_2$, $\forall x_1 \in \Omega_1$. This is called the section property .*

Proof.

Let $\mathscr{A} := \{A \in \mathscr{F}_1 \times \mathscr{F}_2 | A$ has the section property$\}$. Then $\phi, \Omega_1 \times \Omega_2 \in \mathscr{A}$. Moreover, all rectangles are in $\mathscr{A}$.

It remains to show that $\mathscr{A}$ is a σ-algebra.

1. $\forall A \in \mathscr{A}, (A^c)_1(x_2) = (A_1(x_2))^c \in \mathscr{F}_1$
2. $\forall A^k \in \mathscr{A}, k \in \mathbb{N}, (\bigcup A^k)_1(x_2) = \bigcup A_1^k(x_2) \in \mathscr{F}_1$

$\implies$ $\mathscr{A}$ is a σ-algebra. We conclude that $\mathscr{A} = \mathscr{F}_1 \times \mathscr{F}_2$. □

Remark 5.1.2. *We can define $\mathscr{F} \times \mathscr{F} \times \mathscr{F}$ on $\Omega_1 \times \Omega_2 \times \Omega_3$ by induction.*

Theorem 5.1.1. *Let $(\Omega_1, \mathscr{F}_1, \mu_1), (\Omega_2, \mathscr{F}_2, \mu_2)$ be two σ-finite measure spaces. Suppose that $A \subset \Omega_1 \times \Omega_2$ is a $\mathscr{F}_1 \times \mathscr{F}_2$-measurable. Then define $u(x) = \mu_2(A_2(x))$, $\forall x \in \Omega_1$, $v(y) = \mu_1(A_1(y))$,$\forall y \in \Omega_2$. We can get u is $\mathscr{F}_1$-measurable, v is $\mathscr{F}_2$-measurable and $\int_{\Omega_1} u(x) d\mu_1 = \int_{\Omega_2} v(y) d\mu_2 = (\mu_1 \times \mu_2)(A) = \int_A d(\mu_1 \times \mu_2)$.*

Proof.

Let $\mathscr{F} = \{A \in \mathscr{F}_1 \times \mathscr{F}_2 \mid \int_{\Omega_1} u(x) d\mu_1 = \int_{\Omega_2} v(y) d\mu_2\} \subset \mathscr{F}_1 \times \mathscr{F}_2$.

It is easy to check that $\mathscr{F}$ is an algebra and a monotone class containing all rectangles. $\Rightarrow \mathscr{F} = \mathscr{M}_\sigma(\mathscr{F}) = \mathscr{A}_\sigma(\mathscr{F}) = \mathscr{F}_1 \times \mathscr{F}_2$. □

Cor. 5.1.1. *Let $(\Omega_i, \mathscr{F}_i, \mu_i)$ $(i = 1, 2, 3)$ be σ-finite measure spaces. $\forall A \subset \Omega_2 \times \Omega_2$, define $RA = \{(x, y) \mid (y, x) \in A\}$. Then if $A \in \mathscr{F}_1 \times \mathscr{F}_2$, then $RA \in \mathscr{F}_2 \times \mathscr{F}_1$.*

1. Commutative.

$$(\mu_1 \times \mu_2)(A) = (\mu_2 \times \mu_1)(RA). \tag{5.4}$$

2. Associative.

$$(\mu_1 \times \mu_2) \times \mu_3 = \mu_1 \times (\mu_2 \times \mu_3). \tag{5.5}$$

Proof.

2. Monotone class + Algebra $\Rightarrow$ σ-algebra.

Let $u : \Omega \to \mathbb{R}_+$ be a $\mathscr{F}$-measurable function in a measure space $(\Omega, \mathscr{F}, \mu)$. $\forall c \geq 0$, we consider the level set

$$E = \{(x,t) \in \Omega \times \mathbb{R}_+ \mid \chi_{\{u(x) \geq t\}} \leq c\}. \tag{5.6}$$

If $c < 0$, then $E = \phi$.
If $c \geq 1$, then $E = \Omega \times \mathbb{R}_+ \in \mathscr{F} \times \mathscr{B}^{\mathbb{R}_+}$.
If $c \in (0,1)$, $G = \{(x,t) \mid (x,t) \in \Omega \times \mathbb{R}_+, 0 \leq t < u(x)\} = E^c \in \mathscr{F} \times \mathscr{B}^{\mathbb{R}_+}$.
Note that $u(x) = \int_0^{u(x)} \, dt = \int_{\mathbb{R}_+} \chi_{[0,u(x))}(t)\, dt$. Then [1]

$$\int_\Omega u(x)\, d\mu = \int_\Omega \left(\int_{\mathbb{R}_+} \chi_{\{u(x)>t\}}(x)\, dm \right) d\mu = (\mu \times m)(G). \tag{5.7}$$

□

Theorem 5.1.2. *Fubin Theoremm*

Consider two σ-finite measure spaces $(\Omega_1, \mathscr{F}_1, \mu_1)$ and $(\Omega_2, \mathscr{F}_2, \mu_2)$.

Let u be an $\mathscr{F}_1 \times \mathscr{F}_2$-measurable function on $\Omega_1 \times \Omega_2$. If u is extended integrable, then

$$\begin{aligned} &\int_{\Omega_1 \times \Omega_2} u(x,y)\ d\mu_1 d\mu_2 \\ &= \int_{\Omega_2} \left(\int_{\Omega_1} u(x,y)\ d\mu_1 \right)\ d\mu_2 = \int_{\Omega_1} \left(\int_{\Omega_2} u(x,y)\ d\mu_2 \right)\ d\mu_1. \end{aligned} \tag{5.8}$$

Proof.

Similarly, $\int_{\Omega_1 \times \Omega_2} u(x,y)\, d\mu_1 d\mu_2 = (\mu_1 \times \mu_2 \times m)(G) = (\mu_1 \times \mu_2 \times m)(G) = \int_{\Omega_1} \left(\int_{\Omega_2} u(x,y)\ d\mu_2 \right)\ d\mu_1 = (\mu_2 \times \mu_1 \times m)(G)$. Where $G := \{(x,y,t) | 0 \leq t < u(x,y)\}$.

□

[1] $\chi_{[0,u(x))}(t) = \chi_{\{u(x)>t\}}(x)$

Chapter 6

Useful inequalities

6.1 Useful inequalities

Convexity

$$v\left(\frac{a+b}{2}\right) \le \frac{1}{2}(v(a)+v(b)), \quad \forall a,b \in \mathbb{R}$$

Def. 6.1.1. *A function $v(x)$ is said to be convex if:*

$$v\left(ta+(1-t)b\right) \le tv(a)+(1-t)v(b), \quad \forall t \in [0,1]$$

Remark 6.1.1.

1. *If $v(x)$ is convex on $[a,b]$ (or $\mathbb{R}$), then v is Lipschitz continuous on (a,b) (or $\mathbb{R}$). i.e., $|v(y)-v(x)| \le L|y-x|.(|v'(\xi)| \le L, \forall \xi \in$ dom(v).)*

2. *If $v(x)$ is convex and $v'(x)$ exists, then $v'(x)$ is non-decreasing.*

Prop. 6.1.1. *Supporting Lines*
Suppose that v is convex, then there exists $s \in \mathbb{R}$ such that:

$$v(y) \ge v(x) + v'(s)(y-x). \tag{6.1}$$

Proof. $v(y)-v(x) = v'(\xi)(y-x)$. $|v'(\xi)| \le L \implies v'(\xi) \le L$ and $v'(\xi) \ge -L$. $\implies v(y)-v(x) \ge -L(y-x)$. □

Theorem 6.1.1. *Jensen's inequality*
Assume that v is convex and bounded on $\mathbb{R}$. Let $u: \Omega \to \mathbb{R}$ be integrable on a measure space $(\Omega, \mathscr{F}, \mu)$. Then

$$v\left(\int_\Omega u d\tilde{\mu}\right) \le \int_\Omega v(u) d\tilde{\mu} \tag{6.2}$$

where $\tilde{\mu} = \frac{1}{\mu(\Omega)}\mu$.

Proof.

From the convexity of v, there exists $s \in \mathbb{R}$ such that $v(y) \geq v(x) + s(y - x)$, $\forall x\ , y \in \mathbb{R}$.

Taking $y = u(\omega)$,$\omega \in \Omega$, $x = \int_\Omega u\ d\tilde{\mu}$ ($\int_\Omega v(u)\ d\tilde{\mu} = v(x) \int_\Omega\ d\tilde{\mu} = v(x)$),

$$\int_\Omega v(u)\ d\tilde{\mu} \leq v\left(\int_\Omega u\ d\tilde{\mu}\right) + s\left(\int_\Omega u\ d\tilde{\mu} - \int_\Omega u d\tilde{\mu}\right). \tag{6.3}$$

□

Prop. 6.1.2. *If v is convex and v'' exists, then $v'' \geq 0$.*

Theorem 6.1.2. *Young's inequality*

$\mathcal{L}$et $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}\ , \forall a\ , b \geq 0. \tag{6.4}$$

Proof. Set $v(x) = e^x$, then $v''(x) > 0 \Rightarrow$ convex.

$$ab = e^{\ln ab} = e^{\frac{1}{p}\ln a^p + \frac{1}{q}\ln b^q} \leq \frac{1}{p}e^{\ln a^p} + \frac{1}{q}e^{\ln b^q} = \frac{a^p}{p} + \frac{b^q}{q}. \tag{6.5}$$

□

Cor. 6.1.1. *Young's inequality with ε*

$$ab \leq \varepsilon a^p + C(\varepsilon)b^q\ , \forall a\ , b \geq 0,\ \varepsilon > 0 \tag{6.6}$$

Proof.

$$(\varepsilon_0 a)(b\frac{1}{\varepsilon_0}) \leq \frac{\varepsilon_0^p a^p}{p} + \frac{b^q \varepsilon_0^{-q}}{q} = \varepsilon a^p + C(\varepsilon)b^q. \tag{6.7}$$

Where $\varepsilon = \varepsilon_0^p/p, C(\varepsilon) = \varepsilon_0^{-q}/q$. □

Theorem 6.1.3. *Hölder inequality*

Assume that $p, q \in [1, +\infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then $\forall u \in L^p(\mu), v \in L^q(\mu)$ on a measure space $(\Omega, \mathscr{F}, \mu)$,

$$\int_\Omega |uv| d\mu \leq \|u\|_{L^p(\mu)} \|v\|_{L^q(\mu)} \tag{6.8}$$

Proof.

It is trivial for $p = 1, q = \infty$ or $q = 1, p = \infty$.

For $p, q \in (1, \infty)$. Reduce: Assume that $\|u\|_{L^p(\mu)} = \|v\|_{L^q(\mu)} = 1$.[1]
By Young's equality,

$$\int_\Omega |uv| d\mu \leq \int_\Omega \frac{1}{p}|u|^p d\mu + \int_\Omega \frac{1}{q}|v|^q d\mu = \frac{1}{p} + \frac{1}{q} = 1. \tag{6.9}$$

$\square$

Cor. 6.1.2. *General Hölder inequality*
Let $p_1, \ldots, p_m \in [1, +\infty)$ with $\sum_{i=1}^m \frac{1}{p_i} = 1$. Assume that $u_k \in L^{p_k}(\mu)$, $k = 1, 2, \ldots, m$. Then

$$\int_\Omega \left| \prod_{k=1}^m u_k \right| d\mu \leq \prod_{k=1}^m \|u_k\|_{L^{p_k}(\mu)}. \tag{6.10}$$

Remark 6.1.2. *If $\mu(\Omega) < +\infty$, and $1 \leq p < q \leq +\infty$, then $L^q(\mu) \subset L^p(\mu)$.*

Proof. $\int_\Omega 1 \cdot |u|^p d\mu \leq \left(\int_\Omega 1^{\frac{1}{1-\frac{p}{q}}} \, d\mu \right)^{1-\frac{p}{q}} \left(\int_\Omega |u|^q d\mu \right)^{\frac{p}{q}} < +\infty.$ $\square$

Cor. 6.1.3. *Interpolation for L^p norms*
Assume that $1 \leq s \leq r \leq t \leq +\infty$, with $\frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{t}$, $\theta \in (0, 1)$. Suppose that $u \in L^s \cap L^t$, then $u \in L^r$ and $\|u\|_{L^r} \leq \|u\|_{L^s}^{\theta} \|u\|_{L^t}^{1-\theta}$.

[1] $\forall u \in L^p, v \in L^q$, by homogeneity, set $\tilde{u} = \frac{u}{\|u\|_{L^p}}, \tilde{v} = \frac{v}{\|v\|_{L^q}} \Rightarrow \|\tilde{u}\|_p = \|\tilde{v}\|_q = 1$.

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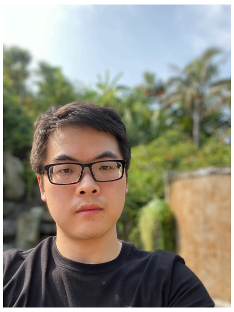
The lecture notes were completed in July 2024. Due to the author's limitations, errors and inadequacies are inevitable. Criticism and corrections are welcome. Please feel free to contact the author via email.

About the Author



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