Lecture 2

Propositional logic II

Possible worlds: collections of facts.
Interpretation: establishing the correspondence between language (sentences) and a world (facts).

Satisfiable sentence: if there is some interpretation in some world for which the sentence is true (i.e. there is a fact in that world corresponding to the sentence).

Unsatisfiable sentence : a sentence which is not satisfiable.
(example: p ^ ~ p)

Tautology: sentence (formula) is always true (regardless of the state of the world). All the lines in the truth table for the formula result in T.
(example: p v ~p)

Inconsistency: sentence (formula) is always false (regardless of the state of the world). All the lines in the truth table for the formula result in F. That is, the sentence is unsatisfiable.

Complexity of the semantic inference in propositional logic using truth-tables.

n: number of propositional variables

2^n lines in the truth table

Therefore, except for fairly simple formulae the truth-table approach becomes too complex.

Furthermore, truth tables are not always necessary:

Suppose we want to determine if the formula

$$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$$

is a tautology (that is, it is always true).

Note that this is an implication, that is, of the form $A \rightarrow B$

A stands for $((p \rightarrow q) \wedge (q \rightarrow r))$
B stands for $(p \rightarrow r)$

We know, from the truth table for the implication that the implication is T except when the antecedent (here A) is T but the consequent (here B) is F.

That is, we now try to find that assignment of truth values to the propositional letters p, q, and r such that the above holds.

This means that <u>A must be T</u> and <u>B must be F</u>.

Since B is an implication, it must be that p is T and r is F.

Since A is a conjunction, this means that $(p \rightarrow q)$ must be T and $(q \rightarrow r)$ must be T.

Each of these are implications and we keep track of the fact that p is T and r is F.

Now, $q \rightarrow r$ is T when r is F , it must be that q must be F.

Next, $p \rightarrow q$ is T and p is T, it must be that q must be T.
CONTRADICTION!!!!

We have the following diagram:

((p → q) ^ (q→r)) → (p→ r)

F

T F

T T T F

T T F F

Another example:

(p v q) → (~q → p)

F

T F

T T F

T q is F p is F

p is T q is T p is F

p is F q is T q is F

Note that ~q → p is the same as ~(~q) v p = q v p = p v q. It is easy to show using truth tables.

PROPOSITIONAL CALCULUS

Syntactic proofs: Use rules of inference based on the form of the argument (not truth values).

The rules of inference mentioned in the previous lecture (IE, ^E, ^I, vE, NE, NI, UR, R) are used to derive a formula.

We can use these rules because we can prove them (by the semantic proof method).

Use the simple turnstile (**|-**) (also read "therefore") to indicate that we want a syntactic proof.

Syntactic proofs will have the following pattern:

- Write the argument to prove.
- Number the premises, including those that will be derived at intermediate steps.
- Derive new formulae based on the premises:
  - indicate which lines (premises have been used)
  - indicate what rule of inference has been used.

Example 1:

p ^ q |- p

Premises	Premise ID	Formula	Premises and Rule used	Remarks
Prem	(1)	p ^ q		This is the premise of the argument
1	(2)	p	1^E	From premise (1) using ^E p is obtained

Example 2:

p, q |- p ^ q

Premises	Premise ID	Formula	Premises and Rule used	Remarks
Prem	(1)	p		This is the 1st premise of the argument
Prem	(2)	q		This is the 2nd premise of the argument
1, 2	(3)	p ^ q	1,2^I	Obtain premise 3 from 1 and 2 by conjunction introduction

Example 3: $p, p \rightarrow q, q \rightarrow r \vdash r$

Premises	Premise ID	Formula	Premises and Rule used	Remarks
Prem	(1)	p		This is the 1st premise of the argument
Prem	(2)	p→q		This is the 2nd premise of the argument
Prem	(3)	q→r		This is the 3rd premise
1,2	(4)	q	1,2IE (→E, Modus Ponens)	Obtain premise 4 from 1 and 2 by Implication Elimination (Modus Ponens)
1,2,3	(5)	r	3,4 (IE)	Obtain premise 5, which is exactly the conclusion of the argument from 1,2,3

Example 4 (reduction to absurd): $p \rightarrow q, \sim q \vdash \sim p$

Premises	Premise ID	Formula	Premises and Rule used	Remarks
Prem	(1)	p→q		This is the 1st premise of the argument
Prem	(2)	~q		This is the 2nd premise of the argument
Prem	(3)	p=~~p		We introduce this premise as the negation of the desired conclusion
1,3	(4)	q	1,3IE (→E, Modus Ponens)	Obtain premise 4 from 1 and 3 by Implication Elimination (Modus Ponens)
1,2,3	(5)	q^~q	2,3 (IE)	Obtain premise 5 by conjunction introduction from 2,3
1,2	(6)	~p	3,5(~I)	Because from p we inferred q^~q we conclude ~p. We also drop premise 3

The rule of ~I (negation introduction) states that if from some formula A as a premise we can derive B ^ ~B (for some B) then we can infer ~A.

Example 5 (Conditional/ implication introduction):

p → q, q → r |- p → r

Premises	Premise ID	Formula	Premises and Rule used	Remarks
Prem	(1)	p→q		This is the 1st premise of the argument
Prem	(2)	q→r		This is the 2nd premise of the argument
Prem	(3)	p		We introduce this premise in preparation to prove →
1,3	(4)	q	1,3IE (→E, Modus Ponens)	Obtain premise 4 from 1 and 3 by Implication Elimination (Modus Ponens)
1,2,3	(5)	r	2,4 (→E)	Obtain premise 5, which is exactly the conclusion of the argument from 1,2,3
1,2	(6)	p→r	3,5(→I)	Assuming p we proved r therefore conclude p→q

The rule →I states that whenever we can derive B having assumed A as a premise we summarize this by A → B.

Similar rules for ←→ or biconditional, introduction, or biconditional elimination.

Example 6 (←→ elimination/introduction: BE, BI)

p ←→ q, q ←→r |- p ←→ r

Premises	Premise ID	Formula	Premises and Rule used	Remarks
Prem	(1)	p←→q		This is the 1st premise of the argument
Prem	(2)	q←→r		This is the 2nd premise of the argument
1	(3)	(p → q) ^ (q →p)	1, BE	Rewrite ←→
1	(4)	p →q	3, ^E	Obtain premise 4 from 3 by Conjunction Elimination
1	(5)	q→p	3, ^E	Obtain premise 5 from 3 by Conjunction Elimination
2	(6)	(q → r) ^ (r →q)	2, BE	Rewrite ←→
2	(7)	q →r	6, ^E	Obtain premise 7 from 6 and 2 by Conjunction Elimination
2	(8)	r →q	6, ^E	Obtain premise 8 from 6 and 2 by Conjunction Elimination
Prem	(9)	p		Introduce this in preparation to prove p→r
1, 9	(10)	q	4,9 →E	
1, 2, 9	(11)	r	7,10, →E	
1, 2	(12)	p→r	9,10,→I	
Prem	(13)	r		Introduce this in preparation to prove r→p
2,13	(14)	q	8,13→E	

1,2,13	(15)	p	5,14→	
1,2	(16)	(r→p)	13,15→I	
1,2	(17)	(p →r) ^ (r → p)	12, 16, ^I	
1,2	(18)	p←→q	17, ←→I	Last step to infer ←→

Exercises

Use syntactic proofs to derive the following arguments:

1. p ^ q |- q ^ p
2. p ^ (q v r) -| |- (p^q) v (p^r)

   (note: A -| |- B means A |- B and B |- A)
3. p → q |- (q →r) → (p →r)
4. p → q v r, q → ~s, r → ~s |- p → ~s
5. r → ~p, ~r → ~q |- ~(p ^ q)
6. p → ~q, p^q |- r