N-body simulation of an open galactic cluster

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Abstract

In this project we look at the Newtonian N-body problem using various numerical methods for solving ordinary differential equations. Specifically of interest is the fourt order Runge-Kutta method and the Velocity-Verlet method.

1 N-body problem

The N-body problem considers N point masses denoted m_i with $i = 1, 2, \dots N$ in an inertial reference frame in three dimensional space $\mathbb{R}^3$. To each point mass m_i we can associate a position vector $\mathbf{q}_i$. According to newtons second law we have that the sum of the forces on the mass is equal to the mass times the acceperation. We have from Newton's law of gravity that the gravitational force felt on m_i by a single mass m_j $(j \neq i)$ is given by

$$\mathbf{F}_{ij} = \frac{G m_i m_j \left(\mathbf{q}_j - \mathbf{q}_i\right)}{\|g_j - q_i\|^3}$$

with G being the gravitational constant and $\|\mathbf{q}_j - \mathbf{q}_i\|$ is the magnitude of the distance between $\mathbf{q}_i$ and $\mathbf{q}_j$. We can now sum over all masses, which gives us the N-body *equations of motion*:

$$m_i \frac{d^2\mathbf{q}_i}{dt^2} = \sum_{j=1, j\neq i}^{N} \frac{G m_i m_j \left(\mathbf{q}_j\right) - \mathbf{q}_i)}{\|\mathbf{q}_j - \mathbf{q}_i\|^3} = \frac{\partial U}{\partial \mathbf{q}_i}$$

where U is the *self-potential* energy

$$U = \sum_{1 \leq i < j \leq N} \frac{G m_i m_j}{\|\mathbf{q}_j - \mathbf{q}_i\|}.$$

We can now define the momentum associated to each mass m_i to be $\mathbf{p}_i = m_i d\mathbf{q}_i / d_t$, which gives us that *Hamilton's equations of motion* for the N-body problem become

$$\frac{d\mathbf{q}_i}{dt} = \frac{\partial H}{\partial \mathbf{p}_i} \qquad\qquad \frac{d\mathbf{p}_i}{dt} = -\frac{\partial H}{\partial \mathbf{q}_i},$$

where $H = T + U$ and T is the kinetic energy given by

$$T = \sum_{i=1}^{N} \frac{\|\mathbf{p}_i\|^2}{2m_i}.$$