N-body simulation of an open galactic cluster

Ivar Haugaløkken Stangeby

November 17, 2015

Abstract

In this project we look at the Newtonian N-body problem using various numerical methods for solving ordinary differential equations. Specifically of interest is the fourt order Runge-Kutta method and the Velocity-Verlet method.

Contents

1 N-body problem

In this section we look at the general form of the N-body problem, as well as examine the special case of $N = 2$, the two-body problem.

1.1 General form

The N-body problem considers N point masses denoted m_i with $i = 1, 2, \dots N$ in an inertial reference frame in three dimensional space $\mathbb{R}^3$. To each point mass m_i we can associate a position vector $\mathbf{q}_i$. According to newtons second law we have that the sum of the forces on the mass is equal to the mass times the acceperation. We have from Newton's law of gravity that the gravitational force felt on m_i by a single mass m_j $(j \neq i)$ is given by

$$\mathbf{F}_{ij} = \frac{G m_i m_j \left(\mathbf{q}_j - \mathbf{q}_i\right)}{\|\mathbf{q}_j - \mathbf{q}_i\|^3}$$

with G being the gravitational constant and $\|\mathbf{q}_j - \mathbf{q}_i\|$ is the magnitude of the distance between $\mathbf{q}_i$ and $\mathbf{q}_j$. We can now sum over all masses, which gives us the N-body *equations of motion*:

$$m_i \frac{d^2\mathbf{q}_i}{dt^2} = \sum_{j=1, j\neq i}^{N} \frac{G m_i m_j \left(\mathbf{q}_j\right) - \mathbf{q}_i)}{\|\mathbf{q}_j - \mathbf{q}_i\|^3} = \frac{\partial U}{\partial \mathbf{q}_i}$$

where U is the *self-potential* energy

$$U = \sum_{1\leq i<j\leq N} \frac{G m_i m_j}{\|\mathbf{q}_j - \mathbf{q}_i\|}.$$

We can now define the momentum associated to each mass m_i to be $\mathbf{p}_i = m_i d\mathbf{q}_i/d_t$, which gives us that *Hamilton's equations of motion* for the N-body problem become

$$\frac{d\mathbf{q}_i}{dt} = \frac{\partial H}{\partial \mathbf{p}_i} \qquad\qquad \frac{d\mathbf{p}_i}{dt} = -\frac{\partial H}{\partial \mathbf{q}_i},$$

where $H = T + U$ and T is the kinetic energy given by

$$T = \sum_{i=1}^{N} \frac{\|\mathbf{p}_i\|^2}{2m_i}.$$

1.2 Two-body problem

If we consider the motion of two bodies, for instance the Earth-Sun system, then:

$$\begin{aligned} \mathbf{F}_{12} &= \frac{G m_1 m_2 \left(\mathbf{q}_2 - \mathbf{q}_1\right)}{\|\mathbf{q}_j - \mathbf{q}_i\|^3} && \text{Earth Sun} \\ \mathbf{F}_{21} &= \frac{G m_2 m_1 \left(\mathbf{q}_1 - \mathbf{q}_2\right)}{\|\mathbf{q}_1 - \mathbf{q}_2\|^3} && \text{Sun Earth} \end{aligned}$$

2 Numerical Methods

We examine the numerical methods used in this project, and look at various properties related to the performance of these methods in N-body simulations. We benchmark using a two-body problem.

2.1 4-th order Runge-Kutta

In the fourth-order Runge-Kutta method, also known as *the classical Runge-Kutta method* we determine the next values the approximations using the value before as well as a weighted average of four increments where each increment is a multiple of the interval and an estimated slope, given by an explicit formula for the first derivative.

2.1.1 Algorithm

Specify an initial value problem as

$$\dot{y} = f(t, y), \quad y(t_0) = y_0$$

where f, t_0 and y_0 is given. Then the fourth order Runge-Kutta method can be described as follows:

for $n = 0$ to $N - 1$ **do**
  $y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
  $t_{n+1} = t_n + h$
end for

where

$$k_1 = f(t_n, y_n), \qquad k_2 = f(t_n + h/2, y_n + k_1 h/2),$$
$$k_3 = f(t_n + h/2, y_n + k_2 h/2), \qquad k_4 = f(t_n + h, y_n + hk_3).$$

2.2 Velocity-Verlet