Problem Set 7 for PDEs

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PROBLEM 1

Set $f(x,t) = \xi^t \exp(ikx)$.

$$\begin{aligned}
&\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0 \\
\Rightarrow\quad & v = -\frac{\partial f/\partial t}{\partial f/\partial x} = -\frac{\ln \xi}{ik} = i\ln(\xi)/k
\end{aligned}$$

$$\begin{aligned}
f(t+dt,x) &= \xi^{t+dt}\exp(ikx) = \xi^t \exp(ikx)\cdot e^{\ln(\xi)dt} \\
&= f(t-dt,x) - \frac{v\,dt}{dx}[f(t,x+dx) - f(t,x-dx)] \\
&= \xi^{t-dt}\exp(ikx) - \frac{v\,dt}{dx}[\xi^t \exp[ik(x+dx)] - \xi^t \exp[ik(x-dx)]] \\
&= \xi^t \exp(ikx)[\xi^{-dt} - \frac{v\,dt}{dx}(e^{ikdx} - e^{-ikdx})]
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\quad & \xi^{dt} = \xi^{-dt} - \frac{v\,dt}{dx}(e^{ikdx} - e^{-ikdx}) \\
\Rightarrow\quad & (\xi^{dt})^2 + \alpha(e^{ikdx} - e^{-ikdx})\xi^{dt} - 1 = 0 \\
\Rightarrow\quad & (\xi^{dt})^2 + 2i\alpha\sin(kdx)\xi^{dt} - 1 = 0
\end{aligned}$$

where $\frac{v\,dt}{dx} = \alpha$

$$\Rightarrow\quad \xi^{dt} = -i\alpha\sin(kdx) \pm \sqrt{1-(\alpha\sin(kdx))^2}$$

If the CFL condition is satisfied $\alpha < 1 \quad\Rightarrow\quad 1-(\alpha\sin(kdx))^2 > 0$ so the second term is the real part $\quad\Rightarrow$

$$|\xi^{dt}|^2 = 1-(\alpha\sin(kdx))^2 + (\alpha\sin(kdx))^2 \equiv 1$$

So the leapfrog scheme preserves energy

PROBLEM 2

part a. A point charge

I set $\rho[0,0] = 1$, and all others are 0. and the potential also defined at the center($[100,100]$). To let $V[0,0] = 1$, I set $V = V - V[n//2, n//2] + 1$ at the end. The potential is shown in fig.1

The potential of several points:

$V[0,0] = 1$

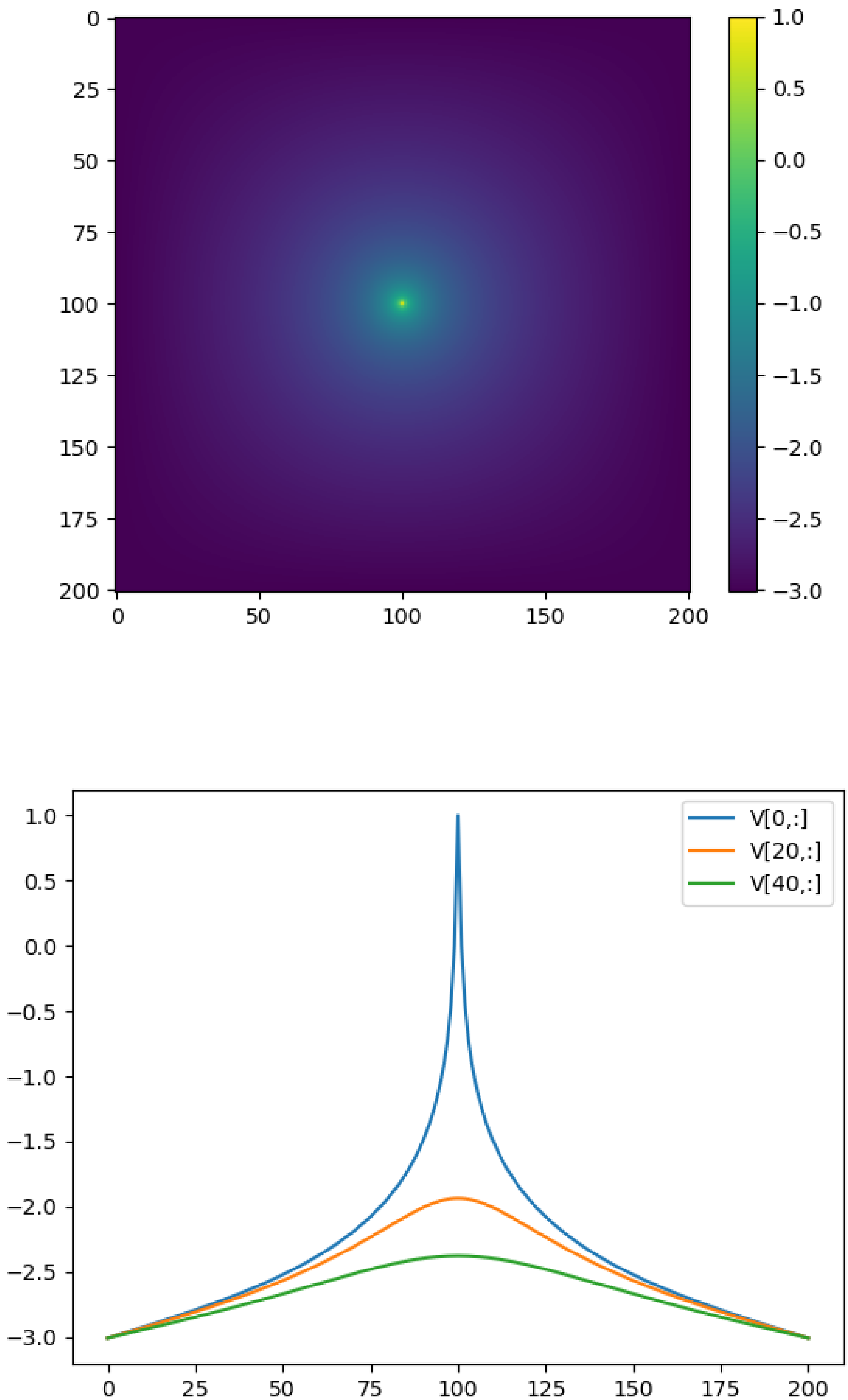
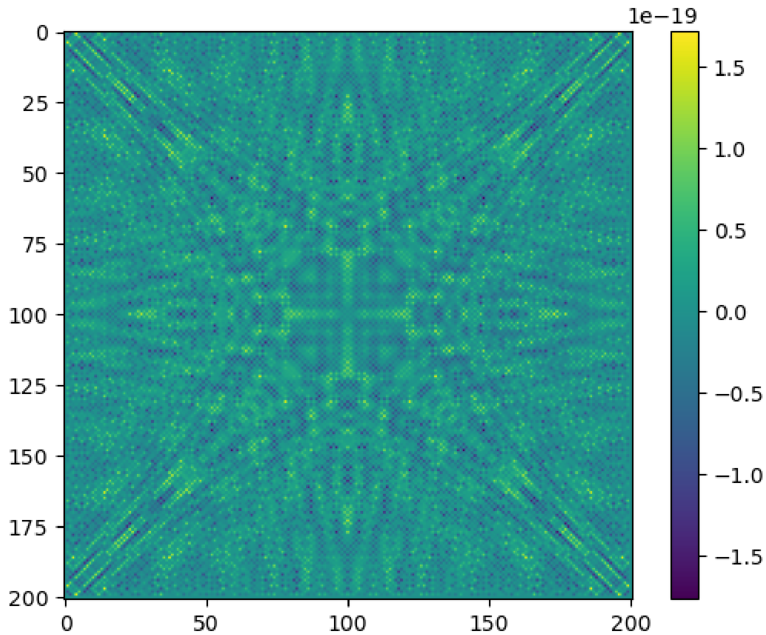


FIG. 1. Potential from a point charge. The top penal is the 2D potential, and the bottom penal shows several section of the potential.



rk.png

FIG. 2. The residue rk at the end.

$V[1,0] = 4.4408e - 15 \sim 0$
$V[2,0] = -0.45352$
$V[5,0] = -1.0516$

Here is the residue and the final rk is shown in fig.2
residue of 0 is 0.25
residue of 100 is 0.0031426538527394444
residue of 200 is 1.724840022148694e-05
residue of 300 is 2.46762839628658e-11
residue of 400 is 5.204556450512147e-19
residue of 500 is 2.9413100149531863e-30
residue of 600 is 5.065917816941024e-35

part b. Charge density

I set the potential equal to 1 except for the boundary (= 0), and the charge density is show in fig.4.

part c. Potential of the above charge

Using the charge density above (in a box with width n=41) and get the potential. We can see that it is really a square potential at the box.

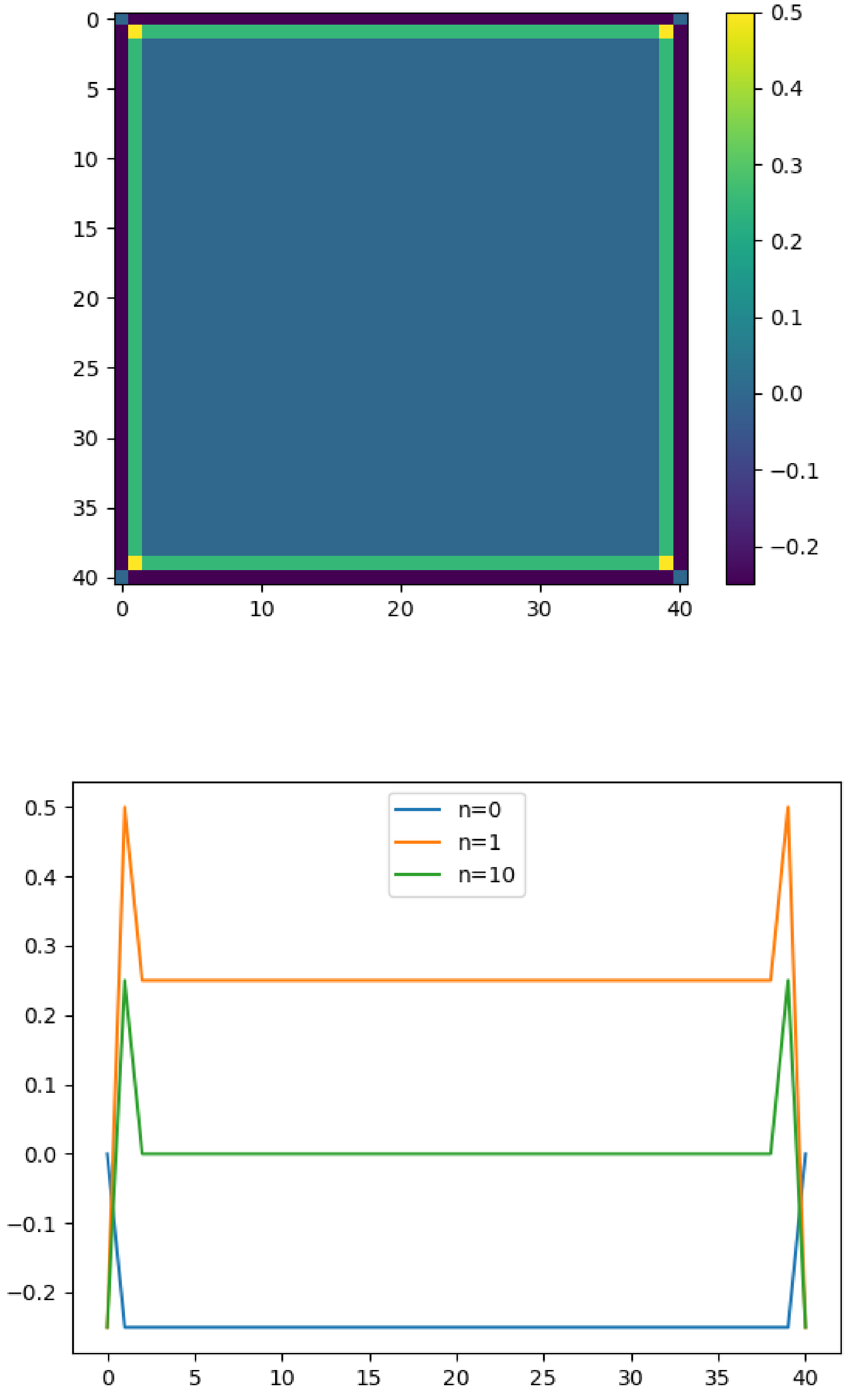


FIG. 3. Charge density of a flat potential. The top penal is the 2D charge density, and the bottom penal shows several section of the charge density.

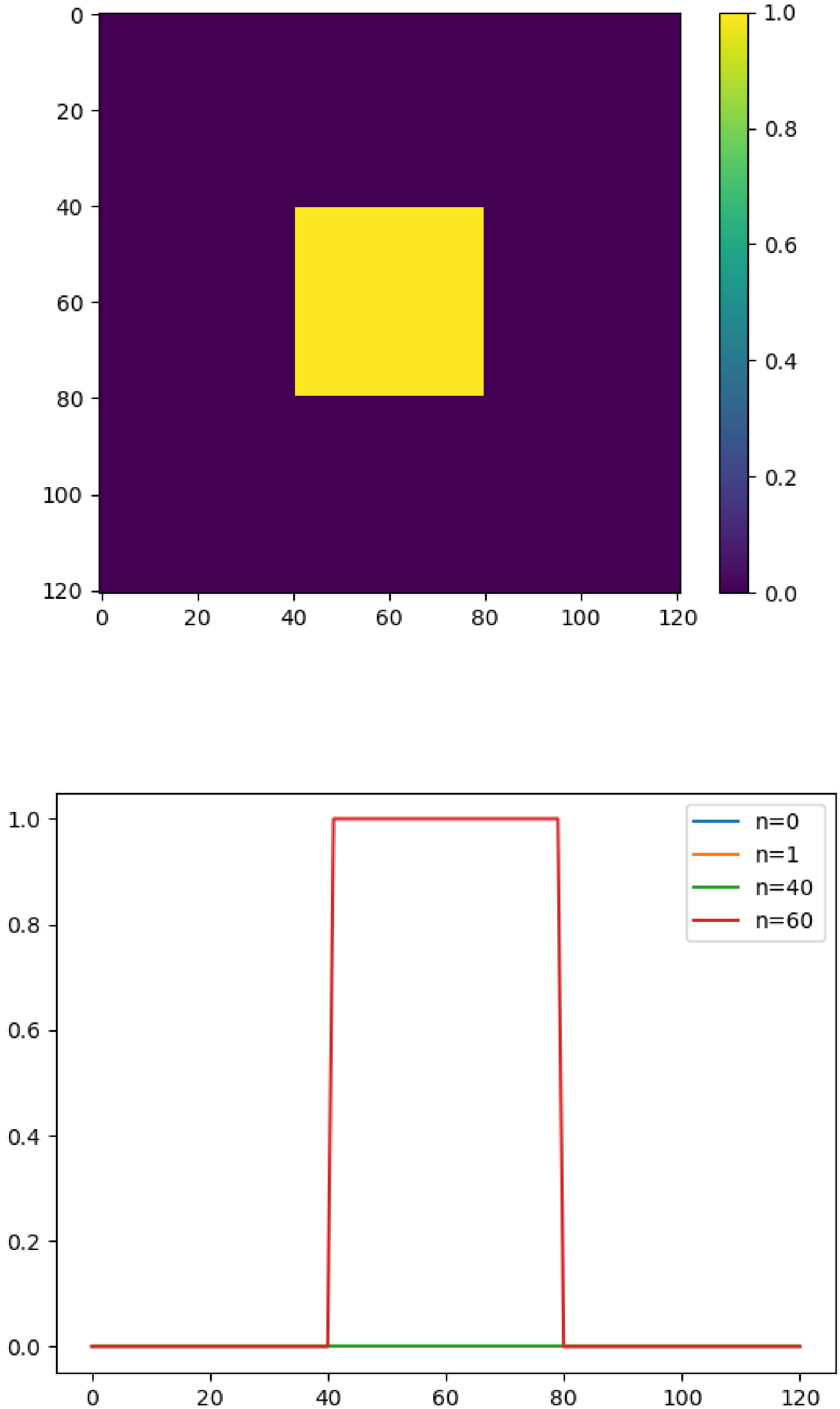


FIG. 4. Charge density of a flat potential. The top penal is the 2D charge density, and the bottom penal shows several section of the charge density.