Problem set 4

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Problem 1

For an array f(x) with N point (x=0,1,...,N-1)
Make a Fourier transition of f(x):

$$F(k) = \sum_{x=0}^{N-1} e^{-2\pi ik\cdot x} f(x) \tag{1}$$
$$\Rightarrow \ f(x) = \sum_{k=0}^{N-1} e^{2\pi ik\cdot x} F(k) \tag{2}$$

And the Fourier transiform of a shift of this array f(x+y) is

$$\begin{aligned} F_y(k) &= \sum_{x=0}^{N-1} e^{-2\pi ik\cdot x/N} f(x+y) \\ &= \sum_{x=0}^{N-1} e^{-2\pi ik\cdot(x'-y)/N} f(x') \\ &= \sum_{x=0}^{N-1} e^{-2\pi ik\cdot x'/N} f(x') \times e^{2\pi ik\cdot y/N} \\ &= F(k) \times G_y(k) \end{aligned}$$

$$f(x-y) = ift[F_y(k)] = ift[dft[f(x)] \times G_y(k)] = [f \otimes g_y](x),$$

where $g_y(x) = ift[G_y(k)] = \sum_{x=0}^{N-1} e^{-2\pi ik\cdot x/N} G_y(k) = \sum_{x=0}^{N-1} e^{-2\pi ik\cdot x/N} e^{2\pi ik\cdot y/N}$
Plot a gaussian that started in the centre of the array shifted by half the array length in fig.1

Problem 2

Make a Fourier transition of f(x) and g(y):

$$F(k) = \sum_{x=0}^{N-1} e^{-2\pi ik\cdot x} f(x) \quad , \quad f(x) = \sum_{k=0}^{N-1} e^{2\pi ik\cdot x} F(k) \tag{3}$$
$$G(k) = \sum_{y=0}^{N-1} e^{-2\pi ik\cdot y} g(y) \quad , \quad g(y) = \sum_{k=0}^{N-1} e^{2\pi ik\cdot y} G(k) \tag{4}$$

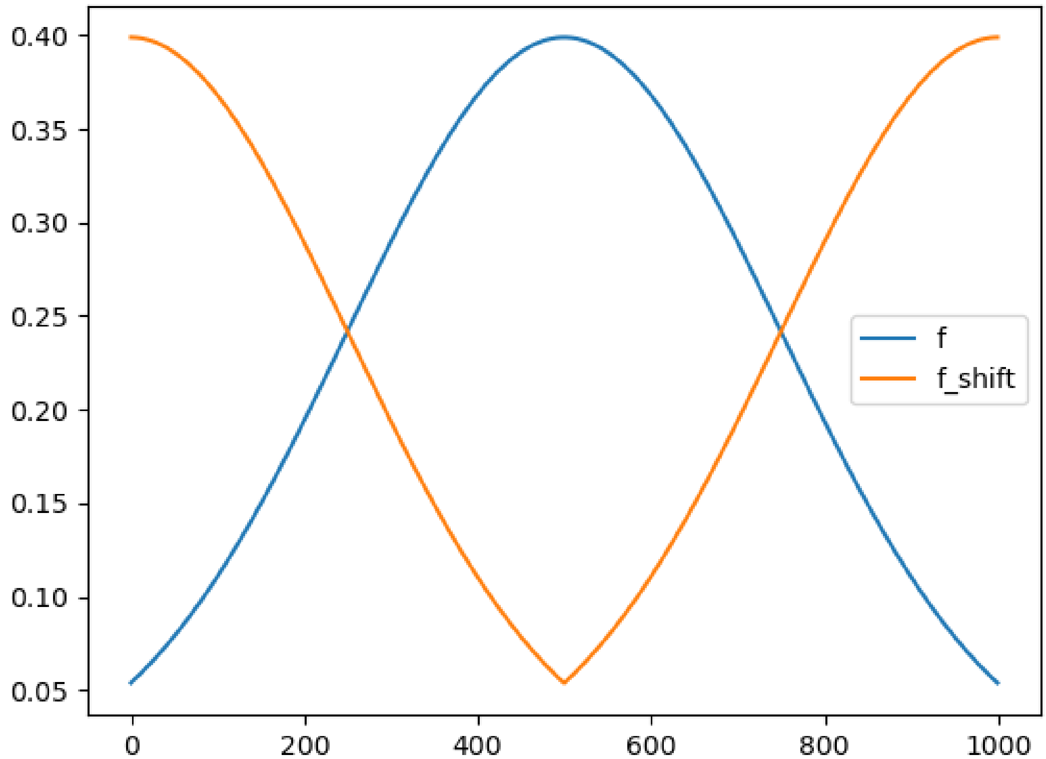

FIG. 1. Plot a gaussian that started in the centre of the array shifted by half the array length.

$$
\begin{aligned}
f * g(y) &= \int f(x)g(x+y)dx \\
&= \int dx \sum_{k=0}^{N-1} e^{2\pi i k\cdot x} F(k) \sum_{k'=0}^{N-1} e^{2\pi i k'\cdot(x+y)} G(k') \\
&= \int dx \sum_{k,k'=0}^{N-1} e^{2\pi i (k+k')\cdot x} F(k)G(k') e^{2\pi i k'\cdot y} \\
&= \sum_{k,k'=0}^{N-1} \frac{1}{N} \delta(k+k') F(k)G(k') e^{2\pi i k'\cdot y} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} F(k)G(-k) e^{-2\pi i k\cdot y} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} F(k)G^*(k) e^{-2\pi i k\cdot y} \\
&= ift(F(k)G^*(k))(y) \\
&= ift(dft(f) * conj(dft(g)))
\end{aligned}
$$

Plot the correlation function of a Gaussian with itself in fig.2

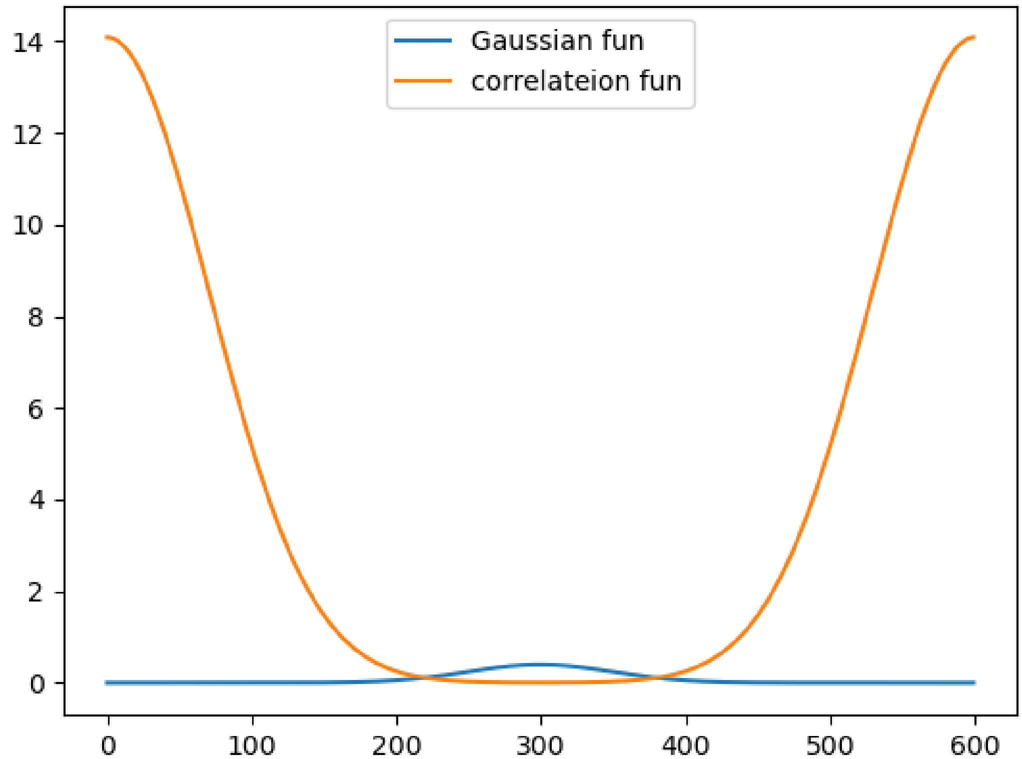

FIG. 2. Plot the correlation function of a Gaussian with itself.

Problem 3

I think here 'take the correlation function of a Gaussian (shifted by an arbitrary amount) with itself': the itself is the shifted Gaussian function, is that right? If so, the result is shown in fig.3. It is amazing that the shift does not affect at all.

If the 'itself' means the unshifted Gausian function, the result is different: the the correlation function shift (shown in fig.4).

Problem 4

The result is show in fig.5

Problem 5

a) show that

$$\sum_{x=0}^{N-1} \exp(-2\pi ikx/N) = \frac{1-\exp(-2\pi ik)}{1-\exp(-2\pi ik/N)} \tag{5}$$

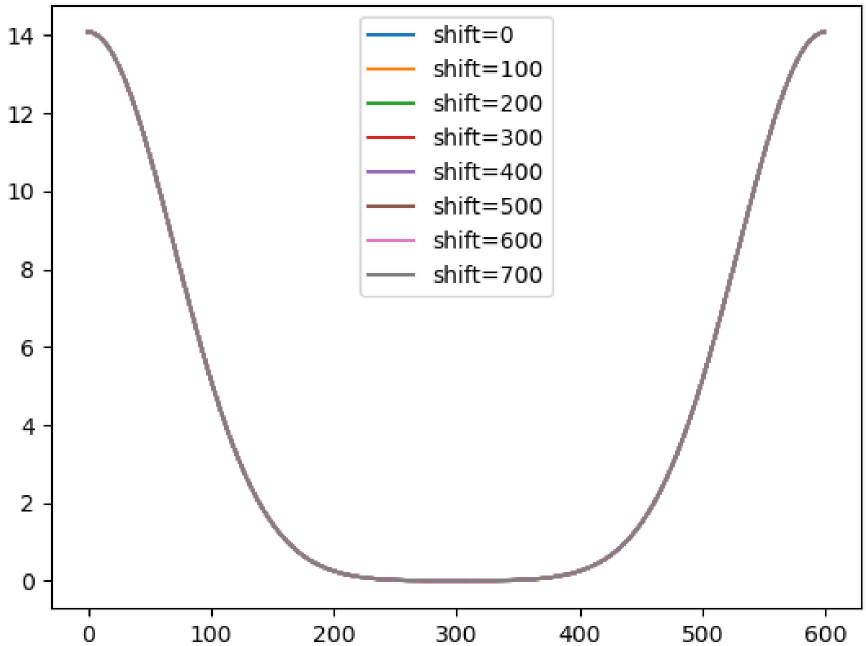

FIG. 3. Compare correlation function with different the shift.

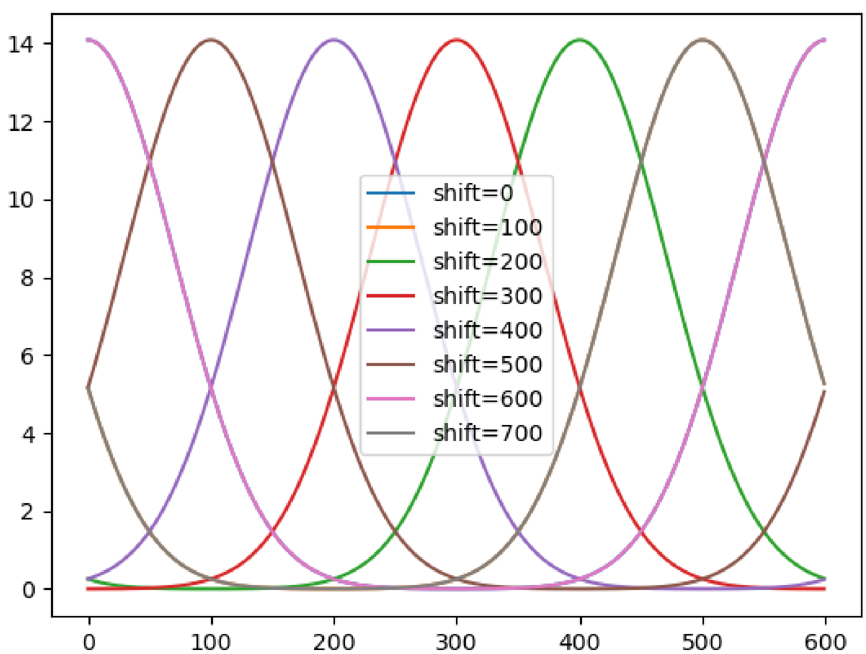

FIG. 4. Compare correlation function with different the shift (only shift 1 Gaussian).

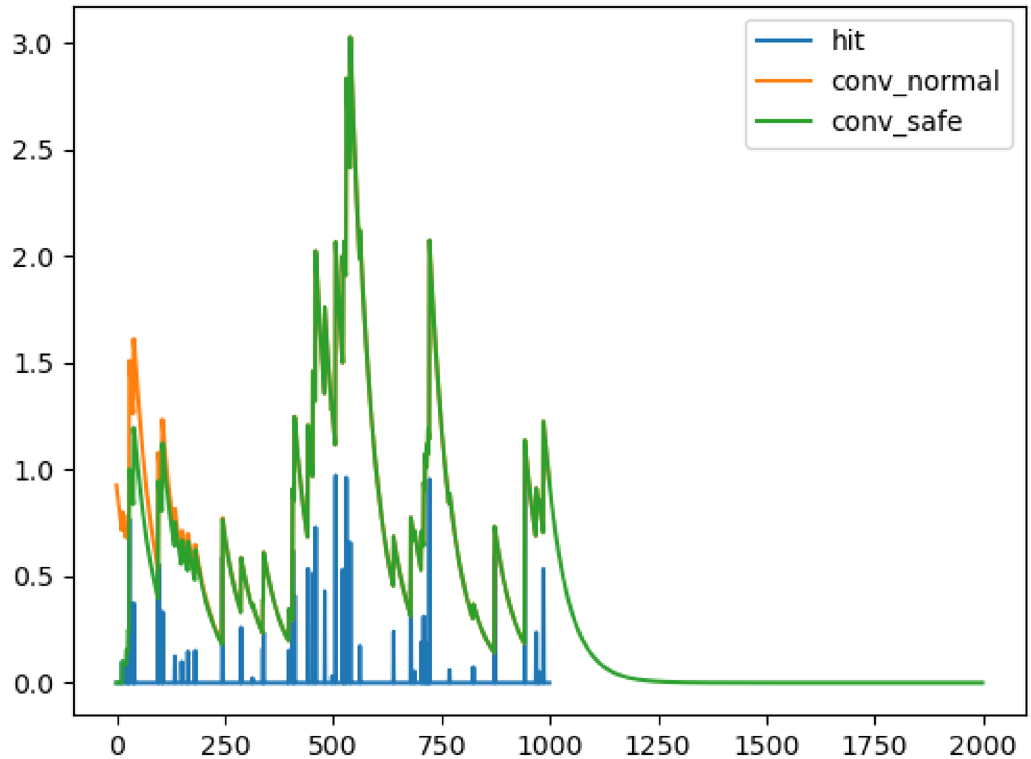

FIG. 5. Show the routine to take the convolution of two arrays without any danger of wrapping around. Take the example we show in the class

Set $\alpha = \exp(-2\pi i k/N)$. Then the lhs is

$$\begin{aligned}\sum_{x=0}^{N-1} \exp(-2\pi i k x/N) &= \sum_{x=0}^{N-1} \exp(-2\pi i k/N)^x = \sum_{x=0}^{N-1} \alpha^x \\ &= \frac{1-\alpha^N}{1-\alpha} = \frac{1-\exp(-2\pi i k)}{1-\exp(-2\pi i k/N)}\end{aligned}$$

$I = \sum_{x=0}^{N-1} \alpha^x$
$\alpha * I = \sum_{x=0}^{N-1} \alpha^{x+1} = \sum_{x=1}^{N} \alpha^x$
$I - \alpha I = (1-\alpha)I = \sum_{x=0}^{N-1} \alpha^x - \sum_{x=1}^{N} \alpha^x = 1 - \alpha^N$
$\Rightarrow \quad I = (1-\alpha^N)/(1-\alpha)$, if $\alpha \neq 1$

b)

1. $k \to 0$: $\alpha = \exp(-2\pi i k/N) \to 1$. So the lhs$\to \sum_{x=0}^{N-1} 1^x = N$
2. for any integer k that is not a multiple of N: This is the DFT of $f(x) \equiv 1$. As show in the class, the result is 0 except $k = nN,\ n \in \mathbb{N}$ (delta function).

c)

1. analytically write down the DFT of a non-integer sine wave.

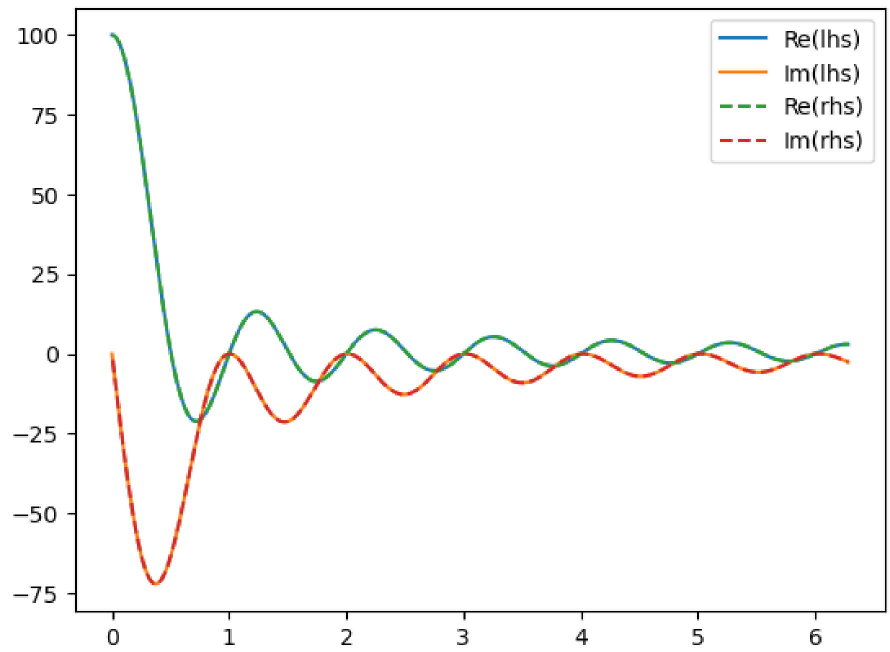

FIG. 6. Verify the result of Problem5 a).

$$\begin{aligned}
DFT(\sin(2\pi k'x/N)) &= \sum_{x=0}^{N-1} \sin(2\pi k'x/N)\exp(-2\pi ikx/N) \\
&= \sum_{x=0}^{N-1} \frac{e^{2\pi ik'x/N} - e^{2\pi ik'x/N}}{2i}\exp(-2\pi ikx/N) \\
&= \sum_{x=0}^{N-1} \frac{1}{2i}[\exp(-2\pi i(kx-k'x)/N) - \exp(-2\pi i(kx+k'x)/N)] \\
&= \frac{1}{2i}\left[\sum_{x=0}^{N-1}\exp(-2\pi i(k-k')x/N) - \sum_{x=0}^{N-1}\exp(-2\pi i(k+k')x/N)\right] \\
&= \frac{1}{2i}\left[\frac{1-\exp(-2\pi i(k-k'))}{1-\exp(-2\pi i(k-k')/N)} - \frac{1-\exp(-2\pi i(k+k'))}{1-\exp(-2\pi i(k+k')/N)}\right]
\end{aligned}$$

Here I set $k' = 23.4$ and the plot is show in fig.7. To some degree, this result is similar to fat delta function at $k \sim k'\ or N-k',\ n \in \mathbb{N}$, but the peak are broaden and the DFT is non-zero even far away the peak.

d) Show that when we multiply by this window, the spectral leakage for a non-integer period sine wave drops dramatically.

The result is show in fig.8. It is real that the window function can cause the peak drops dramatically to 0. But the price is that the shape of peak changes.

e)

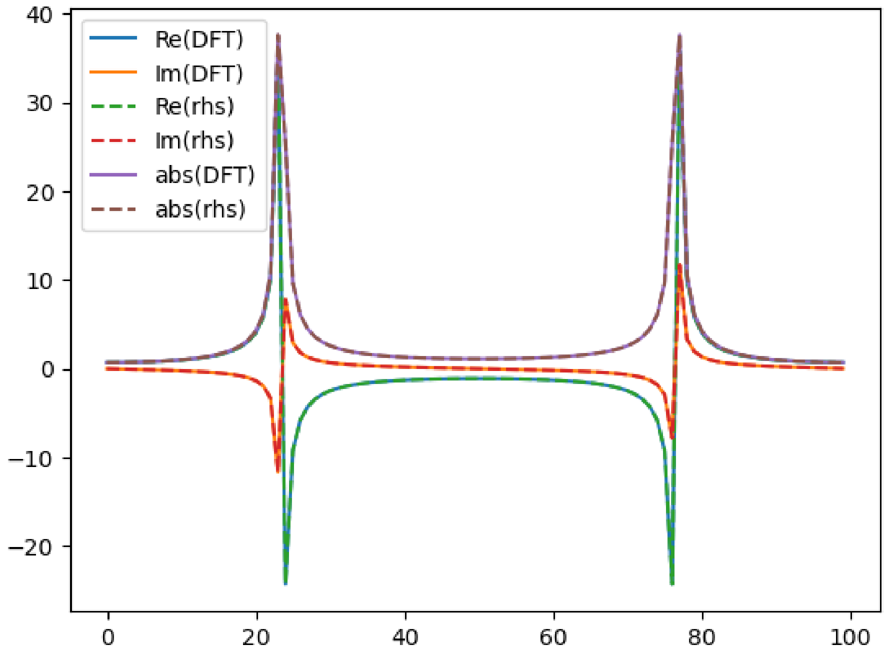

FIG. 7. The my analytic estimate of the DFT to the real one of a non-integer sine wave (The two are identical). Here I set k'=23.4.

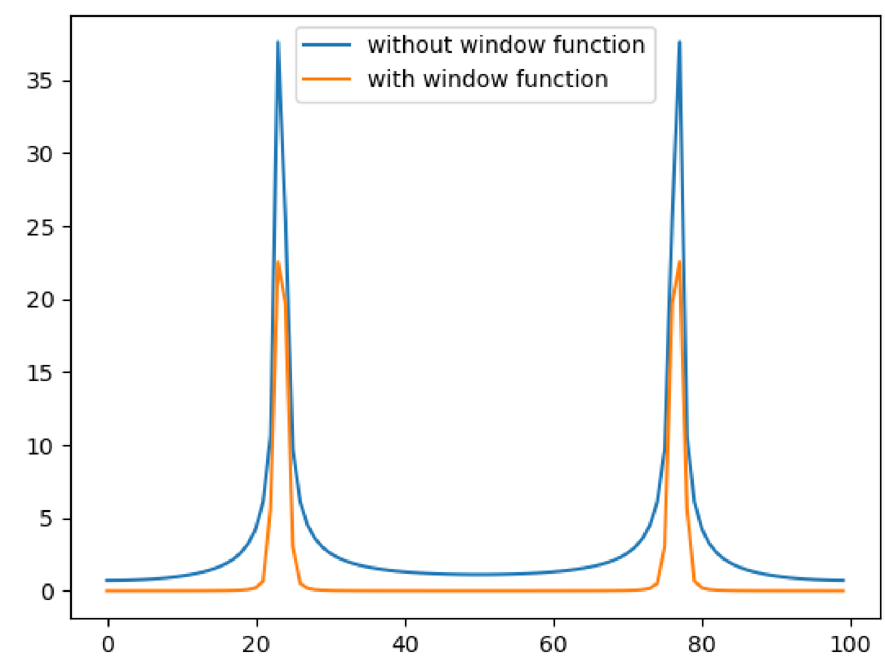

FIG. 8. Compare the DFT with and without window function. k'=23.4.

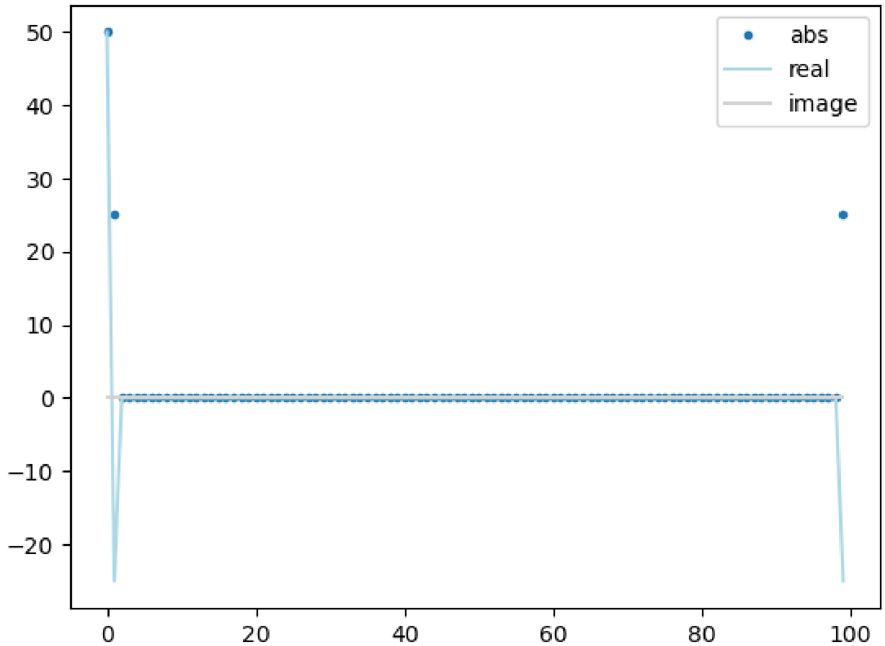

FIG. 9. Fourier transform of the window function. N=100.

1. Fourier transform of the window:

$$
\begin{aligned}
&DFT[0.5-0.5\cos(2\pi x/N)](k)\\
=&\sum_{x=0}^{N-1}[0.5-0.5\cos(2\pi x/N)]\exp(-2\pi ikx/N)\\
=&\sum_{x=0}^{N-1}\left[0.5-\frac{e^{2\pi ix/N}+e^{2\pi ix/N}}{4}\right]\exp(-2\pi ikx/N)\\
=&0.5\sum_{x=0}^{N-1}\exp(-2\pi ikx/N)-\sum_{x=0}^{N-1}\frac{1}{4}[\exp(-2\pi i(kx-x)/N)+\exp(-2\pi i(kx+x)/N)]\\
=&\frac{1}{2}\frac{1-\exp(-2\pi ik)}{1-\exp(-2\pi ik/N)}-\frac{1}{4}\left[\frac{1-\exp(-2\pi i(k-1))}{1-\exp(-2\pi i(k-1)/N)}+\frac{1-\exp(-2\pi i(k+1))}{1-\exp(-2\pi i(k+1)/N)}\right]\\
=&\frac{1-A^{Nk}}{2-2A^{k}}-\frac{1}{4}\left[\frac{1-A^{N(k-1)}}{1-A^{k-1}}+\frac{1-A^{N(k+1)}}{1-A^{k+1}}\right]
\end{aligned}
$$

where $A=\exp(-2\pi ix/N)$

The numerical result is show in fig.9, here I set N=100. And I list the first and last four terms of the DFT: The error is approxmately 1e-15

- First four terms: [5.0000000e+01+0.00000000e+00j -2.5000000e+01+2.41473508e-15j -1.0904149e-16+1.31757938e-15j]

- Last four terms: [4.17403164e-16+1.24406761e-15j -1.51210335e-15-7.15809022e-16j -1.09041490e-16-1.31757938e-15j -2.50000000e+01-2.03250507e-15j]

So $W(k)\equiv$DFT(window)=[N/2, -N/4, 0,..., 0, -N/4]

2. Get the windowed Fourier transform (denoted by $F_w(k)$) by appropriate combinations of each point in the unwindowed Fourier transform and its immediate neighbors:

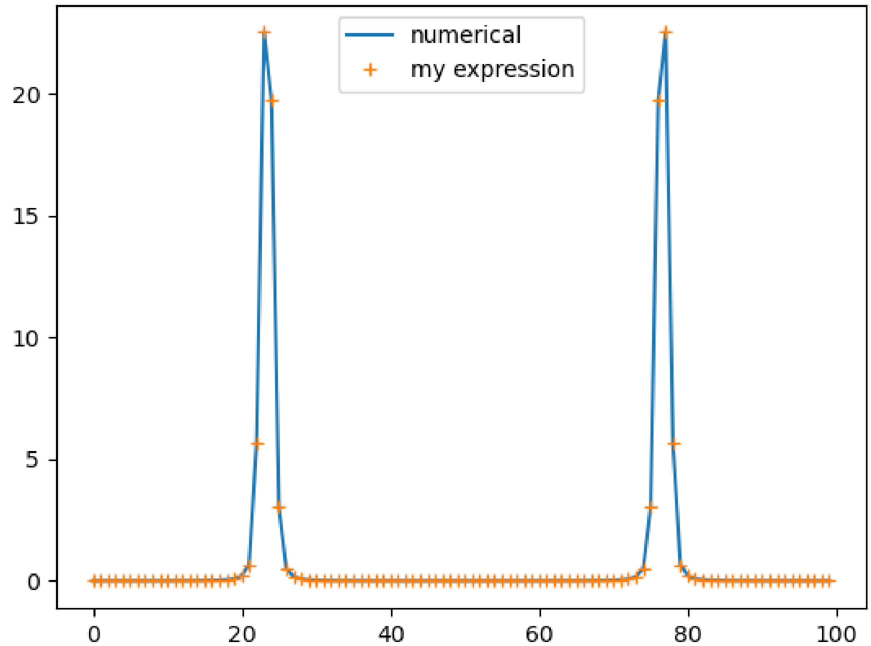

FIG. 10. Fourier transform of the window function. N=100.

The multiple in real space corresponds to convolution in Fourier space.

$$\begin{aligned}
F_w(k) &= \frac{1}{N}\sum_{k'=0}^{N-1} W(k')F(k-k') \\
&= \frac{1}{N}\sum_{k'=-1}^{1} W(k')F(k-k') \\
&= \frac{1}{N}W(-1)F(k-1) + W(0)F(k) + W(1)F(k+1) \\
&= -\frac{1}{4}F(k-1) + \frac{1}{2}F(k) - \frac{1}{4}F(k+1)
\end{aligned}$$

$$\Rightarrow \quad f_w(x) = IFT(F_w(k))(x) = IFT\left[-\frac{N}{4}F(k-1) + \frac{N}{2}F(k) - \frac{N}{4}F(k+1)\right](x)$$

I compare the result with DFT of the windowed function in fig.10 so this expression is right.