1. (a)

I did this homework with Luning Zhao.

We work on our homework seperately, but we discuss when meeting problems.

Homework is fine.

(b) I certify that all solutions are entirely in my words. and that I have not looked at another student's solutions. I have credited all external sources in this write up.

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3. Linear Algebra

(a) $M = uv^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

eigenvalues λ and eigenvectors x should satisfy:

$$Mx = \lambda x$$
$$(M - \lambda I)x = 0$$

In order to have non-zero x, $|M - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 3 \\ 4 & 6-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(6-\lambda) - 12 = 0$$

$$12 - 8\lambda + \lambda^2 - 12 = 0$$

$$\lambda^2 - 8\lambda = 0$$

so $\lambda_1 = 8 \quad \lambda_2 = 0$

① $\lambda_1 = 8 \quad \begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \begin{cases} -6x_1 + 3x_2 = 0 \\ 4x_1 - 2x_2 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 1 \\ x_2 = 2 \end{cases}$

② $\lambda_2 = 0 \quad \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \begin{cases} 2x_1 + 3x_2 = 0 \\ 4x_1 + 6x_2 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 3 \\ x_2 = -2 \end{cases}$

so, when $\lambda = 8$, $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; when $\lambda = 0$ $x = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$: the 1st row $\times 2$ = 2nd row,

so the first and 2nd row are linear dependant.

so $R = 1$

$$\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 2 \times 6 - 4 \times 3 = 0.$$

The null space dimension = rank (M) = 1

(c) ~~To compute eigenvalues and eigenvectors:~~

$$N = \begin{bmatrix} p_1 \\ \vdots \\ p_d \end{bmatrix} [q_1 \cdots q_d] = \begin{bmatrix} p_1 q_1 & \cdots & p_1 q_d \\ & \vdots & \\ p_n q_1 & \cdots & p_d q_d \end{bmatrix}$$

$\cancel{(N - \lambda I)\, x = 0}$.

$$\begin{bmatrix} p_1 q_1 & p_1 q_2 & \cdots & p_1 q_d \\ p_2 q_1 & p_2 q_2 & \cdots & p_2 q_d \\ \vdots & & & \\ p_n q_1 & p_n q_2 & \cdots & p_n q_1 \end{bmatrix} \xrightarrow[\text{linear dependent}]{\text{all rows are}} \begin{bmatrix} p_1 q_1 & p_1 q_2 & \cdots & p_1 q_d \\ 0 & 0 & & 0 \\ & & & \\ 0 & 0 & & 0 \end{bmatrix}$$

exp. row $n \times$ row $1 \times \frac{p_n}{p_1}$

so, $R = 1$, and $|N| = 0$. (it's true for all not full rank Matrix)

the dimension of nullspace of $M = d - 1$.

To compute eigenvalues and eigenvectors

① $(N - \lambda I)x = 0$

To have non-zero solution for x, $|N - \lambda I| = 0$

so when $\lambda = 0$, $|N| = 0$, $|N - \lambda I| = 0$.

In this case, $Nx = \begin{bmatrix} p_1 q_1 & p_1 q_2 & \cdots & p_1 q_d \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix} = 0$

$$q_1 x_1 + q_2 x_2 + \cdots + q_d x_d = 0$$

Solution is: $\begin{bmatrix} 1 \\ -\frac{q_1}{q_2} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -\frac{q_1}{q_3} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdots \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ -\frac{q_d}{q_1} \end{bmatrix}$

② $N = p \cdot q^T \Rightarrow Np = (p \cdot q^T) p = p(q^T \cdot p)$

Since $q^T \cdot p = q_1 p_1 + q_2 p_2 + \cdots + q_n p_n =$ scalar.

$Np = (q^T p) p$

so $\lambda = q_1 p_1 + q_2 p_2 + \cdots + q_n p_n$.

eigenvector $= p = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$

4. Linear Regression and Adversarial Noise.

(a) from simple OLS, we know:

$$w_1 = \frac{\Sigma x_i y_i - \frac{1}{n}\Sigma x_i \Sigma y_i}{\Sigma x_i^2 - \frac{1}{n}(\Sigma x_i)^2} \quad \text{[cite : OLS in wiki]}$$

$$w_2 = \bar{y} - w_1 \bar{x}.$$

so given w_1, only one w_2 will be calculated.

Suppose the data that is changed is $\tilde{y}_a = y_a + \varepsilon_a$.

then $$\hat{w}_1 = \frac{\Sigma x_i y_i + \varepsilon_a x_a - \frac{1}{n}\Sigma x_i \Sigma y_i - \frac{\varepsilon_a}{n}\Sigma x_i}{\Sigma x_i^2 - \frac{1}{n}(\Sigma x_i)^2}$$

$$= w_1 + \frac{\varepsilon_a x_a - \varepsilon_a \bar{x}}{\Sigma x_i^2 - \frac{1}{n}(\Sigma x_i)^2}$$

If the adversary wants $\hat{w}_1 = w_1^*$,

$$\varepsilon_a = \frac{(w_1^* - w_1)\left[\Sigma x_i^2 - \frac{1}{n}(\Sigma x_i)^2\right]}{x_a - \bar{x}}.$$

When $x_a = \bar{x}$, $\varepsilon_a = \infty$, so no solution for ε_a.

So the adversary cannot always fool us by setting one ε_i.

(b) Suppose the data that is changed is $\tilde{y_a} = y_a + \varepsilon_a$

$$\tilde{y_b} = y_b + \varepsilon_b.$$

then $$\hat{w}_1 = \frac{\sum x_i y_i + \varepsilon_a x_a + \varepsilon_b x_b - \frac{1}{n}\sum x_i \sum y_i - \frac{\varepsilon_a + \varepsilon_b}{n}\sum x_i}{\sum x_i^2 - \frac{1}{n}(\sum x_i)^2}$$

$$= w_1 + \frac{\varepsilon_a x_a + \varepsilon_b x_b - (\varepsilon_a + \varepsilon_b)\bar{x}}{\sum x_i^2 - \frac{1}{n}(\sum x_i)^2} = w_1^*$$

$$(x_a - \bar{x})\varepsilon_a + (x_b - \bar{x})\varepsilon_b = (w_1^* - w_1)(\sum x_i^2 - \frac{1}{n}(\sum x_i)^2)$$

Because $x_a \neq x_b$, one of them can't be zero.

Let's assume $x_a - \bar{x} \neq 0$.

Then we can get: $\varepsilon_a = \dfrac{(w_1^* - w_1)(\sum x_i^2 - \frac{1}{n}(\sum x_i)^2) - (x_b - \bar{x})\varepsilon_b}{x_a - \bar{x}}.$

~~if $x_b \neq \bar{x}$,~~ given any ε_b, we will an ε_a.

So, the adversary can always fool us by setting two $\varepsilon_a, \varepsilon_b$.

(c) If you want to change a model with two parameters, you need to modify at least two ~~parameters~~ data points. Similarly, if you want to change a model with n parameters, you need to modify at least n data points.

5. (a) I've taken linear Algebra in undergrad.

(b) No

(c) I've taken Probality theory.

(d) I've learned vector calculation in linear Algebra.

6. Under what circumstances, does $e^{A+B} = e^A \cdot e^B$?

(A, B are matrics).

$$e^{A+B} = 1 + (A+B) + \frac{1}{2!}(A+B)^2 + \cdots$$

$$= 1 + A + B + \frac{1}{2}(A^2 + AB + BA + B^2) + \cdots \quad ①$$

$$e^A \cdot e^B = (1 + A + \frac{A^2}{2} + \cdots)(1 + B + \frac{B^2}{2} + \cdots)$$

$$= 1 + A + B + AB + \frac{A^2}{2} + \frac{B^2}{2} \quad ②$$

Compare ① and ②, $AB = \frac{1}{2}AB + \frac{1}{2}BA$

so $\frac{1}{2}AB = \frac{1}{2}BA$

so $AB = BA$.