Kayles Game on Graphs

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Introduction

The *Kayles Game on graphs* is a **combinatorial two-player game** played on a finite graph in which players alternately take the following moves:

- a player may take a single edge
- a player may take a vertex of positive degree and all edges terminal to that vertex

When a player has no legal moves, that player loses the game.

Since Kayles Game on graphs is a finite impartial game[1], every game state is a winning state for one of the players. This allows us to define the following:

- **N-Position:** A graph where the next player has a winning strategy.
- **P-Position:** A graph where the previous player has a winning strategy.

A graph is an N-position if and only if the graph can be transformed into a P-position by a single move.

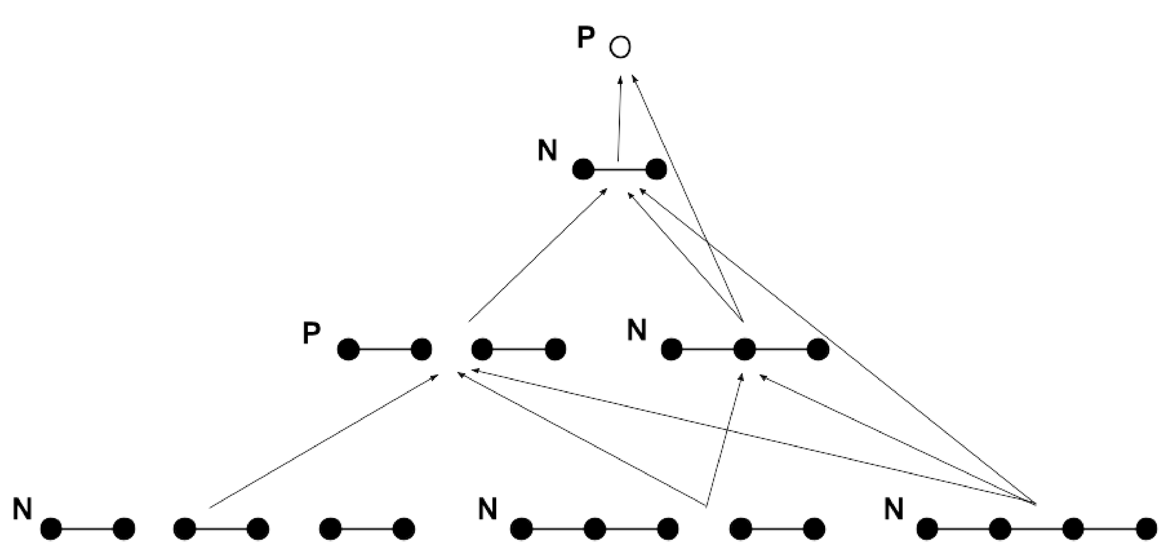

Fig. 1: Paths of at most edges labelled as N- or P-positions

A *complete bipartite graph* is a graph whose vertices can be partitioned into sets A and B so that the set of edges in the graph is $A \times B$. That is, an edge exists between two vertices if and only if those vertices come from different partitions.
Let n and m be positive integers. We label the complete bipartite graph with partitions of size n and m as $K_{n,m}$.

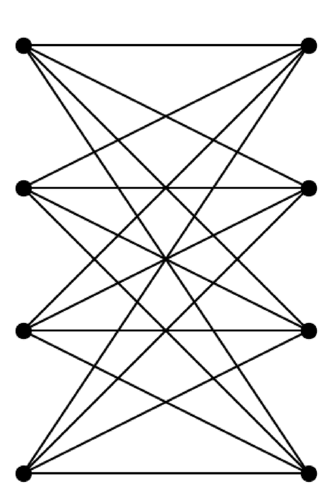

Fig. 2: $K_{4,4}$

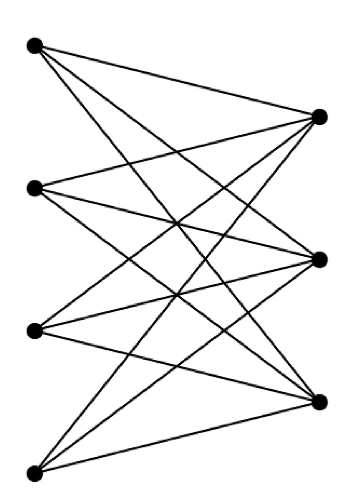

Fig. 3: $K_{4,3}$

Results

Using a Rust program, we generated the following table for the P- and N-positions of the complete bipartite graph $K_{n,m}$ for positive integers n, m:

Kmn	1	2	3	4	5	6	7	8	9	10
1	N	N	N	N	N	N	N	N	N	N
2	N	P	N	P	N	P	N	P	N	P
3	N	N	N	N	P	N	P	N	P	N
4	N	P	N	P	N	P	N	P	N	
5	N	N	P	N	N	N	P			
6	N	P	N	P	N	P				
7	N	N	P	N	P					
8	N	P	N	P						
9	N	N	P	N						
10	N	P	N							

Fig. 4: Table of states for $K_{n,m}$

We make the following claims about the graph $K_{n,m}$:

- $K_{n,m}$ is a P-position if n, m are both even;
- $K_{n,m}$ is an N-position if exactly one of n, m is even.

More generally, we claim the following. Let $G = (V, E)$ denote a graph so that each vertex has a positive degree. Then G is a P-position if there exists a map $f : V \to V$ so that the following hold:

- f is an involution, that is, $f^2 = \mathrm{id}$;
- f has no fixed points, that is, $f(v) \neq v$ for all $v \in V$;
- f is a graph isomorphism, that is, if $(u, v) \in E$ then $(f(u), f(v)) \in E$;
- $(u, f(u)) \notin E$ for any $u \in V$.

We can represent the game on $K_{n,m}$ as an n by m matrix of all 1's. Then the moves of the game become the following:

- a player may change a 1 to a 0
- a player may choose an entire row or column that contains a 1, and change all the 1's to 0's

This allows us to find patterns in the matrix as well as the graph.

	0	1	2
3	1	1	1
4	1	1	1

	0	1	2
3	1	0	1
4	1	1	1

	0	1	2
3	1	0	1
4	1	0	1

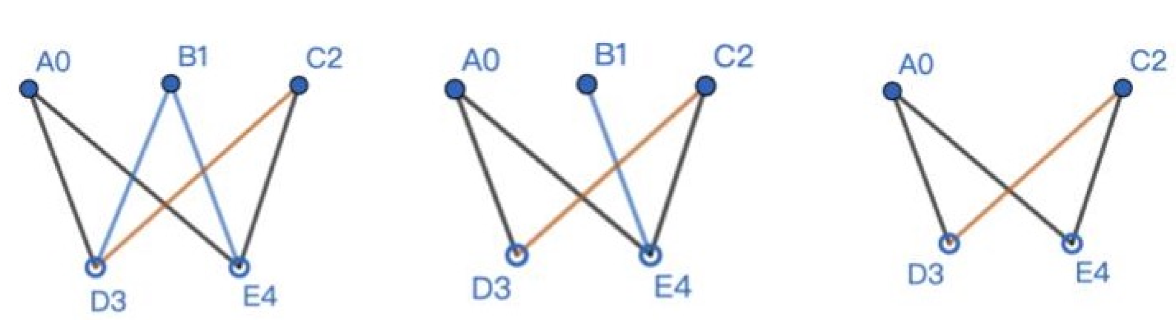

Fig. 5: Matrix representation of $K_{2,3}$ and two sequential moves

Conjectures

We make the following conjectures about the complete bipartite graph $K_{n,m}$:

- if $n = m$ are odd, the graph $K_{n,m}$ is an N-position;
- if n, m are distinct and both odd greater than 1, the graph $K_{n,m}$ is a P-position.

Future Plans

Future research on this topic would involve proving our conjectures described above.

Here are some open problems for the future:

- What do P-position graphs look like?
  In particular, our claim involving the existence of a "symmetry" involution on the vertices describes a type of invariant on graphs that guarantees a P-position. Can we find and describe other types of invariants on graphs that guarantee a P-position?
- Is the problem of determining whether $K_{n,m}$ is a P-position NP-hard?
  This would mean that at least one of our conjectures above cannot be proven. More generally, we can ask this question about all graphs.
- If we extend Kayles Game on graphs to include edge contraction (identifying two vertices connected by an edge), what do P-positions look like in this new game?

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References

[1] Julien Lemoine and Simon Viennot. *Nimbers are inevitable.* 2010. arXiv: 1011.5841 [math.CO]. URL: https://arxiv.org/abs/1011.5841.