Kayles Game on Graphs

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1 Introduction

The *Kayles Game on graphs* is a **combinatorial two-player game** played on a finite graph in which players alternately take the following moves:

- a player may take a single edge
- a player may take a vertex of positive degree and all edges terminal to that vertex

When a player has no legal moves, that player loses the game.

Since Kayles Game on graphs is a finite impartial game[1], every game state is a winning state for one of the players. This allows us to define the following:

- **N-Position:** A graph where the next player has a winning strategy.
- **P-Position:** A graph where the previous player has a winning strategy.

A graph is an N-position if and only if the graph can be transformed into a P-position by a single move.

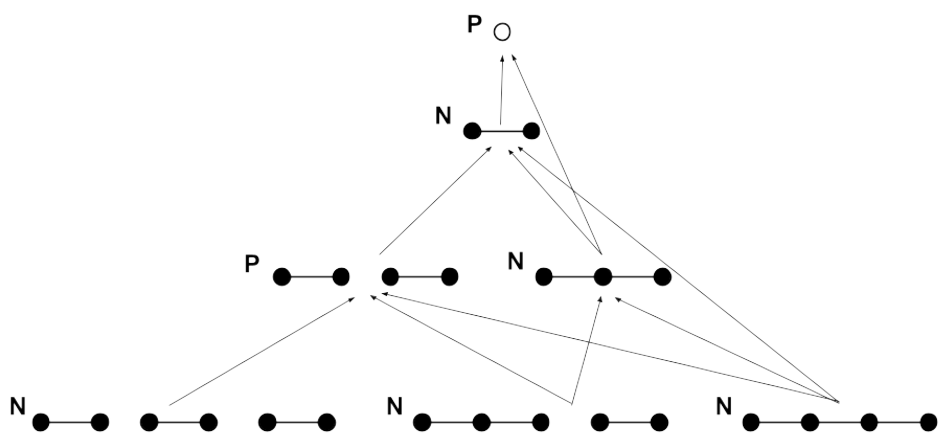

Figure 1: Paths of at most edges labelled as N- or P-positions

For positive integers n and m, a *complete bipartite graph* $K_{n,m}$ is a graph whose vertices can be partitioned into sets A and B so that $|A| = n$, $|B| = m$, and the set of edges in the graph is $A \times B$. That is, an edge exists between two vertices if and only if those vertices come from different parts.

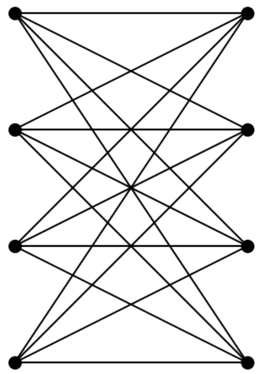

Figure 2: $K_{4,4}$

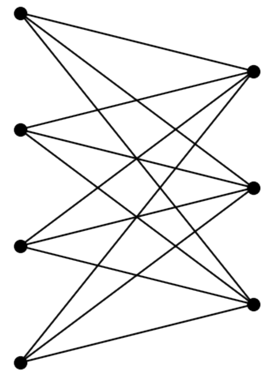

Figure 3: $K_{4,3}$

The goal of the project is to investigate the Kayles game on complete bipartite graphs.

2 Methodology

We developed a Python program that determines the N/P-positions for arbitrary graphs. Using this program, we were able to search for patterns in various families of graphs.

The program uses the Python package NetworkX (`https://networkx.org/`) for various graph functionality. Due to the expensive and recursive nature of the problem, the N/P-positions are stored to a database resembling a hash table. This avoids having to recompute N/P-positions from scratch. We have computed around 120,000 N/P positions over dozens of hours of computation time on an AMD Ryzen 9 5900X laptop CPU. Our source code can be found here: `https://github.com/jkolbly/kayles-mxm`

Additionally, we developed a Rust program that performs a similar role using parallel computation. Due to the Rust program's relative speed, this program was used for more computationally intensive tasks. The code can be found here: `https://github.com/Huaiyuan-Jing/MXM_Kayler-s_Game`

As a visual alternative, we represented the game on $K_{n,m}$ as an $n \times m$ matrix of all 1's. The moves on $K_{n,m}$ are equivalent to the following moves on the $n \times m$ matrices:

- a player may change a 1 to a 0
- a player may choose an entire row or column that contains a 1, and change all the 1's to 0's

This allows us to find patterns in the matrix as well as the graph.

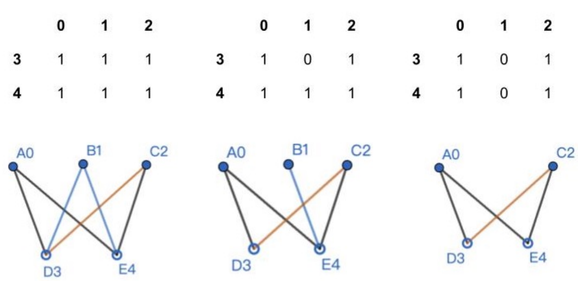

Figure 4: Matrix representation of $K_{2,3}$ and two sequential moves

3 Results

Using a Rust program, we generated the following table for the P- and N-positions of the complete bipartite graph $K_{n,m}$ for positive integers n, m:

Kmn	1	2	3	4	5	6	7	8	9	10
1	N	N	N	N	N	N	N	N	N	N
2	N	P	N	P	N	P	N	P	N	P
3	N	N	N	N	P	N	P	N	P	N
4	N	P	N	P	N	P	N	P	N	
5	N	N	P	N	N	N	P			
6	N	P	N	P	N	P				
7	N	N	P	N	P					
8	N	P	N	P						
9	N	N	P	N						
10	N	P	N							

Figure 5: Table of states for $K_{n,m}$

Next we present a proposition that is useful in determining if a graph is a P-position.

Proposition 3.1. *Let $G = (V, E)$ be a graph with all vertices of positive degree. Suppose $f : V \to V$ satisfies the following:*

- *f is an involution. That is, $f \circ f = f^2 = id_V$;*
- *f has no fixed points. That is, for each vertex $v \in V$, $f(v) \neq v$;*
- *f is a graph automorphism. That is, for each $(u, v) \in E$, $(f(u), f(v)) \in E$;*
- *For each vertex $v \in V$, $(v, f(v)) \notin E$.*

Then G is a P-position. In particular, the following strategy causes player 1 to lose:

- *whenever player 1 takes an edge $(u, v) \in E$, player 2 takes the edge $(f(u), f(v))$;*
- *whenever player 1 takes a vertex $v \in V$, player 2 takes the vertex $f(v)$.*

Before we prove Proposition 3.1, we present some lemmas that are useful.

Lemma 3.2. *Let G denote a graph that satisfies Proposition 3.1. The strategy presented in the proposition is the winning strategy for player 2.*

Proof. If player 1 takes edge $(u, v) \in E$ then since $f(u) \neq u$, $(f(u), f(v)) \neq (u, v)$, so since only the edge (u, v) was removed, edge $(f(u), f(v))$ is available to be removed by player 2.

If player 1 takes vertex $v \in V$ then since $(v, f(v)) \notin E$, the degree of $f(v)$ was not decreased by removing v. So, $f(v)$ has positive degree, so it is legal for player 2 to take vertex $f(v)$. $\square$

Let $G' = (V', E')$ be the graph after both players have made one move following the above strategy, where vertices of degree 0 are removed. Let $f' = f|_{V'}$.

Lemma 3.3. *For each $w \in V$, the following are equivalent: (i) $w \in V'$; (ii) $f(w) \in V'$.*

Proof. Let $w \in V$.

If player 1 takes edge (u, v) then we have two cases:

1. If $w \in \{u, v, f(u), f(v)\}$ then $f(w) \in \{u, v, f(u), f(v)\}$, so the degrees of w and $f(w)$ are each decreased by 1. Since graph isomorphisms preserve degree, we have:

$$\deg_{G'}(w) = \deg_G(w) - 1 = \deg_G(f(w)) - 1 = \deg_{G'}(f(w))$$

2. If $w \notin \{u, v, f(u), f(v)\}$ then $f(w) \notin \{u, v, f(u), f(v)\}$, so the degrees of w and $f(w)$ are unchanged. Thus:

$$\deg_{G'}(w) = \deg_G(w) = \deg_G(f(w)) = \deg_{G'}(f(w))$$

Thus, $w \in V'$ iff $\deg_{G'}(w) > 0$ iff $\deg_{G'}(f(w)) > 0$ iff $f(w) \in V'$.

If player 1 takes vertex v then $(w, v) \in E$ iff $(f(w), f(v)) \in E$ and $(w, f(v)) \in E$ iff $(f(w), v) \in E$. Thus, the degrees of w and $f(w)$ decrease by the same amount (0, 1, or 2), so $w \in V'$ iff $f(w) \in V'$. □

Lemma 3.4. *f' satisfies the listed properties.*

Proof. By Lemma 3.3, for each $w \in V'$ we have $f'(w) = f(w) \in V'$ and $f'(f'(w)) = f(f(w)) = w$. So, f' is an involution.

f' has no fixed points because for $w \in V'$, $f'(w) = f(w) \neq w$.

If player 1 took edge $(u, v) \in E$ then for each edge $(w, x) \in E'$ we have that $(w, x) \neq (u, v)$ because $(u, v) \notin E'$, so $(f'(w), f'(x)) = (f(w), f(x)) \neq (f(u), f(v))$ because automorphisms are injective. We also have that $(w, x) \neq (f(u), f(v))$ because $(f(u), f(v)) \notin E'$, so $(f'(w), f'(x)) = (f(w), f(x)) \neq (u, v)$. Thus, since the only edges removed were (u, v) and $(f(u), f(v))$ and $(f'(w), f'(x))$ is distinct from both (u, v) and $(f(u), f(v))$, we have that $(f'(w), f'(x)) \in E'$.

If player 1 took vertex $v \in V$ then for each edge $(w, x) \in E'$ we have that $w, x \notin \{v, f(v)\}$, so $f(w), f(x) \notin \{v, f(v)\}$, so $(f'(w), f'(x)) = (f(w), f(x)) \in E'$.

In either case, f' is an automorphism.

For each $w \in V'$, $(v, f'(v)) = (v, f(v)) \notin E$, so $(v, f'(v)) \notin E'$. □

Proof of Proposition 3.1. By Lemmas 3.2, 3.4 and induction on the turn number, player 2 always has a legal move by following the given strategy, so player 2 never loses. Thus, player 1 loses, so G is a P-position. □

Corollary 3.5. *The following statements hold about the complete bipartite graph $K_{n,m}$:*

- *$K_{n,m}$ is a P-position if n, m are both even;*
- *$K_{n,m}$ is an N-position if exactly one of n, m is even.*

4 Conjectures

We make the following conjectures about the complete bipartite graph $K_{n,m}$:

- if $n = m$ are odd, the graph $K_{n,m}$ is an N-position;
- if n, m are distinct and both odd greater than 1, the graph $K_{n,m}$ is a P-position.

5 Future Work

Future research on this topic would involve proving our conjectures described above.

Here are some other open problems for the future:

- What do P-position graphs look like?

  In particular, our claim involving the existence of a "symmetry" involution on the vertices describes a type of invariant on graphs that guarantees a P-position. Can we find and describe other types of invariants on graphs that guarantee a P-position?

- Is the problem of determining whether $K_{n,m}$ is a P-position NP-hard?

  This would mean that at least one of our conjectures above cannot be proven. More generally, we can ask this question about all graphs.

- If we extend Kayles Game on graphs to include edge contraction (identifying two vertices connected by an edge), what do P-positions look like in this new game?

6 Acknowledgements

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7 References

[1] Julien Lemoine and Simon Viennot. *Nimbers are inevitable*. 2010. arXiv: 1011.5841 [math.CO]. URL: https://arxiv.org/abs/1011.5841.