STAT5002 Weekly Independent Exercises - solution

Sheet 4 - Week 7

STAT5002

1 Confidence intervals (with known population SD)

1.1

A sample of size 100 from a population with known variance $\sigma^2 = 25$ produces a sample mean of 75. Applying the 68%-95%-99.7% rule, construct a 95% confidence interval for the population mean μ. State the assumption required to construct this confidence interval.

`Answer:`

We are given:
- Sample size: $n = 100$
- Sample mean: $\bar{x} = 75$
- Known variance: $\sigma^2 = 25$, so the population standard deviation is $\sigma = 5$

The standard error of the sample mean is:

$$SE(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{5}{\sqrt{100}} = \frac{5}{10} = 0.5$$

Assuming the Central Limit Theorem (CLT) holds – which is reasonable since $n = 100$ is sufficiently large – the sampling distribution of $\bar{X}$ is approximately normal with mean μ and standard deviation $SE(\bar{X}) = 0.5$.

Using the 68%-95%-99.7% rule (approximation), a 95% confidence interval for the population mean μ is given by:

$$\bar{x} \pm 2 \times SE(\bar{X}) = 75 \pm 2 \times 0.5 = [74, 76]$$

Here, the multiplier 2 is used because approximately 95% of the values under a normal distribution lie within ± 2 standard deviations from the mean.

1.2

A sample of size 900 from a population with known variance $\sigma^2 = 9$ produces a sample mean of 11. Construct a 99% confidence interval for the population mean μ. You may use the outputs of the following R code for this question.

```r
round(qnorm(c(0.985, 0.99, 0.995)), 2)
```

```
[1] 2.17 2.33 2.58
```

`Answer:`

We are given:
- Sample size: $n = 900$
- Sample mean: $\bar{x} = 11$
- Known variance: $\sigma^2 = 9$, so the population standard deviation is $\sigma = 3$

The standard error of the sample mean is:

$$SE(\bar{X}) = \frac{\sigma}{\sqrt{n}} = \frac{3}{\sqrt{900}} = \frac{3}{30} = 0.1$$

Assuming the Central Limit Theorem (CLT) holds – which is reasonable since $n = 900$ is sufficiently large – the sampling distribution of $\bar{X}$ is approximately normal with mean μ and standard deviation $SE(\bar{X}) = 0.1$.

For a 99% confidence interval, we use the value corresponding to `qnorm(0.995)=2.58`, since a 99% confidence level corresponds to an upper tail area of $0.01/2 = 0.005$. Thus, a 99% confidence interval for the population mean μ is given by:

$$\bar{x} \pm 2.58 \times SE(\bar{X}) = 11 \pm 2.58 \times 0.1 = [10.742, 11.258]$$