Time evolution operator and Wick theorem

Time evolution operator

$$|\Psi_I(t)\rangle = U(t,t_0)|\Psi_I(t_0)\rangle$$ **definition**

$$\begin{aligned}|\Psi_I(t)\rangle &= e^{i\frac{H_0 t}{\hbar}}|\Psi_S(t)\rangle = e^{i\frac{H_0 t}{\hbar}}e^{-i\frac{H}{\hbar}(t-t_0)}|\Psi_S(t_0)\rangle \\ &= e^{i\frac{H_0 t}{\hbar}}e^{-i\frac{H}{\hbar}(t-t_0)}e^{-i\frac{H_0 t_0}{\hbar}}|\Psi_I(t_0)\rangle\end{aligned}$$

$$\boxed{U(t,t_0) = e^{i\frac{H_0 t}{\hbar}}e^{-i\frac{H(t-t_0)}{\hbar}}e^{-i\frac{H_0 t_0}{\hbar}}}$$

some properties

$$\begin{cases} U(t_0,t_0) = 1 \\ U^+(t,t_0)U(t,t_0) = U(t,t_0)U^+(t,t_0) = 1 \\ U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3)\end{cases}$$

$U^{-1}(t,t_0)$

$$i\hbar\frac{\partial}{\partial t}|\Psi_I(t)\rangle = H_1(t)|\Psi_I(t)\rangle \quad \Rightarrow \quad i\hbar\frac{\partial}{\partial t}U(t,t_0)|\Psi_I(t_0)\rangle = H_1(t)U(t,t_0)|\Psi_I(t_0)\rangle$$

$$\boxed{i\hbar\frac{\partial}{\partial t}U(t,t_0) = H_1(t)U(t,t_0)}$$

Time evolution operator

integrating from t_0 to t $$U(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt' H_1(t')U(t',t_0)$$

$$U(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt' H_1(t')\left[1 - \frac{i}{\hbar}\int_{t_0}^{t'} dt'' H_1(t'')[1 - \ldots]\right]$$
$$U(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt' H_1(t') + (\frac{-i}{\hbar})^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' H_1(t')H_1(t'') + \ldots$$

$$t > t'$$

$$\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' H_1(t')H_1(t'') = 1/2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' H_1(t')H_1(t'') + 1/2 \int_{t_0}^{t} dt'' \int_{t_0}^{t''} dt' H_1(t'')H_1(t')$$

$$t' > t'' \qquad\qquad t'' > t'$$

(Fetter-Walecka, pgs 54-58)

Time evolution operator

$$\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' H_1(t') H_1(t'') =$$

$$1/2 \quad \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' H_1(t') H_1(t'') + 1/2 \int_{t_0}^{t} dt' \int_{t'}^{t} dt'' H_1(t'') H_1(t') =$$

$$1/2 \quad \int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' \left[H_1(t') H_1(t'') \theta(t' - t'') + H_1(t'') H_1(t') \theta(t'' - t') \right] =$$

$$1/2 \quad \int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' T \left[H_1(t') H_1(t'') \right]$$

Θ step function

$$\boxed{U(t, t_0) = \sum_{n=0}^{\infty} \left(\frac{-i}{\hbar} \right)^n \frac{1}{n!} \int_{t_0}^{t} dt_1 \ldots \int_{t_0}^{t} dt_n T \left[H_1, (t_1) \ldots H_1(t_n) \right]}$$

T time evolution operator:
reassembles operators respect to the time variable from left to right (decreasing order)

Time evolution operator

Adiabatic turning on of the interaction: *description of the eigenstates of a system of interacting particles in terms of the eigenstates of a system of non-interacting particles*

$$H = H_0 + e^{-\epsilon|t|}H_1$$

$$\lim_{t\to 0} H = H_0 + H_1$$

$$\lim_{t\to\pm\infty} H = H_0$$

$$|\Psi_I(t)\rangle = U_\epsilon(t,t_0)|\Psi_I(t_0)\rangle$$

results should be ε-independent

$$U_\epsilon(t,t_0) = \sum_{n=0}^{\infty}\left(\frac{-i}{\hbar}\right)^n \frac{1}{n!}\int_{t_0}^{t} dt_1 \ldots \int_{t_0}^{t} dt_n$$

$$e^{-\epsilon|[t_1|+|t_2|+\ldots]}T[H_1(t_1)\ldots H_1(t_n)]$$

Time evolution operator

$$i\hbar\frac{\partial}{\partial t}|\Psi_I(t)\rangle = H_1(t)|\Psi_I(t)\rangle \quad \longrightarrow \quad \boxed{i\hbar\frac{\partial}{\partial t}|\Psi_I(t)\rangle = e^{-\epsilon|t|}H_1(t)|\Psi_I(t)\rangle \to_{t\to\pm\infty} 0}$$

$$|\Psi_I(t)\rangle = U_\epsilon(t,-\infty)|\Phi_0\rangle$$

In the limit $t \Rightarrow \pm\infty$ the full Hamiltonian is reduced to H_0

Without interaction $|\psi_I\rangle$ would be equal to $|\varphi_0\rangle$

As time increases, interaction is turned on until t = 0 when is completely acting on the system

$$\begin{aligned} |\Psi_H(t)\rangle_{t\to 0} = \lim_{t\to 0} e^{i\frac{Ht}{\hbar}}|\Psi_S(t)\rangle &= |\Psi_S(0)\rangle \\ |\Psi_I(t)\rangle_{t\to 0} = \lim_{t\to 0} e^{i\frac{H_0 t}{\hbar}}|\Psi_S(t)\rangle &= |\Psi_S(0)\rangle \\ |\Psi_H(0)\rangle = |\Psi_I(0)\rangle &= |\Psi_S(0)\rangle \end{aligned}$$

Time evolution operator

$$
\begin{aligned}
|\Psi_H(t)\rangle_{t\to 0} = \lim_{t\to 0} e^{i\frac{Ht}{\hbar}}|\Psi_S(t)\rangle &= |\Psi_S(0)\rangle \\
|\Psi_I(t)\rangle_{t\to 0} = \lim_{t\to 0} e^{i\frac{H_0 t}{\hbar}}|\Psi_S(t)\rangle &= |\Psi_S(0)\rangle \\
|\Psi_H(0)\rangle = |\Psi_I(0)\rangle &= |\Psi_S(0)\rangle
\end{aligned}
$$

$$
\boxed{|\Psi_H(0)\rangle = |\Psi_I(0)\rangle = U_\epsilon(0,-\infty)|\Phi_0\rangle}
$$

This equation describes the eigenstate of an interacting system in terms of the eigenstate of a non-interacting system described by H_0. The result will be physically significant only if $\lim_{\varepsilon\to 0}$ is finite.

Wick theorem

Time ordering operator T $T[ABC\ldots]$

se $t_{n+1} > t_n$

$$T\left[a(t_3)a^+(t_1)a^+(t_2)\right] = -a(t_3)a^+(t_2)a^+(t_1)$$
$$T\left[a(t_2)a^+(t_1)a^+(t_3)\right] = a^+(t_3)a(t_2)a^+(t_1)$$

Normal ordered product N: *reassembles operators in such a way that the expectation value on the vacuum is zero.*

creation operator $\langle 0|a^+ = 0$

destruction operator $a|0\rangle = 0$

$$N[a_1a_2^+a_3a_4^+] = -a_2^+a_4^+a_1a_3$$

Wick theorem

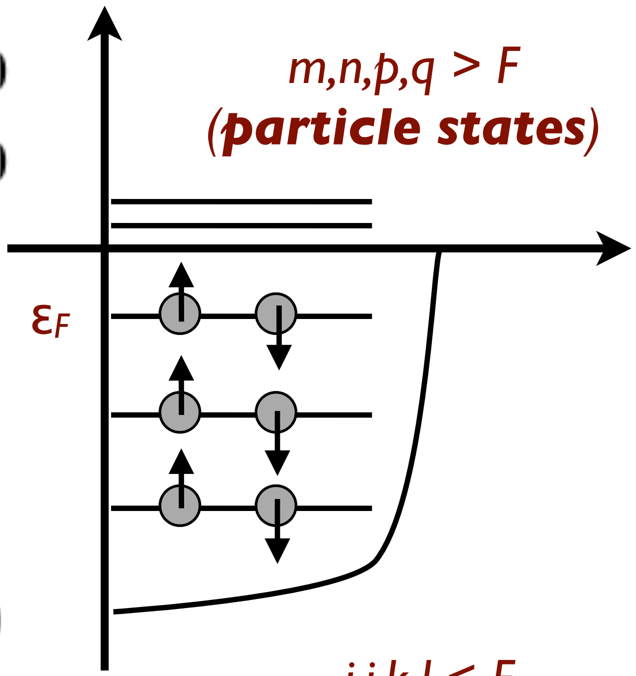
Normal ordered product N: *reassembles operators in such a way that the expectation value on the vacuum is zero. One can define different kind of vacuums*

$$|0\rangle \longrightarrow |\Phi_0\rangle$$

ground state in which all states of lower energies are occupied up to the Fermi level

$$a_m|\Phi_0\rangle = 0$$
$$a_m^+|\Phi_0\rangle \neq 0$$

m,n,p,q > F
*(**particle states**)*



$$N[a_m a_j^+ a_j a_m^+] = \quad a_j a_m^+ a_m a_j^+$$

The vacuum expectation value on Φ_0 is zero

$$a_j|\Phi_0\rangle \neq 0$$
$$a_j^+|\Phi_0\rangle = 0$$

i,j,k,l < F
*(**hole states**)*

Wick theorem

Contractions: $$A^{\alpha} B^{\alpha} \equiv T[AB] - N[AB]$$

Example: if operators are defined at the same time T[...] = [...]

$$a_m^{+\,\alpha} a_i^{\,\alpha} = T[a_m^+ a_i] - N[a_m^+ a_i] = a_m^+ a_i - a_m^+ a_i = 0$$

the result is not an operator but a complex number

It can be proved that performing a contraction is equivalent to the vacuum expectation value

$$\langle \Phi_0 | AB | \Phi_0 \rangle = \langle \Phi_0 | A^{\alpha} B^{\alpha} | \Phi_0 \rangle + \langle \Phi_0 | \underbrace{N[AB]}_{= 0} | \Phi_0 \rangle = A^{\alpha} B^{\alpha} \langle \Phi_0 | \Phi_0 \rangle$$

$$\langle \Phi_0 | T[AB] | \Phi_0 \rangle$$

Never forget to include phases form permutations

$$A^{\alpha} B^{\beta} C^{\alpha} D E F^{\beta} = -A^{\alpha} C^{\alpha} B^{\beta} F^{\beta} D E$$

Wick theorem

Wick Theorem

a string of operators can be written as a sum of normal ordered products in which all possible contractions are considered

$$
\begin{aligned}
T[ABC\ldots Z] = N[ABC\ldots Z] \quad &+ \quad N[A^{\alpha}B^{\alpha}\ldots Z] + N[A^{\alpha}BC^{\alpha}\ldots Z] \\
&+ \quad N[A^{\alpha}B^{\alpha}C^{\beta}\ldots Z] + N[A^{\alpha}B^{\beta}C^{\alpha}\ldots Z] \\
&+ \quad N[A^{\alpha}B^{\alpha}C^{\beta}\ldots Z] + .
\end{aligned}
$$

Wick theorem

Example:

$$
\begin{aligned}
ABCD = N[ABCD] \quad &+ \quad N[A^\alpha B^\alpha CD] + N[A^\alpha BC^\alpha D] + N[A^\alpha BCD^\alpha] \\
&+ \quad N[AB^\alpha C^\alpha D] + N[A^\alpha BCD^\alpha] + N[ABC^\alpha D^\alpha] \\
&+ \quad N[A^\alpha B^\alpha C^\beta D^\beta] + N[A^\alpha B^\beta C^\alpha D^\beta] + N[A^\alpha B^\beta C^\beta D^\alpha] = \\
= N[ABCD] \quad &+ \quad A^\alpha B^\alpha N[CD] - A^\alpha C^\alpha N[BD] + A^\alpha D^\alpha N[BC] \\
&+ \quad B^\alpha C^\alpha N[AD] - B^\alpha D^\alpha N[AC] + C^\alpha D^\alpha N[AB] \\
&+ \quad A^\alpha B^\alpha C^\beta D^\beta - A^\alpha C^\alpha B^\beta D^\beta + A^\alpha D^\alpha B^\beta C^\beta
\end{aligned}
$$

The vacuum expectation value of this string of operators, given a particular ground state, is reduced to the sum of the terms completely contracted. The terms including N products are zero by definition