# The Similarity Renormalization Group

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**Abstract**

Quantum field theories require a cutoff to regulate divergences that result from local interactions, and yet physical results can not depend on the value of this cutoff. The renormalization group employs a transformation that changes the cutoff to isolate hamiltonians that produce cutoff-independent eigenvalues. The similarity renormalization group is based on similarity transformations that regulate off-diagonal matrix elements, forcing the hamiltonian towards a band-diagonal form as the cutoff is lowered. This avoids pathologies that plagued tradition transformations acting on hamiltonians, making it possible to produce a well-behaved perturbative approximation of renormalized hamiltonians in asymptotically free theories. We employ a simple two-dimensional delta function example to illustrate this new renormalization technique.

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# 1   Introduction

Early attempts to combine quantum mechanics and special relativity led to the consideration of local interactions, which are consistent with causality and avoid signals that propagate faster than light. Local interactions lead to divergences in perturbation theory, whose discovery caused some of the best theorists in the world to question the foundations of quantum mechanics. Eventual successes at fitting precise atomic experimental data led to the universal acceptance of renormalization recipes that were acknowledged to make little sense [1]. Initially the perturbative renormalization of QED required theorists to match perturbative expansions in powers of a bare and physical electronic charge [2], but the bare charge clearly diverges logarithmically in QED and the success of an expansion in powers of such a coupling was mysterious at best [3].

The first steps towards making sense of renormalization theory were taken in the 1950's with the invention of the perturbative renormalization group [4, 5, 6, 7, 8], although serious investigators found the theory was still plagued by non-convergent sums because QED is not asymptotically free. The development of Wilson's renormalization group formalism [9, 10] and the discovery of asymptotic freedom [11] allowed physicists to produce a logically reasonable picture of renormalization in which perturbative expansions at any high energy scale can be matched with one another, with no necessity to deal with intermediate expansions in powers of a large parameter.

In this pedagogical article we take advantage of the fact that the divergences in field theory result entirely from local interactions. To understand the most important aspects of renormalization theory requires only a background in nonrelativistic quantum mechanics, because as has been long known the divergences of field theory are directly encountered when one tries to impose locality on the Schrödinger equation. In this case the interactions we consider that are local at all scales are delta functions and derivatives of delta functions. These divergences can be regulated by the introduction of a cutoff, and the artificial effects of this cutoff must be removed by renormalization. The simplicity of the one-body Schrödinger equation makes it possible to renormalize the theory exactly, disentangling the effects of locality from the complicated many-body effects and symmetry constraints encountered in realistic field theories. There is a large literature on the subject [12]-[29], primarily pedagogical.

The similarity renormalization group (SRG) is a very recent development invented by Stan Głazek and Ken Wilson [30, 31], and independently by Franz Wegner [32]. We do not review the applications of this method, which are growing in number.

In the SRG, as in Wilson's original renormalization group formalism [33, 34], transformations that explicitly run the cutoff are developed. These transformations are the group elements that give the renormalization group its name.

In his earliest work [33, 34] Wilson exploited a transformation originally invented by Claude Bloch [35]. It uses a cutoff on the states themselves, and as the cutoff is lowered, states are removed from the Hilbert space. If the hamiltonian is viewed as a matrix, these cutoffs can be seen as limiting the size of this matrix and the transformation reduces this size, as illustrated in Fig. 1a. Wilson introduced a rescaling operation to allow transformed hamiltonians to be compared with initial hamiltonians, despite the fact that they act on different spaces; however, the Bloch transformation is ill-defined and even in perturbation theory it leads to artificial divergences. These divergences come from the small energy dif-
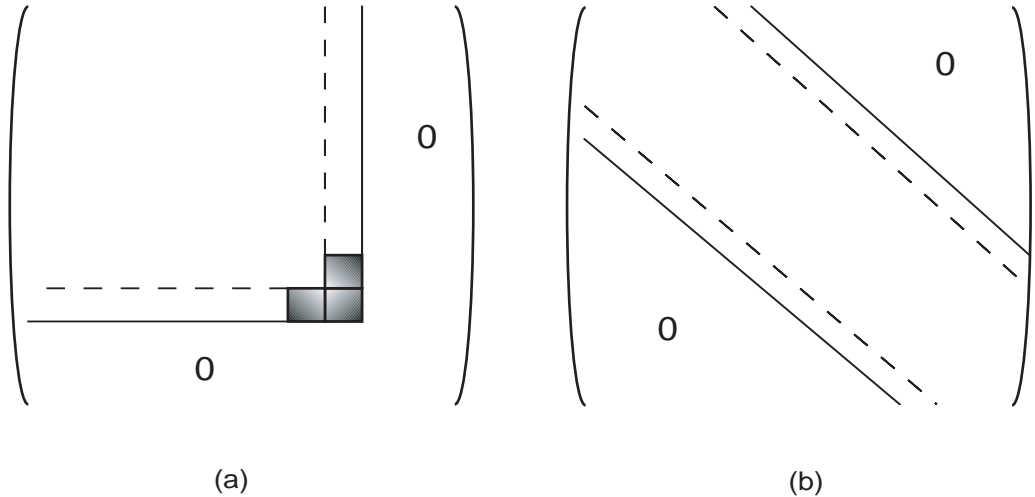
Figure 1: Two ways to run a cutoff on free energy. In (a) a cutoff on the magnitude of the energy is lowered from the solid to the dashed lines, with problems resulting from the removed shaded region. In (b) a cutoff on how far off diagonal matrix elements appear is lowered from the dashed to the solid lines.

ferences between states retained and states removed by the transformation, and they appear in the form of small energy denominators in the perturbative expansion of the transformed hamiltonian. These small energy denominator problems led Wilson to abandon the hamiltonian formulation of field theory in favor of path integral formulations, but the virtues of the hamiltonian formulation over the path integral formulation for many problems remains.

The breakthrough provided by the SRG is that the transformations are typically unitary, making them well-defined, and they run a cutoff on energy differences rather than on individual states, as illustrated in Fig. 1b. Again viewing the hamiltonian as a large matrix, these cutoffs limit the off-diagonal matrix elements and as they are reduced the hamiltonian is forced towards diagonal form. The perturbative expansion for transformed hamiltonians contains no small energy denominators, so the expansion breaks down only when interactions become sufficiently strong, in which case perturbation theory should fail in any case.

Although the SRG has not yet been applied to a wide range of problems, it may be an important new tool both for attacking field theories and non-relativistic many-body problems.

When the SRG is used with coupling coherence [36, 37], which we explain below, it allows us to construct effective theories with the same number of free parameters as the underlying 'fundamental' theory. For the delta-function example there is one fundamental parameter, the strength of the regulated delta-function as the cutoff is removed. In the SRG with coupling coherence, there is only one fundamental coupling and all new couplings are perturbative functions of the fundamental coupling that are given by coupling coherence.

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It is the renormalization group flow of the added couplings, and a boundary condition that they vanish when the fundamental coupling is taken to zero, that fixes their dependence on the fundamental coupling.

The examples we use in this article do not illustrate non-Gaussian fixed points, so their scaling properties are driven by naive dimensional analysis. However, we will see that even in these cases scaling behavior of effective hamiltonians derived using a perturbative similarity renormalization group can be very complicated. We will see that in the perturbative SRG there are errors arising from the approximate treatment of the fundamental running coupling and the approximate treatment of the relation between this coupling and the new couplings of irrelevant operators.

In a realistic calculation the marginal coupling, which corresponds to the strength of the regulated delta function, would be fit to data. In order to clearly illustrate the logarithmic errors that result from using the perturbative SRG equations, we approximate this marginal coupling in this article rather than renormalizing it nonperturbatively by fitting data. The strengths of the irrelevant operators, which correspond to derivatives of the regulated delta function, are approximated using expansions in powers of the approximate running coupling that are fixed by coupling coherence. The approximate running coupling differs from the exact running coupling by inverse powers of logarithms of the cutoff, and the error analysis for the binding energy displays the resultant inverse logarithmic errors in addition to power-law errors seen in all approximate renormalization group calculations. In addition there are errors in the strengths of the irrelevant operators resulting from using a truncated expansion in powers of the running coupling and an approximate running coupling, both of which introduce inverse logarithmic errors in addition to the power-law errors normally seen.

The utility of the renormalization group rests on our ability to accurately determine and control the magnitude of errors resulting from the artificial cutoff. For perturbative calculations this issue is not critical, but in all field theories (and in our example) a scale is reached where the coupling becomes large and a non-perturbative calculation must be done. The renormalization group allows us to eliminate as much perturbative physics as possible (*i.e.*, lower the cutoff as far as possible in an asymptotically free theory), so that the essential degrees of freedom that couple non-perturbatively can be isolated.

# 2   Similarity Renormalization Group

In this section we review the general formulation of the SRG developed by Głazek and Wilson [30, 31] and a specific transformation developed by Wegner [32]. The reader may wish to skip the general formulation on a first reading.

## 2.1   Głazek-Wilson Formulation

Consider a system described by a hamiltonian written in the form

$$H = h + V \ , \tag{1}$$

where $h$ is the free hamiltonian and $V$ is an interaction.

In general, the hamiltonian can couple states of all energy scales and such couplings can be a source of ultraviolet divergences. The goal of the SRG is to obtain an effective hamiltonian

in which the couplings between high and low-energy states are removed, while avoiding any problems from small energy denominators in effective interactions. The procedure is implemented by a unitary transformation that generates effective interactions that reproduce the effects of the eliminated couplings. The effective hamiltonian cannot produce ultraviolet divergences at any order in perturbation theory as long as its matrix elements are finite.

In our discussion we will use the basis of eigenstates of the free hamiltonian,

$$h|i> = \epsilon_i|i> . \tag{2}$$

We start by defining a bare hamiltonian, $H_\Lambda$, regulated by a very large cutoff $\Lambda$ (here with dimensions of energy) on the change in free energy at the interaction vertices,

$$
\begin{aligned}
H_\Lambda &\equiv h + V_\Lambda , & (3) \\
V_\Lambda &\equiv f_\Lambda \overline{V}_\Lambda , & (4) \\
\overline{V}_\Lambda &\equiv v + H_\Lambda^{ct} , & (5)
\end{aligned}
$$

where $f_\Lambda$ is a "similarity function", $\overline{V}_\Lambda$ is defined as the reduced interaction and $H_\Lambda^{ct}$ are counterterms that must be determined through the process of renormalization in order to remove $\Lambda$ dependence in physical quantities.

The similarity function $f_\Lambda$ regulates the hamiltonian by suppressing matrix elements between free states with significantly large energy difference and acts in the following way:

$$
\begin{aligned}
< i|f_\Lambda H_\Lambda|j > &\equiv \epsilon_i\,\delta_{ij} + f_\Lambda(\epsilon_i - \epsilon_j) < i|\overline{V}_\Lambda|j > \\
&\equiv \epsilon_i\,\delta_{ij} + f_{\Lambda ij}\overline{V}_{\Lambda ij}.
\end{aligned}
\tag{6}
$$

Typically, the similarity function is chosen to be a smooth function satisfying

$$
\begin{aligned}
(i)&f_\Lambda(\epsilon_i - \epsilon_j) \to 1, \text{ when } |\epsilon_i - \epsilon_j| << \Lambda , \\
(ii)&f_\Lambda(\epsilon_i - \epsilon_j) \to 0, \text{ when } |\epsilon_i - \epsilon_j| >> \Lambda .
\end{aligned}
\tag{7}
$$

In several papers the similarity function has been chosen to be a step function. Although useful for doing analytic calculations, such a choice can lead to pathologies.

The similarity transformation is defined to act on the bare regulated Hamiltonian, $H_\Lambda$, lowering the cutoff down to a scale $\lambda$:

$$H_\lambda \equiv U(\lambda, \Lambda)\, H_\Lambda\, U^\dagger(\lambda, \Lambda) . \tag{8}$$

The renormalized Hamiltonian can be written in the general form

$$
\begin{aligned}
H_\lambda &\equiv h + V_\lambda , & (9) \\
V_\lambda &\equiv f_\lambda\, \overline{V}_\lambda . & (10)
\end{aligned}
$$

The transformation is unitary, so $H_\Lambda$ and $H_\lambda$ produce the same spectra for observables. Also, if an exact transformation is implemented, the physical predictions using the renormalized Hamiltonian must be independent of the cutoff $\lambda$ and $H_\Lambda^{ct}$ is chosen so that they also become independent of $\Lambda$ as $\Lambda \to \infty$.

The unitarity condition is given by:

$$U(\lambda, \Lambda)\, U^\dagger(\lambda, \Lambda) \equiv U^\dagger(\lambda, \Lambda)\, U(\lambda, \Lambda) \equiv 1 . \tag{11}$$

The similarity transformation $U$ can be defined in terms of an anti-hermitian operator $T_\lambda$ ($T_\lambda^\dagger = -T_\lambda$) which generates infinitesimal changes of the cutoff energy scale,

$$U(\lambda, \Lambda) \equiv \mathcal{T} \exp\left(\int_\lambda^\Lambda T_{\lambda'}\, d\lambda'\right) , \tag{12}$$

where $\mathcal{T}$ orders operators from left to right in order of *increasing* energy scale $\lambda'$. Using

$$T_\lambda = U(\lambda, \Lambda)\, \frac{dU^\dagger(\lambda, \Lambda)}{d\lambda} = -\frac{dU(\lambda, \Lambda)}{d\lambda}\, U^\dagger(\lambda, \Lambda) , \tag{13}$$

and the unitarity condition Eq. (11), we can write Eq. (8) in a differential form,

$$\frac{dH_\lambda}{d\lambda} = [H_\lambda, T_\lambda] . \tag{14}$$

This is a first-order differential equation, which is solved with the boundary condition $H_\lambda|_{\lambda \to \Lambda} \equiv H_\Lambda$. The bare Hamiltonian is typically given by the canonical Hamiltonian plus counterterms that must be uniquely fixed to complete the renormalization.

The operator $T_\lambda$ is defined by specifying how $\overline{V}_\lambda$ and $h$ depend on the cutoff scale $\lambda$. For simplicity in this article, we demand that $h$ is independent of $\lambda$, although this may not lead to an increasingly diagonal effective hamiltonian in all cases. We also demand that no small energy denominators can appear in the hamiltonian. These constraints are implemented by the conditions

$$\frac{dh}{d\lambda} \equiv 0 , \tag{15}$$

$$\frac{d\overline{V}_\lambda}{d\lambda} \equiv [V_\lambda, T_\lambda] . \tag{16}$$

To obtain the renormalized Hamiltonian perturbatively, expand

$$\overline{V}_\lambda = \overline{V}_\lambda^{(1)} + \overline{V}_\lambda^{(2)} + \cdots , \tag{17}$$

$$T_\lambda = T_\lambda^{(1)} + T_\lambda^{(2)} + \cdots , \tag{18}$$

$$H_\Lambda^{ct} = H_\Lambda^{(2),ct} + H_\Lambda^{(3),ct} + \cdots , \tag{19}$$

where the superscripts denote the order in the original interaction, $V$. A general form of these effective interactions is

$$\overline{V}_\lambda^{(i)} = -\sum_{j,k=1}^\infty \delta_{(j+k,i)} \int_\lambda^\Lambda d\lambda' \, [V_{\lambda'}^{(j)}, T_{\lambda'}^{(k)}] + H_\Lambda^{(i),ct} , \tag{20}$$

for $i = 2, 3, \cdots$, with $\overline{V}_\lambda^{(1)} = v$. For instance, the explicit form of the second-order effective interaction $\overline{V}_\lambda^{(2)}$ is

$$\overline{V}_{\lambda ij}^{(2)} = \sum_k V_{ik} V_{kj} \left( \frac{g_{ikj}^{(\lambda\Lambda)}}{\Delta_{ik}} + \frac{g_{jki}^{(\lambda\Lambda)}}{\Delta_{jk}} \right) + H_{\Lambda ij}^{(2),ct} , \tag{21}$$

where

$$g_{ikj}^{(\lambda\Lambda)} \equiv \int_\lambda^\Lambda d\lambda' \, f_{\lambda'jk} \, \frac{df_{\lambda'ki}}{d\lambda'} \, , \tag{22}$$

$$\Delta_{ij} = \epsilon_i - \epsilon_j \, . \tag{23}$$

The counterterms $H_\Lambda^{(n),ct}$ can be determined order-by-order using the idea of coupling-coherence [36, 37]. This is implemented by requiring the hamiltonian to reproduce itself in form under the similarity transformation, the only change being explicit dependence on the running cutoff in the operators and the implicit cutoff dependence in a finite number of independent running couplings. All other couplings depend on the cutoff only through their dependence on the independent couplings. In general, we also demand the dependent couplings to vanish when the independent couplings are taken to zero; i.e, the interactions are turned off. If the only independent coupling in the theory is $\alpha_\lambda$, the renormalized hamiltonian can be written as an expansion in powers of this coupling:

$$H_\lambda = h + \alpha_\lambda \mathcal{O}^{(1)} + \alpha_\lambda^2 \mathcal{O}^{(2)} + \dots \, . \tag{24}$$

In this way, the effective hamiltonian obtained using the similarity transformation is completely determined by the underlying theory. The procedure can be extended to arbitrarily high orders, although it becomes increasingly complex both analytically and numerically.

## 2.2  Wegner Formulation

The Wegner formulation of the SRG [32] is defined in a very elegant way in terms of a flow equation analogous to the SRG Equation in the Głazek-Wilson formalism [30, 31],

$$\frac{dH_s}{ds} = [H_s, T_s] \, . \tag{25}$$

Here the hamiltonian $H_s = h + v_s$ evolves with a flow parameter $s$ that ranges from 0 to $\infty$. The flow-parameter has dimensions $1/(\text{energy})^2$ and is given in terms of the similarity cutoff $\lambda$ by $s = 1/\lambda^2$.

In Wegner's scheme the similarity transformation in defined by an explicit form for the generator of the similarity transformation, $T_s = [H_s, H_0]$, which corresponds to the choice of a gaussian similarity function with uniform width. In the original formulation, Wegner advocates the inclusion of the full diagonal part of the hamiltonian at scale $s$ in $H_0$. For a perturbative calculation of $H_s$, we can use the free hamiltonian, $H_0 = h$. With this choice, the flow equation for the hamiltonian is given by

$$\frac{dH_s}{ds} = [H_s, [H_s, h]] \, . \tag{26}$$

The reduced interaction, $\overline{V}_{sij}$ (the interaction with the gaussian similarity function factored out) is defined by

$$V_{sij} = f_{sij} \, \overline{V}_{sij} \, , \tag{27}$$

$$f_{sij} = e^{-s\Delta_{ij}^2} \, . \tag{28}$$

Assuming that the free hamiltonian is independent of $s$, we obtain the flow equation for the reduced interaction,

$$\frac{d\overline{V}_{sij}}{ds} = \sum_k (\Delta_{ik} + \Delta_{jk}) \ \overline{V}_{sik} \ \overline{V}_{skj} \ e^{-2s\Delta_{ik}\Delta_{jk}} \ , \tag{29}$$

where we use $\Delta_{ij}^2 - \Delta_{ik}^2 - \Delta_{jk}^2 = -2\Delta_{ik}\Delta_{jk}$. We should emphasize that this is an exact equation.

To solve this equation we impose a boundary condition, $H_s|_{s \to s_0} \equiv H_{s_0}$. Then, we make a perturbative expansion,

$$\overline{V}_s = \overline{V}_s^{(1)} + \overline{V}_s^{(2)} + \cdots \ , \tag{30}$$

where the superscript implies the order in the bare interaction $\overline{V}_{s_0}$. It is important to observe that counterterms are implicit in the bare interaction and can be determined in the renormalization process using coupling coherence.

At first order we have

$$\frac{d\overline{V}_{sij}^{(1)}}{ds} = 0 \ , \tag{31}$$

which implies

$$\overline{V}_{sij}^{(1)} = \overline{V}_{s_0 ij} \ , \tag{32}$$

where $s$ is the final scale. Because of the dimensions of the flow parameter we have $s > s_0$, corresponding to a smaller cutoff. The "no cutoff limit" corresponds to $s_0 \longrightarrow 0$.

At second order we have

$$\frac{d\overline{V}_{sij}^{(2)}}{ds} = \sum_k (\Delta_{ik} + \Delta_{jk}) \ \overline{V}_{s_0 ik} \ \overline{V}_{s_0 kj} \ e^{-2s\Delta_{ik}\Delta_{jk}} \ . \tag{33}$$

Integrating, we obtain

$$\overline{V}_{sij}^{(2)} = \frac{1}{2} \sum_k \overline{V}_{s_0 ik} \ \overline{V}_{s_0 kj} \ \left( \frac{1}{\Delta_{ik}} + \frac{1}{\Delta_{jk}} \right) \times$$
$$\times \left[ e^{-2s_0\Delta_{ik}\Delta_{jk}} - e^{-2s\Delta_{ik}\Delta_{jk}} \right] \ . \tag{34}$$

By construction, the Wegner transformation is unitary and avoids small energy denominators. The Wegner transformation is one of the Głazek-Wilson transformations, with the similarity function chosen to be $f_{\lambda ij} = e^{-\Delta_{ij}^2/\lambda^2}$.

## 2.3 Strategy

In our applications of the SRG we use Wegner's transformation. The renormalized hamiltonian for the non-relativistic delta-function potential in D-dimensions is given by

$$H_\lambda(\mathbf{p}, \mathbf{p}') = p^2 \delta^{(D)}(\mathbf{p} - \mathbf{p}') + e^{-\frac{(p^2 - p'^2)^2}{\lambda^4}} \left[ \bar{V}_\lambda^{(1)}(\mathbf{p}, \mathbf{p}') + \bar{V}_\lambda^{(2)}(\mathbf{p}, \mathbf{p}') + ... \right] \ , \tag{35}$$

where

$$\bar{V}_\lambda^{(1)}(\mathbf{p}, \mathbf{p}') = -\frac{\alpha_{\lambda,i}}{(2\pi)^D} , \tag{36}$$

$$\bar{V}_\lambda^{(2)}(\mathbf{p}, \mathbf{p}') = \alpha_{\lambda,i}^2 \, F_s^{(2)}(\mathbf{p}, \mathbf{p}') , \tag{37}$$

$$\bar{V}_\lambda^{(n)}(\mathbf{p}, \mathbf{p}') = \alpha_{\lambda,i}^n \, F_s^{(n)}(\mathbf{p}, \mathbf{p}') . \tag{38}$$

Here $\lambda$ is a momentum cutoff (as opposed to the energy cutoff discussed above) related to the flow parameter by $s = 1/\lambda^4$. The index $i$ denotes the order of the calculation for the running coupling.

The renormalized hamiltonian can be used to compute eigenvalues and eigenstates. Since the hamiltonian is derived perturbatively we expect cutoff dependent errors in the observables. Formally, we can regroup the terms in the renormalized hamiltonian and write it as a momentum expansion, and the expansion parameters are analytic functions of the running coupling $\alpha_\lambda$. Expanding the operators $F_s^{(n)}(\mathbf{p}, \mathbf{p}')$ in powers of $p^2/\lambda^2$ we obtain

$$F_s^{(n)}(\mathbf{p}, \mathbf{p}') = z_0 + z_2 \, \frac{(p^2 + p'^2)}{2\lambda^2} + z_4 \, \frac{(p^4 + p'^2)}{4\lambda^4} + z_4' \, \frac{p^2 p'^2}{2\lambda^4} + \cdots , \tag{39}$$

where the $z_i's$ are constants. Regrouping the terms we obtain

$$H_\lambda(p, p') = p^2 \delta^{(D)}(\mathbf{p} - \mathbf{p}') + e^{-\frac{(p^2 - p'^2)^2}{\lambda^4}} \left[ g_0(\alpha_\lambda) + g_2(\alpha_\lambda) \frac{(p^2 + p'^2)}{2\lambda^2} + \cdots \right] , \tag{40}$$

where

$$g_i(\alpha_\lambda) = a_i \, \alpha_\lambda + b_i \, \alpha_\lambda^2 + \cdots . \tag{41}$$

We can identify three interdependent sources of errors in the perturbative similarity renormalization group when the hamiltonian given by Eq. (40) is truncated and used to compute a physical quantity:

a) errors introduced by the truncation of the hamiltonian at a given order in $p^2/\lambda^2$;

b) errors introduced by the truncation of the hamiltonian at a given order in the running coupling $\alpha_{\lambda,i}$, which correspond to the use of an approximation for the functions $g_i$;

c) errors introduced by the approximation for the running coupling $\alpha_{\lambda,i}$.

In the actual calculation using the hamiltonian given by Eq. (35) errors of type (a) do not appear directly because we do not truncate the operators that appear in the hamiltonian. However, errors of type (b) can be understood as coming from approximating the couplings in front of the operators in Eq. (40). Errors of type (c) appear in our calculations only because we do not fit the canonical coupling to data at each scale, but fix it at a given scale and evolve it perturbatively from that scale. The strategy we would use for a realistic theory (*e.g.*, QED and QCD) is the following:

1) Obtain the renormalized hamiltonian using the similarity transformation and coupling-coherence, truncating the hamiltonian at a given order in powers of $\alpha_{\lambda,i}$.

2) Fix the coupling $\alpha_\lambda$ by fitting an observable (*e.g.*, a bound-state energy).

3) Evaluate other observables (*e.g.*, scattering phase shifts).

As pointed out before, the evaluation of scattering observables with the similarity hamiltonian with standard techniques is complicated and so in our examples we focus on the bound state errors. We fix the coupling at some scale using a given renormalization prescription and use the flow-equation to obtain the coupling as a function of the cutoff $\lambda$ to a given order. We then perform a sequence of bound-state calculations with better approximations for the hamiltonian such that the errors in the bound-state energy are systematically reduced. Once the sources of errors are identified, it becomes relatively simple to analyze order-by-order how such errors scale with the cutoff $\lambda$. In principle, to completely eliminate the errors proportional to some power $m$ in the momentum expansion we should use the similarity hamiltonian with the exact running coupling (renormalized to all orders) and include the contributions up to $\mathcal{O}(p^m/\lambda^m)$ coming from all effective interactions (all orders in $\alpha_\lambda$). Some details of this scaling analysis are presented later for the specific examples we work out. We emphasize again that in a realistic calculation we would fit the coupling $\alpha_\lambda$ to an observable. This nonperturbative renormalization eliminates the dominant source of errors we display in SRG calculations in this paper. We choose to renormalize the coupling perturbatively in this paper because the only observable we compute is the single bound state energy of a delta-function potential, and fitting this energy would prevent us from displaying errors.

# 3   Two-Dimensional Delta-function Potential

We now consider the case of two nonrelativistic particles in two dimensions interacting via an attractive Dirac delta-function potential. The Schrödinger equation for relative motion in position space (with $\hbar = 1$), can be written as:

$$-\nabla_{\mathbf{r}}^2\Psi(\mathbf{r}) - \alpha_0\ \delta^{(2)}(\mathbf{r})\ \Psi(\mathbf{r}) = E\ \Psi(\mathbf{r})\ . \tag{42}$$

Both the delta-function potential in two dimensions and the kinetic energy operator scale as $1/r^2$, therefore, the coupling $\alpha_0$ is dimensionless. As a consequence, the hamiltonian is scale invariant (*i.e.*, there is no intrinsic energy scale) and we can anticipate the presence of logarithmic ultraviolet divergences, analogous to those appearing in QED and QCD. The problem requires renormalization. In this subsection we present the standard method that produces an exact solution analytically, using simple regularization and renormalization schemes [18].

## 3.1   Exact Solution

We start with the Schrödinger equation in momentum space,

$$p^2\ \Phi(\mathbf{p}) - \frac{\alpha_0}{(2\pi)^2}\ \int d^2q\ \Phi(\mathbf{q}) = E\ \Phi(\mathbf{p})\ , \tag{43}$$

where $\Phi(\mathbf{p})$ is the Fourier transform of the position space wave-function,

$$\Phi(\mathbf{p}) = \frac{1}{2\pi}\int d^2r\ \Psi(\mathbf{r})\ e^{-i\mathbf{p}\cdot\mathbf{r}}\ . \tag{44}$$

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As a consequence of scale invariance, if there is any negative energy solution to Eq. (43) then it will admit solutions for any $E < 0$. This corresponds to a continuum of bound states with energies extending down to $-\infty$, so the system is not bounded from below. By rearranging the terms in the Schrödinger equation we obtain

$$\Phi(\mathbf{p}) = \frac{\alpha_0}{2\pi} \frac{\Psi(0)}{(p^2 + E_0)} \, , \tag{45}$$

where $\Psi(0)$ is the position space wave-function at the origin and $E_0 > 0$ is the binding energy.

To obtain the eigenvalue condition for the binding energy, we can integrate both sides of Eq. (45):

$$1 = \frac{\alpha_0}{2\pi} \int_0^\infty dp \, p \, \frac{1}{(p^2 + E_0)} \, . \tag{46}$$

The integral on the r.h.s. diverges logarithmically, so the problem is ill-defined.

The conventional way to deal with this problem is renormalization. First, we regulate the integral with a momentum cutoff, obtaining

$$1 = \frac{\alpha_0}{2\pi} \int_0^\Lambda dp \, p \, \frac{1}{(p^2 + E_0)} = \frac{\alpha_0}{4\pi} \ln\left(1 + \frac{\Lambda^2}{E_0}\right) \, , \tag{47}$$

so that

$$E_0 = \frac{\Lambda^2}{e^{-\frac{4\pi}{\alpha_0}} - 1} \, . \tag{48}$$

Clearly, if the coupling $\alpha_0$ is fixed then $E_0 \to \infty$ as $\Lambda \to \infty$. In order to eliminate the divergence and produce a finite, well-defined bound state we can renormalize the theory by demanding that the coupling runs with the cutoff $\Lambda$ in such a way that the binding energy remains fixed as the cutoff is removed:

$$\alpha_0 \to \alpha_\Lambda = \frac{4\pi}{\ln\left(1 + \frac{\Lambda^2}{E_0}\right)} \, . \tag{49}$$

The dimensionless renormalized running coupling $\alpha_\Lambda$ that characterizes the strength of the interaction is therefore replaced by a new (dimensionful) parameter $E_0 > 0$, the binding energy of the system. This is a simple example of dimensional transmutation [38]: even though the original "bare" hamiltonian is scale invariant, the renormalization procedure leads to a scale that characterizes the physical observables. Note that $E_0$ can be chosen arbitrarily, fixing the energy scale of the underlying (renormalized) theory. It is also interesting to note that the renormalized running coupling $\alpha_\Lambda$ vanishes as $\Lambda \to \infty$ and so the theory is asymptotically free.

This renormalized hamiltonian can be used to compute other observables. The usual prescription for the calculations is to obtain the solutions with the cutoff in place and then take the limit as the momentum cutoff is removed to $\infty$. If an exact calculation can be implemented, the final results should be independent of the regularization and renormalization schemes. As an example, we calculate the scattering wave function,

$$\Phi_k(\mathbf{p}) = \delta^{(2)}(\mathbf{p} - \mathbf{k}) + \frac{\alpha_\Lambda}{2\pi} \frac{\Psi(0)}{(p^2 - k^2 - i\,\epsilon)} \, , \tag{50}$$

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where $k = \sqrt{E}$. Integrating both sides over $\mathbf{p}$ with a cutoff $\Lambda$ in place, we obtain

$$\Psi(0) = \frac{1}{2\pi} \left[ 1 - \frac{\alpha_\Lambda}{4\pi} \ln \left( 1 + \frac{\Lambda^2}{-k^2 - i\epsilon} \right) \right]^{-1} ; \tag{51}$$

thus,

$$\alpha_\Lambda \ \Psi(0) = \frac{1}{2\pi} \left[ \frac{1}{4\pi} \ln \left( 1 + \frac{\Lambda^2}{E_0} \right) - \frac{1}{4\pi} \ln \left( 1 + \frac{\Lambda^2}{-k^2 - i\epsilon} \right) \right]^{-1} . \tag{52}$$

In the limit $\Lambda \to \infty$ we obtain:

$$\alpha_\Lambda \ \Psi(0) = \frac{2}{\ln \left( \frac{k^2}{E_0} \right) - i \ \pi} . \tag{53}$$

The resulting scattering wave function is then given by

$$\Phi_k(\mathbf{p}) = \delta^{(2)}(\mathbf{p} - \mathbf{k}) + \frac{1}{2\pi} \frac{2}{(p^2 - k^2 - i\epsilon)} \left[ \ln \left( \frac{k^2}{E_0} \right) - i \ \pi \right]^{-1} . \tag{54}$$

It is important to note that only S-wave scattering occurs, corresponding to zero angular momentum states. For the higher waves the centrifugal barrier completely screens the delta-function potential and the non-zero angular momentum scattering states are free states.

The same prescription can be used to evaluate the T-matrix or the K-matrix. For the T-matrix, the Lippmann-Schwinger equation with the renormalized potential is given by:

$$T(\mathbf{p}, \mathbf{p}'; k) = V(\mathbf{p}, \mathbf{p}') + \int d^2q \ \frac{V(\mathbf{p}, \mathbf{q})}{k^2 - q^2 + i\epsilon} \ T(\mathbf{q}, \mathbf{p}'; k) . \tag{55}$$

Since only S-wave scattering takes place we can integrate out the angular variable, obtaining

$$T^{(l=0)}(p, p'; k) = V^{(l=0)}(p, p') + \int_0^\Lambda dq \ q \ \frac{V^{(l=0)}(p, q)}{k^2 - q^2 + i\epsilon} \ T^{(l=0)}(q, p'; k) , \tag{56}$$

where

$$V^{(l=0)}(p, p') = -\frac{\alpha_\Lambda}{2\pi} . \tag{57}$$

The Lippmann-Schwinger equation for the "on-shell" T-matrix is given by:

$$T^{(l=0)}(k) = -\frac{\alpha_\Lambda}{2\pi} - \frac{\alpha_\Lambda}{2\pi} \ T^{(l=0)}(k) \int_0^\Lambda dq \ q \ \frac{1}{k^2 - q^2 + i\epsilon}. \tag{58}$$

Solving this equation and taking the limit $\Lambda \to \infty$, we obtain the exact "on-shell" T-matrix:

$$T_0(k) = -\frac{2}{\ln \left( \frac{k^2}{E_0} \right) - i \ \pi} . \tag{59}$$

Here and in what follows we drop the superscript and use the subscript 0 to denote the exact quantities.

In the same way, the S-wave Lippmann-Schwinger equation for the K-matrix is given by

$$K(p, p'; k) = V(p, p') + \mathcal{P} \int_0^\Lambda dq \ q \ \frac{V(p, q)}{k^2 - q^2} \ K(q, p'; k) , \tag{60}$$

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and the exact "on-shell" K-matrix is given by

$$K_0(k) = -\frac{2}{\ln\left(\frac{k^2}{E_0}\right)} \; . \tag{61}$$

The "on-shell" K-matrix and T-matrix are related by

$$K_0(k) = \frac{T_0(k)}{1 - \frac{i\pi}{2}T_0(k)} \; . \tag{62}$$

Using either

$$k \cot \delta_0(k) - ik = -\frac{2}{\pi}\frac{k}{1}\frac{1}{T_0(k)} \; , \tag{63}$$

or

$$k \cot \delta_0(k) = -\frac{2}{\pi}\frac{k}{1}\frac{1}{K_0(k)} \; , \tag{64}$$

we can obtain the exact phase-shifts:

$$\cot \delta_0 = \frac{1}{\pi} \ln\left(\frac{k^2}{E_0}\right) \; . \tag{65}$$

## 3.2 Similarity Renormalization Group Approach

In the two-dimensional case the canonical hamiltonian in momentum space with a delta-function potential can be written as

$$H(\mathbf{p}, \mathbf{p}') = h(\mathbf{p}, \mathbf{p}') + V(\mathbf{p}, \mathbf{p}') \; , \tag{66}$$

where $h(\mathbf{p}, \mathbf{p}') = p^2 \delta^{(2)}(\mathbf{p}-\mathbf{p}')$ corresponds to the free hamiltonian and $V(\mathbf{p}, \mathbf{p}') = -\alpha_0/(2\pi)^2$ corresponds to the Fourier transform of the delta-function potential.

Integrating out the angular variable, the flow equation obtained with Wegner's transformation in terms of matrix elements in the basis of free states is given by

$$\frac{dV_s(p,p')}{ds} = -(p^2 - p'^2)^2 \, V_s(p,p') - \int_0^\infty dk \, k \, (2k^2 - p^2 - p'^2) \, V_s(p,k) \, V_s(k,p') \; . \tag{67}$$

In principle, we can set the boundary condition at $s = 0$ (no cutoff), i.e,

$$H_{s=0}(p,p') = H(p,p') = p^2 \delta^{(1)}(p-p') - \frac{\alpha_0}{2\pi} \; . \tag{68}$$

However, the hamiltonian with no cutoff produces logarithmic divergences, requiring renormalization. As we will see, the boundary condition must be imposed at some other point, leading to dimensional transmutation [38]. The reduced interaction $\bar{V}_s(p,p')$ is defined such that

$$V_s(p,p') = e^{-s(p^2 - p'^2)^2} \, \bar{V}_s(p,p') \; . \tag{69}$$

Assuming that $h$ is cutoff independent we obtain the flow equation for the reduced interaction,

$$\frac{d\bar{V}_s}{ds} = -e^{-2s\,p^2 p'^2} \int_0^\infty dk \, k \, (2k^2 - p^2 - p'^2) \, e^{-2s[k^4 - k^2(p^2 + p'^2)]}$$
$$\times \bar{V}_s(p,k) \, \bar{V}_s(k,p') \; . \tag{70}$$

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This equation is solved using a perturbative expansion, starting with

$$\bar{V}_s^{(1)}(p, p') = -\frac{\alpha_s}{2\pi} \ . \tag{71}$$

We assume a coupling-coherent solution in the form of an expansion in powers of $\alpha_s/2\pi$, satisfying the constraint that the operators $F_s^{(n)}(p, p')$ vanish when $p = p' = 0$,

$$\bar{V}_s(p, p') = -\frac{\alpha_s}{2\pi} + \sum_{n=2}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^n F_s^{(n)}(p, p') \ . \tag{72}$$

Note that the expansion parameter is $\alpha_s/2\pi$.

Using the solution Eq. (72) in Eq. (70) we obtain

$$
\begin{aligned}
\frac{d\bar{V}_s}{ds} &= -\frac{1}{(2\pi)^2} \frac{d\alpha_s}{ds} + \sum_{n=2}^{\infty} \frac{1}{(2\pi)^n} \left[ n\, \alpha_s^{n-1} \frac{d\alpha_s}{ds} F_s^{(n)}(p, p') + \alpha_s^n \frac{dF_s^{(n)}(p, p')}{ds} \right] \\
&= \int_0^\infty dk\, k\, (2k^2 - p^2 - p'^2)\, e^{-2s[p^2 p'^2 + k^4 - k^2(p^2 + p'^2)]} \\
&\quad \times \left[ -\frac{\alpha_s}{2\pi} + \sum_{n=2}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^n F_s^{(n)}(p, k) \right] \left[ -\frac{\alpha_s}{2\pi} + \sum_{m=2}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^m F_s^{(m)}(k, p') \right] \ . \tag{73}
\end{aligned}
$$

This equation is solved iteratively order-by-order in $\alpha_s/2\pi$. Again, if $\alpha_s/2\pi$ is small the operator $\bar{V}_s^{(1)}(p, p')$ can be identified as the dominant term in the expansion of $\bar{V}_s(p, p')$ in powers of $p$ and $p'$. In the $D = 2$ case this operator corresponds to a marginal operator (since the coupling is dimensionless and there is no implicit mass scale). The higher-order terms correspond to irrelevant operators.

At second-order we have

$$-\frac{1}{2\pi} \frac{d\alpha_s}{ds} + \frac{1}{(2\pi)^2} \alpha_s^2 \frac{dF_s^{(2)}(p, p')}{ds} = -\alpha_s^2 \, I_s^{(2)}(p, p') \ , \tag{74}$$

where

$$
\begin{aligned}
I_s^{(2)}(p, p') &= \frac{1}{(2\pi)^2} \int_0^\infty dk\, k\, (2k^2 - p^2 - p'^2)\, e^{-2s[p^2 p'^2 + k^4 - k^2(p^2 + p'^2)]} \\
&= \frac{1}{(2\pi)^2} \frac{e^{-2s\, p^2 p'^2}}{4s} \ . \tag{75}
\end{aligned}
$$

The equation for $\alpha_s$ is obtained by taking the limit $(p, p') \to 0$,

$$\frac{1}{2\pi} \frac{d\alpha_s}{ds} = \alpha_s^2 \, I_s^{(2)}(0, 0) \ , \tag{76}$$

where

$$I_s^{(2)}(0, 0) = \frac{1}{(2\pi)^2} \frac{1}{4s} \ . \tag{77}$$

Integrating Eq. (76) from $s_0$ to $s$,

$$\alpha_{s,2} = \frac{\alpha_{s_0}}{1 - \frac{\alpha_{s_0}}{8\pi} \ln\left(\frac{s}{s_0}\right)} \ . \tag{78}$$

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In terms of the cutoff $\lambda$ we obtain

$$\alpha_{\lambda,2} = \frac{\alpha_{\lambda_0}}{1 + \frac{\alpha_{\lambda_0}}{2\pi} \ln\left(\frac{\lambda}{\lambda_0}\right)} . \tag{79}$$

In principle, knowing the value of $\alpha_{s_0}$ for a given $s_0$ we can determine the running coupling $\alpha_s$ for any $s$. Since we cannot choose $s_0 = 0$ ($\lambda_0 = \infty$), to use Eq. (78) we must specify a renormalization prescription that allows us to fix the coupling at some finite non-zero value of $s_0$. We discuss this issue in detail later in this subsection.

The equation for $F_s^{(2)}(p, p')$ is given by

$$\frac{1}{(2\pi)^2} \frac{dF_s^{(2)}(p, p')}{ds} = I_s^{(2)}(0, 0) - I_s^{(2)}(p, p') . \tag{80}$$

Integrating from $s_0$ to $s$ we obtain

$$
\begin{aligned}
F_s^{(2)}(p, p') &= \int_{s_0}^{s} ds' \, \frac{\left[1 - e^{-2s' \, p^2 p'^2}\right]}{4s'} \\
&= \frac{1}{4}\left[\ln\left(\frac{s}{s_0}\right) - \mathrm{Ei}(-2s \, p^2 \, p'^2) + \mathrm{Ei}(-2s_0 \, p^2 \, p'^2)\right] .
\end{aligned} \tag{81}
$$

Insisting that $F_s^{(2)}(p, p') = 0$ when $p, p' = 0$ we obtain

$$F_s^{(2)}(p, p') = \frac{1}{4}\left[\gamma + \ln(2s \, p^2 \, p'^2) - \mathrm{Ei}(-2s \, p^2 \, p'^2)\right] . \tag{82}$$

At third-order we have

$$
\begin{aligned}
-\frac{1}{2\pi} \frac{d\alpha_s}{ds} + \frac{1}{(2\pi)^2} \alpha_s^2 \frac{dF_s^{(2)}(p, p')}{ds} &+ \frac{2}{(2\pi)^2} \alpha_s \frac{d\alpha_s}{ds} F_s^{(2)}(p, p') + \frac{1}{(2\pi)^3} \alpha_s^3 \frac{dF_s^{(3)}(p, p')}{ds} \\
&= -\alpha_s^2 I_s^{(2)}(p, p') + \alpha_s^3 I_s^{(3)}(p, p') ,
\end{aligned} \tag{83}
$$

where

$$
\begin{aligned}
I_s^{(3)}(p, p') &= \frac{1}{(2\pi)^3} \int_0^{\infty} dk \, k \, (2k^2 - p^2 - p'^2) \, e^{-2s[p^2 p'^2 + k^4 - k^2(p^2 + p'^2)]} \\
&\quad \times \left[F_s^{(2)}(p, k) + F_s^{(2)}(k, p')\right] .
\end{aligned} \tag{84}
$$

In the limit $p, p' \to 0$ we obtain:

$$\frac{1}{2\pi} \frac{d\alpha_s}{ds} = \alpha_s^2 I_s^{(2)}(0, 0) - \alpha_s^3 I_s^{(3)}(0, 0) , \tag{85}$$

where

$$I_s^{(3)}(0, 0) = \frac{1}{(2\pi)^3} \int_0^{\infty} dk \, 2k^3 e^{-2sk^4} \left[F_s^{(2)}(0, k) + F_s^{(2)}(k, 0)\right] . \tag{86}$$

Since $F_s^{(2)}(k, 0) = F_s^{(2)}(0, k) = 0$, the term proportional to $\alpha_s^3$ in Eq. ( 85) is zero.

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The equation for $F_s^{(3)}(p, p')$ is given by

$$\frac{1}{(2\pi)^3} \frac{dF_s^{(3)}(p, p')}{ds} = -\frac{1}{\pi} I_s^{(2)}(0,0) \ F_s^{(2)}(p, p') + I_s^{(3)}(p, p') \ . \tag{87}$$

To obtain $F_s^{(3)}(p, p')$ the integrals in $k$ and $s$ must be evaluated numerically.

At fourth-order we obtain

$$-\frac{1}{2\pi} \frac{d\alpha_s}{ds} \ + \ \frac{1}{(2\pi)^2} \alpha_s^2 \frac{dF_s^{(2)}(p, p')}{ds} + \frac{2}{(2\pi)^2} \alpha_s \frac{d\alpha_s}{ds} F_s^{(2)}(p, p')$$

$$+ \ \frac{1}{(2\pi)^3} \alpha_s^3 \frac{dF_s^{(3)}(p, p')}{ds} + \frac{3}{(2\pi)^3} \alpha_s^2 \frac{d\alpha_s}{ds} F_s^{(3)}(p, p') + \frac{1}{(2\pi)^4} \alpha_s^4 \frac{dF_s^{(4)}(p, p')}{ds}$$

$$= \ -\alpha_s^2 \ I_s^{(2)}(p, p') + \alpha_s^3 \ I_s^{(3)}(p, p') + \alpha_s^4 \ I_s^{(4)}(p, p') \ , \tag{88}$$

where

$$I_s^{(4)}(p, p') \ = \ \frac{1}{(2\pi)^4} \int_0^\infty dk \ k \ (2k^2 - p^2 - p'^2) \ e^{-2s[p^2 p'^2 + k^4 - k^2(p^2 + p'^2)]}$$

$$\times \ \left[ F_s^{(3)}(p, k) + F_s^{(3)}(k, p') + F_s^{(2)}(p, k) F_s^{(2)}(k, p') \right] \ . \tag{89}$$

In the limit $p, p' \to 0$ we obtain:

$$\frac{1}{2\pi} \frac{d\alpha_s}{ds} = \alpha_s^2 \ I_s^{(2)}(0,0) - \alpha_s^4 \ I_s^{(4)}(0,0) \ , \tag{90}$$

where

$$I_s^{(4)}(0,0) = \frac{1}{2\pi} \int_0^\infty dk \ k \ 2k^2 e^{-2sk^4} \left[ F_s^{(3)}(0, k) + F_s^{(3)}(k, 0) \right] \ . \tag{91}$$

For dimensional reasons Eq. (90) takes the form

$$\frac{d\alpha_s}{ds} = \frac{B_2}{s} \alpha_s^2 - \frac{B_4}{s} \alpha_s^4 \ , \tag{92}$$

where $B_2 = \frac{1}{8\pi}$ and $B_4$ can be obtained by evaluating $I_s^{(4)}(0,0)$ (numerically) for $s = 1$. In terms of the cutoff $\lambda$ we obtain:

$$\frac{d\alpha_\lambda}{d\lambda} = \frac{1}{\lambda^4} B_2 \ \alpha_\lambda^2 - \frac{1}{\lambda^4} B_4 \ \alpha_\lambda^4 \ . \tag{93}$$

Integration of Eq. (90) leads to a transcendental equation which is solved numerically in order to obtain the running coupling $\alpha_{s,4}$.

Qualitatively, the errors are expected to be a combination of inverse powers of $\lambda$ and powers (or inverse powers) of $\ln(\lambda)$ coming from the perturbative expansion in powers of $\alpha_\lambda$ for the coefficients of the irrelevant operators and the perturbative approximation for $\alpha_\lambda$.

As pointed out above, to completely determine the renormalized hamiltonian at a given order we need to specify the coupling at some scale $\lambda_0$. The simplest way is to choose a value for the 'exact' $\alpha_{\lambda_0}$. Formally, this fixes the underlying theory; i.e., if we had the exact hamiltonian (to all orders) we could obtain the exact values for all observables. However, since

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the hamiltonian is derived perturbatively and must be truncated at some order in practical calculations, we can only obtain approximate cutoff-dependent results for the observables. Moreover, in this case the errors cannot be directly evaluated, since the exact values remain unknown. As an example, we calculate the bound-state energy choosing $\alpha_{\lambda_0} = 1.45$ at $\lambda_0 = 100$. In Fig. 2 we show the binding-energy as a function of the cutoff $\lambda$ using the following approximations for the interaction:

(a) marginal operator with coupling ($\alpha_{\lambda_0}$),

$$V_\lambda(p, p') = -\frac{\alpha_{\lambda_0}}{2\pi} e^{-\frac{(p^2 - p'^2)^2}{\lambda^4}} \; ; \tag{94}$$

(b) marginal operator with running coupling renormalized to second-order ($\alpha_{\lambda,2}$),

$$V_\lambda(p, p') = -\frac{\alpha_{\lambda,2}}{2\pi} e^{-\frac{(p^2 - p'^2)^2}{\lambda^4}} \; ; \tag{95}$$

(c) marginal operator plus second-order irrelevant operator with running coupling renormalized to second-order ($\alpha_{\lambda,2}, F_\lambda^{(2)}$),

$$V_\lambda(p, p') = \left[ -\frac{\alpha_{\lambda,2}}{2\pi} + \left(\frac{\alpha_{\lambda,2}}{2\pi}\right)^2 F_\lambda^{(2)}(p, p') \right] e^{-\frac{(p^2 - p'^2)^2}{\lambda^4}} \; ; \tag{96}$$

(d) marginal operator with running coupling renormalized to fourth-order ($\alpha_{\lambda,4}$),

$$V_\lambda(p, p') = -\frac{\alpha_{\lambda,4}}{2\pi} e^{-\frac{(p^2 - p'^2)^2}{\lambda^4}} \; ; \tag{97}$$

(e) marginal operator plus second-order irrelevant operator with running coupling renormalized to fourth-order $\alpha_{\lambda,4}, F_\lambda^{(2)}$),

$$V_\lambda(p, p') = \left[ -\frac{\alpha_{\lambda,4}}{2\pi} + \left(\frac{\alpha_{\lambda,4}}{2\pi}\right)^2 F_\lambda^{(2)}(p, p') \right] e^{-\frac{(p^2 - p'^2)^2}{\lambda^4}} \; . \tag{98}$$

We see that as the approximation is improved the cutoff dependence is reduced. As $\lambda \to \infty$ all curves should approach the same binding-energy, which corresponds to the exact value, and as $\lambda$ becomes small the perturbative approximation breaks down.

A similar prescription is to find $\alpha_{\lambda_0}$ at $\lambda_0$ that produces a given binding-energy, $E_0$. Since the fitting is implemented using a truncated hamiltonian, this $\alpha_{\lambda_0}$ is an approximation that becomes more accurate if we use a larger $\lambda_0$ and/or include higher order operators. Although in this case we can evaluate the errors, the scaling analysis becomes complicated as $\lambda \to \lambda_0$ because at this point we force the energy to be the exact value and so the error is zero.

As an example, we calculate the bound-state energy when the coupling is fixed at $\lambda_0 = 100$ to give $E_0 = 1$. In Fig. 3 we show the 'errors' in the binding energy using the same approximations listed above for the potential. As expected, all error lines drop abruptly to zero when $\lambda \to \lambda_0$, where the running coupling is chosen to fit what we define to be the exact binding energy. Away from this point we can analyze the errors. With the hamiltonian (a)
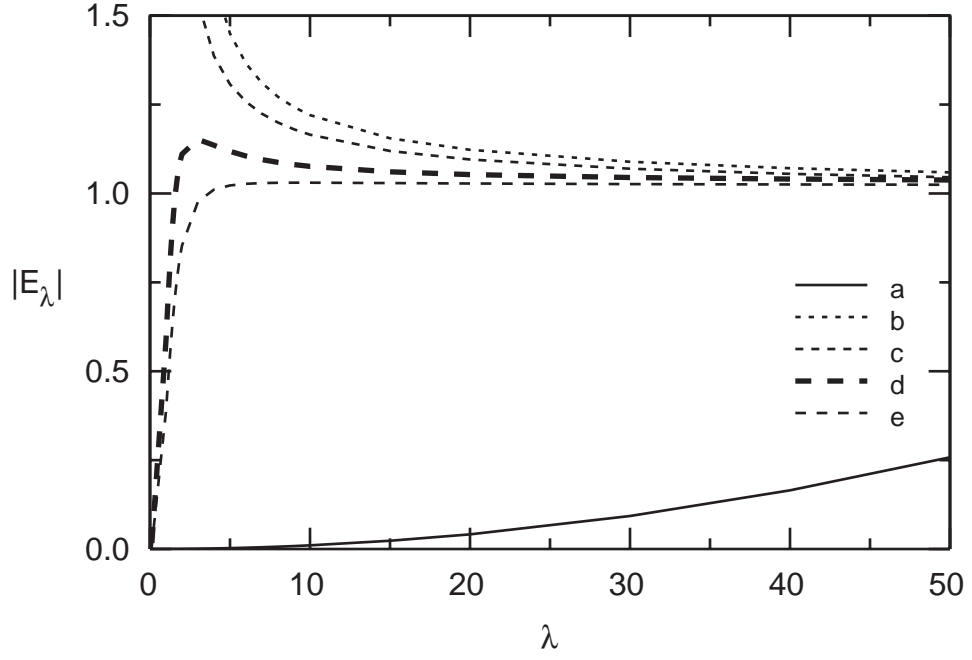
Figure 2: The binding energy for the two-dimensional delta-function potential with various approximations for the SRG hamiltonian. The exact theory is fixed by choosing $\alpha_0 = 1.45$ at $\lambda_0 = 100$.
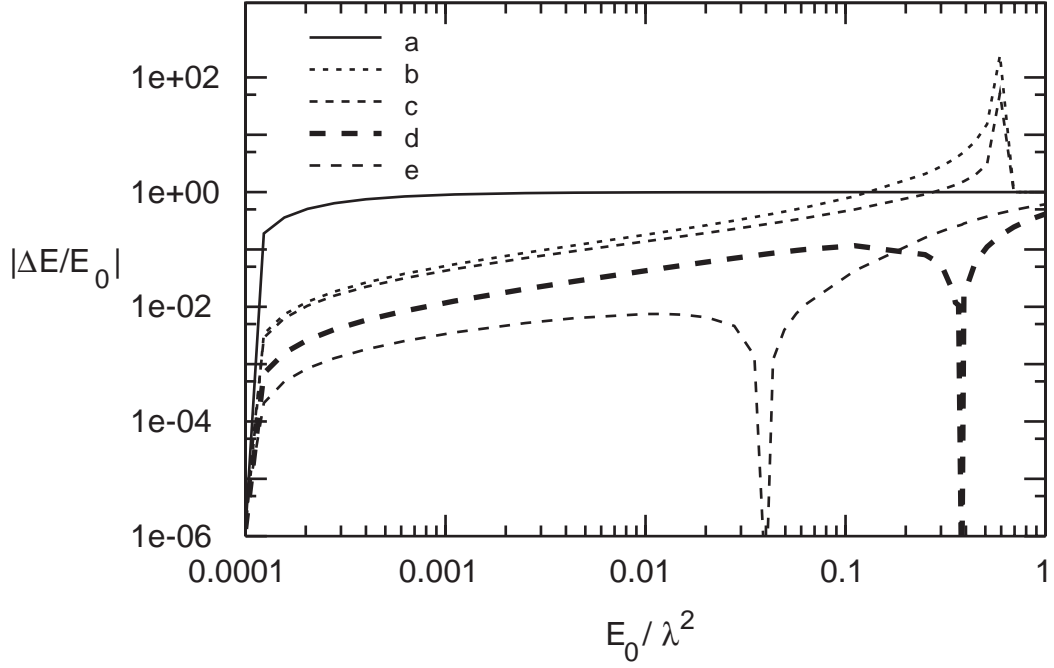
Figure 3: The SRG errors in the binding energy for the two-dimensional delta-function potential using various approximations for the SRG hamiltonian. The exact theory is fixed by choosing $E_0 = 1$ at $\lambda_0 = 100$.

(unrenormalized) we obtain a dominant error that scales like $\ln(\lambda_0/\lambda)$, corresponding to the leading order error. With the renormalized hamiltonian (b) the dominant errors scale like $[\ln(\lambda_0/\lambda)]^{-2}$, indicating the elimination of the leading order logarithmic errors. With the hamiltonian (c) there is a small shift, but no significant change in the error scaling. The added irrelevant operator may remove errors of order $(\lambda_0/\lambda)^2$, but these are smaller than the remaining $[\ln(\lambda_0/\lambda)]^{-2}$ errors. With hamiltonians (d) and (e) in the range of intermediate cutoffs ($E_0 << \lambda^2 << \lambda_0^2$) there is also only a shift in the errors. The dips in (d) and (e) correspond to values of $\lambda$ where the binding energy equals the exact value.

This behavior is a perturbative artifact that can be understood in the following way. Consider the Schrödinger equation with potential (d). Rescaling the momenta $p \rightarrow \lambda\tilde{p}$ we obtain

$$\tilde{p}^2\tilde{\Phi}(\tilde{p}) - \frac{\alpha_{\lambda,4}}{2\pi} \int_0^\infty d\tilde{q}\, \tilde{q}\, e^{(\tilde{p}^2 - \tilde{q}^2)}\, \tilde{\Phi}(\tilde{q}) = \frac{E_\lambda}{\lambda^2}\, \tilde{\Phi}(\tilde{p})\,. \tag{99}$$

and

$$E_\lambda = \lambda^2\, \frac{\left[\int_0^\infty d\tilde{p}\, \tilde{p}\, \left(\tilde{p}^2\, |\tilde{\Phi}(\tilde{p})|^2\right) - \frac{\alpha_{\lambda,4}}{2\pi} \int_0^\infty d\tilde{p}\, \tilde{p} \int_0^\infty d\tilde{q}\, \tilde{q}\, e^{(\tilde{p}^2 - \tilde{q}^2)}\, \tilde{\Phi}(\tilde{p})\, \tilde{\Phi}(\tilde{q})\right]}{\int_0^\infty d\tilde{p}\, \tilde{p}\, |\tilde{\Phi}(\tilde{p})|^2}\,. \tag{100}$$

As shown in Fig. 4, the coupling renormalized to fourth-order, $\alpha_{\lambda,4}$, approximately freezes for small $\lambda$ and as a consequence the bound-state energy scales like $E_\lambda \simeq \lambda^2 \times$ constant
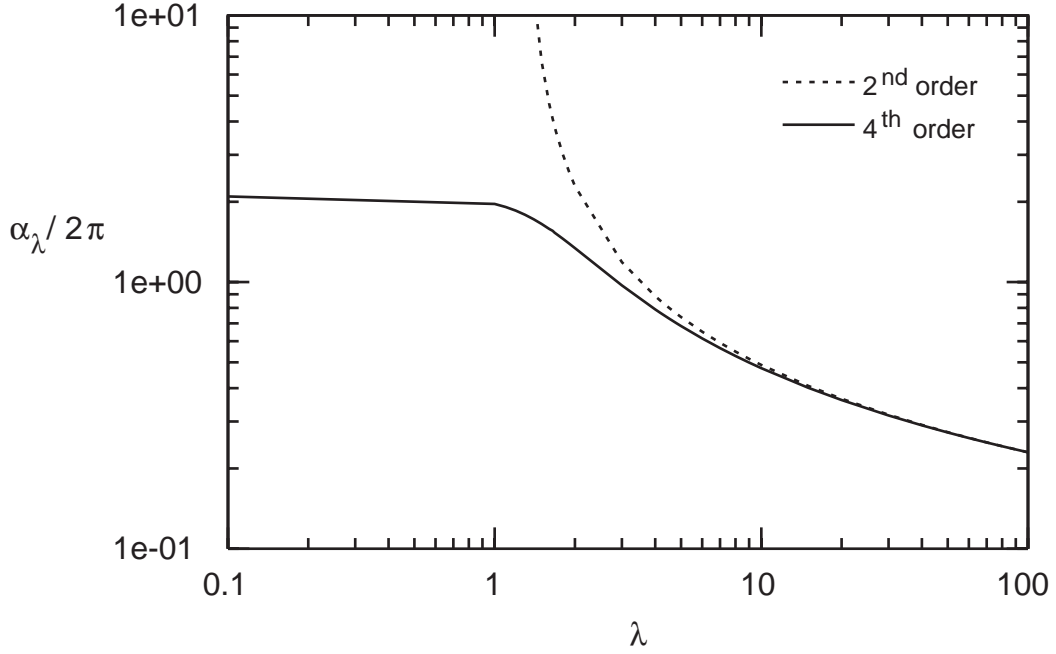
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Figure 4: The SRG running coupling for the two-dimensional delta-function potential obtained with $\alpha_{\lambda_0}$ at $\lambda_0 = 100$ fixed to fit $E_0 = 1$.

eventually becoming equal to the exact value and then deviating again. With the hamiltonian (e) the behavior is similar, with the dip occurring at a different value of $\lambda$ because of the irrelevant operator. For small values of $\lambda$ the lines converge, indicating the breakdown of the perturbative expansion.

An alternative prescription is to use the potential derived in subsection 5.1 as the starting point for the similarity transformation. We introduce a large momentum cutoff $\Lambda$, define

$$\alpha_{s_0=0} = \alpha_\Lambda = \frac{4\pi}{\ln\left(1 + \frac{\Lambda^2}{E_0}\right)} \ , \tag{101}$$

and set all irrelevant operators to zero at $s_0 = 0$. Note that the coupling $\alpha_{\lambda_0}$ is fixed at $\lambda_0 = \infty$ by fitting the exact binding energy. With this definition the similarity hamiltonian with no similarity cutoff becomes well-defined and we can set all of the similarity transformation boundary conditions at $s = 0$. The previous derivation remains essentially the same. The only modification is that all integrals over momentum are cut off ($p \leq \Lambda$).

At second-order we have

$$-\frac{1}{2\pi}\frac{d\alpha_{s,\Lambda}}{ds} + \frac{1}{(2\pi)^2}\alpha_{s,\Lambda}^2 \frac{dF_{s,\Lambda}^{(2)}(p,p')}{ds} = -\alpha_{s,\Lambda}^2\ I_{s,\Lambda}^{(2)}(p,p') \tag{102}$$

where

$$I_{s,\Lambda}^{(2)}(p,p') \quad = \quad \frac{1}{(2\pi)^2}\int_0^\Lambda dk\ k\ (2k^2 - p^2 - p'^2)\ e^{-2s[p^2p'^2+k^4-k^2(p^2+p'^2)]}$$

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$$= \frac{1}{(2\pi)^2} \frac{e^{-2s\,p^2p'^2}}{4s} \; . \tag{103}$$

The resulting second-order running coupling and irrelevant operator are given respectively by

$$\alpha_{s,\Lambda,2} = \frac{\alpha_\Lambda}{1 - \frac{\alpha_\Lambda}{8\pi}\left[\gamma + \ln\left(2s\Lambda^4\right) - \mathrm{Ei}\left(-2s\Lambda^4\right)\right]} \tag{104}$$

and

$$
\begin{aligned}
F_{s,\Lambda}^{(2)}(p,p') &= \frac{1}{4}\left[\gamma + \ln(2s\,p^2\,p'^2) - \mathrm{Ei}(-2s\,p^2\,p'^2)\right] \\
&+ \frac{1}{4}\left[\gamma + \ln(2s\Lambda^4) - \mathrm{Ei}(-2s\Lambda^4)\right] \\
&- \frac{1}{4}\left[\gamma + \ln\left(s\left[(p^2-\Lambda^2)^2 + (p'^2-\Lambda^2)^2 - (p^2-p'^2)^2\right]\right)\right. \\
&\qquad\left. -\mathrm{Ei}\left(-s\left[(p^2-\Lambda^2)^2 + (p'^2-\Lambda^2)^2 - (p^2-p'^2)^2\right]\right)\right] \; . \tag{105}
\end{aligned}
$$

At third-order we have

$$
\begin{aligned}
-\frac{1}{2\pi}\frac{d\alpha_{s,\Lambda}}{ds} &+ \frac{1}{(2\pi)^2}\alpha_{s,\Lambda}^2 \frac{dF_{s,\Lambda}^{(2)}(p,p')}{ds} + \frac{2}{(2\pi)^2}\alpha_{s,\Lambda}\frac{d\alpha_{s,\Lambda}}{ds}F_{s,\Lambda}^{(2)}(p,p') \\
&+ \frac{1}{(2\pi)^3}\alpha_{s,\Lambda}^3 \frac{dF_{s,\Lambda}^{(3)}(p,p')}{ds} = -\alpha_{s,\Lambda}^2 I_{s,\Lambda}^{(2)}(p,p') + \alpha_{s,\Lambda}^3 I_{s,\Lambda}^{(3)}(p,p') \; , \tag{106}
\end{aligned}
$$

where

$$
\begin{aligned}
I_{s,\Lambda}^{(3)}(p,p') &= \frac{1}{(2\pi)^3}\int_0^\Lambda dk\, k\,(2k^2 - p^2 - p'^2)\, e^{-2s[p^2p'^2+k^4-k^2(p^2+p'^2)]} \\
&\times \left[F_s^{(2)}(p,k) + F_s^{(2)}(k,p')\right] \; . \tag{107}
\end{aligned}
$$

In this case,

$$
\begin{aligned}
F_{s,\Lambda}^{(2)}(0,k) &= \frac{1}{4}\left[\gamma + \ln(2s\Lambda^4) - \mathrm{Ei}(-2s\Lambda^4)\right] \\
&= \frac{1}{4}\left[\gamma + \ln(2s\Lambda^4 - 2sk^2\Lambda^2) - \mathrm{Ei}(-2s\Lambda^4 + 2sk^2\Lambda^2)\right] \; , \tag{108}
\end{aligned}
$$

and so

$$
\begin{aligned}
I_{s,\Lambda}^{(3)}(0,0) &= \frac{1}{(2\pi)^3}\frac{1}{4}\int_0^\Lambda dk\, 4k^3 e^{-2sk^4}\left(\left[\gamma + \ln(2s\Lambda^4) - \mathrm{Ei}(-2s\Lambda^4)\right]\right. \\
&- \left.\left[\gamma + \ln(2s\Lambda^4 - 2sk^2\Lambda^2) - \mathrm{Ei}(-2s\Lambda^4 + 2sk^2\Lambda^2)\right]\right) \; . \tag{109}
\end{aligned}
$$

Since $I_{s,\Lambda}^{(3)}(0,0) \neq 0$ the term proportional to $\alpha_{s,\Lambda}^3$ in Eq. (85) does not vanish. To obtain $\alpha_{s,\Lambda,3}$ we evaluate $I_{s,\Lambda}^{(3)}(0,0)$ and solve Eq. (85) numerically.

The equation for $F_{s,\Lambda}^{(3)}(p,p')$ is given by

$$\frac{1}{(2\pi)^3}\frac{dF_{s,\Lambda}^{(3)}(p,p')}{ds} = -\frac{1}{\pi}I_{s,\Lambda}^{(2)}(0,0)\, F_{s,\lambda}^{(2)}(p,p') + I_{s,\Lambda}^{(3)}(p,p') \; . \tag{110}$$

To obtain $F_{s,\Lambda}^{(3)}(p,p')$ the integrals over $k$ and $s$ must be evaluated numerically. In the limit $s\Lambda^4 \to \infty$ with $s$ fixed at some non-zero value

$$F_{s,\Lambda}^{(2)}(p,p') \to \frac{1}{4}\left[\gamma + \ln(2s\,p^2\,p'^2) - \mathrm{Ei}(-2s\,p^2\,p'^2)\right]\;,\tag{111}$$

$$I_{s,\Lambda}^{(3)}(0,0) \to 0\;,\tag{112}$$

and $\alpha_{s,\Lambda}$ becomes indeterminate, requiring the coupling to be fixed at some $s_0 \neq 0$. In this way, we recover the result of the previous prescription.

Although less trivial, this prescription allows a more transparent error analysis. We can also extend the calculation to larger values of the similarity cutoff, $\lambda$, and analyze the errors in a different scaling regime. In Fig. 5 we show the errors in the binding-energy obtained using the following approximations for the potential with $\Lambda = 50$:

(a) marginal operator with coupling $(\alpha_{\lambda_0})$,

$$V_{\lambda,\Lambda}(p,p') = -\frac{\alpha_\Lambda}{2\pi}\,e^{-\frac{(p^2-p'^2)^2}{\lambda^4}}\;;\tag{113}$$

(b) marginal operator with running coupling renormalized to second-order $(\alpha_{\lambda,\Lambda,2})$,

$$V_{\lambda,\Lambda}(p,p') = -\frac{\alpha_{\lambda,\Lambda,2}}{2\pi}\,e^{-\frac{(p^2-p'^2)^2}{\lambda^4}}\;;\tag{114}$$

(c) marginal operator plus second-order irrelevant operator with running coupling renormalized to second-order $(\alpha_{\lambda,\Lambda,2}, F_\lambda^{(2)})$,

$$V_{\lambda,\Lambda}(p,p') = \left[-\frac{\alpha_{\lambda,\Lambda,2}}{2\pi} + \left(\frac{\alpha_{\lambda,\Lambda,2}}{2\pi}\right)^2 F_{\lambda,\Lambda}^{(2)}(p,p')\right]e^{-\frac{(p^2-p'^2)^2}{\lambda^4}}\;;\tag{115}$$

(d) marginal operator with running coupling renormalized to third-order $(\alpha_{\lambda,\Lambda,3})$,

$$V_{\lambda,\Lambda}(p,p') = -\frac{\alpha_{\lambda,\Lambda,3}}{2\pi}\,e^{-\frac{(p^2-p'^2)^2}{\lambda^4}}\;;\tag{116}$$

(e) marginal operator plus second-order irrelevant operator with running coupling renormalized to third-order $\alpha_{\lambda,\Lambda,3}, F_\lambda^{(2)})$,

$$V_{\lambda,\Lambda}(p,p') = \left[-\frac{\alpha_{\lambda,\Lambda,3}}{2\pi} + \left(\frac{\alpha_{\lambda,\Lambda,3}}{2\pi}\right)^2 F_{\lambda,\Lambda}^{(2)}(p,p')\right]e^{-\frac{(p^2-p'^2)^2}{\lambda^4}}\;.\tag{117}$$

There are clearly two distinct scaling regions when an additional large momentum cutoff $\Lambda$ is placed on the initial matrix and a similarity cutoff is then applied. When the similarity cutoff is larger than $\Lambda$, we see power-law improvement resulting from the addition of irrelevant operators. Curves (a), (b), and (d) all have the same slope. None of these hamiltonians contains irrelevant operators, but the marginal coupling differs in each. All results become exact as the similarity cutoff goes to infinity, and these curves are close to one another because the coupling runs little in this region. Curves (c) and (e) show that there is a power-law
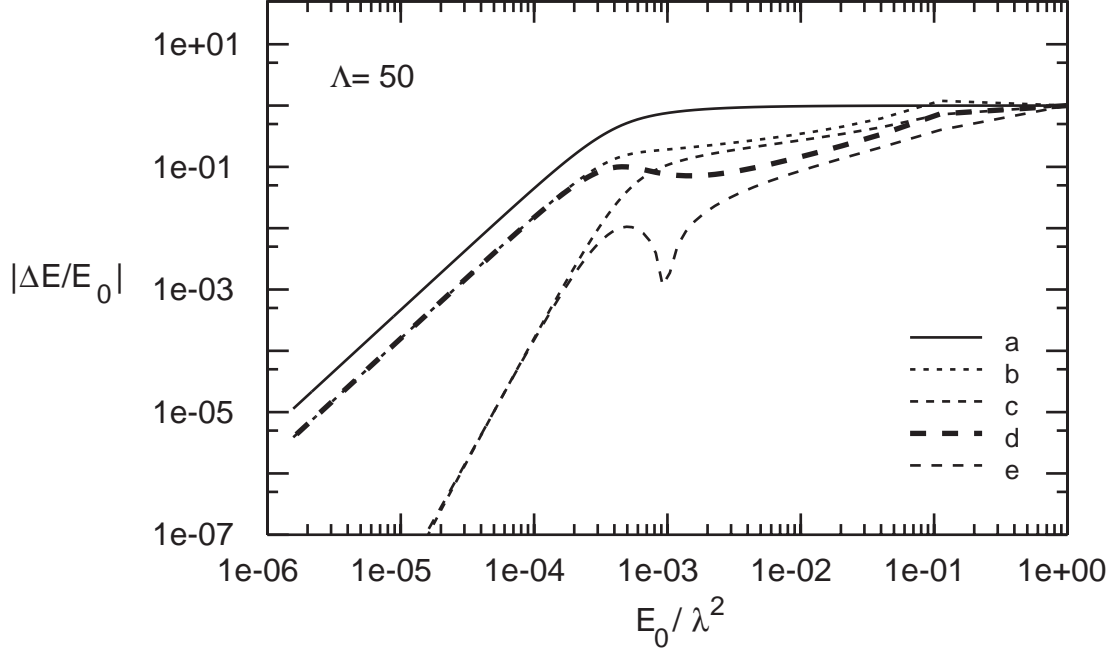
Figure 5: The SRG errors in the binding energy for the two-dimensional delta-function potential using various approximations for the similarity hamiltonian. The exact theory is fixed by regulating the "bare hamiltonian" using a sharp momentum cutoff, $\Lambda$, and letting the bare coupling depend on $\Lambda$ such that the binding energy is fixed. We use $\Lambda = 50$ and $E_0 = 1$.

improvement when irrelevant operators are added, and that once again when the similarity cutoff is larger than $\Lambda$ an improvement in the running coupling makes little difference. Even though the coupling in front of this operator is approximated by the first term in an expansion in powers of the running coupling, the coupling is sufficiently small that this approximation works well and the operator eliminates most of the leading power-law error in curves (a), (b), and (d).

When the similarity cutoff become smaller than $\Lambda$ we see a crossover to a more complicated scaling regime that resembles the SRG scaling discussed above. The error displayed by curve (a) approaches 100%, while the running coupling introduced in curve (b) reduces the error to an inverse logarithm. Improving the running coupling in curve (d) further reduces the error, and we see that curve (c) crosses curve (d) at a point where improving the running coupling becomes more important than adding irrelevant operators. As above, the best results require us to both improve the running coupling by adding third-order corrections and add the second-order irrelevant operators. In no case do we achieve power-law improvement, because as we have discussed there are always residual inverse logarithmic errors. Had we fit the running coupling to data, as we would do in a realistic calculation, we would obtain power-law improvement and the residual error would be proportional to an inverse power of the cutoff times an inverse power of the logarithm of the cutoff.

In Fig. 6 we show the running coupling at 2nd and 3rd order. Although the 3rd order corrections are small for all $\lambda$ and vanish when $\Lambda \to \infty$, the improvement resulting from this correction in Fig. 5 is significant.

The scaling behavior with a large momentum cutoff $\Lambda$ in place is complicated, but it is fairly straightforward to understand it and to find a sequence of approximations that systematically improve the non-perturbative results. The calculations become increasingly complicated, but at each order one must improve the running coupling, or fit it to data, and add higher order irrelevant operators. In a field theory we need to let $\Lambda \to \infty$ and study the scaling behavior of the theory in the regime where $\lambda \ll \Lambda$. Although we do not display a compete set of figures, in Fig. 7 we show what happens to the running coupling as $\Lambda$ is increased, with the bound state energy fixed at one.

As is evident in the exact solution, as $\Lambda$ increases the coupling decreases. When $\lambda \gg \Lambda$, the coupling runs slowly and stays near its asymptotic value. As $\lambda$ approaches $\Lambda$ the coupling begins to run noticeably, and when $\lambda$ becomes much less than $\Lambda$ the coupling approaches a universal curve that is insensitive to its asymptotic value. Plots of the error in the binding energy for various approximations and different values of $\Lambda$ closely resemble Fig. 11, with two scaling regimes whose boundary is $\lambda = \Lambda$.

We close this section by reminding the reader that in all of these calculations there is only one free parameter. In a realistic calculation we would fit this parameter to a binding energy and we would expect to see residual errors in other observables that is inversely proportional to powers of the cutoff and logarithms of the cutoff.

# 4   Conclusions

We have illustrated the similarity renormalization group method for producing effective cutoff hamiltonians using the two-dimensional delta-function potential. We have shown that the SRG with coupling coherence leads to errors that scale as inverse powers of the cutoff and
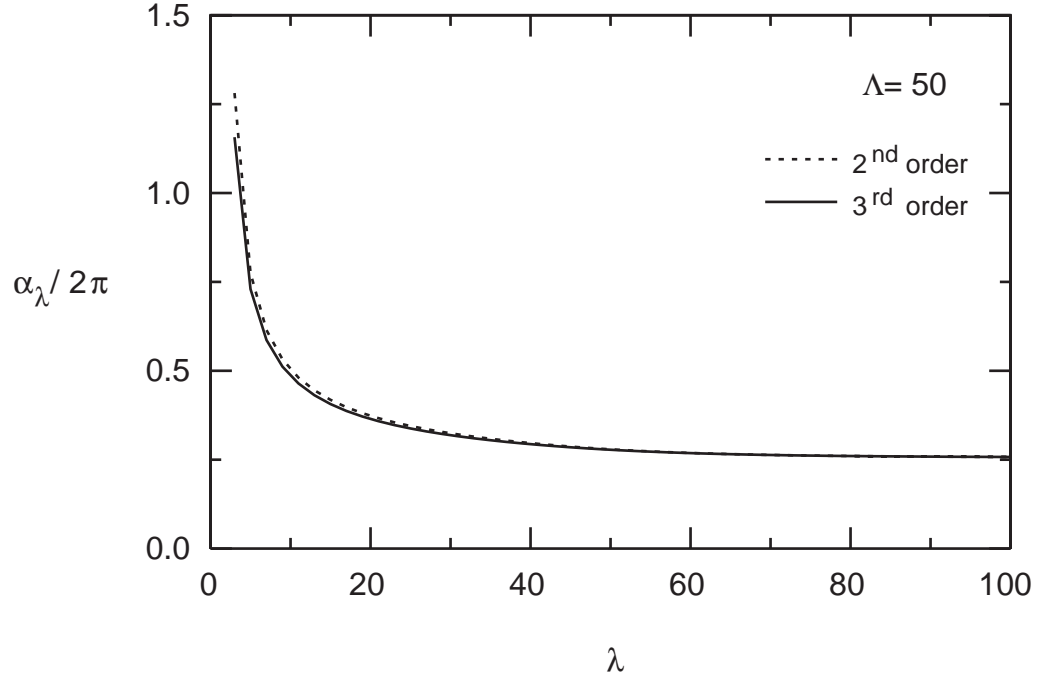
Figure 6: The SRG running coupling for the two-dimensional delta-function potential renormalized to second and third-order obtained with $\alpha_{\lambda_0=\infty} \to \alpha_\Lambda = 4\pi/\ln\left(1 + \frac{\Lambda^2}{E_0}\right)$.
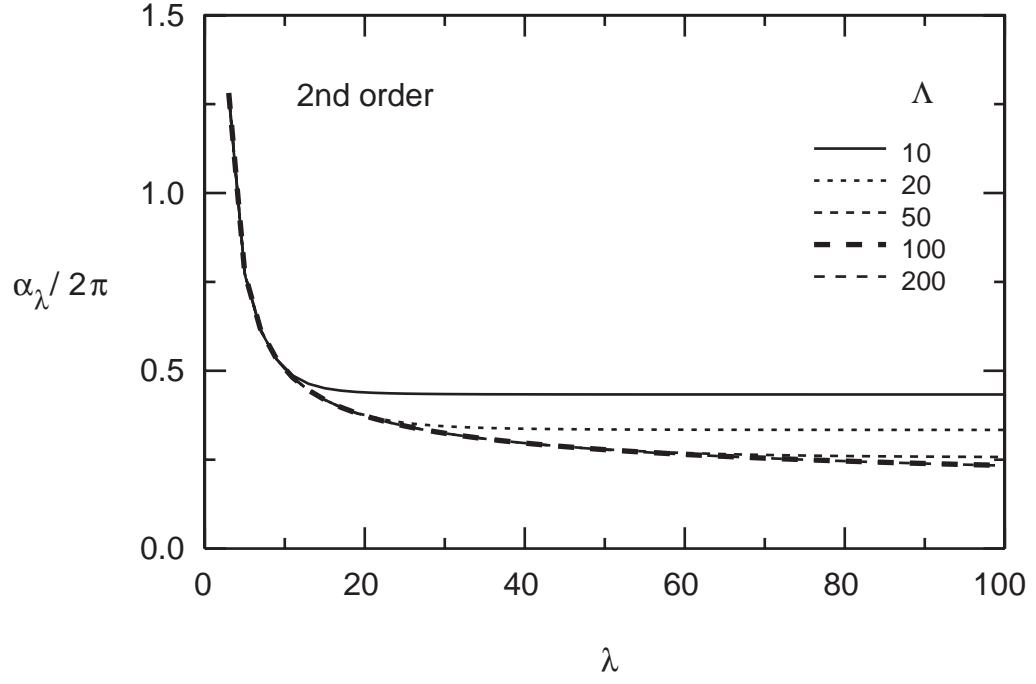
Figure 7: The SRG running coupling for the two-dimensional delta-function potential renormalized to second-order obtained with $\alpha_{\lambda_0=\infty} \to \alpha_\Lambda = 4\pi/\ln\left(1 + \frac{\Lambda^2}{E_0}\right)$.

inverse logarithms of the cutoff. The SRG with coupling coherence requires the same number of parameters as the underlying 'fundamental' theory, but the cost is exponentially increasing algebraic complexity to remove errors that contain inverse powers of logarithms of the cutoff.

# 5   Acknowledgments

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