these nuclei by the mean field approach in a rotation about a symmetry axis, the same effect shows up in the fact that (as discussed in Fig. 3.18) aligned levels with maximal α-values and oblate density distribution are shifted downwards in energy because of their large overlap with the oblate core.

Both methods—shell model calculations in a spherical basis based on angular momentum coupling techniques and cranking calculations in a slightly deformed well—are certainly hard to compare. The spherical shell model is certainly much better (though it takes more effort) as long as one can assume the spherical core to be inert, otherwise one has to take into account multiparticle–multihole configurations. It is therefore limited to the very close vicinity of magic nuclei. On the other side, the cranking model treats the residual interaction by a deformed well and is therefore much easier to handle over a wide range of nuclei. It allows, however, only very qualitative predictions, since it violates conservation of angular momentum and is microscopically derived only in the regions of well deformed nuclei with rotations perpendicular to the symmetry axis (see Sec. 11.4).

CHAPTER 4

Nuclear Forces

4.1 Introduction

Up to now we have only taken into account the forces acting between nucleons in a very qualitative way. We have used some of their properties —such as their short range and saturation character—to explain the volume and surface terms in the liquid drop model. We have also assumed that they give rise to an average single-particle potential. By adjusting a few parameters, we are able to reproduce a large quantity of experimental data. The success of these phenomenological models gives us the confidence to go a step further and investigate the nuclear many-body problem from a more microscopic point of view. In particular, we wish to apply the techniques of modern many-body theory.

The starting point for all these considerations is obviously the two-body interaction between nucleons. There are three basic assumptions in this concept:

(i) Dynamical mesonic degrees of freedom can be neglected and the nucleus can be described as a system of A nucleons whose interaction can be represented by a potential.
(ii) Relativistic effects are negligible.
(iii) Only two-body forces are important.

Even with these rather drastic assumptions, we immediately run into two difficulties when we try to proceed in the way we have discussed:

(i). There exists no derivation of the *nucleon–nucleon force* from first principles. Though this should be possible in principle with the modern theory of gauge fields for quarks and gluons, attempts in this direction are only in their infancy [De 78]. On the other hand, theories that start out from an effective Lagrangian for interacting mesons and nucleons have recently proved quite successful (see, for example, [CLL 73, LLR 75, DSB 77, Vi 78, Ho 80]). The basic ingredient is the pion–nucleon coupling constant, which is known from experiment. The nucleon–nucleon force is obtained without a free parameter for particle distances greater than 0.8 fm. The part from 0 to 0.8 fm is represented by a phenomenological potential containing six parameters in each isospin channel. Excellent fits to the measured nucleon–nucleon phase shifts are achieved.

The potentials used until now have been almost entirely phenomenological (besides the Yukawa part resulting from the one-pion exchange; see below) and contain up to about 50 parameters. The experimental phase shifts are also very well reproduced with these potentials. On the basis of these forces, which we shall discuss very briefly in Section 4.2.2, we should be able to apply the method of many-body theory and to derive the phenomenological properties discussed in the preceding chapters in a quantitative way.

(ii). There is, however, a second difficulty in nuclear theory. These bare nuclear forces are, from a numerical point of view, very ill behaved. They show strong repulsion at short distances (hard core) and cannot be treated straightforwardly by the usual many-body techniques. For instance, they are too strong to be treated by perturbation theory and the hard core makes a direct self-consistent field approach (see Chap. 5), for example, impossible. In fact, the nucleons within a nucleus do not feel the bare nucleon–nucleon interaction. Taking into account that they interact with one another in the presence of many other nucleons permits one to introduce an *effective nucleon–nucleon interaction*, which is rather well behaved and allows application of the usual many-body methods, such as Hartree–Fock theory (Chap. 5). Much work has been done to derive this effective interaction from the bare nucleon–nucleon force. We will in Sec. 4.3 discuss the basic ideas that have been used to achieve this goal. As a result, a great deal has been learned about the properties and the structure of these effective forces. However, in order to reproduce experimental data quantitatively one still needs phenomenological renormalization parameters, and from this point of view the theory is not yet very satisfying.

In most of the so-called microscopic descriptions of the nucleus one uses *phenomenological effective forces*, which are constructed on the basis of these considerations, but depend on some parameters that are adjusted to fit experimental data.

In this chapter we do not want to go into such attempts to derive the bare nucleon–nucleon force [BJ 76a]. In the second section we will discuss some invariance principles that should be obeyed by the bare forces and which already allow one to draw some conclusions about their analytical structure. In the third section, we briefly present the microscopic description of effective interactions and discuss their properties and their field of application. The fourth section presents a number of phenomenological potentials that have been used to represent the residual interaction between the nucleons moving in a given average potential.

We wish to emphasize that in this chapter we are only dealing with the nuclear interaction. The *Coulomb interaction* has to be treated separately. When a comparison with experimental data is required one has always to subtract Coulomb effects first.

4.2 The Bare Nucleon–Nucleon Force

4.2.1 General Properties of a Two-Body Force

The most general quantum mechanical two-body potential V is completely specified by its matrix elements between two-body states (in a coordinate representation $|\mathbf{r}_1, s_1, t_1; \mathbf{r}_2, s_2, t_2\rangle$; where $s_i = \pm\frac{1}{2}$ and $t_i = \pm\frac{1}{2}$ are spin and isospin coordinates) as:

$$\langle \mathbf{r}_1' s_1' t_1' \mathbf{r}_2' s_2' t_2' | V | \mathbf{r}_1 s_1 t_1 \mathbf{r}_2 s_2 t_2 \rangle . \tag{4.1}$$

The space of two-particle states $|\mathbf{r}_1, s_1, t_1; \mathbf{r}_2, s_2, t_2\rangle$ is a product space of the coordinate wave functions $|\mathbf{r}_1\rangle$ and $|\mathbf{r}_2\rangle$ and the spin and isospin vectors $|s_1\rangle$, $|s_2\rangle$ and $|t_1\rangle$, $|t_2\rangle$. Since any operator in the spin space of one particle can be represented as a linear combination of the spin matrices $\sigma_1, \sigma_2, \sigma_3$ and the unity matrix $\sigma_0 = 1$, the most general form of the operator V is

$$V = \sum_{i,k=0}^{3} V_{ik} \sigma_i^{(1)} \sigma_k^{(2)} . \tag{4.2}$$

The function V_{ik} also depends analogously on the isospin operators $\boldsymbol{\tau}^{(1)}$ and $\boldsymbol{\tau}^{(2)}$. In addition to this isospin dependence, the V_{ik} are, in general, integral operators in coordinate space

$$V|\mathbf{r}_1 \mathbf{r}_2\rangle = \int V(\mathbf{r}_1', \mathbf{r}_2', \mathbf{r}_1, \mathbf{r}_2) |\mathbf{r}_1' \mathbf{r}_2'\rangle \, d^3 r_1' \, d^3 r_2' . \tag{4.3}$$

In the special case in which $V(\mathbf{r}_1', \mathbf{r}_2', \mathbf{r}_1, \mathbf{r}_2)$ has the form

$$V(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_1', \mathbf{r}_2') = \delta(\mathbf{r}_1 - \mathbf{r}_1') \delta(\mathbf{r}_2 - \mathbf{r}_2') V(\mathbf{r}_1, \mathbf{r}_2) \tag{4.4}$$

V is called a *local potential*, and we have

$$V|\mathbf{r}_1 \mathbf{r}_2\rangle = V(\mathbf{r}_1, \mathbf{r}_2) |\mathbf{r}_1 \mathbf{r}_2\rangle . \tag{4.5}$$

In this case the interaction between the two particles depends only on the points $\mathbf{r}_1$ and $\mathbf{r}_2$ (and eventually on the spin and isospin). It does not, for instance, depend on the velocity of the particles.

We shall show that, in general, nonlocal potentials correspond to a velocity dependence. We therefore expand*

$$|\mathbf{r}_1', \mathbf{r}_2'\rangle = |\mathbf{r}_1\mathbf{r}_2\rangle + (\mathbf{r}_1' - \mathbf{r}_1)\frac{\partial}{\partial \mathbf{r}_1}|\mathbf{r}_1\mathbf{r}_2\rangle + (\mathbf{r}_2' - \mathbf{r}_2)\frac{\partial}{\partial \mathbf{r}_2}|\mathbf{r}_1\mathbf{r}_2\rangle + \cdots$$

$$=: \exp\left\{(\mathbf{r}_1' - \mathbf{r}_1)\frac{\partial}{\partial \mathbf{r}_1} + (\mathbf{r}_2' - \mathbf{r}_2)\frac{\partial}{\partial \mathbf{r}_2}\right\} : |\mathbf{r}_1\mathbf{r}_2\rangle \tag{4.6}$$

and get, from (4.3),

$$V|\mathbf{r}_1\mathbf{r}_2\rangle = \int V(\mathbf{r}_1', \mathbf{r}_2', \mathbf{r}_1, \mathbf{r}_2)\exp\left\{\frac{i}{\hbar}(\mathbf{r}_1' - \mathbf{r}_1)\mathbf{p}_1 + \frac{i}{\hbar}(\mathbf{r}_2' - \mathbf{r}_2)\mathbf{p}_2\right\}|\mathbf{r}_1\mathbf{r}_2\rangle\, d^3r_1'\, d^3r_2'$$

$$= \tilde{V}(\mathbf{r}_1, \mathbf{p}_1, \mathbf{r}_2, \mathbf{p}_2)|\mathbf{r}_1\mathbf{r}_2\rangle. \tag{4.7}$$

This means that the most general potential can be represented by Eq. (4.2) where the V_{ik} are operators in coordinate space of the form (4.7) (for reasons of simplicity we neglected the isospin dependence).

In the following we investigate the symmetry properties of such potentials $V(1,2) = V(\mathbf{r}_1, \mathbf{p}_1, \boldsymbol{\sigma}^{(1)}, \boldsymbol{\tau}^{(1)}, \mathbf{r}_2, \mathbf{p}_2, \boldsymbol{\sigma}^{(2)}, \boldsymbol{\tau}^{(2)})$. The form of this general function can, however, be restricted by requiring the imposition of a number of symmetries.

In particular, we require the following eight symmetries:

(i) Hermeticity

(ii) Invariance under an exchange of the coordinates

$$V(1,2) = V(2,1). \tag{4.8}$$

This property is strongly connected with the symmetry of the two-particle wave function $|1\,2\rangle$. Since nucleons are fermions, they have to be totally antisymmetric. For example, if we take a product wave function built out of ordinary space, a spin and an isospin part

$$\langle \mathbf{r}_1 s_1 t_1, \mathbf{r}_2 s_2 t_2 \,|\, 12\rangle = \varphi(\mathbf{r}_1, \mathbf{r}_2)\chi(s_1, s_2)\zeta(t_1, t_2) \tag{4.9}$$

we have four combinations compatible with the Pauli principle, which are characterized by the symmetry of the coordinate space and spin part (Table 4.1). The isospin component is determined in each case by requiring the antisymmetry of the total wave function (4.9).

*: : means *normal ordering*, i.e., the derivatives $\partial/\partial \mathbf{r}_i$ should not act on the coordinates $\mathbf{r}_i$ in the expansion of the exponent.

Table 4.1 Characterization of the symmetries of the two-particle state (4.9)

φ	χ	abbreviation	ζ
even	singlet	es	+
even	triplet	et	−
odd	singlet	os	−
odd	triplet	ot	+

(iii) Translational invariance. The potential depends on the relative coordinate $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ only

$$V(1,2) = V(\mathbf{r}, \mathbf{p}_1, \boldsymbol{\sigma}^{(1)}, \boldsymbol{\tau}^{(1)}, \mathbf{p}_2, \boldsymbol{\sigma}^{(2)}, \boldsymbol{\tau}^{(2)}). \tag{4.10}$$

(iv) Galilean invariance. The potential is not changed by a transformation to a system which moves with constant velocity, that is, it depends only on the relative momentum $\mathbf{p} = \frac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2)$:

$$V(1,2) = V(\mathbf{r}, \mathbf{p}, \boldsymbol{\sigma}^{(1)}, \boldsymbol{\tau}^{(1)}, \boldsymbol{\sigma}^{(2)}, \boldsymbol{\tau}^{(2)}). \tag{4.11}$$

(v) Invariance under space reflection. Contrary to the weak interaction, there is no parity violation for strong interactions:

$$V(\mathbf{r}, \mathbf{p}, \boldsymbol{\sigma}^{(1)}, \boldsymbol{\tau}^{(1)}, \boldsymbol{\sigma}^{(2)}, \boldsymbol{\tau}^{(2)}) = V(-\mathbf{r}, -\mathbf{p}, \boldsymbol{\sigma}^{(1)}, \boldsymbol{\tau}^{(1)}, \boldsymbol{\sigma}^{(2)}, \boldsymbol{\tau}^{(2)}). \tag{4.12}$$

(vi) Time reversal invariance guarantees that the equations of motion do not depend on the direction in which the time evolves (for details, see [Me 61])

$$V(\mathbf{r}, \mathbf{p}, \boldsymbol{\sigma}^{(1)}, \boldsymbol{\tau}^{(1)}, \boldsymbol{\sigma}^{(2)}, \boldsymbol{\tau}^{(2)}) = V(\mathbf{r}, -\mathbf{p}, -\boldsymbol{\sigma}^{(1)}, \boldsymbol{\tau}^{(1)}, -\boldsymbol{\sigma}^{(2)}, \boldsymbol{\tau}^{(2)}). \tag{4.13}$$

(vii) Rotational invariance in coordinate space. Rotations in three-dimensional coordinate space act not only on the vectors $\mathbf{r}$ and $\mathbf{p}$ but also on the spin matrices $\boldsymbol{\sigma} = 2\cdot\mathbf{s}$. With respect to spin, the operator V has the form (4.2). It has to be a scalar under a rotation in coordinate space, which means in particular that V_{00} has to be a scalar. There exist three independent scalars which we can construct from the two vectors $\mathbf{r}$ and $\mathbf{p}$, namely $\mathbf{r}^2, \mathbf{p}^2$ and $\mathbf{r}\cdot\mathbf{p} + \mathbf{p}\cdot\mathbf{r}$. However, the latter expression can only appear quadratically because of time reversal invariance (vi). It is more convenient to express $(\mathbf{rp} + \mathbf{pr})^2$ through $\mathbf{r}^2, \mathbf{p}^2$ and $\mathbf{L}^2 = (\mathbf{r}\times\mathbf{p})^2$. V_{00} can then be written as a function of $\mathbf{r}^2, \mathbf{p}^2$ and $\mathbf{L}^2$. Because of invariance (ii) and (v) we find

$$V(\mathbf{r}, \mathbf{p}, \boldsymbol{\sigma}^{(1)}, \boldsymbol{\sigma}^{(2)}) = V(\mathbf{r}, \mathbf{p}, \boldsymbol{\sigma}^{(2)}, \boldsymbol{\sigma}^{(1)}). \tag{4.14}$$

The terms in (4.2) that are linear in $\boldsymbol{\sigma}^{(i)}$ therefore depend only on

$$\mathbf{S} = \tfrac{1}{2}(\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)}). \tag{4.15}$$

To form a scalar, $\mathbf{S}$ has to be multiplied by a vector, which is invariant under space reflection. Only $\mathbf{L}$ fulfils this requirement

$$\mathbf{L}\cdot\mathbf{S} = \tfrac{1}{2}\mathbf{L}(\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)}) \tag{4.16}$$

The quadratic terms in $\boldsymbol{\sigma}$ in Eq. (4.2) form a tensor. It can be decomposed into a scalar $\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}$, a vector $\boldsymbol{\sigma}^{(1)}\times\boldsymbol{\sigma}^{(2)}$, and a symmetric tensor with vanishing trace $(\sigma_i^{(1)}\sigma_k^{(2)} + \sigma_k^{(1)}\sigma_i^{(2)})(1 - \frac{1}{3}\delta_{ik})$. Because of (4.14), $\boldsymbol{\sigma}^{(1)}\times\boldsymbol{\sigma}^{(2)}$ cannot appear. As shown by Okubo and Marshak [OM 58], the only possible independent combinations are

$$\begin{aligned}&\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)},\ (\mathbf{r}\boldsymbol{\sigma}^{(1)})(\mathbf{r}\boldsymbol{\sigma}^{(2)}),\ (\mathbf{p}\boldsymbol{\sigma}^{(1)})(\mathbf{p}\boldsymbol{\sigma}^{(2)}),\\ &(\mathbf{L}\boldsymbol{\sigma}^{(1)})(\mathbf{L}\boldsymbol{\sigma}^{(2)}) + (\mathbf{L}\boldsymbol{\sigma}^{(2)})(\mathbf{L}\boldsymbol{\sigma}^{(1)}).\end{aligned} \tag{4.17}$$

Each of these terms can be multiplied by an arbitrary function of $\mathbf{r}^2, \mathbf{p}^2$ and $\mathbf{L}^2$.

(viii) Rotational invariance in isospin space. Within the isospin formalism, protons

form a doublet (see Sec. 2.6.3) with isospin $\frac{1}{2}$. The two-dimensional representation of the rotational group reproduces all their transformation properties. Rotations within the isospin space (as long as they are not around the 3-axis) produce mixtures of protons and neutrons. Rotational invariance of the nuclear force therefore means the same as charge independence, that is, the proton–proton interaction has the same strength as the neutron–neutron interaction. This has been confirmed by nucleon–nucleon scattering experiments as well as by the symmetry properties of mirror nuclei (e.g., He^3 and H^3). Mathematically speaking, this means that the nucleon–nucleon interaction $V(1,2)$ commutes with the operators of the total isospin

$$\mathbf{T}=\mathbf{t}_1+\mathbf{t}_2. \tag{4.18}$$

Eigenstates can then be constructed of $\mathbf{T}^2, \mathbf{T}_3$ with eigenvalues $T=0$, $T_3=0$ and $T=1$, $T_3=-1, 0, +1$ (isospin singlet and isospin triplet). Charge invariance means that $\mathbf{T}^2$ commutes with the operator of the nuclear force. Therefore, the interactions in $T=1$ states have to be the same (*pp*, *nn*, or symmetric *pn* states). However, they may be different in the $T=0$ state (antisymmetric *pn* system).

The formalism of isospin matrices $\tau=(\tau_1, \tau_2, \tau_3)$ is identical to that of regular spin. Since there is no other vector in isospin space, the only isospin invariant combination of the isospin matrices corresponding to particle 1 and 2 is

$$V_0+V_\tau \tau^{(1)}\tau^{(2)}. \tag{4.19}$$

The functions V_0 and V_τ depend on the remaining coordinates such as $\mathbf{r}, \mathbf{p}, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2$, as we have already discussed.

Not all of the combinations possible from the symmetric point of view have been used to describe the nuclear force. We will mention here only the most important terms:

(i). Among the local forces, which do not depend on the velocity, the *central force* is the most important. It depends only on the distance r between the nucleons:

$$V_C(1,2)=V_0(r)+V_\sigma(r)\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)}+V_\tau(r)\tau^{(1)}\tau^{(2)}+V_{\sigma\tau}(r)\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)}\tau^{(1)}\tau^{(2)}. \tag{4.20}$$

(ii). The only remaining local part is the *Tensor force*

$$V_T(1,2)=\left[V_{T_0}(r)+V_{T_\tau}(r)\tau^{(1)}\tau^{(2)}\right]\cdot S_{12} \tag{4.21}$$

with

$$S_{12}=\frac{3}{r^2}(\boldsymbol{\sigma}^{(1)}\mathbf{r})(\boldsymbol{\sigma}^{(2)}\mathbf{r})-\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)}.$$

The term $-\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)}$ is added to make sure that an average of $V_T(1,2)$ taken over all directions of $\mathbf{r}$ vanishes:

$$\int V_T(1,2)\,d\Omega=0 \quad \text{where} \quad \mathbf{r}=(r,\Omega). \tag{4.22}$$

nucleon–nucleon potential is given by the quadrupole moment of the deuteron, which cannot be explained by pure central forces.

(iii). The most important nonlocal term is the *two-body spin orbit interaction*

$$V_{LS}=V_{LS}(r)\mathbf{L}\cdot\mathbf{S}. \tag{4.23}$$

As we shall see in Chap. 5, such a two-body spin orbit potential causes the one-body spin orbit term in the average single-particle nuclear potential, used to explain the magic numbers in nuclei.

(iv). One sometimes also uses a *second-order spin orbit interaction*:

$$V_{LL}=V_{LL}(r)\left\{(\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)})\mathbf{L}^2-\frac{1}{2}\left[(\boldsymbol{\sigma}^{(1)}\mathbf{L})(\boldsymbol{\sigma}^{(2)}\mathbf{L})+(\boldsymbol{\sigma}^{(2)}\mathbf{L})(\boldsymbol{\sigma}^{(1)}\mathbf{L})\right]\right\}. \tag{4.24}$$

4.2.2 The Structure of the Nucleon–Nucleon Interaction

The central force (4.20) is the most important part of the nucleon–nucleon interaction. It can also be represented in terms of exchange or projection operators.

The operators

$$P^\sigma=\frac{1}{2}(1+\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)}), \qquad P^\tau=\frac{1}{2}(1+\tau^{(1)}\tau^{(2)}) \tag{4.25}$$

exchange the spin and isospin coordinates, respectively, in a wave function. For instance, we can apply P^σ to the wave function (4.9) and obtain

$$P^\sigma\varphi(\mathbf{r}_1\mathbf{r}_2)\chi(s_1,s_2)\zeta(t_1,t_2)=\varphi(\mathbf{r}_1,\mathbf{r}_2)\chi(s_2,s_1)\zeta(t_1,t_2). \tag{4.26}$$

This is easy to understand by using the operator of the total spin $\mathbf{S}$ (4.15). The eigenstates of $\mathbf{S}^2$ are singlet and triplet states and we find:

$$P^\sigma=\frac{1}{2}\left(1+2(\mathbf{S}^2-\mathbf{s}^{(1)^2}-\mathbf{s}^{(2)^2})\right)=S(S+1)-1=\begin{cases}1 & \text{for triplet}\\ -1 & \text{for singlet.}\end{cases} \tag{4.27}$$

We can also define an operator P^r which exchanges the spacial coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$ of the particles.* Since the wave function has to be antisymmetric under the exchange of all coordinates of the particles 1 and 2, the Pauli principle may be written in the form

$$P^rP^\sigma P^\tau=-1. \tag{4.28}$$

We can therefore express the operator $P^\tau=-P^rP^\sigma$ and eliminate the products $\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)}$ and $\tau^{(1)}\tau^{(2)}$ in Eq. (4.20).

*The operator P^r can be represented by a nonlocal operator in coordinate space, viz:

$$V(\mathbf{r})P^r\psi(\mathbf{r}_1,\mathbf{r}_2)=\int V(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{r}_1'-\mathbf{r}_2)\delta(\mathbf{r}_2'-\mathbf{r}_1)\psi(\mathbf{r}_1',\mathbf{r}_2')\,d^3r_1'\,d^3r_2'.$$