Thursday, 14 . Nov

# Example Sheet 04

---

## Exercise 1: Inferences and validity

Prove (by arguing about valuation functions) or refute (by giving a counterexample) the following meta-logical statement:

If $\Rightarrow\phi\vee\psi$, then $\Rightarrow\phi$ or $\Rightarrow\psi$.

Hints:

Validity of meta-logical statements instead of objects(SL formulas)(in assignments sheet 03).

Considering the definition of Logical Inference we had last week and interpreting the given statement.

Prove the valid ones: consider the possible methods of conducting proof we discussed last week. (Induction, deduction, indirect proof: the idea of negation then contradiction)

Provide counter examples for invalid ones: come up with some certain formulas as counter examples.

---

## Exercise 2: Truth trees for inferences

Prove or refute the following inferences using the calculus of proof trees. For the inferences that do not hold, give the counter valuation function associated with each open branch.

$p\wedge p,\ q\rightarrow r \vdash_{TT} p\vee r$

Hints:

Draw truth trees.

Comparing to the trees we draw last week(find strategies for drawing truth trees in example sheet 03), we have more than on assumptions this week.

=> root node become more complicated.

Therefore there is more than one way to draw the tree, since more assumptions can be used here. (Examples on slide 7 and 8)

Consider the strategies(fins in example sheet 03) we discussed last week and refer to the examples on slides(page 7 and 8) this week, which illustrates the difference from the tree  starts from one assumption.

# Exercise 3: Understanding truth trees

Hints:

Inferences (with a finite set of premises; from now on we tacitly assume that premise sets are finite) can always be transformed into tautologies using the deduction theorem

Inferences can also directly be proven using truth trees though:
-premises are assumed to be true
-conclusion is assumed to be false

**Theorem**
Let $\phi_1, \ldots, \phi_n$ be formulas of statement logic. $\psi$ follows logically from the premises $\phi_1, \ldots, \phi_n$ if and only if every branch of a truth tree which starts with $\phi_1, \ldots, \phi_n$ and $\neg\psi$ and only uses the known rules can be closed with an "x" because every formula occurs in it both in negated and non-negated form.

# Exercise 4: Natural deduction

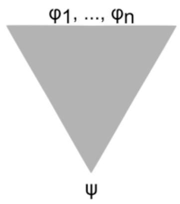
Prove the following derivability claims by presenting natural deduction derivations.

$p \rightarrow q, q \rightarrow r \vdash_{ND} \neg p \vee r$

Hints:

## What is natural deduction?

**Definition**
If it is possible to construct a proof of the form




according to the rules of natural deduction and ϕ1, . . . , ϕn are the undischarged assumptions, then ψ is derivable from ϕ1,...,ϕn+k, i.e.

$$\phi_1, \ldots, \phi_{n+k} \vdash \psi$$

## How to construct natural deduction?

**Calculus** of natural deduction:
-syntactic calculus: only the syntactic form of the formula matters (so the calculus of truth trees is also syntactic, despite its name)
-two central issues for each operator O:

> *Use introduction rule when you want to introduce some operator, mostly when such operator exits in conclusion.
> *Use elimination rule when you want to eliminate unwanted operator, for construction purpose.

For each operator(negation, conjunction, disjunction, implication and equivalence) of statement logic, there are one ore two introduction rules and one or two elimination rules

In addition to the mentioned above, we introduce an auxiliary symbol ⊥ (“**falsum**”), with the intended meaning that for all valuation functions V , V (⊥) = 0.

There is also a pair of introduction and elimination rule for “falsum”.

## Further notes on natural deduction:

*-assumptions (leaves of the tree) can be chosen freely*
*-trees are extended /two trees are combined by applying introduction rules or elimination rules*
*-for each operator of statement logic, there are one ore two introduction rules and one or two elimination rules*
*-the linear order of the premises of a rule is irrelevant*
*-some rules require assumptions to be discharged (marked with an indexed overline)*
*-“discharging an assumption ϕ” means to mark a (possibly empty) set of occurrences of ϕ*

*Thinking:*

So far we have learned some confusingly similar thing, some of which are basically the same thing with different name (from the perspective of semantics and syntactics respectively), while some of which are significantly different ideas that are to be differentiated from one another.

$\Rightarrow$ vs. $\vdash$

Notation: we use $\vdash$ (rather than $\Rightarrow$) for syntactically derived inferences

Terminology:
syntactically proven formulas are called theorems (which is the counterpart to the semantic notion of a tautology)

If the conclusion $\phi$ can be syntactically derived from the premises M, then $\phi$ is derivable from M (counterpart to the semantic notion “follows logically”)

**Theorem** $M \vdash \phi$ iff $M \Rightarrow \phi$

*Also:*

Syntactical idea “Falsum($\perp$)” vs. Semantical idea contradiction

*Further thinking:*

$\rightarrow$ vs. $\rightsquigarrow$ vs. $\Rightarrow$

A strict implication as logical operator.

A denotation for natural language expression “if”.

A logical inference(a syntactical correspondence of entailment).