

Statistical Natural Language Processing

A refresher on probability theory

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Why probability theory?

But it must be recognized that the notion 'probability of a sentence' is an entirely useless one, under any known interpretation of this term. — Chomsky (1968)

Short answer: practice proved otherwise.

Slightly long answer

- Many linguistic phenomena are better explained as tendencies, rather than fixed rules
- Probability theory captures many characteristics of (human) cognition, language is not an exception

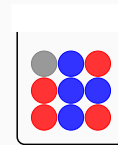
What is probability?

- Probability is a measure of (un)certainity
- We quantify the probability of an event with a number between 0 and 1
 - 0 the event is impossible
 - 0.5 the event is as likely to happen as it is not
 - 1 the event is certain
- The set of all possible *outcomes* of a trial is called *sample space* (Ω)
- An *event* (E) is a set of outcomes

Axioms of probability state that

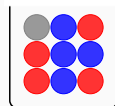
- $P(E) \in \mathbb{R}, P(E) \geq 0$
- $P(\Omega) = 1$
- For *disjoint* events E_1 and E_2 , $P(E_1 \cup E_2) = P(E_1) + P(E_2)$

What you should already know



- $P(\{\bullet\}) = 4/9$
- $P(\{\bullet\}) = 4/9$
- $P(\{\bullet\}) = 1/9$
- $P(\{\bullet, \bullet\}) = 8/9$
- $P(\{\bullet, \bullet, \bullet\}) = 1$
- $P(\{\bullet\bullet\}) = 16/81$
- $P(\{\bullet\bullet\}) = 16/81$
- $P(\{\bullet\bullet\}) = 4/81$
- $P(\{\bullet\bullet\}) = 1/81$
- $P(\{\bullet\bullet, \bullet\bullet\}) = 20/81$

Where do probabilities come from



Axioms of probability do not specify how to assign probabilities to events.

Two major (rival) ways of assigning probabilities to events are

- Frequentist (objective) probabilities: probability of an event is its relative frequency (in the limit)
- Bayesian (subjective) probabilities: probabilities are degrees of belief

Random variables

- A random variable is a variable whose value is subject to uncertainties
- A random variable is always a number
- Think of a random variable as mapping between the outcomes of a trial to (a vector of) real numbers (a real valued function on the sample space)
- Example outcomes of uncertain experiments
 - height or weight of a person
 - length of a word randomly chosen from a corpus
 - whether an email is spam or not
 - the first word of a book, or first word uttered by a baby

Note: not all of these are numbers

Random variables

mapping outcomes to real numbers

- Continuous
 - frequency of a sound signal: 100.5, 220.3, 4321.3 ...
- Discrete
 - Number of words in a sentence: 2, 5, 10, ...
 - Whether a review is negative or positive:

| Outcome | Negative | Positive |
|---------|----------|----------|
| Value | 0 | 1 |

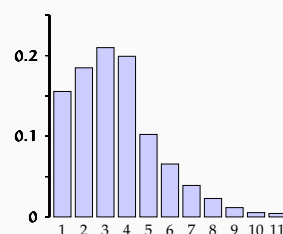
– The POS tag of a word:

| Outcome | Noun | Verb | Adj | Adv | ... |
|---------|-------|-------|-------|-------|-----|
| Value | 1 | 2 | 3 | 4 | ... |
| ...or | 10000 | 01000 | 00100 | 00010 | ... |

Probability mass function

Example: probabilities for sentence length in words

- Probability mass function (PMF)* of a *discrete* random variable (X) maps every possible (x) value to its probability ($P(X = x)$).



| x | P(X = x) |
|----|----------|
| 1 | 0.155 |
| 2 | 0.185 |
| 3 | 0.210 |
| 4 | 0.194 |
| 5 | 0.102 |
| 6 | 0.066 |
| 7 | 0.039 |
| 8 | 0.023 |
| 9 | 0.012 |
| 10 | 0.005 |
| 11 | 0.004 |

Populations, distributions, samples

- A probability distribution characterizes a random variable
- We can define a distribution with a vector or table of probabilities, if we have a finite sample space
- Otherwise, we use (parametric) functions to map the (infinite) set of outcomes to probabilities
- Probability distributions characterize possibly infinite *populations*
- In most cases we have to work with *samples*

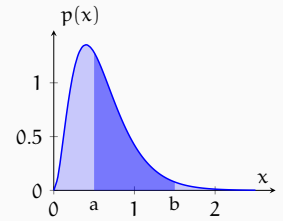
A sample from the distribution on the previous slide:

[1, 2, 2, 3, 3, 3, 4, 4, 5, 7, 11]

[[IMAGE DISCARDED DUE TO `tikz/external/mode=list and make']]

Probability density function (PDF)

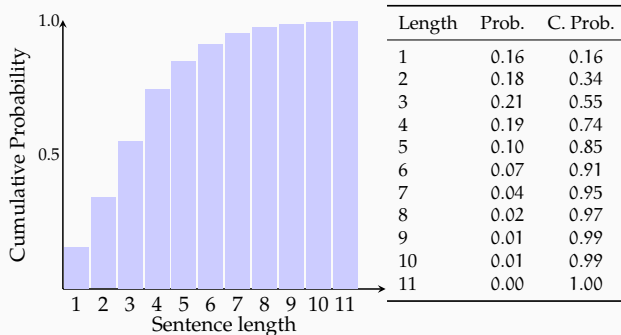
- Continuous variables have *probability density functions*
- $p(x)$ is not a probability (note the notation: we use lowercase p for PDF)
- Area under $p(x)$ sums to 1
- $P(X = x) = 0$
- Non zero probabilities are possible for ranges:



$$P(a \leq x \leq b) = \int_a^b p(x) dx$$

Cumulative distribution function

- $F_X(x) = P(X \leq x)$



Expected value

- Expected value (mean) of a random variable X is,

$$E[X] = \mu = \sum_{i=1}^n P(x_i) x_i = P(x_1) x_1 + P(x_2) x_2 + \dots + P(x_n) x_n$$

- More generally, expected value of a function of X is

$$E[f(X)] = \sum_x P(x) f(x)$$

- Expected value is a measure of central tendency
- Note: it is not the 'most likely' value
- Expected value is linear

$$E[aX + bY] = aE[X] + bE[Y]$$

Variance and standard deviation

- **Variance** of a random variable X is,

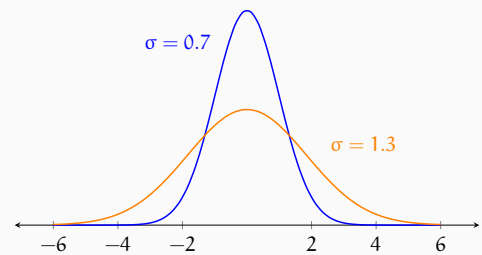
$$\text{Var}(X) = \sigma^2 = \sum_{i=1}^n P(x_i) (x_i - \mu)^2 = E[X^2] - (E[X])^2$$

- It is a measure of spread, divergence from the central tendency
- The square root of variance is called **standard deviation**

$$\sigma = \sqrt{\left(\sum_{i=1}^n P(x_i) x_i^2 \right) - \mu^2}$$

- Standard deviation is in the same units as the values of the random variable
- Variance is not linear: $\sigma_{X+Y}^2 \neq \sigma_X^2 + \sigma_Y^2$ (neither the σ)

Example: two distributions with different variances



Short divergence: Chebyshev's inequality

For any probability distribution, and $k > 1$,

$$P(|x - \mu| > k\sigma) \leq \frac{1}{k^2}$$

| Distance from μ | 2σ | 3σ | 5σ | 10σ | 100σ |
|---------------------|-----------|-----------|-----------|------------|-------------|
| Probability | 0.25 | 0.11 | 0.04 | 0.01 | 0.0001 |

This also shows why standardizing values of random variables,

$$z = \frac{x - \mu}{\sigma}$$

makes sense (the normalized quantity is often called the **z-score**).

Median and mode of a random variable

Median is the mid-point of a distribution. Median of a random variable is defined as the number m that satisfies

$$P(X \leq m) \geq \frac{1}{2} \quad \text{and} \quad P(X \geq m) \geq \frac{1}{2}$$

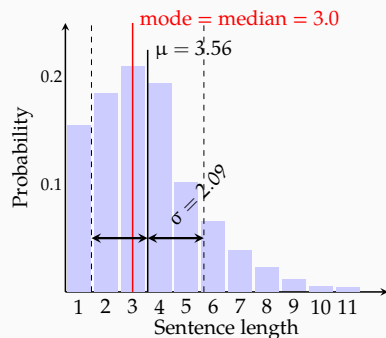
- Median of 1, 4, 5, 8, 10 is 5
- Median of 1, 4, 5, 7, 8, 10 is 6

Mode is the value that occurs most often in the data.

- Modes appear as peaks in probability mass (or density) functions
- Mode of 1, 4, 4, 8, 10 is 4
- Modes of 1, 4, 4, 8, 9, 9 are 4 and 9

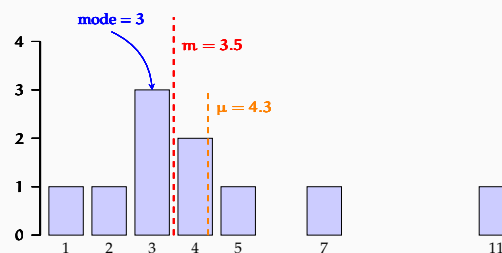
Mode, median, mean, standard deviation

Visualization on sentence length example

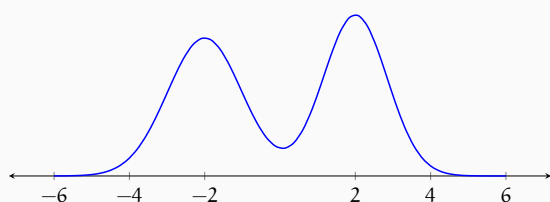


Mode, median, mean

sensitivity to extreme values



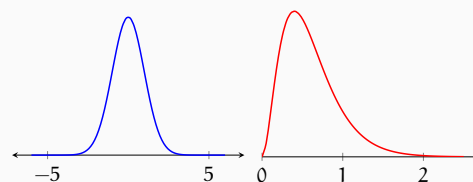
Multimodal distributions



- A distribution is multimodal if it has multiple modes
- Multimodal distributions often indicate confounding variables

Skew

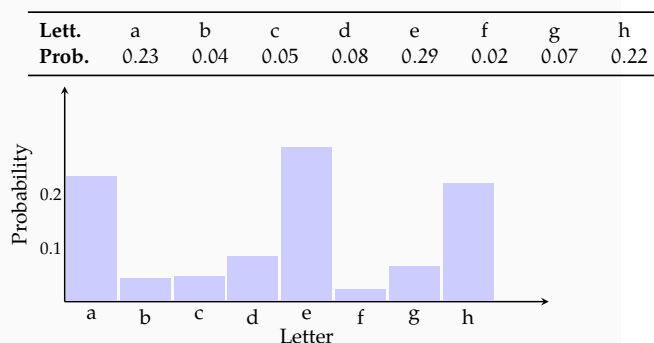
- Another important property of a probability distribution is its *skew*
- **symmetric** distributions have no skew
- **positively skewed** distributions have a long *tail* on the right
- **negatively skewed** distributions have a long left tail



Another example distribution

A probability distribution over letters

- An alphabet with 8 letters and their probabilities of occurrence;



Probability distributions

- A distributions on a finite set of outcomes can be defined by a vector (or table) of probabilities
- Some random variables (approximately) follow a distribution that can be parametrized with a number of parameters
- For example, Gaussian (or normal) distribution is conventionally parametrized by its mean (μ) and variance (σ^2)
- Common notation we use for indicating that a variable X follows a particular distribution is

$$X \sim \text{Normal}(\mu, \sigma^2) \quad \text{or} \quad X \sim \mathcal{N}(\mu, \sigma^2).$$

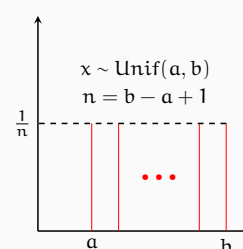
- For the rest of this lecture, we will revise some of the important probability distributions

Probability distributions (cont)

- A probability distribution is called *univariate* if it was defined on scalars
- *multivariate* probability distributions are defined on vectors
- Probability distributions are abstract mathematical objects (functions that map events/outcomes to probabilities)
- A probability distribution is generative device: it can generate samples
- In most problems, we only have access to a *samples*
- Learning (or *inference*) is often cast as finding an (approximate) distribution from a sample

Uniform distribution (discrete)

- A uniform distribution assigns equal probabilities to all values in range $[a, b]$, where a and b are the parameters of the distribution
- Probabilities of the values outside range is 0
- $\mu = \frac{b+a}{2}$
- $\sigma^2 = \frac{(b-a+1)^2-1}{12}$
- There is also an analogous continuous uniform distribution



Bernoulli distribution

Bernoulli distribution characterizes simple random experiments with two outcomes

- Coin flip: heads or tails
- Spam detection: spam or not
- Predicting gender: female or male

We denote (arbitrarily) one of the possible values with 1 (often called a success), the other with 0 (often called a failure)

$$P(X = 1) = p$$

$$P(X = 0) = 1 - p$$

$$P(X = k) = p^k(1 - p)^{1-k}$$

$$\mu_X = p$$

$$\sigma_X^2 = p(1 - p)$$

Binomial distribution

Binomial distribution is a generalization of Bernoulli distribution to n trials, the value of the random variable is the number of 'successes' in the experiment

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

$$\mu_X = np$$

$$\sigma_X^2 = np(1 - p)$$

Remember that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Categorical distribution

- Extension of Bernoulli to k mutually exclusive outcomes
- For any k -way event, the probability distribution is parametrized by k parameters p_1, \dots, p_k ($k - 1$ independent parameters) where

$$\sum_{i=1}^k p_i = 1$$

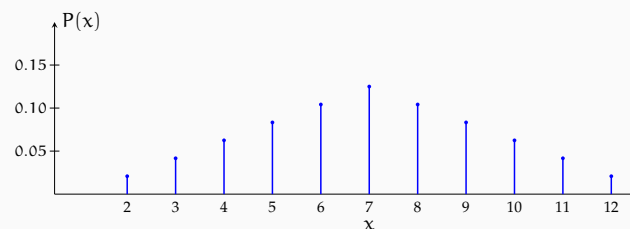
$$E[x_i] = p_i$$

$$\text{Var}(x_i) = p_i(1 - p_i)$$

- Similar to Bernoulli-binomial generalization, *multinomial* distribution is the generalization of categorical distribution to n trials

Categorical distribution example

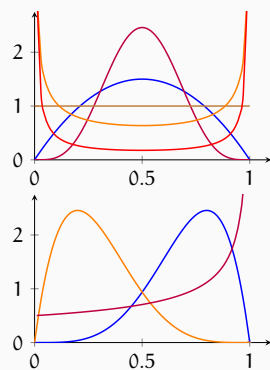
sum of the outcomes from roll of two fair dice



Beta distribution

- Beta distribution is defined in range $[0, 1]$
- It is characterized by two parameters α and β

$$p(x) = \frac{x^{\alpha-1} (1-x)^{\beta-1}}{\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}}$$



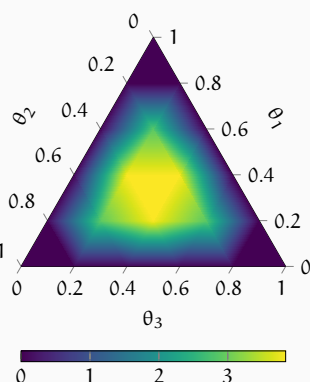
Beta distribution

where do we use it

- A common use is the random variables whose values are probabilities
- Particularly important in Bayesian methods as a conjugate prior of Bernoulli and Binomial distributions
- The *Dirichlet distribution* generalizes Beta distribution to k -dimensional vectors whose components are in range $(0, 1)$ and $\|x\|_1 = 1$.
- Dirichlet distribution is used often in NLP, e.g., *latent Dirichlet allocation* is a well know method for topic modeling

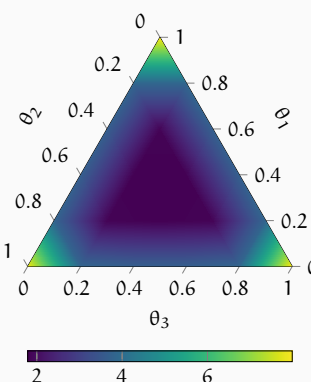
Example Dirichlet distributions

$\theta = (2, 2, 2)$



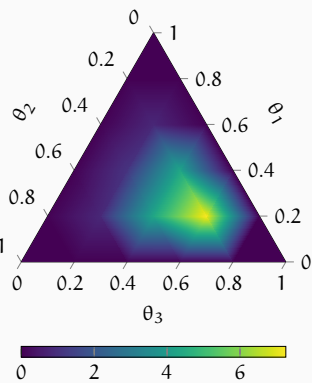
Example Dirichlet distributions

$\theta = (0.8, 0.8, 0.8)$



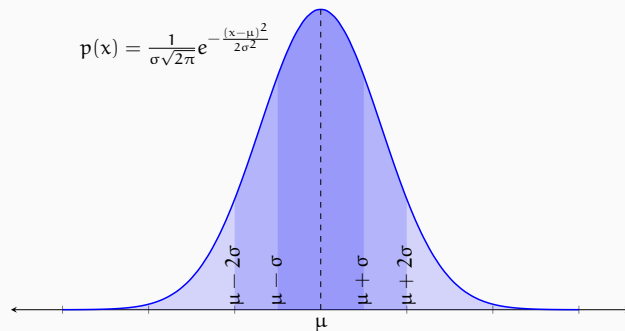
Example Dirichlet distributions

$\theta = (2, 2, 4)$



Gaussian (normal) distribution

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



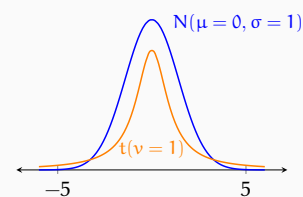
Short detour: central limit theorem

Central limit theorem states that the sum of a large number of independent and identically distributed variables (i.i.d.) is normally distributed.

- Expected value (average) of means of samples from any distribution will be distributed normally
- Many (inference) methods in statistics and machine learning work because of this fact

Student's t-distribution

- T-distribution is another important distribution
- It is similar to normal distribution, but it has heavier tails
- It has one parameter: *degree of freedom* (ν)



Joint and marginal probability

Two or more random variables form a *joint probability distribution*.

An example with letter bigrams:

| | a | b | c | d | e | f | g | h | |
|---|------|------|------|------|------|------|------|------|------|
| a | 0.04 | 0.02 | 0.02 | 0.03 | 0.05 | 0.01 | 0.02 | 0.06 | 0.23 |
| b | 0.01 | 0.00 | 0.00 | 0.00 | 0.01 | 0.00 | 0.00 | 0.01 | 0.04 |
| c | 0.02 | 0.00 | 0.00 | 0.00 | 0.01 | 0.00 | 0.00 | 0.01 | 0.05 |
| d | 0.02 | 0.00 | 0.00 | 0.01 | 0.02 | 0.00 | 0.01 | 0.02 | 0.08 |
| e | 0.06 | 0.02 | 0.01 | 0.03 | 0.08 | 0.01 | 0.01 | 0.07 | 0.29 |
| f | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.00 | 0.00 | 0.01 | 0.02 |
| g | 0.01 | 0.00 | 0.00 | 0.01 | 0.02 | 0.00 | 0.01 | 0.02 | 0.07 |
| h | 0.08 | 0.00 | 0.00 | 0.01 | 0.10 | 0.00 | 0.01 | 0.02 | 0.22 |
| | 0.23 | 0.04 | 0.05 | 0.08 | 0.29 | 0.02 | 0.07 | 0.22 | |

Expected values of joint distributions

$$E[f(X, Y)] = \sum_x \sum_y P(x, y) f(x, y)$$

$$\mu_X = E[X] = \sum_x \sum_y P(x, y) x$$

$$\mu_Y = E[Y] = \sum_x \sum_y P(x, y) y$$

We can simplify the notation by vector notation, for $\mu = (\mu_X, \mu_Y)$,

$$\mu = \sum_{x \in XY} x P(x)$$

where vector x ranges over all possible combinations of the values of random variables X and Y .

Variances of joint distributions

$$\sigma_X^2 = \sum_x \sum_y P(x, y) (x - \mu_X)^2$$

$$\sigma_Y^2 = \sum_x \sum_y P(x, y) (y - \mu_Y)^2$$

$$\sigma_{XY} = \sum_x \sum_y P(x, y) (x - \mu_X)(y - \mu_Y)$$

- The last quantity is called *covariance* which indicates whether the two variables vary together or not

Again, using vector/matrix notation we can define the *covariance matrix* (Σ) as

$$\Sigma = E[(x - \mu)^2]$$

Covariance and the covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{YX} & \sigma_Y^2 \end{bmatrix}$$

- The main diagonal of the covariance matrix contains the variances of the individual variables
- Non-diagonal entries are the covariances of the corresponding variables
- Covariance matrix is symmetric ($\sigma_{XY} = \sigma_{YX}$)
- For a joint distribution of k variables we have a covariance matrix of size $k \times k$

Correlation

Correlation is a normalized version of covariance

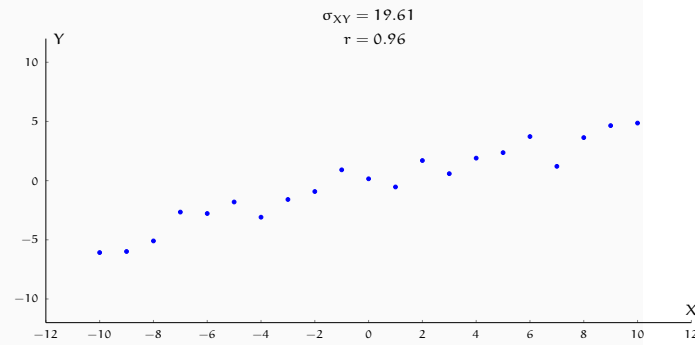
$$r = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

Correlation coefficient (r) takes values between -1 and 1

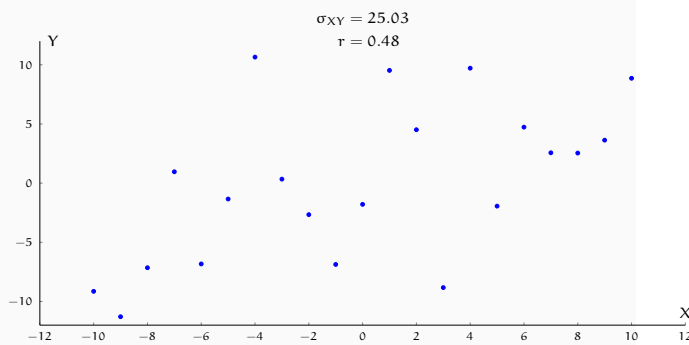
- 1 Perfect positive correlation.
- $(0, 1)$ positive correlation: x increases as y increases.
- 0 No correlation, variables are independent.
- $(-1, 0)$ negative correlation: x decreases as y increases.
- -1 Perfect negative correlation.

Note: like covariance, correlation is a symmetric measure.

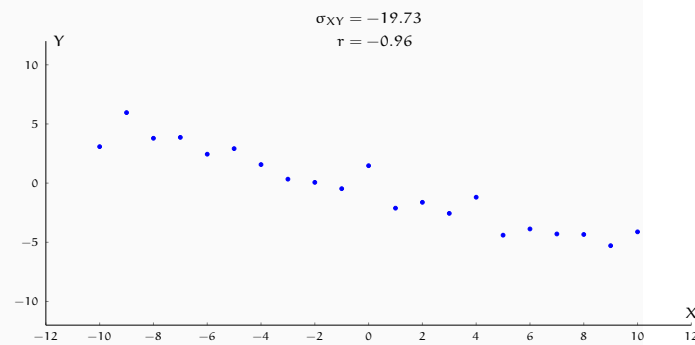
Correlation: visualization (1)



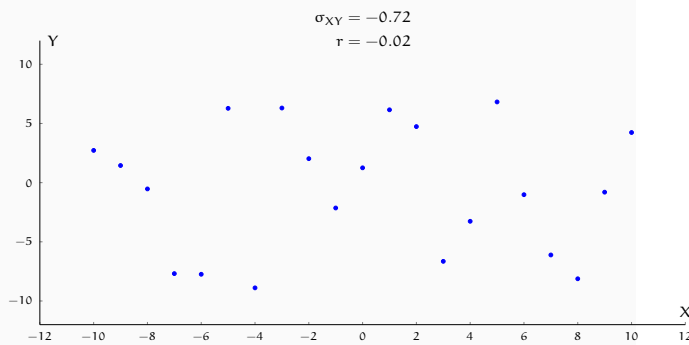
Correlation: visualization (2)



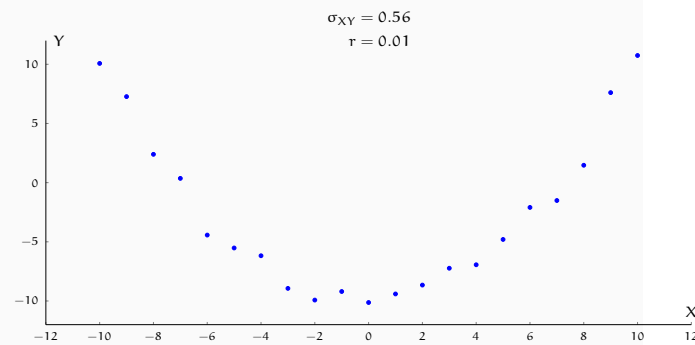
Correlation: visualization (3)



Correlation: visualization (4)



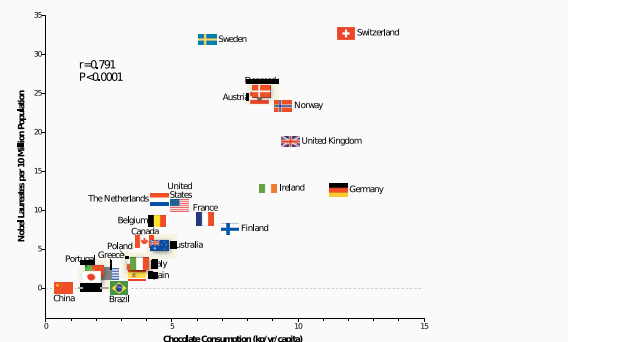
Correlation: visualization (5)



Correlation and independence

- Statistical (in)dependence is an important concept (in ML)
- The correlation (or covariance) of independent random variables is 0
- The reverse is not true: 0 correlation does not imply independence
- Correlation measures a linear dependence (relationship) between two variables, a non-linear dependence is not measured by correlation

Short divergence: correlation and causation



From Messerli (2012).

Conditional probability

In our letter bigram example, given that we know that the first letter is **e**, what is the probability of second letter being **d**?

| | a | b | c | d | e | f | g | h | |
|---|------|------|------|------|------|------|------|------|------|
| a | 0.04 | 0.02 | 0.02 | 0.03 | 0.05 | 0.01 | 0.02 | 0.06 | 0.23 |
| b | 0.01 | 0.00 | 0.00 | 0.00 | 0.01 | 0.00 | 0.00 | 0.01 | 0.04 |
| c | 0.02 | 0.00 | 0.00 | 0.00 | 0.01 | 0.00 | 0.00 | 0.01 | 0.05 |
| d | 0.02 | 0.00 | 0.00 | 0.01 | 0.02 | 0.00 | 0.01 | 0.02 | 0.08 |
| e | 0.06 | 0.02 | 0.01 | 0.03 | 0.08 | 0.01 | 0.01 | 0.07 | 0.29 |
| f | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.00 | 0.00 | 0.01 | 0.02 |
| g | 0.01 | 0.00 | 0.00 | 0.01 | 0.02 | 0.00 | 0.01 | 0.02 | 0.07 |
| h | 0.08 | 0.00 | 0.00 | 0.01 | 0.10 | 0.00 | 0.01 | 0.02 | 0.22 |
| | 0.23 | 0.04 | 0.05 | 0.08 | 0.29 | 0.02 | 0.07 | 0.22 | |

$$P(L_1 = e, L_2 = d) = 0.025940365$$

$$P(L_1 = e) = 0.28605090$$

$$P(L_2 = d | L_1 = e) = \frac{P(L_1 = e, L_2 = d)}{P(L_1 = e)}$$

Bayes' rule

$$P(X | Y) = \frac{P(Y | X)P(X)}{P(Y)}$$

- This is a direct result of the axioms of the probability theory
- It is often useful as it 'inverts' the conditional probabilities
- The term $P(X)$, is called **prior**
- The term $P(Y | X)$, is called **likelihood**
- The term $P(X | Y)$, is called **posterior**

Chain rule

We rewrite the relation between the joint and the conditional probability as

$$P(X, Y) = P(X | Y)P(Y)$$

We can also write the same quantity as,

$$P(X, Y) = P(Y | X)P(X)$$

For more than two variables, one can write

$$P(X, Y, Z) = P(Z | X, Y)P(Y | X)P(X) = P(X | Y, Z)P(Y | Z)P(Z) = \dots$$

In general, for any number of random variables, we can write

$$P(X_1, X_2, \dots, X_n) = P(X_1 | X_2, \dots, X_n)P(X_2, \dots, X_n)$$

Continuous random variables

some reminders

The rules and quantities we discussed above apply to continuous random variables with some differences

- For continuous variables, $P(X = x) = 0$
- We cannot talk about probability of the variable being equal to a single real number
- But we can define probabilities of ranges
- For all formulas we have seen so far, replace summation with integrals
- Probability of a range:

$$P(a < X < b) = \int_a^b p(x) dx$$

Conditional probability (2)

In terms of probability mass (or density) functions,

$$P(X | Y) = \frac{P(X, Y)}{P(Y)}$$

If two variables are **independent**, knowing the outcome of one does not affect the probability of the other variable:

$$P(X | Y) = P(X) \quad P(X, Y) = P(X)P(Y)$$

More notes on notation/interpretation:

$P(X = x, Y = y)$ Probability that $X = x$ and $Y = y$ at the same time (joint probability)

$P(Y = y)$ Probability of $Y = y$, for any value of X ($\sum_{x \in X} P(X = x, Y = y)$) (marginal probability)

$P(X = x | Y = y)$ Probability of $X = x$, given $Y = y$ (conditional probability)

Example application of Bayes' rule

We use a test t to determine whether a patient has COVID-19 (c)

- If a patient has c test is positive 99% of the time:
 $P(t | c) = 0.99$
- What is the probability that a patient has c given t ?
- ...or more correctly, can you calculate this probability?
- We need to know two more quantities. Let's assume
 $P(c) = 0.01$ and $P(t | \neg c) = 0.1$

$$P(c | t) = \frac{P(t | c)P(c)}{P(t)} = \frac{P(t | c)P(c)}{P(t | c)P(c) + P(t | \neg c)P(\neg c)} = 0.09$$

Conditional independence

If two random variables are conditionally independent:

$$P(X, Y | Z) = P(X | Z)P(Y | Z)$$

This is often used for simplifying the statistical models. For example in spam filtering with *naïve Bayes* classifier, we are interested in

$$P(w_1, w_2, w_3 | \text{spam}) = P(w_1 | w_2, w_3, \text{spam})P(w_2 | w_3, \text{spam})P(w_3 | \text{spam})$$

with the assumption that occurrences of words are independent of each other given we know the email is spam or not,

$$P(w_1, w_2, w_3 | \text{spam}) = P(w_1 | \text{spam})P(w_2 | \text{spam})P(w_3 | \text{spam})$$

Multivariate continuous random variables

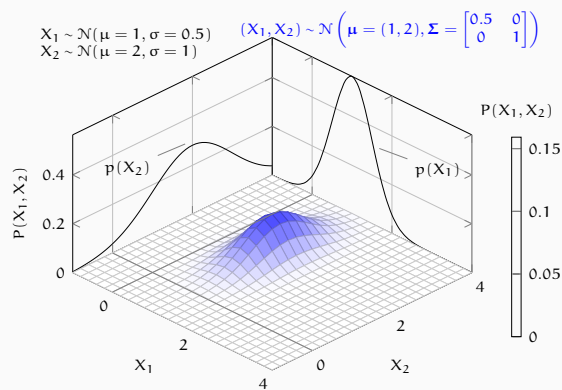
- Joint probability density

$$p(X, Y) = p(X | Y)p(Y) = p(Y | X)p(X)$$

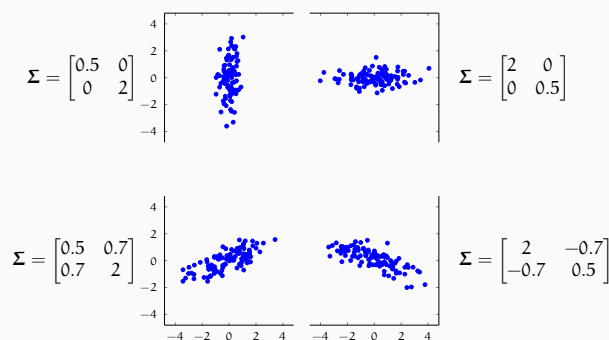
- Marginal probability

$$P(X) = \int_{-\infty}^{\infty} p(x, y) dy$$

Multivariate Gaussian distribution



Samples from bi-variate normal distributions



Summary: some keywords

- Probability, sample space, outcome, event
- Random variables: discrete and continuous
- Probability mass function
- Probability density function
- Cumulative distribution function
- Expected value
- Variance / standard deviation
- Median and mode
- Skewness of a distribution
- Joint and marginal probabilities
- Covariance, correlation
- Conditional probability
- Bayes' rule
- Chain rule
- Some well-known probability distributions:

| | |
|-------------|-------------|
| Bernoulli | binomial |
| categorical | multinomial |
| beta | Dirichlet |
| Gaussian | Student's t |

Next

Wed Information theory
 Mon ML Intro / regression
 Wed Classification

References and further reading

- MacKay (2003) covers most of the topics discussed in a way quite relevant to machine learning. The complete book is available freely online (see the link below)
- See Grinstead and Snell (2012) a more conventional introduction to probability theory. This book is also freely available
- For an influential, but not quite conventional approach, see Jaynes (2007)



Chomsky, Noam (1968). "Quine's empirical assumptions". In: *Synthese* 19.1, pp. 53–68. doi: 10.1007/BF00668049.



Grinstead, Charles Miller and James Laurie Snell (2012). *Introduction to probability*. American Mathematical Society. ISBN: 9780821894149. URL: http://www.dartmouth.edu/~chance/teaching_aids/books_articles/probability_book/book.html.



Jaynes, Edwin T (2007). *Probability Theory: The Logic of Science*. Ed. by G. Larry Bretthorst. Cambridge University Press. ISBN: 978-05-2159-271-0.



MacKay, David J. C. (2003). *Information Theory, Inference and Learning Algorithms*. Cambridge University Press. ISBN: 978-05-2164-298-9. URL: <http://www.inference.phy.cam.ac.uk/itprnn/book.html>.

References and further reading (cont.)



Messerli, Franz H (2012). "Chocolate consumption, cognitive function, and Nobel laureates". In: *The New England journal of medicine* 367.16, pp. 1562–1564.