Linear Algebra Done Right

complex nos - invented so we can $\sqrt{\ }$ -ve nos
, assume $\sqrt{-1} = i$

complex no = ordered pair where $a, b \in R \rightarrow a + bi$

1. commutative $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$
   for all $\alpha, \beta \in C$

2. associative $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$

3. identity $\lambda + 0 = \lambda$ $\lambda \times 1 = \lambda$

4. additive inverse $\alpha + \beta = 0$

5. multiplicative inverse $\alpha\beta = 1$

6. distributive

Exercise 1A

1. show $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in C$

Let $\alpha = (\alpha_1, \alpha_2 \ldots \alpha_n) \in C$

$\beta = (\beta_1, \beta_2 \ldots \beta_n) \in C$

$$\alpha + \beta = (\alpha_1, \alpha_2 \ldots \alpha_n) + (\beta_1, \beta_2 \ldots \beta_n)$$
$$= (\alpha_1 + \beta_1, \alpha_2 + \beta_2 \ldots n)$$
$$= (\beta_1 + \alpha_1, \beta_2 + \alpha_2 \ldots n)$$
$$= (\beta_1, \beta_2 \ldots \beta_n) + (\alpha_1, \alpha_2 \ldots \alpha_n)$$

$$= \beta + \alpha$$

(OR)

Say $\alpha = (a+bi)$ $\quad a,b \in C$
$\quad \beta = (c+di)$ $\quad c,d \in C$

$$\alpha + \beta = (a+bi) + (c+di)$$
$$= (a+c) + i(b+d)$$

$$= (c+a) + i(d+b)$$
$$= (c+di) + (a+bi)$$
$$= \beta + \alpha$$

2. Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta, \lambda \in C$.

$\alpha = a + bi$ $\quad a,b \in C$
$\beta = c + di$ $\quad c,d \in C$
$\lambda = e + fi$ $\quad e,f \in C$

$$(\alpha + \beta) + \lambda = (a+bi + c+di) + (e+fi)$$
$$= (a+c + i(b+d)) + (e+fi)$$
$$= (a+c + e) + i(b+d+f)$$

$$\alpha + (\beta + \lambda) = (a+bi) + (c+di + e + fi)$$
$$= (a+bi) + c+e + i(d+f)$$

$$= (a+c+e) + i(b+d+f)$$

$\therefore$ LHS $=$ RHS

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3. $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$

$\alpha = a + bi \qquad \beta = c + di \qquad \lambda = e + fi$

$(\alpha\beta)\lambda = [(a+bi)(c+di)](e+fi)$

$i^2 = -1$

$= [ac + adi + bci - bd](e + fi)$

$= [(ac - bd) + i(ad + bc)](e + fi)$

$= ace + acfi + adei - adf + bcei - bcf - bde - bdfi$

$= (ace - bde - adf - bcf) + (acf + ade + bce - bdf)i$

$\alpha(\beta\lambda) = (a+bi)[(c+di)(e+fi)]$

$= (ace - bde - adf - bcf) + (acf + ade + bce - bdf)i$

5. show for every $\alpha \in \mathbb{C}$, exists a unique $\beta \in \mathbb{C}$ such that $\alpha + \beta = 0$.

$\alpha = a + bi \qquad \beta = c + di$

$\alpha + \beta = 0$

$(a+bi) + (c+di) = 0$

$(a+c) + i(b+d) = 0$

$a + c = 0 \qquad b + d = 0$

$a = -c \qquad b = -d$

$\therefore \alpha + \beta = (-c + c) + i(-d + d)$

$= 0 + i0$

$= 0$

For uniqueness

$\alpha + \beta_1 = 0 \quad (\text{if } \beta_1 = -\alpha)$

Take β_2

$\alpha + \beta_2 = 0$

$(a+bi) + (e+fi) = 0$

$(a+e) + i(b+f) = 0$

$a = -e \qquad b = -f$

$(a+bi) + (-a-bi) = 0$

basically

$\alpha + -\alpha = 0$

so $\beta_1 = \beta_2$, $\therefore$ unique

b. Show $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha\beta = 1$.

$\alpha = a + bi \qquad \beta = c + di$

$\alpha\beta = 1$

$(a+bi)(c+di) = 1$

$ac + adi + bci - bd = 1$

$(ac - bd) + i(ad + bc) = 1 + 0i$

$ac - bd = 1 \rightarrow (1)$

$ad + bc = 0 \rightarrow (2)$

Solving (remember $\alpha \neq 0$ $\therefore$ a ~~and~~ or b $\neq 0$)

$ad = -bc \rightarrow (3)$ from (2)

$d = \frac{-bc}{a} \rightarrow (4)$

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Sub to (1)

$$ac - b\left(\frac{-bc}{a}\right) = 1$$

$$ac + \frac{b^2c}{a} = 1$$

$$\frac{a^2c + b^2c}{a} = 1$$

$$\frac{c(a^2+b^2)}{a} = 1$$

$$c(a^2+b^2) = a$$

$$c = \frac{a}{a^2+b^2}$$

Sub to (4)

$$d = \frac{-b\left(\frac{a}{a^2+b^2}\right)}{a}$$

$$= \frac{-b}{a^2+b^2}$$

From (1) $ac - bd = 1$

base case, $a = 0$

$$-bd = 1$$

$$d = \frac{-1}{b}$$

$bc = 0 \rightarrow$ from (2)

$$c = 0$$

$$c = \frac{a}{a^2+b^2} = 0 \qquad d = \frac{-b}{a^2+b^2} = \frac{-b}{b^2} = \frac{-1}{b}$$

$\therefore$ consistent and solution exist.