Subspaces

subset U of V is subspace | linear subspace if U is also a vector space that follows addition, scalar mul and additive identity like V

3 conditions

1. additive identity
   $0 \in U$
2. closed under addition
   $u, w \in U \quad u + w \in U$
3. closed under scalar multiplication
   $a \in F \quad u \in U \quad au \in U$

Sum of subspaces

suppose $V_1, \dots V_m$ are subspaces of V, the sum is denoted by set of all possible sums of elements

$$V_1 + \dots V_m = \{ v_1 + \dots v_m : v_1 \in V_1 \dots v_m \in V_m \}$$

read as "such that"

sum of subspace is the smallest containing subset

1. proving sum of subspace is a subspace of V (easy)

2. $W_1, W_2 \longrightarrow$ subspaces
   $W_1 + W_2$ is smallest subspace

~~some $v \in W_1 + W_2$~~ $\quad \therefore \; v \in W_1 \quad v \in W_2$

understand, $W_1 + \dots W_n$ is a subspace containing $W_1 \dots W_n$.

now if you have another subspace U

$\therefore W_1 \subseteq U \quad W_2 \subseteq U$

~~U ⊆ subspace containg $W_1, \ldots W_n$~~

U is a subspace closed under addition.

$w_1 + w_2 \in U$

$\therefore W_1 + W_2 \subseteq U$

U contains both W_1 and W_2, so every subspace contains $W_1 + W_2$

$\therefore$ smallest.

direct sums

Suppose $V_1, V_2 \ldots V_m$ are subspaces of V.

every element of $V_1 + V_2 \ldots V_m$ can be written as $v_1 + \ldots + v_m$ as the only way.

where $v_k \in V_k$.

has a symbol $\oplus$

conditions for a direct sum

1. suppose V_1, V_2 are subspaces of V, then $V_1 + V_2$ is $\oplus$ only if the only way to write 0 as a sum $v_1 + v_2$ where $v_k \in V_k$ is to take each $v_k = 0$

2. $U + W$ is direct sum $\iff U \cap W = \{0\}$

"if and only if"

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Q. sum that is not direct sum

$V_1 = \{(x,y,0) \in F^3 : x, y \in F\}$

$V_2 = \{(0,0,z) \in F^3 : z \in F\}$

$V_3 = \{(0,y,y) \in F^3 : y \in F\}$

$F^3 = V_1 + V_2 + V_3$

$(x,y,z) = (x,y,0) + (0,0,z) + (0,y,y)$

$F^3 \neq V_1 \oplus V_2 \oplus V_3$

$(0,0,0) = (0,-1,0) + (0,0,-1) + (0,1,1)$
$= (0,0,0) + (0,0,0) + (0,0,0)$

Exercise 1C

1. For each subset of F^3, determine if its a subspace of F^3.

* need to check for

① additive identity

② closure under addition

③ closure under scalar multiplication

a) $\{(x_1, x_2, x_3) \in F^3 : x_1 + 2x_2 + 3x_3 = 0\}$,

① $0 + 0 + 0 = 0$ ∴ identity

② take $u = \{u_1, 2u_2, 3u_3\}$ $v = \{v_1, 2v_2, 3v_3\}$

adding them should be within same space.

$(u_1 + v_1) + 2(u_2 + v_2) + 3(u_3 + v_3)$

or

$(u_1 + 2u_2 + 3u_3) + (v_1 + 2v_2 + 3v_3)$
$= 0 + 0$
$= 0$

∴ yes, closure under add

③ yes, take any scalar λ, then mult, you get $\lambda \times 0 = 0$

$\therefore$ yes, its a subspace.

b) $\{(x_1, x_2, x_3) \in F^3 : x_1 + 2x_2 + 3x_3 = 4\}$

① $0 \neq 4$ $\therefore$ no.

c) $\{(x_1, x_2, x_3) \in F^3 : x_1 x_2 x_3 = 0\}$

① yes

② $u = (u_1, u_2, u_3)$ $v = (v_1, v_2, v_3)$

where $u_1 u_2 u_3 = 0$ $v_1 v_2 v_3 = 0$

so atleast one of the comp is 0

take $(1, 0, 1)$ $(0, 1, 1)$

~~[illegible]~~ = ~~[illegible]~~

add both

$(1, 1, 2) \neq 0$

$\therefore$ not closed

5. is R^2 a subspace of complex vector space C^2?

① yes

② yes

③ Take complex scalar $\lambda = ai + b$

$u_i = (u_1, u_2)$

$\lambda(u_1, u_2) = (ai + b)u_1, (ai + b)u_2$

= not a real no $\therefore$ not closed

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a) b. is $\{(a,b,c) \in R^3 : a^3 = b^3\}$ a subspace of R^3?

① yes $0^3 = 0^3$

② $(a_1 + a_2)^3 = (b_1 + b_2)^3$

take a pair $(1,-1,0)$ $(10,10,0)$

$1^3 = (-1)^3$ $10^3 = 10^3$

take addition

$11 \neq 9$

$(1+10)$ $(-1+10)$

$11^3 \neq 9^3$

$\therefore$ not closed

b) $(a,b,c) \in C^3 : a^3 = b^3\}$ a subspace of C^3?

$(a+ib, \; b+ic, \; c+id)$

① yes

② $(a+ib)^3 = (c+id)^3$

idts

7. if U is nonempty subset of R^2 such that U closed under addition and taking additive inverse, then U is subspace of R^2 (prove or not)

① zero vector exists

additive inverse = $v = -v$

closed under add = $v - v = 0 \in U$

② closed under addition - written

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if you take a scalar like $\frac{3}{2}$

$\frac{3}{2}(u, v) = (1.5, 1.5) \notin R^2$

not R^2

$\therefore$ false