$T \circ S : X \to Z$: composition of T with S

$T \circ S : T(S(x))$

- Proof for composition is linear transformation

$T \circ S(x+y) = T(S(x+y)) = T(S(x) + S(y))$
$= T(S(x)) + T(S(y))$
$= T \circ S(x) + T \circ S(y)$

$T \circ S(cx) = T(S(cx)) = T(cS(x))$
$= cT(S(x))$
$= c\, T \circ S(x)$

- motivating matrix multiplications

we know linear trans is matrix vector product

$T(x) = Bx$ $\quad$ $S(x) = Ax$

$l \times m$ $\quad$ $m \times n$

$T \circ S(x) = T(S(x)$
$= T(Ax)$

$T \circ S(x) = T(S(x))$
$= B(Ax)$
$= Cx$

$I_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ upto n.

$C = \left[B\left(A\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) + B\left(A\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) \right]$

$= \left[Ba_1 \cdot Ba_2 \cdot Ba_3 \right]$

* matrix mul is associative

$H(x) = Ax \qquad G(x) = Bx \qquad F(x) = Cx$

$$(H \circ G) \circ F(x) = H(G(F(x)))$$
$$= H \circ (G \circ F(x))$$
$$= H((G \circ F)(x))$$

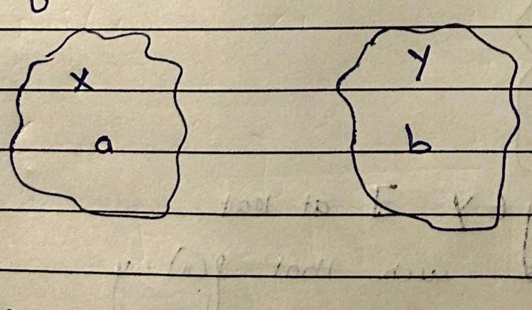
* matrix mul is distributive

$B = [b_1 \quad b_2 \quad b_3] \qquad C = [c_1 \quad c_2 \quad c_3]$

$$A(B+C) = A\begin{bmatrix} b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{bmatrix}$$
$$= A\begin{bmatrix} b_1 + b_2 + b_3 \end{bmatrix} + A\begin{bmatrix} c_1 + c_2 + c_3 \end{bmatrix}$$
$$= AB + AC$$

Inverse of a function

$f : X \to Y$



identity fn

$I_X : X \to X$

$I_X(a) = a$

$I_Y(b) = b$

f is invertible $\iff$ there exists a function $f^{-1} : Y \to X$ such that $f^{-1} \circ f = I_X$ and $f \circ f^{-1} = I_Y$

$f: X \to Y$ if f is invertible, is f^{-1} unique?

proof

assume it has 2 inverse

$g: Y \to X$

$g \circ f = I_x$

$f \circ g = I_y$

$h: Y \to X$

$h \circ f = I_x$

$f \circ h = I_y$

$g = I_x \circ g$ → basically same place

$= h \circ f \circ g$

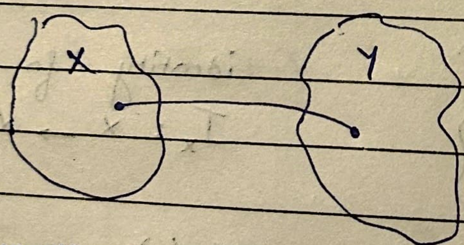
$= h \circ I_y$

$= h$

$\therefore g = h$ unique inverse

Surjective and injective

$f: X \to Y$



surjective – every $y \in Y$ $\exists$ at least one $x \in X$ such that $f(x) = y$

X		Y
1	→	A
2	→	B
3	→	C

✓

X		Y
1	→	A
2	↗	B

✗

B not mapped

injective   one to one

X → Y

1 → A

2 → B

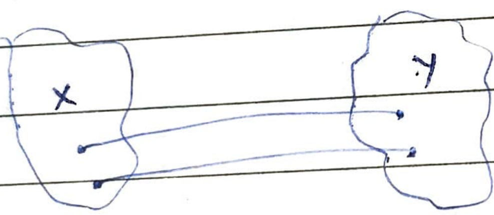
3 → C

✓

for any $y \in Y$ at most one $x \in X$

* Relating invertibility to onto and one-to one

$f : X \to Y$ invertible $\iff$ $\overbrace{\text{for every } y \in Y}^{\text{onto}}$ (co-domain) there exists $\underbrace{\text{UNIQUE}}_{\text{one to one}}$ $x \in X$ (domain) such that $f(x) = y$.



$\therefore$ f should be surjective and injective.