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# Chapter - 3
## Linear Maps

a linear map from V to W is a function $T: V \to W$ with properties -

① additivity $T(u+v) = Tu + Tv \quad \forall\ u, v \in V$
② homogeneity $T(\lambda v) = \lambda(Tv) \quad \forall\ \lambda \in F$ and all $v \in V$.

linear transformation / linear map denoted as $T(v)$ or $Tv$

* Suppose $v_1 \ldots v_n$ is basis of V and $w_1 \ldots w_n \in W$. Then there exists a unique linear map $T: V \to W$ such that -

$$Tv_k = w_k \quad \text{for each } k = 1 \ldots n$$

Since $v_1, v_2 \ldots v_n$ is basis, it means V is a unique linear combination $v_1 = a_1v_1 + a_2v_2 \ldots a_nv_n$

$$T(v) = T(a_1v_1 + a_2v_2 \ldots a_nv_n)$$
$$= a_1T(v_1) \ldots a_nT(v_n)$$

respects both addition and homogenity $\therefore$ a linear map.

Proof for uniqueness
say we have 2 maps ~~T, S~~ $T, S: V \to W$ such that $Tv_k = w_k$ and $Sv_k = w_k$

Take another map $U = T - S$
$U(v) = T(v) - S(v) = w_k - w_k = 0$

not clear

* addition and scalar multiplication on $\mathcal{L}(V,W)$
Suppose $S, T \in \mathcal{L}(V,W)$ and $\lambda \in F$, the sum $S+T$ and $\lambda T$ are linear maps from $V$ to $W$ defined as -

$(S+T)(v) = S(v) + T(v)$ $\quad$ $(\lambda T)(v) = \lambda T(v)$
$\forall\, v \in V$.

* $\mathcal{L}(V,W)$ is a vector space

properties of product of linear maps

1. NOT commutative $ST \neq TS$
2. associative $(T_1 T_2) T_3 = T_1 (T_2 T_3)$
3. identity $TI = IT = T$ whenever $T \in \mathcal{L}(V,W)$
4. distributive $(S_1 + S_2) T = S_1 T + S_2 T$

linear maps take 0 to 0.
$T(0) = 0$

Exercise 3A

1. suppose $b, c \in \mathbb{R}$, define $T: \mathbb{R}^3 \to \mathbb{R}^2$
$T(x,y,z) = (2x - 4y + 3z + b,\ 6x + cxyz)$
show $T$ is linear if and only if $b = c = 0$.

1. additivity $T(u+v) = T(u) + T(v)$
$u = (x_1, y_1, z_1)$ $\quad$ $v = (x_2, y_2, z_2)$

$T(u+v) = T(x_1+x_2,\ y_1+y_2,\ z_1+z_2) =$
$2(x_1+x_2) - 4(y_1+y_2) + 3(z_1+z_2) + b($
$x_1 + x_2) + c(x_1+x_2)(y_1+y_2)(z_1+z_2)$

$T(u) + T(v) = (2x_1 - 4y_1 + 3z_1 + b), bx_1 + c x_1 y_1 z_1) +$
$(2x_2 - 4y_2 + 3z_2 + b, bx_2 + c x_2 y_2 z_2)$

$b = 2b$

$\therefore b = 0$

## Null spaces and injectivity

For $T \in \mathcal{L}(V, W)$, null space of $T$ is subset of $V$ consisting of vectors that $T$ maps to $0$

$$\text{null } T = \{v \in V : Tv = 0\}$$

* $T \in \mathcal{L}(V, W)$, then null $T$ is subspace of $V$.

$T$ is a linear map. $T(0) = 0$, $0 \in$ null $T$

suppose $u, v \in$ null $T$, then

$T(u+v) = 0$

$T(u) + T(v) = 0 + 0 = 0$

$\therefore u+v \in$ null $T$, closed under addition.

$T(\lambda u) = \lambda(T(u)) = \lambda 0 = 0$

$\lambda u \in$ null $T$ $\therefore$ closed under scalar mul.

$\therefore$ null $T$ is subspace of $V$.

* a function $T: V \to W$ is injective if $Tu = Tv$ implies $u = v$.

* injectivity <=> null space equals $\{0\}$

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## Range and surjectivity

For $T \in \mathcal{L}(V,W)$, range of $T$ is subset of $W$ that contains vectors equal to $Tv$ for some $v \in V$

$$\text{range } T = \{Tv : v \in V\}$$

* range is a subspace.

a function $T : V \to W$ is surjective if its range equals $W$.

* Suppose $V$ is finite dimensional and $T \in \mathcal{L}(V,W)$ then range $T$ is finite dimensional and

$$\dim V = \dim \text{null } T + \dim \text{range } T$$

## Representing linear map by matrix

if $v_1, v_2 \ldots v_n$ is basis of $V$ and $T : V \to W$ is linear, then values of $Tv_1, \ldots Tv_n$ determine the values of $T$ on arbitrary vectors in $V$

For the matrices part, use supplementary Sal or other lectures

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invertible linear maps

a linear map $T \in L(V, W)$ is invertible if there exist a linear map $S \in L(V, W)$ such that $ST$ equals identity operator on $V$ and $TS$ = identity on $W$

a linear map $S \in L(V, W)$ where $ST = I$ and $TS = I$ is called inverse of $T$.

an isomorphic is an invertible linear map.

Learn linear maps with external materials. hard to follow.

Chapter - 4 polynomials

$z = a + bi$
real part denoted by $Re_z = a$
imaginary part $Im_z = b$

$$z = Re z + (Im z) i$$

complex conjugate of $z \in C$ = $\bar{z} = Re z - (Im z) i$
absolute value of $z = |z| = \sqrt{(Re z)^2 + (Im z)^2}$

* sum of $z$ and $\bar{z}$ = $z + \bar{z} = 2 Re z$
* $z - \bar{z} = 2 (Im z) i$
* $z \times \bar{z} = |z|^2$
* $\overline{w+z} = \bar{w} + \bar{z}$ and $\overline{wz} = \bar{w} \times \bar{z}$
* $\bar{\bar{z}} = z$
* $|Re z| \leq |z|$ $|Im z| \leq |z|$
* $|\bar{z}| = z$
* $|w+z| \leq |w| + |z|$
* $|wz| = |w| |z|$