free variable

$x_1 + 2x_2 \quad + 3x_4 = 2$

$\quad x_3 - 2x_4 = 5$

pivot

$x_1 = 2 - 2x_2 - 3x_4$ (you can sub in for x_2 and x_4)

$x_3 = 5 + 2x_4$

- Proof for transformation is invertible $\Leftrightarrow$ $rref(A) = I_n$

$T: \mathbb{R}^n \to \mathbb{R}^m$, is T invertible?

$T(x) = \underset{m \times n}{A}\vec{x}$

we know for invertible, it has to be -

- onto $\rightarrow$ rank$(A) = m$
- one to one $\rightarrow$ rank$(A) = n$

rank$(A) = m = n$, $\therefore$ it must be a square matrix $(n \times n)$

$\underset{n \times n}{A} = \begin{bmatrix} \vec{a_1} & \vec{a_2} & \vec{a_3} \end{bmatrix}$ all of these are basis with a pivot.

which means

$rref(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= I_n$

* proof for inverse is a linear transformation

$T: R^n \to R^n$

T is invertible $\exists$ some transformation T^{-1} such that

$T \circ T^{-1} = I_{R^n}$

$T^{-1} \circ T = I_{R^n}$

$T(x+y) = T(x) + T(y)$

$T(cx) = c\,T(x)$

$(T \circ T^{-1})(a+b) = T \circ T^{-1}(a) + T \circ T^{-1}(b)$

$T(T^{-1}(a+b)) = T(T^{-1}(a)) + T(T^{-1}(b))$

$= T(T^{-1}(a) + T^{-1}(b))$

apply inverse

$T^{-1}(T(T^{-1}(a+b))) = T^{-1}(T(T^{-1}(a) + T^{-1}(b)))$

$= (T^{-1} \circ T)(T^{-1}(a+b)) = (T^{-1} \circ T)(T^{-1}(a) + T^{-1}(b))$

$= I_n(T^{-1}(a+b)) = I_n(T^{-1}(a) + T^{-1}(b))$

(1) $= T^{-1}(a+b) = T^{-1}(a) + T^{-1}(b)$

$(T \circ T^{-1})(ca) = ca$

$= c\,(T \circ T^{-1})\,a$

$T(T^{-1}(ca)) = c\,T(T^{-1}(a))$

apply inverse

$T^{-1}(T(T^{-1}(ca))) = T^{-1}(T(T^{-1}(a))\,c)$

(2) $= T^{-1}(ca) = c\,T^{-1}(a)$

the inverse cancel out

since its a linear transformation, it can expressed as a matrix vector.

$T^{-}(x) = A^{-1}(x)$

$T(x) = A(x)$

$T^{-1} \circ T = I_{R^n}$ $\quad$ $T \circ T^{-1} = I_{R^n}$

$T^{-1} \circ T(x) = A^{-1} A(x)$

$= I_{R^n}(x)$

$= I(x)$

- deriving method for determining inverses

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix}$$

each row op is a linear transformation

Do row echelon form

$$\rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

$$T: \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \rightarrow \begin{bmatrix} a_1 \\ a_2 + a_1 \\ a_3 - a_1 \end{bmatrix}$$

$T(x) = Sx$

take an identity and apply this transformation to get S

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = S$$

also

$$S\begin{bmatrix}1\\-1\\1\end{bmatrix} \quad S\begin{bmatrix}-1\\2\\1\end{bmatrix} \quad S\begin{bmatrix}-1\\3\\4\end{bmatrix} = \begin{bmatrix}1&0&0\\1&1&0\\-1&0&1\end{bmatrix}\begin{bmatrix}1&-1&-1\\-1&2&3\\1&1&4\end{bmatrix}$$

$$= \begin{bmatrix}1&-1&-1\\0&1&2\\0&2&5\end{bmatrix}$$

$S_1 A$ $\qquad\qquad$ $S_2 S_1 A$

$$\begin{bmatrix}1&-1&-1\\0&1&2\\0&2&5\end{bmatrix} \rightarrow \begin{bmatrix}1&0&1\\0&1&2\\0&0&1\end{bmatrix}$$

$$T_2\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_1+x_2\\x_2\\x_3-2x_2\end{bmatrix}$$

$S_3 S_2 S_1 A = I$

$$\rightarrow \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \qquad T_3\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}x_1-x_3\\x_2-2x_2\\x_3\end{bmatrix}$$

since the rref$(A) = I$, it invertible.

we know $A^{-1}A = I$ $\qquad$ also $S_3 S_2 S_1 A = I$

$\therefore A^{-1} = S_3 S_2 S_1$

So you can setup

$$[A \mid I] \rightarrow \text{row op} \rightarrow \text{row op} \rightarrow [I \mid A^{-1}]$$

* figuring formula for 2x2 matrix.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & ad-bc & -c & a \end{array}\right] \qquad T_1 \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ a c_2 - c c_1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} (ad-bc)a & 0 & ad & -ab \\ 0 & ad-bc & -c & a \end{array}\right] \qquad T_2 \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{(ad-bc)c_1 - bc_2}{c_1}$$

$$\left[\begin{array}{cc|cc} \frac{ad}{(ad-bc)a} & \frac{-ab}{(ad-bc)a} & 1 & 0 \\ \frac{-c}{ad-bc} & \frac{d}{ad-bc} & 0 & 1 \end{array}\right] \qquad T_3 \begin{bmatrix} \frac{1}{(ad-bc)a} c_1 \\ \frac{1}{(ad-bc)} c_2 \end{bmatrix}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$\therefore$ inverse for 2x2 is not defined if $ad - bc = 0$

basically the determinant

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$