3 x 3 matrix

$$A_{3\times3} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Generalizing to a nxn matrix using recursion

$$A_{n\times n} = \begin{bmatrix} a_{11} & a_{21} & a_{1n} \\ a_{21} & \ddots & \vdots \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

A_{ij} = $(n-1)(n-1)$ matrix you get if you ignore i^{th} row and j^{th} column.

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) \ldots a_{1n}\det(A_{1n})$$

you keep recurring A till you get base 2×2 matrix, then solve.

Rule of Sarrus

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - ge)$$

$$= aei - afh - bdi + fgh + cdh - gce$$

$$= aei + fgh + cdh - afh - dhi - gce$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$$

You can solve this way also

* determinant when row multiplied by scalar

$$\begin{vmatrix} a & b \\ kc & kd \end{vmatrix} = kad - kbc = k(ad - bc) = k \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

multiply only one row

$$kA = \begin{vmatrix} ka & kb \\ kc & kd \end{vmatrix} = k^2(ad - bc)$$

For the general $n \times n$

$$\underset{n \times n}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{i1} & a_{i2} & & a_{in} \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$$

$$\det A = (-1)^{i+1} a_{i1} \det(A_{i1}) + (-1)^{i+2} a_{i2} \det(A_{i2})$$

$$+ \cdots + a_{in} \det(A_{in}) (-1)^{i+n}$$

$$= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

$$A'_{n\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ ka_{i1} & ka_{i2} & ka_{in} \\ a_{n1} & a_{n2} & a_{nn} \end{bmatrix}$$

$$\det(A') = \sum_{j=1}^{n} (-1)^{i+j} k a_{ij} \det(A_{ij})$$

$$= k \sum_{j}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

- determinant when row is added

$$X = \begin{bmatrix} a & b \\ x_1 & x_2 \end{bmatrix} \quad Y = \begin{bmatrix} a & b \\ y_1 & y_2 \end{bmatrix} \quad Z = \begin{bmatrix} a & b \\ x_1+y_1 & x_2+y_2 \end{bmatrix}$$

$$|X| = ax_2 - bx_1 \qquad |Y| = ay_2 - by_1$$

$$|Z| = a(x_2+y_2) - b(x_1+y_1)$$

$$= ax_2 + ay_2 - bx_1 - by_1$$

$$= \underbrace{ax_2 - bx_1}_{|X|} + \underbrace{ay_2 - by_1}_{|Y|}$$

happens if the matrices are same and only if one row is the sum.

Generalizing to $n\times n$

$$\det(X) = \sum_{j=1}^{n} (-1)^{i+j} x_j \det(A_{ij})$$

$$+$$

$$\det(Y) = \sum_{j=1}^{n} (-1)^{i+j} y_j \det(A_{ij})$$

$$= \det(Z) = \sum_{j=1}^{n} (-1)^{i+j} (x_j + y_j) \det(A_{ij})$$

* duplicate row determinant

$$\underset{n\times n}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & & a_{in} \\ a_{i1} & & & a_{jn} \\ a_{j1} & & & a_{nn} \\ a_{n1} & & & \end{bmatrix}$$

$$r_i = \begin{bmatrix} a_{i1} & a_{i2} & a_{in} \end{bmatrix}$$

$$\underset{n\times n}{A} = \begin{bmatrix} r_1 \\ r_2 \\ r_i \\ r_j \\ r_n \end{bmatrix}$$ swap i and j $$S_{ij} = \begin{bmatrix} r_1 \\ r_2 \\ r_j \\ r_i \\ r_n \end{bmatrix}$$

$$\det(S) = -\det(A)$$

if $i = j$
$$\det(S) = \det(A)$$

there is only one case where $\det(x) = -\det(A)$
if $x = 0$

remember matrix invertible if $\text{rref}(x)$ is I_n

duplicate rows $\longrightarrow$ not $\text{rref}(x) \neq I_n$
not invertible
$= \det(A) = 0$

- upper triangular determinant

$$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = ad$$

$$B = \begin{vmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{vmatrix} = a \begin{vmatrix} d & e \\ 0 & f \end{vmatrix} - 0 \begin{vmatrix} b & c \\ 0 & f \end{vmatrix} + 0 \begin{vmatrix} b & c \\ d & e \end{vmatrix}$$

$$= adf$$

to general case $n \times n$

$$\underset{n \times n}{A} = \det(A) = a_{11}\, a_{22} \cdots a_{nn}$$