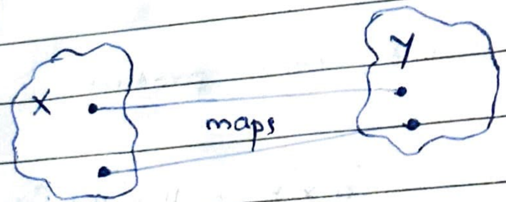
Functions

$f : X \to Y$

function is a relation b/w members of one set and members of another set.



X is the domain.

Y is co-domain.

Range - subset of codomain that function actually maps to.

$g : R^2 \to R$

$g(x_1, x_2) = 2$

domain R^2

codomain R

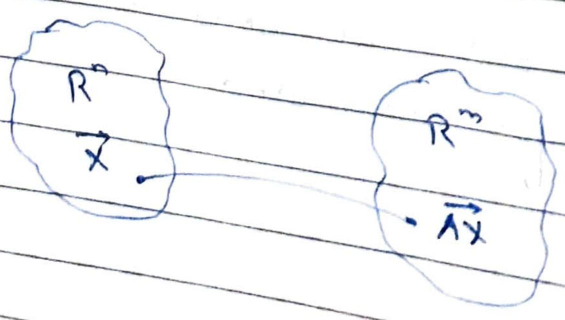
range 2

- Matrix vector products are linear transformations

$$\underset{m \times n}{A} = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \cdots & \vec{v_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 v_1 + x_2 v_2 \ldots \ldots x_n v_n \in R^m$$

$T : R^n \to R^m$

$T(\vec{x}) = A\vec{x}$



$$B = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$$

$T: R^2 \rightarrow R^2$

$T(x) = Bx$

$$= \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 \\ 3x_1 + 4x_2 \end{bmatrix}$$

$T(x) = (2x_1 - x_2, 3x_1 + 4x_2)$

constraints for linear transformation

$T(a+b) = T(a) + T(b)$

$T(ca) = cT(a)$

proof

$$Ax = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix} = x_1 v_1 + x_2 v_2$$

(1) $\underset{m \times n}{A}(a+b) = A \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_n + b_n \end{bmatrix} = A(a_1 + b_1) + A(a_2 + b_2)$

$= v_1(a_1 + b_1) + v_2(a_2 + b_2)$

$\ldots$

$= (a_1 v_1 + a_2 v_2 \ldots a_n v_n) + (b_1 v_1 + b_2 v_2 \ldots b_n v_n)$

$$= A \begin{bmatrix} a_1 \\ a_2 \\ a_n \end{bmatrix} + B \begin{bmatrix} b_1 \\ b_2 \\ b_n \end{bmatrix}$$

(2) $A(c\vec{a}) = \begin{bmatrix} \vec{v_1} & \vec{v_2} & \vec{v_n} \end{bmatrix} \begin{bmatrix} c\vec{a_1} \\ c\vec{a_2} \\ c\vec{a_n} \end{bmatrix} = ca_1 v_1 + ca_2 v_2$

$= c(a_1 v_1 + a_2 v_2 \ldots a_n v_n)$

$= c A \vec{a}$

- Any linear transformation can be represented as matrix vector product.

$$I_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\quad\vec{e_1} \quad \vec{e_2} \quad \vec{e_3}$

$$I_n \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\{e_1, e_2, e_n$ standard basis for R^n.

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_n \end{bmatrix} = a_1 e_1 + a_2 e_2 \ldots a_n e_n$$

$$= \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ a_2 \\ 0 \\ 0 \end{bmatrix} + \ldots \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_n \end{bmatrix}$$

$\vec{x} = x_1 e_1 + x_2 e_2 \ldots x_n e_n$

any linear tran

$T(x) = T(x_1 e_1 + x_2 e_2 \ldots x_n e_n)$

$= T(x_1 e_1) + T(x_2 e_2) \ldots T(x_n e_n)$

$= x_1 T(e_1) + x_2 T(e_2) \ldots x_n T(e_n)$

$$T(x) = \begin{bmatrix} Te_1 & Te_2 & Te_n \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

matrix vector

Example, represent this as matrix vector

$T: R^2 \to R^3$

$$T(x_1, x_2) = (x_1 + 3x_2,\ 5x_2 - x_1,\ 4x_1 - x_2)$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 3x_2 \\ 5x_2 - x_1 \\ 4x_1 - x_2 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} + T\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & 5 \\ 4 & 1 \end{bmatrix}$$

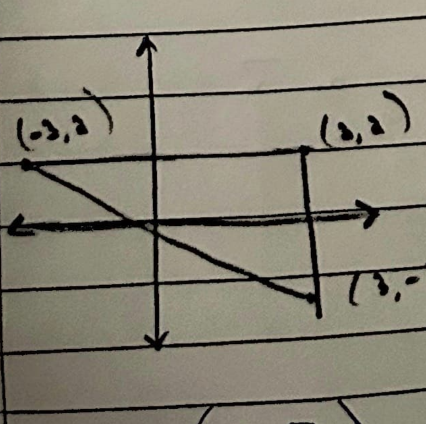
$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 3 \\ -1 & 5 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

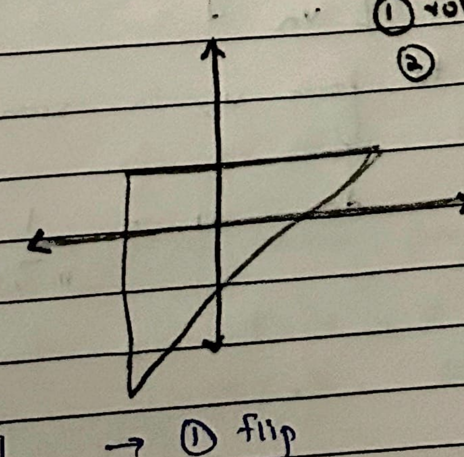
Scaling and reflections

Take 3 points $\begin{bmatrix} 3 \\ 2 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

(1) rotate

(2) scale





$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ 2y \end{bmatrix}$$

→ (1) flip

→ (2) stretch by 2

Since R^2, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} + T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

1 $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

2 $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

3 $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$

Unit vector $\rightarrow$ has length of 1.

$u \in R^n$ $\quad \|u\| = \sqrt{u_1^2 + u_2^2 \ldots u_n^2}$

unit vector : $\|u\| = 1$

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_n \end{bmatrix}$ which is not unit vector,

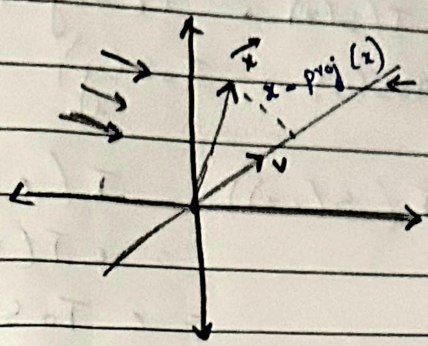
to form u, in same direction as $\vec{v}$

do $u = \dfrac{1}{\|\vec{v}\|} \times \|\vec{v}\|$

eg $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ $\quad \|\vec{v}\| = \sqrt{1+4+1} = \sqrt{6}$

$$\vec{u} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

Projections



proj(x) = shadow of x falling on L

proj(x) = some vector in L where x - proj(x) is orthogonal to L.

(dot product is 0)

$$(x - cv) \cdot v = 0$$

$$xv - cvv = 0$$

$$xv = cvv$$

$$c = \frac{xv}{vv}$$

$$\text{proj}(x) = cv = \left(\frac{xv}{vv}\right) v$$

Composition of linear transformations

basically transform one space, then transform this space again.

$S : X \to Y$

$x \in \mathbb{R}^n \quad y \in \mathbb{R}^m$

we know linear transformation is matrix vector product

$T : Y \to Z$

$S(x) = Ax$

$T(x) = B(x)$