Zeros of polynomials

function $p : F \to F$ is called polynomial of degree m if there exists $a_0, a_1, \ldots a_m \in F$ with $a_m \neq 0$ such that -

$p(z) = a_0 + a_1 z + \ldots a_m z^m$

a number Z is zero of a polynomial $p \in P(F)$ if $p(\lambda) = 0$

Pick this up after Khan Academy refresher.

Vector dot product and vector length

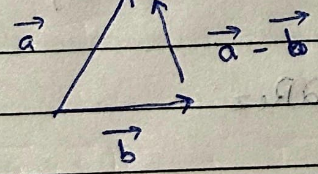
length $= \|\vec{a}\| = \sqrt{a_1^2 + a_2^2 \ldots a_n^2}$ (across all dims)

$\|\vec{a}\|^2 = \vec{a} \times \vec{a}$, basically

$$\vec{a} \times \vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_n \end{bmatrix} = a_1^2 + a_2^2 \ldots a_n^2$$

angle b/w vectors

$\|\vec{a}\|$ = length



$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

$\|\vec{a}\| = \|\vec{b} + (\vec{a} - \vec{b})\| \leq \|\vec{b}\| + \|\vec{a} - \vec{b}\|$

$\|\vec{b}\| = \|\vec{a} + (\vec{b} - \vec{a})\| \leq \|\vec{a}\| + \|\vec{b} - \vec{a}\|$

$\|\vec{a} - \vec{b}\| = \|\vec{a} + (-\vec{b})\| \leq \|\vec{a}\| + \|-\vec{b}\|$

$(\vec{a} \times \vec{b}) = \|\vec{a}\| \, \|\vec{b}\| \cos\theta$

c is scalar, if

$\vec{a} = c\vec{b}$ $c > 0 \Rightarrow \theta = 0$

$c < 0 \Rightarrow \theta = 180^\circ$

perpendicular vectors

$a \perp b$ if $\theta = 90^\circ$

$a \cdot b = \|\vec{a}\| \, |b| \cos 90^\circ$

$= a \cdot b = 0$

$a, b \perp er$, then dot product is 0.

cross product

dot product $a, b \in R^2 \Rightarrow$ scalar

cross product only for $R^3 \Rightarrow$ vector

$$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \vec{a} \times \vec{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

$\downarrow$

orthogonal to a and b

Show orthogonal.

$$\begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$= (a_2 b_3 a_1 - a_1 a_3 b_2) + (a_2 a_3 b_1 - a_1 a_2 b_3) +$
$(a_1 b_2 a_3) - a_2 a_3 b_1)$

$= 0$ (all of them cancel each other out)

This is the same case with b also (multiply and verify).

Cross product and sin proof

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

To prove $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin\theta$

we know $\vec{a}.\vec{b} = \|a\| \|b\| \cos\theta$

$$\|\vec{a} \times \vec{b}\|^2 = (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2$$

$$= a_2^2 b_3^2 - 2 a_2 a_3 b_2 b_3 + a_3^2 b_2^2 + \ldots$$

$$= a_1^2 (b_2^2 + b_3^2) + a_2^2 (b_1^2 + b_3^2) + a_3^2 (b_1^2 + b_2^2)$$

$$- 2 (a_2 a_3 b_2 b_3 + a_1 a_3 b_1 b_3 + a_1 a_2 b_1 b_2) \rightarrow (1)$$

we know

$\|\vec{a}\| \|\vec{b}\| \cos\theta = \vec{a}.\vec{b}$

squaring

$$\|\vec{a}\|^2 \|\vec{b}\|^2 \cos^2\theta = (\vec{a}.\vec{b})^2$$

$$= (a_1 b_1 + a_2 b_2 + a_3 b_3)^2$$

$$= (a_1 b_1 + a_2 b_2 + a_3 b_3)(a_1 b_1 + a_2 b_2 + a_3 b_3)$$

$$= a_1^2 b_1^2 + a_2^2 b_2^2 + a_3^2 b_3^2 + 2 (a_1 a_2 b_1 b_2 + a_1 a_3 b_1 b_3 + a_2 a_3 b_2 b_3) \rightarrow (2)$$

$$\|\vec{a} \times \vec{b}\|^2 + \|\vec{a}\|^2 \|\vec{b}\|^2 \cos^2\theta = (1) + (2)$$

$$= a_1^2 (b_1^2 + b_2^2 + b_3^2) + a_2^2 (b_1^2 + b_2^2 + b_3^2) + a_3^2 (b_1^2 + b_2^2 + b_3^2)$$

$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$

$$= \|\vec{a}\|^2 \|\vec{b}\|^2$$

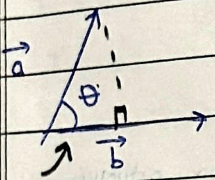
$$\|\vec{a}\times\vec{b}\|^2 = \|\vec{a}\|^2\|\vec{b}\|^2 - \|\vec{a}\|^2\|\vec{b}\|^2\cos^2\theta$$
$$= \|\vec{a}\|^2\|\vec{b}\|^2(1-\cos^2\theta)$$
$$= \|\vec{a}\|^2\|\vec{b}\|^2\sin^2\theta$$

$$\therefore \|\vec{a}\times\vec{b}\| = \|\vec{a}\|\,\|\vec{b}\|\sin\theta$$

Intuition for dot product and cross product

$\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$

$\vec{a}\times\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\sin\theta$



$\cos\theta = \dfrac{adj}{hyp}$

$adj = \cos\theta\,\vec{a}$

$\sin\theta = \dfrac{op}{hyp}$

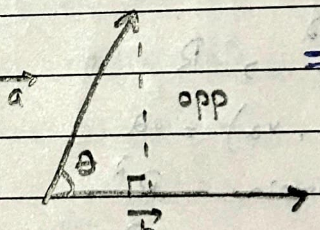
$= \dfrac{op}{\|\vec{a}\|}$

$op = \|\vec{a}\|\sin\theta$

adj

$\|\vec{b}\|\,\|\vec{a}\|\cos\theta$

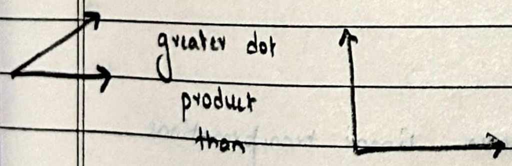
$\|\vec{b}\|$ adj

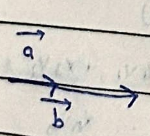


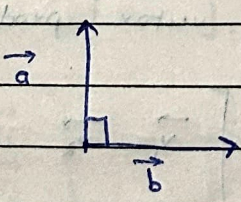
$= \|\vec{b}\|$ opp

basically how much of those vectors are moving in same direction.

basically $\vec{b}$ times what part is per direction to $\vec{b}$







$|\vec{a}\times\vec{b}|$

$= \|\vec{a}\|\,\|\vec{b}\|$

maxed out

$\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|$

maxed out

and vice versa for each other