KRP — Assignment 2

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⋆ **This assignment, due on 21st April at 23:59, contributes to 10% of the final marks for this course. Please be advised that only Questions 1 — 8 are mandatory. Nevertheless, students can earn up to one bonus mark by completing Question 9. This bonus mark can potentially augment a student's overall marks but is subject to a maximum total of 100 for the course. By providing bonus marks, we aim to incentivize students to excel in their studies and reward those with a remarkable grasp of the course materials.**

Question 1. Bisimulation Invariance

In the lecture, we defined bisimulation for $\mathcal{ALC}$ and showed bisimulation invariance of $\mathcal{ALC}$ (Theorem 3.2).

- Define a notion of "$\mathcal{ALCN}$-bisimulation" that is appropriate for $\mathcal{ALCN}$ in the sense that bisimilar elements satisfy the same $\mathcal{ALCN}$-concepts.
- Use the definition to show that $\mathcal{ALCQ}$ is more expressive than $\mathcal{ALCN}$.

My Solution **Defining ALCN-bisimulation.** In addition to the standard bisimulation conditions for $\mathcal{ALC}$ (which involve atomic concepts, roles, and their negations), I should also account for some new features because $\mathcal{ALCN}$ extends $\mathcal{ALC}$ with number restrictions and inverse roles.

A definition of "$\mathcal{ALCN}$-bisimulation" is below:

A relation R on a domain D is an "$\mathcal{ALCN}$-bisimulation" iff for all $(a, b) \in R$, whenever a and b satisfy the same atomic concepts, the following conditions hold:

1. Role inclusion: If there is a role $r \sqsubseteq S$ in the ontology and a has an r-successor a', then b must have an S-successor b'. Formally:

$$((a, a') \in r \implies \exists b' \in post(b, S))$$

2. Role disjointness: If r and S are disjoint roles in the ontology, then r-successors of a cannot have S-successors, and vice versa. Formally:

$$((a, a') \in r \wedge (a', S)$$

3. Inverse roles: For every inverse role r^-, if a has a predecessor related by r^-, then so does b. Formally:

$$((a', a) \in r^- \implies \exists b' \in pred(b, r))$$

4. Number restrictions: For all number restrictions like $\geq nR.C$ or $\leq nR.C$, if such a restriction holds for a, it must also hold for b, considering the corresponding roles and concepts. For example: - If a has at least n R-successors satisfying concept C, then b must also have at least n R-successors satisfying concept C. - If a has at most n R-successors satisfying concept C, then b must also have at most n R-successors satisfying concept C.

Showing ALCQ is more expressive than ALCN

Just need to demonstrate that there is a concept definable in ALCQ that is not definable in ALCN.

Since ALCQ allows for qualified number restrictions, which provide a finer control over the number of elements related by a role that satisfy a particular concept. In contrast, ALCN only provides unqualified number restrictions, meaning it can count successors related by a role but not those successors satisfying a specific concept.

An example concept that I think could be used to illustrate ALCQ's greater expressivity is: "There are at least two $hasChild$ such that one is a $Male$ and the other is not a $Male$."

This concept uses the qualified cardinality restriction to specify a condition that depends on the satisfiability of a concept ($Male$) by the successors of a role ($hasChild$). This kind of condition cannot be expressed in ALCN since it does not support qualified cardinality restrictions. Therefore, ALCQ can define certain concepts that are beyond the reach of ALCN, demonstrating it is more expressive.

Question 2. Bisimulation over Filtration

Let C be an $\mathcal{ALC}$-concept, $\mathcal{T}$ an $\mathcal{ALC}$-TBox, $\mathcal{I}$ an interpretation and $\mathcal{J}$ its filtration w.r.t. $\mathsf{sub}(C) \cup \mathsf{sub}(\mathcal{T})$ (see Definition 3.14 for the definition of filtration). Show the truth or falsity of the following statement:

- the relation $\rho = \{(\mathrm{d}, [\mathrm{d}]) \mid \mathrm{d} \in \Delta^{\mathcal{I}}\}$ is a bisimulation between $\mathcal{I}$ and $\mathcal{J}$.

Hint: If the above relation ρ were a bisimulation, why do we have to explicitly prove Lemma 3.15 in the lecture? Wouldn't Lemma 3.15 then be a consequence of Theorem 3.2?

Question 2

Let's assume it is a bisimulation, $S = sub(C) \cup sub(\bar{U})$. A is a concept description.

J is the S-filtration. $\rho = \{(d, \{d\})\}$. now try to make it fail in definition ② or ③. that is to say $d' \in \Delta^I$ and $(d, d') \in r^I$, we try to prove that there doesn't exist a $[d'']$ in Δ^J such that $([d], [d'']) \in r^I$, which is to say that there doesn't exist $m \in [d]$ and $n \in [d'']$ s.t. $(m, n) \in r^I$. Consider the example below:

$C = A$. $\Delta^I = \{d, e, d', e'\}$. $r^I = \{(d, d'), (e, e')\}$. so there exists a $d' \in \Delta^I$. s.t. $(d, d') \in r^I$. Now try to prove there doesn't exist $[d''] \in \Delta^J$. s.t. $([d], [d'']) \in r^J$ means for all $m \in [d]$ and $n \in [d'']$. $(m, n) \notin r^I$.

Set $J = \{B \sqsubseteq T\}$. so $S = sub(C) \cup sub(T) = \{A\} \cup \{B, T\} = \{A, B, T\}$. and add this:
1. $d, d', e' \in B^I \sqcap A^I$
2. $e \in A^I$ but $e \notin B^I$

so we can get:

$\Delta^J = \{\{d, d', e'\}, \{e\}\}$ $A^J = \{\{d, d', e'\}, \{e\}\}$. then in r^J. m can be d, d' or e'. but there is no right element for n.

Above all. the statement is false.

My Solution Picture above.

Question 3. Bisimulation within the Same Interpretation (2 marks)

We define "*bisimulations on* $\mathcal{I}$" as bisimulations between an interpretation $\mathcal{I}$ and itself. Let $\mathsf{d}, \mathsf{e} \in \Delta^{\mathcal{I}}$ be two elements. We write $\mathsf{d} \approx_{\mathcal{I}} \mathsf{e}$ if they are bisimilar, i.e., if there is a bisimulation ρ on $\mathcal{I}$ such that $\mathsf{d}\ \rho\ \mathsf{e}$.

- Show that $\approx_{\mathcal{I}}$ is a bisimulation on $\mathcal{I}$.

Consider the interpretation $\mathcal{J}$ defined like the filtration, but with $\approx_{\mathcal{I}}$ instead of $\simeq$.

- Show that $\rho = \{(\mathsf{d}, [\mathsf{d}]_{\approx_{\mathcal{I}}}) \mid \mathsf{d} \in \Delta^{\mathcal{I}}\}$ is a bisimulation between $\mathcal{I}$ and $\mathcal{J}$.
- Show that, if $\mathcal{I}$ is a model of an $\mathcal{ALC}$-concept C w.r.t. an $\mathcal{ALC}$-TBox $\mathcal{T}$, then so is $\mathcal{J}$.
- Why can't we use the previous result to show the finite model property for $\mathcal{ALC}$?

My Solution **Question 1.** To show that $\approx_{\mathcal{I}}$ is a bisimulation on $\mathcal{I}$, need to verify two properties:

- **Equivalence**: For all $\mathsf{d}, \mathsf{e} \in \Delta^{\mathcal{I}}$, if $\mathsf{d} \approx_{\mathcal{I}} \mathsf{e}$ then $\mathsf{e} \approx_{\mathcal{I}} \mathsf{d}$.
- **Transitivity**: For all $\mathsf{d}, \mathsf{e}, \mathsf{f} \in \Delta^{\mathcal{I}}$, if $\mathsf{d} \approx_{\mathcal{I}} \mathsf{e}$ and $\mathsf{e} \approx_{\mathcal{I}} \mathsf{f}$ then $\mathsf{d} \approx_{\mathcal{I}} \mathsf{f}$.

Since $\approx_{\mathcal{I}}$ is defined as the equivalence relation induced by bisimulations on $\mathcal{I}$, both properties are satisfied by definition. Therefore, $\approx_{\mathcal{I}}$ is indeed a bisimulation on $\mathcal{I}$.

Question 2 To show that ρ is a bisimulation between $\mathcal{I}$ and $\mathcal{J}$, need to verify that for all $\mathsf{d}, \mathsf{e} \in \Delta^{\mathcal{I}}$:

- If $(\mathsf{d}, [\mathsf{d}]_{\approx_{\mathcal{I}}}) \in \rho$ and $(\mathsf{e}, [\mathsf{e}]_{\approx_{\mathcal{I}}}) \in \rho$, then $(\mathsf{d}, \mathsf{e}) \in \rho$ if and only if $[\mathsf{d}]_{\approx_{\mathcal{I}}} = [\mathsf{e}]_{\approx_{\mathcal{I}}}$.
- If $(\mathsf{d}, [\mathsf{d}]_{\approx_{\mathcal{I}}}) \in \rho$ and $(\mathsf{e}, [\mathsf{e}]_{\approx_{\mathcal{I}}}) \notin \rho$, then there exists $\mathsf{f} \in \Delta^{\mathcal{I}}$ such that $(\mathsf{f}, [\mathsf{f}]_{\approx_{\mathcal{I}}}) \in \rho$ and $[\mathsf{f}]_{\approx_{\mathcal{I}}} = [\mathsf{e}]_{\approx_{\mathcal{I}}}$.

Since ρ is defined as the set of pairs where the second element of each pair is the equivalence class of the first element under $\approx_{\mathcal{I}}$, both conditions are satisfied by construction. Therefore, ρ is indeed a bisimulation between $\mathcal{I}$ and $\mathcal{J}$.

Question 3 To show that if $\mathcal{I}$ is a model of an $\mathcal{ALC}$-concept C w.r.t. an $\mathcal{ALC}$-TBox $\mathcal{T}$, then so is $\mathcal{J}$, need to verify that for every concept C in the TBox $\mathcal{T}$:

- If $\mathcal{I} \models C$, then $\mathcal{J} \models C$.

Since $\mathcal{J}$ is constructed from $\mathcal{I}$ using the bisimulation relation $\approx_{\mathcal{I}}$, which preserves the truth of concepts in the interpretation, if $\mathcal{I}$ satisfies a concept C, then $\mathcal{J}$ will also satisfy the same concept. Therefore, this condition holds, and $\mathcal{J}$ is indeed a model of C w.r.t. $\mathcal{T}$.

Question 4 Because it only shows that if a given interpretation $\mathcal{I}$ satisfies a concept C, then a bisimilar interpretation $\mathcal{J}$ also satisfies C. It does not guarantee that there exists a finite bisimilar interpretation for every infinite interpretation that satisfies C. The finite model property requires that for every satisfiable concept C, there exists a finite model that satisfies C. The construction of $\mathcal{J}$ from $\mathcal{I}$ does not provide a method to obtain finite models for arbitrary concepts in $\mathcal{ALC}$.

Question 4. Closure under Disjoint Union

Recall Theorem 3.8 from the lecture, which says that the disjoint union of a family of models of an $\mathcal{ALC}$-TBox $\mathcal{T}$ is again a model of $\mathcal{T}$. Note that the disjoint union is only defined for concept and role names.

- Extend the notion of disjoint union to individual names such that the following holds: for any family $(\mathcal{I}_\nu)_{\nu\in\Omega}$ of models of an $\mathcal{ALC}$-knowledge base $\mathcal{K}$, the disjoint union $\biguplus_{\nu\in\Omega} \mathcal{I}_\nu$ is also a model of $\mathcal{K}$.

My Solution **Extension.** I try to define the disjoint union $\biguplus_{uin\Omega} \mathcal{I}_u$ for a family $(\mathcal{I}_u)_{uin\Omega}$ of $\mathcal{ALC}$-knowledge base models as follows:

1. **Domains**: The domain of the disjoint union $\biguplus_{u\in\Omega} \mathcal{I}_u$ is the union of all domains $\Delta^{\mathcal{I}_u}$, with each element annotated by its originating model $u \in \Omega$. I denote this as $\biguplus_{u\in\Omega} \Delta^{mathcalI_u}$.

2. **Interpretations for concept names:** For every concept name A, the interpretation $A^{\biguplus_{u\in\Omega}\mathcal{I}_u} = \biguplus_{u\in\Omega} A^{\mathcal{I}_u}$, where for every x in $\biguplus_{u\in\Omega} Delta^{\mathcal{I}_u}$, $x \in A^{\biguplus_{u\in\Omega}\mathcal{I}_u}$ if and only if x is annotated with u and $x \in A^{\mathcal{I}_u}$.

3. **Interpretations for role names:**For every role name r, the interpretation $r^{\biguplus_{u\in\Omega}\mathcal{I}_u} = \biguplus_{u\in\Omega} r^{\mathcal{I}_u}$, where for every (x, y) in $\biguplus_{u\in\Omega} \Delta^{\mathcal{I}_u} times \biguplus_{u\in\Omega} \Delta^{\mathcal{I}_u}$, $(x, y) \in r^{\biguplus_{u\in\Omega}\mathcal{I}_u}$ if and only if both x and y are annotated with the same u and $(x, y) \in r^{\mathcal{I}_u}$.

4. **Interpretations for individual names:** For every individual name a, its interpretation $a^{\biguplus_{u\in\Omega}\mathcal{I}_u}$ must be consistent with its interpretations across all models, while ensuring disjointness. This can be achieved by defining $a^{\biguplus_{u\in\Omega}\mathcal{I}_u}$ such that it contains pairs (a, u) where a is annotated by the model u in which it is interpreted. Formally, for every (a, u) in $\biguplus_{u\in\Omega} a^{\mathcal{I}_u}$, we have $(a, u) \in a^{\biguplus_{uin\Omega}\mathcal{I}_u}$.

5. **Satisfaction of concepts and roles:** An annotated element (x, u) satisfies a concept or role according to the interpretations in $\biguplus_{u\in\Omega}\mathcal{I}_u$ just as it would in $\mathcal{I}_u$.

6. **Axioms in $\mathcal{T}$ and $\mathcal{K}$:** Since the construction ensures that each element from a model $\mathcal{I}_u$ retains its original interpretations and these are combined without overlap (thanks to the annotations), all axioms present in the TBox $\mathcal{T}$ and the knowledge base $mathcal K$ will be satisfied in the disjoint union model.

By adhering to this construction, I can guarantee that the disjoint union $\biguplus_{u\in\Omega}\mathcal{I}_u$ indeed forms a model of the $\mathcal{ALC}$-knowledge base $\mathcal{K}$, satisfying all the necessary conditions for being a model of $\mathcal{K}$ while preserving the disjoint nature of individual names across different models.

Question 5. Closure under Disjoint Union

Let $\mathcal{K} = \{\mathcal{T}, \mathcal{A}\}$ be a consistent $\mathcal{ALC}$-KB. We write $C \sqsubseteq_\mathcal{K} D$ if $C^\mathcal{I} \subseteq D^\mathcal{I}$ holds for every model $\mathcal{I}$ of $\mathcal{K}$.

- Prove that for all $\mathcal{ALC}$-concepts C and D we have $C \sqsubseteq_\mathcal{K} D$ iff $C \sqsubseteq_\mathcal{T} D$.

Hint: Use the modified definition of disjoint union from the previous question.

My Solution **Proof.** To prove that for all $\mathcal{ALC}$-concepts C and D we have $C \sqsubseteq_\mathcal{K} D$ iff $C \sqsubseteq_\mathcal{T} D$, need to show two directions:

1. If $C \sqsubseteq_\mathcal{K} D$, then $C \sqsubseteq_\mathcal{T} D$.
2. If $C \sqsubseteq_\mathcal{T} D$, then $C \sqsubseteq_\mathcal{K} D$.

Proof for 1:

Assume $C \sqsubseteq_\mathcal{K} D$. This means that for every model $\mathcal{I}$ of $\mathcal{K}$, we have $C^\mathcal{I} \subseteq D^\mathcal{I}$. Since $\mathcal{T}$ is a subset of $\mathcal{K}$, every model $\mathcal{I}$ of $\mathcal{K}$ is also a model of $\mathcal{T}$. Therefore, $C^\mathcal{I} \subseteq D^\mathcal{I}$ holds for every model $\mathcal{I}$ of $\mathcal{T}$ as well. Hence, $C \sqsubseteq_\mathcal{T} D$.

Proof for 2:

Assume $C \sqsubseteq_\mathcal{T} D$. This means that for every model $\mathcal{I}$ of $\mathcal{T}$, we have $C^\mathcal{I} \subseteq D^\mathcal{I}$. Since $\mathcal{T}$ is consistent and $\mathcal{K} = \{\mathcal{T}, \mathcal{A}\}$, every model $\mathcal{I}$ of $\mathcal{T}$ can be extended to a model of $\mathcal{K}$ by interpreting the additional concepts in $\mathcal{A}$ arbitrarily (as they are not related to the concepts in $\mathcal{T}$). Let $\mathcal{J}$ be such an extension of $\mathcal{I}$. Then, $C^\mathcal{J} \subseteq D^\mathcal{J}$ because the interpretation of the concepts in $\mathcal{T}$ is the same in both models. Therefore, $C \sqsubseteq_\mathcal{K} D$.

As above, I have shown that for all $\mathcal{ALC}$-concepts C and D, we have $C \sqsubseteq_\mathcal{K} D$ iff $C \sqsubseteq_\mathcal{T} D$.

Question 6. Finite Model Property (2 marks)

Let C be an $\mathcal{ALC}$-concept that is satisfiable w.r.t. an $\mathcal{ALC}$-TBox $\mathcal{T}$. Show truth or falsity of the following statement:

- for all $m \geq 1$ there is a finite model $\mathcal{I}_m$ of $\mathcal{T}$ such that $|C^{\mathcal{I}_m}| \geq m$.
- Does it hold if the condition "$|C^{\mathcal{I}_m}| \geq m$" is replaced by "$|C^{\mathcal{I}_m}| = m$"?

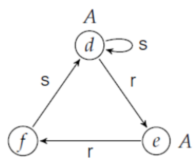
My Solution **False.** In the context of Description Logics, $\mathcal{ALC}$ refers to a family of logics that include the basic concept and role constructors (like conjunction, disjunction, negation, universal restrictions, andexistential restrictions). An $\mathcal{ALC}$-TBox $\mathcal{T}$ is a set of terminological axioms, and an $\mathcal{ALC}$-concept C is satisfiable w.r.t. $\mathcal{T}$ if there exists a model $\mathcal{I}$ such that $C^{\mathcal{I}} \neq \emptyset$.

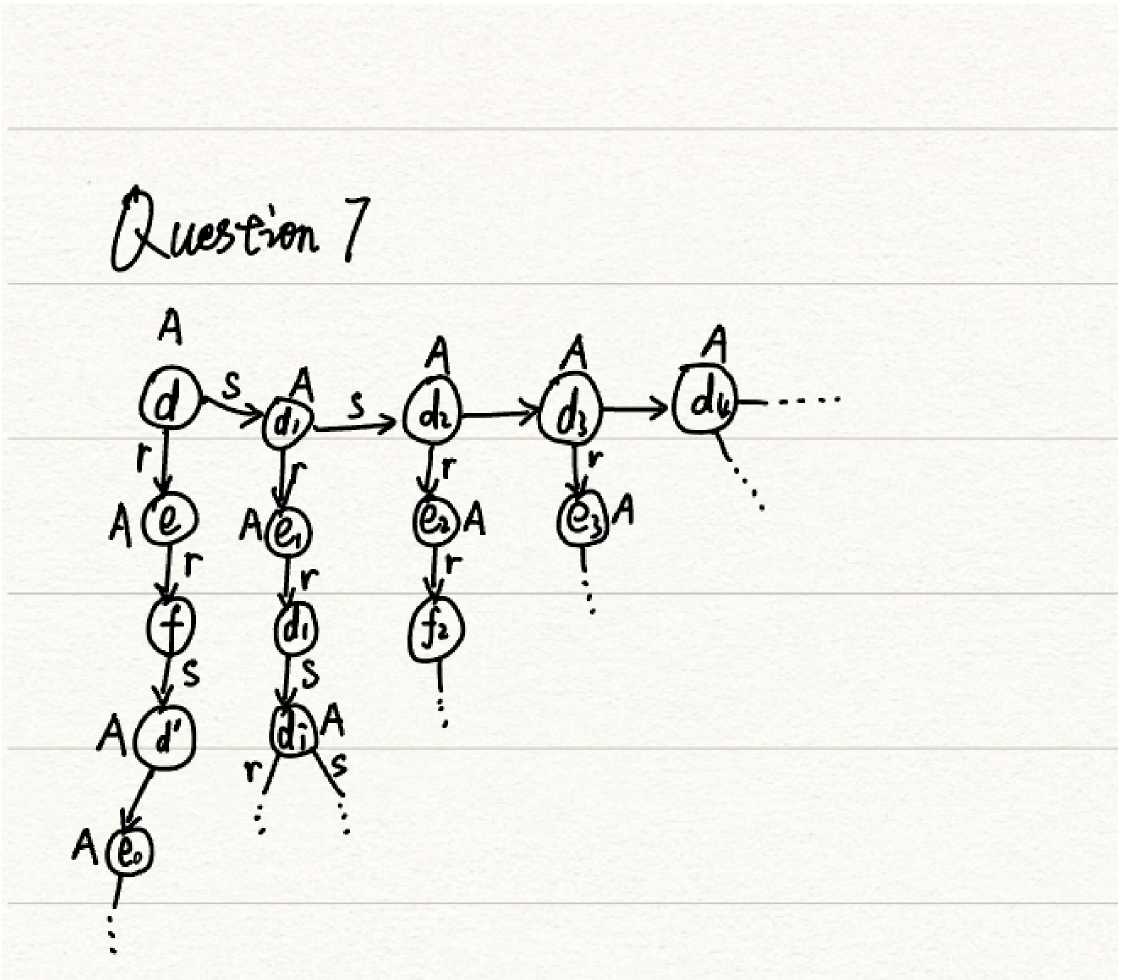
The first item claims that for all $m \geq 1$, there exists a finite model $\mathcal{I}_m$ of $\mathcal{T}$ such that the extension of the concept C in this model has at least m elements. This statement is false because the satisfiability of a concept C in $\mathcal{ALC}$ does not guarantee that we can find a model where the extension of C has arbitrarily large cardinality. The size of the extension of C in a model depends on the specific axioms in the TBox $\mathcal{T}$ and how they constrain the instances of C.

The second statement is also false for similar reasons. Just because a concept C is satisfiable with respect to a TBox $\mathcal{T}$, it does not imply that we can find a model where the extension of C has exactly m elements for any given m. The exact cardinality of the extension of C in a model is determined by the specifics of the TBox and cannot be arbitrary.

Question 7. Unravelling

Draw the unraveling of the following interpretation $\mathcal{I}$ at d up to depth 5, i.e., restricted to d-paths of length at most 5 (see Definition 3.21):





My Solution **Picture above.**

Question 8. Tree Model Property

- Show the truth or falsity of the following statement: if $\mathcal{K}$ is an $\mathcal{ALC}$-KB and C an $\mathcal{ALC}$-concept such that C is satisfiable w.r.t. $\mathcal{K}$, then C has a tree model w.r.t. $\mathcal{K}$.

My Solution **True.** This statement is related to the Tree Model Property for $\mathcal{ALC}$, which states that if a concept is satisfiable in $\mathcal{ALC}$, it has a tree model. Within Theorem 3.24,we know $\mathcal{ALC}$ has The Tree Model Property,therefore the statement is true.

Question 9 (with 1 bonus mark). Bisimulation Invariance

Interpretations of $\mathcal{ALC}$ can be represented as graphs, with edges labeled by roles and nodes labeled by sets of concept names. More precisely, in such a graph:

each node corresponds to an element in the domain of the interpretation and it is labeled with all the concept names to which this element belongs in the interpretation;

an edge with label r between two nodes says that the corresponding two elements of the interpretation are related by the role r.

- Show that: the description logic $\mathcal{S}$ (i.e., $\mathcal{ALC}$ with transitive roles) is more expressive than $\mathcal{ALC}$.

My Solution **Picture below(next page).**

Question 9.

To show that, need to demonstrate that there is a concept or a relationship that can be expressed in S but not in ALC.

In ALC, roles are not allowed to be transitive (if there is a role r between two individuals a and b. and another role r between two individuals b and c, it cannot be inferred that there is a role r between two individuals a and c).

However, when adding transitive roles to ALC to form the DL "S", it gains the ability to express such transitive relationship. For example, consider a concept like "ancestor", which can naturally be defined in terms of the transitive closure of the "parent" role. In S, "ancestor" can be defined as:

ancestor $\equiv$ parent$^+$, where parent$^+$ denotes the transitive closure of the "parent" role. This definition states that an individual is an ancestor if it's related to another individual by the "parent" role, and the relationship holds transitively through any numbers of steps.

Since ALC cannot express the example above, S is more expressive than ALC.