Logistic Regression

With logistic regression, one of the things we might want is for the hypothesis $h_\theta(x) \in [0, 1]$. We'll say that our hypothesis function is equal to some function of g w.r.t $\theta^T x$ which can be defined as

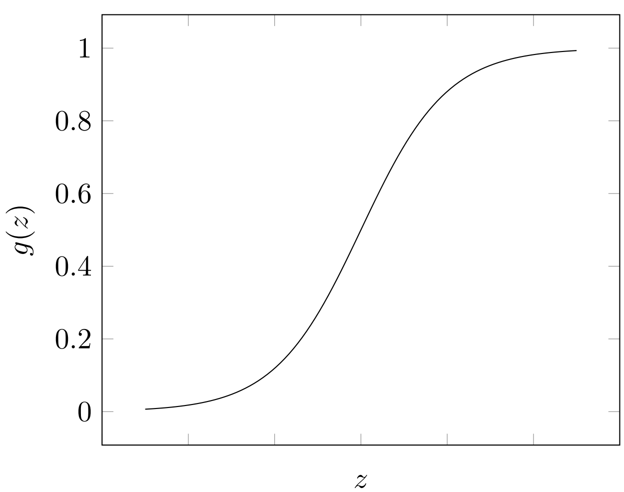
$$\frac{1}{1 + e^{-\theta^T x}}$$

This is actually just a form of the function $g(z)$, where

$$g(z) = \frac{1}{1 + e^{-z}}$$

The graph of this function is a "sigmoid" or a "logistic" curve. This forces the output values to only exist between 0 and 1.

Logistic Curve



If we say that $h_\theta(x) = \theta^T x$ and force it to follow a logistic curve to constrain the output to the range $[0, 1]$, then

$$P(y = 1 \mid x; \theta) = h_\theta(x)$$

$$P(y = 0 \mid x; \theta) = 1 - h_\theta(x)$$

where $y = \{0, 1\}$. This can be rewritten to combine these two equations as

$$P(y \mid x; \theta) = h(x)^y (1 - h(x))^{1-y}$$

which uses the fact that $n^0 = 1$ and $n^1 = n$ to force the output to be $1 - h(x)$ when $y = 0$ and $h(x)$ when $y = 1$. We can then define the likelihood of the parameters $\mathcal{L}(\theta)$ as

$$\prod_{i=1}^{m} h_\theta\left(x^{(i)}\right)^{y^{(i)}} \left(1 - h_\theta\left(x^{(i)}\right)\right)^{1-y^{(i)}}$$

We then define the log likelihood to be $\ell(\theta) = \log \mathcal{L}(\theta)$ which is equivalent to

$$\sum_{i=1}^{m} \left[y^{(i)} \log \left(h_\theta \left(x^{(i)} \right) \right) + \left(1 - y^{(i)} \right) \log \left(1 - h_\theta \left(x^{(i)} \right) \right) \right]$$

We want to choose θ to maximize $\ell(\theta)$ following the maximum likelihood estimation. For this, we can use the batch gradient ascent algorithm

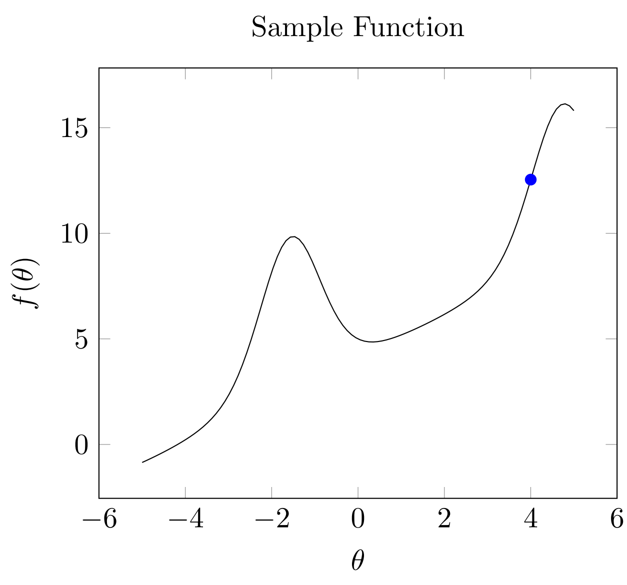
$$\theta_j = \theta_j + \alpha \frac{\partial}{\partial \theta} \ell(\theta)$$

which is similar to the batch gradient descent we used in the first lesson. The difference here is that we are trying to maximize $\ell(\theta)$ instead of minimize $J(\theta)$, which is why we are now adding to the parameters instead of subtracting. Instead of trying to reach a local minimum, we want to find the local maximum. Plugging in our features, target, and hypothesis function, the algorithm becomes

$$\theta_j = \theta_j + \alpha \sum_{i=1}^{m} \left(y^{(i)} - h_\theta \left(x^{(i)} \right) \right) x_j^{(i)}$$

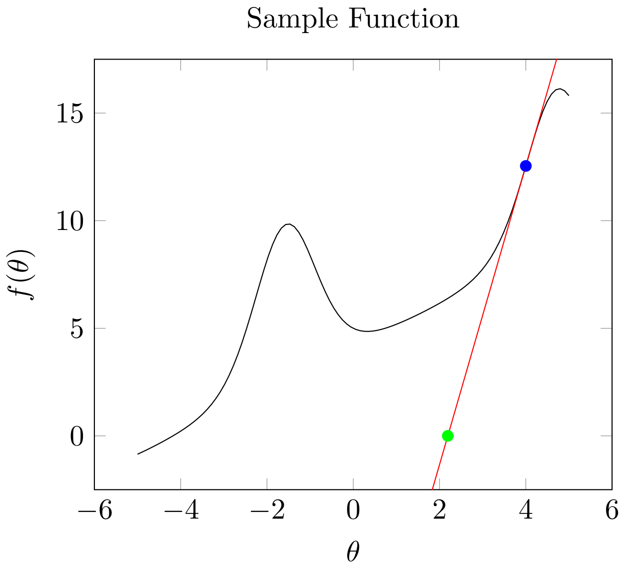
However, gradient ascent takes quite a few iterations to converge.

Say we have some function f and want to find a θ such that $f(\theta) = 0$. What this means is that we want to maximize $\ell(\theta)$ so that $\ell'(\theta) = 0$. Below is a sample function f we will use to demonstrate Newton's method.

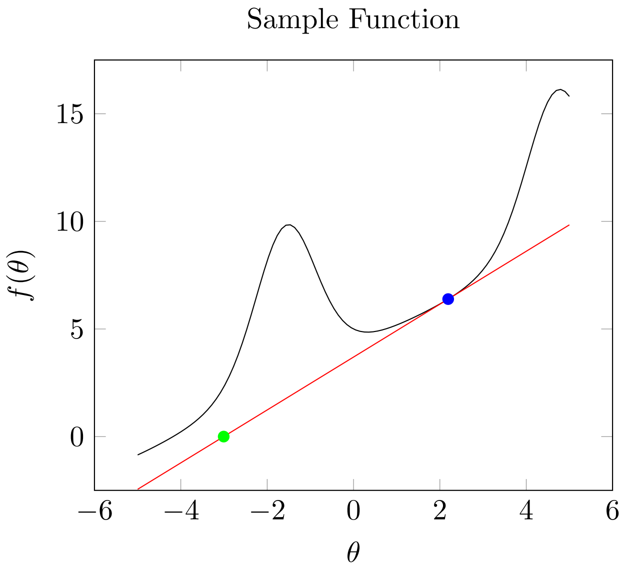


Let's say $(4, 12.543)$ is a point on this curve. This will be our first point representing $\theta^{(0)}$. This is the blue point you see in the graph.

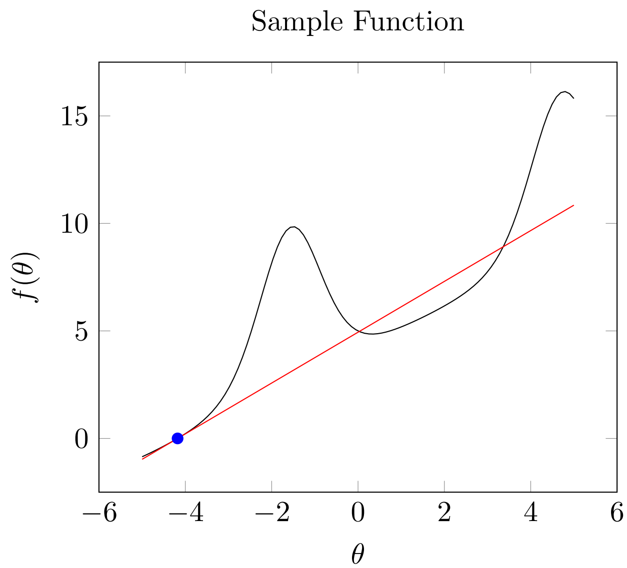
The next thing we want to do is find a line tangent to to f at this point; in other words, the derivative of f at $\theta^{(0)}$.



Newton's method then sets $\theta^{(1)}$ to where our tangent line $t(\theta) = 0$, which is the green point in the above graph $(2.192, 0)$. We repeat our finding of the tangent line and look for where it crosses that horizontal axis.



This continues until we reach a $\theta^{(i)}$ for which $f(\theta) = 0$.



As we can see, this causes our θ to converge to -4.179. Numerically, we know that $\theta^{(i+1)} = \theta^{(i)} - \Delta$ where Δ is the horizontal distance between iterations of θ. We know that the slope of tangent line is $f'\left(\theta^{(i)}\right)$, the change in θ is Δ, and the change in f is $f\left(\theta^{(i)}\right)$. This relationship can be modeled as the slope formula

$$f'\left(\theta^{(i)}\right) = \frac{f\left(\theta^{(i)}\right)}{\Delta}$$

and then rearranged to solve for Δ

$$\Delta = \frac{f\left(\theta^{(i)}\right)}{f'\left(\theta^{(i)}\right)}$$

which allows us to conclude that the algorithm for Newton's method is

$$\theta^{(i+1)} = \theta^{(i)} - \frac{f\left(\theta^{(i)}\right)}{f'\left(\theta^{(i)}\right)}$$

where $f(\theta) = \ell'(\theta)$.