Improper Integrals

(Feel free to comment, markup, and chat on this study guide.)

Theorems

Integral Comparison Test: Suppose there exists two improper integrals $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ and $f(x) \leq g(x)$ for all $x \in [a, b]$. Then if $\int_a^b g(x)dx$ converges, $\int_a^b f(x)dx$ and if $\int_a^b f(x)dx$ diverges, so does $\int_a^b g(x)dx$.

Examples

Example 1: Compute $\int_1^{\infty} \frac{dx}{x-2}$.

Solution 1: This integral has two improper points. The obvious one is that the upper limit is infinite. A less obvious one is that since the integrand is undefined at $x = 2$, it is discontinuous and therefore improper there. Thus we'll split the integral like so.

$\lim\limits_{a \to 2^-} \int_1^a \frac{dx}{x-2} + \lim\limits_{b \to 2^+} \int_b^3 \frac{dx}{x-2} + \lim\limits_{c \to \infty} \int_3^c \frac{dx}{x-2}$ (Remember that the first two limits are one sided because the area is being accumulated up until the discontinuity from that side).

Now we'll compute each integral independently.

Starting with the first one:

$\lim\limits_{a \to 2^-} \int_1^a \frac{dx}{x-2} = \lim\limits_{a \to 2^-} ln\,|a-2| - ln\,1$. As the argument of the logarithm grows smaller and smaller, that expression will grow to negative infinity, and so our first integral diverges. If one of our integrals diverges, that means the entire integral diverges. Thus $\int_1^{\infty} \frac{dx}{x-2}$ diverges.

Recap

This Study Guide went over:

- Improper Integrals and Testing for Divergence

Now attempt the practice problems on the next page!

Practice Problems

The problems will ramp up in difficulty.

1) $\int_{-1}^{1} \frac{dx}{x^2}$
2) $\int_{0}^{\infty} x \ln x \, dx$
3) $\int_{-\infty}^{\infty} \frac{e^x(x-1)}{e} \, dx$
4) $\int_{-1}^{2} \frac{dx}{x(x^3-1)}$
5) $\int_{1}^{2} \frac{dx}{(x-2)^2(x-1)}$
6) $\lim_{x \to 0^+} \frac{x}{\int_0^x \frac{dt}{e^t}}$
7) $\int_{1}^{2} \frac{x^3}{\sqrt{x^2-1}} dx$
8) $\lim_{x \to \infty} \frac{x}{e^x} \int_1^x \frac{dt}{t}$
9) $\lim_{x \to 0^+} x * \int_x^1 \frac{dt}{t \ln t}$ (Hint: View hint #1)
10) $\lim_{x \to 0^+} x * \int_x^{\frac{1}{e}} \frac{dt}{t \ln t}$ (Hint: View hint #1)

Hints to Selected Problems

Hint 1: Observe that it is possible for the integral to converge to a real value. In this case, jumping directly to L'Hopital's rule is incorrect.

Answer Key

1) Diverges
2) Diverges
3) Diverges
4) Diverges
5) Diverges
6) 1
7) $2\sqrt{3}$
8) 0
9) Undefined (due to attempting to find the derivative of infinity)
10) 0

Extra Practice Problems (Improper Integral Problem Bank)

Feel free to comment your answers to many of these well-known problems on the next page!
Also, comment in problems that you would like to see!
Sources: AoPS, Putnam, HMMT, StudyShed editors, and others.

1) $\int_{0}^{1} x * ln(1-x^2)dx$

2) $\lim_{x\to 0} \frac{x}{1-e^{x^2}}(\int_{0}^{x} e^{t^2}dt)$

3) $\int_{0}^{\infty} e^{-\sqrt{x}}dx$

4) $\int_{0}^{\infty} xe^{-x^2}dx$

5) $\int_{0}^{\infty} x^2e^{-x}dx$

6) $\int_{0}^{1} \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$

7) $\int_{1}^{\infty} \frac{dx}{e^{2+x}+e^{2-x}}$

8) $\int_{2}^{\infty} \frac{x^2dx}{\sqrt{x^2-4x+4}}$

9) $\int_{0}^{\infty} (\frac{x^2}{x-3} + \sqrt{x})dx$

10) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} tan\ x\ dx$

11) $\int_{-\infty}^{\infty} 3x^2(x+1)(x^2-x+1)(e^{-x^6-2x^3})dx$

12) $\int_{0}^{1}((tan\ x)^x * (2x\ csc\ (2x) + ln(tan\ x)))dx$ (Hint: View Hint #3)

13) $\int_{0}^{4} \frac{2x^3-128}{\sqrt{x}-2}dx$

14) $\lim_{x\to 0}((\lim_{a\to -1} x^a) * \int_{0}^{x^2} \sqrt{1+t^n}dt)$, where n is an integer.

<u>Extra Practice Problems Answers</u>

Leave the answers to your problems here!

4) 0.5
5) 2
6) 5 - 6ln(2)
7) (pi - 2arctan(e)) / e^2
8) Diverges
9) Diverges
10) Diverges
12) $\tan 1 - 1$
13) approx 815.543 (28544/35)
14) 0