<u>Applications of the Derivative</u>

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<u>Theorems</u>

The Mean Value Theorem: Suppose f is a continuous function on $[a,\ b]$ and a differentiable function on $(a,\ b)$. Then there exists a $c \in (a,\ b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Rolle's Theorem: Suppose f is a continuous function on $[a,\ b]$ and a differentiable function on $(a,\ b)$. Further assume that $f(a) = f(b)$. Then there exists a $c \in (a,\ b)$ such that $f'(c) = 0$.

The Extreme Value Theorem: Suppose f is a continuous function on $[a,\ b]$. If this is true, f attains one maximum and one minimum over $[a,\ b]$.

L'Hopital's Rule: $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ if $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ has the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Indeterminate forms: $\frac{0}{0},\ \frac{\infty}{\infty},\ 1^{\infty},\ \infty^0,\ 0 * \infty,\ \infty - \infty,\ 0^0$

Examples

Example 1: Use the tangent line at $x = 1$ to approximate $f(1.1)$ for $f(x) = x^2$.

Solution 1: This type of problem is known as a **tangent-line** approximation problem.

The overall strategy for this problem is to find the equation of the tangent line, and allow it to approximate the function for values of x near the point of tangency.

First, let's begin with finding the tangent line to f at 1. $f'(x) = 2x$, $f'(1) = 2$. So the tangent line has slope 2. Now $f(1) = 1$. Therefore, we have enough information to find the tangent line.

Using point-slope form: $y - 1 = 2(x - 1)$

$y - 1 = 2x - 2$

$y = 2x - 1$

Now, we can let $y \approx f(x)$ for x near 1.

Thus $f(1.1) \approx y(1.1) = 1.2$.

Example 2: Find all extrema of $f(x) = x^3 - 3x$.

All extrema implies **local** and **global/absolute** extrema. Local extrema are points that are higher or lower than surrounding points (turning points). Global extrema are the maximum and minimum of the entire function.

We'll first begin with local extrema.

Beginning with finding $f'(x) = 3x^2 - 3$.

Then, finding the critical numbers of f, $f'(x) = 0 \Rightarrow x = -1,\ 1$.

Now, applying the **first derivative test**, which states to test the behavior of the derivative in between the critical numbers, we get:

On the interval $(-\infty,\ -1)$, f' is positive.

On the interval $(-1,\ 1)$, f' is negative.

Finally, on the interval $(1,\ \infty)$, f' is positive.

Therefore, by the first derivative test, since f' changes from positive to negative at $x = -1$, f changes from increasing to decreasing and f has a local maximum there.

Similarly, since f' changes from negative to positive at $x = 1$, f changes from decreasing to increasing and f has a local minimum there.

To analyze global extrema, we consider the local extrema, and then analyze the endpoints. In this case, since we are considering the entire function, we consider the end behavior of the function.

Here, as x tends to positive infinity, so does $f(x)$.

And as x tends to negative infinity, so does $f(x)$.

Therefore, the function has no global extrema, since there is no concrete minimum or maximum that f takes.

Example 3: Evaluate $\lim_{x \to 0^+} x \ln x$.

Solution 3: This is a prime example of a limit that requires L'Hopital's rule. However, it is not in one of the two indeterminate forms that are required for L'Hopital's rule. Thus, we will use some clever algebra to do so:

$x \ln x = \frac{\ln x}{\frac{1}{x}}$.

Now, $\lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}}$ is of the indeterminate form $\frac{\infty}{\infty}$ and so we apply L'Hopital's Rule

$$\lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x \to 0^+} -x = 0$$

Example 4: Apply the Mean Value Theorem to the function $f(x) = 2x^2$ over $(0,\ 2)$.

Solution 4: Since f is continuous on $[0,\ 2]$ and differentiable on $(0,\ 2)$ by the Mean Value Theorem, there exists a $c \in (0,\ 2)$ such that $f'(c) = \frac{f(2)-f(0)}{2-0}$.

Evaluating $\frac{f(2)-f(0)}{2-0}$ we get 4.

Thus, $f'(c) = 4$

$f'(x) = 4x$

Thus, $4c = 4$

$c = 1$

Example 5: Find the maximum value of xy if $2y + 2x = 2$.

Solution 5: We want to maximize (global maximum of) $f(x,\ y) = xy$, but this is a multivariable function. To reduce the number of variables, we can write y in terms of x.

$2y = 2 - 2x$

$y = 1 - x$

Making a substitution, we get $f(x) = x(1 - x) = x - x^2$.

Taking the derivative of this, we get $f'(x) = 1 - 2x$.

$f'(x) = 0$ implies that $x = \frac{1}{2}$.

Applying the first derivative test to calculate extrema, we get that:

f' is positive for $x < \frac{1}{2}$ and negative for $x > \frac{1}{2}$.

Thus, f has a local maximum at $x = \frac{1}{2}$.

Analyzing the end behavior of $f(x)$, we can claim that f tends to negative infinity as x tends to both positive and negative infinity, so $x = \frac{1}{2}$ is a global maximum.

Now, finding the value of y, we get $y = 1 - \frac{1}{2} = \frac{1}{2}$, and $xy = \frac{1}{4}$.

Example 6: Analyze $f(x) = x^3 - 27x$.

Solution 6: To analyze a function, we will find where it increases, decreases, turning points, areas where f is concave up and down, and inflection points.

Sidenote: An explanation covering rectilinear particle motion will be covered here as well through the sidenotes. Rectilinear motion describes the motion of a particle moving left and right on a number line. Generally, f represents the position of a particle, f' the velocity, f'' the acceleration, and so on.

To begin let's write out f, f' and f''.

$f(x) = x^3 - 27x$

$f'(x) = 3x^2 - 27$

$f''(x) = 6x$

To determine where f is increasing and decreasing, we use the values of where f' is positive or negative, respectively.

Sidenote: This corresponds to where a particle's velocity is negative or positive

An easier way to do it is the first derivative test.

Finding the critical numbers of f, we get $f'(x) = 0,\ x = 3,\ -3$. f' is never undefined.

On the interval $(-\infty,\ -3)$, f' is positive.

On the interval $(-3,\ 3)$, f' is negative.

Finally, on the interval $(3,\ \infty)$, f' is positive.

Thus, f is increasing on $(-\infty,\ -3) \cup (3,\ \infty)$, and decreasing on $(-3,\ 3)$. It has turning points at $x =-3$ *and* 3 since $f'(x) = 0$ there and f' switches signs there. (Note that turning points can occur when f' is undefined as well! The sign just has to switch.)

Sidenote: Turning points represent the moments where a particle is non-moving. However, a particle can be non-moving even when not at a turning point. The #1 way to determine a point where the particle isn't moving is to use the fact that f' represents the velocity. Thus the velocity is 0 when $f'(x) = 0$.

To determine concavity, we will use the 2nd derivative.

Applying the first derivative test on f', we get:

$f''(x) = 6x$

$f''(x) = 0$ means that $x = 0$. f'' is never undefined.

To the left of 0 f'' is negative, and to the right, f'' is positive.

Functions are concave up when $f''(x) > 0$ and concave down when $f''(x) < 0$. (An intuitive way to think about concavity is that a concave up function makes a smiley face and a concave down function makes a frowny face).

Thus, f is concave down over $(-\infty,\ 0)$ and concave up over $(0,\ \infty)$.

Sidenote: Concavity is correspondent to acceleration. Concave up regions are areas where acceleration is positive, and concave down regions are areas where acceleration is negative.

Sidenote: Speed and velocity are two independent things (speed mathematically and physically is not exactly the same thing as the absolute value of velocity). But for the sake of most calculus problems, all that's needed to be known is that speed is increasing when velocity and

acceleration share the same sign and decreasing when velocity and acceleration are of different signs.
A function has an inflection point when it changes concavity (this, like a turning point, is when f'' is undefined or 0 and then changes signs on either side of the point).
Again, using the fact that our second derivative was $f''(x) = 6x$, we see that $f''(x) = 0$ at $x = 0$.
Then, from our work with finding concavity, we know that $f''(x)$ switches signs at $x = 0$.
Thus, we can claim that $x = 0$ is an inflection point of f.
Sidenote: Inflection points are areas where the acceleration is 0.
And that is the function analyzed!

Recap
This Study Guide went over:

- Tangent Line Approximations
- The MVT and the EVT
- Rolle's Theorem
- Function Analysis
- Concavity and Inflection
- Optimization
- Rectilinear Motion

Now attempt the practice problems on the next page!

Practice Problems

The problems will ramp up in difficulty.

1) Find all local extrema of $f(x) = x^3 - 3x^2 + 3x$.
2) Find the absolute extrema of $f(x) = e^x - 1$ on $[-2. 3]$
3) Find an interval to $f(x) = x^2 - 3x + 2$ such that $f(a) = f(b) = 0$, and f satisfies Rolle's Theorem.
4) Apply the mean value theorem to $f(x) = \frac{2}{x}$ on $[1,\ 4]$.
5) Analyze $f(x) = x^3 - 6x^2 + 12x$.
6) Compute each limit:
    a) $\lim_{x \to 0}(\frac{\sin x}{x})$
    b) $\lim_{x \to 0^+} x^x$
    c) $\lim_{x \to \frac{\pi}{2}} \frac{\arctan x}{\operatorname{arccsc} x}$
    d) $\lim_{x \to 0^+} \sin x * \ln(x)$
7) Approximate $f(1.1)$ for $f(x) = x^4 - 3x^2 + x - 5$. State if the approximation is larger or smaller than the actual value (do not find the actual value).
8) If $x^2 + y^2 = 9$, find the maximum value for xy^2.
9) A particle moves with position given by $f(x) = x^3 - 6x^2 + 12x$.
    a) Find the point(s) where the particle is stationary.
    b) Find the interval(s) where the particle's speed is increasing.
    c) Find the interval(s) where the particle's acceleration is negative.

Answer Key

1) No local extrema. (The point where $f'(x) = 0,\ x = 1$ is a critical number, but f' is positive on either side).
2) No absolute extrema. (The function does approach -1 but only as the function approaches negative infinity).
3) $a = 1,\ b = 2$
4) $c = 2$
5) f is always increasing (acceptable answer: f is increasing on $(-\infty,\ 2) \cup (2,\ \infty)$). f has no turning points. f is concave up on $(2,\ \infty)$ and concave down on $(-\infty,\ 2)$. f has an inflection point at $x = 2$.
6) Answers below
   a) 1
   b) 1
   c) $\frac{1}{arccsc\left(\frac{\pi}{2}\right)}$
   d) 0
7) $f(1.1) \approx -6.1$
8) 9.547 (about)
9) Answers below
   a) The particle is not moving at $x = 2$ since $f'(2) = 0$.
   b) The particle's speed is increasing on $(2,\ \infty)$ since f' *and* f'' are both positive.
   c) The particle's acceleration is negative on $(-\infty,\ 2)$ since f' is decreasing (f'' is negative).