<u>Limits</u>

(Feel free to comment, markup, and chat on this study guide.)

<u>Theorems</u>

The Squeeze Theorem: Suppose $f(x),\ g(x),\ \textit{and}\ h(x)$ are functions that are continuous over a certain interval, and $f(x) \le g(x) \le h(x)$ over the entire interval. If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} h(x) = L$ then $\lim_{x\to a} g(x) = L$.

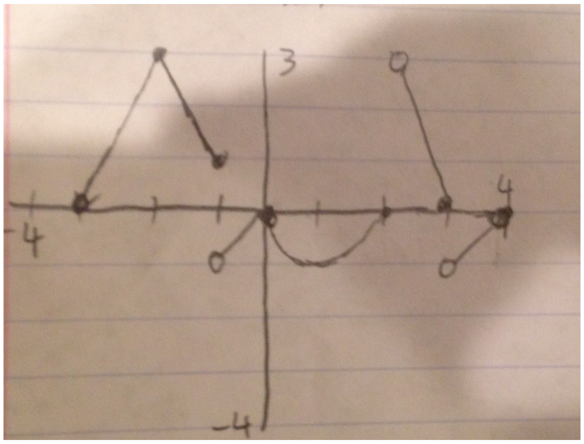
The Epsilon-Delta Definition of a Limit: If $\lim_{x\to c} f(x) = L$, then for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $|x - c| < \delta$.

$\lim_{x\to 0} \frac{\sin x}{x} = 1$

$\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

Examples

Examples 1-3 use the graph of $f(x)$ shown below.



Example 1: $\lim_{x\to 1} f(x)$

Solution 1: As x approaches 1 from both sides, y approaches -1. Therefore $\lim_{x\to 1} f(x) =- 1$

Example 2 $\lim_{x\to -2^+} f(x)$

Solution 2: As x approaches -2 from the right side, y approaches 3. Therefore $\lim_{x\to 2^+} f(x) = 3$

Example 3: $\lim_{x\to -2^-} f(x)$

Solution 3: As x approaches -2 from the left side, y approaches 3. Therefore $\lim_{x\to 2^-} f(x) = 3$

Example 4: Find: $\lim_{x\to 2} \frac{|x-2|}{x-2}$

Solution 4: $\lim_{x\to 2} \frac{|x-2|}{x-2}$ takes the indeterminate form $\frac{0}{0}$, so all we can do is split the absolute value into two cases and work for each one.

Case I:

For $x > 2$:

$\lim_{x\to 2} \frac{x-2}{x-2} = \lim_{x\to 2}(1) = 1$

Case II:

For $x < 2$

$\lim_{x\to 2} \frac{-(x-2)}{x-2} = \lim_{x\to 2}(-1) =- 1$

Since the limit from the RHS $\neq$ the limit from the LHS, we can safely say that:

$\lim_{x \to 2} \frac{|x-2|}{x-2}$ DNE.

Example 5: Find: $\lim_{x \to 0} \frac{\sin 2x}{x}$

Solution 5: Firstly, the indeterminate form is $\frac{0}{0}$.

We need to get the limit to look like $\frac{\sin 1*(variable\ a)}{1*(variable\ a)}$. We can do so by multiplying the numerator and denominator by 2. This is essentially multiplication by 1.

$\lim_{x \to 0} \frac{2\sin 2x}{2x} = \lim_{x \to 0} 2(\frac{\sin 2x}{2x})$

Now, by limit properties:

$2 \lim_{x \to 0} \frac{\sin 2x}{2x}$

Since the definition works for only $\frac{\sin 1*(variable\ a)}{1*(variable\ a)} = 1$, with no coefficients, we can make a valid substitution, where: $u = 2x$. As $x \to 0,\ u \to 0$ so:

$2 \lim_{u \to 0} \frac{\sin u}{u} = 2$.

Example 6: Prove: $\lim_{x \to 3}(x-5) = \ -2$

Solution 6: Let's Start. Make sure you write the following statement:

Proof:

One must prove that $\forall\ \varepsilon > 0,\ \exists\ \delta > 0,\ \ni\ |(x-5)+2| < \varepsilon,$ *whenever* $|x-3| < \delta$.

Our first objective in this proof is to transform the inequality with epsilon into the inequality with delta.

$|(x-5)+2| < \varepsilon$

$|x-3| < \varepsilon$

Now, we have gotten the LHS of the epsilon inequality the same as the LHS of the delta inequality.

Thus, we can safely say: $\delta = \varepsilon$.

For the second part of this proof, we need to now show that if we start with the delta inequality, that if we make substitutions defined in the first part of the proof, we will end up with $|f(x) - L| < \varepsilon$, and if we do, the limit exists.

So, starting with the delta inequality:

$|x-3| < \delta$

Here, we can make a substitution:

$|x-3| < \varepsilon$

Now, we can rewrite inside the inequality.

$|(x-5)+2| < \varepsilon$

This is in the form $|f(x) - L| < \varepsilon$, so our proof is complete.

□

The square marks the end of our proof.

Example 7: Prove: $\lim_{x \to 5}(x^2 - 3x) = 10$

Solution 7: We must obviously start with this:

Proof:

One must prove that $\forall\ \varepsilon > 0,\ \exists\ \delta > 0,\ \ni\ \left|(x^2 - 3x) - 10\right| < \varepsilon,$ *whenever* $|x - 5| < \delta$.

Let's start by changing the epsilon inequality into the delta inequality so we can find epsilon in terms of delta.

$\left|(x^2 - 3x) - 10\right| < \varepsilon$

$\left|x^2 - 3x - 10\right| < \varepsilon$

$|(x - 5)(x + 2)| < \varepsilon$

$|x - 5| < \frac{\varepsilon}{|x+2|}$

Wait! We can't have variables in our expression for delta and epsilon.

At this point, we progress to a second step of the proof reserved for variables in the epsilon term. Graphically, this means that in the points $(c,\ L + \varepsilon)$ and $(b,\ L - \varepsilon)$, c and b are not symmetric about *a*, and thus we must pick the point c or b that is closer to a in order for the values we pick correspond to a value epsilon units from L.

To do so, let's restrict delta to be small, specifically: $\delta < 1$. By making delta under 1, delta satisfies the requirement of being really small. Now we can make a substitution.

$|x - 5| < 1$

$4 < x < 6$

Now, we need to somehow transform that inequality into the denominator of epsilon, in this case $|x + 2|$.

$6 < x + 2 < 8$

$6 < |x + 2| < 8$

At this point we need to stop and think. What is the minimum value that we can make epsilon, do that delta is minimized.

If we replace $|x + 2|$ with 8 in the denominator, we are sure to minimize delta.

Now, we take the minimum of the two values delta can be:

$\delta = min\{1,\ \frac{\varepsilon}{8}\}$. We picked 1 for one of the two terms we need to take the minimum of because we initially restricted $\delta < 1$. $\frac{\varepsilon}{8}$ is less than 1 so:

$\delta = \frac{\varepsilon}{8}$

Now for the final part of the proof:

$|x - 5| < \delta$

$|x - 5| < \frac{\varepsilon}{8}$

If we multiply the RHS by 8 and the LHS by $|x + 2|$, the inequality remains true, because $|x + 2| < 8$.

$\left|x^2 - 3x - 10\right| < \varepsilon$
$\left|(x^2 - 3x) - 10\right| < \varepsilon$
Therefore: $|f(x) - L| < \varepsilon$. □

Example 8: Evaluate the limit: $\lim\limits_{x\to\infty} \frac{\sin x}{2x-5}$.
Solution 8: We can use the squeeze theorem to evaluate this.
Using the range of $\sin x$, we can squeeze $\sin x$ in between 2 numbers: $-1 < \sin x < 1$.
Dividing by $2x - 5$, we get $\frac{-1}{2x-5} < \frac{\sin x}{2x-5} < \frac{1}{2x-5}$.
Now we take the limit of each side of the inequality.
$\lim\limits_{x\to\infty} \frac{-\frac{1}{x}}{2-\frac{5}{x}} < \lim\limits_{x\to\infty} \frac{\sin x}{2x-5} < \lim\limits_{x\to\infty} \frac{\frac{1}{x}}{2-\frac{5}{x}}$
$0 < \lim\limits_{x\to\infty} \frac{\sin x}{2x-5} < 0$
Therefore by the Squeeze Theorem, $\lim\limits_{x\to\infty} \frac{\sin x}{2x-5} = 0$.

Recap
This Study Guide went over:
- Limit Fundamentals and Properties
- One-Sided Limits
- Trigonometric Limits
- Proving Limits using the Epsilon-Delta Definition
- The Squeeze Theorem

If you need detailed help, take a look at this video!
Now attempt the practice problems on the next page!

Practice Problems

1) Evaluate each limit.
    a) $\lim_{x\to 2} \frac{x-2}{x^2-4}$
    b) $\lim_{x\to 2^+} \frac{3}{|x-2|}$
    c) $\lim_{x\to 3} \frac{3-3\cos 3x}{x}$
    d) $\lim_{x\to 0} \frac{\sin(2x)}{3x}$
    e) $\lim_{x\to \infty} \sin x$
    f) $\lim_{x\to \frac{\pi}{4}} \frac{\sin x - \cos x}{\sec x - \csc x}$
    g) $\lim_{x\to 0} \frac{\sin(\cos x)}{x}$
2) Prove that $\lim_{x\to 2} (3x-5) = 1$ using the Epsilon-Delta Definition of a Limit.
3) Prove that $\lim_{x\to 1} \frac{1}{x} = 1$ using the Epsilon-Delta Definition of a Limit.
4) Suppose $\lim_{x\to \infty} \frac{ax+\cos(2x)}{1-2x^2} = L$, where L is a real number. Find all possible value(s) of a, and the value(s) of L.

Answer Key

1) Answers Below
    a) $\frac{1}{4}$
    b) ∞
    c) 0
    d) $\frac{2}{3}$
    e) DNE
    f) $\frac{1}{2}$
    g) 0
2) Use the example for tips
3) Use the example for tips
4) For all a, $L = 0$