<u>The Fundamental Theorem of Calculus</u>

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<u>Theorems</u>

The First Fundamental Theorem of Calculus: If f is a continuous function and $g'(x) = f(x)$, then $\int_a^b f(x)dx = g(b) - g(a)$, where $b > a$.

The Second Fundamental Theorem of Calculus: If f is a continuous and integrable function and x_0 is in the domain of f, define the area function g such that $g(x) = \int_{x_0}^{x} f(t)dt$. Then $g'(x) = f(x)$.

Essentially, differentiation and integration are inverse operations.

Examples

Example 1: $g(x) = \int_2^x (2t^2)dt$. Find the relative minimum/maximum, and points of inflection of g.

Solution 1: To find the absolute minimum and maximum we need the first derivative of g. By the 2nd Fundamental Theorem of Calculus. $g'(x) = 2x^2$.

g' is never undefined and $g'(x) = 0$ when $x = 0$. Sign-testing this, we get that g' is always positive, meaning that g never changes signs and thus has no relative extrema.

For points of inflection, we need the second derivative of g, which is $4x$. Sign-testing this, we get that the inflection point is at x is 0, since $g''(0) = 0$ and g'' changes from negative to positive at $g''(0)$. To find the inflection POINT, we plug in 0 for g.

$$g(0) = \int_2^0 (2t^2)dt = -\int_0^2 (2t^2)dt.$$

We can evaluate the integral using the 1st Fundamental Theorem of Calculus.

$g(0) = \frac{-2}{3}t^3$ evaluated from 0 to 2.

$g(0) = \frac{-2}{3} * 8 - 0 = - = \frac{-16}{3}$.

Therefore, the point of inflection is $(0, \frac{-16}{3})$

Example 2: A particle's velocity is modeled by the function $v(t) = 2t - 2\ \{0 \le t < 5\}$ and $t^2 - 13\ \{5 \le x < 6\}$. Find the position at $t = 6$ if the particle's position at $t = 3$ was 7.

Solution 2: The position of the particle at $t = 6$ is the position of the particle at $t = 3$ plus the displacement of the particle from $t = 3$ *to* 6. Therefore, using an integral, we can write:

$$s(6) = s(3) + \int_3^6 v(t)dt.$$

We know $s(3) = 7$ so some of our work becomes easy.

$$s(6) = 7 + \int_3^6 v(t)dt.$$

But since v is piecewise, we need to split this into two integrals.

$$s(6) = 7 + \int_3^5 (2t - 2)dt + \int_5^6 (t^2 - 13)dt$$

Using the First Fundamental Theorem of Calculus, we can evaluate everything.

The first integral is $t^2 - 2t$ evaluated from 3 to 5, which is $15 - 6 = 9$

The second integral is $\frac{1}{3}t^3 - 13t$ evaluated from 5 to 6, which is $-6 + \frac{70}{3} = \frac{52}{3}$.

Therefore, the position of the particle at $t = 6$ is $s(6) = \frac{100}{3}$.

Example 3: Let $f(x) = \int_{e^x}^{1} t\ dt$. Find $f'(x)$.

Solution 3: By the First and Second Fundamental Theorem of Calculus, as well as the chain rule,
$f'(x) = \frac{1}{x} * \frac{-1}{x^2} - e^x * e^x$
$f'(x) = \frac{-1}{x^3} - e^{2x}$.

Recap

This Study Guide went over:

- The first and second parts to the Fundamental Theorem of Calculus
- Analyzing Integral defined functions

Now attempt the practice problems on the next page!

Practice Problems

The problems will ramp-up in difficulty.

1) If $f(x) = cos\, x$, find the exact area under f on the interval $[0,\ \frac{\pi}{2}]$.
2) If $f(x) = \int_{\pi}^{x} sin\, t\, dt$ on $[0,\ 2\pi]$
    a) Find $f'(x)$.
    b) Find the inflection points of f.
    c) Find the inflection points of f'.
    d) Find the value of $f(3\pi)$
3) Let $f(x) = x^2 + 4x + 4$.
    a) Find the zeros of f.
    b) Let $g(x) = ln(f(x))$. Find the zeros of g.
    c) Let $h(t)$ be defined by $h(t) = \int_{0}^{t} g(x)dx$ for $t > -2$.. Find the x-coordinates of any turning points of h.
    d) Find where h is both concave down and increasing.
    e) Let $j(t) = \int_{t}^{t^2} f(x)dx$. Find $j(5) + h'(-2)$.
    f) Let $k(x) = f(x)$ for $[-2,\ 2]$, and $m(x) = \int_{0}^{2x-2} k(t)dt$. Find the range of m.

Answer Key

1) 1
2) Answers Below
    a) $f'(x) = sin\ x$
    b) $x = \frac{\pi}{2},\ \frac{3\pi}{2}$
    c) $x = 0,\ \pi,\ 2\pi$
    d) 0
3) Answers Below
    a) $x = -2$
    b) $x = -3,\ -1$
    c) f has turning points at $x = -1.737,\ 0$
    d) Never over the given interval.
    e) 6449.439
    f) Range: $[-24,\ 18\frac{2}{3}]$