Infinite Series

(Feel free to comment, markup, and chat on this study guide.)

Convergence Tests (Tests in Italics are not formal tests)

Telescoping Series Test
Geometric Series Test
nth Term Test
Integral Test
p-Series Test
Direct Comparison Test
Limit Comparison Test
Alternating Series Test
Ratio Test
Root Test
Absolute Convergence Test
Sequence of Partial Sums Test

Also Chart of the Series Tests and their uses are on the next page

Given a series $\sum_{n=k}^{\infty} a_n$, where $k > 0$, how do we tell if it converges?

Test Name	Preconditions	What does it show?	Conditions for Convergence	Conditions for Divergence	Conditions for Inconclusivity
Telescoping Series Test	a_n must be a decomposable rational function with linear factors.	Convergence ONLY	If the sum of the numerators is 0, then the series converges.	CANNOT be used for divergence.	If the sum of the numerators is not 0, then the test is inconclusive.
Geometric Series Test	a_n must in the form $a_0(k)^n$	Convergence and Divergence	If $\lvert k\rvert < 1$, the series converges.	If $\lvert k\rvert \geq 1$, the series diverges.	Never Inconclusive.
nth Term Test	a_n must be defined for all $n > k$.	Divergence ONLY	CANNOT be used for convergence.	If $\lim_{n\to\infty} a_n \neq 0$, the series diverges.	If $\lim_{n\to\infty} a_n = 0$, the test is inconclusive.
Integral Test	$f(x) = a_n$ must be decreasing, continuous, and positive for all $x = n > k$.	Convergence and Divergence	If $\int_k^{\infty} f(x)dx = L$, where L is positive and finite, then the series converges.	If $\int_k^{\infty} f(x)dx$ diverges, then the series diverges.	If $\int_k^{\infty} f(x)dx$, is invaluable but not divergent, then the test is inconclusive.
p-Series Test	a_n must be in the form $\frac{b}{cn^p}$.	Convergence and Divergence	If $p > 1$, the series converges.	If $p \leq 1$, the series diverges.	Never Inconclusive.
Direct Comparison Test	a_n, b_n must be positive and defined for all $n > k$.	Convergence and Divergence	If a second series $\sum_{n=k}^{\infty} b_n$ is known to converge, resembles $\sum_{n=k}^{\infty} a_n$, and	If a second series $\sum_{n=k}^{\infty} b_n$ is known to diverge, resembles $\sum_{n=k}^{\infty} a_n$, and	Never Inconclusive.

			$a_n < b_n$ for all $n > k$ (plus or minus an integer less than or greater than k by a small amount), then the series $\sum_{n=k}^{\infty} a_n$ converges.	$a_n > b_n$ for all $n > k$ (plus or minus an integer less than or greater than k by a small amount), then the series $\sum_{n=k}^{\infty} a_n$ diverges.	
Limit Comparison Test ** Limit Comparison Test has more cases that will be dealt with later in the guide.	a_n, b_n must be defined for all $n > k$.	Convergence and Divergence	If a second series $\sum_{n=k}^{\infty} b$ is known to converge, resembles $\sum_{n=k}^{\infty} a_n$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, where where L is positive and finite, the series $\sum_{n=k}^{\infty} a_n$ converges.	If a second series $\sum_{n=k}^{\infty} b$ is known to diverge, resembles $\sum_{n=k}^{\infty} a_n$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, where where L is positive and finite, the series $\sum_{n=k}^{\infty} a_n$ diverges.	Never Inconclusive.
Alternating Series Test	a_n must equal $(-1)^{n+m} b_n$ and b_n must be positive and defined for all $n > k$. Also, $n + m >- 1$.	Convergence ONLY	Two conditions must be met: I. $\lim_{n\to\infty} b_n = 0$ II. $b_{n+1} < b_n$	CANNOT be used for divergence.	If Condition I fails, the series diverges by the *nth* Term Test. If Condition II fails, the test is inconclusive.
Ratio Test	a_n must be defined for	**Absolute**	If	If	If

	all $n > k$.	Convergence and Divergence	$\lim_{n\to\infty}\left\|\frac{a_{n+1}}{a_n}\right\| < 1$, and is finite, the series converges.	$\lim_{n\to\infty}\left\|\frac{a_{n+1}}{a_n}\right\| > 1$, the series diverges..	$\lim_{n\to\infty}\left\|\frac{a_{n+1}}{a_n}\right\| = 1$, the test is inconclusive (it is possible that the series may also be conditionally convergent).
Root Test	a_n must be defined for all $n > k$.	**Absolute** Convergence and Divergence	If $\lim_{n\to\infty}\sqrt[n]{a_n} < 1$, and is finite, the series converges.	If $\lim_{n\to\infty}\sqrt[n]{a_n} > 1$, the series diverges.	If $\lim_{n\to\infty}\sqrt[n]{a_n} = 1$, the test is inconclusive (it is possible that the series may also be conditionally convergent).
Absolute Convergence Test	a_n must be defined for all $n > k$.	Convergence ONLY	If $\sum_{n=k}^{\infty}\|a_n\|$ converges, then $\sum_{n=k}^{\infty}a_n$ also converges.	CANNOT be used for divergence.	If $\sum_{n=k}^{\infty}\|a_n\|$ diverges, then the test is inconclusive.
Sequence of Partial Sums	a_n must be defined for all $n > k$.	Convergence and Divergence	If $\lim_{n\to\infty}S_n = L$ (where S_n is the *nth* partial sum of a_n), where L is finite, the series converges.	If $\lim_{n\to\infty}S_n = \infty$ (where S_n is the *nth* partial sum of a_n), the series diverges.	Never Inconclusive.

<u>Recap</u>

This Study Guide went over:

- Series Convergence Tests

Now attempt the practice problems on the next page!

Practice Problems

Instructions: For each set of problems there are 10 series on the left column, and 10 tests on the right column. Each test is used at exactly once. Match the series on the left to the test on the right, then work out to see if the series converges or diverges, showing all of your work.

Question 1

Series	**Test**
$\sum_{n=1}^{\infty} \frac{1}{n^2+4^n}$	Telescoping Series Test
$\sum_{n=1}^{\infty} \frac{(2n+1)^{2n}}{n^n}$	*nth* Term Test
$\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2}$	Geometric Series Test
$\sum_{n=1}^{\infty} \frac{1}{n^2+3n}$	Integral Test
$\sum_{n=1}^{\infty} \frac{n^2}{2}(\frac{6}{5})^n$	*p*-series Test
$\sum_{n=1}^{\infty} \frac{1}{19n}$	Direct Comparison Test
$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{ln\ n}}$	Limit Comparison Test
$\sum_{n=1}^{\infty} \frac{2^n n!}{(n+2)!}$	Alternating Series Test
$\sum_{n=1}^{\infty} n\ sin(\frac{1}{n})$	Ratio Test
$\sum_{n=1}^{\infty} (\frac{1}{4^n})(\frac{3}{2})^{2n}$	Root Test

Question 2

Series	**Test**
$\sum_{n=1}^{\infty} \frac{2n+1}{3n^3+1}$	Telescoping Series Test
$\sum_{n=1}^{\infty} \frac{3^n+4^n}{7^n}$	*nth* Term Test
$\sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n-1}$	Geometric Series Test
$\sum_{n=1}^{\infty} \frac{1}{n^2+3n}$	Integral Test
$\sum_{n=1}^{\infty} \frac{1}{n^2+4n+3}$	*p*-series Test
$\sum_{n=1}^{\infty} \frac{2n}{2n^e n^2}$	Direct Comparison Test
$\sum_{n=2}^{\infty} \frac{n^2}{n\sqrt{\ln n}}$	Limit Comparison Test
$\sum_{n=1}^{\infty} \frac{(-1)^n(\pi)^{2n+1}}{2n+1}$	Alternating Series Test
$\sum_{n=1}^{\infty} \frac{1}{2^n}\left(\frac{3}{2}\right)^n \frac{6}{5^n}$	Ratio Test
$\sum_{n=1}^{\infty} \frac{2^n n^n}{n6^n}$	Root Test

Answer Key

1. Answers Below
    a. Limit Comparison Test, Converges
    b. Root Test, Diverges
    c. Alternating Series Test, Converges
    d. Telescoping Series Test, Converges
    e. Direct Comparison Test, Diverges
    f. *p*-series Test, Diverges
    g. Integral Test, Diverges
    h. Ratio Test, Diverges
    i. *nth* Term Test, Diverges
    j. Geometric Series Test, Converges
2. Answers Below
    a. Limit Comparison Test, Converges
    b. Direct Comparison Test, Converges
    c. Alternating Series Test, Converges
    d. Integral Test, Converges
    e. Telescoping Series Test, Converges
    f. *p*-series Test, Converges
    g. *nth* Term Test, Diverges
    h. Ratio Test, Converges
    i. Geometric Series Test, Converges
    j. Root Test, Diverges