Calculus of Polar and Parametric Functions

(Feel free to comment, markup, and chat on this study guide.)

This study guide is split into multiple sections.

Contents

Parametric Motion

Theorems and Formulas

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(dy/dx)}{dx/dt}$$

Speed: $s = \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}$

Arc Length: $L = \int_a^b \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}\,dt$

Examples

Example 1: Find $\frac{dx}{dt}$ and $\frac{dy}{dx}$ for the parametric curve given by $s(t) = \ < t,\ t^2 >$. Then find $\frac{d^2y}{dx^2}$ and state the areas where the curve is concave up.

Solution 1: $\frac{dx}{dt}$ is easy, the derivative of $x(t) = t$ is just $x'(t) = 1$.

Using the formula for $\frac{dy}{dx}$, we can quickly get $y'(t) = 2t$ and use the expression $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ to claim that $\frac{dy}{dx} = 2t$.

Using the formula for $\frac{d^2y}{dx^2}$, we differentiate $\frac{dy}{dx}$ to get the numerator of $\frac{d^2y}{dx^2}$ to be 2. We've found that the denominator is 1 so $\frac{d^2y}{dx^2} = 2$ and due to this, we can claim that the curve is always concave up.

Example 2: Find the speed of a particle travelling along $s(t) = \ < t,\ t^2 >$ at $t = 2$. Then find the distance travelled along the time interval $t \in (1,\ 2)$.

Using the formula of speed, we first need $x'(2)$ and $y'(2)$

$x'(t) = 2t,\ x'(2) = 4$, $y'(t) = y'(2) = 1$

Now, using the formula for speed: $sp(2) = \sqrt{4^2 + 1^2} = \sqrt{17}$.

Finally, we need to compute the arc length. Using the formula $L = \int_a^b \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}$, we get that

$L = \int_1^2 \sqrt{4t^2 + 1}dt$. This is a classic trig sub integral, (we'll use the substitution $t = \frac{1}{2}tan\ \theta$), and after performing the calculus, we get $L = 3.168$.

Recap

This Study Guide went over:

- Parametric Derivatives
- Speed and Arc Length

Now attempt the practice problems on the next page!

Practice Problems

The problems will ramp-up in difficulty.

1) Eliminate the Parameter, state the trace, and the domain. Graph **1a and 1b** only.
   a) $x = 4t - 3,\ y = 7t + 2$
   b) $x = t^2,\ y = t^3$
   c) $x = sin^2 t,\ y = cos^2 t$
   d) $x = tan\ t,\ y = cot\ t$
   e) $x = sin(\frac{t}{2}),\ y = cos\ 2t$
   f) $x = 1 + cos\ t,\ y = tan\ t + sin\ t$
2) A parametric curve is given by $c(t) = \ < t^2,\ e^t >$.
   a) Find the tangent line to the curve at $x = 9$.
   b) Find the speed of the particle at $x = 9$.
3) A parametric curve is given by $c(t) = \ < \frac{1}{t},\ \frac{-1}{3t^3} >,\ t > 0$.
   a) Find the arc length of the curve from $t = 1\ to\ t = 3$.
   b) Find the values for which c is concave down.
   c) Find all inflection points of c.
   d) How is the particle moving at time $t = 3$?
4) A parametric curve is given by $c(t) = \ < \frac{1}{2}t^2,\ y(t) >$. $y'(t)$ is in the form b. If the arc length of *c(t)* from 0 to 2 is 10, write an equation in terms of only b (no integrals or *t's*) that can be used to find $y'(t)$.
5) A parametric curve is given by $C = \ < x(t),\ y(t) >$. Both $x'(t)$ and $y'(t)$ are in the form $at + b$. C has a horizontal tangent line at $t = 5$.
   a) If $x'(t) = 2t - 5$, and $\frac{dy}{dx} = 4$ at $t = 2$, find $y'(2)$.
   b) Find $y'(t)$.
   c) Find $x(t)$ and $y(t)$, where $x(0) = y(0) = 0$.
6) A parametric curve is given by $C = \ < x(t),\ y(t) >$. Both $x'(t)$ and $y'(t)$ are in the form $t + b$. C has a vertical tangent line at $t = 2$, and $y'(t) = t - 1$. Find the arc length of C from 1 to 2.

Answer Key

1) Answers Below
   a) $y = \frac{7x+29}{4}$
   b) $y = x^{\frac{3}{2}}$
   c) $x + y = 1$
   d) $y = \frac{1}{x}$
   e) $y = 8x^4 - 8x^2 - 1$
   f) $y^2 = \frac{x^4-2x^3-2x^2+2}{x^2-2x+1}$
2) Answers Below
   a) $y - 8103.09 = 450.171(x - 81)$
   b) 8103.104
3) Answers Below
   a) $L = 1.297$
   b) C is always concave down.
   c) No inflection points
   d) Up and to the left.
4) $b^2(sec\ 2\ tan\ 2 + \frac{1}{2}ln|sec\ 2 + tan\ 2|) = 10$
5) Answers Below
   a) $y'(2) =- 4$
   b) $y(t) = \frac{4}{3}t - \frac{20}{3}$
   c) $x(t) = t^2 - 5t$, $y(t) = \frac{2}{3}t^2 - \frac{20}{3}t$
6) 0.812

Polar Functions

Theorems and Formulas

$x = r\cos\theta,\ y = r\sin\theta,\ \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

$\frac{y}{x} = \tan\theta$

$x^2 + y^2 = r^2$

$A = \frac{1}{2}\int_a^b (r(\theta))^2 d\theta$

Area between two curves: $A = \frac{1}{2}\int_a^b [(f_o(\theta))^2 - (f_i(\theta))^2] d\theta$

There exist two derivatives for polar curves, $\frac{dr}{d\theta}$ (rate of change of distance from the origin) and $\frac{dy}{dx}$ (slope).

Examples

Example 1: Determine if, at $\theta = \frac{\pi}{3}$, is the distance from the origin to $r = 2\sin\theta$ increasing, and if the tangent line has a positive slope.

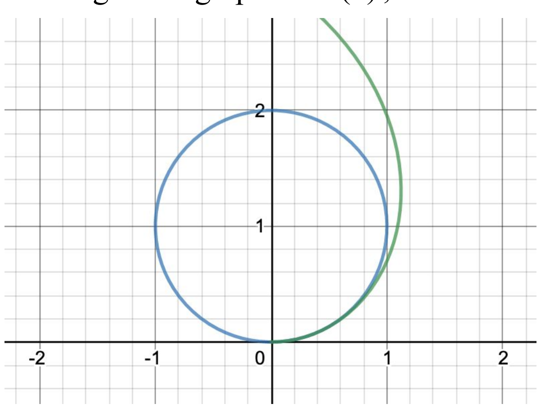
Solution 1: First, to determine the rate at which distance from the origin to the curve is changing, we need $\frac{dr}{d\theta}$.

$\frac{dr}{d\theta} = 2\cos\theta$. Plugging in $\frac{\pi}{3}$, we get $\frac{dr}{d\theta} = \frac{1}{2} > 0$, and so the distance from the origin to the curve is increasing.

Now, we need to find the slope, which means $\frac{dy}{dx}$. Finding $y(\theta)$ *and* $x(\theta)$, we get $x(\theta) = \sin 2\theta$, and $y(\theta) = 2\sin^2\theta$. $y'(\theta) = 2\sin 2\theta$ and $x'(\theta) = 2\cos 2\theta$. Thus $\frac{dy}{dx} = 2\tan 2\theta$, and at $\theta = \frac{\pi}{3}$, we get $2\tan\frac{2\pi}{3}$, which is clearly negative.

Example 2: Find the area between the rays $\theta = \frac{\pi}{6},\ \theta = \frac{\pi}{3}$, outside $r = 2\sin\theta$ and inside of $r = 2\theta$.

Looking at the graph for $r(\theta)$, we see:



This shows us that $2\theta > 2\sin\theta$.

Thus, when we set up out integral, we need 2θ to be the outer function.

$$A = \frac{1}{2}\int_{\pi/6}^{\pi/3} (2\theta)^2 - (2\,sin\,\theta)^2 d\theta\ .$$

Integrating this expression, we get 0.146

<u>Recap</u>

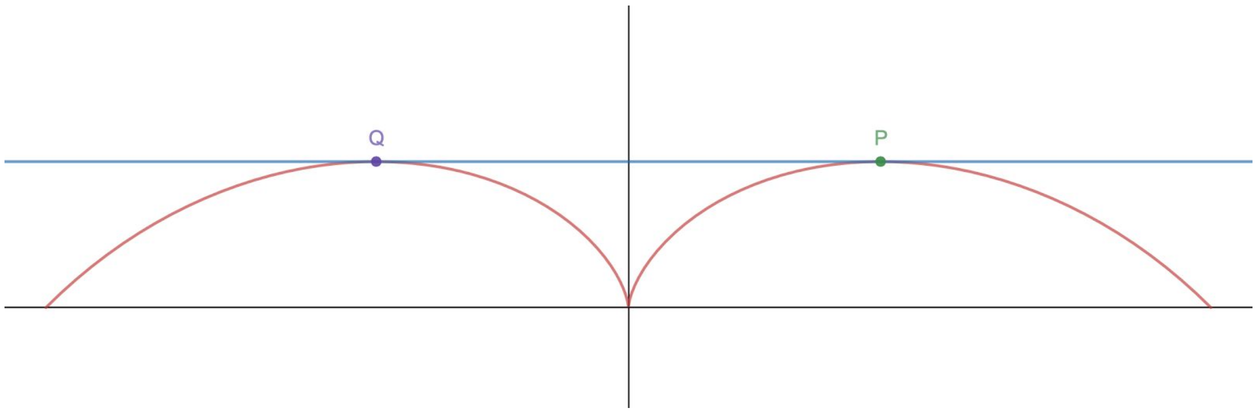
This Study Guide went over:

- Polar Derivatives
- Polar Area

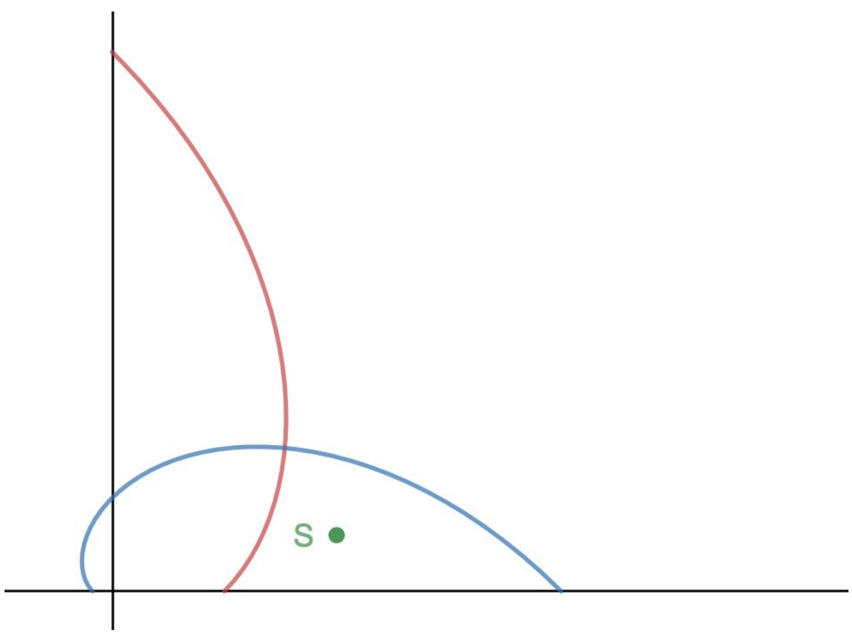
Now attempt the practice problems on the next page!

Practice Problems

1) A curve is given by the equation $(x^2 + y^2)^2 = 2x^2 y$.
   a) Convert the equation from rectangular coordinates to polar coordinates.
   b) Find the maximum possible value for r over the interval $0 \le \theta \le \pi$.
2) Below is the graph of the polar function $r = 4 - 4\ sin\ \theta$ defined over the interval $0 \le \theta \le \pi$. The line PQ is tangent to the curve at points P and Q, and is parallel to the $x - axis$.



   a) Find the coordinates, in polar form, for points P and Q.
   b) Find the area between the red and blue curves, and between points P and Q.
3) Two curves are graphed below. The curve $r = e^{\theta}$ is in red, and the curve $r = 4e^{-\theta}$ is in blue. Both graphs are given over the interval $0 \le \theta \le \pi$.



   a) Find the point of intersection of the two graphs.

b) Point S labels a region between the two graphs. Find the area of the region that contains point S.

<u>Answer Key</u>

1) Answers Below
   a) $r = \sin 2\theta$
   b) 1
2) Answers Below
   a) $\theta = \frac{\pi}{6},\ \frac{5\pi}{6}$
   b) $A = 1.696$
3) Answers Below
   a) $\theta = \ln 2$
   b) 4.5