The Derivative

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Theorems

The Inverse Function Rule: If $g(x)$ is the inverse function of $f(x)$, then $g'(x) = \frac{1}{f'(g(x))}$

The Limit Definition of the Derivative: $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

The Alternate Definition of the Derivative: $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$

The Implicit Differentiation Rule: If y is a function of x, then $\frac{d}{dx}(y) = y' = \frac{dy}{dx}$.

More generally, If m is a function of p, then the derivative of m with respect to p is $\frac{dm}{dp}$.

Product Rule: $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

Quotient Rule: $(\frac{f(x)}{g(x)})' = \frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}$

Examples

Example 1: Find the derivative of $|x-1|+2x$. State the interval where the derivative is continuous.

Solution 1: First, let's use the limit definition of the derivative to find the derivative.

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$f'(x) = \lim_{h\to 0} \frac{|x+h-1|+2(x+h)-(|x-1|+2x)}{h}$

$f'(x) = \lim_{h\to 0} \frac{|x+h-1|+2x+2h-|x-1|-2x}{h}$

$f'(x) = \lim_{h\to 0} \frac{|x+h-1|+2h-|x-1|}{h}$

$f'(x) = \lim_{h\to 0} \frac{|x+h-1|+2h-|x-1|}{h}$

$f'(x) = \lim_{h\to 0} \frac{|x|+|h|+|-1|+2h-|x|-|-1|}{h}$

$f'(x) = \lim_{h\to 0} \frac{|h|+2h}{h}$

Split the limit into its one sided limits here.

$f'(x) = \lim_{h\to 0^+} \frac{h+2h}{h}$

$f'(x) = 3$

$f'(x) = \lim_{h\to 0^-} \frac{|h|+2h}{h}$

$f'(x) = 1$

So the derivative can be written as a piecewise function:

$f'(x) = 3,\ when\ x > 1,\ 1,\ when\ x < 1$.

However, notice that the limits don't match. So the derivative doesn't exist at $x = 1$.

The continuity of the derivative is ***GENERALLY*** the same thing as the differentiability (the points where f can have a derivative) of the original function. Therefore, when we are trying to determine the continuity of the derivative, we are simply asking for the values where the derivative is defined, and where the original function can be differentiated.

Clearly the derivative is defined for all $x > 1\ and\ x < 1$. However, the one-sided limits of the derivative do not match, therefore the derivative is not defined at $x = 1$. Therefore the original function f is differentiable at $x > 1\ and\ x < 1$.

Example 2: Identify tangent and normal lines to $f(x) = sin\,x$ at $x = 0$.

Solution 2: Use the alternate definition of the derivative to find the slope of the tangent line at 1.

$f'(0) = \lim_{x\to 0} \frac{sin\,x - 0}{x-0}$

$f'(0) = \lim_{x\to 0} \frac{sin\,x}{x}$

$f'(0) = 1$

$sin(0) = 0$, so the equation of the tangent line in point-slope form is $(y-0) = 1(x-0)$

The slope of the normal line is the inverse reciprocal, -1.
Therefore, the equation of the normal line in point-slope form is $(y-0) = -1(x-0)$

Example 3: Estimate $f(1.1)$ for $f(x) = x^2 + 2x$.
Solution 3: Instead of plugging in 1.1, we can use a linear approximation with the tangent line at $x = 1$ for an easy estimation.
We can use the power rule to evaluate this derivative fast.
$f'(x) = 2x + 2$
$f'(1) = 4$
$f(1) = 3$
Therefore, the equation of the tangent line at 1 is $y - 3 = 4(x-1)$
Now we can use this as a **linear approximation** to estimate values close to 1.
We can substitute in 1.1, getting $y - 3 = 4(0.1)$
$y = 3.4$
$f(1.1) \approx 3.4$
$f(1.1) = 3.41$ actually, so this approximation is pretty good.

Example 4: Differentiate $\sqrt{2x}$.
Solution 4: This can not be differentiated normally, we need to apply the chain rule.
Let's make a substitution, $u = 2x$
We can differentiate u with respect to x, getting $\frac{du}{dx} = 2$
Now the original function is $\sqrt{u}$
Differentiating with respect to u, we get $\frac{dy}{du}\frac{1}{2\sqrt{u}}$
Then, $\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx}$
$\frac{dy}{dx} = \frac{2}{2\sqrt{u}}$
Now, we simplify, and write the derivative in terms of x.
$\frac{dy}{dx} = \frac{1}{\sqrt{2x}}$

Example 5: Differentiate $f(x) = sin^3(sin^2(x))$
Solution 5: Let's rewrite this as $(sin(sin(x))^2)^3$
Now, we repeatedly use the chain rule.
$f'(x) = 3sin^2(sin^2(x))(cos(sin^2 2x))(sin2x)$

Example 6: Suppose $f(x) = x^3 + x + 1$, and $g(x)$ is the inverse of $f(x)$. Find $g'(-1)$.
Solution 6: This is a classic application of the inverse-function rule.
We want $g'(-1)$, and by the inverse-function rule, we know $g'(-1) = \frac{1}{f'(g(-1))}$.
But what's $g(-1)$? We could try inverting $f(x)$, but we'll soon see its ridiculously tough.

Instead, we'll use a common property of inverse functions, which is that if $f(x) = L$, then $f^{-1}(L) = x$. Here, $g(x)$ is the inverse, and we want $g(-1)$, so we'll let $g(-1) = L$.
That means $f(L) = -1$, and we have an equation to solve!

$L^3 + L + 1 = -1$

$L^3 + L + 2 = 0$

Now, we'll use our algebra skills (plugging in integers) to find one root of this function.
Trying $L = -1$, we see that the equation is satisfied. Thus, we know $L = -1$, and so $g(-1) = -1$.
Now, we plug -1 into our inverse-function formula, getting $g'(-1) = \frac{1}{f'(-1)}$.
$f'(x) = 3x^2 + 1$, so $f'(-1) = 4$
Thus, $g'(-1) = \frac{1}{4}$.

Example 7: Implicitly differentiate: $y \sin y = x \sin x$, if y is a function of x.
Solution 7: **Implicit differentiation** involves the use of the chain rule. Essentially, what is going on is that we have an equation which consists of a variable, and a function of that variable.
Recall that $f(g(x))$'s derivative is $f'(g(x))g'(x)$. In this case, $y = f(g(x))$, since y is a function of x. Therefore, whenever we differentiate a variable that is a function of another, we differentiate normally, and then differentiate the variable by chain rule.
In this case, since y is a function of x, the derivative of y with respect to x will be $\frac{dy}{dx}$ (or its other notationally equivalent forms).
Now, applying implicit differentiation and product rule to the left hand side, we need to differentiate $y \sin y$, which is the derivative of y times $\sin y$ plus the derivative of $\sin y$ times y
The derivative of y is $\frac{dy}{dx}$, and the derivative of $\sin y$ is $\cos y$. But since y is a function of x, we need to apply chain rule, and so we take the derivative of the argument of $\sin y$, which is the derivative of y, which, as we established, is $\frac{dy}{dx}$.
Thus, the complete derivative of the left hand side, using product rule, is $\frac{dy}{dx}\sin y + \frac{dy}{dx}y \cos y$.
We'll factor out a $\frac{dy}{dx}$ to get $\frac{dy}{dx}(\sin y + y \cos y)$.
Now, we apply product rule on the right hand side, however, since the variable is x, which is not a function of anything, we don't need to apply implicit chain rule.

Example 8: A 12-inch tall cone is being filled with water. At any point in the cone, the diameter and height are the same. At the time when the radius is 1, the height of the cone is changing at a rate of 0.2 inches per second. What is the rate of change of the volume of the cone when the height is 2 inches?
Solution 8: These types of problems are known as **related rates** problems.
We'll begin with the volume of a cone, $V = \frac{1}{3}\pi r^2 h$. We want the rate of change of the volume with respect to time, so we'll need the derivative of the above formula.

Note that every variable in here is a function of time, so everything will be differentiated implicitly. Remember product rule!
$\frac{dV}{dt} = \frac{1}{3}\pi(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt})$.
Now, since we know that the height is the same as the diameter, this means the height is the same as two times the radius. This means that $h = 2r$. Differentiating this, we get $\frac{dh}{dt} = 2\frac{dr}{dt}$. We also know that the time at when the height is 2 inches is the same as when the radius is 1 inch due to this property. Thus, since at this time, the rate of change of the height with respect to time ($\frac{dh}{dt}$) is 0.2, we can use this equation to determine $\frac{dr}{dt}$.
Using this equation:
$0.2 = 2\frac{dr}{dt}$
$\frac{dr}{dt} = 0.1$
Now, we know enough information to compute the rate of change of the volume.
$\frac{dV}{dt} = \frac{1}{3}\pi((1)^2 * 0.2 + 2 * 1 * 2 * 0.1)$.
$\frac{dV}{dt} = \frac{\pi}{5}$
The rate of change of the volume of water with respect to time is $\frac{\pi}{5}$ cubic inches per second.

Recap

This Study Guide went over:

- Derivative Fundamentals and Properties
- Derivative Rules
- Inverse Differentiation
- Implicit Differentiation
- Related Rates

Now attempt the practice problems on the next page!

Practice Problems

The questions will ramp-up in difficulty.

1) Use the limit definition to differentiate: $f(x) = (x+1)^2$
2) Convert: $\lim_{x \to \frac{\pi}{6}} \frac{sin(x) - \frac{1}{2}}{x - \frac{\pi}{6}}$ to the limit definition of the derivative.
3) Evaluate: $\lim_{h \to 0} \frac{m(p+h) - m(p)}{h} * \lim_{h \to 0} \frac{g(m+h) - g(m)}{h}$
4) Evaluate: $\lim_{x \to 3} \frac{x^3 + 4x - 39}{x - 3}$
5) Use the alternate definition of the derivative to find the slope of the normal line to $f(x) = sin(\frac{\pi}{2} - x)$ at $x = \frac{\pi}{6}$
6) Let $f(x) = g(h(x))$ and $h(x) = i(x)j(x)$. Evaluate $f'(2x)$
7) State where $|sin\ x|$ is differentiable.
8) Differentiate each of the following functions:
   a) $f(x) = ln\ (ax * bx)$
   b) $f(x) = arcsec(arcsin(2x))$
   c) $f(x) = x^x$
   d) $f(x) = x^{arcsin\ x}$
   e) If $f(2x) = arcsin(2x\ *\ ln\ x)$, differentiate $f(x)$.
   f) $g(x)$, where $f(x) = x^2 - 1$, and g is the inverse of f.
9) Suppose $g(x)$ is the inverse of $f(x) = x^4 - 2x^2 + 4$. Find $g'(3)$.
10) Differentiate each implicity.
    a) $\frac{dy}{dx}$ of $x^2 - 2y = 3$.
    b) $\frac{dy}{dx}$ of $2x^2 - arcsin(y) = 3$
    c) $\frac{dy}{dx}$ of $ln(xy) - ln(x) - ln(3e^y) = 2y$
11) Given the unit hyperbola
    a) Find all possible values where the tangent line to the unit hyperbola has a slope of 1.
    b) Show that the unit hyperbola never has a horizontal tangent line.
12) Let $V = \frac{4}{3}\pi r^3$.
    a) What is this the formula for?
    b) Find the rate of change of volume with respect to time.
    c) Suppose Zerth inflates the solid at a rate of $\pi\ in^3/sec$. At what rate does the surface area of this solid change when the volume is $36\pi\ in^3$?
13) A rocket is flying very fast directly upwards. Zerth is standing at a relatively safe distance of 1 kilometer away. The rocket's height is changing at a rate of $1\ km/sec$, when time $t = 2\ seconds$. The rocket is 2 kilometers high at this time.
    a) Find the rate of change of the diagonal distance between Zerth and the rocket at the time $t = 2\ seconds$.

b) Find the rate of change of the angle formed between the ground and the diagonal when $t = 2\ seconds$.

Answer Key

1) $f'(x) = 2x + 2$
2) $\lim_{h \to 0} \frac{sin(\frac{\pi}{6}+h) - sin\frac{\pi}{6}}{h}$
3) $m'(p) * g'(m)$
4) 31
5) 2
6) $f'(2x) = g'(h(2x)) * [2i'(2x)j(2x) + 2i(2x)j'(2x)]$
7) Differentiable everywhere but $x = n\pi$, where $n \in Z$.
8) Answers below:
    a) $f'(x) = \frac{2}{x}$
    b) $f'(x) = \frac{2}{arcsin(2x)\sqrt{arcsin^2(2x)-1}\sqrt{1-4x^2}}$
    c) $f'(x) = x^x(1 + ln\ x)$
    d) $f'(x) = x^{arcsin\ x}(\frac{ln\ x}{\sqrt{1-x^2}} + \frac{arcsin\ x}{x})$
    e) $f'(x) = \frac{1}{\sqrt{1-(x\ ln\ \frac{x}{2})^2}} * (\frac{1}{2} + ln\frac{x}{2})$
    f) $g'(x) = \frac{1}{2\sqrt{x+1}}$
9) Derivative undefined (infinity)
10) Answers below:
    a) $\frac{dy}{dx} = x$
    b) $\frac{dy}{dx} = 4x\sqrt{1 - y^2}$
    c) $\frac{dy}{dx} = 0$
11) Answers below:
    a) Never
    b) For this equation to have a horizontal tangent line, the derivative must equal 0, thus $x = 0$. But x can not be 0 for this graph because otherwise y^2 must be negative.
12) Answers below:
    a) Volume of a sphere
    b) $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$
    c) $\frac{dS}{dt} = 2\pi\ in^2/sec$
13) Answers below:
    a) $\frac{dz}{dt} = \frac{4\sqrt{5}}{5} km/sec$
    b) $\frac{d\theta}{dt} = \frac{-3}{5} rad/sec$