<u>Integration Methods</u>

(Feel free to comment, markup, and chat on this study guide.)

This study guide is split into multiple sections

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Integration by Substitution

Theorems

None here!

Examples

Example 1: Integrate $\int(\sqrt{sin\ x} * cos\ x)dx$

Solution 1: Let $u = sin\ x$. Then $\frac{du}{dx} = cos\ x$, and $du = cos\ x * dx$.

Therefore we get the integral: $\int(\sqrt{u})du$, which is integrable by the power rule to give $\frac{2}{3}u^{3/2} + C$

Back substituting gives us: $\frac{2}{3}sin\ x\sqrt{sin\ x} + C$

Example 2: Find the value of $\int_1^3(C + f(z))dz$ if $\int_1^2 f(2x-1)dx = 7$, and $C > 0$.

Solution 2: Split the first integral: $\int_1^3 Cdz + \int_1^3 f(z)dz$.

Then, in the second integral, make a substitution, $z = 2x - 1$, $dz = 2\ dx$, $\frac{dz}{2} = dx$, and the limits of integration change to $1\ (2(1) - 1 = 1)$ and $3\ (2(2) - 1)$.

The second integral now reads: $\frac{1}{2}\int_1^3 f(z)dz = 7$.

Therefore, $\int_1^3 f(z)dz = 14$.

Now, we just back substitute into our original expression, realizing that $z\ and\ x$ are dummy variables and are interchangeable.

Therefore, our answer is $2C + 14$.

Recap

This Study Guide went over:

- Integration by u-substitution
- The idea of a dummy variable

Now attempt the practice problems on the next page!

Practice Problems

The problems will ramp up in difficulty.

1) Integrate each of the following:
    a) $f(x) = sin(2x)$
    b) $f(x) = sin^2 x$
    c) $f(x) = \frac{2}{x}$
    d) $f(x) = cos(sin\ x) * cos\ x$
    e) $f(x) = \frac{x}{\sqrt{x^2-1}}$
    f) $f(x) = tan\ x + cot\ x$
2) Integrate each of the following:
    a) $f(x) = sin(3x)$ from 0 to π.
    b) $f(x) = \frac{ln\ x}{x}$ from 1 to e.
    c) $f(x) = \frac{3x^2}{x^3+1}$ from 1 to 2.
3) If $\int_0^1 (2 + f(x))dx = 5$, find $\int_0^3 f(\frac{x}{3})dx$.

Answer Key

1) Answers Below
   a) $\frac{-1}{2}cos(2x) + C$
   b) $\frac{1}{2}x - \frac{1}{4}sin(2x) + C$
   c) $2\ ln|x| + C$
   d) $sin(sin\ x) + C$
   e) $\sqrt{x^2 - 1} + C$
   f) $ln|sec\ x| + ln|sin\ x| + C$
2) Answers Below
   a) $\frac{2}{3}$
   b) $\frac{1}{2}$
   c) $ln\ 9 - ln\ 2$
3) 3

Other Integration Methods

Theorems

Integration by Parts: $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$

Trig Substitution Rules:

$\int \frac{dx}{\sqrt{a^2-x^2}}$ uses the substitution $x = a \sin \theta$ (the denominator is in the form $1 - \sin^2 x$)

$\int \frac{dx}{a^2+x^2}$ uses the substitution $x = a \tan \theta$ (the denominator is in the form $1 + \tan^2 x$)

$\int \frac{dx}{\sqrt{x^2-a^2}}$ uses the substitution $x = a \sec \theta$ (the denominator is in the form $\sec^2 x - 1$)

Examples

Example 1: Evaluate $\int xe^x$

Solution 1: A u-substitution doesn't work here, so we try integration by parts.
Since we know the derivative of x, let $f(x) = x$ and since we know the integral of e^x, let $g'(x) = e^x$.
Applying integration by parts:

$$\int xe^x dx = xe^x - \int e^x dx$$

$$= xe^x - e^x + C$$

Example 2: Evaluate $\int \frac{1}{\sqrt{1-x^2}} dx$.

Solution 2: This is in the form of one of the integrals that use a trigonometric substitution.
Since it appears to be in the form of the trigonometric identity $sin^2 x + cos^2 x = 1$ (here it's in the form $1 - sin^2 x$). Thus, we'll make the substitution $x = sin\,\theta$. (If this were a definite integral, make sure to change the limits of integration!)

$$x = sin\,\theta,\ dx = cos\,\theta\, d\theta$$

$$\int \frac{1}{\sqrt{1-x^2}} dx$$

$$= \int \frac{cos\,\theta}{\sqrt{1-sin^2\theta}} d\theta$$

$= \int d\theta$ (we can disregard absolute values in

$$= \theta + C$$

Inverting our substitution, we get that $\theta = arcsin\ x$
Thus our answer is $arcsin\ x + C$ (you should have recognized the integrand from the beginning!)
Note that if there were a trig function in the resulting θ answer, you would need to draw a triangle to determine the answer.

Example 3: Evaluate $\int \frac{dx}{x^2-1}$.

Solution 3: This is a classic example of an integral with partial fractions.
First, we'll need to split $\frac{1}{x^2-1}$ via partial fraction decomposition.
The decomposition is $\frac{1}{2}(\frac{1}{x-1} - \frac{1}{x+1})$, and so our integrand simplifies to:

$\int \frac{1}{2}(\frac{dx}{x-1} - \frac{dx}{x+1}) = \frac{1}{2}(\int \frac{dx}{x-1} - \int \frac{dx}{x+1})$.

These integrals are fundamental.

$\frac{1}{2}(\int \frac{dx}{x-1} - \int \frac{dx}{x+1}) = \frac{1}{2}(ln|x-1| - ln|x+1| + D)$

Combining logarithms:
$\frac{1}{2}ln|\frac{x-1}{x+1}| + C$.

Recap

This Study Guide went over:

- Integration by Parts
- Integration by Trig Sub
- Integration by Partial Fractions

If you want to review partial fractions, check out this guide.
Now attempt the practice problems on the next page!

Problems

1) Evaluate each integral

   a) $\int x \sin x \, dx$

   b) $\int x^2 e^x \, dx$

   c) $\int \sec^3 x \, dx$ (This is a very famous integral!)

   d) $\int \frac{x^2}{\sqrt{x^2-4}} \, dx$

   e) $\int \sin^2(\sqrt{x}) \, dx$

   f) $\int \sqrt{\frac{1}{x^2} + \frac{1}{x^4}} dx$

   g) $\int \frac{dx}{x^3+1}$

2) Evaluate each integral

   a) $\int_1^2 x \ln x \, dx$

   b) $\int_1^{\sqrt{3}} \frac{x^3}{4x^2+1} dx$

Answer Key

1) Answers Below
   a) $-x\ cos\ x + sin\ x + C$
   b) $x^2e^x - 2xe^x + 2e^x + C$
   c) $\frac{1}{2}\ ln\,|sec\ x + tan\ x| + \frac{1}{2}sec\ x\ tan\ x + C$
   d) $4\ ln\,|\frac{x+\sqrt{x^2-4}}{2}| + x\sqrt{x^2-4} + C$
   e) $\frac{1}{2}x + \frac{1}{2}\sqrt{x}\ sin\ (2\sqrt{x}) + \frac{1}{4}cos\ (2\sqrt{x}) + C$
   f) $\frac{-1}{6}ln(x^2 - x + 1) - \frac{1}{3}ln\ |x+1| + \frac{\sqrt{3}}{6}arctan(\frac{2x-1}{\sqrt{3}}) + C$
2) Answers Below
   a) 0.636
   b) 0.220