<u>The Integral and its Applications</u>

(Feel free to comment, markup, and chat on this study guide.)

This study guide is split into multiple sections

<u>Contents</u>

Riemann Sums and Introduction to the Integral

Theorems

General forms of Riemann Sums:

Left: $\sum_{i=1}^{n} f(a + (i - 1)\Delta x)$

Right: $\sum_{i=1}^{n} f(a + i\Delta x)$

Midpoint: $\sum_{i=1}^{n} f(a + (i - \frac{1}{2})\Delta x)$

$\Delta x = \frac{b-a}{n}$

The Integral follows the same rules as that of summations.

Examples

Example 1: Use a midpoint Riemann sum with 2 subintervals to approximate the area under the curve of the function $f(x) = sin\ x$ over the interval $[0,\ \pi]$.

Solution 1: First, we need to set up our Riemann sum. 2 subintervals notifies us that each block will be $\pi/2$ units long. The midpoint sum tells us that the two heights are the midpoint of each interval. Therefore, the area under the curve is approximated by:

$Area \approx base * height$

$Area \approx \frac{\pi}{2}(f(\frac{\pi}{4}) + f(\frac{3\pi}{4}))$

$Area \approx \frac{\pi\sqrt{2}}{2}$.

Example 2: Use a right Riemann sum to determine the area of $f(x) = x^3$ over the interval $[1, 2]$.

The formula for a Right Riemann Sum is the following:

$Area = \Delta x * \sum_{i=1}^{n}(f(a + i\Delta x)$, $\Delta x = \frac{b-a}{n}$, n is the number of rectangles.

Since we are getting the exact area, we are going to have an infinite number of rectangles, meaning that their base lengths approach 0. Therefore, we are eventually going to have to take the limit of the sum as n approaches infinity (*Why? As we add more and more rectangles over an interval, the error gets smaller and smaller since each rectangle approximates its partition better and better. The rectangle's height is much more focused and much more accurate over a smaller interval. Since we are cramming more and more rectangles, the base length gets smaller. This entire operation results in the indeterminate form* $0 * \infty$ *without simplification, and that is why we fully simplify the function before taking the limit. To visualize this, take y = x and try drawing more and more rectangles over the interval* [0,1]. *You'll notice that the following note is true.*)

In this case, Δx is $\frac{2-1}{n} = \frac{1}{n}$, $a = 1$.

Therefore, what we are attempting to evaluate is the following:

$$Area = \int_1^2 x^3 dx = \lim_{n\to\infty}(\Delta x * \sum_{i=1}^{n}(f(a + i\Delta x))$$

$$= \lim_{n\to\infty}(\frac{1}{n} * \sum_{i=1}^{n}(f(1 + \frac{i}{n})))$$

$$= \lim_{n\to\infty}(\frac{1}{n} * \sum_{i=1}^{n}((1 + \frac{i}{n})^3))$$

$$= \lim_{n\to\infty}(\frac{1}{n} * \sum_{i=1}^{n}(1 + \frac{3i}{n} + \frac{3i^2}{n^2} + \frac{i^3}{n^3}))$$

$$= \lim_{n\to\infty}(\frac{1}{n}\sum_{i=1}^{n} 1 + \frac{1}{n}\sum_{i=1}^{n}\frac{3i}{n} + \frac{1}{n}\sum_{i=1}^{n}\frac{3i^2}{n^2} + \frac{1}{n}\sum_{i=1}^{n}\frac{i^3}{n^3})$$

$$= \lim_{n\to\infty}(\frac{1}{n}\sum_{i=1}^{n} 1 + \frac{3}{n^2}\sum_{i=1}^{n} i + \frac{3}{n^3}\sum_{i=1}^{n} i^2 + \frac{1}{n^4}\sum_{i=1}^{n} i^3)$$

$$= \lim_{n\to\infty}(1 + \frac{3(n)(n+1)}{2n^2} + \frac{3(n)(n+1)(2n+1)}{6n^3} + \frac{(n^2)(n+1)^2}{4n^4})$$

$= 1 + \frac{3}{2} + 1 + \frac{1}{4}$
$= \frac{15}{4}$

Example 3: Use the limit of a Riemann Sum to expand $\int_1^3 x^2 dx$. Use any type of sum.

Solution 3: While a Right Riemann Sum is easier, for the sake of example, we will use a left Riemann Sum.

$Area = \int_1^3 x^2 dx = \lim_{n\to\infty}(\Delta x * \sum_{i=1}^{n}(f(a + (i-1)\Delta x))$ (Formula for a left Riemann Sum)

$Area = \lim_{n\to\infty}(\frac{2}{n} * \sum_{i=1}^{n}(f(1 + \frac{2(i-1)}{n})))$

$= \lim_{n\to\infty}(\frac{2}{n} * \sum_{i=1}^{n}((1 + \frac{2(i-1)}{n})^2))$

$= \lim_{n\to\infty}(\frac{2}{n} * \sum_{i=1}^{n}(1 + \frac{4(i-1)}{n} + \frac{4(i-1)^2}{n^2}))$

Example 4: Write $\lim_{n\to\infty} \frac{2}{n}(\frac{2}{4} + \frac{2}{(2+(2/n))^2} + \frac{2}{(2+(4/n))^2} + ... + \frac{2}{(4-(n/n))^2})$ as an integral.

Solution 4: Notice that this is a Left Riemann Sum. Therefore, we can simplify this.

$\lim_{n\to\infty} \frac{2}{n}(\sum_{i=1}^{n}(\frac{2}{(2+(2(i-1)/n))^2}))$

$\lim_{n\to\infty} \frac{2}{n}(\sum_{i=1}^{n} f(2 + \frac{2(i-1)}{n})$

$\int_2^4 \frac{2}{x^2} dx$.

<u>Recap</u>

This Study Guide went over:

- Integral Fundamentals and Properties
- Integral Rules
- Riemann Sum Approximations
- Integrals as a Riemann Sum

Now attempt the practice problems on the next page!

Practice Problems

The problems will ramp-up in difficulty.

1) Use a left Riemann sum with 2 subintervals to approximate $f(x) = e^x$ on the interval $[0,\ 2]$.
2) Use a right Riemann sum with four subintervals of unequal length to evaluate the area of f on the interval $[0,\ 11]$ if $f(0) = 3, f(6) = 8, f(11) = 5, f(1) = 2, f(3) = 5, f(1.5) = 5$.
3) Evaluate $\int_1^2 x^2\, dx$ using a right Riemann Sum.
4) Evaluate $\int_1^0 x^3 dx$ using a left Riemann Sum.
5) Use a midpoint Riemann sum with 3 even subintervals to find the area under one bump of $f(x) = sin\ x$. Compare that to the result of the integral $\int_{\pi/2}^{3\pi/2} f(x)dx$ approximated with 3 subintervals.

Answer Key

1) $1+e$
2) 61
3) $\frac{8}{3}$
4) $-\frac{1}{4}$
5) $\frac{2\pi}{3}$, 0

Area, Volume, and Average Value

Theorems

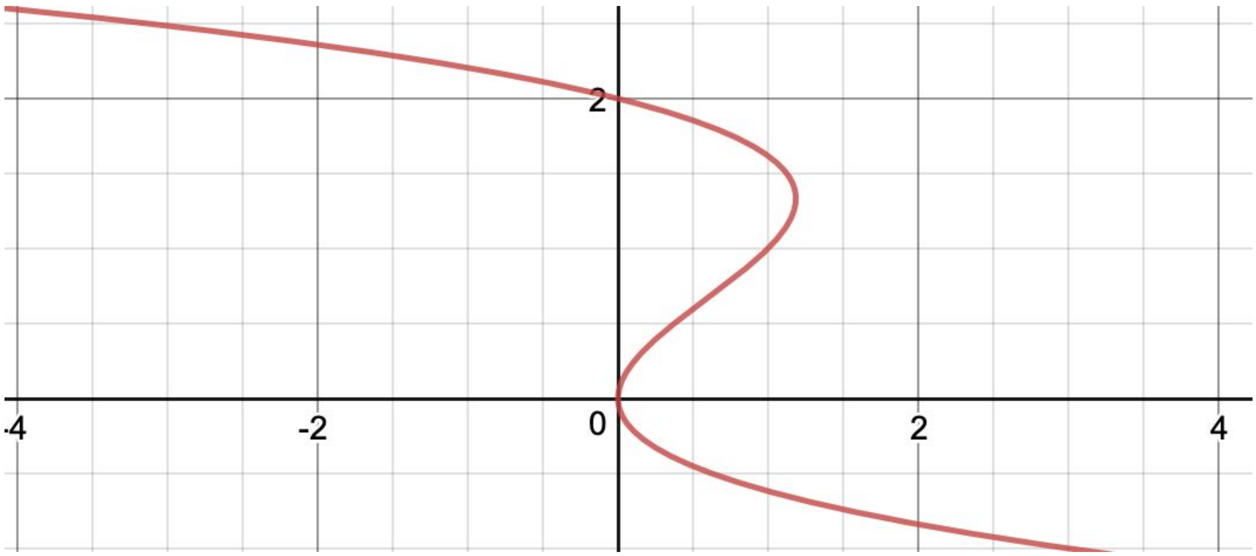
Average Value of a function: $f_a(x) = \frac{1}{b-a}\int_a^b f'(x)dx$

Volume of a Solid of Revolution: $V = \pi\int_a^b (f(x))^2 dx$

Examples

Example 1: Find the area under the curve of $f(y) = 2y^2 - y^3$ on the interval $[0,\ 2]$.

Solution 1: Taking a look at the graph:



We can see that we are going to integrate with respect to the y-axis.

Therefore, we can set-up the integral and use the y-values for the bounds.

$Area = \int_0^2 (2y^2 - y^3)dy$.

Now, we can use power rule to evaluate easily.

$Area = \frac{2}{3}y^3 - \frac{1}{4}y^4$ evaluated from 0 to 2.

Using the First Fundamental Theorem, we get $Area = \frac{2}{3} * 8 - \frac{1}{4} * 16 = \frac{4}{3}$.

Example 2: Find the area between $f(x) = sin\ x$ and $g(x) = tan\ x$ on the interval $[\frac{3\pi}{4},\ \frac{5\pi}{4}]$

The general integration formula for the area between two curves is $\int_a^b f(x) - g(x)$, where $f(x) > g(x)$. However, we need to watch out for points where $sin\ x$ changes from being greater than $tan\ x$ to being less than $tan\ x$.

We can find this point out easily by solving $sin\ x = tan\ x$

$sin\ x\ cos\ x = sin\ x$

$sin\ x\ cos\ x - sin\ x = 0$

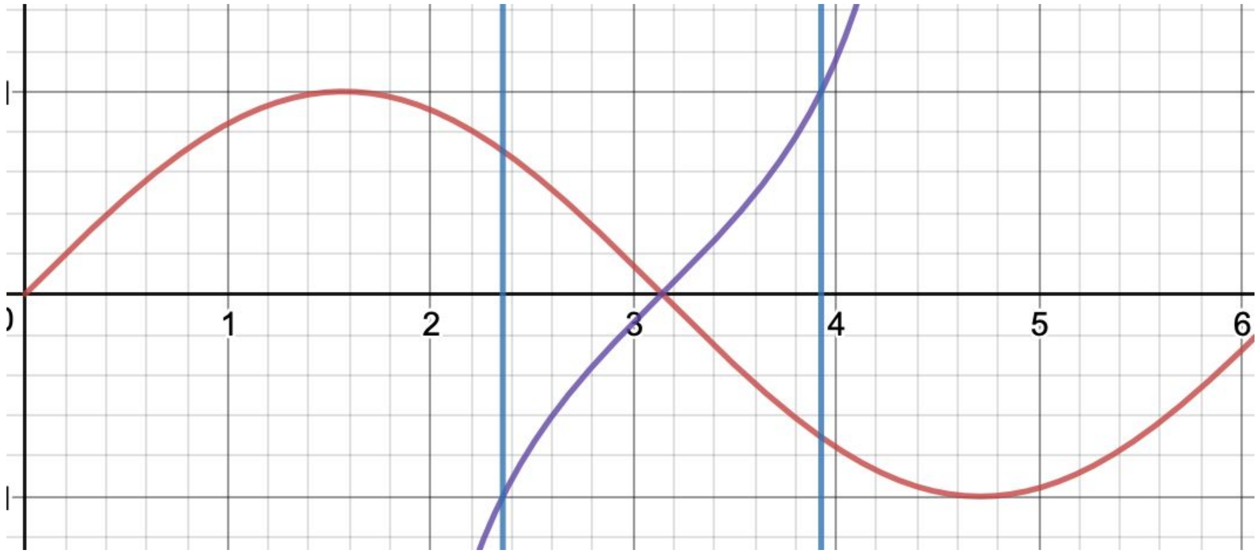
$sin\ x(cos\ x\ - 1) = 0$

$sin\ x\ = 0\ \rightarrow x = n\pi$

$cos\ x = 1\ \rightarrow x = 2n\pi$

In our interval, the solution we want is π.

Now, graphically, we can see that sin x switches from being greater than tan x to being less than tan x at a point, and we've determined this point to be π (always mathematically show where the intersection happens, if not you can EASILY get trolled)



Now, we set up 2 integrals, one for the area where $sin\,x > tan\,x$ and one for the area where $tan\,x > sin\,x$.

The total area is: $Area = \int_{\frac{3\pi}{4}}^{\pi} (sin\,x - tan\,x)dx + \int_{\pi}^{\frac{5\pi}{4}} (tan\,x - sin\,x)dx$

Using the 1st FTC to evaluate, the first integral is $cos - sec^2x$ from $\frac{3\pi}{4}$ $to\ \pi$, and the second integral is $sec^2x - cos\,x$ from $\pi\ to\ \frac{5\pi}{4}$.

At this point, plugging in values and checking if anything cancels would be a good approach:

$cos(\pi) - sec^2(\pi) - cos(\frac{3\pi}{4}) + sec^2(\frac{3\pi}{4}) + sec^2(\frac{5\pi}{4}) - cos(\frac{5\pi}{4}) - sec^2(\pi) + cos(\pi)$

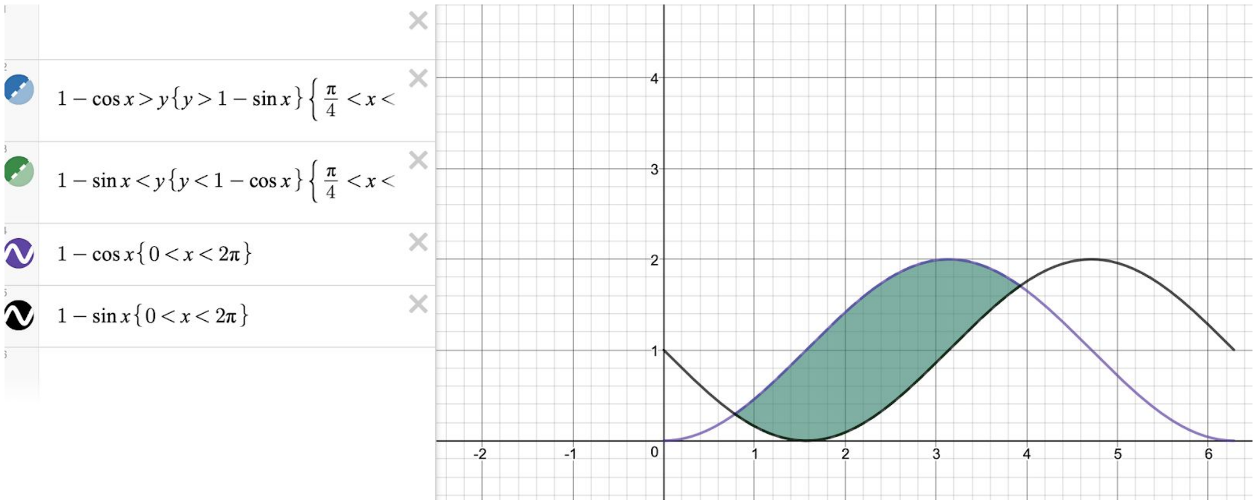
Nothing cancels, so it's time to hard evaluate

$Area =- 1 - 1 + \frac{\sqrt{2}}{2} + 2 + 2 + \frac{\sqrt{2}}{2} - 1 - 1$

$Area = \sqrt{2}$

Example 3: Setup and evaluate an integral which provides the volume of a figure whose cross-sections are perpendicular to the x-axis and consist of squares whose side lengths are the difference between $1 - cos\,x$ and $1 - sin\,x$ from π/4 to 5π/4.

Solution 3: Let's look at the graph of the region.



Therefore, each side length will be upper minus lower, that is $(1 - cos\ x) - (1 - sin\ x)$, which is $sin\ x - cos\ x$

Volume is the area of the cross section multiplied by the height. We take a cross section and drag it along the height.
However, note that an integral is essentially volume, where each cross section is a line with length $f(x)$ and width dx. Therefore, if we were to square this length, we would get the area of a square.

Therefore, the volume by cross-section is $\int_a^b (area)dx$, where dx is a very small height. Adding and accumulating all of these infinitely thin cross-sections gives us the area.

In this case, the area of our cross-section is $(y_1(x) - y_2(x))^2$ (we square it since $y_1(x) - y_2(x)$ is the side length of a square).

Our integral is thus $\int_{\pi/4}^{5\pi/4} (sin\ x - cos\ x)dx \approx 2.828$

Example 4: Find the average value of $f(x) = \frac{1}{x^2+1}$ over the interval $[0, 1]$.

Solution 4: Let's just use the formula for average value: $\frac{1}{b-a}\int_a^b f(x)dx$.

Average Value $= \int_0^1 (x^2 + 1)^{-1} dx$

$= arctan\ x$ from 0 to 1
$= \frac{\pi}{4}$

Example 5: The area between $f(x) = \sqrt[3]{x}$, $g(x) = 0$, and $x = 1$ is revolved around $y = -1$. Set up an integral that yields the volume of the resulting figure.

Solution 5: Note that when we revolve a figure, its cross-section will be πr^2, where r is the distance from the axis of rotation to the curve. However, in this case, after revolution, there will be a hole in the figure due to there being a part of the area revolved that will not form a shape. Therefore, to determine the radius of the figure, we subtract the radius of the hole from the radius of the entire figure (Graph the equations above to see what's being described).

We can write the volume of this figure like so: $V = \pi r_o{}^2 h - \pi r_i{}^2 h$. Here $r^2 h$ can be written as $\int_a^b (f(x) - axis)^2 dx$, where dx is the height.

We are comparing functions relative to y-value (upper vs. lower), so we integrate with respect to x in this scenario.

The larger radius is the distance from the upper curve or $f(x)$ to the axis of revolution and thus the volume of the outer figure is $\pi \int_0^1 (\sqrt[3]{x} - (-1))^2 dx$.
The smaller radius is the distance from the lower curve or $g(x)$ to the axis of revolution, which is $\pi \int_0^1 (0 - (-1))^2 dx$
We can combine the integrands here, since everything is the same, giving us:
$$Volume = \pi \int_0^1 ((\sqrt[3]{x} - (-1))^2 - (0 - (-1))^2)dx$$

Recap
This Study Guide went over:
- Average Value
- Volume by Revolution and Cross Section
- Area between Multiple Curves
- Integration over the y-axis

Now attempt the practice problems on the next page!
Practice Problems
The problems will ramp up in difficulty.

1) Evaluate each integral:

a) $f(x) = x^2$ using a Right Riemann Sum from $[0,\ 1]$

b) $f(x) = x^2$ from $[0,\ 1]$

c) $f(x) = x^3 - cos\ x$ from $[0,\ \pi]$

d) $f(y) = 2y - 3y^2$ from $[0,\ 2]$

e) $x = y(y-2)$ in the area to the left of the $y - axis$ with respect to y (not asking for area)

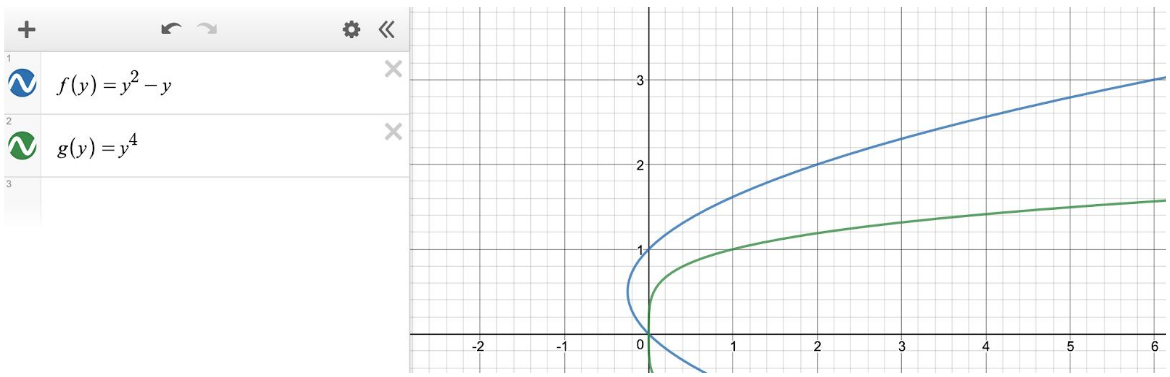
f) $x = cos\ y$ from $[0,\ \pi]$ with respect to y.

2) Evaluate the area between the curves. For parts $a,\ b,\ and\ e$ graph the functions.

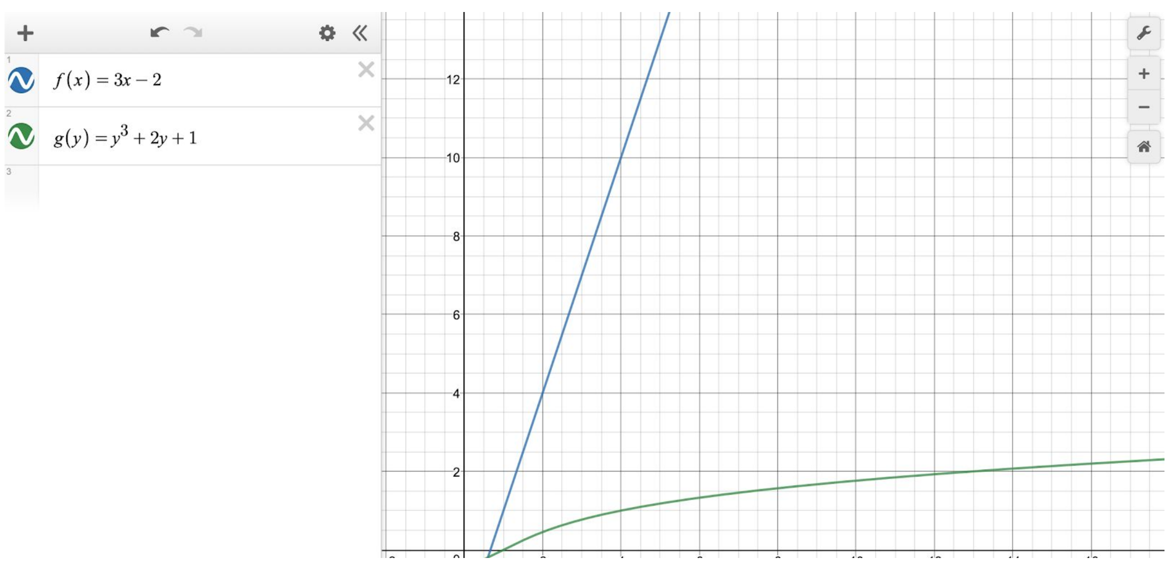
a) $f(x) = x^2$ and $g(x) = x^3$ from $[0, 1]$

b) $f(x) = x^3$ and $g(x) = -x^3$ from $[-1, 1]$

c) $f(y) = y^2 - y$ and $g(y) = y^4$ from $[0,\ 1]$



d) $f(x) = 3x - 2$ and $g(y) = y^3 + 2y + 1$ from $y \in [0,\ 1]$



e) $f(x) = sec\ \frac{x}{2}$ and $g(x) = sin\ x$ from $[0,\ \frac{\pi}{2}]$ (Graphing Calc OK)

3) Find the average value of $f(x) = x^2$ from 0 to 3.

4) Suppose $f(x) = x^3$ from $0 < x < 1$, $0 < y < 1$

a) Write an integral expression giving the volume of a solid whose cross sections are equilateral triangles with side lengths perpendicular to the y-axis and along the area between f and $x = 1$.

b) f is revolved about $x = 1$. Write an integral expression to find the volume.

5) StudyShed decided to hold a grand fundraiser in order to make some money to add something to their nonexistent monetary total. The CEO decided to sell donuts. Each donut had a height of 2 inches. The hole in the center of the donut was 2 inch and from edge to edge, the donut was 6 inches.

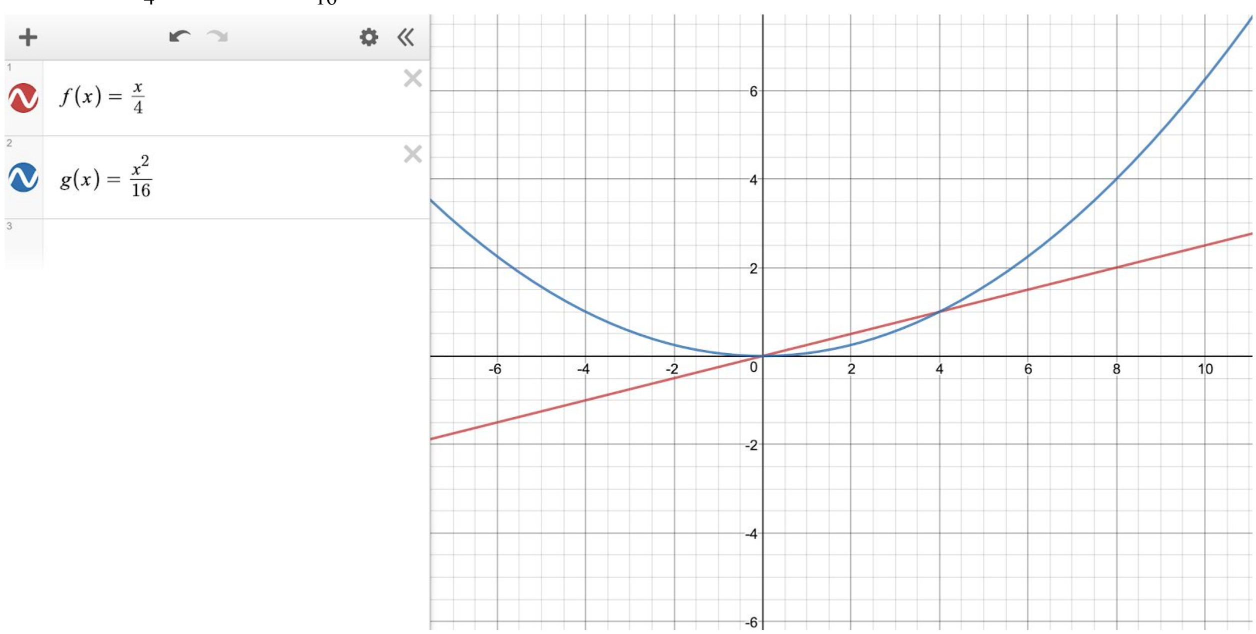
a) Find the volume of 1 donut.

b) A prism box can hold x donuts and has width 16 inches, length 18 inches, and height 4 inches. Find the leftover volume and the value of x.

c) A box takes \$1 to make and each donut takes \$0.50 to make. The money made by selling n donuts bought at one time is modeled by the function $f(n) = ln(e^n) + 0.1n - 0.1(n^2)$. Find the maximum possible profits made in one purchase, the number of boxes needed for the purchase, and the amount of extra space left over. If the maximum profits can be made using an infinite number of donuts, explain why.

d) One of the StudyShed editors is a fatty and loves donuts. He ate one donut in x bites. He ate half the donut on the first bite, and on each subsequent bite, he ate half as much compared to the previous. How much of the donut did he eat after the first four bites?

6) Let $f(x) = \frac{x}{4}$ and $g(x) = \frac{x^2}{16}$ for $0 \leq x \leq 8$



a) Let $j(x)$ be the function representing the area between $f(x)$ and $g(x)$. Write j using a piecewise function and integrals.

b) Show that j is continuous.

c) Find the domain and range for $j(x)$.

d) Find where j is differentiable:

  i) Once

  ii) Twice

e) Find the local extrema for $j(x)$.

f) Find where j is concave down.

Practice Problems (with Graphing Calculator)

A graphing calculator is required for these problems.

1) Suppose $f(x) = cos\ x - e^x$ and $g(x) = e^{-x} - e^x$.
    a) Find the intersection points of the two functions
    b) Suppose R is the region between $f\ and\ g$. Find the area of R.
    c) A solid has cross sections perpendicular to the x axis. The cross sections are circles with diameter along R. Find the volume of the solid.
    d) The area R is revolved about the x-axis. Find the volume of the solid.

Answer Key

1) Answers Below
    a) $\frac{1}{3}$
    b) $\frac{1}{3}$
    c) 24.35
    d) -4
    e) $\frac{-4}{3}$
    f) 0
2) Answers Below
    a) $\frac{1}{12}$
    b) $\frac{1}{2}$
    c) $\frac{11}{30}$
    d) 0.367
    e) 0.763
3) 3
4) Answers Below
    a) $Volume = \frac{\sqrt{3}}{4}\int_0^1 (x^3)^2 dx$
    b) $Volume = \pi\int_0^1 (x^3)^2 dx$
5) Answers Below
    a) $4\pi^2\ in^3$
    b) $1152 - 48\pi^2\ in^3$
    c) $3\ donuts,\ 1152 - 12\pi^2\ in^3$ of space left
    d) $\frac{15\pi^2}{4}\ in^3$
6) Answers Below
    a) $j(x) = \{\int_0^x (f(t) - g(t))dt\ for\ 0 \le x \le 4\}$ and
    $\{\int_0^4 (f(t) - g(t))dt\ +\ \int_4^x (g(t) - f(t))dt\ for\ 4 \le x \le 8\}$
    b) Since $f(x) = g(x)$ at $x = 4$, we use the limit test at $x = 4$. After testing the two sided limits, the limits are equal, $j(x) = \frac{2}{3}$, thus j is continuous. It is continuous everywhere else since f and g are integrable, thus their difference is integrable.
    c) $D_j = 0 \le x \le 8\ R_j = 0 \le x \le 4$
    d) j is differentiable:
        i) Once everywhere

ii) Twice everywhere but $x = 4$

e) j has no local extrema

f) j is concave down on $2 \le x < 4$ since $j'(x)$ is decreasing.

<u>Answer Key (GC Questions)</u>

1) Answers Below
    a) $x_1 = 0,\ x_2 = 1.2926957$
    b) 0.236
    c) 0.041
    d) 1.889