<u>Differential Equations</u>

(Feel free to comment, markup, and chat on this study guide.)

<u>Theorems</u>

The logistic equation: $\frac{dy}{dx} = ky(1 - \frac{y}{L})$, where L is the population maximum. It solves to $y = \frac{L}{1+Ae^{-kt}}$, where $A,\ k$ are constants. The logistic equation has the greatest rate of change at $y = \frac{L}{2}$ and switches concavities there.

Euler's Method: $y_n = y_{n-1} + y'\,dx$ (this is probably not the best way to understand Euler's Method but it's here regardless)

Examples

Example 1: Solve the differential equation $\frac{dy}{dx} = xy$, when $x = 1$ and $y = 1$. Use the interval to determine the largest open interval for when the solution holds.

Solution 1: The separation of variables is similar to isolating like terms in algebra. We make sure that the variables on each side of the equation are the same.

Thus, we cross multiply to get $\frac{dy}{x} = x\,dx$. Each side is integrable with respect to its own variable.

Now, we integrate both sides to get $ln|y| + C_1 = \frac{x^2}{2} + C_2$. We can combine the constants to get $ln\,|y| = x^2 + C$.

Now, we use the initial condition and substitute to solve for C.

$ln\ 1 = 1^2 + C$

$-1 = C$

Therefore, the equation is $ln|y| = x^2 - 1$.

Raising both sides with base e we get: $|y| = e^{x^2-1}$, giving us

$y = \pm\, e^{x^2-1}$. Again, plugging in $x = 1,\ y = 1$ we get $1 = \pm\, e^0$, showing that we need to take the positive half.

There are no restrictions on both the initial differential equation and the solution, therefore the largest open interval with the solution is negative infinity to infinity.

Therefore, the particular solution to this differential equation is: $y = e^{x^2-1}\ for\ (-\infty,\ \infty)$.

Example 2: Provide 9 points that will allow you to graph a slope field for the differential equation $\frac{dy}{dx} = x^2y$. Then solve the differential equation for $y(0) = 1$ and in general.

Solution 2:

We'll make a table:

x	y	$\frac{dy}{dx}$
-1	-1	-1
0	-1	0
1	-1	-1
-1	0	0
0	0	0
1	0	0
-1	1	1

0	1	0
1	1	1

After this,we simply draw small lines (about 1/4 inch) to form **isoclines**. These isoclines create the slope field. To draw a certain solution, we just follow the isoclines.

Now, to solve the differential equation, we use separation of variables.

$\frac{dy}{dx} = x^2y$
$\frac{dy}{y} = x^2dx$

$\int \frac{dy}{y} = \int x^2dx$

$ln\,|y| = \frac{x^3}{3} + C$
From here, we are going to split the process. First, we are going to solve for the particular solution.
Since $y(0) = 1$, we get $ln\ 1 = 0 + C$
$C = 0$
Thus, our equation is $ln\,|y| = \frac{x^3}{3}$
Solving, we get:
$|y| = e^{\frac{x^3}{3}}$
$y = \pm\, e^{\frac{x^3}{3}}$.
Checking with our initial condition again to see which half we take, we realize we need the positive half.
Thus, our particular solution is $y = e^{\frac{x^3}{3}}$
Now, for the general solution, we continue the process as normal, but we keep the +C.
$ln\,|y| = \frac{x^3}{3} + C$
$|y| = e^{\frac{x^3}{3}+C}$
$|y| = e^{\frac{x^3}{3}}e^C$
Here, e^C is a constant, so we need to replace it with another constant, A.

$|y| = Ae^{\frac{x^3}{3}}$
$y = \pm\, Ae^{\frac{x^3}{3}}$
$y = Be^{\frac{x^3}{3}}$ is our general solution.

Example 3: Suppose $\frac{dy}{dx} = 2x - 2xy$, with initial condition $y(0) = 1$. Use Euler's Method with three equal steps to estimate $y(3)$ and determine the error in your estimate.

Solution 3: Euler's Method works as follows.
The two differentials *dx* and *dy* represent small changes in *x* and *y*, respectively. The differential equation essentially now says that the change in y divided by the change in x is $2x - 2xy$. Note that if we multiply dx to the RHS, we get that the change in *y* is $2x - 2xy$ times the change in *x*. If we are given a condition and the change in *x*, we can iteratively find the change in *y*. That is essentially what Euler's Method describes.

Here, we are given that we are going from 0 to 3 in three steps. Therefore, $dx = 1$.
Now, $dy = (2x - 2xy)(dx)$, and we are ready to start Euler's Method.

Remember, that the *x* and *y* that we plug in to generate *dy* are the values from the previous step.

$y = y_0 + dy$, so essentially, the rigorous formula for Euler's Method is $y_n = y_{n-1} + (\frac{dy}{dx})(dx)$.
We'll create a table as follows:

x	y
0	1
1	$y_n = 1 + (2(0) - 2(0)(1))(1) = 1$
2	$y_n = 1 + (2(1) - 2(1)(1)) = 1$
3	$y_n = 1 + (2(2) - (2)(2)(1)) = 1$

Therefore, by our estimate, $y(3) \approx 1$.
Now, to find error, we solve our differential equation with initial condition $y(0) = 1$

$\frac{dy}{dx} = 2x(1 - y)$

$\int \frac{dy}{1-y} = \int 2x\, dx$

$-\ln|1 - y| = x^2 + C$

From here, we can immediately claim that this initial condition does not exist, since $ln(0)$ is undefined. Therefore, the Euler's Method error is infinite.
This question gave a great example on how Euler's Method may give a number even though the equation has no solution at the condition.

Recap

This Study Guide went over:

- Solving Differential Equations
- Slope Fields
- Euler’s Method
- Separation of Variables

Now attempt the practice problems on the next page!

Practice Problems

The problems will ramp up in difficulty.

Note for question 1, write the largest open interval where the solution holds.

1) Solve the differential equation $\frac{dy}{dx} = x$ where $x = y = 1$.
2) Solve the differential equation $\frac{dy}{dx} = \frac{y}{ln\, y * \sqrt{1-x^2}}$ where $x = 0$, $y = e$.
3) Solve the differential equation $\frac{dy}{dx} = \frac{tan^4 x}{2}$ where $x = 0,\ y = 1$.
4) Given the logistic equation $\frac{dy}{dx} = 2y - \frac{y^2}{40}$.
    a) What is the carrying capacity/maximum population.
    b) If $y(0) = 5$, solve for y.
    c) When is the population growth increasing the fastest?
    d) If $y(0) = 85$, what is $\lim\limits_{x \to \infty} y(x)$? Does it approach this value from the top or the bottom?
5) Solve each differential equation:
    a) $\frac{dy}{dx} 2x = y$
    b) $dy = dx$
    c) $\frac{dy}{dx} = 0$
6) Solve the differential equation: $\frac{dy}{dx} = \frac{2x}{y}$ Then generate a slope field for any 9 points that lie on the graph.
7) Generate a 9 point slope field for $\frac{dy}{dx} = 2x - y$ and explain why this equation is not separable. Graph the equation going through $(1, 0)$.
8) Suppose $\frac{dy}{dx} = 2x$, where $y(1) = 2$. Using Euler's Method with a step size of 1, find $y(4)$.
9) Suppose $\frac{d^2y}{dx^2} = 2xy$, where $y'(1) = 4$. Using Euler's Method with two equal steps, determine if $y = f(x)$ is increasing at $x = 3$.
10) Suppose $2x - xy = 4y$.
    a) When $x = 4$, find y.
    b) Estimate the value of y when $x = 6$ using Euler's Method. Use two steps of equal length.

Practice Problems (with Graphing Calculator)

A graphing calculator is required for this problem.

1) Suppose $f(x) = x^2 - 3x + x^3$.
   a) Find $f(1)$ and $f'(1)$.
   b) Use a tangent line to approximate the value of $f(1.2)$. State if the approximation is larger or smaller than the actual value of $f(1.2)$.
   c) Use an integral expression to relate $f'(x)$ with $f(x)$ and write an expression for $f(1.2)$.
   d) Use a Right Riemann Sum with 2 subintervals of equal length to approximate $f(1.2)$.
   e) Use Euler's Method with two equal steps to approximate $f(1.2)$. State if the approximation is larger or smaller than the actual value of $f(1.2)$.
   f) Now find $f(1.2)$ using a graphing calculator. Which gave the best approximation? (BONUS: Can you explain why?)

Answer Key

1) $y = \frac{1}{2}x^2 + \frac{1}{2}$, $x \in (-\infty,\ \infty)$
2) $y = e^{\sqrt{2\ arcsin(x) + 1}}$
3) $y = \frac{1}{6}tan^3(x) + \frac{1}{2}x - \frac{1}{2}tan\ x + 1$
4) Answers Below
   a) 80
   b) $y = \frac{80}{1+15e^{-2x}}$
   c) When $y = 40$.
   d) 80 , from the top
5) Answers Below
   a) $y = C\sqrt{x}$
   b) $y = x + C$
   c) $y = C$
6) $y = \pm\sqrt{2x^2 + C}$
7) The equation is not separable since the function on the RHS is multivariable. If anyone can generate slope fields via a computer, please comment a link to your slope field on this question! Thanks for helping us out!
8) $y(4) \approx 14$
9) $y = f(x)$ is increasing at $x = 3$ since $y' = f'(x) > 0$ at $x = 3$.
10) Answers Below
   a) $y = 1$
   b) $y(6) \approx \frac{11}{9}$

Answer Key (Graphing Calculator Problems)

1) Answers Below
   a) $f'(1) = 2$
   b) $f(1.2) \approx -0.6$, this estimate is less than the actual value since f is concave up at $x = 1$.
   c) $f(1.2) = f(1) + \int_{1}^{1.2} f'(x)dx$
   d) $f(1.2) \approx -0.345$
   e) $f(1.2) \approx -0.517$
   f) $f(1.2) = -0.432$ (This is because $f''(1) = 8$ meaning that the function is extremely concave up. This results in anything using a tangent line over an extension to be wildly off, and so the Riemann sum is the best here)