<u>Continuity</u>

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<u>Theorems</u>

The Intermediate Value Theorem: If $f(x)$ is continuous over an interval $[a,\ b]$, and $f(a)$ and $f(b)$ are defined as real numbers, then:

I. Every value between $f(a)$ and $f(b)$ exists.
II. For some $c \in [a,\ b]$, $f(c) \in [f(a),\ f(b)]$

Definition of continuity: *If a function is continuous at c:*

I. $f(c)$ exists and is a real number
II. $\lim_{x \to c} f(x)$ exists and is a real number.
III. $\lim_{x \to c} f(x) = f(c)$

A function is continuous over a domain $x \in [a,\ b]$ if it is continuous at the endpoints, and continuous over the subdomain $x \in (a,\ b)$.

Examples

Example 1: State whether $f(x) = |x - 1| + 2x$ is continuous at 1.

Solution 1: Using the definition of continuity:

I. $f(1) = 2$

II. $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = L = 2$.

III. $\lim_{x \to 1} f(x) = f(1) = 2$

Thus, the function is continuous at 1.

Example 2: Prove that $f(x) = x^3 - 3x^2 + 1$ has at least one real root and at most three real roots.

Solution 2: This is a basic use-case of the Intermediate Value Theorem. First, we must show that the function has at least one real root, which is a value of x that makes $f(x) = 0$.

This prompts us to find an a and b that allows us to apply the IVT.

$f(-1) =- 1 - 3 + 1 =- 3$

$f(3) = 27 - 27 + 1 = 1$.

Also, since f is a polynomial, it is continuous everywhere, so it is continuous over $x \in [-1, 3]$.

Now, we apply the IVT.

Since f is continuous over $x \in [-1, 3]$, and $f(-1) =- 3$, and $f(3) = 1$, by the Intermediate Value Theorem, every value between -3 and 1 must exist, so there must be at least one value c over $[a, b]$ such that $f(c) = 0$. Thus, f has at least one real root.

Now, we can claim that f has a maximum of 3 roots since by the Fundamental Theorem of Algebra, an nth degree polynomial has at most n real roots. Since f is a third degree polynomial, it must have at most 3 real roots.

Example 3: Prove that $f(x) = x^4$ is not one-to-one.

Solution 3: A function is one-to-one if for every x there exists a unique $f(x) = c$, and for every $f(x) = c$, there exists only one x in the domain of f that allows $f(x) = c$.

To show if a function is one-to-one simply take the parent function (which is normal defined as one-to-one or not one to one), and then modify it until it matches $f(x)$. If you want to show if a function is not one-to-one, find a counterexample.

In this case, a counterexample is quite easy.

$f(-1) = 1$

$f(1) = 1$

Since there are two values of x that allow $f(x) = 1$, the function f is not one-to-one.

Recap

This Study Guide went over:

- The definition of continuity

- The Intermediate Value Theorem
- One-to-One Functions

If you need detailed help, take a look at this video!

Now attempt the practice problems on the next page!

Practice Problems

1) For each problem, show that f is continuous at the given value c. If not, explain why, and find the discontinuity. If the function has a hole, patch the hole by writing f as a piecewise function.
   a) $f(x) = x^2,\ x = 2$
   b) $f(x) = sin\, x,\ x = 1$
   c) $f(x) = ln\, x,\ x = 0$
   d) $f(x) = \frac{sin\, x}{x},\ x = 0$
2) For each problem, prove that the function has at least one real root.
   a) $f(x) = x^2$
   b) $f(x) = e^{2x} - 3x$
3) Show that $f(x) = e^x$ and $g(x) = sin\, x$ intersect at least once.
4) Prove that for two unique c, $f(c) = 7$, if $f(2) = 9$, $f(3) = 4$, and $f(5) = 11$, If its not possible, state why.
5) Prove that $f(x) = x^3 + 1$ is one-to-one.

Answer Key

1) Answer Key
    a) Continuous
    b) Continuous
    c) Infinite Discontinuity
    d) Hole, Patch with $f(0) = 1$
2) Refer to Example 2 to get an idea on how to do these types of problems!
3) Essentially you are solving $e^x = sin\, x$, meaning that $e^x - sin\, x = 0$. Now, you've written this problem to show that $e^x - sin\, x = 0$ has at least one real solution, which is like Example 2!
4) f was not stated to be continuous, and therefore we can not claim anything.
5) If $x_1 \neq x_2$ then $x^3{}_1 \neq x^3{}_2$, and $x^3{}_1 + 1 \neq x^3{}_2 + 1$.