Taylor and Maclaurin Series

(Feel free to comment, markup, and chat on this study guide.)

Theorems

The general form of a Taylor Series: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$. If $c = 0$, this series is a Maclaurin Series.

The Taylor Coefficient: $a_n = \frac{f^{(n)}(c)}{n!}$

The Alternating Series Error Bound: $|S - S_n| < a_{n+1}$ (this is an upper bound to the error)

The Lagrange Error Bound/Taylor's Theorem: $|S - S_n| < \frac{M}{(n+1)!}(x-c)^{n+1}$, where $M \geq max(f^{(n+1)}(z))$ for $z \in [c,\ x]$. (this is also an upper bound to the error)

Common Maclaurin Series:

$$sin\ x = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1}}{(2n+1)!}$$

$$cos\ x = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n}}{(2n)!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$arctan\ x = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1}}{(2n+1)}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{n+1}}{n+1}$$

$$(1+x)^k = \sum_{n=0}^{\infty} \frac{k!}{n!(n-k)!} x^n = 1 + kx + \frac{k(k-1)}{2}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + ...$$

Examples

Example 1: Write the Maclaurin Series for $f(x) = sin(2x)$. Then find the 22nd derivative of f evaluated at 0.

Solution 1: We know that the Maclaurin Series for $sin\, x$ is $\sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1}}{(2n+1)!}$. Simply replacing the x with $2x$ should do the trick, and we'll get $\sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}$. Now, to find $f^{(22)}(0)$, we simply use the coefficient rule, $a_n = \frac{f^{(n)}(c)}{n!}$. We are looking for a_{22}, this should be the term that matches up with x^{22}. However, note that in the Maclaurin Series for $sin(2x)$ (and $sin\, x$), all the terms that occur in the series are odd. Therefore, there is no x^{22} term, meaning that $a_{22} = 0$, implying that $f^{(22)}(0) = 0$.

Example 2: Determine the radius and interval of convergence for the Taylor Series:
$f(x) = \sum_{n=0}^{\infty} \frac{(n+1)!}{2^n n!}(x-3)^n$.

Solution 2: To begin, let's use the ratio test.

$\lim_{n\to\infty} \left| \frac{(n+2)!\,(x-3)^{n+1}}{2^{n+1}(n+1)!} * \frac{n!*2^n}{(n+1)!\,(x-3)^n} \right| < 1$.

$\lim_{n\to\infty} \left| \frac{(n+2)}{2} * \frac{1}{(n+1)}(x-3) \right| < 1$

$(x-3) \lim_{n\to\infty} \left| \frac{n+2}{n+1} \right| < 2$

$(x-3) < 2$

$1 < x < 5$.

The radius of convergence tells us the distance from the center (3) to one of the endpoints of the interval. Looking at our rough interval, we can see that the radius $r = 2$.

Now to determine our final interval of convergence, we test the endpoints of our rough interval.

$f(1) = \sum_{n=0}^{\infty} \frac{(n+1)!}{2^n n!}(1-3)^n = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!}{n!} = \sum_{n=0}^{\infty}(-1)^n (n+1)$. This diverges by the *nth* term test.

Similarly, $f(5) = \sum_{n=0}^{\infty} \frac{(n+1)!}{2^n n!}(5-3)^n = \sum_{n=0}^{\infty} \frac{(n+1)!}{n!} = \sum_{n=0}^{\infty}(n+1)$. This also diverges by the *nth* term test.

Thus the interval of convergence is $1 < x < 5$.

Example 3: Use a third-degree Maclaurin Polynomial to estimate $f(1)$ for $f(x) = e^x$. Use the lagrange error bound to find an upper bound to the error in your estimate.

The third degree Maclaurin Polynomial for e^x is $P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$.

$f(1) \approx P_3(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} = \frac{8}{3}$.

The Lagrange Error Bound states that $|S - S_n| < \frac{M}{(n+1)!}(x-c)^{n+1}$. Since $n = 3$, $|S - S_3| < \frac{M}{(4)!}(1-0)^4$. Now we need to determine M.
Making appropriate substitutions:
$M \geq max(f^{(4)}(z))$ for $z \in [0, 1]$. (Remember that M can NOT be significantly larger than the actual value)
The fourth derivative of e^x is e^x, and this attains a maximum at $x = 1$. Thus, we will let $M = 3$ since we know $3 > e^1$.
Therefore, $|S - S_3| < \frac{3}{(4)!}(1-0)^4 = \frac{1}{8}$, which is our upper bound.
Our actual error is 0.0516 so our approximation was actually quite good, and below the upper bound.

Recap

This Study Guide went over:

- Writing Taylor Series
- Radius and Interval of Convergence
- Error

Video on Error: Video (Part 1)
Video on Maclaurin Series: Video
Now attempt the practice problems on the next page!

Practice Problems

1) Write Maclaurin Series for each function centered at the appropriate value.
   a) $f(x) = e^{-x}$
   b) $f(x) = ln\ (1 + 2x)$
   c) $f(x) = arctan\ x + arctan\ 4x$
   d) $f(x) = sin(x + \frac{\pi}{4})$
   e) $f(x) = \frac{7}{4-8x^2}$
   f) $f(x) = ln(\frac{1+x}{1-x})$
2) Write Taylor Polynomials of given degree with the given center.
   a) $f(x) = sin(x),\ c = 1,\ n = 2$
   b) $f(x) = e^x sin\ x,\ c = 0,\ n = 3$
   c) $f(x) = ln(1 + x),\ c = -2,\ n = 2$
3) Find the interval of convergence of the Maclaurin Series for $ln(1 + x)$.
4) Suppose $S = \sum_{n=0}^{\infty} 6(x)^n$. When $S = 9$, the sum is changing at a rate of $\frac{1}{2}$ units per second. Find the rate of change of x at this moment.
5) The area enclosed by $x = 0,\ x = 1,\ and\ f(x) = sin(x^2)$ is revolved about the $x - axis$. Estimate the volume of this solid with an error less than $\frac{1}{1000}$.
6) Suppose $\frac{dy}{dx} = xy$, and $y(0) = 1$. Using a Maclaurin Series, approximate $y(1)$ with an error less than $\frac{1}{100}$.
7) Solve the equation $x^x - x^{x-1} - x^{x-2} - x^{x-3} - ... = 0$ over the real numbers.

Answer Key

1) Answers Below
   a) $e^x = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$
   b) $\sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{n+1}}{n+1}$
   c) $\sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1}}{(2n+1)} + \sum_{n=0}^{\infty} \frac{(-1)^n (4x)^{2n+1}}{(2n+1)}$
   d) $\frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1}}{(2n+1)!} + \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n}}{(2n)!}$
   e) $\frac{7}{4} \sum_{n=0}^{\infty} (2x^2)^n$
   f) $\sum_{n=0}^{\infty} \frac{(-1)^n (x)^{n+1}}{n+1} - \sum_{n=0}^{\infty} \frac{(-1)(x)^{n+1}}{n+1}$
2) Answers Below
   a) $sin\ 1 + cos\ 1(x-1) - \frac{sin\ 1}{2!}(x-1)^2$
   b) $x + x^2 + \frac{x^3}{3}$
   c) $ln(-1)$ is not real.
3) $-1 < x \leq 1$
4) $\frac{dx}{dt} = \frac{1}{27}$
5) $\frac{\pi}{2} - \frac{\pi}{2}(1 - \frac{4}{10} + \frac{16}{216} - \frac{64}{9360})$
6) $\frac{633}{384}$
7) $x = 2$