Fundamentals of Planetary Science Homework 3

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1 The raw materials to form planets and how do we know them

1.1

- Metal: Fe
- Rock: $MgSiO_3$, Mg_2SiO_4
- Ice: H_2O, CH_4, NH_3
- Gas: H_2, He

By abundance measurement using spectral lines and the analysis of CI Chondrite.

1.2 Terrestrial planets

Rock and mental, terrestrial planets have smaller radius but larger mass. They are high density.

1.3 Gas planets

Gas, H_2 and He. Having the largest radius and larger distanse from Solar.

1.4 Ice planets

Ice, H_2O, CH_4, NH_3. Having neutral density.

2 The relation between gravitational monemt J_2 and the moment of inertia

$$\begin{aligned}
J_2 &= -\frac{1}{Ma^2}\int r'^2 P_2(\cos\theta)\rho(r')\,d^3r' = -\frac{1}{Ma^2}\int r'^2\frac{1}{2}(3\cos^2\theta'-1)\rho(r')\,d^3r' \\
&= -\frac{1}{Ma^2}\int\left(\frac{3z^2}{2}-\frac{x^2+y^2+z^2}{2}\right)\rho(r')\,d^3r' \\
&= -\frac{1}{Ma^2}\int\left(\frac{z^2+x^2}{2}+\frac{z^2+y^2}{2}-\left(x^2+y^2\right)\right)\rho(r')\,d^3r' \\
&= -\frac{1}{Ma^2}\left(\frac{B}{2}+\frac{A}{2}-C\right)
\end{aligned}$$

$$Ma^2J_2 = C-\frac{1}{2}(A+B) = C-A$$

C implies the moment of inertia about the z-axis, which is the largest moment of inertia.

A implies the moment of inertia about the x (or y) -axis, which is the smallest moment of inertia.

C-A implies the difference between these two moments of inertia, implying the oblateness of the planet.

Considering the $q \equiv \frac{\omega^2a^3}{GM}$, $J_2 = \frac{q}{2}$, a more homogeneous planet have a larger J_2.

3 Using a simple two-layer interior model determine the relative core radius od the Moon and Mercury

$$\rho = \rho_c \alpha^3 + \rho_m (1-\alpha^3), \quad 1 = \frac{\rho_c}{\rho}\alpha^3 + \frac{\rho_m}{\rho}(1-\alpha^3), \quad \frac{\rho_c}{\rho}\alpha^3 = 1 - \frac{\rho_m}{\rho}(1-\alpha^3)$$

$$\begin{aligned}\tilde{C} &= \frac{2}{5}\left(\frac{\rho_c}{\rho}\alpha^5 + \frac{\rho_m}{\rho}(1-\alpha^5)\right) = \frac{2}{5}\left(\left(1 - \frac{\rho_m}{\rho}(1-\alpha^3)\right)\alpha^2 + \frac{\rho_m}{\rho}(1-\alpha^5)\right) \\ &= \frac{2}{5}\left(\alpha^2 - \frac{\rho_m}{\rho}\alpha^2 + \frac{\rho_m}{\rho}\right)\end{aligned}$$

$$\alpha = \left(\frac{5\tilde{C}}{2} - \frac{\rho_m}{\rho}\right)^{\frac{1}{2}}\left(1 - \frac{\rho_m}{\rho}\right)^{-\frac{1}{2}}$$

	Mean density (g/cm^3)	**Moment of inertia**	**Mantle density (g/cm^3)**	**Core radius (rel.)**
Moon	3.344	0.395	3.3	0.2236
Mercury	5.429	0.349	3.5	0.8007

Table 1: Physical parameters for the Moon and Mercury

4 Derive the density distribution of the planet in the hydrostatic equilibrium, assuming the planetary interior follow the equation of polytrope $P = K\rho^2$

$$P = K\rho^2, \quad \frac{dP}{dr} = 2K\rho(r)\frac{d\rho}{dr}\, and \quad \frac{dP}{dr} = -\rho(r)g(r)$$

$$2K\frac{d\rho}{dr} = -g(r) = -\frac{G}{r^2}\int_0^r 4\pi\rho(x)x^2\,dx$$

$$k^2 \equiv \frac{2\pi G}{G}, \quad \frac{d}{dr}(r^2\frac{d\rho}{dr}) = -k^2 r^2 \rho$$

$$2r\frac{d\rho}{dr} + r^2\frac{d^2\rho}{dr^2} + k^2r^2\rho = 0, \quad r\frac{d^2\rho}{dr^2} + 2\frac{d\rho}{dr} + k^2 r\rho = 0$$

$$\begin{aligned}0 &= \mathcal{L}(r\rho'') + 2\mathcal{L}(\rho') + k^2\mathcal{L}(r\rho) = -\mathcal{L}'(\rho'') + 2\mathcal{L}(\rho') - k^2\mathcal{L}'(r\rho) \\ &= -(s^2P - s\rho_0 - \rho_0')' + 2(sP - \rho_0) - k^2P' = -2sP - s^2P' + \rho_0 + 2sP - 2\rho_0 - k^2P'\end{aligned}$$

$$-(s^2+k^2)P' = \rho_0$$

$$\frac{dP}{ds} = -\frac{\rho_0}{s^2+k^2}, \quad P = -\frac{\rho_0}{k}\arctan\frac{P}{k} + c$$

$$\rho = \mathcal{L}^{-1}(P) = -\rho_0 \left(\frac{\pi}{2}\delta(kr) - \frac{sinkr}{kr} + c\delta(r) \right)$$

when $r = R$, $\rho_R = 0$, $kR = \pi, R = \sqrt{\frac{\pi K}{2G}}$

$$M = \int_0^R \rho_r \cdot 4\pi r^2 \, dr = \int_0^R \rho_0 \frac{\sin kr}{kr} \cdot 4\pi r^2 \, dr = \frac{4\pi^2 \rho_0}{k^3} = \frac{4\pi^2 \rho_0 R^3}{\pi^3}$$

$$\rho_0 = \frac{M\pi}{4R^3}, \quad \rho_r = \frac{M\pi}{4R^3}\frac{\sin kr}{kr}$$

$$\begin{aligned} I &= \frac{2}{3}\int r^2 \, dm = \frac{2}{3}\int_0^R r^2 \rho_r \cdot 4\pi r^2 \, dr \\ &= \frac{2\pi^2 M}{3kR^3}\int_0^R \sin kr \cdot r^3 \, dr = \frac{2\pi^2 M}{3k^5 R^3}\int_0^\pi \theta^3 \sin\theta \, d\theta \\ &= \frac{2MR^2}{3\pi^3}(\pi^3 - 6\pi) \end{aligned}$$

$\frac{I}{MR^2} = \frac{2}{3\pi^3}(\pi^3 - 6\pi) = 0.261 > 0.25(Jupiter) > 0.24(Saturn)$

Jupiter and Saturn have a dense core.

5 Compare the estimated heat flux through conducting alone

$$\begin{aligned} &\Delta T = 3000K - 300K = 2700K, \quad \Delta z = 2.9 \times 10^6 m \\ &q = -k\frac{\Delta T}{\Delta z} = -3.724 \times 10^{-3} W m^{-2} \ll -90 \times 10^{-3} W m^{-2} \end{aligned}$$

Convection takes place to transport internal heat out.

6 Rayleigh number

$$R_a = \frac{g\alpha \Delta T D^3}{\kappa\nu}, \quad F_B = V\rho g\alpha\Delta T$$

The Rayleigh number represents the ratio of buoyancy force to the resisting effects of viscosity and thermal conduction. A higher R_a indicates that buoyancy forces dominate, making convection more likely, whereas a lower R_a suggests that viscous and diffusive effects suppress fluid motion, favoring heat transfer by conduction.

	g (m/s^2)	α (K^{-1})	ΔT (K)	D (km)	κ (m^2/s)	ν (m^2/s)	R_a
Mantle	9.8	3×10^{-5}	1000	2500	10^{-4}	10^{17}	4.59×10^5
Outer Core	7.7	3×10^{-5}	2000	$5150 - 2900$	10^{-4}	10^{-6}	5.26×10^{28}

Table 2: Physical parameters for the mantle and outer core (transposed)

The Rayleigh numbers for both Earth's mantle and outer core are significantly higher than the critical value of $R_{ac} = 1700$, indicating that convection is not only possible but is a dominant mechanism in both regions.