**Fundamentals of Planetary Science Homework 2**

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# Kepler's Laws Derived from Newton's Laws

Starting from the Lagrangian formalism:

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2$$

$$V = -\frac{GMm}{r}$$

$$\mathcal{L} = T - V = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{GMm}{r}$$

Applying the Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\partial \mathcal{L}}{\partial r}$$

$$m\ddot{r} = mr\dot{\theta}^2 - \frac{GMm}{r^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial \theta}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\theta}) = 0 \Rightarrow mr^2\dot{\theta} = L \quad \text{(Conservation of angular momentum)}$$

Hamiltonian:

$$\begin{aligned} H &= \sum \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L} \\ &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{GMm}{r} \\ &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\frac{L^2}{m^2r^4} - \frac{GMm}{r} \\ &= \frac{1}{2}m\dot{r}^2 + \left(\frac{L^2}{2mr^2} - \frac{GMm}{r}\right) = T + U_{\text{eff}} \end{aligned}$$

Using energy conservation $E = H$:

$$\frac{dr}{dt} = \sqrt{\frac{2}{m}}\sqrt{E-U}$$

$$dt = \frac{d\theta}{\dot{\theta}} = \frac{mr^2 d\theta}{L}$$

$$\frac{dr}{\sqrt{\frac{2}{m}(E-U_{\text{eff}})}} = \frac{mr^2 d\theta}{L}$$

Transforming variables:

$$
\begin{aligned}
d\theta &= \frac{L}{m}\frac{\frac{dr}{r^2}}{\sqrt{\frac{2E}{m} - \frac{L^2}{m^2r^2} + \frac{2GM}{r}}} \\
&= \frac{L}{m}\frac{-d\frac{1}{r}}{\sqrt{-\frac{L^2}{m^2}\left(\frac{1}{r} - \frac{2GMm^2}{L^2}\frac{1}{r}\right) + \frac{2E}{m}}} \\
&= \frac{L}{m}\frac{-d\frac{1}{r}}{\sqrt{-\frac{L^2}{m^2}\left(\frac{1}{r} - \frac{GMm^2}{L^2}\right)^2 + \frac{G^2M^2m^4}{L^4} + \frac{2E}{m}}} \\
&= \frac{-d\frac{1}{r}}{\sqrt{-(\frac{1}{r} - \frac{GMm^2}{L^2})^2 + \frac{G^2M^2m^4}{L^4} + \frac{2Em}{L^2}}} \\
&= \frac{-du}{\sqrt{A^2 - u^2}}
\end{aligned}
$$

Let $u = A\cos\varphi$, we get the conic solution:

$$d\theta = \frac{\sin\varphi d\varphi}{\sin\varphi}$$

$$\theta = arc\cos\varphi$$

$$\cos\theta = \cos\varphi = \frac{\frac{1}{r} - \frac{GMm^2}{L^2}}{\sqrt{\frac{G^2M^2m^4}{L^4} + \frac{2Em}{L^2}}}$$

$$
\begin{aligned}
r &= \frac{1}{\frac{GMm^2}{L^2} + \sqrt{\frac{G^2M^2m^4}{L^4} + \frac{2Em}{L^2}}\cos\theta} \\
&= \frac{\frac{L^2}{GMm^2}}{1 + \sqrt{1 + \frac{2EL^2}{G^2M^2m^3}}\cos\theta} \qquad \text{(First Law)} \\
&= \frac{p}{1 + e\cos\theta}
\end{aligned}
$$

For ellipse:

$$\frac{p}{1+e} = a - c$$

$$\frac{p}{1-e} = a + c$$

$$2a = \frac{p}{1+e} + \frac{p}{1-e} = p\frac{2}{1-e^2} \Rightarrow a = \frac{p}{1-e^2}$$

$$b = \sqrt{a^2 - a^2e^2} = a\sqrt{1-e^2} = \frac{p}{\sqrt{1-e^2}}$$

For period:

$$P = \frac{L^2}{GMm^2}$$

$$e = \sqrt{1 + \frac{2EL^2}{G^2M^2m^3}}$$

$$mr^2\dot{\theta} = L$$

$$\frac{1}{2}r^2\dot{\theta} = \frac{L}{2m} = \frac{dS}{dt} = \frac{\pi ab}{T} \quad \text{(Second law)}$$

$$T = \frac{\pi ab \cdot 2m}{L} = \frac{2\pi m a^2\sqrt{1-e^2}}{L} = \frac{2\pi m \cdot a^{3/2}}{\sqrt{GM} \cdot m}$$

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM} \quad \text{(Third Law)}$$