Fundamentals of Signal Processing and Data Analysis
Homework 2

12311405 纪浩正

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1 A Digital Filter Given by the Difference Equation

We are given a digital filter described by:

$$y[n] = x[n] + 0.1\,y[n-1], \quad \text{with } y[-1] = 0$$

We are to find the impulse response and the step response of this system.

1.1 Recurrence Expansion

$$y[n] = x[n] + 0.1\,y[n-1] \tag{1}$$
$$y[n-1] = x[n-1] + 0.1\,y[n-2] \tag{2}$$

Subtracting (2) from (1):

$$y[n] - y[n-1] = x[n] - x[n-1] + 0.1\,(y[n-1] - y[n-2]) \tag{3}$$

1.2 The Impulse Response

Define the unit impulse:

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Substitute $x[n] = \delta[n]$ into the difference equation:

$$y[n] = \delta[n] + 0.1\,y[n-1]$$

When $n = 0$:

$$y[0] = 1$$

When $n > 0$, $\delta[n] = 0$, so:

$$y[n] = 0.1\,y[n-1] = 0.1^n$$

For $n < 0$, from the initial condition $y[-1] = 0$, and backward recursion, we find that:

$$y[n] = 0 \quad \text{for all } n < 0$$

Thus, the impulse response is:

$$h[n] = y[n] = \begin{cases} 0.1^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

1.3 The Step Response

1st Method (Recursive)

Define the step function:

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Using the difference equation with $x[n] = u[n]$:

$$\begin{aligned} y[0] &= u[0] + 0.1\,y[-1] = 1 \\ y[1] &= u[1] + 0.1\,y[0] = 1 + 0.1 = 1.1 \\ y[1] - y[0] &= 0.1 \end{aligned}$$

For $n > 1$, since $x[n] = x[n-1] = 1$, equation (3) becomes:

$$y[n] - y[n-1] = 0.1\,(y[n-1] - y[n-2])$$

This is a geometric progression:

$$y[n] - y[n-1] = 0.1^n, \quad n \geq 1$$

So the total change from $y[0]$ is:

$$y[n] - y[0] = \sum_{k=1}^{n} 0.1^k = \frac{1 - 0.1^n}{9}$$

Thus:

$$y[n] = y[0] + \frac{1 - 0.1^n}{9} = 1 + \frac{1 - 0.1^n}{9} = \frac{10 - 0.1^n}{9}$$

Therefore, the step response is:

$$y[n] = \begin{cases} \frac{10-0.1^n}{9}, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

2nd Method (Convolution)

Use the impulse response $h[n] = 0.1^n$ for $n \geq 0$, and zero otherwise.

By convolution:

$$\begin{aligned} y[n] = u[n] * h[n] &= \sum_{i=-\infty}^{\infty} u[i]\, h[n-i] \\ &= \sum_{i=0}^{n} h[n-i] = \sum_{k=0}^{n} 0.1^k \\ &= \frac{1 - 0.1^{n+1}}{1 - 0.1} = \frac{10 - 0.1^{n+1}}{9} \end{aligned}$$

Note: This matches the previous result with a one-index shift.

Final result:

$$y[n] = \begin{cases} \frac{10-0.1^{n+1}}{9}, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

2 Proof That Two Cascaded LTI Systems Yield the Same Output

We aim to prove that cascading two LTI systems with impulse responses $h_1[n]$ and $h_2[n]$ produces the same output regardless of their order:

$$(x[n] * h_1[n]) * h_2[n] = (x[n] * h_2[n]) * h_1[n]$$

Step 1: Use Commutativity of Convolution

$$(x[n] * h_1[n]) * h_2[n] = (h_1[n] * x[n]) * h_2[n]$$

Step 2: Prove Associativity of Convolution

We now show that:

$$(h_1[n] * x[n]) * h_2[n] = h_1[n] * (x[n] * h_2[n])$$

Compute the left-hand side:

$$\begin{aligned} (h_1[n] * x[n]) * h_2[n] &= \left(\sum_{\tau=-\infty}^{\infty} h_1[\tau]\, x[n-\tau] \right) * h_2[n] \\ &= \sum_{t=-\infty}^{\infty} \left(\sum_{\tau=-\infty}^{\infty} h_1[\tau]\, x[t-\tau] \right) h_2[n-t] \\ &= \sum_{\tau=-\infty}^{\infty} h_1[\tau] \sum_{t=-\infty}^{\infty} x[t-\tau]\, h_2[n-t] \end{aligned}$$

Make a change of variable: let $\eta = t - \tau$, so $t = \eta + \tau$:

$$\begin{aligned}
&= \sum_{\tau=-\infty}^{\infty} h_1[\tau] \sum_{\eta=-\infty}^{\infty} x[\eta]\, h_2[n - (\eta + \tau)] \\
&= \sum_{\tau=-\infty}^{\infty} h_1[\tau] \left(\sum_{\eta=-\infty}^{\infty} x[\eta]\, h_2[n - \tau - \eta] \right) \\
&= \sum_{\tau=-\infty}^{\infty} h_1[\tau] \, (x[n-\tau] * h_2[n-\tau]) \\
&= h_1[n] * (x[n] * h_2[n])
\end{aligned}$$

Step 3: Use Commutativity Again

$$h_1[n] * (x[n] * h_2[n]) = (x[n] * h_2[n]) * h_1[n]$$

Conclusion:

Combining all steps:

$$(x[n] * h_1[n]) * h_2[n] = (x[n] * h_2[n]) * h_1[n]$$

Therefore, the outputs of the two LTI system configurations are the same, confirming that cascading LTI systems is order-independent due to the commutativity and associativity of convolution.