Fundamentals of Space Science and Technology Homework 1

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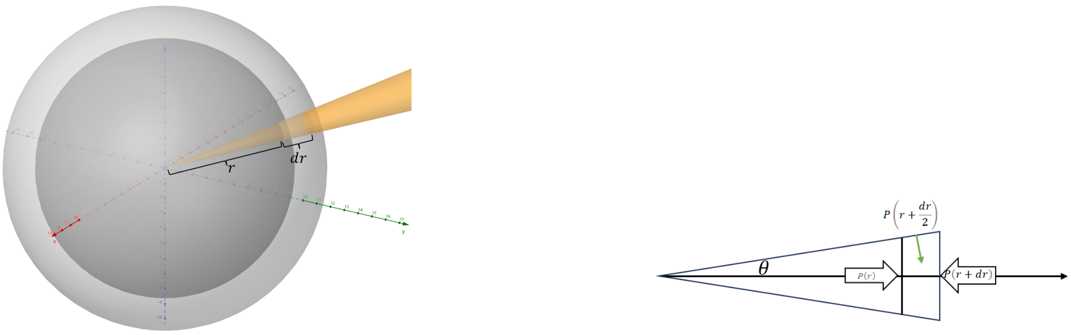
1 Derive the Parker solar wind model and discuss why only the supersonic solution with $C = -3$ is the correct solution for the solar wind.

1.1 Following the key steps to drive the Parker solar Wind model equation.



(a) Spherical coordinate

(b) Pressure distribution

Consider a conical element of the solar wind with half-angle θ. The solid angle subtended by the cone is

$$\Omega = 2\pi\left(1-\cos\theta\right) \approx \pi\theta^2, \tag{1}$$

where we used the small-angle approximations

$$\cos\theta \approx 1-\frac{\theta^2}{2}, \quad \sin\theta \approx \theta.$$

The cross-sectional area at a distance r is therefore

$$S(r) = r^2\Omega = \pi r^2\theta^2. \tag{2}$$

The lateral surface area of the conical element between r and $r+\mathrm{d}r$ is approximated by

$$S_l = 2\pi\left(\frac{(r+\mathrm{d}r)\theta + r\theta}{2}\right)\mathrm{d}r \approx 2\pi r\theta\,\mathrm{d}r, \tag{3}$$

neglecting higher-order terms in $\mathrm{d}r$.

Applying the momentum theorem to the conical element yields

$$P_{(r)}S_{(r)} - P_{(r+\mathrm{d}r)}S_{(r+\mathrm{d}r)} + P_{\left(r+\frac{\mathrm{d}r}{2}\right)}S_l\sin\theta - \int_r^{r+\mathrm{d}r}\frac{GMmn\,S(r)}{r^2}\,\mathrm{d}r = \int_r^{r+\mathrm{d}r} nm\,S(r)\frac{\mathrm{d}v}{\mathrm{d}t}\,\mathrm{d}r. \tag{4}$$

$$\begin{aligned}\text{L.H.S.} &= -\pi\theta^2r^2\,\mathrm{d}P - 2\pi\theta^2 P\,r\,\mathrm{d}r \\ &\quad + \left(P+\frac{\mathrm{d}P}{2}\right)2\pi r\theta^2\,\mathrm{d}r - GMmn\pi\theta^2\,\mathrm{d}r \\ &\approx -\pi\theta^2r^2\,\mathrm{d}P - GMmn\pi\theta^2\,\mathrm{d}r. \end{aligned}\tag{5}$$

$$\text{R.H.S.} = nm\pi r^2\theta^2 \frac{dv}{dt} dr = nm\pi r^2\theta^2 v \frac{dv}{dr} dr \tag{6}$$

Cancelling the common factor $\pi\theta^2$ and rearranging, we obtain:

$$-\frac{\mathrm{d}P}{\mathrm{d}r} - \frac{GMmn}{r^2} = nm\, v \frac{\mathrm{d}v}{\mathrm{d}r}. \tag{7}$$

Dividing by nm, the equation becomes:

$$\frac{1}{nm} \frac{\mathrm{d}P}{\mathrm{d}r} + \frac{GM}{r^2} + v \frac{\mathrm{d}v}{\mathrm{d}r} = 0. \tag{8}$$

For an ionized gas in the solar corona, the ideal gas law is given by

$$P = 2nk_BT. \tag{9}$$

Thus, we can write:

$$\frac{\mathrm{d}(2nk_BT)}{nm\,\mathrm{d}r} + \frac{GM}{r^2} + v \frac{\mathrm{d}v}{\mathrm{d}r} = 0. \tag{10}$$

Assuming the temperature T is invariant and using the mass conservation equation

$$4\pi r^2 nv = 4\pi R_0^2 n_0 v_0, \tag{11}$$

we obtain the density profile:

$$n(r) = n_0 \frac{R_0^2 v_0}{r^2 v}. \tag{12}$$

$$\frac{2k_BT\mathrm{d}n}{nm\mathrm{d}r} + \frac{GM}{r^2} + v\frac{\mathrm{d}v}{\mathrm{d}r} = 0 \tag{13}$$

$$\frac{2k_BTvr^2}{n_0v_0R_0^2m\mathrm{d}r} n_0v_0R_0^2 \left(\frac{-\mathrm{d}v}{v^2r^2} + \frac{-2\mathrm{d}r}{vr^3} \right) + \frac{GM}{r^2} + v\frac{\mathrm{d}v}{\mathrm{d}r} = 0 \tag{14}$$

$$\frac{2k_BTvr^2}{m\mathrm{d}r} \left(\frac{-\mathrm{d}v}{v^2r^2} + \frac{-2\mathrm{d}r}{vr^3} \right) + \frac{GM}{r^2} + v\frac{\mathrm{d}v}{\mathrm{d}r} = 0 \tag{15}$$

where the critical speed and radius are defined by

$$v_c^2 = \frac{2k_BT}{m}, \quad r_c = \frac{GMm}{4k_BT}. \tag{16}$$

Rearranging, we have:

$$\left(\frac{v^2}{v_c^2} - 1 \right) \frac{\mathrm{d}v}{v} = 2 \left(1 - \frac{r_c}{r} \right) \frac{\mathrm{d}r}{r}. \tag{17}$$

Then, we proceed with the following steps:

$$\left(\frac{e^{2\ln v}}{v_c^2} - 1 \right) \mathrm{d}\ln v = 2 \left(1 - r_c e^{-\ln r} \right) \mathrm{d}\ln r, \tag{18}$$

$$\frac{e^{2\ln v}}{2v_c^2} - \ln v = 2 \left(\ln r + \frac{2r_c}{r} \right) + \text{Const}, \tag{19}$$

$$\frac{v^2}{2v_c^2} - \ln\frac{v}{v_c} = 2 \left(\ln\frac{r}{r_c} + \frac{2r_c}{r} \right) + \frac{C}{2}, \tag{20}$$

$$\frac{v^2}{v_c^2} - \ln\left(\frac{v}{v_c} \right)^2 = 4\ln\frac{r}{r_c} + 4\frac{r_c}{r} + C. \tag{21}$$

1.2 Discuss why only the supersonic solution with $C = -3$ is the correct solution for the solar wind.

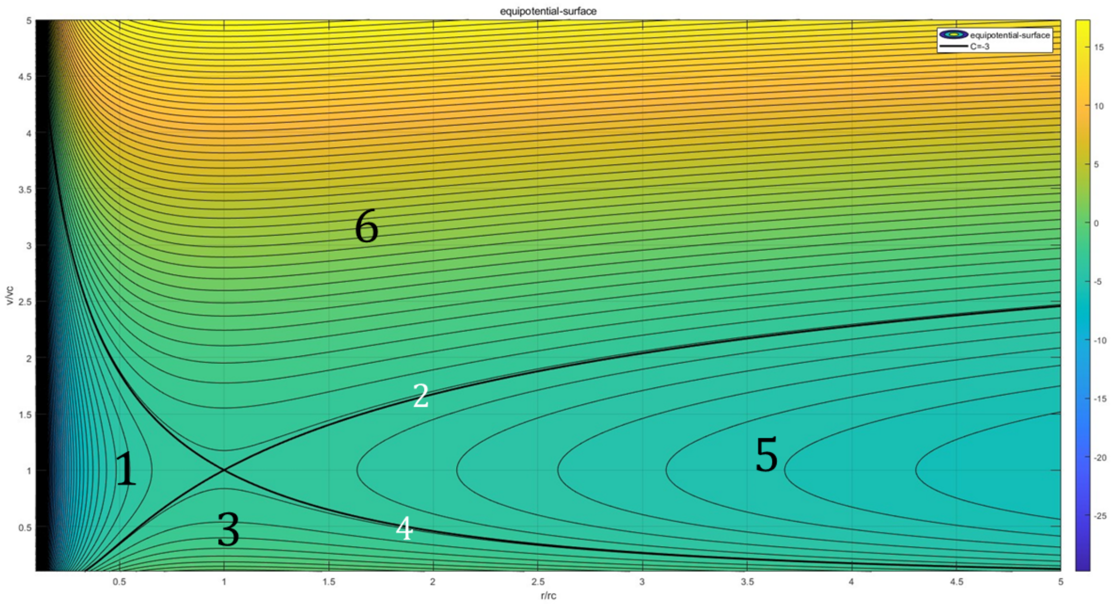


Figure 2: Orbit of Solar Wind

- Section 5 ($C < -3$): The case does not satisfy the initial condition, as the gas was ejected from the corona.
- Sections 1 ($C < -3$), 6 ($C > -3$) and Curve 4 ($C = -3$): As $r \to 0$, $v \to \infty$. The case inconsistent with the physics model, which requires v to be finite.
- For $r \to \infty$, if $v < v_c$, the original equation(21) will be simplify to $-ln\left(\frac{v}{v_c}\right)^2 = 4ln\frac{r}{r_c}$, $v = \frac{v_c r_c^2}{r^2}$, and $n \approx const.$ Region 3 ($C > -3$) satisfies the physical model and represents the solar breeze.
- For $r \to \infty$, if $v > v_c$, the origin equation(21) will be simplify to $\frac{v^2}{v_c^2} = 4ln\frac{r}{r_c}$, $v = 2v_c\left[ln\frac{r}{r_c}\right]^{1/2}$. Curve 2 ($C = -3$) satisfies the physical model and represents the solar wind.

2 Estimate the solar wind velocities and the angles between the solar wind magnetic field line and the planetary orbits using Parker model.

The solar wind velocity is given by:

$$v = 2v_c\left[\ln\left(\frac{r}{r_c}\right)\right]^{1/2} \tag{22}$$

where:

- v is the solar wind velocity (km/s),
- $v_c = \sqrt{\frac{2k_BT}{m}} = 128.433$ km/s, where $T = 10^6$K and $m = 1.674 \times 10^{-26}$kg
- r is the planetary distance from the Sun (km),
- $r_c = \frac{GMm}{4k_BT} = 4.014 \times 10^6$ km (critical distance), where $M = 1.989 \times 10^{30}$ kg.

The angle between the solar wind velocity and planetary orbit is:

$$\theta = \arctan\left(\frac{v_{\text{solar wind}}}{r \cdot \omega_{\text{sun}}}\right) \tag{23}$$

where:

- θ is the angle (degrees),
- $\omega_{\text{sun}} = 2.85 \times 10^{-6}$ rad/s (Sun's angular velocity).

Using the given formulas, we calculate the solar wind velocity and the angle at different planetary distances.

Planet	Distance (km)	Solar Wind Velocity (km/s)	Angle (°)
Mercury	57.9×10^6	419.6	68.53
Earth	149.6×10^6	488.6	48.89
Mars	227.9×10^6	516.2	38.48
Jupiter	778.5×10^6	589.5	14.88
Neptune	4500×10^6	680.7	3.04

Table 1: Solar Wind Velocity and Angle at Different Planetary Orbits

3 Appendix

```
x = linspace(0.1, 5, 1000);
y = linspace(0.1, 5, 1000);

[X, Y] = meshgrid(x, y);
Z = Y.^2 - log(Y.^2) - 4*log(X) - 4./X;

figure;
contourf(X, Y, Z, 100);
title('equipotential-surface');
xlabel('r/rc');
ylabel('v/vc');
colorbar;

hold on;
contour(X, Y, Z, [-3, -3], 'k', 'LineWidth', 2);
legend('equipotential-surface', 'C=-3');
grid on;

%%
vc = 128.433;
rc = 4.014e6;
omega_sun = 2.85e-6;

planets = {'Mercury', 'Earth', 'Mars', 'Jupiter', 'Neptune'};
distances = [57.9e6, 149.6e6, 227.9e6, 778.5e6, 4500e6];
```

```
solar_wind_velocity = zeros(size(distances));
angles = zeros(size(distances));

for i = 1:length(distances)
    r = distances(i);

    v = 2 * vc * sqrt(log(r / rc));
    solar_wind_velocity(i) = v;

    theta = atan(v / (r * omega_sun)) * (180 / pi);
    angles(i) = theta;
end

disp('Planet  | Distance (km)   | Solar Wind Velocity (km/s) | Angle (degrees)');
disp('-----------------------------------------------------');
for i = 1:length(planets)
    fprintf('%-8s | %.1f            | %.1f                     | %.2f\n', ...
        planets{i}, distances(i), solar_wind_velocity(i), angles(i));
end
```