Fundamentals of Space Science and Technology Homework 3

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1 Estimating the Jovian Magnetopause Standoff Distance

Given the Earth's magnetopause standoff distance of 10 Earth radii, Jupiter's dipole moment is 20,000 times larger than Earth's, Jupiter's radius is 11 Earth radii, and Jupiter is 5 times farther from the Sun than Earth, estimate the standoff distance of Jupiter's magnetopause.

Assume that

- the Earth's magnetopause standoff distance is $r_{\mathrm{E}} = 10\, R_E$,
- Jupiter's dipole moment is $m_J = 2 \times 10^4\, m_E$,
- Jupiter's radius is $R_J = 11\, R_E$,
- Jupiter orbits at $d_J = 5\, d_E$ from the Sun.

In general the standoff distance scales as

$$r \;\propto\; \left(\frac{2\, B^2}{\mu_0\, \rho_{\mathrm{sw}}\, v_{\mathrm{sw}}^2}\right)^{1/6} R,$$

so that

$$r_{\mathrm{E}} = \left(\frac{2\, B_E^2}{\mu_0\, \rho_{\mathrm{sw,E}}\, v_{\mathrm{sw,E}}^2}\right)^{1/6} R_E = 10\, R_E,$$

$$r_J = \left(\frac{2\, B_J^2}{\mu_0\, \rho_{\mathrm{sw,J}}\, v_{\mathrm{sw,J}}^2}\right)^{1/6} R_J.$$

Planet	Distance from Sun (km)	Solar Wind Speed (km/s)	Angle (°)
Earth	149.6×10^6	488.6	48.89
Jupiter	778.5×10^6	589.5	14.88

Table 1: Solar wind speed and angle at Earth's and Jupiter's orbits

We note from Table 1 that the solar-wind speed changes very little between Earth and Jupiter, so we set

$$v_{\mathrm{sw,J}} \approx v_{\mathrm{sw,E}}.$$

Using the inverse-square law,

$$\rho_{\mathrm{sw,J}} = \frac{\rho_{\mathrm{sw,E}}}{5^2} = \frac{\rho_{\mathrm{sw,E}}}{25},$$

and since

$$B_J \propto \frac{m_J}{R_J^3} = \frac{2 \times 10^4}{11^3}\, B_E,$$

it follows that

$$r_J = \left(\frac{2\big(2 \times 10^4/11^3\big)^2 B_E^2}{\mu_0\, (1/25)\, \rho_{\mathrm{sw,E}}\, v_{\mathrm{sw,E}}^2}\right)^{1/6} 11\, R_E = 464.16\, R_E \approx 42.2\, R_J.$$

Thus Jupiter's magnetopause standoff distance is approximately $464.16\, R_E$ (or about $42.2\, R_J$).

2 Problem 2: Magnetic Field Line Plotting

(a) Assuming Earth's magnetic field is a dipole, plot the magnetic field lines.

(b) Considering a planar magnetopause model with a mirror surface located 10 Earth radii from Earth's center, model the Earth's magnetic field as the superposition of two mirror dipoles and plot the magnetic field lines. Briefly describe your method and include the calculation code used to obtain your results.

2.1 $1st$ method

$$B_r(r,\theta) = \frac{\mu_0}{4\pi}\frac{2\,m\,\cos\theta}{r^3} = -\frac{\partial U}{\partial r}, \tag{1}$$

$$B_\theta(r,\theta) = \frac{\mu_0}{4\pi}\frac{m\,\sin\theta}{r^3} = -\frac{1}{r}\frac{\partial U}{\partial \theta}, \tag{2}$$

$$U(r,\theta) = \frac{\mu_0}{4\pi}\frac{m\,\cos\theta}{r^2} = \frac{\mu_0}{4\pi}\frac{m\,r\,\cos\theta}{r^3} = \frac{\mu_0}{4\pi}\frac{m\,y}{r^3}. \tag{3}$$

For Earth dipole located at $(x,y)=(0,0)$ the pseudo-potential is

$$U(x,y) = \frac{\mu_0\,m}{4\pi}\left(\frac{y}{r^3}\right)$$

The magnetic field vector follows from the gradient of U:

$$\mathbf{B}(x,y) = \nabla U = \big(-\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y}\big).$$

The magnetic field line satisfies:

$$\frac{dy}{dx} = \frac{2y}{3x} - \frac{x}{3y}$$

For two identical dipoles located at $(x,y)=(-d,0)$ and $(+d,0)$, the pseudo-potential is

$$U(x,y) = \frac{\mu_0\,m}{4\pi}\left(\frac{y}{r_1^3} + \frac{y}{r_2^3}\right), \quad r_1 = \sqrt{(x+d)^2+y^2}, \quad r_2 = \sqrt{(x-d)^2+y^2}.$$

The magnetic field vector follows from the gradient of U:

$$\mathbf{B}(x,y) = \nabla U = \big(-\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y}\big).$$

The magnetic field line satisfies:

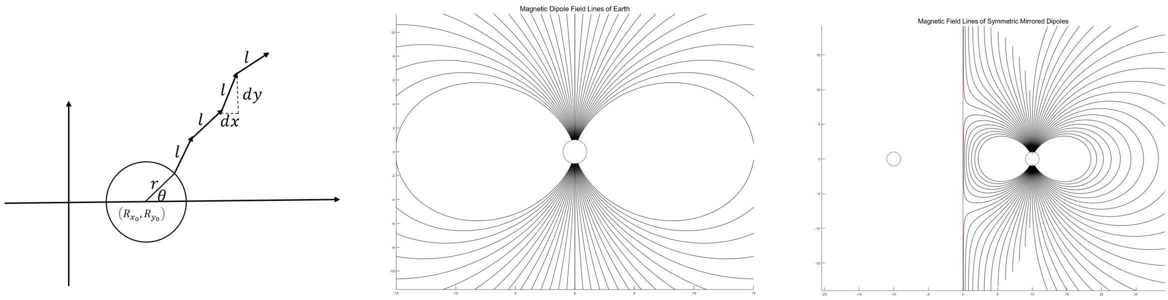
$$\frac{dy}{dx} = \frac{B_y}{B_x} = \frac{\frac{1}{r_1^5}\left(2y^2-(x-10)^2\right) + \frac{1}{r_2^5}\left(2y^2-(x-10)^2\right)}{\frac{3y(x-10)}{r_1^5} + \frac{3y(x-10)}{r_2^5}}$$

We use a recursive algorithm to trace the magnetic field lines, where l is the incremental step size:

$$x_{n+1} = x_n + l\cdot\frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}, \quad y_{n+1} = y_n + l\cdot\frac{\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}.$$

The initial point is set as:

$$x_1 = R_{x_0} + r\cos\theta, \quad y_1 = R_{y_0} + r\sin\theta.$$



(a) Algorithm Diagram **(b)** Dipole Field Lines of Earth **(c)** Symmetric Mirrored Dipoles

Figure 1: Algorithm diagram and magnetic field configurations

2.2 $2nd$ Method: Using the Magnetic Vector Potential

For magnetic field lines described by (x, y), the derivative is given by:

$$\frac{dy}{dx} = \frac{B_y}{B_x}$$

which can be rearranged as:

$$-B_x\,dy + B_y\,dx = 0$$

For magnetic dipole, we can compute the magnetic vector potential $\vec{A}$:

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3} = \frac{\mu_0 m}{4\pi}\frac{\hat{y}\times(x\hat{x}+y\hat{y})}{r^3} = -\frac{\mu_0 m}{4\pi}\frac{x}{r^3}\hat{z}$$

The magnetic field is given by the curl of the vector potential:

$$\vec{B} = \nabla\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & A_z \end{vmatrix} = \frac{\partial A_z}{\partial y}\hat{i} - \frac{\partial A_z}{\partial x}\hat{j}$$

$$B_x = \frac{\partial A_z}{\partial y}, \quad B_y = -\frac{\partial A_z}{\partial x}$$

Substituting into the equation $-B_x\,dy + B_y\,dx = 0$:

$$-\frac{\partial A_z}{\partial y}\,dy - \frac{\partial A_z}{\partial x}\,dx = 0 \quad \Rightarrow \quad dA_z = 0$$

Therefore, the magnetic field lines lie along the contours of constant A_z:

$$A_z = \text{const}$$

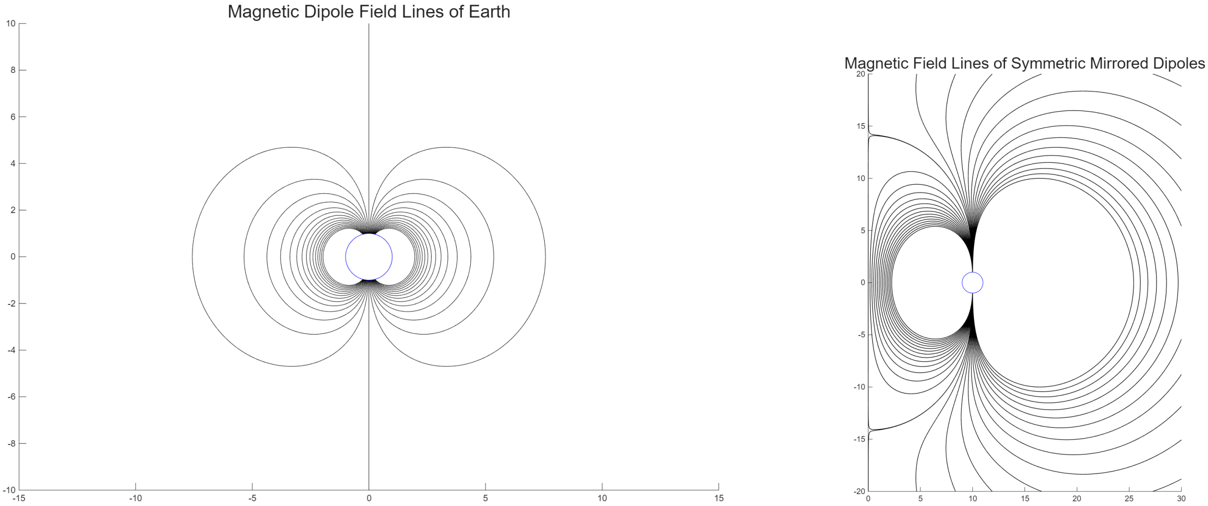
For a single dipole:

$$\frac{x}{r^3} = \text{const}$$

For the mirror identical dipoles located at $(\pm d, 0)$**:**

$$\frac{x+d}{r_1^3} + \frac{x-d}{r_2^3} = \text{const}$$

This expression defines the magnetic field lines for the two-dipole system.



(a) Dipole Field Lines of Earth

(b) Symmetric Mirrored Dipoles

Figure 2: Magnetic field configurations using $2nd$Method

3 Appendix

3.1 $1st$ method

```
xm=15; ym=10; x=linspace(-xm,xm,500); y=linspace(-ym,ym,500);
[X,Y]=meshgrid(x,y); X(X==0)=1e-6; Y(Y==0)=1e-6;

N=5000; xt=zeros(1,N); yt=zeros(1,N); l=0.01;

figure(1); xlim([-xm xm]); ylim([-ym ym]);
rectangle('Position', [0-1, 0-1, 2, 2], 'Curvature', [1, 1], ...
          'EdgeColor', 'b', 'FaceColor', 'none'); % First dipole (blue circle)
hold on;

for jj=75:1:90
    theta=jj*pi/180; xt(1)=cos(theta); yt(1)=sin(theta);
    for ii=1:N-1
        k=2*yt(ii)./3./xt(ii)-xt(ii)/3./yt(ii);
        xt(ii+1)=xt(ii)+l*(1+k.^2)^(-0.5);
        yt(ii+1)=yt(ii)+l*k*(1+k.^2)^(-0.5);
    end
    plot(xt,yt,'k'); plot(-xt,yt,'k');
    plot(xt,-yt,'k'); plot(-xt,-yt,'k');
end
title('Magnetic Dipole Field Lines of Earth','FontSize',20);
```

```
axis equal;

%%
figure(2); xlim([-2*xm 2*xm]); ylim([-ym ym]);
rectangle('Position', [10-1, 0-1, 2, 2], 'Curvature', [1, 1], ...
          'EdgeColor', 'b', 'FaceColor', 'none'); % First dipole (blue
             circle)
hold on;
rectangle('Position', [-10-1, 0-1, 2, 2], 'Curvature', [1, 1], ...
          'EdgeColor', 'b', 'FaceColor', 'none'); % First dipole (blue
             circle)
plot([0; 0], [-2*ym; 2*ym],'r');
for jj=70:1:90
    theta=jj*pi/180; xt(1)=10+cos(theta); yt(1)=sin(theta);
    for ii=1:N-1
        r1=((xt(ii)-10)^2+yt(ii)^2)^0.5;
        r2=((xt(ii)+10)^2+yt(ii)^2)^0.5;
        k=((2*yt(ii)^2-(xt(ii)-10)^2)/r1^5+(2*yt(ii)^2-(xt(ii)+10)^2)/
           r2^5) / (3*yt(ii)*(xt(ii)-10)/r1^5+3*yt(ii)*(xt(ii)+10)/r2
           ^5);
        xt(ii+1)=xt(ii)+1*(1+k^2)^(-0.5);
        yt(ii+1)=yt(ii)+1*k*(1+k^2)^(-0.5);
    end
    plot(xt,yt,'k'); plot(xt,-yt,'k');
end

for jj=90:1:110
    theta=jj*pi/180; xt(1)=10+cos(theta); yt(1)=sin(theta);
    for ii=1:N-1
        r1=((xt(ii)-10)^2+yt(ii)^2)^0.5;
        r2=((xt(ii)+10)^2+yt(ii)^2)^0.5;
        k=((2*yt(ii)^2-(xt(ii)-10)^2)/r1^5+(2*yt(ii)^2-(xt(ii)+10)^2)/
           r2^5) / (3*yt(ii)*(xt(ii)-10)/r1^5+3*yt(ii)*(xt(ii)+10)/r2
           ^5);
        xt(ii+1)=xt(ii)-1*(1+k^2)^(-0.5);
        yt(ii+1)=yt(ii)-1*k*(1+k^2)^(-0.5);
    end
    plot(xt,yt,'k'); plot(xt,-yt,'k');
end
title('Magnetic Field Lines of Symmetric Mirrored Dipoles', 'FontSize',
    20);
axis equal;
```

3.2 *2nd* method

```
    clc;clear;close all;
xm=15; ym=10;
x=linspace(-xm,2*xm,500);
y=linspace(-2*ym,2*ym,500);
[X,Y]=meshgrid(x,y);

% Avoid division by zero at the origin
X(X==0)=1e-6;
Y(Y==0)=1e-6;

%% Figure 1: Magnetic dipole field
figure;
xlim([-xm xm]); ylim([-ym ym]);
```

```
hold on;

% Plot field lines of a single dipole centered at origin
for jj=75:1:105
    theta=jj*pi/180;
    x0=cos(theta);
    y0=sin(theta);
    U0=x0/(x0^2+y0^2)^1.5;  % reference potential for field line
    r=sqrt(X.^2+Y.^2);
    U=X./r.^3;  % magnetic field line equation for dipole
    contour(X,Y,U,[U0 U0],'LineColor','k');
end

% Draw the central dipole as a circle
rectangle('Position', [0-1, 0-1, 2, 2], 'Curvature', [1, 1], ...
    'EdgeColor', 'b', 'FaceColor', 'w');

title('Magnetic Dipole Field Lines of Earth','FontSize',20);
axis equal;
axis([-xm xm -ym ym]);

%% Figure 2: Two symmetric dipoles (mirror image)
figure;
xlim([0 2*xm]); ylim([-2*ym 2*ym]);
hold on;

% Calculate combined potential from dipoles at x = +-10
r1=sqrt((X+10).^2+Y.^2);
r2=sqrt((X-10).^2+Y.^2);
U=(X+10)./r1.^3+(X-10)./r2.^3;

% Contour levels for visualizing field lines
neg_levels=linspace(-1e-2,-1e-6,15);
pos_levels=linspace(1e-6,5e-3,15);
contour(X,Y,U,[neg_levels pos_levels],'LineWidth',0.7,'LineColor','k');

% Draw the Earth dipole as a circle
rectangle('Position', [10-1, 0-1, 2, 2], 'Curvature', [1, 1], ...
          'EdgeColor', 'b', 'FaceColor', 'w');

title('Magnetic Field Lines of Symmetric Mirrored Dipoles', 'FontSize',
    20);
axis equal;
axis([0 2*xm -2*ym 2*ym]);
```