Fundamentals of Signal Processing and Data Analysis
Homework 4

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June 13, 2025

1 Frequency Response of a Rectangular Window Filter

Suppose the impulse response of a digital filter is defined by

$$h(n) = \begin{cases} 1, & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

Find its frequency response:

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n} = \sum_{n=0}^{N-1} e^{-j\omega n} \\ &= \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} \\ &= \frac{e^{-j\omega N/2} \left(e^{j\omega N/2} - e^{-j\omega N/2} \right)}{e^{-j\omega/2} \left(e^{j\omega/2} - e^{-j\omega/2} \right)} \\ &= e^{-j\omega \frac{N-1}{2}} \cdot \frac{\sin(\omega N/2)}{\sin(\omega/2)} \end{aligned}$$

Magnitude response:

$$\left| H(e^{j\omega}) \right| = \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

2 Trigonometric Form of the Fourier Series

Deduce the trigonometric Fourier series of $x(t)$:

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} C_k e^{ik\omega_0 t} \\
&= C_0 + \sum_{k=1}^{\infty} \left(C_k e^{ik\omega_0 t} + C_{-k} e^{-ik\omega_0 t} \right) \\
&= C_0 + \sum_{k=1}^{\infty} \left[(C_k + C_{-k}) \cos(k\omega_0 t) + i \left(C_k - C_{-k} \right) \sin(k\omega_0 t) \right]
\end{aligned}
$$

Since $x(t) \in \mathbb{R}$, it follows:

$$
C_k + C_{-k} \in \mathbb{R}, \quad C_k - C_{-k} \in \mathbb{I}, \quad C_{-k} = \overline{C_k}
$$

Thus:

$$
\begin{aligned}
x(t) &= C_0 + 2 \sum_{k=1}^{\infty} \left(\operatorname{Re} C_k \cos\left(k\omega_0 t\right) + i \operatorname{Im} C_k \sin\left(k\omega_0 t\right) \right) \\
&= C_0 + 2 \sum_{k=1}^{\infty} \left(\operatorname{Re} C_k \cos\left(k\omega_0 t\right) - \operatorname{Im} C_k \sin\left(k\omega_0 t\right) \right) \\
&= C_0 + 2 \sum_{k=1}^{\infty} \left(|C_k| \cos\angle C_k \cos\left(k\omega_0 t\right) - |C_k| \sin\angle C_k \sin\left(k\omega_0 t\right) \right) \\
&= C_0 + 2 \sum_{k=1}^{\infty} |C_k| \cos\left(k\omega_0 t + \angle C_k\right)
\end{aligned}
$$

3 Gibbs Phenomenon Demonstration

3.1 Theoretical Proof of the Gibbs Phenomenon

We consider the function:

$$
f(t) = \begin{cases} 1 & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}
$$

The Fourier series is:

$$
f_N(t) = \frac{1}{2} + \sum_{n=0}^{N} \frac{2(-1)^n}{(2n+1)\pi} \cos\left((2n+1)\pi t\right)
$$

Let:

$$
f_N(t) = \frac{1}{2} + g_N(t)
$$

Then the derivative of $g_N(t)$ is:

$$
g_N'(t) = \sum_{n=0}^{N} 2(-1)^{n+1} \sin\left((2n+1)\pi t\right)
$$

$$= 2\sum_{n=0}^{N} \sin\left((2n+1)\pi t + (n+1)\pi\right)$$

$$= 2\ \mathrm{Im}\sum_{n=0}^{N} e^{i(2\pi t+\pi)n}\cdot e^{i(\pi t+\pi)}$$

$$= 2\ \mathrm{Im}\left(\frac{1-e^{i(2\pi t+\pi)(N+1)}}{1-e^{i(2\pi t+\pi)}}\cdot e^{i(\pi t+\pi)n}\right)$$

$$= 2\ \mathrm{Im}\left(\frac{e^{i\frac{(2\pi t+\pi)(N+1)}{2}}}{e^{i\frac{(2\pi t+\pi)}{2}}}\cdot\frac{e^{-i\frac{(2\pi t+\pi)(N+1)}{2}}-e^{i\frac{(2\pi t+\pi)(N+1)}{2}}}{e^{-i\frac{(2\pi t+\pi)}{2}}-e^{i\frac{(2\pi t+\pi)}{2}}}\cdot e^{i(\pi t+\pi)}\right)$$

$$= 2\ \mathrm{Im}\left(e^{i\left(\frac{(2\pi t+\pi)(N+1)}{2}+\frac{\pi}{2}\right)}\cdot\frac{\sin\left(\frac{(2\pi t+\pi)(N+1)}{2}\right)}{\sin\left(\frac{2\pi t+\pi}{2}\right)}\right)$$

$$= 2\cdot\frac{\sin\left(\left(\pi t+\frac{\pi}{2}\right)(N+1)+\frac{\pi}{2}\right)\cdot\sin\left(\left(\pi t+\frac{\pi}{2}\right)(N+1)\right)}{\sin\left(\frac{2\pi t+\pi}{2}\right)} = 0$$

Then:

$$\sin\left(\left(\pi t+\frac{\pi}{2}\right)(N+1)+\frac{\pi}{2}\right)\cdot\sin\left(\left(\pi t+\frac{\pi}{2}\right)(N+1)\right) = 0$$

$$\cos\left(\left(\pi t+\frac{\pi}{2}\right)(N+1)\right)\cdot\sin\left(\left(\pi t+\frac{\pi}{2}\right)(N+1)\right) = 0$$

$$\sin\left((2\pi t+\pi)(N+1)\right) = 0$$

$$(2\pi t+\pi)(N+1) = h\pi \quad\Rightarrow\quad t = \frac{h}{2(N+1)}-\frac{1}{2}$$

Gibbs phenomenon appears at $t = \pm\frac{1}{2}$. For instance, take $h = 1$,such that $t \to -\frac{1}{2}$ then:

$$t = \frac{1}{2(N+1)}-\frac{1}{2}$$

Evaluate the overshoot:

$$f_N(t) = \frac{1}{2}+\sum_{n=0}^{N}\frac{2(-1)^n}{(2n+1)\pi}\cos\left((2n+1)\pi\left(\frac{1}{2(N+1)}-\frac{1}{2}\right)\right)$$

$$= \frac{1}{2}+\sum_{n=0}^{N}\frac{2(-1)^n}{(2n+1)\pi}\cos\left(\frac{(2n+1)\pi}{2(N+1)}-n\pi-\frac{\pi}{2}\right)$$

$$= \frac{1}{2}+\sum_{n=0}^{N}\frac{2(-1)^n}{(2n+1)\pi}\sin\left(\frac{(2n+1)\pi}{2(N+1)}-n\pi\right)$$

$$= \frac{1}{2}+\sum_{n=0}^{N}\frac{2(-1)^{2n}}{(2n+1)\pi}\sin\left(\frac{(2n+1)\pi}{2(N+1)}\right)$$

$$= \frac{1}{2}+\sum_{n=0}^{N}\frac{\sin\left(\frac{n\pi}{N+1}+\frac{\pi}{2(N+1)}\right)}{n\pi+\frac{\pi}{2}}$$

$$= \frac{1}{2} + \frac{1}{\pi} \sum_{n=0}^{N} \frac{\sin\left(\frac{n\pi}{N+1} + \frac{\pi}{2(N+1)}\right)}{\frac{n\pi}{N+1} + \frac{\pi}{2} \cdot \frac{1}{N+1}} \cdot \frac{\pi}{N+1}$$

As $N \to \infty$, the sum approaches:

$$f_N(t) \to \frac{1}{2} + \frac{1}{\pi} \int_0^{\pi} \frac{\sin x}{x}\, dx = \frac{1}{2} + \frac{1}{\pi} \mathrm{Si}(\pi)$$

Therefore, the overshoot ratio is:

$$\eta = \frac{\frac{1}{2} + \frac{1}{\pi}\mathrm{Si}(\pi) - 1}{1} = \frac{\mathrm{Si}(\pi)}{\pi} - \frac{1}{2} \approx 0.0895 = 8.95\%$$

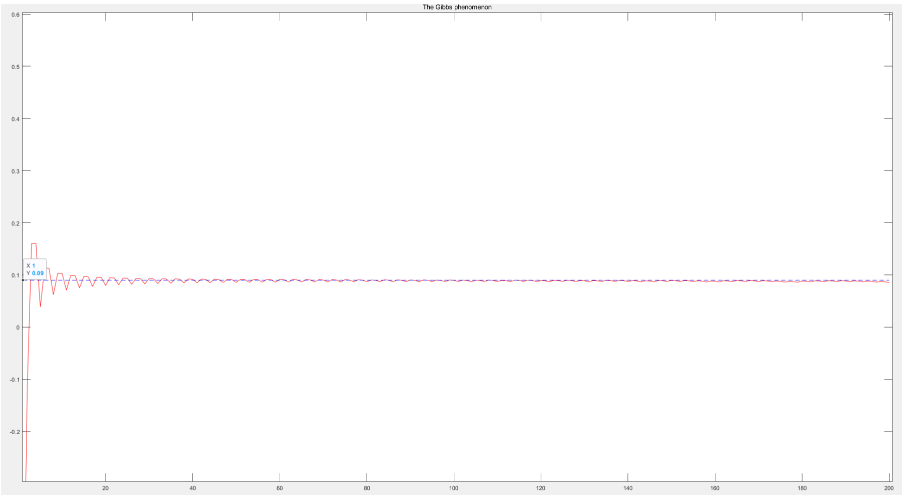

Figure 1: The Gibbs Phenomenon

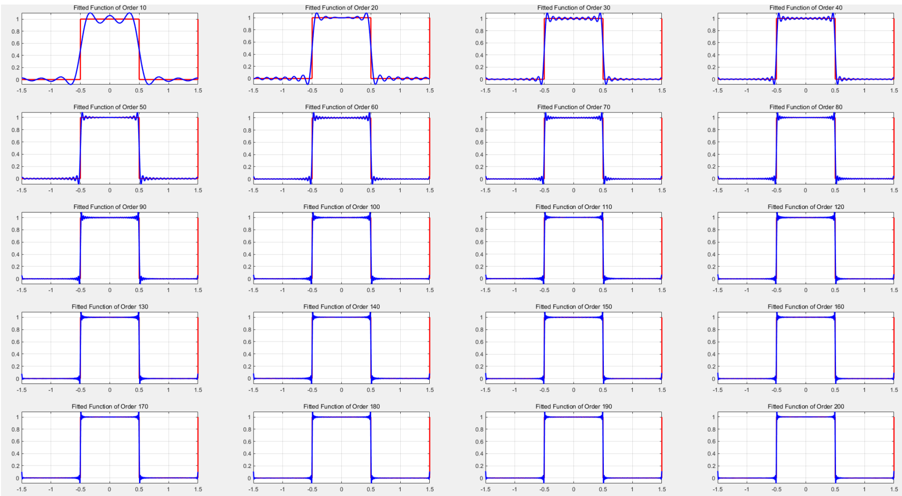

Figure 2: Fourier Approximation of Square Wave

3.2 MATLAB Code for Fourier Series Approximation

Reference the file titled "Fourier Transform Pre.pdf".

```
%% Variable allocation
n = 200; % Fitting order
maxgibbis = zeros(1, n); % Maximum value of the fitting function in each iteration
a = -1.5; b = 1.5; % Define the interval
N = 3e3; delta = (b - a) / N; x_lin = transpose(a:delta:b); % Sample points
N = N + 1;

% Basis functions: cosine
createfun = @(ii, x) cos((ii - 1) * 2 * pi / (b - a) * (x - (b + a) / 2));

vec_fun = zeros(N, n);
for ii = 1:n
    vec_fun(:, ii) = createfun(ii, x_lin);
end

% Square wave target function
orifun = @(x) 0.5 + 0.5 * square(pi * (x + 0.5));
ori_fun = orifun(x_lin);

% Gram-Schmidt orthogonalization
normal_fun = vec_fun;
coeff = sum(delta .* normal_fun(:, 1).^2);
normal_fun(:, 1) = normal_fun(:, 1) / sqrt(coeff);

for ii = 2:n
    for jj = 1:ii-1
        coeff = sum(normal_fun(:, jj) .* normal_fun(:, ii)) * delta;
        normal_fun(:, ii) = normal_fun(:, ii) - coeff * normal_fun(:, jj);
    end
    coeff = sum(delta .* normal_fun(:, ii).^2);
    normal_fun(:, ii) = normal_fun(:, ii) / sqrt(coeff);
end

% Fitting process
coeff = zeros(n, 1);
fit_fun = zeros(N, 1);
figure; rows = ceil(n / 40); cols = 4;

for ii = 1:n
    coeff(ii) = sum(ori_fun .* normal_fun(:, ii)) * delta;
    fit_fun = fit_fun + coeff(ii) * normal_fun(:, ii);
    maxgibbis(ii) = max(fit_fun);
    if mod(ii, 10) == 0
```

```
        subplot(rows, cols, ii / 10);
        plot(x_lin, ori_fun, 'r', 'LineWidth', 2);
        hold on;
        plot(x_lin, fit_fun, 'b', 'LineWidth', 2);
        title(['Fitted Function of Order ' num2str(ii)]);
        grid on;
    end
end

% Gibbs phenomenon plot
figure;
plot(1:n, maxgibbis - 1, 'r'); hold on;
y_constant = 0.09;
plot([1, n], [y_constant, y_constant], 'b--');
title('The Gibbs Phenomenon');
```