Fundamentals of Signal Processing and Data Analysis Homework 3

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1 Fine the Laplace transform of $f\left(t\right)$

1.1 $f\left(t\right) = e^{-at}cos\omega t$

$$
\begin{aligned}
\frac{\mathbf{d}^2 cos\omega t}{\mathbf{d}t^2} &= -\omega^2 cos\omega t \\
\mathcal{L}\left(\frac{\mathbf{d}^2 cos\omega t}{\mathbf{d}t^2}\right) &= \mathcal{L}\left(-\omega^2 cos\omega t\right) \\
s^2\mathcal{L}\left(cos\omega t\right) - s &= -\omega^2\mathcal{L}\left(cos\omega t\right) \\
\mathcal{L}\left(cos\omega t\right) &= \frac{s}{s^2+\omega^2}
\end{aligned}
$$

By complex shifting, $\mathcal{L}\left(e^{at}f\left(t\right)\right) = \mathcal{L}_{s+a}\left(f\left(t\right)\right)$,

$$
\mathcal{L}\left(e^{-at}cos\omega t\right) = \frac{s+a}{\left(s+a\right)^2+\omega^2}
$$

1.2 $f\left(t\right) = e^{-at}sin\omega t$

$$
\begin{aligned}
\frac{\mathbf{d}^2 sin\omega t}{\mathbf{d}t^2} &= -\omega^2 sin\omega t \\
\mathcal{L}\left(\frac{\mathbf{d}^2 sin\omega t}{\mathbf{d}t^2}\right) &= \mathcal{L}\left(-\omega^2 sin\omega t\right) \\
s^2\mathcal{L}\left(sin\omega t\right) - \omega &= -\omega^2\mathcal{L}\left(cos\omega t\right) \\
\mathcal{L}\left(sin\omega t\right) &= \frac{\omega}{s^2+\omega^2}
\end{aligned}
$$

By complex shifting, $\mathcal{L}\left(e^{at}f\left(t\right)\right) = \mathcal{L}_{s+a}\left(f\left(t\right)\right)$,

$$
\mathcal{L}\left(e^{-at}sin\omega t\right) = \frac{\omega}{\left(s+a\right)^2+\omega^2}
$$

2 Solution of the Given Ordinary Differential Equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 4\frac{\mathrm{d}y}{\mathrm{d}t} + 6y = 2, \quad y(0) = 0, \quad y'(0) = 0.$$

Applying the Laplace transform to both sides:

$$\mathcal{L}\left(\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 4\frac{\mathrm{d}y}{\mathrm{d}t} + 6y\right) = \mathcal{L}(2),$$

then,

$$s^2\mathcal{L}(y) - sy(0) - y'(0) + 4s\mathcal{L}(y) - 4y(0) + 6\mathcal{L}(y) = \frac{2}{s}.$$

Substituting the given initial conditions:

$$(s^2 + 4s + 6)\mathcal{L}(y) = \frac{2}{s},$$

which leads to the transformed function:

$$\begin{aligned}\mathcal{L}(y) &= \frac{2}{s} \cdot \frac{1}{s^2 + 4s + 6}. \\ &= \frac{\sqrt{2}}{s} \cdot \frac{\sqrt{2}}{(s+2)^2 + (\sqrt{2})^2}.\end{aligned}$$

Applying the convolution theorem:

$$\mathcal{L}(y) = \mathcal{L}(\sqrt{2}) \cdot \mathcal{L}\left(e^{-2t}\sin(\sqrt{2}t)\right).$$

$$y = \int_0^t \mathcal{L}^{-1}\left(\mathcal{L}(\sqrt{2})\right)_{t=\tau-t} \mathcal{L}^{-1}\left(\mathcal{L}\left(e^{-2t}\sin(\sqrt{2}t)\right)\right)_{t=\tau} \mathbf{d}\tau$$

$$\begin{aligned}y &= \int_0^t \sqrt{2}e^{-2\tau}\sin(\sqrt{2}\tau)\mathbf{d}\tau = \sqrt{2}\,\mathrm{Im}\int_0^t e^{-2\tau+i\sqrt{2}\tau}\mathbf{d}\tau \\ &= \sqrt{2}\,\mathrm{Im}\left[\frac{e^{-2\tau+i\sqrt{2}\tau}}{-2+i\sqrt{2}}\right]_0^t. \\ &= \mathrm{Im}\left(\frac{-\sqrt{2}-i}{3}\left(e^{-2t}\cos(\sqrt{2}t) + ie^{-2t}\sin(\sqrt{2}t) - 1\right)\right). \\ &= \frac{1 - \sqrt{2}e^{-2t}\sin(\sqrt{2}t) - e^{-2t}\cos(\sqrt{2}t)}{3}.\end{aligned}$$