A brief introduction to Convolution

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ESS 206 Course Presention

March 12, 2025

Contents

What is Convolution?

What is a system?

Figure: General System

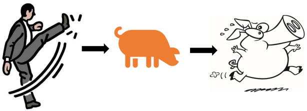

Figure: example

A simple but useful system & Some necessary assumptions

Figure: example

Assumption

- *Assumption 1: The system is linearity.*
- *Assumption 2: The system is time-invariant.*
  $\Leftrightarrow$ *The system is a LTI D-T system.*

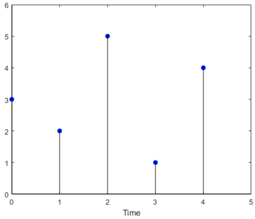

Figure: Signal input

Additivity:separate the signals

$T(u+v) = Tu + Tv$

Homogeneity: process the signal as a multiple of the impulse signal

$T(\lambda v) = \lambda T(v)$

Additivity:superposition all the response

$Tu + Tv = T(u+v)$

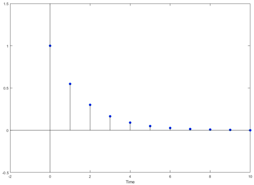

Figure: Impulse response

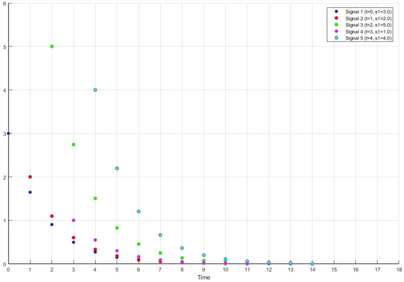

Figure: Signal response

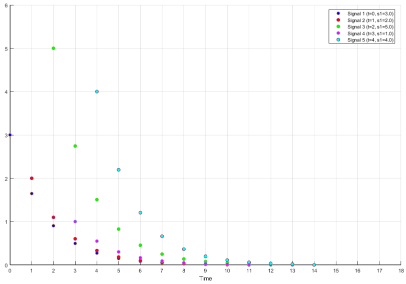

Figure: Response output

$$
\begin{aligned}
Y[0] &= S[0] \times R[0] \\
Y[1] &= S[0] \times R[1] + S[1] \times R[0] \\
Y[2] &= S[0] \times R[2] + S[1] \times R[1] + S[2] \times R[0] \\
&\vdots \\
Y[n] &= \sum_{i=0}^{n} S[i] \times R[n-i]
\end{aligned}
$$

Convolution formula

- Discrete Convolution

$$Y\left[n\right]=\sum_{i=0}^{n}S\left[i\right]\times R\left[n-i\right]$$

- Continuous Convolution

$$Y\left(t\right)=\int_{0}^{\infty}S\left(\tau\right)\times R\left(t-\tau\right)d\tau$$

A simple example

Cauchy Product $\mathbb{V}.\mathbb{S}.$ Convolution

Convolution:

$$Y[n] = \sum_{i=0}^{n} S[i] \times R[n-i]$$

Cauchy Product:

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)$$

$$\begin{aligned}\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k x^k b_{n-k} x^{n-k}\right) &= \sum_{n=0}^{\infty} x^n \left(\sum_{k=0}^{n} a_k b_{n-k}\right) \\ &= \sum_{n=0}^{\infty} x^n \left(a(n) * b(n)\right)\end{aligned}$$

When $x = 10, \forall i \quad a_i, b_i \in \{0,1,2,3,4,5,6,7,8,9\}$

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} x^n \left(a\left(n\right) * b\left(n\right)\right)$$

$$\overline{a_n a_{n-1} \ldots a_2 a_1 a_0} \times \overline{b_n b_{n-1} \ldots b_2 b_1 b_0} = \sum_{n=0}^{\infty} 10^n \left(a\left(n\right) * b\left(n\right)\right)$$

	$\widetilde{a_3}$	$\widetilde{a_2}$	$\widetilde{a_1}$	$\widetilde{a_0}$	a_3	a_2	a_1	a_0
$\widetilde{b_3}$								0
$\widetilde{b_2}$							0	0
$\widetilde{b_1}$						0	0	0
$\widetilde{b_0}$					0	0	0	0
b_3				0	(3,3)	(3,2)	(3,1)	(3,0)
b_2			0	0	(2,3)	(2,2)	(2,1)	(2,0)
b_1		0	0	0	(1,3)	(1,2)	(1,3)	(1,3)
b_0	0	0	0	0	(0,3)	(0,2)	(0,1)	(0,0)

Calculation Method by traditional method $n = 10^2$

```
a = randi([0, 9], 1, n);
b = randi([0, 9], 1, n);

tic;
product1=0;
sum=0;
for ii=1:length(a)
    for jj=1:length(b)
        sum=sum+a(ii)*b(jj)*10^(length(b)-jj);
    end
    product1=product1+sum*10^(length(a)-ii);
    sum=0;
end
```

iteration:$n * n$
Algorithmic Complexity: $O(n^2)$

Calculation by Convolution

```
A=zeros(1,length(a));
a=[a A];
b=[b A];
tic;
product2=0;
convolution=0;
for ii=1:length(a)
    for jj=1:ii
        convolution=convolution+a(jj)*b(ii-jj+1);
    end
    product2=product2+convolution*10^(length(A)+length(A)-1-ii);
    convolution=0;
end
```

iteration:$1 + 2 + 3 + \cdots + (2n - 1) + 2n = n * (2n - 1)$
Algorithmic Complexity: $O(2n^2)$

More steps, but less time!

Comparison of Running Times:

Run Number	Running Time of Method 1 (seconds)	Running Time of Method 2 (seconds)	Ratio
1	0.000936	0.000064	14.7186
2	0.000939	0.000062	15.0465
3	0.000932	0.000062	15.0306
4	0.000931	0.000062	14.9630
5	0.000935	0.000071	13.2450
6	0.000941	0.000063	14.9825
7	0.000929	0.000062	14.9614
8	0.000937	0.000063	14.9856
9	0.000935	0.000076	12.3567
10	0.000945	0.000063	14.9242
11	0.000928	0.000062	15.0146
12	0.000928	0.000062	14.9613
13	0.000929	0.000062	14.9549
14	0.000929	0.000063	14.7882
15	0.000927	0.000062	14.9084
16	0.000928	0.000062	14.9952
17	0.000967	0.000062	15.6505
18	0.000928	0.000062	15.0129
19	0.000927	0.000062	14.9935
20	0.000928	0.000062	14.9661

Figure: Comparison of Running Times

Reference

The End

Reference

- Wikipedia contributors. "Generating function." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 26 Dec. 2024. Web. 10 Mar. 2025.
- Wikipedia contributors. "Convolution." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia, 23 Jan. 2025. Web. 10 Mar. 2025.
- But what is a convolution?
- Linear Algebra Done Right