

Computational Homework11 Report

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1 Fixed-point method

1.1 Introduction

The fixed-point iteration method is a powerful numerical technique for finding the roots of a system of non-linear equations. Consider a system of two non-linear equations in two variables, x and y :

$$f_0(x, y) = 0$$

$$g_0(x, y) = 0$$

In this problem, we aim to find the roots of the specific system:

$$f_0(x, y) = 4x^2 + y^2 - 4 = 0 \quad (1)$$

$$g_0(x, y) = x + y - \sin(x - y) = 0 \quad (2)$$

The core functions for these equations are defined in Python as:

```
def f0(x,y):  
    return 4*x**2+y**2-4  
def g0(x,y):  
    return x+y-np.sin(x-y)
```

To apply the fixed-point method, we must transform this system into an equivalent fixed-point problem of the form:

$$x = F(x, y)$$

$$y = G(x, y)$$

The choice of iteration functions F and G , as well as the iterative update strategy, are crucial for convergence. We will explore two different ways of defining $F(x, y)$ and $G(x, y)$ (Scheme 1 and Scheme 2), and two corresponding iterative update methods (Original/Jacobi-style and Improved/Gauss-Seidel style).

Defining Iteration Functions

We consider two schemes for defining the iteration functions:

Scheme 1: Simple Rearrangement (Used in Original Iteration) This scheme uses iteration functions derived from a simple algebraic rearrangement of the original equations.

$$F_1(x, y) = x + 1 - x^2 - \frac{y^2}{4} \quad (3)$$

$$G_1(x, y) = \sin(x - y) - x \quad (4)$$

In the Python implementation, these are typically defined as:

```
def f(x,y):  
    return x+1-x**2-y**2/4  
def g(x,y):  
    return np.sin(x-y)-x
```

Scheme 2: Alternative Rearrangement (Used in Improved Iteration) In this scheme, we reformulate the first equation (1) to directly solve for x , assuming $x > 0$. This typically leads to better convergence properties.

$$F_2(x, y) = \sqrt{1 - \frac{y^2}{4}} \quad (5)$$

The second iteration function $G_2(x, y)$ remains the same as $G_1(x, y)$ (from the second original equation):

$$G_2(x, y) = \sin(x - y) - x \quad (6)$$

The Python functions for this scheme would be:

```
def new_f(x,y):  
    return np.sqrt(1-y**2/4)  
def g(x,y):  
    return np.sin(x-y)-x
```

Iterative Update Strategies

We apply two distinct fixed-point iteration strategies:

Method A: Original Fixed Point Method (Jacobi-style) This method employs a Jacobi-style update, where both x_{k+1} and y_{k+1} are computed exclusively from values at step k :

$$x_{k+1} = F(x_k, y_k)$$

$$y_{k+1} = G(x_k, y_k)$$

This method can be used with either function Scheme 1 or Scheme 2. The Python function ‘fp’ (when ‘mode=0’) implements this:

```
def fp(f,g,x,y,mode=0,count=0,tol=1e-6):
    if mode==0:
        x_new=f(x,y)
        y_new=g(x,y)
        d=(x-x_new)**2+(y-y_new)**2
        if count>=300:
            print(f"Reached maximum iterations ({count}). Returning current approximation.")
            return x_new, y_new
        elif d<tol**2:
            print(f"Converged in {count + 1} iterations.")
            return x_new, y_new
        else:
            return fp(f,g,x_new,y_new,mode,count+1)

def fixed_point(f,g,x0,y0,mode=0):
    return fp(f,g,x0,y0,mode)
```

Method B: Improved Fixed Point Method (Gauss-Seidel style) This method utilizes a Gauss-Seidel style update. The newly computed x_{k+1} is immediately used when calculating y_{k+1} :

$$x_{k+1} = F(x_k, y_k)$$

$$y_{k+1} = G(x_{k+1}, y_k)$$

This method is generally expected to converge faster than the Jacobi style if it converges at all. This method can also be used with either function Scheme 1 or Scheme 2. The Python function ‘fp’ (when ‘mode=1’) implements this:

```
def fp(f,g,x,y,mode=0,count=0,tol=1e-6):
    # ... (code for mode=0 omitted for brevity, as provided in the original text)
    else: # This implicitly handles mode=1 in the full function
        x_new=f(x,y)
        y_new=g(x_new,y)
```

```

d=(x-x_new)**2+(y-y_new)**2
if count>=300:
    print(f"Reached maximum iterations ({count}). Returning current approximation.")
    return x_new, y_new
elif d<tol**2:
    print(f"Converged in {count + 1} iterations.")
    return x_new, y_new
else:
    return fp(f,g,x_new,y_new,mode,count+1)

def fixed_point(f,g,x0,y0,mode=0):
    return fp(f,g,x0,y0,mode)

```

This method corresponds to ‘mode=1’ in our implementation. The iterative process for all scenarios begins with an initial guess (x_0, y_0) and continues until the squared Euclidean distance between successive approximations, $(x_{k+1} - x_k)^2 + (y_{k+1} - y_k)^2$, falls below a specified tolerance squared (e.g., tol^2), or a maximum number of iterations is reached. The convergence properties vary significantly between these different choices of iteration functions and update styles.

1.2 Result

The fixed-point iteration was performed using an initial guess $(x_0, y_0) = (1.0, 0.0)$, a tolerance of 1×10^{-6} for the squared distance, and a maximum of 300 iterations.

Comparison and Summary of Results

A comparative overview of the four scenarios, derived from the Python output in Figure 1, is presented in Table 1.

Table 1 Comparison of Fixed-Point Iteration Results. *Max Residual refers to the maximum absolute value among $f_0(x, y)$ and $g_0(x, y)$ at the approximated root.

Metric	Original		Improved	
	Scheme A	Scheme B	Scheme A	Scheme B
Iterations	160	15	>300 (Diverged)	17
Approx. Root(x, y)	(0.9986066, -0.1055308)	(0.9986070, -0.1055307)	(1.1667598, -0.2881905)	(0.9986070, -0.1055307)
Max Residual*	2.44×10^{-6}	3.33×10^{-7}	1.53×10^0	3.42×10^{-7}

The experiments unequivocally demonstrate the critical influence of both the chosen iteration functions and the update strategy on the convergence of the fixed-point method.

```
PS C:\Users\20369> & E:/anaconda3/envs/py310/python.exe "c:/Users/20369/Homework/homework11/homework11root.py"
##### Original fixed point method #####

--- Without appropriate F(x,y) ---
Converged in 160 iterations.

Final approximated root: (0.998606630, -0.105530847)

--- Verification using original system (f0(x,y)=0, g0(x,y)=0) ---
f0: -2.43729369e-06
g0: -6.87129325e-07
--- With appropriate F(x,y) ---
Converged in 15 iterations.

Final approximated root: (0.998606959, -0.105530728)

--- Verification using original system (f0(x,y)=0, g0(x,y)=0) ---
f0: 1.69431615e-07
g0: -3.33045869e-07

##### Improved fixed point method #####

--- Without appropriate F(x,y) ---
Reached maximum iterations (300). Returning current approximation.

Final approximated root: (1.166759810, -0.288190498)

--- Verification using original system (f0(x,y)=0, g0(x,y)=0) ---
f0: 1.52836758e+00
g0: -1.14728039e-01
--- With appropriate F(x,y) ---
Converged in 17 iterations.

Final approximated root: (0.998606958, -0.105530734)

--- Verification using original system (f0(x,y)=0, g0(x,y)=0) ---
f0: 1.60578685e-07
g0: -3.42295646e-07
PS C:\Users\20369>
```

Figure 1 Detailed Python Output for All Fixed-Point Iteration Scenarios.

2 Matrix Norms and Condition Numbers

This section details the calculation of different matrix norms and their corresponding condition numbers for the given matrix \mathbf{A} . The matrix under consideration is:

$$\mathbf{A} = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

2.1 Matrix Norms

Matrix norms measure the "size" of a matrix. We consider the L_1 , L_2 , and L_∞ norms.

2.1.1 The L_1 Norm (Column Sum Norm)

The L_1 norm of a matrix \mathbf{A} is the maximum absolute column sum:

$$\|\mathbf{A}\|_1 = \max_j \sum_{i=1}^n |a_{ij}|$$

For matrix \mathbf{A} :

$$\text{Column 1 sum: } |-2| + |1| + |0| = 3$$

$$\text{Column 2 sum: } |1| + |-2| + |1| = 4$$

$$\text{Column 3 sum: } |0| + |1| + |-2| = 3$$

Thus, $\|\mathbf{A}\|_1 = \max(3, 4, 3) = 4$.

2.1.2 The L_∞ Norm (Row Sum Norm)

The L_∞ norm of a matrix \mathbf{A} is the maximum absolute row sum:

$$\|\mathbf{A}\|_\infty = \max_i \sum_{j=1}^m |a_{ij}|$$

For matrix \mathbf{A} :

$$\text{Row 1 sum: } |-2| + |1| + |0| = 3$$

$$\text{Row 2 sum: } |1| + |-2| + |1| = 4$$

$$\text{Row 3 sum: } |0| + |1| + |-2| = 3$$

Thus, $\|\mathbf{A}\|_\infty = \max(3, 4, 3) = 4$.

2.1.3 The L_2 Norm (Spectral Norm)

The L_2 norm of a matrix \mathbf{A} is defined as:

$$\|\mathbf{A}\|_2 = \sqrt{\lambda_{\max}(\mathbf{A}^T \mathbf{A})}$$

where $\lambda_{\max}(\mathbf{A}^T \mathbf{A})$ is the largest eigenvalue of $\mathbf{A}^T \mathbf{A}$. For a symmetric matrix, $\mathbf{A}^T = \mathbf{A}$, so $\mathbf{A}^T \mathbf{A} = \mathbf{A}^2$. A symmetric matrix \mathbf{A} can be spectrally decomposed as $\mathbf{A} = \mathbf{U} \mathbf{\Lambda} \mathbf{U}^T$, where \mathbf{U} is an orthogonal matrix ($\mathbf{U}^T \mathbf{U} = \mathbf{I}$) and $\mathbf{\Lambda}$ is a diagonal matrix containing the eigenvalues of \mathbf{A} on its diagonal. Then, we can find the spectral decomposition of \mathbf{A}^2 :

$$\mathbf{A}^2 = (\mathbf{U} \mathbf{\Lambda} \mathbf{U}^T)(\mathbf{U} \mathbf{\Lambda} \mathbf{U}^T) = \mathbf{U} \mathbf{\Lambda} (\mathbf{U}^T \mathbf{U}) \mathbf{\Lambda} \mathbf{U}^T = \mathbf{U} \mathbf{\Lambda} \mathbf{\Lambda} \mathbf{U}^T = \mathbf{U} \mathbf{\Lambda}^2 \mathbf{U}^T$$

The eigenvalues of \mathbf{A}^2 are the diagonal entries of $\mathbf{\Lambda}^2$, which are λ_k^2 , where λ_k are the eigenvalues of \mathbf{A} . Therefore, the largest eigenvalue of $\mathbf{A}^T \mathbf{A} = \mathbf{A}^2$ is $\max_k(\lambda_k^2)$. Substituting this into the definition of the L_2 norm:

$$\|\mathbf{A}\|_2 = \sqrt{\max_k(\lambda_k^2)} = \max_k \sqrt{\lambda_k^2} = \max_k |\lambda_k|$$

Thus, for a symmetric matrix, the L_2 norm simplifies to its largest absolute eigenvalue.

The eigenvalues of \mathbf{A} are found by solving $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$. For the given matrix \mathbf{A} :

$$\det \begin{pmatrix} -2 - \lambda & 1 & 0 \\ 1 & -2 - \lambda & 1 \\ 0 & 1 & -2 - \lambda \end{pmatrix} = 0$$

Solving this characteristic equation yields the eigenvalues:

$$\lambda_1 = -2$$

$$\lambda_2 = -2 + \sqrt{2} \approx -0.586$$

$$\lambda_3 = -2 - \sqrt{2} \approx -3.414$$

The absolute values of the eigenvalues are: $|\lambda_1| = 2$, $|\lambda_2| \approx 0.586$, $|\lambda_3| \approx 3.414$. Therefore, $\|\mathbf{A}\|_2 = \max(|\lambda_k|) = |-2 - \sqrt{2}| = 2 + \sqrt{2} \approx 3.414$.

2.2 Condition Numbers

The condition number $\kappa(\mathbf{A})$ measures the sensitivity of the solution of a linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ to input perturbations. It is defined as:

$$\kappa(\mathbf{A}) = \|\mathbf{A}\| \cdot \|\mathbf{A}^{-1}\|$$

We first need the inverse matrix \mathbf{A}^{-1} . The inverse of \mathbf{A} is:

$$\mathbf{A}^{-1} = \begin{pmatrix} -3/4 & -1/2 & -1/4 \\ -1/2 & -1 & -1/2 \\ -1/4 & -1/2 & -3/4 \end{pmatrix}$$

2.2.1 Condition Number in L_1 Norm, $\kappa_1(\mathbf{A})$

First, calculate $\|\mathbf{A}^{-1}\|_1$:

$$\text{Column 1 sum: } |-3/4| + |-1/2| + |-1/4| = 1.5$$

$$\text{Column 2 sum: } |-1/2| + |-1| + |-1/2| = 2$$

$$\text{Column 3 sum: } |-1/4| + |-1/2| + |-3/4| = 1.5$$

So, $\|\mathbf{A}^{-1}\|_1 = \max(1.5, 2, 1.5) = 2$. Then, $\kappa_1(\mathbf{A}) = \|\mathbf{A}\|_1 \cdot \|\mathbf{A}^{-1}\|_1 = 4 \cdot 2 = 8$.

2.2.2 Condition Number in L_∞ Norm, $\kappa_\infty(\mathbf{A})$

First, calculate $\|\mathbf{A}^{-1}\|_\infty$:

$$\text{Row 1 sum: } |-3/4| + |-1/2| + |-1/4| = 1.5$$

$$\text{Row 2 sum: } |-1/2| + |-1| + |-1/2| = 2$$

$$\text{Row 3 sum: } |-1/4| + |-1/2| + |-3/4| = 1.5$$

So, $\|\mathbf{A}^{-1}\|_\infty = \max(1.5, 2, 1.5) = 2$. Then, $\kappa_\infty(\mathbf{A}) = \|\mathbf{A}\|_\infty \cdot \|\mathbf{A}^{-1}\|_\infty = 4 \cdot 2 = 8$.

2.2.3 Condition Number in L_2 Norm, $\kappa_2(\mathbf{A})$

The condition number in the L_2 norm is defined as:

$$\kappa_2(\mathbf{A}) = \|\mathbf{A}\|_2 \cdot \|\mathbf{A}^{-1}\|_2$$

Now let's consider \mathbf{A}^{-1} . If \mathbf{A} is invertible, its inverse can be expressed as:

$$\mathbf{A}^{-1} = (\mathbf{U}\mathbf{\Lambda}\mathbf{U}^T)^{-1} = (\mathbf{U}^T)^{-1}\mathbf{\Lambda}^{-1}\mathbf{U}^{-1}$$

Since \mathbf{U} is orthogonal, $\mathbf{U}^{-1} = \mathbf{U}^T$ and $(\mathbf{U}^T)^{-1} = \mathbf{U}$. Thus:

$$\mathbf{A}^{-1} = \mathbf{U}\mathbf{\Lambda}^{-1}\mathbf{U}^T$$

Applying the L_2 norm definition for symmetric matrices to \mathbf{A}^{-1} :

$$\|\mathbf{A}^{-1}\|_2 = \max_k |\lambda_k(\mathbf{A}^{-1})| = \max_k \left| \frac{1}{\lambda_k(\mathbf{A})} \right| = \frac{1}{\min_{k, \lambda_k \neq 0} |\lambda_k(\mathbf{A})|}$$

$$\kappa_2(\mathbf{A}) = \|\mathbf{A}\|_2 \cdot \|\mathbf{A}^{-1}\|_2 = \left(\max_k |\lambda_k(\mathbf{A})| \right) \cdot \left(\frac{1}{\min_{k, \lambda_k \neq 0} |\lambda_k(\mathbf{A})|} \right) = \frac{\max_k |\lambda_k(\mathbf{A})|}{\min_{k, \lambda_k \neq 0} |\lambda_k(\mathbf{A})|}$$

Using the eigenvalues calculated previously: $\max |\lambda_k| = 2 + \sqrt{2}$ $\min |\lambda_k| = 2 - \sqrt{2}$ Thus,

$$\kappa_2(\mathbf{A}) = \frac{2 + \sqrt{2}}{2 - \sqrt{2}} = \frac{(2 + \sqrt{2})(2 + \sqrt{2})}{(2 - \sqrt{2})(2 + \sqrt{2})} = \frac{4 + 4\sqrt{2} + 2}{4 - 2} = \frac{6 + 4\sqrt{2}}{2} = 3 + 2\sqrt{2} \approx 5.828$$

2.3 Summary of Results

The calculated matrix norms and condition numbers for \mathbf{A} are:

- $\|\mathbf{A}\|_1 = 4$
- $\|\mathbf{A}\|_\infty = 4$
- $\|\mathbf{A}\|_2 = 2 + \sqrt{2} \approx 3.414$
- $\kappa_1(\mathbf{A}) = 8$
- $\kappa_\infty(\mathbf{A}) = 8$
- $\kappa_2(\mathbf{A}) = 3 + 2\sqrt{2} \approx 5.828$

These condition numbers indicate that the matrix \mathbf{A} is relatively well-conditioned.

3 Gaussian Elimination

The linear system $\mathbf{Ax} = \mathbf{b}$ is given by:

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 1 & 9 \\ 1 & 2 & 6 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

We form the augmented matrix $[\mathbf{A}|\mathbf{b}]$:

$$\left(\begin{array}{ccc|c} 2 & 3 & 4 & 0 \\ 1 & 1 & 9 & 2 \\ 1 & 2 & 6 & 1 \end{array} \right)$$

3.1 Forward Elimination

Step 1: Eliminate x_1

Using $a_{11} = 2$ as the pivot.

- $R_2 \leftarrow R_2 - (1/2)R_1: \begin{pmatrix} 1 & 1 & 9 & | & 2 \end{pmatrix} - 0.5 \begin{pmatrix} 2 & 3 & 4 & | & 0 \end{pmatrix} = \begin{pmatrix} 0 & -0.5 & 7 & | & 2 \end{pmatrix}$
- $R_3 \leftarrow R_3 - (1/2)R_1: \begin{pmatrix} 1 & 2 & 6 & | & 1 \end{pmatrix} - 0.5 \begin{pmatrix} 2 & 3 & 4 & | & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0.5 & 4 & | & 1 \end{pmatrix}$

The matrix becomes:

$$\begin{pmatrix} 2 & 3 & 4 & | & 0 \\ 0 & -0.5 & 7 & | & 2 \\ 0 & 0.5 & 4 & | & 1 \end{pmatrix}$$

Step 2: Eliminate x_2

Using $a_{22} = -0.5$ as the pivot (no row swap, despite $a_{32} = 0.5$ having the same magnitude).

- $R_3 \leftarrow R_3 + R_2: \begin{pmatrix} 0 & 0.5 & 4 & | & 1 \end{pmatrix} + \begin{pmatrix} 0 & -0.5 & 7 & | & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 11 & | & 3 \end{pmatrix}$

The matrix is now in upper triangular form:

$$\begin{pmatrix} 2 & 3 & 4 & | & 0 \\ 0 & -0.5 & 7 & | & 2 \\ 0 & 0 & 11 & | & 3 \end{pmatrix}$$

3.2 Backward Substitution

From the upper triangular matrix, we have the system:

$$2x_1 + 3x_2 + 4x_3 = 0$$

$$-0.5x_2 + 7x_3 = 2$$

$$11x_3 = 3$$

Solving from bottom up:

- $x_3 = 3/11$
- $-0.5x_2 + 7(3/11) = 2 \Rightarrow -0.5x_2 = 2 - 21/11 = 1/11 \Rightarrow x_2 = (1/11)/(-0.5) = -2/11$
- $2x_1 + 3(-2/11) + 4(3/11) = 0 \Rightarrow 2x_1 - 6/11 + 12/11 = 0 \Rightarrow 2x_1 + 6/11 = 0 \Rightarrow x_1 = -3/11$

3.3 Solution

The solution to the linear system is:

$$\mathbf{x} = \begin{pmatrix} -3/11 \\ -2/11 \\ 3/11 \end{pmatrix}$$