

Computational Homework12 Report

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1 Solving $\mathbf{Ax} = \mathbf{b}$ with $\mathbf{A} = \mathbf{LDL}^T$ Decomposition

Given the matrix \mathbf{A} :

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

Since $\mathbf{A} = \mathbf{A}^T$, we assume $\mathbf{A} = \mathbf{LDL}^T$. We define \mathbf{L} as a unit lower triangular matrix and \mathbf{D} as a diagonal matrix:

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}.$$

Then $\mathbf{A} = \mathbf{LDL}^T$ can be written as:

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{pmatrix} \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \begin{pmatrix} 1 & \ell_{21} & \ell_{31} \\ 0 & 1 & \ell_{32} \\ 0 & 0 & 1 \end{pmatrix}.$$

Comparing the elements of \mathbf{A} with \mathbf{LDL}^T :

$$d_1 = 3$$

$$d_1 \ell_{21} = 2 \implies 3\ell_{21} = 2 \implies \ell_{21} = \frac{2}{3}$$

$$d_1 \ell_{31} = 1 \implies 3\ell_{31} = 1 \implies \ell_{31} = \frac{1}{3}$$

$$d_1 \ell_{21}^2 + d_2 = 2 \implies 3\left(\frac{2}{3}\right)^2 + d_2 = 2 \implies d_2 = \frac{2}{3}$$

$$d_1 \ell_{31} \ell_{21} + d_2 \ell_{32} = 0 \implies 3\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + \frac{2}{3}\ell_{32} = 0 \implies \ell_{32} = -1$$

$$d_1 \ell_{31}^2 + d_2 \ell_{32}^2 + d_3 = 3 \implies 3\left(\frac{1}{3}\right)^2 + \frac{2}{3}(-1)^2 + d_3 = 3 \implies d_3 = 2$$

So, the decomposition is:

$$\mathbf{A} = \mathbf{LDL}^T = \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\mathbf{U} = \mathbf{DL}^T = \begin{pmatrix} 3 & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 0 & \frac{2}{3} & -\frac{2}{3} \\ 0 & 0 & 2 \end{pmatrix}.$$

Therefore, $\mathbf{A} = \mathbf{LU}$.

Given $\mathbf{b} = \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}$.

1) **Solve $\mathbf{Ly} = \mathbf{b}$:**

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}$$

$$y_1 = 5$$

$$y_2 = 3 - \frac{2}{3}y_1 = 3 - \frac{2}{3}(5) = 3 - \frac{10}{3} = -\frac{1}{3}$$

$$y_3 = 4 - \frac{1}{3}y_1 - (-y_2) = 4 - \frac{1}{3}(5) + y_2 = 4 - \frac{5}{3} - \frac{1}{3} = 4 - \frac{6}{3} = 4 - 2 = 2$$

So, $\mathbf{y} = \begin{pmatrix} 5 \\ -\frac{1}{3} \\ 2 \end{pmatrix}$.

2) **Solve $\mathbf{Ux} = \mathbf{y}$:**

$$\begin{pmatrix} 3 & 2 & 1 \\ 0 & \frac{2}{3} & -\frac{2}{3} \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ -\frac{1}{3} \\ 2 \end{pmatrix}$$

$$2x_3 = 2 \implies x_3 = 1$$

$$\frac{2}{3}x_2 - \frac{2}{3}x_3 = -\frac{1}{3} \implies \frac{2}{3}x_2 - \frac{2}{3}(1) = -\frac{1}{3} \implies \frac{2}{3}x_2 = -\frac{1}{3} + \frac{2}{3} = \frac{1}{3} \implies x_2 = \frac{1}{2}$$

$$3x_1 + 2x_2 + x_3 = 5 \implies 3x_1 + 2\left(\frac{1}{2}\right) + 1 = 5 \implies 3x_1 + 1 + 1 = 5 \implies 3x_1 = 3 \implies x_1 = 1$$

The solution vector \mathbf{x} is:

$$\mathbf{x} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 1 \end{pmatrix}.$$

2 Solving $\mathbf{Ax} = \mathbf{b}$ with LU Decomposition

Given the linear system :

$$\begin{pmatrix} 2 & 3 & 4 \\ 1 & 1 & 9 \\ 1 & 2 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{Ax} = \mathbf{b}$$

We perform **LU** decomposition on matrix \mathbf{A} to factorize it into lower triangular matrix \mathbf{L} and upper triangular matrix \mathbf{U} :

$$\mathbf{A} = \mathbf{LU}$$

First, we find the **LU** decomposition:

$$\mathbf{L}_1 = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{pmatrix}, \quad \mathbf{L}_1\mathbf{A} = \begin{pmatrix} 2 & 3 & 4 \\ 0 & -\frac{1}{2} & 7 \\ 0 & \frac{1}{2} & 4 \end{pmatrix}$$

$$\mathbf{L}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \mathbf{U} = \mathbf{L}_2\mathbf{L}_1\mathbf{A} = \begin{pmatrix} 2 & 3 & 4 \\ 0 & -\frac{1}{2} & 7 \\ 0 & 0 & 11 \end{pmatrix}$$

$$\mathbf{L} = \mathbf{L}_2\mathbf{L}_1 = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

Now, solving $\mathbf{Ax} = \mathbf{b}$ by using the relation $\mathbf{A} = \mathbf{LU}$, we have:

$$\mathbf{L}\mathbf{Ax} = \mathbf{U}\mathbf{x}$$

Let $\mathbf{b} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$, then:

$$\mathbf{Lb} = \mathbf{U}\mathbf{x}$$

Solving for \mathbf{Lb} :

$$\mathbf{Lb} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$

Next, solve for \mathbf{x} by solving $\mathbf{Ux} = \mathbf{b}$:

$$\begin{pmatrix} 2 & 3 & 4 \\ 0 & -\frac{1}{2} & 7 \\ 0 & 0 & 11 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$$

Solving the system:

$$x_3 = \frac{3}{11}$$
$$x_2 = \frac{2 - 7x_3}{-\frac{1}{2}} = \frac{2 - 7\left(\frac{3}{11}\right)}{-\frac{1}{2}} = \frac{2 - \frac{21}{11}}{-\frac{1}{2}} = \frac{\frac{22-21}{11}}{-\frac{1}{2}} = -\frac{2}{11}$$

Finally:

$$x_1 = \frac{0 - 3x_2 - 4x_3}{2} = \frac{0 - 3\left(-\frac{2}{11}\right) - 4\left(\frac{3}{11}\right)}{2} = \frac{0 + \frac{6}{11} - \frac{12}{11}}{2} = -\frac{3}{11}$$

The solution vector is:

$$\mathbf{x} = \begin{pmatrix} -\frac{3}{11} \\ -\frac{2}{11} \\ \frac{3}{11} \end{pmatrix}.$$