Fundamentals of Planetary Science Homework 5

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1 Explain the composition of terrestrial planets

Whether the planet can contain the Hydrogen and Helium depends on whether the thermal motion of its molecules can overcome the planet's gravitational pull. The escape velocity at the surface of a planet is given by:

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}}$$

where G is the gravitational constant, M is the planet's mass, and R is its radius.

The root-mean-square (rms) speed of gas molecules in thermal equilibrium at temperature T is:

$$v_{\text{rms}} = \sqrt{\frac{3kT}{m}}$$

where k is the Boltzmann constant and m is the mass of a single molecule.

Empirically, it is considered that a gas can be retained if its molecular rms speed is less than a fraction β of the escape velocity:

$$v_{\text{rms}} \leq \beta\, v_{\text{esc}}$$

A common choice is $\beta = \frac{1}{6}$, corresponding to a very low fraction of molecules in the Maxwellian tail exceeding escape speed.

By combining the two expressions, we derive a threshold for the minimum molecular mass required for gravitational retention:

$$m_{\text{min}} = \frac{3kTR}{2GM\beta^2}$$

Although Earth's surface temperature is approximately 288 K, atmospheric escape primarily occurs in the outermost layer, the exosphere, where collisions are rare and molecules can travel freely. Due to absorption of solar ultraviolet and X-ray radiation, the exosphere can reach temperatures of $1000 \sim 2000\,\text{K}$. For this analysis, we adopt $T = 1000\,\text{K}$ as a representative value.

Using Earth's parameters:

$$M = 5.972 \times 10^{24}\,\text{kg}, \quad R = 6.371 \times 10^{6}\,\text{m}, \quad \beta = \frac{1}{6}$$

Substituting into the formula:

$$m_{\text{min}} = \frac{3 \cdot (1.380649 \times 10^{-23}) \cdot 1000 \cdot (6.371 \times 10^{6})}{2 \cdot (6.67430 \times 10^{-11}) \cdot (5.972 \times 10^{24}) \cdot \left(\frac{1}{6}\right)^2} \approx 1.19 \times 10^{-26}\,\text{kg}$$

The masses of hydrogen and helium are:

$$\mu_{\text{H}_2} = 3.347 \times 10^{-27}\,\text{kg}, \quad \mu_{\text{He}} = 6.646 \times 10^{-27}\,\text{kg}$$

Since both are significantly below the minimum mass , they are unable to be retained in Earth's exosphere. This explains the absence of hydrogen and helium in Earth's atomspheres.

This conclusion generalizes to other terrestrial planets:

Planet	**Mass (kg)**	**Radius (m)**	**Exosphere Temp (K)**	**Min Capture Mass (kg)**
Mercury	3.3011×10^{23}	2.4397×10^{6}	800	6.61×10^{-26}
Venus	4.8675×10^{24}	6.0518×10^{6}	1500	2.08×10^{-26}
Earth	5.9720×10^{24}	6.3710×10^{6}	1000	1.19×10^{-26}
Mars	6.4171×10^{23}	3.3895×10^{6}	800	4.72×10^{-26}
Moon	7.3420×10^{22}	1.7371×10^{6}	600	1.59×10^{-25}

Table 1: Minimum molecular mass (in kg) required for atmospheric retention on terrestrial planets, based on the criterion $v_{\text{rms}} \leq \beta v_{\text{esc}}$ with $\beta = \frac{1}{6}$.

2 The maximum thermal radiation intensity from Plank's radiation law

We begin with Planck's radiation law in terms of frequency:

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \cdot \frac{1}{e^{h\nu/kT} - 1}$$

To find the frequency ν at which the intensity is maximized, we differentiate B_ν with respect to ν and set the derivative equal to zero:

$$\frac{dB_\nu}{d\nu} = \frac{2h}{c^2} \left[3\nu^2 \cdot \frac{1}{e^{h\nu/kT} - 1} + \nu^3 \cdot \frac{d}{d\nu} \left(\frac{1}{e^{h\nu/kT} - 1} \right) \right] = 0$$

$$\frac{dB_\nu}{d\nu} = \frac{2h}{c^2} \left[\frac{3\nu^2}{e^{h\nu/kT} - 1} - \nu^3 \cdot \frac{h}{kT} \cdot \frac{e^{h\nu/kT}}{(e^{h\nu/kT} - 1)^2} \right] = 0$$

Now, let us define a dimensionless variable:

$$x = \frac{h\nu}{kT} \quad \Rightarrow \quad \nu = \frac{kT}{h} x$$

Substitute into the equation:

$$\frac{3}{e^x - 1} - \frac{xe^x}{(e^x - 1)^2} = 0$$

Multiply both sides by $(e^x - 1)^2$:

$$3(e^x - 1) - xe^x = 0$$

This transcendental equation has a numerical solution:

$$x = \frac{h\nu_{\text{max}}}{kT} \approx 2.82144$$

Solving for ν_{max}:

$$\nu_{\text{max}} = \frac{xkT}{h} = \frac{2.82144 \cdot kT}{h}$$

Using physical constants:

$$k = 1.380649 \times 10^{-23}\ \text{J/K}, \quad h = 6.62607015 \times 10^{-34}\ \text{J} \cdot \text{s}$$

$$\nu_{\text{max}} = \frac{2.82144 \cdot 1.380649 \times 10^{-23}}{6.62607015 \times 10^{-34}} \cdot T \approx 5.88 \times 10^{10} \cdot T \quad (\text{Hz})$$

Final result:

$$\boxed{\nu_{\max} \approx 5.88 \times 10^{10} \cdot T \quad \text{Hz}}$$

Temperature T (K)	$\nu_{\max}$ (Hz)	$\lambda_{\max}$ (m)	Spectrum Region
5700	3.35×10^{14}	8.96×10^{-7}	Visible light
300	1.76×10^{13}	1.70×10^{-5}	Infrared

Table 2: Peak frequency and wavelength of thermal radiation at different temperatures, using $\nu_{\max} = 5.88 \times 10^{10} \cdot T$ and $\lambda_{\max} = c/\nu_{\max}$.

3 Pressure scale height

Starting from the hydrostatic equilibrium equation:

$$\frac{dP}{dz} = -\rho g$$

Use the ideal gas law $P = \rho \frac{RT}{m}$ to substitute for ρ:

$$\frac{dP}{dz} = -\left(\frac{Pm}{RT}\right) g = -P \cdot \frac{Mg}{RT}$$

Comparing with the exponential pressure law:

$$\frac{dP}{dz} = -\frac{P}{H}$$
$$P = P_0 e^{-\frac{z}{H}}$$

we identify the scale height as:

$$\boxed{H = \frac{RT}{mg}}$$

where:

- $R = 8.314\,\mathrm{J/(mol \cdot K)}$ is the gas constant,
- T is temperature in K,
- M is molar mass in kg/mol,
- g is gravitational acceleration in $\mathrm{m/s^2}$.

Planet	m (g/mol)	g (m/s^2)	T (K)	H (m)	Main Gas
Venus	44.0	8.87	737	15,701	CO_2
Mars	44.0	3.71	210	10,696	CO_2
Jupiter	2.3	24.79	165	24,061	H_2/He mix

Table 3: Atmospheric scale height $H = \frac{RT}{Mg}$ for selected planets.

4 Equilibrium temperature

We assume the planet absorbs solar radiation and re-emits it as blackbody radiation. The absorbed power is:

$$(1 - A) \cdot \pi R^2 \cdot S$$

where:

- A is the Bond albedo,
- R is the planet radius,
- S is the solar constant at the planet's orbit.

The planet emits thermal radiation over its full surface area:

$$4\pi R^2 \cdot \varepsilon\sigma T^4$$

At radiative equilibrium (absorbed = emitted), and assuming emissivity $\varepsilon = 1$, we get:

$$(1 - A) \cdot \pi R^2 \cdot S = 4\pi R^2 \cdot \sigma T^4$$

Cancel πR^2 from both sides:

$$(1 - A)S = 4\sigma T^4 \quad \Rightarrow \quad T = \left(\frac{(1 - A)S}{4\sigma}\right)^{1/4}$$

Planet	**Albedo A**	**Solar Constant S (W/m^2)**	**Equilibrium Temperature T (K)**
Venus	0.76	2620	229.46
Earth	0.30	1360	254.53
Mars	0.25	594	210.52

Table 4: Equilibrium temperatures of Venus, Earth, and Mars calculated using $T = \left(\frac{(1-A)S}{4\sigma}\right)^{1/4}$.