Fundamentals of Signal Processing and Data Analysis
Homework 6

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June 13, 2025

1 Z-Transform of the LCCDE

Find the Z-transform of the system described by the linear constant-coefficient difference equation

$$y[n] \;=\; -\sum_{k=1}^{N} a_k \, y[n-k] \;+\; \sum_{k=0}^{M} b_k \, x[n-k].$$

$$\sum_{n=-\infty}^{\infty} y[n] \, z^{-n} = -\sum_{k=1}^{N} a_k \sum_{n=-\infty}^{\infty} y[n-k] \, z^{-n} + \sum_{k=0}^{M} b_k \sum_{n=-\infty}^{\infty} x[n-k] \, z^{-n}$$

$$Y(z) = -\sum_{k=1}^{N} a_k \sum_{m=-\infty}^{\infty} y[m] \, z^{-(m+k)} + \sum_{k=0}^{M} b_k \sum_{m=-\infty}^{\infty} x[m] \, z^{-(m+k)}$$

$$Y(z) = -Y(z) \sum_{k=1}^{N} a_k z^{-k} + X(z) \sum_{k=0}^{M} b_k z^{-k}$$

$$Y(z)\Big(1 + \sum_{k=1}^{N} a_k z^{-k}\Big) = X(z) \sum_{k=0}^{M} b_k z^{-k}$$

$$H(z) \;=\; \frac{Y(z)}{X(z)} = \frac{\displaystyle\sum_{k=0}^{M} b_k z^{-k}}{1 + \displaystyle\sum_{k=1}^{N} a_k z^{-k}}.$$

2 DT System Interpretation

2.1 Time-Domain Representation

Difference Equation:

$$y[n] \;=\; -\sum_{i=1}^{p} a_i \, y[n-i] \;+\; \sum_{i=0}^{q} b_i \, x[n-i].$$

Block Diagram: Replace each delay by a z^{-1} block, each coefficient by a gain block, and sum. This yields the standard direct-form structure, algebraically equivalent to the above recursion.

Impulse Response: For $x[n] = \delta[n]$,

$$y[n] = (\delta * h)[n] = h[n].$$

By LTI theory, for general $x[n]$,

$$y[n] = (h * x)[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k].$$

2.2 Z-Domain Representation

Taking the Z-transform (zero initial conditions) of the difference equation gives

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_q z^{-q}}{1 + a_1 z^{-1} + \cdots + a_p z^{-p}}.$$

- *Difference Eq.* $\leftrightarrow$ *Transfer Fn.*: via ZT.
- *Impulse Resp.* $\leftrightarrow$ *Transfer Fn.*: $H(z) = \mathcal{Z}\{h[n]\}$.
- *Poles & Zeros*: roots of denominator and numerator.

2.3 Frequency-Domain (DTFT)

Evaluating $H(z)$ on the unit circle ($z = e^{j\omega}$) yields the DTFT:

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}},$$

whose magnitude $|H(e^{j\omega})|$ and phase $\angle H(e^{j\omega})$ describe gain and phase vs. ω.

2.4 Analysis Workflow

1. *Implement* in time via the difference equation or block diagram.
2. *Transform* to Z-domain to obtain $H(z)$.
3. *Analyze* via pole–zero plots and $H(e^{j\omega})$.
4. *Convert back* (inverse ZT/DTFT) to recover $h[n]$ or the original recursion.