**Fundamentals of Space Science and Technology Homework 2**

**Student:** 纪浩正, jihz2023@mail.sustech.edu.cn

Consider a gasdynamic flow normal to a simple shock wave as shown on class. Assume that $\gamma = \frac{5}{3}$ and that the flow is time-independent.

# 1 Derive the Density Ratio Across the Shock

Using the time-independent gasdynamic theory to verify that the density jump across the shock wave is given by:

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)M^2}{(\gamma-1)M^2+2}$$

where:

$$M = \frac{u_1}{v_s} \quad \text{and} \quad v_s = \left(\frac{\gamma p_1}{\rho_1}\right)^{1/2}$$

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By mass conservation

$$\rho_1 u_1 = \rho_2 u_2 := a$$

By impluse theorem

$$\begin{aligned} P_1 S\Delta t - P_2 S\Delta t &= \rho_1 u_1 \Delta t\,(u_2 - u_1) \\ P_1 + \rho_1 u_1^2 &= P_2 + \rho_1 u_1 u_2 \\ P_1 + \rho_1 u_1^2 &= P_2 + \rho_2 u_1^2 := b \end{aligned}$$

By energy conservation

$$\begin{aligned} \left(\frac{1}{2}u_1^2 + \epsilon_1 + \frac{P1}{\rho_1}\right)\rho_1 u_1 &= \left(\frac{1}{2}u_2^2 + \epsilon_2 + \frac{P2}{\rho_2}\right)\rho_2 u_2 \\ \frac{1}{2}u_1^2 + \epsilon_1 + \frac{P1}{\rho_1} &= \frac{1}{2}u_2^2 + \epsilon_2 + \frac{P2}{\rho_2} \end{aligned}$$

Here $\epsilon = \dfrac{1}{\gamma-1}\dfrac{P}{\rho}$

$$\frac{1}{2}u_1^2 + \frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1} = \frac{1}{2}u_2^2 + \frac{\gamma}{\gamma-1}\frac{P_2}{\rho_2} := c$$

$$\begin{cases} \rho_2 u_2 = a \\ P_2 + \rho_2 u_2^2 = b \end{cases} \Rightarrow \begin{cases} \rho_2 u_2 = a \\ P_2 + a u_2 = b \end{cases} \Rightarrow \begin{cases} \rho_2 = \dfrac{a}{u_2} \\ P_2 = b - a u_2 \end{cases}$$

$$\frac{1}{2}u_2^2 + \frac{\gamma}{\gamma-1}\frac{P_2}{\rho_2} = c$$

$$\frac{1}{2}u_2^2 + \frac{\gamma}{\gamma-1}\frac{b-au_2}{\frac{a}{u_2}} = c$$

$$\frac{1}{2}u^2 + \frac{\gamma}{\gamma-1}\frac{bu_2}{a} - \frac{\gamma}{\gamma-1}u_2^2 = c$$

$$\frac{\gamma+1}{2(\gamma-1)}u_2^2 - \frac{\gamma}{\gamma-1}\frac{b}{a}u_2 + c = 0$$

Similarily, we have

$$\frac{\gamma+1}{2(\gamma-1)}u_1^2 - \frac{\gamma}{\gamma-1}\frac{b}{a}u_1 + c = 0$$

Then, $u_1, u_2$ are the roots of the equation:

$$\frac{\gamma+1}{2(\gamma-1)}u^2 - \frac{\gamma}{\gamma-1}\frac{b}{a}u + c = 0$$

We must have

$$u_1u_2 = \frac{C}{A}$$

$$u_1u_2 = \frac{c}{\frac{\gamma+1}{2(\gamma-1)}} = \frac{2(\gamma-1)}{\gamma+1}\left(\frac{1}{2}u_1^2 + \frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1}\right)$$

$$\begin{aligned}\frac{u_1}{u_2} &= \frac{u_1^2}{\frac{2(\gamma-1)}{\gamma+1}\left(\frac{1}{2}u_1^2 + \frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1}\right)} \\ &= \frac{(\gamma+1)u_1^2}{(\gamma-1)u_1^2 + 2\gamma\frac{P_1}{\rho_1}} \\ &= \frac{(\gamma+1)\frac{u_1^2\rho_1}{\gamma P_1}}{(\gamma-1)\frac{u_1^2\rho_1}{\gamma P_1} + 2}\end{aligned}$$

$M := \left(\frac{\rho_1}{\gamma P_1}\right)^{\frac{1}{2}} u_1, \quad v_{s1} = \left(\frac{\gamma P_1}{\rho_1}\right)^{\frac{1}{2}}$

$$\frac{u_1}{u_2} = \frac{(\gamma+1)M^2}{(\gamma-1)M^2+2}$$

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma+1)M^2}{(\gamma-1)M^2+2} := h$$

$$
\begin{aligned}
\frac{P_2}{P_1} &= \frac{P_1 + \rho_1 u_1^2 - \rho_2 u_2^2}{P_1} \\
&= \frac{P_1 + \rho_1 u_1^2 - \dfrac{\rho_1 u_2^2}{h}}{P_1} \\
&= \frac{P_1 + \rho_1 u_1^2 - \rho_1 u_1^2 \dfrac{(\gamma - 1) M^2 + 2}{(\gamma + 1) M^2}}{P_1} \\
&= \frac{2\gamma M^2 - (\gamma - 1)}{\gamma + 1}
\end{aligned}
$$

## 2 Show that if $M > 1$, then $\frac{u_2}{v_s} < 1$, which means supersonic flow becomes subsonic across the shock.

Since $M > 1$, then

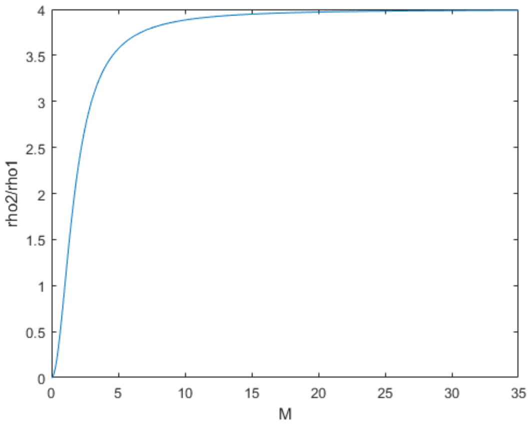
$$
\begin{aligned}
\gamma + 1 &< \gamma M^2 + M^2 \\
\gamma M^2 - M^2 + 2 &< 2\gamma M^2 - \gamma + 1 \\
\left((\gamma - 1) M^2 + 2\right)^{\frac{1}{2}} &< \left(2\gamma M^2 - (\gamma - 1)\right)^{\frac{1}{2}} \\
\left((\gamma - 1) M^2 + 2\right) &< \left(2\gamma M^2 - (\gamma - 1)\right)^{\frac{1}{2}} \left((\gamma - 1) M^2 + 2\right)^{\frac{1}{2}} = \frac{\left(2\gamma M^2 - (\gamma - 1)\right)^{\frac{1}{2}}}{\frac{1}{((\gamma-1)M^2+2)^{\frac{1}{2}}}} \\
u_1 \frac{\left((\gamma - 1) M^2 + 2\right)}{(\gamma + 1) M^2} &< \left( \frac{u_1^2 \dfrac{\left(2\gamma M^2 - (\gamma - 1)\right)}{\gamma + 1}}{M^2 \dfrac{(\gamma + 1) M^2}{(\gamma - 1) M^2 + 2}} \right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\textbf{Since} \frac{\gamma P_1}{\rho_1} = \frac{u_1^2}{M^2}, \quad u_1 \frac{\left((\gamma - 1) M^2 + 2\right)}{(\gamma + 1) M^2} < \left( \frac{\gamma P_1 \dfrac{\left(2\gamma M^2 - (\gamma - 1)\right)}{\gamma + 1}}{\rho_1 \dfrac{(\gamma + 1) M^2}{(\gamma - 1) M^2 + 2}} \right)^{\frac{1}{2}}
$$

$$
u_2 < \left( \frac{\gamma P_2}{\rho_2} \right)^{\frac{1}{2}}
$$

## 3 Plot $\frac{\rho_2}{\rho_1}$ as a function of $M$.

$$
\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1) M^2}{(\gamma - 1) M^2 + 2} = \frac{\dfrac{8}{3} M^2}{\dfrac{2}{3} M^2 + 2}
$$




```
f=@(x) (5/3+1).*x.^2./((5/3-1).*x.^2+2);
x=linspace(0,35,1000);
plot(x,f(x));
xlabel('M');
ylabel('rho2/rho1');
```