

# Topology Notes - Spring 2022

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# 1 General Topology

## 1.1 August 30, 2022

**Definition 1.** Let  $X$  be a set. A topology  $\tau$  on  $X$  is a family of subsets (called open subsets) such that:

1.  $\emptyset, X \in \tau$
2. Finite intersections of open subsets are open
3. Arbitrary unions of open subsets are open

**Definition 2.** Let  $f : X \rightarrow Y$ , where  $X$  and  $Y$  are topological spaces.  $f$  is continuous if the preimage of every open set is an open set,

**Proposition 1.** If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are continuous maps, then  $g \circ f$  is continuous.

*Proof.* If  $U$  is open in  $Z$ , then  $g^{-1}(U)$  is open in  $Y$ . Since  $f$  is continuous,  $f^{-1}(g^{-1}(U))$  is open. Thus,  $U$  open implying  $(g \circ f)^{-1}(U)$  is open gives us that  $g \circ f$  is continuous.  $\square$

**How should we construct topological spaces?:**

**Definition 3.** A basis for a topology is a collection  $\mathcal{B}$  of subsets of a set  $X$  such that:

- $X$  is the union of the elements of  $\mathcal{B}$
- if  $B_1, B_2 \in \mathcal{B}$ , then for every  $x \in B_1 \cap B_2$ , there is a  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$

We define the open sets as arbitrary unions of elements of  $\mathcal{B}$ .

By the first property,  $\emptyset, X \in \mathcal{B}$ . Arbitrary unions are in the topology by definition. Now, suppose  $B_1, B_2 \in \mathcal{B}$ , then  $B_1 \cap B_2 = \bigcup_{x \in B_1 \cap B_2} B_3(x)$ , where  $B_3$  is as prescribed in the second property. We then get by induction that finite intersections are in the topology.

**Example 1.** Consider  $\{(a, b) | a < b, a, b \in \mathbb{R}\}$ . This forms a basis for the standard topology on  $\mathbb{R}$ .

**Example 2.** Consider  $\{(a, b) | a < b, a, b \in \mathbb{Q}\}$ . This ALSO forms a basis for the standard topology on  $\mathbb{R}$ .

**Proposition 2.**  $f : X \rightarrow Y$  is continuous if the preimage of every basis element is open.

**Example 3.** Consider  $C[a, b]$ . Consider the function  $\phi : C[a, b] \rightarrow \mathbb{R}$  such that  $\phi(f) = \int_a^b f(x) dx$  is continuous

**Example 4.** Consider  $L^p(\mathbb{R})$ , where  $p \geq 1$ . Consider  $\phi : L^p(\mathbb{R}) \rightarrow \mathbb{R}$ , where  $\phi(f) = \int_{-\infty}^{\infty} (f(x))^p dx$ .  $\phi$  is continuous.

**Definition 4.** Let  $(X, \tau)$  be a topological space. Consider  $Y \subseteq X$ . We define  $\tau_Y = \{U \cap Y | U \in \tau\}$  to be the subspace topology. The axioms for a topology are easily checked here.

**Proposition 3.** Let  $(X, \tau)$  be a topological space. Let  $\mathcal{B}$  be a family of subsets that satisfy the conditions of a basis. If every element of  $\mathcal{B}$  is open and if for every open set  $U$  and every  $x \in U$ , there is  $B \in \mathcal{B}$  such that  $x \in B \subseteq U$ , then  $\mathcal{B}$  is a basis for  $\tau$

*Proof.* For every  $U \in \tau$ , write  $U = \bigcup_{x \in U} B(x)$ , where  $B$  is prescribed as above.  $\square$

**Example 5.** Consider  $\{(a, b) | a < b, a, b \in \mathbb{R}\}$ . This is the lower limit topology. Note that  $[1, 2)$  is not a union of open intervals, so this is not the standard one.

$$(a, b) = \bigcup_{n=1}^{\infty} [a + \frac{1}{n}, b)$$

**Definition 5.**  $\tau'$  is finer than  $\tau$  if  $U \in \tau \implies U \in \tau'$ . We say here that  $\tau$  is coarser than  $\tau'$ . For instance, the lower limit topology is finer than the standard topology.

**Example 6.** Consider  $\mathcal{B} = \{V_n(a) \mid a \in \mathbb{R}^n, n > 0\}$ .

**Example 7.** Consider  $C[a, b]$ . The basis consists of the sets:

$$\{g \mid \sup_{x \in [a, b]} |f(x) - g(x)| < r\}, \quad f, g \in C[a, b], \quad r > 0$$

**Example 8.** Consider  $L^p(\mathbb{R})$ , for  $p \geq 1$ . The basis consists of the sets:

$$\{g \mid \left( \int_{-\infty}^{\infty} |f - g|^p dx \right)^{\frac{1}{p}} < r\}, \quad f \in L^p(\mathbb{R}), \quad n > 0$$

It is of note that these are great examples of Banach Spaces

## 1.2 September 1, 2022

Consider the following diagram for the product topology:

$$\begin{array}{ccc} Z & \xrightarrow{f} & \prod_{\alpha \in A} X_\alpha \\ & \searrow f_\alpha & \downarrow \pi_\alpha \\ & & X_\alpha \end{array}$$

Note that  $f_\alpha = \pi_\alpha \circ f$ .  $f$  is continuous iff  $f_\alpha$  is continuous for all  $\alpha$ . We want each of the projections  $\pi_\alpha$  to be continuous.

Note that  $\pi_\alpha^{-1}(U_\alpha)$  is open for every open  $U_\alpha \subseteq X_\alpha$ , because:

$$\pi^{-1}(U_\alpha) = U_\alpha \times \prod_{\beta \neq \alpha} X_\beta$$

Hence, a basis for this topology are sets of the form:

$$U_{\alpha_1} \times U_{\alpha_2} \times \cdots \times U_{\alpha_n} \times \prod_{\beta \neq \alpha_i} X_\beta \quad U_{\alpha_i} \in X_{\alpha_i} \text{ open}$$

**Proposition 4.**  $f$  is continuous if and only if  $f_\alpha$  is continuous for every  $\alpha$ .

*Proof.* The forward direction is trivial. Since  $\pi_\alpha$  is continuous,  $f_\alpha = f \circ \pi_\alpha$  is a composition of continuous maps, and is thus continuous. Now, let  $f_\alpha$  be continuous for every  $\alpha$ . We only need to check that the preimage of a basis element is an open set.

$$\begin{aligned} f^{-1} \left( U_{\alpha_1} \times U_{\alpha_2} \times \cdots \times U_{\alpha_n} \times \prod_{\beta \neq \alpha_i} X_\beta \right) &= f^{-1} \left( \bigcap_{i=1}^n \left( U_{\alpha_i} \times \prod_{\beta \neq \alpha_i} X_\beta \right) \right) \\ &= \bigcap_{i=1}^n f^{-1} \left( U_{\alpha_i} \times \prod_{\beta \neq \alpha_i} X_\beta \right) \\ &= \bigcap_{i=1}^n f_{\alpha_i}^{-1}(U_{\alpha_i}) \end{aligned}$$

This is open because the preimages of  $U_\alpha$  by  $f_\alpha$  are open, and finite intersections of open sets are open  $\square$

**Example 9.** Consider  $C[a, b] \subseteq \prod_{x \in [a, b]} \mathbb{R}$ . This topology has as basis sets of the form:

$$V_{f, x_1, \dots, x_n, \epsilon} = \{g : |f(x_i) - g(x_i)| < \epsilon, i \in \mathbb{N}\}$$

This is called the weak\* topology on  $C[a, b]$ .

**Example 10.** Consider  $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ . we can define this to have multiplication, addition, or division defined on  $\mathbb{R} \times \mathbb{R} \setminus \{0\}$ .

**Proposition 5.** If  $X$  is a topological space and  $f, g : X \rightarrow \mathbb{R}$  (with maybe  $g : X \rightarrow \mathbb{R} \setminus \{0\}$ ) are continuous functions, then  $f + g, f \cdot g, \frac{f}{g}$  are continuous.

*Proof.* These are compositions of continuous functions. We consider:

$$X \xrightarrow{(f, g)} \mathbb{R} \times \mathbb{R} \xrightarrow{+} \mathbb{R}$$

The same happens for multiplication and division. □

### Disjoint Unions of Topological Spaces:

Consider  $X_\alpha, \alpha \in A, X_\alpha \cap X_\beta = \emptyset$ . We then denote the disjoint union as  $\bigsqcup_{\alpha \in A} X_\alpha$ . Consider the following diagram:

$$\begin{array}{ccc} \bigsqcup_{\alpha \in A} X_\alpha & \xrightarrow{f} & Y \\ \uparrow & \nearrow f_\alpha & \\ X_\alpha & & \end{array}$$

Similarly as before,  $f$  is continuous if and only if  $f_\alpha$  is continuous for all  $\alpha$ . A topology consists of the sets that are unions of  $U_\alpha$  where  $U_\alpha$  are open in  $X_\alpha$ . If the  $X_\alpha$ 's are not disjoint, then we might run into trouble in the overlap!

**Quotient Spaces** Consider a topological space  $X$ . Take  $f : X \rightarrow Y$  to be a surjective function. The open sets of  $Y$  are of the form  $f(U)$  where  $U$  is open in  $X$ .

Another perspective: Consider a topological space  $X$ . Take an equivalence relation on  $X$ . Denote by  $\hat{x}$  the equivalence class of  $x$ . Let:

$$Y = \{\hat{x} \mid x \in X\}$$

Now, we consider the function  $f : X \rightarrow Y$  where  $f(x) = \hat{x}$ , and we use the notation  $X/\sim$  to represent this quotient space. Now, consider the following diagram:

$$\begin{array}{ccc} X & \xleftarrow{g \circ f} & Z \\ f \downarrow & \nwarrow g & \\ Y & & \end{array}$$

We want  $g$  to be continuous if and only if  $g \circ f$  is continuous, and this is the reason for our construction of the quotient space.

**Proposition 6.** The above is a topology on  $Y$ .

*Proof.* (a)  $Y = f(X)$  is open, and  $\emptyset = f(\emptyset)$  is open

(b)  $f(U_1 \cap U_2 \cap \cdots \cap U_n)$  is a finite intersection of  $f(U_i)$ 's, so this is open

(c) If  $f(U_\alpha)$  is open, for all  $\alpha$ , then it's easy to see that the images commute with the union, and so arbitrary unions are open. □

**Example 11.** Take  $f : \mathbb{R} \rightarrow \mathbb{C}$ . This may not be a surjection, but it is certainly a surjection onto the image by construction. Let  $f(x) = e^{2\pi i x}$ . The image is  $\{z : |z| = 1\} =: S^1$  (the 1-dimensional sphere). We can induce a topology on  $S^1$  by using the standard topology on  $\mathbb{R}$ , or by inducing from the topology on  $\mathbb{C}$ . The induced topology is the same as the quotient topology. This can be resolved by thinking of the standard convention where  $S^1 = \mathbb{R}/\mathbb{Z}$ .

**Example 12.** Consider  $S^2$ , the sphere in  $\mathbb{R}^3$ . Consider the inclusion of  $S^2 \subseteq \mathbb{R}^3$ , and consider the induced topology on  $S^2$ . We can consider two points on the sphere to be equivalent as follows:

$$(x, y, z) \sim (x', y', z') \iff (x = x', y = y', z = z') \vee (x = -x', y = -y', z = -z')$$

Let  $\mathbb{R}P^2 = S^2 / \sim$ , and consider the quotient topology. This space is essentially a cap being put on top of a Möbius band. By definition, this is the two-dimensional projective plane. Another way to construct  $\mathbb{R}P^2$ :

consider  $\mathbb{R}^3 \setminus \{(0, 0, 0)\} \subseteq \mathbb{R}^3$  using the standard topology. We take the following equivalence:

$$(x, y, z) \sim (x', y', z') \iff \exists \lambda \neq 0 : x = \lambda x', y = \lambda y', z = \lambda z'$$

Then,  $(\mathbb{R}^3 \setminus \{(0, 0, 0)\}) / \sim$  is exactly  $\mathbb{R}P^2$ . We can then generalize this construction to  $\mathbb{R}P^n$  using the same procedure, but we can also have  $\mathbb{C}P^n$  to be equivalence classes of points in  $\mathbb{C}^{n+1} \setminus \{(0, 0, \dots, 0)\}$ , with the same equivalence as before, except now  $\lambda \in \mathbb{C}$ . For instance  $\mathbb{C}P^1$  is the Riemann sphere.

**Example 13.** Consider the unit square with corners  $(0, 0), (0, 1), (1, 0), (1, 1)$ . We induce an equivalence relation by  $(x, 1) \sim (x, 0)$  for every  $x$ , and  $(1, y) \sim (0, y)$  for every  $y$ . This generates a torus. This comes from the induced topology from the inclusion into  $\mathbb{R}^2$ , so the torus is the quotient topology  $\mathbb{R}^2 / \mathbb{Z}^2$ . The torus can also be described as  $S^1 \times S^1$ .

**Definition 6.**  $f : X \rightarrow Y$  is a homeomorphism if  $f$  is continuous, invertible, and  $f^{-1}$  is continuous.

**Example 14.**  $([0, 1] \times [0, 1]) / \sim$  is homeomorphic to  $S^1 \times S^1$ , where  $\sim$  is the equivalence relation defined in Example 11.

**Example 15.** Consider  $[1, 2], [3, 4] \subseteq \mathbb{R}$  with the induced topology. Then, take the disjoint union  $[1, 2] \sqcup [3, 4]$ . Then, we take the quotient where  $1 \sim 2 \sim 3 \sim 4$ . This gives us a figure 8, where the circles touch at  $\{1, 2, 3, 4\}$ .

**Example 16.** Let  $X$  be a topological space. Consider  $[-1, 1]$  with the induced topology. Consider  $X \times [-1, 1]$  with the product topology. We then consider  $(x, 1) \sim (y, 1)$  and  $(x, -1) \sim (y, -1)$  for every  $x$  and  $y$ , and take the quotient topology. This then gives a two-sided cone with a circular middle, called the suspension  $\Sigma X$ . If we consider  $S^n$ , then  $\Sigma S^n \cong S^{n+1}$ .

### 1.3 September 6, 2022

Thus far, as a recap, we have covered the following constructions:

- Induced Topology
- Product Topology
- Disjoint Sum
- Quotients

We now discuss metric spaces.

**Definition 7.** A distance on a set  $X$  is a function  $d : X \times X \rightarrow [0, \infty)$  such that:

- (a)  $d(x, y) = 0 \iff x = y$
- (b)  $d(x, y) = d(y, x)$
- (c)  $d(x, z) \leq d(x, y) + d(y, z)$

A metric is then another word we use instead of “distance”.  $(X, d)$  is thus termed a metric space.

**Example 17.**  $(\mathbb{R}^n, d)$ , where  $d(\vec{x}, \vec{y}) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$

**Example 18.**  $(\mathbb{R}^n, d)$ , where  $d(\vec{x}, \vec{y}) = \max_{1 \leq i \leq n} |x_i - y_i|$

**Example 19.**  $([0, 1]^A, d)$ , where  $(x_\alpha)_\beta, (y_\alpha)_\beta \in \prod_{\beta \in B} [0, 1]$ , and:

$$d((x_\alpha)_\alpha, (y_\alpha)_\alpha) = \sup_\alpha |x_\alpha - y_\alpha|$$

A basis for the topology induced by the metric consists of the balls (for  $x \in X$  and  $\epsilon > 0$ ):

$$B(x, \epsilon) = \{y | d(x, y) < \epsilon\}$$

We now prove that these open balls form a basis.

*Proof.* First, we must check that these cover the entire space. However, it is clear that  $X = \bigcup_{x \in X} B(x, \epsilon)$ .

For another condition, we see that if  $B_1 = B(x_1, \epsilon_1)$  and  $B_2 = B(x_2, \epsilon_2)$  are elements of the basis, and if  $x \in B_1 \cap B_2$ , we have to show that there is a  $B_3 = B(x_3, \epsilon_3)$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ . We can construct this geometrically such that we pick  $x = x_3$  and  $\epsilon_3 < \min(\epsilon_1 - d(x, x_1), \epsilon_2 - d(x, x_2))$ . Then, if  $y \in B_3$ , we have that:

$$\begin{aligned} d(y, x_1) &< d(y, x) + d(x, x_1) \\ &< \epsilon_3 + d(x, x_1) \\ &< \epsilon_1 - d(x, x_1) + d(x, x_1) \\ &= \epsilon_1 \end{aligned}$$

Thus,  $y \in B_1$ , and by the exact same argument,  $y \in B_2$ , so we have indeed proven that this does form a basis.  $\square$

We now note that open sets are unions of balls.

**Example 20.** On  $\mathbb{R}^n$ , the standard metric and the max metric from Examples 17 and 18 induce the same topology.

**Proposition 7.** If two metrics have the property that for every ball of the first and every element of it, there is a ball of the second topology centered at this element and included in the first ball, then the second topology is finer than the first.

$$x \in B'(x, \epsilon') \subseteq B(y, \epsilon), \quad \forall x \in B(y, \epsilon)$$

In Example 20, a disc can be written as a union of squares, and vice versa.

**Example 21.** If  $(X, d)$  is a metric space, define:

$$\bar{d}(x, y) = \min(d(x, y), 1)$$

Then,  $\bar{d}$  is a metric that induces the same topology.

We first check that  $\bar{d}$  is a metric by ensuring that it satisfies the triangle inequality. If  $d(x, y), d(y, z), d(x, z) < 1$ , then:

$$\begin{aligned} \bar{d}(x, z) &= d(x, z) \\ &\leq d(x, y) + d(y, z) \\ &= \bar{d}(x, y) + \bar{d}(y, z) \end{aligned}$$

We have a similar case where  $d(x, y)$  or  $d(y, z) > 1$ , because we then have  $1 \leq \bar{d}(x, y) + \bar{d}(y, z)$  holds. Finally, if  $d(x, y), d(y, z) < 1$ , and  $d(x, z) > 1$ , then:

$$\bar{d}(x, z) \leq d(x, z) < d(x, y) + d(y, z) = \bar{d}(x, y) + \bar{d}(y, z)$$

**Theorem 1.** If  $(X, d), (Y, d')$  are metric spaces, then  $f : X \rightarrow Y$  is continuous at  $x \in X$  if  $\forall \epsilon > 0, \exists \delta > 0$  such that  $f(B(x, \delta)) \subseteq B(f(x), \epsilon)$ .

**Definition 8.**  $f : X \rightarrow Y$  is called uniformly continuous if  $\forall \epsilon > 0, \exists \delta > 0$  such that if  $d(x, y) < \delta$ , then  $d'(f(x), f(y)) < \epsilon$ .

**Definition 9.** A sequence  $(x_n)_{n \geq 1}$  in the metric space  $(X, d)$  is called Cauchy if  $\forall \epsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that for all  $m, n \geq N$ ,  $d(x_m, x_n) < \epsilon$ .

**Definition 10.** A metric space is called complete if every Cauchy sequence is convergent.

**Definition 11.** A normed vector space  $V$  is a vector space endowed with a function  $\|\cdot\| : V \rightarrow [0, \infty)$  such that:

$$(a) \quad \|x\| = 0 \iff x = 0$$

$$(b) \quad \|\lambda x\| = |\lambda| \|x\|, \quad \lambda \in \mathbb{R} \vee \mathbb{C}$$

$$(c) \quad \|x + y\| \leq \|x\| + \|y\|$$

We define  $d(x, y) = \|x - y\|$ .

**Definition 12.** A Banach space is a complete normed vector space.

**Example 22.**  $C[0, 1]$ ,  $L^p$  spaces, Sobolev spaces, Hardy spaces, Bergman spaces.

## Manifolds:

**Definition 13** (Veblen). A manifold  $M$  is a Hausdorff topological space endowed with an onto function

$$f : \bigsqcup_{\alpha \in A} U_{\alpha} \rightarrow M$$

such that for every  $\alpha$ ,  $U_{\alpha}$  is an open set in some  $\mathbb{R}^n$ , where  $n$  is the same for all  $\alpha$ , and  $f|_{U_{\alpha}}$  is a homeomorphism onto the image for each  $\alpha$ .

We can define charts here as something of the form:

$$f_{\alpha} = f|_{U_{\alpha}} : U_{\alpha} \rightarrow f(U_{\alpha}) \subseteq M$$

A collection of charts is then an atlas. Also, we get compositions where:

$$f_{\beta}^{-1} \circ f_{\alpha} : U_{\alpha} \rightarrow U_{\beta}$$

**Definition 14.** We call a topological space Hausdorff if for every  $x \neq y$ , there are open sets  $U, V$  such that  $x \in U$  and  $y \in V$ , and  $U \cap V = \emptyset$ .

- If  $f_{\beta}^{-1} \circ f_{\alpha}$  is smooth, then the manifold is called smooth.
- If  $\mathbb{R}^n$  is replaced by  $\mathbb{C}^n$  and  $f_{\beta}^{-1} \circ f_{\alpha}$  is holomorphic or analytic then  $M$  is called complex.

**Example 23.** The circle  $S^1 = \{z : |z| = 1\}$ . Define  $f : \mathbb{R} \rightarrow \mathbb{C}$  such that  $f(t) = e^{2\pi it}$ . Let  $U_1 = (-\pi, \pi)$  and  $U_2 = (0, 2\pi)$ . Now, we consider a function  $g : U_1 \sqcup U_2 \rightarrow \mathbb{C}$ , where  $g(t) = e^{2\pi it}$ . We can consider  $f(U_1) \cap f(U_2)$  as the overlap, and so  $f_1^{-1}(f(U_1) \cap f(U_2)) = (\pi, 0) \cup (0, \pi)$ , and  $f_2^{-1}(f(U_1) \cap f(U_2)) = (\pi, 2\pi) \cup (0, \pi)$ .

Then,  $f_2^{-1} \circ f_1 : (-\pi, 0) \cup (0, \pi) \rightarrow (\pi, 2\pi) \cup (0, \pi)$  is as follows:

$$f_2^{-1} \circ f_1(t) = \begin{cases} t & , t \in (0, \pi) \\ t + 2\pi & , t \in (-\pi, 0) \end{cases}$$

**Theorem 2.** If  $M_1, M_2$  are manifolds of dimensions  $n_1, n_2$ , then  $M_1 \times M_2$  is a manifold of dimension  $n_1 + n_2$ .

Consider:

$$\begin{aligned} f &: \bigsqcup U_{\alpha} \rightarrow M_1 \\ g &: \bigsqcup V_{\beta} \rightarrow M_2 \\ f \times g &: \bigsqcup U_{\alpha} \times V_{\beta} \rightarrow M_1 \times M_2 \\ (f \times g)_{\alpha\beta} &= (f_{\alpha}, g_{\beta}) \end{aligned}$$

**Example 24.**  $(S^1)^{\times n}$  is a manifold. In the  $n = 2$  case, we get  $(-\pi, \pi)^2 \sqcup (-\pi, \pi) \times (0, 2\pi) \sqcup (0, 2\pi) \times (-\pi, \pi) \sqcup (0, 2\pi)^2$  as our domain, where  $f(t, s) = (e^{2\pi it}, e^{2\pi is})$ , and this gives us the torus.



## 1.4 September 8, 2022

**Example 25.** Consider  $\mathbb{R}P^2 = \{\hat{x} : x \in \mathbb{R}^3 \setminus \{0\}, x \sim y \iff x = \lambda y, \lambda \neq 0\}$ .  $\mathbb{R}P^2$  is a manifold, with the following three charts:

$$U_1 = U_2 = U_3 = \mathbb{R}^2$$

$$f : \bigsqcup_{i=1}^3 U_i \rightarrow \mathbb{R}P^2$$

$$f_1(x_1, x_2) = [1 : x_1 : x_2]$$

$$f_2(x_1, x_2) = [x_1 : 1 : x_2]$$

$$f_3(x_1, x_2) = [x_1 : x_2 : 1]$$

These maps are one-to-one, and onto. We note that  $f_2^{-1} \circ f_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is the map  $(x_1, x_2) \xrightarrow{f_1} [1 : x_1 : x_2] = \left[\frac{1}{x_1} : 1 : \frac{x_2}{x_1}\right] \xrightarrow{f_2^{-1}} \left(\frac{1}{x_1}, \frac{x_2}{x_1}\right)$

**Example 26.** Consider  $\mathbb{C}P^1$ . We have that  $U_1 = U_2 = \mathbb{C}$ . We define maps as:

$$f_1 : U_1 \rightarrow \mathbb{C}P^1$$

$$f_1(z) = [1 : z]$$

$$f_2 : U_2 \rightarrow \mathbb{C}P^1$$

$$f_2(z) \rightarrow [z : 1]$$

$$f_2^{-1} \circ f_1 : z \rightarrow [1 : z] \rightarrow \left[\frac{1}{z} : 1\right] \rightarrow \frac{1}{z}$$

**Definition 15** (Poincaré). A smooth manifold is a subspace of  $\mathbb{R}^n$  that is locally the graph of a smooth function.

**Example 27.** Consider  $S^2$ . Consider the graph  $f(x_1, x_2) = \pm\sqrt{1 - x_1^2 - x_2^2}$ , and similar considerations for the two other orientations. These graphs act as charts from the plane to the sphere.

Now, we return to closed sets!

**Example 28.**  $[a, b], [a, \infty)$

**Example 29.**

$$C = [0, 1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right)$$

This set consists of all numbers in  $[0, 1]$  that admit a ternary representation with only 0's and 2's. Consider a function:

$$f : C \rightarrow [0, 1]^2$$

$$f(0.a_1a_2a_3\cdots) = (0.\frac{a_1}{2}\frac{a_3}{2}\cdots, 0.\frac{a_2}{2}\frac{a_4}{2}\cdots)$$

This function is then continuous and onto from  $C$  to  $[0, 1]^2$ . By Tietze's Extension Theorem, we get an extension  $\tilde{f} : [0, 1] \rightarrow [0, 1]^2$  which is continuous and onto. This was induced by G. Peano.

**Example 30.** Sierpiński Triangle.

**Example 31.** In the discrete topology, every set is clopen

**Example 32.**  $\mathbb{Q} \subseteq \mathbb{R}$ ,  $(a, b)$  are clopen in  $\mathbb{Q}$ .

**Example 33.** In  $\mathbb{R}^n$ , consider  $\overline{B}(x, \epsilon) = \{y : d(x, y) \leq \epsilon\}$  as the closed balls.

**Example 34.** The Zariski topology, where  $f(z_1, z_2, \dots, z_n) \in \mathbb{C}[z_1, \dots, z_n]$ , and:

$$V(f) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : f(z_1, \dots, z_n) = 0\}$$

These are called algebraic sets, or varieties if  $f$  is irreducible, and these are the closed sets of the Zariski topology. Unions of algebraic sets are the closed sets of the Zariski topology.

**Proposition 8** (MAYBE ON EXAM 1). (1) If  $Y$  is a subspace of  $X$ , then  $A \subseteq Y$  is closed if and only if  $A = B \cap Y$  where  $B \subseteq X$  is closed.

(2) Let  $A \subseteq Y \subseteq X$ . If  $A$  is closed in  $Y$  and  $Y$  is closed in  $X$ , then  $A$  is closed in  $X$ .

(3) If  $A$  is closed in  $X$  and  $B$  is closed in  $Y$ , then  $A \times B$  is closed in  $X \times Y$ .

(4) If  $A_\alpha$  is closed in  $X_\alpha$  for  $\alpha \in I$ , then  $\prod_{\alpha \in I} A_\alpha$  is closed in  $\prod_{\alpha \in I} X_\alpha$  with the product topology.

*Proof.* (1)  $Y \setminus A$  is open. So there is a  $U$  such that  $Y \setminus A = Y \cap U$ . Let  $B = X \setminus U$ . Then

$$A = Y \setminus (Y \setminus A) = Y \setminus (Y \cap U) = Y \setminus U = Y \cap (X \setminus U) = B \cap Y$$

Now, if  $A = B \cap Y$ , then  $Y \setminus A = (X \setminus B) \cap Y$ .

(2)  $A$  closed in  $Y$  and  $Y$  closed in  $X$  gives us that  $X \setminus Y = U$  is open in  $X$ . Then,  $A = Y \cap (X \setminus V)$ . Thus  $X \setminus A = V \cup U$ , so since this is a union of two open sets, it is open, and thus  $A$  is closed in  $X$ .

(3)  $(X \times Y) \setminus (A \times B) = (X \times (Y \setminus B)) \cup ((X \setminus A) \times Y)$ . These sets are all open, so  $A \times B$  is closed.

(4) Induction! Do the same thing as above as unions.

$$\left( \prod_{\alpha \in I} X_\alpha \right) \setminus \left( \prod_{\alpha \in I} A_\alpha \right) = \bigcup_{\alpha \in I} \left( \left( \prod_{\beta \neq \alpha} X_\beta \right) \times (X_\alpha \setminus A_\alpha) \right)$$

□

**Proposition 9.** Let  $X, Y$  be topological spaces. Then,  $f : X \rightarrow Y$  is continuous if and only if the preimage of every closed set is closed.

*Proof.*

$$f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$$

□

**Definition 16.** The closure of a set  $S$  is the smallest closed set containing  $S$ , denoted  $\overline{S}$ .

**Definition 17.** The interior of a set  $S$  is the largest open set contained in  $S$ , denoted  $\text{Int}(S)$ .

**Example 35.**  $\overline{\mathbb{Q}} = \mathbb{R}$ .  $\text{Int}(\mathbb{Q}) = \emptyset$

**Example 36.**  $B(x, r) = \{y : d(x, y) < r\} \implies \overline{B(x, r)} = \{y : d(x, y) \leq r\}$ .

**Lemma 1.** Let  $X$  be a topological space, and  $A \subseteq X$ . Then,  $\overline{X \setminus A} = X \setminus \text{Int}(A)$

*Proof.*

$$X \setminus A \subseteq X \setminus \text{Int}(A) \implies \overline{X \setminus A} \subseteq \overline{X \setminus \text{Int}(A)} = X \setminus \text{Int}(A)$$

Now, let  $x \in X \setminus \overline{(X \setminus A)}$ . Then there is an open set  $U$  such that  $x \in U$  and  $U \cap \overline{(X \setminus A)} = \emptyset$ . Thus,  $U \cap (X \setminus A) = \emptyset$ , so  $U \subseteq A$ . Therefore,  $x \in \text{Int}(A)$ . Hence,  $x \notin X \setminus \text{Int}(A)$ , so  $X \setminus \text{Int}(A) \subseteq \overline{X \setminus A}$ , and we have the double inclusion as desired. □

**Theorem 3.** Let  $A \subseteq X$ . Then,  $x \in \overline{A}$  if and only if for every open set  $U$  such that  $x \in U$ , we have  $U \cap A \neq \emptyset$ .

*Proof.* First, we wish to show that If  $x \in \overline{A}$  and  $U$  is open where  $x \in U$ , then  $U \cap A \neq \emptyset$ . If, for the sake of contradiction,  $U \cap A = \emptyset$ , then  $U \subseteq \text{Int}(X \setminus A)$ , so  $X \setminus \text{Int}(X \setminus A) = \overline{X \setminus (X \setminus A)} = \overline{A}$ , which does not contain  $x$ , and so we get a contradiction.

Conversely, if  $x$  has the property that every open set  $U$  containing  $x$  intersects  $A$ , then  $x \notin \text{Int}(X \setminus A)$ . So  $x \in X \setminus \text{Int}(X \setminus A) = \overline{X \setminus (X \setminus A)} = \overline{A}$ . Thus, we are done.  $\square$

**Proposition 10** (Midterm maybe). 1) Let  $Y \subseteq X$ , with the subspace topology. If  $A \subseteq Y$ , then let us denote  $\overline{A}_X$  as the closure of  $A$  in  $X$ . Then, the closure of  $A$  in  $Y$  is  $\overline{A}_X \cap Y$ .

2) Taking the same conditions as above, if  $Y$  is closed in  $X$ , then the closure of  $A$  in  $X$  and  $Y$  is the same.

$$3) \prod_{\alpha \in I} \overline{A}_\alpha = \overline{\prod_{\alpha \in I} A_\alpha}$$

*Proof. 1):*

Let  $x \in \overline{A}_X \cap Y$ . Then, for every open set  $U$  in  $X$  containing  $x$ ,  $U \cap A \neq \emptyset$ . So  $(U \cap Y) \cup A \neq \emptyset$ . But  $U \cap Y$  with  $U$  open in  $X$  are all open subsets of  $Y$ . By Theorem 3,  $x$  is in the closure of  $A$  in  $Y$ . Thus,  $\overline{A}_X \cap Y \subseteq \overline{A}_Y$ .

Conversely,  $\overline{A}_X \cap Y$  is closed and contains  $A$ , so the closure of  $A$  in  $Y$  is contained in  $\overline{A}_X \cap Y$ .

2):

if  $Y$  is closed, then  $\overline{A}_X \subseteq \overline{Y} = Y$ . So  $\overline{A}_X \cap Y = \overline{A}_X$ .

3):

Let  $x = (x_\alpha)_\alpha \in \prod_{\alpha \in I} \overline{A}_\alpha$ . Let  $\prod_{j=1}^n U_{\alpha_j} \times \prod_{\beta \neq \alpha_j} X_\beta$ , or  $\prod_{\alpha \in I} U_\alpha$  such that  $x_\alpha \in U_\alpha$ . Then, by Theorem 3,  $U_\alpha \cap A_\alpha \neq \emptyset$ , for all  $\alpha$ , and  $X_\beta \cap A_\beta \neq \emptyset$ , for all  $\beta \neq \alpha_j$ . It follows that in either case, the open set intersects  $\prod_{\alpha \in I} A_\alpha$ . Thus,  $\prod_{\alpha \in I} \overline{A}_\alpha \subseteq \overline{\prod_{\alpha \in I} A_\alpha}$ . Conversely, let  $x \in \overline{\prod_{\alpha \in I} A_\alpha}$ . Then, every open set containing  $x$  intersects  $\prod_{\alpha \in I} A_\alpha$ . Choose  $U$  of either form as expressed in the previous direction, and we get coordinate-wise nonempty intersections. Vary  $U_\alpha$ 's to make them be any open set that contains  $x_\alpha$ . You then obtain the reverse inclusion, and thus we are done.  $\square$

This result does not hold for interiors as seen in the following examples:

**Example 37.**  $\mathbb{Q} \subseteq \mathbb{R}$ .  $\text{Int}(\mathbb{Q}) \neq \emptyset$ , but the interior of  $\mathbb{Q}$  in  $\mathbb{Q}$  is  $\emptyset$ .

However, the third case of the proposition works for interiors in the box topology but not the product topology.

**Example 38.**  $[0, 1] \subseteq \mathbb{R}$ ,  $\text{Int}([0, 1]) = (0, 1)$ . However:

$$\underbrace{\prod_{i=1}^{\infty} (0, 1)}_{\text{not an open set}} \subseteq \prod_{i=1}^{\infty} \mathbb{R}$$

**Proposition 11.** *Let  $X, Y$  be topological spaces.  $f : X \rightarrow Y$  is continuous if and only if for every subset  $A \subseteq X$ , we have  $f(\overline{A}) \subseteq \overline{f(A)}$*

*Proof.* Assume  $f$  is continuous. Then  $f^{-1}(Y \setminus \overline{f(A)})$  is open. This is the complement of  $f^{-1}(\overline{f(A)})$ , so this set is open. Then  $A \subseteq f^{-1}(\overline{f(A)})$ , so  $\overline{A} \subseteq f^{-1}(\overline{f(A)}) \implies f(\overline{A}) \subseteq \overline{f(A)}$ .

Conversely, let  $B \subseteq Y$  be open. we want to show  $f^{-1}(B)$  is open. Let  $A = X \setminus f^{-1}(B)$ . Then:

$$f(\overline{A}) \subseteq \overline{f(A)} = \overline{f(X \setminus f^{-1}(B))} = \overline{Y \setminus B} = Y \setminus B$$

Thus,  $f(\overline{A}) \subseteq Y \setminus B$ , but  $A = X \setminus f^{-1}(B)$ , so  $A = \overline{A}$ , and we are done  $\square$

**Definition 18.**  $x$  is a limit point for  $A$  if for every  $U$  open such that  $x \in U$ , there is  $x' \neq x$  such that  $x' \in A \cap U$ . We denote  $A'$  to be the set of limit points of  $A$

**Example 39.**

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$$

$A$  has only 1 limit point, namely 0.

**Proposition 12.**

$$\overline{A} = A \cup A'$$

**Corollary 1.** *A subset of a topological space is closed if and only if it contains all of its limit points.*

**Definition 19.** In an arbitrary topological space, one says that a sequence is convergent to  $x \in X$  if for every neighbourhood  $V$  of  $x$ , there is an  $N$  such that for all  $n \geq N$ ,  $x_n \in V$

Note: IN the Zariski topology, a sequence with no constant subsequence converges to every point in the space. The definition is fine in metric spaces. In every metric space, the limit is unique.

**Proposition 13.** *In metric spaces, the limit is unique.*

*Proof.* If  $x_1, x_2$  are both limits of a sequence, consider the open ball with radius  $\frac{d(x_1, x_2)}{2}$ . All but finitely many terms lie in each of the disjoint balls.  $\square$

**Lemma 2** (Sequence Lemma). *Let  $X$  be a metric space.*

(a)  $x \in \overline{A}$  if and only if there is a sequence in  $A$  converging to  $x$ .

(b)  $x \in A'$  if and only if there is a sequence of points in  $A$  that does not eventually become constant that converges to  $x$ .

*Proof.* Recall that  $\overline{A} = A \cup A'$ . If  $x \in A$ , then  $x_n = x$  for all  $n$  converges to  $x$ . Thus, it suffices to prove part (b).

Let  $x \in A'$ . For  $n \in \mathbb{N}$ , consider  $B(x, \frac{1}{n})$ . Since  $x \in A'$ , there is  $x_n \in A \cap B(x, \frac{1}{n})$  such that  $x_n \neq x$ . Now, let  $\epsilon > 0$ , and choose  $K(\epsilon)$  so that  $\frac{1}{K(\epsilon)} < \epsilon$ . Then, for  $n \geq K(\epsilon)$ ,  $x_n \in B(x, \frac{1}{n}) \subseteq B(x, \frac{1}{K(\epsilon)}) \subseteq B(x, \epsilon)$ . Conversely, if  $x_n \rightarrow x$ ,  $x_n \in A$ ,  $x_n$  not eventually constant. We consider an open set  $U$  containing  $x$ . Let  $B(x, \epsilon) \subseteq U$ . There is a  $K(\epsilon)$  such that for all  $n \geq K(\epsilon)$ ,  $x_n \in B(x, \epsilon)$ . From these, we can choose a term that is not  $x$ . The definition of  $a'$  is thus fulfilled, and so we are done  $\square$

**Theorem 4.** *Let  $X, Y$  be metric spaces. Then,  $f : X \rightarrow Y$  is continuous if and only if for every convergent sequence  $x_n \in X$ , the sequence  $f(x_n)$  is convergent in  $Y$*

*Proof.* For  $x_n \rightarrow x$ , consider  $B(f(x), \epsilon)$ . Then  $f^{-1}(B(f(x), \epsilon))$  is an open neighbourhood of  $x$ . Thus, there is an integer  $K(\epsilon) \in \mathbb{N}$  such that for all  $n \geq K(\epsilon)$ ,  $x_n \in f^{-1}(B(f(x), \epsilon))$ . Thus, for all  $n \geq K(\epsilon)$ ,  $f(x_n) \in B(f(x), \epsilon)$ , so  $f(x_n)$  is convergent in  $Y$ .

Conversely, assume that  $x_n$  converges to  $x$ . Consider  $x_1, x, x_2, x, x_3, x, \dots$ . This converges to  $x$ . Then  $f(x_1), f(x), f(x_2), f(x), \dots$  converges by hypothesis. Since it has a constant subsequence equal to  $f(x)$ , the sequence itself converges to  $f(x)$ . The conclusion then follows from Proposition 11.  $\square$

**Definition 20.**  $X$  is Hausdorff if for every  $x, y \in X$ ,  $x \neq y$  there are open sets  $U, V$  such that  $x \in U$ ,  $y \in V$ ,  $U \cap V = \emptyset$ .

**Theorem 5.** If  $X$  and  $Y$  are homeomorphic, and  $X$  is Hausdorff, then  $Y$  is Hausdorff.

*Proof.* Consider  $h : Y \rightarrow X$ , a homeomorphism. Let  $x, y \in Y$ . Then  $f(x) \neq f(y)$ . There are disjoint open sets  $U$  and  $V$  in  $X$  containing these 2 points separately. The open sets  $f^{-1}(U)$  and  $f^{-1}(V)$  contain  $x$  and  $y$  respectively and are disjoint.  $\square$

**Remark 1.** Hausdorff is a topological property.

**Example 40.** Consider  $\mathbb{C}^n$  with the standard topology and  $\mathbb{C}^n$  with the Zariski Topology. These are NOT homeomorphic. One is Hausdorff and the other is not.

**Definition 21.** A topological space  $X$  is not connected if there exist two open sets  $U, V$  such that  $U \cup V = X$  and  $U \cap V = \emptyset$ .

**Example 41.**  $(-\infty, 0) \cup (0, \infty)$  is not connected.

If a space is not “not connected”, then it is connected.

**Example 42.**  $\mathbb{Q}$  is not connected.

**Proposition 14.** (a) If  $A, B \subseteq X$  are disjoint subsets such that  $A \cup B = X$  and neither of these sets contains a limit point of the other. Then, they form a separation of  $X$ .

(b) If  $U, V$  form a separation of  $X$  and if  $Y \subseteq X$ ,  $Y$  connected, then  $Y$  lies entirely inside  $U$  or  $V$ .

*Proof.* (a):

Since neither contain limit points of the other,  $\bar{A} = A$  and  $\bar{B} = B$ , so  $A$  and  $B$  are closed. Then,  $A = X \setminus B$  and  $B = X \setminus A$  are open, and form a separation.

(b):

Consider  $Y \cap U$  and  $Y \cap V$ .  $\overline{Y \cap U} \subseteq U$ , and  $\overline{Y \cap V} = V$ . Thus,  $\overline{Y \cap U} \cap \overline{Y \cap V} = \emptyset$ , so  $Y$  is entirely in  $U$ .  $\square$

**Proposition 15.** 1) The union of the collection of connected sets that share one point is connected.

*Proof.* Let us assume that  $X_\alpha$ ,  $\alpha \in A$  are connected,  $x \in X_\alpha$  for all  $\alpha$ . Let us assume there is separation  $U \cup V = \bigcup_{\alpha \in A} X_\alpha$ . Then, there is a  $y \in \bigcup_{\alpha \in A} X_\alpha$  such that  $x, y$  are in different sets  $U, V$ . However,  $y \in X_\alpha$  for some  $\alpha$ , and  $U \cap X_\alpha \mid V \cap X_\alpha$  are open, disjoint and  $(U \cap X_\alpha) \cup (V \cap X_\alpha) = X_\alpha$ .  $x$  is in one,  $y$  is in the other. This then forms a separation of  $X_\alpha$ , and we get a contradiction.  $\square$