Topology Notes - Spring 2022

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Contents

1 General Topology

1.1 August 30, 2022

Definition 1. *Let X be a set. A topology τ on X is a family of subsets (called open subsets) such that:*

1. $\emptyset, X \in \tau$
2. *Finite intersections of open subsets are open*
3. *Arbitrary unions of open subsets are open*

Definition 2. *Let $f : X \to Y$, where X and Y are topological spaces. f is continuous if the preimage of every open set is an open set,*

Proposition 1. *If $f : X \to Y$ and $g : Y \to Z$ are continuous maps, then $g \circ f$ is continuous.*

Proof. If U is open in Z, then $g^{-1}(U)$ is open in Y. Since f is continuous, $f^{-1}(g^{-1}(U))$ is open. Thus, U open implying $(g \circ f)^{-1}(U)$ is open gives us that $g \circ f$ is continuous. □

How should we construct topological spaces?:

Definition 3. *A basis for a topology is a collection $\mathcal{B}$ of subsets of a set X such that:*

- *X is the union of the elements of $\mathcal{B}$*
- *if $B_1, B_2 \in \mathcal{B}$, then for every $x \in B_1 \cap B_2$, there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$*

We define the open sets as arbitrary unions of elements of $\mathcal{B}$.

By the first property, $\emptyset, X \in \mathcal{B}$. Arbitrary unions are in the topology by definition. Now, suppose $B_1, B_2 \in \mathcal{B}$, then $B_1 \cap B_2 = \bigcup\limits_{x \in B_1 \cap B_2} B_3(x)$, where B_3 is as prescribed in the second property. We then get by induction that finite intersections are in the topology.

Example 1. *Consider $\{(a, b) | a < b,\ a, b \in \mathbb{R}\}$. This forms a basis for the standard topology on $\mathbb{R}$.*

Example 2. *Consider $\{(a, b) | a < b\ a, b \in \mathbb{Q}\}$. This ALSO forms a basis for the standard topology on $\mathbb{R}$.*

Proposition 2. *$f : X \to Y$ is continuous if the preimage of every basis element is open.*

Example 3. *Consider $C[a, b]$. Consider the function $\phi : C[a, b] \to \mathbb{R}$ such that $\phi(f) = \int_a^b f(x)\ dx$ is continuous*

Example 4. *Consider $L^p(\mathbb{R})$, where $p \geq 1$. Consider $\phi : L^p(\mathbb{R}) \to \mathbb{R}$, where $\phi(f) = \int_{-\infty}^{\infty} (f(x))^p\ dx$. ϕ is continuous.*

Definition 4. *Let (X, τ) be a topological space. Consider $Y \subseteq X$. We define $\tau_Y = \{U \cap Y |\ U \in \tau\}$ to be the subspace topology. The axioms for a topology are easily checked here.*

Proposition 3. *Let (X, τ) be a topological space. Let $\mathcal{B}$ be a family of subsets that satisfy the conditions of a basis. If every element of $\mathcal{B}$ is open and if for every open set U and every $x \in U$, there is $B \in \mathcal{B}$ such that $x \in B \subseteq U$, then B is a basis for τ*

Proof. For every $U \in \tau$, write $U = \bigcup\limits_{x \in U} B(x)$, where B is prescribed as above. □

Example 5. *Consider $\{[a, b) |\ a < b, a, b \in \mathbb{R}\}$. This is the lower limit topology. Note that $[1, 2)$ is not a union of open intervals, so this is not the standard one.*

$$(a, b) = \bigcup_{n=1}^{\infty} [a + \frac{1}{n}, b)$$

Definition 5. *τ' is finer than τ if $U \in \tau \implies U \in \tau'$. We say here that τ is coarser than τ'. For instance, the lower limit topology is finer than the standard topology.*

Example 6. *Consider $\mathcal{B} = \{V_n(a)|\ a \in \mathbb{R}^n, n > 0\}$.*

Example 7. *Consider $C[a, b]$. The basis consists of the sets:*

$$\{g|\ \sup_{x\in[a,b]} |f(x) - g(x)| < r\}, \qquad f, g \in C[a, b],\ r > 0$$

Example 8. *Consider $L^p(\mathbb{R})$, for $p \geq 1$.The basis consists of the sets:*

$$\{g| \left(\int_{-\infty}^{\infty} |f - g|^p\ dx\right)^{\frac{1}{p}} < r\}, \qquad f \in L^p(\mathbb{R}),\ n > 0$$

It is of note that these are great examples of Banach Spaces

1.2 September 1, 2022

Consider the following diagram for the product topology:

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[dr, "f_\alpha"'] & \prod_{\alpha\in A} X_\alpha \arrow[d, "\pi_\alpha"] \\ & X_\alpha \end{tikzcd}$$

Note that $f_\alpha = \pi_\alpha \circ f$. f is continuous iff f_α is continuous for all α. We want each of the projections π_α to be continuous.

Note that $\pi_\alpha^{-1}(U_\alpha)$ is open for every open $U_\alpha \subseteq X_\alpha$, because:

$$\pi^{-1}(U_\alpha) = U_\alpha \times \prod_{\beta\neq\alpha} X_\beta$$

Hence, a basis for this topology are sets of the form:

$$U_{\alpha_1} \times U_{\alpha_2} \times \cdots \times U_{\alpha_n} \times \prod_{\beta\neq\alpha_i} X_\beta \qquad U_{\alpha_i} \in X_{\alpha_i} \text{ open}$$

Proposition 4. *f is continuous if and only if f_α is continuous for every α.*

Proof. The forward direction is trivial. Since π_α is continuous, $f_\alpha = f \circ \pi_\alpha$ is a composition of continuous maps, and is thus continuous. Now, let f_α be continuous for every α. We only need to check that the preimage of a basis element is an open set.

$$\begin{aligned} f^{-1}\left(U_{\alpha_1} \times U_{\alpha_2} \times \cdots \times U_{\alpha_n} \times \prod_{\beta\neq\alpha_i} X_\beta\right) &= f^{-1}\left(\bigcap_{i=1}^{n}\left(U_{\alpha_i} \times \prod_{\beta\neq\alpha_i} X_\beta\right)\right) \\ &= \bigcap_{i=1}^{n} f^{-1}\left(U_{\alpha_i} \times \prod_{\beta\neq\alpha_i} X_\beta\right) \\ &= \bigcap_{i=1}^{n} f_{\alpha_i}^{-1}(U_{\alpha_i}) \end{aligned}$$

This is open because the preimages of U_α by f_α are open, and finite intersections of open sets are open $\square$

Example 9. *Consider* $C[a,b] \subseteq \prod_{x\in[a,b]} \mathbb{R}$*. This topology has as basis sets of the form:*

$$V_{f,x_1,\cdots,x_n,\epsilon} = \{g : \ |f(x_i) - g(x_i)| < \epsilon, \ i \in \mathbb{N}\}$$

*This is called the weak** *topology on* $C[a,b]$*.*

Example 10. *Consider* $\mathbb{R}\times\mathbb{R} \to \mathbb{R}$*. we can define this to have multiplication, addition, or division defined on* $\mathbb{R}\times\mathbb{R}\setminus\{0\}$*.*

Proposition 5. *If* X *is a topological space and* $f, g : X \to \mathbb{R}$ *(with maybe* $g : X \to \mathbb{R}\setminus\{0\}$*) are continuous functions, then* $f+g, f\cdot g, \frac{f}{g}$ *are continuous.*

Proof. These are compositions of continuous functions. We consider:

$$X \xrightarrow{(f,g)} \mathbb{R}\times\mathbb{R} \xrightarrow{+} \mathbb{R}$$

The same happens for multiplication and division. □

Disjoint Unions of Topological Spaces:

Consider $X_\alpha, \alpha \in A$, $X_\alpha \cap X_\beta = \emptyset$. We then denote the disjoint union as $\bigsqcup_{\alpha\in A} X_\alpha$. Consider the following diagram:

$$\begin{tikzcd} \bigsqcup_{\alpha\in A} X_\alpha \arrow[r, "f"] & Y \\ X_\alpha \arrow[u, hook] \arrow[ur, "f_\alpha"'] & \end{tikzcd}$$

Similarly as before, f is continuous if and only if f_α is continuous for all α. A topology consists of the sets that are unions of U_α where U_α are open in X_α. If the X_α's are not disjoint, then we might run into trouble in the overlap!

Quotient Spaces Consider a topological space X. Take $f : X \to Y$ to be a surjective function. THe open sets of Y are of the form $f(U)$ where U is open in X.

Another perspective: Consider a topological space X. Take an equivalence relation on X. Denote by $\hat{x}$ the equivalence class of x. Let:

$$Y = \{\hat{x} | \ x \in X\}$$

Now, we consider the function $f : X \to Y$ where $f(x) = \hat{x}$., and we use the notation $X/\sim$ to represent this quotient space. Now, consider the following diagram:

$$\begin{tikzcd} X \arrow[d, "f"'] & Z \arrow[l, "g\circ f"'] \arrow[dl, "g"] \\ Y & \end{tikzcd}$$

We want g to be continuous if and only if $g \circ f$ is continuous, and this is the reason for our construction of the quotient space.

Proposition 6. *The above is a topology on* Y*.*

Proof. (a) $Y = f(x)$ is open, and $\emptyset = f(\emptyset)$ is open

(b) $f(U_1 \cap U_2 \cap \cdots \cap U_n)$ is a finite intersection of $f(U_i)$'s, so this is open

(c) If $f(U_\alpha)$ is open, for all α, then it's easy to see that the images commute with the union, and so arbitrary unions are open.

$\square$

Example 11. *Take $f : \mathbb{R} \to \mathbb{C}$. This may not be a surjection, but it is certainly a surjection onto the image by construction. Let $f(x) = e^{2\pi i x}$. The image is $\{z : |z| = 1\} =: S^1$ (the 1-dimensional sphere). We can induce a topology on S^1 by using the standard topology on $\mathbb{R}$, or by inducing from the topology on $\mathbb{C}$. The induced topology is the same as the quotient topology. This can be resolved by thinking of the standard convention where $S^1 = \mathbb{R}/\mathbb{Z}$.*

Example 12. *Consider S^2, the sphere in $\mathbb{R}^3$. Consider the inclusion of $S^2 \subseteq \mathbb{R}^3$, and consider the induced topology on S^2. We can consider two points on the sphere to be equivalent as follows:*

$$(x, y, z) \sim (x', y', z') \iff (x = x', y = y', z = z') \vee (x = -x', y = -y', z = -z')$$

Let $\mathbb{R}P^2 = S^2/\sim$, and consider the quotient topology. This space is essentially a cap being put on top of a Möbius band. By definition, this is the two-dimensional projective plane. Another way to construct $\mathbb{R}P^2$:

consider $\mathbb{R}^3 \setminus \{(0,0,0)\} \subseteq \mathbb{R}^3$ using the standard topology. We take the following equivlence:

$$(x, y, z) \sim (x', y', z') \iff \exists \lambda \neq 0 : \ x = \lambda x', y = \lambda y', z = \lambda z'$$

Then, $(\mathbb{R}^3 \setminus \{(0,0,0)\})/\sim$ is exactly $\mathbb{R}P^2$. We can then generalize this construction to $\mathbb{R}P^n$ using the same procedure, but we can also have $\mathbb{C}P^n$ to be equivalence classes of points in $\mathbb{C}^{n+1} \setminus \{(0, 0, \cdots, 0)\}$, with the same equivalence as before, except now $\lambda \in \mathbb{C}$. For instance $\mathbb{C}P^1$ is the Riemann sphere.

Example 13. *Consider the unit square with corners $(0,0), (0,1), (1,0), (1,1)$. We induce an equivalence relation by $(x,1) \sim (x,0)$ for every x, and $(1,y) \sim (0,y)$ for every y. This generates a torus. This comes from the induced topology from the inclusion into $\mathbb{R}^2$, so the torus is the quotient topology $\mathbb{R}^2/\mathbb{Z}^2$. The torus can also be described as $s^1 \times S^1$.*

Definition 6. *$f : X \to Y$ is a <u>homeomorphism</u> if f is continuous, invertible, and f^{-1} is continuous.*

Example 14. *$([0,1] \times [0,1])/\sim$ is homeomorphic to $S^1 \times S^1$, where $\sim$ is the equivalence relation defined in Example 11.*

Example 15. *Consider $[1,2], [3,4] \subseteq \mathbb{R}$ with the induced topology. Then, take the disjoint union $[1,2] \sqcup [3,4]$. Then, we take the quotient where $1 \sim 2 \sim 3 \sim 4$. This gives us a figure 8, where the circles touch at $\{1,2,3,4\}$.*

Example 16. *Let X be a topological space. Consider $[-1,1]$ with the induced topology. Consider $X \times [-1,1]$ with the product topology. We then consider $(x,1) \sim (y,1)$ and $(x,-1) \sim (y,-1)$ for every x and y, and take the quotient topology. This then gives a two-sided cone with a circular middle, called the suspension ΣX. If we consider S^n, then $\Sigma S^n \cong S^{n+1}$.*

1.3 September 6, 2022

Thus far, as a recap, we have covered the following constructions:

- Induced Topology
- Product Topology
- Disjoint Sum
- Quotients

We now discuss metric spaces.

Definition 7. *A distance on a set X is a function $d : X \times X \to [0, \infty)$ such that:*

(a) $d(x, y) = 0 \iff x = y$

(b) $d(x, y) = d(y, x)$

(c) $d(x, z) \leq d(x, y) + d(y, z)$

A metric is then another word we use instead of "distance". (X, d) is thus termed a metric space.

Example 17. $(\mathbb{R}^n, d)$*, where* $d(\vec{x}, \vec{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$

Example 18. $(\mathbb{R}^n, d)$*, where* $d(\vec{x}, \vec{y}) = \max_{1 \leq i \leq n} |x_i - y_i|$

Example 19. $([0,1]^A, d)$*, where* $(x_\alpha)_\beta, (y_\alpha)_\beta \in \prod_{\beta \in B} [0, 1]$*, and:*

$$d((x_\alpha)_\alpha, (y_\alpha)_\alpha) = \sup_\alpha |x_\alpha - y_\alpha|$$

A basis for the topology induced by the metric consists of the balls (for $x \in X$ and $\epsilon > 0$):

$$B(x, \epsilon) = \{y | d(x, y) < \epsilon\}$$

We now prove that these open balls form a basis.

Proof. First, we must check that these cover the entire space. However, it is clear that $X = \bigcup_{x \in X} B(x, \epsilon)$. For another condition, we see that if $B_1 = B(x_1, \epsilon_1)$ and $B_2 = B(x_2, \epsilon_2)$ are elements of the basis, and if $x \in B_1 \cap B_2$, we have to show that there is a $B_3 = B(x_3, \epsilon_3)$ such that $x \in B_3 \subseteq B_1 \cap B_2$. We can construct this geometrically such that we pick $x = x_3$ and $\epsilon_3 < \min(\epsilon_1 - d(x, x_1), \epsilon_2 - d(x, x_2))$. Then, if $y \in B_3$, we have that:

$$\begin{aligned} d(y, x_1) &< d(y, x) + d(x, x_1) \\ &< \epsilon_3 + d(x, x_1) \\ &< \epsilon_1 - d(x, x_1) + d(x, x_1) \\ &= \epsilon_1 \end{aligned}$$

Thus, $y \in B_1$, and by the exact same argument, $y \in B_2$, so we have indeed proven that this does form a basis. $\square$

We now note that open sets are unions of balls.

Example 20. *On $\mathbb{R}^n$, the standard metric and the max metric from Examples 17 and 18 induce the same topology.*

Proposition 7. *If two metrics have the property that for every ball of the first and every element of it, there is a ball of the second topology centered at this element and included in the first ball, then the second topology is finer than the first.*

$$x \in B'(x, \epsilon') \subseteq B(y, \epsilon), \qquad \forall x \in B(y, \epsilon)$$

In Example 20, a disc can be written as a union of squares, and vice versa.

Example 21. *If (X, d) is a metric space, define:*

$$\overline{d}(x, y) = \min\left(d(x, y), 1\right)$$

Then, $\overline{d}$ is a metric that induces the same topology.

We first check that $\overline{d}$ is a metric by ensuring that it satisfies the triangle inequality. If $d(x, y), d(y, z), d(x, z) < 1$, then:

$$\begin{aligned} \overline{d}(x, z) &= d(x, z) \\ &\leq d(x, y) + d(y, z) \\ &= \overline{d}(x, y) + \overline{d}(y, z) \end{aligned}$$

We have a similar case where $d(x, y)$ or $d(y, z) > 1$, because we then have $1 \leq \overline{d}(x, y) + \overline{d}(y, z)$ holds. Finally, if $d(x, y), d(y, z) < 1$, and $d(x, z) > 1$, then:

$$\overline{d}(x, z) \leq d(x, z) < d(x, y) + d(y, z) = \overline{d}(x, y) + \overline{d}(y, z)$$

Theorem 1. *If (X, d), (Y, d') are metric spaces, then $f : X \to Y$ is continuous at $x \in X$ if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $f(B(x, \delta)) \subseteq B(f(x), \epsilon)$.*

Definition 8. *$f : X \to Y$ is called uniformly continuous if $\forall \epsilon > 0$, $\exists \delta > 0$ such that if $d(x, y) < \delta$, then $d'(f(X), f(y)) < \epsilon$.*

Definition 9. *A sequence $(x_n)_{n \geq 1}$ in the metric space (X, d) is called Cauchy if $\forall \epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $m, n \geq N$, $d(x_m, x_n) < \epsilon$.*

Definition 10. *A metric space is called complete if every Cauchy sequence is convergent.*

Definition 11. *A normed vector space V is a vector space endowed with a function $\| \cdot \| : V \to [0, \infty)$ such that:*

(a) $\|x\| = 0 \iff x = 0$

(b) $\|\lambda x\| = |\lambda| \|x\|, \qquad \lambda \in \mathbb{R} \vee \mathbb{C}$

(c) $\|x + y\| \leq \|x\| + \|y\|$

We define $d(x, y) = \|x - y\|$.

Definition 12. *A Banach space is a complete normed vector space.*

Example 22. *$C[0, 1]$, L^p spaces, Sobolev spaces, Hardy spaces, Bergman spaces.*

Manifolds:

Definition 13 (Veblen)**.** *A manifold M is a Hausdorff topological space endowed with an onto function*

$$f : \bigsqcup_{\alpha \in A} U_\alpha \to M$$

such that for every α, U_α is an open set in some $\mathbb{R}^n$, where n is the same for all α, and $f|U_\alpha$ is a homeomorphism onto the image for each α.

We can define charts here as something of the form:

$$f_\alpha = f|U_\alpha : U_\alpha \to f(U_\alpha) \subseteq M$$

A collection of charts is then an atlas. Also, we get compositions where:

$$f_\beta^{-1} \circ f_\alpha : U_\alpha \to U_\beta$$

Definition 14. *We call a topological space Hausdorff if for every $x \neq y$, there are open sets U, V such that $x \in U$ and $y \in V$, and $U \cap V = \emptyset$.*

- If $f_\beta^{-1} \circ f_\alpha$ is smooth, then the manifold is called smooth.
- If $\mathbb{R}^n$ is replaced by $\mathbb{C}^n$ and $f_\beta^{-1} \circ f_\alpha$ is holomorphic or analytic then M is called complex.

Example 23. *The circle $S^1 = \{z : |z| = 1\}$. Define $f : \mathbb{R} \to \mathbb{C}$ such that $f(t) = e^{2\pi i t}$. Let $U_1 = (-\pi, \pi)$ and $U_2 = (0, 2\pi)$. Now, we consider a function $g : U_1 \sqcup U_2 \to \mathbb{C}$, where $g(t) = e^{2\pi i t}$. We can consider $f(U_1) \cap f(U_2)$ as the overlap, and so $f_1^{-1}(f(U_1) \cap f(U_2)) = (= \pi, 0) \cup (0, \pi)$, and $f_2^{-1}(f(U_1) \cap f(U_2)) = (\pi, 2\pi) \cup (0, \pi)$.*

Then, $f_2^{-1} \circ f_1 : (-\pi, 0) \cup (0, \pi) \to (\pi, 2\pi) \cup (0, \pi)$ is as follows:

$$f_2^{-1} \circ f_1(t) = \begin{cases} t & , t \in (0, \pi) \\ t + 2\pi & , t \in (-\pi, 0) \end{cases}$$

Theorem 2. *If M_1, M_2 are manifolds of dimensions n_1, n_2, then $M_1 \times M_2$ is a manifold of dimension $n_1 + n_2$.*

Consider:

$$f : \bigsqcup U_\alpha \to M_1$$

$$g : \bigsqcup V_\beta \to M_2$$

$$f \times g : \bigsqcup U_\alpha \times V_\beta \to M_1 \times M_2$$

$$(f \times g)_{\alpha\beta} = (f_\alpha, g_\beta)$$

Example 24. *$(S^1)^{\times n}$ is a manifold. In the $n = 2$ case, we get $(-\pi, \pi)^2 \sqcup (-\pi, \pi) \times (0, 2\pi) \sqcup (0, 2\pi) \times (-\pi, \pi) \sqcup (0, 2\pi)^2$ as our domain, where $f(t, s) = (e^{2\pi i t}, e^{2\pi i s})$, and this gives us the torus.*

1.4 September 8, 2022

Example 25. *Consider* $\mathbb{R}P^2 = \{\hat{x} : \ x \in \mathbb{R}^3 \setminus \{0\}, x \sim y \iff x = \lambda y, \lambda \neq 0\}$. $\mathbb{R}P^2$ *is a manifold, with the following three charts:*

$$U_1 = U_2 = U_3 = \mathbb{R}^2$$

$$f : \bigsqcup_{i=1}^{3} U_i \to \mathbb{R}P^2$$

$$f_1(x_1, x_2) = [1 : x_1 : x_2]$$

$$f_2(x_1, x_2) = [x_1 : 1 : x_2]$$

$$f_3(x_1, x_2) = [x_1 : x_2 : 1]$$

These maps are one-to-one, and onto. We note that $f_2^{-1} \circ f_1 : \mathbb{R}^2 \to \mathbb{R}^2$ *is the map* $(x_1, x_2) \xrightarrow{f_1} [1 : x_1 : x_2] = \left[\frac{1}{x_1} : 1 : \frac{x_2}{x_1}\right] \xrightarrow{f_2^{-1}} \left(\frac{1}{x_1}, \frac{x_2}{x_1}\right)$

Example 26. *Consider* $\mathbb{C}P^1$. *We have that* $U_1 = U_2 = \mathbb{C}$. *We define maps as:*

$$f_1 : U_1 \to \mathbb{C}P^1$$

$$f_1(z) = [1 : z]$$

$$f_2 : U_2 \to \mathbb{C}P^1$$

$$f_2(z) \to [z : 1]$$

$$f_2^{-1} \circ f_1 : z \to [1 : z] \to \left[\frac{1}{z} : 1\right] \to \frac{1}{z}$$

Definition 15 (Poincaré)**.** *A smooth manifold is a subspace of* $\mathbb{R}^n$ *that is locally the graph of a smooth function.*

Example 27. *Consider* S^2. *Consider the graph* $f(x_1, x_2) = \pm\sqrt{1 - x_1^2 - x_2^2}$, *and similar considerations for the two other orientations. These graphs act as charts from the plane to the sphere.*

Now, we return to closed sets!

Example 28. $[a, b], [a, \infty)$

Example 29.

$$C = [0, 1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right)$$

This set consists of all numbers in $[0, 1]$ *that admit a ternary representation with only 0's and 2's. Consider a function:*

$$f : C \to [0, 1]^2$$

$$f(0.a_1a_2a_3\cdots) = (0.\frac{a_1}{2}\frac{a_3}{2}\cdots, 0.\frac{a_2}{2}\frac{a_4}{2}\cdots)$$

This function is then continuous and onto from C *to* $[0, 1]^2$. *By Tietze's Extension Theorem, we get an extention* $\tilde{f} : [0, 1] \to [0, 1]^2$ *which is continuous and onto. This was induced by G. Peano.*

Example 30. *Sierpiński Triangle.*

Example 31. *In the discrete topology, every set is clopen*

Example 32. $\mathbb{Q} \subseteq \mathbb{R}$, (a, b) *are clopen in* $\mathbb{Q}$.

Example 33. *In* $\mathbb{R}^n$, *consider* $\overline{B}(x,\epsilon) = \{y : \ d(x,y) \leq \epsilon\}$ *as the closed balls.*

Example 34. *The Zariski topology, where* $f(z_1, z_2, \cdots, z_n) \in \mathbb{C}[z_1, \cdots, z_n]$, *and:*

$$V(f) = \{(z_1, \cdots, z_n) \in \mathbb{C}^n : \ f(z_1, \cdots, z_n) = 0\}$$

These are called algebraic sets, or varieties if f *is irreducible, and these are the closed sets of the Zariski topology. Unions of algebraic sets are the closed sets of the Zariski topology.*

Proposition 8 (MAYBE ON EXAM 1)**.** *(1) If* Y *is a subspace of* X, *then* $A \subseteq Y$ *is closed if and only if* $A = B \cap Y$ *where* $B \subseteq X$ *is closed.*

(2) Let $A \subseteq Y \subseteq X$. *If* A *is closed in* Y *and* Y *is closed in* X, *then* A *is closed in* X.

(3) If A *is closed in* X *and* B *is closed in* Y, *then* $A \times B$ *is closed in* $X \times Y$.

(4) If A_α *is closed in* X_α *for* $\alpha \in I$, *then* $\prod_{\alpha \in I} A_\alpha$ *is closed in* $\prod_{\alpha \in I} X_\alpha$ *with the product topology.*

Proof. (1) $Y \setminus A$ is open. So there is a U such that $Y \setminus A = Y \cap U$. Let $B = X \setminus U$. Then

$$A = Y \setminus (Y \setminus A) = Y \setminus (Y \cap U) = Y \setminus U = Y \cap (X \setminus U) = B \cap Y$$

Now, if $A = B \cap Y$, then $Y \setminus A = (X \setminus B) \cap Y$.

(2) A closed in Y and Y closed in X gives us that $X \setminus Y = U$ is open in X. Then, $A = Y \cap (X \setminus V)$. Thus $X \setminus A = V \cup U$, so since this is a union of two open sets, it is open, and thus A is closed in X.

(3) $(X \times Y) \setminus (A \times B) = (X \times (Y \setminus B)) \cup ((X \setminus A) \times Y)$. These sets are all open, so $A \times B$ is closed.

(4) Induction! Do the same thing as above as unions.

$$\left(\prod_{\alpha \in I} X_\alpha\right) \setminus \left(\prod_{\alpha \in I} A_\alpha\right) = \bigcup_{\alpha \in I} \left(\left(\prod_{\beta \neq \alpha} X_\beta\right) \times (X_\alpha \setminus A_\alpha)\right)$$

□

Proposition 9. *Let* X, Y *be topological spaces. Then,* $f : X \to Y$ *is continuous if and only if the preimage of every closed set is closed.*

Proof.

$$f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$$

□

Definition 16. *The* <u>closure</u> *of a set* S *is the smallest closed set containing* S, *denoted* $\overline{S}$.

Definition 17. *The* <u>interior</u> *of a set* S *is the largest open set contained in* S, *denoted* $Int(S)$.

Example 35. $\overline{\mathbb{Q}} - \mathbb{R}$. $Int(\mathbb{Q}) = \emptyset$

Example 36. $B(x,r) = \{y : \ d(x,y) < r\} \implies \overline{B(x,r)} = \{y : \ d(x,y) \leq r\}$.

Lemma 1. *Let* X *be a topological space, and* $A \subseteq X$. *Then,* $\overline{X \setminus A} = X \setminus Int(A)$

Proof.

$$X \setminus A \subseteq X \setminus Int(A) \implies \overline{X \setminus A} \subseteq \overline{X \setminus Int(A)} = X \setminus Int(A)$$

Now, let $x \in X \setminus (\overline{X \setminus A})$. Then there is an open set U such that $x \in U$ and $U \cap (\overline{X \setminus A}) = \emptyset$. Thus, $U \cap (X \setminus A) = \emptyset$, so $U \subseteq A$. Therefore, $x \in Int(A)$. Hence, $x \notin X \setminus Int(A)$, so $X \setminus Int(A) \subseteq \overline{X \setminus A}$, and we have the double inclusion as desired. □

Theorem 3. *Let $A \subseteq X$. Then, $x \in \overline{A}$ if and only if for every open set U such that $x \in U$, we have $U \cap A \neq \emptyset$.*

Proof. First, we wish to show that If $x \in \overline{A}$ and U is open where $x \in U$, then $U \cap A \neq \emptyset$. If, for the sake of contradiciotn, $U \cap A = \emptyset$, then $U \subseteq Int(X \setminus A)$, so $X \setminus Int(X \setminus A) = \overline{X \setminus (X \setminus A)} = \overline{A}$, which does not contain x, and so we get a contradiction.

Conversely, if x has the property that every open set U containing x intersects A, then $x \notin Int(X \setminus A)$. So $x \in X \setminus Int(X \setminus A) = \overline{X \setminus (X \setminus A)} = \overline{A}$. Thus, we are done. □

Proposition 10 (Midterm maybe)**.** *1) Let $Y \subseteq X$, with the subspace topology. If $A \subseteq Y$, then let us denote $\overline{A}_X$ as the closure of A in X. Then, the closure of A in Y is $\overline{A}_X \cap Y$.*

2) Taking the same conditions as above, if Y is closed in X, then the closure of A in X and Y is the same.

3) $\prod_{\alpha \in I} \overline{A}_\alpha = \overline{\prod_{\alpha \in I} A_\alpha}$

Proof. **1):**

Let $x \in \overline{A}_X \cap Y$. Then, for every open set U in X containing x, $U \cap A \neq \emptyset$. So $(U \cap Y) \cup A \neq \emptyset$. But $U \cap Y$ with U open in X are all open subsets of Y. By Theorem 3, x is in the closure of A in Y. Thus, $\overline{A}_X \cap Y \subseteq \overline{A}_Y$.

Conversely, $\overline{A}_X \cap Y$ is closed and contains A, so the closure of A in Y is contained in $\overline{A}_X \cap Y$.

2):

if Y is closed, then $\overline{A}_X \subseteq \overline{Y} = Y$. So $\overline{A}_X \cap Y = \overline{A}_X$.

3):

Let $x = (x_\alpha)_\alpha \in \prod_{\alpha \in I} \overline{A}_\alpha$. Let $\prod_{j=1}^{n} U_{\alpha_j} \times \prod_{\beta \neq \alpha_j} X_\beta$, or $\prod_{\alpha \in I} U_\alpha$ such that $x_\alpha \in U_\alpha$. Then, by Theorem 3, $U_\alpha \cap A_\alpha \neq \emptyset$, for all α, and $X_\beta \cap A_\beta \neq \emptyset$, for all $\beta \neq \alpha$. It follows that in either case, the open set intersects $\prod A_\alpha$. Thus, $\prod \overline{A}_\alpha \subseteq \overline{\prod A_\alpha}$. Conversely, let $x \in \overline{\prod A_\alpha}$. Then, every open set containing x intersects $\prod A_\alpha$. Choose U of either form as expressed in the previous direction, and we get coordinate-wise nonempty intersections. Vary U_α's to make them be any open set that contains x_α. You then obtain the reverse inclusion, and thus we are done. □

This result does not hold for interiors as seen in the following examples:

Example 37. $\mathbb{Q} \subseteq \mathbb{R}$. *$Int * \mathbb{Q}) \neq \emptyset$, but the interior of $\mathbb{Q}$ in $\mathbb{Q}$ is $\mathbb{Q}$.*

However, the third case of the proposition works for interiors in the box topology but not the product topology.

Example 38. $[0,1] \subseteq \mathbb{R}$, $Int([0,1]) = (0,1)$. *However:*

$$\underbrace{\prod_{i=1}^{\infty}(0,1)}_{\text{not an open set}} \subseteq \prod_{i=1}^{\infty} \mathbb{R}$$

Proposition 11. *Let X, Y be topological spaces. $f : X \to Y$ is continuous if and only if for every subset $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$*

Proof. Assume f is continuous. Then $f^{-1}(Y \setminus \overline{f(A)})$ is open. This is the complement of $f^{-1}(\overline{f(A)})$, so this set is open. Then $A \subseteq f^{-1}(\overline{f(A)})$, so $\overline{A} \subseteq f^{-1}(\overline{f(A)}) \implies f(\overline{A} \subseteq \overline{f(A)}$.

Conversely, let $B \subseteq Y$ be open. we want to show $f^{-1}(B)$ is open. Let $A = X \setminus f^{-1}(B)$. Then:

$$f(\overline{A}) \subseteq \overline{f(A)} = \overline{f(X \setminus f^{-1}(B))} = \overline{Y \setminus B} = Y \setminus B$$

Thus, $f(\overline{A} \subseteq Y \setminus B$, but $A = X \setminus f^{-1}(B)$, so $A = \overline{A}$, and we are done ☐

Definition 18. *x is a limit point for A if for every U open such that $x \in U$, there is $x' \neq x$ such that $x' \in A \cap U$. We denote A' to be the set of limit points of A*

Example 39.

$$A = \{\frac{1}{n} : \ n \in \mathbb{N}\}$$

A has only 1 limit point, namely 0.

Proposition 12.

$$\overline{A} = A \cup A'$$

Corollary 1. *A subset of a topological space is closed if and only if it contains all of its limit points.*

Definition 19. *In an arbitrary topological space, one says that a sequence is convergent to $x \in X$ if for every neighbourhood V of x, there is an N such that for all $n \geq N$, $x_n \in V$*

Note: IN the Zariski topology, a sequence with no constant subsequence converges to every point in the space. The definition is fine in metric spaces. In every metric space, the limit is unique.

Proposition 13. *In metric spaces, the limit is unique.*

Proof. If x_1, x_2 are both limits of a sequence, consider the open ball with radius $\frac{d(x_1,x_2)}{2}$. All but finitely many terms lie in each of the disjoint balls. ☐

Lemma 2 (Sequence Lemma)**.** *Let X be a metric space.*

(a) $x \in \overline{A}$ if and only if there is a sequence in A converging to x.

(b) $x \in A'$ if and only if there is a sequence of points in A that does not eventually become constant that converges to x.

Proof. Recall that $\overline{A} = A \cup A'$. If $x \in A$, then $x_n = x$ for all n converges to x. Thus, it suffices to prove part (b).

Let $x \in A'$. For $n \in \mathbb{N}$, consider $B(x, \frac{1}{n})$. Since $x \in A'$, there is $x_n \in A \cap B(x, \frac{1}{n}$ such that $x_n \neq x$. Now, let $\epsilon > 0$, and choose $K(\epsilon)$ so that $\frac{1}{K(\epsilon)} < \epsilon$. Then, for $n \geq K(\epsilon)$, $x_n \in B(x, \frac{1}{n}) \subseteq B(x, \frac{1}{K(\epsilon)} \subseteq B(x, \epsilon)$. Convsersely, if $x_n \to x$, $x_n \in A$, x_n not eventually constant. We consider an open set U containing x. Let $B(x, \epsilon) \subseteq U$. There is a $K(\epsilon)$ such that for all $n \geq K(\epsilon)$, $x_n \in B(x, \epsilon)$. From these, we can choose a term that is not x. The definition of a' is thus fulfilled, and so we are done ☐

Theorem 4. *Let X, Y be metric spaces. Then, $f : X \to Y$ is continuous if and only if for every convergent sequence $x_n \in X$, the sequence $f(x_n)$ is convergent in Y*

Proof. For $x_n \to x$, consider $B(f(x), \epsilon)$. Then $f^{-1}(B(f(x), x))$ is an open neighbourhood of x. Thus, there is an integer $K(\epsilon) \in \mathbb{N}$ such that for all $n \geq K(\epsilon)$, $x_n \in f^{-1}(B(f(x), \epsilon))$. Thus, for all $n \geq K(\epsilon)$, $f(x_n) \in B(f(x), \epsilon))$, so $f(x_n)$ is convergent in Y.

Conversely, assume that x_n converges to x. Consider $x_1, x, x_2, x, x_3, x \cdots$. This converges to x. Then $f(x_1), f(x), f(x_2), f(x), \cdots$ converges by hypothesis. Since it has a constant subsequence equal to $f(x)$, the sequence itself converges to $f(x)$. The conclusion then follows from Proposition 11. ☐

Definition 20. *X is Hausdorff if for every $x, y \in X$, $x \neq y$ there are open sets U, V such that $x \in U$, $y \in V$, $U \cap V = \emptyset$.*

Theorem 5. *If X and Y are homeomorphic, and X is Hausdorff, then Y is Hausdorff.*

Proof. Consider $h : Y \to X$, a homeomorphism. Let $x, y \in Y$. Then $f(x) \neq f(y)$. There are disjoint open sets U and V in X containing these 2 points separately. THe open sets $f^{-1}(U)$ and $f^{-1}(V)$ contain x an y respectively and are disjoint. ☐

Remark 1. *Hausdorff is a topological property.*

Example 40. *Consider $\mathbb{C}^n$ with the standard topology and $\mathbb{C}^n$ with the Zariski Topology. These are NOT homeomorphic. One is Hausdorff and the other is not.*

Definition 21. *A topological space X is not connected if there exist two open sets U, V such that $U \cup V = X$ and $U \cap V = \emptyset$.*

Example 41. $(-\infty, 0) \cup (0, \infty)$ *is not connected.*

If a space is not "not connected", then it is connected.

Example 42. $\mathbb{Q}$ *is not connected.*

Proposition 14. *(a) If $A, B \subseteq X$ are disjoint subsets such that $A \cup B = X$ and neither of these sets contains a limit point of the other. Then, they form a separation of X.*

(b) If U, V form a separation of X and if $Y \subseteq X$, Y connected, then Y lies entirely inside U or V.

Proof. **(a)**:

Since neither contain limit points of the other, $\overline{A} = A$ and $\overline{B} = B$, so A and B are closed. Then, $A = X \setminus B$ and $B = X \setminus A$ are open, and form a separation.

(b):

Consider $\cap U$ and $Y \cap V$. $\overline{Y \cap U} \subseteq U$., and $\overline{Y \cap V} = V$. Thus, $\overline{Y \cap U} \cap \overline{Y \cap V} = \emptyset$, so Y is entirely in A. ☐

Proposition 15. *1) The union of the collection of connected sets that share one point is connected.*

Proof. Let us assume that X_α, $\alpha \in A$ are connected, $x \in X_\alpha$ for all α. Let us assume there is separation $U \cup V = \cup_{\alpha \in A} X_\alpha$. Then, there is a $y \in \bigcup_{\alpha \in A} X_\alpha$ such that x, y are in different sets U, V. However, $y \in X_\alpha$ for some α, and $U \cap X_\alpha$¡ $V \cap X_\alpha$ are open, disjoint and $(U \cap X_\alpha) \cup (V \cap X_\alpha) = X_\alpha$. x is in one, y is in the other. This then forms a separation of X_α, and we get a contradiction. ☐

Theorem 6. *If X is a connected top. sp. and if $f : X \to Y$ is continuous, then $f(X)$ is connected.*

Proof. By contradiction. Assume that $f(X)$ has a separation. Then there exist disjoint open sets $U, V \in f(X)$ such that $U \cup V = f(X)$. Then

$$f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V) = (f^{-1} \circ f)(X) = X$$

and f being continuous implies that the preimages of U and V are both open, and furthermore $U \cap V = \varnothing \implies f^{-1}(U) \cap f^{-1}(V) = \varnothing$, so these sets form a separation of X, contradiction. $\square$

Proposition 16. *1. The union of connected sets that have one point in common is connected*

2. If A and B are spaces such that, $\bar{A} = B$, then B is connected iff A is connected.

3. The product of connected spaces is connected in the product topology

Proof. **1.** (Already done, see above)
2. (Already done, see above)
3.
We start with the finite case:
Let X, Y be connected. Choose $x_0 \in X$ and $y_0 \in Y$. Then $(X \times \{y_0\}) \cup (\{x_0\} \times Y)$ is connected, being the union of the connected spaces $X \times \{y_0\}$ and $\{x_0\} \times Y$, sharing (x_0, y_0).
Next, the union

$$(X \times \{y_0\}) \cup (\{x\} \times Y)$$

is connected for any $x \in X$ by the same argument, since the two share (x, y_0). Hence

$$X \times Y = \cup_{x \in X}((X \times \{y_0\}) \cup (\{x_0\} \times Y))$$

is connected since all sets in the union share (x_0, y_0).
Now for the infinite case. Let us consider:

$$\prod_{\alpha \in I}(X_\alpha)$$

where I is some infinite family, and where X_α is connected for all $\alpha \in I$. Choose any point $(a_\alpha)_{\alpha \in I}$. Consider sets of the form:

$$X_{\alpha_1} \times X_{\alpha_2} \times \cdots \times X_{\alpha_n} \times \prod_{\beta \neq \alpha}(a_\beta)$$

They are connected, and contain (a_α). Then

$$\cup_{\alpha_1, \alpha_2, \cdots, \alpha_n \in I}(X_{\alpha_1} \times X_{\alpha_2} \times \cdots \times X_{\alpha_n} \times \prod_{\beta \neq \alpha}\{a_\beta\})$$

is connected. We will show this set is dense in $\prod_{\alpha \in I}(X_\alpha)$.
Let $(x_\alpha)_{\alpha \in I} \subset \prod X_\alpha$. Let U be an open neighborhood of (x_α) in the product topology. The claim is that $U \cap A \neq \varnothing$. Consider a basis element

$$U_{\alpha_1} \times U_{\alpha_2} \times \cdots \times U_{\alpha_n} \times \prod_{\beta \neq \alpha_i}(X_\beta)$$

be a basis element inside U containing (x_α). It contains the point

$$\{x_{\alpha_1}\} \times \{x_{\alpha_2}\} \times \cdots \times \{x_{\alpha_N}\} \times \prod_{\beta \neq \alpha_i}\{a_\beta\}$$

This point lies in

$$X_{\alpha_1} \times X_{\alpha_2} \times \cdots \times X_{\alpha_n} \times \prod_{\beta \neq \alpha_i}\{a_\beta\}$$

Therefore, A is connected, and the closure $\bar{A} = \prod_{\alpha \in I} X_\alpha$ so the latter is connected $\square$

Definition 22. *A connected component of a topological space X is a maximal (under set inclusion) connected subspace*

Theorem 7. *Every topological space can be partitioned into connected components*

Proof. Define a relation on points in X: for all $x, y \in X$, let $x \sim y$ if y is in the connected component of x. Then this relation is in fact an equivalence relation, which partitions X into equivalence classes, which are in fact connected components. $\square$

Remark 2. *The number of connected components of a topological space X is a numerical invariant modulo homeomorphisms.*

Definition 23. *A topological space X is locally connected if every point $x \in X$ and every open neighborhood $U(x)$ containing x, there is a connected open set V such that $x \in V \subset U$.*

Proposition 17. *A topological space X is locally connected iff the connected components of every open set are open.*

Proof. Suppose the space X is locally connected. If U is an open set, the for every $x \in U$, then there is a connected open set $V_x \subset U$ with $x \in V_x$. Let C be the connected component of U containing x. Then $V_x \subset C$ because if it weren't, then $C \subsetneq C \cup V_x \subset U$, contradicting the maximality of C. Thus, C is exactly the union $C = \cup_{x \in C} V_x \subset U$, and is therefore open. Conversely, let X be such that the connected components of every open set are open. Then, given an open set and a point $x \in U$, the connected component of U containing x is open, so $x \in V \in U$, which is exactly the requirement for local connectedness. $\square$

Not all connected spaces are locally connected

Example 43. *The comb*

Theorem 8. *connected sets in $\mathbb{R}$ A subset $U \subset \mathbb{R}$ is connected if and only if it is a point, interval, or R.*

Proof. If $A \subset R$ is none of these, then there are $a, b \in A$ and $c \notin A$ such that $a < c < b$. But then $(-\infty, c) \cup (c, \infty)$ is a separation of A.
Conversely, assume A is not a point or $\mathbb{R}$, so let it be an interval. Set $A = U \cup V$, and $U \cap V = \varnothing$, where U, V are both open. Then U, V are both closed. Let $a \in U$, $b \in V$, $a < b$. Define $c = \sup\{x \in U, x < b\}$ Then $c \in \bar{U} \cap \bar{V}$ (contradiction). $\square$

Example 44. *S^1 is connected: $f : \mathbb{R} \to S^1 : f(t) = e^{it}$ is a continuous function mapping a connected top space, $\mathbb{R}$ onto S^1. However, $\mathbb{R}$ and S^1 are not homeomorphic: let g be a homeomorphism between them. Then consider $g(\mathbb{R} \setminus \{0\}) = S^1 \setminus \{g(0)\}$, but $\mathbb{R} \setminus \{0\}$ is disconnected, while $S^1 \setminus \{g(0)\}$ remains connected. (contradiction)*