

# MATH 5362 Homework II

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## 1 Problem I

**Proposition 1.1.** *Let  $K = \mathbb{Q}(\theta)$  where  $\theta^3 + 11\theta - 4 = 0$ . Then  $\frac{\theta^2 - \theta}{2} \in \mathcal{O}_K$*

*Proof.*

$$\begin{aligned}\theta^3 &= 4 - 11\theta \\ x &:= \frac{1}{2}(\theta^2 - \theta) \\ x^2 &= \frac{1}{4}(\theta^4 - 2\theta^3 + \theta^2) \\ &= \frac{1}{4}(\theta(4 - 11\theta) - 2(4 - 11\theta) + \theta^2) \\ &= \frac{1}{4}(4\theta - 11\theta^2 - 8 + 22\theta + \theta^2) \\ &= \frac{1}{4}(-10\theta^2 + 26\theta - 8) \\ &= \frac{1}{2}(-5\theta^2 + 13\theta - 4) \\ x^2 + 5x &= \frac{1}{2}(-5\theta^2 + 13\theta - 4 + 5\theta^2 - 5\theta) \\ &= \frac{1}{2}(8\theta - 4) = 4\theta - 2 \\ x^3 &= \frac{1}{4}(-5\theta^4 + 13\theta^3 - 4\theta^2 + 5\theta^3 - 13\theta^2 + 4\theta) \\ &= \frac{1}{4}(-5\theta^4 + 18\theta^3 - 17\theta^2 + 4\theta) \\ &= \frac{1}{4}(-5\theta(4 - 11\theta) + 18(4 - 11\theta) - 17\theta^2 + 4\theta) \\ &= \frac{1}{4}(-20\theta + 55\theta^2 + 72 - 198\theta - 17\theta^2 + 4\theta) \\ &= \frac{1}{4}(38\theta^2 - 214\theta + 72) \\ &= \frac{1}{2}(19\theta^2 - 107\theta + 36)\end{aligned}$$

$$\begin{aligned}
x^3 - 19x &= \frac{1}{2}(19\theta^2 - 107\theta + 36 - (19\theta^2 - 19\theta)) \\
&= \frac{1}{2}(-88\theta + 36) \\
&= (-44\theta + 18) \\
x^3 - 19x + 11(x^2 + 5x) &= (-44\theta + 18) + 11(4\theta - 2) \\
x^3 + 11x^2 + 36x &= -44\theta + 18 + 44\theta - 22 \\
&= -4 \\
x^3 + 11x^2 + 36x + 4 &= 0
\end{aligned}$$

□

## 2 Problem II

**Proposition 2.1.** *Let  $K = \mathbb{Q}(\theta)$  where  $\theta^3 - 4\theta + 2 = 0$ . Let  $\alpha = \theta + \theta^2$ . Then  $D(\alpha) = -148$*

*Proof.*

$$\begin{aligned}
\alpha^2 &= \theta^4 + 2\theta^3 + \theta^2 \\
&= \theta(4\theta - 2) + 2(4\theta - 2) + \theta^2 \\
&= 4\theta^2 - 2\theta + 8\theta - 4 + \theta^2 \\
&= 5\theta^2 + 6\theta - 4 \\
\alpha^2 - 5\alpha &= \theta - 4 \\
\alpha^3 &= 5\theta^4 + 6\theta^3 - 4\theta^2 + 5\theta^3 + 6\theta^2 - 4\theta \\
&= 5\theta^4 + 11\theta^3 + 2\theta^2 - 4\theta \\
&= 5\theta(4\theta - 2) + 11(4\theta - 2) + 2\theta^2 - 4\theta \\
&= 20\theta^2 - 10\theta + 44\theta - 22 + 2\theta^2 - 4\theta \\
&= 22\theta^2 + 30\theta - 22 \\
\alpha^3 - 22\alpha &= 22\theta^2 + 30\theta - 22 - (22\alpha^2 + 22\alpha) \\
&= 8\alpha - 22 \\
\alpha^3 - 22\alpha - 8(\alpha^2 - 5\alpha) &= 8\alpha - 22 - 8(\alpha - 4) \\
&= 10 \\
\alpha^3 - 8\alpha^2 + 18\alpha - 10 &= 0
\end{aligned}$$

Now, clearly  $p(x) := x^3 - 8x^2 + 18x - 10$  is monic. It is irreducible via the rational roots theorem. Thus, it is the minimal polynomial of  $\alpha$ . Furthermore, let

$$p'(x) = 3x^2 - 16x + 18$$

be the formal derivative of  $p$ . Then:

$$\begin{aligned}
p'(\alpha) &= 3\alpha^2 - 16\alpha + 18 \\
&= 3(\theta + \theta^2)^2 - 16(\theta + \theta^2) + 18 \\
&= 3(\theta^4 + 2\theta^3 + \theta^2) - 16(\theta + \theta^2) + 18 \\
&= 3\theta^4 + 6\theta^3 + 3\theta^2 - 16\theta - 16\theta^2 + 18 \\
&= 3\theta^4 + 6\theta^3 - 13\theta^2 - 16\theta + 18 \\
&= 3\theta(4\theta - 2) + 6(4\theta - 2) - 13\theta^2 - 16\theta + 18 \\
&= 12\theta^2 - 6\theta + 24\theta - 12 - 13\theta^2 - 16\theta + 18 \\
&= -\theta^2 + 2\theta + 6 \\
\theta \cdot p'(\alpha) &= -(\theta^3) + 2\theta^2 + 6\theta \\
&= -(4\theta - 2) + 2\theta^2 + 6\theta \\
&= 2\theta^2 + 2\theta + 2 \\
\theta^2 \cdot p'(\alpha) &= 2\theta^3 + 2\theta^2 + 2\theta \\
&= 2(4\theta - 2) + 2\theta^2 + 2\theta \\
&= 2\theta^2 + 10\theta - 4
\end{aligned}$$

Now,  $\{1, \theta, \theta^2\}$  is a basis for  $\mathbb{Q}(\theta)$ . This means that the norm  $N_{\mathbb{Q}(\theta)}(p'(\alpha))$  is in fact the determinant of a matrix as follows:

$$N_{\mathbb{Q}(\theta)}(p'(\alpha)) = \det \begin{vmatrix} -1 & 2 & 6 \\ 2 & 2 & 2 \\ 2 & 10 & -4 \end{vmatrix} = 148$$

To conclude, we use the following formula,

$$\begin{aligned}
D(\alpha) &= (-1)^{\binom{n}{2}} N_K(p'(\alpha)) \\
&= (-1)^3 (148) \\
&= -148
\end{aligned}$$

□

### 3 Problem III

**Proposition 3.1.** *For  $p$  an odd prime, the discriminant of the cyclotomic field  $\mathbb{Q}(\zeta_p)$  equals  $(-1)^{\frac{p-1}{2}} p^{p-2}$*

*Proof.* The discriminant of a general cyclotomic field  $\mathbb{Q}(\zeta_p)$  is given by □