

Topology Notes - Spring 2022

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1 General Topology

1.1 August 30, 2022

Definition 1. Let X be a set. A topology τ on X is a family of subsets (called open subsets) such that:

1. $\emptyset, X \in \tau$
2. Finite intersections of open subsets are open
3. Arbitrary unions of open subsets are open

Definition 2. Let $f : X \rightarrow Y$, where X and Y are topological spaces. f is continuous if the preimage of every open set is an open set,

Proposition 1. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous maps, then $g \circ f$ is continuous.

Proof. If U is open in Z , then $g^{-1}(U)$ is open in Y . Since f is continuous, $f^{-1}(g^{-1}(U))$ is open. Thus, U open implying $(g \circ f)^{-1}(U)$ is open gives us that $g \circ f$ is continuous. \square

How should we construct topological spaces?:

Definition 3. A basis for a topology is a collection \mathcal{B} of subsets of a set X such that:

- X is the union of the elements of \mathcal{B}
- if $B_1, B_2 \in \mathcal{B}$, then for every $x \in B_1 \cap B_2$, there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

We define the open sets as arbitrary unions of elements of \mathcal{B} .

By the first property, $\emptyset, X \in \mathcal{B}$. Arbitrary unions are in the topology by definition. Now, suppose $B_1, B_2 \in \mathcal{B}$, then $B_1 \cap B_2 = \bigcup_{x \in B_1 \cap B_2} B_3(x)$, where B_3 is as prescribed in the second property. We then get by induction that finite intersections are in the topology.

Example 1. Consider $\{(a, b) | a < b, a, b \in \mathbb{R}\}$. This forms a basis for the standard topology on \mathbb{R} .

Example 2. Consider $\{(a, b) | a < b, a, b \in \mathbb{Q}\}$. This ALSO forms a basis for the standard topology on \mathbb{R} .

Proposition 2. $f : X \rightarrow Y$ is continuous if the preimage of every basis element is open.

Example 3. Consider $C[a, b]$. Consider the function $\phi : C[a, b] \rightarrow \mathbb{R}$ such that $\phi(f) = \int_a^b f(x) dx$ is continuous

Example 4. Consider $L^p(\mathbb{R})$, where $p \geq 1$. Consider $\phi : L^p(\mathbb{R}) \rightarrow \mathbb{R}$, where $\phi(f) = \int_{-\infty}^{\infty} (f(x))^p dx$. ϕ is continuous.

Definition 4. Let (X, τ) be a topological space. Consider $Y \subseteq X$. We define $\tau_Y = \{U \cap Y | U \in \tau\}$ to be the subspace topology. The axioms for a topology are easily checked here.

Proposition 3. Let (X, τ) be a topological space. Let \mathcal{B} be a family of subsets that satisfy the conditions of a basis. If every element of \mathcal{B} is open and if for every open set U and every $x \in U$, there is $B \in \mathcal{B}$ such that $x \in B \subseteq U$, then \mathcal{B} is a basis for τ

Proof. For every $U \in \tau$, write $U = \bigcup_{x \in U} B(x)$, where B is prescribed as above. \square

Example 5. Consider $\{(a, b) | a < b, a, b \in \mathbb{R}\}$. This is the lower limit topology. Note that $[1, 2)$ is not a union of open intervals, so this is not the standard one.

$$(a, b) = \bigcup_{n=1}^{\infty} [a + \frac{1}{n}, b)$$

Definition 5. τ' is finer than τ if $U \in \tau \implies U \in \tau'$. We say here that τ is coarser than τ' . For instance, the lower limit topology is finer than the standard topology.

Example 6. Consider $\mathcal{B} = \{V_n(a) \mid a \in \mathbb{R}^n, n > 0\}$.

Example 7. Consider $C[a, b]$. The basis consists of the sets:

$$\{g \mid \sup_{x \in [a, b]} |f(x) - g(x)| < r\}, \quad f, g \in C[a, b], \quad r > 0$$

Example 8. Consider $L^p(\mathbb{R})$, for $p \geq 1$. The basis consists of the sets:

$$\{g \mid \left(\int_{-\infty}^{\infty} |f - g|^p dx \right)^{\frac{1}{p}} < r\}, \quad f \in L^p(\mathbb{R}), \quad n > 0$$

It is of note that these are great examples of Banach Spaces

1.2 September 1, 2022

Consider the following diagram for the product topology:

$$\begin{array}{ccc} Z & \xrightarrow{f} & \prod_{\alpha \in A} X_\alpha \\ & \searrow f_\alpha & \downarrow \pi_\alpha \\ & & X_\alpha \end{array}$$

Note that $f_\alpha = \pi_\alpha \circ f$. f is continuous iff f_α is continuous for all α . We want each of the projections π_α to be continuous.

Note that $\pi_\alpha^{-1}(U_\alpha)$ is open for every open $U_\alpha \subseteq X_\alpha$, because:

$$\pi^{-1}(U_\alpha) = U_\alpha \times \prod_{\beta \neq \alpha} X_\beta$$

Hence, a basis for this topology are sets of the form:

$$U_{\alpha_1} \times U_{\alpha_2} \times \cdots \times U_{\alpha_n} \times \prod_{\beta \neq \alpha_i} X_\beta \quad U_{\alpha_i} \in X_{\alpha_i} \text{ open}$$

Proposition 4. f is continuous if and only if f_α is continuous for every α .

Proof. The forward direction is trivial. Since π_α is continuous, $f_\alpha = f \circ \pi_\alpha$ is a composition of continuous maps, and is thus continuous. Now, let f_α be continuous for every α . We only need to check that the preimage of a basis element is an open set.

$$\begin{aligned} f^{-1} \left(U_{\alpha_1} \times U_{\alpha_2} \times \cdots \times U_{\alpha_n} \times \prod_{\beta \neq \alpha_i} X_\beta \right) &= f^{-1} \left(\bigcap_{i=1}^n \left(U_{\alpha_i} \times \prod_{\beta \neq \alpha_i} X_\beta \right) \right) \\ &= \bigcap_{i=1}^n f^{-1} \left(U_{\alpha_i} \times \prod_{\beta \neq \alpha_i} X_\beta \right) \\ &= \bigcap_{i=1}^n f_{\alpha_i}^{-1}(U_{\alpha_i}) \end{aligned}$$

This is open because the preimages of U_α by f_α are open, and finite intersections of open sets are open \square

Example 9. Consider $C[a, b] \subseteq \prod_{x \in [a, b]} \mathbb{R}$. This topology has as basis sets of the form:

$$V_{f, x_1, \dots, x_n, \epsilon} = \{g : |f(x_i) - g(x_i)| < \epsilon, i \in \mathbb{N}\}$$

This is called the weak* topology on $C[a, b]$.

Example 10. Consider $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. we can define this to have multiplication, addition, or division defined on $\mathbb{R} \times \mathbb{R} \setminus \{0\}$.

Proposition 5. If X is a topological space and $f, g : X \rightarrow \mathbb{R}$ (with maybe $g : X \rightarrow \mathbb{R} \setminus \{0\}$) are continuous functions, then $f + g, f \cdot g, \frac{f}{g}$ are continuous.

Proof. These are compositions of continuous functions. We consider:

$$X \xrightarrow{(f, g)} \mathbb{R} \times \mathbb{R} \xrightarrow{+} \mathbb{R}$$

The same happens for multiplication and division. □

Disjoint Unions of Topological Spaces:

Consider $X_\alpha, \alpha \in A, X_\alpha \cap X_\beta = \emptyset$. We then denote the disjoint union as $\bigsqcup_{\alpha \in A} X_\alpha$. Consider the following diagram:

$$\begin{array}{ccc} \bigsqcup_{\alpha \in A} X_\alpha & \xrightarrow{f} & Y \\ \uparrow & \nearrow f_\alpha & \\ X_\alpha & & \end{array}$$

Similarly as before, f is continuous if and only if f_α is continuous for all α . A topology consists of the sets that are unions of U_α where U_α are open in X_α . If the X_α 's are not disjoint, then we might run into trouble in the overlap!

Quotient Spaces Consider a topological space X . Take $f : X \rightarrow Y$ to be a surjective function. The open sets of Y are of the form $f(U)$ where U is open in X .

Another perspective: Consider a topological space X . Take an equivalence relation on X . Denote by \hat{x} the equivalence class of x . Let:

$$Y = \{\hat{x} \mid x \in X\}$$

Now, we consider the function $f : X \rightarrow Y$ where $f(x) = \hat{x}$, and we use the notation X/\sim to represent this quotient space. Now, consider the following diagram:

$$\begin{array}{ccc} X & \xleftarrow{g \circ f} & Z \\ f \downarrow & \nwarrow g & \\ Y & & \end{array}$$

We want g to be continuous if and only if $g \circ f$ is continuous, and this is the reason for our construction of the quotient space.

Proposition 6. The above is a topology on Y .

Proof. (a) $Y = f(X)$ is open, and $\emptyset = f(\emptyset)$ is open

(b) $f(U_1 \cap U_2 \cap \cdots \cap U_n)$ is a finite intersection of $f(U_i)$'s, so this is open

(c) If $f(U_\alpha)$ is open, for all α , then it's easy to see that the images commute with the union, and so arbitrary unions are open. □

Example 11. Take $f : \mathbb{R} \rightarrow \mathbb{C}$. This may not be a surjection, but it is certainly a surjection onto the image by construction. Let $f(x) = e^{2\pi i x}$. The image is $\{z : |z| = 1\} =: S^1$ (the 1-dimensional sphere). We can induce a topology on S^1 by using the standard topology on \mathbb{R} , or by inducing from the topology on \mathbb{C} . The induced topology is the same as the quotient topology. This can be resolved by thinking of the standard convention where $S^1 = \mathbb{R}/\mathbb{Z}$.

Example 12. Consider S^2 , the sphere in \mathbb{R}^3 . Consider the inclusion of $S^2 \subseteq \mathbb{R}^3$, and consider the induced topology on S^2 . We can consider two points on the sphere to be equivalent as follows:

$$(x, y, z) \sim (x', y', z') \iff (x = x', y = y', z = z') \vee (x = -x', y = -y', z = -z')$$

Let $\mathbb{R}P^2 = S^2 / \sim$, and consider the quotient topology. This space is essentially a cap being put on top of a Möbius band. By definition, this is the two-dimensional projective plane. Another way to construct $\mathbb{R}P^2$:

consider $\mathbb{R}^3 \setminus \{(0, 0, 0)\} \subseteq \mathbb{R}^3$ using the standard topology. We take the following equivalence:

$$(x, y, z) \sim (x', y', z') \iff \exists \lambda \neq 0 : x = \lambda x', y = \lambda y', z = \lambda z'$$

Then, $(\mathbb{R}^3 \setminus \{(0, 0, 0)\}) / \sim$ is exactly $\mathbb{R}P^2$. We can then generalize this construction to $\mathbb{R}P^n$ using the same procedure, but we can also have $\mathbb{C}P^n$ to be equivalence classes of points in $\mathbb{C}^{n+1} \setminus \{(0, 0, \dots, 0)\}$, with the same equivalence as before, except now $\lambda \in \mathbb{C}$. For instance $\mathbb{C}P^1$ is the Riemann sphere.

Example 13. Consider the unit square with corners $(0, 0), (0, 1), (1, 0), (1, 1)$. We induce an equivalence relation by $(x, 1) \sim (x, 0)$ for every x , and $(1, y) \sim (0, y)$ for every y . This generates a torus. This comes from the induced topology from the inclusion into \mathbb{R}^2 , so the torus is the quotient topology $\mathbb{R}^2 / \mathbb{Z}^2$. The torus can also be described as $S^1 \times S^1$.

Definition 6. $f : X \rightarrow Y$ is a homeomorphism if f is continuous, invertible, and f^{-1} is continuous.

Example 14. $([0, 1] \times [0, 1]) / \sim$ is homeomorphic to $S^1 \times S^1$, where \sim is the equivalence relation defined in Example 11.

Example 15. Consider $[1, 2], [3, 4] \subseteq \mathbb{R}$ with the induced topology. Then, take the disjoint union $[1, 2] \sqcup [3, 4]$. Then, we take the quotient where $1 \sim 2 \sim 3 \sim 4$. This gives us a figure 8, where the circles touch at $\{1, 2, 3, 4\}$.

Example 16. Let X be a topological space. Consider $[-1, 1]$ with the induced topology. Consider $X \times [-1, 1]$ with the product topology. We then consider $(x, 1) \sim (y, 1)$ and $(x, -1) \sim (y, -1)$ for every x and y , and take the quotient topology. This then gives a two-sided cone with a circular middle, called the suspension ΣX . If we consider S^n , then $\Sigma S^n \cong S^{n+1}$.

1.3 September 6, 2022

Thus far, as a recap, we have covered the following constructions:

- Induced Topology
- Product Topology
- Disjoint Sum
- Quotients

We now discuss metric spaces.

Definition 7. A distance on a set X is a function $d : X \times X \rightarrow [0, \infty)$ such that:

- (a) $d(x, y) = 0 \iff x = y$
- (b) $d(x, y) = d(y, x)$
- (c) $d(x, z) \leq d(x, y) + d(y, z)$

A metric is then another word we use instead of “distance”. (X, d) is thus termed a metric space.

Example 17. (\mathbb{R}^n, d) , where $d(\vec{x}, \vec{y}) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$

Example 18. (\mathbb{R}^n, d) , where $d(\vec{x}, \vec{y}) = \max_{1 \leq i \leq n} |x_i - y_i|$

Example 19. $([0, 1]^A, d)$, where $(x_\alpha)_\beta, (y_\alpha)_\beta \in \prod_{\beta \in B} [0, 1]$, and:

$$d((x_\alpha)_\alpha, (y_\alpha)_\alpha) = \sup_\alpha |x_\alpha - y_\alpha|$$

A basis for the topology induced by the metric consists of the balls (for $x \in X$ and $\epsilon > 0$):

$$B(x, \epsilon) = \{y | d(x, y) < \epsilon\}$$

We now prove that these open balls form a basis.

Proof. First, we must check that these cover the entire space. However, it is clear that $X = \bigcup_{x \in X} B(x, \epsilon)$.

For another condition, we see that if $B_1 = B(x_1, \epsilon_1)$ and $B_2 = B(x_2, \epsilon_2)$ are elements of the basis, and if $x \in B_1 \cap B_2$, we have to show that there is a $B_3 = B(x_3, \epsilon_3)$ such that $x \in B_3 \subseteq B_1 \cap B_2$. We can construct this geometrically such that we pick $x = x_3$ and $\epsilon_3 < \min(\epsilon_1 - d(x, x_1), \epsilon_2 - d(x, x_2))$. Then, if $y \in B_3$, we have that:

$$\begin{aligned} d(y, x_1) &< d(y, x) + d(x, x_1) \\ &< \epsilon_3 + d(x, x_1) \\ &< \epsilon_1 - d(x, x_1) + d(x, x_1) \\ &= \epsilon_1 \end{aligned}$$

Thus, $y \in B_1$, and by the exact same argument, $y \in B_2$, so we have indeed proven that this does form a basis. \square

We now note that open sets are unions of balls.

Example 20. On \mathbb{R}^n , the standard metric and the max metric from Examples 17 and 18 induce the same topology.

Proposition 7. If two metrics have the property that for every ball of the first and every element of it, there is a ball of the second topology centered at this element and included in the first ball, then the second topology is finer than the first.

$$x \in B'(x, \epsilon') \subseteq B(y, \epsilon), \quad \forall x \in B(y, \epsilon)$$

In Example 20, a disc can be written as a union of squares, and vice versa.

Example 21. If (X, d) is a metric space, define:

$$\bar{d}(x, y) = \min(d(x, y), 1)$$

Then, \bar{d} is a metric that induces the same topology.

We first check that \bar{d} is a metric by ensuring that it satisfies the triangle inequality. If $d(x, y), d(y, z), d(x, z) < 1$, then:

$$\begin{aligned} \bar{d}(x, z) &= d(x, z) \\ &\leq d(x, y) + d(y, z) \\ &= \bar{d}(x, y) + \bar{d}(y, z) \end{aligned}$$

We have a similar case where $d(x, y)$ or $d(y, z) > 1$, because we then have $1 \leq \bar{d}(x, y) + \bar{d}(y, z)$ holds. Finally, if $d(x, y), d(y, z) < 1$, and $d(x, z) > 1$, then:

$$\bar{d}(x, z) \leq d(x, z) < d(x, y) + d(y, z) = \bar{d}(x, y) + \bar{d}(y, z)$$

Theorem 1. If $(X, d), (Y, d')$ are metric spaces, then $f : X \rightarrow Y$ is continuous at $x \in X$ if $\forall \epsilon > 0, \exists \delta > 0$ such that $f(B(x, \delta)) \subseteq B(f(x), \epsilon)$.

Definition 8. $f : X \rightarrow Y$ is called uniformly continuous if $\forall \epsilon > 0, \exists \delta > 0$ such that if $d(x, y) < \delta$, then $d'(f(x), f(y)) < \epsilon$.

Definition 9. A sequence $(x_n)_{n \geq 1}$ in the metric space (X, d) is called Cauchy if $\forall \epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $m, n \geq N$, $d(x_m, x_n) < \epsilon$.

Definition 10. A metric space is called complete if every Cauchy sequence is convergent.

Definition 11. A normed vector space V is a vector space endowed with a function $\|\cdot\| : V \rightarrow [0, \infty)$ such that:

$$(a) \quad \|x\| = 0 \iff x = 0$$

$$(b) \quad \|\lambda x\| = |\lambda| \|x\|, \quad \lambda \in \mathbb{R} \vee \mathbb{C}$$

$$(c) \quad \|x + y\| \leq \|x\| + \|y\|$$

We define $d(x, y) = \|x - y\|$.

Definition 12. A Banach space is a complete normed vector space.

Example 22. $C[0, 1]$, L^p spaces, Sobolev spaces, Hardy spaces, Bergman spaces.

Manifolds:

Definition 13 (Veblen). A manifold M is a Hausdorff topological space endowed with an onto function

$$f : \bigsqcup_{\alpha \in A} U_{\alpha} \rightarrow M$$

such that for every α , U_{α} is an open set in some \mathbb{R}^n , where n is the same for all α , and $f|_{U_{\alpha}}$ is a homeomorphism onto the image for each α .

We can define charts here as something of the form:

$$f_{\alpha} = f|_{U_{\alpha}} : U_{\alpha} \rightarrow f(U_{\alpha}) \subseteq M$$

A collection of charts is then an atlas. Also, we get compositions where:

$$f_{\beta}^{-1} \circ f_{\alpha} : U_{\alpha} \rightarrow U_{\beta}$$

Definition 14. We call a topological space Hausdorff if for every $x \neq y$, there are open sets U, V such that $x \in U$ and $y \in V$, and $U \cap V = \emptyset$.

- If $f_{\beta}^{-1} \circ f_{\alpha}$ is smooth, then the manifold is called smooth.
- If \mathbb{R}^n is replaced by \mathbb{C}^n and $f_{\beta}^{-1} \circ f_{\alpha}$ is holomorphic or analytic then M is called complex.

Example 23. The circle $S^1 = \{z : |z| = 1\}$. Define $f : \mathbb{R} \rightarrow \mathbb{C}$ such that $f(t) = e^{2\pi it}$. Let $U_1 = (-\pi, \pi)$ and $U_2 = (0, 2\pi)$. Now, we consider a function $g : U_1 \sqcup U_2 \rightarrow \mathbb{C}$, where $g(t) = e^{2\pi it}$. We can consider $f(U_1) \cap f(U_2)$ as the overlap, and so $f_1^{-1}(f(U_1) \cap f(U_2)) = (\pi, 0) \cup (0, \pi)$, and $f_2^{-1}(f(U_1) \cap f(U_2)) = (\pi, 2\pi) \cup (0, \pi)$.

Then, $f_2^{-1} \circ f_1 : (-\pi, 0) \cup (0, \pi) \rightarrow (\pi, 2\pi) \cup (0, \pi)$ is as follows:

$$f_2^{-1} \circ f_1(t) = \begin{cases} t & , t \in (0, \pi) \\ t + 2\pi & , t \in (-\pi, 0) \end{cases}$$

Theorem 2. If M_1, M_2 are manifolds of dimensions n_1, n_2 , then $M_1 \times M_2$ is a manifold of dimension $n_1 + n_2$.

Consider:

$$\begin{aligned} f &: \bigsqcup U_{\alpha} \rightarrow M_1 \\ g &: \bigsqcup V_{\beta} \rightarrow M_2 \\ f \times g &: \bigsqcup U_{\alpha} \times V_{\beta} \rightarrow M_1 \times M_2 \\ (f \times g)_{\alpha\beta} &= (f_{\alpha}, g_{\beta}) \end{aligned}$$

Example 24. $(S^1)^{\times n}$ is a manifold. In the $n = 2$ case, we get $(-\pi, \pi)^2 \sqcup (-\pi, \pi) \times (0, 2\pi) \sqcup (0, 2\pi) \times (-\pi, \pi) \sqcup (0, 2\pi)^2$ as our domain, where $f(t, s) = (e^{2\pi it}, e^{2\pi is})$, and this gives us the torus.

1.4 September 8, 2022

Example 25. Consider $\mathbb{R}P^2 = \{\hat{x} : x \in \mathbb{R}^3 \setminus \{0\}, x \sim y \iff x = \lambda y, \lambda \neq 0\}$. $\mathbb{R}P^2$ is a manifold, with the following three charts:

$$U_1 = U_2 = U_3 = \mathbb{R}^2$$

$$f : \bigsqcup_{i=1}^3 U_i \rightarrow \mathbb{R}P^2$$

$$f_1(x_1, x_2) = [1 : x_1 : x_2]$$

$$f_2(x_1, x_2) = [x_1 : 1 : x_2]$$

$$f_3(x_1, x_2) = [x_1 : x_2 : 1]$$

These maps are one-to-one, and onto. We note that $f_2^{-1} \circ f_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the map $(x_1, x_2) \xrightarrow{f_1} [1 : x_1 : x_2] = \left[\frac{1}{x_1} : 1 : \frac{x_2}{x_1}\right] \xrightarrow{f_2^{-1}} \left(\frac{1}{x_1}, \frac{x_2}{x_1}\right)$

Example 26. Consider $\mathbb{C}P^1$. We have that $U_1 = U_2 = \mathbb{C}$. We define maps as:

$$f_1 : U_1 \rightarrow \mathbb{C}P^1$$

$$f_1(z) = [1 : z]$$

$$f_2 : U_2 \rightarrow \mathbb{C}P^1$$

$$f_2(z) \rightarrow [z : 1]$$

$$f_2^{-1} \circ f_1 : z \rightarrow [1 : z] \rightarrow \left[\frac{1}{z} : 1\right] \rightarrow \frac{1}{z}$$

Definition 15 (Poincaré). A smooth manifold is a subspace of \mathbb{R}^n that is locally the graph of a smooth function.

Example 27. Consider S^2 . Consider the graph $f(x_1, x_2) = \pm\sqrt{1 - x_1^2 - x_2^2}$, and similar considerations for the two other orientations. These graphs act as charts from the plane to the sphere.

Now, we return to closed sets!

Example 28. $[a, b], [a, \infty)$

Example 29.

$$C = [0, 1] \setminus \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right)$$

This set consists of all numbers in $[0, 1]$ that admit a ternary representation with only 0's and 2's. Consider a function:

$$f : C \rightarrow [0, 1]^2$$

$$f(0.a_1a_2a_3\cdots) = (0.\frac{a_1}{2}\frac{a_3}{2}\cdots, 0.\frac{a_2}{2}\frac{a_4}{2}\cdots)$$

This function is then continuous and onto from C to $[0, 1]^2$. By Tietze's Extension Theorem, we get an extension $\tilde{f} : [0, 1] \rightarrow [0, 1]^2$ which is continuous and onto. This was induced by G. Peano.

Example 30. Sierpiński Triangle.

Example 31. In the discrete topology, every set is clopen

Example 32. $\mathbb{Q} \subseteq \mathbb{R}$, (a, b) are clopen in \mathbb{Q} .

Example 33. In \mathbb{R}^n , consider $\overline{B}(x, \epsilon) = \{y : d(x, y) \leq \epsilon\}$ as the closed balls.

Example 34. The Zariski topology, where $f(z_1, z_2, \dots, z_n) \in \mathbb{C}[z_1, \dots, z_n]$, and:

$$V(f) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : f(z_1, \dots, z_n) = 0\}$$

These are called algebraic sets, or varieties if f is irreducible, and these are the closed sets of the Zariski topology. Unions of algebraic sets are the closed sets of the Zariski topology.

Proposition 8 (MAYBE ON EXAM 1). (1) If Y is a subspace of X , then $A \subseteq Y$ is closed if and only if $A = B \cap Y$ where $B \subseteq X$ is closed.

(2) Let $A \subseteq Y \subseteq X$. If A is closed in Y and Y is closed in X , then A is closed in X .

(3) If A is closed in X and B is closed in Y , then $A \times B$ is closed in $X \times Y$.

(4) If A_α is closed in X_α for $\alpha \in I$, then $\prod_{\alpha \in I} A_\alpha$ is closed in $\prod_{\alpha \in I} X_\alpha$ with the product topology.

Proof. (1) $Y \setminus A$ is open. So there is a U such that $Y \setminus A = Y \cap U$. Let $B = X \setminus U$. Then

$$A = Y \setminus (Y \setminus A) = Y \setminus (Y \cap U) = Y \setminus U = Y \cap (X \setminus U) = B \cap Y$$

Now, if $A = B \cap Y$, then $Y \setminus A = (X \setminus B) \cap Y$.

(2) A closed in Y and Y closed in X gives us that $X \setminus Y = U$ is open in X . Then, $A = Y \cap (X \setminus V)$. Thus $X \setminus A = V \cup U$, so since this is a union of two open sets, it is open, and thus A is closed in X .

(3) $(X \times Y) \setminus (A \times B) = (X \times (Y \setminus B)) \cup ((X \setminus A) \times Y)$. These sets are all open, so $A \times B$ is closed.

(4) Induction! Do the same thing as above as unions.

$$\left(\prod_{\alpha \in I} X_\alpha \right) \setminus \left(\prod_{\alpha \in I} A_\alpha \right) = \bigcup_{\alpha \in I} \left(\left(\prod_{\beta \neq \alpha} X_\beta \right) \times (X_\alpha \setminus A_\alpha) \right)$$

□

Proposition 9. Let X, Y be topological spaces. Then, $f : X \rightarrow Y$ is continuous if and only if the preimage of every closed set is closed.

Proof.

$$f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$$

□

Definition 16. The closure of a set S is the smallest closed set containing S , denoted \overline{S} .

Definition 17. The interior of a set S is the largest open set contained in S , denoted $\text{Int}(S)$.

Example 35. $\overline{\mathbb{Q}} - \mathbb{R}$. $\text{Int}(\mathbb{Q}) = \emptyset$

Example 36. $B(x, r) = \{y : d(x, y) < r\} \implies \overline{B(x, r)} = \{y : d(x, y) \leq r\}$.

Lemma 1. Let X be a topological space, and $A \subseteq X$. Then, $\overline{X \setminus A} = X \setminus \text{Int}(A)$

Proof.

$$X \setminus A \subseteq X \setminus \text{Int}(A) \implies \overline{X \setminus A} \subseteq \overline{X \setminus \text{Int}(A)} = X \setminus \text{Int}(A)$$

Now, let $x \in X \setminus \overline{(X \setminus A)}$. Then there is an open set U such that $x \in U$ and $U \cap \overline{(X \setminus A)} = \emptyset$. Thus, $U \cap (X \setminus A) = \emptyset$, so $U \subseteq A$. Therefore, $x \in \text{Int}(A)$. Hence, $x \notin X \setminus \text{Int}(A)$, so $X \setminus \text{Int}(A) \subseteq \overline{X \setminus A}$, and we have the double inclusion as desired. □

Theorem 3. Let $A \subseteq X$. Then, $x \in \overline{A}$ if and only if for every open set U such that $x \in U$, we have $U \cap A \neq \emptyset$.

Proof. First, we wish to show that If $x \in \overline{A}$ and U is open where $x \in U$, then $U \cap A \neq \emptyset$. If, for the sake of contradiction, $U \cap A = \emptyset$, then $U \subseteq \text{Int}(X \setminus A)$, so $X \setminus \text{Int}(X \setminus A) = \overline{X \setminus (X \setminus A)} = \overline{A}$, which does not contain x , and so we get a contradiction.

Conversely, if x has the property that every open set U containing x intersects A , then $x \notin \text{Int}(X \setminus A)$. So $x \in X \setminus \text{Int}(X \setminus A) = \overline{X \setminus (X \setminus A)} = \overline{A}$. Thus, we are done. \square

Proposition 10 (Midterm maybe). 1) Let $Y \subseteq X$, with the subspace topology. If $A \subseteq Y$, then let us denote \overline{A}_X as the closure of A in X . Then, the closure of A in Y is $\overline{A}_X \cap Y$.

2) Taking the same conditions as above, if Y is closed in X , then the closure of A in X and Y is the same.

$$3) \prod_{\alpha \in I} \overline{A}_\alpha = \overline{\prod_{\alpha \in I} A_\alpha}$$

Proof. 1):

Let $x \in \overline{A}_X \cap Y$. Then, for every open set U in X containing x , $U \cap A \neq \emptyset$. So $(U \cap Y) \cup A \neq \emptyset$. But $U \cap Y$ with U open in X are all open subsets of Y . By Theorem 3, x is in the closure of A in Y . Thus, $\overline{A}_X \cap Y \subseteq \overline{A}_Y$.

Conversely, $\overline{A}_X \cap Y$ is closed and contains A , so the closure of A in Y is contained in $\overline{A}_X \cap Y$.

2):

if Y is closed, then $\overline{A}_X \subseteq \overline{Y} = Y$. So $\overline{A}_X \cap Y = \overline{A}_X$.

3):

Let $x = (x_\alpha)_\alpha \in \prod_{\alpha \in I} \overline{A}_\alpha$. Let $\prod_{j=1}^n U_{\alpha_j} \times \prod_{\beta \neq \alpha_j} X_\beta$, or $\prod_{\alpha \in I} U_\alpha$ such that $x_\alpha \in U_\alpha$. Then, by Theorem 3, $U_\alpha \cap A_\alpha \neq \emptyset$, for all α , and $X_\beta \cap A_\beta \neq \emptyset$, for all $\beta \neq \alpha_j$. It follows that in either case, the open set intersects $\prod_{\alpha \in I} A_\alpha$. Thus, $\prod_{\alpha \in I} \overline{A}_\alpha \subseteq \overline{\prod_{\alpha \in I} A_\alpha}$. Conversely, let $x \in \overline{\prod_{\alpha \in I} A_\alpha}$. Then, every open set containing x intersects $\prod_{\alpha \in I} A_\alpha$. Choose U of either form as expressed in the previous direction, and we get coordinate-wise nonempty intersections. Vary U_α 's to make them be any open set that contains x_α . You then obtain the reverse inclusion, and thus we are done. \square

This result does not hold for interiors as seen in the following examples:

Example 37. $\mathbb{Q} \subseteq \mathbb{R}$. $\text{Int}(\mathbb{Q}) \neq \emptyset$, but the interior of \mathbb{Q} in \mathbb{Q} is \emptyset .

However, the third case of the proposition works for interiors in the box topology but not the product topology.

Example 38. $[0, 1] \subseteq \mathbb{R}$, $\text{Int}([0, 1]) = (0, 1)$. However:

$$\underbrace{\prod_{i=1}^{\infty} (0, 1)}_{\text{not an open set}} \subseteq \prod_{i=1}^{\infty} \mathbb{R}$$

Proposition 11. *Let X, Y be topological spaces. $f : X \rightarrow Y$ is continuous if and only if for every subset $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$*

Proof. Assume f is continuous. Then $f^{-1}(Y \setminus \overline{f(A)})$ is open. This is the complement of $f^{-1}(\overline{f(A)})$, so this set is open. Then $A \subseteq f^{-1}(\overline{f(A)})$, so $\overline{A} \subseteq f^{-1}(\overline{f(A)}) \implies f(\overline{A}) \subseteq \overline{f(A)}$.

Conversely, let $B \subseteq Y$ be open. we want to show $f^{-1}(B)$ is open. Let $A = X \setminus f^{-1}(B)$. Then:

$$f(\overline{A}) \subseteq \overline{f(A)} = \overline{f(X \setminus f^{-1}(B))} = \overline{Y \setminus B} = Y \setminus B$$

Thus, $f(\overline{A}) \subseteq Y \setminus B$, but $A = X \setminus f^{-1}(B)$, so $A = \overline{A}$, and we are done \square

Definition 18. *x is a limit point for A if for every U open such that $x \in U$, there is $x' \neq x$ such that $x' \in A \cap U$. We denote A' to be the set of limit points of A*

Example 39.

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$$

A has only 1 limit point, namely 0.

Proposition 12.

$$\overline{A} = A \cup A'$$

Corollary 1. *A subset of a topological space is closed if and only if it contains all of its limit points.*

Definition 19. *In an arbitrary topological space, one says that a sequence is convergent to $x \in X$ if for every neighbourhood V of x , there is an N such that for all $n \geq N$, $x_n \in V$*

Note: IN the Zariski topology, a sequence with no constant subsequence converges to every point in the space. The definition is fine in metric spaces. In every metric space, the limit is unique.

Proposition 13. *In metric spaces, the limit is unique.*

Proof. If x_1, x_2 are both limits of a sequence, consider the open ball with radius $\frac{d(x_1, x_2)}{2}$. All but finitely many terms lie in each of the disjoint balls. \square

Lemma 2 (Sequence Lemma). *Let X be a metric space.*

(a) $x \in \overline{A}$ if and only if there is a sequence in A converging to x .

(b) $x \in A'$ if and only if there is a sequence of points in A that does not eventually become constant that converges to x .

Proof. Recall that $\overline{A} = A \cup A'$. If $x \in A$, then $x_n = x$ for all n converges to x . Thus, it suffices to prove part (b).

Let $x \in A'$. For $n \in \mathbb{N}$, consider $B(x, \frac{1}{n})$. Since $x \in A'$, there is $x_n \in A \cap B(x, \frac{1}{n})$ such that $x_n \neq x$. Now, let $\epsilon > 0$, and choose $K(\epsilon)$ so that $\frac{1}{K(\epsilon)} < \epsilon$. Then, for $n \geq K(\epsilon)$, $x_n \in B(x, \frac{1}{n}) \subseteq B(x, \frac{1}{K(\epsilon)}) \subseteq B(x, \epsilon)$. Conversely, if $x_n \rightarrow x$, $x_n \in A$, x_n not eventually constant. We consider an open set U containing x . Let $B(x, \epsilon) \subseteq U$. There is a $K(\epsilon)$ such that for all $n \geq K(\epsilon)$, $x_n \in B(x, \epsilon)$. From these, we can choose a term that is not x . The definition of a' is thus fulfilled, and so we are done \square

Theorem 4. *Let X, Y be metric spaces. Then, $f : X \rightarrow Y$ is continuous if and only if for every convergent sequence $x_n \in X$, the sequence $f(x_n)$ is convergent in Y*

Proof. For $x_n \rightarrow x$, consider $B(f(x), \epsilon)$. Then $f^{-1}(B(f(x), \epsilon))$ is an open neighbourhood of x . Thus, there is an integer $K(\epsilon) \in \mathbb{N}$ such that for all $n \geq K(\epsilon)$, $x_n \in f^{-1}(B(f(x), \epsilon))$. Thus, for all $n \geq K(\epsilon)$, $f(x_n) \in B(f(x), \epsilon)$, so $f(x_n)$ is convergent in Y .

Conversely, assume that x_n converges to x . Consider $x_1, x, x_2, x, x_3, x, \dots$. This converges to x . Then $f(x_1), f(x), f(x_2), f(x), \dots$ converges by hypothesis. Since it has a constant subsequence equal to $f(x)$, the sequence itself converges to $f(x)$. The conclusion then follows from Proposition 11. \square

Definition 20. X is Hausdorff if for every $x, y \in X$, $x \neq y$ there are open sets U, V such that $x \in U$, $y \in V$, $U \cap V = \emptyset$.

Theorem 5. If X and Y are homeomorphic, and X is Hausdorff, then Y is Hausdorff.

Proof. Consider $h : Y \rightarrow X$, a homeomorphism. Let $x, y \in Y$. Then $f(x) \neq f(y)$. There are disjoint open sets U and V in X containing these 2 points separately. The open sets $f^{-1}(U)$ and $f^{-1}(V)$ contain x and y respectively and are disjoint. \square

Remark 1. Hausdorff is a topological property.

Example 40. Consider \mathbb{C}^n with the standard topology and \mathbb{C}^n with the Zariski Topology. These are NOT homeomorphic. One is Hausdorff and the other is not.

Definition 21. A topological space X is not connected if there exist two open sets U, V such that $U \cup V = X$ and $U \cap V = \emptyset$.

Example 41. $(-\infty, 0) \cup (0, \infty)$ is not connected.

If a space is not “not connected”, then it is connected.

Example 42. \mathbb{Q} is not connected.

Proposition 14. (a) If $A, B \subseteq X$ are disjoint subsets such that $A \cup B = X$ and neither of these sets contains a limit point of the other. Then, they form a separation of X .

(b) If U, V form a separation of X and if $Y \subseteq X$, Y connected, then Y lies entirely inside U or V .

Proof. (a):

Since neither contain limit points of the other, $\bar{A} = A$ and $\bar{B} = B$, so A and B are closed. Then, $A = X \setminus B$ and $B = X \setminus A$ are open, and form a separation.

(b):

Consider $Y \cap U$ and $Y \cap V$. $\overline{Y \cap U} \subseteq U$, and $\overline{Y \cap V} \subseteq V$. Thus, $\overline{Y \cap U} \cap \overline{Y \cap V} = \emptyset$, so Y is entirely in U or V . \square

Proposition 15. 1) The union of the collection of connected sets that share one point is connected.

Proof. Let us assume that X_α , $\alpha \in A$ are connected, $x \in X_\alpha$ for all α . Let us assume there is separation $U \cup V = \bigcup_{\alpha \in A} X_\alpha$. Then, there is a $y \in \bigcup_{\alpha \in A} X_\alpha$ such that x, y are in different sets U, V . However, $y \in X_\alpha$ for some α , and $U \cap X_\alpha \mid V \cap X_\alpha$ are open, disjoint and $(U \cap X_\alpha) \cup (V \cap X_\alpha) = X_\alpha$. x is in one, y is in the other. This then forms a separation of X_α , and we get a contradiction. \square