Category Theory Problem Sets and Solutions

Fall 2022

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1.1 Problem 4 - Orin Gotchey

1.1.1 Measurable Spaces as a Category

Definition 1.2. *σ-algebras Let X be a set. Let Ω be any subset of $\mathcal{P}(X)$ satisfying the following conditions:*

- $X \in \Omega_X$
- *For each $E \in \Omega$, $X \backslash E \in \Omega_X$*
- *For any index $I : \mathbb{N} \to \Omega_X$, $(\cup_{n\in\mathbb{N}} I(n)) \in \Omega_X$*

Ω is called a σ-algebra on X, the pair (X, Ω_X) a measurable space, the elements of Ω_X the measurable subsets of X.

It follows immediately that $\varnothing \in \Omega_X$, and that Ω_X is closed under countable intersection.

Definition 1.3. *Measurable MapsThe maps $f : X \to Y$ between measurable spaces which have the following property:*

$$\forall E \subset X : (f(E) \in \Sigma \implies E \in \Omega)$$

are called measurable maps or measurable functions.

Let `Meas` be the category specified as follows:

- Objects are the measurable spaces (X, Ω)
- Morphisms are the measurable functions.

Then, given any morphism $f : X \to Y$,

$$f \circ \mathtt{Id}_X = f$$

$$\mathtt{Id}_Y \circ f = f$$

Associativity follows from the fact that the composition of functions on the underlying sets is associative. Given two composable morphisms, say $f : X \to Y$ and $g : Y \to Z$, consider the composition $g \circ f : X \to Z$, and let $\gamma \in \Omega_Z$. Then:

$$g^{-1}(\gamma) \in \Omega_Y$$

$$f^{-1}(g^{-1}(\gamma)) = (g \circ f)^{-1}(\gamma) \in \Omega_X$$

. Thus, we have that $\mathtt{Meas}$ is a category.

1.3.1 Enhanced Measurable Spaces

Let (X, Ω_X) be a topological space. Then $\mathcal{P}(X)$ forms a Boolean commutative ring with the operations $\cap$ and $\triangle$ as multiplication and addition, respectively, and of which Ω_X is a subring. Define an *enhanced measurable space* as a triple (X, Ω_X, N_X), where (X, Ω_X) form a measurable space, and N_X is a σ-ideal of Ω_X (recall: a *σ-ideal* is an ideal which is closed under *countable* addition). A *negligible set* in X is some subset of N_X.
The *measurable maps* $f : (X, \Omega_X, N_X) \to (Y, \Omega_Y, N_Y)$ are maps of sets: $f : X_f \to Y$, where $X_f \subset X$, such that f obeys the following conditions (which are verified for the identity maps in the subpoints where $X = Y$):

1. The set $X \backslash X_f$ is negligible
   - $X = X_{\mathtt{Id}_X}$ and $X \backslash X_{\mathtt{Id}_X} = \varnothing \in N_X$, by definition of ideal.
2. For any $m_y \in \Omega_Y$, there exists a set m_x such that $f^{-1}(m_y) \triangle m_x$ is negligible
   - Given m_x, $\mathtt{Id}_X^{-1}(m_x) \triangle m_X = m_x \triangle m_x = \varnothing \in N_X$
3. For any $n_y \in N_Y$, the set $f^{-1}(n_y)$ is negligible.
   - $\mathtt{Id}_X(n_x) = n_x$

We cannot define composition of morphisms strictly as composition of underlying maps, because there is no guarantee, e.g., for two maps between enhanced measurable spaces $f : X \to Y$, $g : Y \to Z$, that $Imf \subset Y_g$. Thus, we restrict the domain of the composition to:

$$X_{g \circ f} := f^{-1}(Y_g)$$

. However, it is clear by inspection that composition of morphisms retains associativity. Then,

$$X \backslash X_{g \circ f} = X \backslash f^{-1}(Y_g) = (f^{-1}(Y \backslash (Y_g)))$$

The negligibility of the above quantity then follows from the definition of f.
Furthermore, given $m_z \in \Omega_Z$, we have that $(g \circ f)^{-1}(m_z) = f^{-1}(g^{-1}(m_z))$. Since g is measurable (*why?*) and since f is presumed to satisfy (2), $g \circ f$ satisfies (2).
(3) is clearly transitive.
Thus, enhanced measurable spaces and measurable maps form a category.

1.3.2 Equality Almost Everywhere

Two parallel morphisms $f, g : (X, \Omega_X, N_X) \to (Y, \Omega_Y, N_Y)$ are "equal almost everywhere" if the set $\{x \in X_f \cap X_G : f(x) \neq g(x)\}$ is negligible. Let "f and g are equal almost everywhere" be denoted $f \sim g$. Claim: $\sim$ defines an equivalence relation.

- Reflexivity: A function differs from itself on the empty set ($\varnothing$), which is negligible (see above)
- Symmetry: Note that the symbols f and g in the definition of equality almost everywhere are symmetric
- Transitivity: If $f \sim g$ and $g \sim h$ for parallel morphisms f, g, and h, then

  $$\{x \in X_f \cap X_h : f(x) \neq h(x)\} \subset (\{x \in X_f \cap X_g : f(x) \neq g(x)\} \cup \{x \in X_g \cap X_h : g(x) \neq h(x)\})$$

  , and N is closed under countable unions and taking subsets, so the left hand side of the above is negligible.

Furthermore, this equivalence relation is compatible with composition. Assume that there are morphisms $f, f' : X \to Y$ and $g, g' : Y \to Z$. such that $f \sim f'$ and $g \sim g'$. We're interested in the set

$$\begin{aligned}\{x \in X_{g \circ f} \cap X_{g' \circ f'} : (g \circ f)(x) \neq (g' \circ f')(x)\} \subset \{x \in X_f \cap X_{f'} : f(x) \neq f'(x)\} \\ \cup f^{-1}(\{y \in Y_g \cap Y_g' : g(y) \neq g'(y)\})\end{aligned}$$

This set is the union of two negligible sets.

1.3.3 Hom-sets mod an Equivalence Relation

Suppose that for every pair of objects X, Y in a category C, we are given (e.g. by the above) an equivalence relation $R_{X,Y}$ on $C(X, Y)$ that is compatible with composition (i.e. if $f \sim_R f'$ and $g \sim_R g'$ then $(g \circ f) \sim_R (g' \circ f')$. We identify all morphisms in C between any two objects X and Y which relate through $R_{X,Y}$. Composition of equivalence classes of $\sim$ does not depend on choice of representative: this is exactly compatibility with $\circ$
Verifying that the proper morphisms are unital and associative are gifted as simple exercises to the reader ;)

1.4 Problem 6 - Alan Bohnert

1.4.1 Question

Fix a category $\mathbf{C}$. A *bimorphism* in $\mathbf{C}$ is a morphism f that is simultaneously a monomorphsim and an epimorphism. Is any isomorphism a bimorphism? Give and example of a category $\mathbf{C}$ and a bimorphism f in $\mathbf{C}$ that is not an isomorphism.

1.4.2 Solution

In any category $\mathbf{C}$ every isomorphism is a bimorphism.

Proof. Let $f : X \to Y$ be an isomorphism in $\mathbf{C}$. Then there exists a morphism $g : Y \to X$ in $\mathbf{C}$ such that

$$gf = \mathrm{Id}_X \text{ and } fg = \mathrm{Id}_Y.$$

To show f is a monomorphism let $h, k : W \rightrightarrows X$ and $fh = fk$. It follows that $gfh = gfk$ for the g given above. Therefore $\mathrm{Id}_X h = \mathrm{Id}_X k$ and so $h = k$ tells us f is a monomorphism.

To show f is an epimorphism let $m, n : Y \rightrightarrows Z$ and $mf = nf$. Composing with the g we know $mfg = nfg$. Consequently $m\mathrm{Id}_Y = n\mathrm{Id}_Y$ and $m = n$ tells us f is an epimorphism. Therefore f is a bimorphism. $\square$

Let $\mathbf{C}$ be the category **Ring** and let $f : \mathbb{Z} \hookrightarrow \mathbb{Q}$ the inclusion map. We claim f is a bimorphism but not an isomorphism.

Proof. To show f is a monomorphism let $h, k : W \rightrightarrows \mathbb{Z}$ and $f \circ h(w) = f \circ k(w)$ $\forall w \in W$. Since f is injective

$$h(w) = f \circ h(w) = f \circ k(w) = k(w).$$

Therefore $h(w) = k(w)$ and f is a monomorphism.

To show f is an epimorphism let $m, n : \mathbb{Q} \rightrightarrows S$ such that $m \circ f(x) = n \circ f(x)$ $\forall x \in \mathbb{Q}$. Since f is injective, we know $m(z) = n(z)$ $\forall z \in \mathbb{Z}$. Seeking a contraction, suppose there exists $\frac{a}{b} \in \mathbb{Q}$ such that $m(\frac{a}{b}) \neq n(\frac{a}{b})$. Given m and n are ring homomorphisms we know

$$m(a)m(b^{-1}) = m(\tfrac{a}{b}) \neq n(\tfrac{a}{b}) = n(a)n(b^{-1}).$$

Given b is an invertible integer and $m(b) = n(b)$ we can multiply on the right and retain the inequality. Thus,

$$m(a)m(b^{-1})m(b) \neq n(a)n(b^{-1})n(b)$$

and as ring homomorphisms we have

$$m(a) = m(a)m(b^{-1}b) \neq n(a)n(b^{-1}b) = n(a).$$

Therefore f is a epimorphism.

To show f is not an isomorphism we note $\frac{1}{3} \in \mathbb{Q}$ has no preimage in $\mathbb{Z}$. □

2 Problem Set 2

Assignments

- Problem 1 - Orin Gotchey
- Problem 2 - James
- Problem 3 - Bradley
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- Problem 6 - Emilio

2.1 Problem 1 - Orin Gotchey

Lemma 2.2. *Existence and Uniqueness of Borel σ-Algebras. Let X be a topological space. Then there exists a unique σ-algebra, Ω on X which contains all open subsets of X and which is the smallest among such σ-algebras with respect to inclusion.*

Proof. Let Σ be the collection of all σ-algebras on X which contain all open subsets of X. Σ contains $\mathcal{P}(X)$, and thus is nonempty. Let

$$\Omega := \cap_{x \in \Sigma} x$$

Clearly, $X \in \Omega$. Given an index $I : \mathbb{N} \to \Omega$, such that for every natural n, $I(n) \in \Omega$, we have that $I(n) \in x$, $\forall x \in \Sigma$, whence it follows that $\cap_{n \in \mathbb{N}} I(n) \in x$, $\forall x \in \Sigma$. Therefore, $\cap_{n \in \mathbb{N}} I(n) \in \Omega$. By a similar argument, for any $E \in \Omega$, $X \backslash E \in \Omega$. Thus, Ω contains all open subsets of X, and is indeed inferior to any other σ-algebra with this property. □

Definition 2.3. *A complex *-algebra A is a complex algebra, equipped with a complex-antilinear operation $*: A \to A$ obeying the following:*

$$(ab)^* = b^*a^*$$

$$1^* = 1$$

$$(a^*)^* = a$$

Definition 2.4. *A complex-valued morphism $f : X \to \mathbb{C}$ (on some topological space X) is called "bounded" if it factors through some bounded subset of $\mathbb{C}$. That is, there exists some subset $C \in \mathbb{C}$ which is contained in some open ball, and some map $\bar{f}$ which makes the following diagram commute:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "\bar{f}"'] & \mathbb{C} \\ C \arrow[ur, "\iota"'] & \end{tikzcd}$$

Lemma 2.5. *Given an enhance measurable set (X, Ω_X, N_X), the set of all bounded morphisms $\{f : (X, \Omega_X, N_X) \to (\mathbb{C}, \Omega_C, \{\varnothing\})\}$ is a complex *-algebra.*

Proof. The zero morphism 0_X acts as the additive identity. Addition, multiplication, and involution are pointwise. Everything else follows by inspection. $\square$

Proposition 2.6. *Together with complex algebra homomorphisms: $f : A \to B$ satisfying $f(a^*) = f(a)^*$, and objects: commutative complex *-algebras, $\mathsf{CAlg}^*_{\mathbb{C}}$ is a category.*

Proof. Let $\forall a, b, c \in \mathsf{Obj}(\mathsf{CAlg}^*_{\mathbb{C}}),\ f \in \mathsf{CAlg}^*_{\mathbb{C}}(a, b),\ g \in \mathsf{CAlg}^*_{\mathbb{C}}(b, c)$ then:

- $\exists \mathtt{id}_a : a \to a$ given by $\mathtt{id}_a(x) = x$ satisfies $\mathtt{id}_a(x^*) = x^* = \mathtt{id}_a(x)^*$, and which is clearly a $\mathbb{C}$-algebra homomorphism
- $g \circ f$ satisfies $(g \circ f)(x)^* = g(f(x))^* = g(f(x)^*) = g(f(x^*)) = g \circ f(x^*)$, and is clearly a $\mathbb{C}$-algebra homomorphism.
- The composition of underlying sets is associative.

$\square$

Let $\mathsf{L}^\infty : \mathsf{PreEMS}^{op} \to \mathsf{CAlg}^*_{\mathbb{C}}$ send an enhanced measurable space to the complex *-algebra of bounded morphisms: $(X, \Omega_X, N_X) \mapsto (\mathrm{L}^\infty(X) : \{\phi : (X, \Omega_X, N_X) \to (\mathbb{C}, \Omega_{\mathbb{C}}, \{\varnothing\}) |\, \phi \,\text{bounded}\})$, and which sends an enhanced measurable morphism $f : (X, \Omega_X, N_X) \to (Y, \Omega_Y, N_Y)$ to

$$\mathsf{L}^\infty(f) : (\mathrm{L}^\infty(Y) : \{\phi : (Y, \Omega_Y, N_Y) \to (\mathbb{C}, \Omega_{\mathbb{C}}, N_{\mathbb{C}})\}) \to (\mathrm{L}^\infty(X) : \{\psi : (X, \Omega_X, N_X) \to (\mathbb{C}, \Omega_{\mathbb{C}}, N_{\mathbb{C}})\})$$

given by:

$$(\mathrm{L}^\infty(f))(\phi) = (\phi \circ f)$$

Proposition 2.7. L^∞ *is a contravariant functor*

Proof. We need to show the following:

1. $\mathrm{L}^\infty(f)$ defines a morphism in $\mathsf{CAlg}^*_{\mathbb{C}}$ i.e. a complex algebra homomorphism which respects involution.
2. L^∞ respects identity
3. L^∞ respects composition

For (1), given an $f : X \to Y$, $\phi,\ \psi \in \mathrm{L}^\infty(Y)$, and $c \in \mathbb{C}$

$$\begin{aligned}
&\mathrm{L}^\infty(f) : \mathrm{L}^\infty(Y) \to \mathrm{L}^\infty(X)\\
&\mathrm{L}^\infty(f)(0_Y) = 0_X\\
&\mathrm{L}^\infty(f)(\phi+\psi) = (\phi+\psi)\circ(f) = (\phi\circ f) + (\psi\circ f) = \mathrm{L}^\infty(f)(\phi) + \mathrm{L}^\infty(f)(\psi)\\
&\mathrm{L}^\infty(f)(\phi\cdot\psi) = (\phi\cdot\psi)\circ f = (\phi\circ f)\cdot(\psi\circ f) = \mathrm{L}^\infty(f)(\phi)\cdot\mathrm{L}^\infty(f)(\psi)\\
&c\cdot\mathrm{L}^\infty(f)(\phi) = c\cdot(\phi\circ f) = (c\cdot\phi)\circ f = \mathrm{L}^\infty(f)(c\cdot\phi)\\
&\mathrm{L}^\infty(f)(\phi^*) = (\phi^*)\circ f = (\phi\circ f)^* = \mathrm{L}^\infty(f)(\phi)^*
\end{aligned}\tag{1}$$

For (2),

$$\mathrm{L}^\infty(\mathtt{id}_X)(\phi) = (\phi\circ\mathtt{id}_X) = \phi \implies \mathrm{L}^\infty(\mathtt{id}_X) = \mathtt{id}_{\mathrm{L}^\infty(X)}\tag{2}$$

For (3), we give two morphisms $f : X \to Y$, $g : Y \to Z$ in PreEMS. Then for any $\phi \in \mathrm{L}^\infty(Z)$

$$\mathrm{L}^\infty(g\circ f)(\phi) = \phi\circ(g\circ f) = (\phi\circ g)\circ f = \mathrm{L}^\infty(f)(\phi\circ g) = \mathrm{L}^\infty(f)(\mathrm{L}^\infty(g)(\phi)) = (\mathrm{L}^\infty(f)\circ\mathrm{L}^\infty(g))(\phi)\tag{3}$$

□

Lemma 2.8. *Let C be a category with an equivalence relation R on its set of morphisms, and let F be some functor from C/R to another category D. Then precomposing with the functor $\mathsf{C} \xrightarrow{\Pi} \mathsf{C}/R$ gives a bijection between functors $\mathsf{C}/R \xrightarrow{F} \mathsf{D}$ and functors $\mathsf{C} \xrightarrow{G} \mathsf{D}$ with the property that $\alpha \sim_R \beta \Rightarrow G(\alpha) = G(\beta)$*

Proof. If $\alpha \sim_R \beta$, then $\Pi(\alpha) = \Pi(\beta)$, so $(F\circ\Pi)(\alpha) = (F\circ\Pi)(\beta)$. For the other direction, let $G : \mathsf{C} \to \mathsf{D}$ and R be given as above. Then, let $F : \mathsf{C}/R \to D$ be identical to G on objects, and let the image of an equivalence class of morhpisms in C/R be the image of any of its representatives under G. This is well-defined, because any two morphisms in the same equivalence class of R will have the same image under G. □

Definition 2.9. *Let R be an equivalence relation defined on the Hom-sets of PreEMS. Define the category $\mathsf{StrictEMS}$, whose objects are the same as PreEMS, and whose morphisms are equivalence classes of morphisms under R sharing domain and codomain.*

It follows from the foregoing observations that $\mathsf{StrictEMS}$ is, in fact, a category. Extend the L^∞ to a functor from $\mathsf{StrictEMS}$ to $\mathsf{CAlg}^*_\mathbb{C}$ as follows:

- Send an EMS (X, Ω_X, N_X) to the set of *equivalence classes* of bounded morphisms from X to $(\mathbb{C}, \Omega_\mathbb{C}, \{\varnothing\})$ (here $\Omega_\mathbb{C}$ denotes the Borel sets of $\mathbb{C}$).
- Send an equivalence class of morphisms

3 Other Problems

- For which pairs of fields (of the same characteristic) does a categorical product exist?
- Prove that the category of connected topological spaces does not have a product.