

# Category Theory Problem Sets and Solutions

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# 1 Problem Set 1

## Assignments

- Problem 1 - Unclaimed
- Problem 2 - Emilio Verdooren
- Problem 3 - Emilio Verdooren
- Problem 4 - Orin Gotchey
- Problem 5 - Unclaimed
- Problem 6 - Alan Bohnert
- Problem 7 - James
- Problem 8 - James
- Problem 9 - Unclaimed
- Problem 10 - Unclaimed
- Problem 11 - Unclaimed

## 1.1 Problem 4 - Orin Gotchey

### 1.1.1 Measurable Spaces as a Category

**Definition 1.2.**  *$\sigma$ -algebras* Let  $X$  be a set. Let  $\Omega$  be any subset of  $\mathcal{P}(X)$  satisfying the following conditions:

- $X \in \Omega_X$
- For each  $E \in \Omega$ ,  $X \setminus E \in \Omega_X$
- For any index  $I : \mathbb{N} \rightarrow \Omega_X$ ,  $(\cup_{n \in \mathbb{N}} I(n)) \in \Omega_X$

$\Omega$  is called a  $\sigma$ -algebra on  $X$ , the pair  $(X, \Omega_X)$  a measurable space, the elements of  $\Omega_X$  the measurable subsets of  $X$ .

It follows immediately that  $\emptyset \in \Omega_X$ , and that  $\Omega_X$  is closed under countable intersection.

**Definition 1.3.** *Measurable Maps* The maps  $f : X \rightarrow Y$  between measurable spaces which have the following property:

$$\forall E \subset X : (f(E) \in \Sigma \implies E \in \Omega)$$

are called measurable maps or measurable functions.

Let **Meas** be the category specified as follows:

- Objects are the measurable spaces  $(X, \Omega)$
- Morphisms are the measurable functions.

Then, given any morphism  $f : X \rightarrow Y$ ,

$$f \circ \text{Id}_X = f$$

$$\text{Id}_Y \circ f = f$$

Associativity follows from the fact that the composition of functions on the underlying sets is associative. Given two composable morphisms, say  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ , consider the composition  $g \circ f : X \rightarrow Z$ , and let  $\gamma \in \Omega_Z$ . Then:

$$g^{-1}(\gamma) \in \Omega_Y$$

$$f^{-1}(g^{-1}(\gamma)) = (g \circ f)^{-1}(\gamma) \in \Omega_X$$

. Thus, we have that **Meas** is a category.

### 1.3.1 Enhanced Measurable Spaces

Let  $(X, \Omega_X)$  be a topological space. Then  $\mathcal{P}(X)$  forms a Boolean commutative ring with the operations  $\cap$  and  $\Delta$  as multiplication and addition, respectively, and of which  $\Omega_X$  is a subring. Define an *enhanced measurable space* as a triple  $(X, \Omega_X, N_X)$ , where  $(X, \Omega_X)$  form a measurable space, and  $N_X$  is a  $\sigma$ -ideal of  $\Omega_X$  (recall: a  $\sigma$ -ideal is an ideal which is closed under *countable* addition). A *negligible set* in  $X$  is some subset of  $N_X$ .

The *measurable maps*  $f : (X, \Omega_X, N_X) \rightarrow (Y, \Omega_Y, N_Y)$  are maps of sets:  $f : X_f \rightarrow Y$ , where  $X_f \subset X$ , such that  $f$  obeys the following conditions (which are verified for the identity maps in the subpoints where  $X = Y$ ):

1. The set  $X \setminus X_f$  is negligible
  - $X = X_{\text{Id}_X}$  and  $X \setminus X_{\text{Id}_X} = \emptyset \in N_X$ , by definition of ideal.
2. For any  $m_y \in \Omega_Y$ , there exists a set  $m_x$  such that  $f^{-1}(m_y) \Delta m_x$  is negligible
  - Given  $m_x$ ,  $\text{Id}_X^{-1}(m_x) \Delta m_X = m_x \Delta m_x = \emptyset \in N_X$
3. For any  $n_y \in N_Y$ , the set  $f^{-1}(n_y)$  is negligible.
  - $\text{Id}_X(n_x) = n_x$

We cannot define composition of morphisms strictly as composition of underlying maps, because there is no guarantee, e.g., for two maps between enhanced measurable spaces  $f : X \rightarrow Y$ ,  $g : Y \rightarrow Z$ , that  $\text{Im}f \subset Y_g$ . Thus, we restrict the domain of the composition to:

$$X_{g \circ f} := f^{-1}(Y_g)$$

. However, it is clear by inspection that composition of morphisms retains associativity. Then,

$$X \setminus X_{g \circ f} = X \setminus f^{-1}(Y_g) = (f^{-1}(Y \setminus (Y_g)))$$

The negligibility of the above quantity then follows from the definition of  $f$ .

Furthermore, given  $m_z \in \Omega_Z$ , we have that  $(g \circ f)^{-1}(m_z) = f^{-1}(g^{-1}(m_z))$ . Since  $g$  is measurable (*why?*) and since  $f$  is presumed to satisfy (2),  $g \circ f$  satisfies (2).

(3) is clearly transitive.

Thus, enhanced measurable spaces and measurable maps form a category.

### 1.3.2 Equality Almost Everywhere

Two parallel morphisms  $f, g : (X, \Omega_X, N_X) \rightarrow (Y, \Omega_Y, N_Y)$  are "equal almost everywhere" if the set  $\{x \in X_f \cap X_g : f(x) \neq g(x)\}$  is negligible. Let "f and g are equal almost everywhere" be denoted  $f \sim g$ . Claim:  $\sim$  defines an equivalence relation.

- Reflexivity: A function differs from itself on the empty set  $(\emptyset)$ , which is negligible (see above)
- Symmetry: Note that the symbols  $f$  and  $g$  in the definition of equality almost everywhere are symmetric
- Transitivity: If  $f \sim g$  and  $g \sim h$  for parallel morphisms  $f, g$ , and  $h$ , then

$$\{x \in X_f \cap X_h : f(x) \neq h(x)\} \subset (\{x \in X_f \cap X_g : f(x) \neq g(x)\} \cup \{x \in X_g \cap X_h : g(x) \neq h(x)\})$$

, and  $N$  is closed under countable unions and taking subsets, so the left hand side of the above is negligible.

Furthermore, this equivalence relation is compatible with composition. Assume that there are morphisms  $f, f' : X \rightarrow Y$  and  $g, g' : Y \rightarrow Z$ . such that  $f \sim f'$  and  $g \sim g'$ . We're interested in the set

$$\begin{aligned} \{x \in X_{g \circ f} \cap X_{g' \circ f'} : (g \circ f)(x) \neq (g' \circ f')(x)\} &\subset \{x \in X_f \cap X_{f'} : f(x) \neq f'(x)\} \\ &\cup f^{-1}(\{y \in Y_g \cap Y_{g'} : g(y) \neq g'(y)\}) \end{aligned}$$

This set is the union of two negligible sets.

### 1.3.3 Hom-sets mod an Equivalence Relation

Suppose that for every pair of objects  $X, Y$  in a category  $C$ , we are given (e.g. by the above) an equivalence relation  $R_{X,Y}$  on  $C(X, Y)$  that is compatible with composition (i.e. if  $f \sim_R f'$  and  $g \sim_R g'$  then  $(g \circ f) \sim_R (g' \circ f')$ ). We identify all morphisms in  $C$  between any two objects  $X$  and  $Y$  which relate through  $R_{X,Y}$ . Composition of equivalence classes of  $\sim$  does not depend on choice of representative: this is exactly compatibility with  $\circ$

Verifying that the proper morphisms are unital and associative are gifted as simple exercises to the reader ;)

## 1.4 Problem 6 - Alan Bohnert

### 1.4.1 Question

Fix a category  $\mathbf{C}$ . A *bimorphism* in  $\mathbf{C}$  is a morphism  $f$  that is simultaneously a monomorphism and an epimorphism. Is any isomorphism a bimorphism? Give an example of a category  $\mathbf{C}$  and a bimorphism  $f$  in  $\mathbf{C}$  that is not an isomorphism.

### 1.4.2 Solution

In any category  $\mathbf{C}$  every isomorphism is a bimorphism.

*Proof.* Let  $f : X \rightarrow Y$  be an isomorphism in  $\mathbf{C}$ . Then there exists a morphism  $g : Y \rightarrow X$  in  $\mathbf{C}$  such that

$$gf = \text{Id}_X \text{ and } fg = \text{Id}_Y.$$

To show  $f$  is a monomorphism let  $h, k : W \rightrightarrows X$  and  $fh = fk$ . It follows that  $gfh = gfk$  for the  $g$  given above. Therefore  $\text{Id}_X h = \text{Id}_X k$  and so  $h = k$  tells us  $f$  is a monomorphism.

To show  $f$  is an epimorphism let  $m, n : Y \rightrightarrows Z$  and  $mf = nf$ . Composing with the  $g$  we know  $mfg = nfg$ . Consequently  $m\text{Id}_Y = n\text{Id}_Y$  and  $m = n$  tells us  $f$  is an epimorphism. Therefore  $f$  is a bimorphism.  $\square$

Let  $\mathbf{C}$  be the category **Ring** and let  $f : \mathbb{Z} \hookrightarrow \mathbb{Q}$  the inclusion map. We claim  $f$  is a bimorphism but not an isomorphism.

*Proof.* To show  $f$  is a monomorphism let  $h, k : W \rightrightarrows \mathbb{Z}$  and  $f \circ h(w) = f \circ k(w) \forall w \in W$ . Since  $f$  is injective

$$h(w) = f \circ h(w) = f \circ k(w) = k(w).$$

Therefore  $h(w) = k(w)$  and  $f$  is a monomorphism.

To show  $f$  is an epimorphism let  $m, n : \mathbb{Q} \rightrightarrows S$  such that  $m \circ f(x) = n \circ f(x) \forall x \in \mathbb{Q}$ . Since  $f$  is injective, we know  $m(z) = n(z) \forall z \in \mathbb{Z}$ . Seeking a contradiction, suppose there exists  $\frac{a}{b} \in \mathbb{Q}$  such that  $m(\frac{a}{b}) \neq n(\frac{a}{b})$ . Given  $m$  and  $n$  are ring homomorphisms we know

$$m(a)m(b^{-1}) = m(\frac{a}{b}) \neq n(\frac{a}{b}) = n(a)n(b^{-1}).$$

Given  $b$  is an invertible integer and  $m(b) = n(b)$  we can multiply on the right and retain the inequality. Thus,

$$m(a)m(b^{-1})m(b) \neq n(a)n(b^{-1})n(b)$$

and as ring homomorphisms we have

$$m(a) = m(a)m(b^{-1}b) \neq n(a)n(b^{-1}b) = n(a).$$

Therefore  $f$  is an epimorphism.

To show  $f$  is not an isomorphism we note  $\frac{1}{3} \in \mathbb{Q}$  has no preimage in  $\mathbb{Z}$ . □

## 2 Problem Set 2

### Assignments

- Problem 1 - Orin Gotchey
- Problem 2 - James
- Problem 3 - Bradley
- Problem 4 - Alan
- Problem 5 - Mason
- Problem 6 - Emilio

### 2.1 Problem 1 - Orin Gotchey

**Lemma 2.2.** *Existence and Uniqueness of Borel  $\sigma$ -Algebras. Let  $X$  be a topological space. Then there exists a unique  $\sigma$ -algebra,  $\Omega$  on  $X$  which contains all open subsets of  $X$  and which is the smallest among such  $\sigma$ -algebras with respect to inclusion.*

*Proof.* Let  $\Sigma$  be the collection of all  $\sigma$ -algebras on  $X$  which contain all open subsets of  $X$ .  $\Sigma$  contains  $\mathcal{P}(X)$ , and thus is nonempty. Let

$$\Omega := \bigcap_{x \in \Sigma} x$$

Clearly,  $X \in \Omega$ . Given an index  $I : \mathbb{N} \rightarrow \Omega$ , such that for every natural  $n$ ,  $I(n) \in \Omega$ , we have that  $I(n) \in x$ ,  $\forall x \in \Sigma$ , whence it follows that  $\bigcap_{n \in \mathbb{N}} I(n) \in x$ ,  $\forall x \in \Sigma$ . Therefore,  $\bigcap_{n \in \mathbb{N}} I(n) \in \Omega$ . By a similar argument, for any  $E \in \Omega$ ,  $X \setminus E \in \Omega$ . Thus,  $\Omega$  contains all open subsets of  $X$ , and is indeed inferior to any other  $\sigma$ -algebra with this property. □

**Definition 2.3.** A complex  $*$ -algebra  $A$  is a complex algebra, equipped with a complex-antilinear operation  $*$  :  $A \rightarrow A$  obeying the following:

$$\begin{aligned}(ab)^* &= b^* a^* \\ 1^* &= 1 \\ (a^*)^* &= a\end{aligned}$$

**Definition 2.4.** A complex-valued morphism  $f : X \rightarrow \mathbb{C}$  (on some topological space  $X$ ) is called "bounded" if it factors through some bounded subset of  $\mathbb{C}$ . That is, there exists some subset  $C \subseteq \mathbb{C}$  which is contained in some open ball, and some map  $\bar{f}$  which makes the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{f} & \mathbb{C} \\ \bar{f} \downarrow & \nearrow \iota & \\ C & & \end{array}$$

**Lemma 2.5.** Given an enhanced measurable set  $(X, \Omega_X, N_X)$ , the set of all bounded morphisms  $\{f : (X, \Omega_X, N_X) \rightarrow (\mathbb{C}, \Omega_{\mathbb{C}}, \{\emptyset\})\}$  is a complex  $*$ -algebra.

*Proof.* The zero morphism  $0_X$  acts as the additive identity. Addition, multiplication, and involution are pointwise. Everything else follows by inspection.  $\square$

**Proposition 2.6.** Together with complex algebra homomorphisms:  $f : A \rightarrow B$  satisfying  $f(a^*) = f(a)^*$ , and objects: commutative complex  $*$ -algebras,  $\mathbf{CAlg}_{\mathbb{C}}^*$  is a category.

*Proof.* Let  $\forall a, b, c \in \mathbf{Obj}(\mathbf{CAlg}_{\mathbb{C}}^*)$ ,  $f \in \mathbf{CAlg}_{\mathbb{C}}^*(a, b)$ ,  $g \in \mathbf{CAlg}_{\mathbb{C}}^*(b, c)$  then:

- $\exists \text{id}_a : a \rightarrow a$  given by  $\text{id}_a(x) = x$  satisfies  $\text{id}_a(x^*) = x^* = \text{id}_a(x)^*$ , and which is clearly a  $\mathbb{C}$ -algebra homomorphism
- $g \circ f$  satisfies  $(g \circ f)(x)^* = g(f(x))^* = g(f(x)^*) = g(f(x^*)) = g \circ f(x^*)$ , and is clearly a  $\mathbb{C}$ -algebra homomorphism.
- The composition of underlying sets is associative.

$\square$

Let  $L^\infty : \mathbf{PreEMS}^{op} \rightarrow \mathbf{CAlg}_{\mathbb{C}}^*$  send an enhanced measurable space to the complex  $*$ -algebra of bounded morphisms:  $(X, \Omega_X, N_X) \mapsto (L^\infty(X) : \{\phi : (X, \Omega_X, N_X) \rightarrow (\mathbb{C}, \Omega_{\mathbb{C}}, \{\emptyset\}) \mid \phi \text{ bounded}\})$ , and which sends an enhanced measurable morphism  $f : (X, \Omega_X, N_X) \rightarrow (Y, \Omega_Y, N_Y)$  to

$$L^\infty(f) : (L^\infty(Y) : \{\phi : (Y, \Omega_Y, N_Y) \rightarrow (\mathbb{C}, \Omega_{\mathbb{C}}, N_{\mathbb{C}})\}) \rightarrow (L^\infty(X) : \{\psi : (X, \Omega_X, N_X) \rightarrow (\mathbb{C}, \Omega_{\mathbb{C}}, N_{\mathbb{C}})\})$$

given by:

$$(L^\infty(f))(\phi) = (\phi \circ f)$$

**Proposition 2.7.**  $L^\infty$  is a contravariant functor

*Proof.* We need to show the following:

1.  $L^\infty(f)$  defines a morphism in  $\mathbf{CAlg}_{\mathbb{C}}^*$  i.e. a complex algebra homomorphism which respects involution.
2.  $L^\infty$  respects identity
3.  $L^\infty$  respects composition

For (1), given an  $f : X \rightarrow Y$ ,  $\phi, \psi \in L^\infty(Y)$ , and  $c \in \mathbb{C}$

$$\begin{aligned}
L^\infty(f) : L^\infty(Y) &\rightarrow L^\infty(X) \\
L^\infty(f)(0_Y) &= 0_X \\
L^\infty(f)(\phi + \psi) &= (\phi + \psi) \circ f = (\phi \circ f) + (\psi \circ f) = L^\infty(f)(\phi) + L^\infty(f)(\psi) \\
L^\infty(f)(\phi \cdot \psi) &= (\phi \cdot \psi) \circ f = (\phi \circ f) \cdot (\psi \circ f) = L^\infty(f)(\phi) \cdot L^\infty(f)(\psi) \\
c \cdot L^\infty(f)(\phi) &= c \cdot (\phi \circ f) = (c \cdot \phi) \circ f = L^\infty(f)(c \cdot \phi) \\
L^\infty(f)(\phi^*) &= (\phi^*) \circ f = (\phi \circ f)^* = L^\infty(f)(\phi)^*
\end{aligned} \tag{1}$$

For (2),

$$L^\infty(\text{id}_X)(\phi) = (\phi \circ \text{id}_X) = \phi \implies L^\infty(\text{id}_X) = \text{id}_{L^\infty(X)} \tag{2}$$

For (3), we give two morphisms  $f : X \rightarrow Y, g : Y \rightarrow Z$  in **PreEMS**. Then for any  $\phi \in L^\infty(Z)$

$$L^\infty(g \circ f)(\phi) = \phi \circ (g \circ f) = (\phi \circ g) \circ f = L^\infty(f)(\phi \circ g) = L^\infty(f)(L^\infty(g)(\phi)) = (L^\infty(f) \circ L^\infty(g))(\phi) \tag{3}$$

□

**Lemma 2.8.** *Let  $\mathcal{C}$  be a category with an equivalence relation  $R$  on its set of morphisms, and let  $F$  be some functor from  $\mathcal{C}/R$  to another category  $\mathcal{D}$ . Then precomposing with the functor  $\mathcal{C} \xrightarrow{\Pi} \mathcal{C}/R$  gives a bijection between functors  $\mathcal{C}/R \xrightarrow{F} \mathcal{D}$  and functors  $\mathcal{C} \xrightarrow{G} \mathcal{D}$  with the property that  $\alpha \sim_R \beta \implies G(\alpha) = G(\beta)$*

*Proof.* If  $\alpha \sim_R \beta$ , then  $\Pi(\alpha) = \Pi(\beta)$ , so  $(F \circ \Pi)(\alpha) = (F \circ \Pi)(\beta)$ . For the other direction, let  $G : \mathcal{C} \rightarrow \mathcal{D}$  and  $R$  be given as above. Then, let  $F : \mathcal{C}/R \rightarrow \mathcal{D}$  be identical to  $G$  on objects, and let the image of an equivalence class of morphisms in  $\mathcal{C}/R$  be the image of any of its representatives under  $G$ . This is well-defined, because any two morphisms in the same equivalence class of  $R$  will have the same image under  $G$ . □

**Definition 2.9.** *Let  $R$  be an equivalence relation defined on the **Hom**-sets of **PreEMS**. Define the category **StrictEMS**, whose objects are the same as **PreEMS**, and whose morphisms are equivalence classes of morphisms under  $R$  sharing domain and codomain.*

It follows from the foregoing observations that **StrictEMS** is, in fact, a category. Extend the  $L^\infty$  to a functor from **StrictEMS** to  $\text{CAlg}_{\mathbb{C}}^*$  as follows:

- Send an enhanced measurable space  $(X, \Omega_X, N_X)$  to the complex \*-algebra of equivalence classes of bounded morphisms from  $X$  to  $\mathbb{C}$
- Send a morphism in **StrictEMS** between two EMS  $X$  and  $Y$  to the morphism from  $L^\infty(X)$  to  $L^\infty(Y)$  which precomposes just like the above functor. The fact that this is well-defined, and indeed is a morphism in the target category follows from the above lemma.

Now, we come to a (perhaps surprising) result that equality almost everywhere is "too strong" for our functor  $L^\infty$ .

**Proposition 2.10.**  *$L^\infty$  is not faithful*

*Proof.* We begin by constructing an enhanced measurable space  $X$ , and an endomorphism  $X \xrightarrow{f} X$ , such that  $f(x) \neq x$  for almost every  $x \in X$ , but for any measurable subset  $E \subset X$ ,  $f^*(E) \triangle E$  is negligible. □

## 2.11 Problem 4 - Alan

### 2.11.1 Construct a functor $\mathbf{Ban}^{op}$ to $\mathbf{Ball}$

Objects in  $\mathbf{Ban}^{op}$  are real (or complex) vector spaces with norms, and morphisms are  $\mathbb{R}$  (or  $\mathbb{C}$ ) linear maps:

$$(X, \|\cdot\|_X) \xrightarrow{f} (Y, \|\cdot\|_Y)$$

such that for all  $x \in X$ ,  $\|f(x)\|_Y \leq \|x\|_X$

Objects in  $\mathbf{Ball}$  are pairs  $(V, B)$  where  $V$  is a Hausdorff, locally convex topological real (complex) vector space, and  $B$  a compact, convex, Hausdorff topological vector subspace of  $V$ , which is balanced. Here, "balanced" means:

$$0 \in B \wedge \forall x \in B \forall t \in \mathbb{R} (|t| \leq 1 \implies tx \in B)$$

For a given object  $(X, \|\cdot\|_X)$  in  $\mathbf{Ban}^{op}$ ,  $X^*$  denotes the space of continuous linear functionals on  $X$  with the weak-\* topology and  $X_{\leq 1}^*$  denotes the subspace of  $X^*$  consisting of functionals of norm at most 1.  $(X^*, X_{\leq 1}^*)$  is an object of  $\mathbf{Ball}$ . Let  $F : \mathbf{Ban}^{op} \rightarrow \mathbf{Ball}$  be the functor which sends  $(X, \|\cdot\|_X)$  to  $(X^*, X_{\leq 1}^*)$  and sends a morphism  $g$  from  $X \rightarrow Y$  to  $X^* \rightarrow Y^*$  to  $F(g) = \circ g$ . The functor  $F$  is encoded in the following commutative diagram:

$$\begin{array}{ccc}
 (X, \|\cdot\|_X) & & (X^*, X_{\leq 1}^*) \\
 \downarrow g & \xrightarrow{F(g) = \circ g} & \uparrow F(g)(y^*) = y^* \circ g \\
 (Y, \|\cdot\|_Y) & & (Y^*, Y_{\leq 1}^*) \\
 \downarrow h & \xrightarrow{F(h) = \circ h} & \uparrow F(h)(x^*) = x^* \circ h \\
 (Z, \|\cdot\|_Z) & & (Z^*, Z_{\leq 1}^*)
 \end{array}$$

From this diagram, we can verify

- That  $F(g)$  is actually a morphism in  $\mathbf{Ball}$
- $F$  is functorial
- $F$  is unital

## 3 Problem Set 3

1. For which pairs of fields (of the same characteristic) does a categorical product exist?
2. Prove that the category of connected topological spaces does not have coproducts.
3. Prove: The category of Banach spaces with *continuous maps* has no infinite coproducts.
4. Prove:
  - The category  $\mathbf{TOSet}$  of totally ordered sets and order-preserving maps does not have coproducts.
  - What about the category  $\mathbf{WOSet}$  of well-ordered sets?
5. Prove: the category  $\mathbf{TopGrp}$  of topological groups and continuous homomorphisms has coproducts.
6. Prove: the category of Lie groups (finite dimensional) does not have coproducts.



7. Investigate products and coproducts in the category  $\mathbf{PG}$  of Hilbert spaces and contractive maps.
8. Express limits in analysis (i.e., in a given metric space) as categorical limits
9. Exhibit colimits of towers in  $\mathbf{Field}$
10. Prove or disprove: Given a tower in  $\mathbf{Man}$  whose morphisms are open embeddings, the colimit exists. Give an example of a tower in  $\mathbf{Man}$  without a colimit.
11. Compute all natural transformations from  $\mathbf{Id}$  to  $**(\text{double-dual})$  as endofunctors on  $\mathbf{Vect}_{\mathbb{T}}$

## Assignments

- Problem 1 - Orin
- Problem 2 - Orin
- Problem 3 - Unassigned
- Problem 4 - JJ
- Problem 5 - Orin
- Problem 6 - Unassigned
- Problem 7 - Unassigned
- Problem 8 - Unassigned
- Problem 9 - Unassigned

### 3.1 Problem 1 - Orin

**Lemma 3.2.** *Any non-trivial field homomorphism is injective and preserves prime subfield up to isomorphism. Any homomorphism between fields of different characteristic must be trivial.*

*Proof.* Let  $f : K \rightarrow S$  be a field homomorphism which is not injective. That is, there exists a nonzero  $s \in S$  such that  $f(s) = 0$ . But then for any  $b \in S$  we have:

$$f(b) = f(b)f(1) = f(b)f(s \cdot s^{-1}) = f(b)f(s)f(s^{-1}) = 0$$

, so  $f$  is trivial. For the second statement, note that a nontrivial field homomorphism must send  $1_{F_1}$  to  $1_{F_2}$ , and so must be an isomorphism of prime subfields. Since field characteristic is an invariant on prime subfields, no nontrivial field homomorphisms exist between fields of different characteristic.  $\square$

**Proposition 3.3.** *If  $K$  and  $L$  are fields such that their pairwise categorical product exists, then they have the same characteristic and both have trivial automorphism groups.*

*Proof.* The first part of the proof follows from the foregoing lemma. For the second part, assume towards a contradiction that  $K \times L$  exists and that one of the fields (denoted, WLOG,  $K$ ) has a nontrivial automorphism. Let  $P$  denote the prime subfield of  $K$  and  $L$ . Then  $P \xrightarrow{f_1} K$  and  $P \xrightarrow{f_2} L$  give rise to another homomorphism  $P \xrightarrow{f} K \times L$ . Consider the homomorphism  $f' : P \xrightarrow{f} K \times L$  given by  $(\gamma \circ f_1) \times (f_2)$ , where  $\gamma$  is any non-trivial field automorphism on  $K$ . By uniqueness,  $f' \neq f$ , but they must both yield  $f_2$  when post-composed with  $K \times L \xrightarrow{\pi_L} L$ . Thus, for some  $x \in P$ ,  $\pi_L \circ (f - f')(x) = 0$  and  $\pi_K \circ (f - f')(x) \neq 0$ . Then  $(f - f')(x) \neq 0$ , so there exists some  $\frac{1}{(f-f')(x)} \in K \times L$ , but  $\pi_L(\frac{1}{(f-f')(x)}) = \frac{1}{0}$ , contradiction.  $\square$

### 3.4 Problem 2 - Orin

**Proposition 3.5.** *The category of connected topological spaces with continuous maps has no coproduct*

*Proof.* The coproduct, if it existed, would have to be isomorphic to the coproduct in the category of regular topological spaces, which are disjoint unions of sets. But no disjoint union of sets is connected. Take a family of topological spaces  $X_\alpha$  over some index set  $A$ . Then consider the coproduct:  $X := \coprod (X_\alpha)_{\alpha \in A}$  together with the inclusion maps:

$$\iota_\alpha : X_\alpha \rightarrow X$$

. For any  $\alpha \in A$ , it becomes clear that:

$$(\cup_{\beta \neq \alpha} (\iota_\beta(X_\beta))) \cup \iota_\alpha(X_\alpha) = X$$

and

$$(\cup_{\beta \neq \alpha} (\iota_\beta(X_\beta))) \cap \iota_\alpha(X_\alpha) = \emptyset$$

. Furthermore, a set in  $X$  is open iff it's preimage under every  $\iota_\alpha$  is open. If  $\omega$  is any given element in  $A$ , then

$$\iota_\omega^*(\cup_{\alpha \in A} X_\alpha) = X_\omega$$

. Since the  $X_\alpha$  are all open in their respective topologies, we have in particular that  $\cup_{\beta \neq \alpha} (\iota_\beta(X_\beta))$  is open, so we have a separation of  $X$  and the proof is complete.  $\square$