Category Theory Problem Sets and Solutions

Fall 2022

September 20, 2022

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1.1 Problem 4 - Orin Gotchey

1.1.1 Measurable Spaces as a Category

Definition 1.2. *σ-algebras Let X be a set. Let Ω be any subset of $\mathcal{P}(X)$ satisfying the following conditions:*

- $X \in \Omega_X$
- *For each $E \in \Omega$, $X \backslash E \in \Omega_X$*
- *For any index $I : \mathbb{N} \to \Omega_X$, $(\cup_{n \in \mathbb{N}} I(n)) \in \Omega_X$*

Ω is called a σ-algebra on X, the pair (X, Ω_X) a measurable space, the elements of Ω_X the measurable subsets of X.

It follows immediately that $\varnothing \in \Omega_X$, and that Ω_X is closed under countable intersection.

Definition 1.3. *Measurable MapsThe maps $f : X \to Y$ between measurable spaces which have the following property:*

$$\forall E \subset X : (f(E) \in \Sigma \implies E \in \Omega)$$

are called measurable maps or measurable functions.

Let `Meas` be the category specified as follows:

- Objects are the measurable spaces (X, Ω)
- Morphisms are the measurable functions.

Then, given any morphism $f : X \to Y$,

$$f \circ \texttt{Id}_X = f$$

$$\texttt{Id}_Y \circ f = f$$

Associativity follows from the fact that the composition of functions on the underlying sets is associative. Given two composable morphisms, say $f : X \to Y$ and $g : Y \to Z$, consider the composition $g \circ f : X \to Z$, and let $\gamma \in \Omega_Z$. Then:

$$g^{-1}(\gamma) \in \Omega_Y$$

$$f^{-1}(g^{-1}(\gamma)) = (g \circ f)^{-1}(\gamma) \in \Omega_X$$

. Thus, we have that `Meas` is a category.

1.3.1 Enhanced Measurable Spaces

Let (X, Ω_X) be a topological space. Then $\mathcal{P}(X)$ forms a Boolean commutative ring with the operations $\cap$ and $\triangle$ as multiplication and addition, respectively, and of which Ω_X is a subring. Define an *enhanced measurable space* as a triple (X, Ω_X, N_X), where (X, Ω_X) form a measurable space, and N_X is a σ-ideal of Ω_X (recall: a *σ-ideal* is an ideal which is closed under *countable* addition). A *negligible set* in X is some subset of N_X.
The *measurable maps* $f : (X, \Omega_X, N_X) \to (Y, \Omega_Y, N_Y)$ are maps of sets: $f : X_f \to Y$, where $X_f \subset X$, such that f obeys the following conditions (which are verified for the identity maps in the subpoints where $X = Y$):

1. The set $X \backslash X_f$ is negligible
    - $X = X_{\texttt{Id}_X}$ and $X \backslash X_{\texttt{Id}_X} = \varnothing \in N_X$, by definition of ideal.
2. For any $m_y \in \Omega_Y$, there exists a set m_x such that $f^{-1}(m_y) \triangle m_x$ is negligible
    - Given m_x, $\texttt{Id}_X^{-1}(m_x) \triangle m_X = m_x \triangle m_x = \varnothing \in N_X$
3. For any $n_y \in N_Y$, the set $f^{-1}(n_y)$ is negligible.
    - $\texttt{Id}_X(n_x) = n_x$

We cannot define composition of morphisms strictly as composition of underlying maps, because there is no guarantee, e.g., for two maps between enhanced measurable spaces $f : X \to Y$, $g : Y \to Z$, that $Imf \subset Y_g$. Thus, we restrict the domain of the composition to:

$$X_{g \circ f} := f^{-1}(Y_g)$$

. However, it is clear by inspection that composition of morphisms retains associativity. Then,

$$X \backslash X_{g \circ f} = X \backslash f^{-1}(Y_g) = (f^{-1}(Y \backslash (Y_g)))$$

The negligibility of the above quantity then follows from the definition of f.
Furthermore, given $m_z \in \Omega_Z$, we have that $(g \circ f)^{-1}(m_z) = f^{-1}(g^{-1}(m_z))$. Since g is measurable (*why?*) and since f is presumed to satisfy (2), $g \circ f$ satisfies (2).
(3) is clearly transitive.
Thus, enhanced measurable spaces and measurable maps form a category.

1.3.2 Equality Almost Everywhere

Two parallel morphisms $f, g : (X, \Omega_X, N_X) \to (Y, \Omega_Y, N_Y)$ are "equal almost everywhere" if the set $\{x \in X_f \cap X_G : f(x) \neq g(x)\}$ is negligible. Let "f and g are equal almost everywhere" be denoted $f \sim g$. Claim: $\sim$ defines an equivalence relation.

- Reflexivity: A function differs from itself on the empty set ($\varnothing$), which is negligible (see above)
- Symmetry: Note that the symbols f and g in the definition of equality almost everywhere are symmetric
- Transitivity: If $f \sim g$ and $g \sim h$ for parallel morphisms f, g, and h, then

  $$\{x \in X_f \cap X_h : f(x) \neq h(x)\} \subset (\{x \in X_f \cap X_g : f(x) \neq g(x)\} \cup \{x \in X_g \cap X_h : g(x) \neq h(x)\})$$

  , and N is closed under countable unions and taking subsets, so the left hand side of the above is negligible.

Furthermore, this equivalence relation is compatible with composition. Assume that there are morphisms $f, f' : X \to Y$ and $g, g' : Y \to Z$. such that $f \sim f'$ and $g \sim g'$. We're interested in the set

$$\begin{aligned}\{x \in X_{g \circ f} \cap X_{g' \circ f'} : (g \circ f)(x) \neq (g' \circ f')(x)\} \subset \{x \in X_f \cap X_{f'} : f(x) \neq f'(x)\} \\ \cup f^{-1}(\{y \in Y_g \cap Y_g' : g(y) \neq g'(y)\})\end{aligned}$$

This set is the union of two negligible sets.

1.3.3 Hom-sets mod an Equivalence Relation

Suppose that for every pair of objects X, Y in a category C, we are given (e.g. by the above) an equivalence relation $R_{X,Y}$ on $C(X, Y)$ that is compatible with composition (i.e. if $f \sim_R f'$ and $g \sim_R g'$ then $(g \circ f) \sim_R (g' \circ f')$. We identify all morphisms in C between any two objects X and Y which relate through $R_{X,Y}$. Composition of equivalence classes of $\sim$ does not depend on choice of representative: this is exactly compatibility with $\circ$
Verifying that the proper morphisms are unital and associative are gifted as simple exercises to the reader ;)

1.4 Problem 6 - Alan Bohnert

1.4.1 Question

Fix a category **C**. A *bimorphism* in **C** is a morphism f that is simultaneously a monomorphsim and an epimorphism. Is any isomorphism a bimorphism? Give and example of a category **C** and a bimorphism f in **C** that is not an isomorphism.

1.4.2 Solution

In any category **C** every isomorphism is a bimorphism.

Proof. Let $f : X \to Y$ be an isomorphism in **C**. Then there exists a morphism $g : Y \to X$ in **C** such that

$$gf = \mathrm{Id}_X \text{ and } fg = \mathrm{Id}_Y.$$

To show f is a monomorphism let $h, k : W \rightrightarrows X$ and $fh = fk$. It follows that $gfh = gfk$ for the g given above. Therefore $\mathrm{Id}_X h = \mathrm{Id}_X k$ and so $h = k$ tells us f is a monomorphism.

To show f is an epimorphism let $m, n : Y \rightrightarrows Z$ and $mf = nf$. Composing with the g we know $mfg = nfg$. Consequently $m\mathrm{Id}_Y = n\mathrm{Id}_Y$ and $m = n$ tells us f is an epimorphism. Therefore f is a bimorphism. ☐

Let **C** be the category **Ring** and let $f : \mathbb{Z} \hookrightarrow \mathbb{Q}$ the inclusion map. We claim f is a bimorphism but not an isomorphism.

Proof. To show f is a monomorphism let $h, k : W \rightrightarrows \mathbb{Z}$ and $f \circ h(w) = f \circ k(w)\ \forall w \in W$. Since f is injective

$$h(w) = f \circ h(w) = f \circ k(w) = k(w).$$

Therefore $h(w) = k(w)$ and f is a monomorphism.

To show f is an epimorphism let $m, n : \mathbb{Q} \rightrightarrows S$ such that $m \circ f(x) = n \circ f(x)\ \forall x \in \mathbb{Q}$. Since f is injective, we know $m(z) = n(z)\ \forall z \in \mathbb{Z}$. Seeking a contraction, suppose there exists $\frac{a}{b} \in \mathbb{Q}$ such that $m(\frac{a}{b}) \neq n(\frac{a}{b})$. Given m and n are ring homomorphisms we know

$$m(a)m(b^{-1}) = m(\tfrac{a}{b}) \neq n(\tfrac{a}{b}) = n(a)n(b^{-1}).$$

Given b is an invertible integer and $m(b) = n(b)$ we can multiply on the right and retain the inequality. Thus,

$$m(a)m(b^{-1})m(b) \neq n(a)n(b^{-1})n(b)$$

and as ring homomorphisms we have

$$m(a) = m(a)m(b^{-1}b) \neq n(a)n(b^{-1}b) = n(a).$$

Therefore f is a epimorphism.

To show f is not an isomorphism we note $\frac{1}{3} \in \mathbb{Q}$ has no preimage in $\mathbb{Z}$. □

2 Problem Set 2

Assignments

- Problem 1 - Orin Gotchey
- Problem 2 - James
- Problem 3 - Bradley
- Problem 4 - Alan
- Problem 5 - James
- Problem 6 - Emilio

2.1 Problem 1 - Orin Gotchey

Lemma 2.2. *Existence and Uniqueness of Borel σ-Algebras. Let X be a topological space. Then there exists a unique σ-algebra, Ω on X which contains all open subsets of X and which is the smallest among such σ-algebras with respect to inclusion.*

Proof. Let Σ be the collection of all σ-algebras on X which contain all open subsets of X. Σ contains $\mathcal{P}(X)$, and thus is nonempty. Let

$$\Omega := \cap_{x \in \Sigma} x$$

Clearly, $X \in \Omega$. Given an index $I : \mathbb{N} \to \Omega$, such that for every natural n, $I(n) \in \Omega$, we have that $I(n) \in x$, $\forall x \in \Sigma$, whence it follows that $\cap_{n \in \mathbb{N}} I(n) \in x$, $\forall x \in \Sigma$. Therefore, $\cap_{n \in \mathbb{N}} I(n) \in \Omega$. By a similar argument, for any $E \in \Omega$, $X \backslash E \in \Omega$. Thus, Ω contains all open subsets of X, and is indeed inferior to any other σ-algebra with this property. □

Definition 2.3. *A complex *-algebra A is a complex algebra, equipped with a complex-antilinear operation $*: A \to A$ obeying the following:*

$$(ab)^* = b^* a^*$$

$$1^* = 1$$

$$(a^*)^* = a$$

Definition 2.4. *A complex-valued morphism $f : X \to \mathbb{C}$ (on some topological space X) is called "bounded" if it factors through some bounded subset of $\mathbb{C}$. That is, there exists some subset $C \in \mathbb{C}$ which is contained in some open ball, and some map $\bar{f}$ which makes the following diagram commute:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, dashed, "\bar{f}"'] & \mathbb{C} \\ C \arrow[ur, "\iota"'] & \end{tikzcd}$$

Lemma 2.5. *Given an enhance measurable set (X, Ω_X, N_X), the set of all bounded morphisms $\{f : (X, \Omega_X, N_X) \to (\mathbb{C}, \Omega_C, \{\varnothing\})\}$ is a complex *-algebra.*

Proof. The zero morphism 0_X acts as the additive identity. Addition, multiplication, and involution are pointwise. Everything else follows by inspection. $\square$

Proposition 2.6. *Together with complex algebra homomorphisms: $f : A \to B$ satisfying $f(a^*) = f(a)^*$, and objects: commutative complex *-algebras, $\mathsf{CAlg}^*_{\mathbb{C}}$ is a category.*

Proof. Let $\forall a, b, c \in \mathsf{Obj}(\mathsf{CAlg}^*_{\mathbb{C}}),\ f \in \mathsf{CAlg}^*_{\mathbb{C}}(a, b),\ g \in \mathsf{CAlg}^*_{\mathbb{C}}(b, c)$ then:

- $\exists \mathtt{id}_a : a \to a$ given by $\mathtt{id}_a(x) = x$ satisfies $\mathtt{id}_a(x^*) = x^* = \mathtt{id}_a(x)^*$, and which is clearly a $\mathbb{C}$-algebra homomorphism
- $g \circ f$ satisfies $(g \circ f)(x)^* = g(f(x))^* = g(f(x)^*) = g(f(x^*)) = g \circ f(x^*)$, and is clearly a $\mathbb{C}$-algebra homomorphism.
- The composition of underlying sets is associative.

$\square$

Let $\mathsf{L}^\infty : \mathsf{PreEMS}^{op} \to \mathsf{CAlg}^*_{\mathbb{C}}$ send an enhanced measurable space to the complex *-algebra of bounded morphisms: $(X, \Omega_X, N_X) \mapsto (\mathrm{L}^\infty(X) : \{\phi : (X, \Omega_X, N_X) \to (\mathbb{C}, \Omega_{\mathbb{C}}, \{\varnothing\}) |\, \phi \,\text{bounded}\})$, and which sends an enhanced measurable morphism $f : (X, \Omega_X, N_X) \to (Y, \Omega_Y, N_Y)$ to

$$\mathsf{L}^\infty(f) : (\mathrm{L}^\infty(Y) : \{\phi : (Y, \Omega_Y, N_Y) \to (\mathbb{C}, \Omega_{\mathbb{C}}, N_{\mathbb{C}})\}) \to (\mathrm{L}^\infty(X) : \{\psi : (X, \Omega_X, N_X) \to (\mathbb{C}, \Omega_{\mathbb{C}}, N_{\mathbb{C}})\})$$

given by:

$$(\mathrm{L}^\infty(f))(\phi) = (\phi \circ f)$$

Proposition 2.7. L^∞ *is a contravariant functor*

Proof. We need to show the following:

1. $\mathrm{L}^\infty(f)$ defines a morphism in $\mathsf{CAlg}^*_{\mathbb{C}}$ i.e. a complex algebra homomorphism which respects involution.
2. L^∞ respects identity
3. L^∞ respects composition

For (1), given an $f : X \to Y$, $\phi,\ \psi \in \mathrm{L}^\infty(Y)$, and $c \in \mathbb{C}$

$$\begin{aligned}
&\mathrm{L}^\infty(f) : \mathrm{L}^\infty(Y) \to \mathrm{L}^\infty(X) \\
&\mathrm{L}^\infty(f)(0_Y) = 0_X \\
&\mathrm{L}^\infty(f)(\phi + \psi) = (\phi + \psi) \circ (f) = (\phi \circ f) + (\psi \circ f) = \mathrm{L}^\infty(f)(\phi) + \mathrm{L}^\infty(f)(\psi) \\
&\mathrm{L}^\infty(f)(\phi \cdot \psi) = (\phi \cdot \psi) \circ f = (\phi \circ f) \cdot (\psi \circ f) = \mathrm{L}^\infty(f)(\phi) \cdot \mathrm{L}^\infty(f)(\psi) \\
&c \cdot \mathrm{L}^\infty(f)(\phi) = c \cdot (\phi \circ f) = (c \cdot \phi) \circ f = \mathrm{L}^\infty(f)(c \cdot \phi) \\
&\mathrm{L}^\infty(f)(\phi^*) = (\phi^*) \circ f = (\phi \circ f)^* = \mathrm{L}^\infty(f)(\phi)^*
\end{aligned} \tag{1}$$

For (2),

$$\mathrm{L}^\infty(\mathtt{id}_X)(\phi) = (\phi \circ \mathtt{id}_X) = \phi \implies \mathrm{L}^\infty(\mathtt{id}_X) = \mathtt{id}_{\mathrm{L}^\infty(X)} \tag{2}$$

For (3), we give two morphisms $f : X \to Y$, $g : Y \to Z$ in PreEMS. Then for any $\phi \in \mathrm{L}^\infty(Z)$

$$\mathrm{L}^\infty(g \circ f)(\phi) = \phi \circ (g \circ f) = (\phi \circ g) \circ f = \mathrm{L}^\infty(f)(\phi \circ g) = \mathrm{L}^\infty(f)(\mathrm{L}^\infty(g)(\phi)) = (\mathrm{L}^\infty(f) \circ \mathrm{L}^\infty(g))(\phi) \tag{3}$$

□

Lemma 2.8. *Let C be a category with an equivalence relation R on its set of morphisms, and let F be some functor from C/R to another category D. Then precomposing with the functor $\mathsf{C} \xrightarrow{\Pi} \mathsf{C}/R$ gives a bijection between functors $\mathsf{C}/R \xrightarrow{F} \mathsf{D}$ and functors $\mathsf{C} \xrightarrow{G} \mathsf{D}$ with the property that $\alpha \sim_R \beta \Rightarrow G(\alpha) = G(\beta)$*

Proof. If $\alpha \sim_R \beta$, then $\Pi(\alpha) = \Pi(\beta)$, so $(F \circ \Pi)(\alpha) = (F \circ \Pi)(\beta)$. For the other direction, let $G : \mathsf{C} \to \mathsf{D}$ and R be given as above. Then, let $F : \mathsf{C}/R \to D$ be identical to G on objects, and let the image of an equivalence class of morhpisms in C/R be the image of any of its representatives under G. This is well-defined, because any two morphisms in the same equivalence class of R will have the same image under G. □

Definition 2.9. *Let R be an equivalence relation defined on the Hom-sets of PreEMS. Define the category $\mathsf{StrictEMS}$, whose objects are the same as PreEMS, and whose morphisms are equivalence classes of morphisms under R sharing domain and codomain.*

It follows from the foregoing observations that $\mathsf{StrictEMS}$ is, in fact, a category. Extend the L^∞ to a functor from $\mathsf{StrictEMS}$ to $\mathsf{CAlg}^*_{\mathbb{C}}$ as follows:

- Send an enhanced measurable space (X, Ω_X, N_X) to the complex *-algebra of bounded morphisms from X to $\mathbb{C}$
- Given a morphism $f : X \to Y$, consider the equivalence class of f in $\mathsf{StrictEMS}$.

2.10 Problem 4 - Alan

2.10.1 Construct a functor Ban^{op} to Ball

Objects in Ban^{op} are real (or complex) vector spaces with norms, and morphisms are $\mathbb{R}$ (or $\mathbb{C}$) linear maps:

$$(X, \|\cdot\|_X) \xrightarrow{f} (Y, \|\cdot\|_Y)$$

such that for all $x \in X$, $\|f(x)\|_Y \leq \|x\|_X$
Objects in Ball are pairs (V, B) where V is a Hausdorff, locally convex topological real (complex) vector space, and B a compact, convex, Hausdorff topological vector subspace of V, which is balanced. Here, "balanced" means:

$$0 \in B \wedge \forall x \in B \forall\, t \in \mathbb{R}(|t| \leq 1 \implies tx \in B)$$

For a given object $(X, \|\cdot\|_X)$ in Ban^{op}, X^* denotes the space of continuous linear functionals on X with the weak-$*$ topology and $X^*_{\leq 1}$ denotes the subspace of X^* consisting of functionals of norm at most 1. $(X^*, X^*_{\leq 1})$ is an object of Ball. Let $F : \mathsf{Ban}^{\mathrm{op}} \to \mathsf{Ball}$ be the functor which sends $(X, \|\cdot\|_X)$ to $(X^*, X^*_{\leq}1)$ and sends a morphism g from $X \to Y$ to $X^* \to Y^*$ to $F(g) = \circ g$. The functor F is encoded in the following commutative diagram:

$$
\begin{tikzcd}[column sep=huge]
(X, \| \cdot \|_X) \arrow[d, "g"', ""{name=G}] & (X^*, X^*_{\leq 1}) \\
(Y, \| \cdot \|_Y) \arrow[d, "h"', ""{name=H}] & (Y^*, Y^*_{\leq 1}) \arrow[u, "F(g)(y^*)=y^*\circ g"', ""{name=FG}] \\
(Z, \| \cdot \|_Z) & (Z^*, Z^*_{\leq 1}) \arrow[u, "F(h)(x^*)=x^*\circ h"', ""{name=FH}]
\arrow[from=G, to=FG, mapsto, "F(g)=-\circ g"']
\arrow[from=H, to=FH, mapsto, "F(h)=-\circ h"']
\end{tikzcd}
$$

From this diagram,

3 Problem Set 3

1. For which pairs of fields (of the same characteristic) does a categorical product exist?
2. Prove that the category of connected topological spaces does not have a product.
3. Prove: The category of Banach Spaces with *continuous maps* has no infinite coproducts.
4. Prove:
   - The category TOSet of totally ordered sets and order-preserving maps does not have coproducts.
   - What about the category. WOSet of well-ordered sets?
5. Prove: the category TopGrp of topological groups and continuous homomorphisms has coproducts.
6. Prove: the category of Lie groups (finite dimensional) does not have coproducts.
7. Investigate products and coproducts in the category PG of Hilbert spaces and contractive maps
8. Express limits in analysis (i.e. in a given metric space) as categorical limits

Assignments

- Problem 1 - Unassigned
- Problem 2 - Unassigned
- Problem 3 - Unassigned
- Problem 4 - JJ
- Problem 5 - Orin
- Problem 6 - Unassigned
- Problem 7 - Unassigned
- Problem 8 - Unassigned