

1 Does it work? Who knows?

Claim: The category \mathbf{Ban} does not coproducts in general

Proof. The problem is equivalent to showing that free objects don't generally exist in \mathbf{Ban} . Assume towards a contradiction that free objects do exist generally in \mathbf{Ban} . Then, in particular, there exists a functor $F : \mathbf{Set} \rightarrow \mathbf{Ban}$, which is left adjoint to the forgetful functor. Furthermore, for any set X , there exists a morphism (the unit of F ?) $i : X \rightarrow |F(X)|$, such that for any Banach space B and set morphism $\Phi : X \rightarrow |B|$, there exists a morphism in $g \in \mathbf{Ban}(F(X), B)$ making the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{i} & |F(X)| \\ & \searrow \Phi & \swarrow |g| \\ & & |B| \end{array}$$

Let $X = \mathbb{N}$, $B = \mathbb{R}$ (with norm the identity). Let $\|\cdot\|$ denote the norm on $F(X)$, and let $\Phi(n) := n\|i(n)\|$. Since g is a morphism in \mathbf{Ban} , it must be a bounded linear operator. In particular, there exists some constant $C \in \mathbb{R}$ such that, for every $x \in F(X)$, the following inequality holds:

$$|g(x)| \leq C\|x\|$$

Letting $x = i(n)$ for some $n \in \mathbb{N}$,

$$|g(i(n))| \leq C\|i(n)\|$$

$$\Phi(n) \leq C\|i(n)\|$$

$$n\|i(n)\| \leq C\|i(n)\|$$

$$n \leq C$$

This would have to be true for all $n \in \mathbb{N}$, which is absurd. □