1 Does it work? Who knows?

Claim: The category Ban *does not coproducts in general*

Proof. The problem is equivalent to showing that free objects don't generally exist in Ban. Assume towards a contradiction that free objects do exist generally in Ban. Then, in particular, there exists a functor $F : \mathsf{Set} \to \mathsf{Ban}$, which is left adjoint to the forgetful functor $|-|$. Furthermore, for any set X, there exists a morphism (the unit of F?) $i : X \to |F(X)|$, such that for any Banach space B and set morphism $\Phi : X \to |B|$, there exists a morphism in $g \in \mathsf{Ban}(F(X), B)$ making the following diagram commute:

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "\Phi"'] & & {|F(X)|} \arrow[dl, "|g|"] \\ & {|B|} & \end{tikzcd}$$

Let $X = \mathbb{N}$, $B = \mathbb{R}$ (with norm the identity). Let $||\cdot||$ denote the norm on $F(X)$, and let $\Phi(n) := n||i(n)||$. Since g is a morphism in Ban, it must be a bounded linear operator. In particular, there exists some constant $C \in \mathbb{R}$ such that, for every $x \in F(X)$, the following inequality holds:

$$|g(x)| \leq C||x||$$

Letting $x = i(n)$ for some $n \in \mathbb{N}$,

$$|g(i(n))| \leq C||i(n)||$$
$$\Phi(n) \leq C||i(n)||$$
$$n||i(n)|| \leq C||i(n)||$$
$$n \leq C$$

This would have to be true for all $n \in \mathbb{N}$, which is absurd. $\square$