Logic in Computer Science - Example

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1 Prove the following Theroems with nature deduction.

1.1 $\neg(p \land q) \dashv\vdash \neg q \lor \neg p$

Proof:
Left to right:

1	$\neg(p \land q)$	premise
2	$p \lor \neg p$	LEM
3	p	assumption
4	q	assumption
5	$p \land q$	$\land$i $3, 4$
6	$\bot$	$\neg$e $1, 5$
7	$\neg q$	$\neg$i $4 - 6$
8	$\neg q \lor \neg p$	$\lor$i$_1$ 7
9	$\neg p$	assumption
10	$\neg q \lor \neg p$	$\lor$i$_2$ 9
11	$\neg q \lor \neg p$	$\lor$e $2, 3 - 8, 9 - 10$

Right to left:

1	$\neg q \lor \neg p$	premise
2	$\neg q$	assumption
3	$p \land q$	assumption
4	q	$\land$e$_2$ 3
5	$\bot$	$\neg$e $2, 4$
6	$\neg(p \land q)$	$\neg$i $3 - 5$
7	$\neg p$	assumption
8	$p \land q$	assumption
9	p	$\land$e$_1$ 8
10	$\bot$	$\neg$e $7, 9$
11	$\neg(p \land q)$	$\neg$i $8 - 10$
12	$\neg(p \land q)$	$\lor$e $1, 2 - 6, 7 - 11$

So $\neg(p \land q) \dashv\vdash \neg q \lor \neg p$.

1.2 $p \to q \dashv\vdash \neg q \to \neg p$

Proof:
Left to right:

1	$p \to q$	premise
2	$\neg q$	assumption
3	$\neg p$	MT $1, 2$
4	$\neg q \to \neg p$	$\to$i $2-3$

Right to left:

1	$\neg q \to \neg p$	premise
2	p	assumption
3	$\neg\neg p$	$\neg\neg$i 2
4	$\neg\neg q$	MT $1, 3$
5	q	$\neg\neg$e 4
6	$p \to q$	$\to$i $2-5$

So $p \to q \dashv\vdash \neg q \to \neg p$.

1.3 $p \land q \to p \dashv\vdash r \lor \neg r$

Proof:
Left to right:

1	$r \lor \neg r$	LEM

Right to left:

1	$p \land q$	assumption
2	p	$\land$e$_1$ 1
3	$p \land q \to p$	$\to$i $1-2$

So $p \land q \to p \dashv\vdash r \lor \neg r$.

2 Prove the following theroems with deduction rules.

2.1 $\forall x(P(x) \to \neg Q(x)) \vdash \neg(\exists x(P(x) \land Q(x)))$

Proof:

	1	$\forall x(P(x) \to \neg Q(x))$	premise
	2	$\exists x(P(x) \land Q(x))$	assumption
x_0	3	$P(x_0) \land Q(x_0)$	assumption
	4	$P(x_0)$	$\land$e$_1$ 3
	5	$Q(x_0)$	$\land$e$_2$ 3
	6	$P(x_0) \to \neg Q(x_0)$	$\forall x$ e 1
	7	$\neg Q(x_0)$	$\to$ e $4, 6$
	8	$\perp$	$\neg$e $5, 7$
	9	$\perp$	$\exists x$ e $2, 3-8$
	10	$\neg(\exists(P(x) \land Q(x)))$	$\neg$i $2-9$

2.2 $\forall x(P(x) \leftrightarrow x = b) \vdash P(b) \wedge \forall x \forall y(P(x) \wedge P(y) \rightarrow x = y)$

Proof:

	1	$\forall x(P(x) \leftrightarrow x = b)$	premise
	2	$b = b$	$=$ i
	3	$b = b \rightarrow P(b)$	$\forall x$ e 1
	4	$P(b)$	$\rightarrow$ e $2, 3$
x_0	5	$P(x_0) \rightarrow x_0 = b$	$\forall x$ e 1
y_0	6	$P(y_0) \rightarrow y_0 = b$	$\forall x$ e 1
	7	$P(x_0) \wedge P(y_0)$	assumption
	8	$P(x_0)$	$\wedge e_1$ 7
	9	$P(y_0)$	$\wedge e_2$ 7
	10	$x_0 = b$	$\rightarrow$ e $5, 8$
	11	$y_0 = b$	$\rightarrow$ e $6, 9$
	12	$x_0 = y_0$	$=$ e $10, 11$
	13	$P(x_0) \wedge P(y_0) \rightarrow x_0 = y_0$	$\rightarrow$ i $7 - 12$
	14	$\forall y(P(x_0) \wedge P(y) \rightarrow x_0 = y)$	$\forall y$ i $6 - 13$
	15	$\forall x \forall y(P(x) \wedge P(y) \rightarrow x = y)$	$\forall x$ i $5 - 14$
	16	$P(b) \wedge \forall x \forall y(P(x) \wedge P(y) \rightarrow x = y)$	$\wedge$i $4, 15$

3 Prove the following theroems with deduction rules.

3.1 $\forall x(P(x) \rightarrow \neg Q(x)) \vdash \neg(\exists x(P(x) \wedge Q(x)))$

Proof:

	1	$\forall x(P(x) \rightarrow \neg Q(x))$	premise
	2	$\exists x(P(x) \wedge Q(x))$	assumption
x_0	3	$P(x_0) \wedge Q(x_0)$	assumption
	4	$P(x_0)$	$\wedge e_1$ 3
	5	$Q(x_0)$	$\wedge e_2$ 3
	6	$P(x_0) \rightarrow \neg Q(x_0)$	$\forall x$ e 1
	7	$\neg Q(x_0)$	$\rightarrow$ e $4, 6$
	8	$\bot$	$\neg$e $5, 7$
	9	$\bot$	$\exists x$ e $2, 3 - 8$
	10	$\neg(\exists(P(x) \wedge Q(x)))$	$\neg$i $2 - 9$

3.2 $\forall x(P(x) \leftrightarrow x = b) \vdash P(b) \wedge \forall x \forall y(P(x) \wedge P(y) \to x = y)$

Proof:

	1	$\forall x(P(x) \leftrightarrow x = b)$	premise
	2	$b = b$	$=$ i
	3	$b = b \to P(b)$	$\forall x$ e 1
	4	$P(b)$	$\to$ e $2, 3$
x_0	5	$P(x_0) \to x_0 = b$	$\forall x$ e 1
y_0	6	$P(y_0) \to y_0 = b$	$\forall x$ e 1
	7	$P(x_0) \wedge P(y_0)$	assumption
	8	$P(x_0)$	$\wedge$e$_1$ 7
	9	$P(y_0)$	$\wedge$e$_2$ 7
	10	$x_0 = b$	$\to$ e $5, 8$
	11	$y_0 = b$	$\to$ e $6, 9$
	12	$x_0 = y_0$	$=$ e $10, 11$
	13	$P(x_0) \wedge P(y_0) \to x_0 = y_0$	$\to$ i $7 - 12$
	14	$\forall y(P(x_0) \wedge P(y) \to x_0 = y)$	$\forall y$ i $6 - 13$
	15	$\forall x \forall y(P(x) \wedge P(y) \to x = y)$	$\forall x$ i $5 - 14$
	16	$P(b) \wedge \forall x \forall y(P(x) \wedge P(y) \to x = y)$	$\wedge$i $4, 15$